# Quintic Ornstein-Uhlenbeck model

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The quintic Ornstein-Uhlenbeck (OU) volatility model is a stochastic volatility model where the volatility process is a polynomial function of degree five of a single OU process with fast mean reversion and large vol-of-vol.

## 1 The Model

Under the pricing measure $\mathbb{Q}$ the dynamics of the stock price $S$ are given by:

$$\begin{aligned}
\mathrm{d}S_t &= (r-q)\mathrm{d}t + \sigma_t S_t \mathrm{d}B_t \\
\sigma_t &= \sqrt{\xi_0(t)}\frac{p(X_t)}{\sqrt{\mathbb{E}[p(X_t)^2]}} \\
\mathrm{d}X_t &= -\left(\frac{1}{2} - H\right)\epsilon^{-1} X_t \mathrm{d}t + \epsilon^{H-1/2}\mathrm{d}W_t \qquad X_0 = 0
\end{aligned}$$

where $B_t$ and $W_t$ are two Brownian motions with correlation parameter $\rho$; $\xi_0 \in L^2([0,T],\mathbb{R}^+)$ for any $T > 0$ is an input curve used to match certain term-structures observed in the market, for instance, the normalization $\sqrt{\mathbb{E}[p(X_t)^2]}$ allows $\xi_0$ to match the market initial forward variance curve since:

$$\mathbb{E}\left[\int_0^t \sigma_s^2 \mathrm{d}s\right] = \int_0^t \xi_0(s)\mathrm{d}s \qquad t \geq 0$$

The fifth grade polynomial $p(x)$ is defined as:

$$p(x) \coloneqq \alpha_0 + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5$$

with non-negative parameters $\alpha_0, \alpha_1, \alpha_3, \alpha_5 \geq 0$ ($\alpha_2 = \alpha_4 = 0$). A polynomial of degree five allows to reproduce the upward slope of the VIX smile. Restricting the coefficients $\alpha$ to be non-negative allows the sign of the ATM skew to be the same as $\rho$, as explained in more detail in [2]. The choice to set $\alpha_2 = \alpha_4 = 0$ allows to reduce the number of parameters to calibrate and doesn't impact in a significant way the results as highlighted in [PAPAER Quintic]. The process $X_t$ that drives the volatility is an OU process where the two parameters $H \in (-\infty, 1/2]$ and $\epsilon > 0$ control the mean-reversion speed through $(1/2 - H)\epsilon^{-1}$ and the vol-of-vol through $\epsilon^{H-1/2}$. For small values of $\epsilon$ we have a fast mean-reversion regime and a large vol-of-vol. Such parametrizations are reminiscent of the fast regimes extensively studied by Fouque et al. [9] which corresponds to the case $H = 0$. They can also be linked to more complex models such as jump models [19,1] for $H \leq -1/2$; and rough volatility models [2,1] for which $H \in (0, 1/2)$ would play the role of the Hurst index. The solution of the OU process is:

$$X_t = \epsilon^{H-1/2} \int_0^t e^{-(1/2-H)\epsilon^{-1}(t-s)} \mathrm{d}s$$

The parameters to calibrate are seven:

$$\Theta \coloneqq \{\rho, H, \epsilon, \alpha_0, \alpha_1, \alpha_3, \alpha_5\}$$

plus the the input curve $\xi_0$, we will use the market initial forward variance curve.

# 2 SPX derivatives

The price SPX derivatives we have to resort to Monte Carlo simulations since there isn't a closed formula. Nevertheless, since $X$ is a OU process it can be simulated exactly instead of approximating it using, for exampler, the Euler scheme which is often inaccurate in a fast mean-reversion regime. To simulate $X$ we first define:

$$\tilde{X}_t \coloneqq X_t \, e^{(1/2-H)\epsilon^{-1}t} = \epsilon^{H-1/2} \int_0^t e^{(1/2-H)\epsilon^{-1}s} \mathrm{d}W_s$$

Thus, $\tilde{X}$ can be simulated recursively by:

$$\tilde{X}_{t_{i+1}} = \tilde{X}_{t_i} + \sqrt{\frac{\epsilon^{2H}}{1-2H}} \left( e^{\frac{1-2H}{\epsilon} t_{i+1}} - e^{\frac{1-2H}{\epsilon} t_i} \right) Y_i$$

where $Y_i$ are i.i.d. standard Gaussian. Naturally, to get back to $X_{t_{i+1}}$ we just have to divide $\tilde{X}_{t_{i+1}}$ by $e^{\frac{1-2H}{\epsilon} t_{i+1}}$. This allows us to to easily vectorize computations. Instead, to simulate the log-process $\log(S)$ we will use the Euler scheme paired with antithetic and control variates, the so called turbocharging method as outlined in [18] that we have used also in the rBergomi model. This means we only need to simulate the part of $\log(S)$ that is $F^{\mathcal{W}}$ measurable, we call this $S^{\mathcal{W}}$ and can be simulated as:

$$\log(S^{\mathcal{W}}_{t_{i+1}}) = \log(S^{\mathcal{W}}_{t_i}) - \frac{1}{2}(\rho\sigma_{t_i})^2(t_{i+1} - t_i) + \rho\sigma_{t_i}\sqrt{t_{i+1} - t_i}Y_i + \rho^2(r-q)(t_{i+1} - t_i)$$

We will use an equi-spaced grid so that calling the time step $h$ the above formula reduces to:

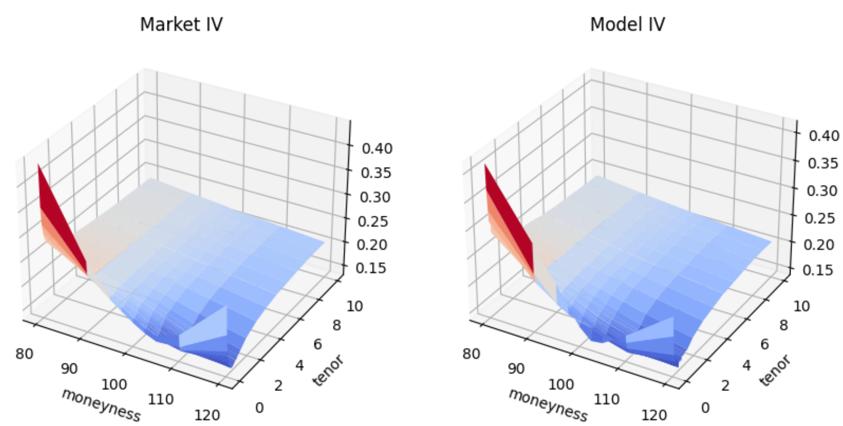
$$\log(S^{\mathcal{W}}_{t_{i+1}}) = \log(S^{\mathcal{W}}_{t_i}) + \left( r - q - \frac{1}{2}\sigma^2_{t_i} \right) \rho^2 h + \rho\sigma_{t_i}\sqrt{h}Y_i$$

## 2.1 Numerical results

We decided to calibrate our model both in a local and global way. In the local case we calibrated a set of parameters for every tenor while in the global approach we calibrated the set of parameters so that it is the best fit considering all the tenors together. We report, for the local case, some of the calibrated parameters in the following table.

| **Tenor** | $\rho$ | H | $\varepsilon$ | $\alpha_0$ | $\alpha_1$ | $\alpha_3$ | $\alpha_5$ |
|---|---|---|---|---|---|---|---|
| 2 weeks | -0.5332 | 0.1034 | 0.1078 | 0.1353 | 0.6193 | 0.0954 | 0.1024 |
| 1 month | -0.6626 | 0.0915 | 0.0171 | 0.9777 | 0.0187 | 0.0343 | 0.1072 |
| 6 months | -0.6968 | -0.0397 | 0.0083 | 1.2204 | 0.0035 | 0.2296 | 0.0462 |
| 1 year | -0.8565 | 0.1903 | 0.0234 | 1.0277 | 0.2463 | 0.0672 | 0.6016 |
| 10 years | -0.6658 | 0.1141 | 0.1173 | 0.9099 | 0.5726 | 0.1604 | 0.1229 |

Comparing our results with the market we obtained a mean relative error of 2.2314%.

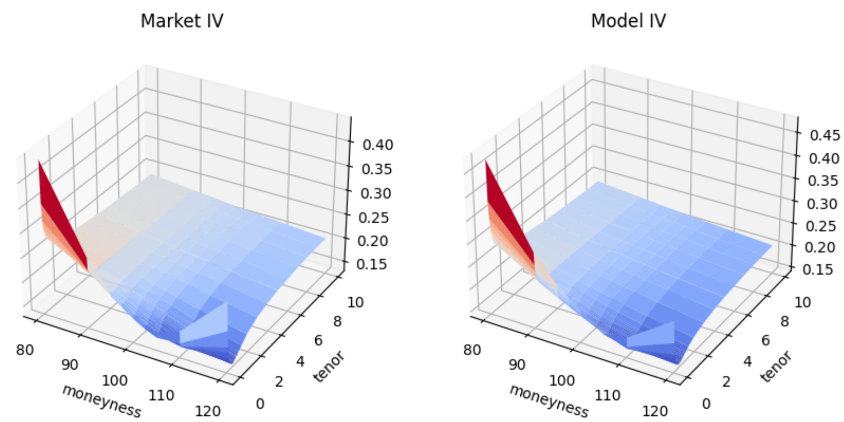



Comparison between market and model IV, local approach

Now we report, for the global approach, the set of the calibrated parameters.

| $\rho$ | H | $\varepsilon$ | $\alpha_0$ | $\alpha_1$ | $\alpha_3$ | $\alpha_5$ |
|---|---|---|---|---|---|---|
| -0.9248 | 0.0731 | 0.0987 | 0.6888 | 0.4998 | 0.0103 | 0.0666 |

In this case we obtained a mean relative error of 2.9693%.




Comparison between market and model IV, global approach

# 3 VIX derivatives

One major advantage of this model is that there is an explicit expression of the VIX. In continuous time the VIX can be expressed as:

$$VIX_T^2 = -\frac{2}{\Delta}\mathbb{E}\Big[\log\left(\frac{S_{T+\Delta}}{S_T}\right)\Big|\mathcal{F}_T\Big] \cdot 100^2 = \frac{100^2}{\Delta}\int_T^{T+\Delta} \xi_T(u)\mathrm{d}u \qquad (4.1)$$

with the usual $\Delta = 30$ days and $\xi_T(u) = \mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ the forward variance curve, that can be explicitly computed. In order to do that we fix $T \leq u$ and rewrite $X$ as:

$$X_u = X_T e^{-(1/2-H)\epsilon^{-1}(u-T)} + \epsilon^{H-1/2}\int_T^u e^{-(1/2-H)\epsilon^{-1}(u-s)}\mathrm{d}W_s =: Z_T^u + G_T^u$$

then, if we define:

$$g(u) := \mathbb{E}[p(X_u)^2]$$

we obtain:

$$\xi_T(u) = \mathbb{E}[\sigma_u^2|\mathcal{F}_T] = \frac{\xi_0(u)}{g(u)}\mathbb{E}\Bigg[\left(\sum_{k=0}^{5}\alpha_k X_u^k\right)^2 \ \Bigg|\ \mathcal{F}_T\Bigg]$$

Defining $\alpha$ the vector $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]$ and indicating with $(\alpha * \alpha)$ the discrete convolution:

$$(\alpha * \alpha)_k = \sum_{j=0}^{k}\alpha_j\alpha_{k-j}$$

we have the following expression:

$$\xi_t(u) = \frac{\xi_0(u)}{g(u)}\mathbb{E}\left[\sum_{k=0}^{10}(\alpha*\alpha)_k X_u^k \;\middle|\; \mathcal{F}_T\right]$$

Using the binomial expansion we can further develop the expression for $\xi_T(u)$ in terms of $Z^u$ and $G^u$ to get:

$$\xi_T(u) = \frac{\xi_0(u)}{g(u)}\sum_{k=0}^{10}\sum_{i=0}^{k}(\alpha*\alpha)_k\binom{k}{i}\left(X_T e^{-(1/2-H)\epsilon^{-1}(u-T)}\right)^i\mathbb{E}\big[(G_T^u)^{k-i}\big] \tag{4.2}$$

where we used both that $Z_T^u$ is $\mathcal{F}_T$-measurable and that $G_T^u$ is independent of $\mathcal{F}_T$. Furthermore, we know that $G_T^u$ is actually a Gaussian random variable:

$$G_T^u \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H}[1-e^{-(1-2H)\epsilon^{-1}(u-T)}]\right)$$

We recall that for a Gaussian variable $Y \sim \mathcal{N}(0,\sigma_Y^2)$ its moments can be computed as:

$$\mathbb{E}[Y^p] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma_Y^p(p-1)!! & \text{if } p \text{ is even} \end{cases} \tag{4.3}$$

where $p!!$ is the double factorial. Therefore we can compute exactly all moments of $G_T^u$. Going back to (4.1) and plugging the expression (4.2) we have that the explicit expression of $VIX_T^2$ is polynomial in $X_T$ and given by:

$$\begin{aligned}
VIX_T^2 &= \frac{100^2}{\Delta}\sum_{k=0}^{10}\sum_{i=0}^{k}(\alpha*\alpha)_k\binom{k}{i}X_T^i\int_T^{T+\Delta}\frac{\xi_0(u)}{g(u)}\mathbb{E}\big[(G_T^u)^{k-i}\big]e^{-(1/2-H)\epsilon^{-1}(u-T)i}\mathrm{d}u \\
&= \frac{100^2}{\Delta}\sum_{i=0}^{10}X_T^i\sum_{k=i}^{10}(\alpha*\alpha)_k\binom{k}{i}\int_T^{T+\Delta}\frac{\xi_0(u)}{g(u)}\mathbb{E}\big[(G_T^u)^{k-i}\big]e^{-(1/2-H)\epsilon^{-1}(u-T)i}\mathrm{d}u \\
&= \frac{100^2}{\Delta}\sum_{i=0}^{10}\beta_i X_T^i \qquad (4.4)
\end{aligned}$$

where we have defined

$$\beta_i := \sum_{k=i}^{10}(\alpha*\alpha)_k\binom{k}{i}\int_T^{T+\Delta}\frac{\xi_0(u)}{g(u)}\mathbb{E}\big[(G_T^u)^{k-i}\big]e^{-(1/2-H)\epsilon^{-1}(u-T)i}\mathrm{d}u$$

We recall that thanks to formula (4.3) we can compute exactly every moment of $G_T^u$. Thanks to the closed from of (4.4) $VIX_T^2$ is a polynomial in $X_T$ that we will denote as $h(X_T)$. Since we have that $X_T$ is Gaussian

$$X_T \sim \mathcal{N}\left(0, \frac{\epsilon^{2H}}{1-2H}\left[1 - e^{-(1-2H)\epsilon^{-1}T}\right]\right)$$

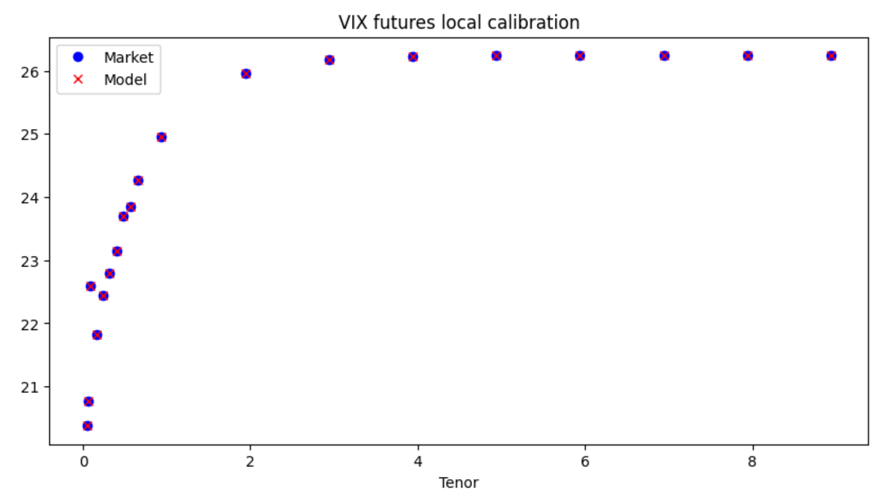
pricing VIX derivatives, with a general payoff function $\Phi$, can be done integrating directly against the standard Gaussian density:

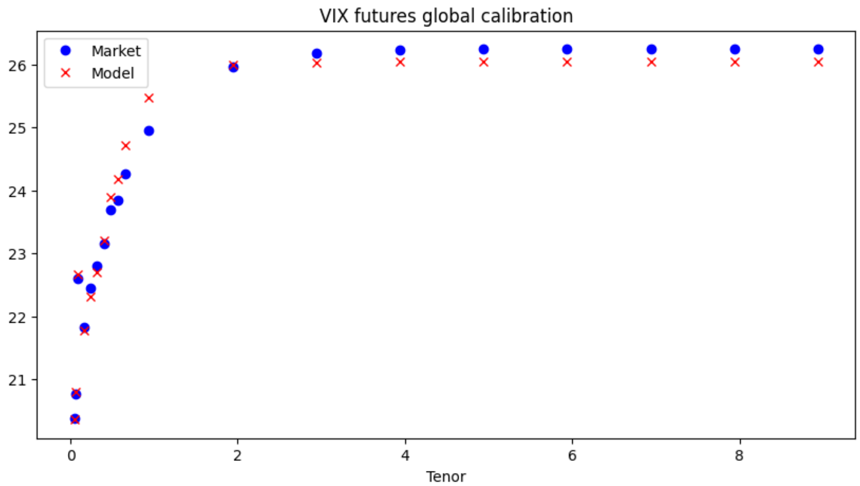
$$\mathbb{E}\big[\Phi(VIX_T)\big] = \mathbb{E}\left[\Phi\big(\sqrt{h(X_T)}\big)\right] = \frac{1}{\sigma_{X_T}\sqrt{2\pi}} \int_{\mathbb{R}} \Phi\big(\sqrt{h(x)}\big) e^{-x^2/\sigma_{X_T}^2} \,\mathrm{d}x$$

This integral can be computed efficiently using a variety of quadrature techniques.

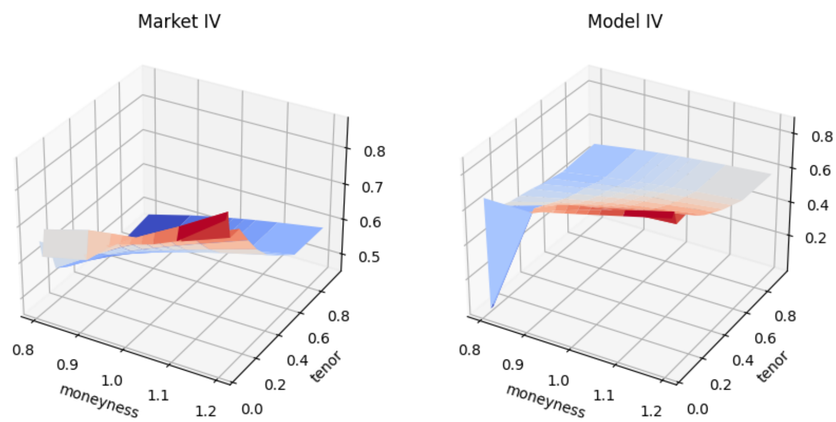
## 3.1 Numerical results

Also in this case we used a local and global approach both for futures and options. First we report the futures results: in the local case the calibration is almost perfect while in the global case we obtained a mean percentage error of 0.6920%.



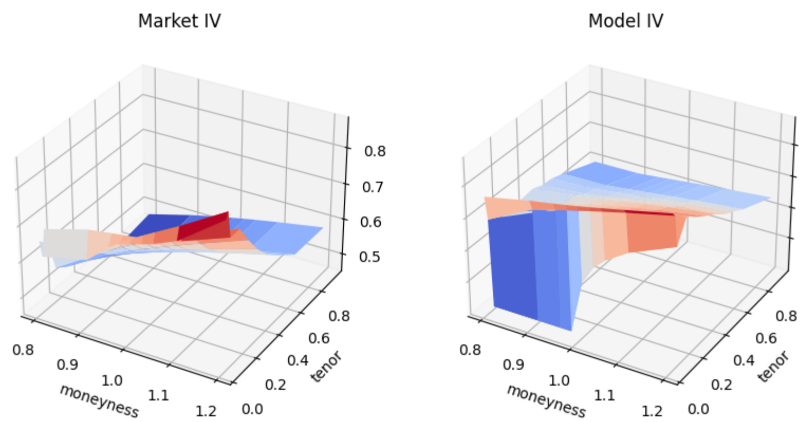



Calibrating the model to the IV of VIX options we discovered, doing the local calibration, that the model isn't suitable for long term-options in fact we obtain a mean percentage error of 31.1995% if we use all the tenors while if we use only the tenors that are less or equal to one year we obtain a mean percentage error of 2.4955%.




Comparison between market and model IV, local approach

For the global approach using only the tenors that are less or equal to one year we obtained a mean percentage error of 8.7636%.




Comparison between market and model IV, global approach

# 4 Joint calibration

We now address the SPX-VIX joint calibration problem, that is the calibration of the model to SPX European options, VIX European options and VIX futures across several maturities. The calibration of VIX futures is necessary as it is used to calculate VIX implied volatility. In fact, recall that the IV is calculated inverting the Black & Scholes formula, that is given a call option price $C_0(K,T)$ with a given strike $K$ and maturity $T$ we find the volatility $\sigma(K,T)$ such that:

$$C_0(K,T) = F_T\Phi(d_1) - K\Phi(d_2)$$

where $F_T$ denotes the futures price: $F_T = \mathbb{E}[S_T] = S_0$ for the SPX and $F_T = \mathbb{E}[VIX_T]$ for the VIX. To jointly calibrated the model we solve the following optimisation problem:

$$\min_{\Theta}\{c_1 f_1(\Theta) + c_2 f_2(\Theta) + c_3 f_3(\Theta)\}$$

where we have defined:

$$f_1(\Theta) := \sqrt{\sum_{i,j}\left(\sigma_{spx}^{\Theta}(K_i,T_j) - \sigma_{spx}^{mkt}(K_i,T_j)\right)^2}$$

$$f_2(\Theta) := \sqrt{\sum_{i,j}\left(\sigma_{vix}^{\Theta}(K_i,T_j) - \sigma_{vix}^{mkt}(K_i,T_j)\right)^2}$$

$$f_3(\Theta) := \sqrt{\sum_{i}\left(F_{vix}^{\Theta}(T_i) - F_{vix}^{mkt}(T_i)\right)^2}$$

So that $f_1$ is the root mean squared error (RMSE) coming from the SPX calibration, $f_2$ is the RMSE coming from the VIX options calibration and $f_3$ is the RMSE coming from the calibration of VIX futures. The constant $c_i$ are positive and reflect what we want to have more importance, in our case we will use $c_1 = c_2 = c_3 = 1$.

### 4.1 Numerical results

For the joint calibration we used only the global approach, for the VIX we used, for both the future and option calibration, only the tenors that are less or equal to one year. Doing so we obtained the following mean percentage error:

- SPX options calibration: 7.7591%;
- VIX futures calibration: 0.4339%;
- VIX options calibration: 18.3786%.

## 5 Bayesian Inverse Problem

## References

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