Dataset

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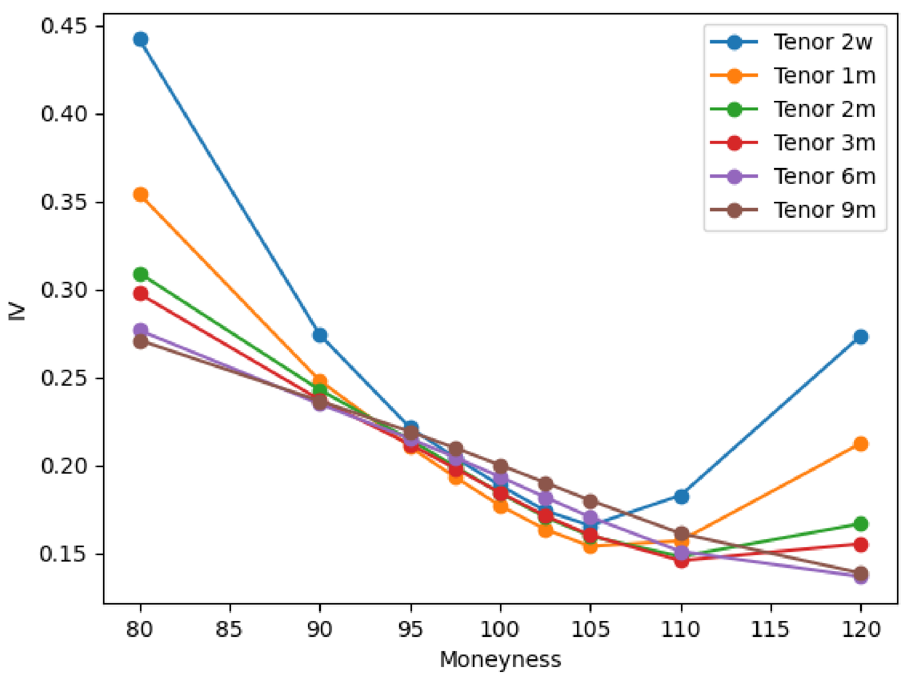
We decided to use the SPX option data of the 23rd January 2023 (source: Bloomberg). Option with moneyness from 80% to 120% and tenor from 2 weeks to almost 10 years. The spot reference was 4017.80. Here below there is the volatility surface for some tenors:



Implied drift term

The options on SPX index are very liquid and rich in structure. Whenever we transform our model equations into the risk-free world to price some contingent claim we have to estimate the drift term. The drift term is the difference between the risk-free rate and the yield (continuously compounded) of the underlying.

In order to estimate that we recall the call-put parity. We will use the following notation:

- $C_{t,K}$ is the value of a call at time t with strike K and tenor T;
- $P_{t,K}$ is the value of a put at time t with strike K and tenor T;
- S_t is the spot value of the underlying;
- r is the risk-free rate (continuously compounded);
- q is the yield of the underlying (continuously compounded).

Then the following hold:

$$C_{t,K} - P_{t,K} = S_t e^{-q(T-t)} - Ke^{-r(T-t)}$$

solving for q we obtain:

$$q = \frac{1}{t-T} \log \left(\frac{C_{t,K} - P_{t,K} + Ke^{-r(T-t)}}{S_t} \right)$$

now this relation still holds for $t = 0$ and we can observe the market price of spot S_0, the call market-price $\hat{C}_{0,K}$ and the put price $\hat{P}_{0,K}$. So the relation now is:

$$q = -\frac{1}{T} \log \left(\frac{\hat{C}_{0,K} - \hat{P}_{0,K} + Ke^{-rT}}{S_0} \right)$$

one can argue now that q can depend on the strike, but this is not true. Indeed, let's suppose that this is true and consider two strikes $K_1 < K_2$ with $q_{K_1} > q_{K_2}$. Buy the following portfolio at time $t = 0$:

$$\frac{1}{S_0}(C_{0,K_1} - P_{0,K_1} + K_1 e^{-rT}) - \frac{1}{S_0}(C_{0,K_2} - P_{0,K_2} + K_2 e^{-rT})$$

due to call-put parity buying this portfolio at time zero gives you money, indeed its value at time 0 is

$$\frac{1}{S_0}(C_{0,K_1} - P_{0,K_1} + K_1 e^{-rT}) - \frac{1}{S_0}(C_{0,K_2} - P_{0,K_2} + K_2 e^{-rT}) = e^{-q_{K_1}T} - e^{-q_{K_2}T} < 0$$

At maturity the value of the portfolio is 0, because $C_{T,K_1} - P_{T,K_1} + K_1 = S_T = C_{T,K_2} - P_{T,K_2} + K_2$. So this strategy is an arbitrage. The same, with obvious modifications, applies if $q_{K_1} < q_{K_2}$. So in conclusion the term q depends only on the tenor. To estimate the risk-free rate we used the values of the USD Swap OIS Fed Funds linearly interpolated for the missing tenors. Here below the data used for the risk-free rate:

$$[INSERIREFIGURA]$$

Here below the estimate for q obtained:

$[INSERIREFIGURA]$

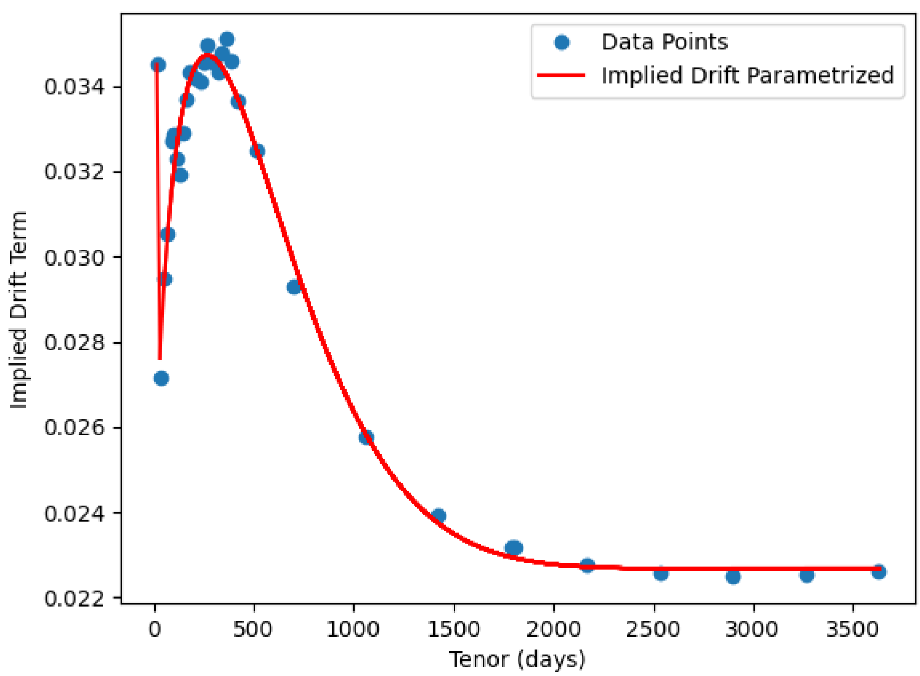
Now, define as T_{max} the maximum tenor for which we have a market data point (in our case 3630 days). We have decided to parametrize, for convenience, the implied drift term $(r-q)$ with the following function of the tenor T:

$$\text{Implied\_Drift(T)} := \begin{cases} a\left(\dfrac{T}{T_{max}}\right)^{b-1}\left(1-\dfrac{T}{T_{max}}\right)^{c-1}+d & 30 \le T < T_{max} \\ d & T \ge T_{max} \end{cases}$$

with $a, b, c, d \in \mathbb{R}^+$. For shorter tenors we will use linear interpolation between the first datapoint that we have (tenor 2 weeks) and the Implied_Drift(30). We used least square method to fit a, b, c and d and obtained:

$$a = 0.13150334 \quad b = 1.67001571 \quad c = 9.41005645 \quad d = 0.02267608$$

and the result is the following:



Forward Variance Curve

The forward variance curve is associated with the fair strike of a variance swap (VS). A VS with maturity T is a contract which pays out the realized variance of a financial underlying, computed as the sum of the squares of daily log-returns, in exchange for a fixed strike called the variance swap variance V_0^T that is determined in such a way that the initial value of the contract is zero. The market instead of quoting the rate V_0^T of a VS quotes its volatility which is the strike K such that:

$$\frac{V_0^T}{T} - K^2 = 0$$

Therefore we define the volatility of a VS with maturity T as:

$$\hat{\sigma}_0^T \coloneqq \sqrt{\frac{V_0^T}{T}}$$

This relates to the forward variance curve since we have:

$$\hat{\sigma}_t^T = \frac{1}{T-t}\int_t^T \xi_0(u)\mathrm{d}u$$

or equivalently:

$$\xi_t(T) = \frac{\mathrm{d}}{\mathrm{d}T}\left[(T-t)\hat{\sigma}_t^T\right]$$

And for the initial forward variance curve it simplifies to:

$$\xi_0(t) = \frac{\mathrm{d}}{\mathrm{d}T}\left[T\hat{\sigma}_0^t\right]$$

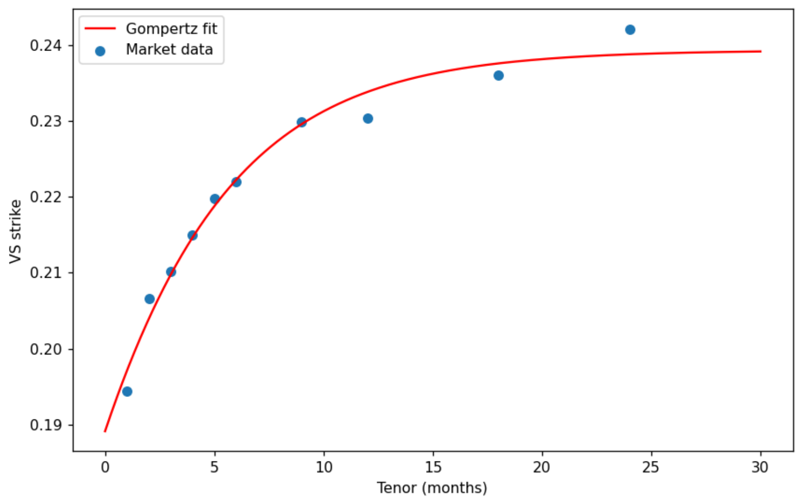
In order to compute the initial forward variance curve we have to find a parametrization for the volatility of the VS. It is reasonable to choose a parameterization with an asymptotic line, our choice is the Gompertz function:

$$\hat{\sigma}_0^t = z_1 e^{-z_2 e^{-z_3 t}}$$

where $z_1 > 0$ is the asymptote, $z_2 > 0$ sets the displacement along the x-axis, i.e. time to maturity, and $z_3 > 0$ sets the growth rate. We used the least squared method to fit the parameters and we obtained:

$$z_1 = 0.23934445541427635 \quad z_2 = 0.23559167522825925 \quad z_3 = 0.19271882494190468$$

and the resulting fit is the following figure.



The initial forward variance curve $\xi_0(t)$ can be obtained as:

$$\xi_0(t) = \left(\hat{\sigma}_0^t\right)^2 + t\frac{\mathrm{d}}{\mathrm{d}t}\left[(\hat{\sigma}_0^t)^2\right]$$

Doing so we obtained the following figure: