EXERCISE 2

- We have solved $A\vec{x}=\vec{b}$ with $A \in \mathbb{R}^{n\times n}$ and is intertible and $\vec{b} \in \mathbb{R}^n$. Since they have used the SQR Algorithm we have the QR decomposition of A.
- So this means we can solve problems of the form $A\vec{x} = \vec{b} \Leftrightarrow \vec{x} = A^{-1}\vec{b}$.
- So if we want to solve such problems we <u>dont</u> have to recalculate Q and R we only need to calculate steps 2 and 3 of the SQR algorithm:

Input $A \in \mathbb{R}^{n\times n}$, $\vec{b} \in \mathbb{R}^n$
Output $\vec{x} \in \mathbb{R}^n$ with $A\vec{x} = \vec{b}$
1. Find QR factorisation of A
→ 2. Solve $Q\vec{y} = \vec{b}$ using Algorithm MV: $y = Q^T\vec{b}$
→ 3. Solve $R\vec{x} = \vec{y}$ using Algorithm BS

So we have now to solve such problem only to use lines 2 and 3, which have number of operations:

step 2 : $2n^2$ operations
step 3 : n^2 operations

$\therefore$ compute cost of solving $A\vec{x} = \vec{b}$ is now $2n^2 + n^2 = 3n^2 = \underline{O(n^2)}$

- Now we want to solve $B\vec{z} = \vec{c}$, and we are given that by Sherman-Morison we have that:

$$B^{-1} = A^{-1} - \frac{A^{-1}\vec{u}\,\vec{v}^T A^{-1}}{1 + \vec{v}^T A^{-1}\vec{u}}$$

- so to solve $B\vec{z} = \vec{c}$ we can use:

$$\vec{z} = B^{-1}\vec{c}$$

$$\therefore\ \vec{z} = \left(A^{-1} - \frac{A^{-1}\vec{u}\,\vec{v}^T A^{-1}}{1+\vec{v}^T A^{-1}\vec{u}} \right) \vec{c}$$

$$\vec{z} = A^{-1}\vec{c} - \frac{A^{-1}\vec{u}\,\vec{v}^T A^{-1}\vec{c}}{1+\vec{v}^T A^{-1}\vec{u}}$$

- Now we can solve $A^{-1}\vec{c}$ using the property from before where we have A's QR factorisation and we use lines 2 and 3 with compute cost $O(n^2)$

we can solve $A^{-1}\vec{c} = \vec{w} \iff A\vec{w} = \vec{c}$
$A^{-1}\vec{u} = \vec{y} \iff A\vec{y} = \vec{u}$

So we can reduce our problem to: with cost $O(n^2)$

$$\vec{z} = \vec{w} - \frac{\vec{y}\,\vec{v}^T\vec{w}}{1-\vec{v}^T\vec{y}}$$

- now we can calculate the 2 dot products $\vec{v}^T\vec{w}$ and $\vec{v}^T\vec{y}$ each with compute cost $O(n)$, however since $n < n^2$ the overall cost is still $O(n^2)$

let $\vec{v}^T\vec{w} = \lambda$, $\vec{v}^T\vec{y} = \mu$ for some scalars λ, μ.

- $\vec{z} = \vec{w} - \frac{\vec{y}\lambda}{1-\mu}$, now $\frac{\lambda}{1-\mu}\vec{y}$ to calculate this will only be 1 division, 1 subtraction, and 1 multiplication to all elements of $\vec{y}$ $\therefore$ $1 + 1 + \sum_{i=1}^{n} 1 = 2+n = O(n)$ cost, as a result the overall cost is still $O(n^2)$

let $\frac{\vec{y}\lambda}{1-\mu} = \vec{\beta}$ $\therefore$ $\vec{z} = \vec{w} - \vec{\beta}$ subtracting one vector from another takes compute cost $O(n)$.

So as a result the overall cost is $O(n^2)$ which comes from solving the equations $A^{-1}\vec{c}$ and $A^{-1}\vec{u}$ and so solving $B\vec{z} = \vec{c}$ $O(n^2)$. Since all the other operations take compute cost $O(n)$ the cost of calulating $B\vec{z} = \vec{c}$ is $O(n^2)$.