MONASH College

MCD4710

Introduction to algorithms and programming

Lecture 10

Invariants

Overview

- Formulate assertions about program states
- Demonstrate that truth of certain assertions is unchanged (invariant) by program (specifically by a loop)
- Relate invariants to computational problem to demonstrate correctness of algorithm

Programs with simple flow are easy to recognise as correct

```
def number_of_days(month, year):
    if month == 2:
        if is_leap_year(year):
            return 29
        else:
            return 28
    elif month in THIRTY_DAYS_MONTH:
        return 30
    else:
        return 31
```

```
def valid_date(day, month, year):
    if month not in VALID_MONTHS:
        return False
    elif day not in range(1, number_of_days(month, year)):
        return False
    else:
        return True
```

…but is this really computing a spanning tree?

```
def spanning_tree(graph):
    """Input : adjacency matrix of graph
       Output: adj. mat. of spanning tree of graph"""
    n = len(graph)
    tree = empty_graph(n)
    conn = {0}
    while len(conn) < n:
        found = False
        for i in conn:
            for j in range(n):
                if j not in conn and graph[i][j]==1:
                    tree[i][j] = 1
                    tree[j][i] = 1
                    conn = conn.add(j)
                    found = True
                    break
            if found:
                break
    return tree
```

Decomposition helps but loops with *re-assignments/mutation* remain tricky

```
def extension(c, g):
    """I: connec. vertices (c), graph 
(g)
       O: extension edge (i, j)"""
    n = len(g)
    for i in vertices:
        for j in range(n):
            if j not in c and g[i][j]:
                return i, j
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    conn = {0}
    while len(conn) < n:
        i, j = extension(conn, graph)
        tree[i][j], tree[j][i] = 1, 1
        conn = conn.add{j}
return tree
```

values behind names change all the time

Outline

- Assertions and invariants
- Analysing Insertion Sort
- Analysing Min Index Selection
- Analysing Prim's Algorithm

Cutting the Chocolate Block

A chocolate block is divided into squares by horizontal and vertical grooves. The object is to cut the chocolate block into individual pieces.

Assume each cut is made on a **single piece** along a groove. How many cuts are needed?

How many cuts does it take to divide the following block into squares?

A. 8
B. 3
C. 24
D. 23
E. None of the above

1. Visit https://flux.qa
2. Log in using your Authcate details (not required if you're already logged in to Monash)
3. Touch the + symbol and enter the code: UF7BD9
4. Answer questions when they pop up.

How many cuts does it take to divide a 100 X 50 block of chocolate?

A. 5000

B. 4999

C. 4900

D. 4950

E. None of the above

1. Visit https://flux.qa
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3. Touch the + symbol and enter the code: UF7BD9
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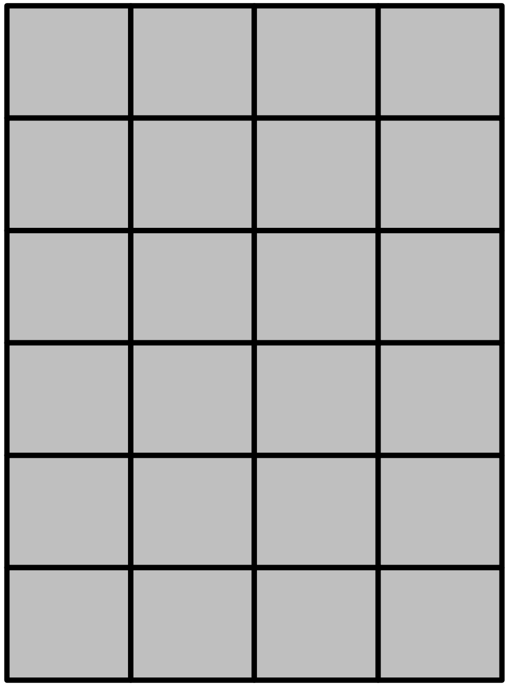
What is the relationship between cuts and number of pieces?



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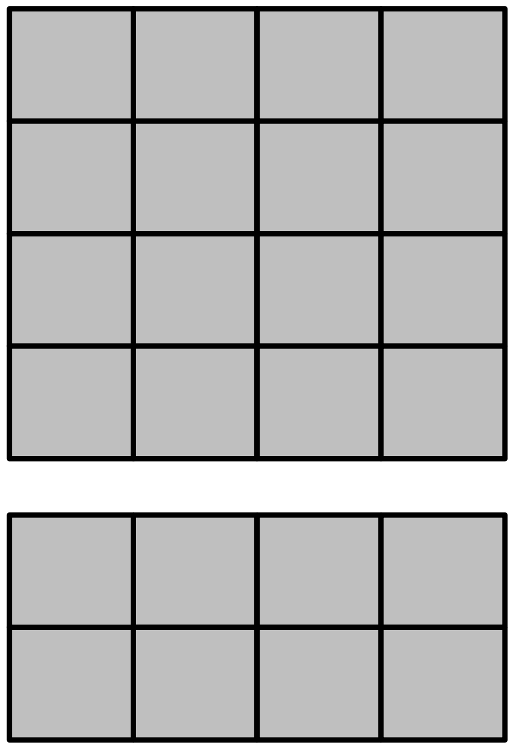
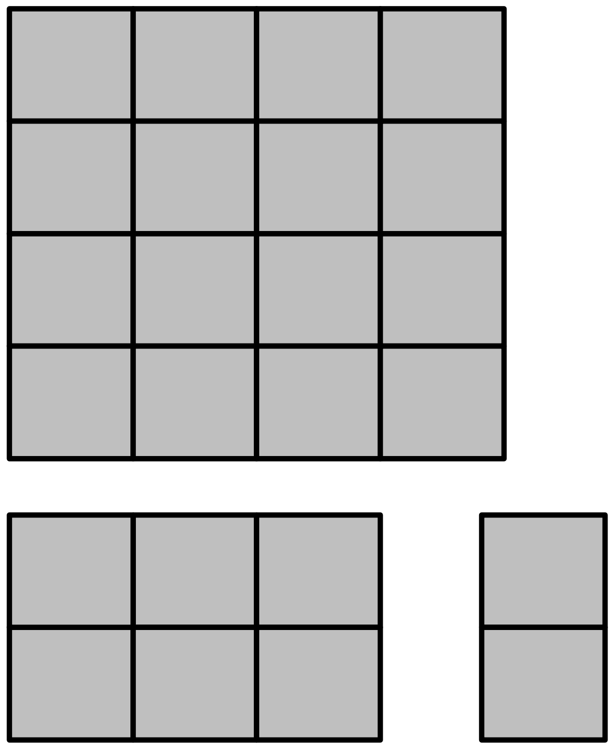


1 cut

2 pieces

What is the relationship between cuts and number of pieces?



2 cuts
3 pieces

Statement
"number of pieces equals number of cuts plus one"
...holds throughout cutting process

Let’s bring this concept into the world of programs

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

Example cutting strategy
(we know it doesn’t matter)

```
def cut_first_possible(pieces):
    for i in range(pieces):
        m, n = pieces[i]
        m, n = max(m,n), min(m,n)
        if m > 1:
            pieces.pop(i)
            pieces.append[(m-1,n), (1,n)]
            break
```

Let's analyse this program by stating *assertions*

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

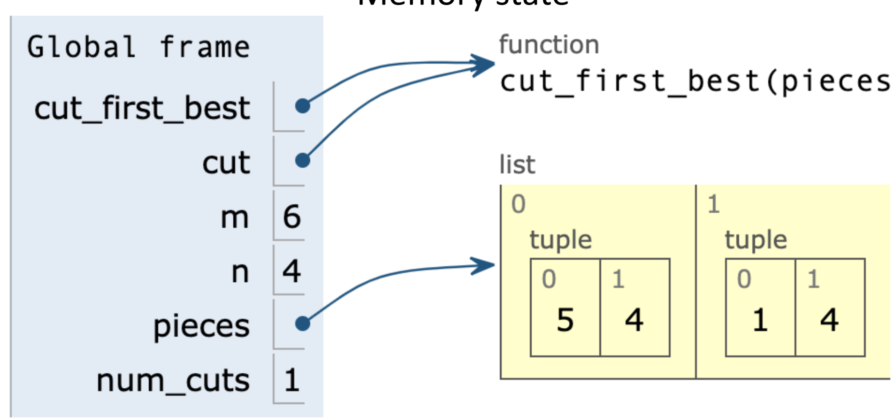
https://goo.gl/Mkvzjm

An **assertion** is a logical statement on a *program (execution) state*.

Instruction pointer

```
cut = cut_first_best
m, n = 6, 4
pieces = [(m, n)]
num_cuts = 0
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

Memory state



Let’s analyse this program by stating *assertions*

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

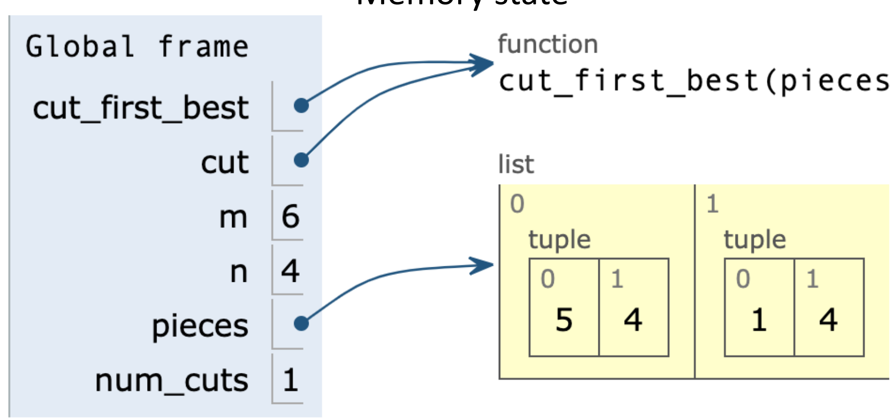
Example:
loop precondition

An **assertion** is a logical statement on a *program (execution) state*.

Instruction pointer

```
cut = cut_first_best
m, n = 6, 4
pieces = [(m, n)]
num_cuts = 0
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

Memory state



What happens during the loop?

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    #len(pieces)==num_cuts+1
    cut(pieces)
    num_cuts += 1
```

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Instruction pointer

```
cut = cut_first_best
m, n = 6, 4
pieces = [(m, n)]
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while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```
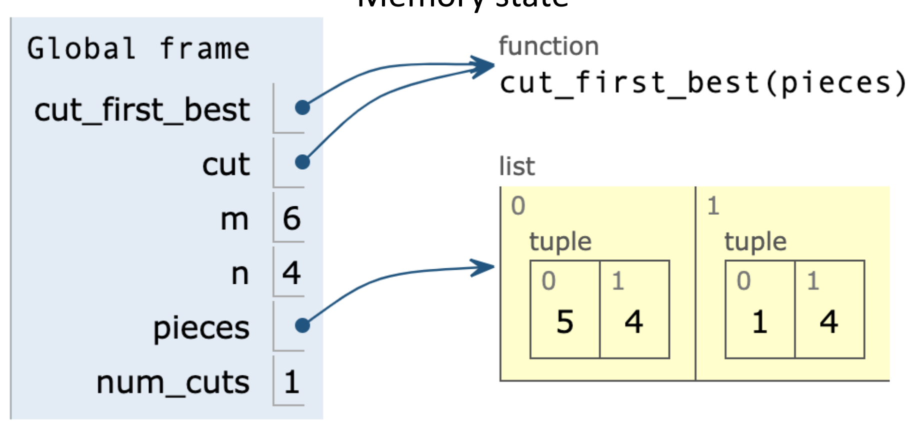
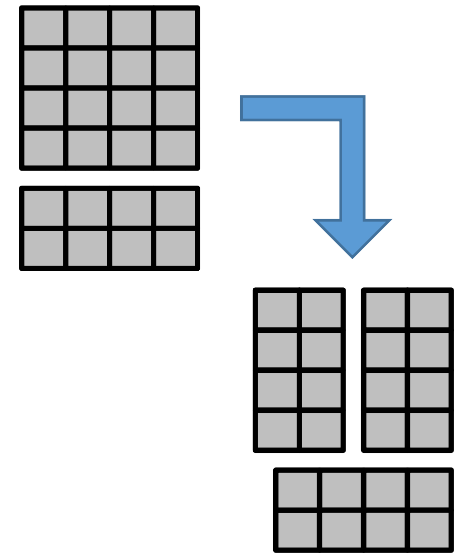

Memory state



What happens during the loop?

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    #len(pieces)==num_cuts+1
    cut(pieces)
    num_cuts += 1
```

after this step, assertion is (temporarily) violated

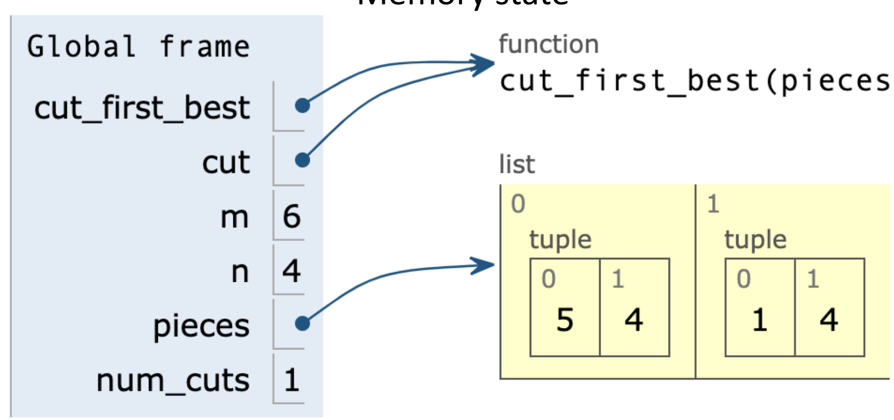


An **assertion** is a logical statement on a *program (execution) state*.

Instruction pointer

```
cut = cut_first_best
m, n = 6, 4
pieces = [(m, n)]
num_cuts = 0
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

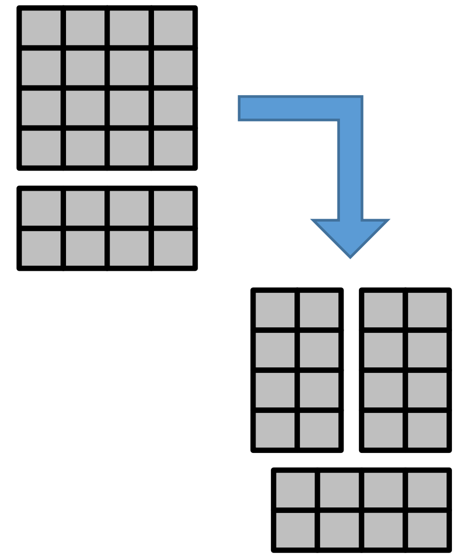
Memory state



What happens during the loop?

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    #len(pieces)==num_cuts+1
    cut(pieces)
    num_cuts += 1
```

after this step it is restored

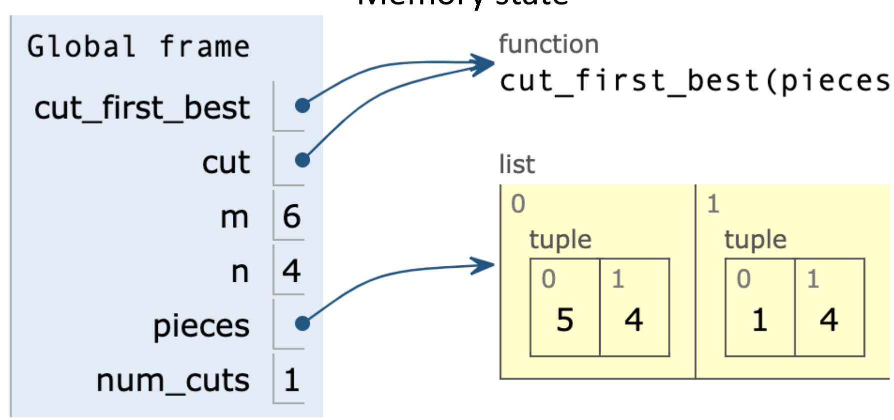


An **assertion** is a logical statement on a *program (execution) state*.

Instruction pointer

```
cut = cut_first_best
m, n = 6, 4
pieces = [(m, n)]
num_cuts = 0
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
```

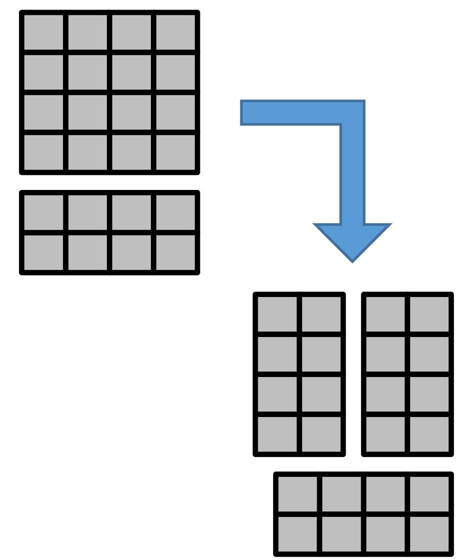
Memory state



What happens during the loop?

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    #len(pieces)==num_cuts+1
    cut(pieces)
    num_cuts += 1
    #len(pieces)==num_cuts+1
```

maintained by loop body



An **assertion** is a logical statement on a *program (execution) state*.

Instruction pointer

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cut = cut_first_best
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num_cuts = 0
while len(pieces)<n*m:
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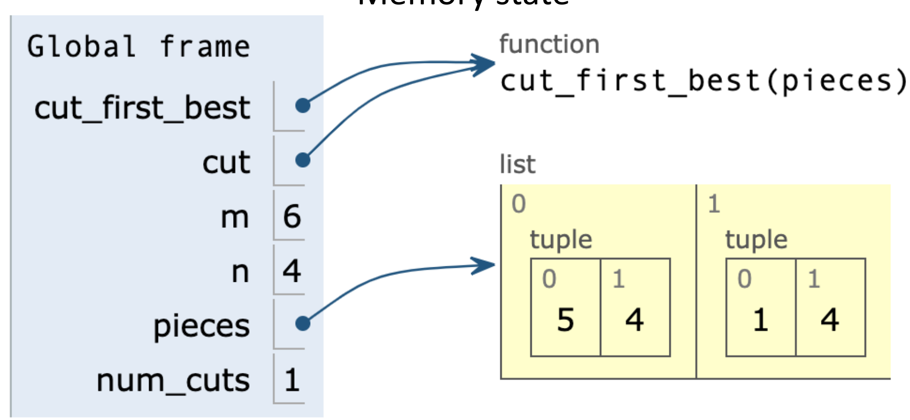
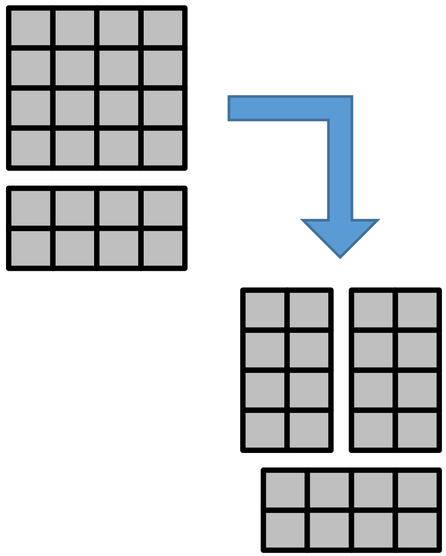
```

Memory state



Loop invariant is an assertion maintained by loop body

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    #INV: len(pieces)==num_cuts+1
    cut(pieces)
    num_cuts += 1
    #INV: len(pieces)==num_cuts+1
```



An **assertion** is a logical statement on a *program (execution) state*.

A **loop invariant** is an assertion inside a loop that is true every time it is reached by the program execution.

[Furia et al., 2014: Loop invariants: analysis, classification, and examples]

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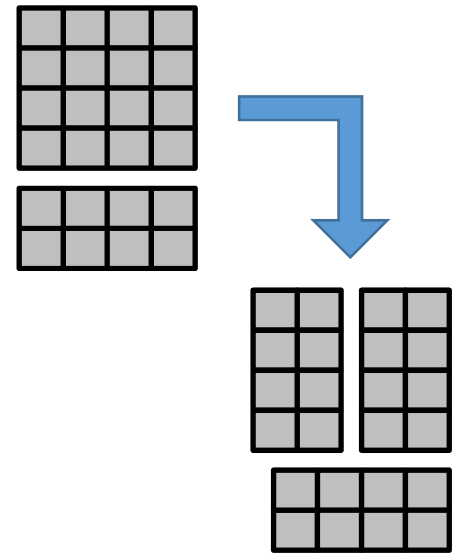
What are useful loop invariants?

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
    #INV: len(pieces)==num_cuts+1
```



An **assertion** is a logical statement on a *program (execution) state*.

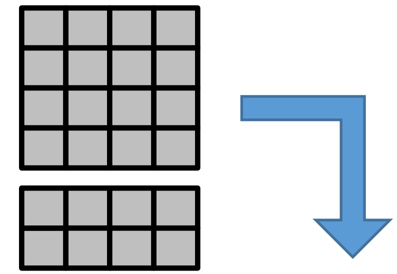
A **loop invariant** is an assertion inside a loop that is true every time it is reached by the program execution.

We want invariants at *end of loop* that together with loop exit condition *"turn into"* desired post-condition.

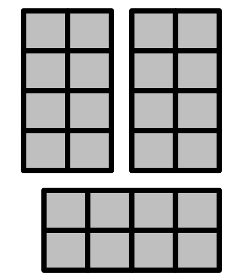
[Furia et al., 2014: Loop invariants: analysis, classification, and examples]

What are useful loop invariants?

```
cut = … #some arbitrary cutting strategy
pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
    #INV: len(pieces)==num_cuts+1
#EXC: len(pieces) == n*m
```



loop exit condition



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What are useful loop invariants?

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pieces = [(m, n)]
num_cuts = 0
#PRC: len(pieces)==num_cuts+1
while len(pieces)<n*m:
    cut(pieces)
    num_cuts += 1
    #INV: len(pieces)==num_cuts+1
#EXC: len(pieces) == n*m
#POC: num_cuts == n*m - 1
```

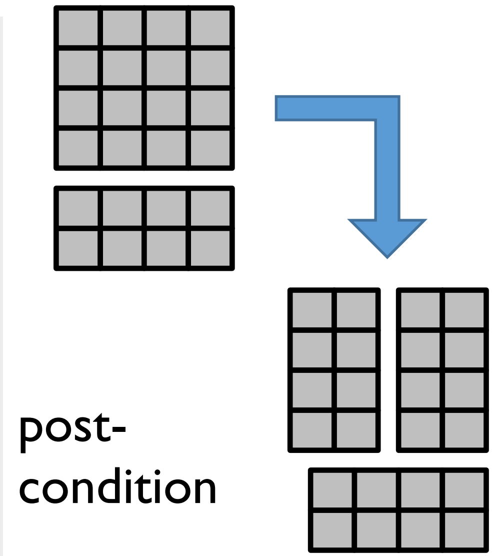

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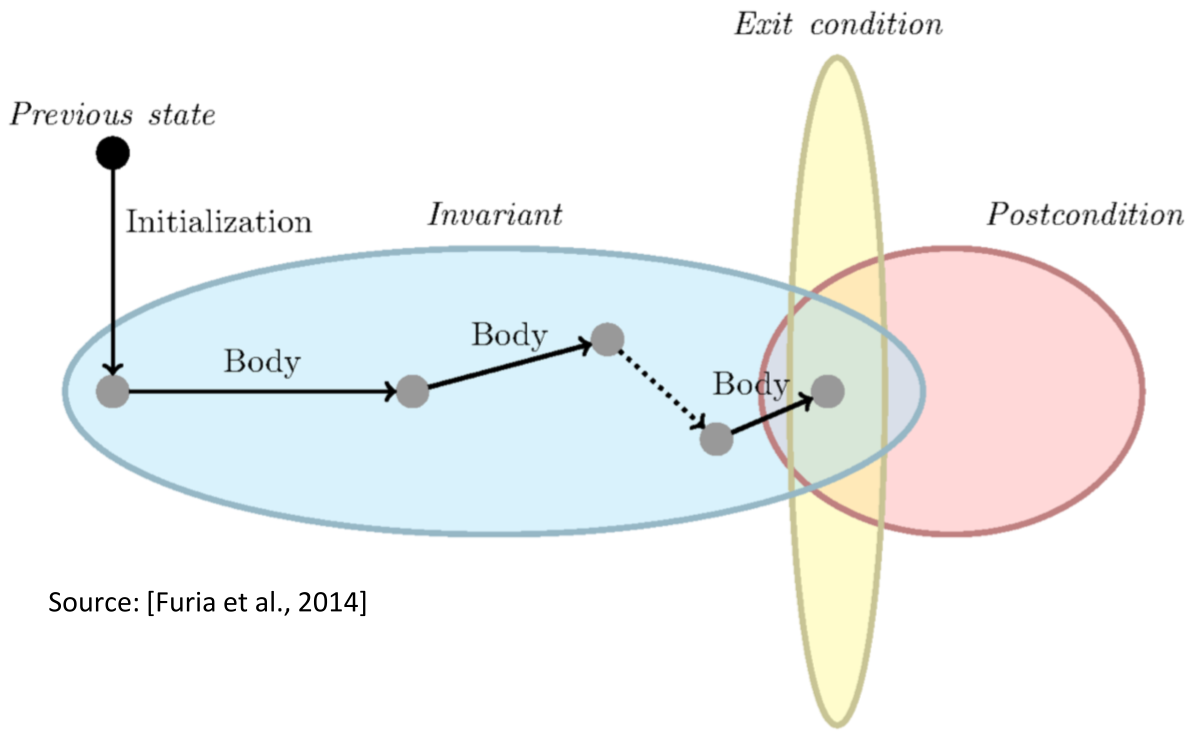
What are useful loop invariants?



Source: [Furia et al., 2014]

We are interested in loop invariants that together with **loop exit condition** *"turn into"* desired **post-condition**.

[Furia et al., 2014: Loop invariants: analysis, classification, and examples]

Outline

- Assertions and invariants
- Analysing Insertion Sort
- Analysing Min Index Selection
- Analysing Prim's Algorithm

Does Insertion Sort always result in a sorted list?

```
def insert(i, lst):
    """accepts: int i and list lst of length n>i>0
               of comp. elements with lst[:i] is sorted
       postcon: lst[:i+1] is sorted"""
    temp = lst[i]
    j = i-1
    while j >= 0 and lst[j] > temp:
        lst[j+1] = lst[j]
        j = j - 1
    lst[j+1] = temp
```

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
               is sorted                              """
    for i in range(1, len(lst)):
        insert(i, lst)
```

Situation at the start of execution

`lst`

?	?	?	?	…													…	?
0	1	2	3	…													…	n-1

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                 """
    for i in range(1, len(lst)):
        insert(i, lst)
```

Loop initialisation

`lst`

?	?	?	?	…													…	?
0	i=1	2	3	…													…	n-1

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                               """
    for i in range(1, len(lst)):
        insert(i, lst)
```

What is true at this point?

lst

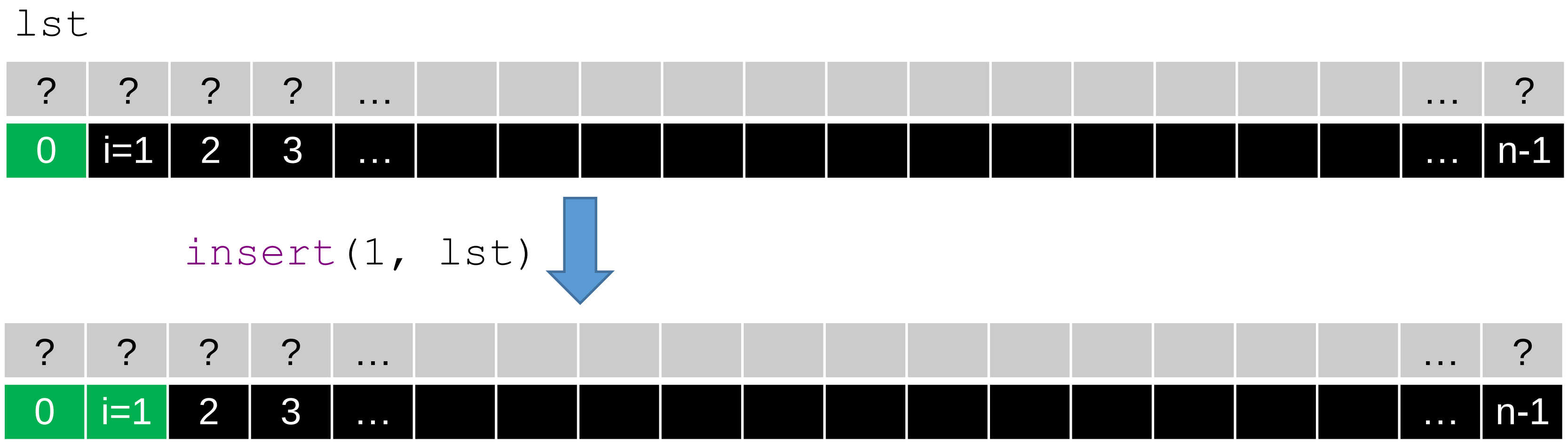
?	?	?	?	…													…	?
0	i=1	2	3	…													…	n-1

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                              """
    for i in range(1, len(lst)):
        # lst[:1] is sorted
        insert(i, lst)
```

Insertion procedure extends sorted range by one


```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                            """
    for i in range(1, len(lst)):
        # lst[:1] is sorted
        insert(i, lst)
```

Insertion procedure extends sorted range by one




```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                               """
    for i in range(1, len(lst)):
        # lst[:1] is sorted
        insert(i, lst)
        # lst[:2] is sorted
```

Hold in first iteration (and further), but not enough do demonstrate post condition

Idea: generalise assertions so that they become stronger every iteration!

These general assertions seem much more useful

lst

?	?	?	?	…												…	?
0	i=1	2	3	…												…	n-1

insert(1, lst)

?	?	?	?	…												…	?
0	i=1	2	3	…												…	n-1

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                """
    for i in range(1, len(lst)):
        # lst[:i] is sorted
        insert(i, lst)
        # lst[:i+1] is sorted
```

But are they preserved by general loop iteration?

`lst`

?	?	?	?	…													…	?
0	i=1	2	3	…													…	n-1

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                  """
    for i in range(1, len(lst)):
        # lst[:i] sorted
        insert(i, lst)
        # lst[:i+1] sorted
```

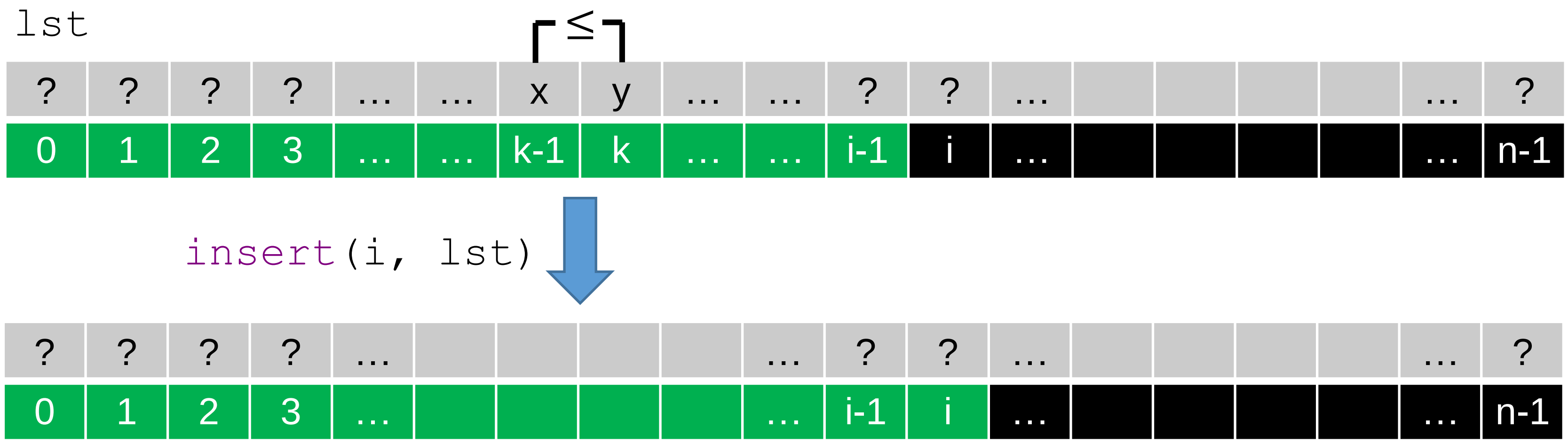
Let’s assume first assertion is true

lst

?	?	?	?	…					…	?	?	…					…	?
0	1	2	3	…					…	i-1	i	…					…	n-1

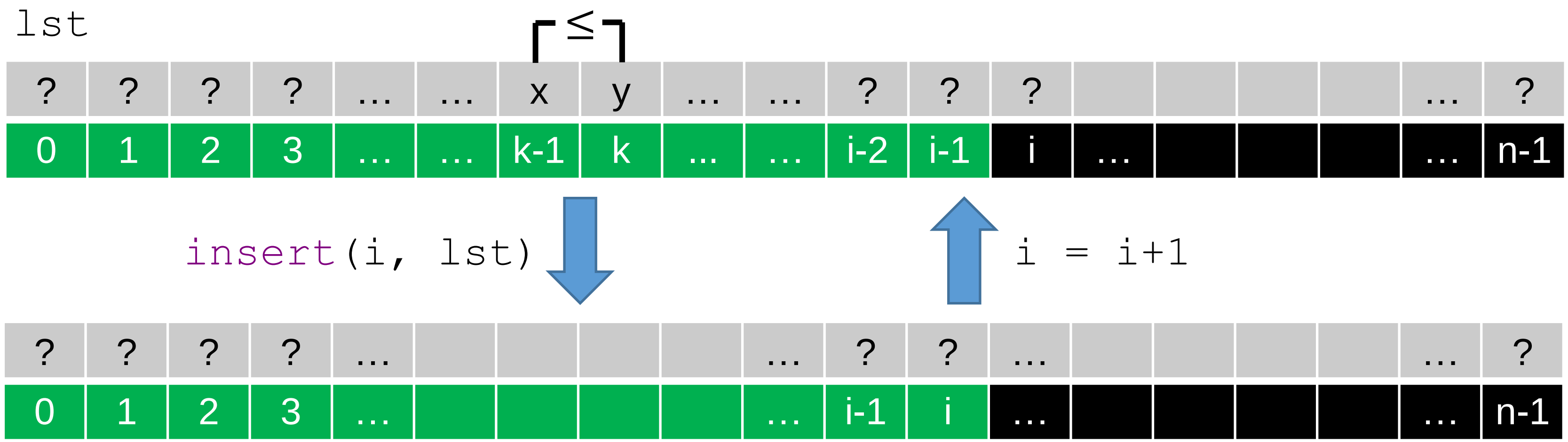
```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                  """
    for i in range(1, len(lst)):
        # lst[:i] sorted
        insert(i, lst)
        # lst[:i+1] sorted
```

Then loop body ensures second assertion



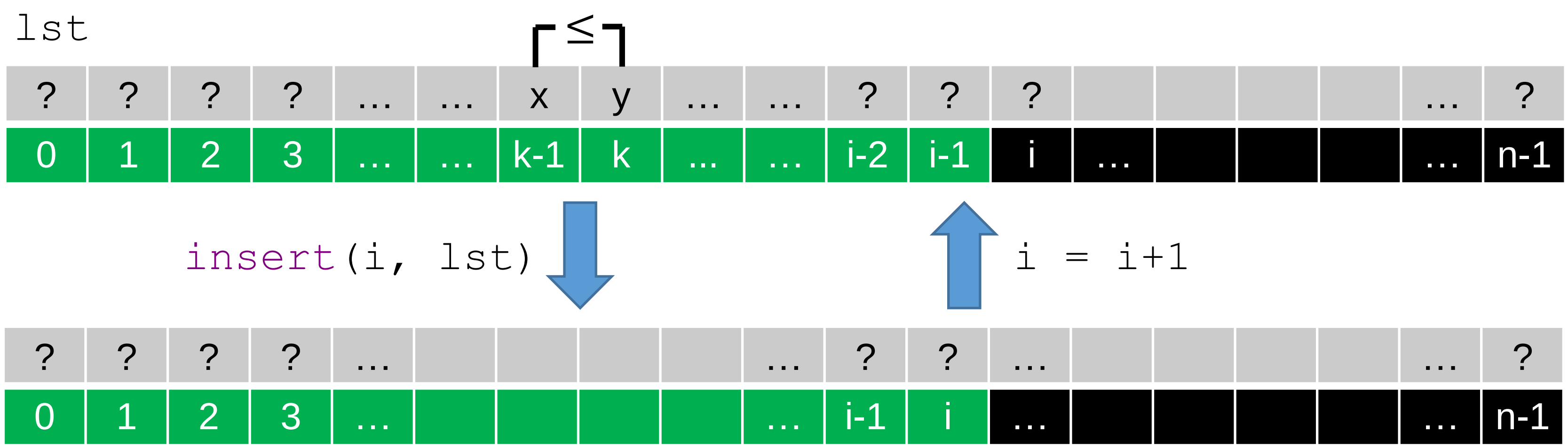

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                  """
    for i in range(1, len(lst)):
        # lst[:i] sorted
        insert(i, lst)
        # lst[:i+1] sorted
```

Which in turn implies first assertion in next iteration!



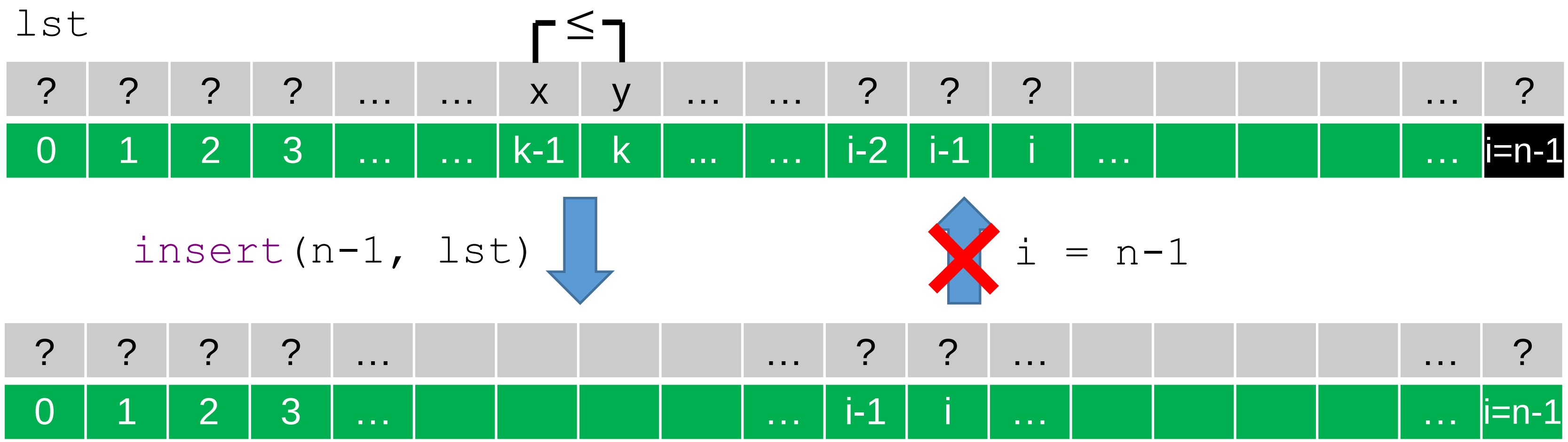

```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                             """
    for i in range(1, len(lst)):
        # lst[:i] sorted
        insert(i, lst)
        # lst[:i+1] sorted
```

Thus these assertions are loop invariants!




```
def insertion_sort(lst):
    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                  """
    for i in range(1, len(lst)):
        #I: lst[:i] sorted
        insert(i, lst)
        #I': lst[:i+1] sorted
```

What happens at end of loop?




```
def insertion_sort(lst):
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       postcon: lst has same elements as on call but
                is sorted                                """
    for i in range(1, len(lst)):
        #I: lst[:i] sorted
        insert(i, lst)
        #I': lst[:i+1] sorted
    #EXC: i = n-1
```
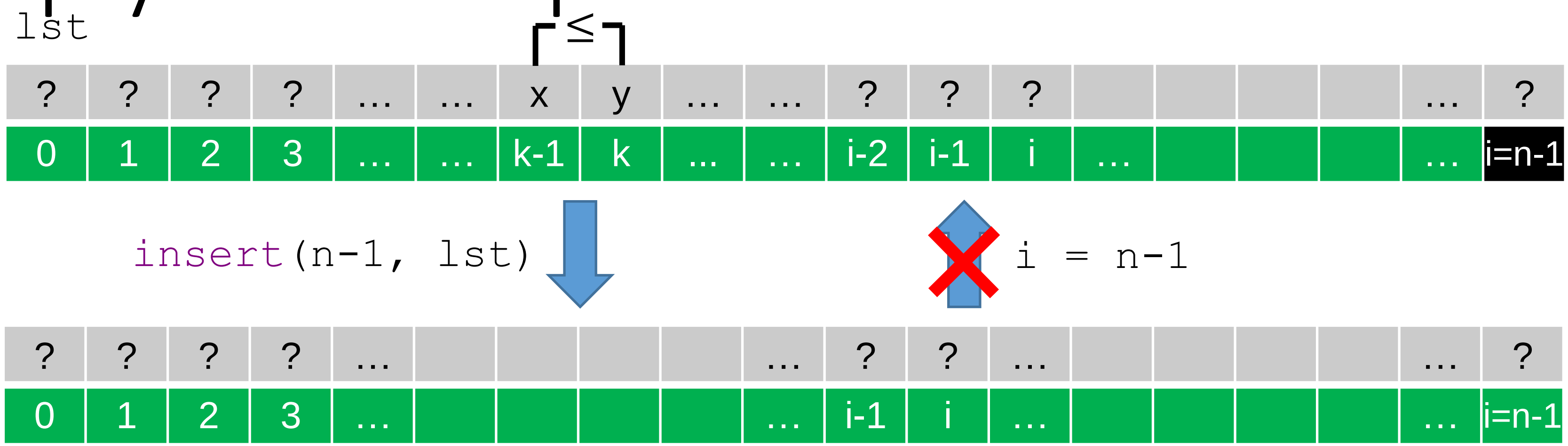

Loop exit condition and invariant imply desired post condition




```
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    """accepts: list lst of length n of comp. elements
       postcon: lst has same elements as on call but
                is sorted                                  """
    for i in range(1, len(lst)):
        #I: lst[i] sorted
        insert(i, lst)
        #I': lst[i+1] sorted
    #EXC: i = n-1
    #POC: lst[:n] sorted
```

Outline

- Assertions and invariants
- Analysing Insertion Sort
- Analysing Min Index Selection
- Analysing Prim's Algorithm

Recap: what is min_index trying to do (formally)?

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: ?
                                                                    """
    k = 0
    for i in range(1, len(lst)):
        if lst[i] < lst[k]:
            k = i
    return k
```

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Recap: what is min_index trying to do (formally)?

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]
"""
    k = 0
    for i in range(1, len(lst)):
        if lst[i] < lst[k]:
            k = i
    return k
```

lst

?	?	?	…	y’	…	x	…	y	…			…					…	?
0	1	2	…	j’	…	k	…	j	…			…					…	n-1

Does min_index function always yield index of minimum value?

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        if lst[i] < lst[k]:
            k = i
    return k
```

lst

?	?	?	…														…	?
0	1	2	…														…	n-1

Situation before reaching loop statement

```
def min_index(lst):
    “””accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]”””
    k = 0
    for i in range(1, len(lst)):
        if lst[i] < lst[k]:
            k = i
    return k
```

lst

?	?	?	…														…	?
k=0	1	2	…														…	n-1

First iteration of loop

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        if lst[i] < lst[k]:
            k = i
    return k
```

lst

?	?	?	…														…	?
k=0	i=1	2	…														…	n-1

What is true at this point?

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # ?
        if lst[i] < lst[k]:
            k = i
    return k
```

`lst`

?	?	?	…														…	?
k=0	i=1	2	…														…	n-1

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Effect of conditional statement

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # ?
        if lst[i] < lst[k]:
            k = i
    return k
```

lst

?	?	?	…														…	?
k=0	i=1	2	…														…	n-1

```
if lst[1] < lst[k]:
    k = 1
```

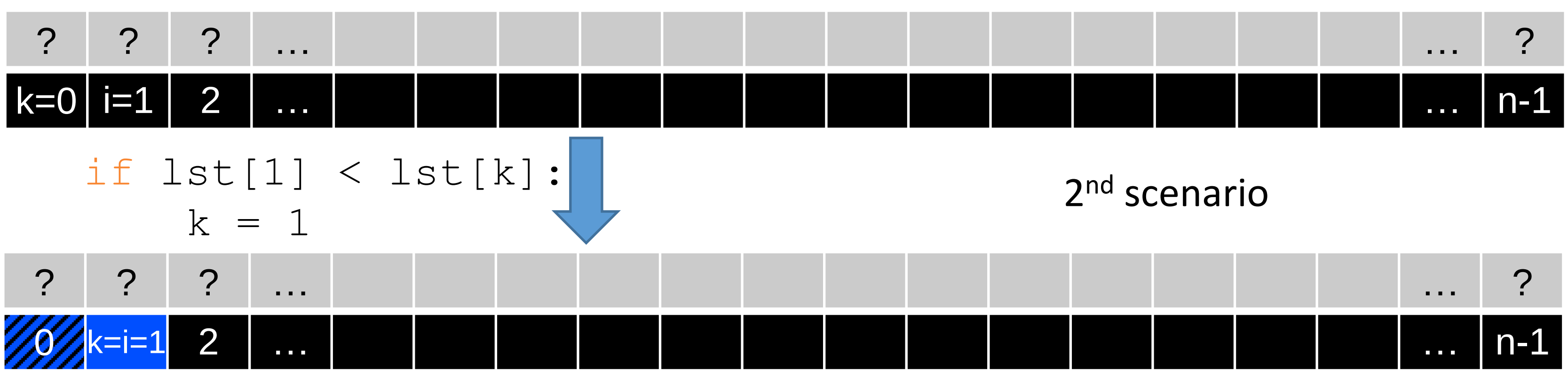
1st scenario

?	?	?	…														…	?
k=0	i=1	2	…														…	n-1

Effect of conditional statement

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # ?
        if lst[i] < lst[k]:
            k = i
    return k
```

lst



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In both cases: k is min index among the small index set {0, 1}

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # ?
        if lst[i] < lst[k]:
            k = i
        # lst[k] <= lst[0] and lst[k]<=lst[1]
    return k
```

lst

?	?	?	…														…	?
k=0	i=1	2	…														…	n-1

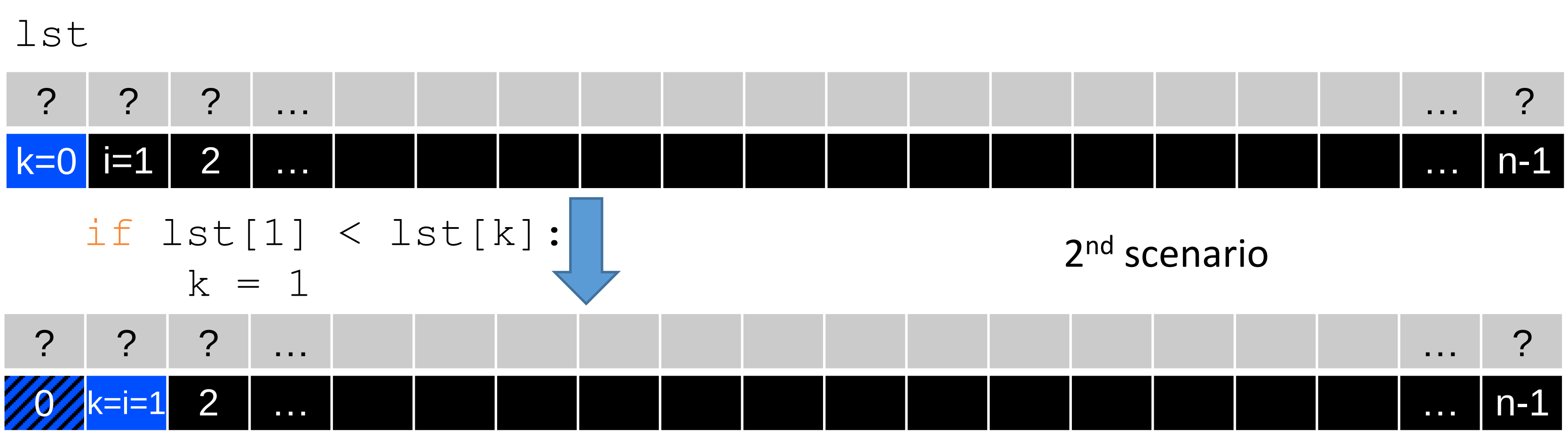
```
if lst[1] < lst[k]:
    k = 1
```

2nd scenario

?	?	?	…														…	?
0	k=i=1	2	…														…	n-1

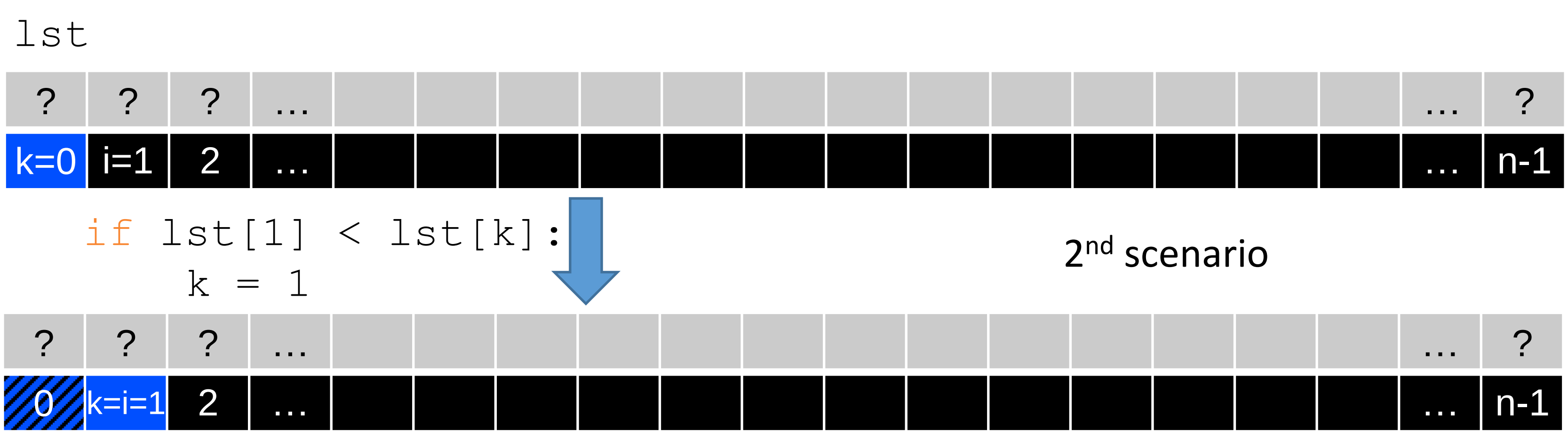
This suggests general pattern

```python
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # lst[k] <= lst[0]
        if lst[i] < lst[k]:
            k = i
        # lst[k] <= lst[0] and lst[k]<=lst[1]
    return k
```



This suggests general pattern

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]:
            k = i
        # for all j in range(i+1): lst[k]<=lst[j]
    return k
```



Let’s consider general loop iteration

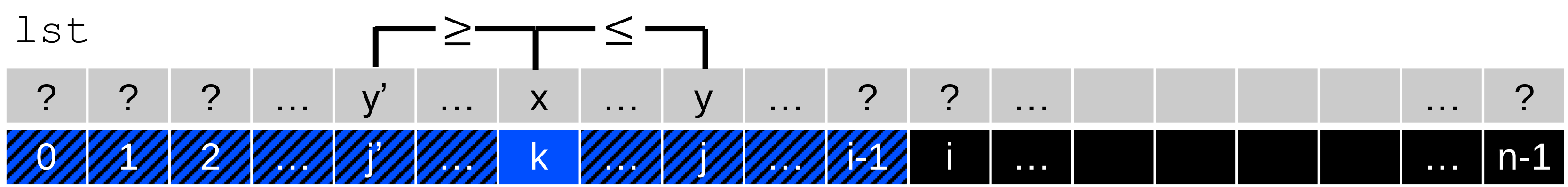
```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]:
            k = i
        # for all j in range(i+1): lst[k]<=lst[j]
    return k
```

`lst`

?	?	?	…							…	?	…					…	?
0	1	2	…							…	i	…					…	n-1

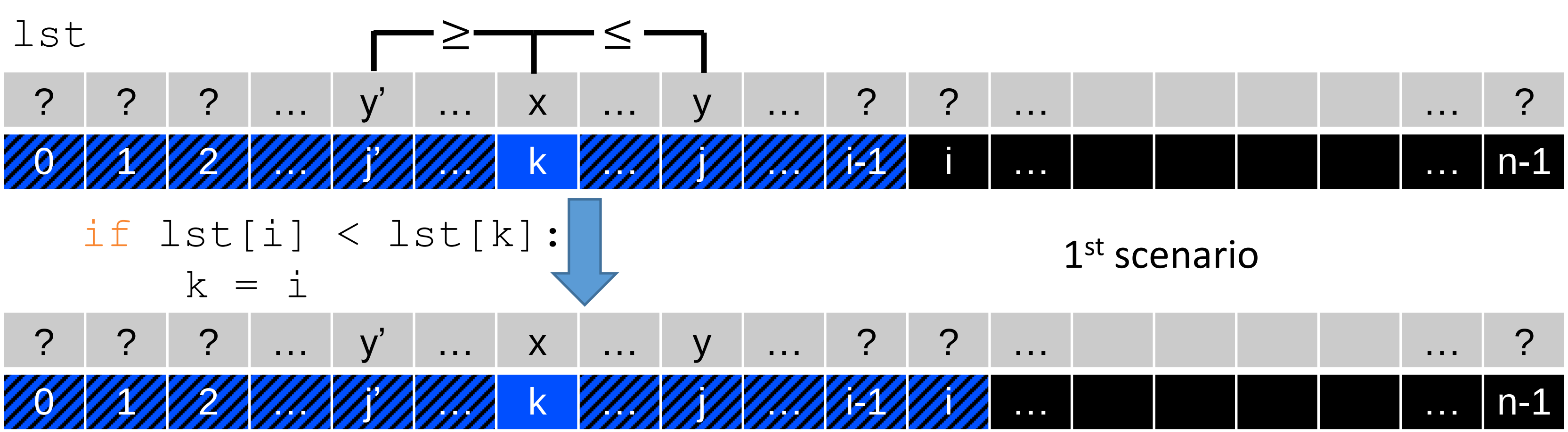
Assume first assertion is true

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]:
            k = i
        # for all j in range(i+1): lst[k]<=lst[j]
    return k
```



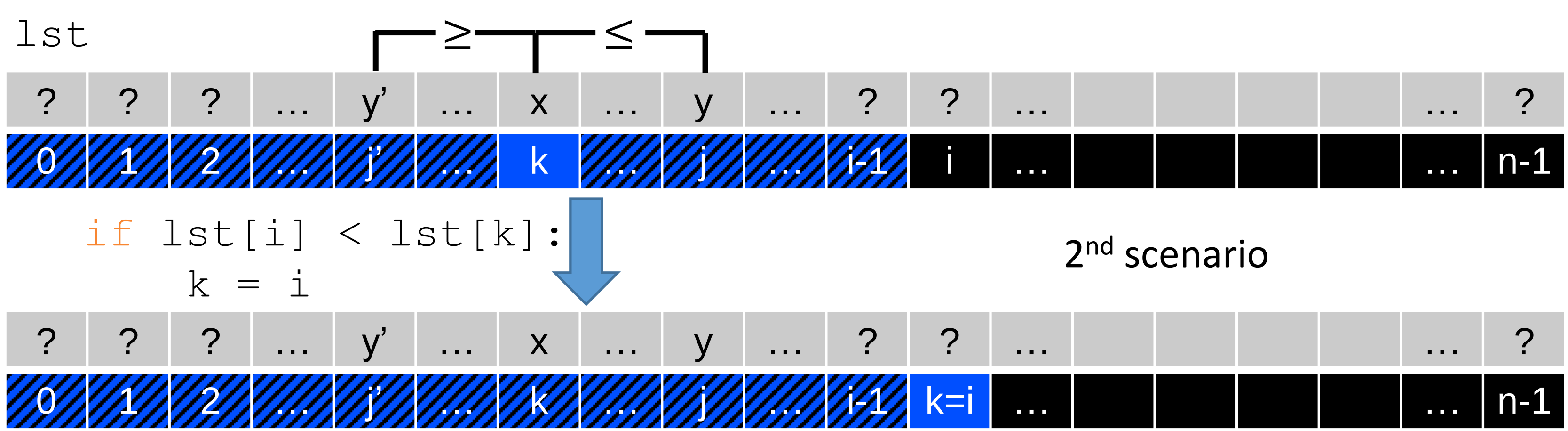
Effect of conditional statement

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]: k = i
        # for all j in range(i+1): lst[k]<=lst[j]
    return k
```



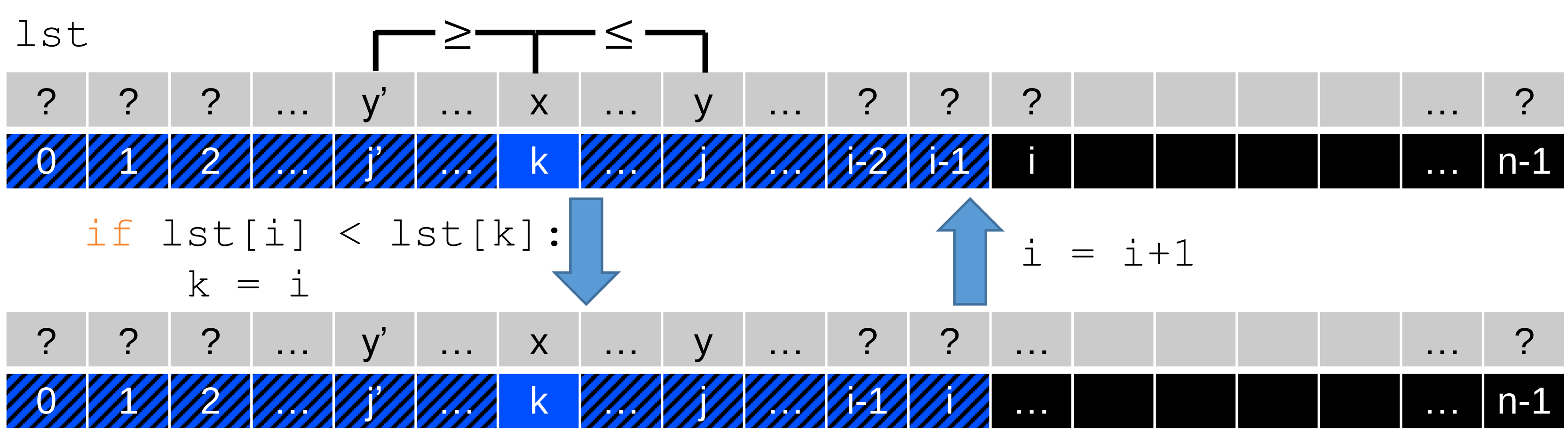
Conditional statement ensures second assertion

```python
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        # for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]: k = i
        # for all j in range(i+1): lst[k]<=lst[j]
    return k
```



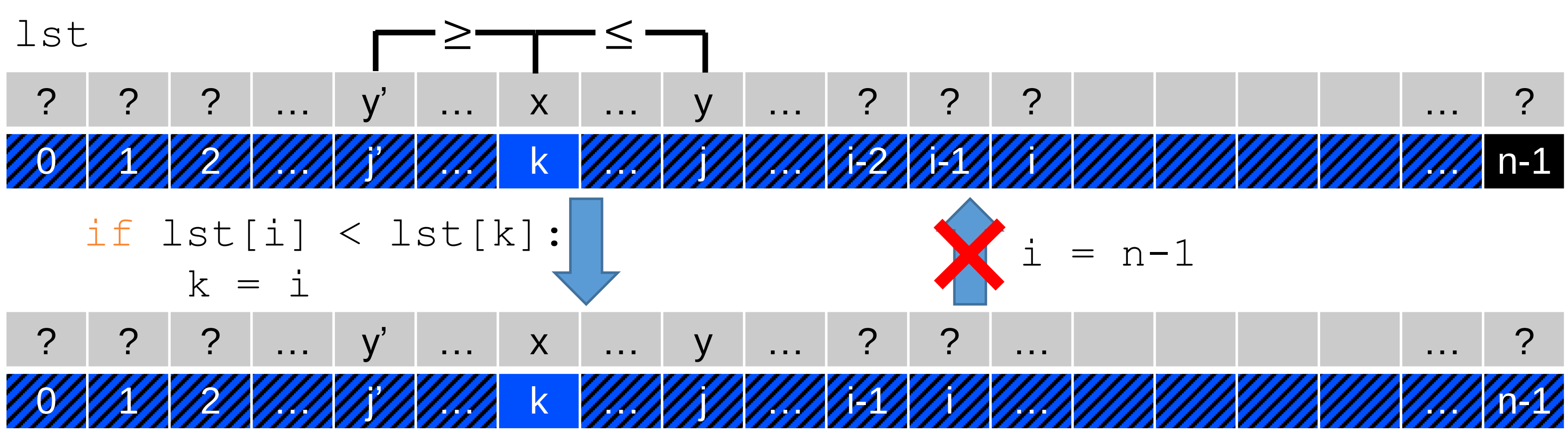
Which in turn assures first assertion in next iteration

```python
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        #I: for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]: k = i
        #I': for all j in range(i+1): lst[k]<=lst[j]
    return k
```



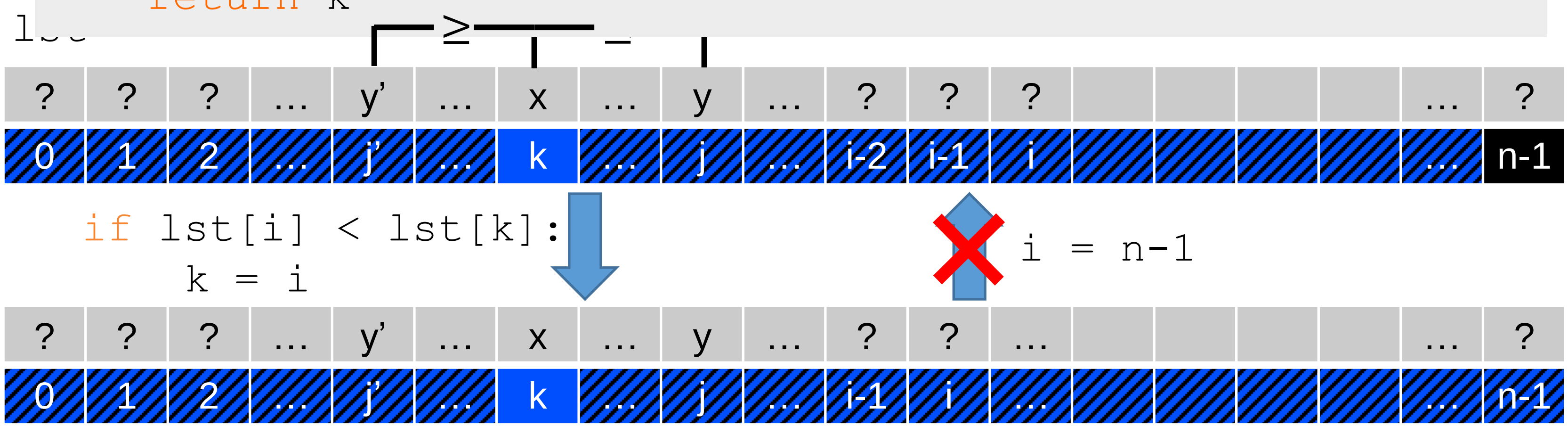
What happens at the end of the loop?

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        #I: for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]: k = i
        #I': for all j in range(i+1): lst[k]<=lst[j]
    #EXC: i = n-1
    return k
```



Again loop exit condition and invariants imply desired post cond.

```
def min_index(lst):
    """accepts: list of length n>0 of comp. elements
       returns: index k in range(n) such that
                for all j in range(n), lst[k]<=lst[j]"""
    k = 0
    for i in range(1, len(lst)):
        #I: for all j in range(i): lst[k]<=lst[j]
        if lst[i] < lst[k]: k = i
        #I': for all j in range(i+1): lst[k]<=lst[j]
    #EXC: i = n-1,
    #POC: for j in range(n): lst[k]<=lst[j]
    return k
```



Selection Sort – The invariants…

```
def selectionSort(aList):
    for k in range(len(aList)-1):
        #INVARIANT: aList[:k] is sorted and represents
        #the k smallest elements in the whole list
        minPos = k
        for current in range(k+1,len(aList)): # Find minimum index
            if aList[current] < aList[minPos]:
                minPos = current # Update new minimum index
        aList[minPos],aList[k] = aList[k],aList[minPos] # Swap
        #INVARIANT: aList[:k+1] is sorted and represents
        #the k+1 smallest elements in the whole list
```

Loop invariants

- Provide a useful invariant in terms of *k*.

```
k = 0
while k < len(my_list):
#my_list[:k] will be replaced with a 1
    my_list[k] = 1
    k+= 1
#my_list[:k] will be replaced with a 1
```

Still *k* because unlike in the FOR loop, *k* is already changed in the While loop

- at the k^{th} iteration, the first *k* elements of the list have been replaced with a 1 i.e. my_list[:k] will be replaced with a 1

Loop invariants

- Provide a useful invariant in terms of *k*.

```
ct = [0] * len(aList)
for k in range(len(aList)):
    for j in range(k):
        if aList[k] == aList[j]:
            ct[k]+= 1
```

- After the *k*th iteration, ct[k] holds the count of the number of times aList[k] occurred in aList[0:k]

Loop invariants

- Provide a useful invariant in terms of *k*.

```
ct = [0] * len(aList)
for k in range(len(aList)):
#ct[k] holds the value of 0
     for j in range(k):
          if aList[k] == aList[j]:
               ct[k]+= 1
#ct[k] holds the count of the number of times aList[k] occurred in aList[0:k]
```

- After the k^{th} iteration, ct[k] holds the count of the number of times aList[k] occurred in aList[0:k]

Outline

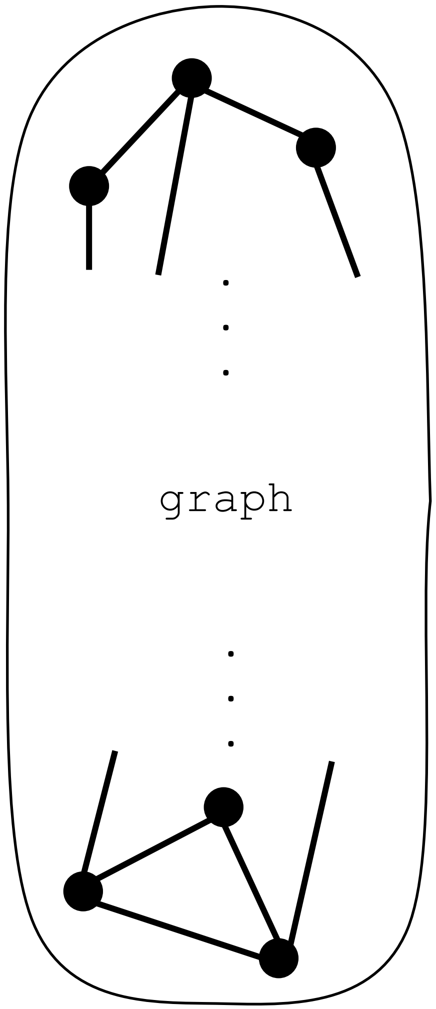
- Assertions and invariants
- Analysing Insertion Sort
- Analysing Min Index Selection
- Analysing Prim's Algorithm

Prim’s algorithm: does it always produce a spanning tree?

```
def extension(con, g):
    “””input: vertices con connected in
g
       output: edge (i,j) of g with i in
               con and j not in con“””
    for i in con:
        for j in range(len(g)):
            if j not in con and g[i][j]:
                return i, j
def spanning_tree(graph):
    “””input: graph given as adj. matrix
       output: spanning tree of graph“””
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```

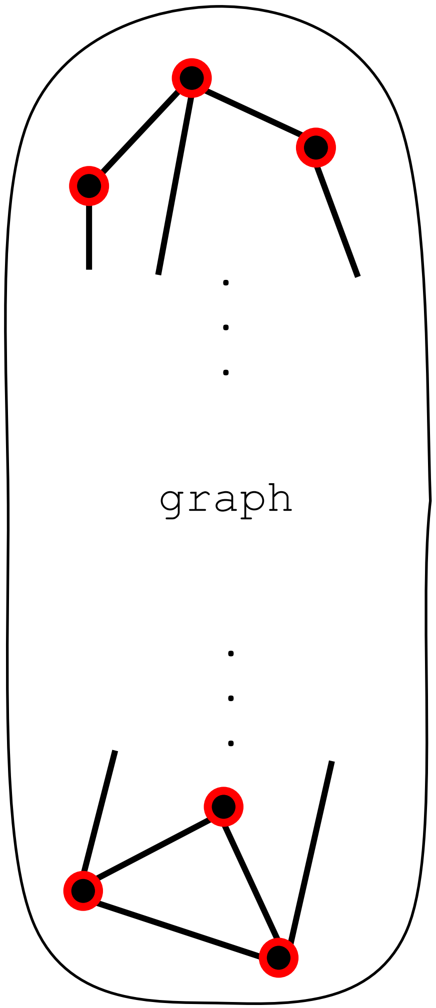
Let us visualise generic input

```
def extension(con, g):
    """input: vertices con connected in
g
       output: edge (i,j) of g with i in
               con and j not in con"""
                     …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



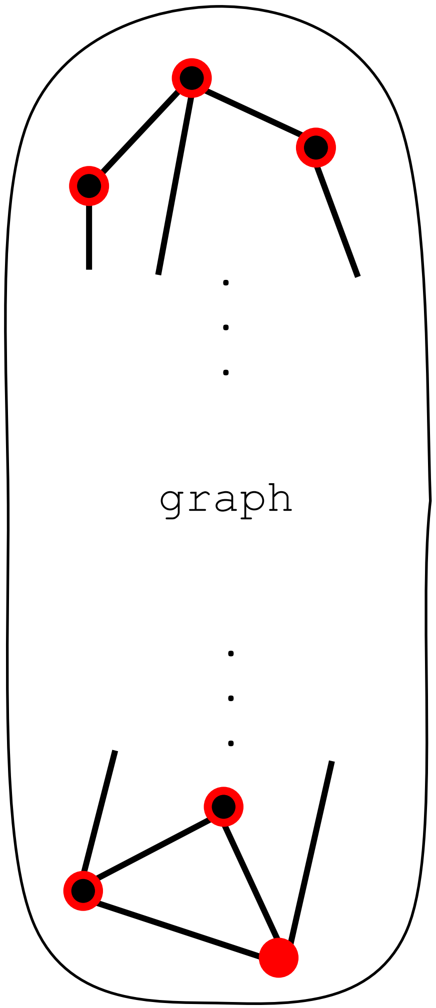
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in
g
       output: edge (i,j) of g with i in
               con and j not in con"""
                    …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



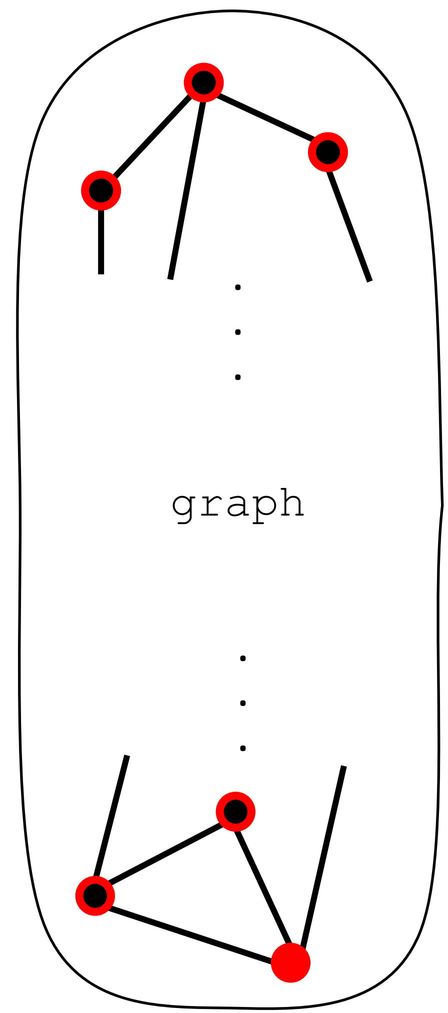
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in
g
       output: edge (i,j) of g with i in
               con and j not in con"""
                    …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



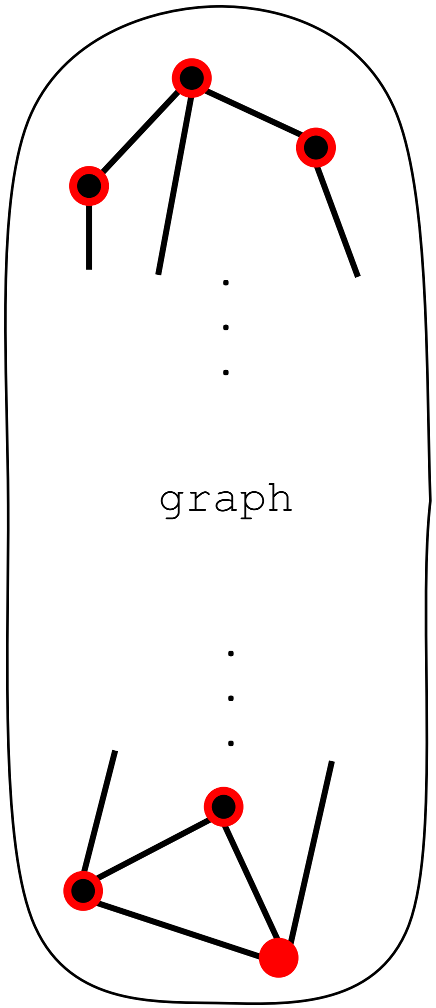
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                    …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



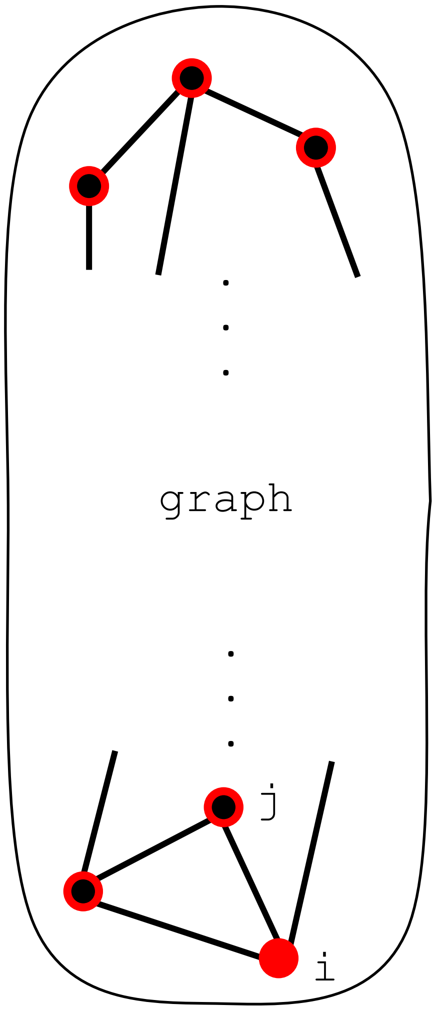
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                        …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



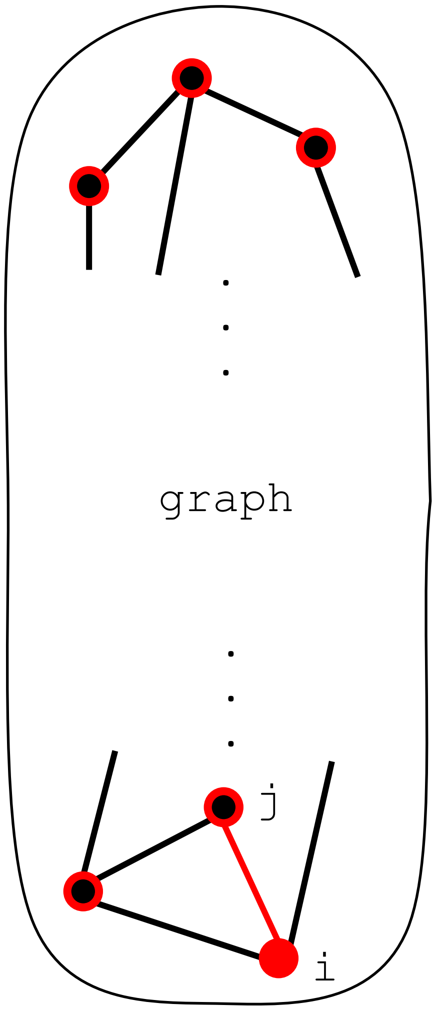
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                       …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



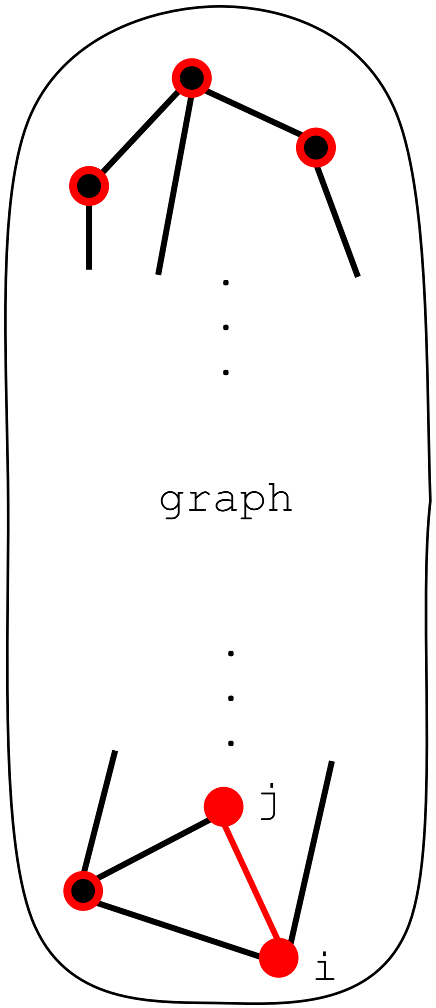
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                    …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



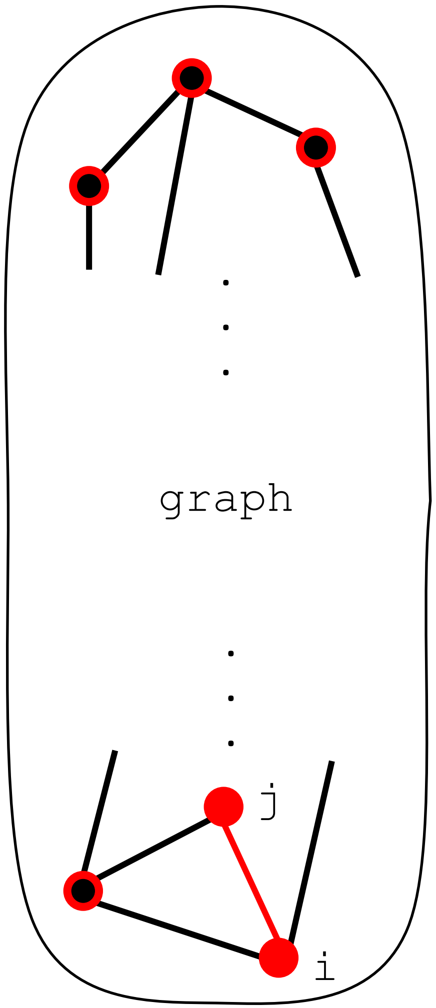
…and analyse what happens during computation

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                        …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
return tree
```



…and analyse what happens during computation

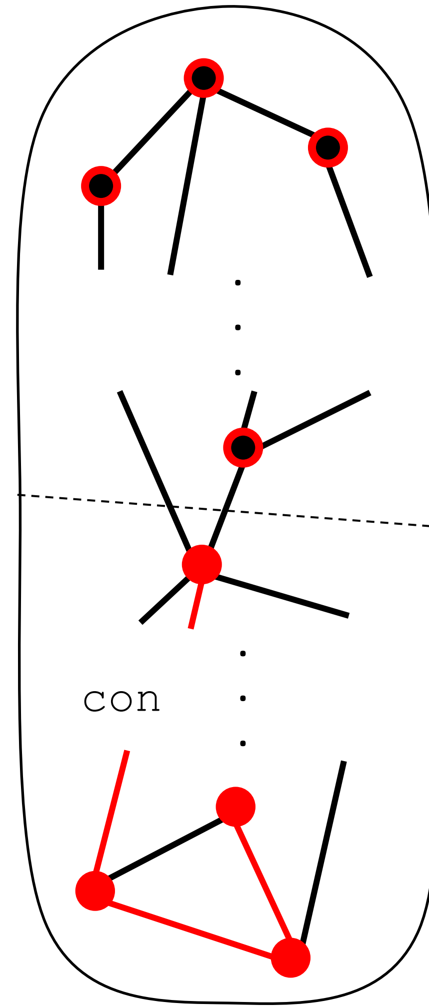
```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                         …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        # vertices in con are connected in tree
return tree
```



Assume con is connected at start of arbitrary loop iteration

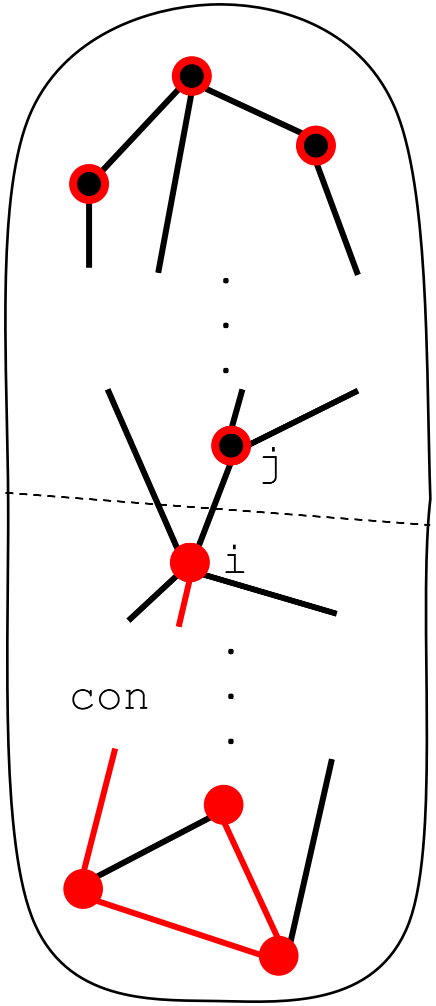
```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                    …

def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        # vertices in con are connected in tree
return tree
```



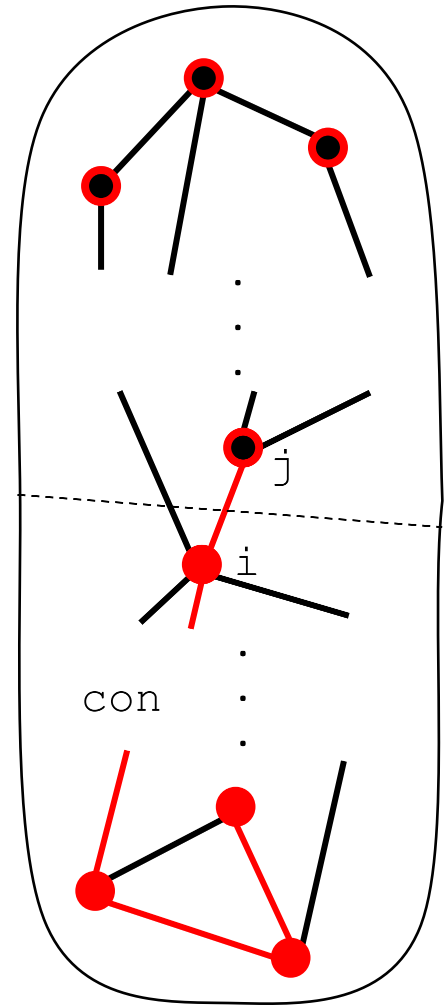
Extension edge bridges connected to not yet connected

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                   …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        # vertices in con are connected in tree
return tree
```



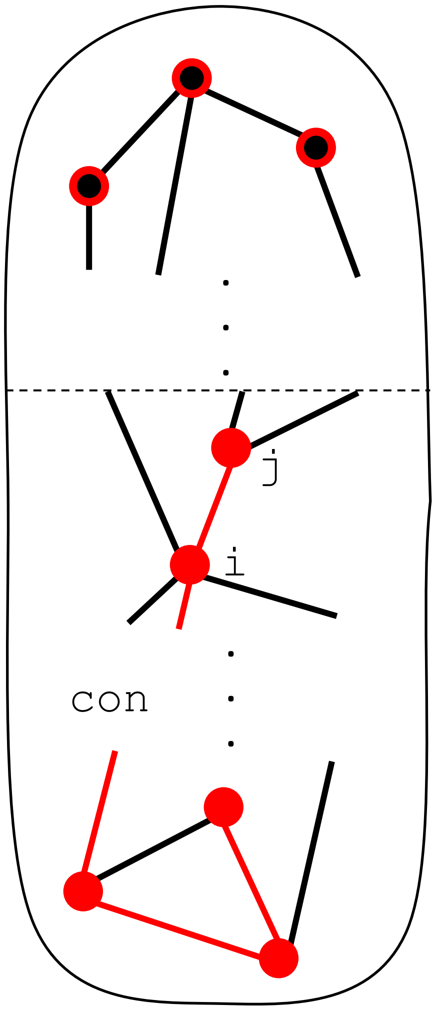
Extension edge bridges connected to not yet connected

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                          …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        # vertices in con are connected in tree
return tree
```



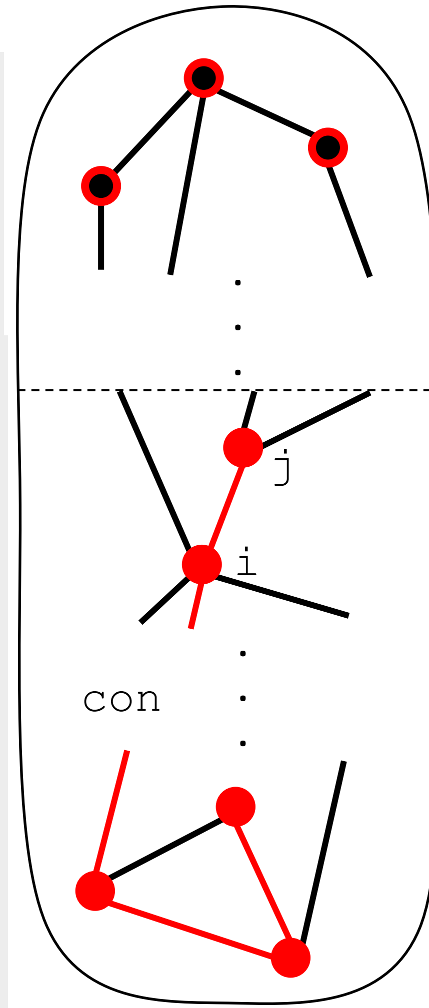
After adding extension edge to tree j is also connected

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                         …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        # vertices in con are connected in tree
return tree
```



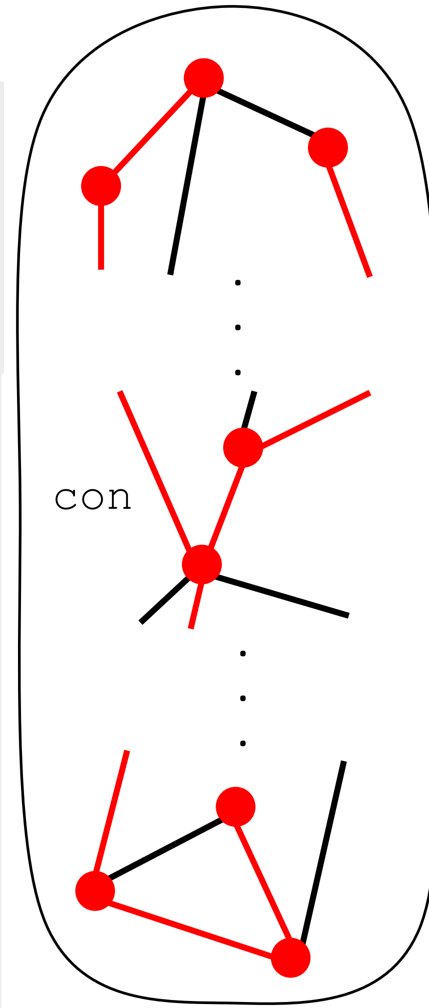
Invariant: con is connected

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                        …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        #I: vertices in con are connected in tree
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I: vertices in con are connected in tree
return tree
```



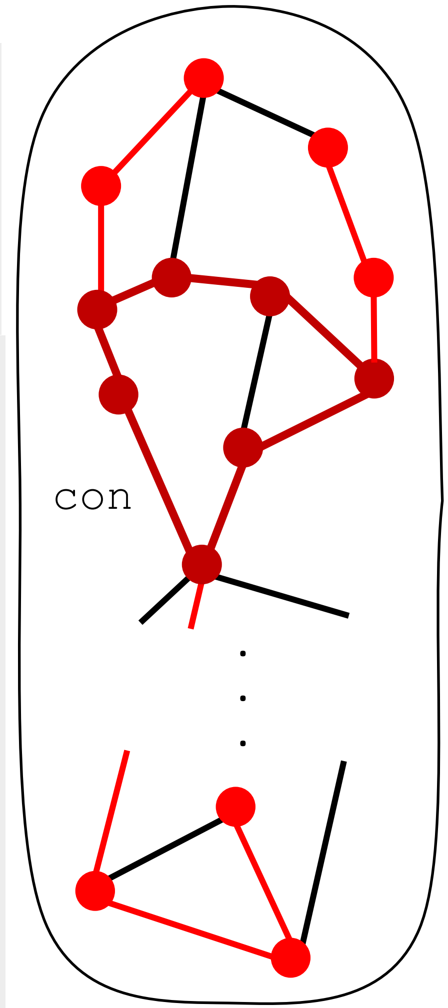
Is this enough to conclude desired post condition?

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                        …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I: vertices in con are connected in tree
    #EXC: len(con) == len(graph)
return tree
```



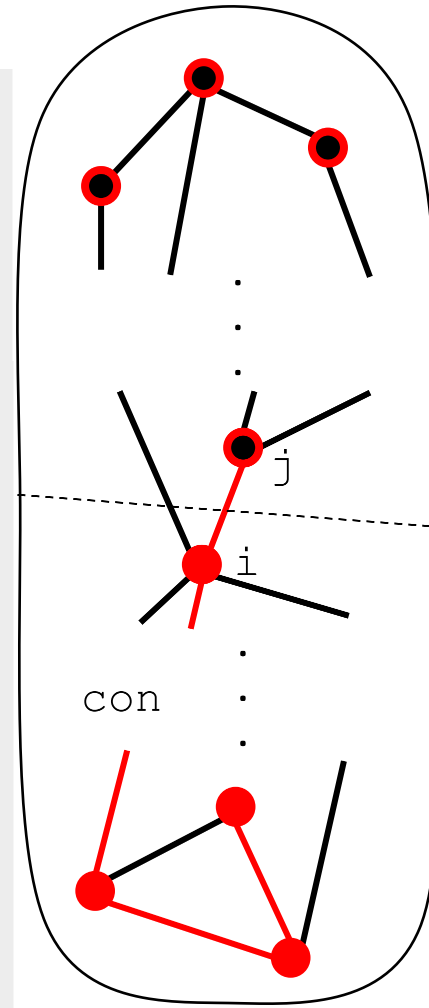
No. Can conclude that the tree is connected, but could contain a cycle

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                         …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I: vertices in con are connected in tree
    #EXC: len(con) == len(graph)
    #POC: all vertices are connected to the tree
return tree
```



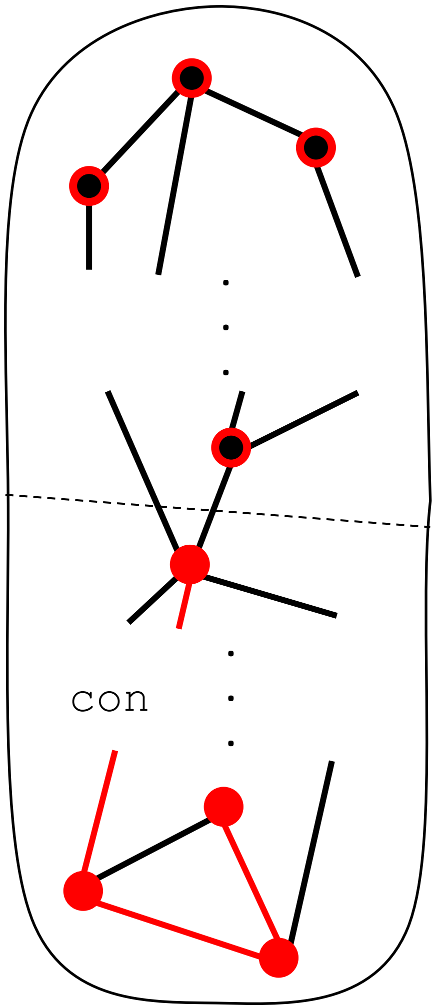
What should the second invariant be?

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                         …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I1: vertices in con are connected in tree
        #I2: ?
return tree
```



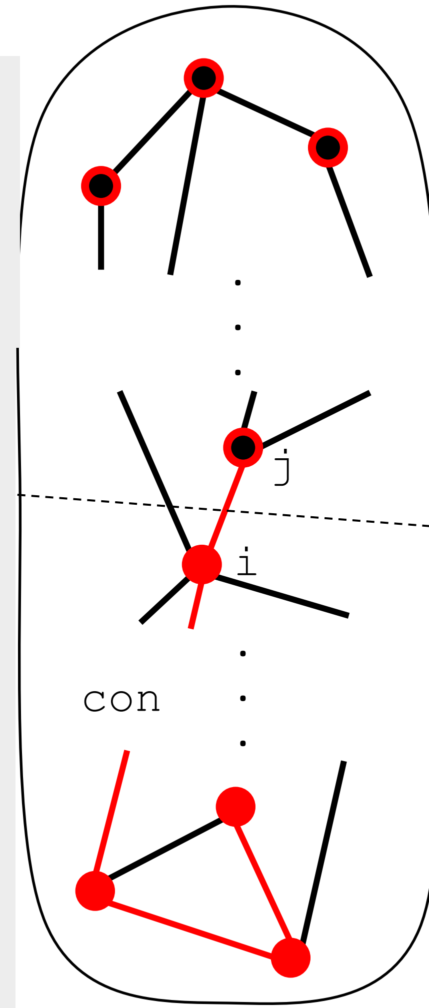
Need to guarantee that we never add a cycle to tree

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                    …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I: vertices in con are connected in tree
        # tree does not contain cycle
return tree
```



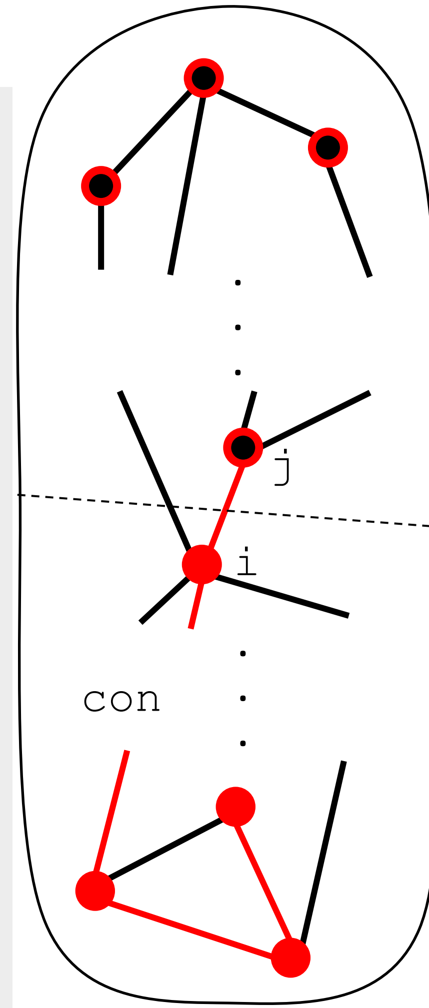
Observation: extension edge never creates a cycle with edges in the tree

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                         …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        # tree does not contain a cycle
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I1: vertices in con are connected in tree
        # tree does not contain a cycle
return tree
```



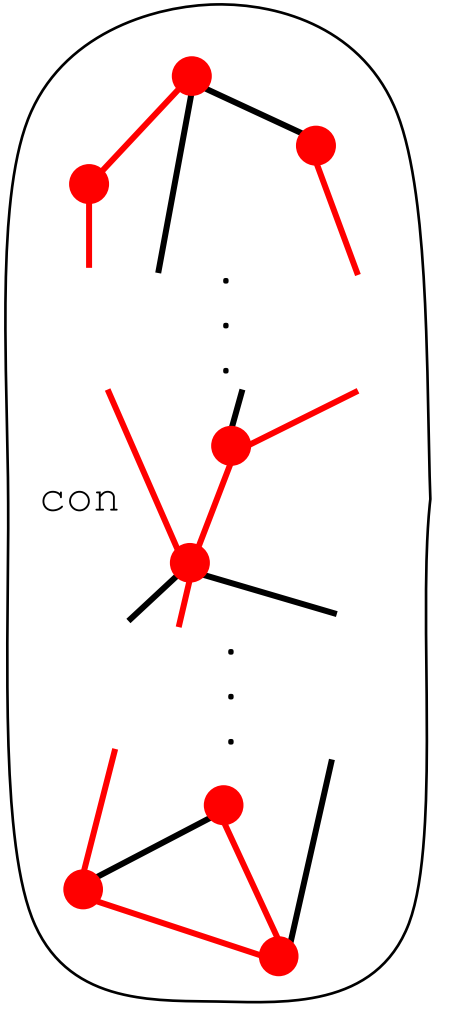
Invariant 2: tree does not contain a cycle

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                        …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I1: vertices in con are connected in tree
        #I2: tree does not contain cycle
return tree
```



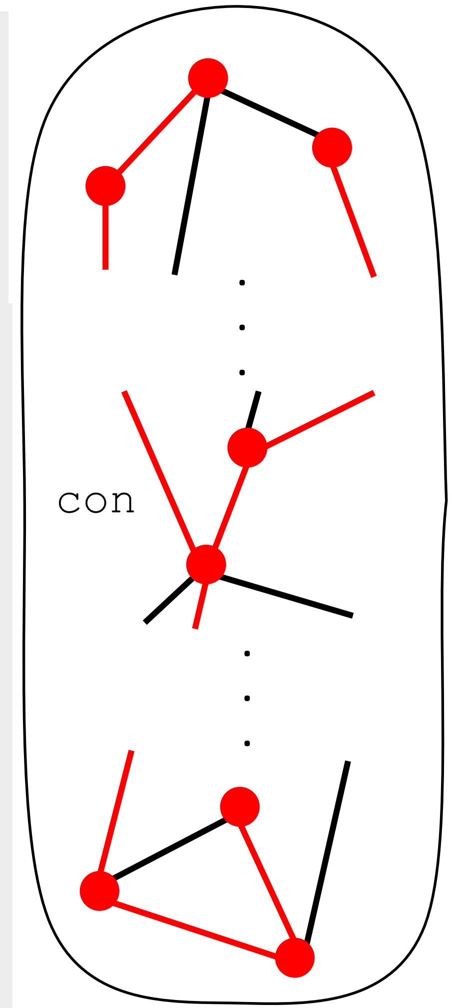
Now we know the tree is connected and without a cycle at loop exit

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                           …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I1: vertices in con are connected in tree
        #I2: tree does not contain cycle
    #EXC: len(con)==len(graph)
return tree
```



…in other words: the tree must be a spanning tree!!

```
def extension(con, g):
    """input: vertices con connected in g
       output: edge (i,j) of g with i in
               con and j not in con"""
                         …
def spanning_tree(graph):
    """input: graph given as adj. matrix
       output: spanning tree of graph"""
    tree = empty_graph(len(graph))
    con = {0}
    while len(con) < len(graph):
        i, j = extension(con, graph)
        tree[i][j], tree[j][i] = 1, 1
        con.add{j}
        #I1: vertices in con are connected in tree
        #I2: tree does not contain cycle
    #EXC: len(con)==len(graph)
    #POC: tree is spanning tree of graph
return tree
```



What have we learnt?

- Use assertions about execution state to reason about programs
- Loop invariants can be used to analyse behaviour of loopy control flows
- Look for invariants that turn into desired post-condition when loop exit condition is true