### Lecture Notes on

# Linear Probing with 5-Independent Hashing

Mikkel Thorup

May 12, 2017

#### Abstract

These lecture notes show that linear probing takes expected constant time if the hash function is 5-independent. This result was first proved by Pagh et al. [STOC'07,SICOMP'09]. The simple proof here is essentially taken from [Pătrașcu and Thorup ICALP'10]. We will also consider a smaller space version of linear probing that may have false positives like Bloom filters.

These lecture notes illustrate the use of higher moments in data structures, and could be used in a course on randomized algorithms.

### 1 k-independence

The concept of k-independence was introduced by Wegman and Carter [21] in FOCS'79 and has been the cornerstone of our understanding of hash functions ever since. A hash function is a random function  $h: [u] \to [t]$  mapping keys to hash values. Here  $[s] = \{0, \ldots, s-1\}$ . We can also think of a h as a random variable distributed over  $[t]^{[u]}$ . We say that h is k-independent if for any distinct keys  $x_0, \ldots, x_{k-1} \in [u]$  and (possibly non-distinct) hash values  $y_0, \ldots, y_{k-1} \in [t]$ , we have  $\Pr[h(x_0) = y_0 \wedge \cdots \wedge h(x_{k-1}) = y_{k-1}] = 1/t^k$ . Equivalently, we can define k-independence via two separate conditions; namely,

- (a) for any distinct keys  $x_0, \ldots, x_{k-1} \in [u]$ , the hash values  $h(x_0), \ldots, h(x_{k-1})$  are independent random variables, that is, for any (possibly non-distinct) hash values  $y_0, \ldots, y_{k-1} \in [t]$  and  $i \in [k]$ ,  $\Pr[h(x_i) = y_i] = \Pr\left[h(x_i) = y_i \mid \bigwedge_{j \in [k] \setminus \{i\}} h(x_j) = y_j\right]$ , and
- (b) for any  $x \in [u]$ , h(x) is uniformly distributed in [t].

As the concept of independence is fundamental to probabilistic analysis, k-independent hash functions are both natural and powerful in algorithm analysis. They allow us to replace the heuristic assumption of truly random hash functions that are uniformly distributed in  $[t]^{[u]}$ , hence needing  $u \lg t$  random bits ( $\lg = \log_2$ ), with real implementable hash functions that are still "independent enough" to yield provable performance guarantees similar to those proved with true randomness. We are then left with the natural goal of understanding the independence required by algorithms.

Once we have proved that k-independence suffices for a hashing-based randomized algorithm, we are free to use any k-independent hash function. The canonical construction of a k-independent

hash function is based on polynomials of degree k-1. Let  $p \ge u$  be prime. Picking random  $a_0, \ldots, a_{k-1} \in [p] = \{0, \ldots, p-1\}$ , the hash function is defined by:

$$h(x) = (a_{k-1}x^{k-1} + \dots + a_1x + a_0) \bmod p$$
 (1)

If we want to limit the range of hash values to [t], we use  $h'(x) = h(x) \mod t$ . This preserves requirement (a) of independence among k hash values. Requirement (b) of uniformity is close to satisfied if  $p \gg t$ . More precisely, for any key  $x \in [p]$  and hash value  $y \in [t]$ , we get  $1/t - 1/p < \Pr[h'(x) = y] < 1/t + 1/p$ .

Sometimes 2-independence suffices. For example, 2-independence implies so-called universality [5]; namely that the probability of two keys x and y colliding with h(x) = h(y) is 1/t; or close to 1/t if the uniformity of (b) is only approximate. Universality implies expected constant time performance of hash tables implemented with chaining. Universality also suffices for the 2-level hashing of Fredman et al. [7], yielding static hash tables with constant query time.

At the other end of the spectrum, when dealing with problems involving n objects,  $O(\lg n)$ -independence suffices in a vast majority of applications. One reason for this is the Chernoff bounds of [18] for k-independent events, whose probability bounds differ from the full-independence Chernoff bound by  $2^{-\Omega(k)}$ . Another reason is that random graphs with  $O(\lg n)$ -independent edges [2] share many of the properties of truly random graphs.

The independence measure has long been central to the study of randomized algorithms. It applies not only to hash functions, but also to pseudo-random number generators viewed as assigning hash values to  $0, 1, 2, \ldots$  For example, [10] considers variants of QuickSort, [1] consider the maximal bucket size for hashing with chaining, and [9, 6] consider Cuckoo hashing. In several cases [1, 6, 10], it is proved that linear transformations  $x \mapsto ((ax + b) \mod p)$  do not suffice for good performance, hence that 2-independence is not in itself sufficient.

Our focus in these notes is linear probing described below.

# 2 Linear probing

Linear probing is a classic implementation of hash tables. It uses a hash function h to map a dynamic set S of keys into an array T of size t > |S|. The entries of T are keys, but we can also see if an entry is "empty". This could be coded, either via an extra bit, or via a distinguished nil-key. We start with an empty set S and all empty locations. When inserting x, if the desired location  $h(x) \in [t]$  is already occupied, the algorithm scans  $h(x) + 1, h(x) + 2, \ldots, t - 1, 0, 1, \ldots$  until an empty location is found, and places x there. Below, for simplicity, we ignore the wrap-around from t - 1 to 0, so a key x is always placed in a location  $i \ge h(x)$ .

To search a key x, the query algorithm starts at h(x) and scans either until it finds x, or runs into an empty position, which certifies that x is not in the hash table. When the query search is unsuccessful, that is, when x is not stored, the query algorithm scans exactly the same locations as an insert of x. A general bound on the query time is hence also a bound on the insertion time.

Deletions are slightly more complicated. The invariant we want to preserve is that if a key x is stored at some location  $i \in [t]$ , then all locations from h(x) to i are filled; for otherwise the above search would not get to x. Suppose now that x is deleted from location i. We then scan locations  $j = i + 1, i + 2, \ldots$  for a key y with  $h(y) \leq i$ . If such a y is found at location j, we move y to location i, but then, recursively, we have to try refilling j, looking for a later key z with  $h(z) \leq j$ . The deletion process terminates when we reach an empty location d, for then the invariant says

that there cannot be a key y at a location j > d with  $h(y) \le d$ . The recursive refillings always visit successive locations, so the total time spent on deleting x is proportional to the number of locations from that of x and to the first empty location. Summing up, we have

**Theorem 1** With linear probing, the time it takes to search, insert, or delete a key x is at most proportional to the number of locations from h(x) to the first empty location.

With n the number of keys and t the size of the table, we call n/t the load of our table. We generally assume that the load is bounded from 1, e.g., that the number of keys is  $n \leq \frac{2}{3}t$ . With a good distribution of keys, we would then hope that the number of locations from h(x) to an empty location is O(1).

This classic data structure is one of the most popular implementations of hash tables, due to its unmatched simplicity and efficiency. The practical use of linear probing dates back at least to 1954 to an assembly program by Samuel, Amdahl, Boehme (c.f. [12]). On modern architectures, access to memory is done in cache lines (of much more than a word), so inspecting a few consecutive values is typically only slightly worse that a single memory access. Even if the scan straddles a cache line, the behavior will still be better than a second random memory access on architectures with prefetching. Empirical evaluations [3, 8, 14] confirm the practical advantage of linear probing over other known schemes, e.g., chaining, but caution [8, 20] that it behaves quite unreliably with weak hash functions. Taken together, these findings form a strong motivation for theoretical analysis.

Linear probing was shown to take expected constant time for any operation in 1963 by Knuth [11], in a report which is now regarded as the birth of algorithm analysis. This analysis, however, assumed a truly random hash function.

A central open question of Wegman and Carter [21] was how linear probing behaves with k-independence. Siegel and Schmidt [17, 19] showed that  $O(\lg n)$ -independence suffices for any operation to take expected constant time. Pagh et al. [13] showed that just 5-independence suffices for expected constant operation time. They also showed that linear transformations do not suffice, hence that 2-independence is not in itself sufficient.

Pătrașcu and Thorup [16] proved that 4-independence is not in itself sufficient for expected constant operation time. They display a concrete combination of keys and a 4-independent random hash function where searching certain keys takes super constant expected time. This shows that the 5-independence result of Pagh et al. [13] is best possible. In fact [16] provided a complete understanding of linear probing with low independence as summarized in Table 1.

Considering loads close to 1, that is load  $(1 - \varepsilon)$ , Pătrașcu and Thorup [15] proved that the expected operation time is  $O(1/\varepsilon^2)$  with 5-independent hashing, matching the bound of Knuth [11] assuming true randomness. The analysis from [15] also works for something called simple tabulation hashing that is we shall return to in Section 3.2.

# 3 Linear probing with 5-independence

Below we present the simplified version of the proof from [15] of the result from [13] that 5-independent hashing suffices for expected constant time with linear probing. For simplicity, we assume that the load is at most  $\frac{2}{3}$ . Thus we study a set S of n keys stored in a linear probing table of size  $t \geq \frac{3}{2}n$ . We assume that t is a power of two.

A crucial concept is a run R which is a maximal interval of filled positions. We have an empty position before R, which means that all keys  $x \in S$  landing in R must also hash into R in the sense

Independence	2	3	4	$\geq 5$
Query time	$\Theta(\sqrt{n})$	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(1)$
Construction time	$\Theta(n \lg n)$	$\Theta(n \lg n)$	$\Theta(n)$	$\Theta(n)$

Table 1: Expected time bounds for linear probing with a poor k-independent hash function. The bounds are worst-case expected, e.g., a lower bound for the query means that there is a concrete combination of stored set, query key, and k-independent hash function with this expected search time while the upper-bound means that this is the worst expected time for any such combination. Construction time refers to the worst-case expected total time for inserting n keys starting from an empty table.

that  $h(x) \in R$ . Also, we must have exactly r = |R| keys hashing to R since the position after R is empty.

By Theorem 1 the time it takes for any operation on a key q is at most proportional to the number of locations from h(x) to the first empty location. We upper bound this number by r+1 where r is the length of the run containing h(q). Here r=0 if the location h(q) is empty. We note that the query key q might itself be in R, and hence be part of the run, e.g., in the case of deletions.

We want to give an expected upper bound on r. In order to limit the number of different events leading to a long run, we focus on dyadic intervals: a (dyadic)  $\ell$ -interval is an interval of length  $2^{\ell}$  of the form  $[i2^{\ell}, (i+1)2^{\ell})$  where  $i \in [t/2^{\ell}]$ . Assuming that the hashing maps S uniformly into [t], we expect  $n2^{\ell}/t \leq \frac{2}{3}2^{\ell}$  keys to hash into a given  $\ell$ -interval I. We say that I is "near-full" if at least  $\frac{3}{4}2^{\ell}$  keys from  $S \setminus \{q\}$  hash into I. We claim that a long run implies that some dyadic interval of similar size is near-full. More precisely,

**Lemma 2** Consider a run R of length  $r \geq 2^{\ell+2}$ . Then one of the first four  $\ell$ -intervals intersecting R must be near-full.

**Proof** Let  $I_0, \ldots, I_3$  be the first four  $\ell$ -intervals intersecting R. Then  $I_0$  may only have its last end-point in R while  $I_1, \ldots, I_3$  are contained in R since  $r \geq 4 \cdot 2^{\ell}$ . In particular, this means that  $L = \left(\bigcup_{i \in [4]} I_i\right) \cap R$  has length at least  $3 \cdot 2^{\ell} + 1$ .

But L is a prefix of R, so all keys landing in L must hash into L. Since L is full, we must have at least  $3 \cdot 2^{\ell} + 1$  keys hashing into L. Even if this includes the query key q, then we conclude that one of our four intervals  $I_i$  must have  $3 \cdot 2^{\ell}/4 \ge \frac{3}{4}2^{\ell}$  keys from  $S \setminus \{q\}$  hashing into it, implying that  $I_i$  is near-full.

Getting back to our original question, we are considering the run R containing the hash of the query q.

**Lemma 3** If the run containing the hash of the query key q is of length  $r \in [2^{\ell+2}, 2^{\ell+3})$ , then one of the following 12 consecutive  $\ell$ -intervals is near-full: the  $\ell$ -interval containing h(q), the 8 nearest  $\ell$ -intervals to its left, and the 3 nearest  $\ell$ -intervals to its right.

**Proof** Let R be the run containing h(q). To apply Lemma 2, we want to show that the first four  $\ell$ -intervals intersecting R has to be among the 12 mentioned in Lemma 3. Since the run R containing h(q) has length less than  $8 \cdot 2^{\ell}$ , the first  $\ell$ -interval intersecting R can be at most 8 before the one containing h(q). The 3 following intervals are then trivially contained among the 12.

For our analysis, in the random choice of the hash function h, we first fix the hash value h(q) of the query key q. Conditioned on this value of h(q), for each  $\ell$ , let  $P_{\ell}$  be an upper-bound on the the probability that any given  $\ell$ -interval is near-full. Then the probability that the run containing h(q) has length  $r \in [2^{\ell+2}, 2^{\ell+3})$  is bounded by  $12P_{\ell}$ . Of course, this only gives us a bound for  $r \geq 4$ . We thus conclude that the expected length of the run containing the hash of the query key q is bounded by

Thus, conditioned on the hash of the query key, for each  $\ell$  we are interested in a bound  $P_{\ell}$  on the probability that any given  $\ell$ -interval is near-full. Then the probability that the run containing h(q) has length  $r \in [2^{\ell+2}, 2^{\ell+3})$  is bounded by  $12P_{\ell}$ . Of course, this only gives us a bound for  $r \geq 4$ . We thus conclude that the expected length of the run containing the hash of the query key q is bounded by

$$3 + \sum_{\ell=0}^{\log_2 t} 2^{\ell+3} \cdot 12P_{\ell} = O\left(1 + \sum_{\ell=0}^{\log_2 t} 2^{\ell} P_{\ell}\right).$$

Combined with Theorem 1, we have now proved

**Theorem 4** Consider storing a set S of keys in a linear probing table of size t where t is a power of two. Conditioned on the hash of a key q, let  $P_{\ell}$  bound the probability that  $\frac{3}{4}2^{\ell}$  keys from  $S \setminus \{q\}$  hash to any given  $\ell$ -interval. Then the expected time to search, insert, or delete q is bounded by

$$O\left(1 + \sum_{\ell=0}^{\log_2 t} 2^{\ell} P_{\ell}\right).$$

We note that Theorem 4 does not mention the size of S. However, as mentioned earlier, with a uniform distribution, the expected number of elements hashing to an  $\ell$ -interval is  $\leq 2^{\ell} |S|/t$ , so for  $P_{\ell}$  to be small, we want this expectation to be significantly smaller than  $\frac{3}{4} 2^{\ell}$ . Assuming  $|S| \leq \frac{2}{3}t$ , the expected number is  $\frac{2}{3} 2^{\ell}$ .

To get constant expected cost for linear probing, we are going to assume that the hash function used is 5-independent. This means that no matter the hash value h(q) of q, conditioned on h(q), the keys from  $S \setminus \{q\}$  are hashed 4-independently. This means that if  $X_x$  is the indicator variable for a key  $x \in S \setminus \{q\}$  hashing to a given interval I, then the variables  $X_x$ ,  $x \in S \setminus \{q\}$  are 4-wise independent.

#### 3.1 Fourth moment bound

The probabilistic tool we shall use here to analyze 4-wise independent variables is a 4<sup>th</sup> moment bound. For  $i \in [n]$ , let  $X_i \in [2] = \{0,1\}$ ,  $p_i = \Pr[X_i = 1] = \operatorname{E}[X_i]$ ,  $X = \sum_{i \in [n]} X_i$ , and  $\mu = \operatorname{E}[X] = \sum_{i \in [n]} p_i$ . Also  $\sigma_i^2 = \operatorname{Var}[X_i] = \operatorname{E}[(X_i - p_i)^2] = p_i(1 - p_i)^2 + (1 - p_i)p_i^2 = p_i - p_i^2$ . As long as the  $X_i$  are pairwise independent, the variance of the sum is the sum of the variances, so we define

$$\sigma^2 = \operatorname{Var}[X] = \sum_{i \in [n]} \operatorname{Var}[X_i] = \sum_{i \in [n]} \sigma_i^2 \le \mu.$$

By Chebyshev's inequality, we have

$$\Pr[|X - \mu| \ge d\sqrt{\mu}] \le \Pr[|X - \mu| \ge d\sigma] \le 1/d^2. \tag{2}$$

We are going to prove a stronger bound if the variables are 4-wise independent and  $\mu \geq 1$  (and which is only stronger if  $d \geq 2$ ).

**Theorem 5** If the variables  $X_0, \ldots, X_{n-1} \in \{0,1\}$  are 4-wise independent,  $X = \sum_{i \in [n]} X_i$ , and  $\mu = E[X] \ge 1$ , then

$$\Pr[|X - \mu| \ge d\sqrt{\mu}] \le 4/d^4.$$

**Proof** Note that  $(X - \mu) = \sum_{i \in [n]} (X_i - p_i)$ . By linearity of expectation, the fourth moment is:

$$E[(X - \mu)^4] = E[\left(\sum_i X_i - p_i\right)^4] = \sum_{i,j,k,l \in [n]} E[(X_i - p_i)(X_j - p_j)(X_k - p_k)(X_l - p_l)].$$

Our goal is to get a good bound on the fourth moment.

Consider a term  $E[(X_i - p_i)(X_j - p_j)(X_k - p_k)(X_l - p_l)]$ . The at most 4 distinct variables are completely independent. Suppose one of them, say,  $X_i$ , appears only once. By definition,  $E[(X_i - p_i)] = 0$ , and since it is independent of the other factors, we get  $E[(X_i - p_i)(X_j - p_j)(X_k - p_k)(X_l - p_l)] = 0$ . We can therefore ignore all terms where any variable appears once. We may therefore assume that each variables appears either twice or 4 times. In terms with variables appearing twice, we have two indices a < b where a is assigned to two of i, j, k, l, while b is assigned to the other two, yielding  $\binom{4}{2}$  combinations based on a < b. Thus we get

$$E[(X - \mu)^{4}] = \sum_{i,j,k,l \in [n]} E[(X_{i} - p_{i})(X_{j} - p_{j})(X_{k} - p_{k})(X_{l} - p_{l})]$$

$$= \sum_{i} E[(X_{i} - p_{i})^{4}] + {4 \choose 2} \sum_{a < b} (E[(X_{a} - p_{a})^{2}]E[(X_{b} - p_{b})^{2}]).$$

Considering any multiplicity  $m = 2, 3, 4, 5, \ldots$ , we have

$$E[(X_i - p_i)^m] \le E[(X_i - p_i)^2] = \sigma_i^2.$$
 (3)

To see this, note that  $X_i, p_i \in [0,1]$ . Hence  $|X_i - p_i| \le 1$ , so  $(X_i - p_i)^{m-2} \le 1$ , and therefore  $(X_i - p_i)^m \le (X_i - p_i)^2$ . Continuing our calculation, we get

$$E[(X - \mu)^4] = \sum_{i} E[(X_i - p_i)^4] + {4 \choose 2} \sum_{a < b} (E[(X_a - p_a)^2] E[(X_b - p_b)^2])$$

$$\leq \sum_{i} \sigma_i^2 + {4 \choose 2} \sum_{a < b} \sigma_a^2 \sigma_b^2$$

$$\leq \sigma^2 + 3 \left(\sum_{i} \sigma_i^2\right)^2$$

$$= \sigma^2 + 3\sigma^4. \tag{4}$$

Since  $\sigma^2 \le \mu$  and  $\mu \ge 1$ , we get

$$E[(X - \mu)^4] \le \mu + 3\mu^2 \le 4\mu^2. \tag{5}$$

which is our desired bound on the fourth moment.

by Markov's inequality,

$$\Pr[|X - \mu| \ge d\sqrt{\mu}] = \Pr[(X - \mu)^4 \ge (d\sqrt{\mu})^4] \le \mathbb{E}[(X - \mu)^4]/(d\sqrt{\mu})^4 \le 4/d^4.$$

This completes the proof of Theorem 5.

We are now ready to prove the 5-independence suffices for linear probing.

**Theorem 6** Suppose we use a 5-independent hash function h to store a set S of n keys in a linear probing table of size  $t \geq \frac{3}{2}n$  where t is a power of two. Then it takes expected constant time to search, insert, or delete a key.

**Proof** First we fix the hash of the query key q. To apply Theorem 4, we need to find a bound  $P_{\ell}$  on the probability that  $\frac{3}{4}2^{\ell}$  keys from  $S \setminus \{q\}$  hash to any given  $\ell$ -interval I. For each key  $x \in S \setminus \{q\}$ , let  $X_x$  be the indicator variable for  $h(x) \in I$ . Then  $X = \sum_{x \in S \setminus \{q\}} X_x$  is the number of keys hashing to I, and the expectation of X is  $\mu = \mathbb{E}[X] = n2^{\ell}/t \leq \frac{2}{3}2^{\ell}$ . Our concern is the event that

$$X \ge \frac{3}{4} 2^{\ell} \implies X - \mu \ge \frac{1}{12} 2^{\ell} > \frac{1}{10} \sqrt{2^{\ell} \mu}.$$

Since h is 5-independent, the  $X_x$  are 4-independent, so by Theorem 5, we get

$$\Pr\left[X \ge \frac{3}{4} \, 2^{\ell}\right] \le 40000/2^{2\ell} = O(1/2^{2\ell}).$$

Thus we can use  $P_{\ell} = O(1/2^{2\ell})$  in Theorem 4, and then we get that the expected operation cost is

$$O\left(1 + \sum_{\ell=0}^{\log_2 t} 2^{\ell} P_{\ell}\right) = O\left(1 + \sum_{\ell=0}^{\log_2 t} 2^{\ell} / 2^{2\ell}\right) = O(1).$$

**Problem 1** Above we assumed that the range of our hash function is [t] where t is a power of two. As suggested in the introduction, we use a hash function based on a degree 4 polynomial over a prime field  $\mathbb{Z}_p$  where  $p \gg 1$ , that is, we pick 5 independent random coefficients  $a_0, \ldots, a_4 \in [p]$ , and define the hash function  $h': [p] \to [t]$  by

$$h'(x) = (a_4x^4 + \dots + a_1x + a_0) \mod p \mod t.$$

Then for any distinct  $x_0, \ldots, x_4$ , the hash values  $h'(x_0), \ldots, h'(x_4)$  are independent. Moreover, we have almost uniformity in the sense that for any  $x \in [p]$  and  $y \in [t]$ , we have  $1/t - 1/p < \Pr[h'(x) = y] < 1/t + 1/p$ .

Prove that Theorem 6 still holds with constant operation time if  $p \geq 24t$ .

**Problem 2** Assuming full randomness, use Chernoff bounds to prove that the longest run in the hash table has length  $O(\log n)$  with probability at least  $1 - 1/n^{10}$ .

Hint. You can use Lemma 2 to prove that if there is run of length  $r \geq 2^{\ell+2}$ , then some  $\ell$ -interval is near-full. You can then pick  $\ell = C \ln n$  for some large enough constant C.

**Problem 3** Using Chebyshev's inequality, show that with 3-independent hashing, the expected operation time is  $O(\log n)$ .

#### 3.2 Fourth moment and simple tabulation hashing

In the preceding analysis we use the 5-independence of the hash function as follows. First we fix the hash of the query key. Conditioned on this fixing, we still have 4-independence in the hashes of the stored keys, and we use this 4-independence to prove the 4th moment bound (5) on the number stored keys hashing to any given interval. This was all we needed about the hash function to conclude that linear probing takes expected constant time per operation.

Pătrașcu and Thorup [15] have proved that something called simple tabulation hashing, that is only 3-independent, within a constant factor provides the same 4th moment bound (5) on the number of stored keys hashing to any given interval conditioned on a fixed hash of the query key. Linear probing therefore also works in expected constant time with simple tabulation. This is important because simple tabulation is 10 times faster than 5-independence implemented with a polynomial as in (1).

Simple tabulation hashing was invented by Zobrist [22] in 1970 for chess computers. The basic idea is to view a key x as consisting of c characters for some constant c, e.g., a 32-bit key could be viewed as consisting of c=4 characters of 8 bits. We initialize c tables  $T_1, \ldots, T_c$  mapping characters to random hash values that are bit-strings of a certain length. A key  $x=(x_1,...,x_c)$  is then hashed to  $T_1[x_1] \oplus \cdots \oplus T_c[x_c]$  where  $\oplus$  denotes bit-wise xor.

#### 4 The k-th moment

The 4<sup>th</sup> moment bound used above generalizes to any even moment. First we need

**Theorem 7** Let  $X_0, \ldots, X_{n-1} \in \{0,1\}$  be k-wise independent variables for some (possibly odd)  $k \geq 2$ . Let  $p_i = \Pr[X_i = 1]$  and  $\sigma_i^2 = \operatorname{Var}[X_i] = p_i - p_i^2$ . Moreover, let  $X = \sum_{i \in [n]} X_i$ ,  $\mu = \operatorname{E}[X] = \sum_{i \in [n]} p_i$ , and  $\sigma^2 = \operatorname{Var}[X] = \sum_{i \in [n]} \sigma_i^2$ . Then

$$E[(X - \mu)^k] \le O(\sigma^2 + \sigma^k) = O(\mu + \mu^{k/2}).$$

**Proof** The proof is a simple generalization of the proof of Theorem 5 up to (4). We have

$$(X - \mu)^k = \sum_{i_0, \dots, i_{k-1} \in [n]} ((X_{i_0} - p_{i_0})(X_{i_1} - p_{i_1}) \cdots (X_{i_{k-1}} - p_{i_{k-1}}))$$

By linearity of expectation,

$$E[(X - \mu)^k] = \sum_{i_0, \dots, i_{k-1} \in [n]} E\left[ \left( (X_{i_0} - p_{i_0})(X_{i_1} - p_{i_1}) \cdots (X_{i_{k-1}} - p_{i_{k-1}}) \right) \right]$$

We now consider a specific term

$$((X_{i_0}-p_{i_0})(X_{i_1}-p_{i_1})\cdots(X_{i_{k-1}}-p_{i_{k-1}}))$$

Let  $j_0 < j_1 < \cdots < j_{c-1}$  be the distinct indices among  $i_0, i_1, \ldots, i_{n-1}$ , and let  $m_h$  be the multiplicity of  $j_h$ . Then

$$((X_{i_0} - p_{i_0})(X_{i_1} - p_{i_1}) \cdots (X_{i_{k-1}} - p_{i_{k-1}}))$$

$$= ((X_{j_0} - p_{j_0})^{m_0} (X_{j_1} - p_{j_1})^{m_1} \cdots (X_{j_{c-1}} - p_{j_{c-1}})^{m_{c-1}}).$$

The product involves at most k different variables so they are all independent, and therefore

Now, for any  $i \in [n]$ ,  $\mathrm{E}[X_i - p_i] = 0$ , so if any multiplicity is 1, the expected value is zero. We therefore only need to count terms where all multiplicities  $m_h$  are at least 2. The sum of multiplicities is  $\sum_{h \in [c]} m_h = k$ , so we conclude that there are  $c \le k/2$  distinct indices  $j_0, \ldots, j_{c-1}$ . Now by (3),

$$\mathrm{E}\left[(X_{j_0} - p_{j_0})^{m_0}\right] \, \mathrm{E}\left[(X_{j_1} - p_{j_1})^{m_1}\right] \cdots \mathrm{E}\left[(X_{j_{c-1}} - p_{j_{c-1}})^{m_{c-1}}\right] \le \sigma_{j_0}^2 \sigma_{j_1}^2 \cdots \sigma_{j_{c-1}}^2.$$

We now want to bound the number tuples  $(i_0, i_1, \ldots, i_{k-1})$  that have the same c distinct indices  $j_0 < j_1 < \cdots < j_{c-1}$ . A crude upper bound is that we have c choices for each  $i_h$ , hence  $c^k$  tuples. We therefore conclude that

$$E[(X - \mu)^{k}] = \sum_{i_{0}, \dots, i_{k-1} \in [n]} E\left[\left((X_{i_{0}} - p_{i_{0}})(X_{i_{1}} - p_{i_{1}}) \cdots (X_{i_{k-1}} - p_{i_{k-1}})\right)\right]$$

$$\leq \sum_{c=1}^{\lfloor k/2 \rfloor} \left(c^{k} \sum_{0 \leq j_{0} < j_{1} < \dots < j_{c-1} < n} \sigma_{j_{0}}^{2} \sigma_{j_{1}}^{2} \cdots \sigma_{j_{c-1}}^{2}\right)$$

$$\leq \sum_{c=1}^{\lfloor k/2 \rfloor} \left(\frac{c^{k}}{c!} \sum_{j_{0}, j_{1}, \dots, j_{c-1} \in [n]} \sigma_{j_{0}}^{2} \sigma_{j_{1}}^{2} \cdots \sigma_{j_{c-1}}^{2}\right)$$

$$\leq \sum_{c=1}^{\lfloor k/2 \rfloor} \left(\frac{c^{k}}{c!} \left(\sum_{j \in [n]} \sigma_{j}^{2}\right)^{c}\right)$$

$$= \sum_{c=1}^{\lfloor k/2 \rfloor} \left(\frac{c^{k}}{c!} \sigma^{2c}\right)$$

$$= O\left(\sigma^{2} + \sigma^{k}\right) = O\left(\mu + \mu^{k/2}\right).$$

Above we used that c, k = O(1) hence that, e.g.,  $c^k = O(1)$ . This completes the proof of Theorem 7.

For even moments, we now get a corresponding error probability bound

Corollary 8 Let  $X_0, \ldots, X_{n-1} \in \{0, 1\}$  be k-wise independent variables for some even constant  $k \geq 2$ . Let  $p_i = \Pr[X_i = 1]$  and  $\sigma_i^2 = \operatorname{Var}[X_i] = p_i - p_i^2$ . Moreover, let  $X = \sum_{i \in [n]} X_i$ ,  $\mu = \operatorname{E}[X] = \sum_{i \in [n]} p_i$ , and  $\sigma^2 = \operatorname{Var}[X] = \sum_{i \in [n]} \sigma_i^2$ . If  $\mu = \Omega(1)$ , then

$$\Pr[|X - \mu| \ge d\sqrt{\mu}] = O(1/d^k).$$

**Proof** By Theorem 7 and Markov's inequality, we get

$$\Pr[|X - \mu| \ge d\sqrt{\mu}] = \Pr[(X - \mu)^k \ge d^k \mu^{k/2}]$$

$$\le \frac{\operatorname{E}[(X - \mu)^k]}{d^k \mu^{k/2}}$$

$$= \frac{O(\mu + \mu^{\lfloor k/2 \rfloor})}{d^k \mu^{k/2}}$$

$$= O(1/d^k).$$

**Problem 4** In the proofs of this section, where and why do we need that (a) k is a constant and (b) that k is even.

## 5 Bloom filters via linear probing

We will now show how we can reduce the space of a linear probing table if we are willing to allow for a small chance of false positives, that is, the table attemps to answer if a query q is in the current stored set S. If it answers "no", then  $q \notin S$ . If  $q \in S$ , then it always answers "yes". However, even if  $q \notin S$ , then with some probability  $\leq P$ , the table may answer "yes". Bloom [4] was the first to suggest creating such a filter using less space than one giving exact answers. Our implementation here, using linear probing, is completely different. The author suggested this use of linear probing to various people in the late 90ties, but it was never written down.

To create a filter, we use a universal hash function  $s:[u] \to [2^b]$ . We call s(x) the signature of x. The point is that s(x) should be much smaller than x, that is,  $b \ll \log_2 u$ . The linear probing array T is now only an array of t signatures. We still use the hash function  $h:[u] \to [t]$  to start the search for a key in the array. Thus, to check if a key q is positive in the filter, we look for s(q) among the signatures in T from location h(q) and onwards until the first empty location. If s(q) is found, we report "yes"; otherwise "no". If we want to include q to the filter, we only do something if s(q) was not found. Then we place s(q) it in the first empty location. Our filter does not support deletion of keys (c.f. Problem 6).

**Theorem 9** Assume that the hash function h and the signature function s are independent, that h is 5-independent, and that s is universal. Then the probability of a false positive on a given key  $q \notin S$  is  $O(1/2^b)$ .

**Proof** The keys from S have been inserted in some given order. Let us assume that h is fixed. Suppose we inserted the keys exactly, that is, not just their signatures, and let X(q) be the set of keys encountered when searching for q, that is, X(q) is the set of keys from h(q) and till the first empty location. Note that X(q) depends only on h, not on s.

In Problem 5 you will argue that if q is a false positive, then s(q) = s(x) for some  $x \in X(q)$ . For every key  $x \in [u] \setminus \{q\}$ , by universality of s, we have  $\Pr[s(x) = s(q)] \leq 1/2^b$ . Since  $q \notin S \supseteq X(q)$ , by union,  $\Pr[\exists x \in X(q) : s(x) = s(q)] \leq |X(q)|/2^b$ . It follows that the probability that q is a false positive is bounded by

$$\sum_{Y \subseteq S} \Pr[X(q) = Y] \cdot |Y|/2^b = E[|X(q)|]/2^b.$$

By Theorem 6, E[|X(q)|] = O(1) when h is 5-independent.

**Problem 5** To complete the proof of Theorem 9, consider a sequence  $x_1, \ldots, x_n$  of distinct keys inserted in an exact linear probing table (as defined in Section 2). Also, let  $x_{i_1}, \ldots, x_{i_m}$  be a subequence of these keys, that is,  $1 \le i_1 < i_2 < \cdots < i_m \le n$ . The task is to prove any fixed  $h: [u] \to [t]$  and any fixed  $j \in [t]$ , that when only the subsequence is inserted, then the sequence of keys encountered from location j and till the first empty location is a subsequence of those encountered when the full sequence is inserted.

Hint. Using induction on n, show that the above statement is preserved when a new key  $x_{n+1}$  is added. Here  $x_{n+1}$  may or may not be part of the subsequence.

The relation to the proof of Theorem 9 is that when we insert keys in a filter, we skip keys whose signatures are found as false postives. This means that only a subsequence of the keys have their signatures inserted. When searching for a key q starting from from location j = h(q), we have thus proved that we only consider (signatures of) a subset of the set X(q) of keys that we would have considered if all keys where inserted. In particular, this means that if we from j encounter a key x with s(x) = s(q), then  $x \in X(q)$  as required for the proof of Theorem 9.

Problem 6 Discuss why we canot support deletions.

**Problem 7** What would happen if we instead used h(s(x)) as the hash function to place or find x? What would be the probability of a false positive?

Sometimes it is faster to generate the hash values and signatures together so that the pairs (h(x), s(x)) are 5-independent while the hash values and signatures are not necessarily independent of each other. An example is if we generate a larger hash value, using high-order bits for h(x) and low-order bits for s(x). In this situation we get a somewhat weaker bound than that in Theorem 9.

**Theorem 10** Assuming that  $x \mapsto (h(x), s(x))$  is 5-independent, the probability of a false positive on a given key  $q \notin S$  is  $O(1/2^{2b/3})$ .

**Proof** Considering the exact insertion of all keys, we consider two cases. Either (a) there is a run of length at least  $2^{b/3}$  around h(q), or (b) there is no such run.

For case (a), we use Lemma 3 together with the bound  $P_{\ell} = O(1/2^{2\ell})$  from the proof of Theorem 4. We get that the probability of getting a run of length at least  $2^{b/3}$  is bounded by

$$\sum_{\ell=b/3-2}^{\infty} 12P_{\ell} = O(1/2^{2b/3}).$$

We now consider case (b). By the statement proved in Problem 5, we know that any signature s(x) considered is from a key x from the set X(q) of keys that we would have considered from j = h(q) if all keys were inserted exactly. With no run of length at least  $2^{b/3}$ , all keys in X(q) must hash

to  $(h(q) - 2^{b/3}, h(q) + 2^{b/3}]$ . Thus, if we get a false positive in case (b), it is because there is a key  $x \in S$  with s(x) = s(q) and  $h(x) \in (h(q) - 2^{b/3}, h(q) + 2^{b/3}]$ . Since (h(x), s(x)) and (h(q), s(q)) are independent, the probability that this happens for x is bounded by  $2^{1+b/3}/(t2^b) = O(1/(n2^{2b/3}))$ , yeilding  $O(1/2^{2b/3})$  when we sum over all n keys in S. By union, the probability of a false positive in case (a) or (b) is bounded by  $O(1/2^{2b/3})$ , as desired.

We note that with the simple tabulation mentioned in Section 3.2, we can put hash-signature pairs as concatenated bit strings in the character tables  $T_1, \ldots, T_c$ . Then  $(h(x), s(x)) = T_1[x_1] \oplus \cdots \oplus T_c[x_c]$ . The nice thing here is that with simple tabulation hashing, the output bits are all completely independent, which means that Theorem 9 applies even though we generate the hash-signature pairs together.

### References

- [1] Noga Alon, Martin Dietzfelbinger, Peter Bro Miltersen, Erez Petrank, and Gábor Tardos. Linear hash functions. J. ACM, 46(5):667–683, 1999.
- [2] Noga Alon and Asaf Nussboim. k-wise independent random graphs. In Proc. 49th IEEE Symposium on Foundations of Computer Science (FOCS), pages 813–822, 2008.
- [3] John R. Black, Charles U. Martel, and Hongbin Qi. Graph and hashing algorithms for modern architectures: Design and performance. In *Proc. 2nd International Workshop on Algorithm Engineering (WAE)*, pages 37–48, 1998.
- [4] Burton H. Bloom. Space/time trade-offs in hash coding with allowable errors. *Communications of the ACM*, 13(7):422–426, 1970.
- [5] Larry Carter and Mark N. Wegman. Universal classes of hash functions. *Journal of Computer and System Sciences*, 18(2):143–154, 1979. Announced at STOC'77.
- [6] Martin Dietzfelbinger and Ulf Schellbach. On risks of using cuckoo hashing with simple universal hash classes. In Proc. 20th ACM/SIAM Symposium on Discrete Algorithms (SODA), pages 795–804, 2009.
- [7] Michael L. Fredman, János Komlós, and Endre Szemerédi. Storing a sparse table with 0(1) worst case access time. *Journal of the ACM*, 31(3):538–544, 1984. Announced at FOCS'82.
- [8] Gregory L. Heileman and Wenbin Luo. How caching affects hashing. In *Proc. 7th Workshop on Algorithm Engineering and Experiments (ALENEX)*, pages 141–154, 2005.
- [9] Daniel M. Kane Jeffery S. Cohen. Bounds on the independence required for cuckoo hashing, 2009. Manuscript.
- [10] Howard J. Karloff and Prabhakar Raghavan. Randomized algorithms and pseudorandom numbers. *Journal of the ACM*, 40(3):454–476, 1993.
- [11] Donald E. Knuth. Notes on open addressing. Unpublished memorandum. See http://citeseer.ist.psu.edu/knuth63notes.html, 1963.

- [12] Donald E. Knuth. The Art of Computer Programming, Volume III: Sorting and Searching. Addison-Wesley, 1973.
- [13] Anna Pagh, Rasmus Pagh, and Milan Ružić. Linear probing with constant independence. SIAM Journal on Computing, 39(3):1107–1120, 2009. Announced at STOC'07.
- [14] Rasmus Pagh and Flemming Friche Rodler. Cuckoo hashing. *Journal of Algorithms*, 51(2):122–144, 2004. Announced at ESA'01.
- [15] Mihai Pătrașcu and Mikkel Thorup. The power of simple tabulation-based hashing. *Journal* of the ACM, 59(3):Article 14, 2012. Announced at STOC'11.
- [16] Mihai Pătrașcu and Mikkel Thorup. On the k-independence required by linear probing and minw ise independence. ACM Transactions on Algorithms, 12(1):Article 8, 2016. Announced at ICALP'10.
- [17] Jeanette P. Schmidt and Alan Siegel. The analysis of closed hashing under limited randomness. In *Proc. 22nd ACM Symposium on Theory of Computing (STOC)*, pages 224–234, 1990.
- [18] Jeanette P. Schmidt, Alan Siegel, and Aravind Srinivasan. Chernoff-Hoeffding bounds for applications with limited independence. SIAM Journal on Discrete Mathematics, 8(2):223–250, 1995. Announced at SODA'93.
- [19] Alan Siegel and Jeanette P. Schmidt. Closed hashing is computable and optimally randomizable with universal hash functions. Technical Report TR1995-687, Courant Institute, New York University, 1995.
- [20] Mikkel Thorup and Yin Zhang. Tabulation-based 5-independent hashing with applications to linear probing and second moment estimation. *SIAM Journal on Computing*, 41(2):293–331, 2012. Announced at SODA'04 and ALENEX'10.
- [21] Mark N. Wegman and Larry Carter. New classes and applications of hash functions. *Journal of Computer and System Sciences*, 22(3):265–279, 1981. Announced at FOCS'79.
- [22] Albert Lindsey Zobrist. A new hashing method with application for game playing. Technical Report 88, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, 1970.