rBergomi

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The rough Bergomi (rBergomi) is a Rough Fractional Stochastic Volatility (RFSV) model. The model has only three, time-independent, parameters and is able to replicate accurately the implied volatility surface dynamics. We will see that we need to use simulation methods to generate option prices as there is no closed form solution and the non-Markovian property of the model doesn't allow a PDE approach. From now on let $(\Omega, \mathbb{F} = \{(F_t)_{t\geq 0}\}, \mathbb{P})$ be a complete filtered probability space, let $\mathbb{P}$ be the *physical-measure* and $T < \infty$ be the right limit of our time horizon.

1 The Model

The rBergomi model is know as a market model, that is to say a financial model consistent with market data. The idea of Bergomi, proposed in [1], is to model the dynamics of the forward variance instead of modelling instantaneous volatility. We will denote the forward variance curve observed at time t with maturity T as $\xi_t(T)$. The forward variance curve observed at time t with maturity T is associated with the fair strike of a variance swap, observed in the same instant t and with the same maturity T, that we will denote as $\sigma_t^2(T)$:

$$\sigma_t^2(T) = \frac{1}{T-t}\int_t^T \xi_t(u)\mathrm{d}u$$

equivalently we have:

$$\xi_t(T) = \frac{\mathrm{d}}{\mathrm{d}T}\big[(T-t)\sigma_t^2(T)\big]$$

1.1 N-Factor Model

In a general N-dimensional setting dictated by the N-dimensional Brownian motion $(W_t^i)_{i=1}^N$ the forward variance $\xi_t(T)$ dynamics are governed by the following SDE:

$$\mathrm{d}\xi_t(u) = \frac{\omega}{\sqrt{\sum_{i,j=1}^N \omega_i\omega_j\rho_{i,j}}}\xi_t(u)\sum_{i=1}^N \omega_i e^{k_i(u-t)}\mathrm{d}W_t^i \tag{1.1}$$

where we have $\mathrm{d}[W_t^i W_t^j]_t = \rho_{i,j}\mathrm{d}t$ and $\omega_i, k_i > 0$. We also have that $\omega > 0$ is the instantaneous volatility of $\xi_t(t)$. In this general setting the solution is given by:

$$\xi_t(T) = \xi_0(T) \exp\left\{\omega \sum_{i=1}^{N} \omega_i e^{-k_i(T-t)} X_t^i - \frac{\omega^2}{2} \sum_{i,j=1}^{N} \omega_i \omega_j e^{-(k_i+k_j)(T-t)} \mathbb{E}\big[X_t^i X_t^j\big]\right\}$$

where the N driven Ornstein-Uhlenbeck (OU) processes $(X_t^i)_{t\geq 0}$ are defined by:

$$\begin{cases} \mathrm{d}X_t^i = -k_i X_t^i \mathrm{d}t + \mathrm{d}W_t^i \\ X_0^i = 0 \end{cases}$$

The instantaneous volatility of $\xi_t(T)$, thanks to (1.1), is:

$$\omega(T-t) = \frac{2\nu}{\sqrt{\sum_{i,j} \omega_i \omega_j \rho_{i,j}}} \sqrt{\sum_{i,j} \omega_i \omega_j \rho_{i,j} e^{-(k_i+k_j)(T-t)}}$$

Where ν is the log-normal volatility of a Variance Swap with vanishing maturity which can be related to the instantaneous volatility of $\xi_t(t)$ by:

$$\omega = 2\nu$$

1.2 One-Factor Model

Now we will restrict our analysis to the mono-dimensional case that is dictated by the Brownian motion $(W_t)_{t\geq 0}$. Now we have, for the forward variance curve, the following dynamics:

$$\mathrm{d}\xi_t(T) = \omega e^{-k(T-t)} \xi_t(T) \mathrm{d}W_t \qquad (1.2)$$

The choice of an exponential decaying volatility function is equivalent to letting an OU process $(X_t)_{t\geq 0}$ dictate the dynamics of the forward variances. The process X_t has to satisfy the following SDE system:

$$\begin{cases} \mathrm{d}X_t = -k X_t \mathrm{d}t + \mathrm{d}W_t \\ X_0 = 0 \end{cases}$$

We can solve the system and find the solution:

$$X_t = \int_0^t e^{-k(t-s)} \mathrm{d}W_s$$

We can also calculate its expected value, its variance and the expected value of the square of the process:

$$\mathbb{E}[X_t] = 0 \qquad \mathbb{V}[X_t] = \frac{1-e^{-2k}}{2k} \qquad \mathbb{E}[X_t^2] = \mathbb{V}[X_t] = \frac{1-e^{-2k}}{2k}$$

Then the solution to the (1.2) SDE is:

$$\xi_t(T) = \xi_0(T)\exp\left\{\omega e^{-k(T-t)}X_t - \frac{\omega^2}{2}e^{-2k(T-t)}\mathbb{E}[X_t^2]\right\}$$

This model is still not flexible enough.

1.3 2-Factors Model

Here we will present the 2-factors model in which we can achieve greater flexibility in the term-structure of volatilities of variances that can be generated. We will use a mixing parameter $\theta \in [0,1]$ and the dynamics become:

$$\begin{cases} \mathrm{d}\xi_t(T) = \omega\alpha_\theta\xi_t(T)\big[(1-\theta)e^{-k_1(T-t)}\mathrm{d}W_t^1 + \theta e^{-k_2(T-t)}\mathrm{d}W_t^2\big] \\ \alpha_\theta = 1/\sqrt{(1-\theta)^2+\theta^2+2\rho_{12}(1-\theta)\theta} \end{cases} \tag{1.3}$$

Where ρ is the correlation between W^1 and W^2 and we have defined the two OU processes X^1 and X^2 given by:

$$\begin{cases} \mathrm{d}X_t^i = -k_i X_t^i \mathrm{d}t + \mathrm{d}W_t^i \\ X_0^i = 0 \end{cases}$$

We will also define an auxiliary gaussian drift-less process:

$$\mathrm{d}x_t^T = \alpha_\theta\big[(1-\theta)e^{-k_1(T-t)}\mathrm{d}W_t^1 + \theta e^{-k_2(T-t)}\mathrm{d}W_t^2\big]$$

whose quadratic variation is given by:

$$\mathrm{d}\langle x^T\rangle_t = \eta^2(T-t)\mathrm{d}t$$

where we have defined:

$$\eta(s) \coloneqq \alpha_\theta\sqrt{(1-\theta)^2e^{-2k_1s}+\theta^2e^{-k_2s}+2\rho_{12}\theta(1-\theta)e^{-(k_1+k_2)s}}$$

Thus substituting in the SDE (1.3) we obtain:

$$\mathrm{d}\xi_t(T) = \omega\xi_t(T)\mathrm{d}x_t^T$$

So the solution is given by:

$$\begin{cases} \xi_t(T) = \xi_0(T)e^{\omega x_t^T - \frac{\omega^2}{2}f(t,T)} \\ f(t,T) = \displaystyle\int_{T-t}^{T}\eta^2(u)\mathrm{d}u \end{cases}$$

We can explicit the value of $f(t,T)$ that is:

$$f(t,T) = \alpha_\theta^2\Big[\frac{(1-\theta)^2}{2k_1}e^{-2k_1(T-t)}(1-e^{-2k_1t}) + \frac{\theta^2}{2k_2}e^{-2k_2(T-t)}(1-e^{-2k_2t}) \\ + 2\theta(1-\theta)\rho_{12}e^{-(k_1+k_2)(T-t)}\frac{1-e^{-(k_1+k_2)t}}{k_1+k_2}\Big]$$

2 The Realized Variance

We will use the Mandelbrot-Vann Ness representation of the fractional Brownian motion to express the increments of the logarithm of realized variance $v = \sigma^2$ as:

$$\begin{aligned}
\log(v_u) - \log(v_t) &= 2\nu C_H (W_u^H - W_t^H) \\
&= 2\nu C_H \left(\int_{-\infty}^{u} (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} - \int_{-\infty}^{t} (t-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} \right) \\
&= 2\nu C_H \left(\int_{t}^{u} (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} + \int_{-\infty}^{t} \left[(u-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right] \mathrm{d}W_s^{\mathbb{P}} \right) \\
&=: 2\nu C_H \left[M_t(u) + Z_t(u) \right]
\end{aligned}$$

We note that $\mathbb{E}[M_t(u)|\mathcal{F}_t] = 0$ and that $Z_t(u)$ is $\mathcal{F}_t$-measurable. If we define $\tilde{W}_t^{\mathbb{P}}$ as:

$$\tilde{W}_t^{\mathbb{P}}(u) := \sqrt{2H} \int_t^u |u-s|^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}}$$

it has the same properties of $M_t(u)$ and defining $\eta := \frac{2\nu C_H}{\sqrt{2H}}$ we have that:

$$\log(v_u) - \log(v_t) = \eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)$$

This gives us:

$$v_u = v_t \exp\left\{ \eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u) \right\}$$

Thanks to the properties, gaussianity in this case, of $\tilde{W}_t^{\mathbb{P}}$ we have:

$$\tilde{W}_t^{\mathbb{P}}(u) \sim \mathcal{N}\left(0, (u-t)^{2H}\right)$$

which gives us that $v_u|\mathcal{F}_t$ is log-normal and thus entails that:

$$\mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] = v_t \exp\left\{ 2\nu C_H Z_t(u) + \frac{1}{2}\eta^2 (u-t)^{2H} \right\}$$

Now we can express the realized variance as:

$$v_u = \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \mathcal{E}(\eta \tilde{W}_t^{\mathbb{P}}(u))$$

where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential.

3 The probability measure change

As observed in [2] the model with two factors is over-parameterized so we will use the 1-factor model. From what we have said up to now we have that, under the physical probability $\mathbb{P}$, the model is expressed as:

$$\begin{cases} \mathrm{d}S_u = \mu_u S_u \mathrm{d}u + \sqrt{v_u} S_u \mathrm{d}Z_u^{\mathbb{P}} \\ v_u = v_t \exp\left\{\eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)\right\} \end{cases}$$

Now we want to change the physical probability measure $\mathbb{P}$ with an equivalent martingale measure $\mathbb{Q}$ in the interval $[t, T]$ in order to price options. To do that we use Girsanov theorem and obtain:

$$\mathrm{d}Z_u^{\mathbb{Q}} = \mathrm{d}Z_u^{\mathbb{P}} + \frac{\mu_u - (r-q)}{\sqrt{v_u}} \mathrm{d}u$$

When we change from $\mathbb{P}$ to $\mathbb{Q}$ we also have to remember that the Brownian motion $W_u^{\mathbb{P}}$, that is used to construct the Volterra-type process $\tilde{W}_u^{\mathbb{P}}$, is correlated with $Z_u^{\mathbb{P}}$ with correlation factor ρ:

$$\mathrm{d}W_u^{\mathbb{P}} = \rho \mathrm{d}Z_u^{\mathbb{P}} + \sqrt{1-\rho^2} \mathrm{d}(Z_u^{\perp})^{\mathbb{P}}$$

where $(Z_u^{\perp})^{\mathbb{P}}$ is independent of $Z_u^{\mathbb{P}}$. A change of measure for $(Z_u^{\perp})^{\mathbb{P}}$ is of the form:

$$(Z_u^{\perp})^{\mathbb{Q}} = (Z_u^{\perp})^{\mathbb{P}} + \gamma_u \mathrm{d}u$$

where γ_u is a suitable process that can be seen as the market price of volatility risk. Now we can express the change in measure also for $W_u^{\mathbb{Q}}$:

$$\begin{aligned} \mathrm{d}W_u^{\mathbb{Q}} &= \rho \mathrm{d}Z_u^{\mathbb{Q}} + \sqrt{1-\rho^2} \mathrm{d}(Z_u^{\perp})^{\mathbb{Q}} \\ &= \mathrm{d}W_u^{\mathbb{P}} + \left(\frac{\mu_u - (r-q)}{\sqrt{v_u}} \rho + \gamma_u \sqrt{1-\rho^2} \right) \mathrm{d}u \\ &= \mathrm{d}W_u^{\mathbb{P}} + \lambda_u \mathrm{d}u \end{aligned}$$

We may now rewrite, assuming that the filtration generated by $W^{\mathbb{P}}$ is the same as the one generated by $W^{\mathbb{Q}}$, the dynamics of the realized variance:

$$\begin{aligned} v_u &= \mathbb{E}^{\mathbb{P}}[v_u | \mathcal{F}_t] \exp\left\{ \eta \sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} - \frac{\eta^2}{2} (u-t)^{2H} \right\} \\ &= \mathbb{E}^{\mathbb{P}}[v_u | \mathcal{F}_t] \mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \exp\left\{ \eta \sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \lambda_s \mathrm{d}s \right\} \\ &= \mathbb{E}^{\mathbb{Q}}[v_u | \mathcal{F}_t] \mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \\ &= \xi_t(u) \mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \end{aligned}$$

where we have

$$\tilde{W}_t^{\mathbb{Q}}(u) := \sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{Q}}$$

The model under $\mathbb{Q}$ is thus expressed as:

$$\begin{cases} \mathrm{d}S_u = (r-q)S_u \mathrm{d}u + \sqrt{v_u} S_u \mathrm{d}Z_u^{\mathbb{Q}} \\ v_u = \xi_t(u)\mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \end{cases}$$

This model is a non-Markovian generalization of the Bergomi model. Specifically, this model is non-Markovian in the instantaneous variance v_t:

$$\mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] \neq \mathbb{E}^{\mathbb{Q}}[v_u|v_t]$$

but is Markovian in the infinite-dimensional state vector:

$$\mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] = \xi_t(u)$$

4 Pricing

Under the pricing measure $\mathbb{Q}$, given the starting time $t_0 = 0$, the scheme to simulate the model is:

$$\begin{cases} S_t = S_0 \exp\left\{(r-q)t - \frac{1}{2}\int_0^t v_u \mathrm{d}u + \int_0^t \sqrt{v_u}\mathrm{d}Z_u^{\mathbb{Q}}\right\} \\ v_t = \xi_0(t)\exp\left\{2\nu C_H \int_0^t (t-u)^{H-\frac{1}{2}}\mathrm{d}W_u^{\mathbb{Q}} - \frac{\nu^2 C_H^2}{H} t^{2H}\right\} \\ \mathrm{d}[Z^{\mathbb{Q}}, W^{\mathbb{Q}}]_t = \rho \mathrm{d}t \end{cases}$$

First we need to simulate the Volterra process using the hybrid scheme introduced in section (...). Then we need to extract the Brownian Motion $W^{\mathbb{Q}}$ that drives the Volterra process and correlate it with $Z^{\mathbb{Q}}$ by the parameter ρ. Lastly, we simulate the stock price process S using the increments. To sum up, in order to simulate the stock price process we have to:

1. fix an equispaced grid $\mathcal{G} = \{t_0 = 0, t_1 = \frac{1}{n}, \ldots, t_{\lfloor nT \rfloor} = \frac{\lfloor nT \rfloor}{n}\}$;
2. simulate the Volterra process $\mathcal{V}_t = \displaystyle\int_0^t (t-u)^{H-\frac{1}{2}}\mathrm{d}W_u^{\mathbb{Q}}$, $t \in \mathcal{G}$, using the hybrid scheme;
3. compute the variance process v using the previously computed Volterra process:

$$v_t = \xi_0(t)\mathcal{E}\big(2\eta C_H \mathcal{V}_t\big) \qquad t \in \mathcal{G}$$

4. extract the path of the Brownian Motion $W^{\mathbb{Q}}$ that drives the Volterra process:

$$\begin{aligned} W_{t_i}^{\mathbb{Q}} &= W_{t_{i-1}}^{\mathbb{Q}} + n^{H-\frac{1}{2}}\big(\mathcal{V}_{t_i} - \mathcal{V}_{t_{i-1}}\big) && i = 1, \ldots, \kappa \\ W_{t_i}^{\mathbb{Q}} &= W_{t_{i-1}}^{\mathbb{Q}} + W_{i-1}^{n} && i > \kappa \end{aligned}$$

where W^n is defined as in (2.2) [fBM hybrid scheme section]

5. correlate the stock price process, driven by $Z^{\mathbb{Q}}$, and the variance process, driven by $W^{\mathbb{Q}}$ through the Volterra process, as:

$$Z^{\mathbb{Q}}_{t_i} - Z^{\mathbb{Q}}_{t_{i-1}} = \rho\big(W^{\mathbb{Q}}_{t_i} - W^{\mathbb{Q}}_{t_{i-1}}\big) + \sqrt{1-\rho^2}\big(W^{\mathbb{Q},\perp}_{t_{i-1}} - W^{\mathbb{Q},\perp}_{t_{i-1}}\big)$$

   where $W^{\mathbb{Q},\perp}$ is a standard Brownian Motion independent of $W^{\mathbb{Q}}$;

6. simulate the stock price process S using the simulated increments as:

$$S_{t_i} = S_{t_{i-1}} \exp\left\{\sqrt{v_{t_{i-1}}}\big(Z^{\mathbb{Q}}_{t_i} - Z^{\mathbb{Q}}_{t_{i-1}}\big) - \frac{v_{t_{i-1}}}{2n} + \frac{r-q}{n}\right\}$$

To price an option at time $t < T$, where T is the maturity, that has payoff $f(S_T)$ we have to calculate the discounted payoff given by:

$$P_t = \mathbb{E}^{\mathbb{Q}}\big[e^{-(r-q)(T-t)} f(S_T) | \mathcal{F}_t\big]$$

To compute this quantity we will use a Monte-Carlo simulation. In practice we will use the implementation devised by McCricked and Pakkanen in [3].

5 Calibration

6 Volatility Skew

7 Forward-Start Options

8 Bayesian Inverse Problem

References

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