rBergomi

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The rough Bergomi (rBergomi) model is a Rough Fractional Stochastic Volatility (RFSV) model. The model has only three, time-independent, parameters and is able to replicate accurately the implied volatility surface dynamics. We will see that we need to use simulation methods to generate option prices as there is no closed form solution and the non-Markovian property of the model doesn't allow a PDE approach. From now on let $(\Omega, \mathbb{F} = \{(F_t)_{t\geq 0}\}, \mathbb{P})$ be a complete filtered probability space, let $\mathbb{P}$ be the *physical-measure* and $T < \infty$ be the right limit of our time horizon.

1 The Model

The rBergomi model is know as a market model, that is to say a financial model consistent with market data. The idea of Bergomi, proposed in [1], is to model the dynamics of the forward variance instead of modelling instantaneous volatility. We will denote the forward variance curve observed at time t with maturity T as $\xi_t(T)$. The forward variance curve observed at time t with maturity T is associated with the fair strike of a variance swap, observed in the same instant t and with the same maturity T, that we will denote as $\sigma_t^2(T)$ and can be expressed via:

$$\sigma_t^2(T) = \frac{1}{T-t}\int_t^T \xi_t(u)\mathrm{d}u$$

equivalently we have:

$$\xi_t(T) = \frac{\mathrm{d}}{\mathrm{d}T}\left[(T-t)\sigma_t^2(T)\right]$$

1.1 N-Factor Model

In a general N-dimensional setting dictated by the N-dimensional Brownian motion $(W_t^i)_{i=1}^N$ the forward variance $\xi_t(T)$ dynamics are governed by the following SDE:

$$\mathrm{d}\xi_t(u) = \frac{\omega}{\sqrt{\sum_{i,j=1}^N \omega_i\omega_j\rho_{i,j}}}\xi_t(u)\sum_{i=1}^N \omega_i e^{k_i(u-t)}\mathrm{d}W_t^i \qquad (1.1)$$

where we have that $\mathrm{d}[W_t^i W_t^j]_t = \rho_{i,j}\mathrm{d}t$ and $\omega_i, k_i > 0$. We also have that $\omega > 0$ is the instantaneous volatility of $\xi_t(t)$. In this general setting the solution is given by:

$$\xi_t(T) = \xi_0(T)\exp\left\{\omega\sum_{i=1}^{N}\omega_i e^{-k_i(T-t)}X_t^i - \frac{\omega^2}{2}\sum_{i,j=1}^{N}\omega_i\omega_j e^{-(k_i+k_j)(T-t)}\mathbb{E}\left[X_t^i X_t^j\right]\right\}$$

where the N driven Ornstein-Uhlenbeck (OU) processes $(X_t^i)_{t\geq 0}$ are defined by:

$$\begin{cases} \mathrm{d}X_t^i = -k_i X_t^i \mathrm{d}t + \mathrm{d}W_t^i \\ X_0^i = 0 \end{cases}$$

The instantaneous volatility of $\xi_t(T)$, thanks to (1.1), is:

$$\omega(T-t) = \frac{2\nu}{\sqrt{\sum_{i,j}\omega_i\omega_j\rho_{i,j}}}\sqrt{\sum_{i,j}\omega_i\omega_j\rho_{i,j}e^{-(k_i+k_j)(T-t)}}$$

Where ν is the log-normal volatility of a Variance Swap with vanishing maturity which can be related to the instantaneous volatility of $\xi_t(t)$ by:

$$\omega = 2\nu$$

1.2 One-Factor Model

Now we will restrict our analysis to the mono-dimensional case that is dictated by the Brownian motion $(W_t)_{t\geq 0}$. Now we have, for the forward variance curve, the following dynamics:

$$\mathrm{d}\xi_t(T) = \omega e^{-k(T-t)}\xi_t(T)\mathrm{d}W_t \qquad (1.2)$$

The choice of an exponential decaying volatility function is equivalent to letting an OU process $(X_t)_{t\geq 0}$ dictate the dynamics of the forward variances. The process X_t has to satisfy the following SDE system:

$$\begin{cases} \mathrm{d}X_t = -kX_t\mathrm{d}t + \mathrm{d}W_t \\ X_0 = 0 \end{cases}$$

We can solve the system and find the solution:

$$X_t = \int_0^t e^{-k(t-s)}\mathrm{d}W_s$$

We can also calculate its expected value, its variance and the expected value of the square of the process:

$$\mathbb{E}[X_t] = 0 \qquad \mathbb{V}[X_t] = \frac{1-e^{-2k}}{2k} \qquad \mathbb{E}[X_t^2] = \mathbb{V}[X_t] = \frac{1-e^{-2k}}{2k}$$

Then the solution to the (1.2) SDE is:

$$\xi_t(T) = \xi_0(T) \exp\left\{\omega e^{-k(T-t)} X_t - \frac{\omega^2}{2} e^{-2k(T-t)} \mathbb{E}[X_t^2]\right\}$$

This model is still not flexible enough to capture simultaneously the forward volatilities term structure and the forward skew. Thus, we should need more factors and the next step is the 2-Factors model.

1.3 2-Factors Model

Here we will present the 2-factors model in which we can achieve greater flexibility in the term-structure of volatilities of variances that can be generated. We will also introduce a mixing parameter $\theta \in [0, 1]$. in this context the dynamics become:

$$\begin{cases} \mathrm{d}\xi_t(T) = \omega\alpha_\theta \xi_t(T)\big[(1-\theta)e^{-k_1(T-t)}\mathrm{d}W_t^1 + \theta e^{-k_2(T-t)}\mathrm{d}W_t^2\big] \\ \alpha_\theta = 1/\sqrt{(1-\theta)^2 + \theta^2 + 2\rho_{12}(1-\theta)\theta} \end{cases} \tag{1.3}$$

Where ρ is the correlation between W^1 and W^2 and we have defined the two OU processes X^1 and X^2 given by:

$$\begin{cases} \mathrm{d}X_t^i = -k_i X_t^i \mathrm{d}t + \mathrm{d}W_t^i \\ X_0^i = 0 \end{cases}$$

We will also introduce the auxiliary Gaussian drift-less process:

$$\mathrm{d}x_t^T = \alpha_\theta\big[(1-\theta)e^{-k_1(T-t)}\mathrm{d}W_t^1 + \theta e^{-k_2(T-t)}\mathrm{d}W_t^2\big]$$

whose quadratic variation is given by:

$$\mathrm{d}\langle x^T\rangle_t = \eta^2(T-t)\mathrm{d}t$$

where we have defined:

$$\eta(s) \coloneqq \alpha_\theta \sqrt{(1-\theta)^2 e^{-2k_1 s} + \theta^2 e^{-k_2 s} + 2\rho_{12}\theta(1-\theta)e^{-(k_1+k_2)s}}$$

Thus, substituting in the SDE (1.3) we obtain the simplyfied form:

$$\mathrm{d}\xi_t(T) = \omega\xi_t(T)\mathrm{d}x_t^T$$

Now we can obtain the solution that is given by:

$$\begin{cases} \xi_t(T) = \xi_0(T) e^{\omega x_t^T - \frac{\omega^2}{2} f(t,T)} \\ f(t,T) = \displaystyle\int_{T-t}^{T} \eta^2(u)\mathrm{d}u \end{cases}$$

We can explicit the value of $f(t,T)$ that is:

$$f(t,T) = \alpha_\theta^2 \bigg[\frac{(1-\theta)^2}{2k_1} e^{-2k_1(T-t)}(1-e^{-2k_1 t}) + \frac{\theta^2}{2k_2} e^{-2k_2(T-t)}(1-e^{-2k_2 t}) + 2\theta(1-\theta)\rho_{12} e^{-(k_1+k_2)(T-t)} \frac{1-e^{-(k_1+k_2)t}}{k_1+k_2} \bigg]$$

2 The Realized Variance

We will use the Mandelbrot-Vann Ness representation of the fractional Brownian motion to express the increments of the logarithm of realized variance $v = \sigma^2$ under the *physical measure* $\mathbb{P}$ as:

$$\begin{aligned}
\log(v_u) - \log(v_t) &= 2\nu C_H (W_u^H - W_t^H) \\
&= 2\nu C_H \bigg(\int_{-\infty}^{u} (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} - \int_{-\infty}^{t} (t-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} \bigg) \\
&= 2\nu C_H \bigg(\int_{t}^{u} (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} + \int_{-\infty}^{t} \big[(u-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \big] \mathrm{d}W_s^{\mathbb{P}} \bigg) \\
&=: 2\nu C_H \big[M_t(u) + Z_t(u) \big]
\end{aligned}$$

We note that $\mathbb{E}[M_t(u)|\mathcal{F}_t] = 0$ and that $Z_t(u)$ is $\mathcal{F}_t$-measurable. If we define $\tilde{W}_t^{\mathbb{P}}$ as:

$$\tilde{W}_t^{\mathbb{P}}(u) := \sqrt{2H} \int_t^u |u-s|^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}}$$

it has the same properties of $M_t(u)$ and defining $\eta := \frac{2\nu C_H}{\sqrt{2H}}$ we have that:

$$\log(v_u) - \log(v_t) = \eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)$$

This gives us:

$$v_u = v_t \exp\big\{ \eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u) \big\}$$

Thanks to the properties, gaussianity in this case, of $\tilde{W}_t^{\mathbb{P}}$ we have:

$$\tilde{W}_t^{\mathbb{P}}(u) \sim \mathcal{N}\big(0, (u-t)^{2H}\big)$$

which gives us that $v_u|\mathcal{F}_t$ is log-normal and thus entails that:

$$\mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] = v_t \exp\bigg\{ 2\nu C_H Z_t(u) + \frac{1}{2}\eta^2 (u-t)^{2H} \bigg\}$$

Now we can express the realized variance as:

$$v_u = \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \mathcal{E}(\eta \tilde{W}_t^{\mathbb{P}}(u))$$

where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential.

3 The probability measure change

As observed in [2] the model with two factors is already over-parameterized so we will use the 1-factor model which, even if in theory is not flexible enough, in practice can achieve good quality results. From what we have said up to now we have that, under the *physical probability* $\mathbb{P}$, the model is expressed as:

$$\begin{cases} \mathrm{d}S_u = \mu_u S_u \mathrm{d}u + \sqrt{v_u} S_u \mathrm{d}Z_u^{\mathbb{P}} \\ v_u = v_t \exp\left\{\eta \tilde{W}_t^{\mathbb{P}}(u) + 2\nu C_H Z_t(u)\right\} \end{cases} \tag{3.1}$$

Now, in order to price options, we want to change the *physical* probability measure $\mathbb{P}$ with an equivalent martingale measure $\mathbb{Q}$ in the interval $[t, T]$. To do that we use Girsanov theorem and obtain:

$$\mathrm{d}Z_u^{\mathbb{Q}} = \mathrm{d}Z_u^{\mathbb{P}} + \frac{\mu_u - (r-q)}{\sqrt{v_u}} \mathrm{d}u$$

When we change from $\mathbb{P}$ to $\mathbb{Q}$ we also have to remember that the Brownian motion $W_u^{\mathbb{P}}$, that is used to construct the Volterra-type process $\tilde{W}_u^{\mathbb{P}}$, is correlated with $Z_u^{\mathbb{P}}$ with correlation factor ρ:

$$\mathrm{d}W_u^{\mathbb{P}} = \rho \mathrm{d}Z_u^{\mathbb{P}} + \sqrt{1-\rho^2} \mathrm{d}(Z_u^{\perp})^{\mathbb{P}}$$

where $(Z_u^{\perp})^{\mathbb{P}}$ is independent of $Z_u^{\mathbb{P}}$. A general change of measure for $(Z_u^{\perp})^{\mathbb{P}}$ is of the form:

$$(Z_u^{\perp})^{\mathbb{Q}} = (Z_u^{\perp})^{\mathbb{P}} + \gamma_u \mathrm{d}u$$

where γ_u is a suitable process that can be seen as the market price of volatility risk. Now we can express the change in measure also for $W_u^{\mathbb{Q}}$:

$$\begin{aligned} \mathrm{d}W_u^{\mathbb{Q}} &= \rho \mathrm{d}Z_u^{\mathbb{Q}} + \sqrt{1-\rho^2} \mathrm{d}(Z_u^{\perp})^{\mathbb{Q}} \\ &= \mathrm{d}W_u^{\mathbb{P}} + \left(\frac{\mu_u - (r-q)}{\sqrt{v_u}} \rho + \gamma_u \sqrt{1-\rho^2} \right) \mathrm{d}u \\ &= \mathrm{d}W_u^{\mathbb{P}} + \lambda_u \mathrm{d}u \end{aligned}$$

We may now rewrite, assuming that the filtration generated by $W^{\mathbb{P}}$ is the same as the one generated by $W^{\mathbb{Q}}$, the dynamics of the realized variance:

$$\begin{aligned} v_u &= \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \exp\left\{ \eta\sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \mathrm{d}W_s^{\mathbb{P}} - \frac{\eta^2}{2}(u-t)^{2H} \right\} \\ &= \mathbb{E}^{\mathbb{P}}[v_u|\mathcal{F}_t] \mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \exp\left\{ \eta\sqrt{2H} \int_t^u (u-s)^{H-\frac{1}{2}} \lambda_s \mathrm{d}s \right\} \\ &= \mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] \mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \\ &= \xi_t(u) \mathcal{E}\big(\eta \tilde{W}_t^{\mathbb{Q}}(u)\big) \end{aligned}$$

where we have

$$\tilde{W}_t^{\mathbb{Q}}(u) := \sqrt{2H}\int_t^u (u-s)^{H-\frac{1}{2}}\mathrm{d}W_s^{\mathbb{Q}}$$

Thus, the model (3.1) under $\mathbb{Q}$ is expressed as:

$$\begin{cases} \mathrm{d}S_u = (r-q)S_u\mathrm{d}u + \sqrt{v_u}S_u\mathrm{d}Z_u^{\mathbb{Q}} \\ v_u = \xi_t(u)\mathcal{E}\big(\eta\tilde{W}_t^{\mathbb{Q}}(u)\big) \end{cases} \tag{3.2}$$

This model is a non-Markovian generalization of the Bergomi model. Specifically, this model is non-Markovian in the instantaneous variance v_t:

$$\mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] \neq \mathbb{E}^{\mathbb{Q}}[v_u|v_t]$$

but is Markovian in the infinite-dimensional state vector:

$$\mathbb{E}^{\mathbb{Q}}[v_u|\mathcal{F}_t] = \xi_t(u)$$

4 Pricing

Under the pricing measure $\mathbb{Q}$, given the starting time $t_0 = 0$, the scheme to simulate the model is:

$$\begin{cases} S_t = S_0\exp\left\{(r-q)t - \frac{1}{2}\int_0^t v_u\mathrm{d}u + \int_0^t \sqrt{v_u}\mathrm{d}Z_u^{\mathbb{Q}}\right\} \\ v_t = \xi_0(t)\exp\left\{2\nu C_H\int_0^t (t-u)^{H-\frac{1}{2}}\mathrm{d}W_u^{\mathbb{Q}} - \frac{\nu^2C_H^2}{H}t^{2H}\right\} \\ \mathrm{d}[Z^{\mathbb{Q}}, W^{\mathbb{Q}}]_t = \rho\mathrm{d}t \end{cases}$$

First we need to simulate the Volterra process using the hybrid scheme introduced in section (...). Then we need to extract the Brownian Motion $W^{\mathbb{Q}}$ that drives the Volterra process and correlate it with $Z^{\mathbb{Q}}$ by the parameter ρ. Lastly, we simulate the stock price process S using the forward Euler scheme. To sum up, in order to simulate the stock price process we have to:

1. fix an equispaced grid $\mathcal{G} = \{t_0 = 0, t_1 = \frac{1}{n}, \dots, t_{\lfloor nT\rfloor} = \frac{\lfloor nT\rfloor}{n}\}$;
2. simulate the Volterra process $\mathcal{V}_t = \displaystyle\int_0^t (t-u)^{H-\frac{1}{2}}\mathrm{d}W_u^{\mathbb{Q}}$, $t \in \mathcal{G}$, using the hybrid scheme;
3. compute the variance process v using the previously computed Volterra process:

$$v_t = \xi_0(t)\mathcal{E}\big(2\eta C_H\mathcal{V}_t\big) \qquad t \in \mathcal{G}$$

4. extract the path of the Brownian Motion $W^{\mathbb{Q}}$ that drives the Volterra process:

$$\begin{aligned} W^{\mathbb{Q}}_{t_i} &= W^{\mathbb{Q}}_{t_{i-1}} + n^{H-\frac{1}{2}}\big(\mathcal{V}_{t_i} - \mathcal{V}_{t_{i-1}}\big) \qquad && i = 1, \dots, \kappa \\ W^{\mathbb{Q}}_{t_i} &= W^{\mathbb{Q}}_{t_{i-1}} + W^n_{i-1} && i > \kappa \end{aligned}$$

   where W^n is defined as in (2.2) [fBM hybrid scheme section]

5. correlate the stock price process, driven by $Z^{\mathbb{Q}}$, and the variance process, driven by $W^{\mathbb{Q}}$ through the Volterra process, as:

$$Z^{\mathbb{Q}}_{t_i} - Z^{\mathbb{Q}}_{t_{i-1}} = \rho\big(W^{\mathbb{Q}}_{t_i} - W^{\mathbb{Q}}_{t_{i-1}}\big) + \sqrt{1-\rho^2}\big(W^{\mathbb{Q},\perp}_{t_{i-1}} - W^{\mathbb{Q},\perp}_{t_{i-1}}\big)$$

   where $W^{\mathbb{Q},\perp}$ is a standard Brownian Motion independent of $W^{\mathbb{Q}}$;

6. simulate the stock price process S using the forward Euler scheme:

$$S_{t_i} = S_{t_{i-1}} + (r-q)S_{t_{i-1}}(t_i - t_{i-1}) + \sqrt{v_{t_{i-1}}}S_{t_{i-1}}\big(Z^{\mathbb{Q}}_{t_i} - Z^{\mathbb{Q}}_{t_{i-1}}\big)$$

To price an option at time $t < T$, where T is the maturity, that has payoff $f(S_T)$ we have to calculate the discounted payoff given by:

$$P_t = \mathbb{E}^{\mathbb{Q}}\big[e^{-(r-q)(T-t)}f(S_T)|\mathcal{F}_t\big]$$

To compute this quantity we will use a Monte-Carlo simulation. In practice we have adapted to our use the implementation devised by McCricked and Pakkanen in [3].

5 Calibration

Since we are using the one-factor rBergomi model when we are calibrating it to market data we are finding the values for the parameters used to simulate the price paths: H, η and ρ. These parameters have direct interpretation: H controls the decay of the term structure of volatility skew for very short expirations whereas the product $\rho\eta$ sets the level of the ATM skew for longer expirations. Keeping the product $\rho\eta$ roughly constant but decreasing ρ (so as to make it more negative) has the effect of pushing the minimum of each smile towards higher strikes. As the initial forward variance curve we will use the parametrization found in **Section [QUELLA SULLA VARIANCE]**.

5.1 Objective Function

To calibrate the model we want to minimize the difference between the market implied volatility and the rBergomi implied volatility. To calculate the rBergomi IV we have to set the parameters, simulate the price paths, compute the price of the option (given a certain strike k and time to maturity τ) and then invert the B&S formula to find the IV given by our simulation, in formula:

$$\sigma_{rB}(k, \tau, H, \eta, \rho) = P_{BS}^{-1}(k, \tau, H, \eta, \rho)$$

We indicate the set of parameters that we have to calibrate as $\Theta := \{H, \eta, \rho\}$ and the set of strikes for a given maturity as K. The problem of minimization is thus:

$$\underset{\Theta}{\operatorname{argmin}} f(K, \tau) := \underset{\Theta}{\operatorname{argmin}} \|\sigma_{rB}(k, \tau) - \sigma_{mkt}(k, \tau)\|^2$$

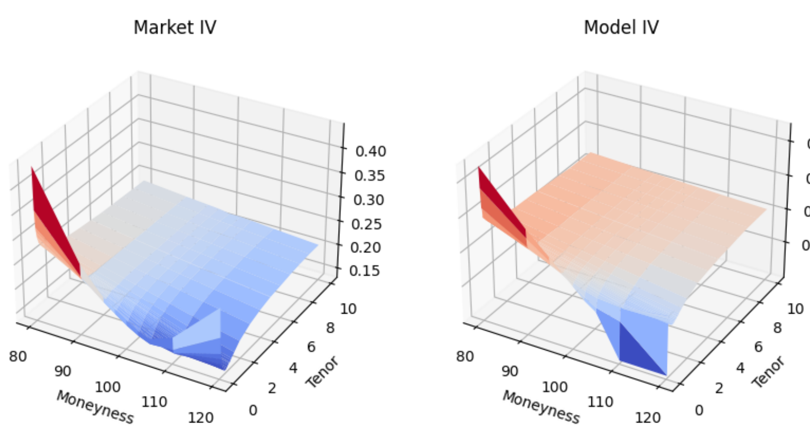
where σ_{mkt} represents the market IV. We calibrate the model for every tenor using the sequential least squares programming algorithm proposed in [4]. We will also calibrate the model in a global way finding just a set of parameters for all the tenors.

5.2 Numerical results

We report some of the calibrated parameters in the following table.

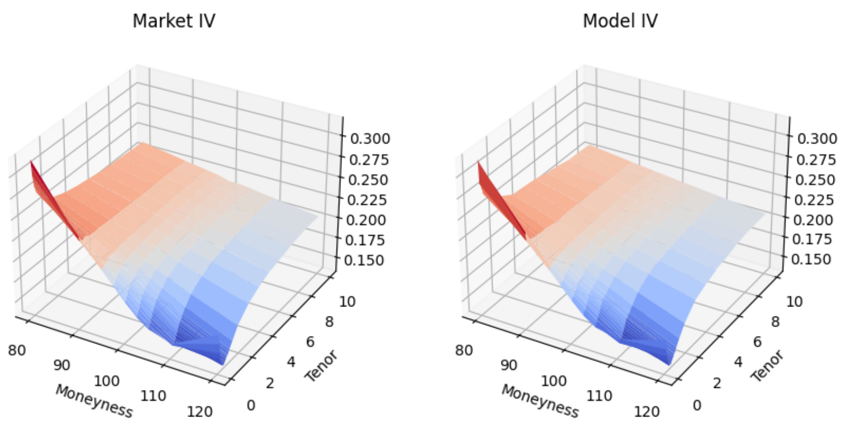
Tenor	**H**	η	ρ
2 weeks	0.0279	1.9077	-0.8583
1 month	0.0328	2.0162	-0.8441
6 months	0.0811	1.9189	-0.8923
1 year	0.0958	1.7628	-0.9368
10 years	0.0466	1.4197	-0.9575

Comparing our results with the market we obtained a mean relative percentage error of 2.2799%.



Comparison between market and model IV, local approach

As we can see the model fails for the first two tenors in the high moneyness range. In fact, if we exclude the first two tenors we obtain a mean relative percentage error of 0.7832%.

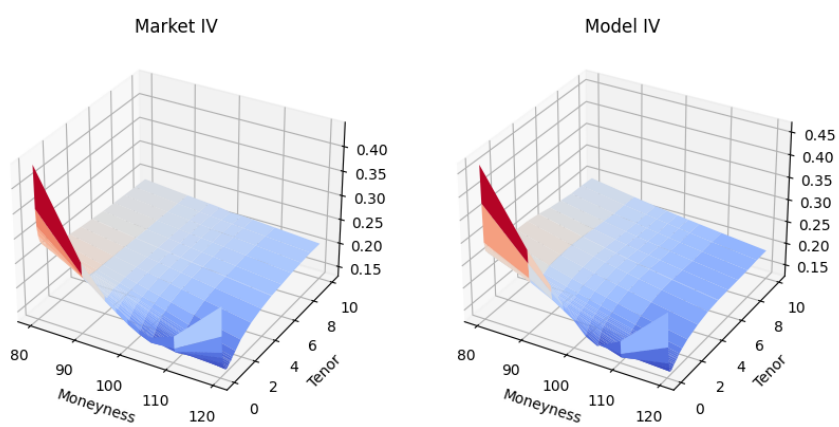


Comparison between market and model IV, local approach restricted

In the global approach the calibrated parameters are:

H	η	ρ
0.0856	1.8906	-0.8978

and the mean relative percentage error that we obtained is 3.1008%.



Comparison between market and model IV, global approach

6 Volatility Skew

The at-the-money (ATM) volatility skew ψ is defined as:

$$\psi(\tau) := \left| \frac{\partial}{\partial k} \hat{\sigma}(k, \tau) \right|_{k=0}$$

where k is the log-moneyness, τ is the time to maturity and $\hat{\sigma}(k, \tau)$ is the implied volatility. The term structure of the market data at-the-money-forward (ATMF) volatility skew Ψ is defined as:

$$\Psi(\tau) := \left| \frac{\partial}{\partial K} \sigma_{mkt}(K, \tau) \right|_{K=F}$$

where K is the strike, F is the forward price and σ_{mkt} is the market implied volatility. It is easy to express the log-moneyness strike k in terms of actual strike K as:

$$k = \log\left(\frac{K}{F}\right)$$

Equating the two implied volatilities $\hat{\sigma}(k, \tau) = \sigma_{mkt}(K, \tau)$ we obtain:

$$\frac{\partial}{\partial K} \sigma_{mkt}(K, \tau) = \frac{\partial}{\partial k} \hat{\sigma}(k, \tau) \frac{\partial k}{\partial K} = \frac{1}{K} \frac{\partial}{\partial k} \hat{\sigma}(k, \tau)$$

Therefore we have mapped data expressed in terms of strike K to data expressed in terms of log-moneyness k. Thus, the ATMF volatility skew can be expressed as:

$$\psi(\tau) = F \left| \frac{\partial}{\partial K} \sigma_{mkt}(K, \tau) \right|_{K=F}$$

6.1 Numerical results

In order to compute the market ATMF volatility skew we will fit a stochastic volatility inspired (SVI) model to the market implied volatility data. The SVI model, presented in [5] and further expanded in [6], is calibrated to the market implied volatility surface using a set of parameters $\lambda = \{a, b, \rho, m, \sigma\}$ such that the total implied variance is expressed, for $k \in \mathbb{R}$, as:

$$\sigma^2_{imp}(k; \lambda) = a + b\left[\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}\right]$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$ and $\sigma > 0$. In addition, if $a + b\sigma\sqrt{1 - \rho^2} \geq 0$ we have that the condition $\sigma^2_{imp}(k; \lambda) > 0$ is obtained. Every parameter has distinct effect:

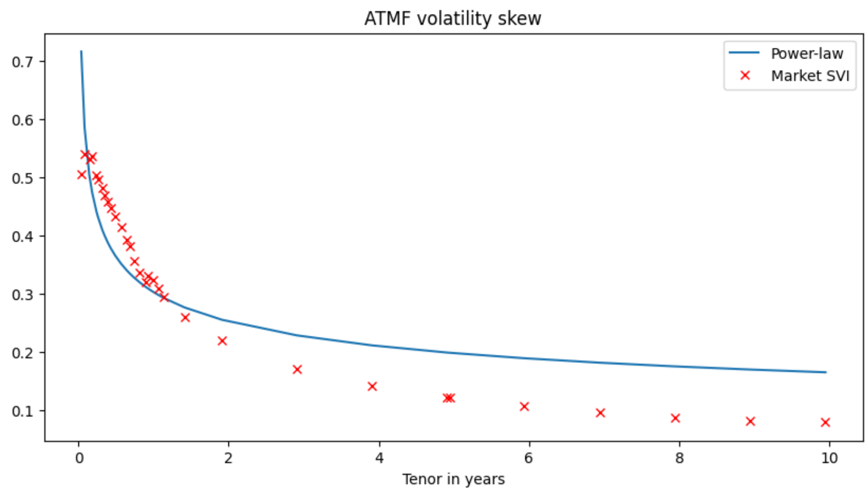
- a controls the general level of variance being a vertical translation of the smile;
- b controls the slopes of both puts and calls, increasing b we will tighten the smile;
- increasing ρ we will have a counter-clockwise rotation of the smile;

- m controls the x-axis position of the smile, increasing m will translate the smile to the right;

- σ controls the ATM curvature of the smile, increasing ρ will reduce the curvature.

It has been shown, for example in [7], that the term structure of the ATMF volatility skew can be approximated by a power-law function of the time to maturity. In this context the function $\tau \to A\tau^{-\alpha}$ is fitted to the market data obtaining the following values:

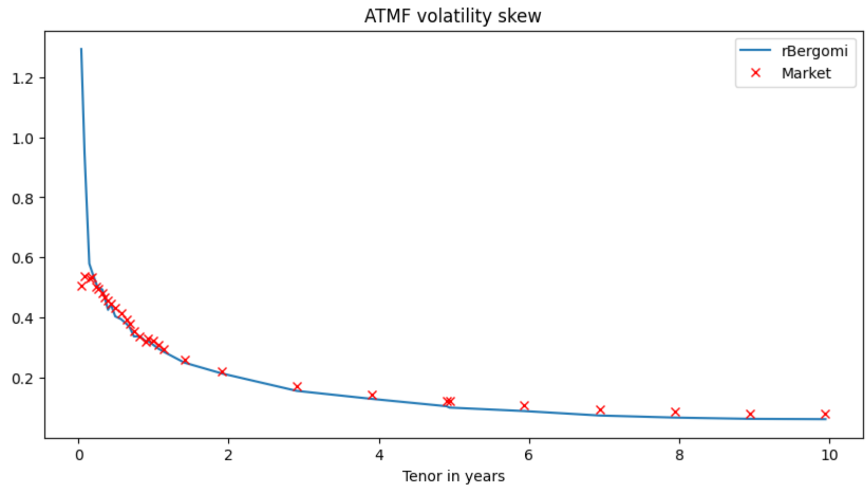
$$A = 0.30287671 \quad \alpha = 0.26347397$$



To compute the ATMF volatility skew given by the rBergomi model we will first calculate the implied volatility given by the model for log-moneyness in the neighbourhood of $k = 0$ and then use the central finite difference approximation. Letting $h > 0$ be the difference step and τ be one of the tenors, the approximation of ψ is given by:

$$\psi(\tau) \approx \left| \frac{\hat{\sigma}_{rB}(h, \tau) - \hat{\sigma}_{rB}(-h, \tau)}{2h} \right|$$

By construction, since the rBergomi model smile should fit the market data also the skew generated by it should fit the market skew obtained by the SVI model. In practice, using $h = 10^{-3}$ for the approximation scheme, we obtained the following fit.



We note that apart from the first two tenors the fit is quite good.

7 Forward-Start Options

Many options can be used to study the sensitivity to forward-smile risk, which is defined as the risk coming from the market future implied volatility and its uncertainty. These options, of which the forward-start options are an example, are priced given the distribution of forward returns in the model, as described in [8]. We remember that the payoff of a forward-start option involves the price of the underlying in two different dates T_1 and T_2 with $T_1 < T_2$. The payoff Φ can be expressed as:

$$\Phi = \left(S_{T_2} - kS_{T_1}\right)^+$$

where $k > 0$ is the moneyness of the option. The forward smile represents the expected future implied volatility for moneyness k: all possible realizations of future smiles are averaged to give $\hat{\sigma}_k^{T_1,T_2}$, that is the implied Black and Scholes volatility for the forward-start option with moneyness k. The instantaneous volatility is time-dependent, due to the forward-start option's nature, whose payoff depends on S_{T_1} and S_{T_2}. The Black and Scholes implied variance for maturity T is given by:

$$\hat{\sigma}_T^2 := \int_t^T \sigma^2(u)\mathrm{d}u$$

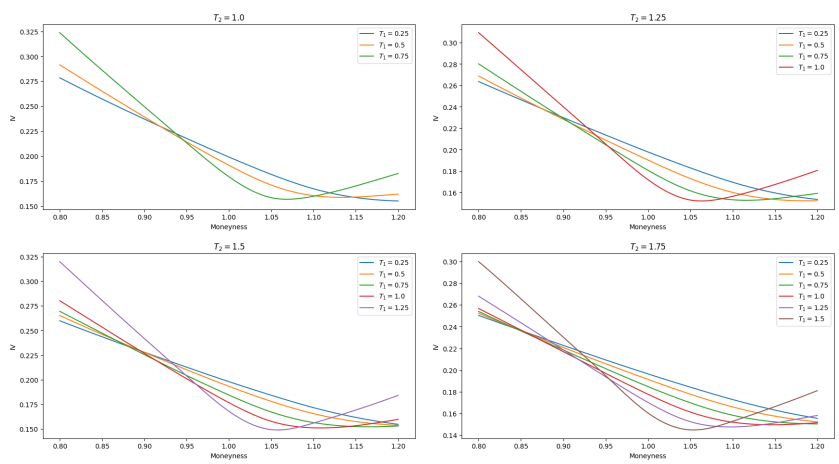
The price of a forward-start option is independent of the underlying price S, but depends on the forward volatility $\hat{\sigma}_{T_1,T_2}$, or the integrated variance over $[T_1, T_2]$, in such a way that:

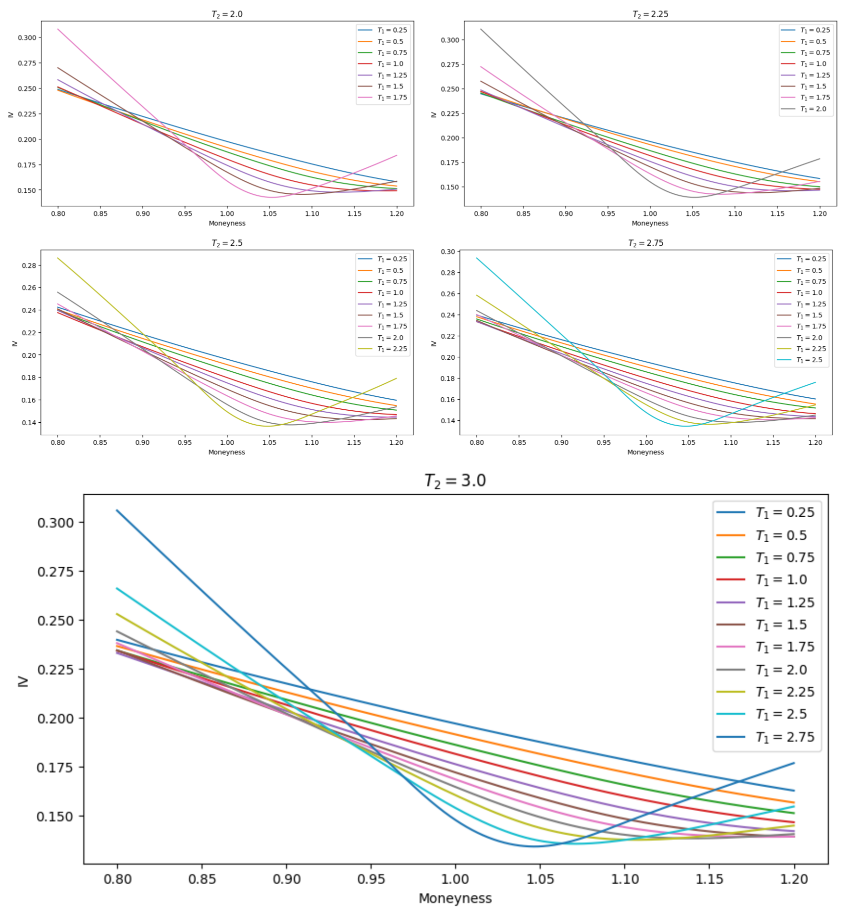
$$
\begin{aligned}
\hat{\sigma}^2_{T_1,T_2} &:= \int_{T_1}^{T_2} \sigma^2(u)\mathrm{d}u \\
&= \frac{(T_2 - t)\hat{\sigma}^2_{T_2} - (T_1 - t)\hat{\sigma}^2_{T_1}}{T_2 - T_1}
\end{aligned}
$$

7.1 Implied Volatility

As for a vanilla option to price a forward-start option in the rBergomi model we have to resort to a Monte-Carlo method. The generation of a price path is executed as explained in **Section 4** and then we extract the values at time T_1 and T_2. The algorithm to compute the implied volatility of a forward-start option under rBergomi is the following:

1. set a range $\{1, 1.25, \ldots, 3\}$ of step 0.25 for the maturity T_2;
2. set a range $\{0.25, \ldots, T_2 - 0.25\}$ of step 0.25 for the starting date T_1;
3. simulate the price path under rBergomi;
4. set a range $\{0.8, 0.85, \ldots, 1.2\}$ of step 0.05 for the moneyness k;
5. compute the payoff of the forward-start option as $(S_{T_2} - kS_{T_1})^+$;
6. compute the price of the forward-start option as the discounted payoff over a Monte-Carlo simulation;
7. compute the corresponding Black and Scholes implied volatility using a root finding method, for example bisection.





8 Parameters stability

To check the stability of the model parameters we decided to test the model using data from other days than the principal one. The dates that we used are: 24/01/2023, 25/01/2023, 26/01/2023, 27/01/2023, 30/01/2023, 06/02/2023, 13/02/2023, 21/02/2023. In doing so we used only options available in all these days and they are 30 in total ranging from few months tenors to almost ten years. We did two tests:

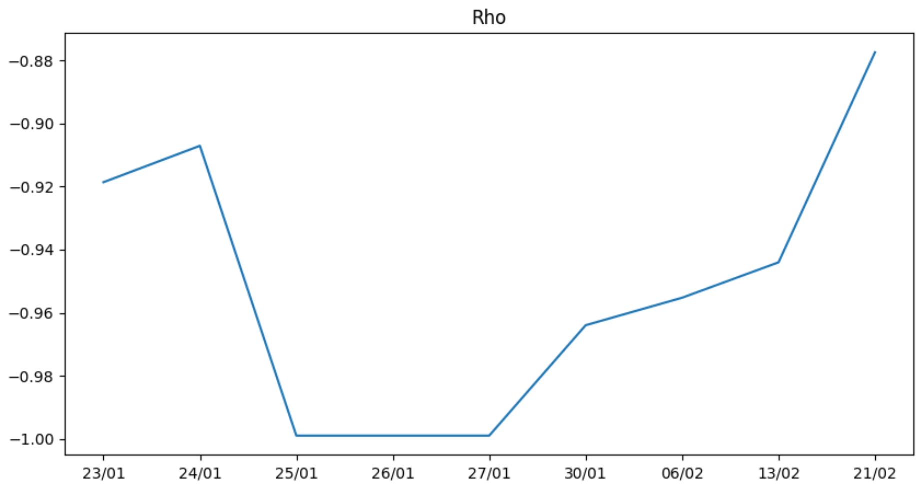
1. using the already calibrated parameters for the principal date we used the model to check the mean percentage error in all the other days comparing it with the error obtained calibrating the model for each day;

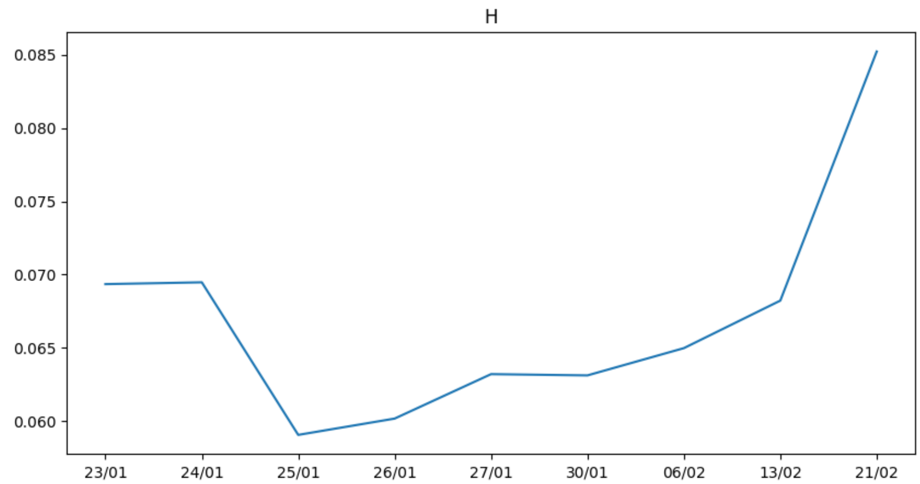
2. we calibrated the model in each day and then analyzed the standard deviation and variance of each parameter.

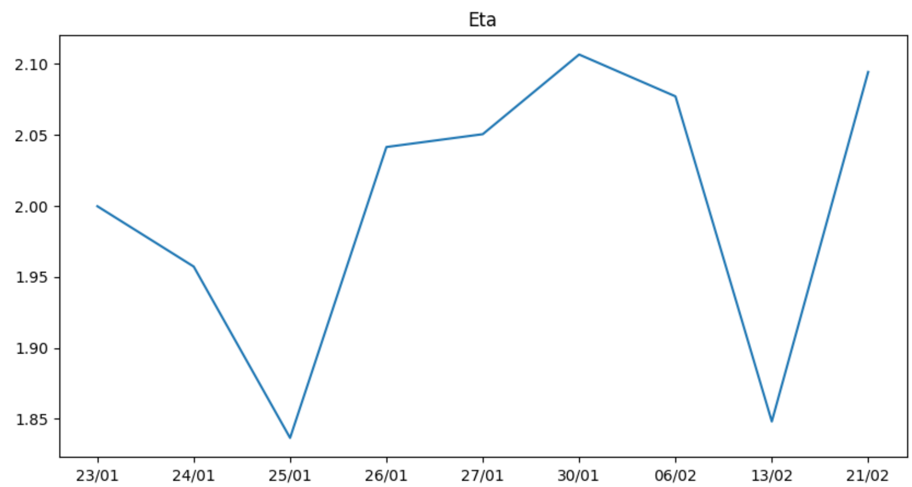
The next table contains the parameters and the error that we obtained for each day.

Day	ρ	H	η	**Error**
23/01	-0.9186	0.0694	1.9998	2.1776%
24/01	-0.9071	0.0695	1.9573	3.5448%
25/01	-0.9999	0.0591	1.8366	5.9528%
26/01	-0.9999	0.0602	2.0415	3.3960%
27/01	-0.9999	0.0632	2.0506	3.0902%
30/01	-0.9640	0.0631	2.1067	4.6805%
06/02	-0.9553	0.0650	2.0772	2.1697%
13/02	-0.9440	0.0682	1.8482	7.8451%
21/02	-0.8774	0.0852	2.0944	2.8109%

A graphical representation of the changes is presented in the next figures.



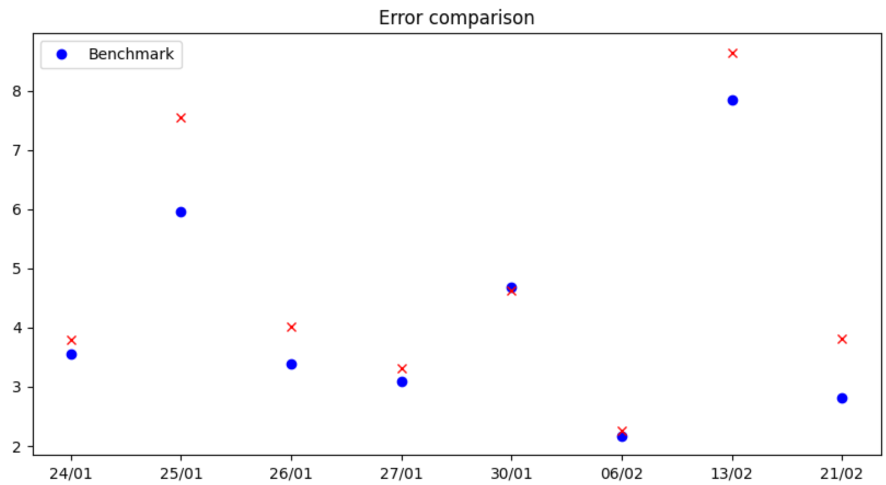




The next table contains the comparison between the benchmark error obtained before and the one obtained using as parameters the one calibrated for the first day (23/01).

Day	**Benchmark**	**Error**
24/01	3.5448%	3.7907%
25/01	5.9528%	7.5450%
26/01	3.3960%	4.0075%
27/01	3.0902%	3.3171%
30/01	4.6805%	4.6345%
06/02	2.1697%	2.2557%
13/02	7.8451%	8.6483%
21/02	2.8109%	3.8043%

The next figure is a graphical comparison.



Analyzing the standard deviation and variance of each parameter we obtained the following results.

Parameter	Standard Deviation	Variance
ρ	0.0415	0.0017
H	0.0074	0.0001
η	0.0955	0.0091

9 Bayesian Inverse Problem

To calibrated the model we used the Approximate Bayesian Computation (ABC) that is a sequential Monte-Carlo method. ABC methods (also called likelihood free inference methods), are a group of techniques developed for inferring posterior distributions in cases where the likelihood function is intractable or costly to evaluate. ABC comes useful when the model used contains unobservable random quantities, which make the likelihood function hard to specify, but data can be simulated from the model. These methods follow a general form:

1. Sample a parameter θ from a prior distribution $\pi(\theta)$.
2. Simulate a data set y using a function that takes θ and returns a data set of the same dimensions as the observed data set y_0.
3. Compare the simulated dataset y with the experimental data set y_0 using a distance function d and a tolerance threshold ε.

In some cases a distance function is computed between two summary statistics $d(S(y^*), S(y_0))$, avoiding the issue of computing distances for entire datasets. As a result we obtain a sample of parameters from a distribution $\pi(\theta|d(y, y_0))$.

If ε is sufficiently small this distribution will be a good approximation of the posterior distribution $\pi(\theta|y_0)$. Sequential monte carlo ABC is a method that iteratively morphs the prior into a posterior by propagating the sampled parameters through a series of proposal distributions, weighting the accepted parameters $\theta^{(i)}$ like:

$$w^{(i)} \propto \frac{\pi(\theta^{(i)})}{\phi(\theta^{(i)})}$$

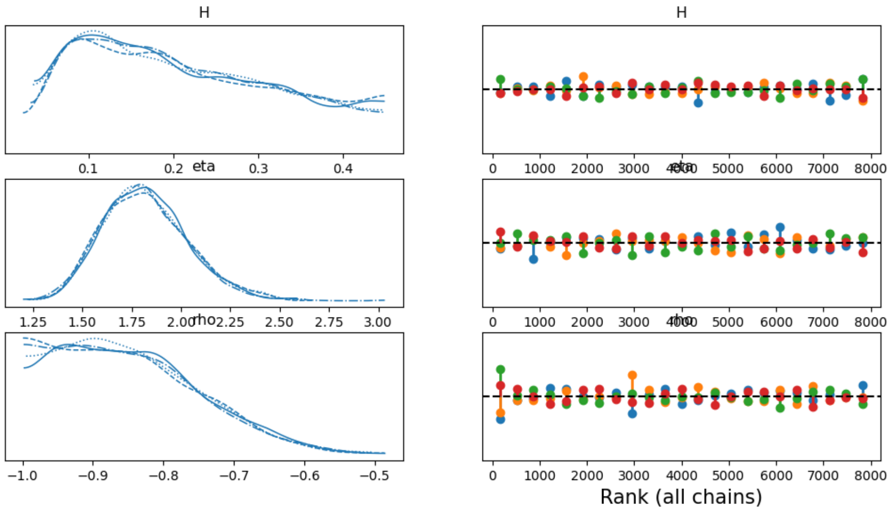
It combines the advantages of traditional SMC, that is the ability to sample from distributions with multiple peaks, but without the need for evaluating the likelihood function.

9.1 Numerical results

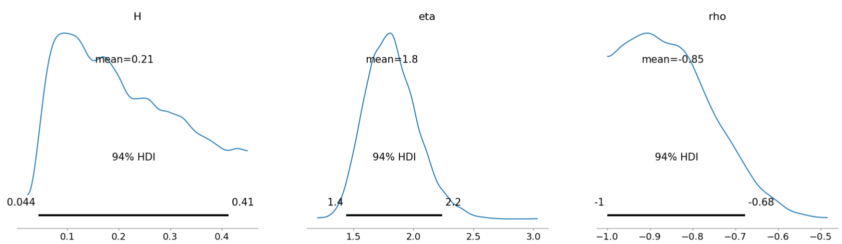
We decided to calibrate the model using only the tenors greater than 1 month and less than 6 months (9 in total). The set of parameters to calibrate is ρ, H and η. We used priors as uninformative as possible:

- $\rho \sim \mathcal{U}(-1, 0)$
- $H \sim \mathcal{U}(0, 0.45)$
- $\eta \sim \mathcal{U}(1, 5)$

We calibrated using the prices instead of the IV, we know that this is not the best, but for time and computational reasons we had to resort to this. We used 4 chains with 2000 samples each, also just for computational and time reasons. Then to test the quality of the calibration we used the maximum of the resulting distribution and plugged it in the model. This first figure represents on the left the distribution obtained from all the 4 chains superimposed, while in the right part we have a graphical representation of the rank of each chain: the vertical lines indicate deviation from the ideal expected value, which is represented with a black dashed line, if it is above we have more samples than expected and vice versa if it is below.



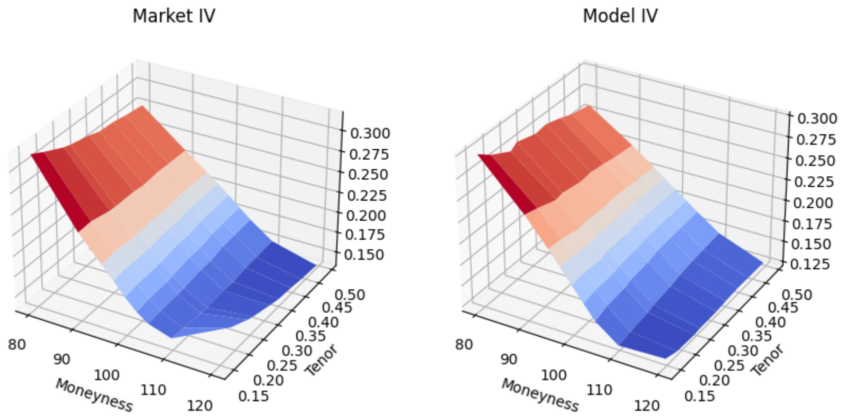
We note that the chains are coherent and the rank plots are good. The next figure represents the posterior density that we have obtained.



To compare the Bayesian calibration error we have to fix the parameters and then compute the error. We decided to use as parameters the maximum a posteriori of the densities that we have obtained, in this case they are:

ρ	H	η
-0.8967	0.1024	1.8348

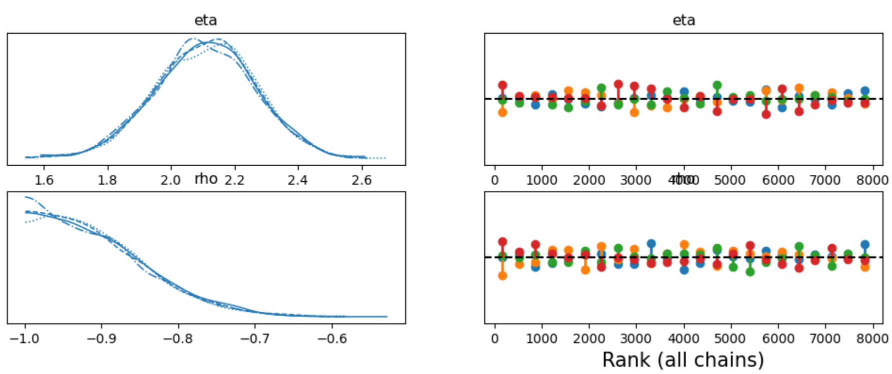
Using these parameters we obtained a mean percentage error of 4.0221%. The next figure contains the comparison between the market and model IV surfaces.



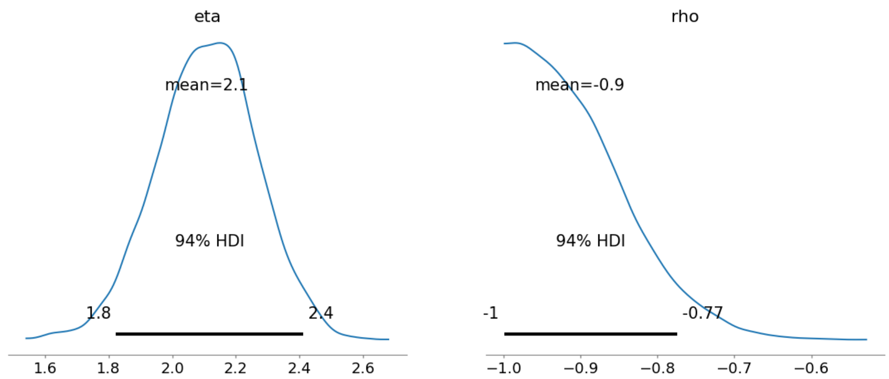
We also decided to do another test. We calibrated the model, with the global approach, using only the tenors that we will use in the Bayesian calibration and we obtained the following set:

ρ	H	η
-0.8831	0.0566	2.1398

obtaining a mean percentage error of 2.1518%. Then we fixed the H parameter and we calibrated the ρ and η parameters in the Bayesian framework.



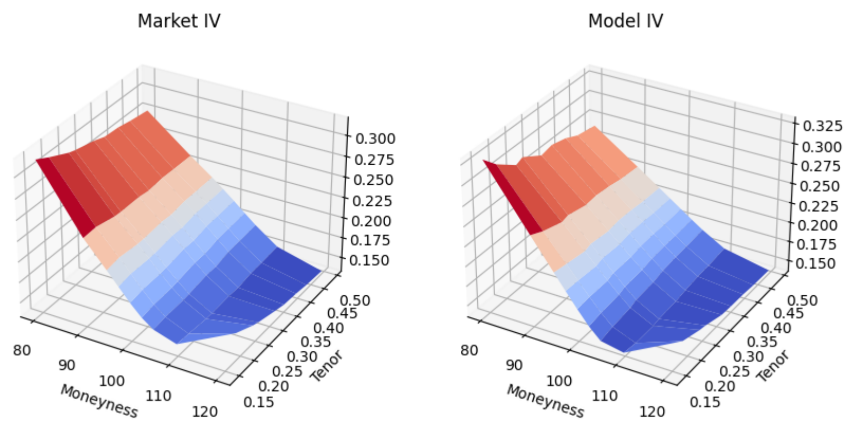
We note that the chains are coherent and the rank plots are good. The next figure represents the posterior density that we have obtained.



The maximum a posteriori of the densities that we have obtained are:

ρ	η
-0.9870	2.1484

Using these parameters and the fixed H we obtained a mean percentage error of 2.0087%. The next figure is the comparison between the market and model IV surfaces.



10 Bayesian Uncertainty Valuation

The last thing that we checked is the uncertainty that we have from our Bayesian calibration. In order to do that we extracted all the samples from our chains, that are the different sets of parameters that the Bayesian framework deemed to be the best and working. Then we calibrated the model using all these sets and computed the mean percentage error of all of these different calibrations, in our case since we have 4 chains with 2000 samples each we have a total of

8000 errors. To put all these results together we decided to weight each error with the inverse of the distance of the parameter set used with respect to the MAP set. We also analyzed the standard deviation and variance of this set of errors. We did these procedures for both the approaches. We first start with the complete set of parameters.

Min	**Max**	**Mean**	**Std**	**Var**
1.5554%	15.4407%	5.2645%	2.1403	4.5807

In the second case, where we have fixed the Hurst parameter H, we obtained the following table.

Min	**Max**	**Mean**	**Std**	**Var**
1.5438%	9.1181%	2.6356%	1.0563	1.1157

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