AGTA - Coursework 2

Exam number: B122217

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1 Question 4

1.1 Question 4a

Give a VCG outcome for this auction. In other words, specify, in that VCG outcome, which bidders will get which of the painting(s), and what price they will each pay for the painting(s) they get. Justify your answer, and show your calculations.

We assume that the bidders bid their true valuations (which we can, since in the VCG mechanism bidding the true valuations is a dominant strategy for all bidders). Then given these valuations, the VCG mechanism firstly picks an outcome that maximizes the sum total valuation of all the bidders, i.e., maximizes the total social welfare of the outcome. This is given by:

$$c\prime \in arg\ max_{c \in C} \sum_{k \in V} v_k(c)$$

Where we have a set $V = \{1, ..., n\}$ of n agents declare their valuation functions, $v_k : C \rightarrow \mathbb{R}_{\geq 0}, i \in V$, over the set C of possible outcomes.

To find the outcomes that maximise this function we fix the outcome for player S and so we just need to find the combinations of the remaining outcomes for players L and B such that they maximise the sum total valuation. Below are the result, when we fix the outcomes for player S. I will now explain how I have found the following sum total valuation and the associated outcome.

Player S	**Sum total Valuation**	**Outcome**
$\{\emptyset\}$	**55**	$S = \{\emptyset\}, L = \{T_1, T_3\}, B = \{T_2\}$
$\{T_1\}$	**55**	$S = \{T_1\}, L = \{\emptyset\}, B = \{T_2, T_3\}$
$\{T_2\}$	53	$S = \{T_2\}, L = \{T_1, T_3\}, B = \{\emptyset\}$
$\{T_3\}$	51	$S = \{T_3\}, L = \{T_1, T_2\}, B = \{\emptyset\}$
$\{T_1, T_2\}$	41	$S = \{T_1, T_2\}, L = \{T_3\}, B = \{\emptyset\}$
$\{T_1, T_3\}$	54	$S = \{T_1, T_3\}, L = \{\emptyset\}, B = \{T_2\}$
$\{T_2, T_3\}$	39	$S = \{T_2, T_3\}, L = \{\emptyset\}, B = \{T_1\}$
$\{T_1, T_2, T_3\}$	54	$S = \{T_1, T_2, T_3\}, L = \{\emptyset\}, B = \{\emptyset\}$

1. **Player S** outcome is $\{\emptyset\}$.
   If this is the case then we know that Players L and B must have between them a combination of all the outcomes $\{T_1, T_2, T_3\}$:

   - Player L outcome: $\emptyset$ and Players B outcome: T_1, T_2, T_3.
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{\emptyset\}) + v_B(\{T_1, T_2, T_3\}) = 0 + 0 + 54 = 54$$
   - Player B outcome: $\emptyset$ and Players L outcome: T_1, T_2, T_3
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_1, T_2, T_3\}) + v_B(\{\emptyset\}) = 0 + 0 + 53 = 53$$
   - Player L outcome: T_1 and Players B outcome: T_2, T_3
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_1\}) + v_B(\{T_2, T_3\}) = 0 + 4 + 39 = 43$$
   - Player L outcome: T_2 and Players B outcome: T_1, T_3
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_2\}) + v_B(\{T_1, T_3\}) = 0 + 7 + 28 = 35$$
   - Player L outcome: T_3 and Players B outcome: T_1, T_2
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_3\}) + v_B(\{T_2, T_3\}) = 0 + 12 + 39 = 51$$
   - Player B outcome: T_1 and Players L outcome: T_2, T_3
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_2, T_3\}) + v_B(\{T_1\}) = 0 + 37 + 10 = 47$$
   - Player B outcome: T_2 and Players L outcome: T_1, T_3
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_1, T_3\}) + v_B(\{T_2\}) = 0 + 37 + 18 = 55$$
   - Player B outcome: T_3 and Players L outcome: T_1, T_2
     The sum total valuation is then given by:
     $$v_S(\{\emptyset\}) + v_L(\{T_1, T_2\}) + v_B(\{T_3\}) = 0 + 38 + 4 = 42$$

   As a result the outcome that maximises the sum total valuation when Player S gets $\{\emptyset\}$, is when Player B outcome: $\{T_2\}$ and Players L outcome: $\{T_1\}$, $\{T_3\}$ where the valuation is 55. Thus, this is the entry in the first row.

2. **Player S** outcome is $\{T_1\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_2, T_3\}$:

- Player L outcome: $\emptyset$ and Player B outcome: T_2, T_3.
  The sum total valuation is then given by:

$$v_S(\{T_1\}) + v_L(\{\emptyset\}) + v_B(\{T_2, T_3\}) = 16 + 0 + 39 = 55$$

- Player B outcome: $\emptyset$ and Player L outcome: T_2, T_3.
  The sum total valuation is then given by:

$$v_S(\{T_1\}) + v_L(\{T_2, T_3\}) + v_B(\{\emptyset\}) = 16 + 37 + 0 = 53$$

- Player L outcome: T_2 and Player B outcome: T_3.
  The sum total valuation is then given by:

$$v_S(\{T_1\}) + v_L(\{T_2\}) + v_B(\{T_3\}) = 16 + 7 + 4 = 27$$

- Player L outcome: T_3 and Players B outcome: T_2.
  The sum total valuation is then given by:

$$v_S(\{T_1\}) + v_L(\{T_3\}) + v_B(\{T_2\}) = 16 + 12 + 18 = 46$$

As a result the outcome that maximises the sum total valuation when Player S gets $\{T_1\}$, is when Player B outcome: $\{T_2, T_3\}$ and Players L outcome: $\{\emptyset\}$ where the valuation is 55. Thus, this is the entry in the second row.

3. **Player S** outcome is $\{T_2\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_1$, $T_3\}$:

- Player L outcome: $\emptyset$ and Player B outcome: T_1, T_3.
  The sum total valuation is then given by:

$$v_S(\{T_2\}) + v_L(\{\emptyset\}) + v_B(\{T_1, T_3\}) = 16 + 0 + 28 = 44$$

- Player B outcome: $\emptyset$ and Player L outcome: T_1, T_3.
  The sum total valuation is then given by:

$$v_S(\{T_2\}) + v_L(\{T_1, T_3\}) + v_B(\{\emptyset\}) = 16 + 37 + 0 = 53$$

- Player L outcome: T_1 and Player B outcome: T_3.
  The sum total valuation is then given by:

$$v_S(\{T_2\}) + v_L(\{T_1\}) + v_B(\{T_3\}) = 16 + 4 + 4 = 24$$

- Player L outcome: T_3 and Players B outcome: T_1.
  The sum total valuation is then given by:

$$v_S(\{T_2\}) + v_L(\{T_3\}) + v_B(\{T_1\}) = 16 + 12 + 10 = 38$$

As a result the outcome that maximises the sum total valuation when Player S gets $\{T_2\}$, is when Player L outcome: $\{T_1, T_3\}$ and Players B outcome: $\{\emptyset\}$ where the valuation is 53. Thus, this is the entry in the third row.

4. **Player S** outcome is $\{T_3\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_1, T_2\}$:

   - Player L outcome: $\emptyset$ and Player B outcome: T_1, T_2.
     The sum total valuation is then given by:
     $$v_S(\{T_3\}) + v_L(\{\emptyset\}) + v_B(\{T_1, T_2\}) = 13 + 0 + 28 = 44$$
   - Player B outcome: $\emptyset$ and Player L outcome: T_1, T_2.
     The sum total valuation is then given by:
     $$v_S(\{T_3\}) + v_L(\{T_1, T_2\}) + v_B(\{\emptyset\}) = 13 + 38 + 0 = 51$$
   - Player L outcome: T_1 and Player B outcome: T_2.
     The sum total valuation is then given by:
     $$v_S(\{T_3\}) + v_L(\{T_1\}) + v_B(\{T_2\}) = 13 + 4 + 18 = 35$$
   - Player L outcome: T_2 and Players B outcome: T_1.
     The sum total valuation is then given by:
     $$v_S(\{T_3\}) + v_L(\{T_2\}) + v_B(\{T_1\}) = 13 + 7 + 4 = 24$$

   As a result the outcome that maximises the sum total valuation when Player S gets $\{T_3\}$, is when Player L outcome: $\{T_1, T_2\}$ and Players B outcome: $\{\emptyset\}$ where the valuation is 51. Thus, this is the entry in the fourth row.

5. **Player S** outcome is $\{T_1, T_2\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_3\}$:

   - Player L outcome: $\emptyset$ and Player B outcome: T_3.
     The sum total valuation is then given by:
     $$v_S(\{T_1, T_2\}) + v_L(\{\emptyset\}) + v_B(\{T_3\}) = 29 + 0 + 4 = 33$$
   - Player B outcome: $\emptyset$ and Player L outcome: T_3.
     The sum total valuation is then given by:
     $$v_S(\{T_1, T_2\}) + v_L(\{T_3\}) + v_B(\{\emptyset\}) = 29 + 12 + 0 = 41$$

   As a result the outcome that maximises the sum total valuation when Player S gets $\{T_1, T_2\}$, is when Player L outcome: $\{T_3\}$ and Players B outcome: $\{\emptyset\}$ where the valuation is 41. Thus, this is the entry in the fifth row.

6. **Player S** outcome is $\{T_1, T_3\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_2\}$:

   - Player L outcome: $\emptyset$ and Player B outcome: T_2.
     The sum total valuation is then given by:
     $$v_S(\{T_1, T_3\}) + v_L(\{\emptyset\}) + v_B(\{T_2\}) = 36 + 0 + 18 = 54$$

- Player B outcome: $\emptyset$ and Player L outcome: T_2.
  The sum total valuation is then given by:
  $$v_S(\{T_1, T_3\}) + v_L(\{T_2\}) + v_B(\{\emptyset\}) = 36 + 7 + 0 = 43$$

As a result the outcome that maximises the sum total valuation when Player S gets $\{T_1, T_3\}$, is when Player L outcome: $\{\emptyset\}$ and Players B outcome: $\{T_2\}$ where the valuation is 41. Thus, this is the entry in the sixth row.

7. **Player S** outcome is $\{T_2, T_3\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{T_1\}$:

   - Player L outcome: $\emptyset$ and Player B outcome: T_1.
     The sum total valuation is then given by:
     $$v_S(\{T_2, T_3\}) + v_L(\{\emptyset\}) + v_B(\{T_1\}) = 29 + 0 + 10 = 39$$
   - Player B outcome: $\emptyset$ and Player L outcome: T_1.
     The sum total valuation is then given by:
     $$v_S(\{T_1, T_2\}) + v_L(\{T_1\}) + v_B(\{\emptyset\}) = 29 + 4 + 0 = 33$$

   As a result the outcome that maximises the sum total valuation when Player S gets $\{T_2, T_3\}$, is when Player L outcome: $\{\emptyset\}$ and Players B outcome: $\{T_1\}$ where the valuation is 39. Thus, this is the entry in the seventh row.

8. **Player S** outcome is $\{T_1, T_2, T_3\}$.
   Then we know that in this case we have that players L and B must have as an outcome some combination of $\{\emptyset\}$ and so the sum total valuation is unique and is equal to
   $$v_S(\{T_1, T_2, T_3\}) + v_L(\{\emptyset\}) + v_B(\{\emptyset\}) = 54 + 0 + 0 = 0$$

As a result we have found the allocations to each of the players, if player S gets allocated items $\{\emptyset\}$ and player L gets allocated $\{T_1, T_3\}$, and player B gets $\{T_2\}$ then the sum total valuation of this outcome is 55.

Likewise if player S gets allocated items $\{T_1\}$ and player L gets allocated $\{\emptyset\}$, and player B gets $\{T_2, T_3\}$ then the sum total valuation of this outcome **also** is 55, this is given by the table.

Furthermore, as we have checked all outcomes that provide the maximum total valuation of any outcome. Therefore, the VCG mechanism picks one of these possible outcomes, and then asks the bidders to pay the VCG payments associated with that outcome.

We know that that the VCG mechanism, as given, does not specify which of these outcomes is to be preferred.

So I will use the outcome $S = \{T_1\}, L = \{\emptyset\}, B = \{T_2, T_3\}$ to calculate the price that each player will pay for each painting. To do this we need to calculate we need to evaluate the price given by:

$$p_i(c\prime) = \left(max_{c \in C} \sum_{k \in V/\{i\}} v_k(c) \right) - \sum_{k \in V/\{i\}} v_k(c\prime)$$

Notice that for a player i the price that he will pay is given by the price paid by i under a specific VCG outcome $c\prime$, we need to calculate the maximum possible sum total valuation of any outcome for all the other bidders and subtract this from this the sum total valuation of all the other bidders under the specific VCG outcome $c\prime$ that we are considering.

The price paid for each player when the outcome $c\prime$ is given by $S = \{T_1\}, L = \{\emptyset\}, B = \{T_2, T_3\}$ is calculated by:

1. Price payed by Player S when the outcome is $c\prime$:

$$\begin{aligned}
p_S(c\prime) &= \left(max_{c \in C} \sum_{k \in V/\{S\}} v_k(c) \right) - \sum_{k \in V/\{S\}} v_k(c\prime) \\
&= (v_L(\{T_1, T_3\}) + v_B(\{T_2\})) - v_L(\{\emptyset\}) + v_B(\{T_2, T_3\} \\
&= 37 + 18 - (0 + 39) \\
&= 16
\end{aligned}$$

2. Price payed by Player L when the outcome is $c\prime$:

$$\begin{aligned}
p_L(c\prime) &= \left(max_{c \in C} \sum_{k \in V/\{L\}} v_k(c) \right) - \sum_{k \in V/\{L\}} v_k(c\prime) \\
&= (v_S(\{T_1\}) + v_B(\{T_2, T_3\})) - v_S(\{T_1\} + v_B(\{T_2, T_3\}) \\
&= 16 + 39 - (16 + 39) \\
&= 0
\end{aligned}$$

3. Price payed by Player B when the outcome is $c\prime$:

$$\begin{aligned}
p_B(c\prime) &= \left(max_{c \in C} \sum_{k \in V/\{B\}} v_k(c) \right) - \sum_{k \in V/\{B\}} v_k(c\prime) \\
&= (v_S(\{T_1, T_2, T_3\}) + v_L(\{\emptyset\})) - v_S(\{T_1\} + v_L(\{\emptyset\}) \\
&= 54 + 0 - (16 + 0) \\
&= 38
\end{aligned}$$

As a result, we have found VCG outcome for this auction. In particular we have that:

1. **Player S** will get painting T_1 and will pay a price of 16×10^5 pounds.
2. **Player L** will get no painting and as we have show will pay a price of 0 pounds.
3. **Player B** will get painting T_2, T_3 and will pay a price of 38×10^5 pounds.

1.2 Question 4b

Is the VCG outcome you have calculated in part (a) unique? Are the VCG prices paid by the player's uniquely determined? Justify your answer, and show your calculations.

As I have showed in the previous section the VCG outcome that I calculated in the previous section is **not** unique. This is because we have found another VCG outcome that has the same sum total valuation, namely 55.

In particular, the other auction that has this valuation is when $S = \{\emptyset\}, L = \{T_1, T_3\}, B = \{T_2\}$, where we can calculate the sum total valuation as:

$$v_S(\{\emptyset\}) + v_L(\{T_1, T_3\}) + v_B(\{T_2\}) = 0 + 37 + 18 = 55$$

So now we need to check if the prices paid by each player uniquely determined. To do this we need to calculate the price that each player pays under this new VCG outcome.

So the price paid for each player is where the outcome $x\prime$ is given by $S = \{\emptyset\}, L = \{T_1, T_3\}, B = \{T_2\}$:

1. Price payed by Player S when the outcome is $x\prime$:

$$\begin{aligned} p_S(x\prime) &= \left(max_{x \in C} \sum_{k \in V/\{S\}} v_k(x) \right) - \sum_{k \in V/\{S\}} v_k(x\prime) \\ &= (v_L(\{T_1, T_3\}) + v_B(\{T_2\})) - v_L(\{T_1, T_3\}) + v_B(\{T_2\}) \\ &= 37 + 18 - (37 + 18) \\ &= 0 \end{aligned}$$

2. Price payed by Player L when the outcome is $x\prime$:

$$\begin{aligned} p_L(x\prime) &= \left(max_{x \in C} \sum_{k \in V/\{L\}} v_k(x) \right) - \sum_{k \in V/\{L\}} v_k(x\prime) \\ &= (v_S(\{T_1\}) + v_B(\{T_2, T_3\})) - v_S(\{\emptyset\} + v_B(\{T_2\}) \\ &= 16 + 39 - (0 + 18) \\ &= 37 \end{aligned}$$

3. Price payed by Player B when the outcome is $c\prime$:

$$\begin{aligned} p_B(x\prime) &= \left(max_{x \in C} \sum_{k \in V/\{B\}} v_k(x) \right) - \sum_{k \in V/\{B\}} v_k(x\prime) \\ &= (v_S(\{T_1, T_2, T_3\}) + v_L(\{\emptyset\})) - v_S(\{\emptyset\} + v_L(\{T_1, T_3\}) \\ &= 54 + 0 - (0 + 37) \\ &= 17 \end{aligned}$$

As a result, we have found VCG outcome for this auction. In particular we have that:

1. **Player S** will get no painting and as we have show will pay a price of 0 pounds.
2. **Player L** will get painting T_1, T_3 and will pay a price of 37×10^5 pounds.
3. **Player B** will get painting T_2 and will pay a price of 17×10^5 pounds.

So to answer the second question, Are the VCG prices paid by the player's uniquely determined, we say that the VCG prices are uniquely determined if with every VCG outcome for that setting every one of those n players has to pay exactly the same VCG price, as with any other VCG outcome.

We can see that we have found to VCG outcomes but the players pay a **different** so as a result the VCG prices paid by the player's are **not** uniquely determined.

1.3 Question 4c

Comment on the wisdom of choosing the VCG mechanism for this or any auction. Do you think it is a good idea to do so? What if instead of this triptych, Christie's wanted to do a simultaneous auction of 20 Andy Warhol paintings, and they knew that at least 30 viable bidders want to bid for (subsets of) those paintings. Would you suggest using the VCG mechanism for such an auction? What alternative auction would you use, and why? Explain, briefly.

We know that by the theory from the notes that the VCG mechanism is incentive compatible (i.e., strategy proof). This means that, declaring their true valuation function v_i is a (weakly) dominant strategy for all players i. However, in may cases a bidders valuation may not be independent. This leads, e.g., to Winner's curse. As a result, information about other people's valuation matters a lot. Since if we know what other people are expected to value then the bidders might change their valuation. That means that it is vulnerable to bidder collusion. If all bidders in Vickrey auction reveal their valuations to each other, they can lower some or all of their valuations, while preserving who wins the auction

The problem of using VCG we have to compute an outcome, $c\prime$, that maximizes the total value $\sum_{i \in P} v_i(c)$. Is that this problem is NP-hard to solve and as a result we will not easily find such a solution. So I would not suggest using the VCG mechanism for this auction.

An alternative that can be used instead of the VCG mechanism is the Generalized Second Price auction. In this case truth telling is not a dominant strategy we are ensure that the minimum equilibrium in which payoffs correspond precisely to the VCG, however if we end up at any other equilibrium then the revenue of the seller is higher than VCG. Thus the auction house can ensure a payoff as least as good as VCG.