Heston

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Let $(\Omega, \mathbb{F} = \{(F_t)_{t \geq 0}\}, \mathbb{P})$ a complete filtered probability space and let call $\mathbb{P}$ the *physical-measure*. Let $T < \infty$ be the right limit of our time horizon from now on.

1 The Model

Given a stock price process $S = (S_t)_{t \geq 0}$ the Heston model, under $\mathbb{P}$, is the following :

$$\begin{cases} \mathrm{d}S_t = \mu S_t \mathrm{d}t + S_t \sqrt{v_t} \mathrm{d}W_t \\ \mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \theta\sqrt{v_t}\mathrm{d}\tilde{W}_t \\ v_0 = \sigma_0^2 \end{cases}$$

where:

- μ is the drift of the stock returns;
- $W = (W_t)_{t \geq 0}$ and $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$ are two correlated Brownian motions with $\mathrm{d}\langle W, \tilde{W} \rangle_t = \rho \mathrm{d}t$ and $\rho \in [-1, 1]$;
- $\sigma_0 > 0$ is the initial volatility;
- $\eta > 0$ is the long run variance;
- $\kappa > 0$ is the mean reversion rate;
- $\theta > 0$ is the volatility of the variance.

The variance process is strictly positive if $2\kappa\eta > \theta^2$. This is known as *Feller condition*.

2 The Valuation Equation

Consider two independent claims which prices, at time t, are given as $U_t = u(t, S_t, v_t)$ and $\tilde{U}_t = \tilde{u}(t, S_t, v_t)$. Suppose that u and $\tilde{u}$ are $\mathcal{C}^1(\mathbb{R}^+)$ w.r.t. the first variable and $\mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^+)$ w.r.t. the last two variables. Consider a portfolio consisting of a long contract U, short Δ shares of the stock and short Δ_1 contracts of $\tilde{U}$. So the value (denoted with Π_t) of our portfolio at time t is equal to:

$$\Pi_t = U_t - \Delta S_t - \Delta_1 \tilde{U}_t$$

then we can apply Itô's lemma and write the dynamics, omitting some subscripts, of the value of the portfolio as:

$$\begin{aligned}
\mathrm{d}\Pi_t = & \left\{ \frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\theta v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 U}{\partial v^2} \right\} \mathrm{d}t \\
& + \left\{ \frac{\partial \tilde{U}}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \tilde{U}}{\partial S^2} + \rho\theta v S \frac{\partial^2 \tilde{U}}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 \tilde{U}}{\partial v^2} \right\} \mathrm{d}t \\
& + \left\{ \frac{\partial U}{\partial S} - \Delta_1 \frac{\partial \tilde{U}}{\partial S} - \Delta \right\} \mathrm{d}S \\
& + \left\{ \frac{\partial U}{\partial v} - \Delta_1 \frac{\partial \tilde{U}}{\partial v} \right\} \mathrm{d}v
\end{aligned}$$

to make our portfolio instantaneously risk-free we must impose the following equations to eliminate the $\mathrm{d}S$ and $\mathrm{d}v$ terms:

$$\begin{cases} \dfrac{\partial U}{\partial S} - \Delta_1 \dfrac{\partial \tilde{U}}{\partial S} - \Delta = 0 \\ \dfrac{\partial U}{\partial v} - \Delta_1 \dfrac{\partial \tilde{U}}{\partial v} = 0 \end{cases} \tag{1}$$

since our portfolio is now risk-free its value must be equal to the value of a portfolio with the same initial value invested in the cash market at risk-free rate r, so:

$$\mathrm{d}\Pi_t = r\Pi_t \mathrm{d}t = r(U_t - \Delta S_t - \Delta_1 \tilde{U}_t)\mathrm{d}t$$

then choosing Δ and Δ_1 as in (1) and collecting all the U terms on the LHS and all the $\tilde{U}$ terms on the RHS we obtain (omitting once again some subscripts):

$$\begin{aligned}
& \frac{\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\theta v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 U}{\partial v^2} + rS \frac{\partial U}{\partial S} - rU}{\frac{\partial U}{\partial v}} \\
& \qquad = \frac{\frac{\partial \tilde{U}}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \tilde{U}}{\partial S^2} + \rho\theta v S \frac{\partial^2 \tilde{U}}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 \tilde{U}}{\partial v^2} + rS \frac{\partial \tilde{U}}{\partial S} - r\tilde{U}}{\frac{\partial \tilde{U}}{\partial v}}
\end{aligned}$$

now since the LHS depends on U and the RHS depends on $\tilde{U}$ then the only way for which this is possible is if they are equal to some function $f = f(t, S_t, v_t)$ which is independent from U and $\tilde{U}$. So we obtain:

$$\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\theta v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 U}{\partial v^2} + rS \frac{\partial U}{\partial S} - rU = -f \frac{\partial U}{\partial v} \tag{2}$$

Equation (2) is called the valuation equation for U. WLOG we can rewrite f as:

$$f(t, S_t, v_t) = \kappa(\eta - v_t) - \sqrt{v_t}\phi(t, S_t, v_t)$$

where ϕ_t is some arbitrary $\mathbb{F}$-adapted function and it is called the *market price of volatility risk*.

NOTE: the stock S is assumed to have yield equals to zero. If the stock in consideration has yield q we can simply switch r with $(r-q)$ in the calculations before.

2.1 The market price of volatility risk

Without assuming that there are two traded independent claims with one of which is dependent from the variance process we cannot a priori fix an $\mathbb{F}$-adapted ϕ. Indeed, we can rewrite $\tilde{W} = \rho W + \sqrt{1-\rho^2} W^{\perp}$ where $W^{\perp}$ is a continuous Brownian motion on the filtration $\mathcal{F}$ such that $\mathrm{d}\langle W, W^{\perp}\rangle = 0$. Then we can rewrite the Heston model under $\mathbb{P}$ as:

$$\begin{cases} \mathrm{d}S_t = \mu S_t \mathrm{d}t + S_t\sqrt{v_t}\mathrm{d}W_t \\ \mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \theta\rho\sqrt{v_t}\mathrm{d}W_t + \theta\sqrt{1-\rho^2}\sqrt{v_t}\mathrm{d}W_t^{\perp} \end{cases}$$

now take $\lambda_t = \lambda(t, S_t, v_t)$ defined as:

$$\lambda_t := \frac{\mu - r}{\sqrt{v_t}}$$

and take ϕ_t $\mathbb{F}$-adapted such that the Novikov's condition is satisfied, i.e.;

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T [\lambda_t^2 + \phi_t^2]\mathrm{d}t\right)\right] < \infty$$

then we have that the process $Z = (Z_t)_{t\in[0,T]}$defined as:

$$Z_t = \mathcal{E}(-\lambda_t W - \phi_t W^{\perp})_t$$

where $\mathcal{E}$ is the Doléans-Dade exponential is a $\mathbb{P}$-martingale, with $E[Z_T] = 1$ and strictly positive. So, using the Radon-Nikodym's theorem we can define a measure $\mathbb{Q}$ on $\mathbb{F}$ as:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{F_t} := Z_t$$

and we have that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on F_T and moreover is martingale equivalent measure. By Girsanov's theorem we can define a 2-dimensional $\mathbb{Q}$-Brownian motion $(W^{\mathbb{Q}}, W^{\mathbb{Q},\perp})$ as:

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \lambda_s \mathrm{d}s \;\text{ and }\; W_t^{\mathbb{Q},\perp} = W_t + \int_0^t \phi_s \mathrm{d}s$$

and under such a $\mathbb{Q}$ we have that the Heston model has the following dynamics:

$$\begin{cases} \mathrm{d}S_t = rS_t\mathrm{d}t + S_t\sqrt{v_t}\mathrm{d}W_t^{\mathbb{Q}} \\ \mathrm{d}v_t = [\kappa(\eta - v_t) - \theta\sqrt{v_t}(\rho\lambda_t + \sqrt{1-\rho^2}\phi_t)]\mathrm{d}t + \theta\sqrt{v_t}\mathrm{d}\tilde{W}_t^{\mathbb{Q}} \\ v_0 = \sigma_0^2 \end{cases}$$

and under this we have that the discounted stock price is a $\mathbb{Q}$ local martingale. However, we can choose any arbitrary ϕ and obtain another $\tilde{\mathbb{Q}}$ with the same properties. But, as in our case, both the claims are actively traded we can infer the ϕ from the market prices of the options, which will determine a $\mathbb{Q}$ and then fix the ϕ. Among all the possible choices from now on we will choose $\phi = 0$. The resulting $\mathbb{Q}^M$ is called the *minimal martingale measure* and Föllmer and Schweizer have proven that, for models with continuous price trajectories, solves the minimization problem:

$$\mathbb{Q}^M = \arg\min_{\mathbb{Q}\in\mathcal{M}} \mathbb{H}(\mathbb{Q}|\mathbb{P})$$

where $\mathcal{M}$ is the set of equivalent martingale measures and $\mathbb{H}$ is the reverse relative entropy. Indeed this is exactly what we obtain if we can completely infer the ϕ from the market prices of the options (because we think that $\mathbb{P}$ contains all the information to reconstruct $\mathbb{Q}$).

NOTE: From now on, to lighten the notation, our Heston model under $\mathbb{Q}$ is written in the following way:

$$\begin{cases} \mathrm{d}S_t = (r-q)S_t\mathrm{d}t + S_t\sqrt{v_t}\mathrm{d}W_t \\ \mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \theta\sqrt{v_t}\mathrm{d}\tilde{W}_t \\ v_0 = \sigma_0^2 \end{cases}$$

with $W = (W_t)_{t\geq 0}$ and $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ are two correlated $\mathbb{Q}$ Brownian motions with $\mathrm{d}\langle W, \tilde{W}\rangle_t = \rho\mathrm{d}t$.

3 The Characteristic Formula

3.1 Derivation of the pseudo-probabilities

Setting the $\phi = 0$ we have that the valuation equation for our model, to price a call option C, becomes:

$$\frac{\partial C}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 C}{\partial S^2} + \rho\theta vS\frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\theta^2 v\frac{\partial^2 C}{\partial v^2} + rS\frac{\partial C}{\partial S} - rC + [\kappa(\eta - v)]\frac{\partial C}{\partial v} = 0$$

if we now substitute with $\tau = T - t$ and $x = \log\left(\frac{S_t e^{(r-q)\tau}}{K}\right)$ in the previous equation we obtain:

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2}v\frac{\partial^2 C}{\partial x^2} + \rho\theta v\frac{\partial^2 C}{\partial x\partial v} + \frac{1}{2}\theta^2 v\frac{\partial^2 C}{\partial v^2} - \frac{1}{2}v\frac{\partial C}{\partial x} + [\kappa(\eta - v)]\frac{\partial C}{\partial v} = 0 \quad (3)$$

according to Duffie, Pan and Singleton (2000), the solution to this equation has the form of:

$$C(x, v, \tau) = K\left\{e^x P_1(x, v, \tau) - P_0(x, v, \tau)\right\} \quad (4)$$

notice how this remind us to the solution of the Black&Scholes' Equation. Moreover, the P_j are absolutely continuous, differentiable and both P_j and P_j' are absolutely integrable on $\mathbb{R}$. Putting (4) into (3) we obtain that P_j with $j = 0, 1$ must satisfy the following PDE:

$$-\frac{\partial P_j}{\partial \tau}+\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}-\left(\frac{1}{2}-j\right)v\frac{\partial P_j}{\partial x}+\frac{1}{2}\theta^2 v\frac{\partial^2 P_j}{\partial v^2}+\rho\theta v\frac{\partial^2 P_j}{\partial x \partial v}+[\kappa(\eta+1)-j\rho\theta v]\frac{\partial P_j}{\partial v}=0$$

with boundary conditions:

$$\lim_{\tau \to 0} P_j(x, v, \tau) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Now take the Fourier transform of P_j as:

$$\hat{P}_j(u, v, \tau) = \int_{\mathbb{R}} e^{-iux} P_j(x, v, \tau)\mathrm{d}x \quad \text{then} \quad \hat{P}_j(u, v, 0) = \frac{1}{iu}$$

and the inverse transform given by:

$$P_j(x, v, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \hat{P}_j(u, v, \tau)\mathrm{d}u$$

thanks to the regularity of P_j we have that $[\widehat{P_j(u)'}] = iu\hat{P}_j(u)$. So, using this property, we can rewrite our equation as:

$$-\left[\frac{1}{2}u^2-\left(\frac{1}{2}-j\right)iu\right]v\hat{P}_j+\left\{\rho\theta iuv+[\kappa(\eta+1)-j\rho\theta v]\right\}\frac{\partial \hat{P}_j}{\partial v}+\frac{1}{2}\theta^2 v\frac{\partial^2 \hat{P}_j}{\partial v^2}=\frac{\partial \hat{P}_j}{\partial \tau}$$

now take two arbitrary functions $f(u, \tau)$, $g(u, \tau)$. The solution should be of the form:

$$\hat{P}_j(u, v, \tau) = \frac{1}{iu} \exp\{f(u, \tau)\eta + g(u, \tau)v\}$$

it follows that:

$$\begin{aligned}
\frac{\partial \hat{P}_j}{\partial \tau} &= \left\{\eta\frac{\partial f}{\partial \tau} + v\frac{\partial g}{\partial \tau}\right\}\hat{P}_j \\
\frac{\partial \hat{P}_j}{\partial v} &= g\hat{P}_j \\
\frac{\partial^2 \hat{P}_j}{\partial v^2} &= g^2\hat{P}_j
\end{aligned}$$

now define:

$$\begin{aligned}
\alpha &= -\frac{u^2}{2} - \frac{iu}{2} + iju \\
\beta &= \kappa - \rho\theta(j + iu) \\
\gamma &= \frac{\theta^2}{2}
\end{aligned}$$

then it must be:

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= \kappa g\\ \frac{\partial g}{\partial \tau} &= \alpha - \beta g - \gamma g^2\\ &= \gamma(g - r_+)(g - r_-)\end{aligned}$$

where $r_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} =: \frac{\beta \pm d}{\theta^2}$. Integrating and putting the terminal condition $f(u,0) = 0$ and $g(u,0) = 0$, we obtain:

$$\begin{aligned} f(u,\tau) &= \kappa \left\{ r_- \tau - \frac{2}{\theta^2} \log\left(\frac{1 - \frac{r_-}{r_+} e^{-d\tau}}{1 - \frac{r_-}{r_+}} \right) \right\} \\ g(u,\tau) &= r_- \frac{1 - e^{-d\tau}}{1 - \frac{r_-}{r_+} e^{-d\tau}} \end{aligned} \tag{5}$$

Taking the inverse transform on $\hat{P}_j$ we obtain finally the form of the pseudo probabilities as:

$$P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{\exp\{f_j(u,\tau)\eta + g_j(u,\tau)v + iux\}}{iu} \right\} \mathrm{d}u \tag{6}$$

3.2 Numerical integration of the complex logarithm

In (5) we decided to take r_- to define f, but we could have taken r_+ and everything would have been "almost" the same (this definition coincides with our previous one only if the imaginary part of the complex logarithm is chosen so that $f(u,\tau)$ is continuous with respect to u). However, we are interested in integrating numerically the characteristic function of the Heston model. We have to keep in mind that the complex logarithm has a branch (we decided to take as branch the semiaxis of negative real numbers in the complex plane), so in order to avoid any kind of discontinuity we want that the characteristic function never crosses the negative real axis on $(0,\infty)$. Albrecher, Mayer, Schoutens, Tistaert (2006) proved the following result when using the FFT-like approach:

Proposition 1. *Whenever the parameters of the Heston model are such that* $\Im\{d(u)\} := \Im\left\{\sqrt{(\rho\theta u i - \kappa)^2 + \theta^2(iu + u^2)}\right\} \neq 0$ *and* $2\kappa\eta \neq \theta^2 n$ *(where* $n \in \mathbb{N}$*), then defining (5) using* r_+ *leads to numerical instabilities for sufficiently large maturities.*

In order to use methods which leverage the frequency domain we usually have to evaluate the characteristic function in $u - (\alpha+1)i$ for positive u. So they also proved the following proposition:

Proposition 2. *Denote with* $\phi(u)$ *the characteristic function of the Heston model obtained using* r_- *in (5). Then* $\forall \alpha > 0$ *and* $\forall u \in (0,\infty)$ *the function* $\phi(u - (\alpha+1)i)$ *does not cross the negative real axis.*

In conclusion if we use r_- in (5) then the characteristic function that we will obtain in the next paragraph is suitable for numerical integration.

3.3 Derivation of the characteristic function

By definition the characteristic function under $\mathbb{Q}$ is defined as:

$$\phi(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iux_T}\middle| x_t = \log\left(\frac{S_t e^{(r-q)\tau}}{K}\right); v_t\right]$$

now thanks to (6) we know that:

$$\mathbf{Pr}_{\mathbb{Q}}(x_T > x_t) = P_0(x_t, v_t, \tau)$$

so, if we define $k = -x_t$ the density is:

$$\begin{aligned} p(k) &= -\frac{\partial P_0}{\partial k} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left\{f_0(s,\tau)\eta + g_0(s,\tau)v_t - isk\right\} \mathrm{d}s \end{aligned}$$

then we have that, using Fubini theorem:

$$\begin{aligned} \phi(u,\tau;x_t,v_t) &= \int_{\mathbb{R}} e^{iuk+iux_t} p(k)\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left\{f_0(s,\tau)\eta + g_0(s,\tau)v_t\right\} e^{iuk} e^{-isk} e^{iux_t} \mathrm{d}s\mathrm{d}k \\ &= \frac{e^{iux_t}}{2\pi}\int_{\mathbb{R}} \exp\left\{f_0(s,\tau)\eta + g_0(s,\tau)v_t\right\} \int_{\mathbb{R}} e^{ik(u-s)}\mathrm{d}k\mathrm{d}s \\ &= e^{iux_t}\int_{\mathbb{R}} \exp\left\{f_0(s,\tau)\eta + g_0(s,\tau)v_t\right\} \delta(u-s)\mathrm{d}s \\ &= \exp\left\{f_0(u,\tau)\eta + g_0(u,\tau)v_t + iu\log\left[S_t e^{(r-q)\tau}\right] - iu\log(K)\right\} \end{aligned}$$

now, for the next section we will rewrite the characteristic function in terms of the log-price (which with a little abuse of notation we will denote with x_t):

$$\phi(u,\tau;x_t,v_t) = \exp\left\{f_0(u,\tau)\eta + g_0(u,\tau)v_t + iux_t + iu(r-q)\tau\right\}$$

4 Fourier Cosine Expansion for Vanilla Options

In order to calibrate our model we need a way to price vanilla options (calls & puts) which usually require integrating the probability density function. However, since we have its Fourier transform (the characteristic function) we can leverage the *Fourier Cosine Expansion* method developed by F. Fang & C.W. Osterlee. The computational speed, especially for plain vanilla options, makes this integration method state-of-the-art for calibration at financial institutions.