Unit_2_Problem_Set

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Unit 2 Problem Set (PS2)

Due: 11 pm CT on Thursday, Feb. 18, 2021
Submission instructions are on Canvas.

All **textbook problems** refer to Chapter 2 problems from *Introduction to Probability*, Second Edition, by Bertsekas and Tsitsiklis. Problems labeled with "(Textbook)" have solutions available at this link or on the noted pages and will be graded for effort and completion; the remaining problems will be graded for effort and correctness.

Show your work. No lonely numerical answers please.

Lesson 2.1: PMFs and Special Distributions

1. Suppose the number of Skittles in a fun-size bag, X, has the PMF given below.

x	13	14	15	16	17
$P(X=x)$	0.07	0.11	0.20	0.60	0.02

   (a) How do we know there aren't any other values in the support of X?
   (b) Calculate $P(X < 16)$.
   (c) Calculate the probability X takes an even value.
   (d) Your friend decides to keep opening bags until she finally finds one of these rare 17-piece bags. Assuming the bags are independent, how likely is it that she must open *at most* 5 bags? *Hint: Use a named distribution.*

2. (Textbook) Ch. 2, Problem 1. *Hint: Draw a tree diagram.*
3. (Textbook) Ch. 2, Problem 2.
4. Consider the experiment of rolling two fair, six-sided dice. Let X be the absolute difference between the dice. For example, if one die shows a 1 and the other shows a 6, X would be 5, regardless of which value was on which die.
   (a) State the PMF of X in a table. *Hint: Draw a grid diagram and start with the support.*
   (b) Plot the PMF of X.
   (c) Find the probability that X is greater than 2.
   (d) Suppose the experiment is repeated 10 times. How likely is it that exactly 4 of those times, X is greater than 2. *Hint: Use a named distribution.*

Lesson 2.2: Functions of RVs, Mean, and Variance

5. Let Y be a random variable with support $\{-3,-2,-1,1\}$. For any y in the support, assume that $p_Y(y) = cy^2$ for some constant c.
   (a) Find c so that the PMF is valid.
   (b) Find the mean (expected value) of Y.
   (c) Find the mean of $3Y$.
   (d) Find the mean of $|Y|$. *Hint: Start with the PMF of* $|Y|$?
6. (Textbook) Consider r.v. X and constants a and b. Let's prove two results from the notes.
   (a) Use the Expected Value Rule for Functions of R.V.s to show that $E[aX+b] = aE[X]+b$.
   (b) Use the definition of variance, along with the Expected Value Rule for Functions of R.V.s, to show that $Var(aX+b) = a^2 Var(X)$.

   Try yourself first. The proofs are on page 86 of the textbook.
7. Reconsider X, the number of Skittles in a fun-size bag, from Problem 1. Its PMF was given as:

x	13	14	15	16	17
$P(X=x)$	0.07	0.11	0.20	0.60	0.02

   (a) Find the expected value **and** variance of X.
   (b) Find the expected value **and** variance of $Y = 20 - 4X$. *Hint: What kind of function is this?*
8. (Textbook) Ch. 2 Problem 20. *Hint: Use a named distribution.*

Lesson 2.3: Joint PMFs, Conditioning, and Independence

9. Suppose $X \sim \text{Pois}(\lambda = 1)$ and $Y \sim \text{Binom}(n=3, p=0.4)$. Also suppose X and Y are independent. Find $P(X+Y=1)$. *Hint: What are the corresponding combination(s) of x and y?*
10. Random variables X and Y have the joint distribution given in the table below.

$P(X=x, Y=y)$	$Y=1$	$Y=2$	$Y=3$
$X=1$	4/20	3/20	2/20
$X=2$	7/20	4/20	

   (a) What value goes in the empty cell? That is, find $P(X=2, Y=3)$.
   (b) Find $E[Y]$. *Hint: Start with its marginal PMF.*
   (c) Find $E[Y \mid X=2]$. *Hint: Start with the conditional PMF.*
   (d) Are X and Y independent? Justify your answer mathematically.
11. (Textbook) Ch. 2, Problem 38.
12. (Textbook) Ch. 2, Problem 41, parts (a) and (b) only.

2.1.

1. a) $\sum_{x \in \{13,14,15,16,17\}} P_x(x) = P_x(X=13) + P_x(X=14) + P_x(X=15) + P_x(X=16) + P_x(X=17)$

$= 0.07 + 0.11 + 0.2 + 0.6 + 0.02$

$= 1.$

Since for all sample space S. $P(S)=1$.
there can't be more values in the support of X.

b). $P(X<16) = \sum_{X<16} P_x(X) = P_x(X=13) + P_x(X=14) + P_x(X=15)$

$= 0.07 + 0.11 + 0.2$

$= 0.38$

c). $P(X\%2=0) = \sum_{x\in\{14,16\}} P_x(x) = P_x(X=14) + P_x(X=16)$

$= 0.11 + 0.6$

$= 0.71.$

d). Geometric Distribution.
The probability of opening at most 5 bags is the sum of probability of getting 17-pieced bag at 1 to 5th trial.

$\sum_{k=1}^{5} P_x(k) = \sum_{k=1}^{5} (1-p)^{k-1} p$

$= 0.02 + 0.98 \times 0.02 + 0.98^2 \times 0.02 + 0.98^3 \times 0.02 + 0.98^4 \times 0.02$

≈ 0.0961

2. PMF.

	lose(0pt)	tie(1pt)	win (2pt).
Game 1	0.6	0.2	0.2
Game 2.	0.3.	0.35	0.35.

G2 \ G1	win	tie	lose
win	4pt 0.07	3pt 0.07	2pt 0.21
tie	3pt 0.07	2pt 0.07	1pt 0.21
lose.	2pt 0.06	1pt 0.06	0pt. 0.18.

PMF for score.

Score s	0	1	2	3	4	>4
$p(S=s)$	0.18	0.27	0.34	0.14	0.07	$p(S>4)=0.$

3. The probability for one random person having the same birthday as you do is $p = 1/365$
The probability that exactly one other guest has the same birthday is $\binom{499}{1}\frac{1}{365}\left(\frac{364}{365}\right)^{498} = 0.3486.$

$\lambda = np = 499/365 \approx 1.367.$

The poisson approx. is $e^{-\lambda}\left(\frac{\lambda^x}{x!}\right) = e^{-\lambda}\lambda$

$= e^{-1.367} \cdot 1.367$

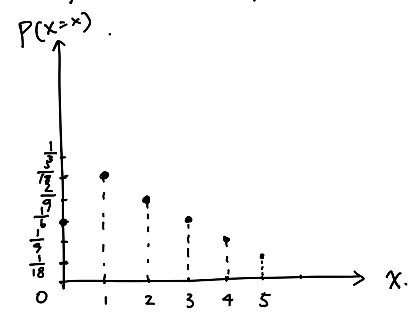
≈ 0.3483

4. a)

Dice 2 \ Dice 1	1	2	3	4	5	6.
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6.	5	4	3	2	1	0

PMF of X	0	1	2	3	4	5
	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

b).



c) $P(X>2) = P(X=3) + P(X=4) + P(X=5)$

$= \frac{1}{6} + \frac{1}{9} + \frac{1}{18}$

$= \frac{1}{3}$

d) Using Binomial distribution.
$X \sim \text{Binom}(n=10, p=\frac{1}{3})$

$P(X=4) = \binom{10}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6.$

$\approx 0.22761.$

5 1). $P_Y(Y) = cy^2$

support of Y is $\{-3,-2,-1,1\}$

$c(1)^2 + c(-1)^2 + c(-2)^2 + c(-3)^2 = 1$

$15c = 1.$

$c = \frac{1}{15}$

2). $E[Y] = P(Y=-1)\times -1 + P(Y=-2)\times -2 + P(Y=-3)\times -3 + P(Y=1)\times 1$

$= (-\frac{1}{15}) + (-\frac{8}{15}) + (-\frac{27}{15}) + (\frac{1}{15})$

$= -\frac{35}{15} = -\frac{7}{3}$

3) $E[3Y] = 3E[Y] = -7$
(According to the linearity of expectation)

4) PMF $|Y|$

	1	2	3
	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{9}{15}$

$E[|Y|] = \frac{2}{15} + \frac{8}{15} + \frac{27}{15} = \frac{37}{15}$

6.1) $E[ax+b]$

$= \sum_x Pr(X=x)(ax+b).$

$= \sum_{x\in X} ax\,Pr(X=x) + \sum_{x\in X} b\,Pr(X=x)$

$= a\sum_{x\in X} x\,Pr(X=x) + b$

$= aE[x] + b.$

2). $Var(ax+b).$

$= \sum_x (ax+b - E[ax+b])^2 p_x(x)$

$= \sum_x (ax+b - aE[x] + b)^2 p_x(x)$

$= a^2 \sum_x E[x] p_x(x)$

$= a^2 Var(x).$

7 1) $E[x] = \sum_x P(X=x)\,x$

$= 13\times 0.07 + 14\times 0.11 + 15\times 0.2 + 16\times 0.6 + 17\times 0.02$

$= 0.91 + 1.54 + 3 + 9.6 + 0.34.$

$= 4.88 + 10.51$

$= 15.39.$

$E[x^2] = 169\times 0.07 + 196\times 0.11 + 225\times 0.2 + 256\times 0.6 + 289\times 0.02$

$= 11.83 + 21.56 + 45 + 153.6. + 5.78$

$= 237.77$

$Var(x) = E[x^2] - (E[x])^2$

$= 237.77 - 236.8521$

$= 0.9179$

2). According to the linearity of expectation.

$E[20-4x] = 20 - 4E[x] = 20 - 4\times 15.39$

$= -41.56$

$Var(20-4x) = (-4)^2 Var(x)$

$= 16 \times 0.9179$

$= 14.6864$

8. Let Y = the number of candy bar needed to get one ticket.

$E[Y] = \frac{1}{p}$

$Var[Y] = (1-p)/p^2$

(This case fits into the geometric distribution model).

9. To get $P(X+Y=1)$:

PMF of Y:	0	1	2	3
	$(0.6)^3$	$\binom{3}{1}0.4\times 0.6^2$	$\binom{3}{2}0.6\times 0.4^2$	$(0.4)^3$

If $X+Y=1$:

case 1: $Y=0$ $X=1$.

$P(X=1) = e^{-1}\frac{1^1}{1} = 0.368$ $P(X=1, Y=0) = 0.079$

$P(Y=0) = 0.216$

case 2: $Y=1$ $X=0$.

$P(X=0) = 0.368$ $P(X=0, Y=1) = 0.159$

$P(Y=1) = 0.432$

$P(X+Y=1) = P(X=1, Y=0) + P(X=0, Y=1)$

$= 0.079 + 0.159$

$= 0.238$

10. 1) $P(X=2, Y=3) = 1 - \frac{4}{20} - \frac{3}{20} - \frac{2}{20} - \frac{7}{20} - \frac{4}{20}$

$= 0.$

2). $P(Y=1) = \frac{11}{20}$

$P(Y=2) = \frac{7}{20}$ $\Rightarrow E[Y] = \sum_Y P(Y=y)y$

$P(Y=3) = \frac{2}{20}$

$= \frac{11}{20}\times 1 + \frac{7\times 2}{20} + \frac{2\times 3}{20}$

$= \frac{31}{20}$

3) $P(Y=1 \mid X=2) = \frac{7}{11}$

$P(Y=2 \mid X=2) = \frac{4}{11}$

$P(Y=3 \mid X=2) = 0.$

$E[Y \mid X=2] = P(Y=2 \mid X=2)\times 2 + P(Y=1 \mid X=2)\times 1$

$= 2\times\frac{4}{11} + 1\times\frac{7}{11}$

$= \frac{15}{11}$

4). $P(X=1) = \frac{9}{20}.$

$P(X=2) = \frac{11}{20}$

If X and Y are independent:

$P(X=x \mid Y=y) = P(X=x).$

$\frac{P(X=x, Y=y)}{P(Y=y)} = P(X=x)$

① $X=1$, $Y=1$.

$P(X=1, Y=1) = \frac{1}{5}$

$P(X=1)P(Y=1) = \frac{11}{20}\times\frac{9}{20}$

$= \frac{99}{400}.$

$\frac{99}{400} \neq \frac{1}{5}$ so X and Y are not independent.

11. a) PMF.

red	0	1	2	3	4.
	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}.$

The number above are calculated with geometric distribution equations.

let X be the number of red lights.

mean: $E[X] = \sum_X P(X=x)X = \frac{1}{16}\times 0 + \frac{1}{4} + \frac{3}{8}\times 2 + \frac{1}{4}\times 3 + \frac{1}{16}\times 4$

$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4}$

$= 2.$

Variance: $Var(x) = E[x^2] - E[x]^2$

$= \left(\frac{1}{4} + \frac{12}{8} + \frac{9}{4} + \frac{16}{16}\right) - 4$

$= 1$

b). $Var(2x) = Var(x)\times 2^2 = 4$ min.

12 1). $E[X] = 5\times 50\times 0.02 = 5$

$P(X=5) = \binom{250}{5}(0.02)^5(0.98)^{245} = 0.17725.$

2). $\lambda = np = 5\times 50\times 0.02 = 5$

the poission distribution is $e^{-\lambda}\left(\frac{\lambda^x}{x!}\right)$

$= e^{-5}\left(\frac{5^5}{5!}\right).$

$= 0.1755$