Digital control system

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1 A servo Machine for antenna azimuth control

It is desired to control the elevation of an antenna designed to track a satellite. The equation of its motion is:

$$J\theta'' + B\theta' = T_{\mathrm{c}} + T_{\mathrm{d}} \tag{1}$$

Where J is the moment of inertia, B is the damping, $\mathrm{T_c}$ is the net torque from the drive motor and $\mathrm{T_d}$ is the disturbance torque due to the wind.If we define: B/J=a, u= $\mathrm{T_c}$/B, $\mathrm{w_d}$= $\mathrm{T_d}$/B the equation reduce to:

$$\frac{1}{a}\theta'' + \theta' = u + w_{\mathrm{d}} \tag{2}$$

and in s-transform domain the ... of the system is:

$$\theta(s) = \frac{1}{s(\frac{s}{a}+1)}[u(s) + w_{\mathrm{d}}(s)] \tag{3}$$

or, without disturbance:

$$\frac{\theta(s)}{u(s)} = \frac{1}{s(\frac{s}{a}+1)} = G(s) \tag{4}$$

In the discrete case with zero-order holder transformation the transfer function become:

$$G(z) = K\frac{(z+b)}{(z-1)(z-e^{-aT})} \tag{5}$$

where:

$$K = \frac{(aT - 1 + e^{-aT})}{a}, \quad b = \frac{1 - e^{-aT} - aTe^{-aT}}{aT - 1 + e^{-aT}} \tag{6}$$

Figure 1: The Universe

2 Specification of the system(cap 7)

To study the system we will assume that $\frac{1}{a} = 10$ sec. The aim of the design is to measure θ_a and compute $\mathrm{T_b}$ so that the error between the angle of the satellite θ_s and the antenna , namely $(\theta_\mathrm{s} - \theta_\mathrm{a})$, is always less than 0.01 rad during tracking. The satellite angle which must be followed may be adequately approximated by a fixed velocity $\theta_\mathrm{s}(t) = (0.01)t$. While the wind torque will be approximated by a step function; so the system become:

$$10\theta'' + \theta' = u + w_\mathrm{d}, \quad y = \theta_\mathrm{a} \tag{7}$$

3 State space equations

From equations (2) and (7) we can easily represent the system in state space equations in continuous time in the canonical form:

$$x' = Fx + Gu, \quad y = Hx \tag{8}$$

we chose as state of the system $x = (\theta \quad \theta')^T$, so the equations become:

$$\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix} + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} w_\mathrm{d} \tag{9}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix} \tag{10}$$

Then we can transform them in their discrete equivalent:

$$x(k+1) = \Phi x(k) + \Gamma u \quad y(k) = Hx(k) \tag{11}$$

It can be done with lot of emulations techniques with differents results, for example applying a ZOH with not delay we have: $\Phi = e^{FT}$, $\Gamma = \int_0^T e^{Fw} G dw$, T is the sampling period. The new matrices are:

$$\Phi = \begin{pmatrix} 1 & 0.9516 \\ 0 & 0.9048 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0.04837 \\ 0.09516 \end{pmatrix} \tag{12}$$

The transfer function of the open loop system is

$$G_1(z) = \frac{0.048(z+0.97)}{(z-1)(z-0.9048)} \tag{13}$$

Hence the system has eigenvalues (1.0, 0.9048), in particular z=1 act like an integrator so the system is not asymptotically stable, but simply stable.

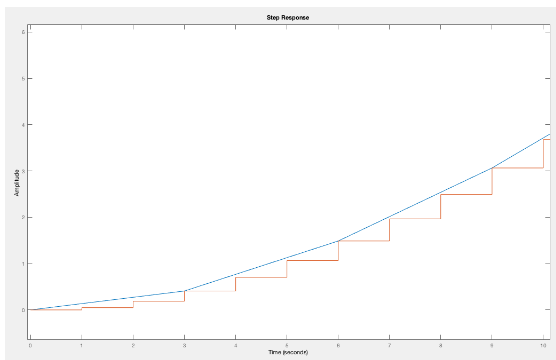


Figure 2: Open loop step rensponse in discrete and continue cases

4 Controllers

We now assume that we have all the states available, so we can proceed to the control-law design: the control law is simply the feedback of a linear combination of all the states, that is,

$$u = -Kx = -(K_1 K_2 ...) \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \end{pmatrix} \tag{14}$$

Substituting in the equation (11) we have:

$$x(k+1) = \Phi x(n) - \Gamma K x(n) \tag{15}$$

The characteristic equation of the controlled(closed loop) system is

$$det|zI - \Phi + k\Gamma| = 0 \tag{16}$$

The control- law design then consists of picking the elements of K so that the roots of (14) are in desirable locations. Given desired root locations, $\beta_1, \beta_2, \beta_3, ...$ the desired control-characteristic equation is

$$\alpha(z) = (z - \beta_1)(z - \beta_2)(z - \beta_3).... = 0 \tag{17}$$

Hence K are obtained by matching coefficients of (14) and (15). In our particular case we have a= 0.1, and we want roots in continuous domain posed in $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, so the correspondent discrete roots are $z = e^{-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}}$.

An alternative, and more convenient way to calculate K is use Ackermann formula

$$K = (0 \; ... \; 0 \; 1)(\Gamma \; \Phi\Gamma \; \Phi^2\Gamma \; ... \; \Phi^{n-1})^{-1}\alpha(\Phi) \tag{18}$$

before calculate the values of K we must first ensure that the system is controllable; To check the controllability of the system we need to calculate the rank of the so called controllability matrix:

$$(\Gamma \; \Phi\Gamma \; \Phi^2\Gamma \; ... \; \Phi^{n-1}) \tag{19}$$

In our system the controllability matrix is:

$$\begin{pmatrix} 0.0484 & 0.1389 \\ 0.0952 & 0.0861 \end{pmatrix} \tag{20}$$

and it is full rank so we can implement the eigenvalues assignment. After closing the loop we have a new Φ_K matrix equal to $\Phi - K\Gamma$:

$$\Phi_K = \begin{pmatrix} 0.7042 & 0.5332 \\ -0.582 & 0.08174 \end{pmatrix} \tag{21}$$

Where K is (6.1157, 8.6494); with these parameters we obtain the eigenvalues desired.

5 Predictor estimator

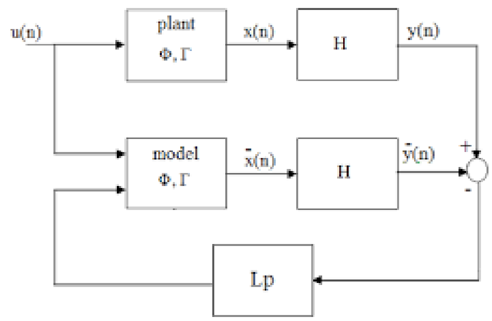
Looking the H matrix we can see that we can not measure all the state, but only θ value. So our pourpose is to reconstruct all the states given measurements of a portion of them. If the state is x, then the estimate is x̂, and the idea is to let u= -Kx̂, replacing the true states by their estimates in the control law.

To estimate the states that we can not measure we build a parallel model with respect to the plant, then we feedback the difference in the prediction between model and plant, as shown in figure.

And its equation is:

$$\hat{x}(n+1) = \Phi\hat{x}(n) + \Gamma u(n) + L[y(n) - H\hat{x}(n)] \tag{22}$$

We will call this a predictor estimator because the estimate, x̂(n+1), is one cycle ahead of the measurement, y(n). Typically L can be chosen so that the



Closed-loop Estimator

system is stable, and to select it we use the same approach seen for design the control law. If we specify the desired estimator(pay attention we L we can not modify the plant root locus, only the estimated one) root locations in the z-plane, L is uniquely determined, provided y is scalar and the system is observable.

As in control design two methods are avaible for computation of L. The first is match the coefficents in the equation

$$|zI - \Phi + LH| = (z - \beta_1)(z - \beta_2)...(z - \beta_n) \tag{23}$$

where β's are the desired estimator root locations. The second is to use Ackermann's formula:

$$L = \alpha(\Phi) \begin{pmatrix} H \\ H\Phi \\ H\Phi^2 \\ . \\ . \\ H\Phi^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix} \tag{24}$$

In our particular case we have $\alpha(z) = z^2$ as requested characteristic function, before calculate the values of L we must first ensure that the system is observable:

To check the observability of the system we need to calculate the rank of the so called observability matrix:

$$\begin{pmatrix} H \\ H\Phi \\ H\Phi^2 \\ . \\ . \\ H\Phi^{n-1} \end{pmatrix} \tag{25}$$

The observability matrix in our system is:

It is full rank so we can implement the observer; if the desired characteristic equation is $\alpha(z) = z^2$ the values of the parameters L_P calculated are $(1.9048, 0.8603)^T$.

Graphs ...

6 Regulator design: estimator and control law

If we take the control law(section 4) and implement it, using an estimated state vector(section 5), the control system can be completed.

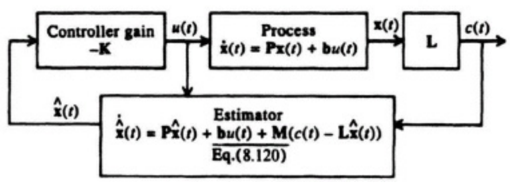


Figure 3: Estimator and controller mechanism

The controlled plant equation become

$$x(n+1) = \Phi x(n) - \Gamma K \hat{x}(n) \tag{26}$$

Its characteristic equation can be written as

$$|zI - \Phi + LH||zI - \Phi + \Gamma K| = \alpha_{estimated}(z) * \alpha_{controller}(z) = 0 \tag{27}$$

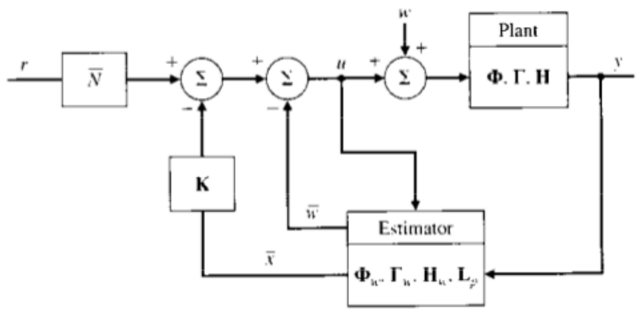
In other words, the characteristic equation roots of the combined system consist of the sum of the estimator roots and the controller roots. The fact that the combined control-estimator system has the same poles as those of the control alone and the estimator alone is a special case of a separation principle by which control and estimation can be designed separately yet used together.

So using L and K calculated in section 4 and section 5 we have already satisfy the regulator structure.

7 Disturbance estimation

Our approach is to estimate the disturbance signal in the estimator and then to use that estimate in the control law so as to force the error to zero. This approach is called **disturbance rejection**. After the estimate, ŵ, converges, the feedback of its value will cancel the actual disturbance, w, and the system will behave in the steady state as if no disturbance were present. If we assume the disturbance constant, as in our case, the resulting model is quite simple:

For purposes of disturbance estimation we augment the system model with the disturbance model:



$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma_u \\ 0 \end{pmatrix} u(k) \tag{28}$$

$$y(k) = \begin{pmatrix} H & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} \tag{29}$$

Nothing that in our particular case $\Gamma_w isequalto \Gamma_u$. In our particular system the state space matrices of the augmented estimator are:

$$\Phi_D = \begin{pmatrix} 1 & 0.9516 & 0.04837 \\ 0 & 0.9048 & 0.09516 \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma_D = \begin{pmatrix} 0.04837 \\ 0.09516 \\ 0 \end{pmatrix} \quad H_D = (1 \quad 0 \quad 0) \tag{30}$$

From them if we desire poles of the estimator in (0, 0.1, 0) we obtain L'_P equal to $(2.8, 2.18, 9.46)_T$

Then the estimation of the state,x̂, is passed to the controller while the estimate of the disturbance, ŵ, is passed to the input to be subtracted.

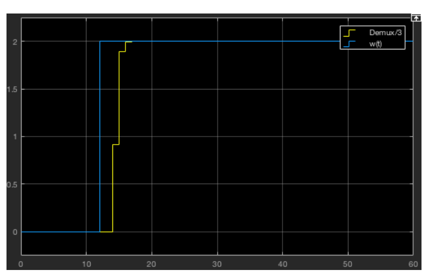


Figure 4: Estimation of a constant disturbance given by the wind gust

8 Reference input

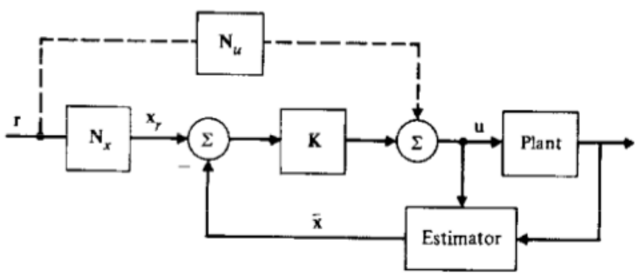
The structure of a reference input consist of a state command matrix $\mathrm{N_x}$ that defines the desired value of the state $\mathrm{x_r}$. We wish to find $\mathrm{N_x}$ such that some system output $\mathrm{y_r{=}H_r}$ x is at a desired reference value. The basic idea in determining $\mathrm{N_x}$ is that it should transform the reference input, r, to a reference state that is an equilibrium one for that r. More specifically, we have defined $\mathrm{N_r}$ so that

$$N_{\mathrm{x}}r = x_{\mathrm{r}} \quad and \quad u = -K(x - x_{\mathrm{r}}) \tag{31}$$

In order to solve all systems types, whether they require a steady-state control input or not, we will include the possibility of a steady state control term that is proportional to the reference input:

$$u_{\mathrm{ss}} = N_{\mathrm{u}}r \tag{32}$$

So the final structure of our system is:



The values of the two command matrices $\mathrm{N_x}$ and $\mathrm{N_u}$ can be easily obtained as :

$$\begin{pmatrix} N_{\mathrm{x}} \\ N_{\mathrm{u}} \end{pmatrix} = \begin{pmatrix} \Phi - I & \Gamma \\ H_{\mathrm{r}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{33}$$

Alternatively calculation of $\mathrm{N_x}$ and $\mathrm{N_u}$ can be carried out by the MATLAB function 'refi.m'. The new input of the system must be passed to the plan and at same time to the estimator. In our project the Eq.(26) we calculated $N_{\mathrm{x}} = (1, 0)^T$ and, $N_u = 6.1157$ the step response become...

9 DeadBeat controller

Nostro controllo ha errore nullo sull gradino ma errore constant sulla rampa, the transfer function of our db controller is

$$G_{DB}(z) = \frac{3.38(z^2 - 0.78z + 0.37)}{(z + 0.97)(z - 1)} \tag{34}$$

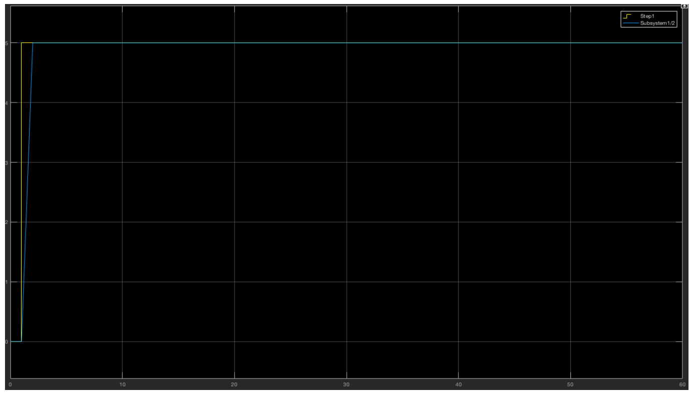


Figure 5: The deadbit step response

10 Conclusion

“I always thought something was fundamentally wrong with the universe” [1]

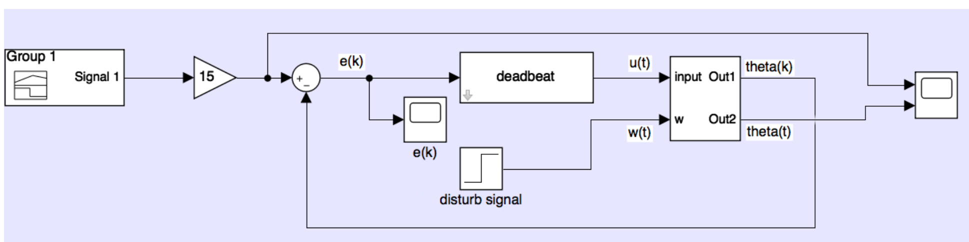


Figure 6: Final scheme

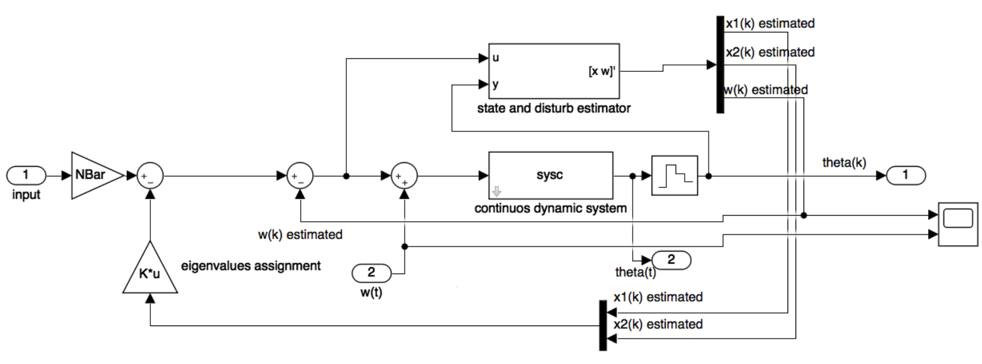


Figure 7: Final estimator

References

[1] D. Adams. *The Hitchhiker's Guide to the Galaxy*. San Val, 1995.