rHeston

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As we have seen the skew of the at the money options in the Heston model does not represent well the skew of real-world at the money options. This, among other reasons, made the researcher and practitioner to think about more sophisticated models. As observed by Gatheral, Jaisson and Rosenbaum the log-volatility in the market behaves like a fractional Brownian motion with small Hurst parameter, so the volatility is rough.

1 Stylized empirical facts

Heston model reproduces several important features of low frequency price data, provides quite reasonable dynamics for the volatility surface and it can be calibrated efficiently. If we want to surpass that we have to build a model which can reproduce the stylized facts of modern electronic markets in the context of high frequency trading. In practice each market behaves tick-by-tick, indeed we receive an update in price by market-makers whenever there is a trade and the movement is discrete and at least of one tick (usually 1 cent). There are 4 main stylized facts that we can observe in market data:

1. Markets are highly endogenous, as showed by Bouchad. This means that most of the orders have no real economic motivation, but are simply the reaction of algorithms to other orders.

2. Markets at high frequency are much more efficient than at lower frequencies, this means that it is much more difficult to find profitable statistical arbitrage strategies.

3. There is some asymmetry in the liquidity on the bid and the ask side of the order book. Indeed, a market-maker is likely to raise the price by less following a buy order than to lower the price following the same size sell order, as seen by Brunnermeier and Pedersen. This is mostly due to the fact that hedging the first position is easier than the second and that market-makers have usually some inventory.

4. A large proportion of transaction is due to big orders, called metaorders, which are not executed at once, but split in time. Indeed, one of the most challenging part of every trading strategies is to execute it in large volumes without moving changing to much the state of the market.

2 Building the model

As in El Euch, Fukasawa & Rosenbaum we will start building a model from Hawkes processes, then slowly including the stylized fact mentioned in the last paragraph and showing that the long-term dynamic of this model will lead to a rough Heston model at the macroscopic scale, in which the leverage effect is still represented.

2.1 Hawkes Processes

Hawkes processes are point processes which are said to be self-exciting, in the sense that the instantaneous jump-probability depends on the location of the past events. In particular we will focus on a bivariate Hawkes process, $(N_t^+, N_t^-)_{t\geq 0}$, where N_t^+ is the number of upward jumps of one tick and N_t^- is the number of downward jumps of one tick, both in the interval $[0, t]$. The probability to get one-tick upward jump in a time $\mathrm{d}t$ is given by $\lambda_t^+\mathrm{d}t$, viceversa by λ_t^-. The array $(\lambda_t^+, \lambda_t^-)$ is called intensity of the process and it is of the form:

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\phi_1(t-s) & \phi_3(t-s)\\ \phi_2(t-s) & \phi_4(t-s)\end{pmatrix} \cdot \begin{pmatrix}\mathrm{d}N_s^+\\ \mathrm{d}N_s^-\end{pmatrix}$$

where μ^+ and μ^- are positive constants and the components of the matrix are positive and locally integrable functions. The process of the prices P_t is given by the difference between the number of upward jumps from time 0 and the number of downward jumps, so:

$$P_t = N_t^+ - N_t^-$$

Each component of the intensity can be decomposed into three parts, for example λ_t^- can be decomposed in:

- μ_t^- which corresponds to the probability that the price will go down because of exogenous reasons;
- $\int_0^t \phi_2(t-s)\mathrm{d}N_s^+$ which corresponds to the probability of a downward jump induced by past upward jumps;
- $\int_0^t \phi_4(t-s)\mathrm{d}N_s^-$ which corresponds to the probability of a downward jump induced by past downward jumps.

Now we will see that, when the ϕ_j have suitable forms the model can reproduce the stylized effects described in the previous section. Moreover, we want to underlying that, due to how the model is built, the price process assumes discrete values, as in the real world.

2.2 Encoding the 2^{nd} property

Since the markets at high frequency are expected to be more efficient then this translate in that, over any period of time, we should have on average the same number of upwards jumps than downwards jumps. This can be translated in:

$$\int_0^t \mathbb{E}[\lambda_s^+]\mathrm{d}s = \mathbb{E}[N_t^+] = \mathbb{E}[N_t^-] = \int_0^t \mathbb{E}[\lambda_s^-]\mathrm{d}s \tag{1}$$

remembering how we have defined λ_t^+ and λ_t^-:

$$\mathbb{E}[\lambda_t^+] = \mu^+ + \int_0^t \phi_1(t-s)\,\mathbb{E}[\lambda_s^+]\mathrm{d}s + \int_0^t \phi_3(t-s)\,\mathbb{E}[\lambda_s^-]\mathrm{d}s$$
$$\mathbb{E}[\lambda_t^-] = \mu^- + \int_0^t \phi_4(t-s)\,\mathbb{E}[\lambda_s^-]\mathrm{d}s + \int_0^t \phi_2(t-s)\,\mathbb{E}[\lambda_s^+]\mathrm{d}s$$

the simplest way to satisfy the equation (1) is to put:

$$\mu^+ = \mu^- \text{ and } \phi_1 + \phi_3 = \phi_2 + \phi_4$$

2.3 Encoding the 3^{rd} property

Market-makers act as liquidity providers, in practice at the beginning they are long inventory, so the ask side is more liquid than the bid side. This translate into the fact that the conditional probability of an upward jump right after an upward jump is smaller than the conditional probability to observe a downward jump after a downward jump. This means that for $t \to 0$ we have:

$$\int_0^t \phi_4(t-s)\mathrm{d}N_s^- > \int_0^t \phi_1(t-s)\mathrm{d}N_s^+$$

or equivalently

$$\int_0^t \phi_2(t-s)\mathrm{d}N_s^+ < \int_0^t \phi_3(t-s)\mathrm{d}N_s^-$$

this can be satisfied in several ways, but we make the strong assumption that exists a constant $\beta > 0$ such that $\phi_3 = \beta\phi_2$. Putting all together we have that the structure of our intensity process is:

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \mu\begin{pmatrix}1\\1\end{pmatrix} + \int_0^t \begin{pmatrix}\phi_1(t-s) & \beta\phi_2(t-s)\\ \phi_2(t-s) & [\phi_1+(\beta-1)\phi_2](t-s)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}N_s^+\\ \mathrm{d}N_s^-\end{pmatrix}$$

2.4 Encoding the 1^{st} property

Markets have an high degree of endogeneity, which means that the proportion of "non-meaningful" orders with respect to the totality of the orders is close to 1. In order to have an intuition on how to include this effect in our model we need to recall dynamical systems. A dynamical system has a stationary point (or equilibrium) if its spectral radius is less than 1, in the same way we have a kernel transition matrix:

$$\int_0^T \begin{pmatrix}\phi_{1,T}(s) & \beta\phi_{2,T}(s)\\ \phi_{2,T}(s) & [\phi_{1,T}+(\beta-1)\phi_{2,T}](s)\end{pmatrix}\mathrm{d}s = \int_0^t \Phi_T(s)\mathrm{d}s$$

we can extend all the functions on $[0,\infty)$ with the constant zero. We refer to them with a tilde. The spectral radius in our case is equal to:

$$\sigma\left(\int_0^\infty \tilde{\Phi}_T(s)\mathrm{d}s\right) = \|\tilde{\phi}_{1,T}\|_1 + \beta\|\tilde{\phi}_{2,T}\|_1$$

Let $(\Omega, \mathbb{F} = \{(F_T)_{T\geq 0}\}, \mathbb{P})$ a complete filtered probability space. We can find a sequence $\{(\tilde{\phi}_{1,T};\tilde{\phi}_{2,T})\}_T$ of couple of positive functions each one in the respective $\mathcal{L}^1(F_T)$ such that:

- $\forall T > 0$ we have $\|\tilde{\phi}_{1,T}\|_1 + \beta\|\tilde{\phi}_{2,T}\|_1 < 1$;
- if $T_2 > T_1$ we have both $\tilde{\phi}_{1,T_2} \geq \tilde{\phi}_{1,T_1}$ and $\tilde{\phi}_{2,T_2} \geq \tilde{\phi}_{2,T_1}$;
- satisfying:
$$\lim_{T\to\infty}\left[\|\tilde{\phi}_{1,T}\|_1 + \beta\|\tilde{\phi}_{2,T}\|_1\right] = 1$$

then there exist a limit to this sequence and we will call that $(\tilde{\phi}_1;\tilde{\phi}_2)$ and, due to continuity of the norm, we have that $\|\tilde{\phi}_1\|_1+\beta\|\tilde{\phi}_2\|_1 = 1$. Then we have built our nearly-unstable system for each $T > 0$ sufficiently large. Moreover notice that, due to the second property of our sequence, we have that the spectral radius, when T is increasing, is also increasing. We will refer to the matrix obtained with $(\tilde{\phi}_1;\tilde{\phi}_2)$ as Φ. The process of the prices up to time $T < \infty$ is now indicated as:

$$P_t^T = N_t^{T,+} - N_t^{T,-}$$

with $N_t^{T,+}$ and $N_t^{T,-}$ with intensity generated by $(\tilde{\phi}_{1,T};\tilde{\phi}_{2,T})$. From now on, we will denote

$$\sigma\left(\int_0^\infty \tilde{\Phi}_T(s)\mathrm{d}s\right) = a_T = \|\tilde{\phi}_{1,T}\|_1 + \beta\|\tilde{\phi}_{2,T}\|_1$$

with a_T constants and $a_T \uparrow 1$. Moreover, we can also construct the Φ_T as $\Phi = a_T\Phi_T$. We will refer to this as **Assumption 1**.

2.5 Encoding the 4^{th} property

As showed by Jaisson and Rosenbaum, the effect of metaorders are reflected in the Hawkes framework by considering the condition that the kernel matrix exibits heavy-tails. In order to encode the metaorders in the framework we need to put some additional assumptions. Let

$$\Psi_T = \sum_{k\geq 1}(\Phi_T)^{*k}$$

where $(\Phi_T)^{*1} = \Phi_T$ and, for $k > 1$, $(\Phi_T)^{*k}(t) = \int_0^t \Phi_T(s)(\Phi_T)^{*(k-1)}(t-s)\mathrm{d}s$. The **Assumption 2** is that Ψ_T is uniformly bounded, Φ is differentiable and the derivative of each component of Φ is bounded and with finite norm 1. In

order to satisfy this assumption is sufficient, but not necessary, that ϕ_1 and ϕ_2 are both non-increasing functions in $\mathcal{L}^1 \cap \mathcal{L}^\infty$ and differentiable.
The last assumption, **Assumption 3**, is that there exist $\alpha \in (1/2, 1)$ and $C > 0$ such that

$$\alpha t^\alpha \int_t^\infty [\phi_1 + \beta\phi_2](s)\mathrm{d}s \xrightarrow[t\to\infty]{} C$$

and moreover, for some $\mu > 0$ and $\lambda^* > 0$,

$$T^\alpha(1 - a_T) \xrightarrow[t\to\infty]{} \lambda^* \text{ and } T^{1-\alpha}\mu_T \xrightarrow[t\to\infty]{} \mu$$

under this three assumptions also the last stylized effect has been encoded in our model.

2.6 From microstructure to macrostructure

If **Assumptions 1,2 and 3** hold then it happens that the asymptotic behaviour of the microstructural model that we have built behaves like an Heston model, more precisely a rough version of it. Indeed, using the same notation as in **Assumption 3**, let:

$$\lambda = \frac{\alpha\lambda^*}{C\Gamma(1-\alpha)}$$

then it holds true the following theorem:

Theorem. *As $T \to \infty$ then the rescaled microscopic price*

$$\sqrt{\frac{1-a_T}{\mu T^\alpha}} P_t^T$$

converges in the sense of finite dimensional laws to the following rough Heston model:

$$P_t = \frac{1}{1 - (\|\phi_1\|_1 - \|\phi_2\|_1)} \sqrt{\frac{2}{\beta+1}} \int_0^t \sqrt{v_s}\mathrm{d}W_s$$

where v_t is the solution to the following rough SDE:

$$v_t = \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1}\lambda(1+\beta-v_s)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\lambda\sqrt{\frac{1+\beta^2}{\lambda^*\mu(1+\beta)}}\sqrt{v_s}\mathrm{d}\tilde{W}_s \right]$$

where $(W, \tilde{W})$ are two correlated Brownian motions with:

$$\mathrm{d}\langle W, \tilde{W}\rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}}\mathrm{d}t$$

furthermore, the process v_t has Hölder regularity $\alpha - 1/2 - \varepsilon$ for each $\varepsilon > 0$.

The proof is really technical and out of our scope. It can be found in El Euch, Fukasawa and Rosenbaum.

3 The rough Heston model

We have seen that the rough Heston model is what arise from taking the limit of Hawkes processes. Here we reparametrize in order to realign the notation to what we are used to in the classical Heston model. Given a stock price process $S = (S_t)_{t\geq 0}$ the rough Heston model, under risk-free probability measure $\mathbb{Q}$, is the following :

$$\begin{cases} \mathrm{d}S_t = (r-q)S_t\mathrm{d}t + S_t\sqrt{v_t}\{\rho\mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}\tilde{W}_t\} \\ v_t = v_0 + \dfrac{\lambda}{\Gamma(H+\frac{1}{2})}\displaystyle\int_0^t \frac{\eta^0(s)-v_s}{(t-s)^{\frac{1}{2}-H}}\mathrm{d}s + \dfrac{\theta}{\Gamma(H+\frac{1}{2})}\displaystyle\int_0^t \frac{\sqrt{v_s}}{(t-s)^{\frac{1}{2}-H}}\mathrm{d}W_s \end{cases}$$

where:

- r is the risk-free rate;
- q is the yield of the underlying;
- $W = (W_t)_{t\geq 0}$ and $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ are two correlated Brownian motions with $\mathrm{d}\langle W, \tilde{W}\rangle_t = \rho\mathrm{d}t$ and $\rho \in [-1,1]$;
- $H \in (0, 1/2)$ is the Hurst exponent of the fractional Brownian motion;
- θ is the volatility of the volatility;
- $\lambda \geq 0$ is a constant representing the "speed" of the mean reversion;
- $\eta^0(\cdot)$ is an F_0-measurable function representing the mean reversion level.

In order to proceed with this chapter we need to define the fractional integral and the fractional derivative. We define the fractional integral of order $\alpha \in (0,1]$ of a function f as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s$$

whenever the integral exists. We define the fractional derivative of order $\alpha \in (0,1]$ of a function f as

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}f(s)\mathrm{d}s$$

whenever it exists.

3.1 Inference of $\lambda\eta^0(\cdot)$ from the forward variance curve

In *Perfect hedging in rough Heston models* El Euch and Rosenbaum showed that there is a link between $\lambda\eta^0(\cdot)$ and the forward variance curve (intended as in the **Dataset** chapter). We know that the forward variance curve can be obtained by differentiation of the variance swap curve. If the asset have a liquid variance swap market then we have it directly, otherwise we can calculate the fair value of a variance swap using an infinite strip of out-of-the-money options. Assume that the forward variance curve $t \to \mathbb{E}[v_t] =: \xi_0(t)$ admits a fractional derivative of order α then $\lambda\eta^0(\cdot)$ can be chosen so that the model is consistent with this market forward variance curve by taking

$$\lambda\eta^0(t) = D^\alpha[\xi_0(\cdot) - v_0](t) - \lambda\xi_0(t) \tag{2}$$

Equation (2) can be obtained by calculating $\mathbb{E}[v_t]$ using the second equation of the rough Heston model, showing that the forward variance curve is locally integrable (so the expected value of the stochastic integral is zero), then fractional differentiate the LHS and RHS and then reorder. Using this fact and assuming that λ is sufficiently small, we can rewrite the dynamic in a compact way as:

$$\begin{cases} \mathrm{d}S_t = (r-q)S_t\mathrm{d}t + S_t\sqrt{v_t}\big\{\rho\mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}\tilde{W}_t\big\} \\ v_t = \xi_0(t) + \dfrac{\theta}{\Gamma(H+\frac{1}{2})}\displaystyle\int_0^t \frac{\sqrt{v_s}}{(t-s)^{\frac{1}{2}-H}}\mathrm{d}W_s \end{cases}$$

the hypothesis that λ must be sufficiently small is sensible since the volatility is *slowly* mean reverting.

4 The Characteristic function

Let $x_t = \log(S_t)$, we wish to obtain the characteristic function of terminal log-spot x_T conditional on the initial price x_0 and the initial forward variance curve $\xi_0(\cdot)$. In mathematical terms:

$$\phi_{rH}(u) = \mathbb{E}_{\mathbb{Q}}\big[e^{iux_T}\big|x_0, \xi_0(\cdot)\big]$$

We need to do a step back and restart from the 2-dimensional Hawkes process presented in the second section of this chapter. Using the same notation as before, let $L(u)$ be the characteristic function of the 2-dimensional Hawkes process N conditional at time t:

$$L(u) = \mathbb{E}\big[e^{iuN_t}\big], \quad u \in \mathbb{R}^2$$

4.1 Rational approximation of the solution

5 Pricing and Hedging

6 Calibration

7 Simulation

References

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