# Gauss Elimination Method for a system of 'n' Linear Equations

In the last class, we saw that Gauss elimination method has a systematic procedure & can be used to develop a computer program directly.

Consider a system of 'n' linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \quad \text{--- (1)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \quad \text{--- (2)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \quad \text{--- (n)}$$

**Forward Elimination :-**

(i) Start from dividing the first equation by the coefficient of $x_1$,

$$x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3 + \cdots + \frac{a_{1n}}{a_{11}}x_n = \frac{b_1}{a_{11}} \quad \text{--- (3)}$$

(ii) Multiply by the coefficient of $x_1$ from eq$^n$(2) :

$$a_{21}x_1 + a_{21}\frac{a_{12}}{a_{11}}x_2 + \cdots + a_{21}\frac{a_{1n}}{a_{11}}x_n = a_{21}\frac{b_1}{a_{11}} \quad \text{--- (4)}$$

(iii) eq$^n$ (2) $-$ eq$^n$ (4)

(Subtract)

$$\underbrace{\left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)}_{a'_{22}} x_2 + \underbrace{\left(a_{23} - a_{21}\frac{a_{13}}{a_{11}}\right)}_{a'_{23}} x_3 + \cdots +$$

$$\underbrace{\left(a_{2n} - a_{21}\frac{a_{1n}}{a_{11}}\right)}_{a'_{2n}} x_n = \underbrace{b_2 - a_{21}\frac{b_1}{a_{11}}}_{b'_2}$$

or, $$a'_{22} x_2 + a'_{23} x_3 + \cdots + a'_{2n} x_n = b'_2 \quad -(5)$$

(iv) Repeat this process, to eliminate the coefficients of $x_1$ from all equations. So after this 1st loop of forward elimination, the system of equations becomes:

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \quad -(6)$$

$$a'_{22} x_2 + \cdots + a'_{2n} x_n = b'_2 \quad -(7)$$

$$a'_{32} x_2 + \cdots + a'_{3n} x_n = b'_3 \quad -(8)$$

$$\vdots$$

$$a'_{n2} x_2 + \cdots + a'_{nn} x_n = b'_n \quad -(n')$$

(v) For the 2nd loop, the terms associated with $x_2$, from eqn (7) to eqn (n') are eliminated. Start with same procedure. Divide $a'_{22}$ throughout the eqⁿ (7) & so on. After the 2nd loop, the system looks like:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \quad — (9)$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \quad — (10)$$

$$a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \quad — (11)$$

$$a''_{43}x_3 + \cdots + a''_{4n}x_n = b''_4 \quad — (12)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a''_{n3}x_3 + \cdots + a''_{nn}x_n = b''_n \quad — (n'')$$

(vi) After $(n-1)$ loops, the system becomes:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \quad — (13)$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \quad — (14)$$

$$a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \quad — (15)$$

$$\vdots$$

$$a^{(n-1)}_{nn}x_n = b^{(n-1)}_n \quad — (n^{n-1})$$

The prime symbols & the values in the superscripts represent the no. of computational loops needed forward elimination method.

<u>Back Substitution</u> :-

From eqⁿ $(n^{n-1})$ the value of $x^n$ can be obtained as,

$$x_n = b^{(n-1)}_n / a^{(n-1)}_{nn} \quad — (16)$$

The unknowns $x_{n-1}, x_{n-2}, \dots x_2, x_1$ can be obtained by using back substitution. The general formula is,

$$\boxed{x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}} \quad \text{—(17)}$$

Now let's see how to implement this in python.

<u>Gauss Jordan Method</u> :- (Dechaumphai. p-73)

It is an extension of Gauss - Jordan method is an extension of the Gauss elimination method. In this method after the elimination process, the eqn becomes:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1^* \\ b_2^* \\ b_3^* \end{bmatrix} \quad \text{— (18)}$$

Read yourself. (Not much important)

<u>Exercise</u>

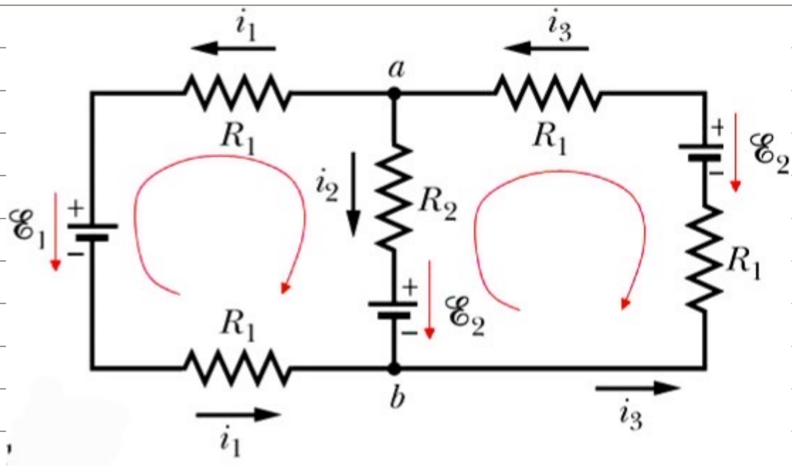
(i) Read the limitations of Gauss elimination method, from Dechaumphai book. And improved methods.

(ii) In the given circuit below,

$\mathcal{E}_1 = 3V$, $\mathcal{E}_2 = 6V$, $R_1 = 20\,\Omega$, $R_2 = 4\,\Omega$

Form a system of Linear equations using KVL & KCL. Solve it by Cramer's rule first. Then find $(i_1, i_2, i_3)$

Solve it in computer by Gauss elimination method.




(Hint : $i_1 = 0.5$ A) → for the above convention.

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