

# 1 Ocean Waves

This document presents a summary of the theory of ocean waves and is mainly based on [1].

The outstanding visible characteristic of an open sea surface is its irregularity. The waves on the surface do not repeat periodically in time or space. Yet, over a wide area and for a period of time, the sea surface maintains a characteristic appearance. Study on wave data have shown that even though the sea surface is irregular, the wave elevation readings is Gaussian in nature and is statistically a constant for a given area for a certain period of time. It is therefore possible to define a sea condition, for a region for a (short) period of time, using statistical parameters such as mean elevation and variance. The mean elevation will however be zero, since wave does not change the water level in the sea. Which means, considering Gaussian distribution for wave elevations, the sea condition can be defined using variance alone.

For the purpose of this research we consider waves generated due to storms, that is waves that are generated by the interaction of wind and water surface. The two main physical process involved in the generation of storm waves are the friction between air and water and the local variation of pressure field due to wind. Even though there are many processes that will affect the growth and propagation of waves, for waves of small amplitude, it is primarily governed by the principle of superposition. So if  $\zeta_1(x, y, t)$  and  $\zeta_2(x, y, t)$  are functions that represent two wave systems then  $\zeta_1(x, y, t) + \zeta_2(x, y, t)$  is also a wave system. However, it should be noted that the assumption regarding the linear superimposability of waves fails when the wave system is too steep and wave breaking occurs.

## 1.1 Regular sea waves

A regular sea wave is a harmonic wave with crests that are infinitely long, parallel and equally spaced and having constant wave heights. The general equation of a regular long crested wave travelling at an angle  $\mu$  to the  $x$ -axis is:

$$\zeta(x, y, t) = \zeta_a \cos[k(x \cos \mu + y \sin \mu) - \omega t + \epsilon] \quad (1)$$

where:

$\zeta_a$  is the wave amplitude at the water surface

$k = \frac{2\pi}{L_w}$  is called the wave number.

$L_w$  is the wave length

$x$  is the distance along the x-coordinate

$y$  is the distance along the y-coordinate

$\mu$  is the direction of propagation of the wave as an angle with respect to positive direction of x-axis.

$\omega$  is the circular frequency of the simple harmonic wave

$t$  is time

$\epsilon$  is a phase angle.

The wave system travels perpendicular to the line of crests with a velocity  $V_c$ . It is assumed that water is incompressible and has zero viscosity. Based on these assumptions the motion of water particle in the wave can be described using a quantity called *velocity potential* which is defined as a function whose negative derivative in any direction yields the velocity component of the fluid in that direction. Given below is a simplified equation for velocity potential:

$$\phi = -\zeta_a V_c \frac{\cosh k(z + h)}{\sinh kh} \sin k(x - V_c t) \quad (2)$$

where:

$h$  is the water depth (distance from sea surface to seabed)

$V_c$  is the wave velocity (or celerity)

$z$  is the vertical distance of the water particle from the surface and is measured positive in the upward direction

For deep water, ie. where  $h \gg \frac{L_w}{2}$ ,

$$\frac{\cosh k(z + h)}{\sinh kh} \approx e^{kz} \quad (3)$$

Substituting equation 3 in equation 2 we get:

$$\phi = -\zeta_a V_c e^{kz} \sin k(x - V_c t) \quad (4)$$

The wave causes variation in the distribution of pressure below the water surface and the equation for variation of pressure head due to waves is:

$$\zeta = \frac{k\zeta_a V_c^2}{g} \frac{\cosh k(z+h)}{\sinh kh} \cos k(x - V_c t) \quad (5)$$

For deep water the above equation can be approximated as:

$$\zeta = \zeta_a e^{kz} \cos k(x - V_c t) \quad (6)$$

For simple harmonic motion:

$$\omega = \frac{2\pi}{T_w} = kV_c \quad (7)$$

where  $T_w$  is the time period of the simple harmonic wave.

Substituting equation 7 in equations 5 and 6:

Pressure head variation due to wave for any water depth:

$$\zeta = \frac{k\zeta_a V_c^2}{g} \frac{\cosh k(z+h)}{\sinh kh} \cos(kx - \omega t) \quad (8)$$

Pressure head variation due to wave for deep water:

$$\zeta = \zeta_a e^{kz} \cos(kx - \omega t) \quad (9)$$

The pressure at any point is given by:

$$p = \rho g(-z + \zeta) \quad (10)$$

$$p = -\rho g z + \zeta_a \rho g e^{kz} \cos(kx - \omega t) \quad (11)$$

The total wave energy per unit area is:

$$E = \frac{1}{2} \rho g \zeta_a^2 \quad (12)$$

The variance, or mean-square value of surface elevation as a function of time is:

$$S = \langle \zeta(t)^2 \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta^2(t) dt \quad (13)$$

For simple harmonic motion of frequency  $\omega$ , the above equation for variance can be written as:

$$S(\omega) = \langle \zeta(t)^2 \rangle = \frac{1}{2} \zeta_a^2 \quad (14)$$

Based on equation 14, equation 12 can be written as:

$$E = \rho g S(\omega) \quad (15)$$

## 1.2 Irregular sea and wave spectrum

Oceanographers have found that an irregular sea wave can be resolved into a sum of regular waves of various length and direction using Fourier Integral techniques. Or alternatively it can be said that the visible system of waves on the sea surface is a result of super positioning of many regular component waves each travelling in different direction with a different amplitude, frequency, wave length and phase. Since the component waves have different direction and celerities, the wave patterns keep changing with time. It is convenient to begin with a simple case of wave pattern observed at a single point (ie.  $x = y = 0$ ) and assuming that all component waves are travelling in the same direction ( $\mu = 0$ ). Based on equation 1, the simplified equation of a compound wave (ie. a wave that consists of many component waves) is:

$$\zeta(t) = \sum_i (\zeta_a)_i \cos(-\omega_i t + \epsilon_i) \quad (16)$$

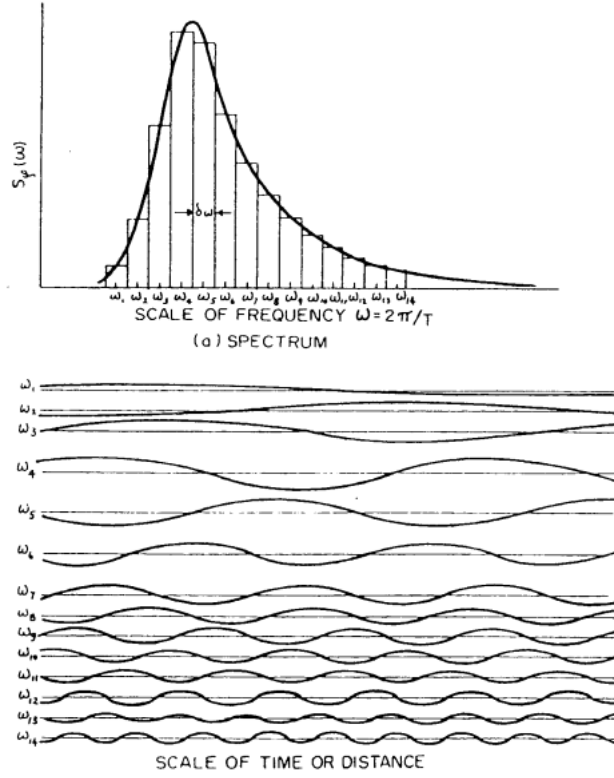


Figure 1: Variance spectrum

It is convenient to define wave components in terms of a function called *variance spectrum*,  $S(\omega)$ . A typical example of a plot of variance spectrum is shown in figure 1. For any particular wave frequency,  $\omega_i$ , the variance of the wave components for a narrow band of frequency,  $\delta\omega$ , centred on  $\omega_i$  is given by:

$$\langle \zeta_i(t)^2 \rangle = S(\omega_i) \delta\omega \quad (17)$$

The total variance of the system is given by:

$$\langle \zeta(t)^2 \rangle = \sum_i \langle \zeta_i(t)^2 \rangle = \int_0^\infty S(\omega) d\omega \quad (18)$$

That is, the total variance of the system is obtained by finding the area under the variance spectrum. Also, as  $\delta\omega \rightarrow 0$ , it means the wave system is composed of only one frequency which means it becomes a simple harmonic wave. The mean value for a wave elevations for a simple harmonic wave is 0 (wave does not increase the mean water level) and the variance is given by:

$$\langle \zeta_i(t)^2 \rangle = \frac{1}{2} (\zeta_a)_i^2 \quad (19)$$

Applying this is equation 17, we get:

$$\frac{1}{2} (\zeta_a)_i^2 = S(\omega_i) \delta\omega \quad (20)$$

$$(\zeta_a)_i = \sqrt{2S(\omega_i) \delta\omega} \quad (21)$$

The above equation is useful to get the amplitude of a component wave within a frequency band.

A fair finite-sum model of a unidirectional sea can be obtained by taking about 20 different frequency bands for a single direction. This is because any particular rectangle in figure 1 represents the variance in that band of frequencies. A regular wave of the indicated finite amplitude would have the same variance as the infinite number of component within that band. Hence, the addition of these components (shown at the bottom of the figure 1) will give a pattern that has the same total variance and closely resemble the record from which the spectrum was obtained.

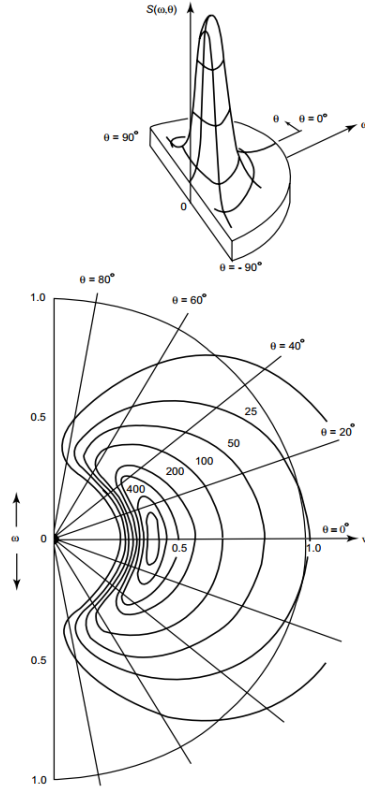


Figure 2: Directional spectrum

A point spectrum does not take into account the direction of propagation of component waves. A *directional spectrum* gives a more complete representation of the sea. Figure 2 show a typical example of a directional spectrum. The general equation for the total wave system of components moving in different direction,  $\mu$  is:

$$\zeta(x, y, t) = \sum_i \sum_j (\zeta_a)_{ij} \cos[k_i(x \cos \mu_j + y \sin \mu_j) - \omega_i t + \epsilon_{ij}] \quad (22)$$

And the total variance of the system is:

$$S = \int_0^\infty \int_0^{2\pi} S(\omega, \mu) d\mu d\omega \quad (23)$$

where  $S(\omega, \mu)$  is the function defining the directional spectrum such as the one shown in figure 2. The following section looks into various spectral families.

### 1.3 Idealized spectral families

This section describes three idealised point spectrum followed by a section which describes how to convert point spectrum into a directional spectrum.

#### 1.3.1 Pierson-Moskowitz spectrum

This spectral form requires only one parameter, wind speed, as input and was developed primarily for oceanographic use. It is intended to represent point spectrum of a fully developed sea, that is fetch and duration are great, and there is no contaminating swell from other areas.

$$S(\omega) = \frac{\alpha g^2}{\omega^5} e^{-\beta \left(\frac{g}{v\omega}\right)^4} \quad (24)$$

where:

$\omega$  is the frequency in radians/sec

$$\alpha = 8.10 \cdot 10^{-3}$$

$$\beta = 0.74$$

$g$  = acceleration due to gravity in  $cm/sec^2$

$V_\omega$  = wind speed in  $cm/sec$  measured 19.5m above the surface.

While its oceanographic importance is great, this spectral model is good only for extreme storm condition and is inappropriate for general use because it is based on the assumption of full developed sea state reached after an extended time period of steady wind with no contaminating swell.

### 1.3.2 Bretschneider spectrum

This spectrum takes two input parameters - wave period and wave height. It has the form:

$$S(\omega) = \frac{A}{\omega^5} e^{\frac{-B}{\omega^4}} \quad (25)$$

where the two parameters  $A$  and  $B$  depend on the modal frequency,  $\omega_m$ , and variance,  $S$ . The modal frequency is

$$\omega_m = \left[ \frac{4}{5} B \right]^{1/4} \quad (26)$$

and total variance is

$$S = A/4B \quad (27)$$

The International Towing Tank Conference (ITTC, 1978) recommends the use of a form of Bretschneider spectrum when more specifically appropriate spectral forms are not known.

$$A = 173 \frac{H_{1/3}^2}{T_1^4} \quad (28)$$

$$B = \frac{691}{T_1^4} \quad (29)$$

$H_{1/3}$  is the significant wave height and  $T_1$  is the time period; both are inputs.

### 1.3.3 JONSWAP spectrum

This spectral function takes two parameters as input - fetch and wind speed. The preceding two spectrum models represent open-ocean wave conditions. However, this might not always be the case where geographic boundaries limit the fetch in generating areas. North Sea is such an area and extensive oceanographic measurements were taken under the Joint North Sea Wave Project (JONSWAP). The spectral function derived from the recorded data is of the form:

$$S(\omega) = \frac{\alpha g}{\omega^5} e^{\left[ -\frac{5}{4} \frac{\omega}{\omega_m} \right]^{-4}} \gamma e^{-\frac{(\omega - \omega_m)^2}{2\sigma^2 \omega_m^2}} \quad (30)$$

where:

$\gamma$  is 3.3

$\sigma$  is 0.07 for  $\omega < \omega_m$  and is 0.09 for  $\omega > \omega_m$

$\alpha$  is  $0.076 \tilde{x}^{-0.22}$

$\omega_m$  is  $2\pi \tilde{f}_m g / V_{W10}$  (modal frequency)

$\tilde{x}$  is  $g^x / V_{W10}^2$

$\tilde{f}_m$  is  $3.5 \tilde{x}^{-0.33}$

$x$  is fetch

$V_{W10}$  is wind speed at 10m (32ft) above sea level

Note: JONSWAP is simply a form of Bretschneider spectrum, multiplied by a frequency dependent factor (the  $\gamma$  term).

[2] provides a more through listing of different wave spectrum.

### 1.3.4 Converting point spectrum to directional spectrum

According to [2] even though there are several unidirectional spectral models, there are only a few multidirectional spectral models. But, for the multidirectional wave models that are available, the data documentation has been lacking or is uncertain. In addition [2] also mentions that for modelling purpose, the directional characteristics of waves are assumed to be uncoupled from their spectral properties, and the spectrum of waves travelling within a given range of headings is taken to be some proportion of that measured at a point. On this basis, the directional spectrum is of the form:

$$S(\omega, \mu) = S(\omega)G(\mu) \quad (31)$$

$G(\mu)$  is called the *spreading function* and is of the form:

$$G(\mu) = F(s) \cos^{2s} \frac{1}{2}(\mu - \mu_1) \quad (32)$$

where:

$\mu_1$  is the predominant wave direction

$s$  is an index that determines the width of the directional spread.

$$F(s) = \frac{2^{2s-1}}{\pi} \frac{\Gamma^2(s+1)}{\Gamma(2s+1)} \quad (33)$$

$\Gamma$  is the Gamma function. The function  $F(s)$  ensures that the total variance of the directional spectrum  $S(\omega, \mu)$  is same as that of the point spectrum  $S(\omega)$  from which it is derived.

**Note:** Could not find appropriate value for the index  $s$  in ITTC recommendations.

DNV-GL-H103 [3] (section 2.2.7) also suggests a similar approach for converting a point spectrum to a directional spectrum.

$$S(\omega, \mu) = S(\omega)G(\mu) \quad (34)$$

$$G(\mu) = \frac{\Gamma(1 + n/2)}{\sqrt{\pi}\Gamma(1/2 + n/2)} \cos^n(\mu - \mu_p) \quad (35)$$

where  $\mu_p$  is the prevailing wind direction.

An alternative to the above two formulation is the 15<sup>th</sup> ITTC (1978) recommendation mentioned in [1].

$$G(\mu) = k \cos^n \mu \quad (36)$$

where:

$$-\pi/2 < \mu < \pi/2$$

$$n = 2$$

$$k = 2/\pi$$

## 2 Simulation of ocean waves

This section presents a rough structure of the classes and how the data are grouped under each class. It also captures the main functions that will be called when simulating the ocean waves. The aim of this documentation is only to provide a rough overview and does not aim to mirror the actual implementation. The source code documentation will contain the most accurate and in-depth description of the classes, its variables.

The ocean is represented by a surface. The contour of the surface is controlled by an  $n \times n$  matrix of points called the *control points*. The sea surface is represented by the class *SeaSurface* (section 2.1) and the control points are represented by the class *SeaSurfaceControlPoint* (section 2.4). Random waves are generated on the surface by the interaction of several regular waves which are represented by the class *RegularWave* (section 2.3). The statistically property of the random sea generated through the regular waves is controlled by a directional spectrum, which is represented by the class *DirectionalSpectrum* (section 2.2). The directional spectrum is a collection of point spectrum for each direction ranging from  $-\pi/2$  to  $\pi/2$ . The directional spectrum contains a table where each row represents a point spectrum for a direction and the columns of the table relate to each frequency.

### 2.1 Class SeaSurface

The class *SeaSurface* is used to define a square patch of the sea surface.

**Member variables in the class:**

- seaPatchSize - size of the square sea patch represented by *SeaSurface*.
- seaSurfaceControPoints - an array of points to control the contour of the sea surface.
- directionalSpectrum - contains a table of variance values for each direction and frequency.
- regularWaves - array of all regular waves on the sea surface.

**Methods in the class:**

- SetCtrlPointMatSize() - sets the number of control points on the surface.
- Init()
  - > Initialise the directional spectrum.
  - > Create the regular waves in the sea as described below:
    - >> for each row in directional spectrum (ie: direction):
      - >>> for each column in directional spectrum (ie: frequency):
        - >>>> Get the variance value from the directional spectrum.
        - >>>> Calculate the amplitude from variance.
        - >>>> Generate a random value for wave phase.
        - >>>> Create a regular wave with the amplitude, phase, frequency and direction.
        - >>>> Insert the regular wave created into the array regularWaves.
- Simulate()
  - > Initialise time = 0.
  - > for each point in the array seaSurfaceControlPoints:
    - >> for each wave in array regularWaves:
      - >>> Calculate displacement due to each wave.
      - >> Sum the displacements for all waves.
      - >> Set SeaSurfaceControlPoint.z = sum of displacements.
      - > Increment time and continue above loop.

### 2.2 Class DirectionalSpectrum

Class to generate a directional spectrum.

**Member variable in the class:**

- countDirections - the number of directions for which point spectrum data will be generated.

- countFrequencies - the number of frequencies bands that will be used for each point spectrum.
- directions - array containing direction angles.
- frequencies - array containing the frequency intervals.
- spectrum - table containing variance values. Each row corresponds to a direction and each column corresponds to a frequency band.

**Methods in the class:**

- SetCountDirections() - sets value for member variable countDirections.
- SetCountFrequencies() - sets value for member variable countFrequencies.
- Init()
  - > Set values for the array directions.
  - > Set values for the array frequencies.
  - > For each angle:
    - >> For each frequency:
      - >>> Calculate variance  $S(\omega)G(\mu)$
      - >>> Enter the variance into the table spectrum.

## 2.3 Class RegularWave

Class to represent a regular wave.

**Member variables in the class:**

- direction - direction of travel of the wave.
- phase - the random generated phase value of the wave.
- frequency - the circular frequency of the regular wave.
- amplitude - the amplitude of the regular wave.

## 2.4 Class SeaSurfaceControlPoint

Class to contain the geographic coordinates of a control point on the sea surface.

**Member variables in the class:**

- x - the x coordinate
- y - the y coordinate
- z - the z coordinate

## 2.5 Steps for simulating the random sea

- Create an object of class SeaSurface.
- Set the sea patch dimension.
- Call SeaSurface.Init() which in turn calls DirectionalSpectrum.Init().
- Call SeaSurface.Simulate().



## References

- [1] E. V. Lewis, “Principles of naval architecture second revision, volume iii,” *Jersey: SNAME*, 1988.
- [2] C. Stansberg, G. Contento, S. W. Hong, M. Irani, S. Ishida, R. Mercier, Y. Wang, J. Wolfram, J. Chaplin, and D. Kriebel, “The specialist committee on waves final report and recommendations to the 23rd ittc,” *Proceedings of the 23rd ITTC*, vol. 2, pp. 505–551, 2002.
- [3] F. DNV, “Recommended practice dnv-rp-h103, modelling and analysis of marine operations,” *Det Norske Veritas*, 2014.