Fisher-Kolmogorov equations for neurodegenerative diseases

Andrea Boin, Giacomo Pauletti, Lorenzo Pettenuzzo

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1 Introduction

The objective of this project is to apply various numerical methods to solve the Fisher-Kolmogorov equations to reproduce the results of the paper [1]. The Fisher-Kolmogorov equations can be used to effectively model the spread of misfolded proteins in the brain, a process associated with numerous neurodegenerative diseases.

1.1 Fisher-Kolmogorov equation

$$\begin{cases} \dfrac{\partial c}{\partial t} - \nabla \cdot (D \nabla c) - \alpha c(1-c) = 0 & \text{in } \Omega \\ D \nabla c \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ c(t=0) = c_0 & \text{in } \Omega \end{cases}$$

c: concentration of the misfolded protein in a zone of the brain ($0 \leq c \leq 1$)
α: constant of concentration growth
D: diffusion coefficient of the misfolded protein.
It can be isotropic (a scalar) or anisotropic (a square matrix).
In case of anisotropic coefficient the term can be computed as:

$$\underline{\mathrm{D}} = d^{\text{ext}} \underline{\mathrm{I}} + d^{\text{axn}} (\mathbf{n} \otimes \mathbf{n})$$

where d^{ext} is the extracellular diffusion term, d^{axn} is the axonal diffusion term and $\mathbf{n}$ the direction of axonal diffusion.
Usually extracellular diffusion is slower than axonal diffusion: $d^{\text{ext}} < d^{\text{axn}}$.
The Fisher-Kolmogorov equation is a **diffusion-reaction** equation with a nonlinear forcing term that can be used to model population growth. In this case it is used to model the spreading of proteins in the brain.
The interested problem is a **nonlinear parabolic PDE** with **Neumann boundary conditions**.

1.2 Mesh

The mesh we used for the simulation is a 3D representation of a hemisphere of the human brain with 21211 points and 42450 cells.
To process the mesh with our software, we did convert the format from *.stl* to *.msh* using **GMSH** with the following procedure:

1. Import the mesh (*.stl*) in GMSH
2. From the left menu, select "geometry → add → volume"
3. Save the new generated *.geo* file
4. Define the 3D mesh: "mesh → define → 3D"
5. Export the file as *.msh*: "file → export → msh"

2 Methods

The weak formulation of the problem is:

$$\int_\Omega \frac{\delta c}{\delta t} v d\Omega + \int_\Omega D \nabla c \nabla v d\Omega - \int_\Omega \alpha c(1-c) v d\Omega = 0$$

We used two methods to discretize the Fisher-Kolmogorov equations:

- A **linearly implicit** scheme in which the linear terms have been treated implicitly while the nonlinear terms explicitly to get rid of nonlinarities.
- A **fully implicit** scheme in which all terms in the equation have been treated implicitly, and then the nonlinear parts have been solved with the Newton method.

2.1 Linearly implicit

2.1.1 Stability and Accuracy

2.2 Fully implicit

2.2.1 Stability and Accuracy

Stability and accuracy analysis of the implemented methods.

3 Results and algorithmic comparation

Results of the project for all the implemented algorithms and graphs.

References

[1] J. Weickenmeier, M. Jucker, A. Goriely, and E. Kuhl. A physics-based model explains the prion-like features of neurodegeneration in Alzheimer's disease, Parkinson's disease, and amyotrophic lateral sclerosis. Journal of the Mechanics and Physics of Solids, 124:264–281, 2019.