Discrete Structures II with Professor David Cash

Description

Provides the background in combinatorics and probability theory required in design and analysis of algorithms, in system analysis, and in other areas of computer science.

- Topics:
  - Counting: Binomial Coefficients, Permutations, Combinations, Partitions.
  - Recurrence Relations and Generating Functions.
  - Discrete Probability:
    - Events and Random Variables;
    - Conditional Probability, Independence;
    - Expectation, Variance, Standard Deviation;
  - Binomial, Poisson and Geometric Distributions;
  - law of large numbers.
  - Some Topics from Graph Theory:
    - Paths,
    - Components,
    - Connectivity,
    - Euler Paths,
    - Hamiltonian Paths,
    - Planar Graphs,
    - Trees.

Syllabus

Course Overview

This course is an introduction to *probability* and *combinatorics*, including their basic mathematical foundations as well as several non-trivial applications of each. The first topic explores how to think about *random processes* in a rigorous and sensical way, and second is about techniques for *counting* the number of objects fitting a given description. As we will see, the techniques involved in both topics are strongly related.

Roughly, we expect to cover the following list of topics. Lecture summaries will be posted at the bottom of this page.

- Review of prerequisites, set theory, countability
- Random experiments, sample spaces, events, probability measures
- Conditional probability, Bayes' Theorem, Independence
- Combinatorics and Counting
- Recurrences, Generating Functions
- Random Variables
- Bernoulli Trials
- Expectation, Variance
- Markov Chains
- Applications of Probability and Combinatorics

General Information

- Instructor: **David Cash**
- **Course website**
- Recommended textbooks:
  - S. Ross, *A First Course in Probability, 8th Edition*
  - H. Rosen, *Discrete Mathematics and its Applications, 7th edition*
- Lecture meetings: Monday and Wednesday 5:00pm -- 6:20pm in Hill 116
- Recitations:
  - Section 1: Monday 6:55pm -- 7:50pm in SEC 220
  - Section 2: Wednesday 6:55pm -- 7:50pm in ARC 110
  - Section 3: Monday 8:25pm -- 9:20pm in Hill 120
- Instructor office hours: Tuesday 1:30pm-3:00pm in Hill 411 and by appointment
- Teaching assistant: TBA (TAB@cs.rutgers.edu)
- Teaching assistant office hours: TBA in TBA and by appointment

Assignments and Grading

Homework will be assigned roughly every 1.5 weeks. There will be one in-class midterm and one final exam. Final grades will be an average of homework grades (30%), midterm grades (30%), and final exam grades (40%).

January 23rd, 2012 - Lecture: Introduction

Topics

- Review of sets and induction.
- Samples spaces and events.

Introduction

- This title of this class is funny because it tells you nothing about it.
- What this class is really about is **discrete probability** and **combinatorics**.
  - Discrete means "not continuous."
  - Probability means about change and percentage.
  - Combinatorics is the study of combining things.
  - More specifically, it is the study of counting the number of discrete objects fitting some description.
- Probability in computer science
  - Random number generator (RNG)
  - Average case algorithms
  - Information security and cryptography
  - Networking, TCP/IP

Formalizaing Probability

- A set is a collection of objects. Ex:
  - $S = a, b, c$
  - $\mathbb{N} = 1, 2, 3, \ldots$ (natural numbers)
  - $\mathbb{Z} = \ldots, -2, -1, 0, 1, 2, \ldots$ (integers)
- Notation:
  - $\emptyset =$
  - Write `{x | `"statement about `x}`" to mean all `x` satisfying statement.
- Relations between sets
  - `A` and `B` are sets.

- $A \subseteq B$ means "`A` is a subset of `B`"
  - $B \subseteq A$ means "`A` is a superset of `B`"
- Operations on sets
  - $A \cup B$ means "union of `A` and `B`"
  - $A \cap B$ means "intersection of `A` and `B`"
  - $A \setminus B$ means "set difference of `A` and `B`" which means everything in `A` that is not in `B`
  - When a "universe" is understood, we can define A^c as the "complement" of `A`
- Set identities

$$A \cup (B \cap C) =$$

$$(A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

  - Prove: $(A \cup B)^c = A^c \cap B^c$
    1. Show: $(A \cup B)^c \subseteq A^c \cap B^c$
    2. Show: $A^c \cap B^c \subseteq (A \cup B)^c$
  - A good way to remember these is to use Venn diagrams.
- DeMorgan's laws
- Proofs by Induction
  - Let's say you want to prove a statement for all natural numbers `n`.
  - For example, for all `n > 0`, $\sum_{i=0}^{n} 2^i = 2^{n+1}$
    - Call the statements $s_0, s_1, s_2, \ldots$
    - Suppose it's not true for all of them.
    - Key obeservation: If they're not all true, then there is a smallest `i` such that S_i is false.
  - A proof by induction works in 2 steps:
    1. Prove statement for first `n`
    2. Give a conditional proof, prove that for all `n`, if S_{n-1} is true, then S_n
- Countability
  - A set `S` is countable if it is finite, or if it can be pun in one-to-one correspondence with the natural numbers.
- Sample space
  - Fix a "sample space" that represents everything that could happen in our random experiment
  - A sample space is always a set
  - Examples of samples spaces
    1. Tossing a coin: $S = (H, T) = (0, 1)$
    2. Tossing two coins: $S = (HH, HT, TH, TT)$
    3. Rolling a die: $S = (1, 2, 3, 4, 5, 6)$
    4. Rolling 2 dice: $S = ((i, j)$| i, j in$(1, 2, 3, 4, 5, 6))$

January 23rd, 2013 - Reading

2.1 Introduction

> In this chapter, we introduce the concept of the probability of an event and then show how probabilities can be computed in certain situations. As a preliminary, however, we need the concept of the sample space and the events of an experiment.

2.2 Sample Space and Events

- The set of all possible outcomes of an experiment is known as the **sample space** of the experiment and is denoted by S. Examples:

1. The sex of a newborn child: S = {g, b}
2. Horse race with seven horses: S = {all 7! permutations of (1, 2, 3, 4, 5, 6, 7)}
3. Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
4. Tossing two dice: S = {(i, j): i, j = 1, 2, 3, 4, 5, 6}

- Any subset E of the sample space is known as an **event**. For the previous examples,
    1. E = {g} would be having a girl.
    2. E = {all outcomes in S beginning with 3} would be 3 winning the race.
    3. E = {(H, H), (H, T)} would be an event where you get heads first.
    4. E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)} would be an event where the sum of the two rolls is seven.
- A **union** of any two events E and F results in an event with all outcomes in both E and F.
- An **intersection** of any two events E and F results in a event with only those outcomes in both E and F.
- If E and F share no outcomes, making their intersection the empty set (denoted by ∅, they are said to be **mutually exclusive**.
- The complement of E is defined as all events not in E that are in the sample space, denoted by E^c.

January 28th, 2013 - Lecture: Probability Measures

Announcements

- Recitation starts this week.
- Homework 1 will be out tonight and due in a week.

Topics

- Operations on events.
- Abstract definition of probability measure.
- Equivalent definition for countable sample spaces.
- Basic consequences of the definition.

Probability

- Sample space = any set
- "Event" = an subset of the sample space
- Fix a sample space s
- Take any events $E \subseteq S, E_2 S$
- Consider the set $E_1 \cup E_2 \subseteq S$
- Example: Flip two coins
    - $S = HH, HT, TH, TT$
    - $E_1 = HT, HH$
    - $E_2 = TH, HH$
    - $E_1 \cup E_2 = HT, HH, TH$ = "heads came up", "either heads came up on the 1st or 2nd toss"
    - $E_1 \cap E_2 = HH$
- Example: Rolling a die
    - $E_1 =$ "die was rolled" = {1, 3, 5}
    - $E_2 =$ "die roll was divisible by 3" = {3, 6}
    - $E_1 \cap E_2 = 3$
- Note: If $E_1 = \emptyset$ and E_2 is any event, then E_1, E_2 are mutually exclusive.
- Complement: If E is an event, then so is $E^c \subseteq s$ means "not E" or "E didn't happen."
    - For the example of the two coins,
        - E = "1st coin was heads" means that $E^c = TH, TT$
- If $E_1 \subseteq E_2$, then what can we say?

- Proposal: If E1 happened then E2 also happened.
- Example: two dice:
  - First event: "a one was rolled" = {1}
  - Second event: "the dice summed to 3" = {(1,2),(2,1)}
  - The second event is a subset of the first event.

Probability measures

- We've been talking about whether or not something happened, but we'll now move to how *likely* it is for something to happen.
- We want to assign to even event a number between zero and one represent how "likely" that event is.
  - We are going to assume things about randomness and predictibility, we quickly get into philosophy otherwise.
- We want a function P that givens the probabilty P(E) for every E ⊆ S
- Example: A coin flip
  - S = {H, T}
  - P({H}) = 1/2
  - P({T}) = 1/2
  - P({H, T}) = 1
  - P(∅) = 0
- Example: Tossing two coins
  - S + {HH, HT, TH, TT}
  - P({HH}) = 1/4 (and for HT, TH, ...)
  - If E ⊆ S, P(E) = $\frac{|E|}{|S|}$
- Definition: A function P is a probability measure if it maps every event E ⊆ S to a number and satisfies the following three rules:
  1. For every E, O \<= P(E) \<= 1
  2. P(S) = 1
  3. Countable additivity
     - For every sequence $E_1, E_2, \ldots$ of event, if the E_i are all mutually exlcusive, so $E_i \cap E_j = \emptyset$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

- Claim: If P is a probabilty measure, then P(∅) = 0
  - Proof: Take $E_1, E_2, \ldots$ where $E_i = \emptyset$ for every i. The sequence is mutually exclusive.
- When S is finite, we can replace (3) with the "finite additivity": > $P\left(\bigcup *i = 1^n E_1\right) = \sum *i = 1^n P(E_i)$
- Definition: If S is countable then we can say that P is discrete.
  - Every sample set in this class will be countable.
- Claim: If P is discrete, then it is completely determined by its value on one-element sets.
- If you know that P({x}) for every $x \in S$, then you know that P(E) for every E ⊆ S
- Write E = $\{x_1, x_2, \ldots\}$ Let $E_i = x_i$
  - The E_i are mutually exclusive.
  - $\bigcup_{i=1}^{\infty} E_1 = E$
  - So by rule 3,

January 28th, 2013 - Reading

2.3 Axioms of Probability

- One way of defining probability is in relative frequency.
- For some event E in sample space S, we define n(E) to be the number of times in the first n repetitions of the experiment that the event E occurs. Therefore, $P(E) = \lim_{x \to \infty} \frac{nE}{n}$
- While intuitive, how do we know that n(E)/n will converge to some constant limiting value that will be the same for each possible sequence of repetitions of the experiment?
  - Proponents of this definitions say that this is an "axiom" of the system, an assumption.
  - Critics say that this is too complicated an assumption.
- Would it not be more reasonable to assume a set of simpler and more self-evident axioms about probability and then attempt to prove that such a constant limiting frequency does in some sense exist?
- Axioms:
  1. $0 \leq P(E) \leq 1$
     - The probability that the outcome of the experiment is an outcome in E is some number between 0 and 1
  2. $P(S) = 1$
     - The outcome will be a point in the sample space S
  3. $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$
     - For any sequence of mutually exclusive events, the probability of at least one of these events occurring is just the sum of their respective probabilities.

2.4 Some simple propositions (through proposition 4.3)

- Proposition 4.1:

$$P(E^c) = 1 - P(E)$$

- Proposition 4.2:

$$If E \subset F, then P(E) \leq P(F).$$

- Proposition 4.3:

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

January 28th, 2013 - Homework 1

1. Let $n \geq 2$ and $A_1, \ldots, A_n$ be sets in some universe S. In this problem we will give a proof by induction of the identity

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$

   1. (5 points) State and prove the base case for an inductive proof, meaning that the identity is true when $n = 2$.
      1. $\left(\bigcap_{i=1}^{2} A_i\right)^c = \bigcup_{i=1}^{2} A_i^c$
      2. $\left(A_1 \bigcap A_2\right)^c = A_1^c \bigcup A_2^c$ is true by DeMorgan's
   2. (5 points) State and prove the inductive step, where one shows that the identity is true for general $n > 2$, assuming it is true for $n - 1$.
      1. $\left(\bigcap *i = 1^n A_i\right)^c = \bigcup *i = 1^n A_i^c$ assume true for n
      2. $\left(\bigcap *i = 1^{n-1} A_i\right)^c = \bigcup *i = 1^{n-1} A_i^c$ induction for n - 1
      3. $\bigcup *i = 1^n A_i^c \cup A * n + 1^c =$

$$\left(\bigcap_{i=1}^{n} A_i \right) \cup A_{n+1}^c$$

4. $\left(\bigcap *i = 1^n A_i\right)^c =$ $$ \left(\bigcap*{i = 1}^n A_i \right) \cup A_{n + 1}^c$

2. Give sample spaces that model the outcomes for the following experiments. You may use a regular expression or other formalisms that you find convenient. (2 points each)
   1. Rolling 3 dice.

$$S = \{(i, j, k) \mid i, j, k \in$$

$$\{1, 2, 3, 4, 5, 6\}\}!,$$

   2. Rolling a die until an even result comes up, or the die is rolled three times.

$$S = \{(i), (j, i), (j, j, \{i, j\}) \mid$$

$$i \in \{1, 2, 4\}, j \in \{1, 3, 5\}\}$$

   3. Tossing a pair of coins until they both come up tails.

$$S = \{(e_1), (e_2, e_1), (e_2, e_2, e_1),$$

$$(e_2, e_2, e_2, \ldots, e_1) \mid$$

$$e_1 = TT, e_2 \in \{HT, TH, HH\}\}$$

   4. Draw 2 balls from an urn which contains 6 balls, each with a distinct label from $\{1, 2, 3, 4, 5, 6\}$.

$$S = \{i, j \mid i, j \in \{1, 2, 3, 4, 5, 6\}$$

$$\wedge i \neq j\}$$

   5. Draw 1 ball from the same urn, then replace it and draw a ball again.

$$S = \{(i, j) \mid$$

$$i, j \in \{1, 2, 3, 4, 5, 6\}\}$$

3. For each of the sample space, describe the events (as sets) $A \cup B$ and $A \cap B$, when A and B are as follows. (2 points each)
   1. A = "5 is rolled exactly twice" and B = "dice values add to an odd number".

$$A \cup B = \{\{i, i, i\}, \{j, j, i\} \mid$$

$$i \in \{1, 3, 5\}, j \in \{2, 4, 6\}\}$$

$$A \cap B = \{\{5, 5, i\} \ \mid$$

$$i \in \{1, 3\}\}$$

   2. A = "1 comes up exactly twice" and B = "3 comes up exactly twice".

$$A \cup B = \{(1, 1, i), (j, 1, 1), (1, j, 1) \mid$$

$$i \in \{2, 3, 4, 5, 6\}$$

$$\wedge j \in \{3, 5\}\}$$

$$\cup\{(3,3,i),(j,3,3),(3,j,3) \mid$$

$$i \in \{1,2,4,5,6\}$$

$$\wedge j \in \{1,5\}\}$$

$$A \cap B = \emptyset$$

3. A = "both coins come up heads at the same time at some point" and B = "both coins come up tails at the same time at some point,"

$$A \cup B = S =$$

$$\{(e_1),(e_2,e_1),(e_2,e_2,e_1),(e_2,e_2,e_2,\ldots,e_1)$$

$$\mid e_1 = TT, e_2 \in \{HT,TH,HH\}\}$$

$$A \cap B = A = \{(e_1,e_2,e_3,\ldots,e_n)$$

$$\mid \ e_n = TT \wedge e_i \in \{HT,TH,HH\}$$

$$\wedge 1 \leq i \leq n-1 \wedge$$

$$n \in N \wedge \exists j e_j = HH$$

$$\wedge 1 \leq j \leq n-1\}$$

4. A = "1 is drawn at least once" and B = "1 is drawn twice".

$$A \cup B = A = \{\{i,j\} \mid$$

$$i,j \in \{1,2,3,4,5,6\} \wedge$$

$$(i \neq j) \wedge (i = 1 \oplus j = 1)\}$$

$$A \cap B = \emptyset$$

5. A = "1 is drawn at least once" and B = "1 is drawn twice"

$$A \cup B = \{\{1,1\},\{1,2\},\{1,3\}$$

$$,\{1,4\},\{1,5\},\{1,6\}\}$$

$$A \cap B = \{\{1,1\}\}$$

January 30th, 2013 - Lecture: Begin counting and its applications

Topics

- Finish proof about probability of a union.
- Counting: The multiplication rule, permutations of n objects and k-out-of-n objects.
- The birthday problem.
- Permutations when some objects are indistinguishable.
- Counting walks on grids.

Introduction

- Claim: If P is a probability measure on S and $E \subseteq S$, $F \subseteq S$ are events, then $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Proof:
    - $P(E \cup F)$
        - The first step in a lot of these proofs is to "disjointify" $E \cup F$
        - The handy way to do this is to write that as $E \cup (E^c \cap F) = P(E) + P(E^c \cap F)$
    - Now we need $P(E^c \cap F) = P(F) - P(E \cap F)$
        - Equivilently, $P(F) = P(E \cap F) + P(E^c \cap F)$. This is true because:
            - $F = (E \cap F) \cup (E^c F)$
            - $E \cap F$ and $E^c \cap F$ are mutually exclusive.

Counting

- Multiplication rule: If I have n_1 choices in an experiment, and if I have for each of my first choices another n_2 choices, and the same n_3 choices, up to n_r. In that setting, then I have $n_1 \times n_2 \times n_3 \times \ldots \times n_r$ total choices.
    - Example: How many license plate numbers are there if they consist of 6 characters, the first six being letters and the last three being numbers? (ABX 161)
        - 26 choices for first letter
        - 26 choices for the second letter
        - 26 choices for the third letter
        - 10 choices for the first number
        - 10 choices for the second number
        - 10 choices for the third number
        - Therefore, 26 x 26 x 26 x 10 x 10 x 10 =
    - Example: What if no repetitions are allowed?
        - 26 possibilities
        - 25 possibilities
        - 24 possibilities
        - 10 possibilities
        - 9 possibilities
        - 8 possibilities
        - 26 x 25 x 24 x 10 x 9 x 8 =
- Permutations: "How many ways can I arrage n distinct objects in a line? (i.e. in order)
    - Answer: Build an arrangement by picking 1st element, then 2nd element, etc, euntil nth.
    - Definition: n! = n(n + 1) ... x 2 x 1
        - 0! = 1 (There is only one way to arrange zero items)
    - What is we only arrange k \< n of the objects?
        - $n - k + 1 = \frac{n!}{(n-k)!}$
    - Example: A class of 6 men and 4 women
        1. How many ways can they be ranked?
            - 10!
        2. What if you rank men and women seperately?
            - 6! + 4!
    - Birthday "Paradox" (Surprise)
        - Assume birthdays are random, you can be born on any day, and ignore February 29th.
        - What is the size of the group required to give us a 50% chance of getting a match?
        - Use your intuition: There are 365 days, so probably around 1/365?
        - You'd expect 100 people, 200 people, once you approach half the number of days.
        - Let's study this using our probability.
        - Take n people, n random birthdays.
        - Sample space will be S = {1, ..., 365}^n
        - $E \subseteq S$ represents "there is at least one match"

- $P(E) = 1 - P(E^c)$
- It's kind of hard to account for E, with multiple matches, all matches, etc.
- The complement of E is "there are no matches."
- How big is E complement?
- $E^c \subseteq S$ is a lists of n numbers without match.
- $|E^c| = 365 \times 364 \times 363 \times (365 - n + 1)$
- $P(E^c) = \frac{|E^c|}{|S|} = \frac{365 \times 364 \times 363 \times (365-n+1)}{365^n}$
- $\approx .507|n = 23$
- $\approx .97|n = 50$
- $\approx > .999999|n = 100$
- Intuition: It's not the number of people that matters, it's the number of pairs, $O(n^2)$ type of growth.

- How many distinct strings can be formed from the letters in "remember"? Answer:
  - There are 8! ways to arrange the letters in total.
  - There are 3! ways to arrange the Es without changing the word.
  - There are 2! ways to arrange the Rs without changing the word.
  - There are 2! ways to arrange the Ms without changing the word.
  - There is 1 way to rearrange the B without changing the word.
  - $\mathrm{R_1 E_1 M_1 E_2 M_2 B_1 E_3 R_2}$
  - $\frac{8!}{3! \times 2! \times 2! \times 1!}$
  - How many times does "remember" show up in the list?
  - Permutations when some elements are interchangable: When putting n objects in order if n_1 are interchangabl and N_2 are interchangable, and ..., n_r are interchangable, the number of permutations $\frac{n!}{n_1!n_2!...n_r!}$

January 30th, 2013 - Recitation

1. Give an inductive proof of the identity: $\left(\bigcup *i = 1^n\right)^c = \bigcap *i = 1^n\left(A_i^c\right)$. You may use the fact that for any two sets A and B, $(A \cup B)^c = A^c \cap B^c$.
2. What are the sample spaces for the following experiments?
   - Throwing three dice
   - Flipping a coin until either (a) "heads" occurs, or (b) the coin is flipped 8 times.
   - Throwing a die until "6" occurs.
   - Drawing 2 balls from an urn, where the urn contains 5 balls, each with a distinct label from 1 to 5.
   - Picking one ball from the same urn as above, and then putting it back into the urn and again picking a ball.
3. Which of the sample spaces from Exercise 2 are finite? Which are countable? (Justify your answers.)
4. For the sample spaces you presented in Exercise 2, describe the events $A \cup B$ and $A \cap B$ when A and B are:
   - A = "at least one 6" and B = "an odd total".
   - A = "heads occurs twice" and B = "tails occurs twice"
   - A = "5 occurs an infinite number of times" and B = "6 occurs"
   - A = "5 is drawn at least once" and B = "5 is drawn twice"
   - A = "5 is drawn at least once" and B = "5 is drawn twice"

January 30th, 2013 - Reading

1.2 The Basic Principle of Counting

> The basic principle of counting will be fundamental to all our work. Loosely put, it states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are mn possible out- comes of the two experiments.

- The basic principle of counting: Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible out- comes of the two experiments.
- The generalized basic principle of counting: If r experiments that are to be performed are such that the first one may result in any of n1 possible outcomes; and if, for each of these n1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \cdot n_2 \ldots n_r$ possible outcomes of the r experiments.

Example 2a

- A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?
- Solution. By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices.

Example 2b

- A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?
- Solution. We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. It then follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

Example 2c

- How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
- Solution. By the generalized version of the basic principle, the answer is $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$.

Example 2d

- How many functions defined on n points are possible if each functional value is either 0 or 1?
- Solution. Let the points be 1,2,...,n. Since $f(i)$ must be either 0 or 1 for each i = 1, 2, . . . , n, it follows that there are 2_n possible functions.

Example 2e

- In Example 2c, how many license plates would be possible if repetition among letters or numbers were prohibited?
- Solution. In this case, there would be $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.

1.3 Permutations

- How many different ordered arrangements of the letters a, b, and c are possible?
- By direct enumeration we see that there are 6, namely, abc, acb, bac, bca, cab, and cba.
- Each arrangement is known as a permutation. Thus, there are 6 possible permutations of a set of 3 objects.

$$n(n-1)(n-2)\ldots 3 \times 2 \times 1 = n!$$

Example 3a

- How many different batting orders are possible for a baseball team consisting of 9 players?
- Solution. There are 9! = 362,880 possible batting orders.

Example 3b

- A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.
  1. How many different rankings are possible?
  2. If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?
- Solution.
  1. Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is 10! = 3,628,800.
  2. Since there are 6! possible rankings of the men among themselves and 4! possi- ble rankings of the women among themselves, it follows from the basic principle that there are (6!)(4!) = (720)(24) = 17,280 possible rankings in this case.

Example 3c

- Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?
- Solution. There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4! 3! 2! 1! possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is 4! 4! 3! 2! 1! = 6912.

Example 3d

- How many different letter arrangements can be formed from the letters PEPPER?
- Solution. We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3P's and the 2E's are distinguished from each other. However, consider any one of these permutations—for instance, P1P2E1P3E2R. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form PPEPER.

Example 3e

- A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?
- Solution: There are $\frac{10!}{4!3!2!1!} = 12600$ possible outcomes

Example 3f

- How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?
- Solution: There are $\frac{9!}{4!3!2!1!} = 1260$ possible outcomes

2.4 Some Simple Propositions (after proposition 4.3)

February 4th, 2013 - Lecture: Combinations

Topics

- Deriving the formula for combinations.
- Basic identities and examples.
- The antenna problem (Ex. 4c of Ross).
- Counting divisions where all sets non-empty and where sets are allowed to be empty.

Combinations

- How many ways can we choose k out of n objects when order does not matter?
- Take S = $\{x_1, \ldots, x_n\}$. How many k-element subsets does S have?
- Call the number of k element subsets C(n,k)
- In general,
  - C(n,1) = n
  - C(n,n) = 1
  - C(n,2) = # of 2-element subsets
- S = $\{x_1, \ldots, x_n\}$
  - From every 2 element list -> n(n - 1)
- Number of subsets will equal

$$\frac{\text{number of 2 element sets}}{2} = \frac{n(n-1)}{2}$$

- C(n,k) for general k (S = $\{x_1, \ldots, x_n\}$)
  - $\frac{n!}{(n-k)!}$ ways to list k-out-of-n elements
  - Every k-element subset is represented k! times in big list
  - Big list has $\frac{n!}{(n-k)!}$ entries
  - $C(n,k) \times k! = \frac{n!}{(n-k)!}$
- Notation: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

$$\binom{n}{1} = \frac{n!}{(n-1)!1!} = n$$

- Claim: $\binom{n}{k} = \binom{n}{n-k}$

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} =$$

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

- The number of possible 5-card poker hands is: $\binom{52}{5} = 2578960$

February 4th, 2013 - Reading

1.4 Combinations (through example 4c)

- How many groups of *r* objects can be formed from a total of *n* objects?
  - How many groups of 3 could be formed from the 5 items *A, B, C, D,* and *E*.
  - Since there are 5 ways to select the initial item, 4 ways to select the next item, and 3 ways to select the final item, there are thus $5 \times 4 \times 3$ ways of selecting the group of 3 (order relevant).

- In this formulation, each group as a set will be counted 6 times.
- The total number of groups that can be formed is:

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

- In general, *n(n - 1) … (n - r + 1)* represents the number of different ways that a group of *r* items could be selected from *n* items when order is relevant.
- It follows that the number of different groups of *r* items that could be formed from a set of *n* items is

$$\frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

- Notation and terminology
  - We define $\binom{n}{r}$, for $r \leq n$ by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

  and say that $\binom{n}{r}$ represents the number of possible combinations of *n* objects taken *r* at a time.

Example 4a

- A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?
- Solution: There are $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible combinations.

Example 4b

- From a group of 5 women and 7 men, (1) how many different committees consisting of 2 women and 3 men can be formed? (2) What if 2 of the men are feuding and refuse to serve on the committee together?
  1. Solution: $\binom{5}{2}\binom{7}{3} = 350$ possible combinations of 2 women and 3 men.
  2. I'll ask about this problem in office hours.

Example 4c

- Consider a set of *n* antennas of which m are defective and *n* − *m* are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?
- Solution:

$$\binom{n-m+1}{m}$$

  possible orders in which there is at least once functional antenna between any two defective ones.

1.6 The Number of Integer Solutions of Equations

February 6th, 2013 - Lecture: Binomial and Multinomial Coefficients

Topics

- Pascal's relationship and Pascal's triangle.
- The binomial theorem and proofs by combinatorial reasoning and induction.

- Some consequences and examples.
- Multinomial coefficients and their applications to dividing groups and counting strings.

Introduction

- For all *n, k >= 0*.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- Proof:
  - $\binom{n}{k}$ = the number of k-elements of subsets of $x_1 \ldots x_n$.
  - Observation: Take any of the $x_1 \ldots x_n$, say x_n . Then every k-element subset either cointains x_n or it doesn't.
  - $\binom{n-1}{k-1}$ is the number that do contain x_n and $\binom{n-1}{k}$ is the number that don't.
  - The number that do contain x_n is $\binom{n-1}{k-1}$.
- Pascal's Triangle
  - Put $\binom{0}{0}$ at the top.
  - The next level is two instances of $\binom{1}{0}$
  - ...

```
                1
            1       2       1
        1       3       3       1
1       4       6       4       1
```

Binomial Theorem

- For all *x, y*, all *n >= 1*,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Intuitive Combinatorial Proof (Simpler)
  - When you expand $(x+y)^n$, it equals $(x+y)(x+y)\ldots(x+y)$"*n*-times"
  - One way to compute all "monomials" in product is to start with *x*, and then pick all the way done, one per time, and repeat the exercise until all combinations are exhausted, and you have every monomial.
  - Which monomials show up in sum?
  - $x^a y^b$ shows up $\binom{n}{a}$
- Proof by induction
  - We've already checked *n = 1*.
  - Now assume true for *n - 1* and prove for *n > 1*

$$(x+y)^n = (x+y)(x+y)^{n-1}$$

$$= (x+y) \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$$

$$= x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} +$$

$$= y\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}$$

- Write *i* for *k + 1*:

$$\sum_{i=1}^{n}\binom{n-1}{i-1}x^i y^{n-1} +$$

$$\sum_{i=1}^{n}\binom{i}{n-1}x^i y^{n-1}$$

$$= x^n + \sum_{i=1}^{n-1}\left[\binom{n-1}{i-1} + \binom{n-1}{i}\right] \times$$

$$x^i y^{n-i} + y^n$$

- Corollary: For any *n >= 1*, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$
  - Proof:

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}$$

  - Intuition: This sum should be the number of wqys to pick a subset of any size from a set of size *n*.
    - To pick a subset, we have *n* decisions where we decide if each x_i is in the subset or not.

Multinomial Coefficients

- Arise from the following type of problem:
  - Given a set of *n* distinct items, you need to divide them up into *r* groups of given sites $n_1, n_2, \ldots, n_r$ where $n_1 + n_2 + \ldots + n_r = n$.
  - How many ways can this be done, for a given *n*, *r*, and n_1 up to n_r where there all add up to *n*.
- This is very close to something we've already seen, and there are two ways of thinking about it.
  1. How many choices for 1st group? Then into the second, then into the third ...

$$\binom{n}{n_1}$$

     ways to choose the first group

$$\binom{n-n_1}{n_2}$$

     ways to choose the second group, etc, and for the r^{th} group:

$$\binom{n-n_1-n_2-\ldots-n_{r-1}}{n_r = 1}$$

     so the multiplication principle tells us that if we multiply these all together, we get our answer. It cancels nicely:

$$= \frac{n!}{(n-n_1)!n!} \cdot \frac{n-n_1}{(n-n_1-n_2)!n!} \ldots$$

and this cancels to

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

- Example: There are 10 TAs in the department, and we need 3 TAs for OSs, 2 for algorithms, and 5 for databases. There there are:

$$\frac{10!}{3!2!5!}$$

ways to assign them, which is 2520.

February 6th, 2013 - Recitation

1. Determine the number of vectors $x_1, x_2, x_3, \ldots, x_n$ such that each x_i is either 0 or 1.

$$\sum_{i=1}^{n} x_i \leq k$$

(constraint)
   - Solution

$$x_i \in \{0, 1\}$$

     - The set should be less than 2^n
     - Three cases:
       1. *n* = *k*: everything should be 1
       2. *n* \|< *k*: everything something 0
       3. *n* > *k*: the only interesting *k*, $n > k$
2. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} +$$

$$\binom{n}{1}\binom{m}{r-1}\ldots+\binom{n}{r}\binom{m}{0}$$

February 6th, 2013 - Reading

1.5 Multinomial Coefficients

- Notation
   - If $n_1 + n_2 + \ldots + n_r = n$, we define $n \choose n_1
     - n_2 + … + n_r $ by

$$\binom{n}{n_1 + n_2 + \ldots + n_r} = \frac{n!}{n_1! + n_2! + \ldots + n_r!}$$

+ Thus, $\binom{n}{n_1+n_2+\ldots+n_r}$ represents the number of possible divisions of *n* distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

- The multinomial theorem

$$(x_1 + x_2 + \ldots + x_r)^n =$$

$$\sum_{(n_1,\ldots,n_r):}$$

February 7th, 2013 - Homework 2

1. (8 points) Prove that $P(A \cup B) \leq P(A) + P(B)$ for any events A and B. Prove the general version by induction, which says that if $A_1, \ldots, A_n$ are events then $P(\bigcup_{i=1}^{n} A_i) \leq P(\sum_{i=1}^{n} A_i)$. When does this inequality become an equality?
   - Base case

     $$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

   - Assume it holds for a general case $(n = k)$

     $$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

     $$P(A_1 \cup A_2 \cup \ldots \cup A_k) \leq P(A_1) + P(A_2) + \ldots + P(A_k)$$

   - Prove that it hold for next case using your assumption.

     $$P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq \sum_{i=1}^{k+1} P(A_i)$$

     $$P(A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}) \leq P(A_1) + P(A_2) + \ldots + P(A_k) + P(A_{k+1})$$

     $$(A_1 \cup A_2 \cup \ldots \cup A_k) = C$$

     $$P(A_1) + P(A_2) + \ldots + P(A_k) = D$$

     $$P(C \cup A_{k+1}) \leq P(D) + P(A_{k+1})$$

   By the base case, this is true, and expression is proven for all events.
2. (4 points) If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{5}$, and $P(A \cup B) = 3/5$, what are $P(A \cap B)$, $P(A^c \cup B)$, and $P(A^c \cap B)$?
   - $P(A \cap B) = \frac{1}{10}$
   - $P(A^c \cup B) = 3/5$ because the probability of A is the same as A^c
   - $P(A^c \cap B) = \frac{1}{10}$
3. (3 points) How many elements are there in the set
   $\{x : 10^7 \leq x \leq 10^8$, and the base 10 representation of x has no digit used twice$\}$?
   - $10^7 = 10000000$, and $10^8 = 100000000$
   - The smallest number possible is 12345678 , the greatest number possible is 98765432
   - For the first number in any element, the first option is {1, 2, 3, 4, 5, 6, 7, 8, 9}.
   - Then for the every second number, it's every number besides the first choice plus 0.
   - The "tree" looks as follows, where $i \neq j \neq k \neq l \neq m \neq n \neq o \neq p$:

     1. $$1ijklmno$$

        where

        $$i, j, k, l, m, n, o \in 0, 2, 3, 4, 5, 6, 7, 8, 9$$

2. $$2ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 3, 4, 5, 6, 7, 8, 9$$

3. $$3ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 2, 4, 5, 6, 7, 8, 9$$

4. $$4ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 2, 3, 5, 6, 7, 8, 9$$

5. $$5ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 2, 3, 4, 6, 7, 8, 9$$

6. $$6ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 2, 3, 4, 5, 7, 8, 9$$

7. $$7ijklmno$$

where $i, j, k, l, m, n, o \in 0, 1, 2, 3, 4, 5, 6, 8, 9$ $

8. $$8ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 2, 3, 4, 5, 6, 7, 9$$

9. $$9ijklmno$$

where

$$i, j, k, l, m, n, o \in 0, 1, 2, 3, 4, 5, 6, 7, 8$$

- Every “node” (a) through (i) is $\binom{9}{1}$.
- When you pick the first number, you still have 9 choices because zero is added. When you choose the second number, you have 8 choices because a number is taken out and non are put back into the set of choices. When you pick your third number, you have 7 choices because a number is taken out and none put back in. ...
- So for integer strings i through o:

$$[i] \quad [j] \quad [k] \quad [l] \quad [m] \quad [n] \quad [o] \quad [p]$$

- The number of elements is equal to

$$\binom{9}{1} \times \binom{9}{1} \times \binom{8}{1} \times \binom{7}{1} \times \binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} \times \binom{3}{1}$$

4. (3 points) An army output has 19 posts to staff using 30 indistinguishable guards. How many ways are there to distribute the guards if no post is left empty?
   - This is a "stars and bars" problem, where the number of "stars" k is equal to 30, and the number of "bars," "bins," or "posts" n is equal to 19.

$$\binom{k-1}{n-1} = \binom{30-1}{19-1} = \binom{29}{18} = 34597290$$

5. (1 point) What is the coefficient of $x^{10}y^{13}$ when $(x+y)^{23}$ is expanded?
   - This problem requires the binomial theorem, where $n = 23$:

$$(x+y)^{23} = \sum_{k=0}^{23} \binom{23}{k} x^k y^{23-k}$$

   - When $k = 10$, $23 - k$ will equal 13.

$$\binom{23}{10} x^{10} y^{13}$$

   - So the coefficient for $x^{10}y^{13}$ will be

$$\binom{23}{10} = 1144066$$

6. (4 points) What is the coefficient of $w^9x^{31}y^4z^{19}$ when $(w+x+y+z)^{63}$ is expanded? How many monomials appear in the expansion?
   - This is a multinomial coefficient problem where $k = 4$ and $n = 63$:

$$(a_1 + a_2 + \ldots + a_k)^n = \sum_{n_1, n_2, \ldots, n_k \geq 0 \; n_1 + n_2 + \ldots + n_k = n} \frac{n!}{n_1! n_2! \ldots n_k!} a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k}$$

   - Set $n_1 = 9$, $n_2 = 31$, $n_3 = 4$, and $n_4 = 19$.

$$\left(\frac{63!}{9!31!4!19!}\right) \times w^9 x^{31} y^4 z^{19}$$

   - Therefore, the coefficient is

$$\frac{63!}{9!31!4!19!}$$

   - The number of monomials in a multinomial coefficient can be expressed as a "stars and bars" problem.

$$\binom{66}{3}$$

7. (6 points) Let p be a prime number and $1 \leq k \leq p-1$. Prove that $\binom{p}{k}$ is a multiple of p. Show that this is not true if p is not prime.

$$\binom{p}{k} = \frac{p(p-1)\times\ldots\times(p-k+1)}{1\times 2\times\ldots\times(k-1)\times k} = p\binom{p-1}{k}$$

$$\binom{4}{2} = 6$$

6 is not a multiple of 4 and is not prime.

8. Verify that for any $n \geq k \geq 1$

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

Then give a combinatorial argument for why this is true.

- Observe that the binomial coefficient with two for all reals:

$$\binom{r}{2} = \frac{r(r-1)}{2}$$

- Apply fact to both sides:

$$\frac{n(n-1)}{2} = \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2}$$

- Multiply everything by two

$$n(n-1) = k(k-1) + 2k(n-k) + (n-k)(n-k-1)$$

- Expand out

$$n(n-1) = k^2 - k + 2kn - 2k^2 + k + k^2 - n - 2kn + n^2$$

- Simplify

$$n(n-1) = -n + n^2$$

- Change form

$$n(n-1) = n(n-1)$$

- Combinatorial argument:
  - $\binom{n}{2}$ is the number of ways we can arrange n objects into groups of 2.
  - Splitting n into 2, one of those groups will be of size k, making the other set of size $n-k$
  - Using the new partitions k and $n-k$, arrange n "things" into groups of two.
  - The first partition is of length k, resulting in $\binom{k}{2}$ ways, and for the second part, we have $n-k$ choose 2 ways, explaining the first and last terms.
  - Now choose 2 "things", one from different groups of 2, we can choose 1 of the k objects and all of $n-k$ "things", which is equal to $k(n-k)$.

February 11th, 2013 - Lecture: Multinomial Theorem, Counting Problems, Inclusion-Exclusion

Topics

- The multinomial theorem and applications

- The "tree method" for various counting problems involving card hands and committees
- Inclusion-exclusion

Lecture

- **Multinomial coefficients**: see reading 1.5 in February 7th reading
  - **Multinomial theorem**: see reading 1.5 in February 7th reading
    - Sum runs over all vectors of non-negative integers that sum to *n*.
    - Example: $(x_1 + x_2 + x_3)^2$ =

$$\sum_{(n_1,n_2,n_3):n_1,n_2,n_3=2} \binom{2}{n_1, n_2, n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

$$= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 +$$

$$\binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 +$$

$$\binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 +$$

$$\binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 +$$

$$\binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 +$$

$$\binom{2}{0,1,1} x_1^0 x_2^1 x_3^1$$

    - Example: What is the coefficient of $x^3 y^4 z$ when $(x + y + z)^1 3$ is expanded? It corresponds to:

$$\binom{13}{3,9,1} = 2860$$

    - Question: How many terms are in the sum? Equivilently, how many monomials are in the expansion? (If *n = 2*, *r = 3*, then 6)
- **More on counting**
  - Deck of cards: 52 cards, 4 suits, 13 ranks, one of each suit/rank combination
  - Number of 5 card hands:

$$\binom{53}{5}$$

  - Number of flushes:
    - 4 different suits for flushes
    - For each suit, the number of flushes is $\binom{13}{5}$
    - Therefore,

$$4 \times \binom{13}{5}$$

- Example: Suppose a committe of *k* people from a group of 7 women and 4 men.
  - How many ways can form the committee if it has:
    1. 3 women and 3 men exactly

$$\binom{7}{3} \times \binom{4}{3}$$

    2. The committee is any size, but it has equal number of men and women. Choices:
        1. 1 each

$$= \binom{7}{1} \times \binom{4}{1} +$$

        2. 2 each

$$\binom{7}{2} \times \binom{4}{2} +$$

        3. 3 each

$$\binom{7}{3} \times \binom{4}{3} +$$

        4. 4 each

$$\binom{7}{4} \times \binom{4}{4}$$

    3. If the commitee has 4 people, at least 2 of which are women?
        - Steps:
            1. Pick the number of women
            2. Pick the women
            3. Pick men for the remaining spots.
        - Possibilities:
            1. 2 women

$$= \binom{7}{2} \times \binom{4}{2} +$$

            2. 3 women

$$\binom{7}{3} \times \binom{4}{1} +$$

            3. 4 women

$$\binom{7}{4} \times \binom{4}{0}$$

- Example: How many hands have exactly 3 aces? "Pick 3 aces, then pick 2 non-aces"

$$\binom{4}{3} \times \binom{48}{2}$$

- Example: How many full houses?
  - Pick the rank of 3-of-a-kind (AAA, 222, ...)
  - Pick the rank of 2-of-a-kind
  - Pick the B-of-a-kind
  - Pick the 2-of-a-kind

$$13 \times 12 \times \binom{4}{3} \times \binom{4}{2}$$

- Probability theory

$$P(A \cup B \cup C) =$$

$$P(A) + P(B) + P(C) -$$

$$P(A \cap B) - P(A \cap C) -$$

$$P(A \cap C) + P(A \cap B \cap C)$$

- If P is a probability measure an $A_1, \ldots A_n$ are events:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) =$$

$$\sum_{i=1}^{n} P(A) - \sum_{i<j} P(A_i \cap A_j) +$$

$$\sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots$$

$$+(-1)^{n+1} P(A_1 \cap \ldots A_n)$$

- Include/Exclude Set Version
  - Some formula, but with set-size intead of P(1)

$$|A_1 \cup \ldots \cup A_n| =$$

$$\sum |A_1| - \sum |A_i \cap A_j$$

$$\sum A_i \cap A_j \cap A_k$$

- Symmetry: For any *i*, |A_i| is the same.

$$A_i = \binom{39}{5}$$

  - For any $i \leq j$, $|A_i \cap A_j|$ is the same

$$\binom{26}{5}$$

February 11th, 2013 - Reading

2.4 Some Simple Propositions (from page 31)

February 13th, 2013 - Lecture

Include/Exclude Formula

- If $E_1, \ldots, E_n$ be events, then

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) =$$

$$\sum_{i=1}^{n} P(E_1) - \sum_{i} \leq jP(E_i \wedge E_j) +$$

$$\sum_{i<j<k} P(E_i \cap E_j \cap E_K - \ldots$$

$$(-1)^{n+1} P(E_i \cap \ldots E_n)$$

de MontMort's Problem (1713)

- Have n students seated in class, no empty seats.
- Students 1 through n.
- Everyone stands up, all are assigned, random seats.
- What's the probability that somebody gets own seat?

Dearrangements

- $E =$ "someone gets own seat"
- $E_i =$ "student i gets own seat"

$$E = E_1 \cup E_2 \cup \ldots \cup E_n$$

Exploit "symmetry"

$$P(E) = P(E_1 \cup E_2 \cup \ldots \cup E_n) =$$

- Observe what is $P(E_1) = \frac{1}{n} = \frac{n-1}{n!}$
- Moreover, $P(E_1) = \frac{1}{n}$ for any i.
- The number of arrangements that put 1 and 2 in seats 1 and 2 over the number of all possible arrangements is

$$\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

- Works for any i and j:

$$P(E_1 \cap E_j \cap E_k) = \frac{1}{n(n-1)(n-2)}$$

Conditional probability

- How we update "beliefs" as new information comes to light.
- Determine for events A, B, P(A/B) =

$$\frac{P(A \cap B)}{P(B)}$$

```
+ When B does not equal zero.
```

Explanation #1

$$P(\{1\})|B)$$

Explanation #2 - "Frequentist"

- We have some experiment we are going to repeat, generating data.

  ```
  01011010 A
  11101011 B
  00010110
  11010100
  ...
  ```

- Count number of times B happens
- Count number of times that A happens.

Example - Flipping two coinis

- S = {HH, TH, HT, TT}, all 1/4
- What's the probabilty of getting 2 heads, given that first toss was heads? It's 1/2
  - A = {HH}
  - B = {HT, HH}

$$P(A|B) = \frac{P(\{HH\})}{\{HT, HH\}}$$

- What's the probability of HH, given that at least one heads came up?
  - A = {HH}
  - B = {HT, TH, HH}

$$P(A|B) = \frac{P(HH)}{P(HT, TH, HH)} =$$

$$\frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{1}{3}$$

Example 2 - Fuses

- Suppose we have 7 fuses, and 5 are working and 2 are broken.
- We want to find the broken ones, we're going to pick them one at a time, and we test.
  - What's the probability that we only need 2 tests?
    - Define E_1 as "first test shows up broken.
    - E_2 as "second ..." ...

Gigerenzer's Experiment

- He did the following experiment:
  - He got the doctors to agree on some numbers.
  - They agreed a given patient (say 40-50 year old woman), you know nothing about her except she's in good health.

    - She has cancer with probability 0.008.
    - They had a cancer test that, if patient has cancer, returns positive .9 rightly.
    - If they don't, it will return positive .07 of they time.
- He asked 24 doctors:
    - Suppose a woman's test is positive, what is the probability that she has cancer?
        - First once said he didn't konw.
        - The other 24 returned this:

```
8 said <= 10%
8 said    50% - 80%
8 said    90%
```

- Lets do some math
    - C = "patient has cancer"
    - T = "test is positive"

$$P(C) = 0.008$$

$$P(T|C) = 0.9$$

$$P(T|C^c) = 0.07$$

- What's the probability of illness given that the test came up positive? $P(C|T) = ?$

$$P(C|T) = \frac{P(C \cap T)}{P(T)}$$

$$P(T|C) = \frac{P(C \cap T)}{P(C)}$$

- Want P(T)
- **Lemma**: For any event T, C

$$P(T) = P(T \cap C) + P(T \cap C^c)$$

- Answer: 9.4%

Intuition, not a proof at all

- Say you take 1000 women,
    - Should have 8 with cancer ill.
    - How may should test positive? \~7 will be positive

February 13th, 2013 - Recitation

1. Prove the following is true:

$$\binom{n}{2} = \binom{k}{2} + K(n-k) + \binom{n-k}{2}$$

    - What is $\binom{n}{2}$?
    - We're going to have to put a *k* into the left hand side.
    - So you have n object, I can select 2 object as n choose 2
    - I could do the same selection of 2 objects by partitioning them into n and n-k objects.

- Or I could select one from one side, etc.
- Since I am selecting two objects, what are the possibilities.
- Both of them could be from the left or right, or one could be on both sides.

2. Prove:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

   - If you expand out the left hand side:

$$1 \times \binom{n}{1} + 2 \times \binom{n}{2} + \ldots + n\binom{n}{n} = n2^{n-1}$$

3. Let n be the number of people and we have to form a commitee of arbitrary size and select a chair from that committee.
   - Method 1

$$k\binom{n}{k}$$

     is the number of ways to select a commite of size k and a chair
     - Hence, the total number of ways to form a commite of arbitrary size is

$$\sum_{k=1}^{n} k\binom{n}{k}$$

   - Method 2
     - We can first select the chair in n-ways. From the remaining n - 1 people we can select a subset in $2^{n-1} ways.

4. Prove:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{1} = 0$$

February 18th, 2013 - Lecture: Bayes's Theorem

Topics:

- Bayes's theorem: Statement, examples, applications to spam filtering and other fields.
- General version of Bayes's theorem and the red/black hat problem.

Boxes and balls

- **Example**: There two boxes, the first contains 2 green balls and 7 res and the second contains 4 green and 3 red. I pick a random box, then a ball from that box. If I pick a red ball, what is the probability that I picked the first box?
  - E = "red ball was picked"
  - F = "1st box was picked

$$P(F|E) = \frac{P(E \cap F)}{P(E)} =$$

$$\frac{\frac{7}{18}}{\frac{38}{63}}$$

(I)

$$P(E \cap F) = P(E|F) \times P(F) =$$

$$\frac{7}{9} \times \frac{1}{2} = \frac{7}{18}$$

(II)

$$P(E) = P(E \cap F) + P(E \cap F^c) =$$

$$\frac{7}{18} + P(E|F^c) \times P(F^c) =$$

$$\frac{7}{18} + \frac{3}{7} \times \frac{1}{2}$$

"Bayesian Reasoning"

- A method for updating "belief"/"uncertainty" based on evidence.
- This is useful when you want to know the probability of some outcome given some evidence when you know the probability of the outcome and you also konw the quality of your evidence, and the probability of the evidence given the outcome, and finally, the probability of the evidence showing up if the outcome didn't happen.
    1. P(outcome|evidence)
    2. P(outcome)
    3. P(evidence|outcome)
    4. P(evidence|outcome^c)

Bayesian Spam Filtering

- Outdated but interesting anyway
- The idea is that you estimate P(Email contains some word w | it's spam)
    - Rolex
    - Viagra
    - Rutgers
    - Cash
    - LaTeX
- Then the probability of the P(Emails contains the same word w | it's not spam)
- P(Email is spam)
- Using these, there is Bayes theorem, which computes P(outcome|evidence) = P(spam | contains w)
- The attack is called Bayesian poisoning
    - You retrain this when you email it everyday.
    - Spammers pass it the words you are interested in, which you mark as spam, and then the ones you aren't interested in eventually pass.

Bayes Theorem

- If E and F are events with $P(E) \neq 0$ and $P(E) \neq 0$, then

$$P(F|E) = \frac{P(E|F) \times P(F)}{P(E|F) \times P(F) + P(E|F^c) \times P(F^c)}$$

1. $P(E \cap F) = P(E|F) \times P(F)$
2. $P(E) = P(E|F) \times P(F) + P(E|F^c) \times P(F^c)$

General version of Bayes theorem

- For multiple outcomes
- Let $F_1 \ldots F_n$ be events that
  1. Are mutually exclusive
  2. $F_1 \cup \ldots \cup F_n = S$
  3. $\forall i P(F_i \neq 0)$
- Then for any j

$$P(F_j|E) =$$

$$\frac{P(E|F_j) \times P(F_j)}{\sum_{i=1}^{n} P(E|F_i) \times P(F_i)}$$

Example with Lizards

- $F_1 \ldots F_n$ are lizards in the family i
- E = "something genome of new lizard"
- Need to estimate the priors

February 18th, 2013 - Reading

Ross 3.3

Rosen 7.3

February 19th, 2013 - Homework 3

1. (6 points) What the probability that a 5 card hand contains exactly 3 spades? What if we condition on the hand containing at least 1 spade?
   - A deck of cards has 52 cards, 4 suits, 13 ranks, one of each suit/rank combination.
   - There are $\binom{52}{5}$ possible hands.
   - There are 13 cards which are spades.
   - Let $E_i =$ "a spade was drawn" and $F_i =$ "anything except a spade was drawn."
   - We want $P(E_1 \cap E_2 \cap E_3 \cap F_1 \cap F_2)$.
   - The probability of E_1 is $\frac{13}{52}$, because thirteen of the cards are spades.
   - Assuming E_1, the probability of E_2 is $\frac{12}{51}$, because there is one less spade and one less card in general.
   - Assuming E_2, the probability of E_3 is $\frac{11}{50}$, because there are now two less spades and two less cards in general.
   - There are now 49 cards in all, 10 of which *are* spades.
   - Assuming E_1 through E_3, the probability of $F_1 = \frac{39}{49}$.
   - Assuming F_1, the probability of $F_2 = \frac{38}{48}$.

$$\frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{39}{49} \times \frac{38}{48} = 0.008154261704$$

- Alternatively, there are 13 choose 3 ways of picking a spade, 39 choose 2 way of picking a “not spade,” and there are 52 choose 5 possible options:

$$\frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} = \frac{286 \times 741}{2598960} = 0.008154261704$$

- For part two, you only have to pick four cards out of 51, because one is a space already.

$$\frac{\binom{13}{2}\binom{39}{2}}{\binom{51}{4}}$$

2. (7 points) Suppose n people each throw a six-sided die. Let A_n be the event that at least two distinct people roll the same number. Calculate $P(A_n)$ for $n = 1, 2, 3, 4, 5, 6, 7$.
   - For A_1, It is impossible for two distinct people to roll the same number if only person rolls.

$$P(A_1) = 0$$

   - For A_2, there are 36 possible throws, but only 6 of could contain two distinct people rolling the same number.

$$P(A_2) = \frac{6}{6^2} = \frac{1}{6} = 0.1666666667$$

   - For A_3 consider the complement, which can be described as “No two distinct people roll the same number.” The probability that all three people roll unique numbers is $1 \times \frac{5}{6} \times \frac{4}{6} \times$,

$$P(A_3) = 1 - P(A_3^c) = 1 - \frac{5}{6} \times \frac{4}{6} = 0.4444444444$$

   - For A_4, consider the complement again, and apply the same reasoning.

$$P(A_4) = 1 - 1 \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} = \frac{13}{18} = 0.7222222222$$

   - For A_5, consider the complement again, and apply the same reasoning.

$$P(A_5) = 1 - \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} = 0.9074074074$$

   - For A_6, consider the complement again. There are $6!$ ways of arranging the integer elements in the string “123456.” This is out of a 6^6 ways of writing a string with integers from one to six.

$$P(A_6) = 1 - \frac{6!}{6^6} = 0.98456790123$$

   - If 7 people roll a dice with 6 sides, it is inevitable that at least two distinct people roll the same number.

$$P(A_7) = 1$$

3. (3 points) Suppose we draw 2 balls at random from an urn that contains 5 distinct balls, each with a different number from $1, 2, 3, 4, 5$, and define the events A and B as $$A = \text{`5 is drawn at least once"} \quad \text{and} \quad B = \text{` 5 is drawn twice"}$$ Compute $P(A)$ and $P(B)$.

- $P(B) = 0$
- $P(A) = \frac{4}{\binom{5}{2}} = 0.4$

4. (4 points) In the previous problem, suppose we place the first ball back in the urn before drawing the second. Compute $P(A), P(B), P(A|B), P(B|A)$ in this version of the experiment.
   - $P(A) = 1 - \left(\frac{4}{5}\right)^2 = .36$
   - $P(B) = \left(\frac{1}{5}\right)^2 = .04$
   - $P(A|B) = 1$
   - $P(B|A) = \frac{1}{5}$
5. (4 points) Suppose 5 percent of cyclists cheat by using illegal doping. The blood test for doping returns positive 98 percent of the people doping and 12 percent who do not. If Lance's test comes back positive, what the probability that he is doping? (Ignoring all other evidence, of course...)
   - $P(C) =$ "the probability a cyclist cheated by doping" $= .05$
   - $P(T|C) =$ "the probability a cyclist giving a positive test if they doped" $= .98$
   - $P(T|C^c) =$ "the probability a cyclist giving a positive test if they *did not* dope" $= .12$
   - We want the probability of a cyclist doping given a positive test, which is $P(C|T)$.

$$P(T|C) = .98 = \frac{P(T \cap C)}{P(C)} = \frac{P(T \cap C)}{.05}$$

$$P(T \cap C) = .049$$

$$P(T \cap C^c) = .12 = \frac{P(T \cap C^c)}{P(C^c)} = \frac{P(T \cap C^c)}{.95}$$

$$P(T \cap C^c) = .114$$

   - For any event T and C,

$$P(T) = P(T \cap C) + P(T \cap C^c)$$

$$P(T) = .049 + .114 = .163$$

$$P(C|T) = \frac{P(C \cap T)}{P(T)} = \frac{.049}{.163} = 0.3006134969 \approx .30$$

   - Intuition:
     - Take 100 cyclists, 5 of them actually doped according to these numbers.
     - If all 100 cyclists are tested, it's very, very likely the 5 will return positive.
     - Of the remaining 95 cyclists, 12 percent of them will also return positive, which makes a little less than 12 cyclists.
     - So for each of the 17 who tested positive, there are five who actually doped, which means each has a $\frac{5}{17} = 0.2941176471 \approx .30$ probability of doping.
6. (6 points) If $A \subseteq B$, can A and B be independent? What we if require that $P(A)$ and $P(B)$ both not equal 0 or 1?
   - A and B are independent if they satisfy the condition $P(A \cap B) = P(A)P(B)$
   - If $A \subseteq B$, it is true that $A \cap B = B$.
   - This means that $P(A \cap B) = P(B)$.
   - In order to be independent when $A \subseteq B$ is true, $P(B)$ must equal $P(A) \cdot P(B)$.
   - The only way anything multiplied by something can equal itself is if that something is one.
   - Therefore, when $A \subseteq B$, A and B can be independent when $P(A) = 1$.
   - Furthermore, being as anything multiplied by zero yields zero, when $A \subset B$, A and B can be independent if $P(B) = 0$.
   - So no.

7. **Extra credit (5 points)** Consider the experiment where two dice are thrown. Let A be the event that the sum of the two dice is 7. For each $i \in 1, 2, 3, 4, 5, 6$ let B_i be the event that at least one i is thrown.
   1. Compute $P(A)$ and $P(A|B_1)$.
      - $P(A) = \frac{6}{6^2} = \frac{1}{6}$
      - $P(A|B_1) = \frac{P(A \cap B_1)}{P(B_1)} = \frac{\frac{3}{\binom{6}{2}}}{\frac{1}{6}+\frac{1}{6}} = 0.6$
   2. Prove that $P(A|B_i) = P(A|B_j)$ for all i and j.
      - Being as the sum has to be seven, there is one and only way to sum to seven for the integers 1 through 6.
      - So it doesn't matter what is rolled on the first roll, the probability "rides on" the second roll being the number the first roll needs to sum to seven.
      - The probability of rolling any given number on a fair die is always $\frac{1}{6}$.
      - Therefore, for all i and j, the probability cannot be anything but $\frac{1}{6}$.
   3. Since you know that some B_i always occurs, does it make sense that $P(A) \neq P(A|B_i)$? (After all, if E is an event with $P(E) = 1$, then for any event F, $P(F|E) = P(F)$. What is going on? Does this seem paradoxical?)

February 20th, 2013 - Lecture: Independence

Topics

- Multiplication rule for conditional probability.
- Independent events: two equivalent definitions and several examples with cards and dice.
- People v. Collins and the prosecutor's fallacy.
- Mutual independence.

Multiplication Rule for Conditional Probability

- **Example**: Hat contains 3 cards, RR, RB, BB. Pick a card, put it on table, see a R side, what's the probability the other side is B?
  - Wrong: 1/2
  - Right: 1/3
- If E_1 through E_2 are events, all P(E_i) do not equal zero,

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) =$$

$$P(E_1) \times P(E_2|E_1) \times \ldots \times P(E_n|E_1 \cap \ldots \cap E_{n-1})$$

  - This might be easier to express as the intersection of smaller events.
  - This is really easy thing to prove.
  - "Proves itself."
  - **Proof**:
    1. $P(E_1) \times \frac{P(E_2 \cap E_1)}{P(E_1)} \times \ldots \times \frac{P(E_1 \cap \ldots \cap E_n)}{P(E_1 \cap \ldots \cap E_{n-1}})$
    2. Cancel. Beautiful cancellation.
  - This allows you to say, what happens when the first once happens? And now, what about the second one? So on so forth.
  - **Example**: What is the probability of a five card hand not containing a pair?
    - E_i = "first i cards in hard do not contain a pair"
    - *Claim*: What we want is $E_1 \cap E_2 \cap \ldots \cap E_5$
    - What's funny about this is that we don't want E_1 through E_4, we are actually only interested in E_5.
    - The reason we do this, however, is because it makes this calculation easier.

- Possibilities:
  1. $P(E_1) = 1$
     - The probaility of the first hand not being a pair is 1, because it's one card.
  2. $P(E_2 | E_1) = \frac{48}{51}$
  3. $P(E_3 | E_1 \cap E_2) = \frac{44}{50}$
  4. $P(E_4 | E_1 \cap \ldots \cap E_3) = \frac{40}{44}$
  5. $P(E_5 | E_1 \cap \ldots \cap E_4) = \frac{36}{48}$
- Multiply these values together to get 50.7%.

Independence

- **Definition**: Say events E and F are independant if $P(E \cap F)
  - P(E) \times P(F) $. + **Equivalent definition**: E and F are independent i(P(F) = 0 or P(E|F) = P(E)$.
  - **Proof**:
    1. Take E and F set $P(E \cap F) = P(E) \times P(F)$.
    2. If $P(F) = 0$, then done.
    3. If not, then $P(E|F) = \frac{P(E \cap F)}{P(F)}$
- **Exercise**: Toss two coins
  - E = "1st coin was H"
  - F = "2nd coin was H"
  - P(E) = P(F) = 1/2
- **Exercise**: Tossing two dice
  - E = "sum of dice is 6"
  - F = "first die was 4"
  - $P(E) = P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = \frac{5}{36}$
  - P(F) = 1/6
  - $P(E \cap F) = P(\{(4,2)\}) = \frac{1}{36}$
  - $P(E) \times P(F) = \frac{5}{36} \times \frac{1}{6}$
- **Exercise**: Suppose a family has 3 kids.
  - E = "family has at least 1 boy and 1 girl"
  - F = "family has at most 1 boy"
  - Are these indepedant?
    - $S = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}$
    - $E = S - \{BBB, GGG\}$
    - $F = \{BGG, GBG, GGB, GGG\}$
    - $E \cap F = \{BGG, GBG, GGB\}$
    - $P(E \cap F) = \frac{3}{8}$
    - $P(E) \times P(F) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$
    - Therefore, independent.

"Independent" and "Mutually Exclusive" are not the same

- **Example**: 2 dice
  - E = "sum was 6"
  - F = "first die was 6"
  - E and F are not mutually exclusive, but not independent.

Prosecutor's Fallacy: People v. Collins (1968)

- The prosectution came up with these numbers:

- The probability of a black man with a beard is 1 in 10
    - The probability of a man with a mustache is 1 in 4
    - White woman with ponytail is 1 in 10
    - So on so forth, 1 in 3, 1 in 10, 1 in 1000
    - So he made this calculation:

$$\frac{1}{10} \times \frac{1}{4} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{3} \times \frac{1}{1000} = \frac{1}{12 \times 10^6}$$

    - *Say* that the population of LA was 24×10^6
    - Then, you'd "expect" to have 2 couples fitting this evidence.
    - P(evidence | innocent) ≠ P(innocent | evidence)

With 3 events

- E is indepedant of F and G
- F is indepedant of G
- Then is E indepedant of F intersect G?
- **Example**: rolling two die
    - E = "sum to 7"
    - F = "first die was 4"
    - G = "second die was 3"

$$P(E|F \cap G) = 1$$

$$P(E) = \frac{1}{6}$$

    - **Definition**: E, F, and G are independant (alternatively "mutually independant") if:

$$P(E \cap F) = P(E) \times P(F)$$

$$P(E \cap G) = P(E) \times P(G)$$

$$P(F \cap G) = P(F) \times P(G)$$

$$P(E \cap F \cap G) = P(E) \times P(F) \times P(G)$$

With n events

- **Definition**: $E_1, \ldots, E_n$ are indepedant if for every subset $I \subseteq \{1, 2, \ldots, n\}$ of at least size 2
- **Example**: Flip n coins

February 20th, 2013 - Recitation

- Proove:

$$\binom{n}{n_1 \cdot n_2 \cdot \ldots \cdot n_r} =$$

[[[{n-1 \choose n_1 - 1 \cdot ... \cdot n_r} + ... + {n-1 \choose n_1 \cdot ... \cdot n*{r-1} \cdot n*{r-1} $$

February 20th, 2013 - Reading

3.3 Bayes' Formula

3.4 Independent Events

- **Equation 4.1**: $P(EF) = P(E)P(F)$
- **Independent events**: Two events E and F are said to be independent if Equation (4.1) holds.
- **Dependent events**: Two events E and F that are not independent are said to be dependent.

Optional: People v. Collions court opinion.

British case involving Sally Clark.

February 25th, 2013 - Lecture

Announcements

1. HW 3 due Wednesday
2. Review on Wednesday
3. Midterm on Monday

Independence of Several Events

- Event are indepedant if $P(EF) = P(E)P(F)$
- **Example**: Three events with dice:
  - E = "sum is 7"
  - F = "1st die is four"
  - G = "2nd die is three"
  - E, F, G are *pairwise independant*
  - But the $P(E|F \cap G) = 1$, not $P(E) = \frac{1}{u}$
- **Definition**: Events $E_1, E_2, \ldots E_n$ are **indepedent** (or **mutually independent**) if for every subset of numbers from 1 through n of size greater than or equal to 2:

$$P\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} P(E_i)$$

  - Equivilently,

$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_n}) = P(E_{i_1}) \cdot P(E_{i_2}) \cdot \ldots \cdot P(E_{i_k})$$

- **Claim**: For any i and $j_i, \ldots, j_k$ (not including i)

$$P(E_i | E_{j_1} \cap \ldots \cap E_{j_k}) = P(E_i)$$

  - **Proof**: This equation equals

$$\frac{P(E_1 \cap E_{j_1} \cap \ldots \cap E_{j_k})}{P(E_{j_1} \cap \ldots \cap E_{j_k})}$$

- **Example**: Suppose we flip a "biased" coin n times

$$P(H) = p,\ P(T) = 1 - p,\ (O < p < 1)$$

1. What is the probability we getting heads all n times?
   - Define $E_i =$ "ith flip was heads."

$$E = E_1 \cap \ldots \cap E_n,\ P(E) = P(E_1) \times \ldots \times P(E_n)$$

2. What is the probability we get at least one H?
   - $F =$ "at least on H"
   - $C(F) =$"all T on n flips"
   - $P(F) = 1 - C(F) = 1 - (1-p)^n$
   - $P(F^c) = (1-p)^n$
3. What is the probability of getting exactly k heads?
   - First find the probability of HHHH...H (k heads) TTTTT...T (n-k tails) (h times)
   - $p \times p \times p \times p \ldots$ (k times) $\times (1-p) \times (1-p) \times \ldots \times (1-p) = p^k (1-p)^{n-k}$
   - How many H/T strings, length n, with k Hs? $\binom{n}{k}$
   - P(exactly k Hs) = $\binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$

Primality Testing

- One way to check if a number is prime is to divide by the square root of number (or something)
- So people found what's called the **Miller-Rabin** test

```
when running MR on x
    if x is not prime
        then MR outputs "not prime" with probability 1/4
        else says "don't know" the other 3/4 times
    if x is prime
        then MR always says "don't know"
```

Algorithmic amplification

- The idea is to run MR n times on x
  - If it ever says "not prime", output "not prime"
  - If not, output "probabily prime"
- What is the probability of outputting "probability prime" if X is not prime
  - $\left(\frac{3}{4}\right)^n$
- **AKS test**

Ramsey Numbers

- Example: "The probabilistic method"
  - Take a complete graph n vertices, color every edge R or B, and you "lose" if you color in a triangle.
  - Avoid making an all red or all blue K_k

```
o----R----o
|\        |
|  \      |
B    R    B
|      \  |
|        \|
o----B----o (or something)
```

- Theorem If n is large enough relative to k, then you can't avoid making all R or B K_k

$$n > R(k)$$

- $R(3) = 6$, $R(4) = 18$ (1979), $R(5)$ is unknown
- **Theorem**: $R(k) \geq n^{\frac{k}{2}}$ when $k > 4$, if $n = 2^{\frac{k}{2}}$, then you can color K_n and avoid a all R or B K_K
    - **Outline**: (Erdos)
        1. Calculate the probability that a random coloring "works", not that is greater than 0
        2. Therefore one exists, and done.
    - Color each edge independently Red or Blue with probability of one half
        - E = "some K_k gets all Red or Blue edges"
        - How many K_K are these in K_K? $\binom{n}{k}$
        - Want:

$$P(E) = P\left(\bigcup_{i=1}^{\binom{n}{k}} E_i\right) < 1$$

$$]

February 25th, 2013 - Lecture: Mutual Independence and Applications

Topics

- Definition of mutual independence
- Examples
- Applications to biased coin flips
    - primality testing
    - the bound on Ramsey numbers

February 25th, 2013 - Reading

Ross 3.4.

Rosen 7.2.

Midterm 1 Review

Review Sheet 1

Set Theory Review

- $x \in A$ means "x is an element of A"; $x \notin A$ means it's not.
- relations between sets
    - $A \subseteq B$ means that if x is in A, then x is in B
    - $A \supseteq B$ means $B \subseteq A$
    - $A = B$ means $A \subseteq B$ and $B \subseteq A$
- operations on sets
    - $A^c = x \in S : x \notin A$ (complement)
    - $\emptyset = S^c$ (the empty set)
    - $A \cap B = x \in S : x \in A \wedge x \in B$ (intersection)

- $A \cup B = x \in S : x \in A \vee x \in B$ (union)
- $A \setminus B = x \in S : x \in A \wedge x \notin B = A \cap B^c$

- set identities
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $(A \cap B)^c = (A^c) \cup (B^c)$ (de Morgan's law)
  - $(A \cup B)^c = (A^c) \cap (B^c)$ (de Morgan's law)

Probability Theory

- Random experiment
  - Idealized or conceptual experiment.
  - It can be useful to imagine that the experiment can be repeated infinitely often under identical conditions, but with different outcomes.
- Sample Space
  - The set of outcomes (elementary events) of a random experiment.
- An event
  - A subset of the sample space of an experiment.
  - If the experiment is performed and the out of $x \in S$ we say that "x occurs."
  - If not, "x does not occur"
- Probability measure
  - A real-valued non-negative function on events in S which satisfy these axioms:
    - $P(S) = 1$
    - $P(A \cup B) = P(A) + P(B)$ whenever $A \cap B = \emptyset$

Conditional Probability

- Conditional probability formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- Hypotheses

$$P(A) = \sum_{i=1}^{n} P(H_1)P(A|H_i)$$

- Bayes' rule

$$P(H_i|A) = \frac{P(A|H_1)P(H_1)}{\sum_{j=1}^{n} P(H_j)P(A|H_j)}$$

Independence

- Events are independent if and only if

$$P(B|A) = P(B)$$

- This means that the probability of B given the information that A has occurred is the original probability of B, so A gives no new information about B's probability. Using the conditional probability formula, for the left-hand side we see that

$$P(A \cap B) \setminus P(A) = P(B)$$

- Multiplying both sides of this equation by $P(A)$ we get the **product law** for independent events

$$P(A \cap B) = P(A)P(B)$$

Axioms of Probability - Self Test Problems and Exercises

1. A cafeteria offers a three-course meal consisting of an entree, a starch, and a dessert. The possible choices are given in the following table:

```
Course          Choices
Entree          Chicken or roast beef
Starch          Pasta or rice or potatoes
Dessert         Ice cream or Jello or apple pie or a peach
```

   A person is to choose one course from each category.
   1. How many outcomes are in the sample space?
      - There are there points a person gets to choose something.
      - At each of those points, there are 2, 3, and 4 choices.
      - So I think it's $2! + 3! + 4!$, but I'm pretty bad at math.
      - In hindsight, that's about putting things in order, and has nothing to do with this problems.
      - It's simply: $2 \cdot 3 \cdot 4$
   2. Let A be the event that ice cream is chosen. How many outcomes are in A?
      - This only says something about the final decision.
      - Therefore, the previous number still holds, with a restriction on the final choice.
      - Therefore, $2 \cdot 3$
   3. Let B be the event that chicken is chosen. How many outcomes are in B?
      - Similar formula.
      - $3 \cdot 4$
   4. List all the outcomes in the event AB.
      1. Chicken
      2. (Pasta|Rice|Potatoes)
      3. Ice cream
         - More formally, {(chicken, x, ice cream) | x ∈ {pasta, rich, potatoes}
   5. Let C be the event that rice is chosen. How many outcomes are in C?
      - $2 \cdot 3$
   6. List all the outcomes in the event ABC.
      - {(Chicken, rice, ice cream)}
2. A customer visiting the suit department of a certain store will:
   - purchase a suit, $.22$
   - purchase a shirt, $.30$
   - purchase a tie, $.28$
   - both a suit and a shirt, $.11$
   - both a suit and a tie, $.14$
   - both a shirt and a tie, $.10$
   - all 3 items with probability $.06$

   What is the probability that a customer purchases
   - none of these items?
      - I think it's too easy for them to just give us the .06 number and then have us calculate the complement. But that's my first guess, .94. (I know this is wrong.)
      - We're interested in the union of purchasing a suit, purchasing a shirt, and purchasing a tie.
      - The probability of buying all of these items is:

$$P(A \cap B \cap C) = .22 + .30 + .28 + .11 + .14 + .10 + .06$$

- Subtract one from this to get the probability of buying none.
- I worry that I would not see this on a test.

- exactly 1 of these items?
  - Add the probabilities that correspond *to* the sample space.
  - Subtract the probabilities that *do not* correspond to this event.

$$.22 + .30 + .28 - .11 - .14 - .10 - .06$$

  - Of course, this is totally wrong.
  - Right, so formally, where A is "purchase suit", B is "purchase shirt", and C is "purchase tie." We want:

$$P(AB \cup AC \cup BC)$$

3. A deck of cards is dealt out. What is the probability that the 14th card dealt is an ace? What is the probability that the first ace occurs on the 14th card?
   - So there are four aces in a deck, meaning there are 48 that aren't.
   - But every card is equally likely to be pulled at any time.
   - This is called **symmetry**.
4. Let A denote the event that the midtown temperature in Los Angeles is 70○F, and let B denote the event that the midtown temperature in New York is 70○F. Also, let C denote the event that the maximum of the midtown temperatures in New York and in Los Angeles is 70○F. If P(A) = .3, P(B) = .4, and P(C) = .2, find the probability that the minimum of the two midtown temperatures is 70○F.
   - A is "the temp of LA is 70F", .3
   - B is "the temp of NY is 70F", .4
   - $P(A \cap B) = .3 + .4 - P(AB)$
   - Call D "the minimum of the two is 70F"
   - Observe that $A \cap B = D \cap C$, and therefore $AB = DC$.
   - $P(D \cap C) = P(D) + .2 - P(DC)$
   - $P(D) + .2 - P(DC) = .3 + .4 - P(AB)$
   - $P(D) + .2 = .3 + .4$
   - $P(D) = .3 + .4 - .2 = .5$
5. An ordinary deck of 52 cards is shuffled. What is the probability that the top four cards have
   1. different denominations?
      - There are $52!$ total possibilities.
      - I think that each of the four top cards can be counted as $\binom{13}{1}$.
      - Wait no.
      - You're picking four cards, ignore the rest of the deck.
      - There are $52 \cdot 51 \cdot 50 \cdot 49$ total ways.
      - Denomiation means 1 or 2 or 3 or ... or King or Ace.
      - After each is picked, there are four less cards that are "available."

$$\frac{52 \cdot 48 \cdot 44 \cdot 40}{52 \cdot 51 \cdot 50 \cdot 49}$$

   2. different suits?
      - Like the last problem, there are $52 \cdot 51 \cdot 50 \cdot 49$ total ways.
      - But this time, each time a card is picked, there are 13 less choices.

$$\frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49}$$

6. Urn A contains 3 red and 3 black balls, whereas urn B contains 4 red and 6 black balls. If a ball is randomly

selected from each urn, what is the probability that the balls will be the same color?
    - R = "both balls were red."
    - B = "both balls were black."
    - There is a probability of zero that both these events happen.
    - $P(R \cup B) = P(R) + P(B) - 0$
    - $P(R) = \frac{3 \cdot 4}{6 \cdot 10}$
    - $P(R) = \frac{3 \cdot 6}{6 \cdot 10}$
7. In a state lottery, a player must choose 8 of the numbers from 1 to
    1. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the $\binom{40}{8}$ combinations, what is the probability that a player has
        - all 8 of the numbers selected by the lottery commission?
        - They can only have one of the combinations.

$$\frac{1}{\binom{40}{8}}$$

        - 7 of the numbers selected by the lottery commission?
        - I think that there are $40 \cdot 39 \ldots \cdot 32$ total combinations.
        - Having one of them would mean, I think,

$$\frac{40 \cdot 39 \ldots \cdot 33}{40 \cdot 39 \ldots \cdot 32}$$

        - Unfortunately, this is very wrong. The better way to think of this is there are 8 choose 7 ways of picking 7 numbers.
        - I can't work out on my own what you need to multiply this by, I think it has something to do with the number of 7 combination choices in a 40 choice pool.
        - at least 6 of the numbers selected by the lottery commission?
8. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size 4 is randomly selected. Find the probability that the committee will consist of
    1. 1 from each class

$$\frac{3 \times 4 \times 4 \times 3}{\binom{14}{4}}$$

        - Every number in the numerator is like "number in class" choose 1, which is just "number in class."
    2. 2 sophomores and 2 juniors

$$\frac{\binom{4}{2} \times \binom{4}{2}}{\binom{14}{4}}$$

        - You have to pick two out of each 4, and then divide it by the entire sample space.
    3. only sophomores or juniors $$\frac{8 \choose 4}{14 \choose 4}
        - It could be all sophomores or all juniors, or a combination thereof, so just count them as one group with 8 choose 4.
9. For a finite set A, let N(A) denote the number of elements in A. Show that:

$$N(A \cup B) = N(A) + N(B) - N(AB)$$

```
     +--------------+    +-------------+
    /                \  /               \
   /                  \/                 \
  /                   /\                  \
 /                   /  \                  \
|                   |    |                  |
|         A         | AB |        B         |
|                   |    |                  |
 \                   \  /                  /
  \                   \/                  /
   \                  /\                 /
    \                /  \               /
     +--------------+    +-------------+
```

- $A \cup B$ is equal to those elements in A plus all the elements in B.
- But that's double counting! Region AB is both in A and B, so you need to substract one instance of AB from the outcome.
- Therefore, $N(A \cup B) = N(A) + N(B) - N(AB)$

10. Consider an experiment that consists of six horses, numbered 1 through 6, running a race, and suppose that the sample space consists of the 6! possible orders in which the horses finish. Let A be the event that the number-1 horse is among the top three finishers, and let B be the event that the number-2 horse comes in second. How many outcomes are in the event $A \cup B$?
    - $A =$ "the number-1 horse is among the top three finishers"
    - $B =$ "B be the event that the number-2 horse comes in second"
    - $A \cup B =$ "the number-1 horse is among the top three finishers and the number-2 horse came second."
    - For B, only one outcome is necessary, that the number-2 horse is second, the rest are totally variable.
    - I think this means that the possible outcomes are therefore:

      $$\frac{5!}{6!}$$

      because you're only taking one of the choices "off the table."
    - Similarly, there are $5!$ ways of specifying the position of horse one.
    - So what is $N(AB)$?
        - There is only one way for the number two horse to be in the second position.
        - There are two options for the first horse to be in the top three, then, in position one and position three.
        - For each of these options, there are 4 choices for the unclaimed spot. $2 \times 4!$

    $$N(A \cup B) = \frac{5!}{6!} + \frac{5!}{6!} - (2 \times 4!)$$

11. A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?
    - This is event is really asking what is the probability that the first four cards are of different suits. (The fifth isn't relevant because it *has* to be a repreated suit.)

    $$\frac{\binom{52}{1}}{52}\frac{\binom{39}{1}}{51}\frac{\binom{26}{1}}{50}\frac{\binom{13}{1}}{49}$$

12. A basketball team consists of 6 frontcourt and 4 backcourt players. If players are divided into roommates at random, what is the probability that there will be exactly two roommate pairs made up of a backcourt and a frontcourt player?

Midterm Review Powerpoint

Counting

- Multiplication principle
  - If we can classify a set of objects by a sequence of decisions, then the (# objects) = (# choices on first decisions) x ... x (# of choices in last decision)
  - Mathematically,

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

- Permutations
  - Number of ways to put n things in order:

$$n(n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$$

  - Number of ways to put k-out-of-n things in order:

$$n(n-1) \times (n-2) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$$

- Permutations with repeats
  - Ways to order n objects, with n_1 alike, ..., n_r alike.

$$\frac{n!}{n_1! \ldots n_r!} = \binom{n}{n_1, \ldots, n_r}$$

- Combinations
  - Ways to choose k-out-of-n things where order doesn't matter:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

- Partitions
  - Number of ways to divide n indistinguishable obects in r non-empty piles:

$$\binom{n-1}{r-1}$$

  - Number of ways to divide n indistinguishable objects in r piles, empty allowed:

$$\binom{n+r-1}{r-1}$$

- Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Combinatorial identities

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Multinomial theorem

$$(x_1 + \ldots + x_r)^n =$$

$$\sum_{0 < n_1 < \ldots < n_r < n} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \ldots x_r^{n_r}$$

  - Sum over all ways to express n as sum of r non-negative integers:

$$0 \leq n_1 \leq \ldots \leq n_r \leq n : n_1 + \ldots + n_r = n$$

PROBABILITY

- Sample spaces and events
  - Sample spaces is a set S
  - Events are subsets of S
  - Intersection of E and F: “Both E and F happen”
  - Union of E and F: “Either E or F or both happen”
  - Complement of E: “E does not happen”
  - E and F are mutually exclusive if intersection of E and F is empty
- Basic identities
  - $P(\emptyset) = 0$
  - $P(E^c) = 1 - P(E)$
  - $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
  - $P(E) = P(E \cap F) + P(E \cap F^c)$
  - If P and F are mutually exclusive, then

$$P(E \cup F) = P(E) + P(F)$$

- Uniform probability measure
  - If all outcomes are equally likely (fair dice, fair coin, random card, poker hand, etc) then,

$$P(E) = \frac{|E|}{|S|}$$

- Conditional probability
  - Probability of E, given that F happened

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

- Bayes's theorem
  - Think E is "evidence" and F is "outcome"

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

- Multiplication rule for conditional probability

$$P(E_1 \cap \ldots \cap E_n) =$$

$$P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)\ldots$$

$$P(E_n | E_1 \cap E_2 \cap \ldots \cap E_{n-1})$$

- Indepedant events
  - E and F are *independant* if

$$P(E \cap F) = P(E) \times P(F)$$

  - Equivilently,

$$P(E|F) = P(E) \vee P(F) = 0$$

- Prosectutor's fallacy

$$P(E|F) \neq P(F|E)$$

Review Problems

1. How many ways can we walk up/right from point A to point B?

```
  +----+----+----+ B
  |    |    |    |
  +----+----+----+
  |    |    |    |
  +----+----+----+
  |    |    |    |
  +----+----+----+
  |    |    |    |
A +----+----+----+
```

   - A clever way to solve this problem is to represent "ups" and "right" as characters in a string.
   - You have to traverse "on the lines" - the strings will have to be seven characters long.
   - You can only go "up" four times in any given run.
   - You can only go "right" 3 times.

$$\binom{7}{3, 4}$$

2. You have 18 non-identical children.
   1. Assign them to 4 possibily empty teams.
      - Again, you can cleverly solve this problem with strings.
      - But I'll do it with sets just to be different.
      - You have four sets, A, B, C, D, each of which can have can have an elements from the set $\{1, \ldots, 18\}$
      - Because each of the four sets can possibly have 18 children in it, there are 4^18 possibilities.
      - Gosh that didn't go as well as planned.
   2. How many ways can 18 identical balls be divided into 4 possibily empty groups.
      - The formula for putting n objects into r possibly empty groups is

$$\binom{n+r-1}{r-1}$$

      - There are 18 identical balls (n) and 4 possibly empty groups (R)

$$\binom{18+4-1}{4-1}$$

- Refer to stars and bars, partitions.

3. How many ways can 18 children be divided into 4 groups so that: Group 1 has 4 children, Group 2 has 6 children, Group 3 has 5 children, Group 4 has 3 children?
4. Suppose we deal a random 5-card poker hand. What is the probability that we get exactly 1 Queen? What about exactly 4 Hearts? Are they independent?
    - $E =$ "get exactly one Queen"
        - There are 52 choose 5 possible hands.
        - There are 4 Queens in every deck.
        - Observe there is symettry.

$$\frac{\binom{4}{1} \times \binom{48}{4}}{\binom{52}{5}}$$

    - $F =$ "exactly four Hearts"
        - There are 52 choose 5 possible hands.
        - There are 13 hearts in every deck.

$$\frac{\binom{13}{4} \times \binom{39}{1}}{\binom{52}{5}}$$

    - Events are independant if $P(E \cap F) = P(E) \times P(F)$
        - What is the probability of getting one Queen and exactly four Hearts?
        - Of course, the Queen *could be* the Heart.
        - So you have to count the instances that the Queen is the fifth card (not heart) as well as those where as the one where *it is* a Heart.
        - There are 12 choose 4 ways of picking an Heart which is *not* a queen, and then 3 choose 1 ways of picking a Queen which *is not* a Heart.
        - There is only one Queen of Hearts, and you need 3 more hearts out of th remaining 12 Hearts after that.
        - Then you need to pick 1 of the remaining 36 cards which are not Hearts *and* not Queens.

$$\frac{\binom{12}{4}\binom{3}{1} + \binom{12}{3}\binom{36}{1}}{\binom{52}{5}}$$

        - I can eyeball that I'd very much doubt this multiplication adds up, and if this were the test, I'd show it. But being as I know it actually doesn't multiply based on the answers, I'm just going to skip the step. "Leave it as an exercise for the reader", as textbooks would say.
5. In a 5-card poker hand, how many ways can we get a 3-of-a-kind? Full house does not count.
    - There are 13 different ranks, and this requires you choose one of them.
    - Of that rank, which has 4 members, you must choose 3 of them.
    - There are 12 ranks left, of which you must choose 2.
    - Because full houses don't count (which I assume is when the other two cards are of the same suit as well), you have to pick 1 from 4, and then 1 from 3 (they *can't* be the same).

$$\binom{13}{1} \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1} \times \binom{3}{1}$$

Conditional Probability and Independence - Self Test Problems and Exercises

1. In a game of bridge, West has no aces. What is the probability of his partner's having (a) no aces? (b) 2 or more aces? (c) What would the probabilities be if West had exactly 1 ace?
   - I would answer this, but like most people under the age of 64, I have no idea how bridge works.
2. The probability that a new car battery functions for over 10,000 miles is .8, the probability that it functions for over 20,000 miles is .4, and the probability that it functions for over 30,000 miles is .1. If a new car battery is still working after 10,000 miles, what is the probability that
   - its total life will exceed 20,000 miles?
     - $E =$ "battery functions for over 10,000 miles"
     - $F =$ "battery functions for over 20,000 miles"
     - $G =$ "battery functions for over 30,000 miles"
     - I think what the question is asking is what is the probability that the the battery functions for 20,000 miles given that it went for 10,000 miles. I don't *quite* have the intuition behind that, but I just kind of "smell" that that is what this problem wants.

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

     - Because $E \in F$, $E \cap F$ is simply F.

$$P(F|E) = \frac{P(F)}{P(E)} = \frac{.4}{.8}$$

   - its additional life will exceed 20,000 miles?
     - This is the probability that the car reaches 30,000 miles given that it's gone both 20,000 and 10,000.

$$P(G|E) = \frac{P(E \cap G)}{P(E)} = \frac{.1}{.8}$$

3. How can 20 balls, 10 white and 10 black, be put into two urns so as to maximize the probability of drawing a white ball if an urn is selected at random and a ball is drawn at random from it?
   - This question is actually pretty cool.
   - I can't quite work out how you'd prove it rigoursly, but the basic idea is that the number will always float around 50% **unless** you do one clever trick.
   - Make it so that one of the urns has only one white ball in it, making it so 100% of the time you at least get one white.
   - Then place the rest in the other, waste none of the white on the "hole in one" so they jack up the probability on the second draw.
4. Urn A contains 2 white balls and 1 black ball, whereas urn B contains 1 white ball and 5 black balls. A ball is drawn at random from urn A and placed in urn B. A ball is then drawn from urn B. It happens to be white. What is the probability that the ball transferred was white?
   - $P(E) =$ "the transferred ball was white"
   - $P(F) =$ "the ball drawn from B was white"

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Homework 1

1. Give sample spaces that model the outcomes for the following experiments. You may use a regular expression or other formalisms that you find convenient.
   1. Rolling 3 dice:

$$\{(i, j, k) \mid i, j, k \in \{1, 2, 3, 4, 5, 6\}\}$$

2. Rolling a die until an even result comes up, or the die is rolled three times:

$$\{(i),(ji),(jji),(jjj\ldots ji) \mid i \in \{2,4,6\} j \in \{1,3,5\}\}$$

3. Tossing a pair of coins until they both come up tails.

$$\{(i),(j,i),(j,j,i),(j,j,j,\ldots,i) \mid i=(TT), j=\{HH,TH,HT\}\}$$

4. Draw 2 balls from an urn which contains 6 balls, each with a distinct label from $\{1,2,3,4,5,6\}$.

$$\{i,j \mid i,j \in \{1,2,3,4,5,6\} \wedge i \neq j\}$$

5. Draw 1 ball from the same urn, then replace it and draw a ball again.

$$\{i,j \mid i,j \in \{1,2,3,4,5,6\}$$

Homework 2

1. If $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{5}$, and $P(A \cup B)=3/5$, find:
   - $P(A \cap B)$
     - The idea is that the probability of both A and B happening is equal to the probability of A happening multiplied by the probability of B happeneing.

$$\frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

   - $P(A^c \cup B)$

$$P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B)$$

$$\frac{1}{2} + \frac{1}{5} - \frac{1}{10} = .6$$

   - $P(A^c \cap B)$
     - The idea is that being as $P(A)=\frac{1}{2}$, the complement is equally likely to happen, because $P(S)=1$ and $P(A)=1-P(A^c)$

$$\frac{1}{10}$$

2. How many elements are there in the set $\{x : 10^7 \leq x \leq 10^8$, and the base 10 representation of x has no digit used twice$\}$?
   - $10^7 = 10000000$, and $10^8 = 100000000$
   - The smallest number possible is 12345678
   - The largest number possible is 98765432
   - For the first number, you have 9 total integers to choose from (it can't be zero).

$$\binom{9}{1}$$

   - For the second number, zero is now an option. So likewise,

$$\binom{9}{1}$$

   - Now, for every value after this one, you still have to choose one, you just have one less choice, all the way down to there being 2 remaining numbers that will be left out of your element.

$$\binom{9}{1}\binom{9}{1}\binom{8}{1}\binom{7}{1}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}$$

3. An army output has 19 posts to staff using 30 indistinguishable guards. How many ways are there to distribute the guards if no post is left empty?
   - Stars and bars.

$$\binom{30-1}{19-1}$$

4. What is the coefficient of $x^{10}y^{13}$ when $(x+y)^{23}$ is expanded?
   - Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

     - $n = 23$
     - $k = 10$

$$(x+y)^2 3 = \binom{23}{10} x^1 0 y^{23-10}$$

5. What is the coefficient of $w^9 x^{31} y^4 z^{19}$ when $(w+x+y+z)^{63}$ is expanded? How many monomials appear in the expansion?
   - This is the multinomial theorem when $r = 4$ and $n = 63$.

$$\binom{63}{4, 9, 19, 32} x_1^4 x_2^9 x_3^{19} x_4^{31}$$

Homework 3

1. What is the probability that a 5 card hand has exactly 3 spades?
   - This problem has symmetry, every card is equally likely at any given point.
   - Therefore the answer can be expressed by taking the primality of the relevant subset over the sample space.
   - How many 5 card hands contain exactly 3 spades?

$$\binom{13}{3}\binom{39}{2}$$

   - The answer is this value over

$$\binom{52}{5}$$

2. What is the probability that a 5 card hand has exactly 3 spades, conditioned on having at least one spade?
   - $P(E|F)$ where $E =$ "having 3 spades," and $F =$ "having a least one spade"

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

   - If you have 3 spades, you definitely have at least one spade.

- Therefore, $E \cap F = E$

$$P(E|F) = \frac{P(E)}{P(F)}$$

- We know $P(F)$, and will calculate $P(E)$ using the complement.
- The complement is that none of the cards are spades.
- There are 13 spades in a deck, so there are 39 choose 5 ways of getting a hand with no spades.
- There are 52 choose 5 total hands, the probability of event E is:

$$1 - \frac{\binom{39}{5}}{\binom{52}{5}}$$

3. Suppose n people each throw a six-sided die. Let A_n be the event that at least two distinct people roll the same number. Calculate $P(A_n)$ for $n = 1, 2, 3, 4, 5, 6, 7$.
    - For n_7, the event that two distinct people roll the same number is inevitable.
    - For the remaining 6 rolls, the complement will be much easier to calculate.
    - What is the likelihood for events 1 through 6 that every roll is unique?
    - n_1 will *always* be unique.
    - n_2 will be unique 5 out of 6 times, as n_1 rolled some number.
    - n_3, similarly, will be unique 4 out of 6 times.
    - n_4 will be unique 3 out of 6 times.
    - n_5 will be unique 2 out of 6 times.
    - n_6 will be unique 1 out of 6 times.
    - (Note that n_7 will never be unique.)

$$P(A_i) = 1 - \prod_{n=0}^{i} \frac{7-i}{6}$$

4. Suppose we draw 2 balls at random from an urn that contains 5 distinct balls, each with a different number from $1, 2, 3, 4, 5$, and define the events A and B as

$$A = \text{"5 is drawn at least once"} \quad \text{and} \quad B = \text{"5 is drawn twice"}$$

    Compute $P(A)$ and $P(B)$.
    - There are four ways to pick five along with another number.

$$P(A) = \frac{4}{\binom{5}{2}} = 0.4$$

    - Observe that there is zero probability of getting *any* number twice.

$$P(B) = 0$$

5. In the previous problem, suppose we place the first ball back in the urn before drawing the second. Compute $P(A), P(B), P(A|B), P(B|A)$ in this version of the experiment.
    - I think it would be helpful, again, to consider the complement.
    - The complement of A, is that 5 is never drawn.
    - Each time you draw, considering that you replace, you have a one in five chance of picking 5.
    - For the complement, then, you have a 4 in five chance of drawing *not* five. (Apparently not that helpful.)

$$P(A) = 1 - \frac{4^2}{5} = .36$$

- Okay I think I messed that one up.
- Much simpler version:
  - There are 5^2 possible drawings.
  - There are 4 ways of drawing 5 first, 4 ways of drawing 5 second, and 1 way of drawing 2 fives which makes for nine ways.

$$P(A) = \frac{9}{25} = .36$$

- For $P(B)$, there are still 25 possible ways of drawing.
- What's changed is that there is now only one way to draw five twice - actually getting it twice.
- Therefore,

$$\frac{1}{25}$$

- For $P(A|B)$, the translation is, "what is the probability of A given than B happened."
- Well if five was drawn twice, it was *definitely* drawn once.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- $A \cap B$ is equal to B.

$$P(A|B) = \frac{P(B)}{P(B)} = 1$$

- For $P(B|A)$, the "translation" is, what is the probability of rolling a five twice given that a five was rolled once?

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

- $B \cap A$ is still B

$$P(B|A) = \frac{\frac{1}{25}}{.36} = \frac{1}{9}$$

6. Suppose 5 percent of cyclists cheat by using illegal doping. The blood test for doping returns positive 98 percent of the people doping and 12 percent who do not. If Lance’s test comes back positive, what the probability that he is doping? (Ignoring all other evidence, of course...)
   - This is a Bayes's theorem type of question.

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

   - Think of E as "evidence" and F as "outcome."
   - Call E "Lance's blood test came back positive."
   - Call F "Lance was doping."
   - Before we know anything else, Lance being a random cyclist, we know the probability of his doping, $P(F) = .05$
   - We also know the probability of Lance's test coming back positive given that he doped, $P(E|F) = .98$
   - We *also* know the probability of Lance's test coming back positive given that he *didn't* dope, $P(E|F^c) = .12$.
   - Finally, we *also* know the probability of Lance *not* doping, $P(F^c) = .95$.
   - We now have all the elements for Bayes's theorem.

$$P(F|E) = \frac{(.98)(.05)}{(.98)(.05) + (.12)(.95)}$$

March 6th, 2013 - Lecture

Random variables

- **Random variables**: A random variable on a sample space, space S is any function from S to $\mathbb{R}$.

$$X \cdot S \to \mathbb{R}$$

- **Example**: Tossing 3 coins

$$S = \{HHH, \ldots TTT\}$$

  - Define $X =$ "number of heads that came up."

$$X \cdot R \to R$$

$$X(HHH) = 3$$

$$X(TTT) = 0$$

  - $Y =$ "number of tails that come up."
  - Then, $Y(w) = 3 - X(w)$ for all $w \in S$.
  - $Z =$ "winnings in game where you win a dollar for each H, lose a dollar for each T."

$$Z(HHH) = 3$$

- If X and Y are RVs (random variables), and $a, b, c \in \mathbb{R}$, then $Z = aX + bY + c$ is a RV.
  - **Indicator random variable** for each event $A \subseteq S$
- **Example**: Suppose we toss b balls into N bins. Define X_i as the number of balls that land in the ith bin (for $i = 1, \ldots, n$). So $X_1 + X_2 =$ numbers of balls in the 1st or 2nd bincs,

$$\forall w \left(\sum_{i=1}^{n} X_i(w) = b \right)$$

- **Example**: When rolling two dice

$$X((i,j)) = i + j$$

$$\to Y((i,j)) = |i - j|$$

Range of a random variable

- **Range of a RV**: The set of values that X takes.
- **Example**: For $X =$ "number H's in 3 win tosses."

$$Range(X) = \{0, 1, 2, 3\}$$

$$Range(X) = \{0, 1, 2, 3\}$$

Partition of a random variable

```
      S
   +-------+    x     +-------+
   |  A_1  | -------> |  a_1  |
   |  A_2  | -------> |  a_2  |
   |  A_3  | -------> |  a_3  |
   |   .   |          |   .   |
   |   .   | -------> |   .   |
   |   .   |          |   .   |
   |  A_k  | -------> |  a_k  |
   +-------+          +-------+
```

$$A_1 = \{w \in S \;:\; X(w) = a_1\}$$

$$A_2 = \{w \in S \;:\; X(w) = a_2\}$$

- Suppose $Range(X) = \{a_1, \ldots a_k\}$,
  - For each i, define $A_i = \{w \in S \;:\; X(w) = a_i\}$ for $i = 1\ldots k$.

$$A_1 \subseteq S$$

  - **Claim**: For all $i \neq j$, $A_i \cap A_j = \emptyset$.
  - **Claim**: Every $w \in S$ is some A_i.
  - This means that $\{A_1, \ldots, A_k\}$ "partition S."
- **Example**: If $x =$ "number of Hs in 3 tosses," what is A_x?

$$Range(X) = \{0, 1, 2, 3\}(k = 4)$$

  - Need $\{A_0, A_1, A_2, A_3\}$ where $A_i = \{w \in S \;:\; X(w) = i\}$

$$A_0 = \{TTT\}$$

- **Example**: Flip a coin until we get H, on n flips

$$S = \{H, TH, \ldots, T^{n-1}H, T^n\}$$

  - $X =$ "number of flips"
  - What is A_x?
    - $Range(X) = \{1, \ldots, n\}$
    - Need $A_1, \ldots, A_n$ where $A_i = X^{-1}(i)$

Frequency Function of a random variable

- Formalize "X takes value I with probability p"
- If $Range(X) = \{a_1, \ldots, a_n\}$, we write

$$P(X = a_i) = P(A_i) = P(w : X(w) = a_i)$$

  - **Frequency function of x** (also probability mass function):

$$f_x(a_i) = P(x = a_i)$$

- **Example**: $X =$ "number of Hs in 3 tosses"
  - $f_x(0) = P(X = 0) = \frac{1}{8}$
  - $f_x(1) = P(X = 1) = \frac{3}{8}$
  - $f_x(2) = P(X = 2) = \frac{3}{8}$
  - $f_x(3) = P(X = 3) = \frac{1}{8}$

Independent random variables

- **Independent**: If X and Y are RVs on S, they are **independant** if for all i, j

$$P(X = i \wedge Y = j)$$

Repeated Indepedent Trials

- Start with a sample space T with probability measure P.
- Formalize repeating n times:

$$S^{(n)} = \{w = (w_1, \ldots, w_2) \ : \ w_i \in T\} = T^n$$

Bernoulli trials

- Sample space $S = \{s, f\}$, "success" and "failure."
    - Define $P(s) = p$, $P(f) = 1 - p$
    - Consider repeat n times $S^{(n)}$ = "all string of length n with s and t"

$$P^{(n)}(w) = P(w_1)P(w_2)\ldots P(w_n)$$

March 11th, 2013 - Reading

4.1 Random Variables

- These quantities of interest, or, more formally, these real-valued functions defined on the sample space, are known as **random variables**.
- The total number of events that occur without specifying the order in which they came up.

Example 1a

- Suppose that our experiment consists of tossing 3 fair coins. If we let Y denote the number of heads that appear, then Y is a random variable taking on one of the values 0, 1, 2, and 3 with respective probabilities

$$P\{Y = 0\} = P\{(T, T, T)\} = \frac{1}{8}$$

$$P\{Y = 1\} = P\{(H, T, T), (T, H, T), (T, T, H), \} = \frac{3}{8}$$

$$P\{Y = 2\} = P\{(H, H, T), (T, H, H), (H, T, H), \} = \frac{3}{8}$$

$$P\{Y = 3\} = P\{(H, H, H)\} = \frac{1}{8}$$

March 11th, 2013 - Lecture: More on Random Variables; Expectation

Introduction

- Topics: More on binomial random variables.
- Geometric and negative binomial random variables.
- Definition of expectation and the alternative formulation.

- Reading: See notes part 3 on Sakai.

Warmup for Homework

- **Example**: Deal a 5 card hand, let $X =$ "the number of aces", $Y =$ "the number of spades", and $Z = X(w) \cdot Y(w)$

$$Range(X) = \{0, 1, 2, 3, 4\}$$

$$Range(Y) = \{0, 1, 2, 3, 4, 5\}$$

$$Range(Z) = Range(XY) = \{0, \ldots, 9\}$$

Are X and Y Independent?

- If indepedent, then dor all $i \in Range(X), j \in Range(Y)$

$$P(X = i \cap Y = j) = P(X = i)P(Y = j)$$

$$P(X = 2 \cap Y = 5) = \emptyset$$

  - But $P(X = 2) > 0, P(Y = 5)$ so not independent.

Binomial Frequency Function

- $S^{(n)} =$ "all strings of length N formed with s and t characters"
- $S^{(n)}(w) = P(w_1) \ldots P(w_n)$ where $w = w_1 \ldots w_n$

$$p^k (q - p)^{n-k}$$

- $S_n =$ "the number of ss in n trials"
- $Range(S_n) = \{0, 1, \ldots n\}$
- $f_{s_n}(k) = \binom{n}{k} P^k (1 - p)^{n-k} = P(S_n = k)$
- Need

$$\sum_{k=0}^{n} f_{s_n}(k) 1$$

$$= \sum_{k=0}^{n} \binom{n}{k} P^k (1 - p)^{n-k} =$$

$$(p + P(1 - p))^n = 1^n = 1$$

- **Example**: An airplane has 200 seats, but we should 202 tickets. Assume passengers fail to show with probability 0.03 *indepedantly*. What is the chance that flight is over full?
  - This "s" is "passenger i fails to show."
  - $X =$ "the number of passengers who failr to show"
  - Event we want is $X = 1 \cup X = 1$

$$P(X = 0) = \binom{202}{0} p^0 (1 - p)^{202-0} = .03$$

$$P(X = 1) = \binom{202}{2} p^1 (1 - p)^{202-1}$$

Repeat Bernoulli trial until success

- $S' = \{s, fs, ffs, \dots\}$
- $P^{(n)} =$ as before
- Define $W_1 =$ "the number of trials until success."
- $Range(W_1) = \{1, 2, \dots\}$
- $A_{w_1} = \{A_1, A_2, \dots\}$, where $A_i = \{W : W_1(w) = i\} \subseteq S$.
- $P(i) = P(W_1 = 1) = P(A_i) = (1-p)^{i-1} p$

$$\sum_{i=1}^{\infty} (1-p)^{i-1}$$

$$= p \cdot \frac{1}{p} =$$

Negative Binomial Frequency Fun

- Repeat Bernoulli unti lwe get k s's

$$S = f * sf * s \dots f * s$$

  - $W_k =$ "number of triials until k s's"
  - $A_{w_k} = \lbrace k, k + 1, ... \rbrace
  - $A_k = \{ss \dots s\}$
  - $A_{k+1} = \{fs \dots s, sfs \dots s, \dots, ss \dots sfs\}$

$$|A_{k+1}| = k$$

Miracle of the linearity of expectation

- For any random variables X and Y, the expectation splits and you sum it.

$$E(X + Y) = E(X) + E(Y)$$

St. Petersburg Paradox

- You pay c dollars
- I flip coin until H. Say it takes i flips.
- I give you 2^{i-1} dollars

March 13th, 2013 - Homework 4

1. (3 points) In the card game bridge we deal 13-card hands to 4 players named North, South, East, and West (all 52 cards are dealt). What is the probability that East and West have no spades?
   - The total number of possibilities in the sample space for this situation is the total number of possible card orders over the number of cards each player is dealt factorial raised to the fourth.

$$\frac{52!}{13!^4}$$

   - There are 13 Spades in a deck of cards, and then 39 non-Spade cards.
   - For the hands of East and West, who collectively represent 26 cards, and who are vying for 39 non-Spade

cards, this is the number of possibilities:

$$\binom{39}{26}$$

- North and South also account for 26 total cards, and they are vying for the remaining cards including Spades. Twenty-six cards were taken out of the deck by East and West.

$$\binom{26}{26} = 1$$

- The probability is therefore:

$$\frac{\binom{39}{13} \times \binom{26}{13} \times \binom{26}{13} \times \binom{13}{13}}{\frac{52!}{13!^4}}$$

2. (4 points) An urn contains $n > 0$ white balls and $m > 0$ black balls. Suppose we draw two balls without replacement. What is the probability that the balls are of the same color? What if we draw them with replacement? Show your work. Which of these probabilities is larger? Briefly explain some intuition for why one should be larger.
   - Without replacement
     - There are $n + m$ balls in all, and we're interested in choosing 2 of them. The sample space is:

$$\binom{m+n}{2}$$

     - Call W "drawing two white balls" and B ``drawing two black balls. Then $W \cup B$ is either of those events happening or both.
     - The events are mutually exclusive, the drawings cannot all be the same for the same drawing.

$$P(W \cup B) = P(W) + P(B)$$

$$= \frac{\binom{m}{2} + \binom{n}{2}}{\binom{m+n}{2}}$$

   - With replacement
     - The events are, again, mutually exclusive.
     - Being as the balls are placed back in the urn, each event is equally likely to occur.
     - Count the number of either color, place it over total balls, and multiply it by itself to represent that the event has to happen twice.
     - Do the same for the other color, add the two values.

$$\left(\frac{m}{m+n}\right)^2 + \left(\frac{n}{m+n}\right)^2$$

3. (4 points) Again consider an urn with $n > 0$ white balls and $m > 0$ black balls. Suppose we draw $r \geq 1$ balls from the urn without replacement. What is the probability that we draw exactly k white balls?
   - The probability of getting a white, where i is the number of total balls already drawn, is

$$\left(\frac{n-i}{m+n-i}\right)$$

   - And the probability of getting a black under the same conditions is

$$\left(\frac{m-i}{m+n-i}\right)$$

- So draw k white balls, and then transfer the counter to drawing the remaining balls to avoid over-counting

$$\left(\prod_{i=0}^{k-1}\frac{n-i}{m+n-i}\right)\times\left(\prod_{i=k}^{r-1}\frac{m-(i-k)}{m+n-i}\right)$$

4. (4 points) A box contains a mixture of cubes and spheres and any of these objects can be either white or black. Suppose the box contains 4 black cubes, 6 black spheres, 6 white cubes, and x white spheres. Consider the experiment of drawing a random object from the box, and let A be the event that a cube is drawn and B be the event that a black object is drawn. If A and B are independent, what is x?
   - The probability of a cube being drawn is the total number of cubes over the total number of cubes and spheres.

$$\frac{4+6}{4+6+6+x}$$

   - The probability of a black object being drawn is similarly the total number of black objects divided by the total number of objects

$$\frac{6+4}{4+6+6+x}$$

   - Consider the complement of $A \cup B$, it is the event that a non-black and non-cube item is draw, which is a white sphere.

$$P(x) = P((A \cup B)^c)$$

5. (10 points total) Two fair dice are rolled. Define the random variables $X =$ the sum of the two rolls, $Y =$ the maximum of the two rolls, $Z =$ the absolute value of the difference of the two rolls and $W = XY$ (i.e., the product of X and Y).
   1. (2 points) What are $\text{Range}(X)$, $\text{Range}(Y)$, $\text{Range}(Z)$ and $\text{Range}(W)$?
      - $\text{Range}(X) = \{2, 3, 4, \ldots, 11, 12\}$
      - $\text{Range}(Y) = \{1, 2, 3, 4, 5, 6\}$
      - $\text{Range}(Z) = \{0, 1, 2, 3, 4, 5\}$
      - $\text{Range}(W) = \{i \times j \ : \ i \in \{2, 3, 4, \ldots, 11, 12\}, j \in \{1, 2, 3, 4, 5, 6\}\}$
   2. (2 points) What are the partitions $\mathcal{A}_X$ and $\mathcal{A}_Z$?
      - $\mathcal{A}_X$
        - $2 = 1 + 1$
        - $3 = 1 + 2$
        - $4 = 1 + 3 = 2 + 2$
        - $5 = 1 + 4 = 3 + 2$
        - $6 = 1 + 5 = 2 + 4 = 3 + 3$
        - $7 = 1 + 6 = 2 + 5 = 3 + 4$
        - $8 = 2 + 6 = 3 + 5 = 4 + 4$
        - $9 = 3 + 6 = 4 + 5$
        - $10 = 4 + 6 = 5 + 5$
        - $11 = 5 + 6$
        - $12 = 6 + 6$
      - $\mathcal{A}_Z$
        - $0 = 1 - 1 = 2 - 2 = 3 - 3 = 4 - 4 = 5 - 5 = 6 - 6$
        - $1 = 6 - 5 = 5 - 4 = 4 - 3 = 3 - 2 = 3 - 2 = 2 - 1$

- $2 = 5 - 4 = 5 - 3 = 4 - 2 = 3 - 1$
- $3 = 6 - 3 = 5 - 2 = 4 - 1$
- $4 = 6 - 2 = 5 - 1$
- $5 = 6 - 1$

March 13th, 2013 - Lecture: Linearity of expectation

Introduction

- Topics:
  - Linearity of expectation.
  - Expectation of binomial, geometric, and negative binomial frequency functions.
- Applications to computing expectations:
  - Birthday problem,
  - balls into bins,
  - and coupon collecting.
- Reading: See notes part 3 on Sakai.

March 25th, 2013 - Lecture: Markov's Inequality; Begin Variance

Introduction

- Topics:
  - Markov's inequality for non-negative random variables.
  - Definition of variance and some calculations of variance.
- Reading: See notes part 4 on Sakai.

March 26th, 2013 - Office Hours

1. Norwa on number one
2. Notes on number two
   - There are no tricks, apply the formula.

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_1)$$

   - What is E(X) for X = minimum of two dice?

$$P(x = 6) = P(\{(6, 6)\}) = \frac{1}{36}$$

$$P(x = 5) = P(\{(5, 6), (6, 5), (5, 5)\}) = \frac{3}{36}$$

3. Notes on number three
   - Find $E(X)$
     1. Find some random variables $X_1, \ldots, X_n$ such that $x = x_1 + \ldots + x_n$
     2. Now $E(X) = E(x_1) + \ldots E(x_n)$
   - We're going to buy x boxes of cereal
   - Call x the number of unique toys
   - We're always stopping at n boxes.

- Imagine there six types of toys, red, green, blue, purple, orange, indigo.
    - When I open my n boxes, I check off the color I got until I get them all.
    - Call x_1 "got a red toy indicator"
    - There's another for each toy, 1 through 6.
4. Notes on number four
    - There m men and w women in a line in a single line, random order, all orders equally likely.
    - You can pick any pair of seats, this pair has formed a couple, if a mw or a wm occur.
    - Use linearity of expectation
    - x = number of couples

    ```
        m w m

    _ _ _ _ _ _ _ _
    1 2 3 4 5 6 7 8
    x_1
    ```

    - $E(X_1) =$ P(couple in a seat 1 and 2)
5. Notes on five
    - What is a Bernoulli trial?
    - So you see this problem there are $2^1 2$ different outcomes, different trials.
        - Big number, don't want to work it out by hand.
    - We know something about this though, it's connected to something we've seen before.

$$E(\frac{S}{12})$$

March 27th, 2013 - Homework 4

1. (10 points total) Two teams x and y are playing each other in the World Series, which is a best-of-seven-game match that ends when one team wins 4 games. Assume that team x wins each game with probability p, and that the outcome of each game constitutes an independent trial.
    1. (0.5 points) What is the probability that x wins the first four games?
        - There is one way for x to win in four, and that's winning four in a row.
        - By the multiplicity principle, we can multiply the chance of winning each game to get the probability of winning all games.

$$p^4$$

    2. (2 points) What is the probability that x wins four games after at most five game have been played?
        - Sample space, strings, x character means x won, y character means y won.

        ```
                xxxx
               yxxxx
               xyxxx
               xxyxx
               xxxyx
        ```

        - The team represented by y cannot win at the end because after 4 x wins, the games halt.
        - P(x wins in less than five games) = $P(\text{xxxx}, \text{yxxxx}, \text{xyxxx}, \text{xxyxx}, \text{xxxyx})$
        - There is a $(1-p)$ chance of y winning.
        - The x team still needs to win 4 games, which has a probability of p^4.
        - Multiplying these two values gets you the likelihood that this happens for any given instance of x winning in less than five games.

- To get all of the possible less than 5 games combinations, multiply by the number there are, which is 4.
- Then, add the probability of just 4 straight victories.

$$(1-p) \times p^4 \times 4 + p^4$$

3. (2 points) What is the probability that x will win four games before y wins four games? (i.e., What is the probability that x wins the Series?)
   - To begin, lets enumerate some possibilities using strings.

$$\text{xxxx}$$

$$\text{yxxxx, xyxxx, xxyxx, xxxyx}$$

$$\text{yyxxxx, yxyxxx}, \ldots, \text{xxxyyx}$$

$$\text{yyyxxxx, yyxyxxx}, \ldots, \text{xxxyyyx}$$

   - These are all the possibilities for x winning. Name these X_1 through X_4 and notice the sum of their probabilities to be the probability that x wins four games.
   - For X_3, there are $\binom{5}{2,3}$ ways of ordering x and y, y has to win twice, and x still has to win four times.
   - You have to subtract off the cases where there is a y at the end,

$$\binom{4}{1,3} \times (1-y)^2 \times p^4$$

   - For X_4, there are $\binom{6}{3,3}$ ways of ordering x and y, y has to win thrice, and x still has to win four times.

$$\binom{5}{2,3} \times (1-y)^3 \times p^4$$

   - Now add 4 straight victories:

$$\binom{4}{1,3} \times (1-y)^2 \times p^4 + \binom{5}{2,3} \times (1-y)^3 \times p^4 + p^4$$

4. (0.5 points) Calculate and simplify your answer in part (c) when $p = 1/2$ and when $p = 2/3$.
   - $p = 1/2 : 1/2$
   - $p = 2/3 : 1808/2187$
5. (1 point) Let X be the random variable that counts the number of games that are played. What is $\text{Range}(X)$?

$$R(X) = 4, 5, 6, 7$$

6. (2 points) What is $P(X = 7)$?

$$\binom{6}{3}(1-p)^3 p^4 + \binom{6}{3} p^3 (1-p)^3 p^4$$

7. (2 points) What is $P(X \geq 6)$?

$$\binom{5}{2}(1-p)^2 p^4 + \binom{5}{2} p^2 (1-p)^4 + \binom{6}{3}(1-p)^3 p^4 + \binom{6}{3} p^3 (1-p)^4$$

2. (4 points) Suppose we roll two fair dice. Let the random variable $X =$ “the minimum of the two dice” and $Y =$ “the

absolute value of the difference of the two dice". Find $E(X)$ and $E(Y)$.

- $E(X)$
  - The formula to solve this is

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_1)$$

  - The sample space has a cardinality of 36, as there 6 choices for each of the two choices.
  - All rolls are equally likely in a fair dice.
  - Starting with the highest element in the range (which is the set containing 1 through 6), there is only one way 6 can be the minimum.

$$P(x = 6) = P(\{(6, 6)\}) = \frac{1}{36}$$

  - There are 3 ways of "getting 5", and that's rolling two fives, and then both variations of a five and a six.

$$P(x = 5) = P(\{(5, 5), (5, 6), (6, 5)\}) = \frac{3}{36}$$

  - Apply the same pattern,

$$P(x = 4) = P(\{(4, 4), (4, 5), (5, 4), (4, 6), (6, 4)\}) = \frac{5}{36}$$

$$P(x = 3) = P(\{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3), (3, 6), (6, 3)\}) = \frac{7}{36}$$

$$P(x = 2) = P(\{(2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (2, 5), (5, 2), (2, 6), (6, 2)) = \frac{9}{36}$$

  - Notice that there 2 more each time.

$$P(x = 1) = \frac{11}{36}$$

  - Now sum them and multiply them by their value (a_i) to find expected value,

$$E(X) = \left(6 \times \frac{1}{36}\right) + \left(5 \times \frac{3}{36}\right) + \left(4 \times \frac{5}{36}\right) + \left(3 \times \frac{7}{36}\right) + \left(2 \times \frac{9}{36}\right) + \frac{11}{36} = 2.5277777778$$

- $E(Y)$
  - The range is $0, 1, 2, 3, 4, 5$.
  - This is another application of the formula

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_1)$$

  - There are 6 ways of getting 0, $6 - 6, 5 - 5, \ldots$. For $i = 0$,

$$0 \times P(X = 0) = 0 \times \frac{6}{36} = 0$$

  - There are 10 ways of getting 1, $6 - 5, 5 - 6, 5 - 4, 4 - 5, 4 - 3, 3 - 4, 3 - 2, 2 - 3, 2 - 1, 1 - 2$. For $i = 1$,

$$1 \times P(X=1) = 1 \times \frac{10}{36}$$

- There are 8 ways of getting 2, $6-4, 4-6, 5-3, 3-5, 4-2, 2-4, 1-5, 5-1$. For $i=2$,

$$2 \times P(X=2) = 2 \times \frac{8}{36} = \frac{4}{9}$$

- There are 6 ways of getting 3, $6-3, 3-6, 5-2, 2-5, 4-1, 1-4$. For $i=3$,

$$3 \times P(X=3) = 3 \times \frac{6}{36} = \frac{2}{3}$$

- There are 4 ways of getting 4, $6-2, 2-6, 5-1, 1-5$. For $i=4$,

$$4 \times P(X=4) = 4 \times \frac{4}{36} = \frac{4}{9}$$

- There are 2 ways of getting 5, $6-1, 1-6$. For $i=5$,

$$5 \times P(X=5) = 5 \times \frac{2}{36} = \frac{5}{18}$$

- We know we've covered the sample space because the sum of the specific instances is the same as the cardinality of the sample space.
- The expected value,

$$E(X) = 0 + \frac{2}{9} + \frac{4}{9} + \frac{2}{3} + \frac{4}{9} + \frac{5}{18} = 1.94 \approx 2$$

3. (4 points) Suppose boxes of cereal are filled with a random prize, each drawn from independently and uniformly from 6 possible prizes. If we buy N boxes of cereal, what is the expected number of distinct prizes we will collect?
   - Let $X_1 \ldots X_6$ be the identifier for each of 6 unique toys

$$E\left(\sum I_{E_i}\right) = \sum_{i=1}^{6} E(I_{E_i}) = 6(1 - \left(5/6)\right)^n$$

4. (4 points) A group of m men and w randomly sit in a single row at a theater. If a man and woman are seated next to each other we say they form a couple. (Couples can overlap, meaning that one person can be a member of two couples.) What is the expected number of couples?
   - Use linearity of expectation for each pair of seats.
   - Let x equal the number of seats.
   - Then, $E(X_1) = P($couple in a seat 1 and 2$)$, $E(X_2) = P($couple in a seat 2 and 3$)$, etc.
   - For any given seat, there is a $\frac{1}{2}$ chance of their being a man or a woman in the seat.
   - The four possibilities are {mw, wm, mm, ww}.
5. (3 points) Suppose an experiment tosses a fair coin twice; the experiment "succeeds" if both tosses were Heads. We repeat this experiment for 12 independent trials. Let N be the random variable that counts the fraction of trials that are successful (so $N = S/12$, where S is the number of successful trials). Find $E(N)$.

$$E(N) = E\left(\frac{S}{12}\right)$$

$$S = x_1 + \ldots + x_1 2$$

$$\frac{1}{12}E(S)$$

$$E(S) = \sum_{a_i \in R(S)} a_i \times P(X_i)$$

$$P(X_i) = \frac{1}{4}$$

$$E(S) = 12 \times E(X_i)$$

$$\frac{1}{12}E(S) = 12 \times \frac{1}{4}$$

$$E(S) = \frac{1}{4}$$

March 27th, 2013 - Lecture: Computing Variances

Introduction

- Topics:
  - Expectation of a function of a random variable.
  - Variance of a sum of independent random variables.
  - Variance of the bernoulli, binomial, geometric, and negative binomial distributions.
- Reading: See notes part 4 on Sakai.

	"story"	$R(X)$	**fx**	E	V
Bernoulli(P)	s/f with prob p	$\{0, 1\}$	$p, 1+p$	p	$p(1-p)$
Binom(n, p)	$\#s$ in n trials	$\{0, \ldots, n\}$	$f_x(i) = \binom{n}{i}p^i(1-p)^{n-i}$	pn	...
Geom(p)	# of trials until first s	$\{1, 2, \ldots\}$	$f_x(i) = (1-p)^{i-1}p$	$\frac{1}{p}$	...
NegBiom(k,p)	# trials until kth s	$\{k, k+1, \ldots\}$	$f_x(i) = \binom{i-1}{k-1}p^k(1-p)^{i-k}$	$k \times \frac{1}{p}$	...

April 1st, 2013 - Lecture: Covariance and Chebyshev's Inequality

Topics

- Variance of a sum of dependent random variables and the definition of covariance.
- Properties of covariance. Independence implies correlation but the converse does not hold.
- Statement of Chebyshev's inequality and a basic application.

Introduction

- If X and Y are indepedant, then

$$V(X+Y) = V(X) + V(Y)$$

- If not, then equality may or may not hold.
- **Example**: A case where equality does not hold.
  - Take $P(X=1) = \frac{1}{2}$, $P(X=0) = \frac{1}{2}$, $x = 1 \rightarrow Y = 0$, $x = 1, Y = 1$.

$$V(Y) = \frac{1}{4}$$

- So

$$V(X)+V(Y)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

- $V(X+Y)$ equals 0? $X+Y$ will always be 1.

$$V(X+Y)<V(X)+V(Y)$$

Variance

- "Swinging together"

$$V(X+Y)<V(X)+V(Y)$$

- "Swings unrelated"

$$V(X+Y)=V(X)+V(Y)$$

- "Swings are opposite each other"

$$V(X+Y)>V(X)+V(Y)$$

- We want to measure "how much they're together or opposite."
- The **covariance of X and Y** is defined to be:

$$Cov(x,y)=E[E(x-E[x])(y-E[y])]$$

- Intuition
  - "Swing together"

$$Cov(X,Y)>0$$

  - "Swings unrelated"

$$Cov(X,Y)=0$$

  - "Swings opposite"

$$Cov(X,Y)<0$$

- **Example** (non rigourous pictures)
  - Examples, x is height and y is shoe size

$$Cov(X,Y)>0$$

```
 v |               .
 a |             . .
 l |          . . .
 u |        . . .
 e |      . . .
   |    . .  .
 y |    . .
   |  .
 0 +-----------------
  0    value of X
```

- Examples, x is temp and y is sales of hot chocolate

$$Cov(X, Y) < 0$$

```
 v | .
 a |  . .
 l |   .  . .
 u |    .  .  .
 e |     .  .  .
   |      .  .  .
 y |        .   .
   |            .
 0 +-----------------
  0    value of X
```

- **Formula**:

$$Cov(X, Y) = E(XY) - E(X) \times E(Y)$$

- **Claim**: If X, Y are indepedant, $Cov(X, Y) = 0$
  - Proof

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

$$= E(X)E(Y) - E(X)E(Y)$$

$$= 0$$

Tchbycheff's Inequility

- Recall Markov's give tail bound base only on $E(X)$
- Chebyshev's give tail bound using $E(X)$ and $V(X)$
- **Chebyshev's Theorem**: Let X be a random variable with $E(X) = \mu$
  - Then for any $\epsilon > 0$.
- **Example**: Roll a fair die 100 times and let Z be the sum and X_1, ..., X_{100} be the outcomes.
  - What's the probability that Z are within 50 of its mean?

April 2nd, 2013 - Office Hours

Number 3

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

- Call z_1 the first roll, z_2 the second.

$$E(XY) = \sum_{w \in S} X(w)Y(w)P(w)$$

- We can do this in principle, it takes fifteen minutes.
- More elegantly, however, take two random variables that represents the rolls, and it's nice because they're indepedant.

$$X = z_1 + z_2$$

$$Y = z_1 - z_2$$

$$XY = (z_1 + z_2)(z_1 - z_2)$$

$$= z_1^2 - z_2^2$$

$$E(XY) = E(z_1^2) - E(z_2^2)$$

$$= 0$$

Banach-Torski Paradox

- There are lots of functions which are not integrable.
- Take the function $f(x)$, which is 0 if x is rational, 1 otherwise.
- $[0, 1] \to \mathbb{R}$

$$\int_0^1 f(x)dx$$

- Ball in 3-space, cut it up into peices and put them in sets, move the set around.
- And you can double the volume! (???)
- Axiom of choice.

> Physics is looking around you and drawing conclusions, but something really meaningful to me is that what you have infallible premises, you have an infallible conclusion. *David Cash*

Zero-knowledge proof

- Zero knowledge proofs are a different way of thinking of proofs.
  - A regular proof involves a prover and verifier.
  - If each step is right, the then conclusion is right, and you've proven something.
- If the statement "x is not prime" (where x is a hundred digit number).
- You want to prove to a server that I know my password, you know I know it, but you don't know my password.

Journal of Craptology: Zero-Knowledge Proofs for Kids

- There's a Waldo, you don't know where he is, but there is one.
- Take a big piece of cardboard, and cut a Waldo sized hole, and show the person Waldo.
- They know there's a Waldo, but have no knowledge about where the Waldo is.

April 3rd, 2013 - Lecture

Public polling

- Say we want to estimate how much of the country will vote independent in the next presidential election.
  - Call n random people and ask what they'll do, with a country of m people.
  - Average response to estimate fraction of country that will vote independent.
  - How big should n be to be wintin $\pm 3\%$ with probability $\geq 95\%$
  - Let x_i $=$ "ith person we call say will vote indepedant."

$$A_n = \frac{x_1 + \ldots + x_n}{n}$$

  - Say a p fraction of the country will vote independant (unknown).

- Want to know:

$$P(|A_n - p| > 0.03) \leq 0.05$$

- Use Cheb.

$$P(|A_n - p| > 0.03) \leq \frac{V(A_n)}{(0.03)^2} \leq \frac{1}{n \times 0.03^2}$$

$$V(A_n) = V(\frac{x_1 + \ldots x_n}{n}) = \frac{1}{n} V(X_1)$$

$$= \frac{p(1-p)}{n}$$

$$\leq \frac{1}{n}$$

- Want $\frac{1}{n \times 0.03^2} \leq 0.05$

Back to counting! Trees

- Counting number of binary trees with n nodes
  1. $n = 0$, $a_0 = 1$
  2. $n = 1$, $a_1 = 1$
  3. $n = 2$, $a_2 = 2$
  4. $n = 3$, $a_3 = 5$
  5. $n = 4$, $a_4 = 14$

$$a_n = \frac{1}{n+1}\binom{2n}{n}$$

Generating functions

- Want to find formula for $a_0, a_1, a_2, \ldots$.
- Write $\{a_i\}_{i=0}^{\infty}$ or $\{a_i\}$ for the sequence.
- **Definition**: The *generating function for* $\{a_i\}$ is the function $A(x)$ defined by

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

- (Treat as "formula sum", meaning ignore convergence)
- Problem sovling recipe for generating functions
- Want to find $a_0, a_1, a_2, \ldots, a_n$
  1. Show that $A(x) has some simple "closed form"

April 8th, 2013 - Lecture

Announcements

- Ungraded HW on Sakai
- New noes on generating functions
- From *Applied Combinatorics* by Alan Tucker

- HW07 online tonight, due next Monday
- Office hours this weeks:
  - 3:00PM to 4:00PM on Thursday
  - 2:00PM to 3:00PM on Friday

Generating functions revisited

- **Question**: How many solutions are there to $z_1 + z_2 = 11$, with z_1, z_2 as non-negative integers.
  - Some mediocre ways:
    - You can list these by hand ...
    - There are 12, you can list them but I won't.
    - It can also be a stars and bars problem, 11 stars, 2 groups, empty allowed.
  - But consider multipling out:

$$(1+x+x^2+\ldots+x^{11})(1+x+x^2+\ldots+x^{11})$$

  - This is going to be:

$$\sum_{x=0}^{22} a_k x^k$$

  - Take a look at this pattern:

$$a_0 = 1|()$$

$$a_1 = 2|(x^0x^1 + x^1x^0 = 2x)$$

$$a_2 = 3|(x^0x^2 + x^1x^1 + x^2x^0 = 3x^2)$$

  - Now consider that $a_1 1 = 12$. This is not a coefficient.
  - Somehow counting the number of polynomials counted this other number we wanted.
  - Generating functions are about doing this counting with shortcuts.
- Considering multiplying out $(1+x+x^2)^4 =$

$$(1+x+x^2)(1+x+x^2)(1+x+x^2)(1+x+x^2)$$

  - How to "expand"?
    - Pick x^0, x^1, x^2 from each group, multiplying choices.
    - Repeat for all possible choices (3^4 of them).
    - Add up results, collect like terms.
    - Each term is of the form:

$$x^{e_1}x^{e_2}x^{e_3}x^{e_4}$$

    - Each $e_i = 0, 1, 2$
    - Expanded version collects for all possible allowed.
- What is the coefficient of x^5 in $(1+x+x^2)^4$.
  - This equals the number of ways x^5 "shows up" in terms.
  - Coefficient is number of ways to wrte $5 = e_1 + e_2 + e_3 + e_4, e_i = 0, 1$
- **Examples**: Find polynomial where $a_k =$ is equal to the number of ways to pick k balls from a pile of 3 green, 2 white, 3 blue, and 3 gold.
  - Stated as "equ problem", this means that $a_k =$ numer of solutions to

$$e_1 + e_2 + e_3 + e_4 = k, e_i \in \{0234\}$$

    - With grreen, white, blue, and gold (respectively)

- **Example**: Same problem, but 7 green, 2 blue, 5 gold.
  - The polynomial is

$$(x^0 + x^1 + \ldots + x^7)(x^0 + x^1)(x^0 + x^1 + \ldots + x^5)$$

- **Example**: Find the polynomial with $a_k =$ "number of ways to pick 6 objects where each one is one of 3 types, but you can't have four of any one type."

April 15th, 2013 - Homework 7

1. (1.5 points each) Find the generating function for the sequence $a_0, a_1, a_2, \ldots$, where a_k is each of the following. Your solution does not need to be closed form.
   1. $a_k =$ the number of solutions to $e_1 + e_2 + e_3 = k$, where $0 \leq e_i \leq 4$ for each i.

$$(1 + x^1 + x^2 + x^3 + x^4)^3$$

   2. $a_k =$ the number of solutions to $e_1 + e_2 + e_3 + e_4 = k$, where $0 \leq e_i < 4$ for each i, e_1 is odd, and e_2 is even.

$$(x^1 + x^3)(x^0 + x^2)(1 + x^1 + x^2 + x^3 + x^4)^2$$

   3. $a_k =$ the number of solutions to $e_1 + e_2 + e_3 + e_4 = k$, where $0 \leq e_i$ for each i.

$$\left(\sum_{i=0}^{\infty} x^i\right)^4$$

   4. $a_k =$ the number of solutions to $e_1 + e_2 + e_3 + e_4 + e_5 = k$, where $0 \leq e_i$ for each i, e_1 and e_3 are odd, and e_2 is even.

$$\left(\sum_{i=0}^{\infty} x^{2i+1}\right)^2 \times \left(\sum_{i=0}^{\infty} x^{2i}\right) \times \left(\sum_{i=0}^{\infty} x^i\right)^2$$

2. (3 points each) Model the following problems using a generation function, which does not need to be in closed form:
   1. Count the number of outcomes of rolling 6 dice that sum to r.

$$(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^6,\ 1 \leq x_i \leq 6$$

   2. Count the number of outcomes of rolling 6 dice that sum to r, where the first three dice are odd and the last three are even.

$$(x_1 + x_3 + x_5)^3 \times (x_2 + x_4 + x_6)^3,\ 1 \leq x_{2i+1} \leq 3, 0 \leq x_{2i} \leq 2$$

   3. Count the number of outcomes of rolling 6 dice that sum to r, where for each i the i-th dice is not equal to i (so the first die is not 1, the second is not 2, and so on).

$$= (x^2 + x^3 + x^4 + x^5 + x^6) \times (x^1 + x^3 + x^4 + x^5 + x^6) \times$$

$$(x^1 + x^2 + x^4 + x^5 + x^6) \times (x^1 + x^2 + x^3 + x^5 + x^6) \times$$

$$(x^1+x^2+x^3+x^4+x^6)\times(x^1+x^2+x^3+x^4+x^5)$$

3. (1.5 points each) Find the following coefficients. Show your work.
   1. The coefficient of x^{10} in the series expansion of $(x^5+x^6+x^7+\cdots)^8$.

      $$\text{DNE}$$

   2. The coefficient of x^{20} in the series expansion of $(x+x^2+x^3+x^4+x^5)(x+x^2+x^3+x^4+\cdots)^5$.

      $$\binom{18}{14}$$

   3. The coefficient of x^{12} in the series expansion of $x^2/(1+x)^8$.

      $$\binom{8+10-1}{10}$$

   4. The coefficient of x^{12} in the series expansion of $1/(1+x^3)^2$.

April 15th, 2013 - Lecture

- **Idea**: To solve $a_1, a_2, \ldots$
  - Part 1: Define generating function

    $$A(x)=\sum_{k=0}^{\infty}$$

    - "Pull out" the initial conditions
    - Substitute recurance relation formula
    - Expand, algebra, etc until you can substitute $A(x)$
    - Get $A(x)=$ "some formula of A(x)"
    - Solve for A(x)
  - Part 2: Find the series expansion
    - Easiest case, like above, $A(x)$ is in the table.
    - Work harder, use partial fraction decomposition

April 17th, 2013 - Notes pt. 5

Generating functions

- Let $a_0, a_1, \ldots$ (or more briefly $\{a_i\}$) denote an infinite sequence of real numbers. Its **generating function** is defined by

  $$A(s)=\sum_{k=0}^{\infty}a_k s^k=a_0+a_1 s+\ldots+a_k s^k+\ldots$$

  - We are not claiming this series converges.
  - In some sense, you can view this as a formalism for infinite series.
  - When the series does converge with a function with algebraic properties, we will make use of this correspondance.
- Facts:

1. $A(0) = a_0$
   - The first element in the sequence.
2. $A(1) = \sum_{k=0}^{\infty} a_k$
   - The sum of the elements.
3. $A`(1) = \sum_{k=1} \infty k a_k s^{k-1}|_{s=1}$
   - Differentiate each term of the sum in (1) and substitute.
4. $A + B = \sum_{k=0}^{\infty} (a_k + b_k) s^k$
   - Sum
5. $A(s)B(s) = a_0 b_0 + (a_0 b_1 + a_1 b_0)s + \ldots + (a_0 b_k + \ldots + a_k b_0)s^k$
   - Multiplication, convolutions

Generating Functions and Recurrence Relations

- Given a recurrence relation for the sequence $\{a_i\}$,
  1. Deduce from it an equation satisfied by the generation function

$$a(x) = \sum_i a_i x^i$$

  2. Solve this equation to get an explicit expression for the generating function.
  3. Extract the coeeficient a_n of x^n from $a(x)$ by expanding $a(x)$ as a power series.
- Alternate steps **wikiHow**
  1. Consider the sequence 2, 5, 14, 41, 122 ... given by this formula:

$$a_0 = 2$$

$$a_n = 3a_{n-1} - 1$$

  2. Write the genrating function of the sequence. A generating function is simply a formula power series where the coeeficient of x^n is the nth term of the sequence.

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

  3. Manipulate the generating function. The objective in this step is to find an equation that will allow us to solve for the generating function $A(x)$. User the formula for the sum a geomtric series.
     - Original formula:

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

     - "Take out" when a is 0:

$$A(x) = 2 + \sum_{k=1}^{\infty} a_k x^k$$

     - Make the first term addition by referring to the original definition of this series.

$$A(x) = 2 + \sum_{k=1}^{\infty} (3a_{k-1} - 1) x^k$$

     - Split the sum

$$A(x) = 2 + \sum_{k=1}^{\infty} 3a_{k-1}x^k - \sum_{k=1}^{\infty} x^k$$

- Now recognize that if you take out the 3 as well as *an* x, you have the original formula.

$$= 2 + 3xA(x) - \frac{x}{1-x}$$

4. Solve for $A(x)$:

$$A(x) = \frac{3x-2}{(3x-1)(1-x)}$$

5. Find the coefficient of x^n using partial fractions or some other method.
6. Write the formual for a_n by identifying the coefficient of x^n in $A(x)$.

$$a_n = \frac{3^{n+1}+1}{2}$$

April 17th, 2013 - Lecture

- Counting number of binary trees on n nodes
  - Let b_n be the number of binary trees given n nodes.
  - $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, etc
  - What does an n node ree "look like"?

Practice Final Exam

1. What is the coefficient of $x^{12}x^9$ when $(x+y)^{21}$ is expanded? How many monomials are in the expansion?
   - This requires the binomial theorem.
   - You plug in 21 for n and 9 for k.

$$\binom{22}{12}$$

2. If $P(A \cap B) < P(A)$, is it always true that $P(A|B) < P(A)$? Either prove it, or disprove it by finding a counterexample.
   - You can disprove this with the values $P(A) = .5$, P(B) = .25$.
3. If we throw n balls into m urns, what is the probability that all of the balls land in exactly 1 urn? At most 2 urns?
   - The number of possible ways to distribute n balls into m possible urns is, according to the partition formula,

$$\binom{n+m-1}{m-1}$$

   - Let this be our sample space.
   - For exactly one urn, how many ways are there to distribute all the balls into one urn?

$$m$$

   - Making our answer m divided by the sample space.
   - How many ways are there to distribute n balls into at most 2 urns? This means that 1 urn is possible, but we're interested in 2 urns *as well*.
   - The number of ways to divide n indistinguishable balls into r possibly empty bins is:

$$\binom{n+r-1}{r-1}$$

- Our answer, therefore is,

$$\frac{m + \binom{n+2-1}{2-1}}{\binom{n+m-1}{m-1}}$$

4. Suppose we shuffle a standard deck of 52 cards and deal a 5 card hand. What is the probability we get a straight? What about a straight flush?
   - Our sample space is the beyond astronmically large:

$$\binom{56}{5}$$

   - Let's look at how many straights there are:

```
A  2  3  4  5  6  7  8  9  10  J  Q  K  A
+-----------+  +-----------+
   +-----------+  +------------+
      +-----------+  +------------+
         +-----------+  +------------+
                +----------+     +-----------+
```

   - I count 10 different ways of getting a straight.
   - So compared to our sample space, you must select 1 of 10 straights, then for each card, you must pick a suit.

$$\frac{\binom{10}{1}\binom{4}{1}^5}{\binom{56}{5}}$$

   - If we're only interested in the case where all of the suits are the same, you can simply take of the 5th power on the suit value.
5. We roll a die 10 times. Let X be the number of time a roll of 1, 2, or 3 comes up. Find the range of X, $P(X=3)$, $E(X)$, and $V(X)$.
   - So we're interested in the frequnecy function where we get "the number of trials until the first success" which is "geometric."
   - Our range, therefore, is from 0 to potentially inifinite.
   - The probability of success for any given trial, because of linearity of expectation, is:

$$\frac{3}{6}$$

   because there are 3 values that represent a sucess and 6 total possible values.

$$P(X=n) = \frac{1}{2}(1-\frac{1}{2})^{n-1}$$

$$E(X) = \frac{1}{\frac{1}{2}}$$

$$V(X) = \frac{1 - \frac{1}{2}}{(\frac{1}{2})^2}$$

6. Suppose again we shuffle a standard deck of 52 cards and deal a five card hand. Let A_i be the event that the i-th card is red.
   1. What is the probability of the event $E =$ "exactly two cards are red"?
      - You can exploit symetry and linearity of expectation here.
      - There are 56 cards in the first drawing, 55 cards in the second drawing.
      - There are 26 red cards in the first drawing, 25 red cards in the second (because we already "got one" in this calculation).
      - Now the rest of the cards have to be not red out of $26, 25, 24$ non-red cards in a deck of $54, 53, 52$.
      - Every fraction must be multiplied by the multiplicative principle.

      $$\binom{5}{2}\left(\frac{26}{56} \times \frac{25}{55} \times \frac{26}{54} \times \frac{25}{53} \times \frac{24}{52}\right)$$

      - Please note that there are in fact 52 cards in a deck.
   2. What is the probability of the event $F =$ "the first two cards are red"?
      - This is like the first question, but we just don't care about the cards after the first two. They could be red, they might not be, whatever the case, we do not have to account for it.

      $$\frac{26}{56} \times \frac{25}{55}$$

   3. Are E and F indepedant? Explain your answer.
      - Two events are indepedant if and only if the probability of their intersection is equal to the product of their individual probabilities. Formally,

      $$P(A \cap B) = P(A) \times P(B)$$

      - What is $A \cap B$? It is both that "the first two cards are red" and "exactly two cards are red." Which is the event that only the first two cards are red.
      - The probability that both of these events happen is not equal to the product of the individual probilities, and therefore this particular tuple of events are dependent on one another.
7. Consider an experiment where we flip three coins. Suppose we repeat this experiment until we get all Heads. Let X be the random variable that is the number of experiments needed. Find $E(X)$ and $V(X)$.
   - Again, we are dealing with a geometric frequency function because we're interested in the number of trials until the first success.
   - The success of any given trial is going to be the cube of the success of a single coin toss,

   $$\left(\frac{1}{2}\right)^3$$

   - Therefore, based on the formula for the geometric frequency function,

   $$P(X = n) = \left(\frac{1}{2}\right)^3 \left(1 - \left(\frac{1}{2}\right)^3\right)^{n-1}$$

   $$E(X) = \frac{1}{(\frac{1}{2})^3}$$

$$V(X) = \frac{1 - \left(\frac{1}{2}\right)^3}{(\left(\left(\frac{1}{2}\right)\right)^3)^2}$$

8. Consider the same experiment as before, except now we flip the coins 100 times. Let W be the random variable representing the number of time you get all Heads. Is the frequency function of W Bernoulli, binomial, geometric, or negative binomial? Find $E(W)$ and $V(W)$. Use Chebyshev's inequality to bound the probability that $|W - E(W)| is more than 10.
   - This is a binomial frequency function, as it represents the number of successes in n trials.
   - The range is limited by n.
   - n is equal to 100, as we are performing the trial that many times.
   - The probability of success in any given trial is still:

$$\left(\frac{1}{2}\right)^3$$

$$P(X = k) = \binom{100}{k}\left(\frac{1}{2}\right)^3\left(1 - \left(\frac{1}{2}\right)^3\right)^k$$

$$E(X) = \left(\frac{1}{2}\right)^3 \times 100$$

$$V(X) = \left(\frac{1}{2}\right)^3\left(1 - \left(\frac{1}{2}\right)^3\right) \times 100$$

9. Consider a group of n married couples which are seated at a rectangular table with n seats on each side. Let X be a random variable that counts the number of married couples that are seated next to each other. Find $E(X)$.
10. Suppose that 51 percent of babies are born girls. Suppose also that there is a prenatal test such that 98 percent of the baby girls come back positive. Use Bayes Theorem to compute the probability that the baby is a girl.
    - Bayes's theorem:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

   - $E =$ "the test for girl is positive"
   - $F =$ "the baby is a girl" = $.51$
   - $P(E|F) =$ "the prenatal test is right" $= .98$

$$\frac{.98 \times .51}{.98 \times .51 + .02 \times .49}$$

11. Give the generating function with the n-coefficient equal to the number of ways to solve $e_1 + e_2 + e_3 = n$ with $e_1, e_2 \geq 0$ and e_3 a multiple of 3. Give a close form version of your function.

$$e_1 = x^0 + x^1 + x^2 + x^3 \ldots$$

$$e_2 = x^0 + x^1 + x^2 \ldots$$

$$e_3 = x^0 + x^3 + x^6 + x^9 \ldots$$

   - Referring the "the chart", not that e_1 and e_2 are of the form:

$$\frac{1}{x-1}$$

- And similarly, e_3 is of the form:

$$\frac{1}{x^3-1}$$

- Yielding the closed form:

$$\left(\frac{1}{1-x}\right)^2\left(\frac{1}{1-x^3}\right)$$

12. Find a formula for a_n, which is defined by the following recurrence relation for all $n > 0$:

$$a_0 = 9$$

$$a_n = 2a_{n-1} + 2$$

$$2a_{n-1}2 + 9 = a_n + a_0$$

$$\sum_{n=0}^{\infty} a_n x^n$$

$$x^n(2a_{n-1}$$

13. Consider the coin flipping game, where player A pays B one dollar for each Heads, and vice versa for each Tails. (The coin is unbiased here.) Let X_1 be the random variable recordin the first time player A is "ahead." Find $P(X_1 \leq 7)$. What is the probability that X is odd? Even?
14. Continuing with the coin flipping game, also define X_2 to record the first time A is up two dollars, Z_1 be the first time any player is up one dollar, and Z_2 the first time any player is up two dollars. Which pairs of random variables from $\{X_1, X_2, Z_1, Z_2\}$ are indepedant? Let $W = Z_2 - Z_1$. Is W indepedant of Z_1? Explain your answers carefully, but explicit calculations are not necessary.