1 Definition of a Manifold

Definition 1.1 (R-ringed space)**.** An R-ringed space is a ringed space (X, O_X) where $O_X \in Sh_{\mathbb{R}-Alg}(X)$.
A locally R-ringed space is a R-ringed space such that $O_{X,x}$ is local $\mathbb{R}$ algebra, then $\kappa(x)$ is always a field extension of $\mathbb{R}$.

Proposition 1.2. Given $U \subseteq \mathbb{R}^n$ open subset, $O_U : V \subseteq U \longrightarrow C^\infty(V, \mathbb{R})$ is a sheaf of $\mathbb{R}$-algebra, with local ring always $\mathbb{R}$. (U, O_U) becomes a locally R-ringed space.

Proposition 1.3. A C^∞ morphism $f : U \subseteq \mathbb{R}^n \to V \subset \mathbb{R}^m$ gives a **local** morphism of locally R-ringed spaces $(f, f^\flat)$, where $f^\flat_W : O_V(W) \to f_* O_U(W) = O_U(f^{-1}(W)) \quad g \mapsto gf$

Definition 1.4 (Euclidean space)**.** A n-Euclidian space ('affine manifold') is a R-ringed space isomorphic to some (U, O_U), where $U \subseteq \mathbb{R}^n$, the maps between n-Euclidean spaces are local morphisms. The category of n-Euclidean spaces is denoted AffMfd^n. The category of all Euclidean spaces is denoted AffMfd

Theorem 1.5. The category AffMfd is equivalent to category of open subset of $\mathbb{R}^n$ for some n with smooth maps.

idea of proof: it suffice to check that the local morphism of open set can be identified as smooth morphism (same as proofing $Hom_{LRS}(SpecR, SpecS) = Hom_{CRing}(R, S)$). The key is that all local morphism $(f, f^\flat)$ comes from some smooth map (which is exactly f).

Now consider map on stalk $f^\sharp_x : O_{Y,f(x)} \to O_{X,x}$, which induces $\overline{f^\sharp_x} : \kappa(f(x)) \to \kappa(x)$, one can check that this gives $g(f(x)) = \overline{g_{f(x)}} \mapsto \overline{f^\flat_V(g)_x} = f^\flat_V(g)(x)$ for $g \in O_Y(V)$, showing that $f^\flat$ is exactly pullback by f. Then f is smooth for pulling back smooth functions to smooth functions.

Note that the 'smallest' skeleton of AffMfd is $\{\coprod_{i \in I} \mathbb{R}^n \mid |I| \leq \aleph_0, n \in \mathbb{N}\}$

Now we do the same thing as defining Scheme.

Definition 1.6 (Premanifold)**.** A premanifold is an locally Euclidean R-ringed space (X, O_X) which is locally Euclidean. In other words $\forall x \in X, \exists U$ *open neiborhood of* x *s.t.* $(U, O_X|_U)$ *is a Euclidian space.* If we assume $(\phi, \phi^\flat) : U \to \phi(U)$, then (U, ϕ) is called a chart.
Maps between premanifolds are local morphisms (i.e. morphism locally acting like morphism of affine manifold)

Definition 1.7 (Open Immersion)**.** X premanifold, $U \subseteq X$, then U inherits structure sheaf from X as $(U, O_X|_U)$, $\alpha : Y \to X$ is open immersion if α factors through $Y \simeq U \hookrightarrow X$. 'Restriction' of morphism is defined as $f|_U = fi$

Note that all Euclidean open (affine open) subset of a premanifold forms a topological basis. A premanifold is locally path connected.

Definition 1.8 (Dimension)**.** The dimension of a Euclidian space X is n if $\exists U \subseteq \mathbb{R}^n$ *s.t.* $X \simeq U$ as locally ringed space. Dimension is well-defined because homeomorphism of spaces induces homeomorphism of connected components and we never have $\mathbb{R}^n \simeq \mathbb{R}^m$ if $m \neq n$.

Proposition 1.9 (Equidimension)**.** Connected Premanifold is always 'integral', explanation: $\forall$ *Euclidean* $U, V \subseteq X, dimU = dimV$

idea of proof: construct set $\Lambda_n = \{V \subseteq X \text{ } Euclidean \text{ } open \mid dimV = n\}$, suppose $\exists n < m \in \mathbb{N}$ s.t. $\Lambda_n, \Lambda_m \neq \emptyset$.

Proof by contradiction, $U_1 := \bigcup_{U \in \Lambda_k, k \leq n} U$, $U_2 := \bigcup_{U \in \Lambda_k, k \geq n+1} U$, then $X = U_1 \cup U_2$ induces $U_1 \cap U_2 \neq \emptyset$. Take $x \in U_1 \cap U_2$, $\exists V_1 \in \Lambda_p, V_2 \in \Lambda_q$ neiborhood of x and $p \leq n < q$, then $V_1 \cap V_2$ has both dimension of p and q which leads to an contradiction.

The whole idea is to prove dimension is somewhat a continuous function valued in $\mathbb{N}$.

A premanifold is direct union of its connected components, where each of which is still premanifold and admits a unique dimension, hence the case can usually be reduced to connected (thus path-connected) premanifolds. We will still accept Nonconnected case, however, for more generality (and convenience when studying Lie group).

In history, scheme was called prescheme and separate scheme was called scheme, hence we will put Hausdorff in the definition of Manifold. Also we will add C_2 (which is equivalent to paracompact and having countable components for Hausdorff premanifolds) to ensure partition of identity.

Definition 1.10 (Manifold)**.** A n-manifold is a Hausforff C_2 premanifold such that each component is of dimension n. Then the dimension of a manifold is well-defined. The category of n-manifold is denoted Mfd^n, the category of manifold is denoted Mfd.

The very first thing is to verify the definition coincide with classical definition.

Lemma 1.11 (Glueing Morphism)**.** X premanifold. Given $X = \bigcup_{i \in I} U_i$ and $\phi_i : U_i \to Y$ satisfying $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$, there exists unique $\phi : X \to Y$ such that $\phi|_{U_i} = \phi_i$

Theorem 1.12. The category of Manifold under our definition is **isomorphic** to the category of manifold under classical definition.

idea of proof: Given topological manifold $(\mathcal{M}, \mathcal{A})$, the structure sheaf is smooth functions, and the atlas gives the covering of Euclidean open sets.

Conversely given ringed space $\mathcal{M}$, take all Euclidean open sets as the atlas. We claim that the atlas is maximal. If (U, ϕ) is compatible with other 'charts', $\phi\phi_i^{-1}$ defines a local morphism $\phi_i(U \cap U_i) \to \phi(U \cap U_i)$, then locally one can define $U \cap U_i \to \phi(U \cap U_i)$ by $(\phi\phi_i^{-1})\phi_i$, which glues to $(\phi, \phi^\flat)$, making U a Euclidean space, then U is already in the atlas.

Then it suffices to determine correspondence of morphism. Going from classical definition to ringed space is rather easy (pullback). Conversely, given a morphism of locally ringed space $(f, f^\flat) : \mathcal{M} \to \mathcal{N}$, by taking Euclidean open set on the target then on the source, one can then proof f is smooth by Theorem 1.5.

References

[1] T. Wedhorn, *Manifolds, sheaves, and cohomology.* Springer, Jul. 2016.

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