Fractional Brownian Motion

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The fractional Brownian Motion (fBM) was first introduced within a Hilbert space framework by Kolmogorov in 1940 in [1], where it was called Wiener Helix. It was further studied by Yaglom in [2]. The name fractional Brownian Motion is due to Mandelbrot and Van Ness, who in 1968 provided in [3] a stochastic integral representation of this process in terms of a standard Brownian motion. From now on we will consider a probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{F}$ is the natural filtration generated by a Brownian Motion.

1 Definition and Properties

We will use the definition of fBM given in [4].

Definition 1. *A fractional Brownian Motion W_t^H of Hurst index $H \in (0,1)$ is a continuous and centered Gaussian process with covariance function*

$$\mathbb{E}\big[W_t^H W_s^H\big] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) \qquad t, s \in \mathbb{R}$$

By **Definition 1** a fBM W_t^H has the following properties:

1. $W_0^H = 0$;
2. $W_t^H \sim \mathcal{N}(0, t^{2H})$, $\forall t \geq 0$;
3. W^H has stationary increments:
$$W_{t+s}^H - W_t^H = W_s^H \quad s \in \mathbb{R}$$
4. W_t^H has $\mathbb{P}$ a.s. continuous trajectories.

Remark 1. *The fBm is divided into three very different families:*

- $0 < H < \frac{1}{2}$ *where two increments of the form* $\big(W_{t+h}^H - W_t^H\big)$ *and* $\big(W_{t+2h}^H - W_{t+h}^H\big)$ *are negatively correlated;*
- $H = \frac{1}{2}$ *then the fBM is actually a standard Brownian Motion and the increments are independent;*
- $\frac{1}{2} < H < 1$ *where two increments of the form* $\big(W_{t+h}^H - W_t^H\big)$ *and* $\big(W_{t+2h}^H - W_{t+h}^H\big)$ *are positively correlated.*

Proposition 1. *A fBM W_t^H admits the following stochastic integral representation:*

$$W_t^H = C_H\left(\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right]\mathrm{d}W_s + \int_0^t (t-s)^{H-\frac{1}{2}}\mathrm{d}W_s\right)$$

where

$$C_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$$

Where Γ is the usual gamma function and W_t is a two-sided Brownian Motion defined on $\mathbb{R}$ as:

$$W_t = \begin{cases} W^1(t) & t \geq 0 \\ W^2(-t) & t < 0 \end{cases}$$

where W^1 and W^2 are two independent Brownian Motion.

Proposition 2. *A fBM W_t^H with Hurst parameter $H \in (0,1)$ is a self-similar process such that, for any $c \geq 0$, it holds:*

$$W_{ct}^H \overset{d}{=} c^H W_t^H$$

Proposition 3. *The paths of a fBM W_t^H with Hurst parameter $H \in (0,1)$ are almost surely locally $(H-\varepsilon)$-Hölder continuous for $\varepsilon \in (0,H)$.*

Proposition 4. *A fBM W_t^H has a monofractal scaling property:*

$$\mathbb{E}\left[|W_{t+\Delta}^H - W_t^H|^q\right] = \mathbb{E}\left[|W_\Delta^H|^q\right] = K_q\Delta^{Hq}$$

where

$$K_q = \int_{-\infty}^{\infty}|x|^q\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\mathrm{d}x$$

We also highlight that a fBM is not a Markov process nor a semi-martingale.

2 Simulation Methods

A great number of methods have been developed to simulate the paths of a fBM. Some of them are exact methods, which are more demanding from a computational standpoint, and other are approximations. We will present an exact method, the Cholesky decomposition, and the hybrid scheme approximation. More methods are presented and analyzed in depth in [4].

2.1 Cholesky Decomposition

This method is based on the so-called Cholesky decomposition of the covariance matrix. We will analyze the case in which we will operate that is when we want to simulate the fBM W_t^H for $t \in [0, T]$. First we discretize the interval using an equi-spaced grid of $n+1$ points $0 = t_0 < t_1 < \ldots < t_n = T$ with time-step $h = \frac{T}{n}$. The covariance structure of our discretization is:

$$\mathbb{E}\big[W_{t_i}^H W_{t_j}^H\big] = \frac{1}{2}\big(t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H}\big) = \frac{h^{2H}}{2}\big(i^{2H} + j^{2H} - |i-j|^{2H}\big)$$

The covariance matrix C is defined element wise as $C_{i,j} = \mathbb{E}\big[W_{t_i}^H W_{t_j}^H\big]$ for $i, j = 1, \ldots, n$. C is a symmetric and positive semi-defined matrix in $\mathbb{R}^{n \times n}$.

$$C = \begin{bmatrix} \mathbb{E}\big[W_{t_1}^H W_{t_1}^H\big] & \cdots & \mathbb{E}\big[W_{t_1}^H W_{t_n}^H\big] \\ \vdots & \ddots & \vdots \\ \mathbb{E}\big[W_{t_n}^H W_{t_1}^H\big] & \cdots & \mathbb{E}\big[W_{t_n}^H W_{t_n}^H\big] \end{bmatrix} \tag{2.1}$$

Hence the usual Cholesky decomposition reads:

$$C = LL^T$$

where L is a lower triangular matrix with real and positive diagonal entries. Now we will draw n independent samples Z_i from a normal distribution $\mathcal{N}(0,1)$. The vector $(0, LZ)$ of size $n+1$ yields a sample path of W_t^H. To summarize, the steps of the method are:

1. choose an equi-spaced grid $\{t_i\}_{i=1,\ldots,n}$ for the interval $[0, T]$;
2. compute the covariance matrix as in (2.1);
3. use the Cholesky decomposition to find the matrix L such that $C = LL^T$;
4. construct a vector Z of n independent realization of a standard normal distribution $\mathcal{N}(0,1)$;
5. compute the path of the fBM as the vector $(0, LZ)$.

We note that the complexity of this method is of the order $\mathcal{O}(n^3)$ which is quite demanding.

2.2 Hybrid Scheme

The following method is an approximation method that was proposed in [5] as a scheme to simulate a Brownian semi-stationary (BSS) process. The class of BSS processes are studied extensively in [6]. For our purpose we will define a BSS process $(Y_t)_{t \in \mathbb{R}}$ as:

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma(s)\mathrm{d}W_s$$

where W_t is a two-sided Brownian motion, g is a deterministic non-negative weight function and σ is a so called càdlàg process. In order to use the hybrid scheme we have to assume that:

1. for some $\alpha \in (-\frac{1}{2}, \frac{1}{2})\backslash\{0\}$ it holds:

   $$g(x) = x^{\alpha} L_g(x) \qquad x \in (0,1]$$

   where $L_g : (0,1] \to [0,\infty)$ is continuously differentiable, slowly varying at 0 and bounded away from 0. Moreover, there exists a constant $d > 0$ such that:

   $$|L_g(x)| \le d(1 + x^{-1}) \qquad \vee \qquad |L_g'(x)| \le d(1 + x^{-1})$$

   where L_g' it's the derivative of L_g;

2. the function g is continuously differentiable in $(0,\infty)$;

3. for some $\beta \in (-\infty, \frac{1}{2})$ it holds:

   $$g(x) = \mathcal{O}(x^{\beta})$$

Since we are only interested in the time intervals that start at 0 we will use a Truncated Brownian semi-stationary (TBSS) process X_t defined as:

$$X_t = \int_{0}^{t} g(t-s)\sigma(s)\mathrm{d}W_s$$

We will use as the discretization grid $\left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{\lfloor nT \rfloor}{n}\right\}$ and we will assume that σ can be taken constant on each interval of the grid. Doing so the TBSS can be approximated as:

$$X_t \simeq \sum_{k=1}^{\lfloor nT \rfloor} \sigma\left(t - \frac{k}{n}\right) \int_{t-\frac{k}{n}}^{t-\frac{k-1}{n}} g(t-s)\mathrm{d}W_s =: X_n(t)$$

For small values of k, say $k \le \kappa$ for a given κ, we can approximate g as in the assumptions:

$$g(t-s) \simeq (t-s)^{\alpha} L_g\left(\frac{k}{n}\right) \qquad (t-s) \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

For larger values of k, say $k > \kappa$, we can approximate g as:

$$g(t-s) \simeq g\left(\frac{b_k}{n}\right)$$

where in [6] the optimal b_k is shown to be:

$$b_k^* = \left(\frac{k^{\alpha+1} - (k-1)^{\alpha+1}}{\alpha+1}\right)^{\frac{1}{\alpha}}$$

Thus, we have that our approximation of the TBSS is composed of two parts:

$$X_n(t) = X_n^1(t) + X_n^2(t)$$

where:

$$X_n^1(t) \coloneqq \sum_{k=1}^{\kappa} L_g\left(\frac{k}{n}\right)\sigma\left(t - \frac{k}{n}\right)\int_{t-\frac{k}{n}}^{t-\frac{k-1}{n}} (t-s)^\alpha \mathrm{d}W_s$$

$$X_n^2(t) \coloneqq \sum_{k=\kappa+1}^{\lfloor nT \rfloor} g\left(\frac{b_k^*}{n}\right)\sigma\left(t - \frac{k}{n}\right)\left(W_{t-\frac{k-1}{n}} - W_{t-\frac{k}{n}}\right)$$

This decomposition tells us that we have to simulate on the grid points $\{\frac{i}{n}\}$:

$$W_{i,j}^n\left(\frac{i}{n}\right) = \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{i+j}{n} - s\right)^\alpha \mathrm{d}W_s \qquad j = 1, \ldots, \kappa$$

$$W_i^n\left(\frac{i}{n}\right) = \int_{\frac{i}{n}}^{\frac{i+1}{n}} \mathrm{d}W_s \qquad (2.2)$$

We will explicit some of the properties of these two processes:

$$\mathrm{Var}\left[W_i^n\right] = \frac{1}{n}$$

$$\mathrm{Var}\left[W_{i,j}^n\right] = \frac{j^{2\alpha+1} - (j-1)^{2\alpha+1}}{(2\alpha+1)n^{2\alpha+1}}$$

$$\mathbb{E}\left[W_{i,j}^n W_j^n\right] = \frac{j^{\alpha+1} - (j-1)^{\alpha+1}}{(\alpha+1)n^{\alpha+1}}\delta_{i,j}$$

where $\delta_{i,j}$ id the Kronecker delta. Thus, simulating a fBM can be seen as simulating a Volterra process of the form:

$$V(t) = \sqrt{2\alpha+1}\int_0^t (t-s)^\alpha \mathrm{d}W_s$$

Defining $\tilde{V}(t) = \frac{V(t)}{\sqrt{2H}}$ and taking:

$$g(s) = s^{H-\frac{1}{2}} \qquad \sigma(s) = 1 \qquad L_g(s) = 1 \qquad s \in (0, T)$$

we have that $\alpha = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})\backslash\{0\}$ and so the Gaussian Volterra process $\tilde{V}(t)$ is a TBSS process that satisfies the assumptions for the use of the hybrid scheme. Choosing $\kappa = 1$ the process is simulated, in the grid points, as:

$$V_n\left(\frac{i}{n}\right) = \sqrt{2\alpha+1}\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n}-s\right)^{\alpha} \mathrm{d}W_s + \sum_{k=2}^{i}\left(\frac{b_k^*}{n}\right)^{\alpha}\left(W_{\frac{i-(k-1)}{n}} - W_{\frac{i-k}{n}}\right)\right)$$

using the covariance structure:

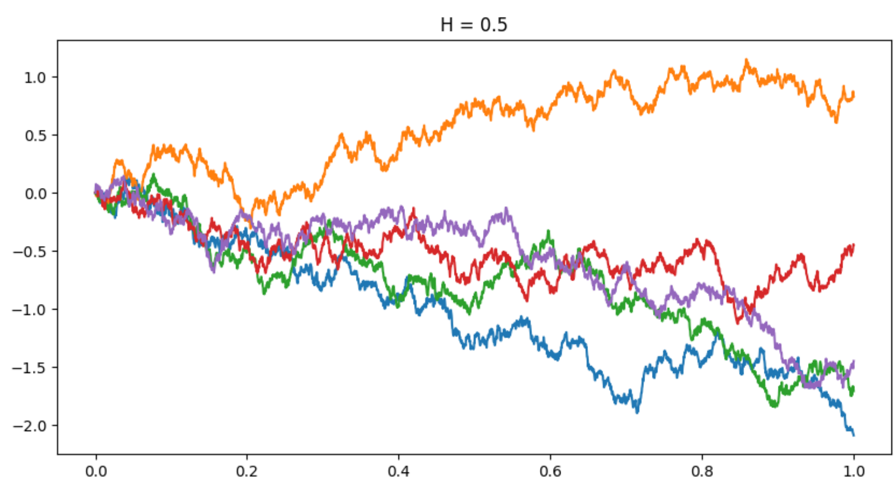
$$\Sigma = \begin{pmatrix} \frac{1}{n} & \frac{1}{(\alpha+1)n^{\alpha+1}} \\ \frac{1}{(\alpha+1)n^{\alpha+1}} & \frac{1}{(2\alpha+1)n^{2\alpha+1}} \end{pmatrix} \qquad (2.2)$$

To summarize, the steps of the hybrid scheme are:

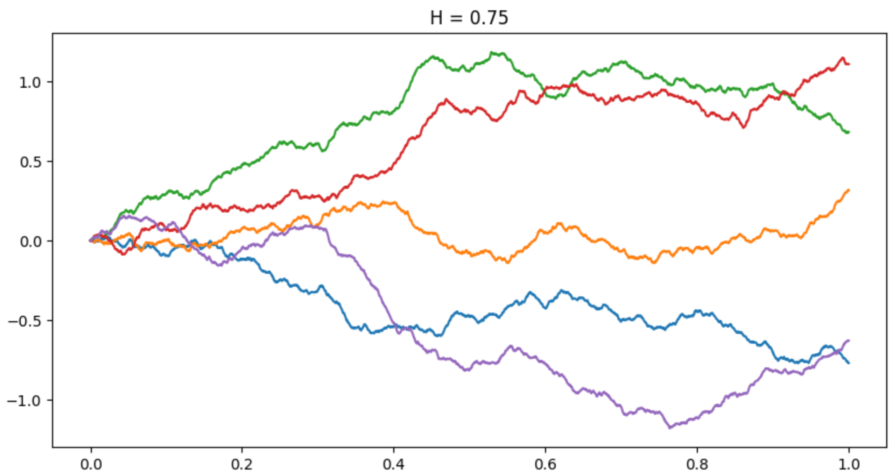
1. compute the covariance matrix Σ as expressed in (2.2);
2. generate a multivariate normal variable $Z = (Z_1, Z_2)$ with mean $\mu = (0, 0)$ and covariance Σ;
3. estimate the first component (the integral one) of V_n using Z_2 since we have $\int_{\frac{i-1}{n}}^{\frac{i}{n}} (\frac{i}{n} - s)^{\alpha} \mathrm{d}W_s \sim \mathcal{N}(0, \frac{1}{(2\alpha+1)n^{2\alpha+1}})$;
4. estimate the second component (the discrete sum) computing $(\frac{b_k^*}{n})^{\alpha}$ and compute the convolution with Z_1;
5. sum the two components and multiply by the factor $\sqrt{2\alpha+1}$.

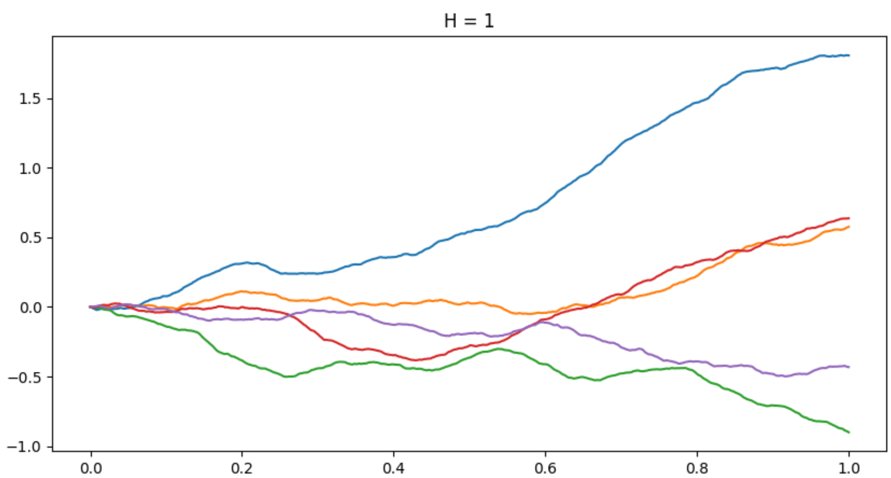
2.3 Simulation results

In this section we present a graphical representation of the effect of the Hurst parameter H on the paths. For all the simulations we have used as the time final time $T = 1$ and as the number of points in the grid we used $n = 2500$. First we present the paths of a standard Brownian Motion.

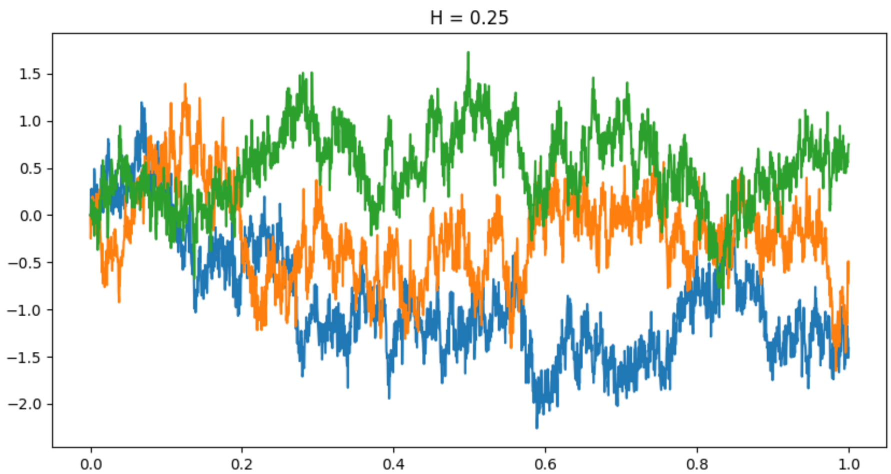


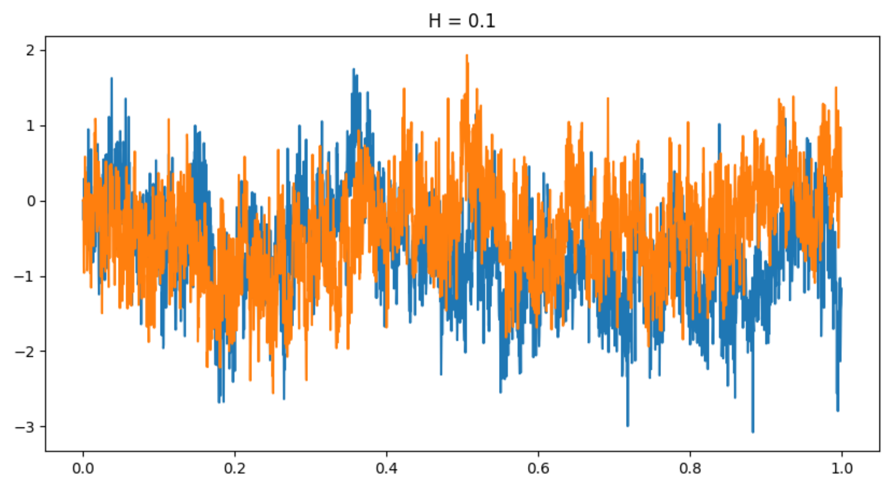
If we take $H > \frac{1}{2}$ we will have smoother trajectories as H keeps increasing as highlighted by the next two figures





While if we take $H < \frac{1}{2}$ we will have progressively rougher paths as H decreases as highlighted by the next two figures.





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