MONASH College

MCD4710

Introduction to algorithms and programming

Lecture 18

Gaussian Elimination

Overview

- Represent problem as system of linear equations
- Represent linear system in matrix/vector form
- Apply and implement backward substitution to solve upper-triangular systems
- Apply and implement forward elimination to transform general system into triangular form

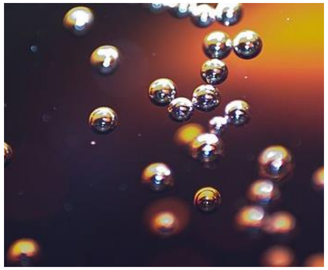
Linear systems of equations

Some Chemistry

ferric oxide
Fe_2O_3

carbon
C

iron
Fe



carbon dioxide
CO_2

Some Chemistry

ferric oxide

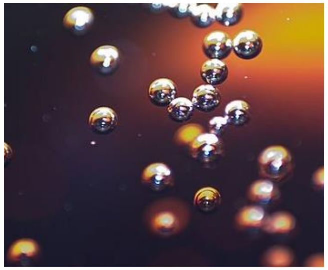
Fe_2O_3

carbon

C

iron

Fe



carbon dioxide

CO_2

Chemical notation

$$Fe_2O_3 + C \rightarrow Fe + CO_2$$

reactants

products

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Some Chemistry

ferric oxide

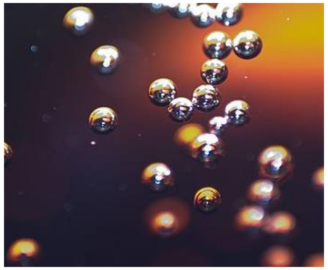
Fe_2O_3

carbon

C

iron

Fe



carbon dioxide

CO_2

Chemical notation

$$Fe_2O_3 + C \rightarrow Fe + CO_2$$

Count atoms

Atom	Reactants	Products
Fe	2	1
O	3	2
C	1	1

Numbers don't add up!

Want to *balance* chemical equation

Unbalanced equation

$$Fe_2O_3 + C \rightarrow Fe + CO_2$$

Balanced equation

$$2Fe_2O_3 + 3C \rightarrow 4Fe + 3CO_2$$

Count atoms

Atom	Reactants	Products
Fe	4	4
O	6	6
C	3	3

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Let's balance equation *algorithmically*

General equation

$$a\mathrm{Fe_2O_3} + b\mathrm{C} \rightarrow c\mathrm{Fe} + d\mathrm{CO_2}$$

Conditions on solution

$$2a = c \qquad \text{(Fe)}$$

Let's balance equation *algorithmically*

General equation

$$aFe_2O_3 + bC \rightarrow cFe + dCO_2$$

Conditions on solution

$$2a = c \qquad \text{(Fe)}$$

$$3a = 2d \qquad \text{(O)}$$

Let's balance equation *algorithmically*

General equation

$$a\mathrm{Fe_2O_3} + b\mathrm{C} \rightarrow c\mathrm{Fe} + d\mathrm{CO_2}$$

Conditions on solution

$$2a = c \qquad (\mathrm{Fe})$$

$$3a = 2d \qquad (\mathrm{O})$$

$$b = d \qquad (\mathrm{C})$$

Let's balance equation *algorithmically*

General equation

$$a\mathrm{Fe_2O_3} + b\mathrm{C} \rightarrow c\mathrm{Fe} + d\mathrm{CO_2}$$

Conditions on solution

$$2a = c \qquad \text{(Fe)}$$

$$3a = 2d \qquad \text{(O)}$$

$$b = d \qquad \text{(C)}$$

Does that uniquely determine solution?

Let’s balance equation *algorithmically*

General equation

$$aFe_2O_3 + bC \rightarrow cFe + dCO_2$$

Conditions on solution

$$2a = c \qquad \text{(Fe)}$$
$$3a = 2d \qquad \text{(O)}$$
$$b = d \qquad \text{(C)}$$

$$4Fe_2O_3 + 6C \rightarrow 8Fe + 6CO_2$$

Atom	Reactants	Products
Fe	8	8
O	12	12
C	6	6

double quantities

$$2Fe_2O_3 + 3C \rightarrow 4Fe + 3CO_2$$

Atom	Reactants	Products
Fe	4	4
O	6	6
C	3	3

Let's balance equation *algorithmically*

General equation

$$aFe_2O_3 + bC \rightarrow cFe + dCO_2$$

Conditions on solution

$$2a = c \qquad \text{(Fe)}$$

$$3a = 2d \qquad \text{(O)}$$

$$b = d \qquad \text{(C)}$$

$$a = 4$$

$$4Fe_2O_3 + 6C \rightarrow 8Fe + 6CO_2$$

Atom	Reactants	Products
Fe	8	8
O	12	12
C	6	6

Fix quantities to fully determine solution

Reached a very general problem

<u>Solve Linear Systems</u>

Input: Set of n linear equations in n variables

Output: Assignment of values to variables satisfying all equations

How to represent this problem in Python?

Finding systematic representation

Solve Linear Systems

Input: Set of n linear equations in n variables

Output: Assignment of values to variables satisfying all equations

$$\begin{aligned} 2a &= c \\ 3a &= 2d \\ b &= d \\ a &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2a - c &= 0 \\ 3a - 2d &= 0 \\ b - d &= 0 \\ a &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2a + 0b - 1c + 0d &= 0 \\ 3a + 0b + 0c - 2d &= 0 \\ 0a + 1b + 0c - 1d &= 0 \\ 1a + 0b + 0c + 0d &= 4 \end{aligned}$$

coefficient matrix $\boldsymbol{A}$

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

right hand side $\boldsymbol{b}$

solution vector $\boldsymbol{x}$

Finding systematic representation

<u>Solve Linear Systems</u>

Input: $n \times n$-matrix $\boldsymbol{A}$ of coefficients, n-dim vector $\boldsymbol{b}$

Output: n-dim vector $\boldsymbol{x}$ such that $\boldsymbol{Ax} = \boldsymbol{b}$

Python table

Python list

coefficient matrix $\boldsymbol{A}$

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

right hand side $\boldsymbol{b}$

solution vector $\boldsymbol{x}$

Simple case: triangular system

Overall strategy: Transform and Conquer Paradigm

another input

or

problem input $\xrightarrow{\text{transform}}$ another representation $\xrightarrow{\text{conquer}}$ solution

or

another problem

What are easy to solve linear systems?

(Upper) triangular systems

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

called *upper triangular*, because only coefficients in main diagonal and above are non-zero

How can we solve such a system?

Back-substitution algorithm

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

$$2c = 6$$

$$c = 6/2$$

Back-substitution algorithm

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

$$b + 3c = 7$$

$$b = 7 - 9$$

Back-substitution algorithm

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

$$a + 2b + c = 2$$

$$a = 2 + 4 - 3$$

Back-substitution algorithm

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

Back-substitution in general

$$\begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} \\ 0 & u_{1,1} & u_{1,2} & u_{1,3} \\ 0 & 0 & u_{2,2} & u_{2,3} \\ 0 & 0 & 0 & u_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

```
def solve_by_back_sub(u,b):
    n = len(b)
    x = n*[0]
    for i in range(n-1, -1, -1):
        # find value x[i]

    return x
```

Back-substitution algorithm

$$\begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} \\ 0 & u_{1,1} & u_{1,2} & u_{1,3} \\ 0 & 0 & u_{2,2} & u_{2,3} \\ 0 & 0 & 0 & u_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$u_{i,i}x_i + \underbrace{(u_{i,i+1}x_{i+1} + \cdots + u_{i,n-1}x_{n-1})}_{s} = b_i$$

$$x_i = (b_i - s)/u_{i,i}$$

```
def solve_by_back_sub(u,b):
    n = len(b)
    x = n*[0]
    for i in range(n-1, -1, -1):
        # find value x[i]

    return x
```

Back-substitution algorithm

$$\begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} \\ 0 & u_{1,1} & u_{1,2} & u_{1,3} \\ 0 & 0 & u_{2,2} & u_{2,3} \\ 0 & 0 & 0 & u_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$u_{i,i}x_i + \underbrace{(u_{i,i+1}x_{i+1} + \cdots + u_{i,n-1}x_{n-1})}_{s} = b_i$$

$$x_i = (b_i - s)/u_{i,i}$$

```
def solve_by_back_sub(u,b):
    n = len(b)
    x = n*[0]
    for i in range(n-1, -1, -1):
        s = sum(x[j] * u[i][j] for j in range(n-1, i, -1))
        x[i] = (b[i] - s)/u[i][i]
    return x
```

Comprehension allows to directly express our formula

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Back-substitution algorithm

```
def solve_by_back_sub(u,b):
    n = len(b)
    x = n*[0]
    for i in range(n-1, -1, -1):
        s = 0
        for j in range(n-1, i, -1):
            s += u[i][j] * x[j]
        x[i] = (b[i] - s)/u[i][i]
    return x
```

$$u_{i,i}x_i + \underbrace{(u_{i,i+1}x_{i+1} + \cdots + u_{i,n-1}x_{n-1})}_{s} = b_i$$

for-loop versus comprehension

```
def solve_by_back_sub(u,b):
    n = len(b)
    x = n*[0]
    for i in range(n-1, -1, -1):
        s = sum(x[j] * u[i][j] for j in range(n-1, i, -1))
        x[i] = (b[i] - s)/u[i][i]
    return x
```

Gaussian Elimination

Carl Friedrich Gauß
1777– 1855

general system → triangular system

$$\begin{bmatrix}1 & 2 & 1\\ -5 & -9 & -2\\ 2 & 3 & 1\end{bmatrix},\begin{bmatrix}2\\ -3\\ 3\end{bmatrix} \xrightarrow[\text{forward elimination}]{\text{transform}} \begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 3\\ 0 & 0 & 2\end{bmatrix},\begin{bmatrix}2\\ 7\\ 6\end{bmatrix} \xrightarrow[\text{back substitution}]{\text{conquer}} \begin{bmatrix}3\\ 2\\ 3\end{bmatrix}$$

How to *eliminate* coefficients?

$$a + 2b + c = 2$$
$$-5a - 9b - 2c = -3$$

$$\Downarrow$$

$$-4a - 7b - c = -1$$

Observation:
Given equations imply (infinitely) many other equations

How to *eliminate* coefficients?

$$a + 2b + c = 2$$
$$-5a - 9b - 2c = -3$$

adding 5 times first equation to second yields equations where a-coefficient is *eliminated*

$$\Downarrow$$

$$b - 3c = 7$$

Observation:
Given equations imply (infinitely) many other equations

How to *eliminate* coefficients?

$$\begin{aligned} a + 2b + \ \ c &= 2 \\ -5a - 9b - 2c &= -3 \\ 2a + 3b + \ \ c &= 3 \end{aligned}$$

$$\Downarrow \ \Uparrow$$

Simpler system but same solutions

$$\begin{aligned} a + 2b + \ \ c &= 2 \\ b - 3c &= 7 \\ 2a + 3b + \ \ c &= 3 \end{aligned}$$

Observation:
Given equations imply (infinitely) many other equations

How to *eliminate* coefficients?

$$\begin{aligned} a + 2b + \ \ c &= 2 \\ -5a - 9b - 2c &= -3 \\ 2a + 3b + \ \ c &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ -5 & -9 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\Downarrow \ \Uparrow \qquad\qquad \Downarrow \ \Uparrow$$

$$\begin{aligned} a + 2b + \ \ c &= 2 \\ b - 3c &= 7 \\ 2a + 3b + \ \ c &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$$

Observations:

- Solution *invariant* when adding multiple of row to another
- Exploit to get simpler but equivalent system

Iteratively eliminate below diagonal coefficients

simplify column 1

$$\begin{bmatrix} 1 & 2 & 1 \\ -5 & -9 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$$

simplify column 2

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

General forward elimination

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & \boxed{a_{1,1}} & a_{1,2} & a_{1,3} \\ 0 & \boxed{a_{2,1}} & a_{2,2} & a_{2,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Subtract q times *pivot* row j from row i

How to choose q such that $a_{i,j}$ will be eliminated?

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        for i in range(j + 1, n):
            # eliminate entry a[i][j]


    return a, b
```

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General forward elimination

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Subtract q times *pivot* row j from row i

$$q = a_{i,j}/a_{j,j}$$

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,2} - qa_{1,2} & a_{2,3} - qa_{1,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 - qb_1 \\ b_3 \end{bmatrix}$$

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        for i in range(j + 1, n):
            # eliminate entry a[i][j]


    return a, b
```

$$a_{i,m} - a_{j,m}\frac{a_{i,j}}{a_{j,j}}$$

$$a_{i,j} - a_{j,j}\frac{a_{i,j}}{a_{j,j}} = 0$$

General forward elimination

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Subtract q times *pivot* row j from row i

$$q = a_{i,j}/a_{j,j}$$

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,2} - q a_{1,2} & a_{2,3} - q a_{1,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 - q b_1 \\ b_3 \end{bmatrix}$$

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        for i in range(j + 1, n):
            q = a[i][j] / a[j][j]
            # subtract q*a[j] from a[i]
            # subtract q*b[j] from b[j]
    return a, b
```

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General forward elimination

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & \boxed{a_{1,1}} & a_{1,2} & a_{1,3} \\ 0 & \boxed{a_{2,1}} & a_{2,2} & a_{2,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Subtract q times *pivot* row j from row i

$$q = a_{i,j}/a_{j,j}$$

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,2} - q a_{1,2} & a_{2,3} - q a_{1,3} \\ 0 & a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 - q b_1 \\ b_3 \end{bmatrix}$$

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        for i in range(j + 1, n):
            q = a[i][j] / a[j][j]
            a[i] = [a[i][m] - q*a[j][m] for m in range(n)]
            b[i] = b[i] - q*b[j]
    return a, b
```

One technical problem remains

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 7 \end{bmatrix}$$

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        for i in range(j + 1, n):
            q = a[i][j] / a[j][j]
            a[i] = [a[i][m] - q*a[j][m] for m in range(n)]
            b[i] = b[i] - q*b[j]
    return a, b
```

Can always flip equations; their order doesn't matter

Need to find pivot row

$$\begin{bmatrix}1 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 1 & -1 & -1\\ 0 & 3 & -2 & 2\end{bmatrix}, \begin{bmatrix}1\\ 1\\ -2\\ 1\end{bmatrix} \Leftrightarrow \begin{bmatrix}1 & 1 & 0 & 1\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 1\\ 0 & 3 & -2 & 2\end{bmatrix}, \begin{bmatrix}1\\ -2\\ 1\\ 1\end{bmatrix}$$

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        # find index k of valid pivot row
        # swap row j and k of system

        for i in range(j + 1, n):
            q = a[i][j] / a[j][j]
            a[i] = [a[i][m] - q*a[j][m] for m in range(n)]
            b[i] = b[i] - q*b[j]
    return a, b
```

Need to find pivot row

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Assuming a non-zero pivot exists

```
def pivot_index(a,j):
    max_abs = abs(a[j][j])
    pindex = j
    for i in range(j+1,len(a)):
        if max_abs < abs(a[i][j]):
            pindex = i
    return pindex
```

```
def triangular(a, b):
    n = len(a)
    for j in range(n):
        k = pivot_index(a, j)
        a[j], a[k] = a[k], a[j]
        b[j], b[k] = b[k], b[j]
        for i in range(j + 1, len(a)):
            q = a[i][j] / a[j][j]
            a[i] = [a[i][m] - q*a[j][m] for m in range(n)]
            b[i] = b[i] - q*b[j]
    return a, b
```

Finishing touches

```
def solve_gauss_elim(a, b):
    u, c = triangular(a, b)
    return solve_backsub(u, c)
```

```
def triangular(a, b):
    u, c = deepcopy(a), deepcopy(b)
    n = len(u)
    for j in range(n):
        k = pivot_index(u, j)
        u[j], u[k] = u[k], u[j]
        c[j], c[k] = c[k], c[j]
        for i in range(j + 1, len(a)):
            q = u[i][j] / u[j][j]
            u[i] = [u[i][m] - q*u[j][m] for m in range(n)]
            c[i] = c[i] - q*c[j]
    return u, c
```

Operate on copies to not destroy input

Brief analysis

```
def solve_gauss_elim(a, b):
    u, c = triangular(a, b)
    return solve_backsub(u, c)
```

```
def triangular(a, b):
    u, c = deepcopy(a), deepcopy(b)
    n = len(u)
    for j in range(n):
        k = pivot_index(u, j)
        u[j], u[k] = u[k], u[j]
        c[j], c[k] = c[k], c[j]
        for i in range(j + 1, len(a)):
            q = u[i][j] / u[j][j]
            u[i] = [u[i][m] - q*u[j][m] for m in range(n)]
            c[i] = c[i] - q*c[j]
            #INV: u, c has same solution(s) as a, b
    return u, c
```

$O(n)$

$O(n^2)$

$O(n^3)$

What is the transform step in Gaussian elimination?

A. Forward elimination
B. Backward elimination
C. Forward substitution
D. Back substitution

1. Visit https://flux.qa
2. Log in using your Authcate details (not required if you're already logged in to Monash)
3. Touch the + symbol and enter the code: HHYBXU
4. Answer questions when they pop up.

What is the conquer step in Gaussian elimination?

A. Forward elimination
B. Backward elimination
C. Forward substitution
D. Back substitution

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Summary

- Systems of linear equations can represent real-world problems with linear dependencies between variables
- Triangular systems can be solved by back-substitution
- General systems (with unique solution) can be transformed into triangular form via forward elimination
- Overall cubic $O(n^3)$ complexity

Important further questions

- What if we can’t find a pivot index?
- What if we have more unknowns than equations?
- What if we have more equations than unknowns?
- What if there is more than one solution?
- What if there is no solution?

Recommended reading

- Levitin, 6.2, Gaussian Elimination, p. 208
- Gilbert Strang, Textbook: Linear Algebra and its Applications

Coming Up

- Combinatorial Optimisation
- Greedy, Brute-force, and Backtracking