AGTA - Coursework 2

Exam number: B122217

March 2021

1 Question 2

1.1 Question 2a

Does this game satisfy "perfect recall"? Explain your answer.

The definition of "perfect recall" is given by: an imperfect information EFG has **perfect recall** if each player i never "forgets" its own sequence of prior actions and information sets. That is an extensive form game has perfect recall if whenever $w, w\prime \in Info_{i,j}$ belong to the same information set, then the "visible history" for player i (sequence of information sets and actions of player i during the play) prior to hitting node w and $w\prime$ must be exactly the same.

As a result, we can see that this game does indeed satisfy perfect recall. This is because for players 1 and 2 we have that for all nodes w of player i comes after a choice c at a previous node x of player i, then every node in the information set that contains w also comes after the same choice c at the information set that contains x.

1.2 Question 2b

Identify all SPNEs in this game, in terms of "behavior strategies". Explain why what you have identified constitutes all SPNEs.

To solve this problem we will use "Backward Induction" algorithm to compute this therefore to do so we first need to identify all the subgames, this has been done in Figure 5, these are highlighted in red. Recall that a subgame is a subtree of the game tree that fully contains all the information sets that any node in that subtree belongs to. Notice that we have imperfect information for player 2 after player 1 plays either M or R, then this cannot be divided into subtrees, since the two nodes belong to the same information set.

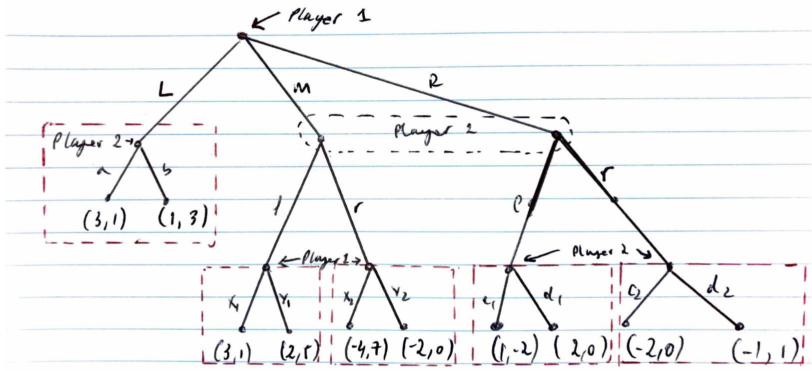


Figure 1: Here we highlight all the subgames of the extensive form game these are highlighted in red, note that the game itself is also a subgame.

As a result, what the next step is to find the SPNE of these games. To do this we need to find the NE of these games. In Figure 2 are all the subgames from left to right, in Figure 5. We shall find all the NE of these subgames, in particular since all these subgames only involve one player we just need to foucs on thier expected pay off E_1 for player one or E_2 for player 2 , that is (E_1, E_2),

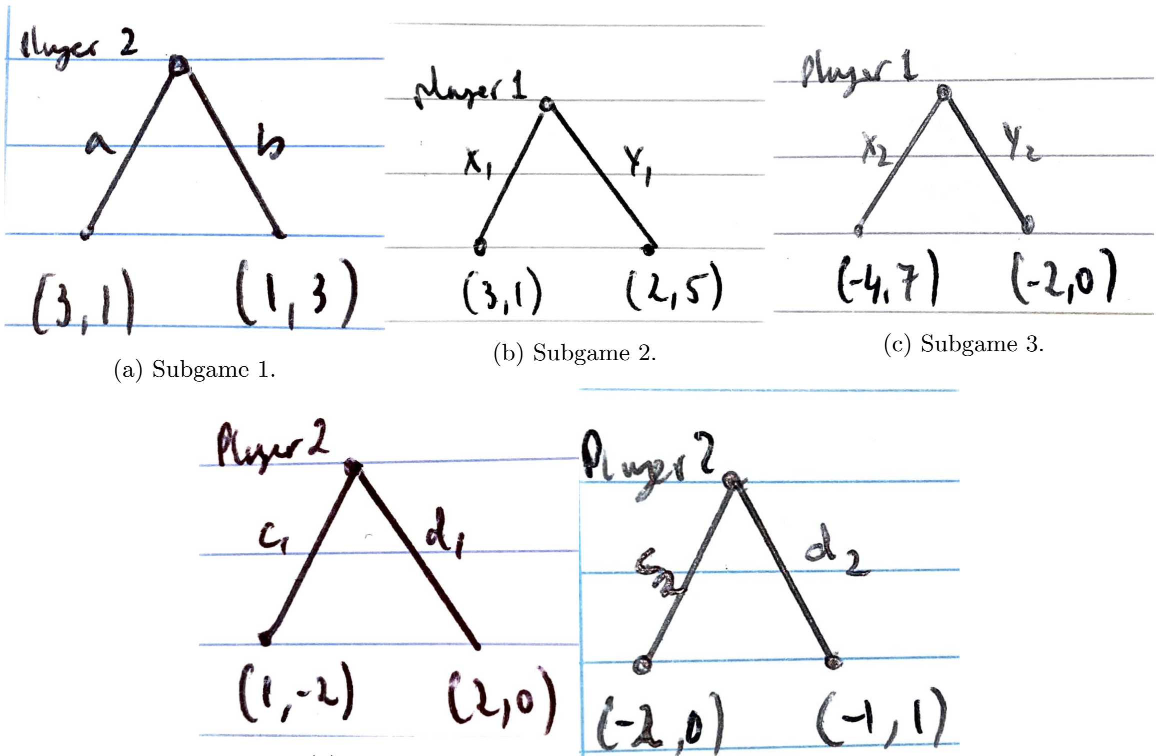
- Subgame 1, (Figure 2 (a))
  In this subgame player 2 has only two actions to take. In particular either action a or b. Since $1 < 3$, then we know that player 2 will always play action b. As a result we have the NE of this subgame, where player 2 plays b and the expected payoff for each player is $(1, 3)$

- Subgame 2, (Figure 2 (b))
  In this subgame player 1, again, has only two actions to take. In particular either action x_1 or x_2. Since $3 > 2$, then we know that player 1 will always play action x_1. As a result we have the NE of this subgame, where player 1 plays x_1 and the expected payoff for each player is $(3, 1)$

- Subgame 3, (Figure 2 (c))
  In this subgame player 1, again, has only two actions to take. In particular either action x_2 or y_2. Since $-4 < -2$, then we know that player 1 will always play action y_2. As a result we have the NE of this subgame, where player 1 plays y_2 and the expected payoff for each player is $(-2, 0)$

- Subgame 4, (Figure 2 (d))
  In this subgame player 2 has only two actions to take. In particular either action c_1 or d_1. Since $-2 < 0$, then we know that player 2 will always play action d_1. As a result we have the NE of this subgame, where player 2 plays d_1 and the expected payoff for each player is $(2, 0)$.

- Subgame 5, (Figure 2 (e))
  In this subgame player 2 has only two actions to take. In particular either action c_2 or d_2. Since $0 < 1$, then we know that player 2 will always play action d_2. As a result we have the NE of this subgame, where player 2 plays d_2 and the expected payoff for each player is $(-1, 1)$



(a) Subgame 1.

(b) Subgame 2.

(c) Subgame 3.

(d) Subgame 4.

(e) Subgame 5.

Figure 2: This figure has all the subgames highlighted in red from Figure 5, from left to right.

After we have found the SPNE of these subgames we can now substitute the subtree by a leaf node which is the payoff expected payoffs for when each player plays the NE in each subtree. In Figure 3 we can see the new tree when we substitute the values of all the subgames we previously calculated.

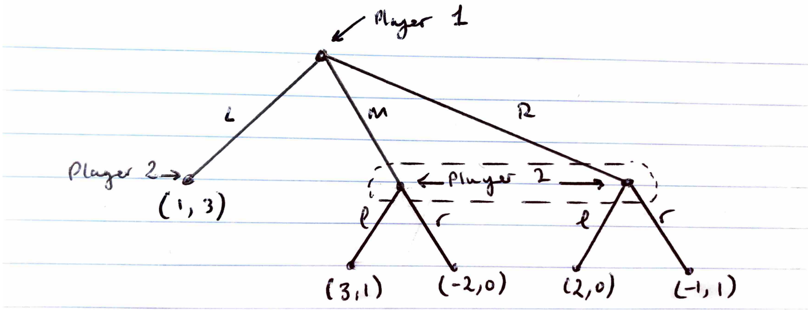


Figure 3: This is the resulting game when we solve all the subgames below it and substitute the expected payoffs in each leaf node.

Notice that in this new game there are **no** subgames, therefore we need to find the NE of the tree in Figure 3 which will result us in finding all the SPNE of the game.

To find the NE of this extensive form game we need to convert it into a strategic form game. Below is the expected payoff table for this game as a strategic form game.

	l	r
L	(1,3)	**(1,3)**
M	**(3,1)**	(-2,0)
R	(2,0)	(-1,1)

The strategies which are highlighted in bold are the pure NE of the game since none of the players can deviate form their strategies to increase their payoffs. This is when player 1 plays M and player 2 plays l and when player 1 plays L and player 2 plays r.

What is left to do is to check if there are any mixed NE of the game. To find the mixed Nash equilibrium of this game we can use the the corollary of the Nash theorem that is, if player 2 is playing against player 1's mixed strategy, both of player 2's pure strategies must be a best response to player 1. Therefore, player 2 expected value should be the same for either pure strategy it plays against players 1 mixed strategy, and vice versa.

To do this I will show that any mixed NE for player 1 only involving strategies M and R. Suppose we are dealing with the following game:

	l	r
M	(3,1)	(-2,0)
R	(2,0)	(-1,1)

Now suppose player 1 plays row 1 with probability p and row 2 with probability $1 - p$. Likewise player 2 plays column 1 with probability r and column 2 with probability $1 - r$.

Then using the corollary to Nash theorem we get for player 2:

$$
\begin{aligned}
3r + -2(1-r) &= 2r - 1(1-r) \\
3r + -2 + 2r &= 2r - 1 + r \\
2r &= 1 \\
r &= \frac{1}{2}, 1 - r = \frac{1}{2}
\end{aligned}
$$

Likewise for player 1 we get:

$$
\begin{aligned}
p &= 1 - p \\
2p &= 1 \\
p &= \frac{1}{2}, 1 - p = \frac{1}{2}
\end{aligned}
$$

Therefore, in this game both players are indifferent to what strategy they play but if we now include the option of playing L, we can see that the expected pay off for player 1 increase from $\frac{1}{2}$ to 1, if player 1 plays pure strategy L. This is because the expected value for player 1 under the mixed profile strategy $x_1 = ((0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is:

$$
\begin{aligned}
U_1(x_1) &= \frac{1}{2}\frac{1}{2}3 + \frac{1}{2}\frac{1}{2} - 2 + \frac{1}{2}\frac{1}{2}2 - \frac{1}{2}\frac{1}{2} \\
&= \frac{3}{4} - \frac{2}{4} + \frac{2}{4} - \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

However, player 2 remains indifferent to what he plays since the expected value remains the same. As a result we have the other NE of the game this is precisely the mixed strategy where player 1 plays L and player 2 is indifferent to what strategy he plays so he plays l and r with probability $\frac{1}{2}$.

However is the the only mixed NE? We can see that if player 1 plays the pure strategy 1, then we can see that for **either** action that player 2 plays he can has the exact same expected pay off. As a result, player 2 is indifferent to what he plays. So we can let player 2 play l with probability p and r with probability $1 - p$, then we have found a a mixed NE for $0 \leq p \leq \frac{1}{2}$, since this game is degenerate.

As a result we have shown that these are indeed **all** the NE of the game. Therefore the NE equilibrium of this game are given by:

- Player 1 plays L and Player 2 plays r
- Player 1 plays M and Player 2 plays l
- Player 1 plays L and Player 2 plays l with probability p and r with probability $1 - p$ for $0 \leq p \leq \frac{1}{2}$.

What is left to do is now SPNEs in this game, we just need to tract back and find the pure strategies that each player played in each of the subgames. We know that; in subgame 1 player 2 plays b, subgame 2 player 1 plays x_1, subgame 3 player 1 plays y_2, subgame 4 player 2 plays d_1 and finally in subgame 5 player 2 plays d_2. These where just calculated before.

As a result the SPNE of the game are given by:

- $[Lx_1y_2, brd_1d_2]$ in behaviour strategies: Player 1: $((1,0,0),(1,0),(0,1))$, Player 2: $((0,1),(0,1),(1,0),(0,1))$
- $[Mx_1y_2, bld_1d_2]$ in behaviour strategies: Player 1: $((0,1,0),(1,0),(0,1))$, Player 2: $((0,1),(1,0),(1,0),(0,1))$
- $[Lx_1y_2, b(rl)d_1d_2]$ in behaviour strategies: Player 1: $((1,0,0),(1,0),(0,1))$, Player 2: $((0,1),(p,1-p),(1,0),(0,1))$ for $0 \leq p \leq \frac{1}{2}$.

Where the behaviour strategies are just a probability distribution over the nodes for each player. Just to make it clear to what action we are taking at each node, I have given this in the following way: for player 1: $((L,M,R),(x_1,y_1),(x_2,y_2))$ and for player 2: $((a,b),(l,r),(c_1,d_1),(c_2,d_2))$. So for example if player 1 plays with probability 1 R with probability 1 x_1 and is plays x_2 with $\frac{1}{2}$ and y_2 $\frac{1}{2}$, then the behaviour strategy for player 1 is given by: $((0,0,1),(1,0),(\frac{1}{2},\frac{1}{2}))$.

1.3 Question 2c

Are there any other Nash equilibria, other than the SPNEs you have identified, in this game? Explain your answer.

There are no other Nash equilibria in this game other that the SPNE I have found.

First of all if we begin with the subgames that we found evaluated in Figure 2, each player must **always** play the action that maximises his outcome in these subgames. As a result, the player will always play these actions in all NE since deviating will strictly decrease his expected payoff. As a result we just need to consider the reduced game:

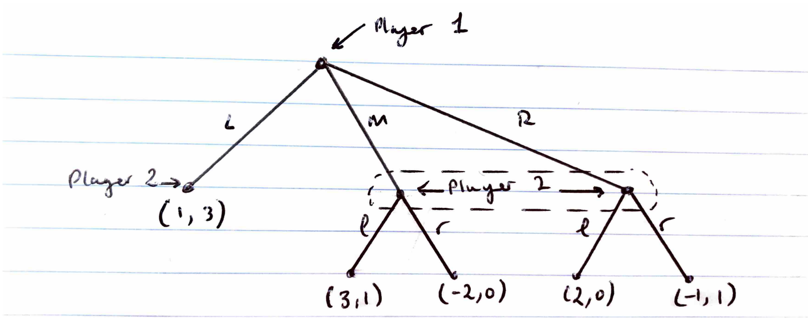


Figure 4: This is the resulting game when we solve all the subgames below it and substitute the expected payoffs in each leaf node.

In the previous example we need to find the SPNE of this game which in turn means that we have found all the NE of the whole game, since if there was another NE then it must have also been a SPNE.

1.4 Question 2d

Which of the equilibria in this game are "credible" and "sequentially rational" (and specifically, which ones do not involve "non-credible threats"). Explain.

The informal definition of "sequentially rational" is that if assuming somehow the play ever ends up at an information set belonging to some player i, then player i will make a "rational" decision at that information set. This means that it will choose a probability distribution on its actions that optimizes its own expected payoff, based on its own "beliefs" about the probabilities of being in each of the nodes of that information set (conditioned on ending up in that information set), and given the behavior strategies of the other players.

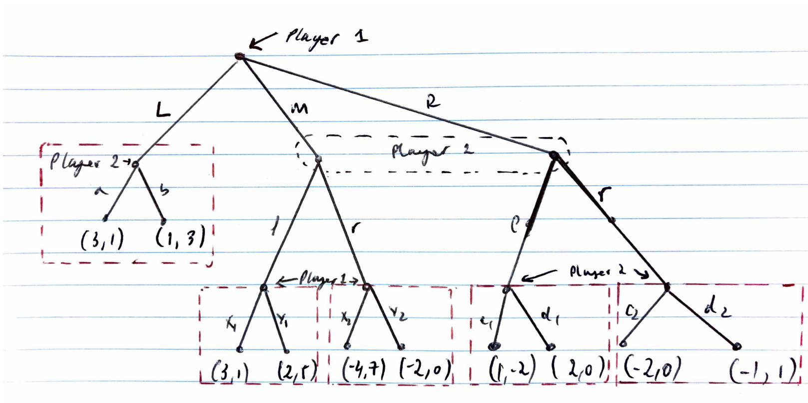


Figure 5: Here we highlight all the subgames of the extensive form game these are highlighted in red, note that the game itself is also a subgame.

We will consider the following SPNE equilibria of the game:

- $[Lx_1y_2, brd_1d_2]$ in behaviour strategies: Player 1: $((1,0,0),(1,0),(0,1))$, Player 2: $((0,1),(0,1),(1,0),(0,1))$
- $[Mx_1y_2, bld_1d_2]$ in behaviour strategies: Player 1: $((0,1,0),(1,0),(0,1))$, Player 2: $((0,1),(1,0),(1,0),(0,1))$
- $[Lx_1y_2, b(rl)d_1d_2]$ in behaviour strategies: Player 1: $((1,0,0),(1,0),(0,1))$, Player 2: $((0,1),(p,1-p),(1,0),(0,1))$ for $0 \leq p \leq \frac{1}{2}$.

We will begin with 1. That is $[Lx_1y_2, brd_1d_2]$ in behaviour strategies: Player 1: $((1,0,0),(1,0),(0,1))$, Player 2: $((0,1),(0,1),(1,0),(0,1))$. We can see that this game involves a "non credible threats" this is because if we assume Player 1 is assuming that player 2 will play r, however we know that if it is indeed the case that player one plays M then player 2 would actually always play l since this will always increase his payoff. As a result this equilibira does involve a non credible threat. Furthermore, this is "reasonable" this is because the actions that both players take the actions that maximise their expected output at every information set given the info they have about the other players belief.

Now I will consider 2. $[Mx_1y_2, bld_1d_2]$ in behaviour strategies: Player 1: $((0,1,0),(1,0),(0,1))$, Player 2: $((0,1),(1,0),(1,0),(0,1))$, again this game has no "non credible threats" and is also "reasonable" this is because if the game where to end up at any information set then the player would have played the same strategies.

Finally we will consider the final game where we have $[Lx_1y_2, b(rl)d_1d_2]$ in behaviour strategies: Player 1: $((1,0,0),(1,0),(0,1))$, Player 2: $((1,0),(p,1-p),(1,0),(0,1))$ for $0 \leq p \leq \frac{1}{2}$. In this case it involves a mixed NE for player 2. In this case we do indeed have a "non credible threat" this is because if the player where to be sequentially rational then if he knows that player 1 plays M then player 2 will never will play l which strictly decreases his expected payoff then the fact that player 2 plays it with probability p suggests that it is indeed a non credible threat. Furthermore, in this equilibrium it is "rational".