AGTA - Coursework 1

Exam number: B122217

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1 Question 4

1.1 Question 4a

One variant of the Farkas Lemma says the following: Farkas Lemma: A linear system of inequalities $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$. Prove this Farkas Lemma with the aid of Fourier-Motzkin elimination. (Hint: One direction of the "if and only if" is easy. For the other direction, use induction on the number of columns of A, using the fact that Fourier-Motzkin elimination "works". Note basically that each round of Fourier-Motzkin elimination can "eliminate one variable" by pre-multiplying a given system of linear inequalities by a non-negative matrix.)

To solve this question I will rewrite Farkas lemma as two statements where exactly only one of these two statements is true at anytime:

1. A linear system of inequalities $Ax \leq b$ has a solution x.
2. There is a vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$

We need to show both directions of the implication.

- $(1 \rightarrow \neg 2)$
  I will first show that if $Ax \leq b$ has a solution x then there is no vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$.

  Suppose that $y \geq 0$, then we know that $y^T Ax \leq y^T b$ since $Ax \leq b$ has a solution.

  Then we have that $y^T Ax - y^T b \leq 0$. Now suppose that:

  1. Suppose $y^T Ax = 0$: Then we have that $-y^T b \leq 0$ so $y^T b \geq 0$, that means that the condition $y^T b < 0$ is not satisfied.
  2. Suppose $y^T b < 0$: Then we have that $y^T Ax \leq y^T b < 0$ so $y^T Ax < 0$, that means that the condition $y^T A = 0$ is not satisfied, since x is a solution which is non zero.

  Therefore, we have shown that there cannot be a solution $y \geq 0$ such that $y^T A = 0$ and $y^T b < 0$. Therefore we have shown the first implication. Now we need to show the other implication.

- $(\neg 1 \rightarrow 2)$
  If A linear system of inequalities $Ax \leq b$ has **no** solution x then there is a vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$.

  To show this I will make use of Fourier-Motzkin elimination. First, we can note that the system $Ax \leq b$ is infeasible due to $(\neg 1)$.

  Therefore, we can now apply Fourier-Motzkin Elimination (FME) on the system to remove all the variables. We can use some positive matrix, lets call it D. We will define D^i to be the matrix used to remove variable i from the system of equations; I will use $D^i \geq 0$ to indicate that each entry is non-negative. From Fourier-Motzkin Elimination we have that $D^n D^{n-1} ... D^1 A \geq 0$ with we let $D = D^n D^{n-1} ... D^1$ we have that $DA = 0$, but from the construction we know that all $D^i \geq 0$ therefore $D \geq 0$.

  Because $Ax \leq b$ is infeasible, there must be some row $y \geq 0$ of D such that $y^T A = 0$, this is because we know that $DA = 0$. Likewise we can see that there must be some vector such that $y^T b < 0$ holds true. Therefore we have found the other implication.

Since we have found both implications we have proved Farka's Lemma.

1.2 Question 4b

Recall that in the Strong Duality Theorem one possible case (case 4, in the theorem as stated on our lecture slides) is that both the primal LP and its dual LP are infeasible. Give an example of a primal LP and its dual LP, for which both are infeasible (and argue why they are both infeasible).

We let A be the matrix defined as:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then we let $x^T = (x_1, x_2)$, $y^T = (y_1, y_2)$, $b^T = (0, -1)$, $c^T = (1, 0)$
The primal and dual form for the following system are equal to:

$$\begin{aligned} &\textbf{Maximize } c^T x \\ &\textbf{Subject to}: \\ &\qquad (Ax)_i \leq b_i \text{ for } i = 1, 2 \\ &\qquad x_i \geq 0, \text{ for } i = 1, 2 \end{aligned}$$

$$\begin{aligned} &\textbf{Minimize } b^T y \\ &\textbf{Subject to}: \\ &\qquad (A^T y)_i \leq c_j, \text{ for } i = 1, 2 \\ &\qquad y_j \geq 0, \text{ for } i = 1, 2 \end{aligned}$$

Now if we substitute the values for the primal we get:

$$\begin{aligned} &\textbf{Maximize } x_1 \\ &\textbf{Subject to}: \\ &\qquad x_1 - x_2 \leq 0 \\ &\qquad -x_1 + x_2 \leq -1 \\ &\qquad x_i \geq 0, \text{ for } i = 1, 2 \end{aligned}$$

Note that we are trying to optimize:

$$x_1 - x_2 \leq 0 \text{ and } -x_1 + x_2 \leq -1$$

, notice that this is the same as:

$$x_1 - x_2 \leq 0 \text{ and } x_1 - x_2 \geq 1$$

As a result there does not exist a solution such that $1 \leq x_1 - x_2 \leq 0$, since $x_1 - x_2$ cannot be both less than 0 and greater than 1 at the same time, this is clearly infeasible. We have shown that the primal is infeasible now we need to show that the dual is too.

Now if we substitute the values for the dual we get:

$$\begin{aligned} &\textbf{Minimize } -y_2 \\ &\textbf{Subject to}: \\ &\qquad y_1 - y_2 \geq 1 \\ &\qquad -y_1 + y_2 \geq 0 \\ &\qquad y_j \geq 0, \text{ for } i = 1, 2 \end{aligned}$$

Note that we are trying to optimize:

$$y_1 - y_2 \geq 1 \text{ and } -y_1 + y_2 \geq 0$$

notice that this is the same as:

$$y_1 - y_2 \geq 1 \text{ and } y_1 - y_2 \leq 0$$

We can see there does not exist a exist a solution such that $1 \leq y_1 - y_2 \leq 0$, so this is clearly infeasible since $y_1 - y_2$ cannot be both greater than 1 and less than 0 at the same time. We have shown that the dual is also infeasible.

Since we have shown that both primal and dual are infeasible, we have completed the example of LP that has this property.