1 Ocean Waves

The outstanding visible characteristic of an open sea surface is its irregularity. The waves on the surface do not repeat periodically in time or space. Yet, over a wide area and for a period of time, the sea surface maintains a characteristic appearance. Study on wave data have shown that even though the sea surface is irregular, the wave elevation readings is Gaussian in nature and is statistically a constant for a given area for a certain period of time. It is therefore possible to define a sea condition, for a region for a (short) period of time, using statistical parameters such as mean elevation and variance. The mean elevation will however be zero, since wave does not change the water level in the sea. Which means, considering Gaussian distribution for wave elevations, the sea condition can be defined using variance alone.

For the purpose of this research we consider waves generated due to storms, that is waves that are generated by the interaction of wind and water surface. The two main physical process involved in the generation of storm waves are the friction between air and water and the local variation of pressure field due to wind. Even though there are many processes that will affect the growth and propagation of waves, for waves of small amplitude, it is primarily governed by the principle of superposition. So if $\zeta_1(x, y, t)$ and $\zeta_2(x, y, t)$ are functions that represent two wave systems then $\zeta_1(x, y, t) + \zeta_2(x, y, t)$ is also a wave system. However, it should be noted that the assumption regarding the linear superimposability of waves fails when the wave system is too steep and wave breaking occurs.

1.1 Regular sea waves

A regular sea wave is a harmonic wave with crests that are infinitely long, parallel and equally spaced and having constant wave heights. The general equation of a regular long crested wave travelling at a angle μ to the x-axis is:

$$\zeta(x, y, t) = \zeta_a \cos[k(x \cos \mu + y \sin \mu) - \omega t + \epsilon] \tag{1}$$

where:
ζ_a is the wave amplitude at the water surface
$k = \frac{2\pi}{L_w}$ is called the wave number.
L_w is the wave length
x is the distance along the x-coordinate
y is the distance along the y-coordinate
μ is the direction of propagation of the wave as an angle with respect to positive direction of x-axis.
ω is the circular frequency of the simple harmonic wave
t is time
ϵ is a phase angle.

The wave system travels perpendicular to the line of crests with a velocity V_c. It is assumed that water is incompressible and has zero viscosity. Based on these assumptions the motion of water particle in the wave can be described using a quantity called *velocity potential* which is defined as a function whose negative derivative in any direction yields the velocity component of the fluid in that direction. Given below is a simplified equation for velocity potential:

$$\phi = -\zeta_a V_c \frac{\cosh k(z + h)}{\sinh kh} \sin k(x - V_c t) \tag{2}$$

where:
h is the water depth (distance from sea surface to seabed)
V_c is the wave velocity (or celerity)
z is the vertical distance of the water particle from the surface and is measured positive in the upward direction

For deep water, ie. where $h \gg \frac{L_w}{2}$,

$$\frac{\cosh k(z + h)}{\sinh kh} \approx e^{kz} \tag{3}$$

Substituting equation 3 in equation 2 we get:

$$\phi = -\zeta_a V_c e^{kz} \sin k(x - V_c t) \tag{4}$$

The wave causes variation in the distribution of pressure below the water surface and the equation for variation of pressure head due to waves is:

$$\zeta = \frac{k\zeta_a {V_c}^2}{g} \frac{\cosh k(z+h)}{\sinh kh} \cos k(x - V_c t) \tag{5}$$

For deep water the above equation can be approximated as:

$$\zeta = \zeta_a e^{kz} \cos k(x - V_c t) \tag{6}$$

For simple harmonic motion:

$$\omega = \frac{2\pi}{T_w} = kV_c \tag{7}$$

where:
T_w is the time period of the simple harmonic wave

Substituting equation 7 in equations 5 and 6:
Pressure head variation due to wave for any water depth:

$$\zeta = \frac{k\zeta_a {V_c}^2}{g} \frac{\cosh k(z+h)}{\sinh kh} \cos(kx - \omega t) \tag{8}$$

Pressure head variation due to wave for deep water:

$$\zeta = \zeta_a e^{kz} \cos(kx - \omega t) \tag{9}$$

The pressure at any point is given by:

$$p = \rho g(-z + \zeta) \tag{10}$$

$$p = -\rho g z + \zeta_a \rho g e^{kz} \cos(kx - \omega t) \tag{11}$$

The total wave energy per unit area is:

$$E = \frac{1}{2} \rho g {\zeta_a}^2 \tag{12}$$

The variance, or mean-square value of surface elevation as a function of time is:

$$S = \langle \zeta(t)^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta^2(t) dt \tag{13}$$

For simple harmonic motion of frequency ω, the above equation for variance can be written as:

$$S(\omega) = \langle \zeta(t)^2 \rangle = \frac{1}{2} {\zeta_a}^2 \tag{14}$$

Based on equation 14, equation 12 can be written as:

$$E = \rho g S(\omega) \tag{15}$$

1.2 Irregular sea and wave spectrum

Oceanographers have found that an irregular sea wave can be resolved into a sum of regular waves of various length and direction using Fourier Integral techniques. Or alternatively it can be said that the visible system of waves on the sea surface is a result of super positioning of many regular component waves each travelling is different direction with a different amplitude, frequency, wave length and phase. Since the component waves have different direction and celerities, the wave patterns keeps changing with time. It is convenient to begin with a simple case of wave pattern observed at a single point (ie. $x = y = 0$) and assuming that all component waves are travelling

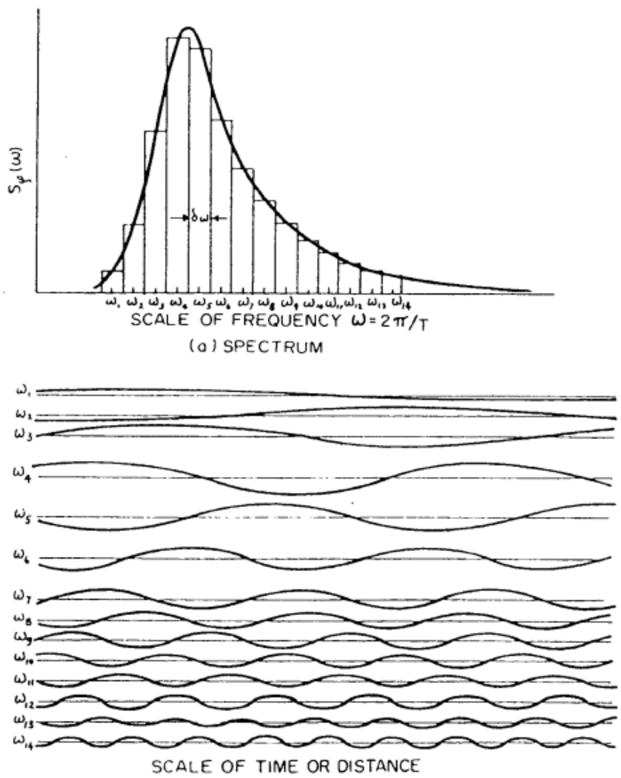

Figure 1: Variance spectrum

in the same direction ($\mu = 0$). Based on equation 1, the simplified equation of a compound wave (ie. a wave that consist of many component waves) is:

$$\zeta(t) = \sum_i (\zeta_a)_i \cos(-\omega_i t + \epsilon_i) \tag{16}$$

It is convenient to define wave components in terms of a function called *variance spectrum,* $S(\omega)$. A typical example of a plot of variance spectrum is shown in figure 1. For any particular wave frequency, ω_i, the variance of the wave components for a narrow band of frequency, $\delta\omega$, centred on ω_i is given by:

$$\langle \zeta_i(t)^2 \rangle = S(\omega_i)\delta\omega \tag{17}$$

The total variance of the system is given by:

$$\langle \zeta(t)^2 \rangle = \sum_i \langle \zeta_i(t)^2 \rangle = \int_0^\infty S(\omega) d\omega \tag{18}$$

That is, the total variance of the system is obtained by finding the area under the variance spectrum. Also, as $\delta\omega \to 0$, it means the wave system is composed of only one frequency which means it becomes a simple harmonic wave. The mean value for a wave elevations for a simple harmonic wave is 0 (wave does not increase the mean water level) and the variance is given by:

$$\langle \zeta_i(t)^2 \rangle = \frac{1}{2} {(\zeta_a)_i}^2 \tag{19}$$

Applying this is equation 17, we get:

$$\frac{1}{2} {(\zeta_a)_i}^2 = S(\omega_i)\delta\omega \tag{20}$$

$$(\zeta_a)_i = \sqrt{2S(\omega_i)\delta\omega} \tag{21}$$

The above equation is useful to get the amplitude of a component wave within a frequency band.

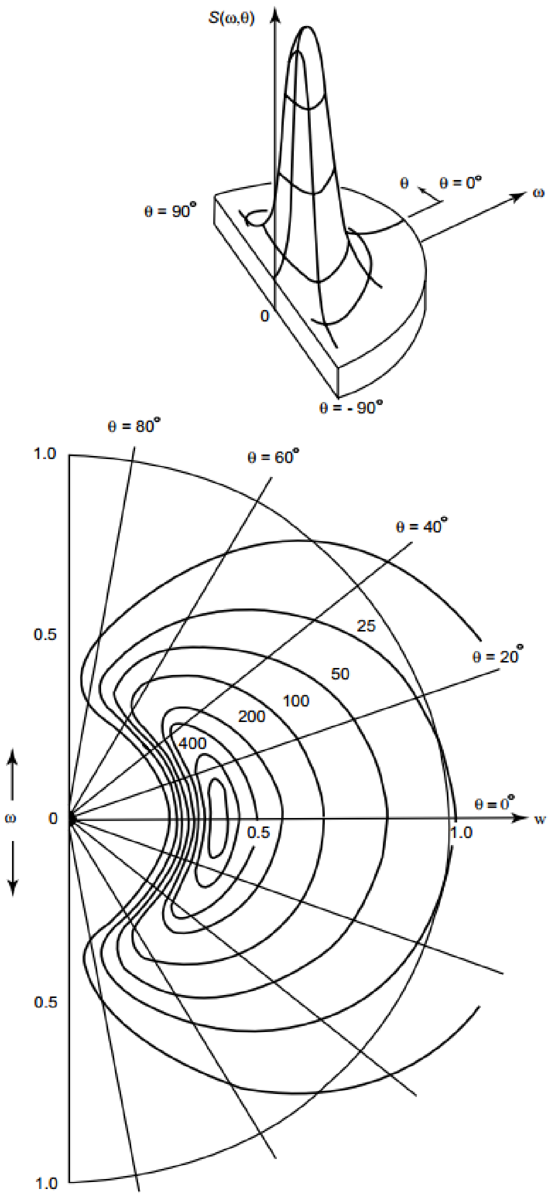

Figure 2: Directional spectrum

A fair finite-sum model of a unidirectional sea can be obtained by taking about 20 different frequency bands for a single direction. This is because any particular rectangle in figure 1 represents the variance in that band of frequencies. A regular wave of the indicated finite amplitude would have the same variance as the infinite number of component within that band. Hence, the addition of these components (shown at the bottom of the figure 1) will give a pattern that has the same total variance and closely resemble the record from which the spectrum was obtained.

A point spectrum does not take into account the direction of propagation of component waves. A *directional spectrum* gives a more complete representation of the sea. Image 2 show a typical example of a directional spectrum.