rBergomi

Nodari Alessandro & Proserpio Lorenzo

The rough Bergomi (rBergomi) is a Rough Fractional Stochastic Volatility (RFSV) model. The model has only three, time-independent, parameters and is able to replicate accurately the implied volatility surface dynamics. We will see that we need to use simulation methods to generate option prices as there is no closed form solution and the non-Markovian property of the model doesn't allow a PDE approach. From now on let $(\Omega, \mathbb{F} = \{(F_t)_{t\geq 0}\}, \mathbb{P})$ be a complete filtered probability space, let $\mathbb{P}$ be the *physical-measure* and $T < \infty$ be the right limit of our time horizon.

1 The Model

The rBergomi model is know as a market model, that is to say a financial model consistent with market data. The idea of Bergomi, proposed in [1], is to model the dynamics of the forward variance instead of modelling instantaneous volatility. We will denote the forward variance curve observed at time t with maturity T as $\xi_t(T)$. The forward variance curve observed at time t with maturity T is associated with the fair strike of a variance swap, observed in the same instant t and with the same maturity T, that we will denote as $\sigma_t^2(T)$:

$$\sigma_t^2(T) = \frac{1}{T-t}\int_t^T \xi_t(u)\mathrm{d}u$$

equivalently we have:

$$\xi_t(T) = \frac{\mathrm{d}}{\mathrm{d}T}\left[(T-t)\sigma_t^2(T)\right]$$

1.1 N-Factor Model

In a general N-dimensional setting dictated by the N-dimensional Brownian motion $(W_t^i)_{i=1}^N$ the forward variance $\xi_t(T)$ dynamics are governed by the following SDE:

$$\mathrm{d}\xi_t(u) = \frac{\omega}{\sqrt{\sum_{i,j=1}^N \omega_i\omega_j\rho_{i,j}}}\xi_t(u)\sum_{i=1}^N \omega_i e^{k_i(u-t)}\mathrm{d}W_t^i \qquad (1.1)$$

where we have $\mathrm{d}[W_t^i W_t^j]_t = \rho_{i,j}\mathrm{d}t$ and $\omega_i, k_i > 0$. We also have that $\omega > 0$ is the instantaneous volatility of $\xi_t(t)$. In this general setting the solution is given by:

$$\xi_t(T) = \xi_0(T)\exp\left\{\omega\sum_{i=1}^{N}\omega_i e^{-k_i(T-t)}X_t^i - \frac{\omega^2}{2}\sum_{i,j=1}^{N}\omega_i\omega_j e^{-(k_i+k_j)(T-t)}\mathbb{E}\left[X_t^i X_t^j\right]\right\}$$

where the N driven Ornstein-Uhlenbeck (OU) processes $(X_t^i)_{t\geq 0}$ are defined by:

$$\begin{cases} \mathrm{d}X_t^i = -k_i X_t^i \mathrm{d}t + \mathrm{d}W_t^i \\ X_0^i = 0 \end{cases}$$

The instantaneous volatility of $\xi_t(T)$, thanks to (1.1), is:

$$\omega(T-t) = \frac{2\nu}{\sqrt{\sum_{i,j}\omega_i\omega_j\rho_{i,j}}}\sqrt{\sum_{i,j}\omega_i\omega_j\rho_{i,j}e^{-(k_i+k_j)(T-t)}}$$

Where ν is the log-normal volatility of a Variance Swap with vanishing maturity which can be related to the instantaneous volatility of $\xi_t(t)$ by:

$$\omega = 2\nu$$

1.2 One-Factor Model

Now we will restrict our analysis to the mono-dimensional case that is dictated by the Brownian motion $(W_t)_{t\geq 0}$. Now we have, for the forward variance curve, the following dynamics:

$$\mathrm{d}\xi_t(T) = \omega e^{-k(T-t)}\xi_t(T)\mathrm{d}W_t \qquad (1.2)$$

The choice of an exponential decaying volatility function is equivalent to letting an OU process $(X_t)_{t\geq 0}$ dictate the dynamics of the forward variances. The process X_t has to satisfy the following SDE system:

$$\begin{cases} \mathrm{d}X_t = -kX_t\mathrm{d}t + \mathrm{d}W_t \\ X_0 = 0 \end{cases}$$

We can solve the system and find the solution:

$$X_t = \int_0^t e^{-k(t-s)}\mathrm{d}W_s$$

We can also calculate its expected value, its variance and the expected value of the square of the process:

$$\mathbb{E}[X_t] = 0 \qquad \mathbb{V}[X_t] = \frac{1-e^{-2k}}{2k} \qquad \mathbb{E}[X_t^2] = \mathbb{V}[X_t] = \frac{1-e^{-2k}}{2k}$$

Then the solution to the (1.2) SDE is:

$$\xi_t(T) = \xi_0(T) \exp\left\{ \omega e^{-k(T-t)} X_t - \frac{\omega^2}{2} e^{-2k(T-t)} \mathbb{E}[X_t^2] \right\}$$

This model is still not flexible enough.

1.3 2-Factors Model

Here we will present the 2-factors model in which we can achieve greater flexibility in the term-structure of volatilities of variances that can be generated. We will use a mixing parameter $\theta \in [0,1]$ and the dynamics become:

$$\begin{cases} \mathrm{d}\xi_t(T) = \omega \alpha_\theta \xi_t(T) \big[(1-\theta) e^{-k_1(T-t)} \mathrm{d}W_t^1 + \theta e^{-k_2(T-t)} \mathrm{d}W_t^2 \big] \\ \alpha_\theta = 1/\sqrt{(1-\theta)^2 + \theta^2 + 2\rho_{12}(1-\theta)\theta} \end{cases} \tag{1.3}$$

Where ρ is the correlation between W^1 and W^2 and we have defined the two OU processes X^1 and X^2 given by:

$$\begin{cases} \mathrm{d}X_t^i = -k_i X_t^i \mathrm{d}t + \mathrm{d}W_t^i \\ X_0^i = 0 \end{cases}$$

We will also define an auxiliary gaussian drift-less process:

$$\mathrm{d}x_t^T = \alpha_\theta \big[(1-\theta) e^{-k_1(T-t)} \mathrm{d}W_t^1 + \theta e^{-k_2(T-t)} \mathrm{d}W_t^2 \big]$$

whose quadratic variation is given by:

$$\mathrm{d}\langle x^T \rangle_t = \eta^2(T-t)\mathrm{d}t$$

where we have defined:

$$\eta(s) := \alpha_\theta \sqrt{(1-\theta)^2 e^{-2k_1 s} + \theta^2 e^{-k_2 s} + 2\rho_{12}\theta(1-\theta) e^{-(k_1+k_2)s}}$$

Thus substituting in the SDE (1.3) we obtain:

$$\mathrm{d}\xi_t(T) = \omega \xi_t(T) \mathrm{d}x_t^T$$

So the solution is given by:

$$\begin{cases} \xi_t(T) = \xi_0(T) e^{\omega x_t^T - \frac{\omega^2}{2} f(t,T)} \\ f(t,T) = \displaystyle\int_{T-t}^{T} \eta^2(u) \mathrm{d}u \end{cases}$$

We can explicit the value of $f(t,T)$ that is:

$$f(t,T) = \alpha_\theta^2 \Bigg[\frac{(1-\theta)^2}{2k_1} e^{-2k_1(T-t)} (1 - e^{-2k_1 t}) + \frac{\theta^2}{2k_2} e^{-2k_2(T-t)} (1 - e^{-2k_2 t}) \\ + 2\theta(1-\theta)\rho_{12} e^{-(k_1+k_2)(T-t)} \frac{1 - e^{-(k_1+k_2)t}}{k_1 + k_2} \Bigg]$$

2 The Realized Variance

We will use the Mandelbrot-Vann Ness representation of the fractional Brownian motion to express the increments of the logarithm of realized variance $v = \sigma^2$ as:

$$\begin{aligned}
\log(v_u) - \log(v_t) &= 2\nu C_H (W_u^H - W_t^H) \\
&= 2\nu C_H \left(\int_{-\infty}^{u} (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} - \int_{-\infty}^{t} (t-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} \right) \\
&= 2\nu C_H \left(\int_{t}^{u} (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} + \int_{-\infty}^{t} \left[(u-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right] \mathrm{d}W_s^{\mathbb{P}} \right) \\
&=: 2\nu C_H \left[M_t(u) + Z_t(u) \right]
\end{aligned}$$

We note that $\mathbb{E}[M_t(u)|\mathcal{F}_t] = 0$ and that $Z_t(u)$ is $\mathcal{F}_t$-measurable. If we define $\tilde{W}_t^{\mathbb{P}}$ as:

$$\tilde{W}_t^{\mathbb{P}}(u) := \sqrt{2H} \int_t^u |u-s|^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}}$$

it has the same properties of $M_t(u)$ and defining $\eta := \frac{2\nu C_H}{\sqrt{2H}}$ we have that:

$$\log(v_u) - \log(v_t) = \eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)$$

This gives us:

$$v_u = v_t \exp\left\{ \eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u) \right\}$$

Thanks to the properties, gaussianity in this case, of $\tilde{W}_t^{\mathbb{P}}$ we have:

$$\tilde{W}_t^{\mathbb{P}}(u) \sim \mathcal{N}\big(0, (u-t)^{2H}\big)$$

which gives us that $v_u|\mathcal{F}_t$ is log-normal and thus entails that:

$$\mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] = v_t \exp\left\{ 2\nu C_H Z_t(u) + \frac{1}{2}\eta^2 (u-t)^{2H} \right\}$$

Now we can express the realized variance as:

$$v_u = \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \mathcal{E}(\eta \tilde{W}_t^{\mathbb{P}}(u))$$

where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential.

3 The probability measure change

As observed in [2] the model with two factors is over-parameterized so we will use the 1-factor model. From what we have said up to now we have that, under the physical probability $\mathbb{P}$, the model is expressed as:

$$\begin{cases} \mathrm{d}S_u = \mu_u S_u \mathrm{d}u + \sqrt{v_u} S_u \mathrm{d}Z_u^{\mathbb{P}} \\ v_u = v_t \exp\left\{\eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)\right\} \end{cases}$$

Now we want to change the physical probability measure $\mathbb{P}$ with an equivalent martingale measure $\mathbb{Q}$ in the interval $[t, T]$ in order to price options. To do that we use Girsanov theorem and obtain:

$$\mathrm{d}Z_u^{\mathbb{Q}} = \mathrm{d}Z_u^{\mathbb{P}} + \frac{\mu_u - (r-q)}{\sqrt{v_u}} \mathrm{d}u$$

When we change from $\mathbb{P}$ to $\mathbb{Q}$ we also have to remember that the Brownian motion $W_u^{\mathbb{P}}$, that is used to construct the Volterra-type process $\tilde{W}_u^{\mathbb{P}}$, is correlated with $Z_u^{\mathbb{P}}$ with correlation factor ρ:

$$\mathrm{d}W_u^{\mathbb{P}} = \rho \mathrm{d}Z_u^{\mathbb{P}} + \sqrt{1-\rho^2} \mathrm{d}(Z_u^{\perp})^{\mathbb{P}}$$

where $(Z_u^{\perp})^{\mathbb{P}}$ is independent of $Z_u^{\mathbb{P}}$. A change of measure for $(Z_u^{\perp})^{\mathbb{P}}$ is of the form:

$$(Z_u^{\perp})^{\mathbb{Q}} = (Z_u^{\perp})^{\mathbb{P}} + \gamma_u \mathrm{d}u$$

where γ_u is a suitable process that can be seen as the market price of volatility risk. Now we can express the change in measure also for $W_u^{\mathbb{Q}}$:

$$\begin{aligned} \mathrm{d}W_u^{\mathbb{Q}} &= \rho \mathrm{d}Z_u^{\mathbb{Q}} + \sqrt{1-\rho^2} \mathrm{d}(Z_u^{\perp})^{\mathbb{Q}} \\ &= \mathrm{d}W_u^{\mathbb{P}} + \left(\frac{\mu_u - (r-q)}{\sqrt{v_u}} \rho + \gamma_u \sqrt{1-\rho^2} \right) \mathrm{d}u \\ &= \mathrm{d}W_u^{\mathbb{P}} + \lambda_u \mathrm{d}u \end{aligned}$$

We may now rewrite, assuming that the filtration generated by $W^{\mathbb{P}}$ is the same as the one generated by $W^{\mathbb{Q}}$, the dynamics of the realized variance:

$$\begin{aligned} v_u &= \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \exp\left\{ \eta\sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} - \frac{\eta^2}{2}(u-t)^{2H} \right\} \\ &= \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \mathcal{E}\left(\eta \tilde{W}_t^{\mathbb{Q}}(u)\right) \exp\left\{ \eta\sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \lambda_s \mathrm{d}s \right\} \\ &= \mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] \mathcal{E}\left(\eta \tilde{W}_t^{\mathbb{Q}}(u)\right) \\ &= \xi_t(u) \mathcal{E}\left(\eta \tilde{W}_t^{\mathbb{Q}}(u)\right) \end{aligned}$$

where we have

$$\tilde{W}_t^{\mathbb{Q}}(u) \coloneqq \sqrt{2H}\int_t^u (u-s)^{H-\frac{1}{2}}\mathrm{d}W_s^{\mathbb{Q}}$$

The model under $\mathbb{Q}$ is thus expressed as:

$$\begin{cases} \mathrm{d}S_u = (r-q)S_u\mathrm{d}u + \sqrt{v_u}S_u\mathrm{d}Z_u^{\mathbb{Q}} \\ v_u = \xi_t(u)\mathcal{E}\big(\eta\tilde{W}_t^{\mathbb{Q}}(u)\big) \end{cases}$$

This model is a non-Markovian generalization of the Bergomi model. Specifically, this model is non-Markovian in the instantaneous variance v_t:

$$\mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] \neq \mathbb{E}^{\mathbb{Q}}[v_u|v_t]$$

but is Markovian in the infinite-dimensional state vector:

$$\mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] = \xi_t(u)$$

4 Pricing

Under the pricing measure $\mathbb{Q}$, given the starting time $t_0 = 0$, the scheme to simulate the model is:

$$\begin{cases} S_t = S_0 \exp\left\{(r-q)t - \frac{1}{2}\int_0^t v_u\mathrm{d}u + \int_0^t \sqrt{v_u}\mathrm{d}Z_u^{\mathbb{Q}}\right\} \\ v_t = \xi_0(t)\exp\left\{2\nu C_H\int_0^t (t-u)^{H-\frac{1}{2}}\mathrm{d}W_u^{\mathbb{Q}} - \frac{\nu^2 C_H^2}{H}t^{2H}\right\} \\ \mathrm{d}[Z^{\mathbb{Q}}, W^{\mathbb{Q}}]_t = \rho\mathrm{d}t \end{cases}$$

First we need to simulate the Volterra process using the hybrid scheme introduced in section (...). Then we need to extract the Brownian Motion $W^{\mathbb{Q}}$ that drives the Volterra process and correlate it with $Z^{\mathbb{Q}}$ by the parameter ρ. Lastly, we simulate the stock price process S using the forward Euler scheme. To sum up, in order to simulate the stock price process we have to:

1. fix an equispaced grid $\mathcal{G} = \{t_0 = 0, t_1 = \frac{1}{n}, \ldots, t_{\lfloor nT \rfloor} = \frac{\lfloor nT \rfloor}{n}\}$;
2. simulate the Volterra process $\mathcal{V}_t = \displaystyle\int_0^t (t-u)^{H-\frac{1}{2}}\mathrm{d}W_u^{\mathbb{Q}}$, $t \in \mathcal{G}$, using the hybrid scheme;
3. compute the variance process v using the previously computed Volterra process:

$$v_t = \xi_0(t)\mathcal{E}\big(2\eta C_H\mathcal{V}_t\big) \qquad t \in \mathcal{G}$$

4. extract the path of the Brownian Motion $W^{\mathbb{Q}}$ that drives the Volterra process:

$$
\begin{aligned}
W^{\mathbb{Q}}_{t_i} &= W^{\mathbb{Q}}_{t_{i-1}} + n^{H-\frac{1}{2}}\big(\mathcal{V}_{t_i} - \mathcal{V}_{t_{i-1}}\big) \qquad && i = 1, \ldots, \kappa \\
W^{\mathbb{Q}}_{t_i} &= W^{\mathbb{Q}}_{t_{i-1}} + W^n_{i-1} && i > \kappa
\end{aligned}
$$

   where W^n is defined as in (2.2) [fBM hybrid scheme section]

5. correlate the stock price process, driven by $Z^{\mathbb{Q}}$, and the variance process, driven by $W^{\mathbb{Q}}$ through the Volterra process, as:

$$Z^{\mathbb{Q}}_{t_i} - Z^{\mathbb{Q}}_{t_{i-1}} = \rho\big(W^{\mathbb{Q}}_{t_i} - W^{\mathbb{Q}}_{t_{i-1}}\big) + \sqrt{1-\rho^2}\big(W^{\mathbb{Q},\perp}_{t_{i-1}} - W^{\mathbb{Q},\perp}_{t_{i-1}}\big)$$

   where $W^{\mathbb{Q},\perp}$ is a standard Brownian Motion independent of $W^{\mathbb{Q}}$;

6. simulate the stock price process S using the forward Euler scheme:

$$S_{t_i} = S_{t_{i-1}} + (r-q)S_{t_{i-1}}(t_i - t_{i-1}) + \sqrt{v_{t_{i-1}}}S_{t_{i-1}}\big(Z^{\mathbb{Q}}_{t_i} - Z^{\mathbb{Q}}_{t_{i-1}}\big)$$

To price an option at time $t < T$, where T is the maturity, that has payoff $f(S_T)$ we have to calculate the discounted payoff given by:

$$P_t = \mathbb{E}^{\mathbb{Q}}\big[e^{-(r-q)(T-t)}f(S_T)|\mathcal{F}_t\big]$$

To compute this quantity we will use a Monte-Carlo simulation. In practice we will use the implementation devised by McCricked and Pakkanen in [3].

5 Calibration

Since we are using the one-factor rBergomi model when we are calibrating it to market data we are finding the values for the parameters used to simulate the price paths: H, η and ρ. These parameters have direct interpretation: H controls the decay of the term structure of volatility skew for very short expirations whereas the product $\rho\eta$ sets the level of the ATM skew for longer expirations. Keeping the product $\rho\eta$ roughly constant but decreasing ρ (so as to make it more negative) has the effect of pushing the minimum of each smile towards higher strikes. As the initial forward variance curve we will use the parametrization found in **Section [QUELLA SULLA VARIANCE]**.

5.1 Objective Function

To calibrate the model we want to minimize the difference between the market implied volatility and the rBergomi implied volatility. To calculate the rBergomi IV we have to set the parameters, simulate the price paths, compute the price of the option (given a certain strike k and time to maturity τ) and then invert the B&S formula to find the IV given by our simulation:

$$\sigma_{rB}(k,\tau,H,\eta,\rho) = P_{BS}^{-1}(k,\tau,H,\eta,\rho)$$

We indicate the set of parameters that we have to calibrate as $\Theta := \{H, \eta, \rho\}$ and the set of strikes for a given maturity as K. The problem of minimization is thus:

$$\underset{\Theta}{\operatorname{argmin}} f(K,\tau) := \underset{\Theta}{\operatorname{argmin}} \|\sigma_{rB}(k,\tau) - \sigma_{mkt}(k,\tau)\|^2$$

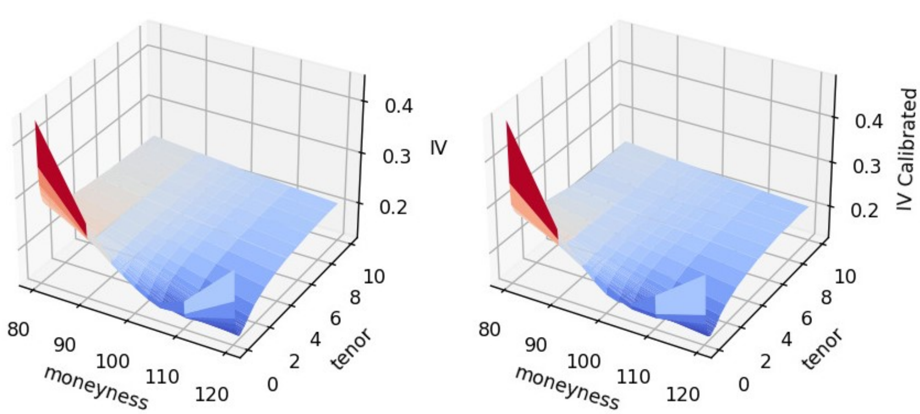
where σ_{mkt} represents the market IV. We calibrate the model for every tenor using the sequential least squares programming algorithm proposed in [4].

5.2 Numerical results

We report some of the calibrated parameters in the following table.

Tenor	**H**	η	ρ
2 weeks	0.0279	1.9077	-0.8583
1 month	0.0328	2.0162	-0.8441
6 months	0.0852	1.951849	-0.8779
1 year	0.0888	1.7579	-0.9836
10 years	0.0562	1.5464	-0.8584

Comparing our results with the market we obtained a relative error of 1.2232%.



Comparison between market and model IV

6 Volatility Skew

The at-the-money (ATM) volatility skew is defined as:

$$\psi(\tau) := \left| \frac{\partial}{\partial k} \hat{\sigma}(k, \tau) \right|_{k=0}$$

where k is the log-moneyness, τ is the time to maturity and $\hat{\sigma}(k, \tau)$ is the implied volatility in terms of k. The term structure of the market data at-the-money-forward (ATMF) volatility skew is defined as:

$$\Psi(\tau) := \left| \frac{\partial}{\partial K} \sigma_{mkt}(K, \tau) \right|_{K=F}$$

where K is the strike, F is the forward price and σ_{mkt} is the market implied volatility. It is easy to express k in terms of K as $k = \log \frac{K}{F}$ and putting $\hat{\sigma}(k, \tau) = \sigma_{mkt}(K, \tau)$ we obtain:

$$\frac{\partial}{\partial K} \sigma_{mkt}(K, \tau) = \frac{\partial}{\partial k} \hat{\sigma}(k, \tau) \frac{\partial k}{\partial K} = \frac{1}{K} \frac{\partial}{\partial k} \hat{\sigma}(k, \tau)$$

so that we have mapped data expressed in terms of strike K to data expressed in terms of log-moneyness k. Thus the ATMF volatility skew can be expressed as:

$$\psi(\tau) = F \left| \frac{\partial}{\partial K} \sigma_{mkt}(K, \tau) \right|_{K=F}$$

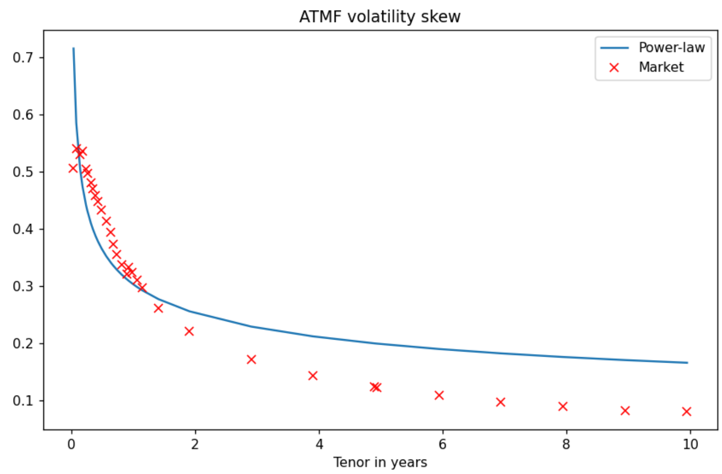
6.1 Numerical results

In order to compute the market ATMF volatility skew we will fit a stochastic volatility inspired (SVI) model to the market implied volatility data. The SVI model, presented in [5] and further expanded in [6], is calibrated to the market implied volatility surface using a set of parameters $\lambda = \{a, b, \rho, m, \sigma\}$ such that the total implied variance is expressed, for $k \in \mathbb{R}$, as:

$$\sigma^2_{imp}(k; \lambda) = a + b\big[\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}\big]$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$ and $\sigma > 0$. In addition we have that $\sigma^2_{imp}(k; \lambda) > 0$ if $a + b\sigma\sqrt{1 - \rho^2} \geq 0$. The term structure of the ATMF volatility skew can be approximated by a power-law function of the time to maturity, as shown in [7]. In this context the function $\tau \to A\tau^{-\alpha} + b$ is fitted to the market data that we have obtaining the following values:

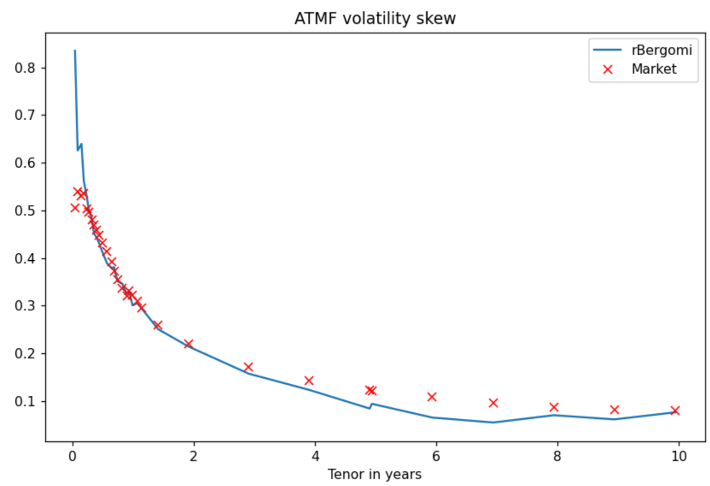
$$A = 0.30263661 \quad \alpha = 0.26363489$$



This function has motivated the use of a power-law kernel function to model volatility, leading to the rBergomi model. Indeed, when volatility is driven by a fBM with Hurst parameter H, then the skew is proportional to $\tau^{H-\frac{1}{2}}$. As α is of order 0.3, H should be of order 0.2, that is close to the values that we obtained in the previous section. To compute the ATMF volatility skew given by the rBergomi model we will first calculate the implied volatility given by the model for log-moneyness in the neighbourhood of $k = 0$ and then use the central finite difference approximation. Letting $h > 0$ be the difference step and τ be one of the tenors the approximation of ψ is given by:

$$\psi(\tau) \approx \left| \frac{\hat{\sigma}_{rB}(h, \tau) - \hat{\sigma}_{rB}(-h, \tau)}{2h} \right|$$

By construction, as the smile generated by the rBergomi model fits the market data also the skew generated/approximated by rBergomi should fit the market skew obtained by the SVI model. In practice, using $h = 10^{-3}$ for the approximation scheme we obtained the following results.



7 Forward-Start Options

The sensitivity to forward-smile risk, described as the risk coming from the market future implied volatility and its uncertainty, is found in many options. These options, of which the forward-start options are an example, are priced given the distribution of forward returns in the model as described in [8]. The payoff of a forward-start option involves the price of the underlying in two different dates T_1 and T_2 with $T_1 < T_2$. The payoff can be expressed as:

$$\left(S_{T_2} - kS_{T_1}\right)^+$$

where $k > 0$ is the moneyness of the contract. The forward smile represents the expected future implied volatility for moneyness k: all possible realizations of future smiles are averaged to give $\hat{\sigma}_k^{T_1,T_2}$, that is the implied Black and Scholes volatility for the forward-start option with moneyness k. The instantaneous volatility is time-dependent, due to the forward-start option's nature, whose payoff depends on S_{T_1} and S_{T_2}. The Black and Scholes implied variance for maturity T is given by:

$$\hat{\sigma}_T^2 := \int_t^T \sigma^2(u)\mathrm{d}u$$

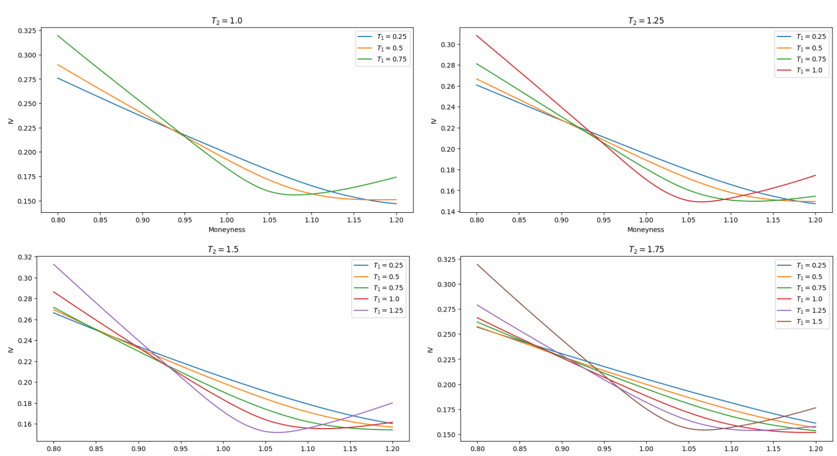
The price of a forward-start option is independent of the underlying price S, but depends on the forward volatility $\hat{\sigma}_{T_1,T_2}$, or the integrated variance over $[T_1, T_2]$, in such a way that:

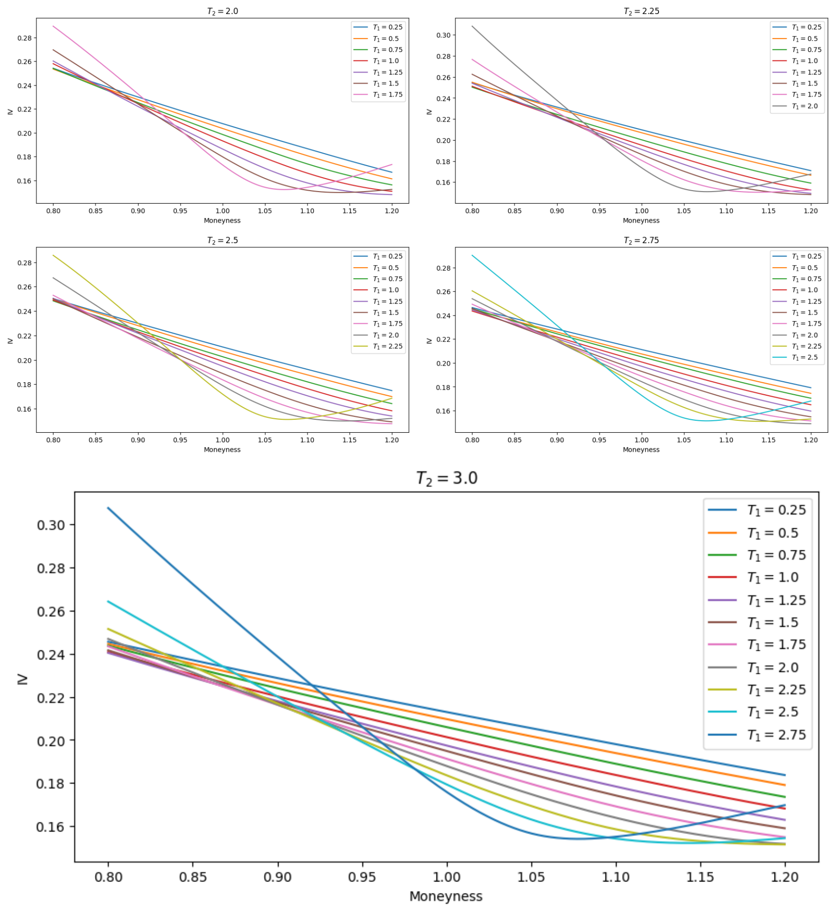
$$\hat{\sigma}^2_{T_1,T_2} := \int_{T_1}^{T_2} \sigma^2(u)\mathrm{d}u$$
$$= \frac{(T_2 - t)\hat{\sigma}^2_{T_2} - (T_1 - t)\hat{\sigma}^2_{T_1}}{T_2 - T_1}$$

7.1 Implied Volatility

As for a vanilla option to price a forward-start option in the rBergomi model we have to resort to a Monte-Carlo method. The generation of a price path is executed as explained in **Section 4** and then we extract the values at time T_1 and T_2. The algorithm to compute the implied volatility of a forward-start option under rBergomi is the following:

1. set a range $\{1, 1.25, \ldots, 3\}$ of step 0.25 for the maturity T_2;
2. set a range $\{0.25, \ldots, T_2 - 0.25\}$ of step 0.25 for the starting date T_1;
3. simulate the price path under rBergomi;
4. set a range $\{0.8, 0.85, \ldots, 1.2\}$ of step 0.05 for the moneyness k;
5. compute the payoff of the forward-start option as $(S_{T_2} - kS_{T_1})^+$;
6. compute the price of the forward-start option as the discounted payoff over a Monte-Carlo simulation;
7. compute the corresponding Black and Scholes implied volatility using a root finding method, for example bisection.





8 Bayesian Inverse Problem

To calibrated the model we used the Approximate Bayesian Computation (ABC) that is a sequential Monte-Carlo method. ABC methods (also called likelihood free inference methods), are a group of techniques developed for inferring posterior distributions in cases where the likelihood function is intractable or costly to evaluate. This does not mean that the likelihood function is not part of the analysis, it just means that we are approximating the likelihood. ABC comes useful when the model used contains unobservable random quantities, which make the likelihood function hard to specify, but data can be simulated from the model. These methods follow a general form:

1. Sample a parameter θ from a prior/proposal distribution $\pi(\theta)$.
2. Simulate a data set y using a function that takes θ and returns a data set of the same dimensions as the observed data set y_0.

3. Compare the simulated dataset y with the experimental data set y_0 using a distance function d and a tolerance threshold ε.

In some cases a distance function is computed between two summary statistics $d(S(y^*), S(y_0))$, avoiding the issue of computing distances for entire datasets. As a result we obtain a sample of parameters from a distribution $\pi(\theta|d(y, y_0))$. If ε is sufficiently small this distribution will be a good approximation of the posterior distribution $\pi(\theta|y_0)$.
Sequential monte carlo ABC is a method that iteratively morphs the prior into a posterior by propagating the sampled parameters through a series of proposal distributions, weighting the accepted parameters $\theta^{(i)}$ like:

$$w^{(i)} \propto \frac{\pi(\theta^{(i)})}{\phi(\theta^{(i)})}$$

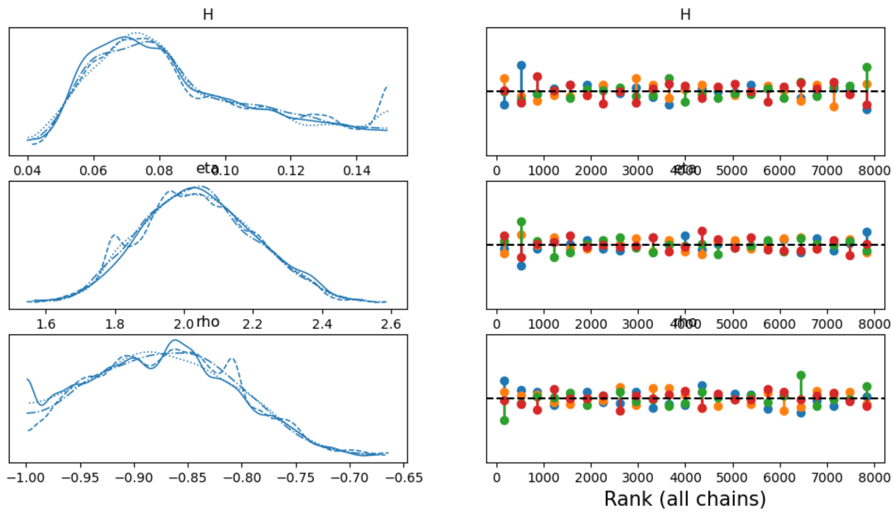
It combines the advantages of traditional SMC, i.e. ability to sample from distributions with multiple peaks, but without the need for evaluating the likelihood function.

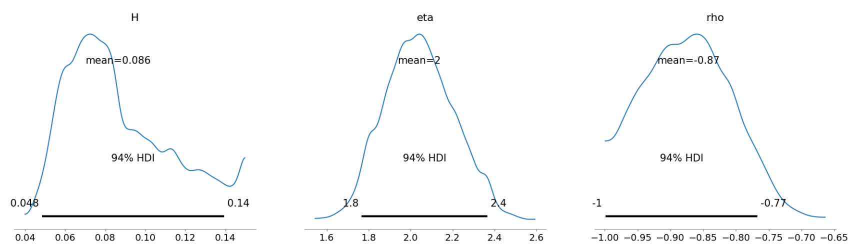
8.1 Numerical results

We decided to calibrate the model using only the tenors from 1 month to 6 months (10 in total). The set of parameters to calibrate is ρ, H and η. We used priors as uninformative as possible:

- $\rho \sim \mathcal{U}(-1, 0)$
- $H \sim \mathcal{U}(0, 0.15)$
- $\eta \sim \mathcal{U}(1, 5)$

We calibrated using the prices instead of the IV. We used 4 chains with 2000 samples each, just for computational and time reasons. Then to test the quality of the calibration we used the maximum of the resulting distribution and plugged it in the model.

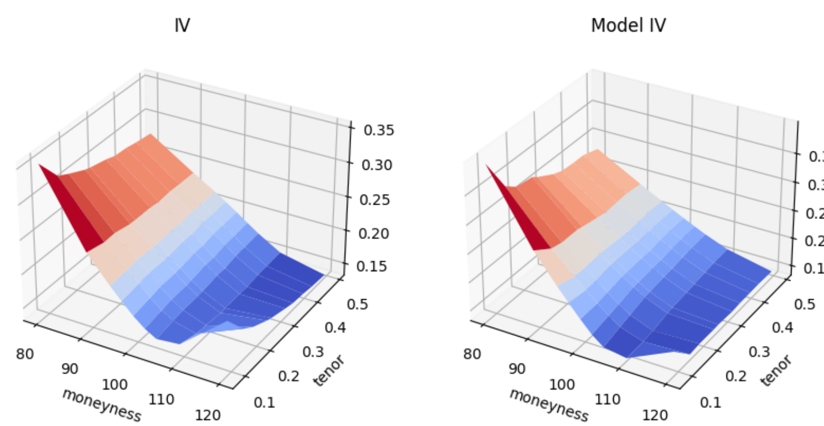




To compare the Bayesian calibration with the deterministic calibration we fixed the parameters value using the maximum a posteriori:

ρ	H	η
-0.8540	0.0669	1.9707

Doing so we obtained a mean percentage error of 3.6005%.



References

1. L. Bergomi: Smile dynamics II. Risk, 10, pp. 67-73, 2005.

2. C. Bayer, P. Friz and J. Gatheral: Pricing under rough volatility. Quantitative Finance 16(6), 887–904, 2016.

3. R. McCrickerd, M. Pakkanen: Turbocharging Monte Carlo pricing for the rough Bergomi model, 2018.

4. D. Kraft. A software package for sequential quadratic programming. Tech. Rep. DFVLR-FB, Vol. 88-28, DLR German Aerospace Center, Institute for Flight Mechanics, Koln, Germany, 1988.

5. J. Gatheral. A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives. Presentation at Global Derivatives, 2004.

6. J. Gatheral and A. Jacquier. Arbitrage-free SVI volatility surfaces. Quantitative Finance, 14(1): 59–71, 2014.

7. J. Gatheral, T. Jaisson and M. Rosenbaum. Volatility is rough. Available at arXiv:1410.3394, 2014.

8. L. Bergomi. Stochastic Volatility Modeling. 2015.