AGTA - Coursework 1

Exam number: B122217

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1 Question 1

Consider the following 2-player strategic game, G:

$$\begin{bmatrix} (1,3) & (7,5) & (5,4) & (9,8) \\ (0,7) & (6,8) & (5,9) & (8,8) \\ (1,1) & (4,0) & (2,1) & (3,1) \\ (1,2) & (8,6) & (6,7) & (7,6) \end{bmatrix}$$

This is a "bimatrix", to be read as follows: Player 1 is the row player, and Player 2 is the column player. If the content of the bimatrix at row i and column j is the pair (a,b), then $u_1(i,j) = a$ and $u_2(i,j) = b$.

1.1 Question 1a

Compute all of the Nash equilibria (NEs) of this game G, together with the expected payoff to each player in each NE. Explain why any profile x that you claim is an NE of G, is indeed an NE of G.

To find all the Nash equilibrium of this game G I will first find all the pure Nash equilibrium of this game. Note that in a Nash equilibrium each player is assumed to know the equilibrium strategies of the other players and no player has anything to gain by changing only their own strategy. To find all pure NE I will go through them exhaustively. To show that a pure strategic profile is not an NE we need to find at least one pure strategy that can increase the payoff for either player 1 or 2.

I will use the notation (a,b) to denote the profile of pure strategies where player 1 plays pure strategy of row a and player 2 plays pure strategy of column b. For example, (1,2) will denote the profile ((1,0,0,0),(0,1,0,0)). Also note that the expected payoff for pure strategies can be easily calculated as the value in the matrix where the row player meets the column player, I will formally explain this subsequently. To calculate the expected payoff for player i using the following equation:

$$U_i(x) = \sum_{k=1}^{n} \sum_{j=1}^{n} x_1(k) x_2(j) u_i(k,j) \tag{1}$$

Where x is a profile for the game, in this game G, $n = 4$

Notice however that for pure strategies $x_1(k)$ and $x_2(j)$ are equal to 0 for all k and j expect for the pure strategy they play with probability 1. As a result for pure strategies one can reduce the expected payoff to:

$$U_i(x) = x_1(k)x_2(j)u_i(k,j) \text{ where } k,j \neq 0 \tag{2}$$
$$= u_i(k,j) \tag{3}$$

As a result, the payoff is exactly the value of picking the value of the matrix at row k and column j. This is because $x_1(k)x_2(j) = 1$ when dealing with pure strategies.

1. (1,1): This is **not** an NE. If player 2 deviates from their pure strategy of playing column 1 to column 4 they can increase their expected payoff from 3 to 8. Therefore, they do indeed gain by changing their strategy. Note that also player 1 (row player) cannot deviate their strategy to any other row since he does not increase his payoff, it will always remain at 1 for any pure strategy he picks.

2. (1,2): This is **not** an NE. If player 2 deviates from their pure strategy of playing column 2 to column 4 they can increase their expected payoff from 5 to 8. Therefore, they do indeed gain by changing their strategy. Likewise for player 1, they again can change their strategy to row 4 to increase their payoff from 7 to 8.

3. (1,3): This is **not** an NE. If player 2 deviates from their pure strategy of playing column 3 to column 4 they can increase their expected payoff from 4 to 8.

4. (1,4): This **is** an NE. Lets first take player 1 (row player) he knows that player 2 will play the pure strategy (0,0,0,1), in this case player 1 cannot unilaterally deviate their strategy since all the expected payoffs are at least as good for pure strategy (1,0,0,0) payoff for player 1. Formally, $U_1(x_{-1};(1,0,0,0)) \geq U_1(x_{-1};\pi_i)$ for all pure strategies π_i. Here x denotes the profile $(1,4) = ((1,0,0,0),(0,0,0,1))$. Likewise for player 2, there is no strategy they can deviate from to increase their expected payoff when they know player 1 will play pure strategy (1,0,0,0). This is because player's 2 best response $U_2(x_{-2};(0,0,0,1)) \geq U_2(x_{-2};\pi_i)$ for all pure strategies π_i for player 2.

5. (2,1): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 2 to any other row and increase their expected payoff from 0 to 1, given that player 2 plays pure strategy (1,0,0,0).

6. (2,2): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 2 to row 4 and increase their expected payoff from 6 to 8, given that player 2 plays pure strategy (0,1,0,0).

7. (2,3): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 2 to row 4 and increase their expected payoff from 5 to 6.

8. (2,4): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 2 to row 1 and increase their expected payoff from 8 to 9.

9. (3,1): This **is** an NE. Lets first take player 1 (row player), he knows that player 2 will play the pure strategy (1,0,0,0), in this case player 1 cannot unilaterally deviate their strategy since all the expected payoffs are at least as good for pure strategy (0,0,1,0) payoff for player 1. Formally, $U_1(x_{-1};(0,0,1,0)) \geq U_1(x_{-1};\pi_i)$ for all pure strategies π_i. Here x denotes the profile $(3,1) = ((0,0,1,0),(1,0,0,0))$. In this case player one cannot increase their payoff to more than 1. Likewise for player 2, there is no strategy they can deviate from to increase their

expected payoff when they know player 1 will play pure strategy (0,0,1,0). This is because player's 2 best response $U_2(x_{-2};(1,0,0,0)) \geq U_2(x_{-2};\pi_i)$ for all pure strategies. Again, player 2 cannot increase their payoff to more than 1.

10. (3,2): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 3 to row 4 and increase their expected payoff from 4 to 8.

11. (3,3): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 3 to row 4 and increase their expected payoff from 2 to 6.

12. (3,4): This is **not** an NE. Player 1 can deviate from their pure strategy of playing row 3 to row 1 and increase their expected payoff from 3 to 9.

13. (4,1): This is **not** an NE. Player 2 can deviate from their pure strategy of playing column 1 to column 3 and increase their expected payoff from 2 to 7.

14. (4,2): This is **not** an NE. Player 2 can deviate from their pure strategy of playing column 2 to column 3 and increase their expected payoff from 6 to 7.

15. (4,3):This **is** an NE. Lets first take player 1 (row player) he knows the that player 2 will play the pure strategy (0,0,1,0), in this case player 1 cannot unilaterally deviate from their strategy since all the expected payoffs are at least as good for pure strategy (0,0,0,1) payoff for player 1. Formally, $U_1(x_{-1};(0,0,0,1)) \geq U_1(x_{-1};\pi_i)$ for all pure strategies π_i. Here x denotes the profile $(4,3) = ((0,0,0,1),(0,0,1,0))$. In this case player one cannot increase its payoff to more than 6 when player 2 is playing pure strategy $(0,0,1,0)$. Likewise for player 2, there is no strategy they can deviate from to increase their expected payoff when they know player 1 will play pure strategy (0,0,0,1). This is because player's 2 best response $U_2(x_{-2};(0,0,1,0)) \geq U_2(x_{-2};\pi_i)$ for all pure strategies available to player 2. In this case player 2 cannot increase their payoff to more than 7.

16. (4,4): This is **not** an NE. Player 2 can deviate from their pure strategy of playing column 4 to column 4 and increase their expected payoff from 6 to 7.

As a result we can conclude that the pure NE for game G are exactly the following strategies:

1. $x_a = ((1,0,0,0),(0,0,0,1))$
2. $x_b = ((0,0,1,0),(1,0,0,0))$
3. $x_c = ((0,0,0,1),(0,0,1,0))$

To calculate the expected payoff $U_i(x)$ for each player $i \in \{1,2\}$ for each strategy using the formula (3):

1. $x_a = ((1,0,0,0),(0,0,0,1))$, $U_1(x_a) = 9$, $U_2(x_a) = 8$
2. $x_b = ((0,0,1,0),(1,0,0,0))$, $U_1(x_b) = 1$, $U_2(x_b) = 1$
3. $x_c = ((0,0,0,1),(0,0,1,0))$, $U_1(x_c) = 6$, $U_2(x_c) = 7$

Now we have found all the pure NE equilibrium's we are left with finding the mixed Nash equilibrium's. That is mixed strategy profiles. In a mixed Nash equilibrium the players must be indifferent between any of the **pure** strategies played with positive probability. If this were not the case, then there is a profitable deviation (play the pure strategy with higher payoff with higher probability).

We can reduce the game G by removing weakly dominated strategies. Since we know that the any equilibrium in the residual game will also be equilibrium in the original game. As a result we can eliminate row 2 since it is weakly dominated by row 4, and we can eliminate column 2 since it is weakly dominated by column 4. The resulting strategic game we get is G':

$$\begin{bmatrix} (1,3) & (5,4) & (9,8) \\ (1,1) & (2,1) & (3,1) \\ (1,2) & (6,7) & (7,6) \end{bmatrix}$$

In this new game we can note that now row 2 is strictly dominated by row 3 and column 1 is weakly dominated by column 3. So we can further reduce the game down to:

$$\begin{bmatrix} (5,4) & (9,8) \\ (6,7) & (7,6) \end{bmatrix}$$

To find the mixed Nash equilibrium of the game we know that by the corollary of the Nash theorem, if player 2 is playing against player 1's mixed strategy, both of player 2's pure strategies must be a best response to player 1. Therefore, player 2 expected value should be the same for either pure strategy it plays against players 1 mixed strategy, and vice versa.

Now suppose player 1 plays row 1 with probability p and row 2 with probability 1-p. Likewise player 2 plays column 1 with probability q and column 2 with probability 1-q.

Then using the corollary to Nash theorem we get:

$$\begin{align} 4p + 7(1-p) &= 8p + 6(1-p) \tag{4} \\ 4p + 7 - 7p &= 8p + 6 - 6p \tag{5} \\ 7 - 3p &= 6 + 2p \tag{6} \\ 1 &= 5p \tag{7} \\ p &= \frac{1}{5}, 1 - p = \frac{4}{5} \tag{8} \end{align}$$

Likewise for player 2 we get:

$$\begin{align} 5q + 9(1-q) &= 6q + 7(1-q) \tag{9} \\ 5q + 9 - 9q &= 6q + 7 - 7q \tag{10} \\ 9 - 4q &= 7 - q \tag{11} \\ 2 &= 3q \tag{12} \\ q &= \frac{2}{3}, 1 - q = \frac{1}{3} \tag{13} \end{align}$$

As a result we have found a NE equilibrium for the residual game. Therefore, the NE in the real game would be equal to: $x_d = ((\frac{1}{5}, 0, 0, \frac{4}{5}), (0, 0, \frac{2}{3}, \frac{1}{3}))$

In this case the expected payoff for each player will be calculated using equation (1).

$$
\begin{aligned}
U_1(x_d) &= \sum_{k=1}^{4}\sum_{j=1}^{4} x_1(k)x_2(j)u_1(i,j) \\
&= x_1(1) * x_2(3)u_1(1,3) + x_1(1) * x_2(4)u_1(1,4) \\
&+ x_1(4) * x_2(3)u_1(4,3) + x_1(4) * x_2(4)u_1(4,4) \\
&= \frac{1}{5}\frac{2}{3}(5) + \frac{1}{5}\frac{1}{3}(9) + \frac{4}{5}\frac{2}{3}(6) + \frac{4}{5}\frac{1}{3}(7) \\
&= \frac{19}{3}
\end{aligned}
$$

$$
\begin{aligned}
U_2(x_d) &= \sum_{k=1}^{4}\sum_{j=1}^{4} x_1(k)x_2(j)u_2(i,j) \\
&= x_1(1) * x_2(3)u_2(1,3) + x_1(1) * x_2(4)u_2(1,4) \\
&+ x_1(4) * x_2(3)u_2(4,3) + x_1(4) * x_2(4)u_2(4,4) \\
&= \frac{1}{5}\frac{2}{3}(4) + \frac{1}{5}\frac{1}{3}(8) + \frac{4}{5}\frac{2}{3}(7) + \frac{4}{5}\frac{1}{3}(6) \\
&= \frac{32}{5}
\end{aligned}
$$

Therefore, we have found a mixed Nash equilibrium. In particular:

$$x_d = \left(\left(\frac{1}{5}, 0, 0, \frac{4}{5}\right), \left(0, 0, \frac{2}{3}, \frac{1}{3}\right)\right) \text{,where the expected payoffs are: } U_1(x_d) = \frac{19}{3}, U_2(x_d) = \frac{32}{5}$$

We are now left to show that these profiles are indeed the **only** NE in this game.

1.1.1 Explain why there are no other (pure or mixed) NEs of G, other than the profile(s) you claim are NE(s) of G.

To do this we remove strictly dominated strategies in order to the reduced game. We notice that a pure strategy may be strictly dominated by a mixed strategy. Therefore, if we can find a mixed strategy that has higher expected payoff than any pure strategy we can eliminate this pure strategy since it will never be in a NE.

We say that a strategy x_i is strictly dominated if there exists some strategy $x_i\prime$ such that $x_i \succ x_i\prime$, that is $U_i(x_{-1}; x_i\prime) > U_i(x_{-1}; x_i)$.

I will first show that the pure strategy of only playing column 2 $x_2 = (0, 1, 0, 0)$ for player 2 is strictly dominated by mixed strategy $x_2\prime = (0, 0, \frac{1}{2}, \frac{1}{2})$. To do this we will compare the expected value of both strategies over all all pure strategies for player 1, this will be sufficient since all player 1's strategies will be a combination of his pure strategies.

1. **When player 1 plays pure strategy (1,0,0,0):**
   Let $y_1 = ((1, 0, 0, 0), x_2)$ and $y_2 = ((1, 0, 0, 0), x_2\prime)$, then we have that the expected value for player 2 is: $U_2(y_1) = 5$ and $U_2(y_2) = \frac{1}{2}4 + \frac{1}{2}8 = 2 + 4 = 6$, therefore, we have that the expected payoff is greater for $x_2\prime$.

2. **When player 1 plays pure strategy (0,1,0,0):**
Let $y_1 = ((0,1,0,0), x_2)$ and $y_2 = ((0,1,0,0), x_2\prime)$, then we have that the expected value for player 2 is: $U_2(y_1) = 8$ and $U_2(y_2) = \frac{1}{2}9 + \frac{1}{2}8 = 4.5 + 4 = 8.5$, therefore, we have that the expected payoff is greater for $x_2\prime$.

3. **When player 1 plays pure strategy (0,0,1,0):**
Let $y_1 = ((0,0,1,0), x_2)$ and $y_2 = ((0,0,1,0), x_2\prime)$, then we have that the expected value for player 2 is: $U_2(y_1) = 0$ and $U_2(y_2) = \frac{1}{2}1 + \frac{1}{2}1 = \frac{1}{2} + \frac{1}{2} = 1$, therefore, we have that the expected payoff is greater for $x_2\prime$.

4. **When player 1 plays pure strategy (0,0,0,1):**
Let $y_1 = ((0,0,0,1), x_2)$ and $y_2 = ((0,0,0,1), x_2\prime)$, then we have that the expected value for player 2 is: $U_2(y_1) = 6$ and $U_2(y_2) = \frac{1}{2}7 + \frac{1}{2}6 = 6.5$, therefore, we have that the expected payoff is greater for $x_2\prime$.

Therefore, we can conclude that the pure strategy x_2 is strictly dominated by $x_2\prime$ since the expected payoff for $x_2\prime$ is greater than x_2 for all pure strategies that player one can play. So we can reduce the game to $G\prime$ by eliminating column 2:

$$\begin{bmatrix} (1,3) & (5,4) & (9,8) \\ (0,7) & (5,9) & (8,8) \\ (1,1) & (2,1) & (3,1) \\ (1,2) & (6,7) & (7,6) \end{bmatrix}$$

We can continue to search for dominated strictly dominated strategies. In this case I will now show that the pure strategy $z_1 = (0,1,0,0)$ for player 1 is strictly dominated by mixed strategy $z_1\prime = (\frac{2}{3}, 0, 0, \frac{1}{3})$. We will calculate the expected payoff for both strategies when player 2 plays their pure strategies, in this new game $G\prime$.

1. **When player 2 plays pure strategy (1,0,0):**
Let $y_1 = (z_1, (1,0,0))$ and $y_2 = (z_1\prime, (1,0,0))$, then we have that the expected value for player 1 is: $U_1(y_1) = 0$ and $U_1(y_2) = \frac{2}{3}1 + \frac{1}{3}1 = 1$, therefore, we have that the expected payoff is greater for $z_1\prime$.

2. **When player 1 plays pure strategy (0,1,0):**
Let $y_1 = (z_1, (0,1,0))$ and $y_2 = (z_1\prime, (0,1,0))$, then we have that the expected value for player 1 is: $U_1(y_1) = 5$ and $U_1(y_2) = \frac{2}{3}5 + \frac{1}{3}6 = \frac{16}{3}$, since $\frac{16}{3} > 5$, we have that the expected payoff is greater for $z_1\prime$.

3. **When player 1 plays pure strategy (0,0,1):**
Let $y_1 = (z_1, (1,0,0))$ and $y_2 = (z_1\prime, (1,0,0))$, then we have that the expected value for player 1 is: $U_1(y_1) = 8$ and $U_1(y_2) = \frac{2}{3}9 + \frac{1}{3}7 = \frac{25}{3}$, since $\frac{25}{3} > 8$, we have that the expected payoff is greater for $z_1\prime$.

Therefore, we can conclude that the pure strategy z_1 is strictly dominated by $z_1\prime$ since the expected payoff for $z_1\prime$ is greater than z_1 for all pure strategies that player one can play. So we can reduce the game $G\prime$ by eliminating row 2, lets call this new game H:

$$\begin{bmatrix} (1,3) & (5,4) & (9,8) \\ (1,1) & (2,1) & (3,1) \\ (1,2) & (6,7) & (7,6) \end{bmatrix}$$

Now all the NE in this new game H are **exactly** the NE in game G, due to the property of strictly dominated strategies.

First we will check that there is **no** mixed strategy that uses the three pure strategies for either player 1 or 2 in this new game H. Let, like before p, q and $1-p-q$ be the probability that player 1 plays row 1, 2, or 3 respectively. We know that by the corollary to Nash theorem that any of players pure strategy is itself a best response to players one mixed strategy $(p, q, 1-p-q)$

Now we have the following three equations for each of expected payoff for player 2 under each pure strategy, we know the expected payoff will have the same value.

1. $3p + q + 2(1-p-q) = 2 + p - q$
2. $4p + q + 7(1-p-q) = 7 - 3p - 6q$
3. $8p + q + 6(1-p-q) = 6 + 2p - 5q$

Now if we let 1. = 2. we get A:

$$\begin{aligned} 2 + p - q &= 7 - 3p - 6q \\ 0 &= 5 - 4p - 5q \\ 4p &= 5 - 5q \end{aligned}$$

Now if we let 1. = 3. we get B:

$$\begin{aligned} 2 + p - q &= 6 + 2p - 5q \\ 0 &= 4 + p - 5q \\ 4q &= 4 + p \end{aligned}$$

And finally we let 2. = 3. we get C:

$$\begin{aligned} 7 - 3p - 6q &= 6 + 2p - 5q \\ 1 - 5p &= q \end{aligned}$$

Now if we substitute C into A we get that: $4p = 5 - 5(1-5p)$ therefore we have that

$$4p = 5 - 5 + 25p \rightarrow 21p = 0 \rightarrow p = 0$$

.

So we now can get the values for q and $1-p-q$, in this case if we substitute into A we get that $q = 1$ therefore the mixed strategy that involves all 3 pure strategy is exactly the pure strategy $(0, 1, 0)$ for player 1.

We can do the same for player 2 to find their pure strategy. Let $p\prime$, $q\prime$ and $1-p\prime-q\prime$ be the probability that player 2 plays column 1, 2, or 3 respectively, again we make use of the corollary to Nash's theorem to get the following equations:

1. $p\prime + 5q\prime + 9(1-p\prime-q\prime) = 9 - 8p\prime - 4q\prime$
2. $p\prime + 2q\prime + 3(1-p\prime-q\prime) = 3 - 2p\prime - q\prime$
3. $p\prime + 6q\prime + 6(1-p\prime-q\prime) = 6 - 5p\prime - q\prime$

If let 2. = 3. we get:

$$\begin{aligned} 3 - 2p\prime - q\prime &= 6 - 5p\prime - q\prime \\ 5p\prime - 2p\prime &= 3 \\ p\prime &= 1 \end{aligned}$$

This means that $q\prime = 0$, and we have found the mixed strategy for player 2 which in this case would be to play pure strategy $(1, 0, 0)$ in game H.

So we have found the **only** profile which has all three pure strategy this is $(0, 1, 0), (1, 0, 0))$ in game H, this profile in game G would be $(0, 0, 1, 0), (1, 0, 0, 0))$ which is exactly the profile x_b we have stated before.

Since we have found the Nash equilibrium with all of the pure strategies we do not need to consider the weakly strategies since they will not be part of any of the remaining NE. Therefore we can reduce the game to since row 2 is weakly dominated by row 3 and column 1 is weakly dominated by column 3, the new game will be called $H\prime$.

$$\begin{bmatrix} (5, 4) & (9, 8) \\ (6, 7) & (7, 6) \end{bmatrix}$$

Notice that $H\prime$ is exactly the same game when we reduced G by only using weakly dominated strategies. We found that the mixed Nash equilibrium is $((\frac{1}{5} \frac{4}{5}), (\frac{2}{3}, \frac{1}{3}))$ where in the original game it is $((\frac{1}{5}, 0, 0, \frac{4}{5}), (0, 0, \frac{2}{3}, \frac{1}{3}))$ which is x_d.

Also, notice that in $H\prime$ we have exactly 2 pure Nash equilibrium the profiles are $((1, 0), (1, 0))$ and $((0, 1), (0, 1))$ which correspond to $((1, 0, 0, 0), (0, 0, 0, 1))$ and $((0, 0, 0, 1), (0, 0, 1, 0))$ in the original game H which are exactly the profiles x_a and x_c we state before.

As a result, we have shown that the profiles we found in the first section are in fact the **unique** NE of this game.

So, to conclude the NEs of game G are with the expected payoffs for Player 1 and Player 2 are:

1. $x_a = ((1, 0, 0, 0), (0, 0, 0, 1))$, $U_1(x_a) = 9$, $U_2(x_a) = 8$
2. $x_b = ((0, 0, 1, 0), (1, 0, 0, 0))$, $U_1(x_b) = 1$, $U_2(x_b) = 1$
3. $x_c = ((0, 0, 0, 1), (0, 0, 1, 0))$, $U_1(x_c) = 6$, $U_2(x_c) = 7$
4. $x_d = \left(\left(\frac{1}{5}, 0, 0, \frac{4}{5}\right), \left(0, 0, \frac{2}{3}, \frac{1}{3}\right)\right)$, $U_1(x_d) = \frac{19}{3}$, $U_2(x_d) = \frac{32}{5}$

1.1.2 Question 1b

For a game H, let NE(H) denote the set of all (pure or mixed) NE's of the game H. For a mixed strategy $x_1 \in X_1$ for player 1, define:

$$g_1(x_1) = \begin{cases} 2 \text{ if } x_{1,1} \geq \frac{1}{2} \\ 1 \text{ otherwise} \end{cases} \tag{14}$$

Recall $\pi_{i,j}$ denotes the j'th pure strategy of player i. Show that there exist a 3-player finite normal form game, $G\prime$, with pure strategy sets $S_1 = S_2 = \{1, 2, 3, 4\}$, and $S_3 = \{1, 2\}$ for the three players, such that:

$$NE(G\prime) = \{(x_1, x_2, \pi_{3,g_1(x_1)}) | (x_1, x_2) \in NE(G)\}$$

We are aiming for the following set of $NE(G\prime)$ using the NE we found in question 1a and using the function $g_1(x_1)$, where the last column is the strategy for player 1.

1. $x_{a\prime} = ((1, 0, 0, 0), (0, 0, 0, 1), (0, 1))$
2. $x_{b\prime} = ((0, 0, 1, 0), (1, 0, 0, 0), (1, 0))$
3. $x_{c\prime} = ((0, 0, 0, 1), (0, 0, 1, 0), (1, 0))$
4. $x_{d\prime} = \left(\left(\frac{1}{5}, 0, 0, \frac{4}{5}\right), \left(0, 0, \frac{2}{3}, \frac{1}{3}\right), (1, 0)\right)$

Therefore to solve this problem we need to define a 3-player finite normal game $G\prime$ since we know that the sets of pure strategy for player 3 is $S_3 = \{1, 2\}$ that means that player 3 can play one of 2 tactics, this can be visualized as two 4x4 matricies where each cell has 3 entries. The payoff is given by: for row i and column j and matrix k is the pair (a, b, c), then $u_1(i, j, k) = a$ and $u_2(i, j, k) = b$ and $u_3(i, j, k) = c$ where c is the payoff to Player 3.

In our case we will define the two 4x4 matrices as game the same as G, this ensures that Players 1 and 2 will have the same NE that they have in game G since what ever Player 3 plays the game remains the same for Players 1 and 2. But we will add an extra slot in every cell for player 3's payoff, this can be visualized as:

$$A_1 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,\_) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,\_) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,\_) & (7,6,\_) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,\_) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,\_) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,\_) & (7,6,\_) \end{bmatrix}$$

Where A_1 is the payoff table if player 1 plays pure Strategy 1 and A_1 if they play pure Strategy 2. In the case above "_" represents an empty slot we need to fill in.

We can now begin to fill in the table progressively. suppose $(x_1, x_2) = x_a$, therefore we have that $x_{a\prime} = ((1, 0, 0, 0), (0, 0, 0, 1), (0, 1))$, however for this to be the case Player 3 cannot deviate his strategy to 1. To do this we augment the tables to:

$$A_1 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,0) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,\_) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,\_) & (7,6,\_) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,1) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,\_) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,\_) & (7,6,\_) \end{bmatrix}$$

We can see that now Players 3 expected payoff is 1 if he picks pure strategy 2 and 0 if he picks 1 therefore he cannot deviate from strategy 2. This is because there only 2 possible choices Player 3 can make, that is pick either matrix 1 or 2. We can do the same for when $(x_1, x_2) = x_b$ and $(x_1, x_2) = x_c$ and we get the following:

$$A_1 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,0) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,1) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,1) & (7,6,\_) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,1) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,0) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,0) & (7,6,\_) \end{bmatrix}$$

This ensures that when $x_{1,1} \geq \frac{1}{2}$ Player 3 picks pure strategy 2. In the case for x_b and x_c we have that $x_{1,1} = 0$ in both cases. We can see, from the tables that the expected payoff for strategy 1 is greater than strategy 2 in both cases for Player 3, so he will pick pure strategy 1.

Finally we need to get the last NE of $G\prime$, that is $x_{d\prime} = \left(\left(\frac{1}{5}, 0, 0, \frac{4}{5}\right), \left(0, 0, \frac{2}{3}, \frac{1}{3}\right), (1, 0)\right)$ we know this is the case since $x_{1,1} < \frac{1}{2}$. Therefore we need to find payoffs for player 3 such that the pure strategy (1,0) is a best response to when player 1 and 2 play the mixed strategy profile $\left(\frac{1}{5}, 0, 0, \frac{4}{5}\right), \left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$.

To find such values we will inspect the expected payoff of this profile and ensure that the pay off is strictly greater for pure strategy (1,0) than for (0,1). Therefore to do this we need to calculate the expected payoff for player 3 for profile $x_{d\prime}$ that is $U_3(x_{d\prime})$.

In the case for 3 player games the expected payoff is given by:

$$U_i(x) = \sum_{l=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} x_1(k) x_2(j) x_1(l) u_i(k, j, l) \tag{15}$$

for this specific game it will be equal to:

$$U_i(x) = \sum_{l=1}^{2} \sum_{k=1}^{4} \sum_{j=1}^{4} x_1(k) x_2(j) x_1(l) u_i(k, j, l) \tag{16}$$

We can notice that for profile $x_{d\prime}$, non-zero values are for when $k = \{1, 4\}$, $j = \{3, 4\}$ and $l = \{1\}$, in the case for l we know that this is the case since $x_3 = (1, 0)$.

So the expected payoff $U_3(x_{d\prime})$ is given by (when we ignore zero values as explained above):

$$x_1(1)x_2(3)x_3(1)u_3(1,3,1) + x_1(4)x_2(3)x_3(1)u_3(4,3,1) + \\ x_1(1)x_2(4)x_3(1)u_3(1,4,1) + x_1(4)x_2(4)x_3(1)u_3(4,4,1)$$

This can be simplified to:

$$\frac{1}{5}\frac{2}{3}1u_3(1,3,1) + \frac{4}{5}\frac{2}{3}1u_3(4,3,1) + \frac{1}{5}\frac{1}{3}1u_3(1,4,1) + \frac{4}{5}\frac{1}{3}1u_3(4,4,1) = \\ \frac{2}{15}u_3(1,3,1) + \frac{8}{15}u_3(4,3,1) + \frac{1}{15}u_3(1,4,1) + \frac{4}{15}u_3(4,4,1)$$

We can now substitute the values we have. Notice however that we have a values for $u_3(4,3,1) = 1$ and $u_3(1,4,1) = 0$. So this simplifies to:

$$\frac{2}{15}a + \frac{8}{15} + \frac{1}{15}0 + \frac{4}{15}b = \\ \frac{2a}{15} + \frac{4b}{15} + \frac{8}{15}$$

When this is the following payoff matrix.

$$A_1 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,a) & (9,8,0) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,1) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,1) & (7,6,b) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,\_) & (9,8,1) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,0) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,0) & (7,6,\_) \end{bmatrix}$$

So we need to find values a and b such that the expected payoff for playing pure strategy $(1,0)$ is greater than playing strategy $(0,1)$.

However, before finding values a and b we need to find the expected payoff of pure strategy $(0,1)$ for player 3 under the current conditions, this is to ensure that pure strategy will $(1,0)$ will have a higher expected probability.

Therefore, we can calculate $U_3((x_{d\prime})_{-3};(0,1))$. We can notice that for profile $(x_{d\prime})_{-3};(0,1)$, the non-zero values for $k=\{1,4\}$, $j=\{3,4\}$ and $l=\{2\}$, in the case for l we know that this is the case since $x_3=(0,1)$.

So the expected payoff $U_3(x_{d\prime})$ is given by (when we ignore zero values as explained above):

$$\begin{gathered} x_1(1)x_2(3)x_3(2)u_3(1,3,2)+x_1(4)x_2(3)x_3(2)u_3(4,3,2)+ \\ x_1(1)x_2(4)x_3(2)u_3(1,4,2)+x_1(4)x_2(4)x_3(2)u_3(4,4,2) \end{gathered}$$

Now we substitute the values in we get:

$$\begin{gathered} \frac{1}{5}\frac{2}{3}1u_3(1,3,2)+\frac{4}{5}\frac{2}{3}1u_3(4,3,2)+\frac{1}{5}\frac{1}{3}1u_3(1,4,2)+\frac{4}{5}\frac{1}{3}1u_3(4,4,2)= \\ \frac{2}{15}u_3(1,3,2)+\frac{8}{15}u_3(4,3,2)+\frac{1}{15}u_3(1,4,2)+\frac{4}{15}u_3(4,4,2)= \\ \frac{2}{15}a\prime+\frac{8}{15}0+\frac{1}{15}1+\frac{4}{15}b\prime= \\ \frac{2a\prime}{15}+\frac{1}{15}+\frac{4b\prime}{15} \end{gathered}$$

When the payoff matrix is given by:

$$A_1 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,a) & (9,8,0) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,1) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,1) & (7,6,b) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,a\prime) & (9,8,1) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,0) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,0) & (7,6,b\prime) \end{bmatrix}$$

Therefore, if we ensure that the expected payoff for player 3 when he plays pure strategy $(1,0)$ is greater than $\frac{2a\prime}{15}+\frac{1}{15}+\frac{4b\prime}{15}$ we are done. To do this we can let $a\prime$ and $b\prime$ equal to 0, so the expected payoff when Player 3 plays pure $(0,1)$ is reduced to $\frac{1}{15}$.

Therefore we need to find a and b such that:

$$\frac{1}{15}<\frac{2a}{15}a+\frac{8}{15}+\frac{4b}{15}$$

Again if we let $a=0$ and $b=0$ we have that:

$$\frac{1}{15}<\frac{8}{15}$$

So we are done, the following payoff table will satisfy the desired $NE(G\prime)$.

Now we can substitute the values for $a, b, a\prime$ and $b\prime$ into the payoff matrix.

$$A_1 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,0) & (9,8,0) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,1) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,1) & (7,6,0) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,\_) & (7,5,\_) & (5,4,0) & (9,8,1) \\ (0,7,\_) & (6,8,\_) & (5,9,\_) & (8,8,\_) \\ (1,1,0) & (4,0,\_) & (2,1,\_) & (3,1,\_) \\ (1,2,\_) & (8,6,\_) & (6,7,0) & (7,6,0) \end{bmatrix}$$

Notice how the there are unfilled slots, we can let them all be equal to zero since this will not change the NE we have found. The pure and mixed NE of the game will be the same, due to the fact we are using the same game G for Players 1 and 2.

The final payoff table is give by A_1 and A_2 below, where the payoff function is given by $u_l(i, j, k)$, where l is the player $\{1, 2, 3\}$, $i \in S_1$,$j \in S_2$ and $k \in S_3$.

$P = \{1, 2, 3\}$ is the set of players and the pure strategy sets are $S_1 = S_2 = \{1, 2, 3, 4\}$, and $S_3 = \{1, 2\}$.

$$A_1 = \begin{bmatrix} (1,3,0) & (7,5,0) & (5,4,0) & (9,8,0) \\ (0,7,0) & (6,8,0) & (5,9,0) & (8,8,0) \\ (1,1,1) & (4,0,0) & (2,1,0) & (3,1,0) \\ (1,2,0) & (8,6,0) & (6,7,1) & (7,6,0) \end{bmatrix}, A_2 = \begin{bmatrix} (1,3,0) & (7,5,0) & (5,4,0) & (9,8,1) \\ (0,7,0) & (6,8,0) & (5,9,0) & (8,8,0) \\ (1,1,0) & (4,0,0) & (2,1,0) & (3,1,0) \\ (1,2,0) & (8,6,0) & (6,7,0) & (7,6,0) \end{bmatrix}$$