Quintic Ornstein-Uhlenbeck model

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The Quintic Ornstein-Uhlenbeck (Quintic) volatility model, introduced in [1], is a stochastic volatility model where the volatility process is defined as a polynomial of degree five of a single Ornstein-Uhlenbeck (OU) process which has a fast mean reversion and a large volatility of volatility (vol-of-vol). This model tries to address the problem of the joint calibration of SPX derivatives and VIX derivatives.

1 The Model

Under the pricing measure $\mathbb{Q}$ the dynamics of the stock price S are given by:

$$\begin{cases} \mathrm{d}S_t & = (r-q)\mathrm{d}t + \sigma_t S_t \mathrm{d}B_t \\ \sigma_t & = \sqrt{\xi_0(t)}\dfrac{p(X_t)}{\sqrt{\mathbb{E}[p(X_t)^2]}} \\ \mathrm{d}X_t & = -\left(\dfrac{1}{2}-H\right)\epsilon^{-1}X_t\mathrm{d}t + \epsilon^{H-1/2}\mathrm{d}W_t \qquad X_0 = 0 \end{cases}$$

where B_t and W_t are two Brownian motions with correlation parameter $\rho \in [-1,1]$. $\xi_0 \in L^2([0,T],\mathbb{R}^+)$ for any $T>0$ is an input curve used to match certain term-structures observed in the market, we will use the forward variance curve since the normalization $\sqrt{\mathbb{E}[p(X_t)^2]}$ allows ξ_0 to match it:

$$\mathbb{E}\left[\int_0^t \sigma_s^2 \mathrm{d}s\right] = \int_0^t \xi_0(s)\mathrm{d}s \qquad t \geq 0$$

The fifth grade polynomial $p(x)$ is defined as:

$$p(x) \coloneqq \alpha_0 + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5$$

with non-negative parameters $\alpha_0, \alpha_1, \alpha_3, \alpha_5 \geq 0$ ($\alpha_2 = \alpha_4 = 0$). The choice of a polynomial of degree five allows to reproduce the upward slope of the VIX smile, while restricting the coefficients α to be non-negative allows the sign of the ATM skew to be the same as ρ, as explained in more detail in [2]. We decided to set $\alpha_2 = \alpha_4 = 0$ in order to reduce the number of parameters to calibrate and this doesn't impact in a significant way the results as highlighted in [1]. The process X_t that drives the volatility is an OU process where the two parameters $H \in (-\infty, 1/2]$ and $\epsilon > 0$ control the mean-reversion speed through $(1/2 - H)\epsilon^{-1}$ and the vol-of-vol through $\epsilon^{H-1/2}$. For small values of ϵ we have a fast mean-reversion regime and a large vol-of-vol. These types of parametrizations can remind of the fast regimes studied in depth in [3] by Fouque. They can also be linked to jump models, studied for example in [4] and [5], when $H \leq -1/2$ and to rough volatility models, for example those presented in [2] and [5], where $H \in (0, 1/2)$ plays the role of the Hurst index. We will restrict our analysis to this last case. The solution of the OU process is:

$$X_t = \epsilon^{H-1/2} \int_0^t e^{-(1/2-H)\epsilon^{-1}(t-s)} \mathrm{d}s$$

Thus, the set of parameters to calibrate is:

$$\Theta := \{\rho, H, \epsilon, \alpha_0, \alpha_1, \alpha_3, \alpha_5\}$$

plus the input curve ξ_0. As said before we will use the market initial forward variance curve parametrized using the Gompertz function as in [VARIANCE CURVE SECTION].

2 SPX derivatives

To price SPX derivatives we have to resort to Monte Carlo simulations since there isn't a closed formula. Nevertheless, since X is a OU process it can be simulated exactly instead of approximating it using, for example, the Euler scheme which is often inaccurate in a fast mean-reversion regime. In order to simulate X we first define the auxiliary process $\tilde{X}$:

$$\tilde{X}_t := X_t \, e^{(1/2-H)\epsilon^{-1}t} = \epsilon^{H-1/2} \int_0^t e^{(1/2-H)\epsilon^{-1}s} \mathrm{d}W_s$$

Thus, $\tilde{X}$ can be simulated recursively by:

$$\tilde{X}_{t_{i+1}} = \tilde{X}_{t_i} + \sqrt{\frac{\epsilon^{2H}}{1-2H}} \left(e^{\frac{1-2H}{\epsilon} t_{i+1}} - e^{\frac{1-2H}{\epsilon} t_i} \right) Y_i$$

where Y_i are i.i.d. standard Gaussian. Naturally, to get back to $X_{t_{i+1}}$ we just have to divide $\tilde{X}_{t_{i+1}}$ by the factor $e^{\frac{1-2H}{\epsilon} t_{i+1}}$. This allows us to to easily vectorize computations. Whereas to simulate the log-process $\log(S)$ we will use the Euler scheme paired with antithetic and control variates, that is the so called turbocharging method as outlined in [18] that we have also used in the rBergomi model. This means that we only need to simulate the part of $\log(S)$ that is $F^{\mathcal{W}}$ measurable, we call this $S^{\mathcal{W}}$, and this can be simulated as:

$$\log(S^{\mathcal{W}}_{t_{i+1}}) = \log(S^{\mathcal{W}}_{t_i}) - \frac{1}{2}(\rho\sigma_{t_i})^2(t_{i+1} - t_i) + \rho\sigma_{t_i}\sqrt{t_{i+1} - t_i} Y_i + \rho^2(r - q)(t_{i+1} - t_i)$$

We will use an equi-spaced grid so that calling the time step h the above formula reduces to:

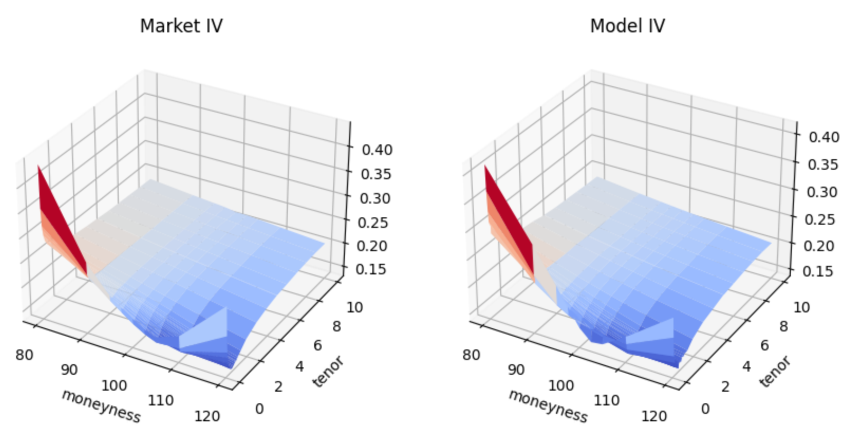
$$\log(S^{\mathcal{W}}_{t_{i+1}}) = \log(S^{\mathcal{W}}_{t_i}) + \left(r - q - \frac{1}{2}\sigma^2_{t_i}\right)\rho^2 h + \rho\sigma_{t_i}\sqrt{h}Y_i$$

2.1 Numerical results

We decided to calibrate our model both in a local and global way, as in the rBergomi set. In the local case we calibrated a set of parameters for every tenor while in the global approach we calibrated the set of parameters so that it is the best fit considering all the tenors together. We report, for the local case, some of the calibrated parameters in the following table.

Tenor	ρ	H	ε	α_0	α_1	α_3	α_5
2 weeks	-0.5332	0.1034	0.1078	0.1353	0.6193	0.0954	0.1024
1 month	-0.6626	0.0915	0.0171	0.9777	0.0187	0.0343	0.1072
6 months	-0.6968	-0.0397	0.0083	1.2204	0.0035	0.2296	0.0462
1 year	-0.8565	0.1903	0.0234	1.0277	0.2463	0.0672	0.6016
10 years	-0.6658	0.1141	0.1173	0.9099	0.5726	0.1604	0.1229

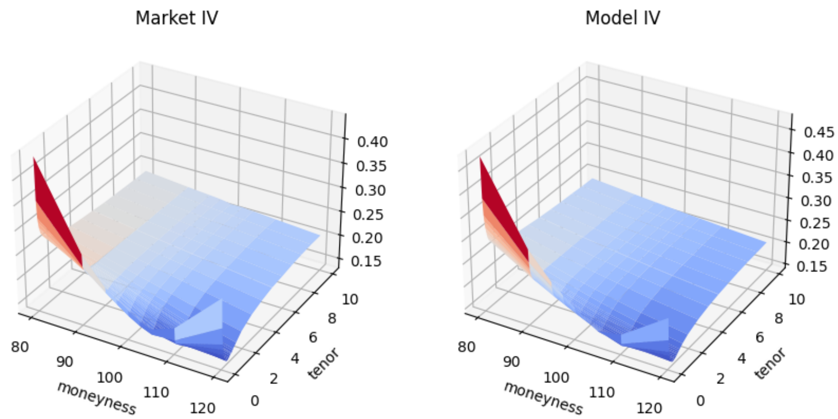
Comparing our results with the market we obtained a mean relative percentage error of 1.9299%. The next figure is the comparison between market and model IV surfaces.



Now we report, for the global approach, the set of the calibrated parameters.

ρ	H	ε	α_0	α_1	α_3	α_5
-0.9468	0.0305	0.1024	0.6101	0.3713	0.0054	0.0394

In this case we obtained a mean relative error of 3.5868%. The next figure is the corresponding comparison between market and model IV surfaces.



3 VIX derivatives

One major advantage of the Quintic OU model is that there is an explicit expression for the VIX. In a continuous time framework the VIX can be expressed as:

$$VIX_T^2 = -\frac{2}{\Delta}\mathbb{E}\left[\log\left(\frac{S_{T+\Delta}}{S_T}\right)\middle|\mathcal{F}_T\right]\cdot 100^2 = \frac{100^2}{\Delta}\int_T^{T+\Delta}\xi_T(u)\mathrm{d}u \tag{4.1}$$

with the usual $\Delta = 30$ days and $\xi_T(u) = \mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ the forward variance curve, that can be explicitly computed. In order to do that we fix $T \leq u$ and rewrite the process X as:

$$X_u = X_T e^{-(1/2-H)\epsilon^{-1}(u-T)} + \epsilon^{H-1/2}\int_T^u e^{-(1/2-H)\epsilon^{-1}(u-s)}\mathrm{d}W_s =: Z_T^u + G_T^u$$

Thus, if we define:

$$g(u) := \mathbb{E}[p(X_u)^2]$$

we obtain the following formula:

$$\xi_T(u) = \mathbb{E}[\sigma_u^2|\mathcal{F}_T] = \frac{\xi_0(u)}{g(u)}\mathbb{E}\left[\left(\sum_{k=0}^{5}\alpha_k X_u^k\right)^2 \,\middle|\, \mathcal{F}_T\right]$$

Defining α the vector $[\alpha_0, \alpha_1, 0, \alpha_3, 0, \alpha_5, 0, 0, \ldots]$ and indicating with $(\alpha * \alpha)$ the discrete convolution:

$$(\alpha * \alpha)_k = \sum_{j=0}^{k} \alpha_j \alpha_{k-j}$$

we have the following expression:

$$\xi_t(u) = \frac{\xi_0(u)}{g(u)} \mathbb{E}\left[\sum_{k=0}^{10} (\alpha * \alpha)_k X_u^k \;\middle|\; \mathcal{F}_T \right]$$

Furthermore, making use of the binomial expansion we can improve the expression for $\xi_T(u)$ in terms of Z^u and G^u so that we have:

$$\xi_T(u) = \frac{\xi_0(u)}{g(u)} \sum_{k=0}^{10} \sum_{i=0}^{k} (\alpha * \alpha)_k \binom{k}{i} \left(X_T e^{-(1/2-H)\epsilon^{-1}(u-T)} \right)^i \mathbb{E}\big[(G_T^u)^{k-i}\big] \tag{4.2}$$

where we used both the fact that Z_T^u is $\mathcal{F}_T$-measurable and the independence of G_T^u from $\mathcal{F}_T$. Moreover, we know that G_T^u is actually a Gaussian random variable.

$$G_T^u \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H}[1 - e^{-(1-2H)\epsilon^{-1}(u-T)}]\right)$$

We recall that the moments of a Gaussian variable $Y \sim \mathcal{N}(0, \sigma_Y^2)$ can be computed as:

$$\mathbb{E}[Y^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sigma_Y^n (n-1)!! & \text{if } n \text{ is even} \end{cases} \tag{4.3}$$

where $n!!$ is the double factorial. Therefore, we have an explicit expression for all the moments of G_T^u. Going back to (4.1) and plugging in expression (4.2) we have that the explicit expression of VIX_T^2 is polynomial in X_T and given by:

$$\begin{aligned} VIX_T^2 &= \frac{100^2}{\Delta} \sum_{k=0}^{10} \sum_{i=0}^{k} (\alpha * \alpha)_k \binom{k}{i} X_T^i \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}\big[(G_T^u)^{k-i}\big] e^{-(1/2-H)\epsilon^{-1}(u-T)i} \mathrm{d}u \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} X_T^i \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}\big[(G_T^u)^{k-i}\big] e^{-(1/2-H)\epsilon^{-1}(u-T)i} \mathrm{d}u \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} \beta_i X_T^i \qquad (4.4) \end{aligned}$$

where we have defined

$$\beta_i := \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}\big[(G_T^u)^{k-i}\big] e^{-(1/2-H)\epsilon^{-1}(u-T)i} \mathrm{d}u$$

We recall that thanks to formula (4.3) we can compute exactly every moment of G^u_T. We note, from formula (4.4), that VIX_T^2 is actually a polynomial in X_T that we will denote with $f(X_T)$. Since we have that X_T is Gaussian:

$$X_T \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H}\left[1 - e^{-(1-2H)\epsilon^{-1}T}\right]\right)$$

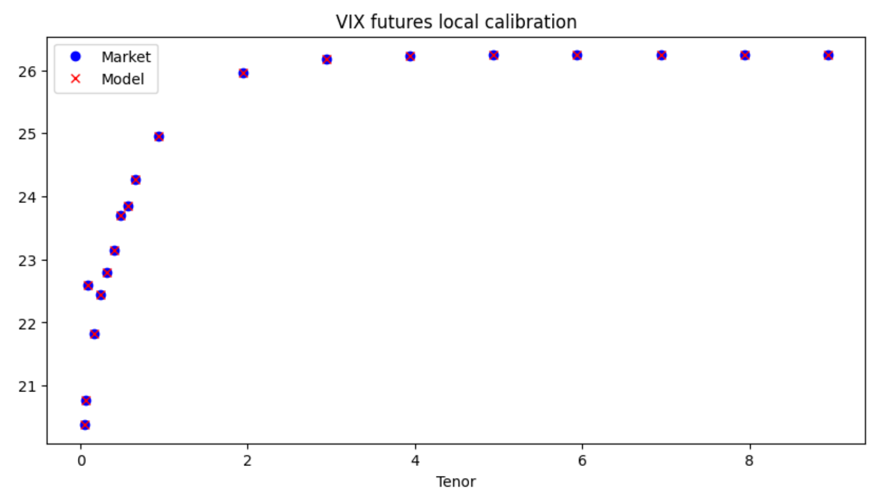
pricing VIX derivatives, with a general payoff function Φ, can be done integrating directly against the standard Gaussian density:

$$\mathbb{E}\left[\Phi(VIX_T)\right] = \mathbb{E}\left[\Phi\left(\sqrt{f(X_T)}\right)\right] = \frac{1}{\sigma_{X_T}\sqrt{2\pi}}\int_{\mathbb{R}} \Phi\left(\sqrt{f(x)}\right) e^{-x^2/\sigma^2_{X_T}}\,\mathrm{d}x$$

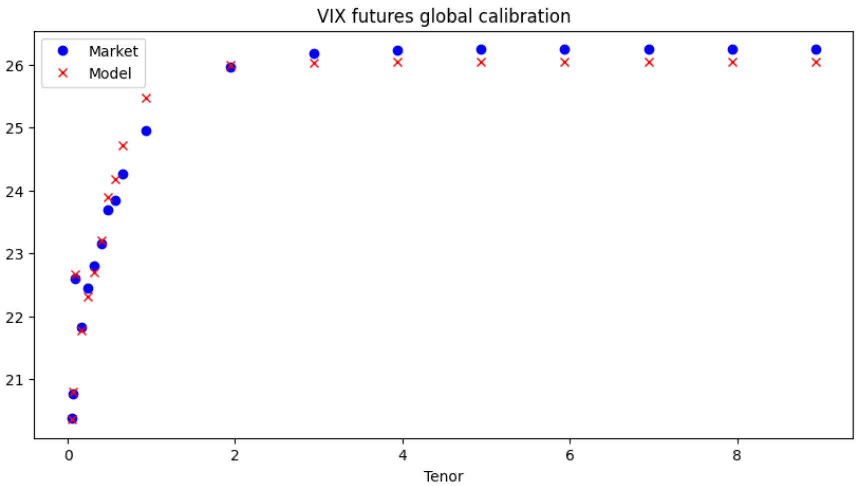
This integral can be computed efficiently using a variety of quadrature techniques.

3.1 Numerical results

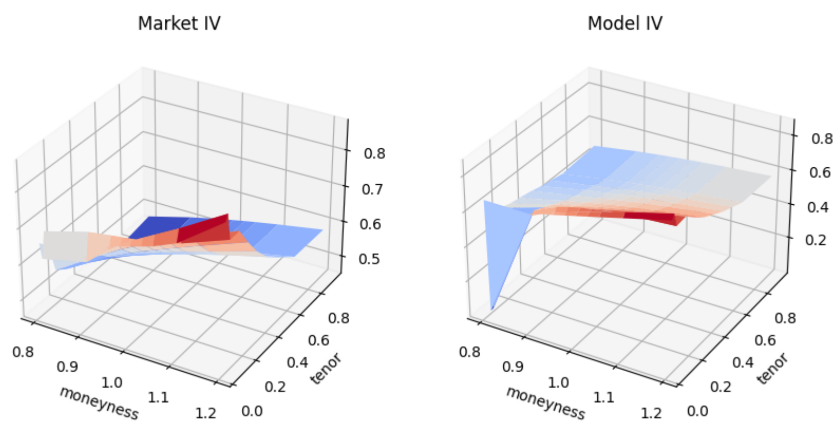
As for the SPX derivatives we used both a local and global approach for futures and options. First we report the future results. In the local case the calibration is almost perfect, while in the global case we obtained a mean percentage error of 0.6920%. The next figure is the comparison between model and market futures in the local approach.



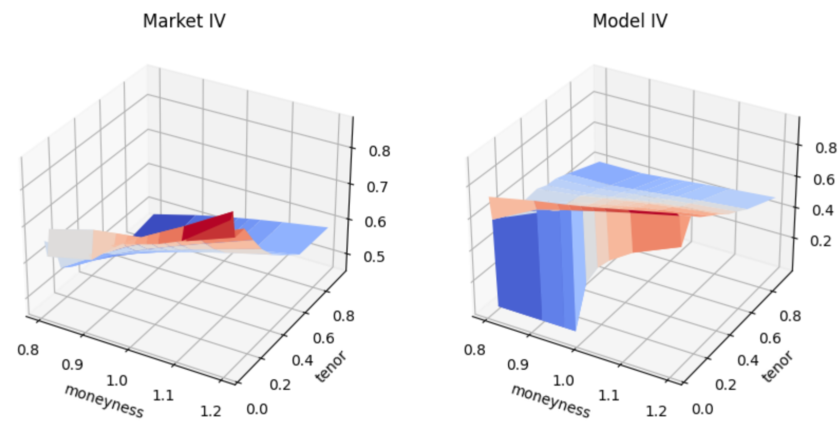
The next figure is the comparison between market and model futures in the global approach.



Calibrating the model to the IV of VIX options we discovered, during the local calibration, that the model isn't suitable for long term-options in fact we obtained a mean relative percentage error of 31.1995% if we use all the tenors, while if we use only the tenors that are less or equal to one year we obtained a mean relative percentage error of 2.4955%. The next figure is the comparison between market and model IV surfaces.



For the global approach using only the tenors that are less or equal to one year we obtained a mean relative percentage error of 8.7636%. The next figure is the comparison between market and model IV surfaces.



We note that the model tends to underestimate the IV surface.

4 Joint calibration

In this section we tackle the joint calibration problem that is the simultaneous calibration of the model to SPX European options, VIX European options and VIX futures across several tenors. To jointly calibrate the model we want to find the solution of the following optimisation problem:

$$\min_{\Theta}\{c_1 f_1(\Theta) + c_2 f_2(\Theta) + c_3 f_3(\Theta)\}$$

where Θ is the set of parameters and we have defined:

$$f_1(\Theta) \coloneqq \sqrt{\sum_{i,j}\left(\sigma_{spx}^{\Theta}(K_i,T_j) - \sigma_{spx}^{mkt}(K_i,T_j)\right)^2}$$

$$f_2(\Theta) \coloneqq \sqrt{\sum_{i,j}\left(\sigma_{vix}^{\Theta}(K_i,T_j) - \sigma_{vix}^{mkt}(K_i,T_j)\right)^2}$$

$$f_3(\Theta) \coloneqq \sqrt{\sum_{i}\left(F_{vix}^{\Theta}(T_i) - F_{vix}^{mkt}(T_i)\right)^2}$$

So that f_1 is the root mean squared error (RMSE) coming from the SPX options calibration, f_2 is the RMSE coming from the VIX options calibration and f_3 is the RMSE coming from the VIX futures calibration. The constants c_i are positive and reflect the weight that we want to give to each particular aspect. In our case we decided to give an equal weight to each part so that we have $c_1 = c_2 = c_3 = 1$.

4.1 Numerical results

For the joint calibration problem we decided to use only the global approach. The tenors that we used for the VIX, for both futures and options, are those which are less or equal to one year. Doing so we obtained the following mean relative percentage error:

- SPX options calibration: 7.7591%;
- VIX futures calibration: 0.4339%;
- VIX options calibration: 18.3786%.

What we can see is that the calibration gives us significantly worse results especially in the SPX and VIX options part. Thus, the model, at least with our data, doesn't seem to solve the joint calibration problem in a satisfying way.

5 SPX Parameters stability

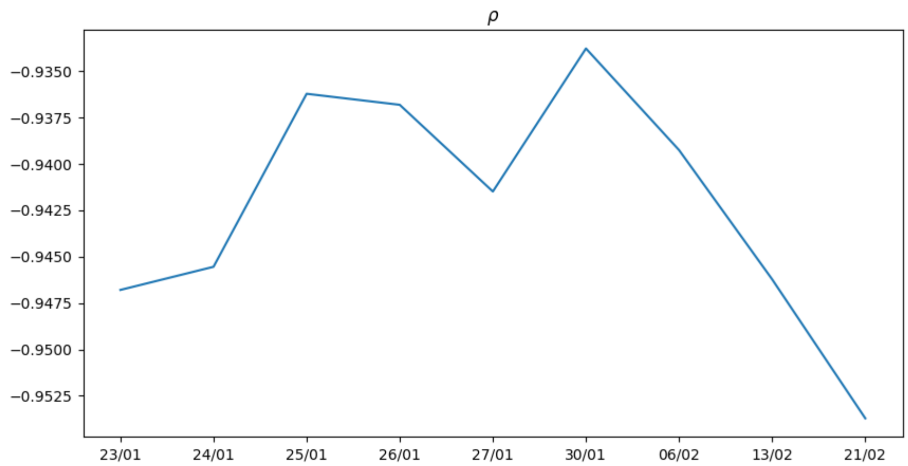
To check the stability of the model parameters we decided to test the model using data from other days than the principal one. The dates that we used are: 24/01/2023, 25/01/2023, 26/01/2023, 27/01/2023, 30/01/2023, 06/02/2023, 13/02/2023, 21/02/2023. In doing so we used only options available in all these days and they are 30 in total ranging from few months tenors to almost ten years. We did two tests:

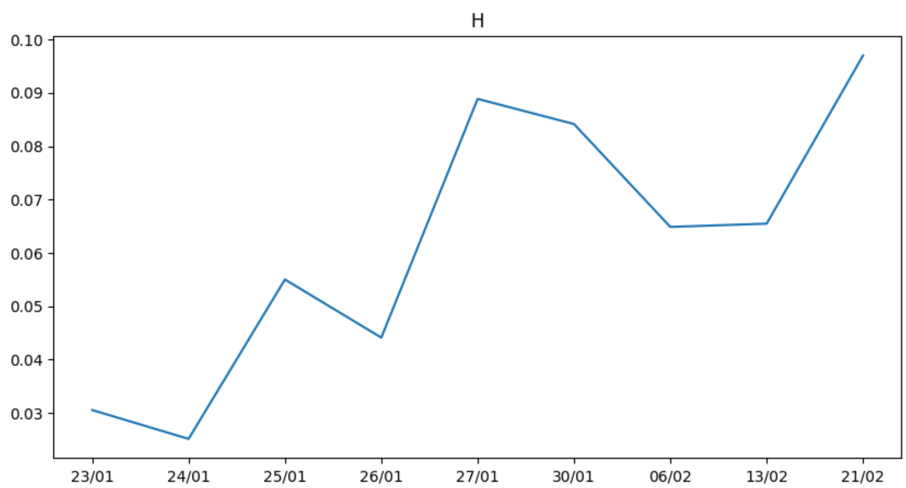
1. using the already calibrated parameters we used the model to check the mean percentage error in all these days comparing them with the error obtained calibrating the model for each;
2. we calibrated the model in each day and then analyzed the standard deviation and variance of the each parameter.

The next table contains the parameters that we obtained for each day.

Day	ρ	H	ε	α_0	α_1	α_3	α_5
23/01	-0.9468	0.0305	0.1024	0.6101	0.3713	0.0054	0.0394
24/01	-0.9455	0.0251	0.1064	0.6128	0.3899	0.0011	0.0373
25/01	-0.9362	0.0550	0.1419	0.5218	0.5026	0.0023	0.0353
26/01	-0.9368	0.0441	0.1255	0.5017	0.5509	0.0013	0.0401
27/01	-0.9415	0.0889	0.1171	0.5626	0.4575	1e-9	0.0553
30/01	-0.9338	0.0842	0.1084	0.5752	0.4485	0.0053	0.0614
06/02	-0.9393	0.0649	0.1009	0.5867	0.4567	0.0002	0.0533
13/02	-0.9462	0.0655	0.1431	0.5715	0.4471	1e-7	0.0405
21/02	-0.9537	0.0970	0.0640	0.6352	0.3648	0.0001	0.0772

A graphical representation of the changes is presented in the next figures.


ρ
−0.9350
−0.9375
−0.9400
−0.9425
−0.9450
−0.9475
−0.9500
−0.9525
23/01
24/01
25/01
26/01
27/01
30/01
06/02
13/02
21/02


H
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
23/01
24/01
25/01
26/01
27/01
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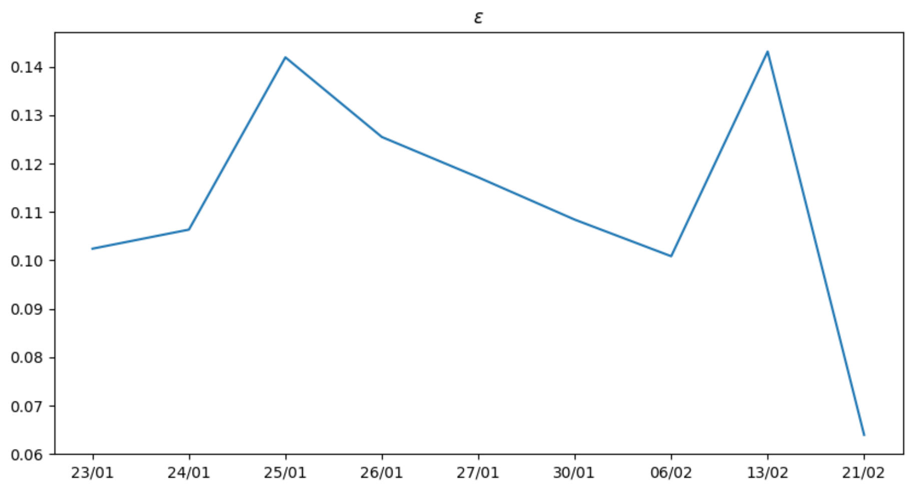
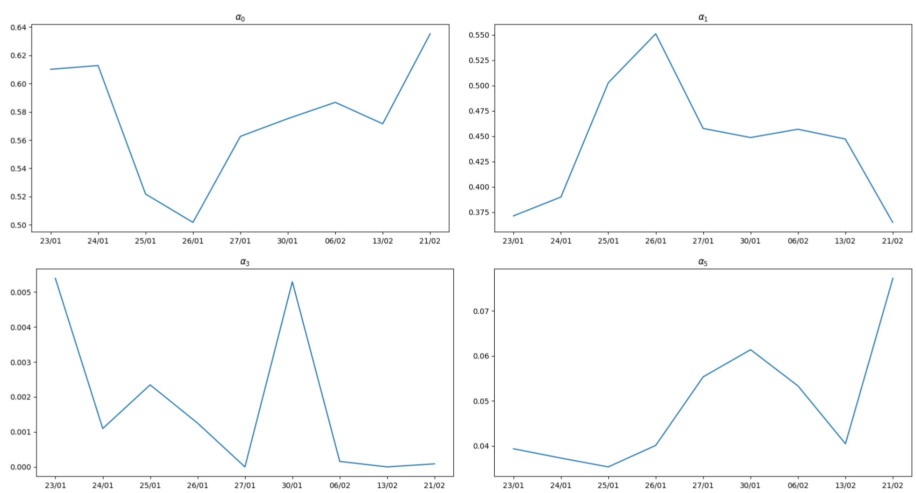
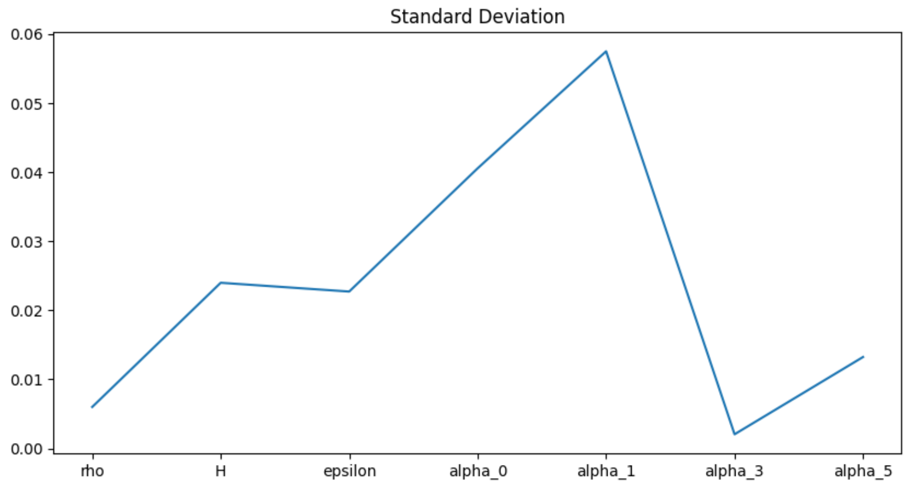
06/02
13/02
21/02

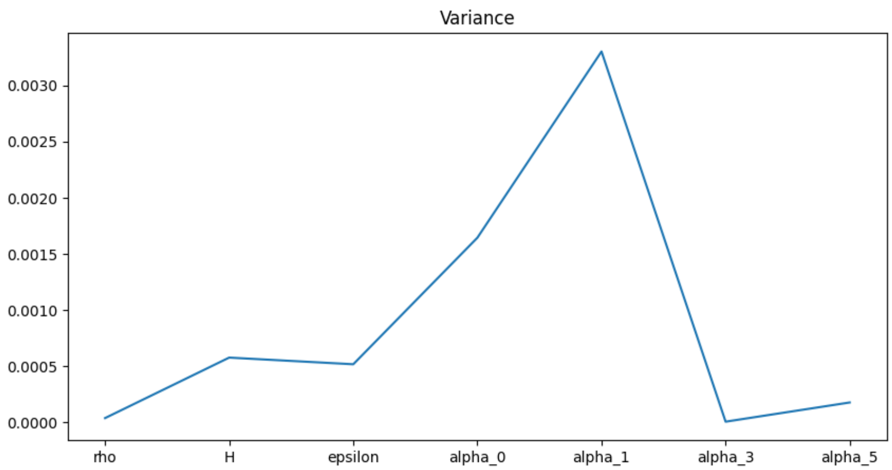




If we look at the standard deviation and variance of each parameter we obtain the following table.

	ρ	H	ε	α_0	α_1	α_3	α_5
Std	0.00602	0.02398	0.02271	0.04054	0.05747	0.00206	0.01324
Var	0.00004	0.00058	0.00052	0.00164	0.00330	4e-6	0.00018

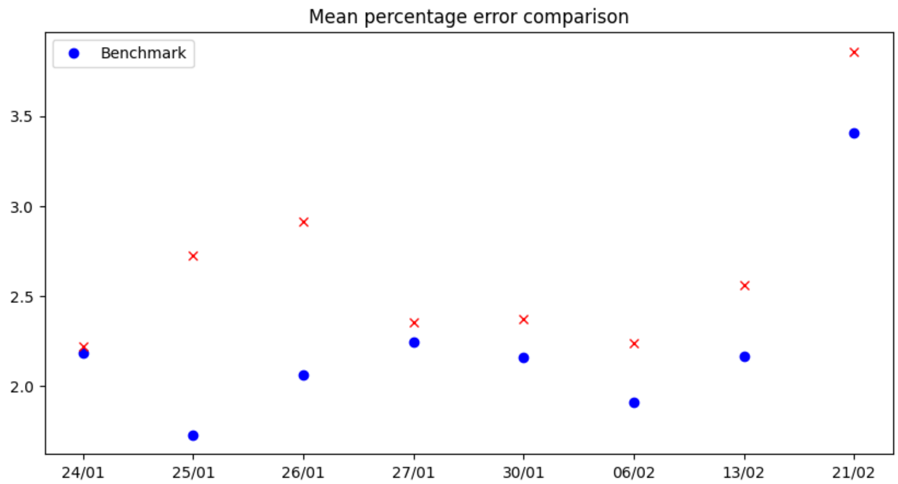




The next table contains the comparison between the benchmark error obtained before and the one obtained using as parameters the one calibrated for the first day (23/01).

Day	**Benchmark**	**Error**
24/01	2.1833%	2.2226%
25/01	1.7296%	2.7234%
26/01	2.0642%	2.9135%
27/01	2.2429%	2.3555%
30/01	2.1602%	2.3735%
06/02	1.9085%	2.2389%
13/02	2.1663%	2.5612%
21/02	3.4100%	3.8596%

The next figure is a graphical comparison.



6 Bayesian Inverse Problem

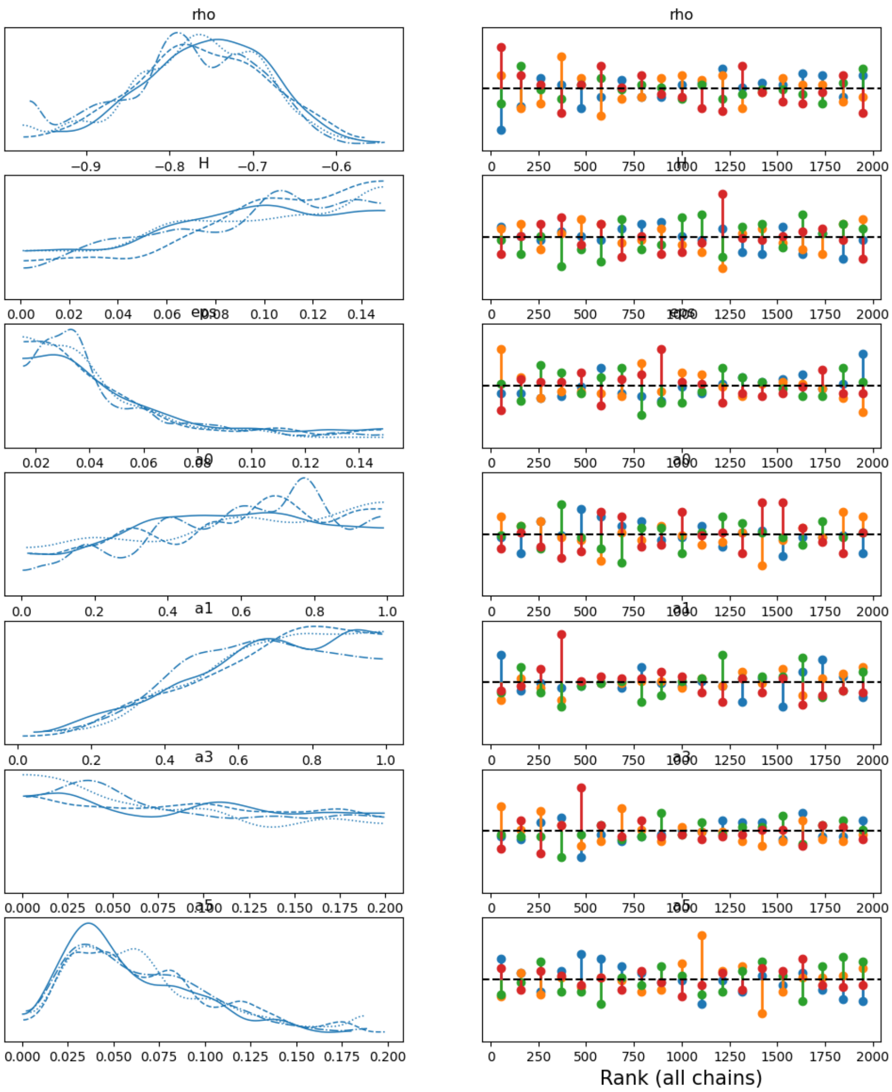
We decided to calibrate the Bayesian model only in the SPX case so that we can compare its results with the other models. The method that we used to do this calibration is again the ABC method. We decided to calibrate the model using only the tenors from 1 month to 6 months (10 in total) and we decided to approach the problem in two ways: calibrating all the parameters and calibrating only the set of parameters comprised of ρ, H and ϵ while using pre-calibrated values of the vector α. We calibrated taking as the reference data the market prices instead of the market IV and we utilized 4 chains with 2000 samples each, we resorted to this escamotage just for computational and time reasons (we are using just a standard laptop).

6.1 Numerical results

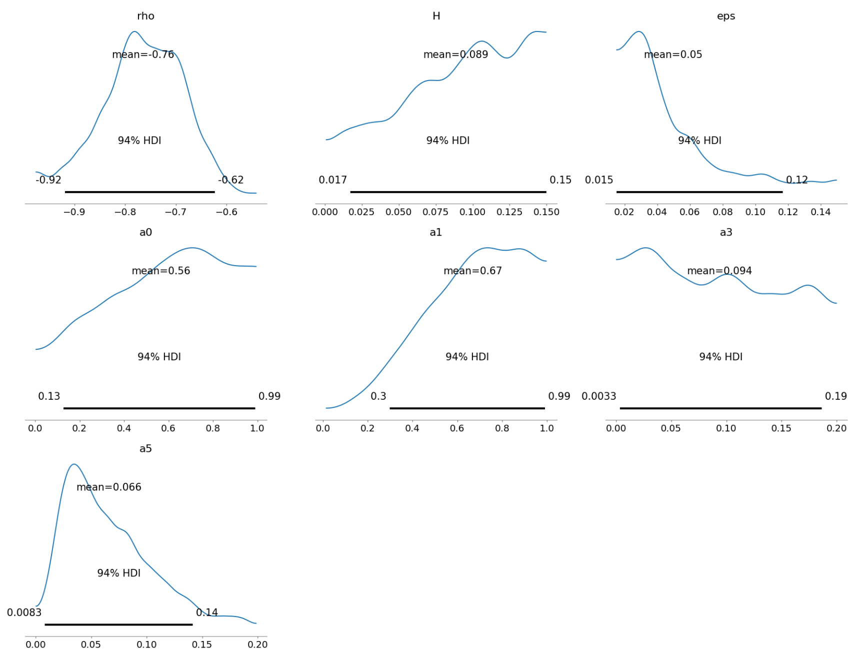
As priors for the parameters we employed:

- $\rho \sim \mathcal{U}(-1, -0.1)$
- $H \sim \mathcal{U}(0, 0.15)$
- $\epsilon \sim \mathcal{U}(0.015, 0.15)$
- $\alpha_0 \sim \mathcal{U}(0, 1)$
- $\alpha_1 \sim \mathcal{U}(0, 1)$
- $\alpha_3 \sim \mathcal{U}(0, 0.2)$
- $\alpha_5 \sim \mathcal{U}(0, 0.2)$

In the first case, where we calibrate all the parameters, we obtained the following rank plots.



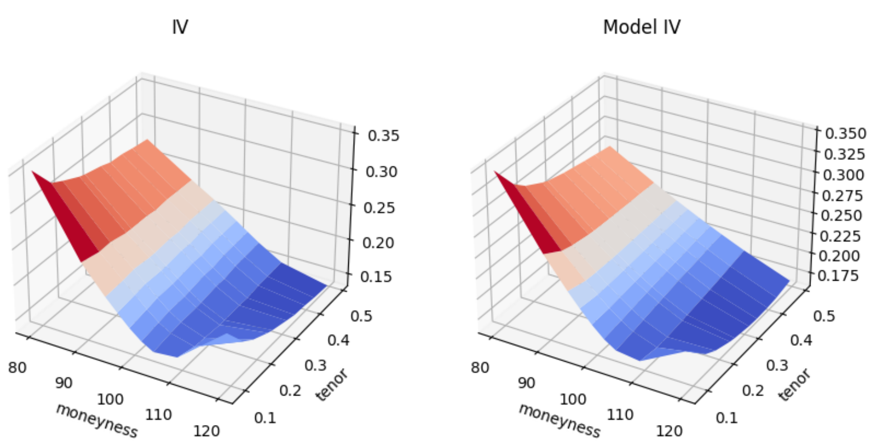
Looking at the chains plots we expect to obtain results that are not very good, since they are not very coherent, while looking at the rank plots we note that we have some problems when sampling. The next figure represents the posterior densities that we have obtained.



To compare the Bayesian calibration with the deterministic calibration we fixed the maximum a posteriori (MAP) values for each parameter:

ρ	H	ϵ	α_0	α_1	α_3	α_5
-0.7761	0.1090	0.0250	0.7630	0.6504	0.0095	0.0332

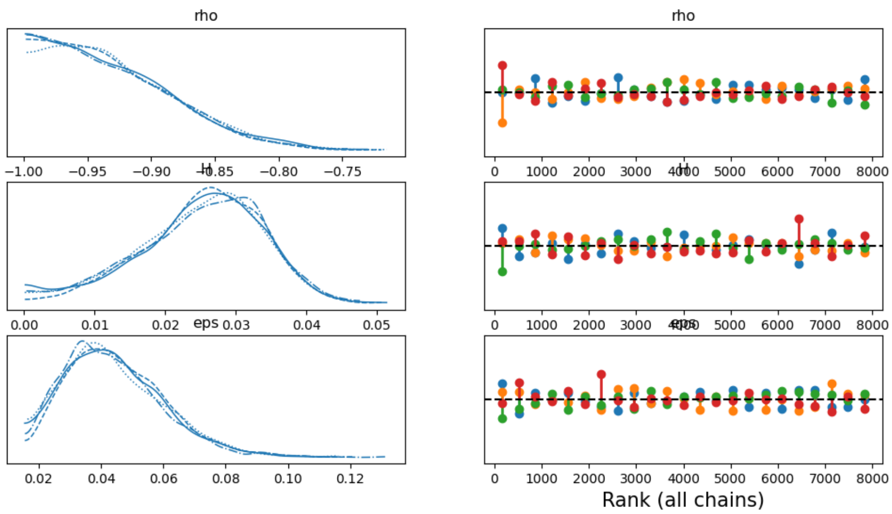
and calibrating the model we achieved a mean relative percentage error of 10.6836%. The next figure is a comparison between the market and model IV surfaces.



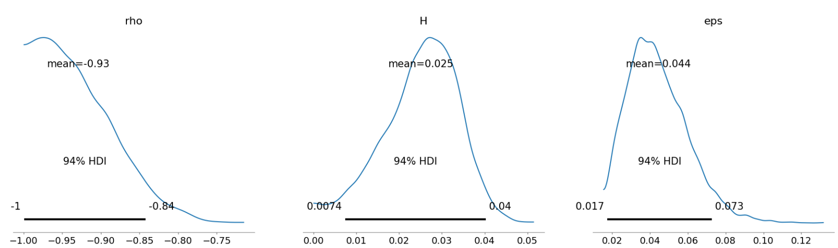
In the second case we fixed the value of the vector α obtained through a deterministic calibration:

$$\alpha_0 = 1.2237 \quad \alpha_1 = 0.4503 \quad \alpha_3 = 0.0361 \quad \alpha_5 = 0.0532$$

Then we calibrated the other three parameters and got the following chain and rank plots.



We note that this time the chains are more coherent and looking at the rank plots we have almost no problems in the sampling. The next figure presents the posterior densities that we got.

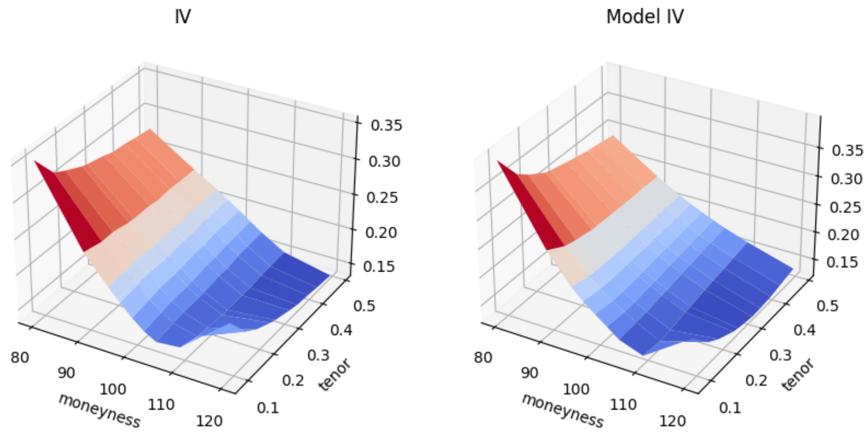


Posterior densities

As before, in order to compare the Bayesian calibration with the deterministic calibration we fixed the MAP values:

ρ	H	ϵ
-0.9635	0.0264	0.0333

and calibrating the model we achieved a mean relative percentage error of 3.7385%. The next figure is the comparison between market and model IV surfaces.



Thus, we can conclude that fixing the α parameter the Bayesian calibration is able to give us some good quality results. Whereas, if the α parameter has to be calibrated the Bayesian framework is not able to give us good results since the set of parameters is not completely independent.

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