# UNIT 5 PROBLEM SET (PS5)

Due: 11 pm CT on Monday, April 19, 2021.
Submission instructions are on Canvas.

All **textbook problems** refer to problems from *An Introduction to Mathematical Statistics and Its Applications*, 6th Edition, by Larsen and Marx. Problems labeled with "(Textbook)" have **solutions available (in the back of the book, on Slader.com, and/or on the specified pages)** and will be graded for effort and completion; the remaining problems will be graded for effort and correctness.

## Lesson 5.1: Inference Framework & Method of Moments

1. Assume the following observations are drawn from a Geometric($p$) distribution: $9, 5, 2, 12, 20, 1$. Use the method of moments to estimate $p$.

2. (Textbook) *This is Example 5.2.6. The solution is on pages 290-291.*

   Suppose that $y_1 = 0.42$, $y_2 = 0.10$, $y_3 = 0.65$, and $y_4 = 0.23$ is a random sample of size 4 from the PDF
   $$f_Y(y;\theta) = \theta y^{\theta-1} \quad \text{for } 0 \leq y \leq 1.$$
   Find the method of moments estimate for $\theta$.

3. (Textbook) Problem 5.2.17 on page 293, but only the MOM parts. That is, find a formula for the MOM estimate **and** calculate the MOM estimate for the given data.

4. The gamma distribution is often used to model the waiting time until a $k^{\text{th}}$ success occurs. It has parameters $k$ and $\lambda$. Its mean is $\mu = k/\lambda$, and its variance is $\sigma^2 = k/\lambda^2$. Assume the following data are drawn from a Gamma($k, \lambda$) distribution:
   $$2.4, \;\; 1.1, \;\; 1.9, \;\; 2.5, \;\; 4.5.$$
   Use MOM to estimate $k$ and $\lambda$.

## Lesson 5.2: Maximum Likelihood Estimation

Note that Problems 5-8 correspond to Problems 1-4 above, in that they are based on the same underlying distributions.

5. Assume the following observations are drawn from a Geometric($p$) distribution: $9, 5, 2, 12, 20, 1$. Use maximum likelihood to estimate $p$.

6. (Textbook) Problem 5.2.7 on page 288. It doesn't explicitly say, but you should use maximum likelihood estimation.

7. (Textbook) Problem 5.2.12 on page 289. *Hint: What happens when the support of the r.v. depends on the parameter? The answer to this problem is with the Problem 5.2.17 answer in the back of the book. Complete solutions for 5.2.12 are available on Slader.com.*

8. The PDF for the gamma distribution is

$$f_X(x;k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} \quad \text{for } x > 0$$

where $\Gamma$ is an extension of the factorial function. Assume the following data are drawn from a Gamma($k, \lambda$) distribution:

2.4, 1.1, 1.9, 2.5, 4.5.

Write out a set of equations that, when solved, give the maximum likelihood estimates for $k$ and $\lambda$. Your answer should be two equations in terms of two unknowns; both equations should have a right hand side "= 0". The term "$\Gamma'(k)$" [the derivative of $\Gamma(k)$] will be in one of these equations; leave this term as is, without simplification.

9. (Textbook) *The solution is in the second paragraph of Example 5.2.1 on page 282.*

Let $X_1 = k_1, X_2 = k_2, ..., X_n = k_n$ be a random sample of size $n$ from the Poisson distribution, $p_X(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, 2, ...$ where $\lambda$ is unknown. Show that $\bar{k}$, the observed sample mean, is the maximum likelihood estimate for $\lambda$.

## Lesson 5.3: Estimator Properties

10. (Textbook) *This is **part of** Example 5.4.2. The solution is on page 310. You **do not** need to do the part about the estimator $Y_{max}$. If you want to, you can find more on $Y_{max}$ in Sec. 3.10.*

In Problem 3 above (textbook problem 5.2.17) you found an estimator of $\theta$. Is it unbiased? If unbiased, show it. If biased, calculate the bias.

11. Let $X_1, X_2, ..., X_n$ be a random sample of size $n$ from the PDF

$$f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0.$$

(a) Find the mean square error (MSE) of $\widehat{\theta}_1 = X_1$.

(b) Find the MSE of $\widehat{\theta}_2 = \overline{X}_n$.

(c) Which estimator ($\widehat{\theta}_1$ or $\widehat{\theta}_2$) is more efficient? Briefly justify your answer.

12. (Textbook) *This is an equivalent setup to Example 5.5.1. The solution is on page 317.*

Consider $X \sim$ Binom($n, p$), where $n$ is known and $p$ is the unknown parameter of interest.

(a) Define $\widehat{p} = \frac{X}{n}$. Show that $\widehat{p}$ is unbiased for $p$.

(b) Recall that we may express a binomial r.v. as the sum of $n$ i.i.d. Bernoulli r.v.s [$X = X_1 + X_2 + \cdots + X_n$ where $X_i \overset{iid}{\sim}$ Bern($p$)]. We may write the Bernoulli PMF in the form

$$p_{X_i}(k;p) = p^k(1-p)^{1-k}, \quad k = 0, 1; \quad 0 < p < 1.$$

Find the Cramér-Rao lower bound for $p_{X_i}(k;p)$. Still considering $\widehat{p} = \frac{X}{n}$, how does $Var(\widehat{p})$ compare with this bound?