Unit 4 Problem Set (PS4)

Due: 11 pm CT on Monday, March 29, 2021.
Submission instructions are on Canvas.

All **textbook problems** refer to problems from *Introduction to Probability*, Second Edition, by Bertsekas and Tsitsiklis. Problems labeled with "(Textbook)" have solutions available at this link and will be graded for effort and completion; problems labeled with "(Completion)" have numerical answers provided at the end of this document and will be graded for effort and completion; the remaining problems will be graded for correctness.

Show your work. No lonely numerical answers please.

To facilitate a quick grading turnaround before Exam 2, most of the problems on this problem set will be graded for completion. *If you do not show work, the problem will not be considered complete.*

Lesson 3.4: Joint PDFs, Independence, Covariance & Correlation

1. Suppose you (simultaneously) buy a used clothes washer and dryer. Let X be how long the washer will last, in days, and let Y be how long the dryer will last, in days. Assume that X and Y are independent, that X follows a uniform distribution from moment of purchase through 400 days, and that Y follows a uniform distribution from moment of purchase through 260 days. *Hint: For the probability calculations, sketch a 2D plot of the joint support.*
   (a) (Completion) State the joint PDF of X and Y. Be sure to include the support.
   (b) (Completion) What is the probability the washer fails before the dryer?
   (c) (Completion) What is the probability the washer lasts at least 31 days longer than the dryer?
   (d) What is the probability the difference in the appliances' lifespans is more than 45 days?
2. (Textbook) Ch. 3, Problem 18 part (a).
3. Prove Property 5 on page 5 of the Lesson 3.4 notes. That is, show that $\rho(X, aY+b) = \rho(X, Y)$ for r.v.s X and Y and scalars $a > 0$ and b. Briefly justify **each** step; a word or short phrase is fine, such as "correlation definition", "covariance property 4", or "algebra". *Hint: Start by using the definition of correlation. You may use the other properties on that notes page without proof, but you need to state which one(s) you use as part of your justification.*
4. (Completion) Consider r.v.s X, Y, and Z. Suppose $Var(X) = 4$, $Var(Y) = 9$, $Var(Z) = 16$, $Cov(X, Y) = -1$, $\rho(X, Z) = 0.3$, and $\rho(Y, 3Z) = -0.9$. *Hint: Use the definitions of covariance and correlation, as well as the properties at the end of the Lesson 3.4 notes.*
   (a) Find $\rho(X, Y)$.
   (b) Find $\rho(Z, Y)$.
   (c) Find $Cov(X, Z)$.
   (d) Find $Cov(X+0.5, 2.5Y)$.
   (e) Find $Cov(X + Y, Y)$.
5. (Textbook) Ch. 4, Problem 18.

Lesson 4.1: Inequalities and Law of Large Numbers

6. (Completion) Let $X \sim$ Exponential($\lambda = 0.5$). For each of the following, use the Markov inequality to find an upper bound on the probability **and** use the exponential PDF or CDF to calculate the exact probability.

   (a) $P(X \geq 0.5)$ (b) $P(X \geq 2)$ (c) $P(X \geq 4)$ (d) $P(X \geq 8)$

7. (Textbook) Ch. 5, Problem 1, parts (a) and (b) only. *Hint: Be careful with units.*

8. Let X_i, $i = 1, ..., n$, be i.i.d. continuous random variables with $Var(X_i) = 2.7$, and consider their mean $\overline{X}_n$. For the values of n below, use Chebyshev's inequality to calculate an upper bound for the probability that $\overline{X}_n$ is at least 0.5 units away from its expected value.

   (a) (Completion) $n = 15$
   (b) $n = 100$
   (c) (Completion) What value does the Law of Large Numbers tell us these probabilities should tend towards as n increases?

Lesson 4.2: Central Limit Theorem

9. (Textbook) *This is Example 5.9. See page 275 for the solution.* We load on a plane 100 packages whose weights are independent r.v.s that are uniformly distributed between 5 and 50 pounds. Use the CLT to find the approximate probability that the total weight will exceed 3000 pounds. Report your answer in Φ notation and provide a final numerical answer.

10. Let $\overline{X}$ be the mean of a random sample of size 50 from a geometric distribution with mean 3.

    (a) (Completion) Calculate $P(2.5 \leq X \leq 4)$. Note that this is asking about a single X, not $\overline{X}$.
    (b) Use the approximate distribution of $\overline{X}$, based on the CLT, to calculate $P(2.5 \leq \overline{X} \leq 4)$.

11. (Completion) The tensile strength X of paper, in pounds per square inch, has $\mu = 30$ and $\sigma = 3$. A random sample of size $n = 100$ is taken from the distribution of tensile strengths. Compute the (approximate) probability that the sample mean $\overline{X}$ is greater than 29.5 pounds per square inch. Provide your answer in Φ notation **and** calculate a final numerical answer.

12. (Completion) Let $S = X_1 + X_2 + \cdots + X_{30}$ be the sum of 30 i.i.d. r.v.s, where each X_i has the distribution defined by PDF $f_X(x) = 1.5x^2$ for $-1 < x < 1$.

    (a) Find $E[X_i]$, the expected value for a single X_i.
    (b) Find $Var(X_i)$, the variance for a single X_i.
    (c) Use the CLT to approximate $P(-0.3 < S < 1.5)$.

13. (Textbook) Ch. 5, Problem 8. *Background: If the roulette is fair, then each number (1 - 36) is equally likely to be the result of the round. Also: It is possible to make this calculation exactly using the binomial distribution, but the problem is asking you to apply the CLT and get an approximate answer.*

14. (Textbook) Ch. 5, Problem 9. *Hint for (b): Refer back to Lesson 2.1 if you need to refresh on the Poisson approximation to the binomial distribution. It's also recommended to model the number of days* ***with*** *a crash, because the Poisson approximation is better when p is small. Since the solutions defined S as the number of days without a crash, they use "50-S" for the number of days with a crash.*

Answers to Problems Marked "Completion"

1. (a) $f_{X,Y}(x,y) = \frac{1}{104000}$ for $0 \leq x \leq 400$, $0 \leq y \leq 260$
   (b) 0.325
   (c) 0.5975
   (d) [graded for correctness]
2. (Textbook)
3. [graded for correctness]
4. (a) $-1/6$ (b) -0.9 (c) 2.4 (d) -2.5 (e) 8
5. (Textbook)
6. (a) 4; 0.7788 (b) 1; 0.3679 (c) 0.5; 0.1353 (d) 0.25; 0.0183
7. (Textbook)
8. (a) 0.72
   (b) [graded for correctness]
   (c) 0
9. (Textbook)
10. (a) 0.2469
    (b) [graded for correctness]
11. $1 - \Phi(-5/3)$ or, equivalently, $\Phi(5/3)$; 0.9522 from applet or 0.95254 from table
12. (a) 0
    (b) 0.6
    (c) ≈ 0.166 (or 0.1647 using table)
13. (Textbook)
14. (Textbook)