

# Heston

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Let  $(\Omega, \mathbb{F} = \{(F_t)_{t \geq 0}\}, \mathbb{P})$  a complete filtered probability space and let call  $\mathbb{P}$  the *physical-measure*. Let  $T < \infty$  be the right limit of our time horizon from now on.

## 1 The Model

Given a stock price process  $S = (S_t)_{t \geq 0}$  the Heston model, under  $\mathbb{P}$ , is the following :

$$\begin{cases} dS_t = \mu S_t dt + S_t \sqrt{v_t} dW_t \\ dv_t = \kappa(\eta - v_t) dt + \theta \sqrt{v_t} d\tilde{W}_t \\ v_0 = \sigma_0^2 \end{cases}$$

where:

- $\mu$  is the drift of the stock returns;
- $W = (W_t)_{t \geq 0}$  and  $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$  are two correlated Brownian motions with  $d\langle W, \tilde{W} \rangle_t = \rho dt$  and  $\rho \in [-1, 1]$ ;
- $\sigma_0 > 0$  is the initial volatility;
- $\eta > 0$  is the long run variance;
- $\kappa > 0$  is the mean reversion rate;
- $\theta > 0$  is the volatility of the variance.

The variance process is strictly positive if  $2\kappa\eta > \theta^2$ . This is known as *Feller condition*.

## 2 The Valuation Equation

Consider two independent claims which prices, at time  $t$ , are given as  $U_t = u(t, S_t, v_t)$  and  $\tilde{U}_t = \tilde{u}(t, S_t, v_t)$ . Suppose that  $u$  and  $\tilde{u}$  are  $\mathcal{C}^1(\mathbb{R}^+)$  w.r.t. the first variable and  $\mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^+)$  w.r.t. the last two variables. Consider a portfolio consisting of a long contract  $U$ , short  $\Delta$  shares of the stock and short  $\Delta_1$  contracts of  $\tilde{U}$ . So the value (denoted with  $\Pi_t$ ) of our portfolio at time  $t$  is equal to:

$$\Pi_t = U_t - \Delta S_t - \Delta_1 \tilde{U}_t$$

then we can apply Itô's lemma and write the dynamics, omitting some subscripts, of the value of the portfolio as:

$$\begin{aligned} d\Pi_t = & \left\{ \frac{\partial U}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\theta vS \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 U}{\partial v^2} \right\} dt \\ & + \left\{ \frac{\partial \tilde{U}}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 \tilde{U}}{\partial S^2} + \rho\theta vS \frac{\partial^2 \tilde{U}}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 \tilde{U}}{\partial v^2} \right\} dt \\ & + \left\{ \frac{\partial U}{\partial S} - \Delta_1 \frac{\partial \tilde{U}}{\partial S} - \Delta \right\} dS \\ & + \left\{ \frac{\partial U}{\partial v} - \Delta_1 \frac{\partial \tilde{U}}{\partial v} \right\} dv \end{aligned}$$

to make our portfolio instantaneously risk-free we must impose the following equations to eliminate the  $dS$  and  $dv$  terms:

$$\begin{cases} \frac{\partial U}{\partial S} - \Delta_1 \frac{\partial \tilde{U}}{\partial S} - \Delta = 0 \\ \frac{\partial U}{\partial v} - \Delta_1 \frac{\partial \tilde{U}}{\partial v} = 0 \end{cases} \quad (1)$$

since our portfolio is now risk-free its value must be equal to the value of a portfolio with the same initial value invested in the cash market at risk-free rate  $r$ , so:

$$d\Pi_t = r\Pi_t dt = r(U_t - \Delta S_t - \Delta_1 \tilde{U}_t) dt$$

then choosing  $\Delta$  and  $\Delta_1$  as in (1) and collecting all the  $U$  terms on the LHS and all the  $\tilde{U}$  terms on the RHS we obtain (omitting once again some subscripts):

$$\begin{aligned} & \frac{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\theta vS \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 U}{\partial v^2} + rS \frac{\partial U}{\partial S} - rU}{\frac{\partial U}{\partial v}} \\ & = \frac{\frac{\partial \tilde{U}}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 \tilde{U}}{\partial S^2} + \rho\theta vS \frac{\partial^2 \tilde{U}}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 \tilde{U}}{\partial v^2} + rS \frac{\partial \tilde{U}}{\partial S} - r\tilde{U}}{\frac{\partial \tilde{U}}{\partial v}} \end{aligned}$$

now since the LHS depends on  $U$  and the RHS depends on  $\tilde{U}$  then the only way for which this is possible is if they are equal to some function  $f = f(t, S_t, v_t)$  which is independent from  $U$  and  $\tilde{U}$ . So we obtain:

$$\frac{\partial U}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 U}{\partial S^2} + \rho\theta vS \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 U}{\partial v^2} + rS \frac{\partial U}{\partial S} - rU = -f \frac{\partial U}{\partial v} \quad (2)$$

Equation (2) is called the valuation equation for  $U$ . WLOG we can rewrite  $f$  as:

$$f(t, S_t, v_t) = \kappa(\eta - v_t) - \sqrt{v_t} \phi(t, S_t, v_t)$$

where  $\phi_t$  is some arbitrary  $\mathbb{F}$ -adapted function and it is called the *market price of volatility risk*.

**NOTE:** the stock  $S$  is assumed to have yield equals to zero. If the stock in consideration has yield  $q$  we can simply switch  $r$  with  $(r - q)$  in the calculations before.

## 2.1 The market price of volatility risk

Without assuming that there are two traded independent claims with one of which is dependent from the variance process we cannot a priori fix an  $\mathbb{F}$ -adapted  $\phi$ . Indeed, we can rewrite  $\tilde{W} = \rho W + \sqrt{1 - \rho^2} W^\perp$  where  $W^\perp$  is a continuous Brownian motion on the filtration  $\mathcal{F}$  such that  $d\langle W, W^\perp \rangle = 0$ . Then we can rewrite the Heston model under  $\mathbb{P}$  as:

$$\begin{cases} dS_t = \mu S_t dt + S_t \sqrt{v_t} dW_t \\ dv_t = \kappa(\eta - v_t) dt + \theta \rho \sqrt{v_t} dW_t + \theta \sqrt{1 - \rho^2} \sqrt{v_t} dW_t^\perp \end{cases}$$

now take  $\lambda_t = \lambda(t, S_t, v_t)$  defined as:

$$\lambda_t := \frac{\mu - r}{\sqrt{v_t}}$$

and take  $\phi_t$   $\mathbb{F}$ -adapted such that the Novikov's condition is satisfied, i.e.;

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T [\lambda_t^2 + \phi_t^2] dt \right) \right] < \infty$$

then we have that the process  $Z = (Z_t)_{t \in [0, T]}$  defined as:

$$Z_t = \mathcal{E}(-\lambda_t W - \phi_t W^\perp)_t$$

where  $\mathcal{E}$  is the Doléans-Dade exponential is a  $\mathbb{P}$ -martingale, with  $E[Z_T] = 1$  and strictly positive. So, using the Radon-Nikodym's theorem we can define a measure  $\mathbb{Q}$  on  $\mathbb{F}$  as:

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} := Z_t$$

and we have that  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $\mathcal{F}_T$  and moreover is martingale equivalent measure. By Girsanov's theorem we can define a 2-dimensional  $\mathbb{Q}$ -Brownian motion  $(W^\mathbb{Q}, W^{\mathbb{Q}, \perp})$  as:

$$W_t^\mathbb{Q} = W_t + \int_0^t \lambda_s ds \quad \text{and} \quad W_t^{\mathbb{Q}, \perp} = W_t + \int_0^t \phi_s ds$$

and under such a  $\mathbb{Q}$  we have that the Heston model has the following dynamics:

$$\begin{cases} dS_t = r S_t dt + S_t \sqrt{v_t} dW_t^\mathbb{Q} \\ dv_t = [\kappa(\eta - v_t) - \theta \sqrt{v_t}(\rho \lambda_t + \sqrt{1 - \rho^2} \phi_t)] dt + \theta \sqrt{v_t} d\tilde{W}_t^\mathbb{Q} \\ v_0 = \sigma_0^2 \end{cases}$$

and under this we have that the discounted stock price is a  $\mathbb{Q}$  local martingale. However, we can choose any arbitrary  $\phi$  and obtain another  $\mathbb{Q}$  with the same properties. But, as in our case, both the claims are actively traded we can infer the  $\phi$  from the market prices of the options, which will determine a  $\mathbb{Q}$  and then fix the  $\phi$ . Among all the possible choices from now on we will choose  $\phi = 0$ . The resulting  $\mathbb{Q}^M$  is called the *minimal martingale measure* and Föllmer and Schweizer have proven that, for models with continuous price trajectories, solves the minimization problem:

$$\mathbb{Q}^M = \arg \min_{\mathbb{Q} \in \mathcal{M}} \mathbb{H}(\mathbb{Q}|\mathbb{P})$$

where  $\mathcal{M}$  is the set of equivalent martingale measures and  $\mathbb{H}$  is the reverse relative entropy. Indeed this is exactly what we obtain if we can completely infer the  $\phi$  from the market prices of the options (because we think that  $\mathbb{P}$  contains all the information to reconstruct  $\mathbb{Q}$ ).

**NOTE:** From now on, to lighten the notation, our Heston model under  $\mathbb{Q}$  is written in the following way:

$$\begin{cases} dS_t = \mu S_t dt + S_t \sqrt{v_t} dW_t \\ dv_t = \kappa(\eta - v_t)dt + \theta \sqrt{v_t} d\tilde{W}_t \\ v_0 = \sigma_0^2 \end{cases}$$

with  $W = (W_t)_{t \geq 0}$  and  $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$  are two correlated  $\mathbb{Q}$  Brownian motions with  $d\langle W, \tilde{W} \rangle_t = \rho dt$ .

## 3 The Characteristic Formula

### 3.1 Derivation of the pseudo-probabilities

Setting the  $\phi = 0$  we have that the valuation equation for our model, to price a call option  $C$ , becomes:

$$\frac{\partial C}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \rho \theta v S \frac{\partial^2 C}{\partial S \partial v} + \frac{1}{2} \theta^2 v \frac{\partial^2 C}{\partial v^2} + r S \frac{\partial C}{\partial S} - r C + [\kappa(\eta - v)] \frac{\partial C}{\partial v} = 0$$

if we now substitute with  $\tau = T - t$  and  $x = \log\left(\frac{S_t e^{(r-q)\tau}}{K}\right)$  in the previous equation we obtain:

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2} v \frac{\partial^2 C}{\partial x^2} + \rho \theta v \frac{\partial^2 C}{\partial x \partial v} + \frac{1}{2} \theta^2 v \frac{\partial^2 C}{\partial v^2} - \frac{1}{2} v \frac{\partial C}{\partial x} + [\kappa(\eta - v)] \frac{\partial C}{\partial v} = 0 \quad (3)$$

according to Duffie, Pan and Singleton (2000), the solution to this equation has the form of:

$$C(x, v, \tau) = K \{ e^x P_1(x, v, \tau) - P_0(x, v, \tau) \} \quad (4)$$

notice how this remind us to the solution of the Black&Scholes' Equation. Moreover, the  $P_j$  are absolutely continuous, differentiable and both  $P_j$  and  $P'_j$  are

absolutely integrable on  $\mathbb{R}$ . Putting (4) into (3) we obtain that  $P_j$  with  $j = 0, 1$  must satisfy the following PDE:

$$-\frac{\partial P_j}{\partial \tau} + \frac{1}{2}v \frac{\partial^2 P_j}{\partial x^2} - \left(\frac{1}{2} - j\right)v \frac{\partial P_j}{\partial x} + \frac{1}{2}\theta^2 v \frac{\partial^2 P_j}{\partial v^2} + \rho\theta v \frac{\partial^2 P_j}{\partial x \partial v} - [\kappa(\eta+1) - j\rho\theta v] \frac{\partial P_j}{\partial v} = 0$$

with boundary conditions:

$$\lim_{\tau \rightarrow 0} P_j(x, v, \tau) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Now take the Fourier transform of  $P_j$  as:

$$\hat{P}_j(u, v, \tau) = \int_{\mathbb{R}} e^{-iux} P_j(x, v, \tau) dx \quad \text{then} \quad \hat{P}_j(u, v, 0) = \frac{1}{iu}$$

and the inverse transform given by:

$$P_j(x, v, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \hat{P}_j(u, v, \tau) du$$

thanks to the regularity of  $P_j$  we have that  $[\widehat{P_j(u)'}] = iu\hat{P}_j(u)$ . So, using this property, we can rewrite our equation as:

$$-\left[\frac{1}{2}u^2 - \left(\frac{1}{2} - j\right)iu\right]v\hat{P}_j + \left\{\rho\theta iuv - [\kappa(\eta+1) - j\rho\theta v]\right\}\frac{\partial \hat{P}_j}{\partial v} + \frac{1}{2}\theta^2 v \frac{\partial^2 \hat{P}_j}{\partial v^2} = \frac{\partial \hat{P}_j}{\partial \tau}$$

now take two arbitrary functions  $f(u, \tau)$ ,  $g(u, \tau)$ . The solution should be of the form:

$$\hat{P}_j(u, v, \tau) = \frac{1}{iu} \exp\{f(u, \tau)\eta + g(u, \tau)v\}$$

it follows that:

$$\begin{aligned} \frac{\partial \hat{P}_j}{\partial \tau} &= \left\{\eta \frac{\partial f}{\partial \tau} + v \frac{\partial g}{\partial \tau}\right\} \hat{P}_j \\ \frac{\partial \hat{P}_j}{\partial v} &= g \hat{P}_j \\ \frac{\partial^2 \hat{P}_j}{\partial v^2} &= g^2 \hat{P}_j \end{aligned}$$

now define:

$$\begin{aligned} \alpha &= -\frac{u^2}{2} - \frac{iu}{2} + iju \\ \beta &= -\kappa - \rho\theta(j + iu) \\ \gamma &= \frac{\theta^2}{2} \end{aligned}$$

then it must be:

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= -\kappa g \\ \frac{\partial g}{\partial \tau} &= \alpha - \beta g - \gamma g^2 \\ &= \gamma(g - r_+)(g - r_-)\end{aligned}$$

where  $r_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} =: \frac{\beta \pm d}{\theta^2}$ . Integrating and putting the terminal condition  $f(u, 0) = 0$  and  $g(u, 0) = 0$ , we obtain:

$$\begin{aligned}f(u, \tau) &= \kappa \left\{ r_- \tau - \frac{2}{\theta^2} \log \left( \frac{1 - \frac{r_-}{r_+} e^{-d\tau}}{1 - \frac{r_-}{r_+}} \right) \right\} \\ g(u, \tau) &= r_- \frac{1 - e^{-d\tau}}{1 - \frac{r_-}{r_+} e^{-d\tau}}\end{aligned}\tag{5}$$

Taking the inverse transform on  $\hat{P}_j$  we obtain finally the form of the pseudo probabilities as:

$$P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{\exp \{f_j(u, \tau)\eta + g_j(u, \tau)v + iux\}}{iu} \right\} du \tag{6}$$

### 3.2 Numerical integration of the complex logarithm

In (5) we decided to take  $r_-$  to define  $f$ , but we could have taken  $r_+$  and everything would have been "almost" the same (this definition coincides with our previous one only if the imaginary part of the complex logarithm is chosen so that  $f(u, \tau)$  is continuous with respect to  $u$ ). However, we are interested in integrating numerically the characteristic function of the Heston model using the FFT approach. We have to keep in mind that the complex logarithm has a branch (we decided to take as branch the semiaxis of negative real numbers in the complex plane), so in order to avoid any kind of discontinuity we want that the characteristic function never crosses the negative real axis on  $(0, \infty)$ . Albrecher, Mayer, Schoutens, Tistaert (2006) proved the following result when using the FFT approach:

**Proposition 1.** *Whenever the parameters of the Heston model are such that  $\Im\{d(u)\} := \Im\{\sqrt{(\rho\theta ui - \kappa)^2 + \theta^2(iu + u^2)}\} \neq 0$  and  $2\kappa\eta \neq \theta^2 n$  (where  $n \in \mathbb{N}$ ), then defining (5) using  $r_+$  leads to numerical instabilities for sufficiently large maturities.*

In order to use the FFT approach of Carr-Madan we have to evaluate the characteristic function in  $u - (\alpha + 1)i$  for positive  $u$ . So they also proved the following proposition:

**Proposition 2.** *Denote with  $\phi(u)$  the characteristic function of the Heston model obtained using  $r_-$  in (5). Then  $\forall \alpha > 0$  and  $\forall u \in (0, \infty)$  the function  $\phi(u - (\alpha + 1)i)$  does not cross the negative real axis.*

In conclusion if we use  $r_-$  in (5) then the characteristic function that we will obtain in the next paragraph is suitable for numerical integration using the FFT method.

### 3.3 Derivation of the characteristic function

By definition the characteristic function under  $\mathbb{Q}$  is defined as:

$$\phi(u) = \mathbb{E}_{\mathbb{Q}} \left[ e^{iux_T} \middle| x_t = \frac{e^{(r-q)\tau} S_t}{K}; v_t \right]$$

now thanks to (6) we know that:

$$\mathbf{Pr}_{\mathbb{Q}}(x_T > x_t) = P_0(x_t, v_t, \tau)$$

so, if we define  $k = -x_t$  the density is:

$$\begin{aligned} p(k) &= -\frac{\partial P_0}{\partial k} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp \{f_0(s, \tau)\eta + g_0(s, \tau)v_t - isk\} ds \end{aligned}$$

then we have that, using Fubini theorem:

$$\begin{aligned} \phi(u; x_t, v_t, \tau) &= \int_{\mathbb{R}} e^{iuk+iu x_0} p(k) dk \\ &= \frac{e^{iu x_0}}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp \{f_0(s, \tau)\eta + g_0(s, \tau)v_t\} e^{iuk} e^{-isk} ds dk \\ &= \frac{e^{iu x_0}}{2\pi} \int_{\mathbb{R}} \exp \{f_0(s, \tau)\eta + g_0(s, \tau)v_t\} \int_{\mathbb{R}} e^{ik(u-s)} dk ds \\ &= e^{iu x_0} \int_{\mathbb{R}} \exp \{f_0(s, \tau)\eta + g_0(s, \tau)v_t\} \delta(u-s) ds \\ &= \exp \{f_0(u, \tau)\eta + g_0(u, \tau)v_t + iux_t\} \end{aligned}$$

## 4 WORK IN PROGRESS