rHeston

Nodari Alessandro & Proserpio Lorenzo

As we have seen the skew of the at the money options in the Heston model does not represent well the skew of real-world at the money options. This, among other reasons, made the researcher and practitioner to think about more sophisticated models. As observed by Gatheral, Jaisson and Rosenbaum the log-volatility in the market behaves like a fractional Brownian motion with small Hurst parameter, so the volatility is rough.

1 Stylized empirical facts

Heston model reproduces several important features of low frequency price data, provides quite reasonable dynamics for the volatility surface and it can be calibrated efficiently. If we want to surpass that we have to build a model which can reproduce the stylized facts of modern electronic markets in the context of high frequency trading. In practice each market behaves tick-by-tick, indeed we receive an update in price by market-makers whenever there is a trade and the movement is discrete and at least of one tick (usually 1 cent). There are 4 main stylized facts that we can observe in market data:

1. Markets are highly endogenous, as showed by Bouchad. This means that most of the orders have no real economic motivation, but are simply the reaction of algorithms to other orders.

2. Markets at high frequency are much more efficient than at lower frequencies, this means that it is much more difficult to find profitable statistical arbitrage strategies.

3. There is some asymmetry in the liquidity on the bid and the ask side of the order book. Indeed, a market-maker is likely to raise the price by less following a buy order than to lower the price following the same size sell order, as seen by Brunnermeier and Pedersen. This is mostly due to the fact that hedging the first position is easier than the second and that market-makers have usually some inventory.

4. A large proportion of transaction is due to big orders, called metaorders, which are not executed at once, but split in time. Indeed, one of the most challenging part of every trading strategies is to execute it in large volumes without moving changing to much the state of the market.

2 Building the model

As in El Euch, Fukasawa & Rosenbaum we will start building a model from Hawkes processes, then slowly including the stylized fact mentioned in the last paragraph and showing that the long-term dynamic of this model will lead to a rough Heston model at the macroscopic scale, in which the leverage effect is still represented.

2.1 Hawkes Processes

Hawkes processes are point processes which are said to be self-exciting, in the sense that the instantaneous jump-probability depends on the location of the past events. In particular we will focus on a bivariate Hawkes process, $(N_t^+, N_t^-)_{t\geq 0}$, where N_t^+ is the number of upward jumps of one tick and N_t^- is the number of downward jumps of one tick, both in the interval $[0, t]$. The probability to get one-tick upward jump in a time $\mathrm{d}t$ is given by $\lambda_t^+ \mathrm{d}t$, viceversa by λ_t^-. The array $(\lambda_t^+, \lambda_t^-)$ is called intensity of the process and it is of the form:

$$\begin{pmatrix} \lambda_t^+ \\ \lambda_t^- \end{pmatrix} = \begin{pmatrix} \mu^+ \\ \mu^- \end{pmatrix} + \int_0^t \begin{pmatrix} \phi_1(t-s) & \phi_3(t-s) \\ \phi_2(t-s) & \phi_4(t-s) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}N_s^+ \\ \mathrm{d}N_s^- \end{pmatrix}$$

where μ^+ and μ^- are positive constants and the components of the matrix are positive and locally integrable functions. The process of the prices P_t is given by the difference between the number of upward jumps from time 0 and the number of downward jumps, so:

$$P_t = N_t^+ - N_t^-$$

Each component of the intensity can be decomposed into three parts, for example λ_t^- can be decomposed in:

- μ_t^- which corresponds to the probability that the price will go down because of exogenous reasons;
- $\int_0^t \phi_2(t-s)\mathrm{d}N_s^+$ which corresponds to the probability of a downward jump induced by past upward jumps;
- $\int_0^t \phi_4(t-s)\mathrm{d}N_s^-$ which corresponds to the probability of a downward jump induced by past downward jumps.

Now we will see that, when the ϕ_j have suitable forms the model can reproduce the stylized effects described in the previous section. Moreover, we want to underlying that, due to how the model is built, the price process assumes discrete values, as in the real world.

2.2 Encoding the 2^{nd} property

Since the markets at high frequency are expected to be more efficient then this translate in that, over any period of time, we should have on average the same number of upwards jumps than downwards jumps. This can be translated in:

$$\int_0^t \mathbb{E}[\lambda_s^+]\mathrm{d}s = \mathbb{E}[N_t^+] = \mathbb{E}[N_t^-] = \int_0^t \mathbb{E}[\lambda_s^-]\mathrm{d}s \tag{1}$$

remembering how we have defined λ_t^+ and λ_t^-:

$$\mathbb{E}[\lambda_t^+] = \mu^+ + \int_0^t \phi_1(t-s)\,\mathbb{E}[\lambda_s^+]\mathrm{d}s + \int_0^t \phi_3(t-s)\,\mathbb{E}[\lambda_s^-]\mathrm{d}s$$
$$\mathbb{E}[\lambda_t^-] = \mu^- + \int_0^t \phi_4(t-s)\,\mathbb{E}[\lambda_s^-]\mathrm{d}s + \int_0^t \phi_2(t-s)\,\mathbb{E}[\lambda_s^+]\mathrm{d}s$$

the simplest way to satisfy the equation (1) is to put:

$$\mu^+ = \mu^- \text{ and } \phi_1 + \phi_3 = \phi_2 + \phi_4$$

2.3 Encoding the 3rd property

Market-makers act as liquidity providers, in practice at the beginning they are long inventory, so the ask side is more liquid than the bid side. This translate into the fact that the conditional probability of an upward jump right after an upward jump is smaller than the conditional probability to observe a downward jump after a downward jump. This means that for $t \to 0$ we have:

$$\int_0^t \phi_4(t-s)\mathrm{d}N_s^- > \int_0^t \phi_1(t-s)\mathrm{d}N_s^+$$

or equivalently

$$\int_0^t \phi_2(t-s)\mathrm{d}N_s^+ < \int_0^t \phi_3(t-s)\mathrm{d}N_s^-$$

this can be satisfied in several ways, but we make the strong assumption that exists a constant $\beta > 0$ such that $\phi_3 = \beta\phi_2$. Putting all together we have that the structure of our intensity process is:

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \mu\begin{pmatrix}1\\1\end{pmatrix} + \int_0^t \begin{pmatrix}\phi_1(t-s) & \beta\phi_2(t-s)\\ \phi_2(t-s) & [\phi_1 + (\beta-1)\phi_2](t-s)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}N_s^+\\ \mathrm{d}N_s^-\end{pmatrix}$$

2.4 Encoding the 1st property

Markets have an high degree of endogeneity, which means that the proportion of "non-meaningful" orders with respect to the totality of the orders is close to 1. In order to have an intuition on how to include this effect in our model we need to recall dynamical systems. A dynamical system has a stationary point (or equilibrium) if its spectral radius is less than 1, in the same way we have a kernel transition matrix:

$$\int_0^t \begin{pmatrix}\phi_{1,t}(s) & \beta\phi_{2,t}(s)\\ \phi_{2,t}(s) & [\phi_{1,t} + (\beta-1)\phi_{2,t}](s)\end{pmatrix}\mathrm{d}s = \int_0^t \Phi_t(s)\mathrm{d}s$$

we can extend all the functions on $[0, \infty)$ with the constant zero. We refer to them with a tilde. The spectral radius in our case is equal to:

$$\sigma\bigg(\int_0^\infty \tilde{\Phi}_t(s)\mathrm{d}s\bigg) = \|\tilde{\phi}_{1,t}\|_1 + \beta\|\tilde{\phi}_{2,t}\|_1$$

Let $(\Omega, \mathbb{F} = \{(F_t)_{t\geq 0}\}, \mathbb{P})$ a complete filtered probability space. We can find a sequence $\big\{\big(\tilde{\phi}_{1,t}; \tilde{\phi}_{2,t}\big)\big\}_t$ of couple of positive functions in $\mathcal{L}^1(F_t)$ such that:

- $\forall t > 0$ we have $\|\tilde{\phi}_{1,t}\|_1 + \beta\|\tilde{\phi}_{2,t}\|_1 < 1$;
- if $t_2 > t_1$ we have both $\tilde{\phi}_{1,t_2} \geq \tilde{\phi}_{1,t_1}$ and $\tilde{\phi}_{2,t_2} \geq \tilde{\phi}_{2,t_1}$;
- satisfying:
$$\lim_{t\to\infty}\Big[\|\tilde{\phi}_{1,t}\|_1 + \beta\|\tilde{\phi}_{2,t}\|_1\Big] = 1$$

then there exist a limit to this sequence and we will call that $(\tilde{\phi}_1; \tilde{\phi}_2)$ and, due to continuity of the norm, we have that $\|\tilde{\phi}_1\|_1 + \beta\|\tilde{\phi}_2\|_1 = 1$. Then we have built our nearly-unstable system for each $t > 0$ sufficiently large. Moreover notice that, due to the second property of our sequence, we have that the spectral radius, when t is increasing, is also increasing. We will refer to the matrix obtained with $(\tilde{\phi}_1; \tilde{\phi}_2)$ as Φ. The process of the prices up to time $T < \infty$ is now indicated as:

$$P_t^T = N_t^{T,+} - N_t^{T,-}$$

with $N_t^{T,+}$ and $N_t^{T,-}$ with intensity generated by $\big(\tilde{\phi}_{1,T}; \tilde{\phi}_{2,T}\big)$.

3 The Characteristic function

4 Pricing and Hedging

5 Rational approximation of the solution

6 Calibration

7 Simulation

References

1. Gatheral, Jaisson, Rosenbaum. Volatility is rough. ArXiv, 2014.

2. Bouchaud, The endogenous dynamics of markets: price impact and feedback loops. ArXiv, 2010.

3. Brunnermeier, Pedersen. Market Liquidity and Funding Liquidity. The Review of Financial Studies, 2009.

4. El Euch, Fukasawa, Rosenbaum. The microstructural foundations of leverage effect and rough volatility. ArXiv, 2016.