

# Quintic Ornstein-Uhlenbeck model

Nodari Alessandro & Proserpio Lorenzo

The quintic Ornstein-Uhlenbeck (OU) volatility model is a stochastic volatility model where the volatility process is a polynomial function of degree five of a single OU process with fast mean reversion and large vol-of-vol.

## 1 The Model

Under the pricing measure  $\mathbb{Q}$  the dynamics of the stock price  $S$  are given by:

$$\begin{aligned} dS_t &= (r - q)dt + \sigma_t S_t dB_t \\ \sigma_t &= \sqrt{\xi_0(t)} \frac{p(X_t)}{\sqrt{\mathbb{E}[p(X_t)^2]}} \\ dX_t &= -\left(\frac{1}{2} - H\right)\epsilon^{-1} X_t dt + \epsilon^{H-1/2} dW_t \quad X_0 = 0 \end{aligned}$$

where  $B_t$  and  $W_t$  are two Brownian motions with correlation parameter  $\rho$ ;  $\xi_0 \in L^2([0, T], \mathbb{R}^+)$  for any  $T > 0$  is an input curve used to match certain term-structures observed in the market, for instance, the normalization  $\sqrt{\mathbb{E}[p(X_t)^2]}$  allows  $\xi_0$  to match the market initial forward variance curve since:

$$\mathbb{E}\left[\int_0^t \sigma_s^2 ds\right] = \int_0^t \xi_0(s) ds \quad t \geq 0$$

The fifth grade polynomial  $p(x)$  is defined as:

$$p(x) := \alpha_0 + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5$$

with non-negative parameters  $\alpha_0, \alpha_1, \alpha_3, \alpha_5 \geq 0$  ( $\alpha_2 = \alpha_4 = 0$ ). A polynomial of degree five allows to reproduce the upward slope of the VIX smile. Restricting the coefficients  $\alpha$  to be non-negative allows the sign of the ATM skew to be the same as  $\rho$ , as explained in more detail in [2]. The choice to set  $\alpha_2 = \alpha_4 = 0$  allows to reduce the number of parameters to calibrate and doesn't impact in a significant way the results as highlighted in [PAPAER Quintic]. The process  $X_t$  that drives the volatility is an OU process where the two parameters  $H \in (-\infty, 1/2]$  and  $\epsilon > 0$  control the mean-reversion speed through  $(1/2 - H)\epsilon^{-1}$  and the vol-of-vol through  $\epsilon^{H-1/2}$ . For small values of  $\epsilon$  we have a fast mean-reversion regime and a large vol-of-vol. Such parametrizations are reminiscent

of the fast regimes extensively studied by Fouque et al. [9] which corresponds to the case  $H = 0$ . They can also be linked to more complex models such as jump models [19,1] for  $H \leq -1/2$ ; and rough volatility models [2,1] for which  $H \in (0, 1/2)$  would play the role of the Hurst index. The solution of the OU process is:

$$X_t = \epsilon^{H-1/2} \int_0^t e^{-(1/2-H)\epsilon^{-1}(t-s)} ds$$

The parameters to calibrate are seven:

$$\Theta := \{\rho, H, \epsilon, \alpha_0, \alpha_1, \alpha_3, \alpha_5\}$$

plus the the input curve  $\xi_0$ , we will use the market initial forward variance curve.

## 2 SPX derivatives

The price SPX derivatives we have to resort to Monte Carlo simulations since there isn't a closed formula. Nevertheless, since  $X$  is a OU process it can be simulated exactly instead of approximating it using, for exemplar, the Euler scheme which is often inaccurate in a fast mean-reversion regime. To simulate  $X$  we first define:

$$\tilde{X}_t := X_t e^{(1/2-H)\epsilon^{-1}t} = \epsilon^{H-1/2} \int_0^t e^{(1/2-H)\epsilon^{-1}s} dW_s$$

Thus,  $\tilde{X}$  can be simulated recursively by:

$$\tilde{X}_{t_{i+1}} = \tilde{X}_{t_i} + \sqrt{\frac{\epsilon^{2H}}{1-2H}} \left( e^{\frac{1-2H}{\epsilon}t_{i+1}} - e^{\frac{1-2H}{\epsilon}t_i} \right) Y_i$$

where  $Y_i$  are i.i.d. standard Gaussian. Naturally, to get back to  $X_{t_{i+1}}$  we just have to divide  $\tilde{X}_{t_{i+1}}$  by  $e^{\frac{1-2H}{\epsilon}t_{i+1}}$ . This allows us to easily vectorize computations. Instead, to simulate the log-process  $\log(S)$  we will use the Euler scheme paired with antithetic and control variates, the so called turbocharging method as outlined in [18] that we have used also in the rBergomi model. This means we only need to simulate the part of  $\log(S)$  that is  $F^{\mathcal{W}}$  measurable, we call this  $S^{\mathcal{W}}$  and can be simulated as:

$$\log(S_{t_{i+1}}^{\mathcal{W}}) = \log(S_{t_i}^{\mathcal{W}}) - \frac{1}{2}(\rho\sigma_{t_i})^2(t_{i+1}-t_i) + \rho\sigma_{t_i}\sqrt{t_{i+1}-t_i}Y_i + \rho^2(r-q)(t_{i+1}-t_i)$$

We will use an equi-spaced grid so that calling the time step  $h$  the above formula reduces to:

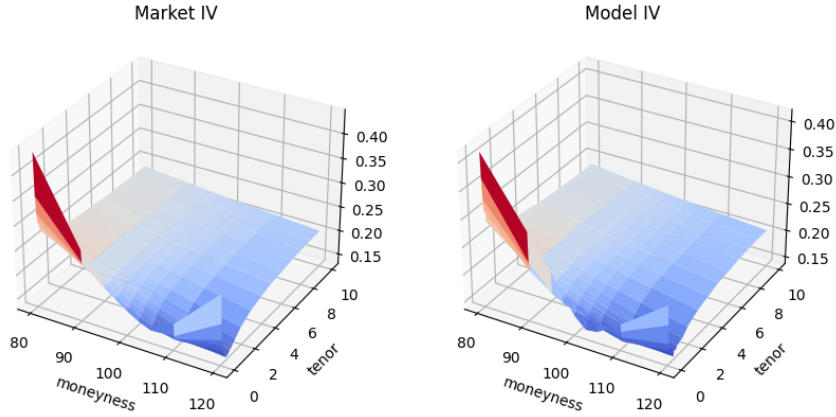
$$\log(S_{t_{i+1}}^{\mathcal{W}}) = \log(S_{t_i}^{\mathcal{W}}) + \left( r - q - \frac{1}{2}\sigma_{t_i}^2 \right) \rho^2 h + \rho\sigma_{t_i}\sqrt{h}Y_i$$

## 2.1 Numerical results

We decided to calibrate our model both in a local and global way. In the local case we calibrated a set of parameters for every tenor while in the global approach we calibrated the set of parameters so that it is the best fit considering all the tenors together. We report, for the local case, some of the calibrated parameters in the following table.

Tenor	$\rho$	H	$\varepsilon$	$\alpha_0$	$\alpha_1$	$\alpha_3$	$\alpha_5$
2 weeks	-0.5332	0.1034	0.1078	0.1353	0.6193	0.0954	0.1024
1 month	-0.6626	0.0915	0.0171	0.9777	0.0187	0.0343	0.1072
6 months	-0.6968	-0.0397	0.0083	1.2204	0.0035	0.2296	0.0462
1 year	-0.8565	0.1903	0.0234	1.0277	0.2463	0.0672	0.6016
10 years	-0.6658	0.1141	0.1173	0.9099	0.5726	0.1604	0.1229

Comparing our results with the market we obtained a mean relative error of 2.2314%.

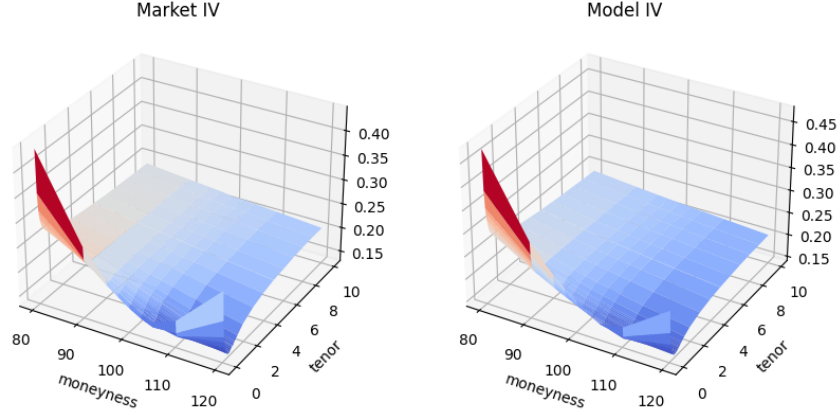


Comparison between market and model IV, local approach

Now we report, for the global approach, the set of the calibrated parameters.

$\rho$	H	$\varepsilon$	$\alpha_0$	$\alpha_1$	$\alpha_3$	$\alpha_5$
-0.9248	0.0731	0.0987	0.6888	0.4998	0.0103	0.0666

In this case we obtained a mean relative error of 2.9693%.



Comparison between market and model IV, global approach

### 3 VIX derivatives

One major advantage of this model is that there is an explicit expression of the VIX. In continuous time the VIX can be expressed as:

$$VIX_T^2 = -\frac{2}{\Delta} \mathbb{E} \left[ \log \left( \frac{S_{T+\Delta}}{S_T} \right) \middle| \mathcal{F}_T \right] \cdot 100^2 = \frac{100^2}{\Delta} \int_T^{T+\Delta} \xi_T(u) du \quad (4.1)$$

with the usual  $\Delta = 30$  days and  $\xi_T(u) = \mathbb{E}[\sigma_u^2 | \mathcal{F}_T]$  the forward variance curve, that can be explicitly computed. In order to do that we fix  $T \leq u$  and rewrite  $X$  as:

$$X_u = X_T e^{-(1/2-H)\epsilon^{-1}(u-T)} + \epsilon^{H-1/2} \int_T^u e^{-(1/2-H)\epsilon^{-1}(u-s)} dW_s =: Z_T^u + G_T^u$$

then, if we define:

$$g(u) := \mathbb{E}[p(X_u)^2]$$

we obtain:

$$\xi_T(u) = \mathbb{E}[\sigma_u^2 | \mathcal{F}_T] = \frac{\xi_0(u)}{g(u)} \mathbb{E} \left[ \left( \sum_{k=0}^5 \alpha_k X_u^k \right)^2 \middle| \mathcal{F}_T \right]$$

Defining  $\alpha$  the vector  $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$  and indicating with  $(\alpha * \alpha)$  the discrete convolution:

$$(\alpha * \alpha)_k = \sum_{j=0}^k \alpha_j \alpha_{k-j}$$

we have the following expression:

$$\xi_t(u) = \frac{\xi_0(u)}{g(u)} \mathbb{E} \left[ \sum_{k=0}^{10} (\alpha * \alpha)_k X_u^k \mid \mathcal{F}_T \right]$$

Using the binomial expansion we can further develop the expression for  $\xi_T(u)$  in terms of  $Z^u$  and  $G^u$  to get:

$$\xi_T(u) = \frac{\xi_0(u)}{g(u)} \sum_{k=0}^{10} \sum_{i=0}^k (\alpha * \alpha)_k \binom{k}{i} \left( X_T e^{-(1/2-H)\epsilon^{-1}(u-T)} \right)^i \mathbb{E}[(G_T^u)^{k-i}] \quad (4.2)$$

where we used both that  $Z_T^u$  is  $\mathcal{F}_T$ -measurable and that  $G_T^u$  is independent of  $\mathcal{F}_T$ . Furthermore, we know that  $G_T^u$  is actually a Gaussian random variable:

$$G_T^u \sim \mathcal{N} \left( 0, \frac{\epsilon^{2H}}{1-2H} [1 - e^{-(1-2H)\epsilon^{-1}(u-T)}] \right)$$

We recall that for a Gaussian variable  $Y \sim \mathcal{N}(0, \sigma_Y^2)$  its moments can be computed as:

$$\mathbb{E}[Y^p] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma_Y^p (p-1)!! & \text{if } p \text{ is even} \end{cases} \quad (4.3)$$

where  $p!!$  is the double factorial. Therefore we can compute exactly all moments of  $G_T^u$ . Going back to (4.1) and plugging the expression (4.2) we have that the explicit expression of  $VIX_T^2$  is polynomial in  $X_T$  and given by:

$$\begin{aligned} VIX_T^2 &= \frac{100^2}{\Delta} \sum_{k=0}^{10} \sum_{i=0}^k (\alpha * \alpha)_k \binom{k}{i} X_T^i \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}[(G_T^u)^{k-i}] e^{-(1/2-H)\epsilon^{-1}(u-T)i} du \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} X_T^i \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}[(G_T^u)^{k-i}] e^{-(1/2-H)\epsilon^{-1}(u-T)i} du \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} \beta_i X_T^i \quad (4.4) \end{aligned}$$

where we have defined

$$\beta_i := \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}[(G_T^u)^{k-i}] e^{-(1/2-H)\epsilon^{-1}(u-T)i} du$$

We recall that thanks to formula (4.3) we can compute exactly every moment of  $G_T^u$ . Thanks to the closed form of (4.4)  $VIX_T^2$  is a polynomial in  $X_T$  that we will denote as  $h(X_T)$ . Since we have that  $X_T$  is Gaussian

$$X_T \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H} [1 - e^{-(1-2H)\epsilon^{-1}T}]\right)$$

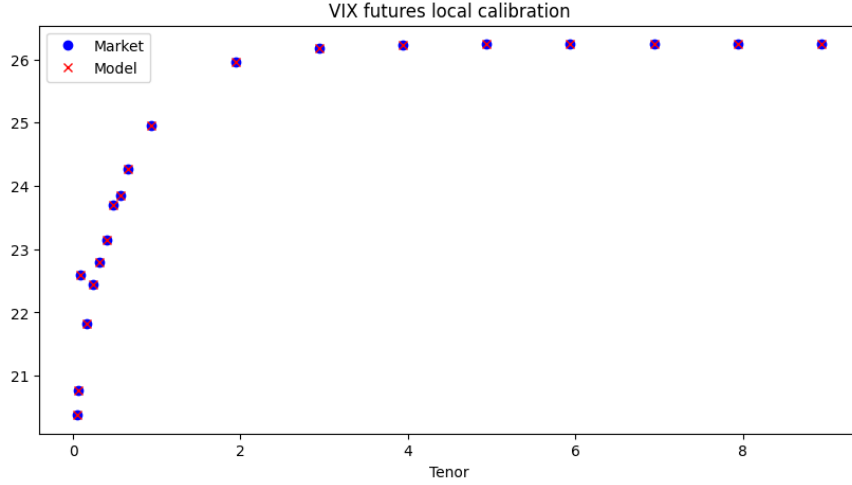
pricing VIX derivatives, with a general payoff function  $\Phi$ , can be done integrating directly against the standard Gaussian density:

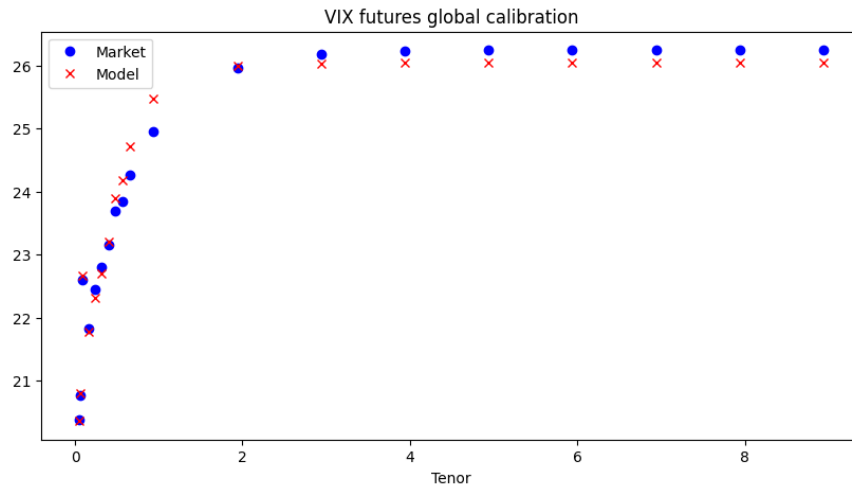
$$\mathbb{E}[\Phi(VIX_T)] = \mathbb{E}\left[\Phi(\sqrt{h(X_T)})\right] = \frac{1}{\sigma_{X_T} \sqrt{2\pi}} \int_{\mathbb{R}} \Phi(\sqrt{h(x)}) e^{-x^2/\sigma_{X_T}^2} dx$$

This integral can be computed efficiently using a variety of quadrature techniques.

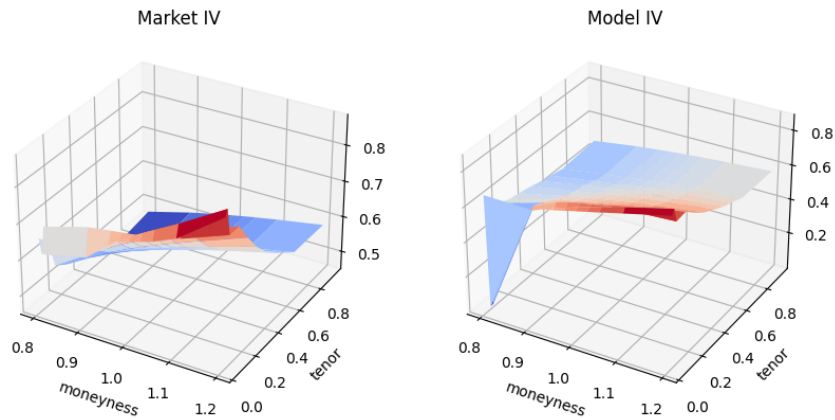
### 3.1 Numerical results

Also in this case we used a local and global approach both for futures and options. First we report the futures results: in the local case the calibration is almost perfect while in the global case we obtained a mean percentage error of 0.6920%.



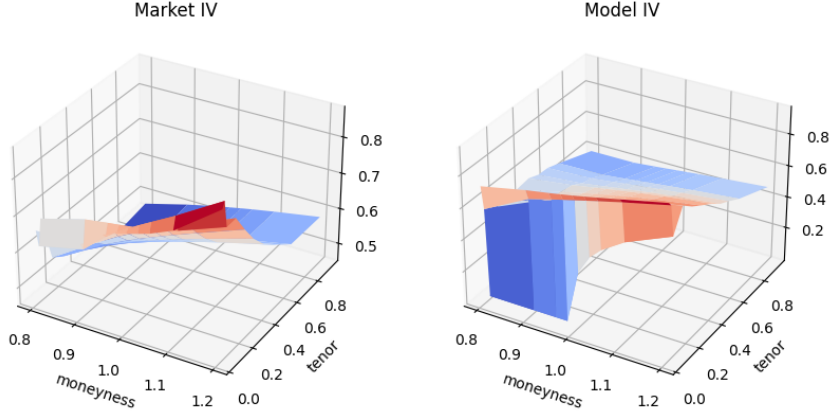


Calibrating the model to the IV of VIX options we discovered, doing the local calibration, that the model isn't suitable for long term-options in fact we obtain a mean percentage error of 31.1995% if we use all the tenors while if we use only the tenors that are less or equal to one year we obtain a mean percentage error of 2.4955%.



Comparison between market and model IV, local approach

For the global approach using only the tenors that are less or equal to one year we obtained a mean percentage error of 8.7636%.



Comparison between market and model IV, global approach

## 4 Joint calibration

We now address the SPX-VIX joint calibration problem, that is the calibration of the model to SPX European options, VIX European options and VIX futures across several maturities. The calibration of VIX futures is necessary as it is used to calculate VIX implied volatility. In fact, recall that the IV is calculated inverting the Black & Scholes formula, that is given a call option price  $C_0(K, T)$  with a given strike  $K$  and maturity  $T$  we find the volatility  $\sigma(K, T)$  such that:

$$C_0(K, T) = F_T \Phi(d_1) - K \Phi(d_2)$$

where  $F_T$  denotes the futures price:  $F_T = \mathbb{E}[S_T] = S_0$  for the SPX and  $F_T = \mathbb{E}[VIX_T]$  for the VIX. To jointly calibrated the model we solve the following optimisation problem:

$$\min_{\Theta} \{c_1 f_1(\Theta) + c_2 f_2(\Theta) + c_3 f_3(\Theta)\}$$

where we have defined:

$$\begin{aligned} f_1(\Theta) &:= \sqrt{\sum_{i,j} (\sigma_{spx}^{\Theta}(K_i, T_j) - \sigma_{spx}^{mkt}(K_i, T_j))^2} \\ f_2(\Theta) &:= \sqrt{\sum_{i,j} (\sigma_{vix}^{\Theta}(K_i, T_j) - \sigma_{vix}^{mkt}(K_i, T_j))^2} \\ f_3(\Theta) &:= \sqrt{\sum_i (F_{vix}^{\Theta}(T_i) - F_{vix}^{mkt}(T_i))^2} \end{aligned}$$

So that  $f_1$  is the root mean squared error (RMSE) coming from the SPX calibration,  $f_2$  is the RMSE coming from the VIX options calibration and  $f_3$  is



the RMSE coming from the calibration of VIX futures. The constant  $c_i$  are positive and reflect what we want to have more importance, in our case we will use  $c_1 = c_2 = c_3 = 1$ .

#### 4.1 Numerical results

For the joint calibration we used only the global approach, for the VIX we used, for both the future and option calibration, only the tenors that are less or equal to one year. Doing so we obtained the following mean percentage error:

- SPX options calibration: 7.7591%;
- VIX futures calibration: 0.4339%;
- VIX options calibration: 18.3786%.

### 5 Bayesian Inverse Problem

To calibrated the model we used the Approximate Bayesian Computation (ABC) that is a sequential Monte-Carlo method. ABC methods (also called likelihood free inference methods), are a group of techniques developed for inferring posterior distributions in cases where the likelihood function is intractable or costly to evaluate. This does not mean that the likelihood function is not part of the analysis, it just means that we are approximating the likelihood. ABC comes useful when the model used contains unobservable random quantities, which make the likelihood function hard to specify, but data can be simulated from the model. These methods follow a general form:

1. Sample a parameter  $\theta$  from a prior/proposal distribution  $\pi(\theta)$ .
2. Simulate a data set  $y$  using a function that takes  $\theta$  and returns a data set of the same dimensions as the observed data set  $y_0$ .
3. Compare the simulated dataset  $y$  with the experimental data set  $y_0$  using a distance function  $d$  and a tolerance threshold  $\varepsilon$ .

In some cases a distance function is computed between two summary statistics  $d(S(y^*), S(y_0))$ , avoiding the issue of computing distances for entire datasets. As a result we obtain a sample of parameters from a distribution  $\pi(\theta|d(y, y_0))$ . If  $\varepsilon$  is sufficiently small this distribution will be a good approximation of the posterior distribution  $\pi(\theta|y_0)$ .

Sequential monte carlo ABC is a method that iteratively morphs the prior into a posterior by propagating the sampled parameters through a series of proposal distributions, weighting the accepted parameters  $\theta^{(i)}$  like:

$$w^{(i)} \propto \frac{\pi(\theta^{(i)})}{\phi(\theta^{(i)})}$$

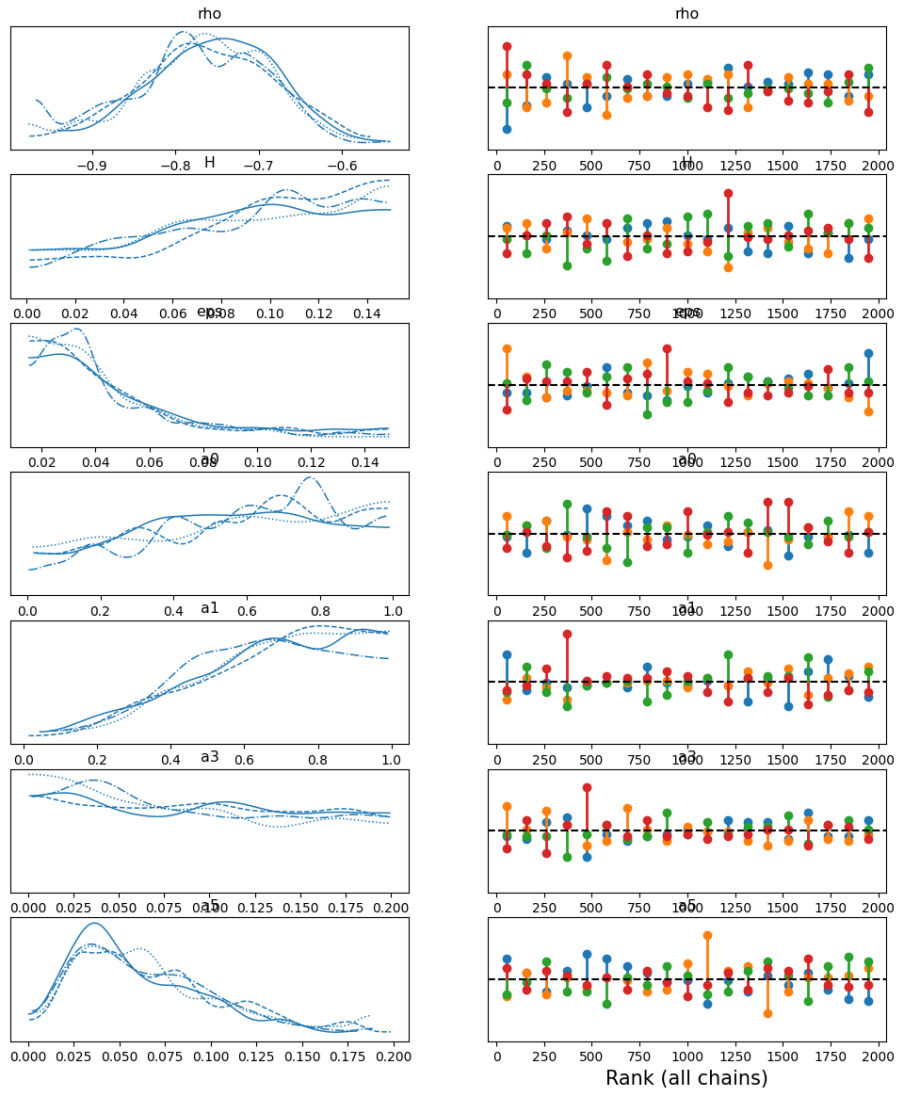
It combines the advantages of traditional SMC, i.e. ability to sample from distributions with multiple peaks, but without the need for evaluating the likelihood function.

## 5.1 Numerical results

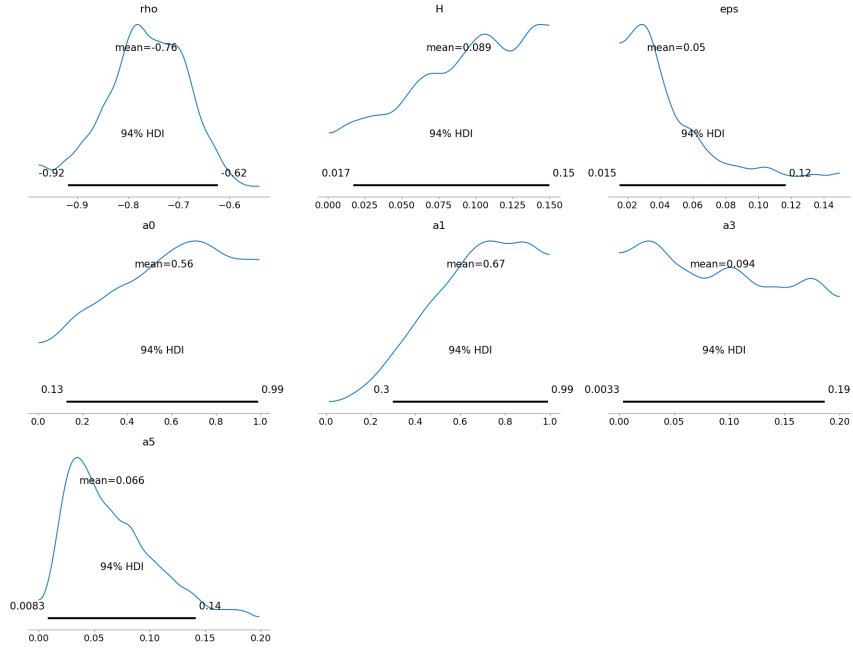
We decided to calibrate the model using only the tenors from 1 month to 6 months (10 in total) and we decided to approach the problem in two ways: calibrating all the parameters and calibrating only the set of parameters comprised of  $\rho$ ,  $H$  and  $\epsilon$  while using the already calibrated values of the vector  $\alpha$ . We used very uninformative priors:

- $\rho \sim \mathcal{U}(-1, -0.1)$
- $H \sim \mathcal{U}(0, -0.15)$
- $\epsilon \sim \mathcal{U}(0.015, 0.15)$
- $\alpha_0 \sim \mathcal{U}(0, 1)$
- $\alpha_1 \sim \mathcal{U}(0, 1)$
- $\alpha_3 \sim \mathcal{U}(0, 0.2)$
- $\alpha_5 \sim \mathcal{U}(0, 0.2)$

We calibrated using the prices instead of the IV. We used 4 chains with 2000 samples each, just for computational and time reasons. Then to test the quality of the calibration we used the maximum of the resulting distribution and plugged it in the model.



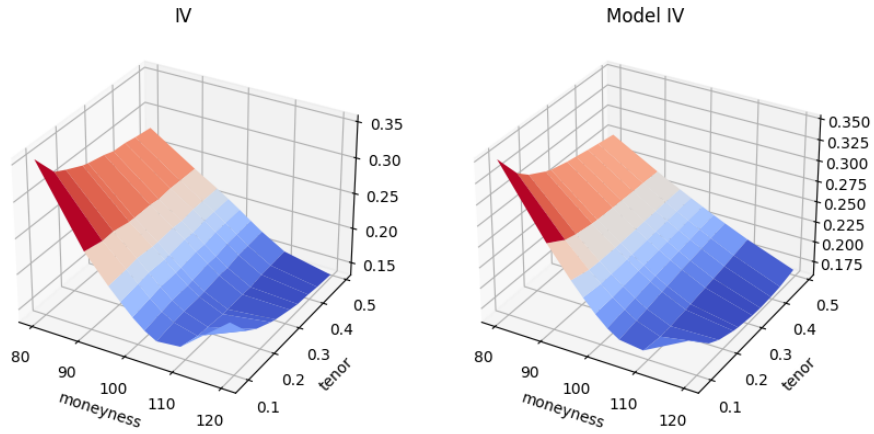
Looking at the chains plots we expect to obtain results that are not very good.



To compare the Bayesian calibration with the deterministic calibration we fixed the parameters value using the maximum a posteriori:

$\rho$	H	$\epsilon$	$\alpha_0$	$\alpha_1$	$\alpha_3$	$\alpha_5$
-0.7761	0.1090	0.0250	0.7630	0.6504	0.0095	0.0332

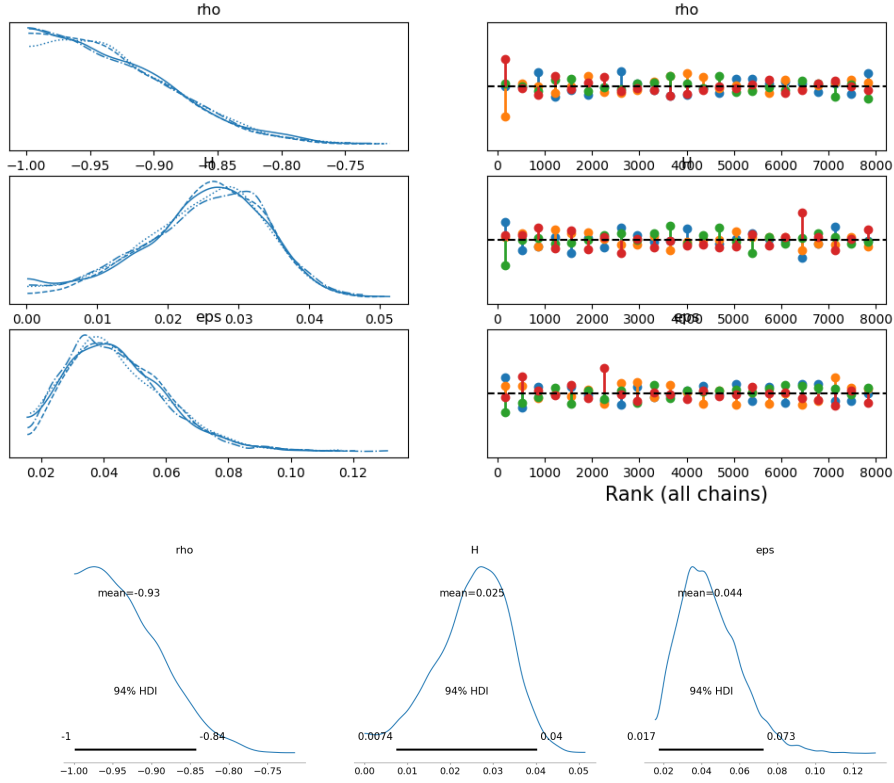
Doing so we obtained a mean percentage error of 10.6836%.



For the second case we fixed the value of  $\alpha$  as:

$$\alpha_0 = 1.2237 \quad \alpha_1 = 0.4503 \quad \alpha_3 = 0.0361 \quad \alpha_5 = 0.0532$$

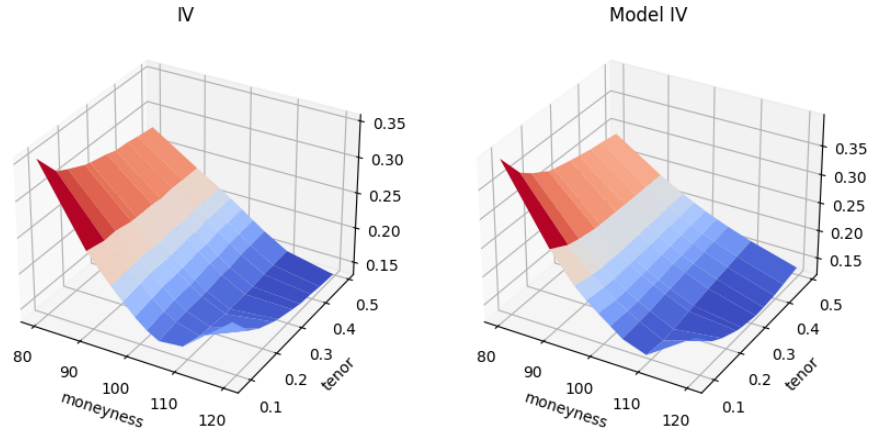
We obtained the following results.



To compare the Bayesian calibration with the deterministic calibration we fixed the parameters value using the maximum a posteriori:

$\rho$	$H$	$\epsilon$
-0.9635	0.0264	0.0333

Doing so we obtained a mean percentage error of 3.7385%.



Thus, we can conclude that fixing the  $\alpha$  parameter the Bayesian calibration is able to give us some good quality results. Whereas, if the  $\alpha$  parameter has to be calibrated the Bayesian framework is not able to give us quality results since the set of parameters is not completely independent.

## References

- 1.
2. (Lintusaari, 2016), (Toni, T., 2008), (Nuñez, Prangle, 2015)