Fractional Brownian Motion

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The fractional Brownian Motion (fBM) was first introduced within a Hilbert space framework by Kolmogorov in 1940 in [1], where it was called Wiener Helix. It was further studied by Yaglom in [2]. The name fractional Brownian Motion is due to Mandelbrot and Van Ness, who in 1968 provided in [3] a stochastic integral representation of this process in terms of a standard Brownian motion. From now on we will consider a probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{F}$ is the natural filtration generated by a Brownian Motion.

1 Definition and Properties

We will use the definition of fBM given in [4].

Definition 1. *A fractional Brownian Motion W_t^H of Hurst index $H \in (0,1)$ is a continuous and centered Gaussian process with covariance function*

$$\mathbb{E}\big[W_t^H W_s^H\big] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) \qquad t, s \in \mathbb{R}$$

By **Definition 1** a fBM W_t^H has the following properties:

1. $W_0^H = 0$;
2. $W_t^H \sim \mathcal{N}(0, t^{2H})$, $\forall t \geq 0$;
3. W^H has stationary increments:
$$W_{t+s}^H - W_t^H = W_s^H \quad s \in \mathbb{R}$$
4. W_t^H has $\mathbb{P}$ a.s. continuous trajectories.

Remark 1. *The fBm is divided into three very different families:*

- $0 < H < \frac{1}{2}$ *where two increments of the form* $\big(W_{t+h}^H - W_t^H\big)$ *and* $\big(W_{t+2h}^H - W_{t+h}^H\big)$ *are negatively correlated;*
- $H = \frac{1}{2}$ *then the fBM is actually a standard Brownian Motion and the increments are independent;*
- $\frac{1}{2} < H < 1$ *where two increments of the form* $\big(W_{t+h}^H - W_t^H\big)$ *and* $\big(W_{t+2h}^H - W_{t+h}^H\big)$ *are positively correlated.*

Proposition 1. *A fBM W_t^H admits the following stochastic integral representation:*

$$W_t^H = C_H \left(\int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] \mathrm{d}W_s + \int_0^t (t-s)^{H-\frac{1}{2}} \mathrm{d}W_s \right)$$

where

$$C_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$$

Where Γ is the usual gamma function and W_t is a two-sided Brownian Motion defined on $\mathbb{R}$ as:

$$W_t = \begin{cases} W^1(t) & t \geq 0 \\ W^2(-t) & t < 0 \end{cases}$$

where W^1 and W^2 are two independent Brownian Motion.

Proposition 2. *A fBM W_t^H with Hurst parameter $H \in (0,1)$ is a self-similar process such that, for any $c \geq 0$, it holds:*

$$W_{ct}^H \stackrel{d}{=} c^H W_t^H$$

Proposition 3. *The paths of a fBM W_t^H with Hurst parameter $H \in (0,1)$ are almost surely locally $(H-\varepsilon)$-Hölder continuous for $\varepsilon \in (0,H)$.*

Proposition 4. *A fBM W_t^H has a monofractal scaling property:*

$$\mathbb{E}\left[|W_{t+\Delta}^H - W_t^H|^q\right] = \mathbb{E}\left[|W_\Delta^H|^q\right] = K_q \Delta^{Hq}$$

where

$$K_q = \int_{-\infty}^{\infty} |x|^q \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x$$

We also highlight that a fBM is not a Markov process nor a semi-martingale.

2 Simulation Methods

A great number of methods have been developed to simulate the paths of a fBM. Some of them are exact methods, which are more demanding, and other are approximations. We will present just an exact method, the Cholesky decomposition, and an hybrid scheme approximation. More methods are presented in [4].

2.1 Cholesky Decomposition

This method is based on the so-called Cholesky decomposition of the covariance matrix. We will analyze the case in which we want to simulate the fBM W_t^H with $t \in [0, T]$. First we discretize the interval using an equi-spaced grid of $n+1$ points $0 = t_0 < t_1 < \ldots < t_n = T$ with time-step $h = \frac{T}{n}$. The covariance structure of our discretization is:

$$\mathbb{E}\big[W_{t_i}^H W_{t_j}^H\big] = \frac{1}{2}\big(t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H}\big) = \frac{h^{2H}}{2}\big(i^{2H} + j^{2H} - |i-j|^{2H}\big)$$

The covariance matrix C defined element wise as $C_{i,j} = \mathbb{E}\big[W_{t_i}^H W_{t_j}^H\big]$ for $i, j = 1, \ldots, n$. C is a symmetric and positive semi-defined matrix in $\mathbb{R}^{n\times n}$.

$$C = \begin{bmatrix} \mathbb{E}\big[W_{t_1}^H W_{t_1}^H\big] & \cdots & \mathbb{E}\big[W_{t_1}^H W_{t_n}^H\big] \\ \vdots & \ddots & \vdots \\ \mathbb{E}\big[W_{t_n}^H W_{t_1}^H\big] & \cdots & \mathbb{E}\big[W_{t_n}^H W_{t_n}^H\big] \end{bmatrix} \tag{2.1}$$

Hence the usual Cholesky decomposition reads:

$$C = LL^T$$

where L is a lower triangular matrix with real and positive diagonal entries. Now we will draw n independent samples Z_i from a normal distribution $\mathcal{N}(0,1)$. The vector $(0, LZ)$ of size $n+1$ yields a sample path of W_t^H. To summarize the steps of the method are:

1. choose an equi-spaced grid $\{t_i\}_{i=1,\ldots,n}$ for the interval $[0, T]$;
2. compute the covariance matrix as in (2.1);
3. use the Cholesky decomposition to find the matrix L such that $C = LL^T$;
4. construct a vector Z of n independent realization of a standard normal distribution $\mathcal{N}(0,1)$;
5. compute the path of the fBM as the vector $(0, LZ)$.

We note that the complexity of this method is of the order $\mathcal{O}(n^3)$.

2.2 Hybrid Scheme

The following method is an approximation method that was proposed in [5] as a scheme to simulate a Brownian semi-stationary (BSS) process. The class of BSS processes are studied in [6]. For our purpose we will define a BSS process $(Y_t)_{t\in\mathbb{R}}$ as:

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma(s)\mathrm{d}W_s$$

where W_t is a two-sided Brownian motion, g is a deterministic non-negative weight function and σ is a so called càdlàg process. In order to use the hybrid scheme we have to assume:

1. for some $\alpha \in (-\frac{1}{2}, \frac{1}{2})\backslash\{0\}$ it holds:

$$g(x) = x^{\alpha} L_g(x) \qquad x \in (0, 1]$$

   where $L_g : (0, 1] \to [0, \infty)$ is continuously differentiable, slowly varying at 0 and bounded away from 0. Moreover, there exists a constant $d > 0$ such that:

$$|L_g(x)| \leq d(1 + x^{-1}) \qquad \vee \qquad |L_g'(x)| \leq d(1 + x^{-1})$$

   where L_g' it's the derivative of L_g;

2. the function g is continuously differentiable in $(0, \infty)$;

3. for some $\beta \in (-\infty, \frac{1}{2})$ it holds:

$$g(x) = \mathcal{O}(x^{\beta})$$

Since we are only interested in the time intervals that start at 0 we will use a Truncated Brownian semi-stationary (TBSS) process X_t defined as:

$$X_t = \int_0^t g(t - s)\sigma(s)\mathrm{d}W_s$$

We will use as the discretization grid $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{\lfloor nT \rfloor}{n}\right\}$ and we will assume that σ can be taken constant on each interval of the grid. Doing so the TBSS can be approximated as:

$$X_t \simeq \sum_{k=1}^{\lfloor nT \rfloor} \sigma\left(t - \frac{k}{n}\right) \int_{t-\frac{k}{n}}^{t-\frac{k-1}{n}} g(t - s)\mathrm{d}W_s =: X_n(t)$$

For small values of k, say $k \leq \kappa$ for a given κ, we can approximate g as in the assumptions:

$$g(t - s) \simeq (t - s)^{\alpha} L_g\left(\frac{k}{n}\right) \qquad (t - s) \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

For larger values of k, say $k > \kappa$, we can approximate g as:

$$g(t - s) \simeq g\left(\frac{b_k}{n}\right)$$

where in [6] the optimal b_k is shown to be:

$$b_k^* = \left(\frac{k^{\alpha+1} - (k-1)^{\alpha+1}}{\alpha + 1}\right)^{\frac{1}{\alpha}}$$

Thus we have that our approximation of the TBSS is composed of two parts:

$$X_n(t) = X_n^1(t) + X_n^2(t)$$

$$X_n^1(t) \coloneqq \sum_{k=1}^{\kappa} L_g\left(\frac{k}{n}\right)\sigma\left(t-\frac{k}{n}\right)\int_{t-\frac{k}{n}}^{t-\frac{k-1}{n}}(t-s)^{\alpha}\mathrm{d}W_s$$

$$X_n^2(t) \coloneqq \sum_{k=\kappa+1}^{\lfloor nT \rfloor} g\left(\frac{b_k^*}{n}\right)\sigma\left(t-\frac{k}{n}\right)\left(W_{t-\frac{k-1}{n}} - W_{t-\frac{k}{n}}\right)$$

This decomposition tells us that we have to simulate on the grid points $\left\{\frac{i}{n}\right\}$:

$$W_{i,j}^n\left(\frac{i}{n}\right) = \int_{\frac{i}{n}}^{\frac{i+1}{n}}\left(\frac{i+j}{n}-s\right)^{\alpha}\mathrm{d}W_s \qquad j=1,\dots,\kappa$$

$$W_i^n\left(\frac{i}{n}\right) = \int_{\frac{i}{n}}^{\frac{i+1}{n}}\mathrm{d}W_s \qquad (2.2)$$

We will explicit some of the properties of these two processes:

$$\mathrm{Var}\left[W_i^n\right] = \frac{1}{n}$$

$$\mathrm{Var}\left[W_{i,j}^n\right] = \frac{j^{2\alpha+1}-(j-1)^{2\alpha+1}}{(2\alpha+1)n^{2\alpha+1}}$$

$$\mathbb{E}\left[W_{i,j}^n W_j^n\right] = \frac{j^{\alpha+1}-(j-1)^{\alpha+1}}{(\alpha+1)n^{\alpha+1}}\delta_{i,j}$$

where $\delta_{i,j}$ id the Kronecker delta. Thus simulating a fBM can be seen as simulating a Volterra process of the form:

$$V(t) = \sqrt{2\alpha+1}\int_0^t (t-s)^{\alpha}\mathrm{d}W_s$$

Defining $\tilde{V}(t) = \frac{V(t)}{\sqrt{2H}}$ and taking:

$$g(s) = s^{H-\frac{1}{2}} \qquad \sigma(s) = 1 \qquad L_g(s) = 1 \qquad s \in (0,T)$$

we have that $\alpha = H - \frac{1}{2} \in (-\frac{1}{2},\frac{1}{2})\backslash\{0\}$ and so the Gaussian Volterra process $\tilde{V}(t)$ is a TBSS process that satisfy the assumptions for the use of the hybrid scheme. Choosing $\kappa = 1$ the process is simulated, in the grid points, as:

$$V_n\left(\frac{i}{n}\right) = \sqrt{2\alpha+1}\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\frac{i}{n}-s\right)^{\alpha}\mathrm{d}W_s + \sum_{k=2}^{i}\left(\frac{b_k^*}{n}\right)^{\alpha}\left(W_{\frac{i-(k-1)}{n}} - W_{\frac{i-k}{n}}\right)\right)$$

using the covariance structure:

$$\Sigma = \begin{pmatrix} \frac{1}{n} & \frac{1}{(\alpha+1)n^{\alpha+1}} \\ \frac{1}{(\alpha+1)n^{\alpha+1}} & \frac{1}{(2\alpha+1)n^{2\alpha+1}} \end{pmatrix} \tag{2.2}$$

To summarize the steps of the hybrid scheme are:

1. compute the covariance matrix Σ as expressed in (2.2);
2. generate a multivariate normal variable $Z = (Z_1, Z_2)$ with mean $\mu = (0, 0)$ and covariance Σ;
3. estimate the first component (the integral one) of V_n using Z_2 since we have $\int_{\frac{i-1}{n}}^{\frac{i}{n}} (\frac{i}{n} - s)^\alpha \mathrm{d}W_s \sim \mathcal{N}(0, \frac{1}{(2\alpha+1)n^{2\alpha+1}})$;
4. estimate the second component (the discrete sum) computing $(\frac{b_k^*}{n})^\alpha$ and compute the convolution with Z_1;
5. sum the two components and multiply by the factor $\sqrt{2\alpha + 1}$.

References

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