Quintic Ornstein-Uhlenbeck model

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The quintic Ornstein-Uhlenbeck (OU) volatility model is a stochastic volatility model where the volatility process is a polynomial function of degree five of a single OU process with fast mean reversion and large vol-of-vol.

1 The Model

Under the pricing measure $\mathbb{Q}$ the dynamics of the stock price S are given by:

$$\begin{aligned}
\mathrm{d}S_t &= (r-q)\mathrm{d}t + \sigma_t S_t \mathrm{d}B_t \\
\sigma_t &= \sqrt{\xi_0(t)}\frac{p(X_t)}{\sqrt{\mathbb{E}[p(X_t)^2]}} \\
\mathrm{d}X_t &= -\left(\frac{1}{2}-H\right)\epsilon^{-1}X_t\mathrm{d}t + \epsilon^{H-1/2}\mathrm{d}W_t \qquad X_0 = 0
\end{aligned}$$

where B_t and W_t are two Brownian motions with correlation parameter ρ; $\xi_0 \in L^2([0,T],\mathbb{R}^+)$ for any $T>0$ is an input curve used to match certain term-structures observed in the market, for instance, the normalization $\sqrt{\mathbb{E}[p(X_t)^2]}$ allows ξ_0 to match the market initial forward variance curve since:

$$\mathbb{E}\left[\int_0^t \sigma_s^2 \mathrm{d}s\right] = \int_0^t \xi_0(s)\mathrm{d}s \qquad t \geq 0$$

The fifth grade polynomial $p(x)$ is defined as:

$$p(x) \coloneqq \alpha_0 + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5$$

with non-negative parameters $\alpha_0, \alpha_1, \alpha_3, \alpha_5 \geq 0$ ($\alpha_2 = \alpha_4 = 0$). A polynomial of degree five allows to reproduce the upward slope of the VIX smile. Restricting the coefficients α to be non-negative allows the sign of the ATM skew to be the same as ρ, as explained in more detail in [2]. The choice to set $\alpha_2 = \alpha_4 = 0$ allows to reduce the number of parameters to calibrate and doesn't impact in a significant way the results as highlighted in [PAPAER Quintic]. The process X_t that drives the volatility is an OU process where the two parameters $H \in (-\infty, 1/2]$ and $\epsilon > 0$ control the mean-reversion speed through $(1/2-H)\epsilon^{-1}$ and the vol-of-vol through $\epsilon^{H-1/2}$. For small values of ϵ we have a fast mean-reversion regime and a large vol-of-vol. Such parametrizations are reminiscent of the fast regimes extensively studied by Fouque et al. [9] which corresponds to the case $H = 0$. They can also be linked to more complex models such as jump models [19,1] for $H \leq -1/2$; and rough volatility models [2,1] for which $H \in (0, 1/2)$ would play the role of the Hurst index. The solution of the OU process is:

$$X_t = \epsilon^{H-1/2} \int_0^t e^{-(1/2-H)\epsilon^{-1}(t-s)} \mathrm{d}s$$

The parameters to calibrate are seven:

$$\Theta \coloneqq \{\rho, H, \epsilon, \alpha_0, \alpha_1, \alpha_3, \alpha_5\}$$

plus the the input curve ξ_0, we will use the market initial forward variance curve.

2 SPX derivatives

The price SPX derivatives we have to resort to Monte Carlo simulations since there isn't a closed formula. Nevertheless, since X is a OU process it can be simulated exactly instead of approximating it using, for exampler, the Euler scheme which is often inaccurate in a fast mean-reversion regime. To simulate X we first define:

$$\tilde{X}_t \coloneqq X_t \; e^{(1/2-H)\epsilon^{-1}t} = \epsilon^{H-1/2} \int_0^t e^{(1/2-H)\epsilon^{-1}s} \mathrm{d}W_s$$

Thus, $\tilde{X}$ can be simulated recursively by:

$$\tilde{X}_{t_{i+1}} = \tilde{X}_{t_i} + \sqrt{\frac{\epsilon^{2H}}{1-2H}} \left(e^{\frac{1-2H}{\epsilon} t_{i+1}} - e^{\frac{1-2H}{\epsilon} t_i} \right) Y_i$$

where Y_i are i.i.d. standard Gaussian. Naturally, to get back to $X_{t_{i+1}}$ we just have to divide $\tilde{X}_{t_{i+1}}$ by $e^{\frac{1-2H}{\epsilon} t_{i+1}}$. This allows us to to easily vectorize computations. Instead, to simulate the log-process $\log(S)$ we will use the Euler scheme paired with antithetic and control variates, the so called turbocharging method as outlined in [18] that we have used also in the rBergomi model. This means we only need to simulate the part of $\log(S)$ that is $F^{\mathcal{W}}$ measurable, we call this $S^{\mathcal{W}}$ and can be simulated as:

$$\log(S^{\mathcal{W}}_{t_{i+1}}) = \log(S^{\mathcal{W}}_{t_i}) - \frac{1}{2}(\rho\sigma_{t_i})^2(t_{i+1}-t_i) + \rho\sigma_{t_i}\sqrt{t_{i+1}-t_i}Y_i + \rho^2(r-q)(t_{i+1}-t_i)$$

We will use an equi-spaced grid so that calling the time step h the above formula reduces to:

$$\log(S^{\mathcal{W}}_{t_{i+1}}) = \log(S^{\mathcal{W}}_{t_i}) + \left(r - q - \frac{1}{2}\sigma^2_{t_i} \right) \rho^2 h + \rho\sigma_{t_i}\sqrt{h}Y_i$$

3 VIX derivatives

One major advantage of this model is that there is an explicit expression of the VIX. In continuous time the VIX can be expressed as:

$$VIX_T^2 = -\frac{2}{\Delta}\mathbb{E}\left[\log\left(\frac{S_{T+\Delta}}{S_T}\right)\middle|\mathcal{F}_T\right]\cdot 100^2 = \frac{100^2}{\Delta}\int_T^{T+\Delta}\xi_T(u)\mathrm{d}u \tag{4.1}$$

with the usual $\Delta = 30$ days and $\xi_T(u) = \mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ the forward variance curve, that can be explicitly computed. In order to do that we fix $T \leq u$ and rewrite X as:

$$X_u = X_T e^{-(1/2-H)\epsilon^{-1}(u-T)} + \epsilon^{H-1/2}\int_T^u e^{-(1/2-H)\epsilon^{-1}(u-s)}\mathrm{d}W_s =: Z_T^u + G_T^u$$

then, if we define:

$$g(u) := \mathbb{E}[p(X_u)^2]$$

we obtain:

$$\xi_T(u) = \mathbb{E}[\sigma_u^2|\mathcal{F}_T] = \frac{\xi_0(u)}{g(u)}\mathbb{E}\left[\left(\sum_{k=0}^{5}\alpha_k X_u^k\right)^2 \,\middle|\, \mathcal{F}_T\right]$$

Defining α the vector $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$ and indicating with $(\alpha * \alpha)$ the discrete convolution:

$$(\alpha * \alpha)_k = \sum_{j=0}^{k}\alpha_j\alpha_{k-j}$$

we have the following expression:

$$\xi_t(u) = \frac{\xi_0(u)}{g(u)}\mathbb{E}\left[\sum_{k=0}^{10}(\alpha*\alpha)_k X_u^k \,\middle|\, \mathcal{F}_T\right]$$

Using the binomial expansion we can further develop the expression for $\xi_T(u)$ in terms of Z^u and G^u to get:

$$\xi_T(u) = \frac{\xi_0(u)}{g(u)}\sum_{k=0}^{10}\sum_{i=0}^{k}(\alpha*\alpha)_k\binom{k}{i}\left(X_T e^{-(1/2-H)\epsilon^{-1}(u-T)}\right)^i\mathbb{E}\big[(G_T^u)^{k-i}\big] \tag{4.2}$$

where we used both that Z_T^u is $\mathcal{F}_T$-measurable and that G_T^u is independent of $\mathcal{F}_T$. Furthermore, we know that G_T^u is actually a Gaussian random variable:

$$G_T^u \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H}[1 - e^{-(1-2H)\epsilon^{-1}(u-T)}]\right)$$

We recall that for a Gaussian variable $Y \sim \mathcal{N}(0, \sigma_Y^2)$ its moments can be computed as:

$$\mathbb{E}[Y^p] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma_Y^p (p-1)!! & \text{if } p \text{ is even} \end{cases} \tag{4.3}$$

where $p!!$ is the double factorial. Therefore we can compute exactly all moments of G_T^u. Going back to (4.1) and plugging the expression (4.2) we have that the explicit expression of VIX_T^2 is polynomial in X_T and given by:

$$\begin{aligned} VIX_T^2 &= \frac{100^2}{\Delta} \sum_{k=0}^{10} \sum_{i=0}^{k} (\alpha * \alpha)_k \binom{k}{i} X_T^i \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}\big[(G_T^u)^{k-i}\big] e^{-(1/2-H)\epsilon^{-1}(u-T)i} \mathrm{d}u \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} X_T^i \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}\big[(G_T^u)^{k-i}\big] e^{-(1/2-H)\epsilon^{-1}(u-T)i} \mathrm{d}u \\ &= \frac{100^2}{\Delta} \sum_{i=0}^{10} \beta_i X_T^i \qquad (4.4) \end{aligned}$$

where we have defined

$$\beta_i := \sum_{k=i}^{10} (\alpha * \alpha)_k \binom{k}{i} \int_T^{T+\Delta} \frac{\xi_0(u)}{g(u)} \mathbb{E}\big[(G_T^u)^{k-i}\big] e^{-(1/2-H)\epsilon^{-1}(u-T)i} \mathrm{d}u$$

We recall that thanks to formula (4.3) we can compute exactly every moment of G_T^u. Thanks to the closed from of (4.4) VIX_T^2 is a polynomial in X_T that we will denote as $h(X_T)$. Since we have that X_T is Gaussian

$$X_T \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H}\big[1 - e^{-(1-2H)\epsilon^{-1}T}\big]\right)$$

pricing VIX derivatives, with a general payoff function Φ, can be done integrating directly against the standard Gaussian density:

$$\mathbb{E}\big[\Phi(VIX_T)\big] = \mathbb{E}\left[\Phi\big(\sqrt{h(X_T)}\big)\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi\big(\sqrt{h(\sigma_{X_T} x)}\big) e^{-x^2/2} \mathrm{d}x$$

This integral can be computed efficiently using a variety of quadrature techniques.

4 Joint calibration

5 Bayesian Inverse Problem

References

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