shallow_water01.f90 – semi-implicit one-dimensional shallow water equations

The model uses Arakawa C-grid staggering. Linearized equation of motion. Bottom friction.

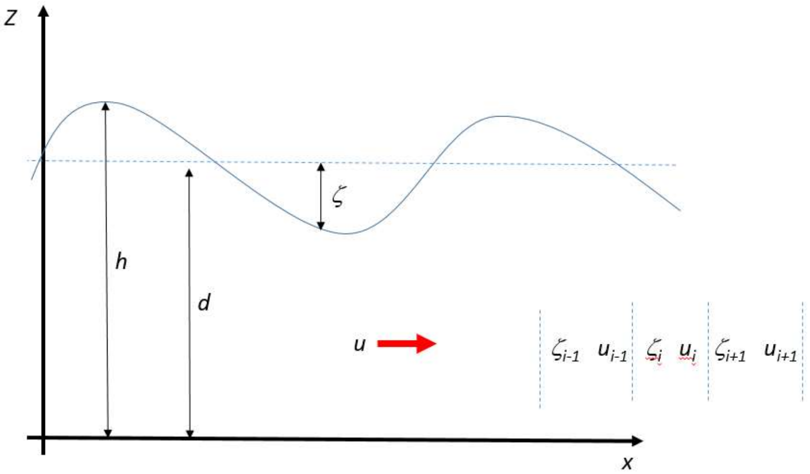

Fig 1: Geometry of the 1-D shallow water problem. Disturbed water depth *h*, un-disturbed water depth *d*, sea surface elevation ζ, horizontal velocity *u*. The lower right corner shows the C-grid staggering.

The momentum equation is formulated on the old (*n*) and new time level (*n+1*). The weight between both levels is w_{imp}.

$$u^{n+1} = u^n - \Delta t\, c_D\, \frac{1}{h}\, u^n |u^n| - \Delta t\, g\, (1 - w_{imp}) \frac{\partial \zeta^n}{\partial x} - \Delta t\, g\, w_{imp} \frac{\partial \zeta^{n+1}}{\partial x} \qquad (1)$$

The same is valid for the equation of continuity:

$$\zeta^{n+1} = \zeta^n - \Delta t\, (1 - w_{imp}) \frac{\partial (h\, u)^n}{\partial x} - \Delta t\, w_{imp} \frac{\partial (h\, u)^{n+1}}{\partial x} \qquad (2)$$

with

$$h = d + \zeta \qquad (3)$$

Our strategy is to form a system of equations for all elements of ζ^{n+1}, solve this system iteratively and finally compute the velocity at the new time step u^{n+1}.

Inserting equation (1) into (2) yields:

$$\zeta^{n+1} = \zeta^n - \Delta t \left(1 - w_{imp}\right) \frac{\partial (h\,u)^n}{\partial x} - \Delta t\, w_{imp} \frac{\partial (h^n\, u^*)}{\partial x} + \Delta t^2\, {w_{imp}}^2\, g\, h\, \frac{\partial^2 \varsigma^{n+1}}{\partial x^2} \tag{4}$$

with the interim solution

$$u^* = u^n - \Delta t\, c_D\, \frac{1}{h}\, u^n |u^n| - \Delta t\, g \left(1 - w_{imp}\right) \frac{\partial \zeta^n}{\partial x} \tag{5}$$

If we define

$$div = -\Delta t \left(1 - w_{imp}\right) \frac{\partial (h\,u)^n}{\partial x} - \Delta t\, w_{imp} \frac{\partial (h^n\, u^*)}{\partial x} \tag{6}$$

then it follows

$$\zeta_i^{n+1} = \frac{1}{1 + c_E + c_W} \left[\zeta^n + div + c_E\, \zeta_{i+1}^{n+1} + c_W\, \zeta_{i-1}^{n+1}\right] \tag{7}$$

with

$$c_E = \begin{cases} \frac{\Delta t^2 w_{imp}^2\, g\, h_E}{\Delta x^2} & if\ h_{i+1} > 0 \\ 0. & otherwise \end{cases} \tag{8}$$

$$c_W = \begin{cases} \frac{\Delta t^2 w_{imp}^2\, g\, h_W}{\Delta x^2} & if\ h_{i-1} > 0 \\ 0. & otherwise \end{cases} \tag{9}$$

with

$$h_E = 0.5\, (h_i + h_{i+1}) \tag{10}$$

$$h_W = 0.5\, (h_i + h_{i-1}) \tag{11}$$

i.e. the water depth at the velocity point u_i and u_{i-1}.

After the system of equations (6) is solved iteratively with the Gauss-Seidel-Method, the velocity at the new time level is computed with

$$u^{n+1} = u^{*} - \Delta t \; g \; w_{imp} \frac{\partial \zeta^{n+1}}{\partial x} \tag{12}$$

Reference:

Backhaus JO (1983) A semi-implicit scheme for the shallow water equations for application to shelf sea modelling. Continental Shelf Research, Vol. 2, No. 4, pp. 243 - 254. https://doi.org/10.1016/0278-4343(82)90020-6