Cox-Ross-Rubinstein model

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Abstract:

This paper provides a comprehensive analysis of the Cox-Ross-Rubinstein (CRR) binomial option pricing model. We begin with an introduction and overview of the CRR model, highlighting its importance in financial mathematics. The assumptions and limitations of the model are then discussed, providing a basis for understanding its practical applications and limitations. The mathematical representation of the CRR model is presented, followed by a detailed derivation of the put price using the put-call parity relationship. We examine the CRR and Jarrow-Rudd (JR) parameterisations, illustrating the differences and implications for pricing European and American options. In the final section, we show how the CRR model, under the assumption of continuously compounding returns in a multi-period scheme, converges to the Black-Scholes model in its limiting case. This convergence highlights the relevance and applicability of the CRR model in various financial contexts, bridging discrete and continuous time frameworks.

1.1 - Introduction and overview of the CRR model

The Cox-Ross-Rubinstein (CRR) model, developed by John Cox, Stephen Ross and Mark Rubenstein in 1979, is a cornerstone in financial theory for option pricing, extending the principles of the Black-Scholes-Merton model into a discrete-time setting.
It is widely known for its adaptability, in fact it excels in scenarios where stock prices display binomial behaviour, making it particularly effective for options on dividend-paying stocks.
Indeed, in this model, dividends can be incorporated by reducing the stock price, by the dividend amount, at each step where a dividend would be paid. This adjustment reflects the drop in stock price that typically occurs when dividends are paid out.
By simulating the future price movements of an underlying asset through a series of discrete steps, in fact, the model provides a clear path for understanding potential price trajectories. Each step in the binomial tree represents a decision point, where the price can move up or down, reflecting realistic market dynamics that are influenced by several different factors.
The model calculates the option's price starting from the final nodes of the tree, where the value of the option is deduced based on the payoff it would yield. Starting from the final nodes, the option values at the previous nodes are calculated based on the values at the immediately following nodes, using a method that will be explained later. By the time this process reaches the first node, which represents the current time, the calculated value represents the fair present value of the option based on all possible future scenarios as simulated by the binomial tree.

2.1 - Core Assumptions of the CRR Model and model's limitations

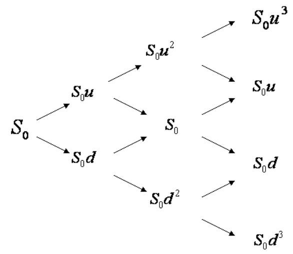
The CRR model is based on several assumptions that determine its methodology and application to option pricing: the asset price follows a binomial process within discrete time frames; each movement of the underlying asset's price is limited to specific upward or downward factors; the probabilities associated to both upward and downward remain constant; Moreover, the CRR model simplifies certain aspects of financial markets by not accounting for transaction costs, taxes, and by assuming a constant risk-free rate throughout the entire option's life.
Despite its practical applications, the Cox-Ross-Rubinstein model has several limitations that can prevent its effectiveness in certain scenarios.
It is a discrete model, unlike continuous models like Black-Scholes, and represents price movements in discrete steps, which can lead to pricing inaccuracies, particularly for options close to expiration or sensitive to small movements in the underlying asset.
The model assumes that volatility remain constant over the option's life, an unrealistic assumption in real markets where volatility can be stochastic and influenced by unpredictable factors. This can affect the accuracy of the model's predictions.
While the model can address options on dividend paying stocks, it does so in a simple way that may not fully capture the effects of dividend announcements or policy changes.
The binomial tree's exponential growth in complexity with increased time steps also raises computational problems, making the model less practical for options with very long durations.
Lastly, although adaptable for pricing American options, which allow for early exercise, the model may not always accurately predict the best time to exercise the option, especially in highly volatile markets.

2.2 - Mathematical representation

The mathematical framework of the CRR model is essential for understanding its functionality and effectiveness in option pricing.
The model starts with an initial stock price S_0, which is positive and set at the beginning. As the model progresses through each discrete time step t (where $1 \leq t \leq T$), it tracks every possible outcome ω in the sample space Ω.
Each node in the binomial tree represents a potential future price of the underlying asset, calculated using the following formula:

$$S_t(\omega) = S_0 \times u^{N_t(\omega)} \times d^{t-N_t(\omega)}$$

where u and d are the up and down factors respectively, and $N_t(\omega)$ is the number of up moves up to time t. This recursive relationship reflects the compound effect of price movements over time, determined by the binomial distribution.



The choice of u and d is important and is generally dependent on the volatility of the underlying asset, ensuring that the tree is a recombining tree, in which up moves followed by down moves return to the original price level.
This recombination is a key feature that simplifies the model by significantly reducing the number of possible price paths to evaluate.
Introducing the risk-free interest rate r, if d < 1+r < u the CRR market model M=(B,S) is arbitrage free and complete. The latter condition guarantee that the ***unique*** arbitrage price of any European contingent claim X can be computed using the risk-neutral valuation formula, given by:

$$\pi(X) = B_t E_t^{\widetilde{P}}(\frac{X}{B_T})$$

Where $\widetilde{P}$ denote the (unique) risk-neutral probability measure. Therefore, in this model, under the conditions described above (namely d<1+r<u), the option pricing problem is completely solved because there is a ***unique*** fair price for it.
Thus, to ensure the above conditions, the probabilities of moving up or down, denoted as p and 1 − p respectively, are not arbitrary, but must coincide with the risk-neutral probabilities, which are given by:

$\widetilde{p} = \frac{1+r-d}{u-d}$ and $1 - \widetilde{p}$.

2.3 - From the Call option Pricing Formula to the put option formula using the put-call parity relationship

The arbitrage price at time t = 0 of the European call option $C_T = (S_T - K)^+$ in the binomial market model M = (B, S) is given by the CRR call pricing formula:

$$C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k (1-\hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k}$$

where:

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \hat{p} = \frac{\tilde{p}u}{1+r}$$

The formula calculates the price of a European call option by subtracting the present value of the cost of exercising the option from the present value of the expected payoffs, ensuring that the option price accurately reflects both the potential gains from price increases and the costs of exercising the option.

In the condition of non arbitrage, namely $d<1+r<u$, to derive the explicit pricing formula for the European put option with payoff $P_T=(K-S_T)^+$ it is possible to utilise the put-call parity equation.

Put-call parity represents a fundamental concept in options pricing, which relates the prices of European call and put options with the same strike price and expiration date.

Put-Call parity arises from arbitrage principles. When the prices of call and put options violate this equation, it means that there is an arbitrage opportunity.

The put-call parity equation, $C_t+KB(t,T)=P_t+S_t$, relates the prices, at the current time t, of call (C_t) and put (P_t) options on the same underlying asset S_t, expiration time T and strike price K. Here $B(t,T)=(1+r)^{-(T-t)}$ is the price at time t of a zero-coupon bond expiring at time T.

In essence, this implies that the sum of the call option price and the present value of the strike price is equal to the sum of the put option price and the current price of the underlying asset.

One may determine the put option price from the call option price by rearranging the put-call parity equation. Solving for P_t, you get:

$P_t = C_t+KB(t,T)-S_t$

This equation allows the put option price to be determined given the call option price, the strike price, the time to expiration, and the current spot price.

2.4 - CRR and Jarrow Rudd parametrization:

The up (u) and down (d) factor represent the factors by which the price of the underlying asset may increase or decrease each time step. Those factors determine the potential price movements of the underlying asset in a binomial tree. At each node of the tree, the price of the asset can go up by a factor u or down by a factor d.

The two factors u and d are typically chosen based on the asset's volatility and time step, and once they are set, the risk-neutral probability p is calculated to ensure no-arbitrage conditions are met.

The CRR parametrization take in account the volatility (σ) of the underlying asset and the length of each time step (Δt). In this case u and d are defined by:

$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}$

The Jarrow Rudd parameterization incorporates the risk-free rate (r) in the calculation of u and d to consider the effects of the interest rate on option prices.

By taking the risk-free rate into account, the JR parameterisation ensures that the binomial model reflects the impact of interest rate changes on option prices, improving the accuracy of the model.

The formula is derived from the equation for continuous geometric Brownian motion.

By discretising this continuous-time process into discrete time steps (Δt), one can approximate the continuous GBM equation and derive the formulas for the u and d factors, which determine the upward and downward movements of the asset price in a binomial tree model.

$$u = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}}$$
$$d = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t-\sigma\sqrt{\Delta t}}$$

2.5 - Difference between European and American options:

Options can be classified according to when they may be exercised. A European option can only be exercised on the expiry date. An American option, on the other hand, can be exercised on any trade date (which may be chosen by the holder) from the time it is written until the expiration date.
Since American options can be exercised at any time before maturity, they tend to be more valuable than European options with the same strike price and expiration date.
European options are easier to value and analyse due to their simplicity of exercise.
However, since European options cannot be exercised until expiration, investors are exposed to potential adverse movements in the price of the underlying asset during the life of the option.
On the other hand, in American options, the possibility of exercising the option early can be advantageous in certain situations.
However, the need to consider the optimal exercise strategy at all times adds complexity to the valuation process, both mathematically and computationally.
In fact, in the CRR model, as we have seen before, European options are typically valued using a closed-form equation derived from risk-neutral pricing principles.
The closed-form equation enables a straightforward estimation of the option value at any time without the necessity of complex, iterative procedures.
Conversely, the valuation of American options necessitates the determination of the optimal exercise strategy at each point in time.
Here's the formula for calculating the price of an American call option in the CRR model:

$$C_{i,j} = \max\left(S_{i,j} - K, e^{-r\Delta t}\left(pC_{i+1,j+1} + (1-p)C_{i+1,j}\right)\right)$$

Where:

- $C_{i,j}$ is the price of the American call option at node (i, j) in the binomial tree.
- $S_{i,j}$ is the price of the underlying asset at node (i, j) in the binomial tree.
- Δt is the time step.

The formula calculates the option price at each node of the tree, starting from the final nodes (at expiration) and moving backward to the initial node. At each node, the profit from immediately exercising the option is compared to the profit from holding the option.
This process is repeated recursively until it reaches the initial node, which represents the current time.

3 – Continuous limit of the Cox-Ross-Rubinstein Model

In this section, we demonstrate that, by appropriately adjusting the parameters of the Cox-Ross-Rubinstein (CRR) model, the fair prices of options in the CRR model converge to those in the renowned Black-Scholes (BS) model as the time steps approach a continuous limit.

The Black-Scholes model is a continuous-time market model with one bond and one stock over the interval [0, T]. Here, we allow T ∈ (0, ∞) to accommodate the fact that the time horizon T in the Black-Scholes model can be any real date or time, not necessarily an integer.
The bond price at any time s is $B_s = e^{rs}$, where r ∈ ℝ is the rate of continuous compounding.
The stock price at any time $s > 0$ is:

$$1)\ \ S_s = S_0\, e^{\left(r-\frac{1}{2}\sigma^2\right)s+\sigma W_s},$$

where σ > 0 is the volatility of the stock, $S_0 > 0$ the initial deterministic value of the stock price, and the process $(W_s)_{s\epsilon[0,T]}$ is a Brownian motion under a probability measure ℚ.
Let's now construct the so-called n^{th} CRR model by dividing the real time interval [0, T] into n equal time steps of length Δt so that any trading date t ∈ {0, 1, ..., n} in such a model corresponds to the real time tΔt ∈ [0, T] in the Black-Scholes model.
To show the convergence of the first model to the second one as $\Delta t \to 0$, we need to require that the bond price $B_t^n = (1 + r_n)^t$ (where $r_n > -1$ is a fixed rate of return for the interval $[0, \mathrm{t}\Delta t\,]$) at time step t in the n^{th} CRR model matches the bond price $B_s = B_{t\Delta t} = e^{rt\Delta t}$ at the real time $s = \mathrm{t}\Delta t$ in the BS model, that is:

$$(1 + r_n)^t = e^{rt\Delta t}$$

The rate r_n satisfying this equation equals $e^{r\Delta t} - 1$ and this is the one that must be used in $B_t^n = (1 + r_n)^t$. Essentially, we are adjusting the CRR parameters so that as the intervals become smaller, the model's output becomes increasingly similar to that of the BS model. With this notation established, we can rewrite the formula for the stock prices in the BS model as:

$$S_{t\Delta t} = S_0\, e^{\left(r-\frac{1}{2}\sigma^2\right)t\Delta t+\,\sigma W_{t\Delta t}} = S_0\, e^{(r-\frac{1}{2}\sigma^2)(t-1)\Delta t\,+\sigma W_{(t-1)\Delta t}}\, e^{(r-\frac{1}{2}\sigma^2)\Delta t\,+\sigma(W_{t\Delta t}\,-W_{(t-1)\Delta t})}$$
$$= S_{(t-1)\Delta t} e^{(r-\frac{1}{2}\sigma^2)\Delta t\,+\sigma(W_{t\Delta t}\,-W_{(t-1)\Delta t})}$$

Now what we have to do is to construct a formula for the stock prices S_t^n in the n^{th} CRR model which approximate those of the stock prices S_s in the BS model. To achieve this objective, Cutland and Roux suggested an approach for estimating the Brownian increments $W_{t\Delta_n} - W_{(t-1)\Delta_n}$ over the real time interval $[(\mathrm{t}-1)\,\Delta t, \mathrm{t}\Delta t]$ by employing a random variable Y_i that assumes only two potential values: $\sqrt{\Delta t}$ and $-\sqrt{\Delta t}$. Therefore we obtain:

$$2)\ \ S_t^n = S_{t-1}^n e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t\pm\sigma\sqrt{\Delta t}} = S_0^n e^{\left(r-\frac{1}{2}\sigma^2\right)t\Delta t+\sigma\sum_i^t Y_i}$$

Where $S_0^n > 0$ is the initial price of the stock and $\sum_i^t Y_i$ is a symmetric random walk that approximates the Brownian motion in equation (1). We will make this final statement more precise in a while.
Note that from this equation, we can derive a more concise notation:

$$S_n^{(n)} = \hat{S}\left(\frac{1}{\sqrt{T}} W_n^{(n)}\right)$$

where $\hat{S}$ is the continuous function

$$\hat{S}(z) := S_0\, e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}z} \quad \text{for } z\epsilon\mathbb{R}.$$

Comparing with equation (1) we obtain the Black-Scholes stock price at the final time T as:

$S_T = \hat{S}\left(\frac{1}{\sqrt{T}} W_T\right)$.

Remembering that the price of a stock in the n^{th} CRR model, given S_{t-1}^n, is:

$$S_t^n = \begin{cases} S_{t-1}^n u_n, \text{if the price goes up.} \\ S_{t-1}^n d_n, \text{if the price goes down.} \end{cases}$$

From (2) we deduce that the parameters representing "up" and "down" movements in the n^{th} CRR model are determined as follows:

$$3) \quad u_n = e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}} \text{ and } d_n = e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}}$$

Which are the ones defined in the Jarrow-Ross parametrization.
The following proposition from Cutland and Roux discusses the absence of arbitrage and the existence of single-step risk-neutral probabilities in the n^{th} CRR model.

- The n^{th} Cox-Ross-Rubinstein model is arbitrage-free if and only if n ≥ N, with N be the smallest integer such that N > $\frac{1}{4}\sigma^2 T$.
- For n ≥ N the unique one-step conditional risk-neutral probabilities for the n^{th} Cox-Ross-Rubinstein model are $(p_n, 1-p_n)$, where:

$$p_n = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{e^{\frac{1}{2}\sigma^2\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

- For n ≥ N the unique equivalent martingale measure $\mathbb{Q}_n$ in the n^{th} Cox-Ross-Rubinstein model satisfies

$$\mathbb{Q}_n\left(\left\{\omega \epsilon \Omega_n : S_n^{(n)} = S_0 u_n^t d_n^{n-t}\right\}\right) = \binom{n}{t} p_n^t (1-p_n)^{n-t}$$

  For t ≤ n.

The first point can be proven by recalling that the n^{th} Cox-Ross-Rubinstein model is arbitrage-free if and only if $d_n < 1 + r_n < u_n$, namely $e^{-\sigma\sqrt{\Delta t}} < e^{\frac{1}{2}\sigma^2\Delta t} < e^{\sigma\sqrt{\Delta t}}$.
By applying logarithms and noting that $\sigma^2\Delta t > 0$, this translates to $\frac{1}{2}\sigma^2\Delta t < \sigma\sqrt{\Delta t}$, which is valid if and only if, remembering that $\Delta t = \frac{T}{n}$ by definition, $N > \frac{1}{4}\sigma^2 T$.
This establishes the initial assertion since N represents the smallest integer greater than $\frac{1}{4}\sigma^2 T$.
The second point is given by the previous point and Proposition 10 discussed in class, while the third point is trivial considering the binomial scheme of the CRR model.
Before proceeding, we need to define the following two concepts:

- Path-Independent Derivative:
  A derivative D is termed path-independent if there exists a payoff function $\widehat{D}$ such that $D = \widehat{D}(S_T)$. Thus, its value depends solely on the number of "up" movements in each step, but remains unaffected by their order.
- Weak Convergence:
  Suppose that $(X_n)_{n=N}^{\infty}$ is a sequence of random variables, where each X_n is defined on a set Ω_n endowed with a probability measure $\mathbb{P}_n$. Let X be a random variable defined on a set Ω endowed with a probability $\mathbb{P}$. The sequence $(X_n)_{n=N}^{\infty}$ converges weakly to X if:

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_n}\left(f(X_n)\right) = \mathbb{E}_{\mathbb{P}}\left(f(X)\right)$$

  For every bounded continuous function f.

3.1 - Theorem

Let $\widehat{D}$ denote a bounded continuous function. For each n ≥ N (with N defined above), let's define the path-independent derivative $D^{(n)}$ within the n^{th} Cox-Ross-Rubinstein model as $D^{(n)}$ $:= \widehat{D}\left(S_n^{(n)}\right)$, and let $D_0^{(n)}$ be its unique fair price at time 0. Then:

$$\lim_{n\to\infty} D_0^{(n)} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(\widehat{D}(S_T)\right)$$

that is,

$$4)\quad \lim_{n\to\infty}(1+\boldsymbol{r_n})^{-n}\sum_{t=0}^{n}\binom{n}{t}p_n^t(1-p_n)^{n-t}\widehat{D}(S_0u_n^td_n^{n-t}) = e^{-rT}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\widehat{D}(\hat{S}(z))\,dz$$

The left-hand side depicts the limit of the Cox-Ross-Rubinstein formula for the fair price of a path-independent derivative at time t=0[1]. Conversely, the right-hand side corresponds to the Black-Scholes formula for such a derivative.

Proof:

The outcome of the aforementioned theorem relies on the weak convergence of $(W_n^{(n)})_{n=N}^{\infty}$ to W_T. We can define a function g as:

$$g(z) := \widehat{D}\left(\hat{S}\left(\tfrac{1}{\sqrt{T}}z\right)\right) \text{ for } z\epsilon\mathbb{R}$$

which is continuous and bounded since $\widehat{D}$ is. The weak convergence of $\left(W_n^{(n)}\right)_{n=N}^{\infty}$ to W_T then implies:

$$\lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}n}\left(\widehat{D}\left(S_n^{(n)}\right)\right) = \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}n}\left(\widehat{D}\left(\hat{S}\left(\tfrac{1}{\sqrt{T}}W_n^{(n)}\right)\right)\right) = \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}n}\left(g\left(W_n^{(n)}\right)\right) = \mathbb{E}_{\mathbb{Q}n}(g(W_T))$$
$$= \mathbb{E}_{\mathbb{Q}}\left(\widehat{D}\left(\hat{S}\left(\tfrac{1}{\sqrt{T}}W_T\right)\right)\right) = \mathbb{E}_{\mathbb{Q}}\left(\widehat{D}(S_T)\right)$$

Given the identity $(1+\boldsymbol{r_n})^n = e^{rT}$, which was required some pages before by choosing $\boldsymbol{r_n}$ properly, it yields:

$$5)\quad \lim_{n\to\infty} D_0^{(n)} = \lim_{n\to\infty}(1+\boldsymbol{r_n})^{-n}\mathbb{E}_{\mathbb{Q}n}\left(\widehat{D}\left(S_n^{(n)}\right)\right) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(\widehat{D}(S_T)\right)$$

Equation (4) readily emerges from the Cox-Ross-Rubinstein formula for the fair price of a path-independent derivative at time t=0 and the property that $\frac{1}{\sqrt{T}}W_T$ [2]is a Gaussian random variable with mean 0 and variance 1 under $\mathbb{Q}$

3.2 - Defining put pricing and deriving call pricing using put-call parity

Let's recall the payoff function of a put option with strike K:

$$\hat{P}(S_T) = [S_T - K]_- = max(K - S_T, 0)\text{ , for } S_T > 0.$$

To ensure continuity and boundedness over $S_T \in \mathbb{R}$ we also define $\hat{P}(S_T) := K$ for $S_T \le 0$. The fair price at time 0 of the put option in the n^{th} Cox-Ross-Rubinstein model is:

$$\textbf{6)}\quad P_0^{(n)} = \boldsymbol{K(1+r_n)^{-n}}\sum_{\boldsymbol{k=\hat{k}}}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{k}}p_n^k(1-p_n)^{n-k} - \boldsymbol{S_0}\sum_{\boldsymbol{k=\hat{k}}}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{k}}\widehat{\boldsymbol{p}^{\boldsymbol{k}}}(\boldsymbol{1-\hat{p}})^{\boldsymbol{n-k}}$$

Where $\hat{p} = \frac{p_n u}{1+r}$ and $\hat{k}$ is the smallest integer such that $K - u^k d^{T-k} S_0 > 0$. According to the preceding theorem, this converges to the Black-Scholes formula:

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[1] $D_0 = (1+\boldsymbol{r_n})^{-n}\sum_{t=0}^{n}\binom{n}{t}p_n^t(1-p_n)^{n-t}\widehat{D}(S_0u_n^td_n^{n-t})$

[2] Remember that, for the properties of a Brownian motion W_S, W_T is a Gaussian random variable under $\mathbb{Q}$ with mean 0 and variance T.

7) $P_0 := e^{-rT}\mathbb{E}_{\mathbb{Q}}([K - S_T]^+) = e^{-rT}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} max\left[K - S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z}, 0\right] dz$

This equation can be simplified further by using the cumulative standard Gaussian distribution function Φ.

$$P_0 = e^{-rT}\, K\Phi(\gamma) - S_0\, \Phi\left(\gamma - \sigma\sqrt{T}\right)$$

where $\gamma := \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S_0} - (r - \frac{1}{2}\sigma^2)T\right)$. This represents the Black-Scholes formula for the price of a put option.

Direct application of the previous theorem to determine the limit of the Cox-Ross-Rubinstein formula for a call option isn't feasible due to the unbounded nature of the call option's payoff function:

$$\lim_{S_T\to\infty}\hat{C}\,(S_T) = \lim_{S_T\to\infty}[S_T - K]^+ = \lim_{S_T\to\infty}(S_T - K) = \infty$$

However, the put-call parity and the convergence of put option prices can be used to demonstrate that the Cox-Ross-Rubinstein call option prices converge to the Black-Scholes price.

For any $n \geq N$ the fair price $C_0^{(n)}$at time 0 of the call option in the n^{th} Cox-Ross-Rubinstein model satisfies:

$$C_0^{(n)} = P_0^{(n)} + S_0 - (1 + r_n)^{-n}K = P_0^{(n)} + S_0 - e^{-rT}K$$

The convergence of $(P_0^{(n)})_{n=N}^{\infty}$ to P_0 then implies that $(C_0^{(n)})_{n=N}^{\infty}$ converges to:

$$P_0 + S_0 - e^{-rT}K = C_0$$

After some computations, we derive:

$$C_0 = S_0\,\Phi\left(\gamma - \sigma\sqrt{T}\right) + e^{-rT}\,K\Phi(\gamma)$$

This represents the call option formula in the Black-Scholes model.

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