AGTA - Coursework 2

Exam number: B122217

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1 Question 3

1.1 Question 3a

Prove that for any pure strategy profile s the following equality holds: $\varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s))$ **where** $\varphi(s) = \sum_{r \in R} \sum_{i=1}^{n_r^{(s)}} d_r(i)$**.**

We first need to assume that we have players $\{1, 2, ..., n\}$ and we have resources $\{1, 2, ..., m\}$, that is we have n players and m resources,

We will now introduce some more definitions:

- $d_r(j)$ is the cost of using resource r if there are j agents simultaneously using r.
- $n_r(s) = |\{i | r \in s_i\}|$ to be the congestion on resource r, for pure strategy profile s. The number of players using resource i.
- $i\prime \in \{1, ..., n\}$ we define $n_r^{(i\prime)}(s) = |\{i | r \in s_i \wedge i \in \{1, ..., i\prime\}\}|$. This is the number of strategies s_i that use resource r up to strategy $i\prime$.

We need to proove that:

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s)) = \sum_{r \in R} \sum_{i=1}^{n_r^{(s)}} d_r(i)$$

To show that this is the case we are given $\sum_{r \in R} \sum_{i=1}^{n_r^{(s)}} d_r(i)$ we know that:

$$\sum_{r \in R} \sum_{i=1}^{n_r^{(s)}} d_r(i) = \sum_{r \in R} \sum_{i=1}^{n} d_r(n_r^{(i)}(s))$$

This is because we know that $n_r^{(s)}$ can take values up to n, this is because it is the congestion on resource r and it can be **at-most** the number of players which in this game is n. Furthermore, we know that $n_r^{(i)}(s) = 0$ when we $r \notin s_i$, so this takes into account for strategies that do not use this resource. Now we have an expression for this value $\varphi(s) = \sum_{r \in R} \sum_{i=1}^{n} d_r(n_r^{(i)}(s))$. We can rewrite this sum in terms explicitly and we get the following sum:

$$
\begin{aligned}
&= d_1(n_1^{(1)}(s)) + \cdots + d_1(n_1^{(n)}(s)) \\
&\quad \vdots \qquad + \ddots + \qquad \vdots \\
&\quad d_1(n_m^{(1)}(s)) + \cdots + d_m(n_m^{(n)}(s))
\end{aligned}
$$

Since we know that we have in fact m resources and n players we know that in this sum we have that each row has n values and we have m rows one for each resource (the outside sum). Therefore, we can change the order of summation to sum over the resources and then over the players. Therefore, we get that:

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in R} d_r(n_r^{(i)}(s))$$

What is left to see is that we do not need to sum over all the individual resources, instead we can sum over the resources used for each player in their strategy that is $r \in s_i$, why is this the case? If we look at the expression $n_r^{(i)}(s)$ we know that it is equal to 0 if player i does not use this resource, therefore we just need to consider when he does. To illustrate this, suppose we have $i = 2$, then we need to find the cost when there are $n_r^{(i)}(s)$ players using it, but notice that this value in this case can only be at most 2 that is both players uses this resources since it is in both of their strategies, now if only one player is using it then it must be player 2 since when $i = 1$ we consider the resource used by player 1. As a result, we can generalize this for more players such that when we reach player n we will be counting if any resource has been used at most n times, this would correspond to the final column of the decomposition of the sum, since we do not consider the value when it is zero. If this resource is only used by player n then we know we need to include it since all other i's have not counted the cost of using this resource.

Therefore we have shown the claim:

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s)) = \sum_{r \in R} \sum_{i=1}^{n_r^{(s)}} d_r(i)$$

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1.2 Question 3b

Compute a pure strategy NE in this atomic network congestion game. Explain why what you have computed is a pure NE.

To find the NE of this game I will begin by assuming that the profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., s-t-paths) is the same for all players and in particular for all player i this pure strategy $\pi_i = s \to v_2 \to t$, that is they follow the path in the middel. As a result the cost to each player of playing this pure strategy is 6+6=12, since there are 3 players on each of the edges. So at the first iteration we have that:

- $\pi_1 = s \to v_2 \to t$ with cost 6+6 = 12.
- $\pi_2 = s \to v_2 \to t$ with cost 6+6 = 12.
- $\pi_3 = s \to v_2 \to t$ with cost 6+6 = 12.

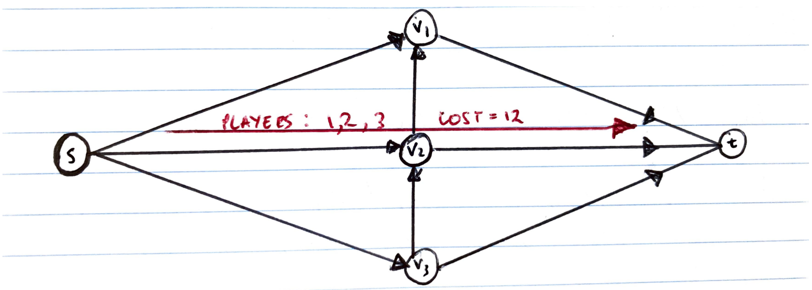

Figure 1: In this position all the players are playing the middle strategy with cost 12.

However, we need to find the NE of this atomic network congestion game. To check that this profile π is **not** a NE, if one of the players can deviate from their pure strategy in order to decrease the cost, then we have found a new profile π where one of the players is deviating from their pure strategy and is minimizing their cost.

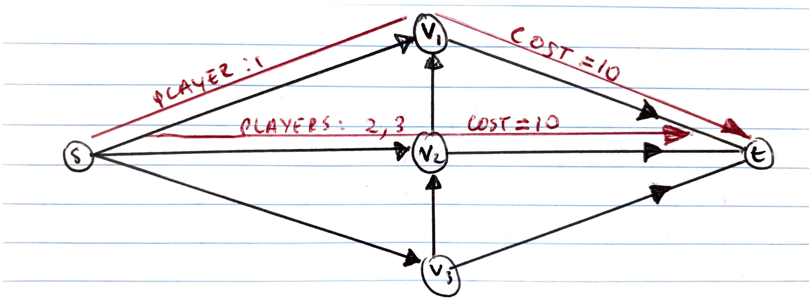

Figure 2: In this case player 1 has now deviated his strategy to $\pi_1 = s \to v_1 \to t$ and player 2 and 3 use the same strategy as in Figure 1.

Suppose that player 1 now instead using strategy $\pi_1 = s \to v_2 \to t$ now choose a new strategy $\pi_1 = s \to v_1 \to t$, that is he goes through vertex v_1 instead of v_2, he can decrease their cost from

12 to 5+5=10 since there is only one player that uses the edges on their directed path from s to t, so now player 1 uses this new path since $10 < 12$. However, if player 1 uses the directed path $s \to v_1 \to t$, this decrease the number of players using the edges on path $s \to v_2 \to t$, now only player 2 and 3 use this path as a result we need to update the cost to each of these players when congestion is 2. So at the second iteration we have the following pure strategy π, this is seen in Figure 2.

- $\pi_1 = s \to v_1 \to t$ with cost 5+5 = 10.
- $\pi_2 = s \to v_2 \to t$ with cost 3+7 = 10.
- $\pi_3 = s \to v_2 \to t$ with cost 3+7 = 10.

Now we need to check if **any** of the players can deviate form their current pure strategies in order to decrease their cost. To do this we need to consider all the possible paths of the game and see if any of the player can take one of these such that they can strictly decrease their cost. The possible paths are:

1. $s \to v_1 \to t$
2. $s \to v_2 \to t$
3. $s \to v_3 \to t$
4. $s \to v_2 \to v_1 \to t$
5. $s \to v_3 \to v_2 \to t$
6. $s \to v_3 \to v_2 \to v_1 \to t$

We will reference to a path by its number. For example, when we say "path 1." then we are referring to $s \to v_1 \to t$.

We will now see if any of the players can deviate form their current pure strategy to use any other of the directed paths to decrease their cost.

1. **Player 1**: $\pi_1 = s \to v_1 \to t$ with cost 5+5 = 10.
   In this case we need to see if player 1 can decrease his cost by using any of the other strategies. We can see that in this case player 1 is using path 1. So we do not need to consider this path since this is his pure strategy. We also do not need to consider path 2. this is because the player in the previous step deviated from this path in order to decrease his cost so we know that this path will have a strictly greater cost. So we need to calculate the cost of paths 3. to 6. when players 2 and 3 are using path 2. (their pure strategy).
   - Path 3. $s \to v_3 \to t$: if player 1 deviates to this path, since their are no other players using this path then the cost of using this path is $4 + 7 = 11$ since $11 > 10$. Player 1 does not deviate from their pure strategy π_1.
   - Path 4. $s \to v_2 \to v_1 \to t$: if player 1 deviates to this path we need to be careful since in this case all the player will be using the edge $s \to v_2$, and all the other edges will only have player 1 using them so as a result, the cost of player 1 is now: 6+1+5 = 12, since $12 > 10$ he does not deviate from this pure strategy π_1.

- Path 5. $s \to v_3 \to v_2 \to t$: if player 1 deviates to this path we need to be careful since in this case all the player will be using the edge $v_2 \to v_3$ but all the other edges will only have player 1 using them. Then the cost of player 1 is now: 4+3+6 = 13, since 13 > 10 he does not deviate from this pure strategy π_1.
- Path 6. $s \to v_3 \to v_2 \to v_1 \to t$: if player 1 deviates to this path, in this case all these edges will exclusively have player 1 using them since, like before player 2 and 3 will be paying their pure strategies π_2 and π_3. Then the cost of player 1 is if he deviates from his current pure strategy to this path is: 4+3+1+5 = 13, since 13 > 10 he does not deviate from this pure strategy π_1.

We have shown that in fact **Player 1** cannot deviate form his pure strategy π_1 since if he does his cost is greater than 10 for all other pure strategies. Now we consider **Player 2**.

2. **Player 2**: $\pi_2 = s \to v_2 \to t$ with cost 3+7 = 10. In this case we can see that player 2 is using is using path 2. We need to see if player 2 can deviate from this path (pure strategy) to decrease his cost to below 10. We will consider the following pure strategies they can deviate to:

- Path 1. $s \to v_1 \to t$: if player 2 deviates to this path in this case this is exactly player 1 pure stratgey π_1 as a result there will be 2 players on these edges. So now the cost to player 2 will be: 5+6 = 11 since 11 > 10. Player 2 does not deviate from their pure strategy π_2.
- Path 3. $s \to v_3 \to v_1 \to t$: if player 2 deviates to this path, we know that their are no other players using this path, so the number of players on these edges will be one. Then the cost of using this path is 4 + 7 = 11 since 11 > 10. Player 2 does not deviate from their pure strategy π_2.
- Path 4. $s \to v_2 \to v_1 \to t$: if player 2 deviates to this path we need to be careful since in this case their will be two players using the edge $v_1 \to t$ but all the other edges will have the same congestion. Then the cost of player 2 is now: 6+1+6 = 13, since 13 > 10 player 2 does not deviate from the pure strategy π_2.
- Path 5. $s \to v_3 \to v_2 \to t$: if player 2 deviates to this Then the cost of player 2 is now: 4+3+7 = 14, since 13 > 10 he does not deviate from this pure strategy π_2.
- Path 6. $s \to v_3 \to v_2 \to v_1 \to t$: if player 2 deviates to this path, in this case two player will be using edge $v_1 \to t$, and all the other edges will only have player 2 using them. Then the cost of player 2 if he deviates from his current pure strategy to use this path is: 4+3+1+6 = 14, since 14 > 10 he does not deviate from this pure strategy π_2.

We have shown that in fact **Player 1** cannot deviate form his pure strategy π_1 since if he does his cost is greater than 10 for all other pure strategies.

3. **Player 3**: $\pi_3 = s \to v_2 \to t$ with cost 3+7 = 10.

Notice that in this case we have that $\pi_2 = \pi_3$ this means that player 3 has the same paths he can deviate to as player 2. But we have shown that player 2 cannot deviate form his pure strategy to decrease his cost that means that player 3 cannot either.

As a result, we have shown that no player can unilaterally decrease their cost by using a different pure strategy. This mean that the profile π:

- $\pi_1 = s \to v_1 \to t$ with cost 5+5 = 10.
- $\pi_2 = s \to v_2 \to t$ with cost 3+7 = 10.
- $\pi_3 = s \to v_2 \to t$ with cost 3+7 = 10.

Is a pure strategy **NE** of this atomic network congestion game.

1.3 Question 3c

Is the pure NE you have computed unique? Explain.

In fact the pure NE that we have calculated is **not** unique. This is because in our example we have let player 1 play the $\pi_1 = s \to v_1 \to t$. But **any** of the players could have played this pure strategy. This means by simply permuting which player plays which pure strategy of the current pure strategy we have found we can find additional pure strategies where it has the same payoff as before the only difference being which player plays which strategy, in particular:

1. This is where player 2 plays $s \to v_1 \to t$ and player 1 and 3 play $s \to v_2 \to t$
   - $\pi_1 = s \to v_2 \to t$ with cost 3+7 = 10.
   - $\pi_2 = s \to v_1 \to t$ with cost 5+5 = 10.
   - $\pi_3 = s \to v_2 \to t$ with cost 3+7 = 10.
2. where player 3 plays $s \to v_1 \to t$ and player 1 and 2 play $s \to v_2 \to t$
   - $\pi_1 = s \to v_2 \to t$ with cost 3+7 = 10.
   - $\pi_2 = s \to v_2 \to t$ with cost 3+7 = 10.
   - $\pi_3 = s \to v_1 \to t$ with cost 5+5 = 10.

These have the exact same cost as the pure strategy we have calculated in question 3b, where the difference being that there is a different player which plays the pure strategy $s \to v_1 \to t$ and the other 2 play $s \to v_2 \to t$.

Furthermore, we can also find another pure NE of this game. In particular where π:

- $\pi_1 = s \to v_1 \to t$ with cost 5+6 = 11.
- $\pi_2 = s \to v_1 \to t$ with cost 5+6 = 11.
- $\pi_3 = s \to v_3 \to t$ with cost 4+7 = 11.

We notice that in this pure NE no player can deviate from their current strategy to get a cost less than 11. We know from the previous NE we calculated that it is only possible to get a cost of 10 if either one player uses strategy $\pi_i = s \to v_1 \to t$ or two players use strategy $\pi_i = s \to v_2 \to t$. Notice that we cannot satisfy this since we have that 2 players in this NE are using path $\pi_i = s \to v_1 \to t$ (so no player can exclusively use this path) and no player is using path $\pi_i = s \to v_2 \to t$, that means if a player deviates to this strategy since these edges have no players which are using them then the payoff will be greater than 10, and so they cannot deviate from this strategy.

As a result we have shown that the pure NE that we have calculated is **not** unique, first due to the fact we can permute the players in this NE and secondly because we found another pure NE with completely different costs.