UNIVERSITY OF ALBERTA
University of Alberta
Computing Science Department
CMPUT 340 - Fall 2020

Assignment 1
Due date: September 17
5 Marks

Introduction to Numerical Methods (CMPUT 340)

How to submit your assignment: You can either answer the questions on a paper, take a photo of the paper, generate a pdf, and then submit the file to eClass. An alternative is to answer the questions on a tablet, generate a pdf of your solutions, and then submit the file to eClass.

1. (1 Mark) Answer the following questions about $\sin(x)$. Recall that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ and $\lim_{x\to 2\pi}\frac{\sin(x)}{x} = 0$.

   a) (0.5 Mark) Compute the condition number of $\sin(x)$.
   The condition number of $\sin(x)$ is $\frac{\cos(x)x}{\sin(x)} = \frac{\cos(x)}{\frac{\sin(x)}{x}}$, which approaches 1 as $x \to 0$ and ∞ as $x \to 2\pi$.

   b) (0.25 Mark) For which values of x the function $\sin(x)$ is well-conditioned?
   $\frac{\cos(x)}{\frac{\sin(x)}{x}}$ approaches 1 as $x \to 0$ so it is well-conditioned around zero.

   c) (0.25 Mark) For which values of x the function $\sin(x)$ is ill-conditioned?
   $\frac{\cos(x)x}{\sin(x)} = \frac{\cos(x)}{\frac{\sin(x)}{x}}$ approaches ∞ as $x \to 2\pi$ so it is ill-conditioned around 2π.

2. (1 Mark) List the numerical problems one could have by solving the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ in a floating-point system.

   Cancellation can happen in $-b \pm \sqrt{b^2 - 4ac}$, depending on the sign of b, and in $b^2 - 4ac$. Overflow and underflow can happen when computing b^2 and $4ac$ if the coefficients are very large.

3. (1.5 Marks) (Heath, 2018)

   a) (0.2 Mark) Using a four-digit decimal arithmetic compute the formula of the area of the planet using the formula $A = 4\pi r^2$ for $r = 6370$ km.
   Given $\pi = 3.1416$ and $r^2 = 4.0577 \times 10^7$ we have $A_1 = 4\pi r^2 = 5.0990 \times 10^8$.

   b) (0.2 Mark) Using the same formula and precision, compute the difference in surface area if the value for the radius is increased by 1 km.

$$
\begin{aligned}
r_1 &= r + h = 6371 \\
r_1^2 &= 4.0590 \times 10^7 \\
A_2 &= 4\pi r^2 = 5.1007 \times 10^8 \\
A_2 - A_1 &= 1.7000 \times 10^5
\end{aligned}
$$

c) (0.2 Mark) Since $dA/dr = 8\pi r$, the change in surface area is approximated by $8\pi rh$, where h is the change in radius. Use this formula, still with four-digit arithmetic, to compute the difference in surface area due to an increase of 1 km in radius. How does the value obtained using this approximate formula compare with that obtained from the "exact" formula in part b?

$dA = 8 \times 3.1416 \times 6370 = 1.6010 \times 10^5$

d) (0.4 Mark) Determine which of the previous two answers is more nearly correct by repeating both computations using higher precision, say, six-digit decimal arithmetic.

(We are looking for ballpark comparisons in this question. There is no need to get the numbers exactly right)

$$\begin{aligned}
\pi &= 3.141593 \\
A_1 &= 4 \times 3.141593 \times 6370^2 = 4.057690 \times 10^7 \\
A_2 &= 4 \times 3.141593 \times 6371^2 = 4.058964 \times 10^7 \\
A_2 - A_1 &= (4 \times 3.141593 \times 6371^2) - (4 \times 3.141593 \times 6370^2) = 1.601081 \times 10^5 \\
dA/dr &= 8 \times 3.141593 \times 6370 = 1.600956 \times 10^5
\end{aligned}$$

Error analysis for the exact model:

$$(b) - (d) = 9891.9$$

Error analysis for the approximate model:

$$(c) - (d) = 4.4$$

The approximate method is more accurate.

e) (0.5 Mark) Explain the results you obtained in parts a-d.

The exact model suffers from cancellation in the very limited precision used, while the approximate model doesn't.

4. (1 Mark) The 2-norm requires one to square all values of a given vector before taking the square root of their sum.

$$||x||_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

A problem with this norm is that the squared value of an entry x_i can overflow. Derive a procedure that prevents overflow from happening. As a hint, think of how you can modify the values x_i while not changing the value of the norm.

We will divide all elements of x by the largest entry of x, i.e., $||x||_\infty$. Let $c = ||x||_\infty$.

$$
\begin{aligned}
||x||_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^{n} \frac{c^2}{c^2} |x_i|^2 \right)^{1/2} \\
&= c \left(\sum_{i=1}^{n} \left(|x_i|/c \right)^2 \right)^{1/2}
\end{aligned}
$$

5. (0.5 Mark) (Heath, 2018) Write the LU factorization for the following matrix. Show both L and U explicitly.

$$
\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
$$

$$
M_2 M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = U
$$

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}
$$