Digital Control System

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1 A servo Machine for antenna azimuth control

It is desired to control the elevation of an antenna designed to track a satellite. The equation of its motion is:

$$J\theta'' + B\theta' = T_{\mathrm{c}} + T_{\mathrm{d}} \tag{1}$$

Where J is the moment of inertia, B is the damping, T_c is the net torque from the drive motor and T_d is the disturbance torque due to the wind. If we define: B/J=a, u= T_c/B, w_d= T_d/B the equation reduce to:

$$\frac{1}{a}\theta'' + \theta' = u + w_{\mathrm{d}} \tag{2}$$

and in s-transform domain the transfer function of the system is:

$$\theta(s) = \frac{1}{s(\frac{s}{a}+1)}[u(s) + w_{\mathrm{d}}(s)] \tag{3}$$

or, without disturbance:

$$\frac{\theta(s)}{u(s)} = \frac{1}{s(\frac{s}{a}+1)} = G(s) \tag{4}$$

In the discrete case with zero-order holder transformation the transfer function becomes:

$$G(z) = K\frac{(z+b)}{(z-1)(z-e^{-aT})} \tag{5}$$

where:

$$K = \frac{(aT - 1 + e^{-aT})}{a}, \quad b = \frac{1 - e^{-aT} - aTe^{-aT}}{aT - 1 + e^{-aT}} \tag{6}$$

2 Specification of the system(cap 7)

To study the system we will assume that $\frac{1}{a} = 10$. The aim of the design is to measure θ_{a} and compute $\mathrm{T_b}$ so that the error between the angle of the satellite θ_{s} and the antenna, namely $(\theta_{\mathrm{s}} - \theta_{\mathrm{a}})$, is always less than 0.01 rad during tracking. The satellite angle which must be followed may be adequately approximated by a fixed velocity $\theta_{\mathrm{s}}(t) = (0.01)t$. While the wind torque will be approximated by a step function; so the system becomes:

$$\frac{1}{a}\theta'' + \theta' = u + w_{\mathrm{d}}, \quad y = \theta_{\mathrm{a}} \tag{7}$$

3 State space equations

From equations (2) and (7) we can easily represent the system in state space equations in continuous time in the canonical form:

$$x' = Fx + Gu, \quad y = Hx \tag{8}$$

we chose as state of the system $x = (\theta \quad \theta')^T$, so the equations become:

$$\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix} + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} w_{\mathrm{d}} \tag{9}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix} \tag{10}$$

Then we can transform them in their discrete equivalent:

$$x(k+1) = \Phi x(k) + \Gamma u \quad y(k) = Hx(k) \tag{11}$$

It can be done with lot of emulations techniques with different results, for example applying a ZOH with not delay we have: $\Phi = e^{FT}$, $\Gamma = \int_0^T e^{Fw} G dw$, where T is the sampling period. The new matrices are:

$$\Phi = \begin{pmatrix} 1 & 0.9516 \\ 0 & 0.9048 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0.04837 \\ 0.09516 \end{pmatrix} \tag{12}$$

The transfer function of the open loop system is

$$G_1(z) = \frac{0.048(z + 0.97)}{(z-1)(z-0.9048)} \tag{13}$$

Hence the system has eigenvalues (1.0, 0.9048), in particular z=1 acts like an integrator so the system is not asymptotically stable, but simply stable.

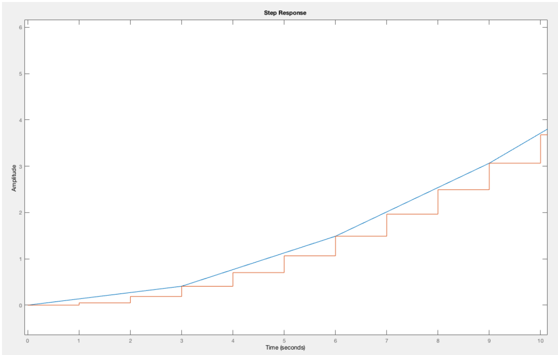


Figure 1: Open loop step rensponse in discrete and continue cases

4 Controllers

We now assume that we have all the states available, so we can proceed to the control-law design: the control law is simply the feedback of a linear combination of all the states, that is,

$$u = -Kx = -(K_1 K_2 ...) \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \end{pmatrix} \tag{14}$$

Substituting in the equation (11) we have:

$$x(k+1) = \Phi x(n) - \Gamma K x(n) \tag{15}$$

The characteristic equation of the controlled(closed loop) system is

$$det|zI - \Phi + k\Gamma| = 0 \tag{16}$$

The control- law design then consists of picking the elements of K so that the roots of (14) are in desirable locations. Given desired root locations $\beta_1, \beta_2, \beta_3, ...$ the desired control-characteristic equation is

$$\alpha(z) = (z - \beta_1)(z - \beta_2)(z - \beta_3).... = 0 \tag{17}$$

Hence K is obtained by matching coefficients of (16) and (17). In our particular case we have a= 0.1, and we want roots in continuous domain posed in $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, so the correspondent discrete roots are $z = e^{-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}}$.

An alternative, and more convenient way to calculate K is use Ackermann formula

$$K = (0 \; ... \; 0 \; 1)(\Gamma \; \Phi\Gamma \; \Phi^2\Gamma \; ... \; \Phi^{n-1})^{-1}\alpha(\Phi) \tag{18}$$

before calculate the values of K we must first ensure that the system is controllable; To check the controllability of the system we need to calculate the rank of the so called controllability matrix:

$$(\Gamma \ \Phi\Gamma \ \Phi^2\Gamma \ ... \ \Phi^{n-1}) \tag{19}$$

In our system the controllability matrix is:

$$\begin{pmatrix} 0.0484 & 0.1389 \\ 0.0952 & 0.0861 \end{pmatrix} \tag{20}$$

and it is full rank so we can implement the eigenvalues assignment. After closing the loop we have a new Φ_K matrix equal to $\Phi - K\Gamma$:

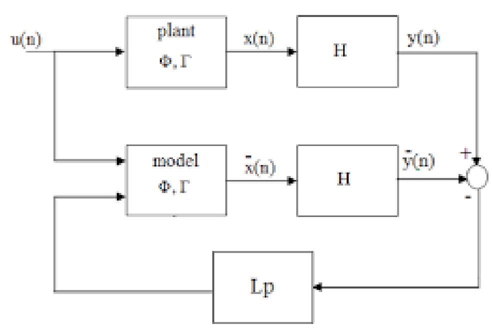
$$\Phi_K = \begin{pmatrix} 0.7042 & 0.5332 \\ -0.582 & 0.08174 \end{pmatrix} \tag{21}$$

Where K is (6.1157, 8.6494); with these parameters we obtain the eigenvalues desired.

5 Predictor estimator

Looking the H matrix we can see that we can not measure all the state, but only θ value. So our pourpose is to reconstruct all the states given measurements of a portion of them. If the state is x, then the estimate is x̂, and the idea is to let u= -Kx̂, replacing the true states by their estimates in the control law.

To estimate the states that we can not measure we build a parallel model with respect to the plant, then we feedback the difference in the prediction between model and plant, as shown in figure.



Closed-loop Estimator

And its equation is:

$$\hat{x}(n+1) = \Phi\hat{x}(n) + \Gamma u(n) + L[y(n) - H\hat{x}(n)] \tag{22}$$

We will call this a predictor estimator because the estimate, x̂(n+1), is one cycle ahead of the measurement, y(n). Typically L can be chosen so that the system is stable, and to select it we use the same approach seen for design the control law. If we specify the desired estimator (pay attention with L we can not modify the plant root locus, only the estimated one) root locations in the z-plane, L is uniquely determined, provided y is scalar and the system is observable.

As in control design two methods are avaible for computation of L. The first is match the coefficents in the equation

$$|zI - \Phi + LH| = (z - \beta_1)(z - \beta_2)...(z - \beta_n) \tag{23}$$

where β's are the desired estimator root locations. The second is to use Ackermann's formula:

$$L = \alpha(\Phi) \begin{pmatrix} H \\ H\Phi \\ H\Phi^2 \\ . \\ . \\ H\Phi^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix} \tag{24}$$

In our particular case we have $\alpha(z) = z^2$ as requested characteristic function, before calculate the values of L we must first ensure that the system is observable:

To check the observability of the system we need to calculate the rank of the so called observability matrix:

$$\begin{pmatrix} H \\ H\Phi \\ H\Phi^2 \\ . \\ . \\ H\Phi^{n-1} \end{pmatrix} \tag{25}$$

The observability matrix in our system is:

$$\begin{pmatrix} 1 & 0 \\ 0.7042 & 0.5332 \end{pmatrix} \tag{26}$$

It is full rank so we can implement the observer; if the desired characteristic equation is $\alpha(z) = z^2$ the values of the parameters L_P calculated are $(1.9048, 0.8603)^T$.

6 Regulator design: estimator and control law

If we take the control law(section 4) and implement it, using an estimated state vector(section 5), the control system can be completed.

The controlled plant equation become

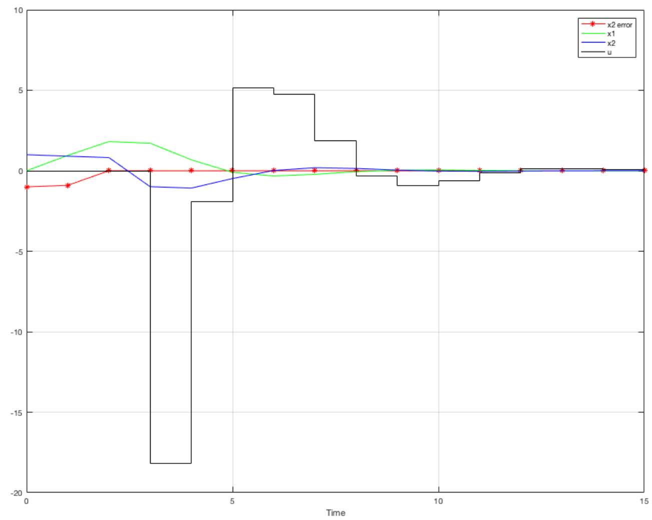


Figure 2: Time history of controlled system with predictor estimator.

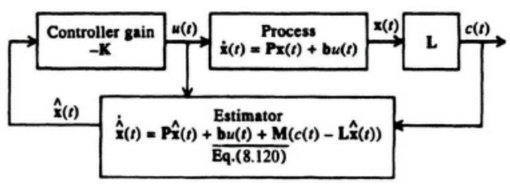


Figure 3: Estimator and controller mechanism

$$x(n+1) = \Phi x(n) - \Gamma K \hat{x}(n) \tag{27}$$

Its characteristic equation can be written as

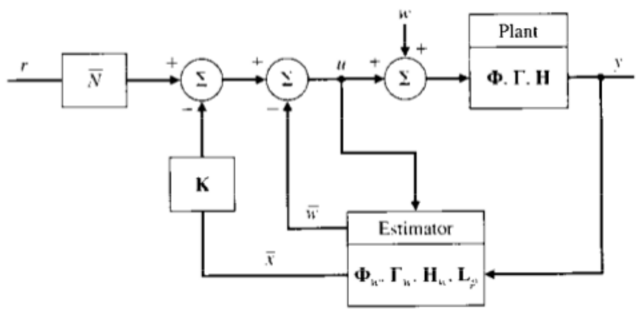
$$|zI - \Phi + LH||zI - \Phi + \Gamma K| = \alpha_{estimated}(z) * \alpha_{controller}(z) = 0 \tag{28}$$

In other words, the characteristic equation roots of the combined system consist of the sum of the estimator roots and the controller roots. The fact that the combined control-estimator system has the same poles as those of the control alone and the estimator alone is a special case of a separation principle by which control and estimation can be designed separately yet used together.

So using L and K calculated in section 4 and section 5 we have already satisfy the regulator structure.

7 Disturbance estimation

Our approach is to estimate the disturbance signal in the estimator and then to use that estimate in the control law so as to force the error to zero. This approach is called **disturbance rejection**. After the estimate, ŵ, converges, the feedback of its value will cancel the actual disturbance, w, and the system will behave in the steady state as if no disturbance were present. If we assume the disturbance constant, as in our case, the resulting model is quite simple:



For purposes of disturbance estimation we augment the system model with the disturbance model:

$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma_u \\ 0 \end{pmatrix} u(k) \tag{29}$$

$$y(k) = \begin{pmatrix} H & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} \tag{30}$$

Nothing that in our particular case Γ_w is equal to Γ_u. In our particular system the state space matrices of the augmented estimator are:

$$\Phi_D = \begin{pmatrix} 1 & 0.9516 & 0.04837 \\ 0 & 0.9048 & 0.09516 \\ 0 & 0 & 1 \end{pmatrix} \quad \Gamma_D = \begin{pmatrix} 0.04837 \\ 0.09516 \\ 0 \end{pmatrix} \quad H_D = (1 \quad 0 \quad 0) \tag{31}$$

From them if we desire poles of the estimator in (0, 0.1, 0) we obtain L'_P equal to $(2.8, 2.18, 9.46)^T$.

Then the estimation of the state, x̂, is passed to the controller while the estimate of the disturbance, ŵ, is passed to the input to be subtracted.

8 Reference input

The structure of a reference input consist of a state command matrix N_x that defines the desired value of the state x_r. We wish to find N_x such that some

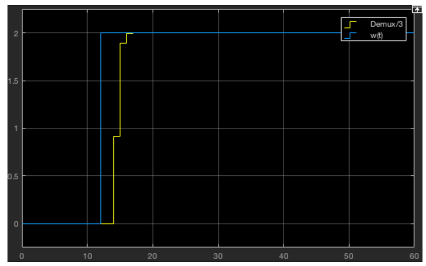


Figure 4: Estimation of a constant disturbance given by the wind gust

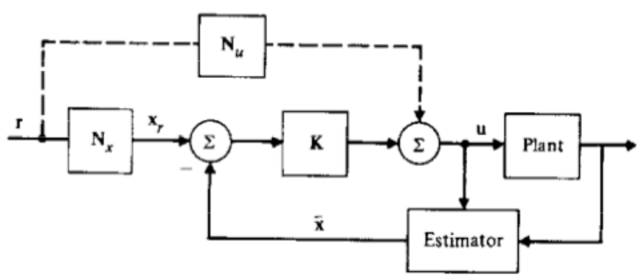
system output $y_r = H_r x$ is at a desired reference value. The basic idea in determining N_x is that it should transform the reference input, r, to a reference state that is an equilibrium one for that r. More specifically, we have defined N_r so that

$$N_x r = x_r \quad and \quad u = -K(x - x_r) \tag{32}$$

In order to solve all systems types, whether they require a steady-state control input or not, we will include the possibility of a steady state control term that is proportional to the reference input:

$$u_{ss} = N_u r \tag{33}$$

So the final structure of our system is:



The values of the two command matrices N_x and N_u can be easily obtained as :

$$\begin{pmatrix} N_x \\ N_u \end{pmatrix} = \begin{pmatrix} \Phi - I & \Gamma \\ H_r & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{34}$$

The new input of the system must be passed to the plan and at same time to the estimator. In our project the we calculated $N_x = (1, 0)^T$ and, $N_u = 6.1157$.

9 DeadBeat controller

In discrete-time control theory, the deadbeat control problem consists of the design of a controller that is able to compute commands to be applied to a system in order to bring the output to the desired steady state in the smallest number of time steps.

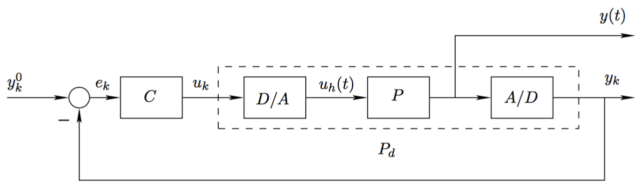


Figure 5: The deadbit schema

The design of this controller is subjected to the consideration of system's specifications, in fact the structure will be different if the components of the system are all discrete or instead mixed so even discrete but also continuous. In our case, we chose to construct a controller that is discrete but with a controlled plant that works on sampled data system. The transfer function of the sampled plant is given by:

$$P(z) = \frac{0.3(z + 0.96)}{z^2 - 0.8 + 0.37} \tag{35}$$

m: number of zeros=1
n: number of poles=2(system is stable since poles are inside the unitary circle)

To satisfy the criteria of a deadbeat control, it is necessary to follow these conditions:
1. The system must have zero steady-state error at sampling instants.

$$W_e(z) = \frac{K_d^2}{K_d + G(z)P(z)} \tag{36}$$

we must impose that the transfer function of the error has one zero at z=1 and that all the poles are placed at the origin.
2. The time to reach final output must be finite and minimum.
3. The controller should be physically realizable, i.e., it should be causal so the order of the numerator must be less or equal to the numerator.
Being n-m = 1 to realize a system with minimum response time it is necessary to assume:

$$C(z) = \frac{K_d}{P(z)(z-1)} = \frac{1}{0.3} \frac{z^2 - 0.8 + 0.37}{(z + 0.96)(z - 1)} \tag{37}$$

And we get the closed loop transfer function as:

$$T(z) = \frac{1}{z} \tag{38}$$

Thus, for unit step input, the output comes out to be:

$$Y(z) = \frac{z}{z(z-1)} = z^{-1} + z^{-2} + ... \tag{39}$$

The output y(k) represents a unit step response where k starts from 1, that is one sample later. In other words, y(k) reaches the desired steady state value in one sampling period without any overshoot and stays there for ever.

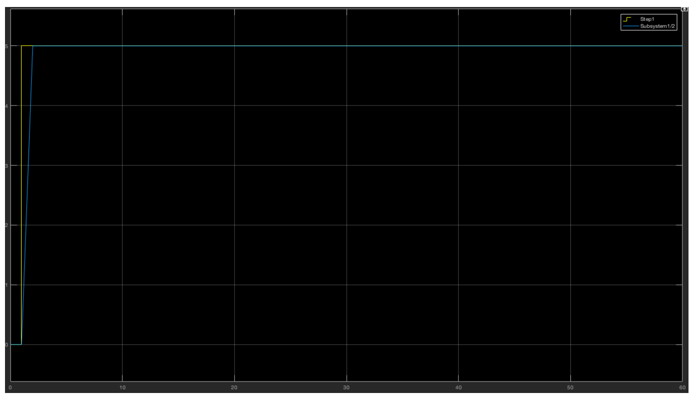


Figure 6: Unit step response

The figure 6 shows the behaviour of the system. It is, sometimes, not very efficient, in fact the response is zero error only in t=hT instants of time and it is impossible influence the dynamic between intervals of the sampling time, this peculiarity causes the phenomena of rippling.

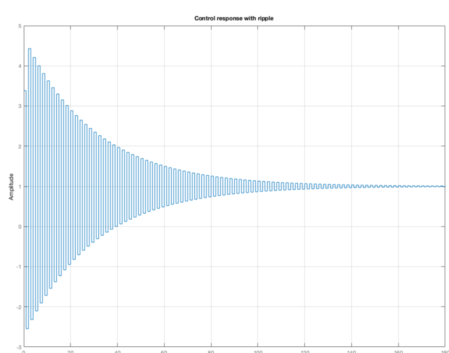


Figure 7: Control signal.

As we can see in figure 7 the control signal oscillates and this effect is undesired because it has effects on the continuous output signal of the system $y(t)$.

So, if we want a system that has steady response without RIPPLES for a step perturbation the controller must be modified. In addition to the conditions listed above we have to impose that also the transfer function from R to U have all its pole at the origin:

$$e'(h) = 0 forh \geq c_0 \quad m'(h) = PK_d forh \geq 0 \tag{40}$$

Where:

$$P = \lim_{s \to 0} \frac{1}{P(s)} \tag{41}$$

Take the plant transfer function P(z) and analyze the poles:

$$P(z) = \frac{0.29(z + 0.97)}{(z^2 - 0.78 + 0.37)} \tag{42}$$

The two poles:

$$z_1 = 0.39 - 0.46i \quad z_2 = 0.39 + 0.46i \tag{43}$$

Since m=1 and n=2, so n-m=1 in this case we have that controller can be structured in this way:

$$P(z) = \frac{b_1 z + b_0}{M(z)} \tag{44}$$

$$C(z) = \frac{M(z) * d_0}{c - c_0} \tag{45}$$

Where:

$$M(z) = \frac{z^2 - 0.78 + 0.36}{z - 1} \quad d_0 = (b_0 + b_1)^{-1} \quad b_0 = 0.28 \quad b_1 = 0.29 \quad c_0 = d_0 * b_0 \tag{46}$$

Finally we have:

$$C(z) = 1.71 \frac{z^2 - 0.78 + 0.36}{(z + 0.49)(z - 1)} \tag{47}$$

What we see in this case is that the control signal, Figure 9, settles in finite time and that also the reference output, Figure 8,settles in finite time. As is possible to see in Fig. 8 the output signal it's going to take two time step to get to the final value, at the instant k=1 assume an intermediate value and finally at k=2 the signal reach the final value. The difference is that in this case the output takes two time step to settle to the final value instead than 1. On the other hand, the control signal has a more desirable behaviour, it is going to settle in two time step as well and the corresponding y(t) now is not being

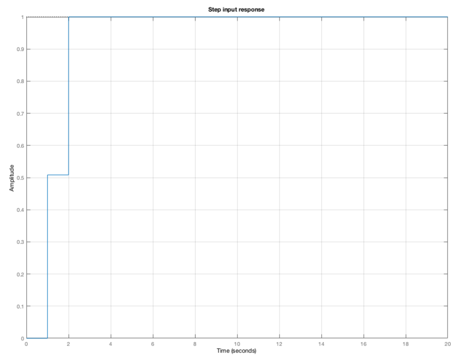


Figure 8: The deadbeat step response

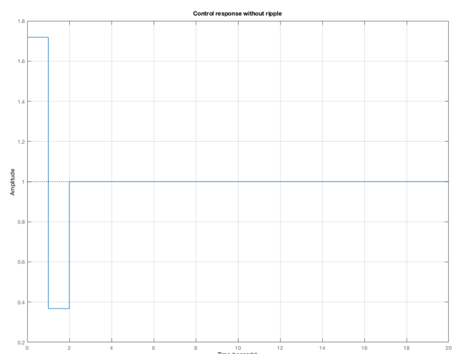


Figure 9: Control signal

excited by an oscillation in the control and so there isn't the inter-sampling ripple effect.

Let now extends the problem of deadbeat controller to another type of input, a ramp input. Then again, following previous steps we can construct rippled-deadbeat controller for ramp input. Considering our ramp input in the z-transform, we have:

$$R(z) = \frac{z}{(z-1)^2} \tag{48}$$

So, we desired that the transfer function of the system M(z) will be structured, as before we did:

$$M(z) = \frac{2z-1}{z^2} \tag{49}$$

In order to obtain a controller that makes the transfer function of the system in closed loop as M(z) and so respect all the specific listed above, zero steady state error in finite steps and zero error at sampling instants, the Deadbeat controller must have this structure:

$$C(z) = \frac{M(z)}{P(z)(1 - M(z))} = \frac{6.76(z - 0.5)(z^2 - 0.78z + 0.37)}{(z + 0.97)(z - 1)^2} \tag{50}$$

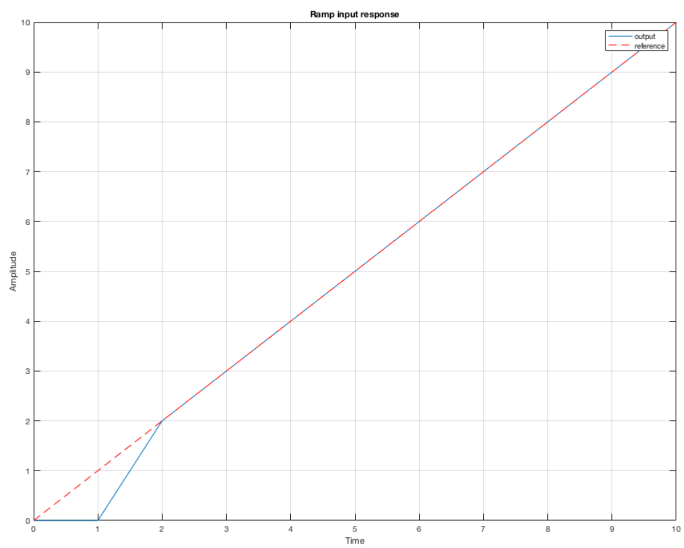


Figure 10: Ramp response

As we can see now in Figure 10, the response of the system is quite good for ramp input. In two instants of time the system reaches the specific signal ramp. The controller output is shown in Fig.11.

9.1 Considerations on deadbeat:

The synthesis procedure used to obtain the deadbeat controller is based on purely theoretical results. The control laws so found sometimes do not find a practical use in reality. In fact, they could lead to overly aggressive control strategies. It is always good to keep in mind that if on the one hand we want high performances, on the other this involves a great control effort. In deadbeat control as you make the time step shorter the magnitude of the control-strikes can become very large, sometimes not achievable by a real actuator. So, the strategy about applying a dead-beat controller without being too aggressive could be to not require a time step too small, in order to avoid a huge request of energy, and instead let your time step to be as big as you can. Since the control forces the output to arrive at the desired value in a finite number of steps, we will go to consider if this number of steps correspond to a time that is acceptable or not for our performance requirements.

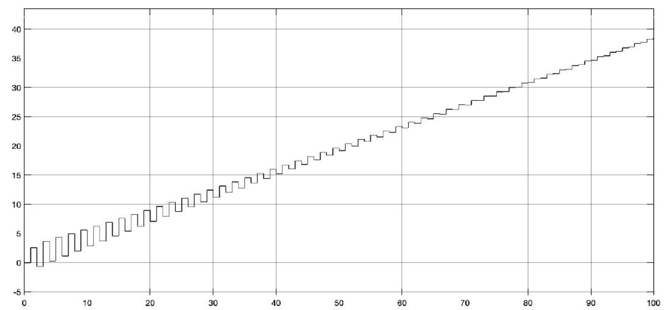


Figure 11: Control signal

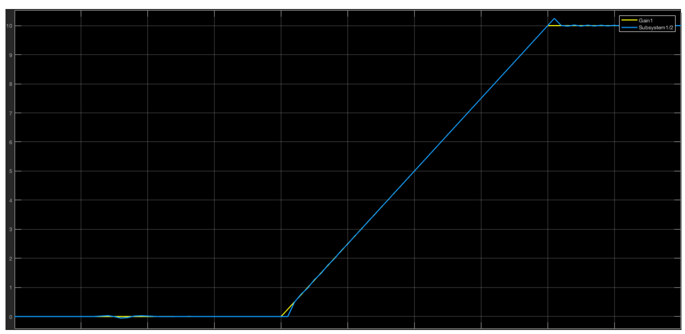


Figure 12: Performance of the controlled system with the deadbeat controller. With the yellow as reference input, the blue as output.

10 Conclusion

In this project starting from the differential equation of a servo motor for an antenna, we transformed the equation in discrete system. We developed the controllers and the estimator, then we link them together and create the regulator. We use a disturbance estimator to correct the noise given by the wind gust. Then we applied the gain scheme to obtain unitary gain on the open loop system. At the end we closed an external control loop using the deadbeat controller in order to achieve a zero error at steady state for a ramp input. In this paper are summarized what we did and the results we obtained.

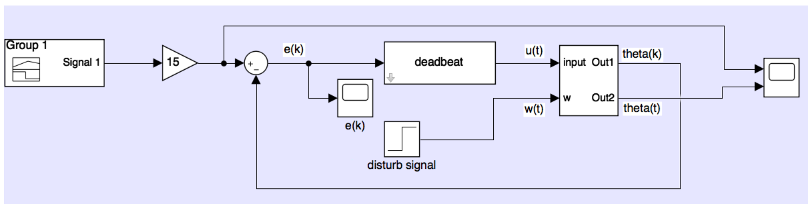


Figure 13: External Loop.

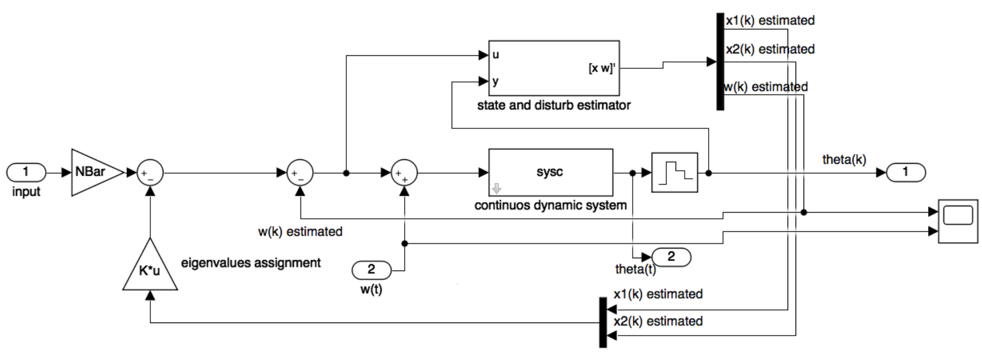


Figure 14: Internal Loop.