AGTA - Coursework 2

Exam number: B122217

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1 Question 1

1.1 Question 1a

Give an example of a pure NE which is not a SPNE, for a finite extensive form game of perfect information.

For this question we will consider the two player extensive form game in Figure 1.

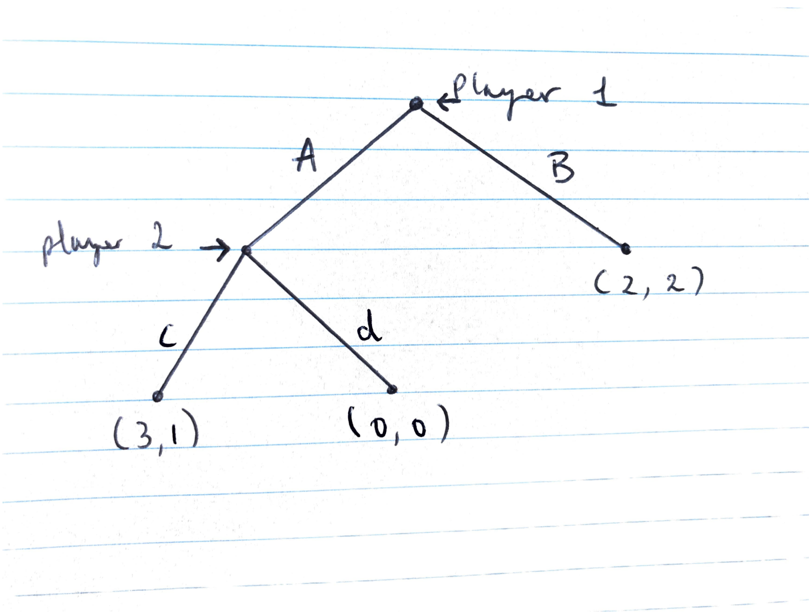

Figure 1: Two player extensive form game for question 1a.

First we will find the SPNE of this game by using "Backwards induction" and then we will find the NE of the strategic game. We will then show that there exists a pure NE in the strategic game that is not a SPNE.

Notice that in this game there are exactly 2 subgames. That is the subgame starting at the node where Player 2 can choose an action and the game it self.

Notice that there is only one action that Player 2 can make to maximise his outcome. In this case he plays action c this is because $1 > 0$. As a result we can reduce the tree to Figure 2 where player 1 can either play A with payoff 3 or B with payoff 2. Since $3 > 2$ then player 1 will always play A.

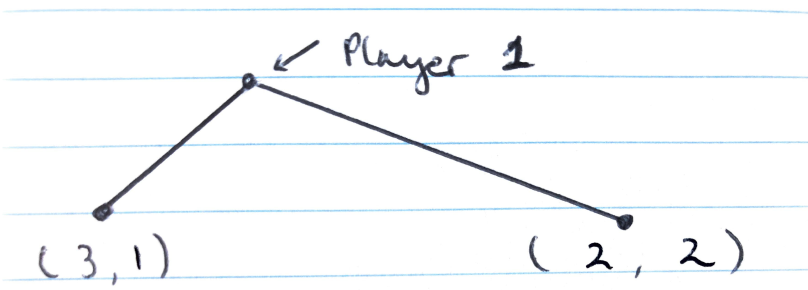

Figure 2: Reduced game after find the SPNE for the first subgame for player 2.

By using "Backwards induction" we have found the SPNE of the game, that is player 1 plays A and player 2 plays c, (A, c).

Now we need to find the NE of this game. This game is given in strategic form is given by the following table:

	c	d
A	(3,1)	(0,0)
B	(2,2)	(2,2)

Where in this case player 1 is the row player and player 2 is the column player. We can easily find the NE of this game, these are precisely the strategies that neither player can increase there payoff by deviating to another strategy.

In this game there are exactly 2 pure NE.

1. The first pure strategy profile being: ((1,0), (1,0)) this is where player 1 plays A and player 2 plays c, we can see this a pure Nash equilibrium since neither player can deviate from there strategy to increase their payoff. Note that pure NE is exactly the SPNE.

2. The second pure strategy profile being: ((0,1), (0,1)) this is where player 1 plays B and player 2 plays d, we can see this a pure Nash equilibrium since neither player can deviate from there strategy to increase their payoff.

As a result, have found a NE, that is player 1 plays B with probability 1 and player 2 plays d with probability 1, which is **not** a SPNE.

1.2 Question 1b

Prove that every finite extensive game of perfect information where there are no chance nodes and where no player gets the same payoff at any two distinct leaves, must have a unique pure-strategy SPNE.

Suppose that we have some pure profile, $s = (s_1, ..., s_n)$, is a pure SPNE. To prove that this is unique if we are in a finite extensive game of perfect information where there are no chance nodes and where no player gets the same payoff at any two distinct leaves We will introduce some definitions. For a game G with game tree T, and for $w \in T$, define the subtree $T_w \in T$, by:

$$T_w = \{w' \in T | w' = ww'' \text{ for } w'' \in \Sigma\}$$

. Since tree is finite, we can just associate payoffs to the leaves. Thus, the subtee T_w , in an obvious way, defines a "subgame", G_w , which is also a perfect information game.

The depth of a node w in T is its length $|w|$ as a string. The depth of tree T is the maximum depth of any node in T. The depth of a game G is the depth of its game tree.

To prove this we will use a proof by induction. We will show that every subgame G_w has a unique pure strategy $s^W = (s_1^W, ..., s_n^W)$, therefore if we show that a game with the maximum depth of the tree has a unique pure strategy we have shown proved the statement.

- **Base case: Depth 0**: In this case we are at a leaf w. there is nothing to show: this is because each player i will gets payoff $u_i(w)$, and the strategies in the SPNE s^0 are "empty", this is because it doesn't matter which player's node w is, since there are no actions to take. Therefore, for depth 0 there must be a unique SPNE which is exactly the "empty" strategy for all players.

- **Base case: Depth 1**: We also need to define the base case when we are in depth one this is in order to show that there must be a unique SPNE. Let $Act(1) = \{a_1', ..., a_r'\}$ be the set of actions available at the root of G_1. We need to show that the subtrees $T_{1aj'}$, for $j = 1, ..., r$, each define a perfect information subgame $G_{1aj'}$, of depth 1 which also have a **unique** SPNE $s^{1a_j'} = (s_1^{1a_j'}, ..., s_n^{1a_j'})$.

  We know that since all the payoffs are unique since all payoff as distinct for any two leaves there must be exactly one pure SPNE for the depth 1, this is because there must be a leaf which has a strictly large payoff than all the others. As a result the players cannot deviate from this their strategies of picking the largest payoff since this pay off is unique. Therefore, there must be a unique action in $Act(1)$ which maximises the payoff $u_i(1a_j')$ for all players at this depth. As a result we have found the unique SPNE for depth 1.

- **Inductive step:** Suppose depth of G_w is $k + 1$. Let $Act(w) = \{a_1', ..., a_r'\}$ be the set of actions available at the root of G_w the subtrees $T_{wa_j'}$, for $j = 1, ..., r$, each define a perfect information subgame $G_{waj'}$, of depth $\leq k$ which have a **unique** SPNE $s^{wa_j'} = (s_1^{wa_j'}, ..., s_n^{wa_j'})$ by **induction**.

  Now we need to show that this holds for the game G_w at depth $k + 1$. Since there are no chance nodes we only need to consider the action that each player can take.

  Let $w \in PI_i$ $i > 0$ that is the root, w, of T_w belongs to player i. Where PI_i is the perfect information set for player i.For $a \in Act(w)$, let $h_i^{wa}(s^{wa})$ be the expected payoff to player i in the subgame G_{wa}.

Let $a' = arg\ max_{a \in Act(w)} h_i^{wa}(s^{wa})$. That is the action that maximises the payoff to player i when we are at node w. However, we know that since we can reduce each subgame $G_{wa'_j}$ to a single leaf and we know that all the payoffs are unique, then it must be the case that a' is unique for all players i, that is there is only one action that can be taken since deviating will result in a smaller payoff.

Therefore, we can define for all players $i' \neq i$, the $s_{i'}^w$ for player i' be the "union" $\cup_{a \in Act(w)} s_{i'}^{wa}$, of its pure strategies in each of the subgames, which we know that all the pure strategies in the subgames are unique by induction.

Likewise, we can define for player i the s_i^w to be $\cup_{a \in Act(w)} s_i^{wa} \cup \{w \to a'\}$ that is the "union" of its pure strategies in each of the subgames including the final action they take at the node w, which in this case is a' which is a unique action.

Therefore we have defined a unique SPNE given by $s^w = (s_1^w, ..., s_n^w)$, where the strategies s_i^w have been defined above. Since, we have show it is true for $k + 1$, and we know it is true for the base cases then by induction we have show it is true for all k, thus completing the proof $\blacksquare$.

1.3 Question 1c

Give an example of a finite extensive form game that contains a pure Nash Equilibrium but does not contain any subgame perfect pure Nash Equilibrium. Justify your answer.

A subgame perfect pure Nash Equilibrium, is a SPNE where all the strategies are pure. Therefore, if we find a SPNE where the only NE are mixed strategies but the whole game has a NE then we have given an example. In this question we will be dealing with the finite extensive form game given in Figure 3. Note that this is a game of imperfect information since we can see this from players 2 information set. Furthermore, we can divide this problem into subgames, the game itself and the subgame which starts where player 1 has to decided between h or t, this is the graph in Figure 4.

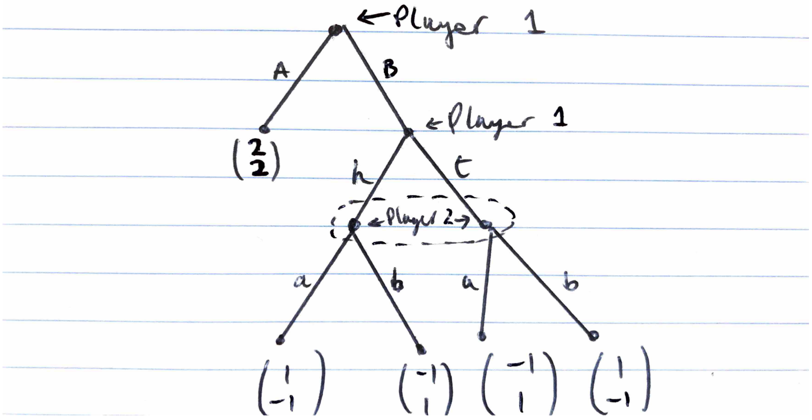

Figure 3: Two player extensive form game for question 1c.

If we consider the first normal-form game that is the normal form representation of the whole extensive-form game, Figure 3. We have the following payoff table:

	a	b
Ah	(2,2)	(2,2)
At	(2,2)	(2,2)
Bh	(1,-1)	(-1,1)
Bt	(-1,1)	(1,-1)

We can see that we have several pure NE in particular: $\{(Ah, a), (Ah, b), (At, a), (At, b)\}$, this is because no player can deviate from their strategies to increase there expected payoff. Therefore, we have shown that this game does indeed have at least one pure NE.

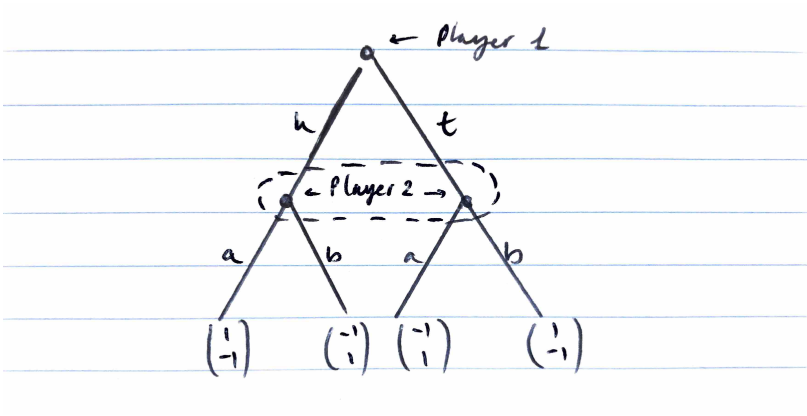

Figure 4: Two player extensive form game for question 1a.

Now we can use "backwards induction" to find the SPNE. We will start by considering the subgame in Figure 4. Notice that we have imperfect information therefore we need to solve this by first converting it into a normal-form game and then finding the NE of the game.

Below we have the payoff table for this subgame:

	a	b
h	(1,-1)	(-1,1)
t	(-1,1)	(1,-1)

Notice that this game has **no pure**, since it is always the case that one of the two players can deviate from their strategies to increase their expected payoff. In fact this game is the matching pennies games. This problem only has one NE and it is the mixed NE where both player, play either tactic with probability $\frac{1}{2}$.

Therefore, since this subgame has no pure NE, then it must be the case that there cannot exist a subgame perfect **pure** Nash Equilibrium, since by definition it must have a pure NE at every subgame of the tree. Therefore, we have found such an example where the game has a NE but no subgame perfect **pure** Nash Equilibrium.