Derivatives
FIN-404

Project Report

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Spring Semester - 2025

Contents

0 Documentation **1**
0.1 Brief history of volatility trading . . . 1
0.2 Purpose and utility of variance derivatives . . . 1
0.3 CBOE volatility index (VIX) . . . 1
0.4 Main variance derivatives with use cases . . . 2
0.5 Distinguishing variance derivatives . . . 2
0.6 Distinguishing VIX futures, variance swaps, and variance futures . . . 3

1 The Carr-Madan formula **3**
1.1 Question 1 . . . 3
1.2 Question 2 . . . 3
1.3 Question 3 . . . 4
1.4 Question 4 . . . 5
1.5 Question 5 . . . 5

2 The VIX index **5**
2.1 Question 1 . . . 5
2.2 Question 2 . . . 5
2.3 Question 3 . . . 6
2.4 Question 4 . . . 6

3 Futures pricing **7**
3.1 Question 1 . . . 7
3.1.1 Deriving and solving the ODE . . . 7
3.2 Question 2 . . . 8
3.2.1 Apply Itô's lemma to derive the PDE . . . 8
3.2.2 Solve PDE . . . 8
3.2.3 Separate variables to get ODEs . . . 9
3.2.4 The ODEs to solve . . . 9
3.2.5 Conclusion . . . 9
3.3 Question 3 . . . 9
3.3.1 Variance futures pricing . . . 9
3.3.2 VIX futures pricing . . . 10
3.4 Question 4 . . . 10
3.4.1 Application to VIX futures prices . . . 10
3.5 Question 5 . . . 11
3.5.1 Relationship between the VIX index and variance futures . . . 11
3.5.2 Arbitrage when this relation fails . . . 11
3.5.3 Case 1: VIX is too high . . . 11
3.5.4 Case 2: VIX is too low . . . 12
3.6 Question 6 . . . 12
3.7 Question 7 . . . 12
3.8 Question 8 . . . 13
3.9 Question 9 . . . 13
3.10 Question 10 . . . 13
3.11 Code . . . 14

0 Documentation

0.1 Brief history of volatility trading

The evolution of financial markets to incorporate explicit volatility trading instruments commenced in the late 1980s and early 1990s. Before this period, the expression of a directional view on market volatility was limited to implicit mechanisms, primarily through the use of options. A fundamental characteristic of a long option position is its inherent dual exposure: sensitivity to the directional movement of the underlying asset and sensitivity to its implied volatility. To extract a pure volatility exposure, market participants were compelled to implement intricate, delta-neutral hedging strategies. These sophisticated portfolio constructions, while theoretically sound, proved to be operationally cumbersome and economically inefficient, due to the continuous rebalancing requirements and associated transaction costs.
The first significant innovation was the emergence of the variance swap in the 1990s. These private, customized contracts allowed two parties to exchange a pre-agreed (fixed) variance level for the actual (floating) realized variance of an underlying asset over a specified period. A landmark moment arrived in 1993 when the CBOE introduced the original VIX Index. However, the true revolution occurred in 2003 when the CBOE revamped the VIX methodology to be based on a wider portfolio of S&P 500 options. This paved the way for exchange-traded derivatives.
The CBOE launched VIX futures in 2004 and VIX options in 2006. More recently, exchanges have also introduced Variance Futures, which settle to the actual realized variance of the index, offering a listed alternative to OTC variance swaps. Today, VIX derivatives are among the most actively traded products in the world, serving as a central hub for managing equity market risk.

0.2 Purpose and utility of variance derivatives

Variance and volatility derivatives exist because investors and risk managers need to isolate and manage the risk of market turbulence itself, independent of the market's direction. Their primary functions are:

- **Hedging:** Equity portfolios are inherently short volatility; their value tends to decrease when market volatility increases. This is often referred to as the negative correlation between index returns and volatility. A long position in a variance derivative can act as an effective hedge against a portfolio's value loss during a market downturn.
- **Speculation:** Volatility can itself be a source of profit. Traders can use variance derivatives to express pure views on the future level of market turbulence. If a trader believes the market is underestimating future risk, they can go long volatility.
- **Diversification:** Because of its strong negative correlation with equity returns, adding a long volatility position to a traditional portfolio can enhance its risk-adjusted returns.

0.3 CBOE volatility index (VIX)

The VIX is often dubbed the "fear gauge." It is a real-time index that represents the market's expectation of 30-day forward-looking volatility of the S&P 500 index [SPX], and is not based on historical price movements (realized volatility). It is calculated using this formula:

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2$$

where T is the time to expiration (years) ; F is the forward price of the SPX ; K_i is the strike price of the i-th option, chosen such that it is a call option if $K_i > K_0$, a put option if $K_i < K_0$, and both

a call and a put option if $K_i = K_0$; $\Delta K_i = (K_{i+1} - K_{i-1})/2$; $Q(K_i)$ is the midpoint of the bid-ask spread for each option with strike K_i ; and finally R is the risk-free rate to expiration.
The VIX index value is the square root of this variance expressed in percentage $VIX = 100 \times \sigma$. For example, a VIX level of 20 corresponds to an expectation of a 20% annualized standard deviation in the S&P 500 over the next 30 days.

0.4 Main variance derivatives with use cases

VIX futures: These are exchange-traded contracts that obligate the buyer to purchase the VIX Index at a predetermined price on a future date. They allow investors to trade on the future level of implied volatility.
Use Case: A hedge fund manager believes the current VIX of 15 is too low. They buy a VIX future at 17, speculating that market fear will rise by the future's expiration.

VIX options: These are options on the VIX Index itself, providing the right to buy or sell (i.e. call or put) the VIX at a specific strike price.
Use Case: An investor wants to protect their portfolio against a sudden, sharp market crash but finds buying VIX futures too costly due to the upward-sloping futures curve (contango). Instead, they could buy an out-of-the-money VIX call option. This provides a cheaper, leveraged way to gain upside exposure to volatility, acting as a form of insurance that pays out only during a significant spike in fear.

Variance swaps: These are OTC contracts where one party pays a fixed amount of variance (the strike) in exchange for receiving the realized variance of the underlying asset over the contract period. Hence Payoff = $(\sigma^2_{\text{Realized}} - \sigma^2_{\text{Strike}}) \times$ Currency amount per unit of variance
Use Case: An institution wants to hedge the exact realized volatility of its S&P 500-linked products for the next quarter, eliminating basis risk related to implied volatility.

Variance futures: These are exchange-traded contracts that settle to the realized variance of an underlying asset over a specific period. They are a listed alternative to OTC variance swaps.
Use Case: An asset manager wants to hedge realized volatility over the next calendar quarter without taking on the counterparty risk of an OTC swap. They buy a variance future today that measures and settles on the S&P 500's realized variance for that future period.

0.5 Distinguishing variance derivatives

Feature	VIX futures	VIX options	Variance swap	Variance futures
What it measures	**Forward implied volatility:** The future expected value of the VIX index.	**Non-linear payoff on forward implied volatility:** The right to buy or sell the VIX index at a future date for a specific price.	**Realized volatility:** The actual volatility that occurs over the contract term.	**Forward realized volatility:** The actual volatility that will occur over a future period.
Underlying	VIX index.	VIX index.	SPX index.	SPX index.
Nature of exposure	Linear bet on the future level of the VIX index (implied).	Leveraged, non-linear (convex) bet on the future level of the VIX index.	Payoff from realized vs fixed variance. Realized volatility exposure.	Linear bet on the future level of realized volatility.
Trading Venue	Exchange-traded (CBOE).	Exchange-traded (CBOE).	Over-the-counter (OTC).	Exchange-traded (CBOE).

0.6 Distinguishing VIX futures, variance swaps, and variance futures

The distinctions between these tradable instruments are critical. The most fundamental dividing line is between implied and realized volatility.
VIX futures are derivatives on an index of implied volatility (the VIX). A position in a VIX future reflects a position on market sentiment and expectations about the future. An investor could be correct that the market will be volatile (high realized volatility) but still lose money on a long VIX futures position if expectations for the next 30 days fall at the time the future expires.
Variance swaps and futures, by contrast, are pure plays on realized volatility. Their settlement value is determined by the statistical variance that actually occurs over a pre-defined period, regardless of market sentiment.

The primary difference between a **variance swap** and a **variance future** lies in their market structure and risk profile. Variance swaps are private, customizable OTC contracts that carry counterparty risk. Variance futures are standardized, exchange-traded instruments cleared through a central counterparty, thus eliminating direct counterparty risk and increasing transparency. Furthermore, swaps are often spot-starting, whereas futures are typically forward-starting, meaning the period over which variance is measured begins at a future date.

1 The Carr-Madan formula

1.1 Question 1

Let $H : \mathbb{R}^+ \to \mathbb{R}$ be a function that is piecewise twice continuously differentiable on the positive real line.
Consider the following integral:

$$\int_{x_0}^{x} H''(k)(x-k)\,dk = [H'(k)(x-k)]_{x_0}^{x} + \int_{x_0}^{x} H'(k)\,dk \quad \text{(by integration by parts)}$$

Evaluating the terms:

$$= -H'(x_0)(x-x_0) + H(x) - H(x_0) \quad \text{(by the Fundamental Theorem of Calculus)}$$

Thus, we have:

$$H(x) = H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{x} H''(k)(x-k)\,dk$$

1.2 Question 2

Since $(x-y) = (x-y)^+ - (y-x)^+$, we can write:

$$\int_{x_0}^{x} H''(k)(x-k)\,dk = \int_{x_0}^{x} H''(k)(x-k)^+\,dk - \int_{x_0}^{x} H''(k)(k-x)^+\,dk = I_1 + I_2$$

Let us now focus on I_1 and I_2.

First, I_1:

$$I_1 = \int_{x_0}^{x} H''(k)(x-k)^+\,dk = \int_{x_0}^{\infty} H''(k)(x-k)^+\,dk - \int_{x}^{\infty} H''(k)(x-k)^+\,dk$$

The second term is zero because $k > x$ implies $(x-k)^+ = 0$.
Hence: $I_1 = \int_{x_0}^{\infty} H''(k)(x-k)^+\,dk$

Next, I_2:

$$I_2 = -\int_{x_0}^{x} H''(k)(k-x)^+\,dk = \int_{0}^{x_0} H''(k)(k-x)^+\,dk - \int_{0}^{x} H''(k)(k-x)^+\,dk$$

The second term is zero because $k < x$ implies $(k-x)^+ = 0$.
Hence: $I_2 = \int_0^{x_0} H''(k)(k-x)^+\,dk$
Thus, we have:

$$H(x) = H(x_0) + H'(x_0)(x-x_0) + \int_{x_0}^{\infty} H''(k)(x-k)^+\,dk + \int_0^{x_0} H''(k)(k-x)^+\,dk$$

1.3 Question 3

The payoff is:

$$H(S_T) = H(x_0) + H'(x_0)(S_T - x_0) + \int_0^{x_0} H''(k)(k-S_T)^+\,dk + \int_{x_0}^{\infty} H''(k)(S_T-k)^+\,dk$$

Since the market is complete (one underlying asset), this payoff is attainable. In other words, there exists a self-financing strategy to replicate it. We can define the portfolio X_t. We have at terminal value $X_T = H(S_T)$.
We therefore have the following (the discounted payoff is a Q-martingale): $X_t = e^{-r(T-t)} E_t^Q[H(S_T)]$
So:

$$X_0 = e^{-rT} H(x_0) + H'(x_0) E_0^Q[e^{-rT}(S_T - x_0)] + \int_{x_0}^{\infty} H''(k) p(0, S_0; k) dk + \int_0^{x_0} H''(k) c(0, S_0; k) dk$$

$$X_0 = e^{-rT}(H(x_0) - H'(x_0)x_0) + H'(x_0) e^{-rT} E_0^Q[S_T] + \int_{x_0}^{\infty} H''(k) p(0, S_0; k) dk + \int_0^{x_0} H''(k) c(0, S_0; k) dk$$

Note that: $S_T = S_t e^{(r-\delta)(T-t) - \frac{1}{2}\int_t^T V_u du + \int_t^T \sqrt{V_u} dB_u^Q}$ and $E_t^Q[S_T] = S_t e^{(r-\delta)(T-t)}$ where r and δ are constant.
The portfolio at time $t=0$ is therefore:

$$X_0 = e^{-rT}(H(x_0) - H'(x_0)x_0) + H'(x_0) e^{-\delta T} S_0 + \int_{x_0}^{\infty} H''(k) p(0, S_0; k) dk + \int_0^{x_0} H''(k) c(0, S_0; k) dk$$

We can replicate this with the following static portfolio :

	At date 0	At date T
Riskless asset	$e^{-rT}(H(x_0) - H'(x_0)x_0)$	$H(x_0) - H'(x_0)x_0$
Underlying asset	$H'(x_0)e^{-\delta T} S_0$	$H'(x_0) S_T$
Put ($k \le x_0$)	$\int_0^{x_0} H''(k) p(0, S_0; k)\,dk$	$\int_0^{x_0} H''(k)(k - S_T)^+\,dk$
Call ($k > x_0$)	$\int_{x_0}^{\infty} H''(k) c(0, S_0; k)\,dk$	$\int_{x_0}^{\infty} H''(k)(S_T - k)^+\,dk$

where $c(t, S_t; k)$ and $p(t, S_t; k)$ represent the prices of a European call and put option, respectively, on the underlying asset S_t, with strike price k.

To summarize:

- The number of units of the underlying asset is $n_0 = e^{-\delta T} H'(x_0)$,
- The amount invested in the riskless asset is $a_0 = e^{-rT}\,(H(x_0) - H'(x_0)x_0)$,
- The number of units of the put and call options with strike k is weighted by $w(k) = H''(k)$.

This strategy theoretically requires purchasing an infinite number of puts and calls.

Recall that the forward price $F_0(T)$ is the strike price k' such that the payoff $E_0^Q[e^{-rT}(S_T - k')] = 0$.
Therefore, if $x_0 = F_0(T)$, then $E_0^Q[e^{-rT}(S_T - k')] = 0$, and we obtain:

$$X_0 = e^{-rT} H(F_0(T)) + \int_{F_0(T)}^{\infty} H''(k) p(0, S_0; k) dk + \int_0^{F_0(T)} H''(k) c(0, S_0; k) dk$$

We can replicate with the following static portfolio:

	At date 0	At date T
Riskless asset	$e^{-rT}H(F_0(T))$	$H(F_0(T))$
Underlying asset	0	0
Put ($k \leq F_0(T)$)	$\int_0^{F_0(T)} H''(k)p(0, S_0; k)\, dk$	$\int_0^{F_0(T)} H''(k)(k - S_T)^+\, dk$
Call ($k > F_0(T)$)	$\int_{F_0(T)}^{\infty} H''(k)c(0, S_0; k)\, dk$	$\int_{F_0(T)}^{\infty} H''(k)(S_T - k)^+\, dk$

We then have $n_0 = 0$, $a_0 = e^{-rT}H(F_0(T)$ and $w(k) = H''(k)$.

1.4 Question 4

We distinguish two cases:
If $p \neq 0$: We have $H(x) = x^p$, $H'(x) = px^{p-1}$, and $H''(x) = p(p-1)x^{p-2}$. The replication strategy is given by:

$$n_0 = px_0^{p-1}, \quad a_0 = e^{-rT}\left(x_0^p - x_0 \cdot p \cdot x_0^{p-1}\right) = e^{-rT}x_0^p(1-p), \quad w(k) = p(p-1)k^{p-2}$$

If $p = 0$: Then $H(x) = 1$, $H'(x) = 0 \Rightarrow n_0 = 0$, $H''(x) = 0 \Rightarrow w(k) = 0$, and $a_0 = e^{-rT}$. Since the payoff is constant (equal to 1) and is independent of S_T, it correspond to a bond $B_t(T)$. So (r constant) we have $B_0(T) = a_0 = e^{-rT}$.

1.5 Question 5

One key limitation of the Carr–Madan result lies in the **availability of strike prices**. The replication strategy derived from the Carr–Madan framework requires a continuum of European options with strikes spanning the entire positive real line. However, in practice, only a finite and discrete set of strike prices is available in financial markets. This discreteness introduces approximation errors in the replication of the desired payoff function.

2 The VIX index

2.1 Question 1

Let's assume that the level of the SPX evolves according to: $\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}dB_t^Q$
where δ and r are constant.
We therefore have: $\int_t^T \frac{dS_s}{S_s} = (r-\delta)(T-t) + \int_t^T \sqrt{V_s}dB_s^Q$
By applying Itô's lemma, we also have: $dlog(S_t) = \frac{\partial log(S_t)}{\partial S_t}dS_t + \frac{\partial^2 log(S_t)}{\partial {S_t}^2}(dS_t)^2$
We can deduce: $dlog(S_t) = \frac{1}{S_t}dS_t + -\frac{1}{2S_t^2}(dS_t)^2 = (r - \delta - \frac{V_t}{2})dt + \sqrt{V_t}dB_t^Q$
Therefore: $log(\frac{S_T}{S_t}) = (r-\delta)(T-t) + \int_t^T \sqrt{V_s}dB_s^Q - \frac{1}{2}\int_t^T V_s ds$
By identification: $log(\frac{S_T}{S_t}) = \int_t^T \frac{dS_s}{S_s} - \frac{1}{2}\int_t^T V_s ds$
By rearranging the terms:

$$x\overline{V}_{t,T} = \frac{1}{2}\int_t^T V_s ds = \int_t^T \frac{dS_s}{S_s} - \log(\frac{S_T}{S_t})$$

where $x = \frac{1}{2}$.

2.2 Question 2

Let's consider: $E_t^Q[\int_t^T \frac{dS_s}{S_s}]$
By the above formula we have: $E_t^Q[\int_t^T \frac{dS_s}{S_s}] = (r-\delta)(T-t) + E_t^Q[\int_t^T \sqrt{V_s}dB_s^Q]$
Since: $E_t^Q[\int_t^T \sqrt{V_s}dB_s^Q] = 0$ (V_t is adapted), we have: $E_t^Q[\int_t^T \frac{dS_s}{S_s}] = \alpha(T-t) = (r-\delta)(T-t)$
where $\alpha = r - \delta$.

2.3 Question 3

Consider a function $H(x) = log(\frac{x}{S_t})$, which is piecewise twice continuously differentiable on $\mathbb{R}^+$, where $x > 0$.
By the Carr-Madan formula, we have:

$$H(x) = H(x_0) + H'(x_0)(x - x_0) + \int_{x_0}^{\infty} H''(k)(x-k)^+ \, dk + \int_0^{x_0} H''(k)(k-x)^+ \, dk$$

Thus, we have: $H(x) = log(\frac{x_0}{S_t}) + \frac{1}{x_0}(x - x_0) - \int_{x_0}^{\infty} \frac{1}{k^2}(x-k)^+ \, dk - \int_0^{x_0} \frac{1}{k^2}(k-x)^+ \, dk$
With $x_0 = K_0$ and $x = S_T$, we have:

$$H(S_T) = log(\frac{K_0}{S_t}) + \frac{1}{K_0}(S_T - K_0) - \int_{K_0}^{\infty} \frac{1}{k^2}(S_T-k)^+ \, dk - \int_0^{K_0} \frac{1}{k^2}(k-S_T)^+ \, dk$$

Since the market is complete, the payoff $H(S_T)$ is attainable, and the discounted payoff is therefore a Q-martingale. We have that:

$$E_t^Q[\frac{H(S_T)}{e^{rT}}] = E_t^Q[e^{-rT}log(\frac{K_0}{S_t}) + \frac{e^{-rT}}{K_0}(S_T - K_0) - \int_{K_0}^{\infty} \frac{1}{k^2}e^{-rT}(S_T-k)^+ \, dk - \int_0^{K_0} \frac{1}{k^2}e^{-rT}(k-S_T)^+ \, dk]$$

$$= e^{-rT}log(\frac{K_0}{S_t}) + \frac{e^{-rT}}{K_0}(E_t^Q[S_T] - K_0) - \int_{K_0}^{\infty} \frac{1}{k^2}E_t^Q[e^{-rT}(S_T-k)^+] \, dk - \int_0^{K_0} \frac{1}{k^2}E_t^Q[e^{-rT}(k-S_T)^+] \, dk$$

Since the discounted payoff of a call (and put) is a Q-martingale, we have:

$$E_t^Q[\frac{H(S_T)}{e^{rT}}] = e^{-rT}log(\frac{K_0}{S_t}) + \frac{e^{-rT}}{K_0}(E_t^Q[S_T] - K_0) - \int_{K_0}^{\infty} \frac{1}{k^2}e^{-rt}Call_t(T,k) \, dk - \int_0^{K_0} \frac{1}{k^2}e^{-rt}Put_t(T,k) \, dk$$

Notice also that since r and δ are constant we have $F_t(T) = E_t^Q[S_T]$.
To conclude we have:

$$E_t^Q[log(\frac{S_T}{S_t})] = E_t^Q[H(S_T)] = e^{rT}E_t^Q[\frac{H(S_T)}{e^{rT}}] = log(\frac{K_0}{S_t}) - (1 - \frac{F_t(T)}{K_0}) - P_t(T, K_0)$$

where: $P_t(T, K_0) = \int_{K_0}^{\infty} \frac{1}{k^2}e^{r(T-t)}Call_t(T,k) \, dk + \int_0^{K_0} \frac{1}{k^2}e^{r(T-t)}Put_t(T,k) \, dk$

2.4 Question 4

By question 1, $x\overline{V}_{t,T} = \int_t^T \frac{dS_s}{S_s} - \log(\frac{S_T}{S_t})$, we can find $E_t^Q[x\overline{V}_{t,T}]$ as follow:

$$E_t^Q[x\overline{V}_{t,T}] = E_t^Q[\int_t^T \frac{dS_s}{S_s}] - E_t^Q[log(\frac{S_T}{S_t})]$$

By replacing by the previous formulas: $xE_t^Q[\overline{V}_{t,T}] = (r-\delta)(T-t) - log(\frac{K_0}{S_t}) + (1 - \frac{F_t(T)}{K_0}) + P_t(T, K_0)$
Noting that since r is constant, we have $F_t(T) = E_t^Q[S_T] = S_te^{(r-\delta)(T-t)}$
So $(r-\delta)(T-t) = log(\frac{F_t(T)}{S_t})$
Therefore: $xE_t^Q[\overline{V}_{t,T}] = log(\frac{F_t(T)}{S_t}) - log(\frac{K_0}{S_t}) + (1 - \frac{F_t(T)}{K_0}) + P_t(T, K_0)$
We conclude that: $xE_t^Q[\overline{V}_{t,T}] = P_t(T, K_0) + log(\frac{F_t(T)}{K_0}) + (1 - \frac{F_t(T)}{K_0})$
We can use the second order Taylor expansion of the log: $log(1+x) \approx x - \frac{x^2}{2}$
Thus we have:

$$xE_t^Q[\overline{V}_{t,T}] = P_t(T, K_0) + log(1 + (\frac{F_t(T)}{K_0} - 1)) + (1 - \frac{F_t(T)}{K_0})$$

$$xE_t^Q[x\overline{V}_{t,T}] \approx P_t(T, K_0) + (\frac{F_t(T)}{K_0} - 1) - \frac{(\frac{F_t(T)}{K_0} - 1)^2}{2} + (1 - \frac{F_t(T)}{K_0})$$

$$xE_t^Q[\overline{V}_{t,T}] \approx P_t(T, K_0) - \frac{(\frac{F_t(T)}{K_0} - 1)^2}{2}$$

The VIX formula is literally the risk-neutral expectation of the integrated variance over the next period (commonly defined as 30 days). This expression involves, in theory, an infinite continuum of call and put option prices on the SPX index. In practice, a discrete set of strikes is used to approximate this integral. Hence, the VIX directly relies on the market's prices of future options, reflecting the market's forward-looking expectation of volatility.
In contrast, the **realized variance** only accounts for past price movements and does not incorporate any forward-looking information from option prices. Therefore, it cannot capture potential upcoming market changes or shocks. The Black-Scholes **implied volatility**, on the other hand, assumes a constant variance and is typically derived from a single call option's price. This approach is limited because it does not incorporate the full range of information embedded in options at multiple strikes and maturities, unlike the VIX formula which is more comprehensive.

3 Futures pricing

3.1 Question 1

We need to show that $\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}(a + bV_T)$ for some constants a and b.

Given the squared volatility process: $dV_t = \lambda(\theta - V_t)dt + \xi\rho\sqrt{V_t}dB_t^Q + \xi\sqrt{|1-\rho^2|}\sqrt{V_t}dZ_t^Q$

We first integrate both sides from time t to some future time s (where $s > t$) and then take the conditional expectation $E^Q[\cdot|V_t] = E_t^Q$ of both sides:

$$E_t^Q[V_s]-E_t^Q[V_t] = E_t^Q[V_s]-V_t = E_t^Q[\int_t^s \lambda(\theta-V_u)du]+E_t^Q[\int_t^s \xi\rho\sqrt{V_u}dB_u^Q]+E_t^Q[\int_t^s \xi\sqrt{|1-\rho^2|}\sqrt{V_u}dZ_u^Q]$$

Since B_t^Q and Z_t^Q are Q-Brownian motions, the stochastic integrals have zero expectation:

$$E_t^Q[V_s] - V_t = E_t^Q[\int_t^s \lambda(\theta - V_u)du] = \int_t^s \lambda(\theta - E_t^Q[V_u])du$$

The last equality follows from Fubini's theorem and linearity of expectation.

3.1.1 Deriving and solving the ODE

Define $m(s) = E_t^Q[V_s]$ for $s \geq t$.
Then our equation above yields: $m(s) - V_t = \int_t^s \lambda(\theta - m(u))du$
Differentiating both sides with respect to s yields the ordinary differential equation:

$$\frac{dm(s)}{ds} = \lambda(\theta - m(s))$$

The initial condition is $m(t) = V_t$, since the expectation at time t equals the known value.
This is a first-order linear differential equation with constant coefficients, which has the general solution: $m(s) = \theta + Ce^{-\lambda(s-t)}$ where C is a constant.
Applying the initial condition $m(t) = V_t$: $V_t = \theta + C \implies C = V_t - \theta$
Therefore: $m(s) = \theta + (V_t - \theta)e^{-\lambda(s-t)}$
Which gives us the conditional expectation: $E_t^Q[V_s] = \theta + (V_t - \theta)e^{-\lambda(s-t)}$
Using this result, we can compute the expected integrated variance at time T using Fubini's theorem:

$$E_T^Q\left[\int_T^{T+\eta} V_u du\right] = \int_T^{T+\eta} E_T^Q[V_u]du = \int_T^{T+\eta} [\theta + (V_T - \theta)e^{-\lambda(u-T)}]du$$

$$= \theta\eta + (V_T - \theta)\left[-\frac{1}{\lambda}e^{-\lambda(u-T)}\right]_T^{T+\eta} = \theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right) + V_T\frac{1-e^{-\lambda\eta}}{\lambda}$$

Identifying Parameters a and b from the problem statement, we know: $\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}E_T^Q\left[\int_T^{T+\eta} V_u du\right]$

Substituting our result: $$\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}\left[\theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right) + V_T\frac{1-e^{-\lambda\eta}}{\lambda}\right]$$

Comparing with the desired form $\frac{1}{\eta}(a + bV_T)$, we identify: $a = \theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right)$ and $b = \frac{1-e^{-\lambda\eta}}{\lambda}$

Therefore, we have shown that $\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}(a + bV_T)$ where a and b are expressed in terms of η, θ, and λ as required.

3.2 Question 2

We aim to show that for some functions c and d in terms of $(\rho, \lambda, \theta, \xi)$:

$$-\log \mathbb{E}_t^Q[e^{-sV_T}] = c(T-t; s) + d(T-t; s)V_t, \quad s > 0$$

Following the hint, we define: $f(t, V_t) = \mathbb{E}_t^Q[e^{-sV_T}]$

Since $f(t, V_t)$ is a Q-martingale, it has zero drift under Q.

3.2.1 Apply Itô's lemma to derive the PDE

Applying Itô's lemma to $f(t, V_t)$: $df(t, V_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial v}dV_t + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}d\langle V\rangle_t$

Substituting the SDE for V_t: $dV_t = \lambda(\theta - V_t)dt + \xi\rho\sqrt{V_t}dB_t^Q + \xi\sqrt{|1-\rho^2|}\sqrt{V_t}dZ_t^Q$

The quadratic variation term is:

$$d\langle V\rangle_t = \xi^2\rho^2 V_t dt + \xi^2|1-\rho^2|V_t dt = \xi^2 V_t[\rho^2 + |1-\rho^2|]dt = \xi^2 V_t dt$$

This simplification occurs because for $\rho \in [-1, 1]$, we have $\rho^2 \leq 1$, which means $1 - \rho^2 \geq 0$.

Therefore $|1-\rho^2| = 1 - \rho^2$, and $\rho^2 + |1-\rho^2| = \rho^2 + (1-\rho^2) = 1$

Since terms in $dtdB_t^Q, dtdt, dtdZ_t^Q, dB_t^Q dZ_t^Q = 0$ we only have terms in $(dB_t^Q)^2, (dZ_t^Q)^2 = dt$

Therefore:

$$df(t, V_t) = \underbrace{[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial v}\lambda(\theta - V_t) + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\xi^2 V_t]}_{\text{drift term}}dt + \frac{\partial f}{\partial v}\underbrace{[\xi\rho\sqrt{V_t}dB_t^Q + \xi\sqrt{|1-\rho^2|}\sqrt{V_t}dZ_t^Q]}_{\text{volatility term}}$$

Since $f(t, V_t)$ is a martingale, the drift term must be zero and this is the PDE that $f(t, v)$ must satisfy:

$$\frac{\partial f}{\partial t} + \lambda(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2 f}{\partial v^2} = 0$$

3.2.2 Solve PDE

We know $-\log \mathbb{E}_t^Q[e^{-sV_T}] = -\log f(t, V_t) = c(T-t; s) + d(T-t; s)V_t, \quad s > 0$

Hence we have the form: $f(t, v) = e^{-(c(T-t;s)+d(T-t;s)v)}$

Let $\tau = T - t \implies d\tau = -dt$ a change of variable.

Hence we have: $f(t, v) = e^{-[c(\tau;s)+d(\tau;s)v]}$ Computing the partial derivatives:

- $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = -\frac{\partial f}{\partial \tau} = (c'(\tau) + d'(\tau)v)\cdot f$
- $\frac{\partial f}{\partial v} = -d(\tau)\cdot f$
- $\frac{\partial^2 f}{\partial v^2} = d(\tau)^2\cdot f$

We then plug these values into our original PDE: $\frac{\partial f}{\partial t} + \lambda(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2 f}{\partial v^2} = 0$

We get: $(c'(\tau) + d'(\tau)v)f + \lambda(\theta - v)(-d(\tau)f) + \frac{1}{2}\xi^2 v d(\tau)^2 f = 0$

Dividing by f (which is non-zero): $c'(\tau) + d'(\tau)v - \lambda\theta d(\tau) + \lambda v d(\tau) + \frac{1}{2}\xi^2 v d(\tau)^2 = 0$

3.2.3 Separate variables to get ODEs

Grouping terms: $[c'(\tau) - \lambda\theta d(\tau)] + v[d'(\tau) + \lambda d(\tau) + \frac{1}{2}\xi^2 d(\tau)^2] = 0$
Since this must hold for all v, both bracket terms must equal zero:

- $c'(\tau) - \lambda\theta d(\tau) = 0 \implies c'(\tau) = \lambda\theta d(\tau) \implies c'(T-t) = -\lambda\theta d(T-t)$
- $d'(\tau) + \lambda d(\tau) + \frac{1}{2}\xi^2 d(\tau)^2 = 0 \implies d'(\tau) = -\lambda d(\tau) - \frac{1}{2}\xi^2 d(\tau)^2$
  $\implies d'(T-t) - \lambda d(T-t) - \frac{1}{2}\xi^2 d(T-t)^2 = 0$

With initial conditions: $c(0; s) = 0$ and $d(0; s) = s$

3.2.4 The ODEs to solve

For function $d(\tau; s)$, we used wolfram alpha to solve this 1st order linear ODE:

$$d(T-t, s) = -\frac{2\lambda s e^{\lambda(T-t)}}{s\xi^2(e^{\lambda(T-t)} - 1) - 2\lambda}$$

For function $c(T-t; s)$,we integrate with respect to t, and we plug the previously found value of $d(T-t, s)$ to find

$$c(T-t, s) = -\lambda\theta \int d(T-t, s)dt = \lambda\theta \int \frac{2\lambda s e^{\lambda(T-t)}}{s\xi^2(e^{\lambda(T-t)} - 1) - 2\lambda} dt$$

since $s, t, T, \lambda, \theta, \xi > 0$ and $c(0, s) = 0$ is our initial condition

$$c(T-t, s) = \frac{2\theta\lambda}{\xi^2} \left(\log((2\lambda) - \left(e^{\lambda(T-t)} - 1\right) s\xi^2) - \log(2\lambda)\right)$$

3.2.5 Conclusion

Given that $f(t, v) = e^{-(c(T-t;s)+d(T-t;s)v)} = \mathbb{E}_t^Q[e^{-sV_T}]$, we can take the negative logarithm of both sides and for $d(\tau)$ and $c(\tau)$ found above:

$$-\log(\mathbb{E}_t^Q[e^{-sV_T}]) = -\log(e^{-(c(T-t;s)+d(T-t;s)V_t)}) = c(T-t; s) + d(T-t; s)V_t$$

We have successfully demonstrated that: $-\log \mathbb{E}_t^Q[e^{-sV_T}] = c(T-t; s) + d(T-t; s)V_t$ for $s > 0$

3.3 Question 3

We need to show that:

- $f_t^{VA}(T) = \frac{10{,}000}{T-t_0} \left(\int_{t_0}^t V_u du + a^*(T-t) + b^*(T-t)V_t\right)$
- $f_t^{VIX}(T) = \frac{100}{\sqrt{\eta}} \mathbb{E}_t^Q \left[\sqrt{a' + b'V_T}\right]$

for some functions a^*, b^* and constants a', b' to be determined.

3.3.1 Variance futures pricing

The futures price at time t is the risk-neutral expectation of the terminal value:

$$f_t^{VA}(T) = \mathbb{E}_t^Q[f^{VA}(T)] = \mathbb{E}_t^Q \left[\frac{10,000}{T-t_0} \int_{t_0}^T V_u du\right]$$

We split the integral into the $\mathcal{F}_t$-measurable part and the remaining part :

$$f_t^{VA}(T) = \frac{10,000}{T-t_0} \left(\int_{t_0}^t V_u du + \mathbb{E}_t^Q \left[\int_t^T V_u du\right]\right)$$

The first term $\int_{t_0}^{t} V_u du$ is known at time t because of the filtration $\mathcal{F}_t$.
For the second term, we use our result from question 1 where we found: $\mathbb{E}_t^Q[V_s] = \theta + (V_t - \theta)e^{-\lambda(s-t)}$
Using the same reasoning as in question 3.1.3, this allows us to compute:

$$\mathbb{E}_t^Q\left[\int_t^T V_u du\right] = \int_t^T \mathbb{E}_t^Q[V_u]du = \int_t^T [\theta + (V_t - \theta)e^{-\lambda(u-t)}]du$$

$$= \theta\left[(T-t) - \frac{1-e^{-\lambda(T-t)}}{\lambda}\right] + \frac{V_t(1-e^{-\lambda(T-t)})}{\lambda}$$

Hence we have the following: $\mathbb{E}_t^Q\left[\int_t^T V_u du\right] = a^*(T-t) + b^*(T-t)V_t$
where: $a^*(T-t) = \theta\left[(T-t) - \frac{1-e^{-\lambda(T-t)}}{\lambda}\right]$ and $b^*(T-t) = \frac{1-e^{-\lambda(T-t)}}{\lambda}$
Therefore:

$$f_t^{VA}(T) = \frac{10,000}{T-t_0}\left(\int_{t_0}^{t} V_u du + a^*(T-t) + b^*(T-t)V_t\right)$$

with the above defined $a^*(T-t)$ and $b^*(T-t)$

3.3.2 VIX futures pricing

From question 1, we know that:$(\frac{VIX_T}{100})^2 = \frac{1}{\eta}(a + bV_T) \implies VIX_T = 100\sqrt{\frac{a+bV_T}{\eta}}$
where: $a = \theta[\eta - \frac{1}{\lambda}(1-e^{-\lambda\eta})]$ and $b = \frac{1}{\lambda}(1-e^{-\lambda\eta})$

The VIX futures price is the risk-neutral expectation:
$f_t^{VIX}(T) = \mathbb{E}_t^Q[VIX_T] = 100\mathbb{E}_t^Q\left[\sqrt{\frac{a+bV_T}{\eta}}\right] = \frac{100}{\sqrt{\eta}}\mathbb{E}_t^Q[\sqrt{a' + b'V_T}]$ with $a' = a$ and $b' = b$

3.4 Question 4

We define from the question: $f(x) = \int_0^\infty \frac{1-e^{-sx}}{\sqrt{s^3}} ds$
Then its derivative, $f'(x)$, is given by: $f'(x) = \int_0^\infty \frac{\partial}{\partial x}\left(\frac{1-e^{-sx}}{\sqrt{s^3}}\right) ds = \int_0^\infty \frac{se^{-sx}}{\sqrt{s^3}} ds = \int_0^\infty \frac{e^{-sx}}{\sqrt{s}} ds$
Using the Laplace transform identity: $\int_0^\infty \frac{e^{-sx}}{\sqrt{s}} ds = \sqrt{\frac{\pi}{x}}, \quad x > 0 \implies f'(x) = \sqrt{\frac{\pi}{x}}$
We then integrate to find f(x): $f(x) = \int f'(x)dx = \int \sqrt{\frac{\pi}{x}} dx = 2\sqrt{\pi} \cdot \sqrt{x} + C$
When $x = 0$:

- $f(0) = \int_0^\infty \frac{1-e^{-s0}}{\sqrt{s^3}} ds = \int_0^\infty \frac{1-1}{\sqrt{s^3}} ds = 0$
- From our expression: $f(0) = 2\sqrt{\pi} \cdot \sqrt{0} + C = 0 + C = C \implies C = 0$

Which implies that: $f(x) = 2\sqrt{\pi} \cdot \sqrt{x} = \int_0^\infty \frac{1-e^{-sx}}{\sqrt{s^3}} ds \implies \sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1-e^{-sx}}{\sqrt{s^3}} ds$
This matches the form requested.

3.4.1 Application to VIX futures prices

Applying this to the VIX futures calculation:

$$f_t^{VIX}(T) = \frac{100}{\sqrt{\eta}}\mathbb{E}_t^Q[\sqrt{a' + b'V_T}] = \frac{100}{\sqrt{\eta}} \cdot \frac{1}{2\sqrt{\pi}}\mathbb{E}_t^Q\left[\int_0^\infty \frac{1 - e^{-s(a'+b'V_T)}}{\sqrt{s^3}} ds\right]$$

By Fubini's theorem (exchanging expectation and integration): $f_t^{VIX}(T) = \frac{50}{\sqrt{\pi\eta}} \int_0^\infty \frac{1-\mathbb{E}_t^Q[e^{-s(a'+b'V_T)}]}{\sqrt{s^3}} ds$
From question 2, we have: $-\log \mathbb{E}_t^Q[e^{-sV_T}] = c(T-t;s) + d(T-t;s)V_t$
This means: $\mathbb{E}_t^Q[e^{-sV_T}] = e^{-c(T-t;s)-d(T-t;s)V_t}$
For our expression $s(a' + b'V_T)$, we decompose: $\mathbb{E}_t^Q[e^{-s(a'+b'V_T)}] = e^{-sa'}\mathbb{E}_t^Q[e^{-sb'V_T}]$

Applying our result from question 2 with sb' instead of s: $\mathbb{E}_t^Q[e^{-sb'V_T}] = e^{-c(T-t;sb')-d(T-t;sb')V_t}$
Therefore: $\mathbb{E}_t^Q[e^{-s(a'+b'V_T)}] = e^{-sa'-c(T-t;sb')-d(T-t;sb')V_t}$
This gives us: $\ell(s, T-t, V_t) = sa' + c(T-t; sb') + d(T-t; sb')V_t$ with the functions c and d having the following closed forms:

- $d(T-t; sb') = \frac{2\lambda sb'}{e^{\lambda(T-t)}(2\lambda+\xi^2 sb')-\xi^2 sb'}$
- $c(T-t; sb') = -\frac{2\theta\lambda}{\xi^2}\left(\lambda(T-t) + \log(2\lambda) - \log(e^{\lambda(T-t)}(2\lambda + \xi^2 sb') - \xi^2 sb')\right)$

Where ℓ is expressed using the correct forms of functions c and d as derived in question 2.
Therefore, the VIX futures price can be computed as: $f_t^{VIX}(T) = \frac{50}{\sqrt{\pi\eta}} \int_0^\infty \frac{1-e^{-\ell(s,T-t,V_t)}}{\sqrt{s^3}} ds$

3.5 Question 5

3.5.1 Relationship between the VIX index and variance futures

To determine the relationship between the VIX index and variance futures with maturity $T - t = \eta$
From the document, we know the VIX index satisfies: $\left(\frac{\text{VIX}_t}{100}\right)^2 = \frac{1}{\eta}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u\, du\right]$
And that variance futures satisfy: $f_t^{VA}(T) = \frac{10{,}000}{T-t_0}\left(\int_{t_0}^t V_u du + \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T V_u du\right]\right)$
For $T = t + \eta$: $f_t^{VA}(t+\eta) = \frac{10{,}000}{t+\eta-t_0}\left(\int_{t_0}^t V_u du + \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u du\right]\right)$
Now, using the VIX formula: $\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u du\right] = \eta\left(\frac{\text{VIX}_t}{100}\right)^2$
Therefore, the relationship is:

$$f_t^{VA}(t+\eta) = \frac{10,000}{t+\eta-t_0}\left(\int_{t_0}^t V_u du + \eta\left(\frac{\text{VIX}_t}{100}\right)^2\right)$$

3.5.2 Arbitrage when this relation fails

If this relation fails, it indicates a market price mismatch and therefore an arbitrage opportunity:

- If VIX is too high relative to variance futures: Sell VIX derivatives (options/futures) and buy variance futures
- If VIX is too low: Buy VIX derivatives and sell variance futures

Let's construct an explicit arbitrage strategy for when the relationship fails.
We'll use: $f_t^{VA}(t+\eta) = \frac{10{,}000}{t+\eta-t_0}\left(\int_{t_0}^t V_u du + \left(\frac{VIX_t}{100}\right)^2 \cdot \eta\right)$.
At $t = t_0$, we have: $f_{t_0}^{VA}(t_0+\eta) = VIX_t^2$

3.5.3 Case 1: VIX is too high

Suppose we have: $f_t^{VA}(t+\eta) < \frac{10{,}000}{t+\eta-t_0}\left(\int_{t_0}^t V_u du + \left(\frac{VIX_t}{100}\right)^2 \cdot \eta\right)$
Then we can create the following arbitrage strategy:

- Sell VIX futures at time t for delivery at $t+\eta$
- Buy variance futures with the same maturity $t+\eta$
- At maturity $t+\eta$:
  - VIX futures settle at $VIX_{t+\eta}$
  - Variance futures settle at $\frac{10{,}000}{t+\eta-t_0}\int_{t_0}^{t+\eta} V_u du$

This arbitrage strategy gives the following cash flows:

- Initial cash flow = 0 (futures contracts don't require premium to enter or exit)
- Terminal profit = $\frac{10{,}000}{t+\eta-t_0}\int_{t_0}^{t+\eta} V_u du - VIX_{t+\eta}^2 \cdot \eta > 0$

This strategy involves zero initial cash flow but yields a strictly positive terminal payoff, hence we have a working arbitrage strategy.

3.5.4 Case 2: VIX is too low

Now suppose: $f_t^{VA}(t+\eta) > \frac{10{,}000}{t+\eta-t_0}\left(\int_{t_0}^{t} V_u du + \left(\frac{VIX_t}{100}\right)^2 \cdot \eta\right)$

We can create the following arbitrage strategy:

- Buy VIX futures at time t for delivery at $t+\eta$
- Sell variance futures with the same maturity $t+\eta$
- At maturity $t+\eta$:
  - VIX futures settle at $VIX_{t+\eta}$
  - Variance futures settle at $\frac{10{,}000}{t+\eta-t_0}\int_{t_0}^{t+\eta} V_u du$

This arbitrage strategy results in the following cash flows:

- Initial cash flow = 0 (futures contracts don't require premium to enter or exit)
- Terminal profit $= VIX_{t+\eta}^2 \cdot \eta - \frac{10{,}000}{t+\eta-t_0}\int_{t_0}^{t+\eta} V_u du > 0$

We have a strategy that cost 0 initial cash flow but with terminal payoff > 0, hence we have a working arbitrage strategy.

3.6 Question 6

The sensitivity of the VIX futures prices versus (λ, θ, ξ) show to be non-linear, as expected from the formulas we derived, with Base Params: $\lambda = 2, \theta = 0.04, \xi = 0.4$

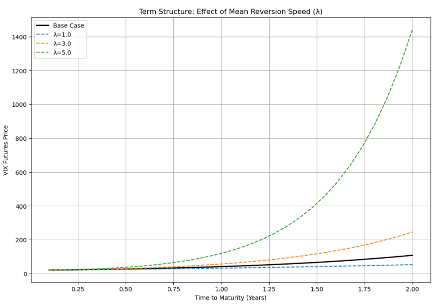

Figure 1: *
(a) Lambda vs Time (VIX)

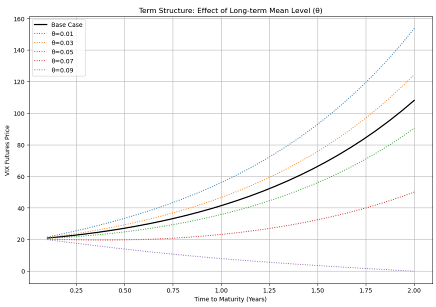

Figure 2: *
(b) Theta vs Time (VIX)

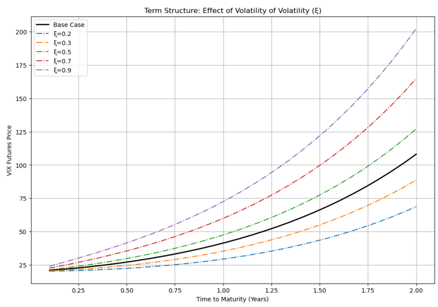

Figure 3: *
(c) Xi vs Time (VIX)

3.7 Question 7

The sensitivity of the Variance futures prices versus (λ, θ, ξ) show to be non-linear,except ξ which has no influence since it does not appear in the variance formula, with Base Params: $V_t = 0.04, \lambda = 2, \theta = 0.08, \xi = 0.4$

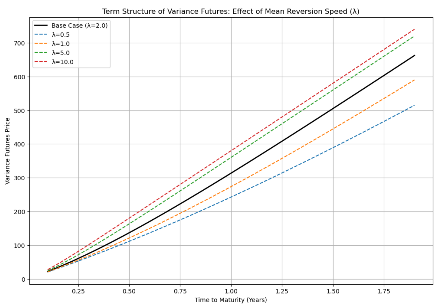

Figure 4: *
(a) Lambda vs Time

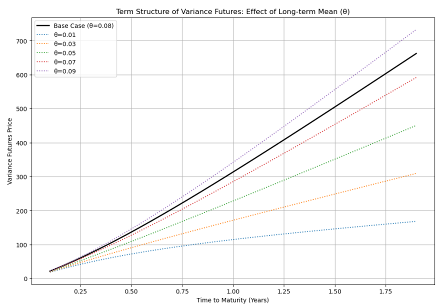

Figure 5: *
(b) Theta vs Time

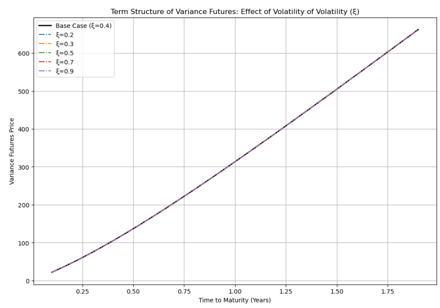

Figure 6: *
(c) Xi vs Time

3.8 Question 8

We used a mean-squared reversion method to compute the best-fit parameters for the market prices. Our model outputed $\lambda = 1.4111, \theta = 0.071, \xi = 0.2837, V_t = 0.049$. The RMSE (Root Mean Squared Error) is satisfying too.

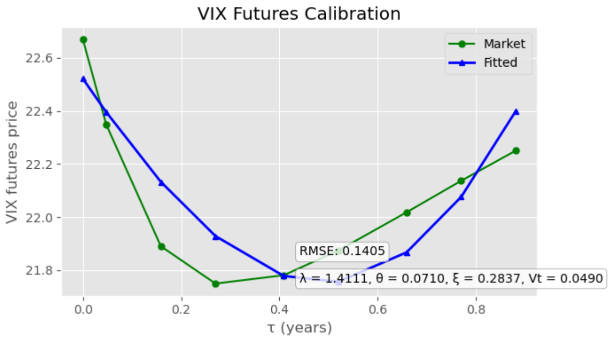

Figure 7: *
(a) VIX Calibration

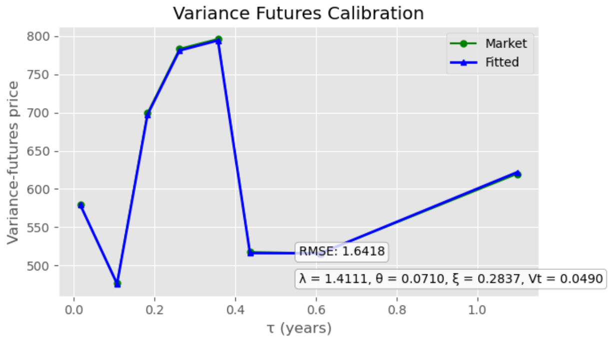

Figure 8: *
(b) Variance Calibration

3.9 Question 9

We have the following equalities from the document and our previous work on part 2:

- SPX Price Process: $\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{V_t} dB_t^Q$
- Squared Volatility Process: $dV_t = \lambda(\theta - V_t)dt + \xi\rho\sqrt{V_t}dB_t^Q + \xi\sqrt{|1-\rho^2|}\sqrt{V_t}dZ_t^Q$
- Log-Price Process: $d\log(S_t) = (r - \delta - \frac{V_t}{2})dt + \sqrt{V_t}dB_t^Q$

Therefore the quadratic covariation between log-returns and variance changes: $d\langle \log(S_t), V_t\rangle = \xi\rho V_t dt$
Meaning that: $\rho = \frac{d\langle \log(S_t), V_t\rangle}{\xi V_t dt}$

Even though we have calibrated V_t and ξ from VIX and variance futures data in the previous questions, these data only provide information about the expected level and term structure of variance. They do not contain information about how variance changes correlate with asset price changes, which is what ρ is.
The correlation parameter ρ shows up in the probability distribution of future asset prices and is not captured by the expectation of integrated variance alone because both

- VIX futures: $\mathbb{E}^Q[\frac{1}{\eta}\int_t^{t+\eta} V_u du]$
- Variance futures: $\frac{10{,}000}{T - t_0}\int_{t_0}^{T} V_u du$

are insensitive to the value of ρ.
To properly calibrate ρ, we need market prices of derivatives that are sensitive to the joint dynamics of price and variance. European SPX options across different strikes provide this information through their implied volatility pattern.

3.10 Question 10

We need to derive the formula for the number of SPX futures and variance futures needed to replicate an SPX call with strike $K = S_0$ and maturity $T = 1$ at time $t = 0$.
We have the implied volatility surface given by:

$$\sigma_t(T, K) = \alpha(t, T, V_t) + \beta(t, T, V_t)\log\left(\frac{K}{S_t}\right) + \gamma(t, T, V_t)\left(\log\left(\frac{K}{S_t}\right)\right)^2$$

For our at-the-money call $K = S_0$ and maturity $T = 1$, we have that $log(\frac{K}{S_0}) = log(1) = 0$
Hence the Implied volatility simplifies to: $\sigma_0(1, S_0) = \alpha(0, 1, V_0)$
To find the replicating portfolio at time 0, we need to match the sensitivities of the call option to both underlying price and variance changes.

The price of the call at time $t = 0$ with maturity $T = 1$ and strike $K = S_0$ is given by the Black-Scholes model: $c(t = 0, T = 1, K = S_0) = e^{-\delta} S_0 \phi(d_+) - S_0 e^{-r} \phi(d_-)$

We now take the partial derivatives of the call price with respect to S_0 and V_0:

- $\frac{\partial C}{\partial S_0} = e^{-\delta} \phi(\frac{r-\delta+\frac{\alpha(0,1,V_0)^2}{2}}{\alpha(0,1,V_0)}) = e^{-\delta}\phi(d_+)$
- $\frac{\partial C}{\partial V_0} = \frac{\partial C}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial V_0} = [e^{-\delta} S_0(\frac{-(r-\delta)}{\alpha(0,1,V_0)} + \frac{1}{2}) \cdot \varphi(\frac{r-\delta}{\alpha(0,1,V_0} + \frac{\alpha(0,1,V_0}{2}) - e^{-r} S_0(\frac{-(r-\delta)}{\alpha(0,1,V_0)} - \frac{1}{2}) \cdot \varphi(\frac{r-\delta}{\alpha(0,1,V_0} - \frac{\alpha(0,1,V_0}{2})] \frac{\partial \alpha}{\partial V_0} = [e^{-\delta} S_0(\frac{-(r-\delta)}{\alpha(0,1,V_0)} + \frac{1}{2}) \cdot \varphi(d_+) - e^{-r} S_0(\frac{-(r-\delta)}{\alpha(0,1,V_0)} - \frac{1}{2}) \cdot \varphi(d_-)] \frac{\partial \alpha}{\partial V_0}$

Where $d_+ = \frac{r-\delta+\frac{\alpha(0,1,V_0)^2}{2}}{\alpha(0,1,V_0)}$, $d_- = \frac{r-\delta-\frac{\alpha(0,1,V_0)^2}{2}}{\alpha(0,1,V_0)}$, ϕ is the standard normal CDF and φ is the standard normal PDF.
Therefore, the number of SPX futures for our replicating portfolio is:

$$n_S = \frac{\partial C}{\partial S_0} = e^{-\delta} \phi(\frac{r - \delta + \frac{\alpha(0,1,V_0)^2}{2}}{\alpha(0, 1, V_0)})$$

Then,we also have the option's sensitivity to variance is given by:

$$\frac{\partial C}{\partial V_0} = \frac{\partial C}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial V_0} = [e^{-\delta} S_0(\frac{-(r - \delta)}{\alpha(0, 1, V_0)} + \frac{1}{2}) \cdot \varphi(d_+) - e^{-r} S_0(\frac{-(r - \delta)}{\alpha(0, 1, V_0)} - \frac{1}{2}) \cdot \varphi(d_-)] \frac{\partial \alpha}{\partial V_0}$$

From our previous work, variance futures price is:

$$f^{VA}(T) = \frac{10,000}{T - t_0} \left(\int_{t_0}^{t} V_u du + a^*(T - t) + b^*(T - t) V_t \right), b^*(T - t) = \frac{1 - e^{-\lambda(T-t)}}{\lambda}$$

At $t = 0, T = 1, t_0 = 0$, we have that: $\frac{\partial f^{VA}}{\partial V_0} = 10,000 \cdot \frac{1-e^{-\lambda}}{\lambda}$
Therefore, the number of variance futures needed is:

$$n_V = \frac{\frac{\partial c}{\partial V_0}}{\frac{\partial f^{VA}}{\partial V_0}} = \frac{[e^{-\delta} S_0(\frac{-(r-\delta)}{\alpha(0,1,V_0)} + \frac{1}{2}) \cdot \varphi(d_+) - e^{-r} S_0(\frac{-(r-\delta)}{\alpha(0,1,V_0)} - \frac{1}{2}) \cdot \varphi(d_-)] \cdot \lambda \frac{\partial \alpha}{\partial V_0}}{10,000 \cdot (1 - e^{-\lambda})}$$

Therefore the initial replicating portfolio for an ATM SPX call with maturity T = 1 at t = 0 consists of:

- SPX futures: n_S contracts
- Variance futures: n_V contracts

3.11 Code

The code for this exercise can be found here: https://github.com/Loris-EPFL/derivatives.