AGTA - Coursework 1

Exam number: B122217

February 2021

1 Question 2

1.1 Question 2a

Consider the 2-player zero-sum game given by the following payoff matrix for player 1 (the row player):

$$\begin{bmatrix} 1 & 2 & 7 & 2 & 4 \\ 0 & 0 & 9 & 6 & 2 \\ 7 & 9 & 4 & 5 & 3 \\ 1 & 4 & 0 & 7 & 9 \\ 9 & 7 & 3 & 8 & 3 \end{bmatrix}$$

Compute both the minimax value for this game, as well as a minimax profile (NE), i.e., "optimal" (i.e., minmaximizer and maxminimizer) strategies for both players 1 and 2, respectively.

To compute this we need to define the following LP problem:

$$\begin{aligned}
&\textbf{Maximize } v \\
&\textbf{Subject to}: \\
&\qquad x_1 + 0x_2 + 7x_3 + 1x_4 + 9x_5 \geq v \\
&\qquad 2x_1 + 0x_2 + 9x_3 + 4x_4 + 7x_5 \geq v \\
&\qquad 7x_1 + 9x_2 + 4x_3 + 0x_4 + 3x_5 \geq v \\
&\qquad 2x_1 + 6x_2 + 5x_3 + 7x_4 + 8x_5 \geq v \\
&\qquad 4x_1 + 2x_2 + 3x_3 + 9x_4 + 3x_5 \geq v \\
&\qquad x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\
&\qquad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0
\end{aligned}$$

Which is equal to:

$$\begin{aligned}
&\textbf{Maximize } v \\
&\textbf{Subject to}: \\
&\qquad v - x_1 - 7x_3 - x_4 - 9x_5 \leq 0 \\
&\qquad v - 2x_1 - 9x_3 - 4x_4 - 7x_5 \leq 0 \\
&\qquad v - 7x_1 - 9x_2 - 4x_3 - 3x_5 \leq 0 \\
&\qquad v - 2x_1 - 6x_2 - 5x_3 - 7x_4 - 8x_5 \leq 0 \\
&\qquad v - 4x_1 - 2x_2 - 3x_3 - 9x_4 - 3x_5 \leq 0 \\
&\qquad x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\
&\qquad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0
\end{aligned}$$

In primal form this is equal to:

$$\begin{aligned}
&\textbf{Maximize } c^T x \\
&\textbf{Subject to}: \\
&\qquad (Bx)_i \leq b_i \text{ for } i = 1,2,3,4,5 \\
&\qquad (Bx)_6 = b_6 \\
&\qquad x_i \geq 0 \text{ for } i = 1,2,3,4,5
\end{aligned}$$

Where we set $c^T = (0,0,0,0,0,1)$, $x^T = (x_1, x_2, x_3, x_4, x_5, v)$, $b^T = (0,0,0,0,0,1)$ and

$$B = \begin{bmatrix} -1 & 0 & -7 & -1 & -9 & 1 \\ -2 & 0 & -9 & -4 & -7 & 1 \\ -7 & -9 & -4 & 0 & -3 & 1 \\ -2 & -6 & -5 & -7 & -8 & 1 \\ -4 & -2 & -3 & -9 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Therefore by using the LP duality therorem we can get the LP problem to find the maxminimizer strategy profile that is the strategy for player 2 we need to use the LP duality theorem, to find the dual LP:

$$\begin{aligned}
&\textbf{Minimize } b^T y \\
&\textbf{Subject to}: \\
&\qquad (B^T y)_i \leq c_j, \text{ for } j = 1,2,3,4,5 \\
&\qquad (B^T y)_6 = c_6 \\
&\qquad y_j \geq 0, \text{ for } j = 1,2,3,4,5
\end{aligned}$$

Where in this case $y^T = (y_1, y_2, y_3, y_4, y_5, v)$

When using Matlab to solve the both of the dual and primal LP. Note that in both of the LP problems $b^T y = c^T x$ by Minimax Theorem.

- **The minimax value for this game** v: $\frac{177}{41}$

- **The minimax profile (NE):**

$$x_1^* = (\frac{18}{41}, 0, 0, \frac{6}{41}, \frac{17}{41})$$
$$x_2^* = (\frac{9}{41}, 0, \frac{40}{123}, 0, \frac{56}{123})$$

1.2 Question 2b

Recall from Lecture 7 on LP duality, the symmetric 2- player zero-sum game, G, for which the (skew-symmetric) payoff matrix (in block form) for player 1 is:

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

Suppose that there exist vectors $x\prime \in R^n$ and $y \in R^m$, such that $Ax\prime < b$, $x\prime \geq 0$, $A^T y\prime > c$ and $y\prime \geq 0$ (Note the two strict inequalities.) Prove that for the game G, every minmaximizer strategy $w = (y^*, x^*, z)$ for player 1 (and hence also every maxminimizer strategy for player 2, since the game is symmetric) has the property that $z > 0$, i.e., the last pure strategy is played with positive probability.

Note that the value of any symmetric 2-player zero-sum game must be equal to zero. This implies, by the minimax theorem, that $Bw \leq 0$. Suppose, for contradiction, that $z = 0$. Then we have the following:

$$Bw = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \begin{bmatrix} y^* \\ x^* \\ 0 \end{bmatrix} = \begin{bmatrix} Ax^* \\ -A^T y^* \\ b^T y^* - c^T x^* \end{bmatrix} \leq 0$$

Therefore we have the following following inequalities:

- $Ax^* \leq 0$
- $-A^T y^* \leq 0$ therefore $A^T y^* \geq 0$
- $b^T y^* - c^T x^* \leq 0$ therefore $b^T y^* \leq c^T x^*$

Now suppose $y^* \neq 0$, then we have that $(y^*)^T (Ax\prime - b) < 0$ since $Ax\prime < b$ and $y^* > 0$.

We can notice that by weak duality we must have that $c^T x^* \leq b^T y^*$, therefore putting this together with $b^T y^* \leq c^T x^*$ we have that $b^T y^* \neq 0$ as a result this must mean that $c^T x^* \neq 0$, so we have that $x^* \neq 0$. Then it follows that $(x^*)^T (A^T y\prime - c) > 0$ since we are given that $A^T y\prime > c$.

We can can put all of this together, and using the fact that $c^T x^* = b^T y^*$ since by Strong duality they must be the same since they are optimal solutions:

$$
\begin{align}
(y^*)^T(Ax\prime - b) < 0 < (x^*)^T(A^T y\prime - c) \tag{1}\\
(y^*)^T(Ax\prime - b) < (x^*)^T(A^T y\prime - c) \tag{2}\\
(y^*)^T Ax\prime - (y^*)^T b < (x^*)^T A^T y\prime - (x^*)^T c \tag{3}\\
(y^*)^T Ax\prime < (x^*)^T A^T y\prime - (x^*)^T c + (y^*)^T b \tag{4}\\
(y^*)^T Ax\prime < (x^*)^T A^T y\prime - 0 \tag{5}\\
(y^*)^T Ax\prime < ((x^*)^T A^T y\prime)^T \tag{6}\\
(y^*)^T Ax\prime < (y\prime)^T Ax^* \tag{7}
\end{align}
$$

However in the last step we see that $(y^*)^T Ax\prime < (y\prime)^T Ax^*$ is a contradiction since we have that y^* is an optimal response. That means there must be **some** $x\prime$ and $y\prime$ such that this equality holds we know this by the Minimax Theorem, we also know that $x\prime$ and $y\prime$ are feasible solutions to the problem, but since they are strictly less than the constant we will never have equality. As a result we that shown that $z \neq 0$ since we have found a contradiction.