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Efficient Estimation for Random Dot Product Graphs via a One-Step Procedure

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ABSTRACT

We propose a one-step procedure to estimate the latent positions in random dot product graphs efficiently. Unlike the classical spectral-based methods, the proposed one-step procedure takes advantage of both the low-rank structure of the expected adjacency matrix and the Bernoulli likelihood information of the sampling model simultaneously. We show that for each vertex, the corresponding row of the one-step estimator (OSE) converges to a multivariate normal distribution after proper scaling and centering up to an orthogonal transformation, with an efficient covariance matrix. The initial estimator for the one-step procedure needs to satisfy the so-called approximate linearization property. The OSE improves the commonly adopted spectral embedding methods in the following sense: Globally for all vertices, it yields an asymptotic sum of squares error no greater than those of the spectral methods, and locally for each vertex, the asymptotic covariance matrix of the corresponding row of the OSE dominates those of the spectral embeddings in spectra. The usefulness of the proposed one-step procedure is demonstrated via numerical examples and the analysis of a real-world Wikipedia graph dataset.

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Approximate linearization property; Asymptotic normality; Bernoulli likelihood information; Latent position estimation; Normalized Laplacian

1. Introduction

Statistical inference on graph data, an important topic in statistics and machine learning, has been pervasive in a variety of application domains, such as social networks (Young and Scheinerman 2007; Girvan and Newman 2002; Wasserman and Faust 1994), brain connectomics (Priebe et al. 2017; Tang et al. 2019), political science (Ward, Stovel, and Sacks 2011), computer networks (Neil et al. 2013; Rubin-Delanchy, Adams, and Heard 2016), etc. Due to the high-dimensional nature and the complex structure of graph data, classical statistical methods typically begin with finding a low-dimensional representation for the vertices in a graph using a collection of points in some Euclidean space, referred to as *latent positions* of the vertices. These latent positions are further used as features for subsequent inference tasks, such as vertex clustering (Sussman et al. 2012) and classification (Sussman, Tang, and Priebe 2014; Tang, Sussman, and Priebe 2013), regression (Mele et al. 2019), and nonparametric graph testing (Tang et al. 2017b).

Hoff, Raftery, and Handcock (2002) proposed the latent position graphs to formalize the idea of latent positions: Each vertex i in the graph is assigned a Euclidean vector $\mathbf{x}_i \in \mathbb{R}^d$, and the occurrence of an edge linking vertices i and j is a Bernoulli random variable with success probability $\kappa(\mathbf{x}_i, \mathbf{x}_j)$, where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$ is a symmetric link function. In this work, we study the random dot product graphs (Young and Scheinerman 2007), a particular class of latent position graphs taking the link function to be the dot product of latent positions: $\kappa(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j$. Random dot product graphs are of special interest due to the following two reasons: First, the adjacency matrix of a random dot product graph can be viewed as the sum of a low-rank matrix and a mean-zero noise matrix, which facilitates the use of low-rank matrix factorization techniques for statistical inference; Second, random dot product graphs are sufficiently flexible as they can approximate general latent position graphs with symmetric positive definite link functions when the dimension d of the latent positions grows with the number of vertices at a certain rate (Tang, Sussman, and Priebe 2013). The readers are referred to the survey article Athreya et al. (2018a) for a thorough review on the recent development of random dot product graphs.

Low-rank matrix factorization methods, or more precisely, spectral-based methods, have been broadly used for estimating latent positions for random dot product graphs due to the low expected rank of the observed adjacency matrix. Sussman, Tang, and Priebe (2014) proposed to estimate latent positions using the eigenvectors associated with the top d eigenvalues of the adjacency matrix. The resulting estimator is referred to as the *adjacency spectral embedding* (ASE). Asymptotic characterization of the global behavior of the ASE for all vertices have been established, including the consistency (Sussman, Tang, and Priebe 2014) and the limit of the sum of squares error (SSE) (Tang et al. 2017a) as the number of vertices goes to infinity. Locally, for each vertex, Athreya et al. (2016) proved that the distribution of the corresponding row of the ASE converges to a mean-zero multivariate normal mixture distribution after proper scaling and centering, up to an orthogonal transformation, as the number of vertices goes to infinity. Another popular spectral-based method is the *Laplacian spectral embedding* (LSE), which computes the eigenvectors of the normalized Laplacian matrix of the adjacency matrix associated with the top d eigenvalues (Rohe, Chatterjee, and Yu 2011). The asymptotic

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theory of the LSE has also been established (Sarkar and Bickel 2015; Tang and Priebe 2018). Notably, Tang and Priebe (2018) showed that each row of the LSE converges to a mean-zero multivariate normal mixture distribution after proper scaling and centering, up to an orthogonal transformation. These theoretical studies of the spectral-based methods lay a solid foundation for the development of subsequent inference tasks, such as vertex clustering (Sussman et al. 2012; Rohe, Chatterjee, and Yu 2011; Sarkar and Bickel 2015), vertex classification (Sussman, Tang, and Priebe 2014; Tang, Sussman, and Priebe 2013), testing between graphs (Tang et al. 2017a, 2017b), and parameter estimation in latent structure random graphs (Athreya et al. 2018b).

Despite the great success of the spectral-based methods for random dot product graphs, it was pointed out in Xie and Xu (2019) that they are formulated in a low-rank matrix factorization fashion, whereas the Bernoulli likelihood information contained in the sampling model has been neglected. A fundamental question remains open: whether or not the adjacency/Laplacian spectral embedding is optimal for estimating latent positions (or the transformation of them) due to the negligence of the likelihood information? In this article, we prove the suboptimality of the ASE by showing that the asymptotic covariance matrix of each row of the ASE is suboptimal. We propose a novel one-step procedure for estimating latent positions, and show that for each vertex, the corresponding row of the proposed one-step estimator (OSE) converges to a multivariate normal distribution after $\sqrt{n}$-scaling and centering at the underlying true latent position, up to an orthogonal transformation. More importantly, the corresponding asymptotic covariance matrix is the same as the maximum likelihood estimator (MLE) as if the rest of the latent positions are known, provided that the procedure is initialized at an estimator satisfying the approximate linearization property, which will be defined later. This phenomenon of the OSE is referred to as the local efficiency, the formal definition of which is provided in Section 3. In particular, we show that the efficient covariance matrix is no greater than the asymptotic covariance matrix of the corresponding row of the ASE in spectra. We also provide an example where the difference between the efficient covariance matrix and the asymptotic covariance matrix of the ASE has at least one negative eigenvalue. Besides, the local efficiency for each vertex, the proposed OSE for latent positions has a smaller SSE than that of the ASE globally for all vertices as well.

The general one-step procedure, which finds a new estimator via a single iteration of the Newton-Raphson update given a $\sqrt{n}$-consistent initial estimator, has been applied to M-estimation theory in classical parametric models to produce an efficient estimator (Van der Vaart 2000). Even when the MLE does not exist (e.g., Gaussian mixture models), the OSE could still be efficient. This motivates us to extend the one-step procedure from classical parametric models to efficient estimation in high-dimensional random graphs, because neither the existence nor the uniqueness of the MLE for random dot product graphs has been established. Unlike the ASE, the proposed one-step procedure takes both the low-rank structure of the mean matrix and the likelihood information of the sampling model into account simultaneously. This work represents, to the best of our knowledge, the first effort in the literature addressing the efficient estimation problem for random dot product graphs.

Moreover, we prove the asymptotic suboptimality of the widely adopted LSE by applying the one-step procedure to construct an estimator for the population version of the LSE and showing that it dominates the LSE in the following sense: Locally for each vertex, the corresponding row of the new estimator converges to a mean-zero multivariate normal distribution after proper scaling and centering, up to an orthogonal transformation, and the asymptotic covariance matrix is no greater than that of the corresponding row of the LSE in spectra; Globally for all vertices, it yields a SSE no greater than that of the LSE.

Recently, there has been substantial progress on generalized random dot product graphs (Rubin-Delanchy et al. 2017), which fall into the category of general latent position graphs as well but allow for a more general link function than random dot product graphs. The link function of a generalized random dot product graph is of the form $\kappa(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^{\mathrm{T}} \mathbf{I}_{p,q} \mathbf{x}_j$, where $\mathbf{I}_{p,q}$ is a diagonal matrix with p ones and q minus ones on its diagonals and p, q are nonnegative integers such that $p + q = d$. This class of random graphs include a broad class of popular network models (e.g., mixed-membership stochastic block models). We remark that the theory and method established in this work can be extended to generalized random dot product graphs as long as p, q are either provided or can be estimated consistently.

The remaining part of the article is structured as follows. We review the background on random dot product graphs and present the limit theorem for the ASE (modified theorem from Athreya et al. 2016) in Section 2.1. The theory for the maximum likelihood estimation of a single latent position with the rest of the latent positions being known, which motivates us to pursue the efficient estimation task, is established in Section 2.2. Section 3 elaborates on the proposed one-step procedure for estimating the entire latent position matrix, establishes its asymptotic theory, and shows that it dominates the ASE as the number of vertices goes to infinity. In Section 4, we apply the proposed one-step procedure to construct an estimator for the population version of the LSE, and show that it dominates the LSE asymptotically. Section 5 demonstrates the usefulness of the proposed one-step procedure via numerical examples and the analysis of a real-world Wikipedia graph data. We conclude the article with a discussion in Section 6.

Notations: The $d \times d$ identity matrix is denoted by $\mathbf{I}_d$ and the vector with all entries being 1 is denoted by the boldface $\mathbf{1}$. We define the notation $[n]$ to be the set of all consecutive positive integers from 1 to n: $[n] := \{1, 2, \ldots, n\}$. The symbols $\lesssim$ and $\gtrsim$ mean the corresponding inequality up to a constant, that is, $a \lesssim b$ ($a \gtrsim b$) if $a \leq Cb$ ($a \geq Cb$) for some constant $C > 0$, and we denote $a \asymp b$ if $a \lesssim b$ and $a \gtrsim b$. The shorthand notation $a \vee b$ denotes the maximum value between a and b, namely, $a \vee b = \max(a, b)$ for any $a, b \in \mathbb{R}$. We use the notation $\mathbb{O}(n, d)$ to denote the set of all orthonormal d-frames in $\mathbb{R}^n$, that is, $\mathbb{O}(n, d) = \{\mathbf{U} \in \mathbb{R}^{n \times d} : \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}_d\}$, where $n \geq d$, and write $\mathbb{O}(d) = \mathbb{O}(d, d)$. The notation $\|\mathbf{x}\|$ is used to denote the Euclidean norm of a vector $\mathbf{x} = [x_1, \ldots, x_d]^{\mathrm{T}} \in \mathbb{R}^d$, that is, $\|\mathbf{x}\| = (\sum_{k=1}^{d} x_k^2)^{1/2}$. For any two vectors $\mathbf{x} = [x_1, \ldots, x_d]^{\mathrm{T}}$ and $\mathbf{y} = [y_1, \ldots, y_d]^{\mathrm{T}}$ in $\mathbb{R}^d$, the inequality $\mathbf{x} \leq \mathbf{y}$ means that $x_k \leq y_k$ for all $k = 1, 2, \ldots, d$. For any two positive semidefinite matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ of the same dimension, the notation $\mathbf{\Sigma}_1 \preceq \mathbf{\Sigma}_2$ ($\mathbf{\Sigma}_1 \succeq \mathbf{\Sigma}_2$) means that $\mathbf{\Sigma}_2 - \mathbf{\Sigma}_1$ ($\mathbf{\Sigma}_1 - \mathbf{\Sigma}_2$) is positive semidefinite, and we say that $\mathbf{\Sigma}_1$ is no greater (no less) than $\mathbf{\Sigma}_2$ in spectra. For

any rectangular matrix $\mathbf{X}$, we use $\sigma_k(\mathbf{X})$ to denote its kth largest singular value. For a matrix $\mathbf{X} = [x_{ik}]_{n\times d}$, we use $\|\mathbf{X}\|_2$ to denote the spectral norm $\|\mathbf{X}\|_2 = \sigma_1(\mathbf{X})$, $\|\mathbf{X}\|_{\mathrm{F}}$ to denote the Frobenius norm $\|\mathbf{X}\|_{\mathrm{F}} = (\sum_{i=1}^{n}\sum_{k=1}^{d} x_{ik}^2)^{1/2}$, and $\|\mathbf{X}\|_{2\to\infty}$ to denote the two-to-infinity norm $\|\mathbf{X}\|_{2\to\infty} = \max_{i\in[n]}(\sum_{k=1}^{d} x_{ik}^2)^{1/2}$.

2. Preliminaries

2.1. Background on Random Dot Product Graphs

Denote $\mathcal{X} = \{\mathbf{x} = [x_1,\ldots,x_d]^{\mathrm{T}} \in \mathbb{R}^d : x_1,\ldots,x_d > 0, \|\mathbf{x}\| < 1\}$ the space of latent positions, and $\mathcal{X}^n = \{\mathbf{X} = [\mathbf{x}_1,\ldots,\mathbf{x}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d} : \mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathcal{X}\}$. For any $\delta \in (0,1/2)$, denote $\mathcal{X}(\delta)$ the set of all $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{x}^{\mathrm{T}}\mathbf{u} \in [\delta, 1-\delta]$ for all $\mathbf{u} \in \mathcal{X}(\delta)$. Given an $n \times d$ matrix $\mathbf{X} = [\mathbf{x}_1,\ldots,\mathbf{x}_n]^{\mathrm{T}} \in \mathcal{X}^n$ and a sparsity factor $\rho_n \in (0,1]$, a symmetric and hollow (*i.e.*, the diagonal entries are zeros) random matrix $\mathbf{A} = [A_{ij}]_{n\times n} \in \{0,1\}^{n\times n}$ is said to be the adjacency matrix of a random dot product graph on n vertices $[n] = \{1,2,\ldots,n\}$ with latent positions $\mathbf{x}_1,\ldots,\mathbf{x}_n$, denoted by $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$, if $A_{ij} \sim \mathrm{Bernoulli}(\rho_n \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)$ independently, $1 \le i < j \le n$. We refer to the matrix $\mathbf{X}$ as the latent position matrix. Namely, the distribution of $\mathbf{A}$ can be written as $p_{\mathbf{X}}(\mathbf{A}) = \prod_{i<j}(\rho_n \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^{A_{ij}}(1-\rho_n \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^{1-A_{ij}}$. When $\rho_n \equiv 1$ for all n, the resulting graph is dense, in the sense that the expected number of edges $\mathbb{E}(\sum_{i<j} A_{ij})$ grows quadratically in n, and when $\rho_n \to 0$ as $n \to \infty$, the corresponding graph is sparse, namely, the expected number of edges is subquadratic in n ($\mathbb{E}(\sum_{i<j} A_{ij}) = o(n^2)$).

The goal of this work is to estimate the latent positions $\mathbf{x}_1,\ldots,\mathbf{x}_n$, which are treated as deterministic parameters. In some cases, the latent positions $\mathbf{x}_1,\ldots,\mathbf{x}_n$ are considered as latent random variables that are independently sampled from some underlying distribution F on $\mathcal{X}$ (see, e.g., Athreya et al. 2016; Sussman, Tang, and Priebe 2014; Tang et al. 2017b; Tang and Priebe 2018). For deterministic latent positions, we require that there exists some cumulative distribution function F on $\mathcal{X}$, such that

$$\sup_{\mathbf{x}\in\mathcal{X}} |F_n(\mathbf{x}) - F(\mathbf{x})| \to 0 \quad \text{as } n \to \infty, \tag{1}$$

where $F_n(\mathbf{x}) = (1/n)\sum_{i=1}^{n} \mathbb{1}\{\mathbf{x}_i \le \mathbf{x}\}$. Condition (1) is similar to the case where $\mathbf{x}_i$'s are random in the following sense: When $\mathbf{x}_1,\ldots,\mathbf{x}_n$ are independent random variables sampled from F, the Glivenko-Cantelli theorem asserts that (1) holds with probability one with respect to the randomness of the infinite iid sequence $(\mathbf{x}_i)_{i=1}^{\infty}$.

Remark 1. The latent position matrix $\mathbf{X}$ can only be identified up to an orthogonal transformation since for any orthogonal matrix $\mathbf{W} \in \mathbb{O}(d)$ and $i,j \in [n]$, $\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j = (\mathbf{W}\mathbf{x}_i)^{\mathrm{T}}(\mathbf{W}\mathbf{x}_j)$. Furthermore, for any $d' > d$ and any latent position matrix $\mathbf{X} \in \mathbb{R}^{n\times d}$, there exists another matrix $\mathbf{X}' \in \mathbb{R}^{n\times d'}$, such that $\mathrm{RDPG}(\mathbf{X})$ and $\mathrm{RDPG}(\mathbf{X}')$ yield the same distribution of $\mathbf{A}$. The latter source of nonidentifiability can be avoided for large n by requiring the second moment matrix $\mathbf{\Delta} = \int_{\mathcal{X}} \mathbf{x}\mathbf{x}^{\mathrm{T}} F(\mathrm{d}\mathbf{x})$ to be non-singular (Tang and Priebe 2018).

Random dot product graphs have connections with the simplest Erdős-Rényi models and the popular stochastic block models. When $F(\mathrm{d}x) = \delta_p(\mathrm{d}x)$, the resulting random dot product graph coincides with an Erdős-Rényi graph, with $(A_{ij})_{i<j}$ being independent $\mathrm{Bernoulli}(p^2)$ random variables. When $F(\mathrm{d}\mathbf{x}) = \sum_{k=1}^{K} \pi_k \delta_{\boldsymbol{\nu}_k}(\mathrm{d}\mathbf{x})$ for $\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_k \in \mathcal{X}$ and $\sum_{k=1}^{K}\pi_k = 1$, there exists a function $\tau : [n] \to [K]$ such that $(1/n)\sum_{i=1}^{n}\mathbb{1}\{\tau(i) = k\} \to \pi_k$ for all $k = 1,2,\ldots,K$ as $n \to \infty$. Denoting $\mathbf{B} = [B_{kl}]_{K\times K} := [\boldsymbol{\nu}_k^{\mathrm{T}}\boldsymbol{\nu}_l]_{K\times K}$ and $\mathbf{x}_i = \boldsymbol{\nu}_{\tau(i)}$, $i \in [n]$, we see that A_{ij} follows $\mathrm{Bernoulli}(B_{\tau(i)\tau(j)}) = \mathrm{Bernoulli}(\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)$ for $i < j$ independently, where $i,j \in [n]$. In this case, the random dot product graph $\mathrm{RDPG}(\mathbf{X})$ with $\mathbf{X} = [\mathbf{x}_1,\ldots,\mathbf{x}_n]^{\mathrm{T}}$ becomes a stochastic block model with a positive semidefinite block probability matrix $\mathbf{B}$ and a cluster assignment function τ.

To estimate the latent positions, Sussman, Tang, and Priebe (2014) proposed to solve the least-square problem

$$\widehat{\mathbf{X}}^{(\mathrm{ASE})} = \arg\min_{\mathbf{X}\in\mathbb{R}^{n\times d}} \|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2. \tag{2}$$

The resulting solution $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ to Equation (2) is referred to as the ASE of $\mathbf{A}$ into $\mathbb{R}^d$. Note that $\mathbb{E}(\mathbf{A})$ is a positive semidefinite low-rank matrix modulus the diagonal entries and $\|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(A_{ij} - \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^2$ is exactly the empirical squared-error loss. Hence, the problem (2) becomes a naive empirical risk minimization problem if we regard $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ as an estimator for $\rho_n^{1/2}\mathbf{X}$, and the solution to Equation (2) can be conveniently computed (Eckart and Young 1936): $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ is the matrix of eigenvectors associated with the top d eigenvalues of $\mathbf{A}$, scaled by the square roots of these eigenvalues.

Sussman, Tang, and Priebe (2014) proved that $\widehat{\mathbf{X}}^{(\mathrm{ASE})} = [\widehat{\mathbf{x}}_1^{(\mathrm{ASE})},\ldots,\widehat{\mathbf{x}}_n^{(\mathrm{ASE})}]^{\mathrm{T}}$ is a consistent estimator for $\rho_n^{1/2}\mathbf{X}$ globally for all vertices: $(1/n)\|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W}_n - \mathbf{X}\|_{\mathrm{F}}^2$ converges to 0 in probability as $n \to \infty$ for a sequence of orthogonal $(\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$. Furthermore, for each fixed vertex $i \in [n]$, the asymptotic distribution of $\widehat{\mathbf{x}}_i^{(\mathrm{ASE})}$ after proper scaling and centering has been established (Athreya et al. 2016; Tang and Priebe 2018) in the case where $\mathbf{x}_1,\ldots,\mathbf{x}_n \overset{\mathrm{iid}}{\sim} F$. The setup in this work is slightly different since we posit that the latent positions are deterministic. To distinguish between an arbitrary element $\mathbf{X} \in \mathcal{X}^n$ and the ground truth, we denote $\mathbf{X}_0$ the true latent position matrix that generates the observed adjacency matrix $\mathbf{A}$.

We modify the limit theorem of the ASE originally presented in Athreya et al. (2016) to accommodate the deterministic setup for $\mathbf{x}_{01},\ldots,\mathbf{x}_{0n}$ and summarize the results in the following theorem. In the current framework, the proof technique for the asymptotic normality of the rows of the ASE is very different from that presented in Athreya et al. (2016) and Tang and Priebe (2018). The proof of Theorem 1 is deferred to supplementary material.

Theorem 1. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n and condition (1) hold for some $\mathbf{X}_0 = [\mathbf{x}_{01},\ldots,\mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$. Suppose either $\rho_n \equiv 1$ for all n or $\rho_n \to 0$ but $(\log n)^4/(n\rho_n) \to 0$ as $n \to \infty$, and denote $\rho = \lim_{n\to\infty}\rho_n$. Let $\widehat{\mathbf{X}}^{(\mathrm{ASE})} = [\widehat{\mathbf{x}}_1^{(\mathrm{ASE})},\ldots,\widehat{\mathbf{x}}_n^{(\mathrm{ASE})}]^{\mathrm{T}}$ be the ASE defined by (2). Denote

$$\mathbf{\Delta} = \int_{\mathcal{X}} \mathbf{x}\mathbf{x}^{\mathrm{T}} F(\mathrm{d}\mathbf{x}),$$

$$\mathbf{\Sigma}(\mathbf{x}) = \mathbf{\Delta}^{-1}\left[\int_{\mathcal{X}} \left\{\mathbf{x}_1^{\mathrm{T}}\mathbf{x}\left(1 - \rho\mathbf{x}_1^{\mathrm{T}}\mathbf{x}\right)\right\}\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}} F(\mathrm{d}\mathbf{x}_1)\right]\mathbf{\Delta}^{-1},$$

and assume that $\mathbf{\Delta}$ and $\mathbf{\Sigma}(\mathbf{x})$ are strictly positive definite for all $\mathbf{x} \in \mathcal{X}$. Then, there exists a sequence of orthogonal matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$, such that

$$\|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2 \xrightarrow{a.s.} \int_{\mathcal{X}} \mathrm{tr}\{\mathbf{\Sigma}(\mathbf{x})\}F(\mathrm{d}\mathbf{x}), \tag{3}$$

and for any fixed index $i \in [n]$,

$$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i^{(\mathrm{ASE})} - \rho_n^{1/2}\mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{\Sigma}(\mathbf{x}_{0i})). \tag{4}$$

In the rest of the article, we drop the subscript n in $\mathbf{W}_n$ for notational simplicity and make the convention that the orthogonal alignment matrix $\mathbf{W}$ implicitly depends on n.

2.2. Motivation: Efficiency in Estimating a Single Latent Position

Theorem 1 suggests the following two properties of the ASE: Globally for all vertices, $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ is a consistent estimator for $\rho_n^{1/2}\mathbf{X}_0$; Locally, for each fixed vertex $i \in [n]$, the distribution of the ith row $\widehat{\mathbf{x}}_i^{(\mathrm{ASE})}$ of $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ after $\sqrt{n}$-scaling and centering at $\rho_n^{1/2}\mathbf{x}_{0i}$, converges to a mean-zero multivariate normal distribution with covariance matrix $\mathbf{\Sigma}(\mathbf{x}_{0i})$, up to a sequence of orthogonal transformations. Nevertheless, it remains open whether the results of Theorem 1 are optimal. In this work, we will propose an estimator $\widehat{\mathbf{X}}$ for $\mathbf{X}_0$ that dominates the ASE asymptotically in the following sense: Globally for all vertices, it yields a smaller asymptotic SSE $\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2$ than (3); Locally for each fixed vertex $i \in [n]$, the corresponding row of $\widehat{\mathbf{X}}$, after $\sqrt{n}$-scaling and centering at $\rho_n^{1/2}\mathbf{x}_{0i}$, also converges to a mean-zero multivariate normal distribution, up to a sequence of orthogonal transformations, but the asymptotic covariance matrix is no greater than $\mathbf{\Sigma}(\mathbf{x}_{0i})$ in spectra.

Before elaborating on the estimator for the entire latent position matrix $\mathbf{X}_0$, we begin with the problem of estimating a single latent position $\mathbf{x}_{0i}$ when the rest of the latent positions are known. The theory established herein motivates the development of the proposed efficient estimation procedure. Specifically, for a fixed $i \in [n]$, we estimate $\mathbf{x}_{0i}$ via the MLE, assuming that the rest of the latent positions $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ are known. For simplicity, we assume that the sparsity factor $\rho_n \equiv 1$ for all n in this subsection. The result is summarized in the following theorem.

Theorem 2. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$ with $\rho_n \equiv 1$ for all n, and condition (1) hold. Suppose that there exists some constant $\delta > 0$ such that $(\mathbf{x}_{0j})_{j=1}^n \subset \mathcal{X}(\delta)$. Let $i \in [n]$ be fixed and consider the problem of estimating $\mathbf{x}_{0i}$ where $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ are known. Further assume that $\mathbf{x}_{0i}$ is in the interior of $\mathcal{X}(\delta)$, and for any $\mathbf{x} \in \mathcal{X}(\delta)$, define $\mathbf{G}(\mathbf{x}) = \int_{\mathcal{X}} \mathbf{x}_1\mathbf{x}_1{}^{\mathrm{T}}\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_1)\}^{-1}F(\mathrm{d}\mathbf{x}_1)$. Then the maximum likelihood estimator $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})} = \arg\max_{\mathbf{x}\in\mathcal{X}(\delta)} \ell_{\mathbf{A}}(\mathbf{x})$ is consistent for $\mathbf{x}_{0i}$, where $\ell_{\mathbf{A}}(\mathbf{x})$ is the log-likelihood function: $\ell_{\mathbf{A}}(\mathbf{x}) = \sum_{j\neq i}\{A_{ij}\log(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}) + (1 - A_{ij})\log(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}$. Furthermore, the following asymptotic normality holds:

$$\sqrt{n}(\widehat{\mathbf{x}}_i^{(\mathrm{MLE})} - \mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{G}(\mathbf{x}_{0i})^{-1}). \tag{5}$$

Furthermore, $\mathbf{\Sigma}(\mathbf{x}) - \mathbf{G}(\mathbf{x})^{-1}$ is always positive semidefinite for all $\mathbf{x} \in \mathcal{X}(\delta)$.

Remark 2. Recall that the cumulative distribution function F is defined on $\mathcal{X}$. Note that under the conditions of Theorem 2, $(\mathbf{x}_{0j})_{j=1}^n \subset \mathcal{X}(\delta)$ for a constant δ that does not depend on n. Therefore, the cumulative distribution function F can be further restricted to the compact subset $\mathcal{X}(\delta)$ of $\mathcal{X}$, and $\mathbf{G}(\mathbf{x})$ can be written as $\int_{\mathcal{X}(\delta)} \mathbf{x}_1\mathbf{x}_1{}^{\mathrm{T}}\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_1)\}^{-1}F(\mathrm{d}\mathbf{x}_1)$ alternatively.

Remark 3. Although the definition of $\mathbf{G}(\mathbf{x})$ given in Theorem 2 is with regard to the case where $\rho_n \equiv 1$ for all n, we remark that it can also be generalized to the case where the sparsity factor $\rho_n \to 0$ as $n \to \infty$ (see Equation (9) in Section 3).

Although the inequality $\mathbf{\Sigma}(\mathbf{x}) \succeq \mathbf{G}(\mathbf{x})^{-1}$ is not strict, we will present an example where there exists at least one negative eigenvalue of $\mathbf{G}(\mathbf{x}_{0i})^{-1} - \mathbf{\Sigma}_n(\mathbf{x}_{0i})$ in Section 3. The conclusion of this example is that the ASE is *inefficient* for estimating the latent position $\mathbf{x}_{0i}$ for vertex i when the rest of the latent positions are known, in contrast to the efficiency of the MLE. The notion of efficiency in estimating a single latent position of a random dot product graph model is slightly subtle, as this special case does not belong to the classical (iid) parametric models. Here, we make the convention that the notion of efficiency is taken in analogy to the case of parametric models. Namely, we say an estimator $\widehat{\mathbf{x}}_i^{(\mathrm{Eff})}$ is asymptotically efficient for estimating a single latent position vector $\mathbf{x}_{0i}$, if $\sqrt{n}(\widehat{\mathbf{x}}_i^{(\mathrm{Eff})} - \mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \mathbf{G}(\mathbf{x}_{0i})^{-1})$. We will see in Section 3 that when all the latent positions are unknown, we can still construct an estimator $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$, such that for each vertex i, $\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \mathbf{G}(\mathbf{x}_{0i})^{-1})$ still holds up to a sequence of orthogonal alignment matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$.

3. Efficient Estimation via a One-step Procedure

The inefficiency of the ASE, indicated by $\mathbf{\Sigma}(\mathbf{x}_{0i}) \succeq \mathbf{G}(\mathbf{x}_{0i})^{-1}$, is due to the fact that the ASE is a least-square estimator not depending on the likelihood function of the sampling model. In contrast, the maximum likelihood estimator $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})}$ utilizes the Bernoulli likelihood function, and this is a main factor for the asymptotic efficiency. For estimating the entire latent position matrix $\mathbf{X}$, one strategy that takes advantage of the likelihood information is the maximum likelihood method. Unfortunately, when all latent positions are unknown, random dot product graphs belong to a curved exponential family rather than a canonical exponential family, and neither the existence nor the uniqueness of the MLE of random dot product graphs has been established. As pointed out in Bickel and Doksum (2015), properties of the MLE in curved exponential families are harder to develop than the canonical ones. Therefore, we seek another approach to find an estimator that is asymptotically equivalent to the MLE. Recall that when $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ are known, the MLE for $\mathbf{x}_{0i}$ is a solution to the estimating equation

$$\mathbf{\Psi}_n(\mathbf{x}) := \frac{1}{n}\sum_{j\neq i}^{n} \frac{(A_{ij} - \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})} = \mathbf{0}.$$

Then, given an "appropriate" initial guess of the solution $\widetilde{\mathbf{x}}_i$, we can perform a one-step Newton-Raphson update to obtain

another estimator $\widehat{\mathbf{x}}_i^{(\mathrm{OS})}$ that is closer to the zero of the estimating equation $\boldsymbol{\Psi}_n$ (see, e.g., Section 5.7 of Van der Vaart 2000):

$$\widehat{\mathbf{x}}_i^{(\mathrm{OS})} = \widetilde{\mathbf{x}}_i + \left\{ \frac{1}{n} \sum_{j \neq i}^{n} \frac{\mathbf{x}_{0j} \mathbf{x}_{0j}^{\mathrm{T}}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}} \mathbf{x}_{0j} (1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \mathbf{x}_{0j})} \right\}^{-1} \left\{ \frac{1}{n} \sum_{j \neq i}^{n} \frac{(A_{ij} - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \mathbf{x}_{0j}) \mathbf{x}_{0j}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}} \mathbf{x}_{0j} (1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \mathbf{x}_{0j})} \right\}. \tag{6}$$

In the case of estimating $\mathbf{x}_{0i}$ with the rest of the latent positions being known, the requirement for $\widetilde{\mathbf{x}}_i$ is that it is $\sqrt{n}$-consistent for $\mathbf{x}_{0i}$, and the resulting OSE $\widehat{\mathbf{x}}_i^{(\mathrm{OS})}$ is as efficient as the maximum likelihood estimator $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})}$. This result is summarized in the following theorem, which is a variation of Theorem 5.45 of Van der Vaart (2000).

Theorem 3. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$ with $\rho_n \equiv 1$ for all n, and assume that the conditions of Theorem 2 hold. Consider the problem of estimating $\mathbf{x}_{0i}$ with $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ being known. Let $\widetilde{\mathbf{x}}_i$ be a $\sqrt{n}$-consistent estimator of $\mathbf{x}_{0i}$, *i.e.*, $\sqrt{n}(\widetilde{\mathbf{x}}_i - \mathbf{x}_{0i}) = O_{\mathbb{P}_0}(1)$. Then $\sqrt{n}(\widehat{\mathbf{x}}_i^{(\mathrm{OS})} - \mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, \mathbf{G}(\mathbf{x}_{0i})^{-1}\right)$.

The above result motivates us to generalize the OSE (6) to the case where the latent positions $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ are all unknown. Let $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}} \in \mathbb{R}^{n \times d}$ be an initial estimator $\widetilde{\mathbf{X}}$ for $\mathbf{X}_0$. An intuitive choice for generalizing the one-step updating scheme (6) to the case of unknown $(\mathbf{x}_{0j})_{j \neq i}$ is to substitute the unknown $\mathbf{x}_{0j}$ by the initial estimator $\widetilde{\mathbf{x}}_j$ for all $j \neq i$ in (6). We define the following one-step estimator $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ for $\mathbf{X}_0$:

$$\widehat{\mathbf{x}}_i = \widetilde{\mathbf{x}}_i + \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{\widetilde{\mathbf{x}}_j \widetilde{\mathbf{x}}_j^{\mathrm{T}}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j (1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)} \right\}^{-1} \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{(A_{ij} - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j) \widetilde{\mathbf{x}}_j}{\widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j (1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)} \right\}, \quad i = 1, 2, \ldots, n. \tag{7}$$

In this case, we require that the initial estimator $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]$ satisfies a finer condition than the $\sqrt{n}$-consistency requirement, referred to as the *approximate linearization property*.

Definition 1 *(Approximate linearization property).* Given $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n, where $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$, an estimator $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ is said to satisfy the approximate linearization property, if for all n, there exists an orthogonal matrix $\mathbf{W} = \mathbf{W}_n \in \mathbb{O}(d)$ and an $n \times d$ matrix $\widetilde{\mathbf{R}} = [\widetilde{\mathbf{R}}_1, \ldots, \widetilde{\mathbf{R}}_n]^{\mathrm{T}}$ with $\|\widetilde{\mathbf{R}}\|_{\mathrm{F}}^2 = O_{\mathbb{P}_0}((n\rho_n)^{-1} (\log n)^{\omega})$ for some $\omega \geq 0$, such that

$$\mathbf{W}^{\mathrm{T}} \widetilde{\mathbf{x}}_i - \rho_n^{1/2} \mathbf{x}_{0i} = \rho_n^{-1/2} \sum_{j=1}^{n} (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j}) \boldsymbol{\zeta}_{ij} + \widetilde{\mathbf{R}}_i, \quad i = 1, 2, \ldots, n, \tag{8}$$

where $\{\boldsymbol{\zeta}_{ij} : i, j \in [n]\}$ is a collection of vectors in $\mathbb{R}^d$ with $\sup_{i,j \in [n]} \|\boldsymbol{\zeta}_{ij}\| \lesssim 1/n$.

The approximate linearization property describes that the deviation of the estimator $\widetilde{\mathbf{X}}$ from $\mathbf{X}_0$ can be approximately controlled by a linear combination of the centered Bernoulli random variables $(A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j})_{i<j}$. It has been shown in Athreya et al. (2016),Tang and Priebe (2018), and Tang et al. (2017a) that the ASE satisfies the approximate linearization property (8) with $\omega = 0$ and $\boldsymbol{\zeta}_{ij}$ being the jth row of $\mathbf{X}_0 (\mathbf{X}_0^{\mathrm{T}} \mathbf{X}_0)^{-1}$, and hence, $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ can be chosen to be an initial estimator for the one-step procedure in practice. Another initial estimator satisfying the approximate linearization property will be given in Theorem 7 using the Laplacian spectral embedding.

We present the complete procedure for obtaining the OSE (7) initialized at the ASE in Algorithm 1.

Algorithm 1 One-step procedure initialized with the ASE

1: **Input:** The adjacency matrix $\mathbf{A} = [A_{ij}]_{n \times n}$ and the embedding dimension d.
2: **Step 1:** Compute the eigen-decomposition of the adjacency matrix:

$$\mathbf{A} = \sum_{i=1}^{n} \widehat{\lambda}_i \widehat{\mathbf{u}}_i \widehat{\mathbf{u}}_i^{\mathrm{T}},$$

where $|\widehat{\lambda}_1| \geq |\widehat{\lambda}_2| \geq \ldots \geq |\widehat{\lambda}_n|$, and $\widehat{\mathbf{u}}_i^{\mathrm{T}} \widehat{\mathbf{u}}_j = \mathbb{1}(i = j)$ for all $i, j \in [n]$.
3: **Step 2:** Compute the ASE

$$\widetilde{\mathbf{X}} = \widehat{\mathbf{X}}^{(\mathrm{ASE})} = \sum_{k=1}^{d} |\widehat{\lambda}_k|^{1/2} \widehat{\mathbf{u}}_k$$

and write $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}} \in \mathbb{R}^{n \times d}$.
4: **Step 3:** For $i = 1, 2, \ldots, n$, compute

$$\widehat{\mathbf{x}}_i = \widetilde{\mathbf{x}}_i + \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{\widetilde{\mathbf{x}}_j \widetilde{\mathbf{x}}_j^{\mathrm{T}}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j (1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)} \right\}^{-1} \left\{ \frac{1}{n} \sum_{j=1}^{n} \frac{(A_{ij} - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j) \widetilde{\mathbf{x}}_j}{\widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j (1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)} \right\}.$$

5: **Output:** The OSE $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$.

The notion of efficiency for random dot product graphs becomes less clear when the number of unknown latent positions grows with the number of vertices. This is because in random dot product graphs, the dimension of the parameter space $\mathcal{X}^n$ grows with the number of vertices, and the definition of the efficiency for classical iid parametric models does not apply. To this end, we introduce the notion of local efficiency for random dot product graphs. The idea is that any row of the estimator $\widehat{\mathbf{X}}$ has the same asymptotic covariance matrix with that of the MLE as if the rest of the latent positions are known.

Definition 2 *(Local efficiency).* Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$, $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n} \in \mathcal{X}(\delta)$ for some $\delta > 0$ that does not depend on n, and either $\rho_n \equiv 1$ or $\rho_n \to 0$. Denote $\rho = \lim_{n \to \infty} \rho_n$. Assume the condition (1) holds. An estimator $\widehat{\mathbf{X}}^{(\mathrm{Eff})} = [\widehat{\mathbf{x}}_1^{(\mathrm{Eff})}, \ldots, \widehat{\mathbf{x}}_n^{(\mathrm{Eff})}]^{\mathrm{T}}$ is said to be a locally efficient estimator for $\mathbf{X}_0$, if there exists a sequence of orthogonal alignment matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty}$, such that for all $i \in [n]$,

$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i^{(\mathrm{Eff})} - \rho_n^{1/2}\mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{G}(\mathbf{x}_{0i})^{-1})$, where $\mathbf{G}$ is a matrix-valued function $\mathbf{G} : \mathcal{X}(\delta) \to \mathbb{R}^{d\times d}$ defined by

$$\mathbf{G}(\mathbf{x}) = \int_{\mathcal{X}} \frac{\mathbf{x}_1\mathbf{x}_1{}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1-\rho\mathbf{x}^{\mathrm{T}}\mathbf{x}_1)} F(\mathrm{d}\mathbf{x}_1). \tag{9}$$

Theorems 4 and 5, which are the main technical results of this article, establish the asymptotic behavior of the one-step estimator (7). In particular, Theorem 5 shows that the OSE $\widehat{\mathbf{X}}$ is locally efficient.

Theorem 4. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$. Assume that condition (1) holds, and there exists some constant $\delta > 0$ that is independent of n such that $(\mathbf{x}_{0i})_{i=1}^n \subset \mathcal{X}(\delta)$. Denote $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ the OSE defined by Equation (7) initialized at an estimator $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ that satisfies the approximate linearization property (8). Denote $\mathbf{G}_n(\mathbf{x}) = (1/n)\sum_{j=1}^n \mathbf{x}_{0j}\mathbf{x}_{0j}{}^{\mathrm{T}}\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}^{-1}$ for any $\mathbf{x} \in \mathcal{X}(\delta)$. If either $\rho_n \equiv 1$ for all n or $\rho_n \to 0$ but $(\log n)^{2(1\vee\omega)}/(n\rho_n^5) \to 0$ as $n \to \infty$, then there exists a sequence of orthogonal matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$ such that

$$\begin{aligned}\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} \\ &\quad + \widehat{\mathbf{R}}_i, \quad i = 1, \ldots, n,\end{aligned} \tag{10}$$

where $\|\widehat{\mathbf{R}}_i\| = O_{\mathbb{P}_0}(n^{-1}\rho_n^{-5/2}(\log n)^{(1\vee\omega)})$ and $\sum_{i=1}^n \|\widehat{\mathbf{R}}_i\|^2 = O_{\mathbb{P}_0}((n\rho_n^5)^{-1}(\log n)^{2(1\vee\omega)})$.

Theorem 5. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$. Assume that the conditions of Theorem 4 hold, and denote $\rho = \lim_{n\to\infty}\rho_n$. Let $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ be the OSE (7) based on an initial estimator $\widetilde{\mathbf{X}}$ that satisfies the approximate linearization property. Then there exists a sequence of orthogonal matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$ such that as $n \to \infty$,

$$\left\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\right\|_{\mathrm{F}}^2 \xrightarrow{\mathbb{P}_0} \int_{\mathcal{X}} \mathrm{tr}\left\{\mathbf{G}(\mathbf{x})^{-1}\right\} F(\mathrm{d}\mathbf{x}), \tag{11}$$

and for each fixed $i \in [n]$,

$$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{G}(\mathbf{x}_{0i})^{-1}), \tag{12}$$

where $\mathbf{G}(\mathbf{x})$ is given by Equation (9).

Since we have already shown that $\mathbf{\Sigma}(\mathbf{x}_{0i}) \succeq \mathbf{G}(\mathbf{x}_{0i})^{-1}$ for all $i \in [n]$, it follows that

$$\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2 - \|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2 \xrightarrow{\mathbb{P}_0} \int_{\mathcal{X}} \mathrm{tr}\{\mathbf{\Sigma}(\mathbf{x}) - \mathbf{G}(\mathbf{x})^{-1}\}F(\mathrm{d}\mathbf{x}) \geq 0,$$

and hence we conclude that the OSE $\widehat{\mathbf{X}}$ improves the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ globally for all vertices asymptotically. Furthermore, for every fixed vertex $i \in [n]$, the ith row of the OSE $\widehat{\mathbf{x}}_i$ is locally efficient by definition, and the corresponding asymptotic covariance matrix is no greater than that of the corresponding row of the ASE in spectra.

Remark 4. Theorem 4 has the following implication: when the graph is dense ($\rho_n \equiv 1$ for all n), one can apply the one-step procedure multiple times, and the resulting estimator still satisfies the approximate linearization property and has the same asymptotic behavior as given by Theorem 4. This multi-step updating strategy is of practical interest for more accurate estimation when the sample size is insufficient for asymptotic approximation.

Proofs sketch for Theorems 4 and 5. The key to the proofs of Theorems 4 and 5 is formula (10). From here, we can apply the logarithmic Sobolev concentration inequality to (10) (see, e.g., Boucheron, Lugosi, and Massart 2013, sec. 6.4) to show that $\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2$ converges in probability to its expectation, which is exactly the quantity on the right-hand side of Equation (11). The asymptotic normality (12) of $\widehat{\mathbf{x}}_i$ can be obtained by directly applying the Lyapunov's central limit theorem to

$$\frac{1}{\sqrt{n\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j},$$

which is a sum of independent random variables. For Equation (10), by construction of the OSE (7) and a Taylor expansion device, we have,

$$\begin{aligned}\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} \\ &\quad + (\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) + \mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{R}_{i1} \\ &\quad + o_{\mathbb{P}_0}(n^{-1/2}),\end{aligned}$$

where $\mathbf{R}_{i1} = -\mathbf{G}_n(\mathbf{x}_{0i})(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) + o_{\mathbb{P}_0}(n^{-1/2})$. Thus, we obtain that

$$\begin{aligned}\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}{}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} \\ &\quad + o_{\mathbb{P}_0}\left(n^{-1/2}\right).\end{aligned}$$

The detailed technical derivation of Equation (10) is deferred to supplementary material.

Remark 5. Theorem 5 asserts that the OSE $\widehat{\mathbf{X}}$ dominates the ASE under the density condition $(n\rho_n^5)^{-1}(\log n)^{2(1\vee\omega)} \to 0$ as $n \to \infty$. When the graph is dense, that is, $\rho_n \equiv 1$ for all n, it is easy to show that this condition holds. When ρ_n^{-1} is a polynomial of $\log n$, indicating that the graph is moderately sparse, this condition still holds. This condition starts to fail when the graph becomes very sparse, for example, $\rho_n^{-1} \asymp n^t$ for some $t \geq 1/5$, in which case a broad range of statistical inference tasks become challenging due to the weak signal.

Remark 6. Theorem 4 requires that the sparsity factor ρ_n is lower bounded by $n^{-1/5}$ times a polynomial factor of $\log n$. This causes the average expected degree to grow at a polynomial rate of n, and the resulting graph is considered as moderately sparse. In contrast, Theorem 1 only requires ρ_n to be lower bounded by n^{-1} times a polynomial factor of $\log n$, and this results in the average expected degree to grow at a polynomial rate of $\log n$, which is a sparser regime than that required by Theorem 4. The stronger density assumption that the average expected degree

is a polynomial factor of n is essential for the proof strategy employed in this work. Nevertheless, we remark that the proof strategy is standard (see, e.g., Section 5.7 of Van der Vaart 2000). In fact, the stronger density assumption stems from the Lipschitz continuity of the Hessian of the average log-likelihood function, which is guaranteed by the continuity of the third derivatives. This is referred to as the *classical conditions* for M-estimators (see, e.g., Van der Vaart 2000, sec. 5.6). Further discussion of the sparsity condition for the one-step estimator (7) is provided in Supplementary Material.

Theorem 5 claims that the asymptotic covariance matrix of any fixed row of the OSE (7) is no greater than that of the ASE in spectra. The following example shows that there exist situations where $\mathbf{G}(\mathbf{x}_{0i})^{-1} - \mathbf{\Sigma}(\mathbf{x}_{0i})$ contains at least one strictly negative eigenvalue. This implies that the OSE dominates the ASE asymptotically.

Example 1. (Two-block stochastic block model). Consider the following two-block stochastic block model, which has also been considered in Tang and Priebe (2018). Let $F = \pi_1\delta_p + \pi_2\delta_q$ be the distribution on $(0, 1)$ giving rise to the latent positions $x_{01}, \ldots, x_{0n}$ via (1), where $p, q \in (0, 1)$ and $p \neq q$. This results in an $n \times n$ adjacency matrix $\mathbf{A}$ drawn from RDPG($\mathbf{X}_0$) with $\mathbf{X}_0 = [x_{01}, \ldots, x_{0n}]^{\mathrm{T}} \in \mathbb{R}^{n\times 1}$. Let $\tau : [n] \to \{1, 2\}$ be a cluster assignment function such that $\tau(i) = 1$ if $x_{0i} = p$, $\tau(i) = 2$ if $x_{0i} = q$, and denote

$$\mathbf{B} = \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix}.$$

Then the distribution of $\mathbf{A}$ can be also regarded as a stochastic block model with a block probability matrix $\mathbf{B}$ and a cluster assignment function τ. Let $\widehat{\mathbf{X}}^{(\mathrm{ASE})} = [\widehat{\mathbf{x}}_1^{(\mathrm{ASE})}, \ldots, \widehat{\mathbf{x}}_n^{(\mathrm{ASE})}]^{\mathrm{T}}$ be the ASE and $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ be the OSE satisfying the conditions of Theorem 4. Using formulas (4) and (12), we obtain:

$$\sqrt{n}(\widehat{\mathbf{x}}_i^{(\mathrm{ASE})} - p) \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, \Sigma(p)\right) \text{ if } x_{0i} = p,$$
$$\sqrt{n}(\widehat{\mathbf{x}}_i^{(\mathrm{ASE})} - q) \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, \Sigma(q)\right) \text{ if } x_{0i} = q,$$

where $\Sigma(p) = \frac{\pi_1 p^4(1-p^2)+\pi_2 pq^3(1-pq)}{(\pi_1 p^2+\pi_2 q^2)^2}$, $\Sigma(q) = \frac{\pi_1 p^3 q(1-pq)+\pi_2 q^4(1-q^2)}{(\pi_1 p^2+\pi_2 q^2)^2}$, and

$$\sqrt{n}(\widehat{\mathbf{x}}_i - p) \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, G(p)^{-1}\right) \text{ if } x_{0i} = p,$$
$$\sqrt{n}(\widehat{\mathbf{x}}_i - q) \xrightarrow{\mathcal{L}} \mathrm{N}\left(0, G(q)^{-1}\right) \text{ if } x_{0i} = q,$$

where $G(p) = \frac{\pi_1 p^2}{p^2(1-p^2)} + \frac{\pi_2 q^2}{pq(1-pq)}$, $G(q) = \frac{\pi_1 p^2}{pq(1-pq)} + \frac{\pi_2 q^2}{q^2(1-q^2)}$. By Cauchy–Schwartz inequality, we see that $G(p)^{-1} \leq \Sigma(p)$ and $G(q)^{-1} \leq \Sigma(q)$ for all $p, q \in (0, 1)$, and in particular, $G(p)^{-1} = \Sigma(p)$ if and only if $q = (1 - p^2)/p$, and $G(q)^{-1} = \Sigma(q)$ if and only if $q = (1/2)(\sqrt{p^2 + 4} - p)$ (recall that $p \neq q$). Namely, the asymptotic variance of the OSE is strictly smaller than that of the ASE for almost every (p, q) pair in $(0, 1)^2 \backslash \{(p, q) : p = q\}$. The comparison of variances between the ASE and the OSE is further visualized in Figure 1 through the relative improvements of the variances $\{\Sigma(p) - G(p)^{-1}\}/G(p)^{-1}$ and $\{\Sigma(q) - G(q)^{-1}\}/G(q)^{-1}$ for different values of p and q.

4. Application to Estimating the Laplacian Matrix

Instead of directly analyzing the adjacency matrix $\mathbf{A}$, another broadly adopted technique for statistical analysis on random graphs is based on the normalized Laplacian of $\mathbf{A}$ (Rohe, Chatterjee, and Yu 2011; Sarkar and Bickel 2015). Formally, given a matrix $\mathbf{M}$ with nonnegative entries and positive row sums, the normalized Laplacian of $\mathbf{M}$, denoted by $\mathcal{L}(\mathbf{M})$, is defined by $(\mathrm{diag}(\mathbf{M1}))^{-1/2}\mathbf{M}(\mathrm{diag}(\mathbf{M1}))^{-1/2}$, Here, for a vector $\mathbf{z}$ =

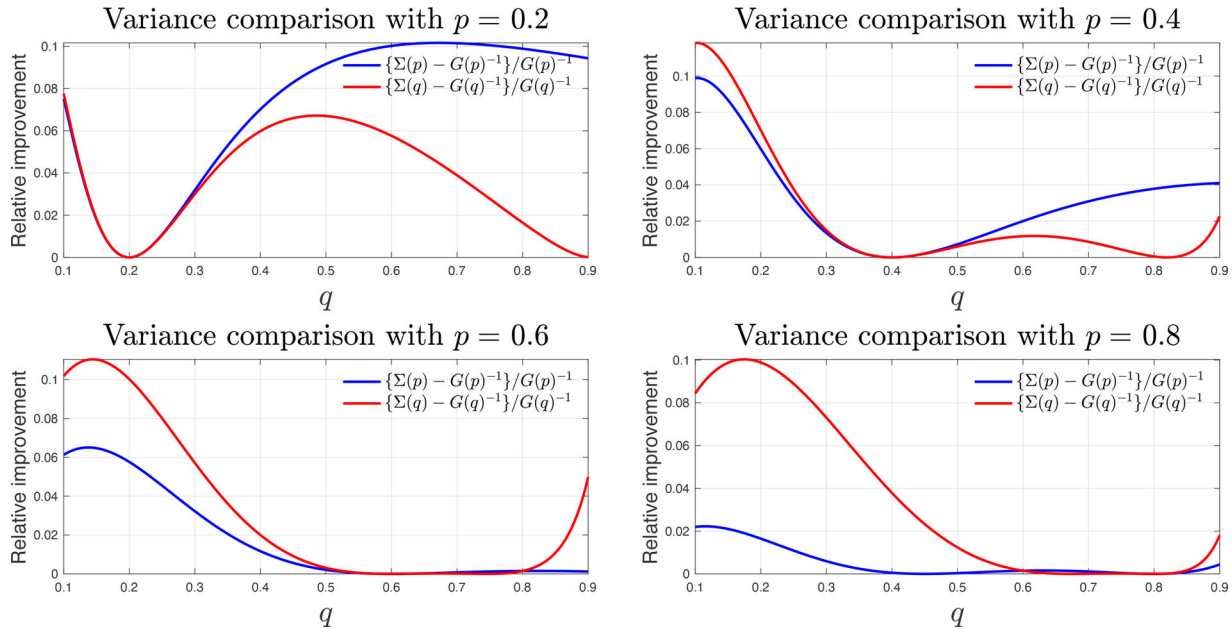

Figure 1. Relative improvements of the one-step estimator variances $\{\Sigma(p) - G(p)^{-1}\}/G(p)^{-1}$ and $\{\Sigma(q) - G(q)^{-1}\}/G(q)^{-1}$ for different values of $p, q \in (0, 1)$ in Example 1. The cluster assignment probabilities are set to $\pi_1 = 0.6$ and $\pi_2 = 0.4$. Note that all the variances $G(p)^{-1}, G(q)^{-1}, \Sigma(p), \Sigma(q)$ depend on both p and q.

$[z_1, \ldots, z_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\mathrm{diag}(\mathbf{z})$ is the $n \times n$ diagonal matrix with $z_1, \ldots, z_n$ being its diagonal entries. We follow the definition of the normalized Laplacian adopted in Tang and Priebe (2018) in contrast to the combinatorial Laplacian $\mathrm{diag}(\mathbf{M1}) - \mathbf{M}$ that has been applied to graph theory (Merris 1994). The (i, j) entry of the normalized Laplacian matrix $\mathcal{L}(\mathbf{A})$ can be interpreted as the connection between vertices i and j normalized by the square roots of the degrees of the two vertices.

Recall that the edge probability matrix $\rho_n \mathbf{X}\mathbf{X}^{\mathrm{T}}$ is positive semidefinite low-rank when $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$ with a sparsity factor ρ_n. Similarly, the normalized Laplacian of $\rho_n \mathbf{X}\mathbf{X}^{\mathrm{T}}$ is also a positive semidefinite low-rank matrix: $\mathcal{L}(\rho_n \mathbf{X}\mathbf{X}^{\mathrm{T}}) = \mathbf{Y}\mathbf{Y}^{\mathrm{T}}$, where $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n]^{\mathrm{T}} \in \mathbb{R}^{n \times d}$, and $\mathbf{y}_i = \mathbf{x}_i (\sum_{j=1}^n \mathbf{x}_i{}^{\mathrm{T}} \mathbf{x}_j)^{-1/2}$. Following the same spirit of the formulation of the ASE through Equation (2), one can analogously define the Laplacian spectral embedding (LSE) $\check{\mathbf{X}}$ of $\mathbf{A}$ into $\mathbb{R}^d$ by solving the least-square problem (Rohe, Chatterjee, and Yu 2011)

$$\check{\mathbf{X}} = \arg\min_{\mathbf{Y} \in \mathbb{R}^{n \times d}} \|\mathcal{L}(\mathbf{A}) - \mathbf{Y}\mathbf{Y}^{\mathrm{T}}\|_{\mathrm{F}}^2. \tag{13}$$

Since the LSE $\check{\mathbf{X}}$ is an estimator for $\mathbf{Y}$, we refer to the $n \times d$ matrix $\mathbf{Y}$ as the population LSE. The estimator $\check{\mathbf{X}}$, which is the LSE of $\mathbf{A}$ into $\mathbb{R}^d$, is also referred to as the sample LSE as opposed to the population LSE $\mathbf{Y}$. Alternatively, the population LSE can be viewed as a transformation $\mathbf{Y} = \mathbf{Y}(\mathbf{X})$ of the latent position matrix $\mathbf{X}$ defined by

$$\mathbf{Y}(\mathbf{X}) = [\mathbf{y}_1(\mathbf{X}), \ldots, \mathbf{y}_n(\mathbf{X})]^{\mathrm{T}}, \quad \mathbf{y}_i = \frac{\mathbf{x}_i}{\sqrt{\sum_{j=1}^n \mathbf{x}_i{}^{\mathrm{T}} \mathbf{x}_j}}, \quad i = 1, \ldots, n. \tag{14}$$

The asymptotic results for the (sample) LSE in random dot product graphs with independent and identically distributed latent positions $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ have been established in Tang and Priebe (2018). In the context of the deterministic latent positions framework adopted in this work, we provide the analogous results for the LSE in Theorem 6. The proof is deferred to Supplementary Material.

Theorem 6. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n \subset \mathbb{R}^{n \times d}$, where $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ satisfy (1). Suppose either $\rho_n \equiv 1$ for all n or $\rho_n \to 0$ but $(\log n)^4/(n\rho_n) \to 0$ as $n \to \infty$, and denote $\rho = \lim_{n\to\infty} \rho_n$. Let $\check{\mathbf{X}} = [\check{\mathbf{x}}_1, \ldots, \check{\mathbf{x}}_n]^{\mathrm{T}}$ be the LSE of $\mathbf{A}$ into $\mathbb{R}^d$ defined by Equation (13). Define the following quantities:

$$\mathbf{Y}_0 = \mathbf{Y}(\mathbf{X}_0), \quad \boldsymbol{\mu} = \int_{\mathcal{X}} \mathbf{x} F(\mathrm{d}\mathbf{x}), \quad \widetilde{\boldsymbol{\Delta}} = \int_{\mathcal{X}} \frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\boldsymbol{\mu}} F(\mathrm{d}\mathbf{x}),$$

$$\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) = \left(\widetilde{\boldsymbol{\Delta}}^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right) \left[\int_{\mathcal{X}} \left\{\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1 - \rho\mathbf{x}^{\mathrm{T}}\mathbf{x}_1)}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}_1)^2} \mathbf{x}_1\mathbf{x}_1{}^{\mathrm{T}}\right\} F(\mathrm{d}\mathbf{x}_1)\right] \left(\widetilde{\boldsymbol{\Delta}}^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}.$$

Then there exists a sequence of orthogonal $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{R}^{d \times d}$ such that as $n \to \infty$,

$$n\rho_n \|\check{\mathbf{X}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2 \xrightarrow{a.s.} \int \mathrm{tr}\{\widetilde{\boldsymbol{\Sigma}}(\mathbf{x})\} F(\mathrm{d}\mathbf{x}). \tag{15}$$

Furthermore, assume the graph model falls into one of the following two regimes:

(i) Dense regime: $\rho_n \equiv 1$ for all n;
(ii) Sparse stochastic block model regime: $\rho_n \to 0$ with $(\log n)^4/(n\rho_n) \to 0$ as $n \to \infty$, and there exists $K \geq d$ linearly independent $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_K \in \mathcal{X}$ and a probability vector $[\pi_1, \ldots, \pi_K]$ with $\sum_{k=1}^K \pi_k = 1$, such that $F(\mathrm{d}\mathbf{x}) = \sum_{k=1}^K \pi_k \delta_{\boldsymbol{\nu}_k}(\mathrm{d}\mathbf{x})$. Namely, the random dot product graph coincides with a stochastic block model.

Then for any fixed $i \in [n]$,

$$n\rho_n^{1/2}(\mathbf{W}^{\mathrm{T}}\check{\mathbf{x}}_i - \mathbf{y}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \widetilde{\boldsymbol{\Sigma}}(\mathbf{x}_{0i})). \tag{16}$$

The LSE can be applied to construct another initial estimator that satisfies the approximate linearization property. This is given in the following theorem.

Theorem 7. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n \subset \mathbb{R}^{n \times d}$, where $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ satisfy Equation (1). Suppose either $\rho_n \equiv 1$ for all n or $\rho_n \to 0$ but $(\log n)^4/(n\rho_n) \to 0$ as $n \to \infty$, and denote $\rho = \lim_{n\to\infty} \rho_n$. Let $\check{\mathbf{X}}$ be the LSE of $\mathbf{A}$ into $\mathbb{R}^d$ defined by Equation (13). Then the estimator $\widetilde{\mathbf{X}} = \mathrm{diag}(\sum_{j=1}^n A_{1j}, \ldots, \sum_{j=1}^n A_{nj})^{1/2}\check{\mathbf{X}}$ satisfies the approximate linearization property.

Similar to the ASE, the LSE is also a least-square type estimator and does not involve the likelihood function. Therefore, to estimate the population LSE $\mathbf{Y}_0 = \mathbf{Y}(\mathbf{X}_0)$ using the Bernoulli likelihood information, we propose the following one-step estimator $\widehat{\mathbf{Y}}$ for $\mathbf{Y}_0$ based on the OSE $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ defined in (7) and an initial estimator $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ that satisfies the approximate linearization property (8):

$$\widehat{\mathbf{Y}} = [\widehat{\mathbf{y}}_1, \ldots, \widehat{\mathbf{y}}_n]^{\mathrm{T}}, \quad \widehat{\mathbf{y}}_i = \frac{\widehat{\mathbf{x}}_i}{\sqrt{\sum_{j=1}^n \widehat{\mathbf{x}}_i{}^{\mathrm{T}}\widetilde{\mathbf{x}}_j}}, \quad i = 1, 2, \ldots, n. \tag{17}$$

In matrix form, we can write $\widehat{\mathbf{Y}} = \{\mathrm{diag}(\widehat{\mathbf{X}}\widetilde{\mathbf{X}}^{\mathrm{T}}\mathbf{1})\}^{-1/2}\widehat{\mathbf{X}}$. The likelihood information is thus absorbed into $\widehat{\mathbf{Y}}$ through the OSE $\widehat{\mathbf{X}}$. We characterize the global and local behavior of the OSE $\widehat{\mathbf{Y}}$ for the population LSE via the following two theorems.

Theorem 8. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$. Assume that the conditions of Theorem 4 hold. Denote $\widehat{\mathbf{Y}} = [\widehat{\mathbf{y}}_1, \ldots, \widehat{\mathbf{y}}_n]^{\mathrm{T}}$ the OSE for the population LSE defined by (17), and $\boldsymbol{\mu}_n = (1/n)\sum_{i=1}^n \mathbf{x}_{0i}$. Then there exists a sequence of orthogonal matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$ such that

$$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{y}}_i - \mathbf{y}_{0i}) = \rho_n^{-1/2} \frac{1}{\sqrt{\boldsymbol{\mu}_n{}^{\mathrm{T}}\mathbf{x}_{0i}}} \left(\mathbf{I}_d - \frac{\mathbf{x}_{0i}\boldsymbol{\mu}_n{}^{\mathrm{T}}}{2\boldsymbol{\mu}_n{}^{\mathrm{T}}\mathbf{x}_{0i}}\right) \left(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}\right) + \mathbf{R}_i^{(\mathrm{L})}, \quad i = 1, 2, \ldots, n,$$

where $\|\mathbf{R}_i^{(\mathrm{L})}\| = O_{\mathbb{P}_0}((n\rho_n^2)^{-1}(\log n)^{1\vee\omega})$ and $\sum_{i=1}^n \|\mathbf{R}_i^{(\mathrm{L})}\|^2 = O_{\mathbb{P}_0}\left((n\rho_n^4)^{-1}(\log n)^{2(1\vee\omega)}\right)$.

Theorem 9. Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$. Assume the conditions of

Theorem 8 hold. Denote $\widehat{\mathbf{Y}} = [\widehat{\mathbf{y}}_1, \ldots, \widehat{\mathbf{y}}_n]^{\mathrm{T}}$ the OSE for the population LSE defined by Equation (17), and

$$\widetilde{\mathbf{G}}(\mathbf{x}) = \frac{1}{(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x})}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)\mathbf{G}(\mathbf{x})^{-1}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}$$

for any $\mathbf{x} \in \mathcal{X}(\delta)$, where $\boldsymbol{\mu} = \int_{\mathcal{X}} \mathbf{x}F(\mathrm{d}\mathbf{x})$ and $\mathbf{G}(\cdot)$ is defined in Equation (9). Then, there exists a sequence of orthogonal matrices $(\mathbf{W})_{n=1}^{\infty} = (\mathbf{W}_n)_{n=1}^{\infty} \subset \mathbb{O}(d)$ such that

$$n\rho_n \left\|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\right\|_{\mathrm{F}}^2 \xrightarrow{\mathbb{P}_0} \int_{\mathcal{X}} \mathrm{tr}\left\{\widetilde{\mathbf{G}}(\mathbf{x})\right\} F(\mathrm{d}\mathbf{x}), \tag{18}$$

and for each fixed $i \in [n]$,

$$n\rho_n^{1/2}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{y}}_i - \mathbf{y}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \widetilde{\mathbf{G}}(\mathbf{x}_{0i})). \tag{19}$$

Furthermore, for any $\mathbf{x} \in \mathcal{X}(\delta)$, $\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) - \widetilde{\mathbf{G}}(\mathbf{x})$ is always positive semidefinite, where the formula for $\widetilde{\boldsymbol{\Sigma}}(\cdot)$ is given in Theorem 6.

Remark 7. The key difference between the assumption of Theorem 8 for the one-step estimator for the population LSE and that of Theorem 6 for the (sample) LSE is that, under the sparse regime (ii), we drop the requirement that F is a finite mixture of point masses and F is allowed to be a general distribution function on $\mathcal{X}^n$, at the cost of a stronger density assumption $(\log n)^{2(1\vee\omega)}/(n\rho_n^4) \to 0$.

In Section 3, it is shown that the OSE $\widehat{\mathbf{X}}$ dominates the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ for estimating $\mathbf{X}_0$ asymptotically. Similarly, since $\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) \succeq \widetilde{\mathbf{G}}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}(\delta)$, it follows that locally for a fixed vertex i, the OSE $\widehat{\mathbf{Y}}$ improves the LSE $\check{\mathbf{X}}$ asymptotically in terms of a smaller asymptotic covariance matrix in spectra. In addition,

$$n\rho_n\|\check{\mathbf{X}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2 - n\rho_n\|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2 \xrightarrow{\mathbb{P}_0} \int_{\mathcal{X}} \mathrm{tr}\{\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) - \widetilde{\mathbf{G}}(\mathbf{x})\}F(\mathrm{d}\mathbf{x}) \geq 0.$$

Namely, the OSE $\widehat{\mathbf{Y}}$ also improves the LSE $\check{\mathbf{X}}$ globally for all vertices in terms of the SSE $\|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2$.

5. Numerical Examples

5.1. A Latent Curve Random Graph Example

In this subsection, we consider a random dot product graph whose latent positions are generated from a curve. Consider a graph with n vertices and latent dimension $d = 1$. The latent position x_{0i} for the ith vertex is set to $x_{0i} = 0.8\sin\{\pi(i-1)/(n-1)\} + 0.1$, where $i \in [n]$. Let $\mathbf{X}_0 = [x_{01}, \ldots, x_{0n}]^{\mathrm{T}}$ and suppose an adjacency matrix $\mathbf{A}$ is generated from RDPG($\mathbf{X}_0$). The four estimators involved are the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$, the one-step estimator $\widehat{\mathbf{X}}$ initialized at the ASE (OSE-A), the LSE $\check{\mathbf{X}}$, and the OSE $\widehat{\mathbf{Y}}$ for the population LSE (OSE-L). We focus on the following objectives:

(i) Comparison between the ASE and the OSE-A, and the comparison between the LSE and the OSE-L. We evaluate the performance of these estimates by computing their SSEs:

$$\mathrm{SSE}_{\mathrm{ASE}} = \inf_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \mathbf{X}_0\|_2^2,$$
$$SSE_{\mathrm{OSE-A}} = \inf_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_2^2,$$
$$\mathrm{SSE}_{\mathrm{LSE}} = \inf_{\mathbf{W}\in\{\pm1\}} \|\check{\mathbf{X}}\mathbf{W} - \mathbf{Y}_0\|_2^2,$$
$$\mathrm{SSE}_{\mathrm{OSE-L}} = \inf_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\|_2^2.$$

(ii) Performance of the vertex-wise confidence intervals (CIs) for the latent positions and the population LSE. The vertex-wise CIs can be derived from Theorems 5 and 9. Let $\widehat{\mathbf{X}} = [\widehat{x}_1, \ldots, \widehat{x}_n]^{\mathrm{T}}$ be the OSE-A. By Theorem 5, $\sqrt{n}(|\widehat{x}_i| - x_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(0, G(x_{0i})^{-1})$, where $G(x_{0i}) = \int x_1\{x_{0i}(1 - x_{0i}x_1)\}^{-1}F(\mathrm{d}x_1)$. To compute a $1-\alpha$ confidence interval for x_{0i}, we need to estimate $G(x_{0i})$ using $\widehat{\mathbf{X}}$ because neither x_{0i} nor the function form of G is accessible from the data. Specifically, let $\widehat{G}(\widehat{x}_i) = (1/n)\sum_{j=1}^n \widehat{x}_j\{\widehat{x}_i(1 - \widehat{x}_i\widehat{x}_j)\}^{-1}$. Then a $1-\alpha$ confidence interval for x_{0i} is given by

$$\left(|\widehat{x}_i| - \frac{q_z(1-\alpha/2)}{\sqrt{\widehat{G}(\widehat{x}_i)n}}, |\widehat{x}_i| + \frac{q_z(1-\alpha/2)}{\sqrt{\widehat{G}(\widehat{x}_i)n}}\right), \tag{20}$$

where $q_z(1-\alpha/2)$ is the $1-\alpha/2$ quantile of the standard normal distribution. Similarly, the asymptotic normality $n(|\widehat{y}_i| - y_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \widetilde{G}(x_{0i}))$ from Theorem 9 can be employed to construct a $1-\alpha$ confidence interval for the coordinate y_{0i} of the population LSE $\mathbf{Y}_0$, where $\widehat{y}_i$ is the ith coordinate of $\widehat{\mathbf{Y}}$. The corresponding asymptotic variance can be estimated by $\{4\widehat{\mu}\widehat{x}_i\widehat{G}(\widehat{x}_i)\}^{-1}$, where $\widehat{\mu} = (1/n)\sum_{j=1}^n \widehat{x}_j$. Therefore, a $1-\alpha$ confidence interval for y_{0i} is given by

$$\left(|\widehat{y}_i| - \frac{q_z(1-\alpha/2)}{\sqrt{4n^2\widehat{\mu}\widehat{x}_i\widehat{G}(\widehat{x}_i)}}, |\widehat{y}_i| + \frac{q_z(1-\alpha/2)}{\sqrt{4n^2\widehat{\mu}\widehat{x}_i\widehat{G}(\widehat{x}_i)}}\right). \tag{21}$$

(iii) Performance of the hypothesis testing as a subsequent inference task. We consider testing hypothesis $H_0 : \mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ against $H_A : \mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_\epsilon)$, where $\mathbf{X}_\epsilon = [\mathbf{X}_0, \epsilon\mathbf{1}]$, $\mathbf{1}$ is the n-dimensional vector of all ones, and $\epsilon \in \{0.001, 0.002, \ldots, 0.01\}$. To compare the impact of the ASE and OSE-A on hypothesis testing, we let $T_{\mathrm{ASE}} = \inf_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$ and $T_{\mathrm{OSE\text{-}A}} = \inf_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$ be the test statistics associated with them. The goal is to explore the powers of T_{ASE} and $T_{\mathrm{OSE\text{-}A}}$ as functions of ϵ.

For Objectives (i) and (ii), we draw 1000 independent adjacency matrices from RDPG($\mathbf{X}_0$) as Monte Carlo replicates. Regarding objective (i), we compute the SSEs across the 1000 Monte Carlo replicates and present the boxplots of $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{OSE-A}}$ and $\mathrm{nSSE}_{\mathrm{LSE}} - \mathrm{nSSE}_{\mathrm{OSE-L}}$ in Figures 2 (a) and (b), respectively. We can see clearly that, for each realization, $\mathrm{SSE}_{\mathrm{OSE-A}} < \mathrm{SSE}_{\mathrm{ASE}}$ and $\mathrm{SSE}_{\mathrm{OSE-L}} < \mathrm{SSE}_{\mathrm{LSE}}$ with large probability. The difference between $\mathrm{SSE}_{\mathrm{ASE}}$ and $\mathrm{SSE}_{\mathrm{OSE-A}}$ and that between $\mathrm{SSE}_{\mathrm{LSE}}$ and $\mathrm{SSE}_{\mathrm{OSE-L}}$ are both statistically significant at level $\alpha = 0.01$. These results support the theory developed

in Sections 3 and 4. In terms of objective (ii), we construct the vertex-wise 95% confidence intervals for both $\mathbf{X}_0$ and $\mathbf{Y}_0$ based on each realization of the adjacency matrix, and compute the corresponding empirical coverage probabilities for each vertex $i \in [n]$. The results are visualized in Figures 2 (c) and (d), respectively. The empirical coverage probabilities concentrate near the nominal coverage probability. We also randomly select one realization of the adjacency matrix and visualize the vertex-wise CIs for $[x_{01}, \ldots, x_{0n}]^{\mathrm{T}}$ and $[y_{01}, \ldots, y_{0n}]^{\mathrm{T}}$ in Figures 2 (e) and (f), respectively, which further consolidate the asymptotic normality of the rows of the OSE-A and the OSE-L developed in Sections 3 and 4.

For objective (iii), we need to determine the null distributions of the test statistics T_{ASE} and T_{OSE}. We compute these null distributions using a Monte Carlo simulation with 1000 independent replicates. The rejection regions for level α tests based on T_{ASE} and $T_{\mathrm{OSE-A}}$ are $R_{\mathrm{ASE}} := \{\mathbf{A} : T_{\mathrm{ASE}} > q_\alpha(\mathrm{ASE})\}$ and $R_{\mathrm{OSE-A}} := \{\mathbf{A} : T_{\mathrm{OSE-A}} > q_\alpha(\mathrm{OSE\text{-}A})\}$, where $q_\alpha(\mathrm{ASE})$ and $q_\alpha(\mathrm{OSE\text{-}A})$ are the $(1-\alpha)$-quantiles of the distributions of T_{ASE} and $T_{\mathrm{OSE-A}}$ under the null hypothesis, respectively. We then compute the powers of the two test statistics under different values of $\epsilon \in \{0.001, 0.002, \ldots, 0.01\}$ using a Monte Carlo simulation with 1000 independent replicates and report the results in Table 1. Due to the improvement of the OSE-A over the ASE, we see clearly that the test based on $T_{\mathrm{OSE-A}}$ is more powerful than that based on T_{ASE}, which shows the usefulness of the proposed OSE-A for hypotheses testing as a subsequent inference task.

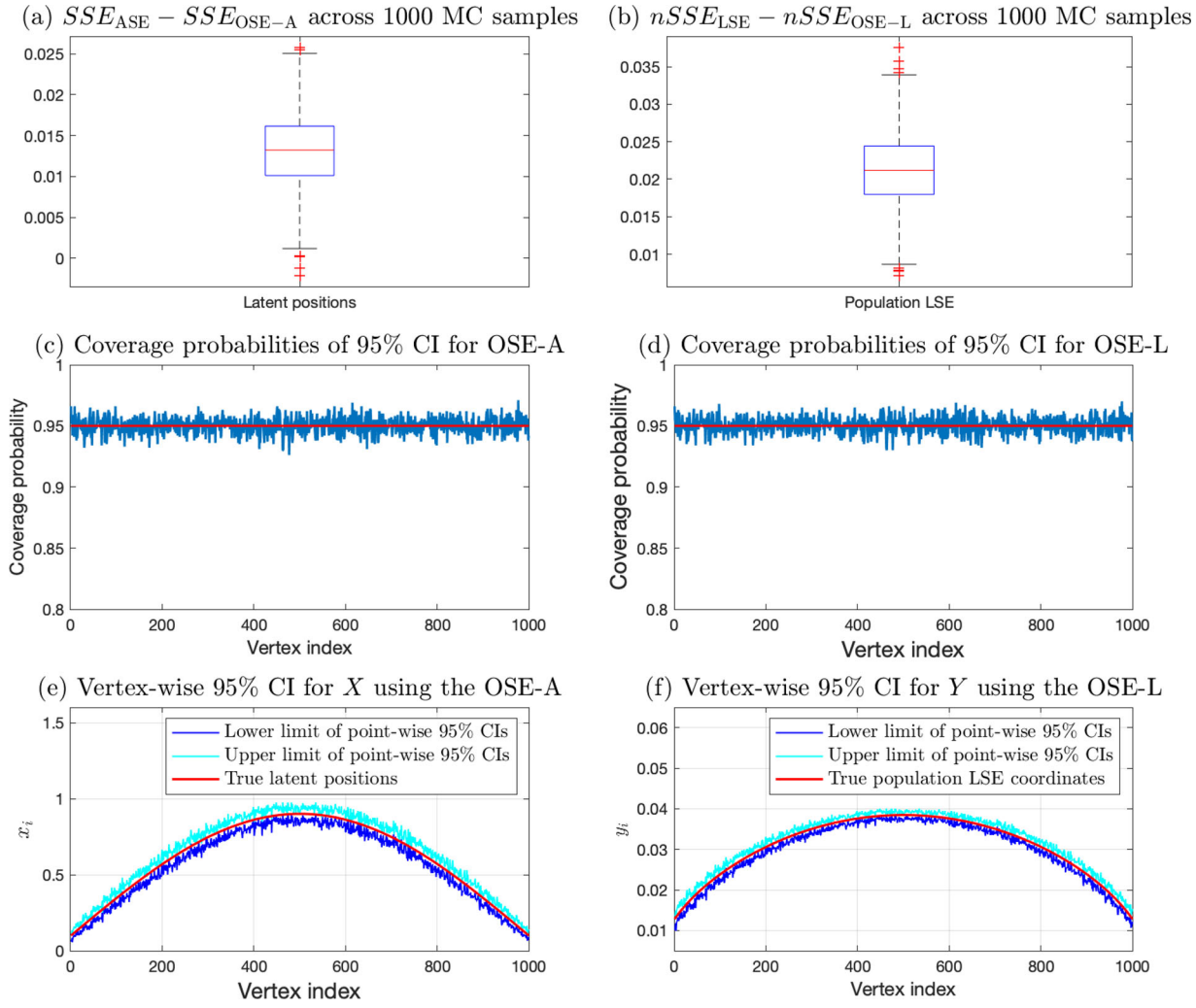

Figure 2. Numerical results for Subsection 5.1: Panels (a) and (b) are the boxplots of $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{OSE-A}}$ and $\mathrm{nSSE}_{\mathrm{LSE}} - \mathrm{nSSE}_{\mathrm{OSE-L}}$ across 1000 Monte Carlo replicates, respectively; Panels (c) and (d) are the coverage probabilities of the vertex-wise confidence intervals for the latent positions $\mathbf{X}_0 = [x_{01}, \ldots, x_{0n}]^{\mathrm{T}}$ and the population LSE $\mathbf{Y}_0 = [y_{01}, \ldots, y_{0n}]^{\mathrm{T}}$, respectively, and the red horizontal lines correspond to the nominal 95% coverage probability; Panels (e) and (f) are the realizations of the vertex-wise 95% confidence intervals for $\mathbf{X}_0 = [x_{01}, \ldots, x_{0n}]^{\mathrm{T}}$ and $\mathbf{Y}_0 = [y_{01}, \ldots, y_{0n}]^{\mathrm{T}}$ from a single draw of $\mathbf{A}$, respectively.

Table 1. Power comparison of T_{ASE} and T_{OSE} for Section 5.1.

ϵ	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
Power of T_{ASE}	0.111	0.097	0.109	0.096	0.122	0.156	0.181	0.288	0.428	0.613
Power of T_{OSE}	0.154	0.137	0.156	0.157	0.208	0.261	0.317	0.437	0.582	0.744

5.2. Comparison With the Method of Maximum Likelihood

This subsection aims at comparing the proposed one-step procedure with the ASE, and a local MLE for the random dot product graph. Although neither the existence nor the uniqueness of the MLE for the random dot product graph has been established, it is always possible to compute a local maximizer of the log-likelihood function using optimization algorithms. We first provide a simple block-coordinate descent method for finding a local maximizer of the log-likelihood function and then implement the algorithm in two concrete simulated examples. The goal is to compare the performance of the resulting estimate with the ASE and the OSE in terms of both the SSEs and the computation time.

Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$ with sparsity factor $\rho_n = 1$ and let $\ell_{\mathbf{A}}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ denote the log-likelihood function of $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}}$. A potential local maximizer of $\ell_{\mathbf{A}}$ can be found using the block-coordinate ascent algorithm in Algorithm 2. Note that within each iteration, Algorithm 2 requires an exact line search along each $\mathbf{x}_i$ direction for all $i = 1, \ldots, n$. This step can be implemented using the Matlab function `fmincon` conveniently.

Algorithm 2 Block-coordinate ascent maximum likelihood

1: **Input:** The adjacency matrix $\mathbf{A} = [A_{ij}]_{n\times n}$ and the embedding dimension d.
2: **Step 1:** Compute the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$
3: **Step 2:** Initialize $\widehat{\mathbf{X}}^{(0)} = \widehat{\mathbf{X}}^{(\mathrm{ASE})}$ and set $t = 0$.
4: **Step 3: While** not converged
5: **For** $i = 1, 2, \ldots, n$

$$\widehat{\mathbf{x}}_i^{(t+1)} \longleftarrow \arg\max_{\mathbf{x}_i} \ell_{\mathbf{A}}\left(\widehat{\mathbf{x}}_1^{(t+1)}, \ldots, \widehat{\mathbf{x}}_{i-1}^{(t+1)}, \mathbf{x}_i, \widehat{\mathbf{x}}_{i+1}^{(t)}, \ldots, \widehat{\mathbf{x}}_n^{(t)}\right).$$

End For
6: **Set** $t \longleftarrow t + 1$.
7: **End While**
8: **Output:** $\widehat{\mathbf{X}}^{(t)} = [\widehat{\mathbf{x}}_1^{(t)}, \ldots, \widehat{\mathbf{x}}_n^{(t)}]^{\mathrm{T}}$.

We next implement Algorithm 2 to Example 1 with $p = 0.6$, $q = 0.4$, $F(\mathrm{d}x) = 0.6\delta_p(\mathrm{d}x) + 0.4\delta_q(\mathrm{d}x)$, and $n = 300$. The same experiment is repeated for 1000 independent Monte Carlo replicates. We report the computation times for the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$, the OSE initialized at the ASE $\widehat{\mathbf{X}}$, and the local maximum likelihood estimate (MLE) $\widehat{\mathbf{X}}^{(\mathrm{MLE})}$ for a single realization in Table 2.

We also compute the SSEs of the three estimates: $\mathrm{SSE}_{\mathrm{ASE}} = \min_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$, $\mathrm{SSE}_{\mathrm{OSE}} = \min_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$, and $\mathrm{SSE}_{\mathrm{MLE}} = \min_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}^{(\mathrm{MLE})}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$. The average SSEs of these estimates across 1000 Monte Carlo replicates are tabulated in Table 2, together with the standard errors. Figure 3 visualizes $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{OSE}}$, $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{MLE}}$, and $\mathrm{SSE}_{\mathrm{OSE}} - \mathrm{SSE}_{\mathrm{MLE}}$ in the three panels, respectively. In particular, $\mathrm{SSE}_{\mathrm{OSE}} - \mathrm{SSE}_{\mathrm{MLE}}$ is mild in this example in contrast to $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{OSE}}$. These numerical results suggest that the improvement from the ASE to the OSE is more significant than the improvement from the OSE to the MLE in terms of the SSEs, whereas the computation cost of the MLE is much higher than that of the OSE and that of the ASE.

We finally consider a simulated example with a comparatively small sample size, which sheds some light to future research direction concerning the method of maximum likelihood in finite-sample problems. The setup is similar to the example in Subsection 5.1. Namely, we consider a 1-dimensional random dot product graph whose latent positions are given by $x_{0i} = 0.8\sin\{\pi(i-1)/(n-1)\} + 0.1$, $i \in [n]$. The number of vertices n is set to 30, and we generate an adjacency matrix $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$, where $\mathbf{X}_0 = [x_{01}, \ldots, x_{0n}]^{\mathrm{T}}$. We compute the ASE, the proposed OSE, and a local MLE using Algorithm 2. We repeat the experiment for 1000 independent Monte Carlo replicates. The computation times for obtaining the ASE, the OSE, and the MLE for a single realization are reported in Table 3, together with the average SSEs and the corresponding standard

Table 2. Computation time and error comparison for Section 5.2: The latent positions are set as in Example 1 with $p = 0.6, q = 0.4$, and the number of vertices is $n = 300$.

Method	ASE	OSE-A	MLE
Computation time in `Matlab`	0.008057 sec	0.01882 sec	26.1634 sec
$\mathrm{SSE} = \inf_{\mathbf{W}\in\{\pm1\}} \|\widehat{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$	0.7178	0.7040	0.7030
Standard error for SSE	0.0019	0.0018	0.0018

Table 3. Computation time and error comparison for Section 5.2: The latent positions are set to $x_{0i} = 0.8\sin\{\pi(i-1)/(n-1)\} + 0.1, i = 1, \ldots, 30$.

Method	ASE	OSE-A	MLE
Computation time in `Matlab`	0.004213 sec	0.001644 sec	0.209708 sec
$\mathrm{SSE} = \|\widehat{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_{\mathrm{F}}^2$	0.4449	0.6260	0.4062
Standard error for SSE	0.0041	0.0079	0.0039

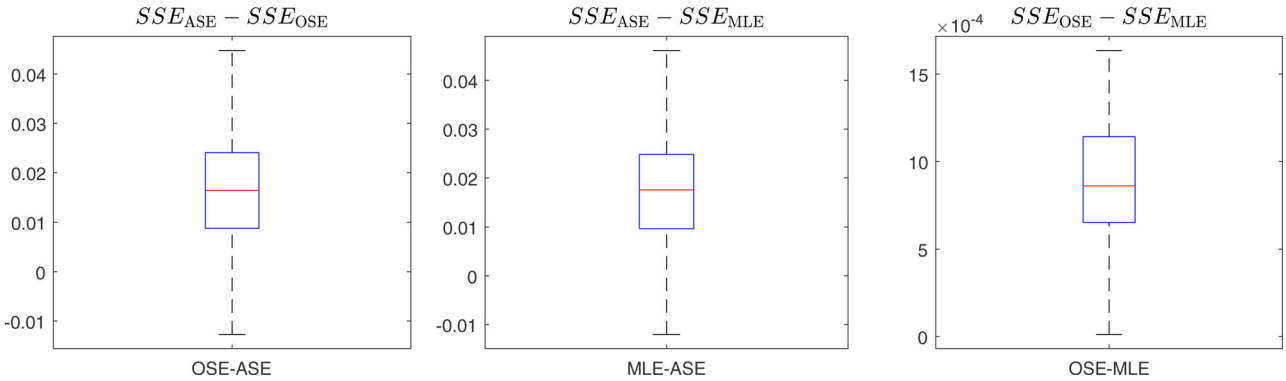

Figure 3. Numerical results for Section 5.2: The boxplots of $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{OSE}}$, $\mathrm{SSE}_{\mathrm{ASE}} - \mathrm{SSE}_{\mathrm{MLE}}$, and $\mathrm{SSE}_{\mathrm{OSE}} - \mathrm{SSE}_{\mathrm{MLE}}$ across 1000 Monte Carlo replicates; The latent positions are set as in Example 1 with $p = 0.6, q = 0.4$, and the number of vertices is $n = 300$.

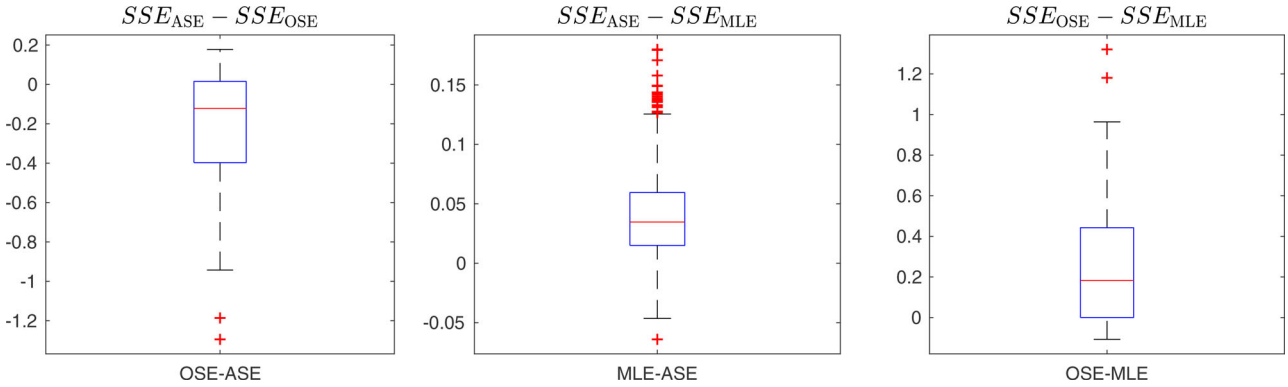

Figure 4. Numerical results for Section 5.2: The boxplots of $SSE_{ASE} - SSE_{OSE}$, $SSE_{ASE} - SSE_{MLE}$, and $SSE_{OSE} - SSE_{MLE}$ across 1000 Monte Carlo replicates; The latent positions are set as $x_{0i} = 0.8 \sin\{\pi(i-1)/(n-1)\} + 0.1, i = 1, \ldots, 30$.

errors across 1000 Monte Carlo replicates. We also visualize $SSE_{ASE} - SSE_{OSE}$, $SSE_{ASE} - SSE_{MLE}$, and $SSE_{OSE} - SSE_{MLE}$ in the three panels of Figure 4, respectively. Observe that in this example, with a relatively small number of vertices $n = 30$, the OSE does not provide improvement over the ASE, whereas the MLE shows significant improvement over the ASE as well as the OSE. The practical performance of the MLE for finite-sample problems is also inspiring for designing a multiple-step procedure that repeats the one-step update multiple times for finding a local MLE. This interesting direction is deferred to future work. Another implication of this experiment is that the practitioners are not recommended to apply the one-step procedure for network data with comparatively small vertices. Instead, it is recommended that a local MLE is used over the one-step estimate and the ASE.

5.3. Wikipedia Graph Data

We finally apply the proposed one-step procedure to a real-world Wikipedia graph dataset, which is available at *http://www.cis.jhu.edu/~parky/Data/data.html*. The Wikipedia graph dataset consists of an adjacency matrix among $n = 1382$ Wikipedia articles that are within two hyperlinks of the article "Algebraic Geometry," and these articles are further manually labeled according to one of the following 6 descriptions: people, places, dates, things, math, and category. To determine a suitable embedding dimension d for the random dot product graph model, we follow the ad hoc approach of Zhu and Ghodsi (2006) and computes

$$\widehat{d} = \arg\max_{d=1,2,\ldots,q} \left\{ \sum_{k=1}^{d} \log f(\sigma_k(\mathbf{A}); \widehat{\mu}_1, \widehat{\sigma}^2) + \sum_{k=d+1}^{q} \log f(\sigma_k(\mathbf{A}); \widehat{\mu}_2, \widehat{\sigma}^2) \right\},$$

where $f(x; \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left\{-(x-\mu)^2/(2\sigma^2)\right\}$ is the normal density with mean μ and variance σ^2, $\mu_1 = \frac{1}{d}\sum_{k=1}^{d} \sigma_k(\mathbf{A})$, $\mu_2 = \frac{1}{p-d}\sum_{k=d+1}^{p} \sigma_k(\mathbf{A})$, $\widehat{\sigma}^2 = \frac{(d-1)s_1^2 + (p-d-1)s_2^2}{p-2}$, s_1^2, s_2^2 are the sample variances of $\{\sigma_k(\mathbf{A})\}_{k=1}^{d}$ and $\{\sigma_k(\mathbf{A})\}_{k=d+1}^{q}$, respectively, and q is an upper bound for the embedding dimension. Here, we select $q = 50$ as a conservative upper bound, resulting in $\widehat{d} = 11$.

Table 4. Wikipedia Graph Data: Rand indices of the GMM-based clustering algorithm applied to the ASE, the LSE, the OSE-A, and the OSE-L, respectively, with the number of clusters being 6, in comparison with the corresponding manual labels.

Method	ASE	LSE	OSE-A	OSE-L
Rand Index	0.7429	0.7350	0.7413	**0.7538**

Table 5. Wikipedia Graph Data: Rand indices of the GMM-based clustering algorithm applied to the ASE, the LSE, the OSE-A, and the OSE-L, respectively, with the number of clusters being 2, in comparison with the corresponding one-versus-all manual labels for the class "Dates".

Method	ASE	LSE	OSE-A	OSE-L
Rand Index	0.5289	0.5097	**0.5432**	0.5313

We next compute the ASE, the LSE, the OSE-A, and the OSE-L, with the embedding dimension $d = 11$, and then apply the GMM-based clustering algorithm to these estimates, with the number of clusters being 6. We next compare the similarity between the manually assigned 6 class labels and these clustering results by computing the respective Rand indices, which are tabulated in Table 4. The results show that the one-step procedure for the population LSE outperforms the rest of the competitors, as it provides the clustering result that is most similar to the original class label assignment among the four methods.

Besides evaluating the performance of the overall clustering for the 6 manually assigned labels, we also focus on the comparison of the article class "Dates" against the rest of the articles specifically. We apply the GMM-based clustering algorithm to the aforementioned four estimates again, but with the number of clusters being 2, and tabulate the Rand indices in Table 5. We can see that the proposed one-step procedure improves the clustering accuracy as well when we focus on the comparison between the article class "Dates" against the rest of the labels. The scatterplots of the first-versus-second dimension of the four estimates are visualized in Figure 5, along with the cluster-specific 95% empirical confidence ellipses in dashed lines.

6. Discussion

In the context of stochastic block models, Gao et al. (2017) proposed a vertex clustering approach that improves the solution

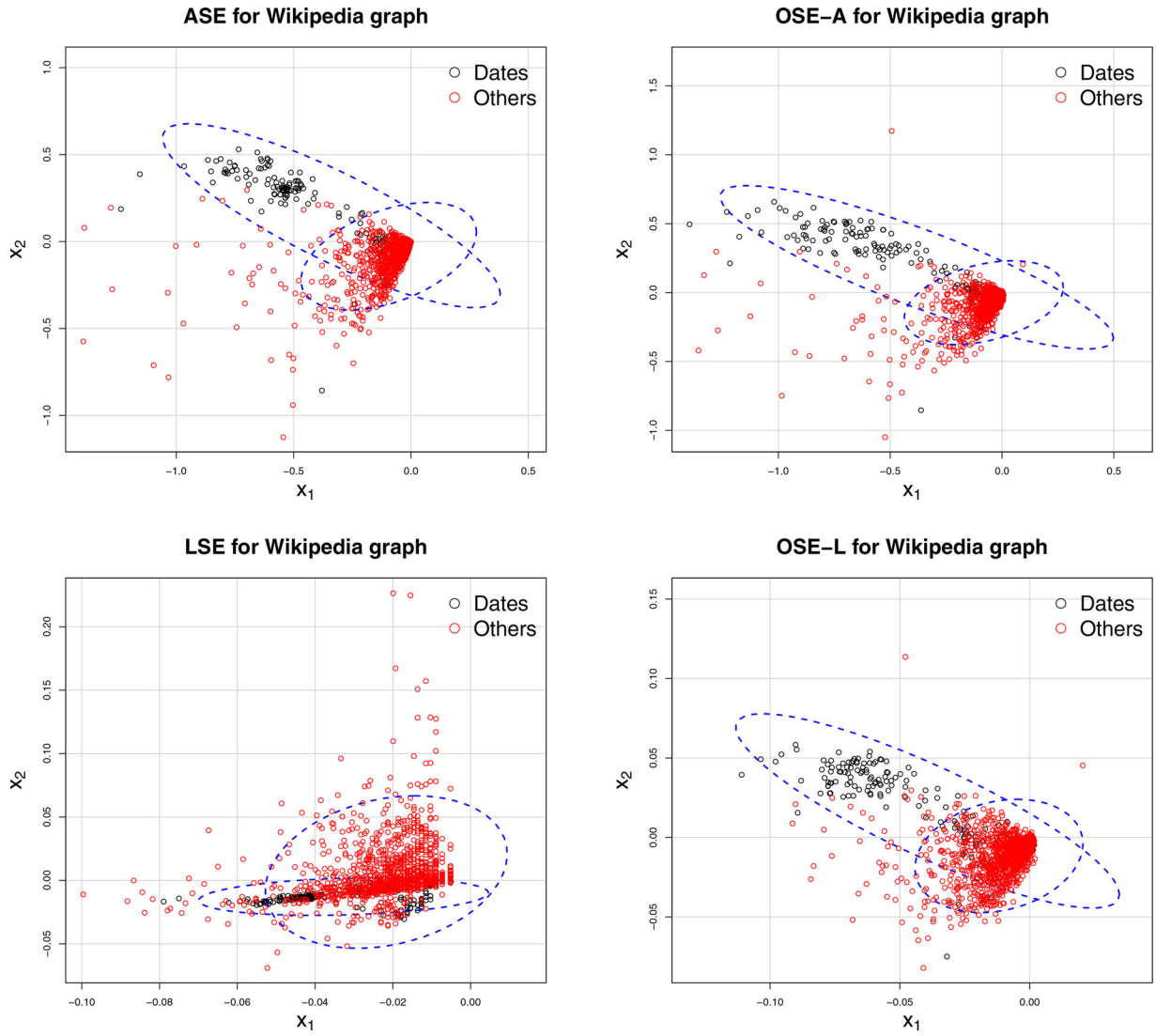

Figure 5. Wikipedia graph data: The scatterplots of the first-versus-second dimension of the four estimates. The scatter points are colored according to whether the articles are in the class "Dates" or the others. The 95% empirical cluster-specific confidence ellipses are displayed by the dashed lines.

provided by the ASE and/or the LSE. The algorithm in Gao et al. (2017) starts from the clustering solution of the ASE/LSE and then refines the cluster assignment of each vertex through the maximization of a penalized Bernoulli likelihood function, where the cluster memberships of the rest of the vertices are fixed at their most recent values. This approach is similar to our one-step procedure for estimating the latent positions in spirit, as both methods are implemented in a vertex-by-vertex optimization fashion with a warm start solution (i.e., the ASE/LSE or the cluster assignment given by them). Our method differs from the method of Gao et al. (2017) in that the proposed one-step procedure aims at maximizing the Bernoulli likelihood function with regard to the continuous-valued latent positions and takes the gradient information of the likelihood function into account, whereas Gao et al. (2017) focused on estimating cluster memberships of vertices, and no gradient information is available due to the discrete nature of the variables of interest.

We assume that the embedding dimension d for the random dot product graph is known throughout the article. The proposed one-step procedure is also valid when the true dimension d for the underlying sampling model is unknown. In this case, the method proceeds by first finding the ASE into $\mathbb{R}^{d'}$ for some $d' \geq 1$ and $d' < d$ (i.e., when the dimension is under-estimated) and then computing the OSE based on d'. Our Theorems 5 and 9 still hold and can be easily proved as suggested by Tang and Priebe (2018). On the other hand, leveraging Bayesian methods when the dimension d is unknown is a promising future direction in light of the recent progress in Bayesian theory and methods for low-rank matrix models with undetermined rank (Bhattacharya and Dunson 2011; Rocková and George 2016) and network models (Caron and Fox 2017; Xie and Xu 2019; Geng, Bhattacharya, and Pati 2019).

We have shown that the one-step procedure produces an estimator enjoying fascinating asymptotic properties both globally for all vertices and locally for each vertex. Nevertheless, for problems with comparatively small sample sizes, we have also shown in a simulation example that the OSEs do not necessarily provide us with better numerical results compared to the classical adjacency/Laplacian spectral embedding. Instead, we have also observed that the method of maximum likelihood, which

is implemented in a block-coordinate ascent algorithm, provides practical improvement over the ASE. Since the one-step procedure only implements a single iteration of the Newton–Raphson algorithm with the observed Hessian matrix replaced by the negative Fisher information matrix, we hope to develop an iterative algorithm for finding a local MLE by repeating the one-step procedure multiple times until convergence. Such an iterative algorithm can be implemented in conjunction with the regularization of the Fisher information matrix and backtracking procedure for finding suitable step sizes to achieve faster convergence (Nocedal and Wright 2006). Furthermore, developing a scalable version of such an algorithm will be highly desirable in the presence of big data and extremely large networks. It will also be useful to explore the statistical properties of the estimator obtained by the iterative algorithm, and to establish its theoretical guarantee. We defer these research topics to future work.

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Supplementary Material

The supplementary material contains a comprehensive list of notations, the proofs of the technical results in Sections 2, Section 3, and Section 4, the behavior of the OSE for positive-definite stochastic block models, further discussion regarding sparse graph models, and additional simulated examples.

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