

Lecture Summer 2020

# **Physical-Optics Modeling**

Prof. Dr. Frank Wyrowski

## **1. Physical-optics modeling: The task**

Connecting field solvers

# Physical-Optics System Modeling: Task

## Optical system

Generalized permittivity:

$$\check{\epsilon}_r(\mathbf{r}, \omega) \stackrel{\text{def}}{=} \epsilon_r(\mathbf{r}, \omega) + i \frac{\sigma(\mathbf{r}, \omega)}{\omega \epsilon_0}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mu_0 \mathbf{H}(\mathbf{r}, \omega)$$

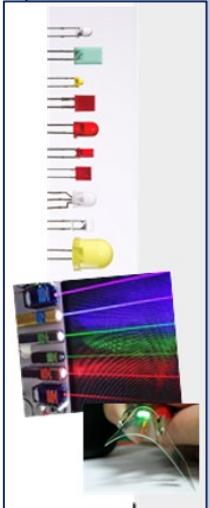
$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -i\omega \epsilon_0 \check{\epsilon}_r(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

$$\nabla \cdot (\check{\epsilon}_r(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)) = 0$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, \omega) = 0$$

- We need to solve Maxwell's equations for a given optical system.
- The system is formally represented by inhomogeneous media which is expressed by the generalized permittivity (here isotropic).

# Physical-Optics System Modeling: Task



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- We need to solve Maxwell's equations for a given optical system.
- The system is formally represented by inhomogeneous media which is expressed by the generalized permittivity (here isotropic).
- We are typically interested in the response of the system on one or more light sources, e.g. the focus of a lens system for an incident laser beam.

# Physical-Optics System Modeling: Task



## Optical system

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- The general permitivity is directly related to the complex refractive index via

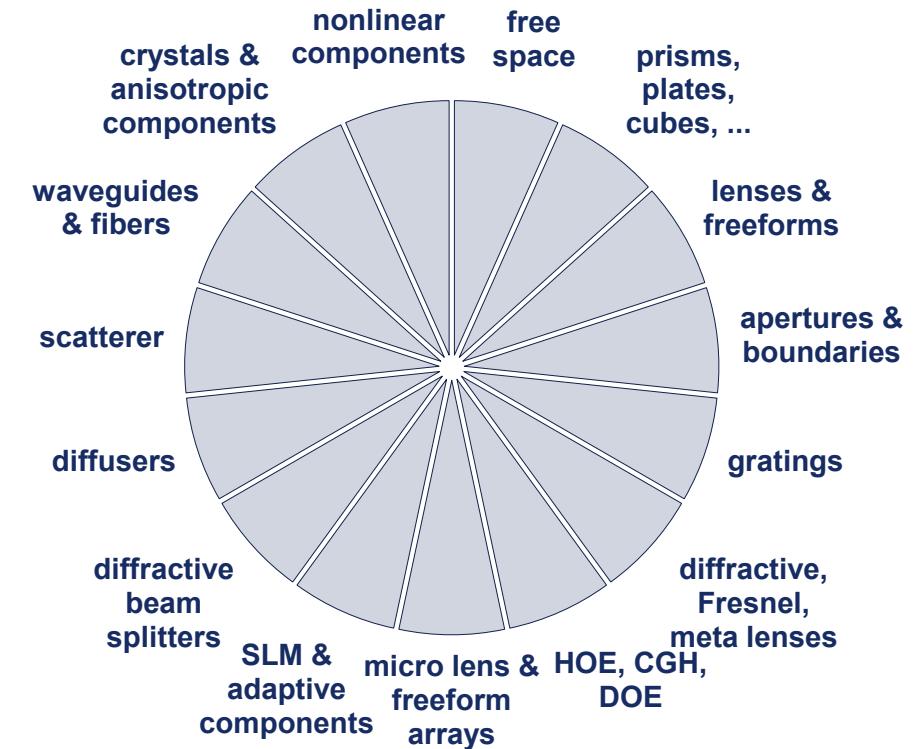
$$\check{n}(\omega) \stackrel{\text{def}}{=} \sqrt{\check{\epsilon}_r(\omega)} = n(\omega) + i n'(\omega).$$

- In practice the refractive index modulation in a system is expressed per optical component.

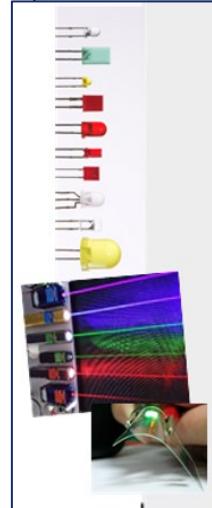
# Physical-Optics System Modeling: Components



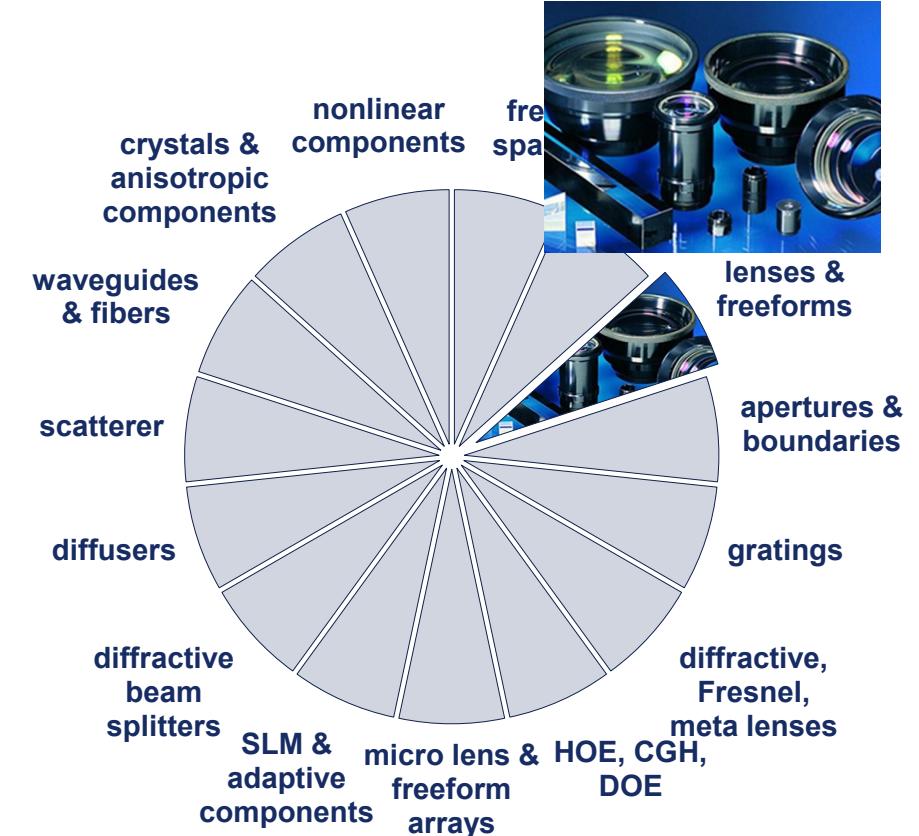
$$\check{n}(\omega) = \sqrt{\check{\epsilon}_r(\omega)}$$



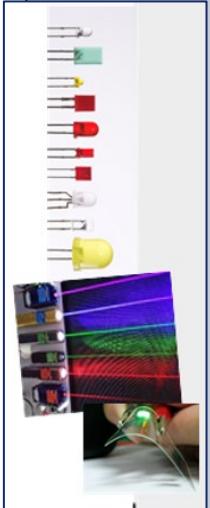
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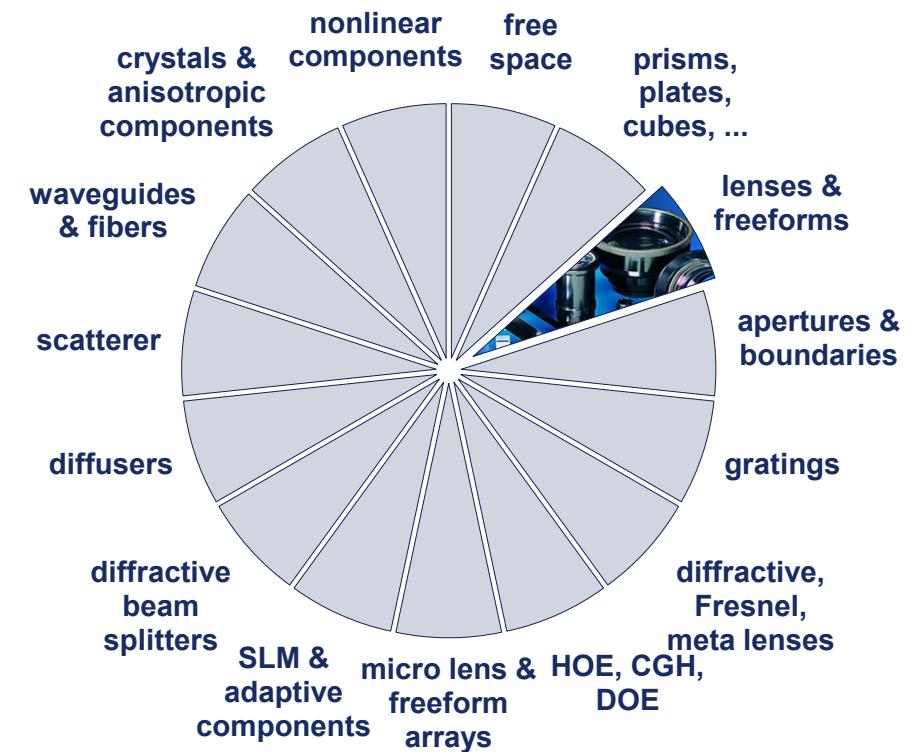
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# Physical-Optics System Modeling: Components



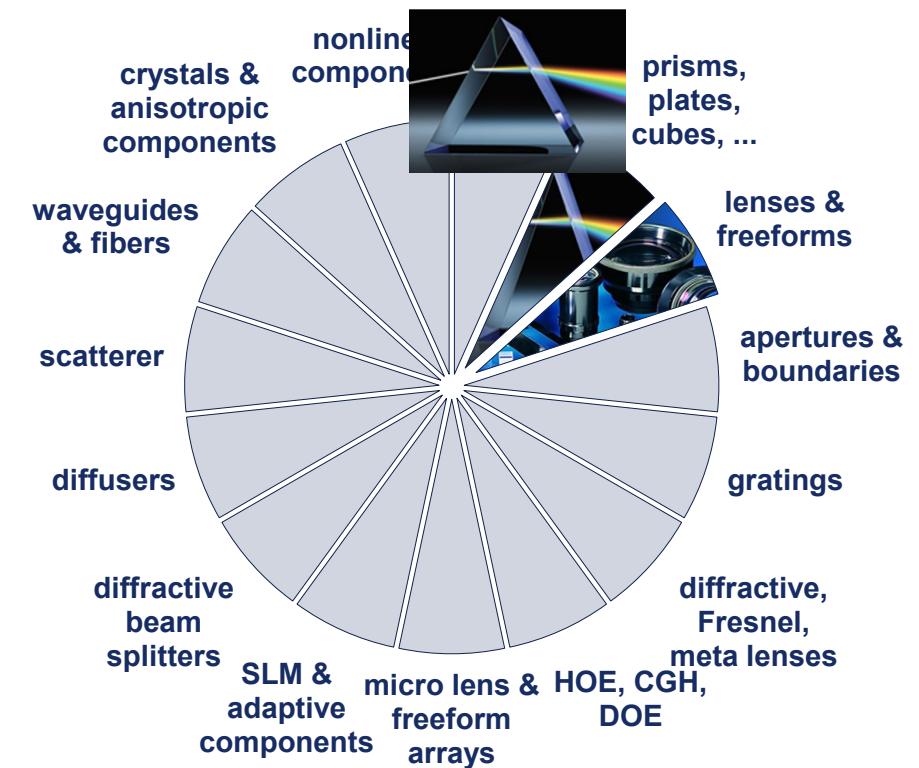
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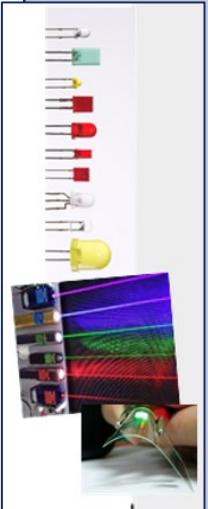
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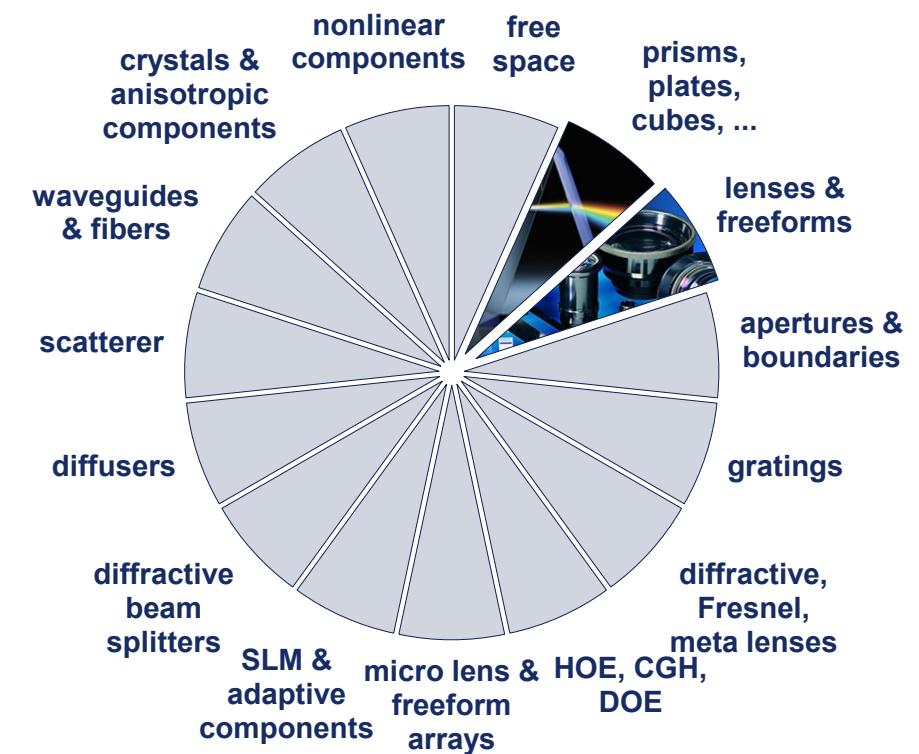
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# Physical-Optics System Modeling: Components



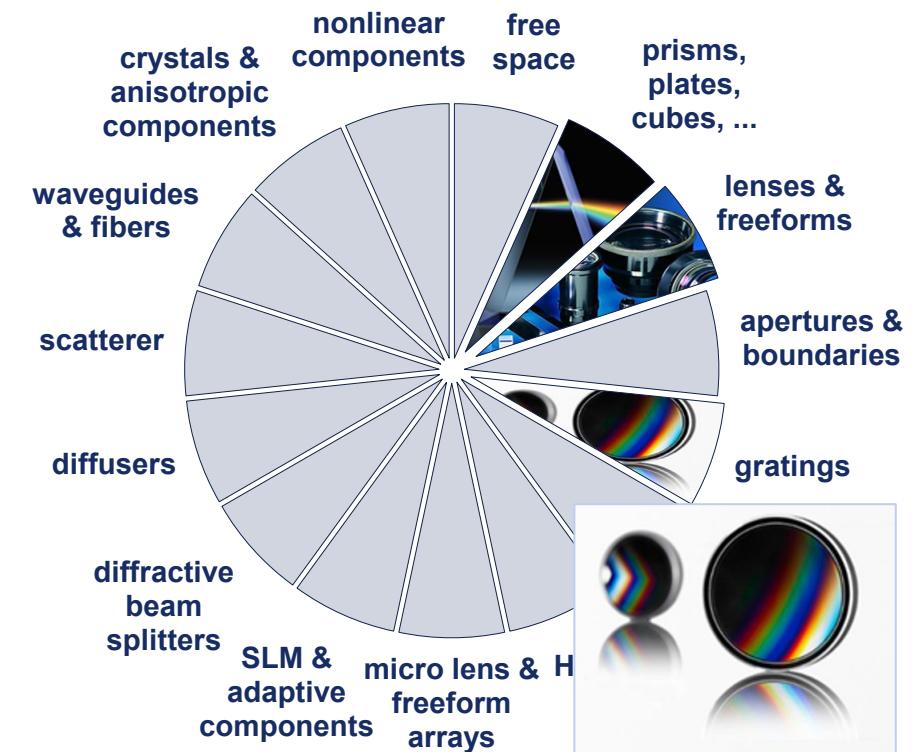
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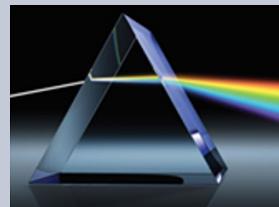
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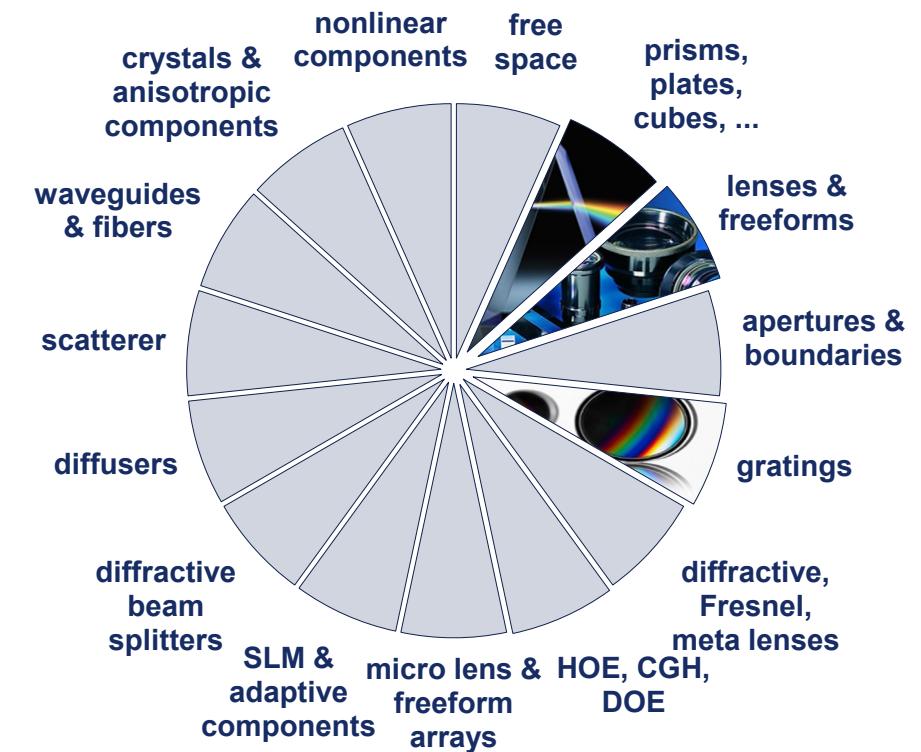
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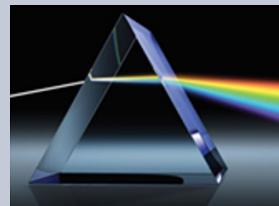
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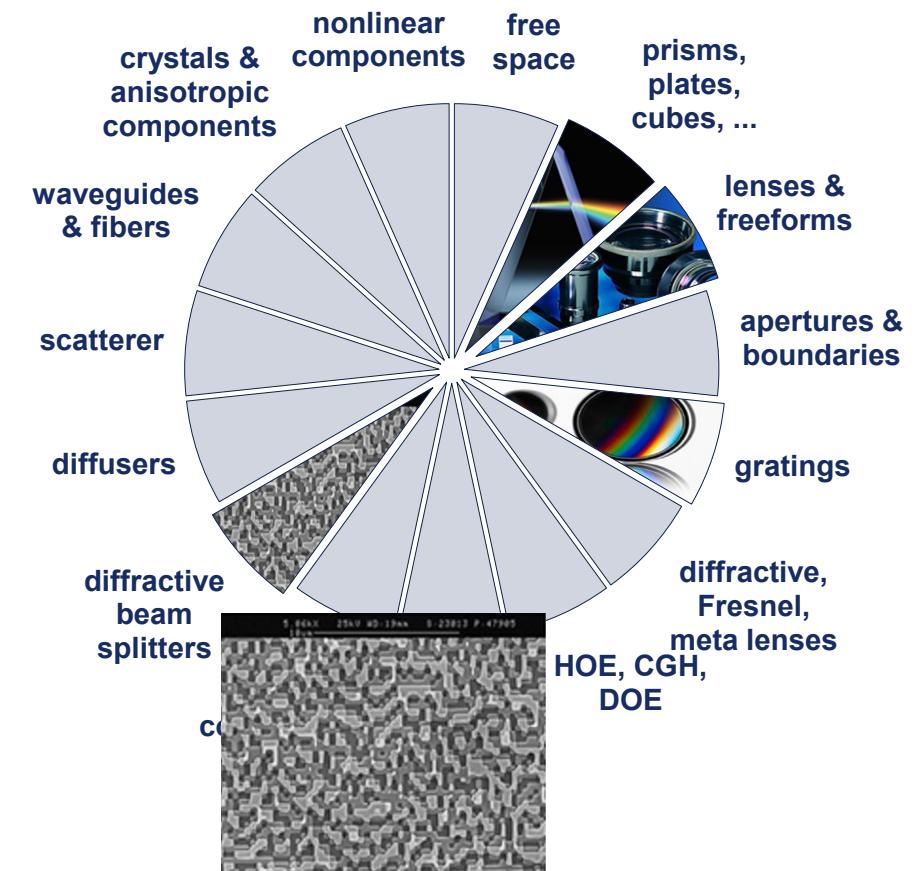
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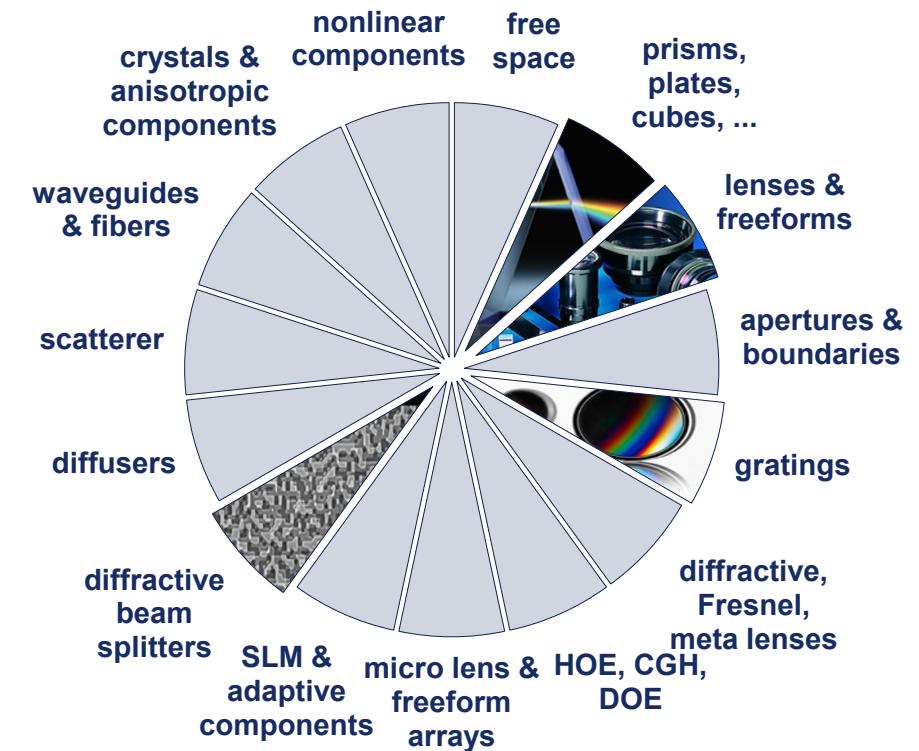
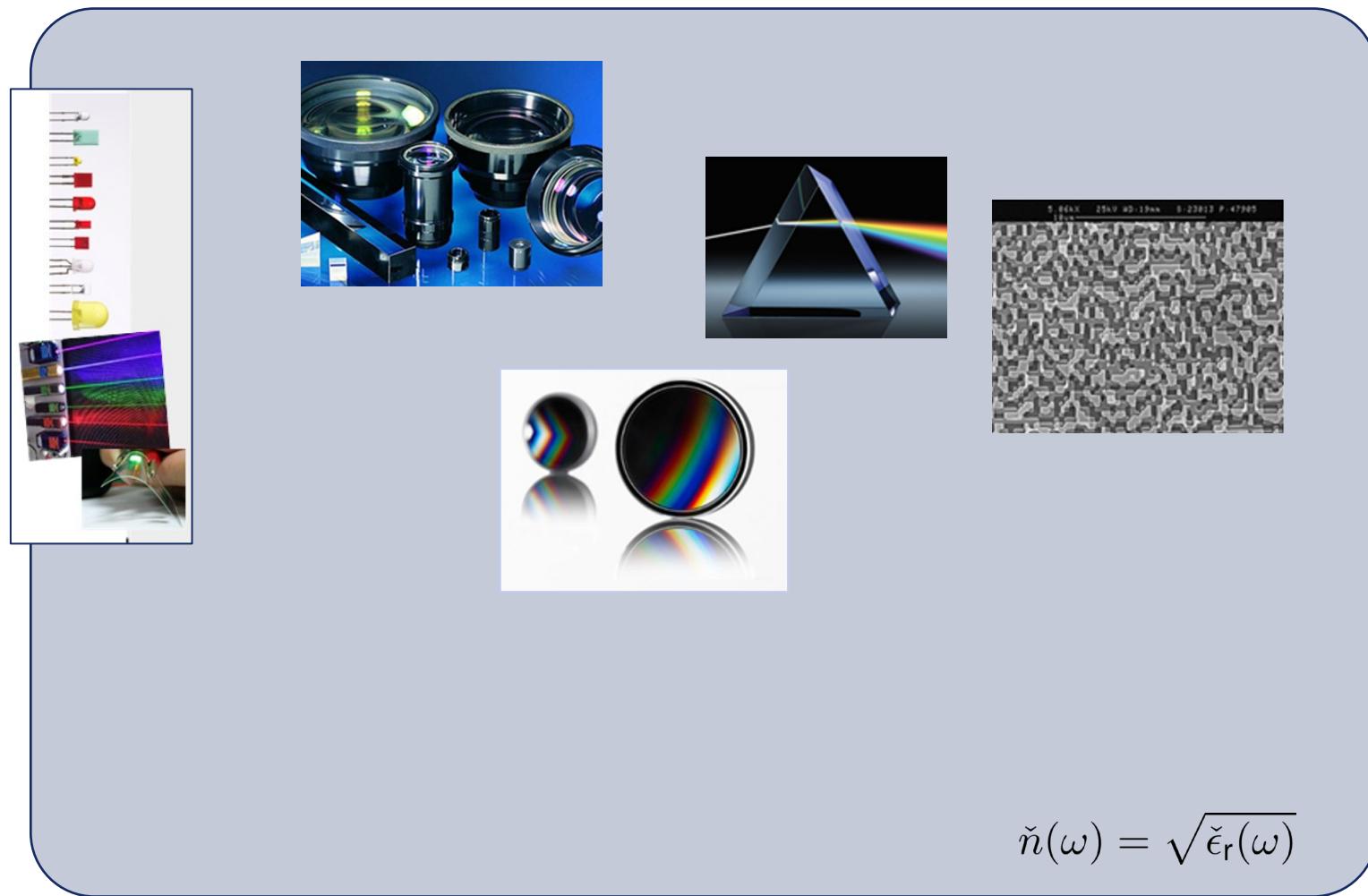
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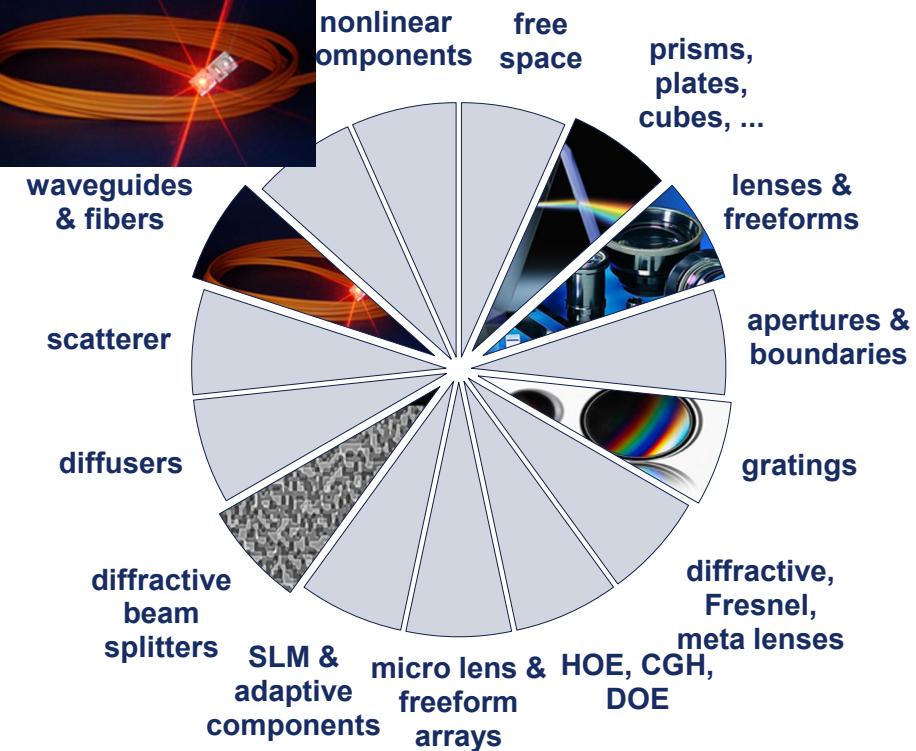
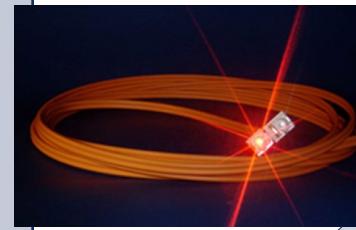
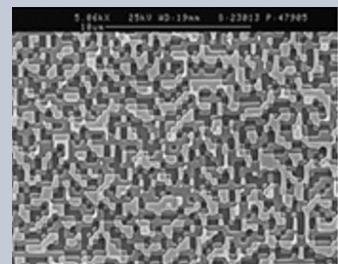
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# Physical-Optics System Modeling: Components

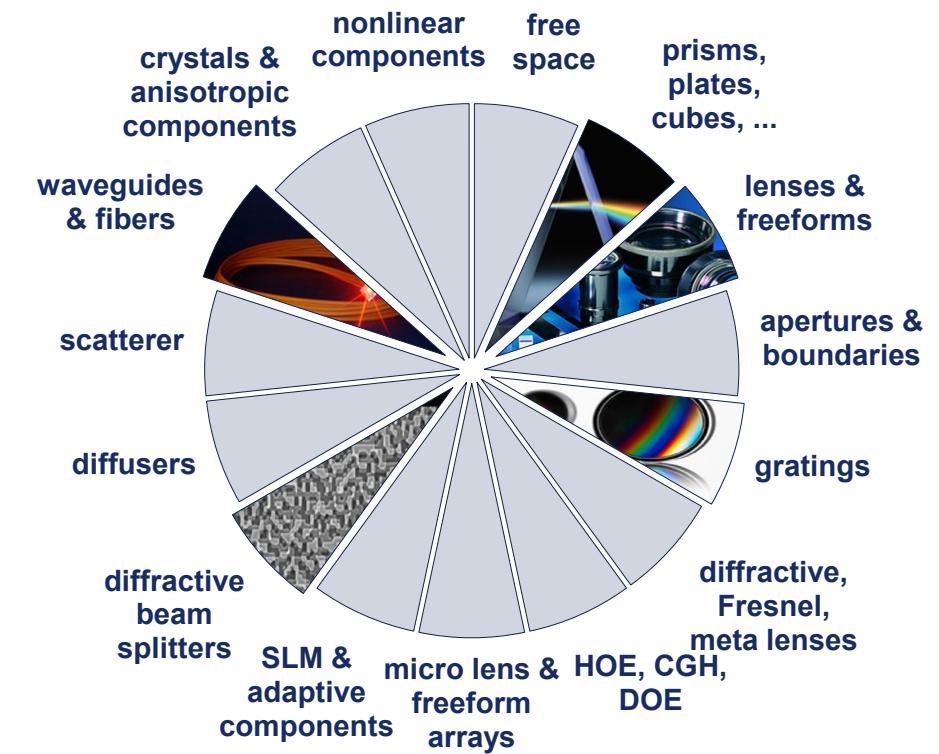
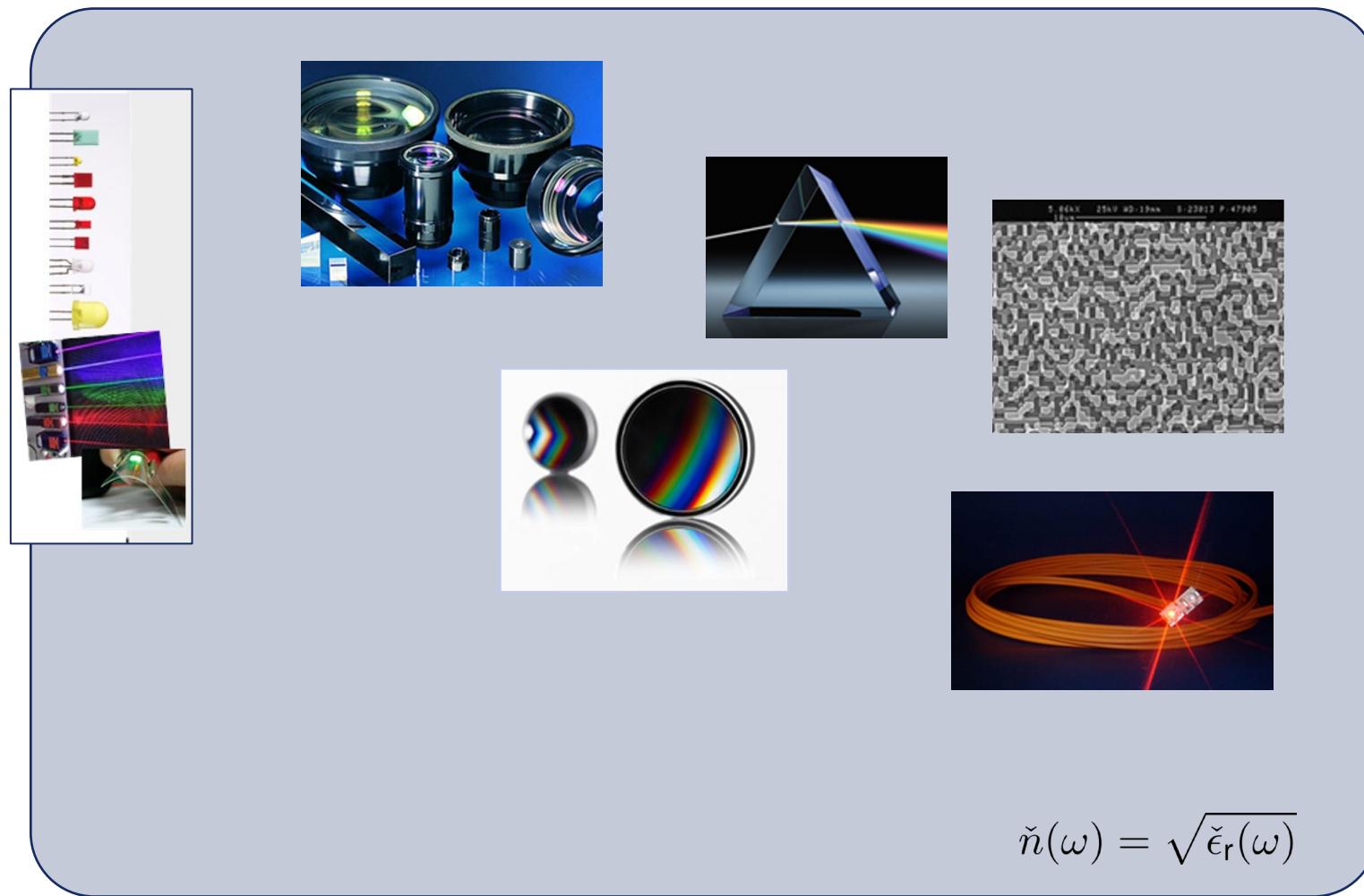


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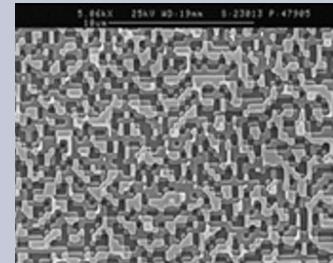
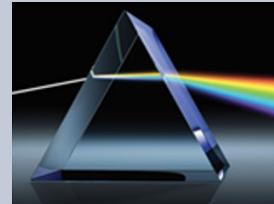


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# Physical-Optics System Modeling: Components

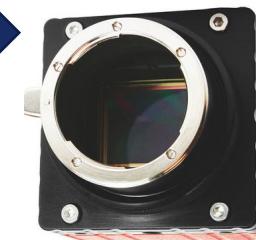


# Physical-Optics System Modeling: Task



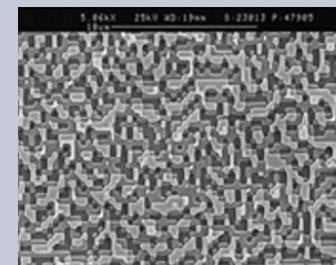
$$\check{n}(\omega) = \sqrt{\check{\epsilon}_r(\omega)}$$

- For a given source field physical-optics modeling should provide the electromagnetic field for further detector evaluation in any region of the system of concern.
- That requires the solution of Maxwell's equations for the specified system and sources.



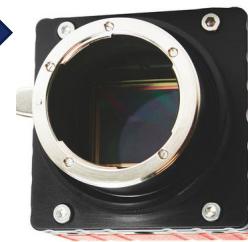
detector

# Physical-Optics System Modeling: Field Solver



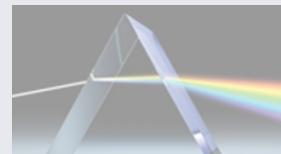
$$\check{n}(\omega) = \sqrt{\check{\epsilon}_r(\omega)}$$

- Algorithms which solve Maxwell's equations are often referred to as field solver.
- We distinguish between rigorous and approximated field solvers.
- Approximated solvers also provide the electromagnetic field, but with some mathematical assumptions which are adapted to the specific modeling situation.



detector

# Physical-Optics System Modeling: Field Solver



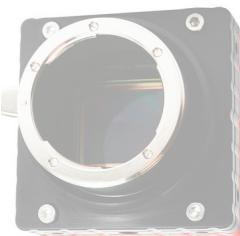
- Algorithms which solve Maxwell's equations are often referred to as Field Solver.
- We distinguish between rigorous

Application of a single field solver, e.g. FDTD\*, is not a realistic approach in system modeling.



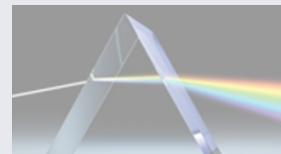
modeling situation.

$$\check{n}(\omega) = \sqrt{\check{\epsilon}_r(\omega)}$$



detector

# Physical-Optics System Modeling: Connecting Solvers



- Algorithms which solve Maxwell's equations are often referred to as Field Solver.
- We distinguish between rigorous

Physical-optics system modeling requires the application and connection of suitable component solvers.



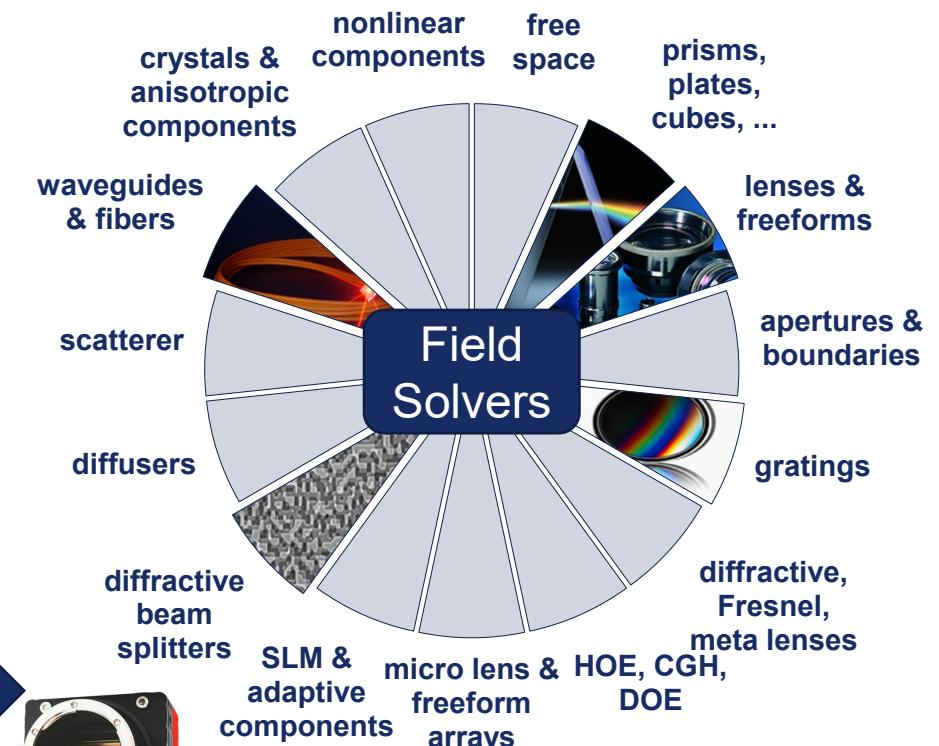
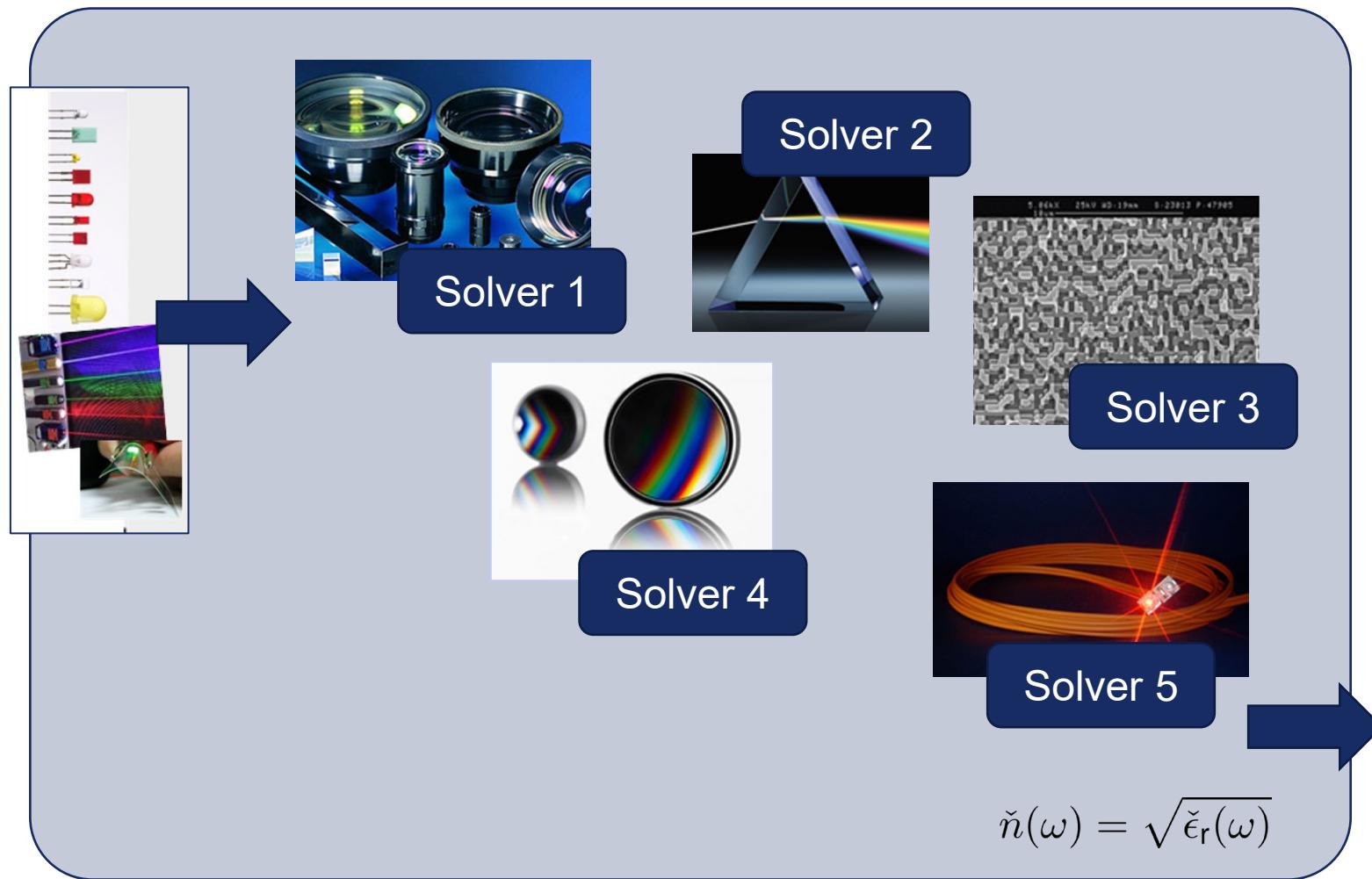
$$\check{n}(\omega) = \sqrt{\check{\epsilon}_r(\omega)}$$

modeling situation.

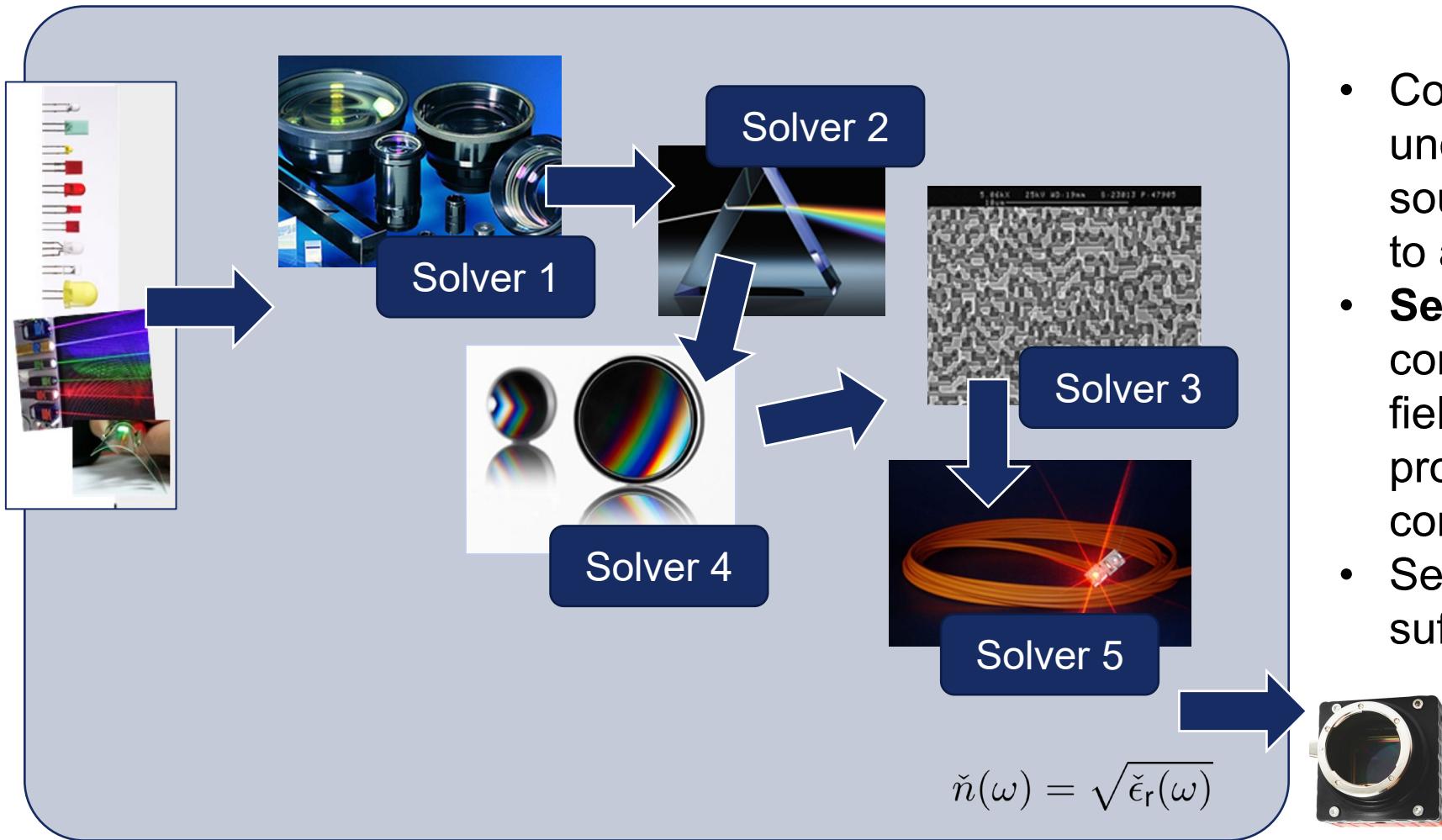


detector

# Physical-Optics System Modeling: Solvers per Component

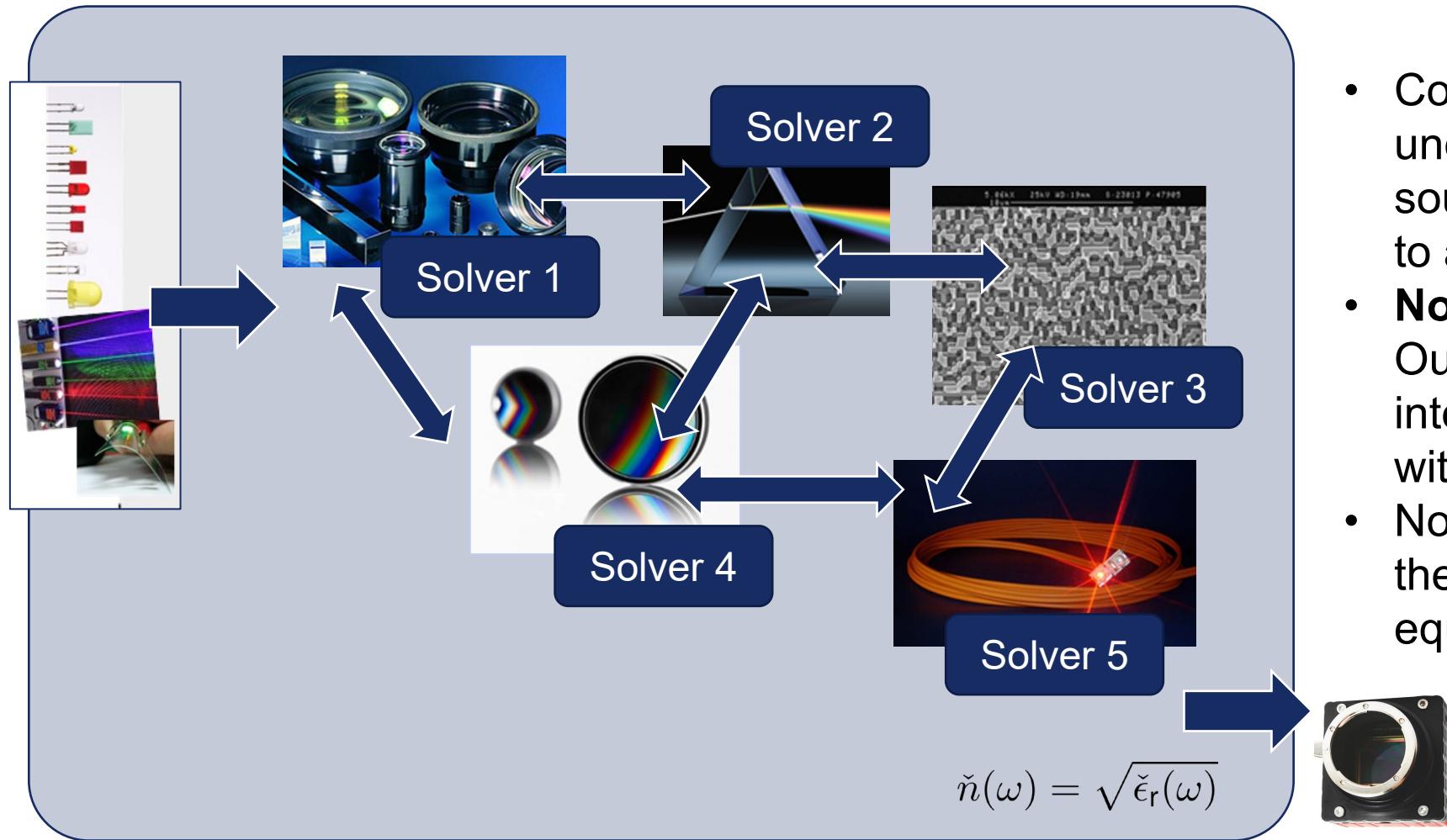


# Physical-Optics System Modeling: Connecting Solvers



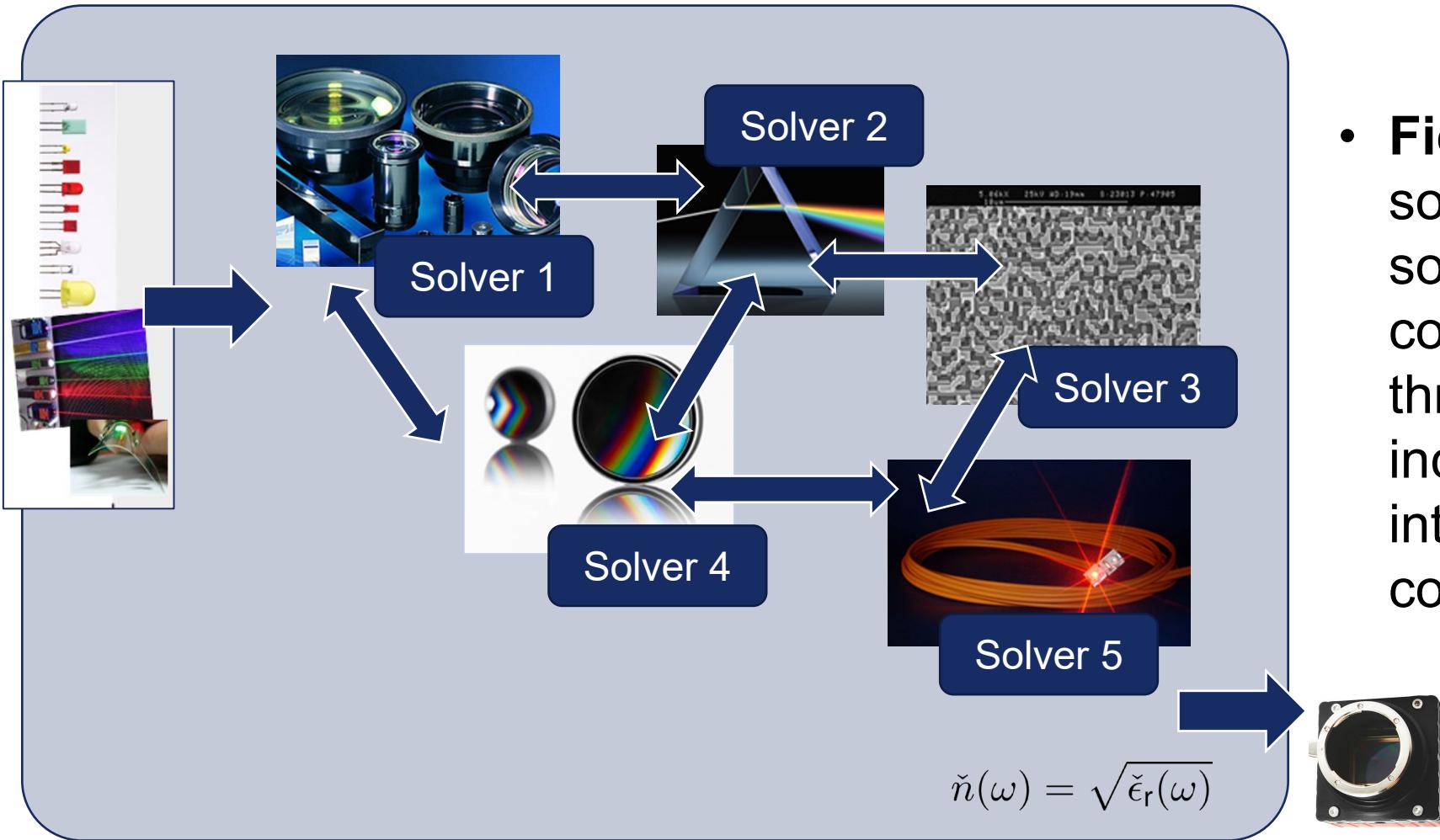
- Connecting solvers can be understood as propagating the source field through the system to all detectors.
- **Sequential modeling:** Each component generates **one** output field per input field which propagates to **one** other component.
- Sequential modeling is often sufficient in optical modeling.

# Physical-Optics System Modeling: Connecting Solvers



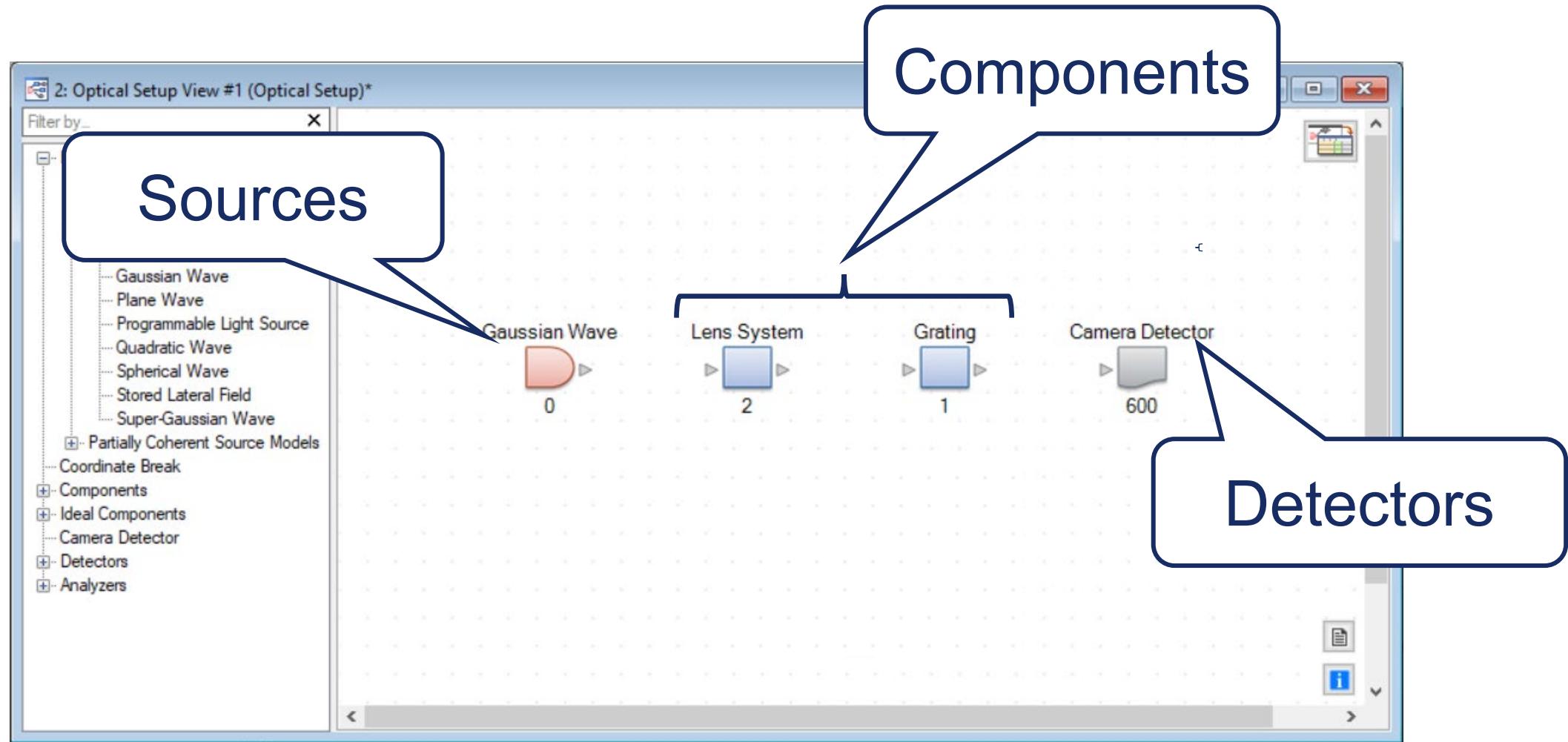
- Connecting solvers can be understood as propagating the source field through the system to all detectors.
- **Non-Sequential modeling:** Output fields per component interact with all other components within reach.
- Non-sequential modeling enables the solution of Maxwell's equations in complex systems.

# Physical-Optics System Modeling: Field Tracing



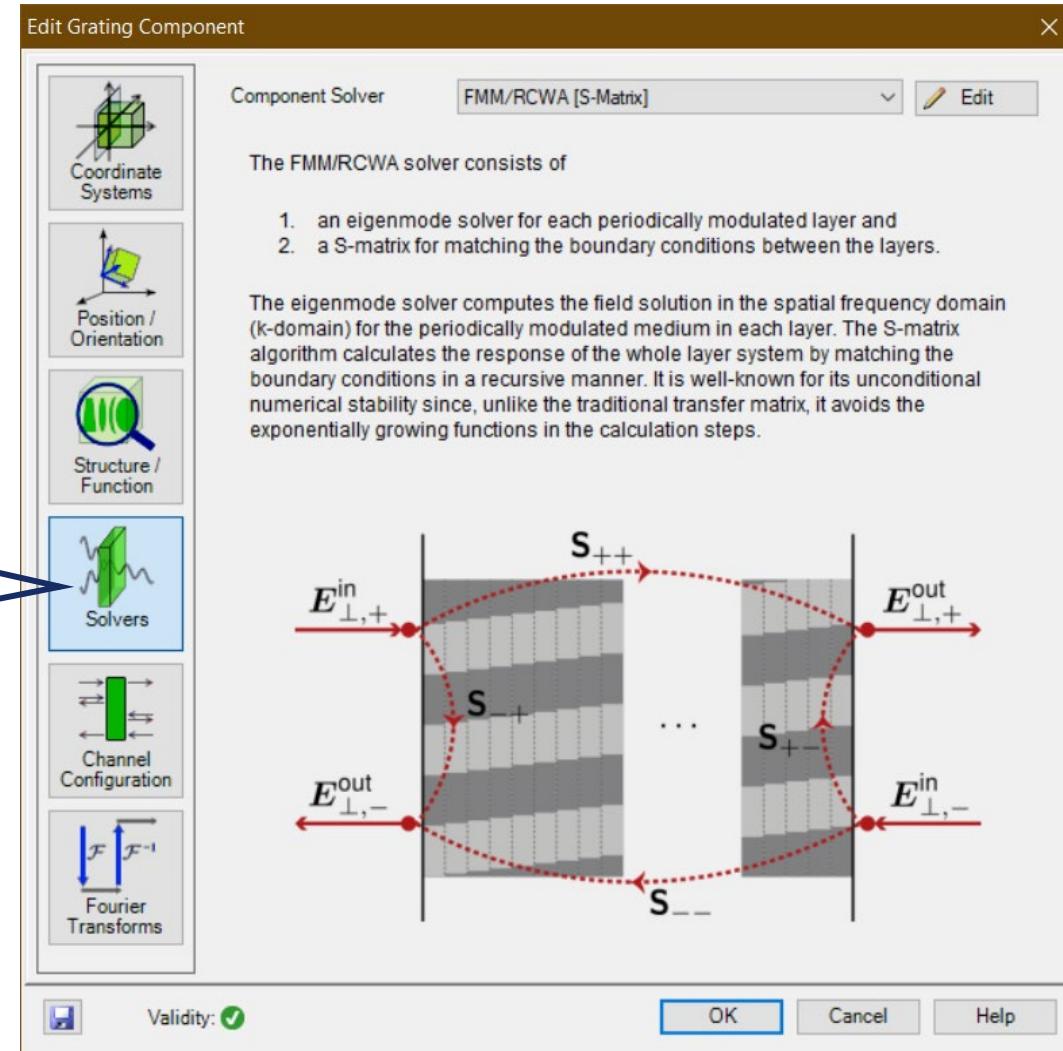
- **Field tracing:** Connecting solvers by propagating the source field(s) and the component responses through the system by including all (selected) interactions between components.

# VirtualLab Fusion (VLF) Field Tracing Technology

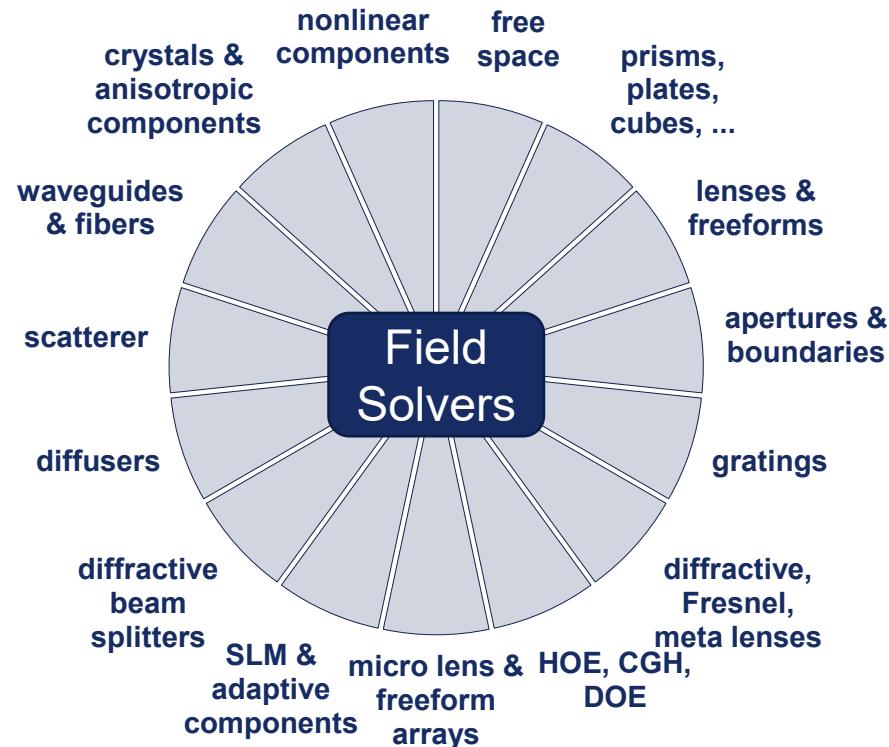


# Solver per Component

## Connecting Field Solvers



# Understanding and Developing Component Field Solvers



- The goal of modern physical-optics modeling is providing a *Digital Twin* to systems in real optical laboratories.
- Field tracing enables physical-optics system modeling.
- Field tracing is based on suitable and fast (enough) solvers for the great variety of components which are used in modern optics & photonics.



Lecture deals with field solvers!

# 1. Physical-optics modeling: The task

## 1 Physical-optics modeling: The task

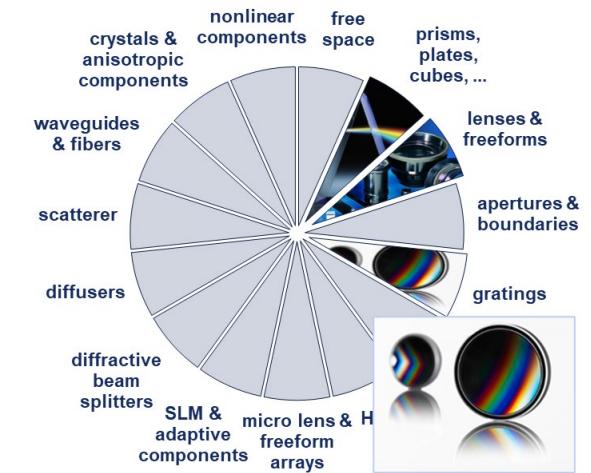
- In optics and photonics we investigate the evolution of electromagnetic fields in homogeneous and inhomogeneous media.
- The inhomogeneity is expressed by the generalized permittivity

$$\check{\epsilon}_r(\mathbf{r}, \omega) := \epsilon_r(\mathbf{r}, \omega) + i \frac{\sigma(\mathbf{r}, \omega)}{\omega \epsilon_0}, \quad (1)$$

or the complex refractive index

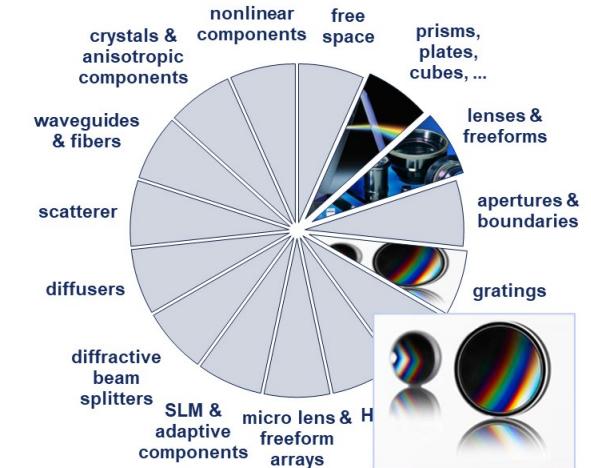
$$\check{n}(\mathbf{r}, \omega) := \sqrt{\check{\epsilon}_r(\mathbf{r}, \omega)} = n(\mathbf{r}, \omega) + i n'(\mathbf{r}, \omega). \quad (2)$$

- Homogeneous media does not show a dependency of  $\mathbf{r}$  and  $\check{\epsilon}_r(\omega)$  follows. The scalar permittivity can be replaced by a permittivity tensor to include anisotropic media.



# 1. Physical-optics modeling: The task

- In system modeling the inhomogeneous regions are organized in **components**. The inhomogeneities often appear in form of jumps between homogeneous media at surfaces of components. But in fibers, GRIN lenses and volume gratings we may also deal with smoothly modulated media.
- In addition, we may have layers with nano-structured permittivity and index variations. Dependent on the type of component we encounter different types of inhomogeneities and geometries.
- In system modeling we consider fields in homogeneous media and their interaction with inhomogeneous regions.
- By superposition of all fields from **sources** and the interactions with components we may observe the field which solves Maxwell's equations in the optical system.
- The field is then available in the system and can be further investigated by **detectors**.



## **2. Vector fields**

Some basic mathematics

# Electric and Magnetic Fields

## 2 Vector fields

In physical optics we mainly deal with the electric and magnetic fields which together represent electromagnetic fields. The electromagnetic field is governed by Maxwell's equations. Mathematically the electric and magnetic fields represent vector fields.

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega\mu_0 \mathbf{H}(\mathbf{r}, \omega)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -i\omega\epsilon_0\check{\epsilon}_r(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega)$$

$$\nabla \cdot (\check{\epsilon}_r(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega)) = 0$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, \omega) = 0$$

### 2.1 Electric and magnetic field

- We distinguish between the real fields and its complex generalizations. The real vector fields in time domain are defined by

$$\bar{\mathbf{E}}^{(r)}(\mathbf{r}, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3, (\mathbf{r}, t) \mapsto \bar{\mathbf{E}}^{(r)}(\mathbf{r}, t) \quad (3)$$

$$\bar{\mathbf{H}}^{(r)}(\mathbf{r}, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3, (\mathbf{r}, t) \mapsto \bar{\mathbf{H}}^{(r)}(\mathbf{r}, t). \quad (4)$$

# Electric and Magnetic Fields

- Its complex generalizations are

$$\bar{\mathbf{E}}(\mathbf{r}, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (\mathbf{r}, t) \mapsto \bar{\mathbf{E}}(\mathbf{r}, t) \quad (5)$$

$$\bar{\mathbf{H}}(\mathbf{r}, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (\mathbf{r}, t) \mapsto \bar{\mathbf{H}}(\mathbf{r}, t). \quad (6)$$

- The complex and real vector fields in time domain are connected via

$$\bar{\mathbf{E}}^{(r)}(\mathbf{r}, t) = \Re \bar{\mathbf{E}}(\mathbf{r}, t) \quad (7)$$

$$\bar{\mathbf{H}}^{(r)}(\mathbf{r}, t) = \Re \bar{\mathbf{H}}(\mathbf{r}, t). \quad (8)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mu_0 \mathbf{H}(\mathbf{r}, \omega)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -i\omega \epsilon_0 \check{\epsilon}_r(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

$$\nabla \cdot (\check{\epsilon}_r(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)) = 0$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, \omega) = 0$$

- In the field equations, e.g. Eq. (3), an explicit description of a function by  $x \mapsto f(x)$  is used, since when applying operators this notation helps to clearly distinguish between operations on a function  $f(x)$  at  $x$  and on the function itself, that is  $x \mapsto f(x)$ .
- Usually fields are discussed in the frequency domain and we ob-

# Electric and Magnetic Fields

tain

$$\mathbf{E}(\mathbf{r}, \omega) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (\mathbf{r}, \omega) \mapsto \mathbf{E}(\mathbf{r}, \omega), \quad (9)$$

$$\mathbf{H}(\mathbf{r}, \omega) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (\mathbf{r}, \omega) \mapsto \mathbf{H}(\mathbf{r}, \omega), \quad (10)$$

via the Fourier transform  $\mathcal{F}_\omega$  of Eq.(69).

- The field components are complex amplitudes and we have for example

$$E_x(\mathbf{r}, \omega) = |E_x(\mathbf{r}, \omega)| \exp(i \arg(E_x(\mathbf{r}, \omega))). \quad (11)$$

- The transversal components of the electric field are often denoted by

$$\mathbf{E}_\perp(\mathbf{r}, \omega) = (E_x(\mathbf{r}, \omega), E_y(\mathbf{r}, \omega))^T. \quad (12)$$

**Fourier Transform**

**3 Fourier transform and fields in k-domain**

**3.1 Fourier transform time-frequency domain**

- For square integrable functions  $\tilde{V}(t)$  the Fourier transform operator  $\mathcal{F}_\omega$  is defined by

$$\mathcal{F}_\omega : \begin{cases} \mathbb{L}_2^2(\mathbb{R}) \rightarrow \mathbb{L}_2^2(\mathbb{R}) \\ \tilde{V} \mapsto \left( \omega \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}(t) \exp(i\omega t) dt \right) \end{cases}, \quad (69)$$

and in compact form we write

$$V(\omega) = \mathcal{F}_\omega(t \mapsto \tilde{V}(t))(\omega) = (\mathcal{F}_\omega \tilde{V})(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}(t) \exp(i\omega t) dt. \quad (70)$$

- It should be emphasized, that the notation  $\mathcal{F}_\omega \tilde{V}(t) = V(\omega)$ , though often used in literature, is mathematically not precisely correct.

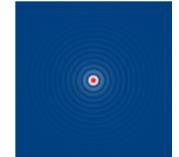
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$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mu_0 \mathbf{H}(\mathbf{r}, \omega)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -i\omega \epsilon_0 \check{\epsilon}_r(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

$$\nabla \cdot (\check{\epsilon}_r(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)) = 0$$

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# Electric and Magnetic Fields

- The function

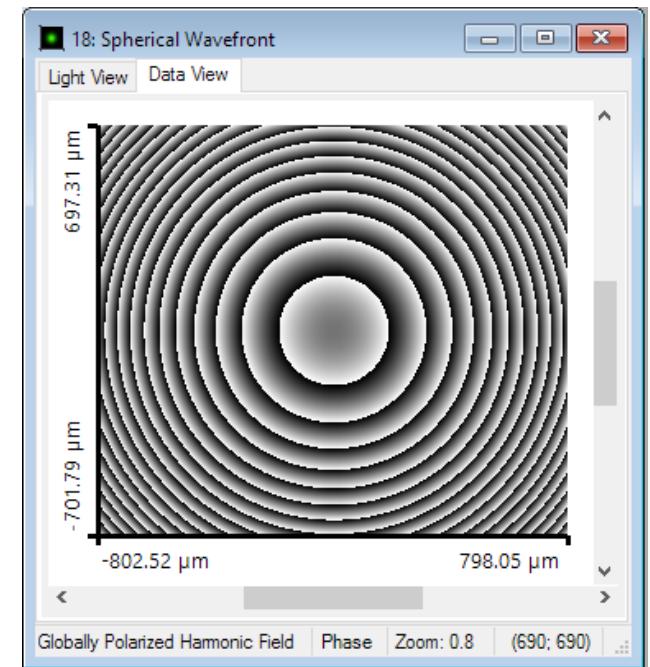
$$V(\mathbf{r}, \omega) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}, (\mathbf{r}, \omega) \mapsto V(\mathbf{r}, \omega) = |V(\mathbf{r}, \omega)| \exp(i\gamma(\mathbf{r}, \omega)) \quad (13)$$

is often applied as a placeholder for an arbitrary field component where suitable for the sake of simplicity.

- All transversal components of the electromagnetic field are sometimes expressed together by

$$\mathbf{V}_\perp := (E_y, E_y, H_x, H_y)^T. \quad (14)$$

- Electromagnetic fields often develop a strong wavefront phase while propagating through optical systems.
- The extraction of a wavefront phase  $\psi(\mathbf{r})$ , which is common for all six field components, helps to discuss and distinguish different



# Wavefront Phase

effects in optical modeling. Thus, we define

$$\mathbf{U}(\mathbf{r}, \omega; E) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (\mathbf{r}, \omega) \mapsto \mathbf{E}(\mathbf{r}, \omega) \exp(-i\psi(\mathbf{r}, \omega)) \quad (15)$$

and

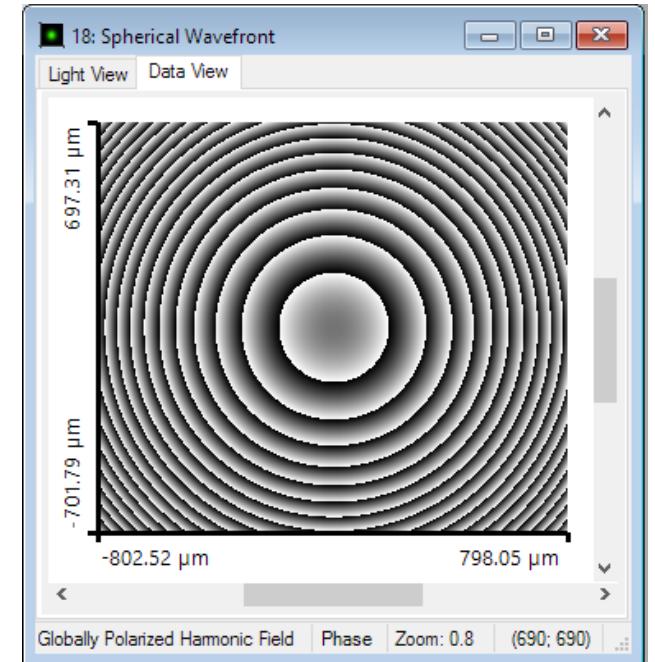
$$\mathbf{U}(\mathbf{r}, \omega; H) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (\mathbf{r}, \omega) \mapsto \mathbf{H}(\mathbf{r}, \omega) \exp(-i\psi(\mathbf{r}, \omega)) \quad (16)$$

with the wavefront phase

$$\psi(\mathbf{r}, \omega) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}, (\mathbf{r}, \omega) \mapsto \psi(\mathbf{r}, \omega). \quad (17)$$

- We refer to  $\exp(i\psi(\mathbf{r}, \omega))$  as the wavefront phase factor and to  $\mathbf{U}(\mathbf{r}, \omega; E)$  and  $\mathbf{U}(\mathbf{r}, \omega; H)$  as the electric and magnetic  $U$ -fields.
- The  $U$ -fields do not include the wavefront phases anymore. The fields can then be written by

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{U}(\mathbf{r}, \omega; E) \exp(i\psi(\mathbf{r}, \omega)) \quad (18)$$



# Wavefront Phase

and

$$\mathbf{H}(\mathbf{r}, \omega) = \mathbf{U}(\mathbf{r}, \omega; H) \exp(i\psi(\mathbf{r}, \omega)). \quad (19)$$

- Per definitions in Eqs. (15) and (16) the magnitude of the  $U$ -fields is identical with the magnitude of the field vectors themselves and the phase of  $U$ -fields follows as the residual phase, e.g.

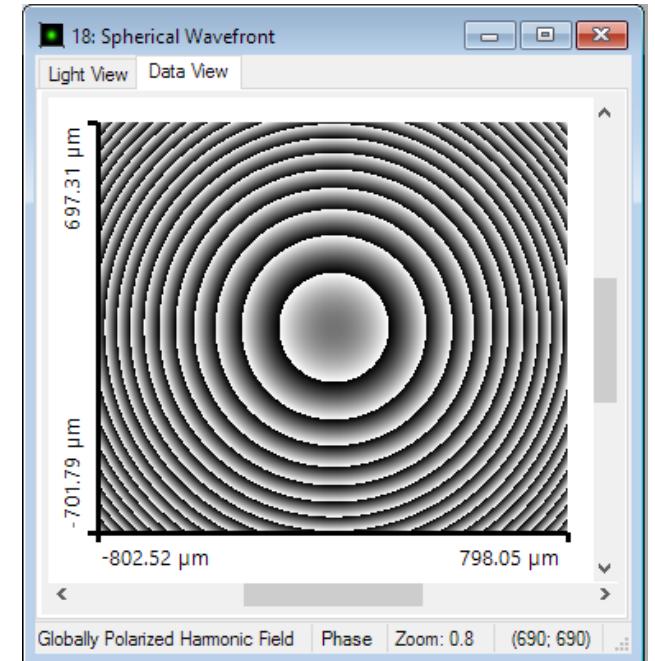
$$\arg[U_y(\mathbf{r}, \omega; H)] = \arg[H_y(\mathbf{r}, \omega)] - \psi(\mathbf{r}, \omega). \quad (20)$$

- For the transversal electric field the extraction of the  $U$ -field leads to

$$\mathbf{E}_\perp(\mathbf{r}, \omega) = \mathbf{U}_\perp(\mathbf{r}, \omega) \exp(i\psi(\mathbf{r}, \omega)). \quad (21)$$

- For an arbitrary component we write

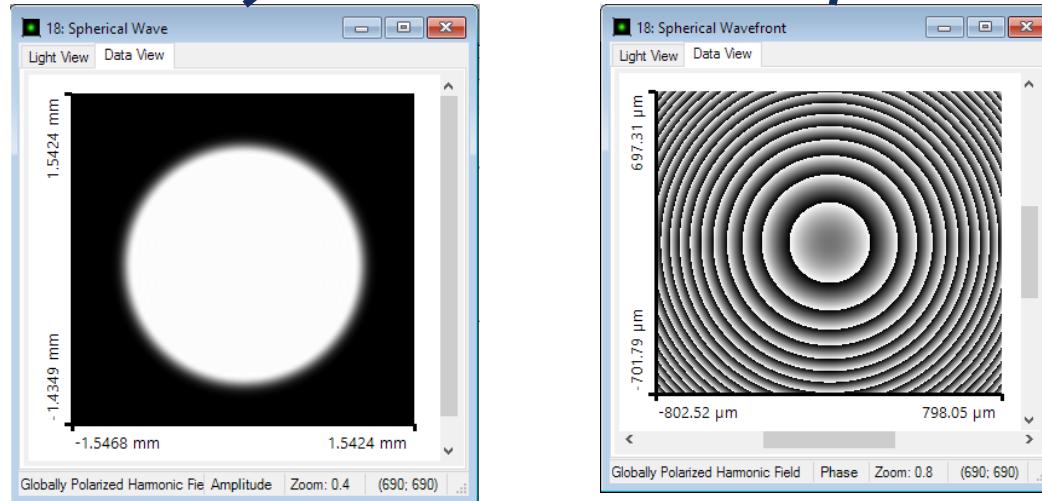
$$V(\mathbf{r}, \omega) = U(\mathbf{r}, \omega) \exp(i\psi(\mathbf{r}, \omega)). \quad (22)$$



Notation Electromagnetic Fields	
Name	Notation
electromagnetic field (time domain) (2)	$\mathbf{E}, \mathbf{H}$
electromagnetic field (frequency domain) (3)	$\mathbf{E}, \mathbf{H}$
arbitrary field components (12)	$E, H$
transversal electromagnetic field (14)	$V, \mathbf{C}$
transversal magnetic field (15)	$\mathbf{B}$
wavefront phase (17)	$\phi$
wavefront phase factor	$\exp(i\phi)$
electric field (18)	$\mathbf{E}(\mathbf{r}, \omega)$
electric A-field (19)	$\mathbf{A}_e(\mathbf{r}, \omega)$
magnetic E-field (20)	$(\mathbf{E}^\perp, \mathbf{B})$
magnetic A-field (21)	$(\mathbf{E}^\perp, \mathbf{A}_e)$
transversal electric / E-field (22)	$V$
transversal electric A-field (23)	$\mathbf{A}_t$

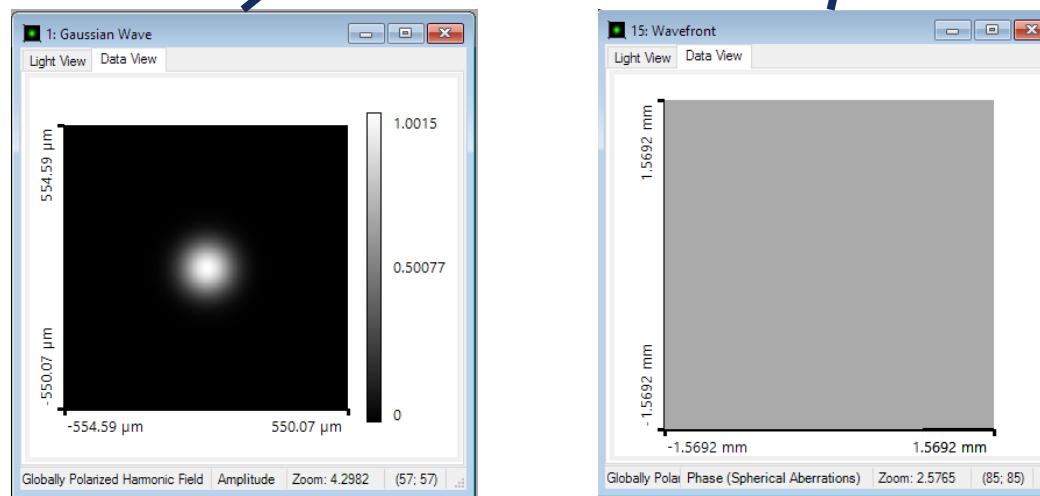
# Wavefront Phase: Spherical Wave

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



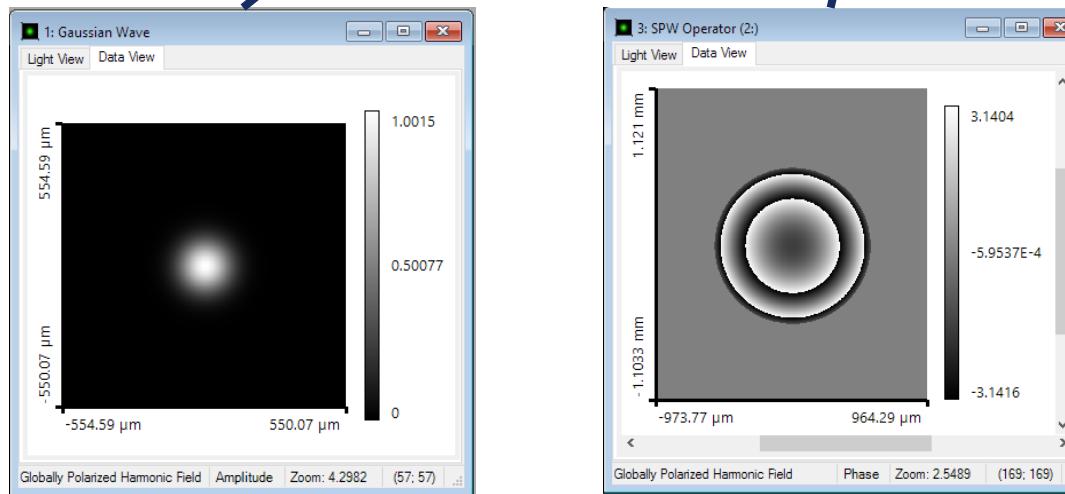
# Wavefront Phase: Gaussian Beam in Its Waist

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



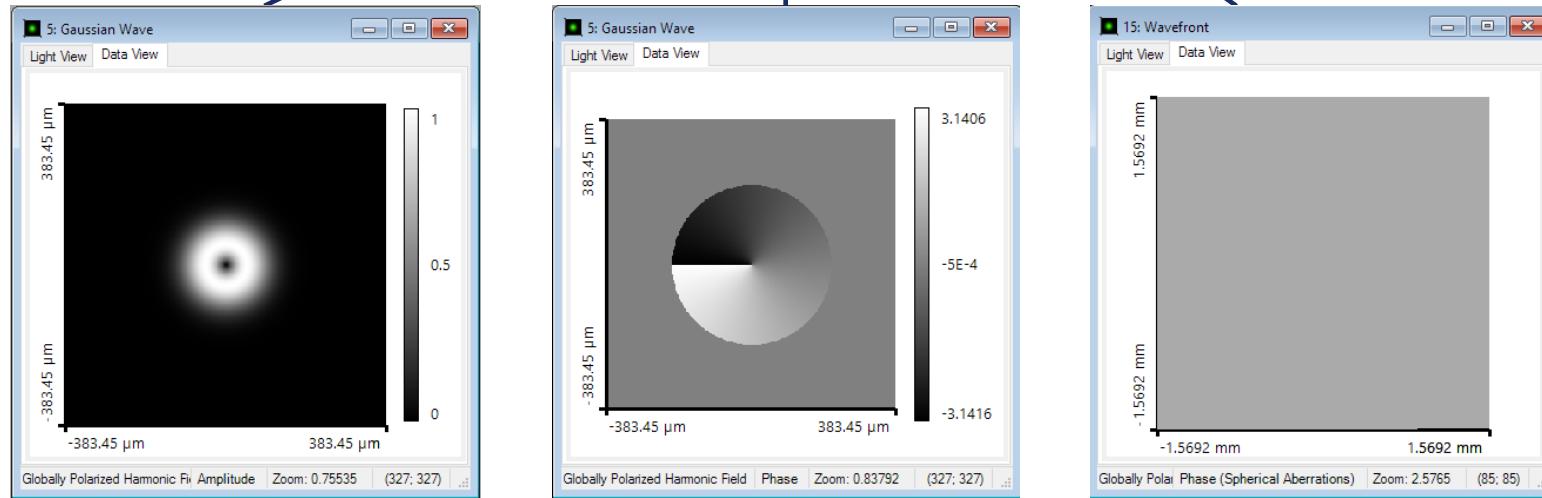
# Wavefront Phase: Gaussian Beam Propagated

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



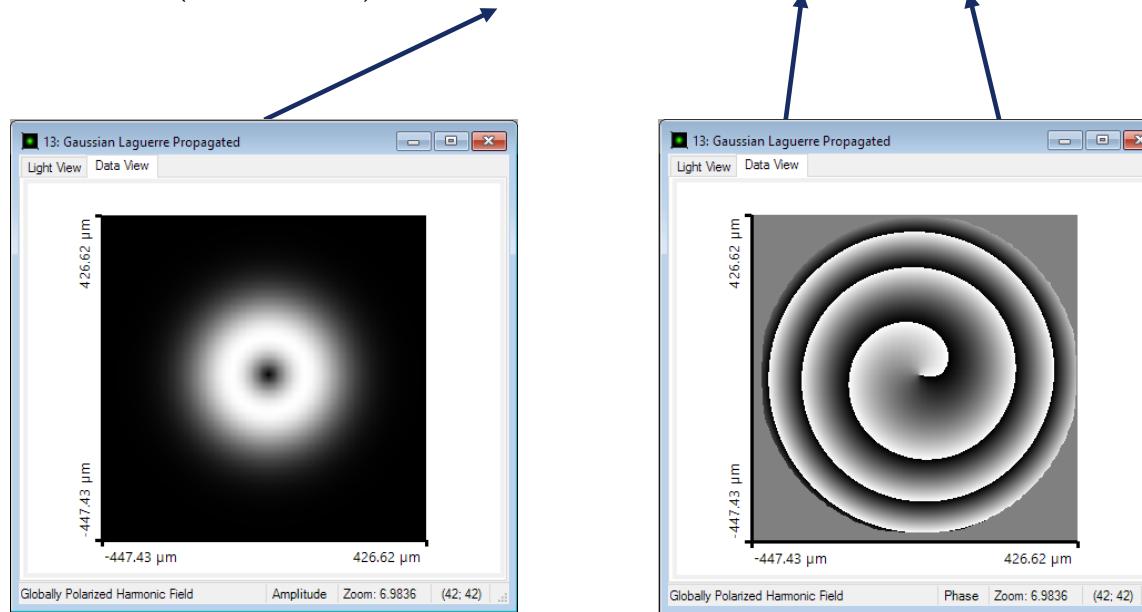
# Example Gaussian Laguerre Beam in Waist

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



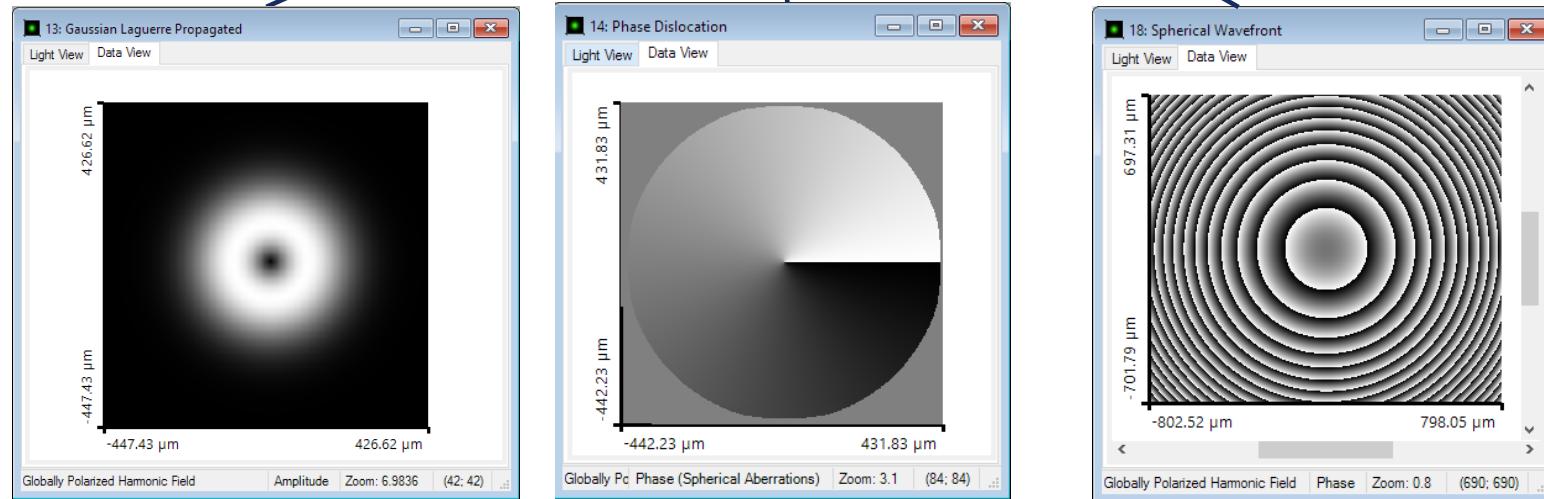
# Example Propagated Gaussian Laguerre Beam

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



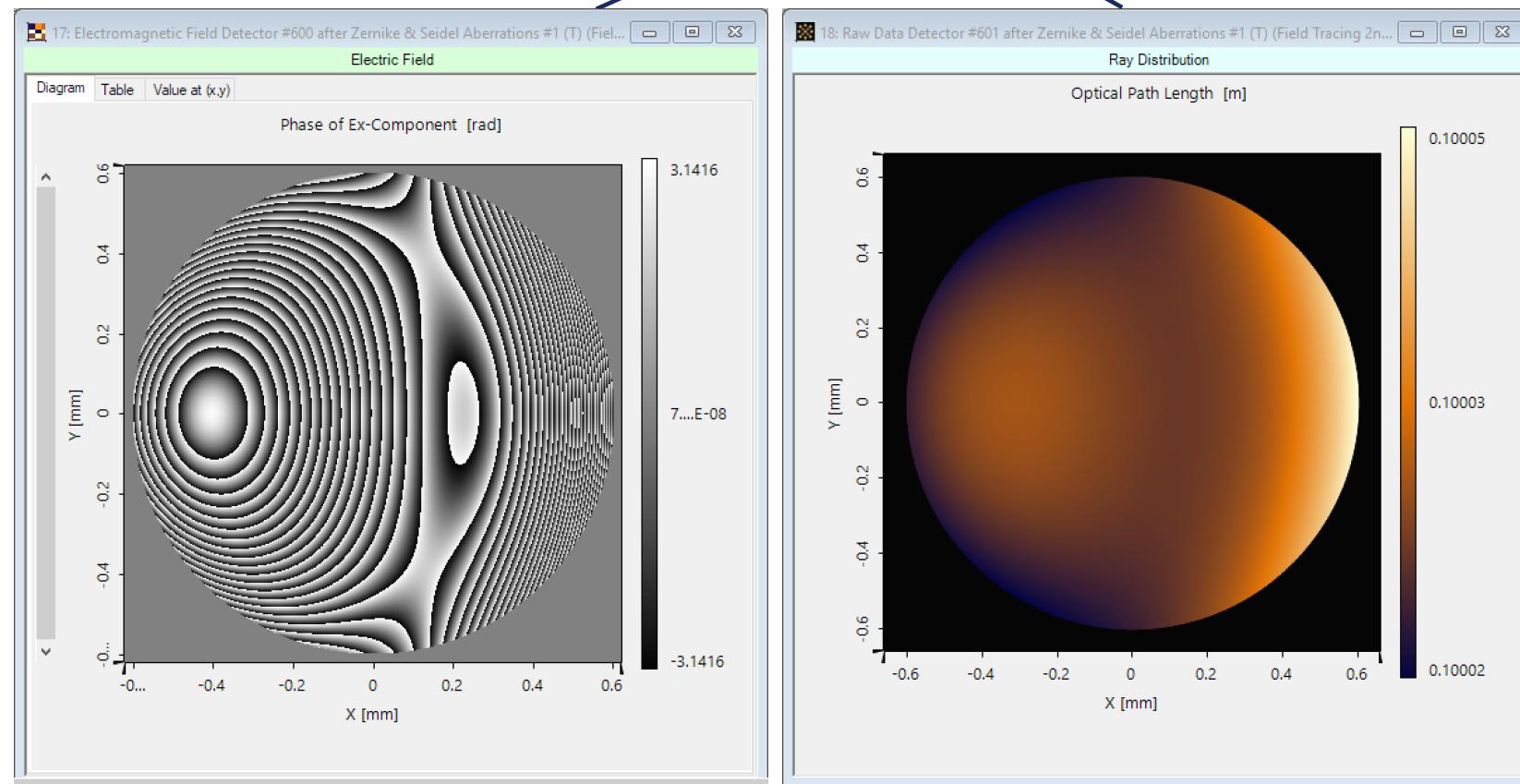
# Example Propagated Gaussian Laguerre Beam

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



# General Example with Aberrations

$$V(x, y) = U(x, y) \exp(i\psi(x, y)) = |V(x, y)| e^{i \arg U(x, y)} e^{i\psi(x, y)}$$



# Notation Electromagnetic Fields

Name	Notation	
	x-domain	k-domain
electric/magnetic field (time domain) (5)	$\bar{E}, \bar{H}$	$\tilde{\bar{E}}, \tilde{\bar{H}}$
electric/magnetic field (frequency domain) (9)	$E, H$	$\tilde{E}, \tilde{H}$
arbitrary field component (13)	$\bar{V}, V$	$\tilde{\bar{V}}, \tilde{V}$
transversal electromagnetic field (14)	$V_{\perp}$	$\tilde{V}_{\perp}$
transversal electric field (12)	$E_{\perp}$	$\tilde{E}_{\perp}$
wavefront phase (17)	$\psi$	$\tilde{\phi}$
wavefront phase factor	$\exp(i\psi)$	$\exp(i\tilde{\phi})$
electric $U$ -field (15)	$U(\dots; E)$	—
electric $\tilde{A}$ -field (75)	—	$\tilde{A}(\dots; E)$
magnetic $U$ -field (16)	$U(\dots; H)$	—
magnetic $\tilde{A}$ -field (76)	—	$\tilde{A}(\dots; H)$
transversal electric $U$ -field (21)	$U_{\perp}$	—
transversal electric $\tilde{A}$ -field (80)	—	$\tilde{A}_{\perp}$

Table 1: Notation and terminology: electromagnetic fields

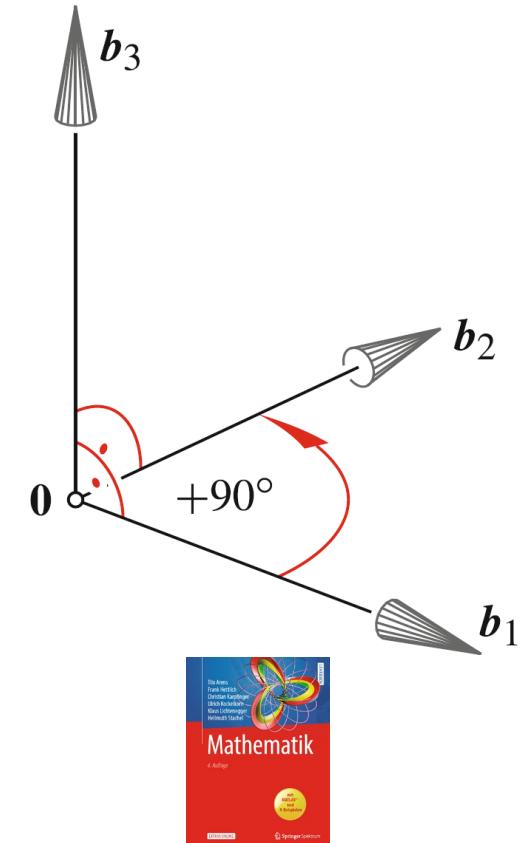
# Coordinate Systems

## 2.2 Coordinate systems

- The mathematical description of electromagnetic vector fields is based on the vectors  $E \in \mathbb{C}^3$ ,  $H \in \mathbb{C}^3$  and  $r \in \mathbb{R}^3$ . As soon as we refer to the components of a vector we need to define a coordinate system.
- In physical optics modeling we often need to change the coordinate system and by that the components of the vectors in the mathematical model.

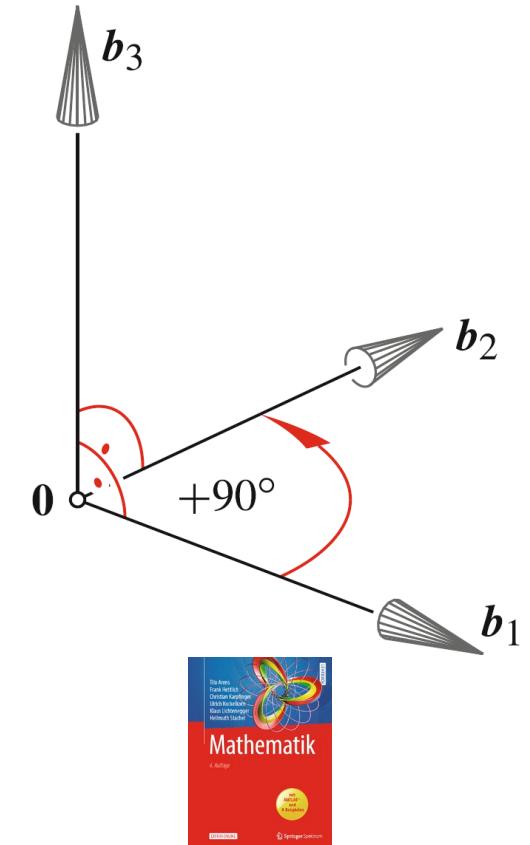
### 2.2.1 Vectors in coordinate systems

- Let us assume we define a coordinate system (CS) in the laboratory in which our optical system is to be described. The system itself consists of sources, optical components, and detectors all adjusted to each other in a suitable form.



# Coordinate Systems

- When we describe the surface in a component we typically do not use the lab CS but a local CS which is more suited to mathematically represent the surface.
- When we investigate a ray which interacts with the surface we may even use a special local CS at the intersection point.
- **In summary:** In optical modeling we start in a lab CS and then define more CS in reference to the lab CS.
- The lab CS can of course be understood with respect to a more global CS.
- When we think of a lab, then we may identify a point  $p$  in space. Together with a reference point  $p_0$  we may associate  $p$  with a position vector  $r$ . It starts in  $p_0$  and ends in  $p$ .
- In the point  $p$  itself we may define a vector, e.g. speed, a direction, or an electric field. So we would have for example  $E(r)$ .



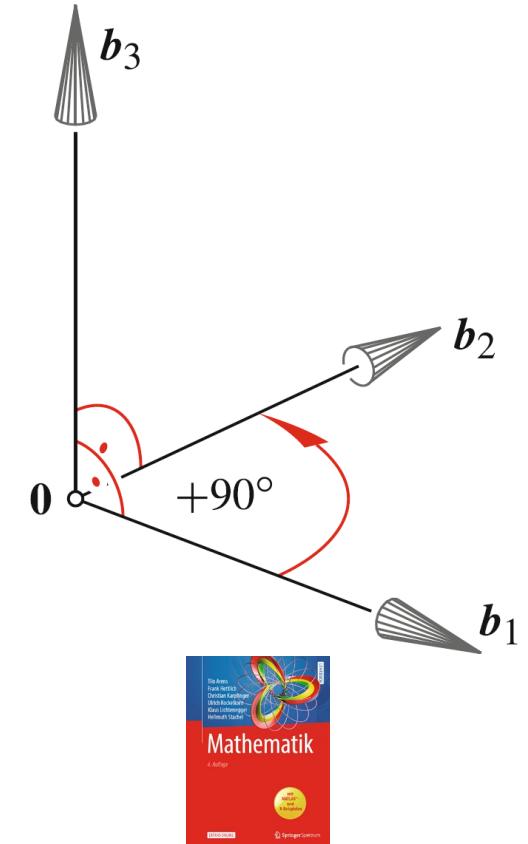
# Coordinate Systems

- In contrast to  $r$  the direction and length of  $E$  in  $p$  is formally free and must be further specified by Maxwell's equations, which already deal with components of the vectors, e.g. by  $\nabla \cdot E(r) = 0$ .
- The use of vector components requires the definition of a coordinate system.
- A coordinate system in  $\mathbb{R}^3$  is defined by three linearly independent basis vectors  $\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$  with  $\mathbf{a}_j \in \mathbb{R}^3$ . The lab CS is denoted by  $\Upsilon^{\text{lab}}$ .
- Any vector  $\mathbf{u} \in \mathbb{R}^3 \vee \mathbf{u} \in \mathbb{C}^3$  can be represented in  $\Upsilon$  by the linear combination

$$\mathbf{u} = u_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + u_3 \mathbf{a}_3 \quad (23)$$

with  $u_j \in \mathbb{R} \vee u_j \in \mathbb{C}$  for  $j = 1, \dots, 3$ .

- For most vectors, like directions and field vectors, the representation in Eq. (23) is sufficient.



# Coordinate Systems

- For a point  $p$  in space, which is represented by the position vector  $r$ , the origin  $p_0$  of the CS must be also specified and we do that by the origin vector  $o$ . For the lab CS we may assume  $o(\Upsilon^{\text{lab}}) = 0$ .
- In general a coordinate system is defined by

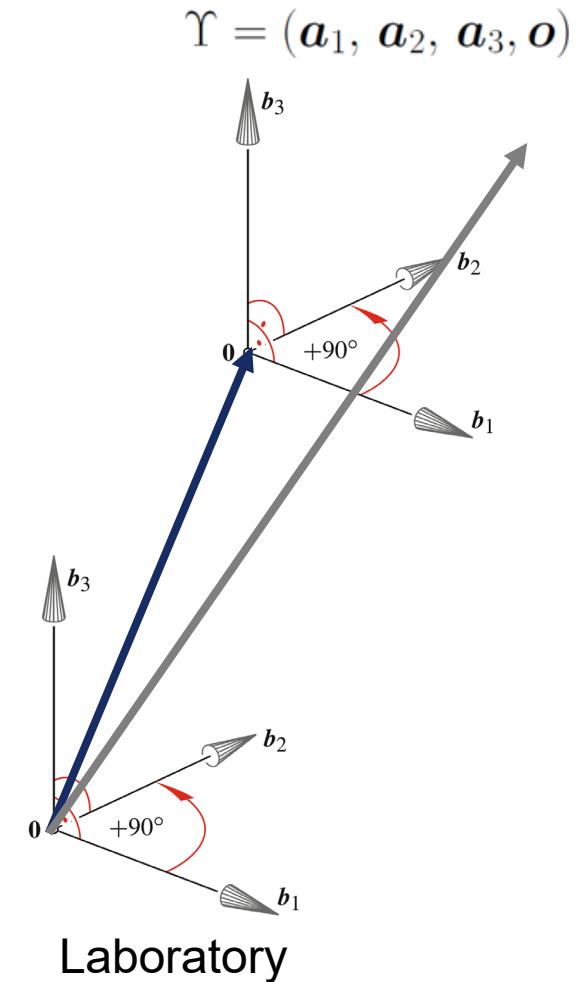
$$\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, o) \quad (24)$$

with the origin  $o$  which is of importance for position vectors, where we have the linear combination

$$\mathbf{r} = o + x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3, . \quad (25)$$

with  $x, y, z \in \mathbb{R}$  for  $j = 1, \dots, 3$ .

- The coefficients  $u_j$  in Eq. (23) are called the components of the vector  $u$  and the coefficients  $x, y, z$  in Eq. (25) are referred to as coordinates of the position  $r$ .



# Coordinate Systems

- The components and coordinates directly depend on the CS. Thus we use the notation

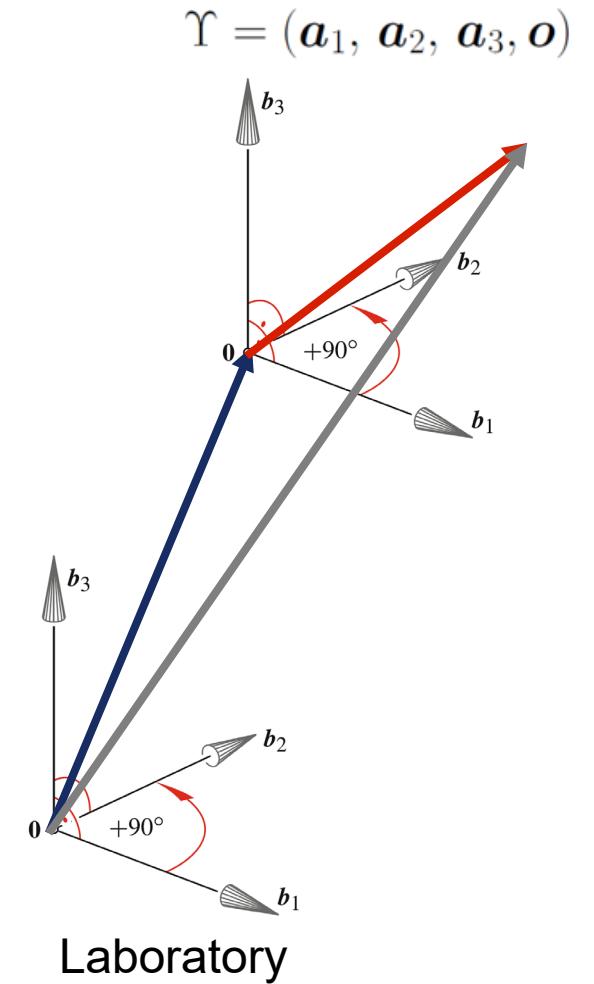
$${}_{\tau}\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \iff \boldsymbol{u} = {}_{\tau}\boldsymbol{u}^T \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}, \quad (26)$$

and

$${}_{\tau}\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \boldsymbol{r} = \boldsymbol{o} + {}_{\tau}\boldsymbol{r}^T \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}, \quad (27)$$

with the right sides of the equations are identical to the linear combinations in Eqs. (23) and (25).

- We apply the notations  ${}_{\tau}\boldsymbol{u}$  and  ${}_{\tau}\boldsymbol{r}$  to indicate the component and coordinate vectors in the CS respectively.



# Coordinate Systems

- If we just write  $u$  and  $r$  we may not have fixed the CS and the components and coordinates are not explicitly specified yet. We apply the notation with CS when that is needed for clarification.
- Cartesian coordinate systems are of special importance and typically used in physical optics modeling. Then the basis vectors are orthonormal, that means

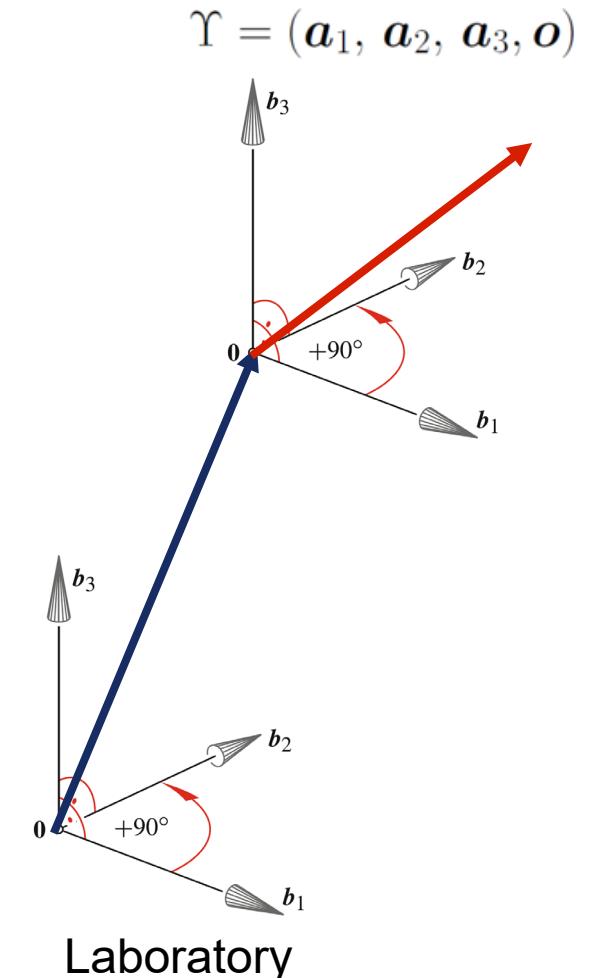
$$\hat{a}_i \cdot \hat{a}_j := \hat{a}_i^T \hat{a}_j = \delta_{ij} \quad (28)$$

with the Kronecker delta function  $\delta_{ij}$  and the scalar product defined by the dot  $\cdot$ .

- We use the hat on  $\hat{a}$  to indicate a normalized vector.
- In a Cartesian CS the calculation of the components and coordinates is simply given by

$${}_{\tau}u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \hat{a}_1 \cdot u \\ \hat{a}_2 \cdot u \\ \hat{a}_3 \cdot u \end{pmatrix}. \quad (29)$$

- That follows directly from the application of Eq. (28) to Eq. (26). Analogously



# Coordinate Systems: Transforming Vectors

we find

$${}_{\tau}r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \hat{a}_1 \cdot (r - o) \\ \hat{a}_2 \cdot (r - o) \\ \hat{a}_3 \cdot (r - o) \end{pmatrix}. \quad (30)$$

## 2.2.2 Transforming vectors between different coordinate systems

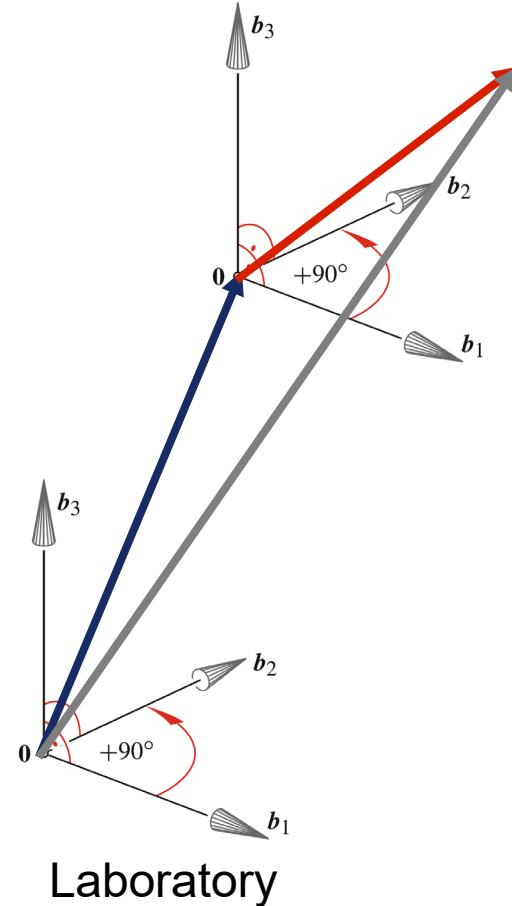
- Each vector can be represented in different CS, e.g. in the systems  $\Upsilon$  and  $\Upsilon'$  by

$$\mathbf{u} = u_1 \mathbf{a}_1(\Upsilon) + u_2 \mathbf{a}_2(\Upsilon) + u_3 \mathbf{a}_3(\Upsilon) = {}_{\tau}\mathbf{u}^T \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix} \quad (31)$$

and

$$\mathbf{u} = u'_1 \mathbf{a}_1(\Upsilon') + u'_2 \mathbf{a}_2(\Upsilon') + u'_3 \mathbf{a}_3(\Upsilon') = ({}_{\tau}\mathbf{u})^T \begin{pmatrix} \mathbf{a}_1(\Upsilon') \\ \mathbf{a}_2(\Upsilon') \\ \mathbf{a}_3(\Upsilon') \end{pmatrix}, \quad (32)$$

$$\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{o})$$



# Coordinate Systems: Transforming Vectors

- We may also express the basis vectors of one system in the one of the others,  
e.g.

$$\mathbf{a}_j(\Upsilon') = a'_{j1}\mathbf{a}_1(\Upsilon) + a'_{j2}\mathbf{a}_2(\Upsilon) + a'_{j3}\mathbf{a}_3(\Upsilon) = ({_r}\mathbf{a}_j(\Upsilon'))^T \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix}, \quad (33)$$

with  $a'_{ij} \in \mathbb{R}$ .

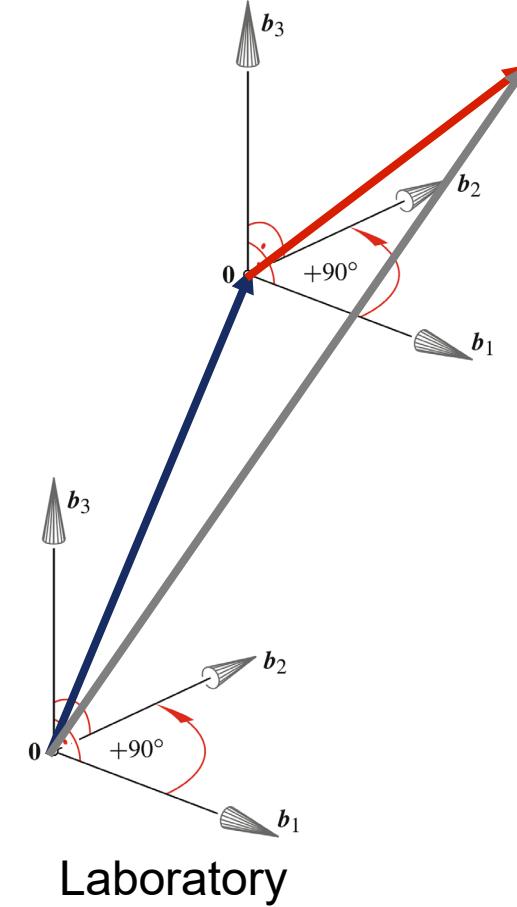
- In matrix form we find:

$$\begin{pmatrix} \mathbf{a}_1(\Upsilon') \\ \mathbf{a}_2(\Upsilon') \\ \mathbf{a}_3(\Upsilon') \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix} \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix} \quad (34)$$

$$= \begin{pmatrix} ({_r}\mathbf{a}_1(\Upsilon'))^T \\ ({_r}\mathbf{a}_2(\Upsilon'))^T \\ ({_r}\mathbf{a}_3(\Upsilon'))^T \end{pmatrix} \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix} \quad (35)$$

$$= {_r}\mathbf{M}_r \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix} \quad (36)$$

$$\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{o})$$



# Coordinate Systems: Transforming Vectors

- The rows of the matrix contain the coordinate vectors of the basis vectors of  $\Upsilon'$  in the CS  $\Upsilon$ . Inserting Eq. (36) into Eq. (32) and comparing with Eq. (31) yields

$${}_{\Upsilon} \mathbf{u}^T = ({}_{\Upsilon} \mathbf{u})^T {}_{\Upsilon'} \mathbf{M}_{\Upsilon} \iff {}_{\Upsilon} \mathbf{u} = ({}_{\Upsilon'} \mathbf{M}_{\Upsilon})^T {}_{\Upsilon'} \mathbf{u} = ({}_{\Upsilon'} \mathbf{T}_{\Upsilon'}) {}_{\Upsilon'} \mathbf{u}, \quad (37)$$

with the definition of the transfer matrix  ${}_{\Upsilon'} \mathbf{T}_{\Upsilon} := ({}_{\Upsilon'} \mathbf{M}_{\Upsilon})^T$ .

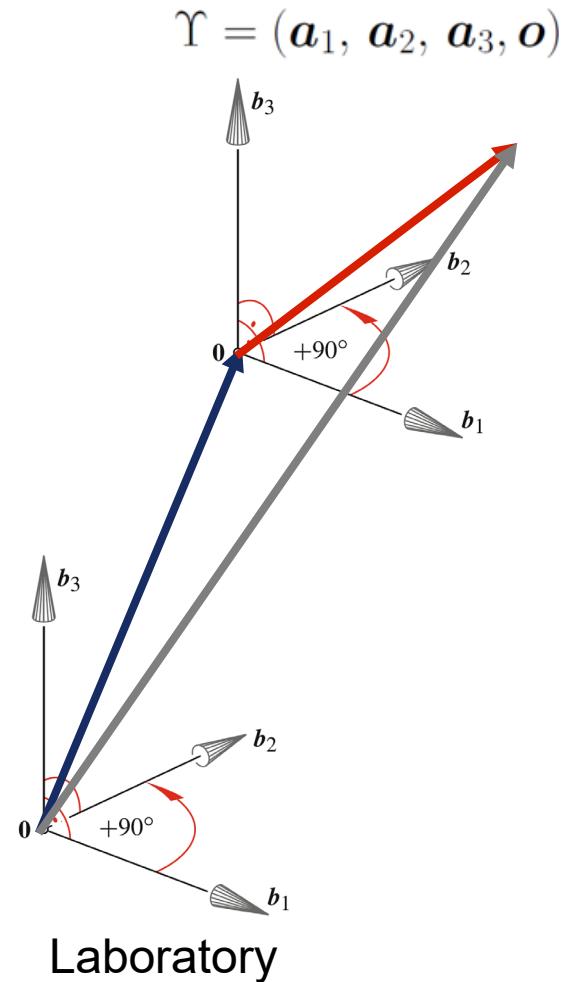
- Analogously we may derive a transfer equation from vectors in  $\Upsilon$  to the one in  $\Upsilon'$ . In summary we have the transfer equations

$${}_{\Upsilon} \mathbf{u} = ({}_{\Upsilon'} \mathbf{T}_{\Upsilon'}) {}_{\Upsilon'} \mathbf{u} \iff \mathbf{u} = ({}_{\Upsilon'} \mathbf{T}_{\Upsilon'}) \mathbf{u}' \quad \text{with} \quad {}_{\Upsilon'} \mathbf{T}_{\Upsilon} := ({}_{\Upsilon} \mathbf{a}_1(\Upsilon') \ {}_{\Upsilon} \mathbf{a}_2(\Upsilon') \ {}_{\Upsilon} \mathbf{a}_3(\Upsilon')) \quad (38)$$

and

$${}_{\Upsilon'} \mathbf{u} = ({}_{\Upsilon} \mathbf{T}_{\Upsilon}) {}_{\Upsilon} \mathbf{u} \iff \mathbf{u}' = ({}_{\Upsilon} \mathbf{T}_{\Upsilon}) \mathbf{u} \quad \text{with} \quad {}_{\Upsilon} \mathbf{T}_{\Upsilon} := ({}_{\Upsilon} \mathbf{a}_1(\Upsilon) \ {}_{\Upsilon} \mathbf{a}_2(\Upsilon) \ {}_{\Upsilon} \mathbf{a}_3(\Upsilon)). \quad (39)$$

- In Eqs. (38) and (39) we also introduced the compact notations  ${}_{\Upsilon} \mathbf{u} =: \mathbf{u}$  and  ${}_{\Upsilon'} \mathbf{u} =: \mathbf{u}'$  which can be used whenever it is clear, that component vectors are considered. The columns of the transfer matrices are the coordinates of the



# Coordinate Systems: Transforming Vectors

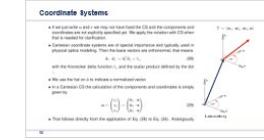
initial basis vectors in the new system.

- According to Eqs. (38) and (39) are the matrices  ${}_{\text{r}}\mathbf{T}_{\text{r}}$  and  ${}_{\text{r}}\mathbf{T}_{\text{r}'}$  the inverse matrices of each other, i.e.

$${}_{\text{r}}\mathbf{T}_{\text{r}} = ({}_{\text{r}}\mathbf{T}_{\text{r}'})^{-1} . \quad (40)$$

- For Cartesian coordinates we may write, according to Eq. (29),

$${}_{\text{r}}\mathbf{T}_{\text{r}} = \begin{pmatrix} \hat{\mathbf{a}}'_1 \cdot \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}'_1 \cdot \hat{\mathbf{a}}_2 & \hat{\mathbf{a}}'_1 \cdot \hat{\mathbf{a}}_3 \\ \hat{\mathbf{a}}'_2 \cdot \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}'_2 \cdot \hat{\mathbf{a}}_2 & \hat{\mathbf{a}}'_2 \cdot \hat{\mathbf{a}}_3 \\ \hat{\mathbf{a}}'_3 \cdot \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}'_3 \cdot \hat{\mathbf{a}}_2 & \hat{\mathbf{a}}'_3 \cdot \hat{\mathbf{a}}_3 \end{pmatrix} \quad (41)$$



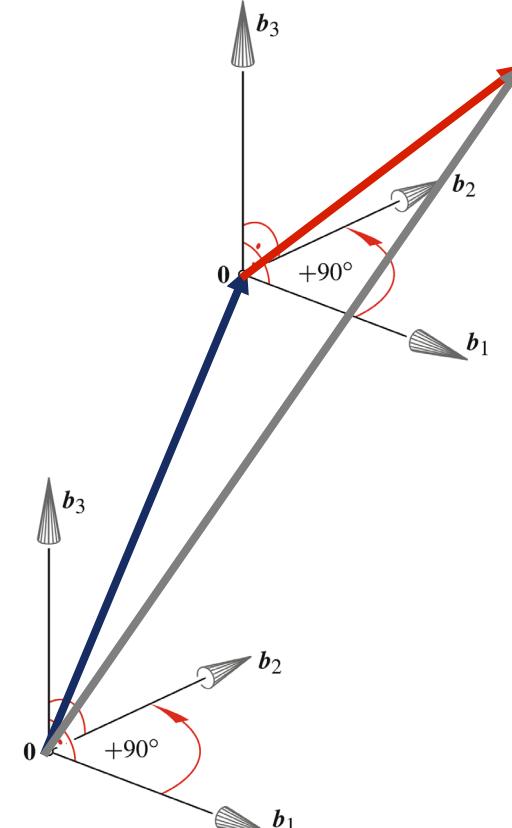
and

$${}_{\text{r}}\mathbf{T}_{\text{r}'} = \begin{pmatrix} \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}'_1 & \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}'_2 & \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}'_3 \\ \hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}'_1 & \hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}'_2 & \hat{\mathbf{a}}_2 \cdot \hat{\mathbf{a}}'_3 \\ \hat{\mathbf{a}}_3 \cdot \hat{\mathbf{a}}'_1 & \hat{\mathbf{a}}_3 \cdot \hat{\mathbf{a}}'_2 & \hat{\mathbf{a}}_3 \cdot \hat{\mathbf{a}}'_3 \end{pmatrix} . \quad (42)$$

- We conclude for Cartesian coordinates

$${}_{\text{r}}\mathbf{T}_{\text{r}'} = ({}_{\text{r}}\mathbf{T}_{\text{r}})^T = ({}_{\text{r}}\mathbf{T}_{\text{r}})^{-1} , \quad (43)$$

$$\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{o})$$



Laboratory

# Coordinate Systems: Transforming Vectors

that means the inverse of a Cartesian transfer matrix can be obtained by a transposition only.

- For the transfer rules of position vectors the origin of the CS must be included.  
For the derivation we start from

$$\mathbf{r} = \mathbf{o}(\Upsilon) + {}_r\mathbf{r}^T \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix} = \mathbf{o}(\Upsilon') + {}_r\mathbf{r}^T \begin{pmatrix} \mathbf{a}_1(\Upsilon') \\ \mathbf{a}_2(\Upsilon') \\ \mathbf{a}_3(\Upsilon') \end{pmatrix} = \mathbf{o}(\Upsilon') + {}_r\mathbf{r}^T {}_r\mathbf{M}_r \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix}. \quad (44)$$

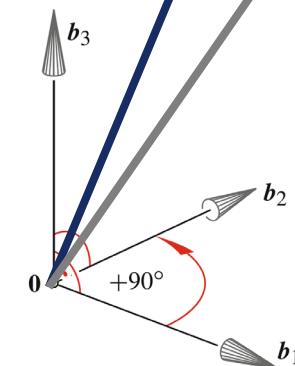
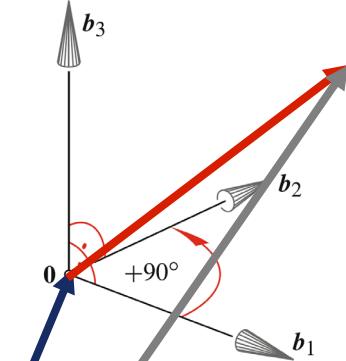
- Next we express  $\mathbf{o}(\Upsilon')$  by Eq. (27) as

$$\mathbf{o}(\Upsilon') = \mathbf{o}(\Upsilon) + ({}_r\mathbf{o}(\Upsilon'))^T \begin{pmatrix} \mathbf{a}_1(\Upsilon) \\ \mathbf{a}_2(\Upsilon) \\ \mathbf{a}_3(\Upsilon) \end{pmatrix}. \quad (45)$$

- Plugging it into Eq. (44) and transposition leads to  ${}_r\mathbf{r} = {}_r\mathbf{o}(\Upsilon') + ({}_r\mathbf{T}_{r'}) {}_r\mathbf{r}$ . From that we conclude

$${}_r\mathbf{r} = {}_{r'}\mathbf{T}_r({}_r\mathbf{r} - {}_r\mathbf{o}(\Upsilon')) \iff \mathbf{r}' = {}_{r'}\mathbf{T}_r(\mathbf{r} - \mathbf{o}(\Upsilon')) \quad (46)$$

$$\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{o})$$



Laboratory

# Coordinate Systems: Transforming Vectors

and for the other direction

$${}_{\text{r}}\mathbf{r} = {}_{\text{r}}\mathbf{T}_{\text{r}'}({}_{\text{r}}\mathbf{r} - {}_{\text{r}}\mathbf{o}(\Upsilon)) \iff \mathbf{r} = {}_{\text{r}}\mathbf{T}_{\text{r}'}(\mathbf{r}' - \mathbf{o}'(\Upsilon)). \quad (47)$$

- The origins of both CS are directly related via

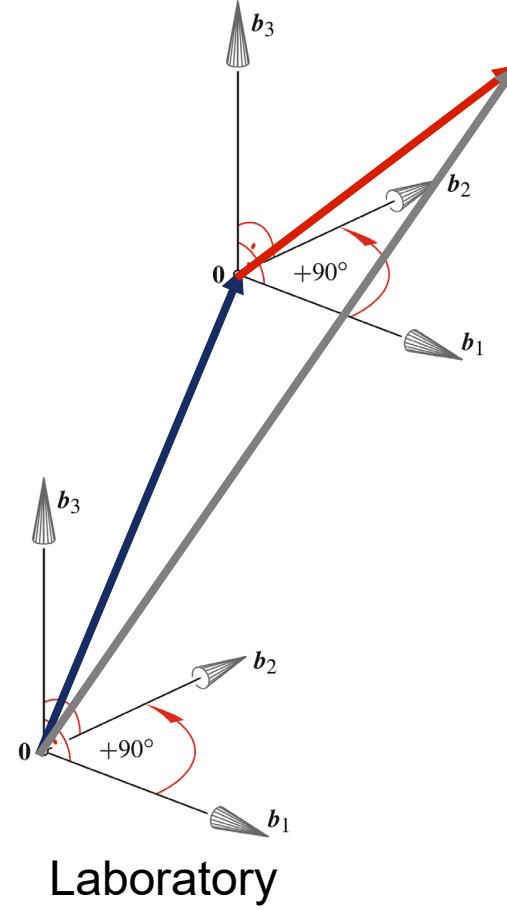
$${}_{\text{r}}\mathbf{o}(\Upsilon) = -{}_{\text{r}}\mathbf{T}_{\text{r}'}{}_{\text{r}}\mathbf{o}(\Upsilon') \iff \mathbf{o}'(\Upsilon) = -{}_{\text{r}}\mathbf{T}_{\text{r}'}\mathbf{o}(\Upsilon') \quad (48)$$

and

$${}_{\text{r}}\mathbf{o}(\Upsilon') = -{}_{\text{r}}\mathbf{T}_{\text{r}'}{}_{\text{r}}\mathbf{o}(\Upsilon) \iff \mathbf{o}(\Upsilon') = -{}_{\text{r}}\mathbf{T}_{\text{r}'}\mathbf{o}(\Upsilon). \quad (49)$$

Equation (48) follows from Eq. (46) for  $\mathbf{r} = \mathbf{o}(\Upsilon)$  because of  ${}_{\text{r}}\mathbf{o}(\Upsilon) = 0$ .

$$\Upsilon = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{o})$$



# Coordinate Systems: Substitution of Vectors

## 2.2.3 Substituting position vector in function

- The transfer equations in Sec. 2.2.2 provide us with the mathematical tools to change coordinate systems when needed.
- When considering electromagnetic fields we change the coordinate system of the field vectors, e.g.  $\mathbf{E}$ , itself and by that the values of its components. But we also change the coordinate system in which we represent the positions in which we observe physical quantities.
- Let us consider the function  $f(\mathbf{r})$  with the location vector  $\mathbf{r} = {}_r\mathbf{r}$ . When we like to change the vector into another CS, then we must substitute  $\mathbf{r}$  by  $\mathbf{r}' = {}_r\mathbf{r}$  according to Eq. (47) and we write

$$f(\mathbf{r})[\mathbf{r} \leftarrow \mathbf{r}(\mathbf{r}')] = f({}_r\mathbf{T}_r(\mathbf{r}' - \mathbf{o}'(\Upsilon))) =: \check{f}(\mathbf{r}'). \quad (50)$$

- For the same position in  $p$  in  $\mathbb{R}^3$  the functions  $f$  and  $\check{f}$  give the

### Coordinate Systems: Transforming Vectors

and for the other direction

$$\mathbf{r} = {}_\Upsilon\mathbf{T}_\Upsilon(\mathbf{r} - {}_\Upsilon\mathbf{o}(\Upsilon)) \iff \mathbf{r} = {}_\Upsilon\mathbf{T}_\Upsilon(\mathbf{r}' - \mathbf{o}'(\Upsilon)). \quad (47)$$

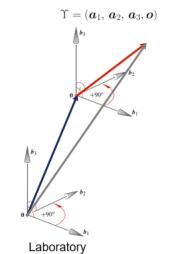
- The origins of both CS are directly related via

$${}_\Upsilon\mathbf{o}(\Upsilon) = -{}_\Upsilon\mathbf{T}_\Upsilon \cdot \mathbf{o}'(\Upsilon) \iff \mathbf{o}'(\Upsilon) = -{}_\Upsilon\mathbf{T}_\Upsilon \cdot \mathbf{o}(\Upsilon) \quad (48)$$

and

$$\mathbf{o}(\Upsilon') = -{}_\Upsilon\mathbf{T}_\Upsilon \cdot \mathbf{o}'(\Upsilon) \iff \mathbf{o}'(\Upsilon') = -{}_\Upsilon\mathbf{T}_\Upsilon \cdot \mathbf{o}'(\Upsilon). \quad (49)$$

Equation (48) follows from Eq. (46) for  $\mathbf{r} = \mathbf{o}(\Upsilon)$  because of  ${}_\Upsilon\mathbf{o}(\Upsilon) = 0$ .



# Coordinate Systems: Substitution of Vectors

same values, since only the position is expressed in different coordinate systems:  $f(\mathbf{r}) = \tilde{f}(\mathbf{r})$ .

- In optics we often deal with the function  $f(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r}$ , with  $\mathbf{k}$  is a direction vector. For this function the substitution of Eq. (50) should be done explicitly for Cartesian coordinate systems next. Then we have

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k}^T (\mathbf{r} - \mathbf{o}'(\Upsilon)) \quad (51)$$

$$\mathbf{k} \cdot \mathbf{r} = ((\mathbf{T}_{\mathbf{r}})^T \mathbf{k})^T \mathbf{r}' - \mathbf{k}^T \mathbf{T}_{\mathbf{r}} \mathbf{o}'(\Upsilon)$$

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k}' \cdot \mathbf{r}' + \mathbf{k} \cdot \mathbf{o}(\Upsilon') \quad (51)$$

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k}' \cdot \mathbf{r}' - \mathbf{k}' \cdot \mathbf{o}'(\Upsilon) \quad (52)$$

- For the conclusions we used Eq. (43) and Eq. (49). Equations (51) and (52) represent explicit examples of Eq. (50).

## Coordinate Systems: Substitution of Vectors

### 2.2.3 Substituting position vector in function

- The transfer equations in Sec. 2.2.2 provide us with the mathematical tools to change coordinate systems when needed.
- When considering electromagnetic fields we change the coordinate system of the field vectors, e.g.  $\mathbf{E}$ , itself and by that the values of its components. But we also change the coordinate system in which we represent the positions in which we observe physical quantities.
- Let us consider the function  $f(\mathbf{r})$  with the location vector  $\mathbf{r} = \mathbf{r}$ . When we like to change the vector into another CS, then we must substitute  $\mathbf{r}$  by  $\mathbf{r}' = \mathbf{r}$  according to Eq. (47) and we write  
$$f(\mathbf{r})|_{[\mathbf{r} \leftarrow \mathbf{r}']} = f(\mathbf{T}_{\mathbf{r}}(\mathbf{r}' - \mathbf{o}'(\Upsilon))) =: \tilde{f}(\mathbf{r}') \quad (50)$$
- For the same position in  $p \in \mathbb{R}^3$  the functions  $f$  and  $\tilde{f}$  give the



57

# Surfaces and Curves

## 2.2.4 Surfaces and curves

- Surfaces, planes, curves and lines are important subsets of  $\mathbb{R}^3$ . They are all defined via a parameter dependency of the position vector  $r$ .
- If we introduce the orthonormal basis vectors  $\hat{x}(S)$ ,  $\hat{y}(S)$  and  $\hat{z}(S)$  we may define a surface by

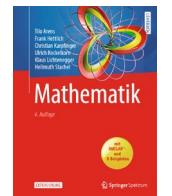
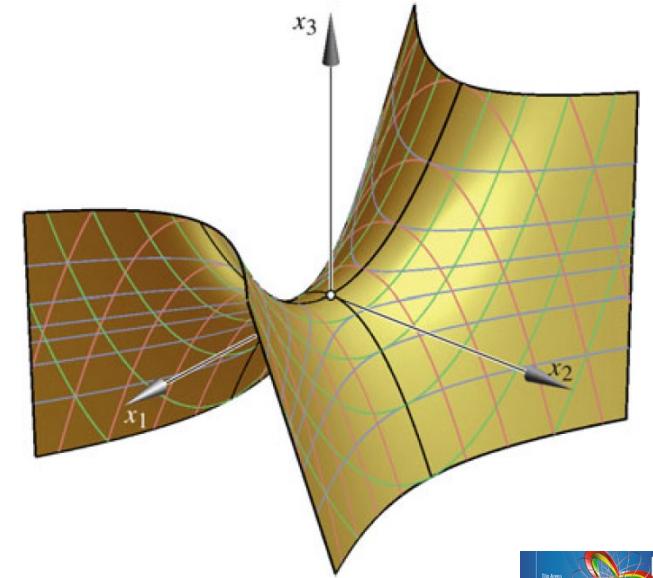
$$S = \{ \mathbf{r} \in \mathbb{R}^3 \mid \mathbf{r}(u, v) : D(S) \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (53)$$

$$(u, v) \mapsto \mathbf{r}_0(S) + x(u, v)\hat{x}(S) + y(u, v)\hat{y}(S) + z(u, v)\hat{z}(S) \}, \quad (54)$$

with  $D(S)$  being the domain of definition of the surface and  $\mathbf{r}_0$  a position vector.

- In a compact notation we write

$$\mathbf{r}^S(u, v) := \mathbf{r}_0(S) + x(u, v)\hat{x}(S) + y(u, v)\hat{y}(S) + z(u, v)\hat{z}(S). \quad (55)$$



# Surfaces and Curves

- Comparison with Eq. (25) suggests to express  $\mathbf{r}^S(u, v)$  in the surface CS which is defined by  $\Upsilon(S) = (\hat{\mathbf{x}}(S), \hat{\mathbf{y}}(S), \hat{\mathbf{z}}(S), \mathbf{r}_0(S))$ . Then, according to Eq. (27), the surface can be described by the coordinate vector

$${}_{\Upsilon(S)}\mathbf{r}^S(u, v) = (x(u, v), y(u, v), z(u, v))^T. \quad (56)$$

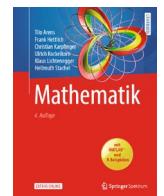
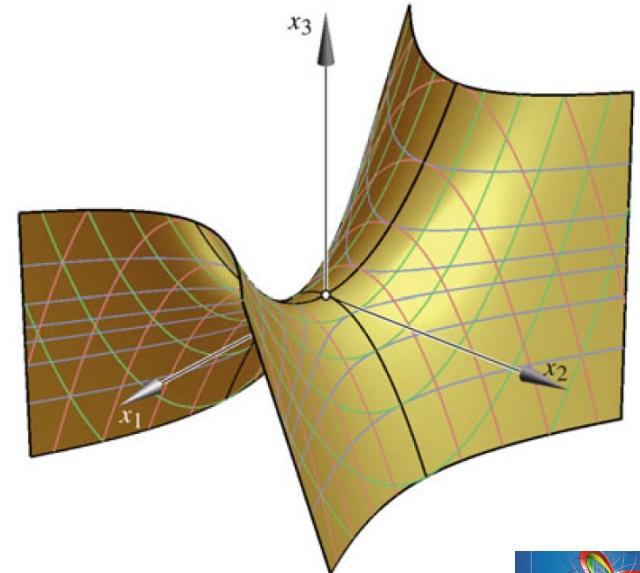
- We are often interested in a special type of surfaces which are related to a height profile  $h(x, y) : D(h) \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ . The surface is defined as

$${}_{\Upsilon(S)}\mathbf{r}^S(x, y) = (x, y, h(x, y))^T \quad (57)$$

and called the graph of the function  $(x, y) \mapsto h(x, y)$ .

- In the simplest case we have  $h(x, y) = 0 \forall (x, y) \in \mathbb{R}^2$ . Then we obtain the planar surface

$${}_{\Upsilon(P)}\mathbf{r}^P(x, y) = (x, y, 0)^T, \quad (58)$$



# Surfaces and Curves

where we have used  $P$  to indicate a plane.

- The notation of a plane in an arbitrary coordinate system is given by

$$\mathbf{r}^P(x, y) := \mathbf{r}_0(P) + x\hat{\mathbf{x}}(P) + y\hat{\mathbf{y}}(P). \quad (59)$$

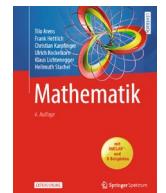
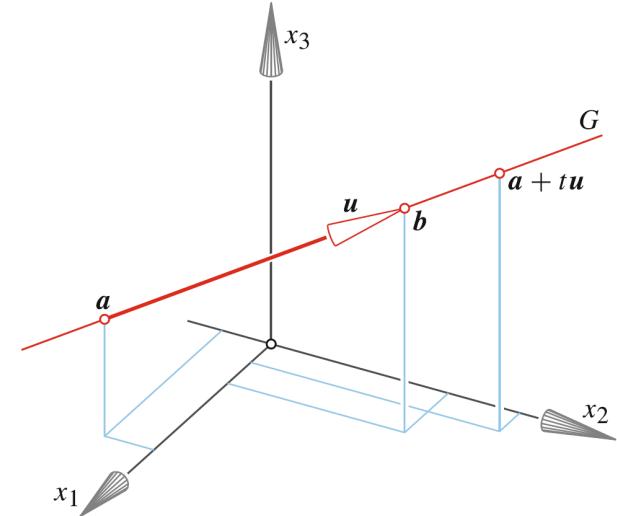
- A curve in  $\mathbb{R}^3$  is defined by

$$\mathbf{r}^K(u) := \mathbf{r}_0 + x(u)\hat{\mathbf{x}} + y(u)\hat{\mathbf{y}} + z(u)\hat{\mathbf{z}}. \quad (60)$$

- A straight line is defined by

$$\mathbf{r}^G(u) := \mathbf{r}_0 + u\hat{\mathbf{s}}, \quad (61)$$

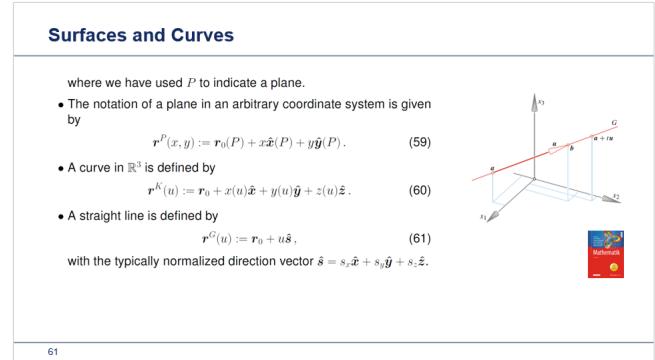
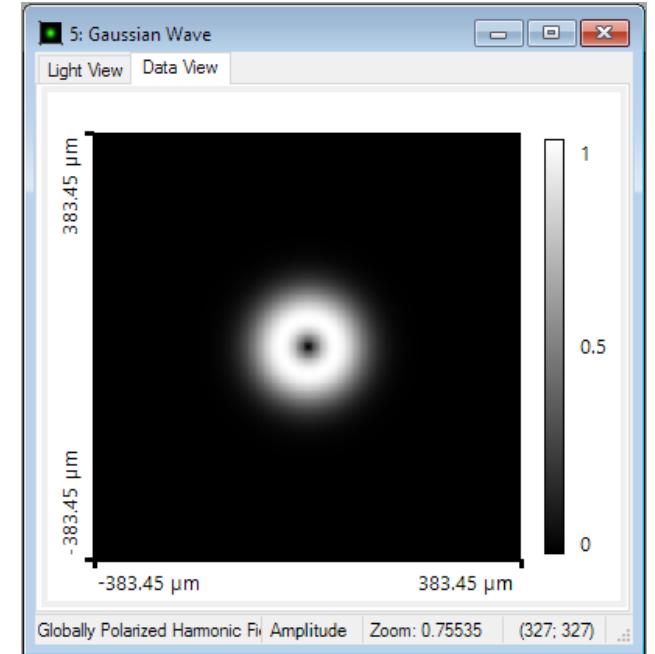
with the typically normalized direction vector  $\hat{\mathbf{s}} = s_x\hat{\mathbf{x}} + s_y\hat{\mathbf{y}} + s_z\hat{\mathbf{z}}$ .



# Vector Fields on Planes

## 2.3 Vector fields on planes

- Fields which are specified on a plane are of special concern in physical-optics modeling. The major reason is the convenient formulation of operators which act on fields in a plane.
- The Fourier transform is a very important example of such an operator. In modeling of real systems such field reference planes have different orientation in different parts of the system.
- Let us consider a complex amplitude  $\check{V}(\mathbf{r})$  (see Eq. (13) with skipped  $\omega$ ). We use the symbol  $\check{V}$  instead of  $V$  in the sense of the use of  $\check{f}$  in Sec. 2.2.3.
- In what follows we like to change the CS of  $\mathbf{r}$  and therefore must change the symbol. On a plane  $P$ , which is expressed by Eq. (59), we have  $\check{V}(\mathbf{r} \in P)$ .
- Wherever possible, we change to the coordinate system of the



# Vector Fields on Planes

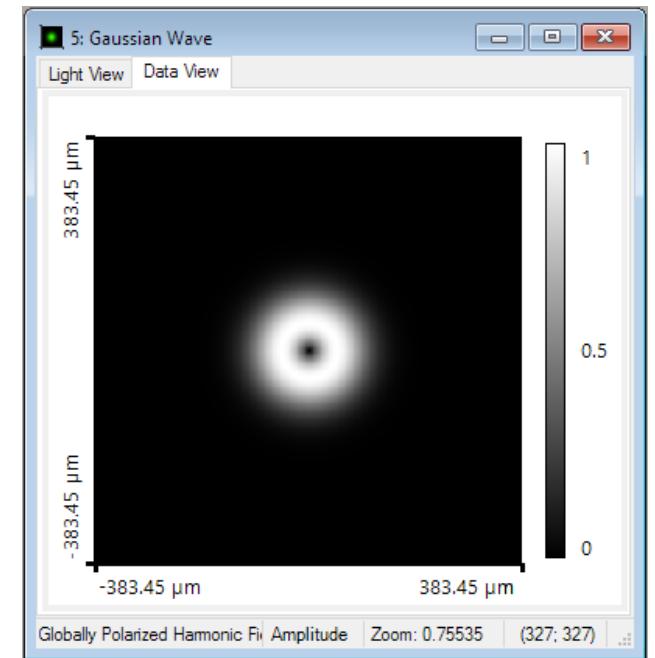
plane, that is  $\Upsilon(P)$ , by substitution  $\check{V}(\mathbf{r} \in P)[\mathbf{r} \leftarrow \mathbf{r}_{(\Upsilon(P)}\mathbf{r})]$  and obtain

$$V(\rho) := V(\mathbf{r}_{(\Upsilon(P)}\mathbf{r} \in P) \quad \text{with} \quad \rho = (x, y). \quad (62)$$

- In this very same CS the field of any parallel plane with distance  $z$  from the reference plane is given by  $V(\rho, z)$ . For a plane  $P'$  which is not parallel to  $P$  it is better to change the CS as well.
- If we consider a vector field  $E(\rho)$  on a plane  $P$ , then the CS is also relevant for the representation of the field component vector  $E$  itself.
- If not explicitly denoted, we assume  $E$  and  $r$  are represented in the same coordinate system  $\Upsilon(P)$ . Thus we have

$$E(\rho) := {}_{\Upsilon(P)}E(\mathbf{r}_{(\Upsilon(P)}\mathbf{r} \in P) \quad \text{or} \quad {}_{\Upsilon'}E(\rho) := {}_{\Upsilon}E(\mathbf{r}_{(\Upsilon(P)}\mathbf{r} \in P), \quad (63)$$

with an arbitrary CS  $\Upsilon'$ .



# Vector Fields on Planes

- The complex amplitudes  $V(\rho)$  are functions of the type

$$V(\rho) : D(V) \rightarrow \mathbb{C}, \quad \rho \mapsto V(\rho), \quad (64)$$

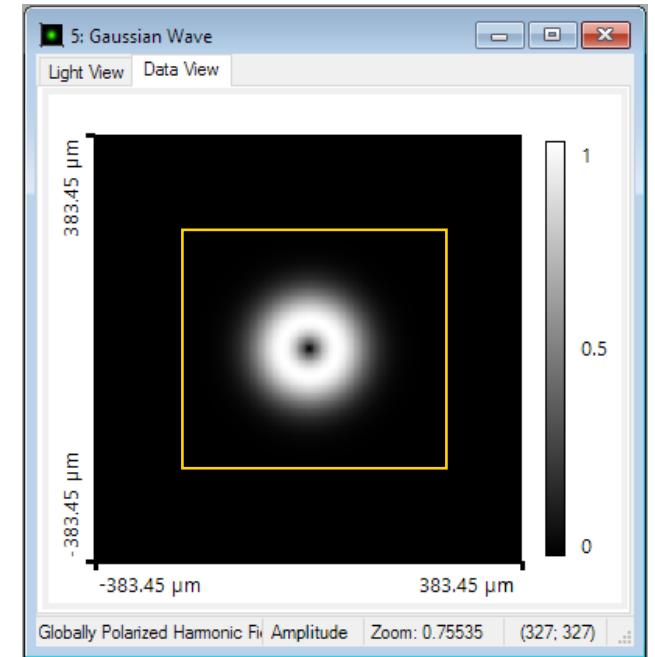
with the domain  $D(V) \subseteq \mathbb{R}^2$ .

- For numerical reasons the domain of the complex amplitudes is typically reduced to a rectangular subdomain

$$X^2(V) := \{\rho \in D(V) \mid -\Delta\rho/2 \leq \rho \leq \Delta\rho/2 \wedge \Delta\rho = (\Delta x, \Delta y)\} \quad (65)$$

with the size  $\Delta\rho$  is defined via

$$\frac{\int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} |V(x, y)|^2 dx dy}{\int \int_{D(V)} |V(x, y)|^2 dx dy} \leq \alpha(\Delta x, \Delta y). \quad (66)$$



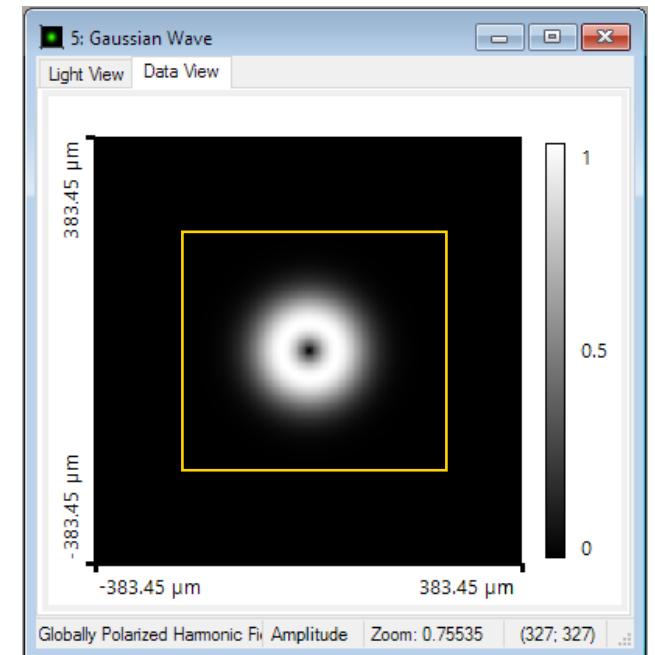
# Vector Fields on Planes

- If the selected value of  $\alpha$  is for example 0.99, it means 99% of the sum of  $|V(\rho)|^2$  originates from inside the subdomain  $X^2(V)$ .
- In practice  $V$  is often not available in  $D(V)$  and thus Eq. (66) is replaced by

$$\frac{\int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} |V(x, y)|^2 dx dy}{\int_{-\Delta x}^{\Delta x} \int_{-\Delta y}^{\Delta y} |V(x, y)|^2 dx dy} \leq \alpha(\Delta x, \Delta y), \quad (67)$$

which enables an iterative specification of the subdomain by a suitable subdomain selection algorithm.

- The concept of subdomains can be extended to any domain, e.g.  $X^3 \in \mathbb{R}^3$ . In practice we refer to e.g.  $X^2$  also as domains, since the field is not known outside of them in numerical calculations.

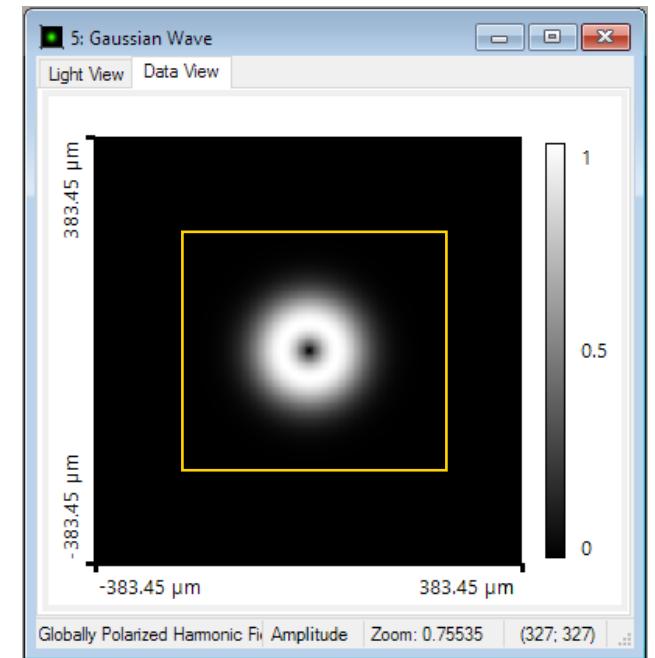


# Vector Fields on Planes

- For vector fields like  $E(\rho)$  we have three components and thus three subdomains. Then we use

$$X^2(E) := \{X^2(E_x), X^2(E_y), X^2(E_z)\} \quad (68)$$

to refer to the subdomains of all components. The extent of the subdomains should be evaluated per component.



### **3. Fourier transform**

Basics and pointwise Fourier transform

# Fourier Transform

## 3 Fourier transform and fields in k-domain

### 3.1 Fourier transform time-frequency domain

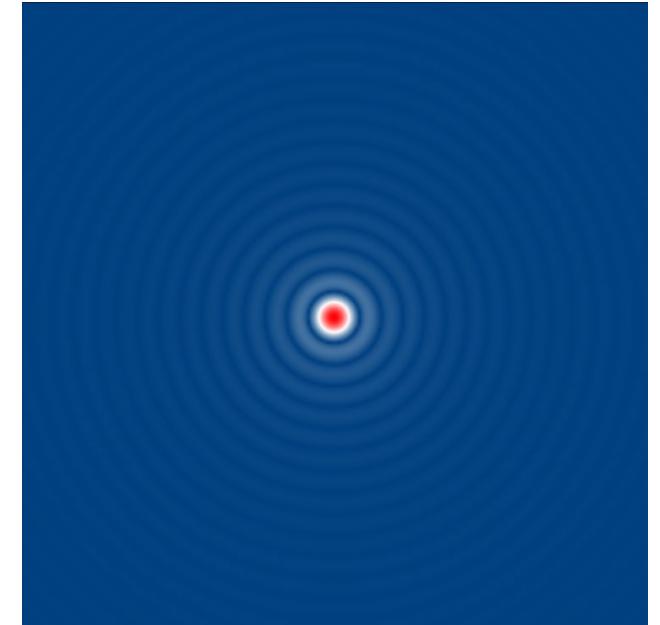
- For square integrable functions  $\bar{V}(t)$  the Fourier transform operator  $\mathcal{F}_\omega$  is defined by

$$\mathcal{F}_\omega : \begin{cases} \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}) \rightarrow \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}) \\ \bar{V} \mapsto \left( \omega \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{V}(t) \exp(i\omega t) dt \right) \end{cases} , \quad (69)$$

and in compact form we write

$$V(\omega) = \mathcal{F}_\omega(t \mapsto \bar{V}(t))(\omega) = (\mathcal{F}_\omega \bar{V})(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{V}(t) \exp(i\omega t) dt . \quad (70)$$

- It should be emphasized, that the notation  $\mathcal{F}_\omega \bar{V}(t) = V(\omega)$ , though often used in literature, is mathematically not precisely correct,



# Fourier Transform

since the Fourier operator acts not on the function  $\bar{V}$  at some time  $t$  but on the function itself.

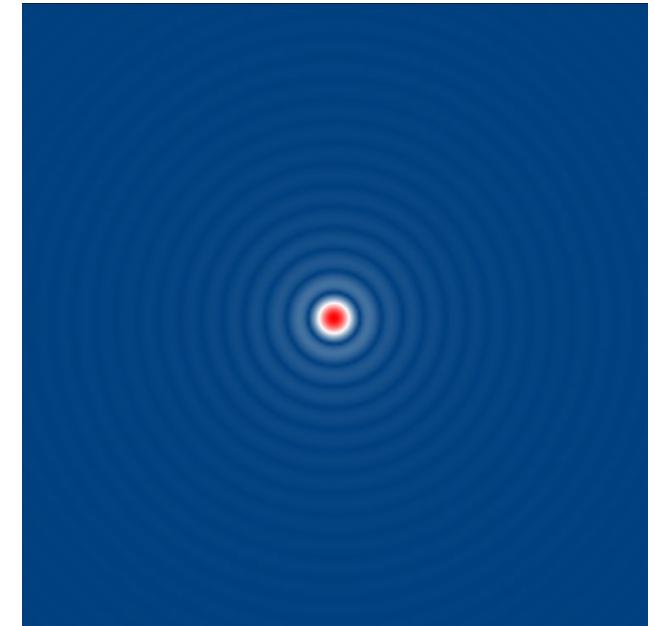
- The inverse Fourier transform reads

$$\bar{V}(t) = \mathcal{F}_\omega^{-1}(\omega \mapsto V(\omega))(t) = (\mathcal{F}_\omega^{-1}V)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(\omega) \exp(-i\omega t) d\omega. \quad (71)$$

- The operations on the variables  $\omega$  and  $t$  are defined in any location  $r$ , which are skipped in the equations before.
- All functions with the bar and without the bar, e.g.  $\bar{f}(t)$  and  $f(\omega)$ , are related by  $\mathcal{F}_\omega$  according to Eqs. (70) and (71) per definition.

## 3.2 Fourier transform x-k-domain

- Let us consider the complex amplitude in one plane according to Eq. (62) and Eq. (64).



# Fourier Transform

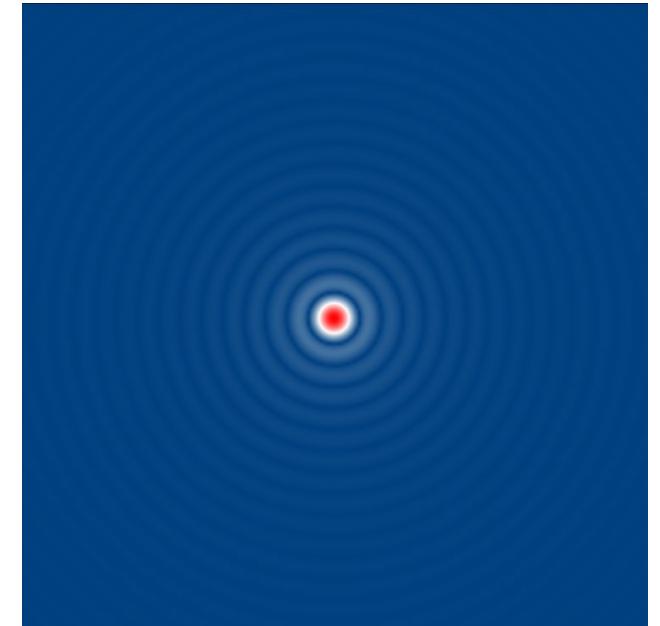
- The Fourier transform operator  $\mathcal{F}_k$  is defined by

$$\mathcal{F}_k : \begin{cases} \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}^2) \rightarrow \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}^2) \\ V \mapsto (\boldsymbol{\kappa} = (k_x, k_y) \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\boldsymbol{\rho}) \exp(-i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) dx dy) \end{cases} \quad (72)$$

and accordingly for the inverse Fourier transform.

- In compact form we write

$$\begin{aligned} \tilde{V}(\boldsymbol{\kappa}) &= \mathcal{F}_k(\boldsymbol{\rho} \mapsto V(\boldsymbol{\rho}))(\boldsymbol{\kappa}) = (\mathcal{F}_k V)(\boldsymbol{\kappa}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\boldsymbol{\rho}) \exp(-i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) dx dy \end{aligned} \quad (73)$$



# Fourier Transform

and for the inverse Fourier transform

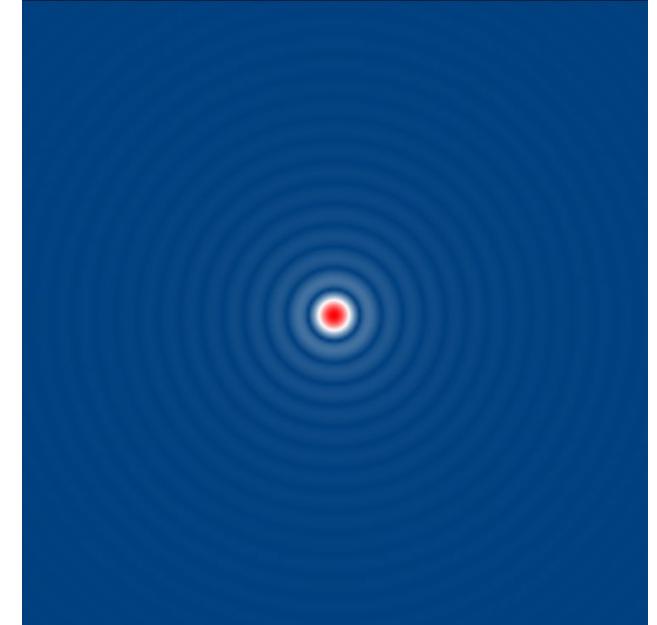
$$\begin{aligned} V(\rho) &= \mathcal{F}_k^{-1}(\kappa \mapsto \tilde{V}(\kappa))(\rho) = (\mathcal{F}_k^{-1}\tilde{V})(\rho) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{V}(\kappa) \exp(i\kappa \cdot \rho) dk_x dk_y. \end{aligned} \quad (74)$$

The  $\omega$ -dependency of the field components is skipped in the equations.

- All functions with the tilde and without the tilde, e.g.  $\tilde{f}(\kappa)$  and  $f(\rho)$ , are related by  $\mathcal{F}_k$  according to Eqs. (73) and (74) per definition.
- Analogously to Eq. (15) and Eq. (16) we define for the fields in the  $k$ -domain (with reference to plane  $P$  in  $x$ -domain)

$$\tilde{A}(\kappa; E) : \mathbb{R}^2 \rightarrow \mathbb{C}^3, \kappa \mapsto \tilde{E}(\kappa) \exp(-i\tilde{\phi}(\kappa)) \quad (75)$$

$$\tilde{A}(\kappa; H) : \mathbb{R}^2 \rightarrow \mathbb{C}^3, \kappa \mapsto \tilde{H}(\kappa) \exp(-i\tilde{\phi}(\kappa)) \quad (76)$$



## Wavefront Phase

effects in optical modeling. Thus, we define

$$U(r, \omega; E) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (r, \omega) \mapsto \mathbf{E}(r, \omega) \exp(-i\psi(r, \omega)) \quad (15)$$

and

$$U(r, \omega; H) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}^3, (r, \omega) \mapsto \mathbf{H}(r, \omega) \exp(-i\psi(r, \omega)) \quad (16)$$

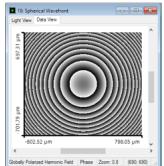
with the wavefront phase

$$\psi(r, \omega) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}, (r, \omega) \mapsto \psi(r, \omega). \quad (17)$$

- We refer to  $\exp(i\psi(r, \omega))$  as the wavefront phase factor and to  $U(r, \omega; E)$  and  $U(r, \omega; H)$  as the electric and magnetic  $U$ -fields.

- The  $U$ -fields do not include the wavefront phases anymore. The fields can then be written by

$$E(r, \omega) = U(r, \omega; E) \exp(i\psi(r, \omega)) \quad (18)$$



# Fourier Transform

with the k-domain wavefront phase

$$\tilde{\phi}(\kappa) : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \kappa \mapsto \tilde{\phi}(\kappa). \quad (77)$$

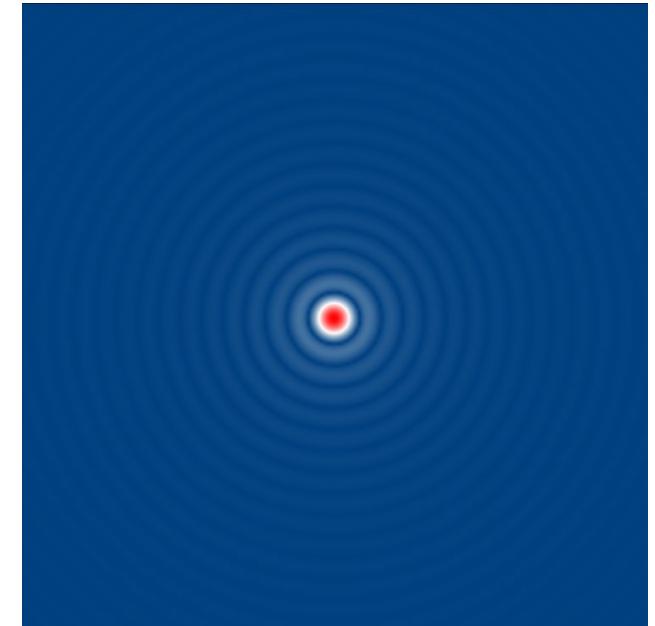
- We refer to  $\exp(i\tilde{\phi}(\kappa))$  as the k-domain wavefront phase factor and to  $\tilde{A}(\kappa; E)$  and  $\tilde{A}(\kappa; H)$  as the electric and magnetic  $\tilde{A}$ -fields respectively.
- The  $\tilde{A}$ -fields do not include the k-domain wavefront phases anymore. The fields in k-domain can then be written by

$$\tilde{E}(\kappa) = \tilde{A}(\kappa; E) \exp(i\tilde{\phi}(\kappa)), \quad (78)$$

$$\tilde{H}(\kappa) = \tilde{A}(\kappa; H) \exp(i\tilde{\phi}(\kappa)). \quad (79)$$

- For the transversal electric field in k-domain the extraction of the  $\tilde{A}$ -field leads to

$$\tilde{E}_\perp(\kappa) = \tilde{A}_\perp(\kappa) \exp(i\tilde{\phi}(\kappa)). \quad (80)$$



# Fourier Transform

- For a general field component we write

$$\tilde{V}(\boldsymbol{\kappa}) = \tilde{A}(\boldsymbol{\kappa}) \exp\left(i\tilde{\phi}(\boldsymbol{\kappa})\right). \quad (81)$$

- An overview of the notation of fields is given in Table 1.

- Analogously to Eq. (65) we may define a subdomain for each component function

$$\tilde{V}(\boldsymbol{\kappa}) : D(\tilde{V}) \subseteq \mathbb{R}^2 \rightarrow \mathbb{C}, \quad \boldsymbol{\kappa} = (k_x, k_y) \mapsto \tilde{V}(\boldsymbol{\kappa}) \quad (82)$$

by

$$K^2(\tilde{V}) := \left\{ \boldsymbol{\kappa} \in D(\tilde{V}) \mid -\Delta\boldsymbol{\kappa}/2 \leq \boldsymbol{\kappa} \leq \Delta\boldsymbol{\kappa}/2 \wedge \Delta\boldsymbol{\kappa} = (\Delta k_x, \Delta k_y) \right\}, \quad (83)$$

Notation Electromagnetic Fields	
Name	Notation
electric/magnetic field (time domain) (5)	$E, H$
electric/magnetic field (frequency domain) (9)	$\tilde{E}, \tilde{H}$
arbitrary field component (13)	$V, V$
transversal electromagnetic field (14)	$V_\perp$
transversal electric field (12)	$E_\perp$
wavefront phase (17)	$\psi$
wavefront phase factor	$\exp(iw)$
electric $U$ -field (15)	$U(\dots; E)$
electric $\tilde{A}$ -field (75)	$\tilde{A}(\dots; E)$
magnetic $U$ -field (16)	$U(\dots; H)$
magnetic $\tilde{A}$ -field (76)	$\tilde{A}(\dots; H)$
transversal electric $U$ -field (21)	$U_\perp$
transversal electric $\tilde{A}$ -field (80)	$\tilde{A}_\perp$

Table 1: Notation and terminology: electromagnetic fields

44

Vector Fields on Planes	
• The complex amplitudes $V(\rho)$ are functions of the type	
$V(\rho) : D(V) \rightarrow \mathbb{C}, \quad \rho \mapsto V(\rho),$	(64)
with the domain $D(V) \subseteq \mathbb{R}^2$ .	
• For numerical reasons the domain of the complex amplitudes is typically reduced to a rectangular subdomain	
$X^2(V) := \{\rho \in D(V) \mid -\Delta\rho/2 \leq \rho \leq \Delta\rho/2 \wedge \Delta\rho = (\Delta x, \Delta y)\}$	(65)
with the size $\Delta\rho$ is defined via	
$\frac{\int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2}  V(x, y) ^2 dx dy}{\int_{D(V)}^{}  V(x, y) ^2 dx dy} \leq \alpha(\Delta x, \Delta y).$	(66)

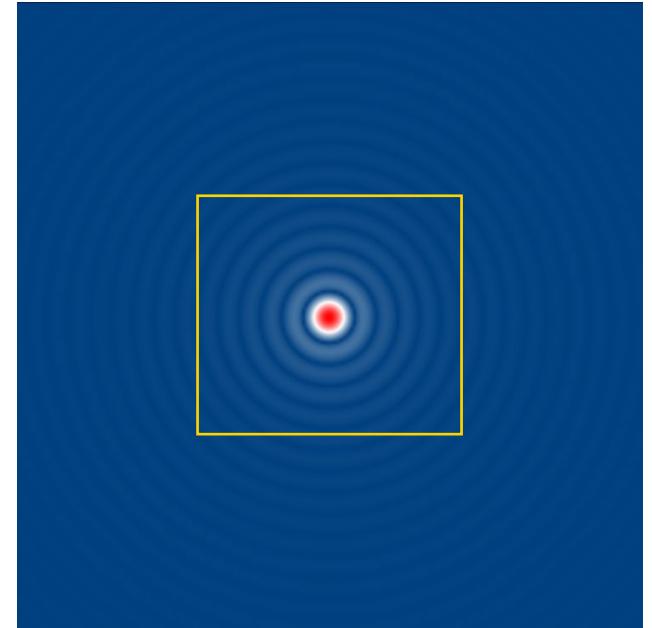
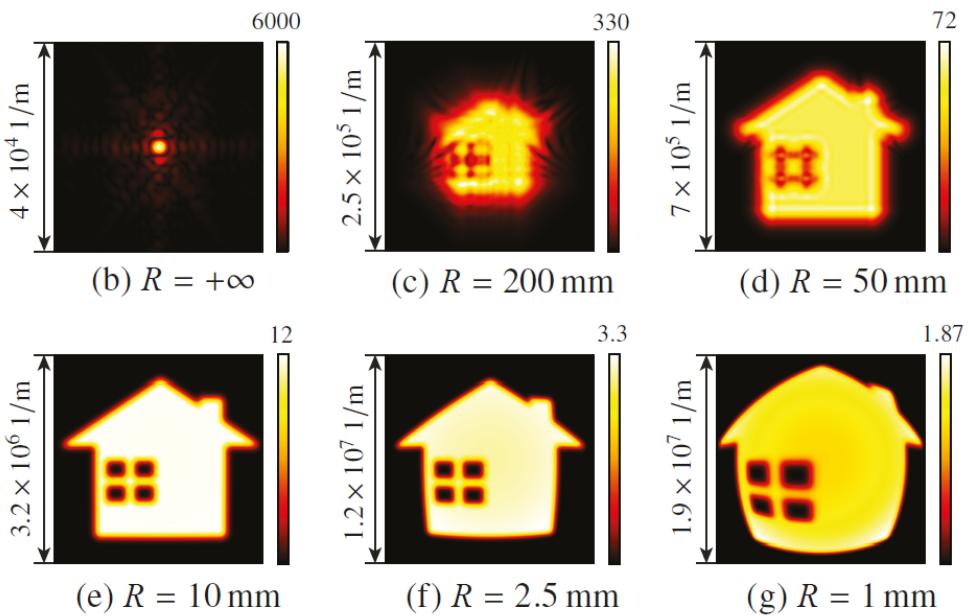
Screenshot of a software interface showing a 2D plot of a complex amplitude  $V(x, y)$  with a central peak and a square boundary. The plot is labeled "Spatiotemporal View".

64

# Fourier Transform

with

$$\frac{\int_{-\Delta k_x/2}^{\Delta k_x/2} \int_{-\Delta k_y/2}^{\Delta k_y/2} |\tilde{V}(k_x, k_y)|^2 dk_x dk_y}{\int_{-\Delta k_x}^{\Delta k_x} \int_{-\Delta k_y}^{\Delta k_y} |\tilde{V}(k_x, k_y)|^2 dk_x dk_y} \leq \alpha(\Delta k_x, \Delta k_y). \quad (84)$$



# Pointwise Fourier Transform

## 3.3 Pointwise Fourier transform x-k-domain

### 3.3.1 Pointwise Fourier transform: formal definition

- Equation (72) defines an integral operator. We will see, that in case of strong (in sense of large local gradients) wavefront phases, the integral operator can be replaced by a pointwise operation, which is formally defined by

$$\mathcal{F}_k^{1:1} : \begin{cases} \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}^2) \rightarrow \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}^2) \\ V \mapsto (\kappa \mapsto (f(\rho)V(\rho))[\rho \leftarrow \rho(\kappa)]) \end{cases}, \quad (85)$$

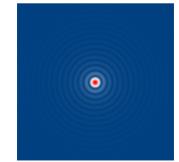
with  $f(\rho) : \mathbb{R}^2 \rightarrow \mathbb{C}$ , the mapping

$$\rho(\kappa) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \kappa \mapsto \rho(\kappa), \quad (86)$$

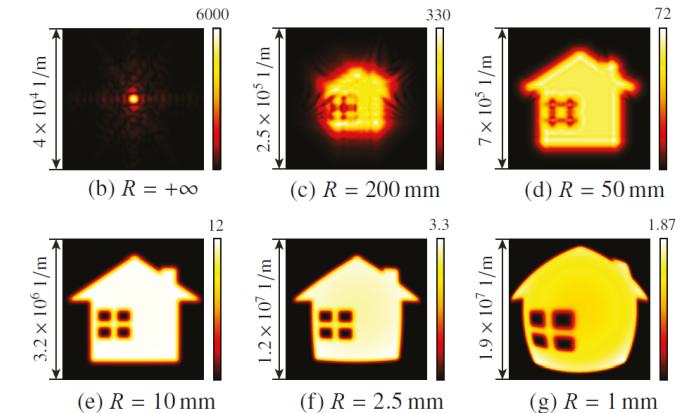
and the substitution  $[\rho \leftarrow \rho(\kappa)]$ .

**Fourier Transform**

- The Fourier transform operator  $\mathcal{F}_k$  is defined by
$$\mathcal{F}_k : \begin{cases} \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}^2) \rightarrow \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}^2) \\ V \mapsto (\kappa = (k_x, k_y) \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\rho) \exp(-i\kappa \cdot \rho) dx dy) \end{cases} \quad (72)$$
and accordingly for the inverse Fourier transform.



70



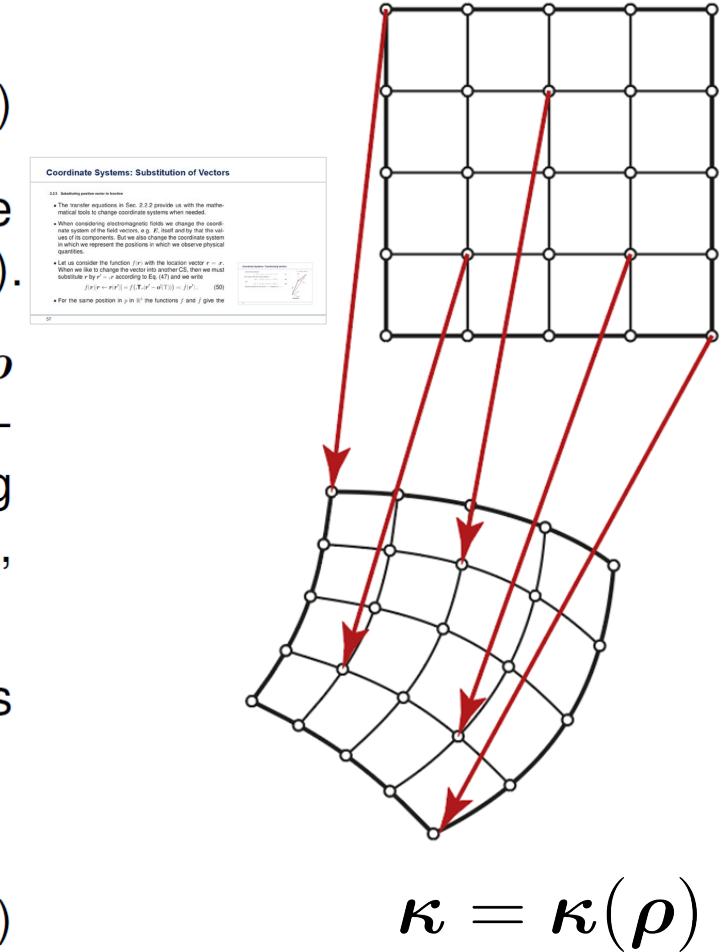
# Pointwise Fourier Transform

- In a compact form we write

$$\tilde{V}(\kappa) = (\mathcal{F}_k^{1:1} V)(\kappa) := (f(\rho)V(\rho))[\rho \leftarrow \rho(\kappa)] =: \check{f}(\kappa)\check{V}(\kappa). \quad (87)$$

- Here we use the notation for substitution and the change of the function symbol, e.g. from  $f$  to  $\check{f}$ , which we also applied in Eq. (50).
- In words Eq. (87) means: The field values  $V$  at some position  $\rho$  are mapped to a specific position  $\kappa(\rho)$  and weighted by  $f(\rho)$  before the mapping or with  $\check{f}(\kappa) : \mathbb{R}^2 \rightarrow \mathbb{C}$  afterwards. The resulting value at  $\kappa(\rho)$  is the value of the Fourier transformed function at  $\kappa$ , that is  $\tilde{V}(\kappa)$ .
- The inverse pointwise Fourier transform in compact notation is given by

$$V(\rho) = ((\mathcal{F}_k^{1:1})^{-1}\tilde{V})(\rho) := (\tilde{F}(\kappa)\tilde{V}(\kappa))[\kappa \leftarrow \kappa(\rho)] =: \check{\tilde{F}}(\rho)\check{V}(\rho), \quad (88)$$



# Pointwise Fourier Transform

with  $\tilde{F}(\kappa), \tilde{F}(\rho) : \mathbb{R}^2 \rightarrow \mathbb{C}$ . Equations (87) and (88) provides formal definitions of the pointwise Fourier transform.

- The coordinate mapping and the complex weight functions  $f$  and  $\tilde{F}$  are not specified yet. That is to be done next.

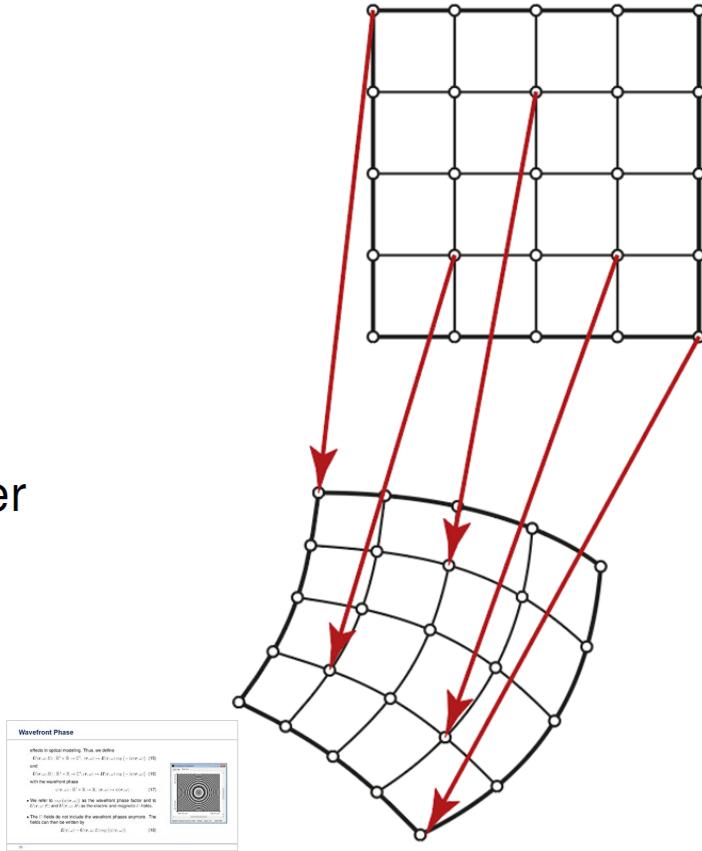
## 3.3.2 Pointwise Fourier transform: derivation

- In this section we like to derive expressions for the pointwise Fourier transform (PFT) of a function

$$V(\rho) = U(\rho) \exp(i\psi(\rho)), \quad (89)$$

where we use the representation of a complex amplitude with extracted wavefront phase as discussed in Sec. 2.1.

- Figure 1 shows the Fourier transform results of a complex amplitude with a special house-shaped function  $U(\rho) = |V(\rho)|$  and

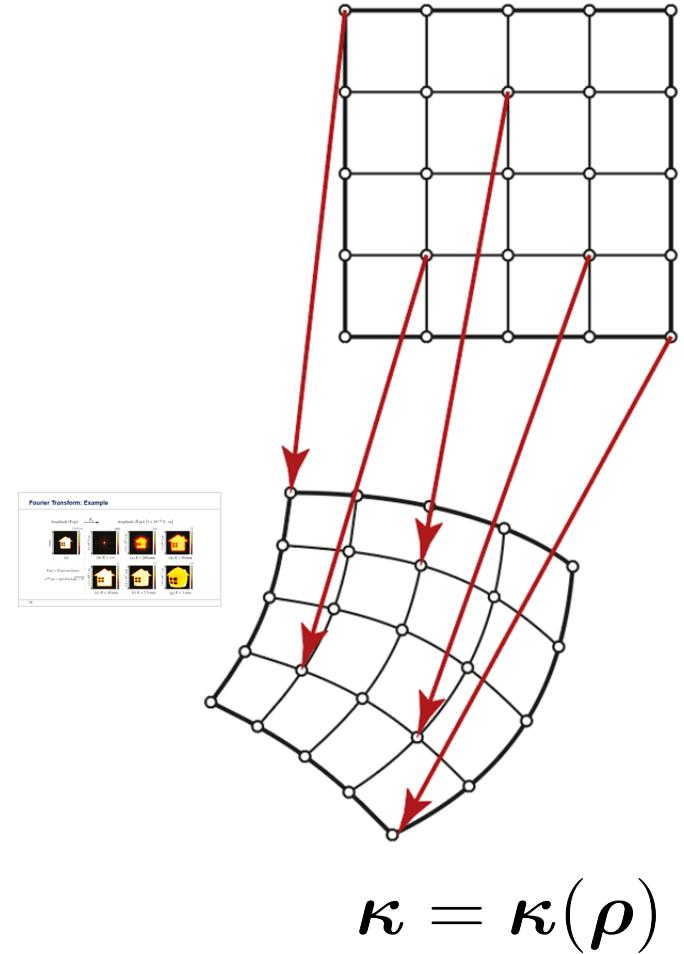


# Pointwise Fourier Transform

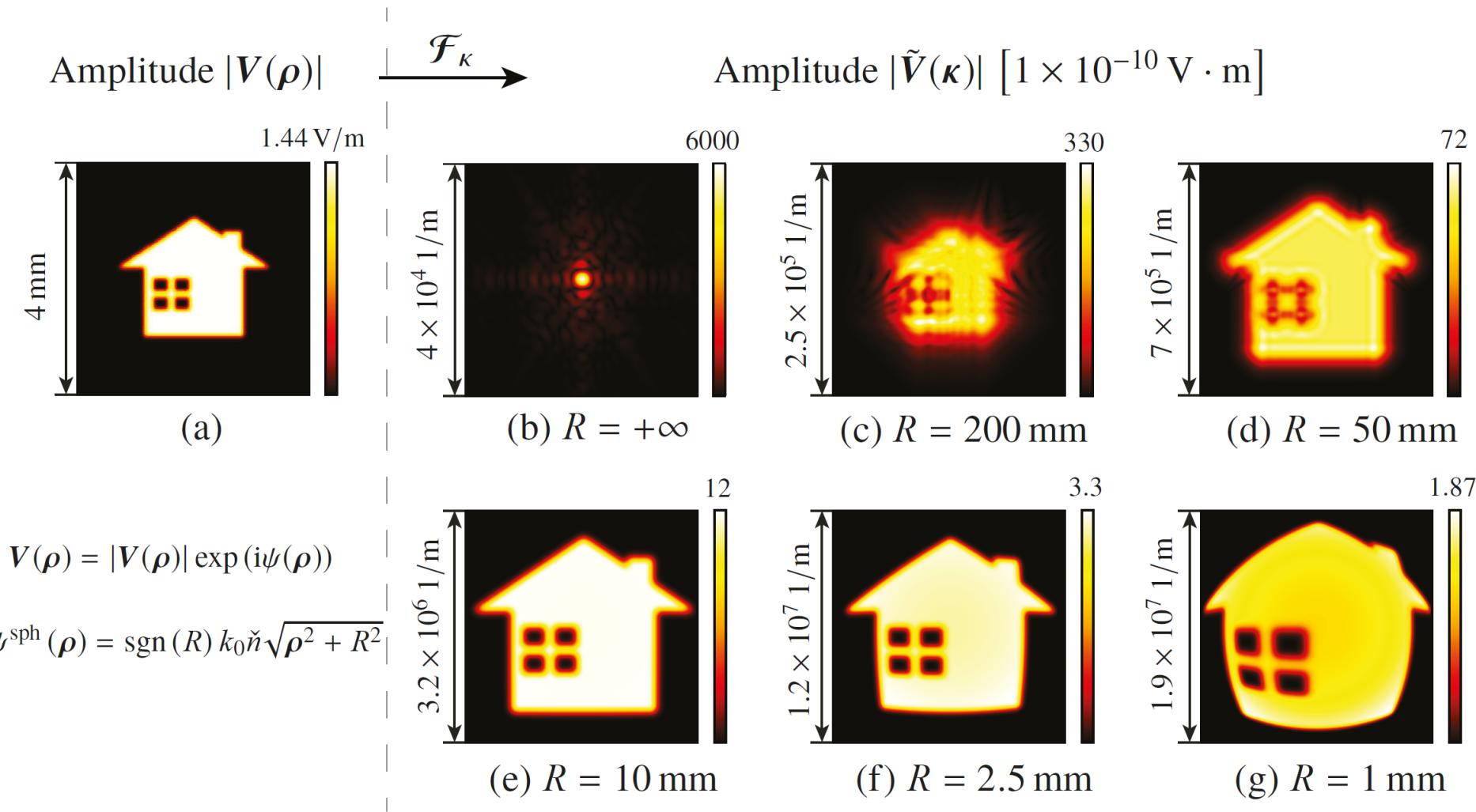
a spherical wavefront phase  $\psi^{\text{sph}}(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r}$ , which becomes in a plane at distance  $z = R$

$$\psi^{\text{sph}}(\rho) := \frac{Rk_0n}{|R|} \sqrt{x^2 + y^2 + R^2} = \frac{Rk_0n}{|R|} r(\rho). \quad (90)$$

- The term  $R/|R|$  cares about the sign of  $R$ . A negative radius of curvature  $R$ , i.e.  $R < 0$ , indicates a convergent spherical wave and  $R > 0$  indicates a divergent wave.
- Figure 1 shows, that for small radius of curvature the Fourier transformed function  $|\tilde{V}(\kappa)|$  looks mainly like a geometrically distorted version of the house-shaped function in (a).
- A geometric distortion is mainly defined by a mapping between the coordinates in  $x$ - and  $k$ -domain. That means for small enough  $R$  we obtain a situation as we formulated in Eq. (87) mathematically.
- What can govern the mapping between the coordinates, that means



# Fourier Transform: Example

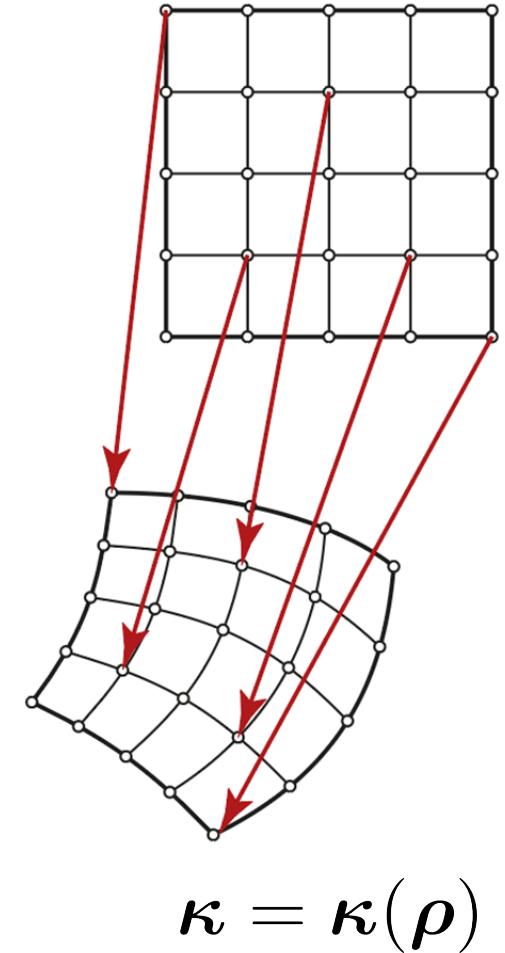


# Pointwise Fourier Transform

a spherical wavefront phase  $\psi^{\text{sph}}(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r}$ , which becomes in a plane at distance  $z = R$

$$\psi^{\text{sph}}(\rho) := \frac{Rk_0 n}{|R|} \sqrt{x^2 + y^2 + R^2} = \frac{Rk_0 n}{|R|} r(\rho). \quad (90)$$

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- What can govern the mapping between the coordinates, that means



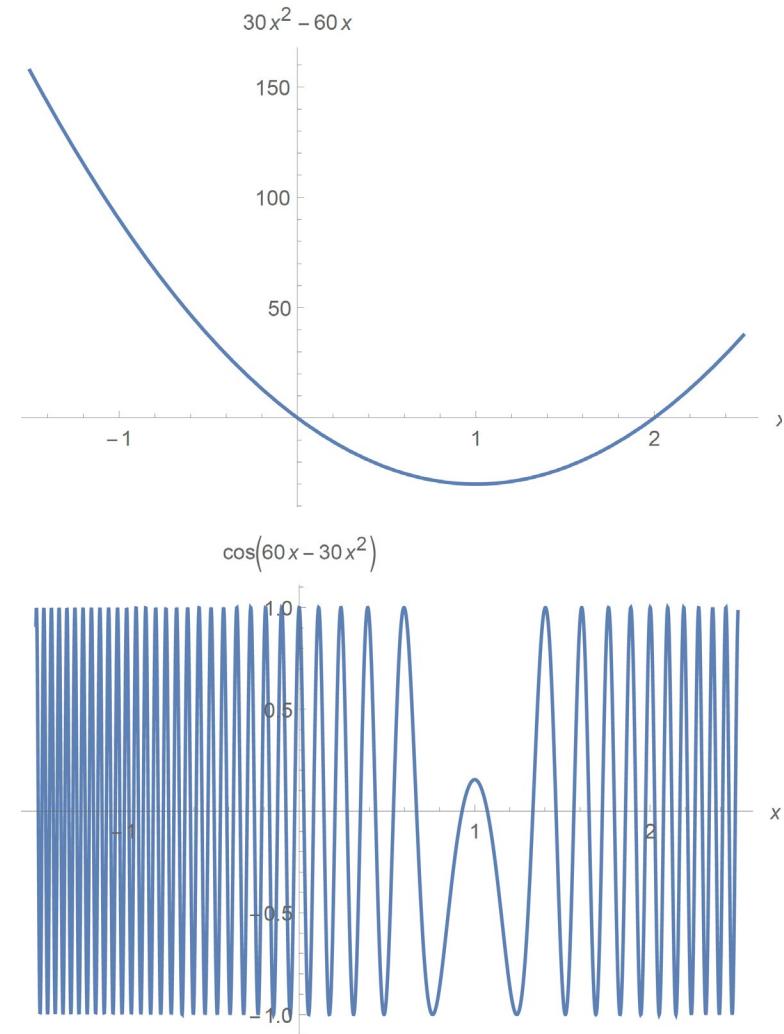
# Pointwise Fourier Transform

$\rho \mapsto \kappa(\rho)$ ?

- To get more insight we write the Fourier transform in Eq. (72) explicitly with the notation of the wavefront phase in Eq. (22) and obtain

$$\tilde{V}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(\rho) \exp(i(\psi(\rho) - \kappa \cdot \rho)) dk_x dk_y. \quad (91)$$

- A 1D example of the phase in the integration kernel is shown in figure. According to the Euler formula we have  $\exp(i(\psi(\rho) - \kappa \cdot \rho)) = \cos((\psi(\rho) - \kappa \cdot \rho)) + i \sin((\psi(\rho) - \kappa \cdot \rho))$ .
- From the figure we see, that the contributions to the integral are to be expected around the critical value  $x' = 1$ . All other regions contribute approximately nothing, since positive and negative parts contribute equally as long as  $U(\rho)$  varies on a lateral distance which is sufficiently larger than the local period of the cosine func-



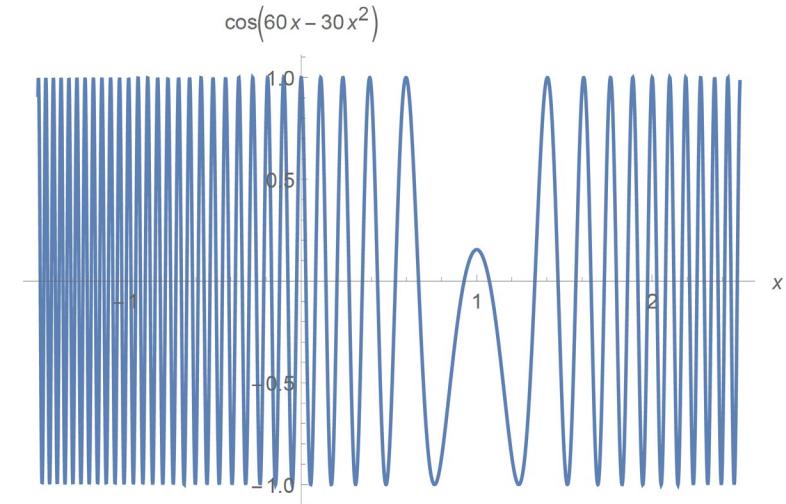
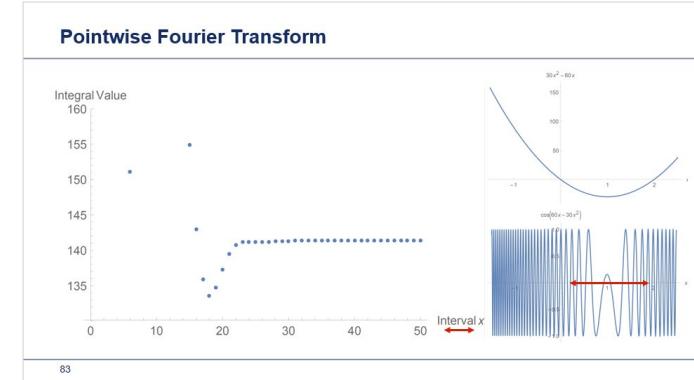
# Pointwise Fourier Transform

tion.

- This behavior of the integral is shown at an example in the figure. It illustrates, that mainly the critical value itself contributes to the integral.
- What does this critical value  $\rho'$  characterize? From the figure we learn, that the condition is  $(\nabla(\psi(\rho) - \kappa \cdot \rho))|_{\rho'} = 0$ , that means the gradient of the phase at the position  $\rho$  in the Fourier kernel should be zero, which is possible by selecting a  $\kappa$  value accordingly.
- From that we conclude the mapping condition[2, 3]

$$\kappa(\rho) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \rho \mapsto \kappa(\rho) = \nabla \psi(\rho). \quad (92)$$

- It provides the connection between the coordinates in the x- and k-domain in case of the pointwise Fourier transform. In practice we know the mapping in both directions.



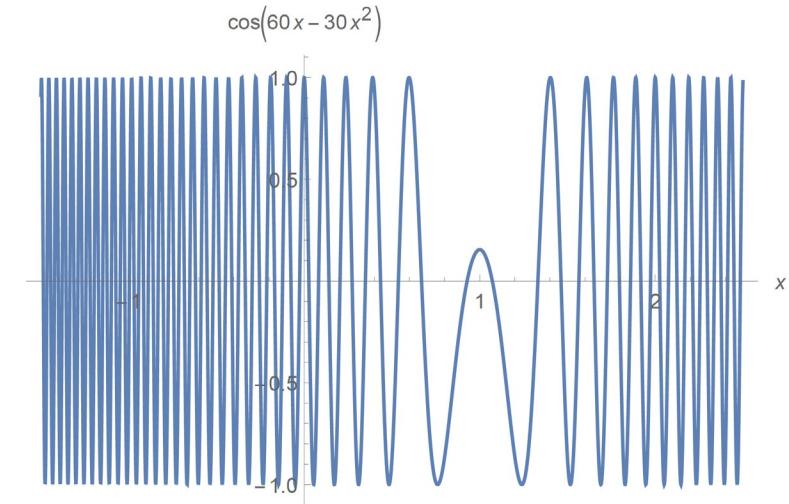
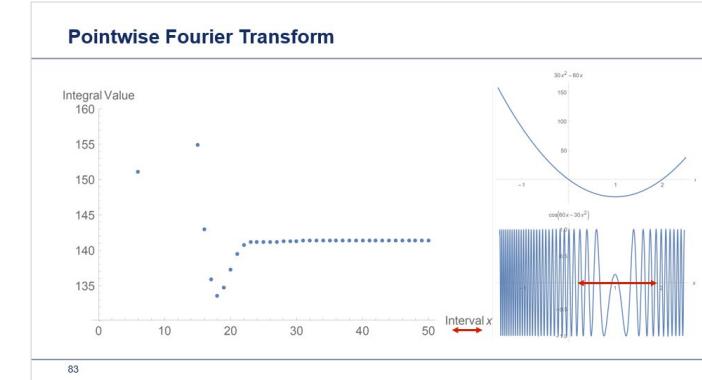
# Pointwise Fourier Transform

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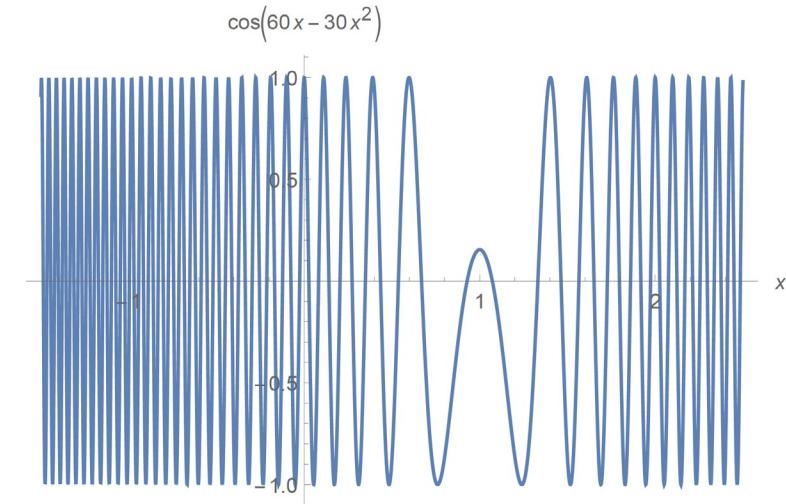
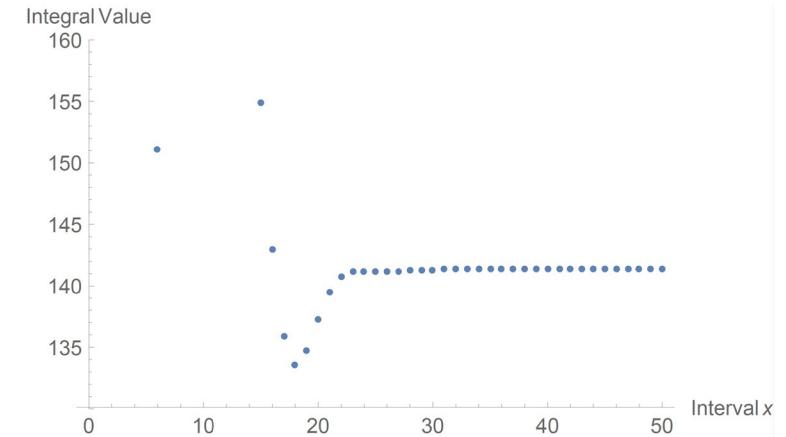
# Pointwise Fourier Transform

- We conclude, that the integral provides a value for  $\tilde{V}(\kappa)$  mainly for the  $\rho$  from Eq. (92) and we obtain as an intermediate result for the PFT at  $\kappa$  the integrand of Eq. (91) itself at the position  $\rho(\kappa)$ , that means

$$\begin{aligned} & \left( U(\rho) \exp(i(\psi(\rho) - \kappa(\rho) \cdot \rho)) \right) [\rho \leftarrow \rho(\kappa)] \\ &= \left( V(\rho) \exp(-i\nabla\psi(\rho) \cdot \rho) \right) [\rho \leftarrow \rho(\kappa)] \end{aligned} \quad (93)$$

- The weight function from Eq. (87) is in Eq. (93) a linear phase factor only. For the full determination of the complex weight function we need additional information on how the Fourier transform changes the amplitude of a function.
- This information comes in form of Parseval's theorem

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(\rho)|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{V}(\kappa)|^2 dk_x dk_y. \quad (94)$$



# Pointwise Fourier Transform

- If we assume a pointwise connection according to Eq. (92), then we may substitute  $\rho$  in the integrand on the left side by  $\rho(\kappa)$  and obtain

$$(|V(\rho)|^2 dx dy)[\rho \leftarrow \rho(\kappa)] = (|V(\rho)|^2)[\rho \leftarrow \rho(\kappa)] |\mathbf{J}_\rho(\kappa)| dk_x dk_y \quad (95)$$

with the determinant of the Jacobian matrix

$$\mathbf{J}_\rho(\kappa) = \begin{pmatrix} \partial_{k_x} x(\kappa) & \partial_{k_x} y(\kappa) \\ \partial_{k_y} x(\kappa) & \partial_{k_y} y(\kappa) \end{pmatrix} \quad (96)$$

and the partial derivative notation  $\partial f / \partial x = \partial_x f$ .

- Here we used the identity

$$dx dy = |\mathbf{J}_\rho(\kappa)| dk_x dk_y. \quad (97)$$

**Pointwise Fourier Transform**

This behavior of the integral is shown at an example in the figure. It illustrates, that mainly the critical value itself contributes to the integral.

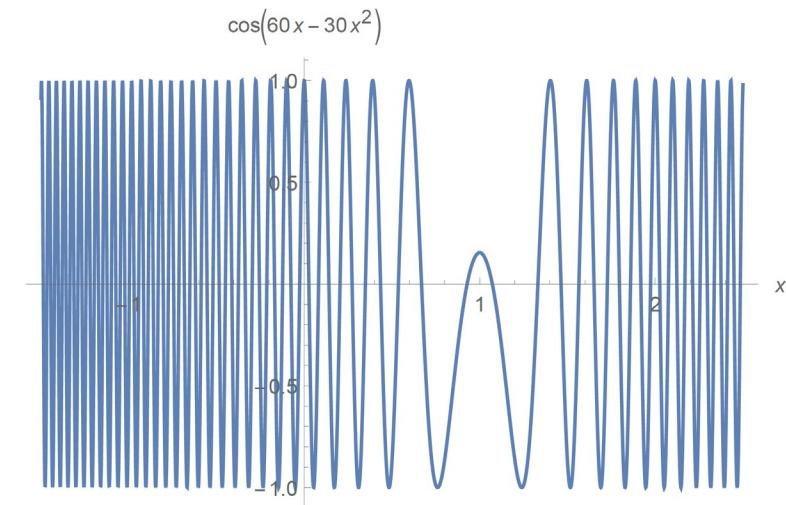
What does this critical value  $\rho^*$  characterize? From the figure we learn, that the condition is  $(\nabla(\psi(\rho) - \kappa \cdot \rho))_{|\rho^*} = 0$ , that means the gradient of the phase at the position  $\rho$  in the Fourier kernel should be zero, which is possible by selecting a  $\kappa$  value accordingly.

From that we conclude the mapping condition [2, 3]

$$\kappa(\rho) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \rho \mapsto \kappa(\rho) = \nabla \psi(\rho). \quad (92)$$

It provides the connection between the coordinates in the x- and k-domain in case of the pointwise Fourier transform. In practice we know the mapping in both directions.

82



# Pointwise Fourier Transform

- Plugging Eq. (95) into the left side of Eq. (94) reveals

$$|\tilde{V}(\kappa)| = \left( |V(\rho)|[\rho \leftarrow \rho(\kappa)] \right) \sqrt{|\mathbf{J}_\rho(\kappa)|}. \quad (98)$$

- Combining this result with Eq. (93) leads to the pointwise Fourier transform

$$\tilde{V}(\kappa) = \left( (V(\rho)\sigma(\rho) \exp(-i\nabla\psi(\rho) \cdot \rho))[\rho \leftarrow \rho(\kappa)] \right) \sqrt{|\mathbf{J}_\rho(\kappa)|} = \check{V}(\kappa) \check{f}(\kappa) \quad (99)$$

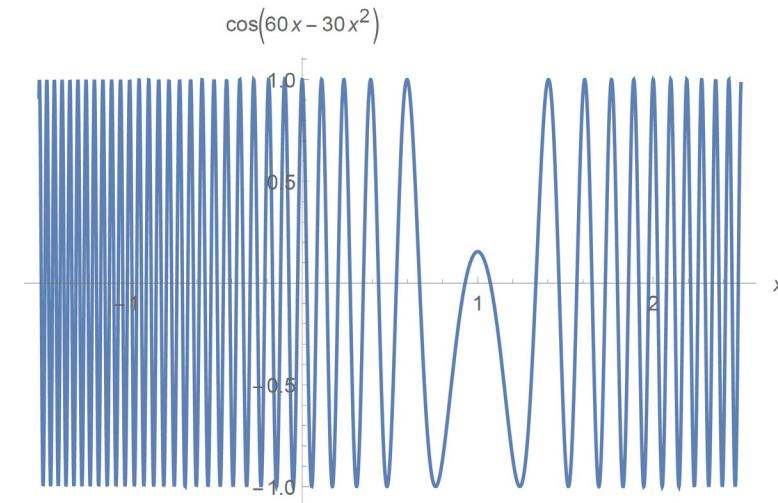
with the notation of Eq. (87).

- The factor  $\sigma(\rho)$  takes into account, that by Parseval's theorem we do not get information about an additional possible phase factor  $\sigma(\rho) = \exp(i\alpha(\rho))$ .
- However, that can be obtained by a more elaborate evaluation of the integral around the critical values. That approach is known as

**Pointwise Fourier Transform**

- We conclude, that the integral provides a value for  $\check{V}(\kappa)$  mainly for the  $\rho$  from Eq. (92) and we obtain as an intermediate result for the PFT at  $\kappa$  the integrand of Eq. (91) itself at the position  $\rho(\kappa)$ , that means
 
$$\begin{aligned} & (U(\rho) \exp(i(\psi(\rho) - \kappa(\rho) \cdot \rho))) [\rho \leftarrow \rho(\kappa)] \\ &= (V(\rho) \exp(-i\nabla\psi(\rho) \cdot \rho)) [\rho \leftarrow \rho(\kappa)] \end{aligned} \quad (93)$$
- The weight function from Eq. (87) is in Eq. (93) a linear phase factor only. For the full determination of the complex weight function we need additional information on how the Fourier transform changes the amplitude of a function.
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$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(\rho)|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\check{V}(\kappa)|^2 dk_x dk_y. \quad (94)$$

85



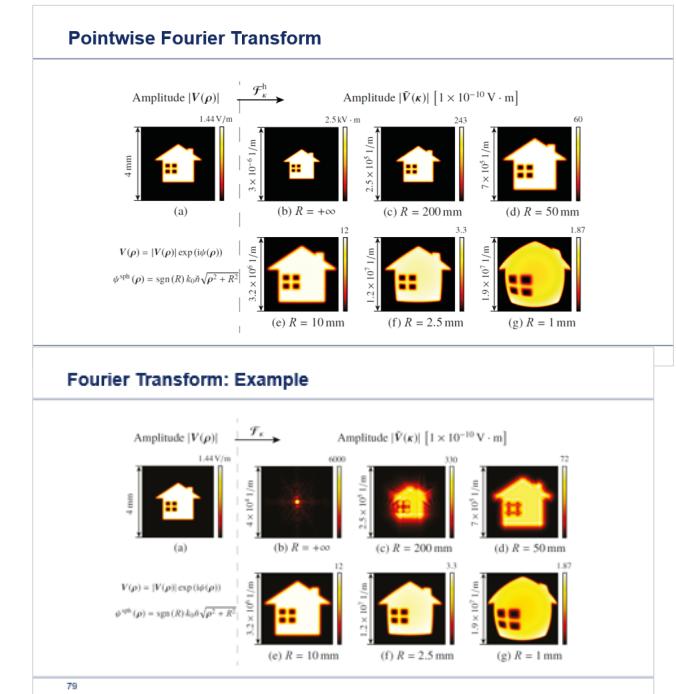
# Pointwise Fourier Transform

the stationary phase integral method [4].

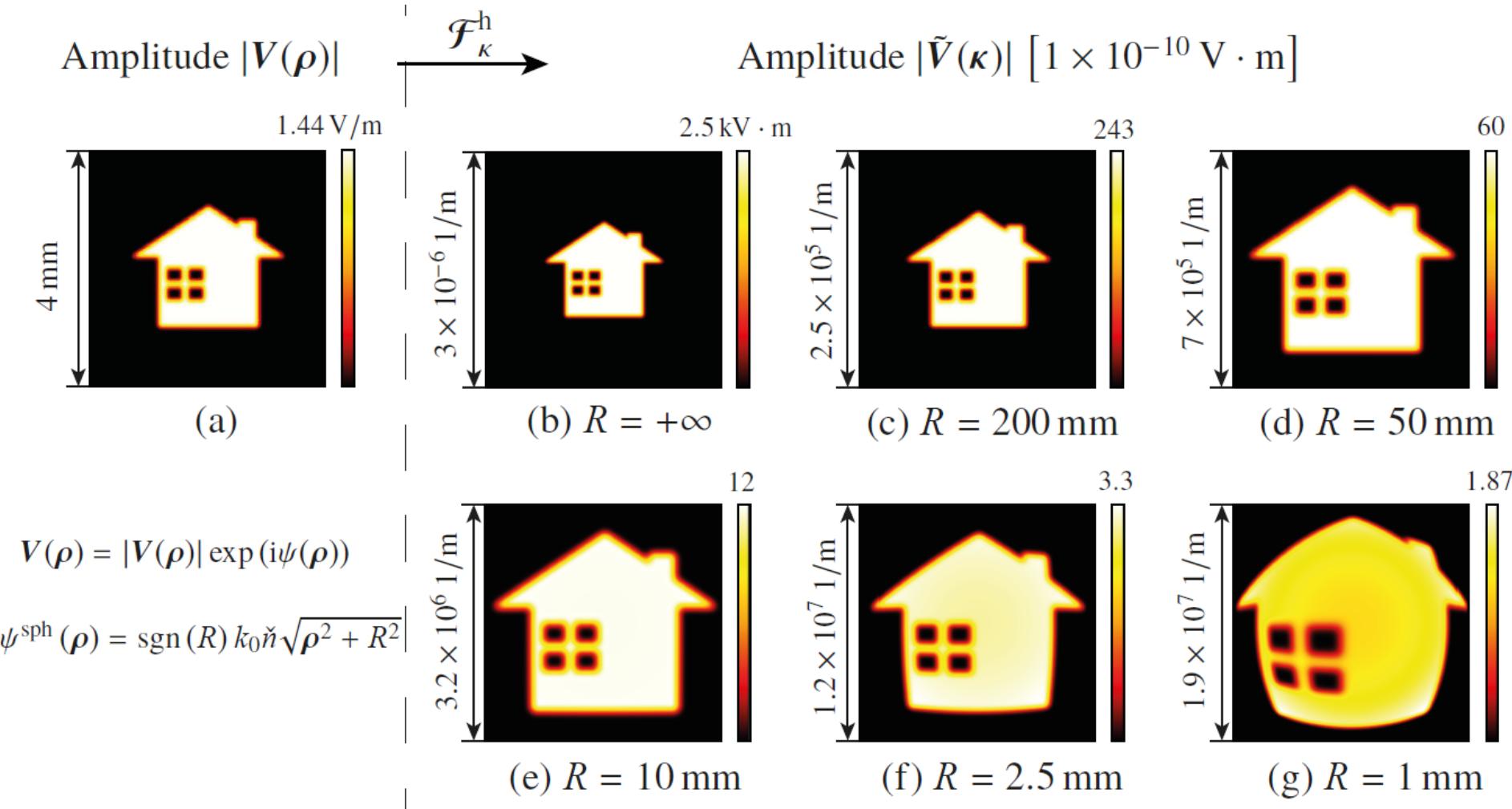
- The resulting factor  $\sigma(\rho)$  is listed in Tab. 3 and can be obtained by evaluating the derivatives of the wavefront phase at any position  $\rho$ .
- If the PFT of Eq. (99) is applied to the  $V(\rho)$  examples which are defined in Fig. 1 we obtain the results shown in Fig. 3.
- Obviously, the pointwise Fourier transform Eq. (99) provides accurate results when the local gradient of the wavefront phase is large enough, which means in the example that  $R$  is small enough.
- In order to evaluate the determinant of  $\mathbf{J}_\rho(\kappa)$  in Eq. (96) the mapping in Eq. (92) must be inverted to obtain  $\rho(\kappa)$ .
- In practice of numerical simulations that is provided in the algorithm. Nevertheless, it is interesting to reformulate Eq. (99) in order to apply the mapping Eq. (92) directly and not its inverse

References			
[1] Zengnian Wang, Oleg Bespalov, Dario, Christian Hirsch, and Peter Wabnitz, "A pointwise Fourier transform for wavefield synthesis," <i>Opt. Express</i> , vol. 18, pp. 1087–1099, 2010.			
[2] Z. Wang, "A pointwise Fourier transform for wavefield synthesis," <i>J. Opt. Soc. Am. A</i> , vol. 27, pp. 104–114, 2010.			
[3] Z. Wang, "A pointwise Fourier transform for wavefield synthesis," <i>J. Opt. Soc. Am. A</i> , vol. 27, pp. 104–114, 2010.			
[4] Z. Wang, "A pointwise Fourier transform for wavefield synthesis," <i>Optical coherence and quantum optics</i> , Springer International Publishing, 2016.			

$\partial_{xx}\psi(\rho), \partial_{yy}\psi(\rho)$	$\partial_{xy}\psi(\rho)$	$ \mathbf{J}_\kappa(\rho) $	factor $\sigma(\rho)$
same sign ( $\pm$ )	small	$> 0$	$\pm i$
one/both = 0	$\neq 0$	$< 0$	1
different sign	any	$< 0$	1
same sign	large	$< 0$	1



# Pointwise Fourier Transform



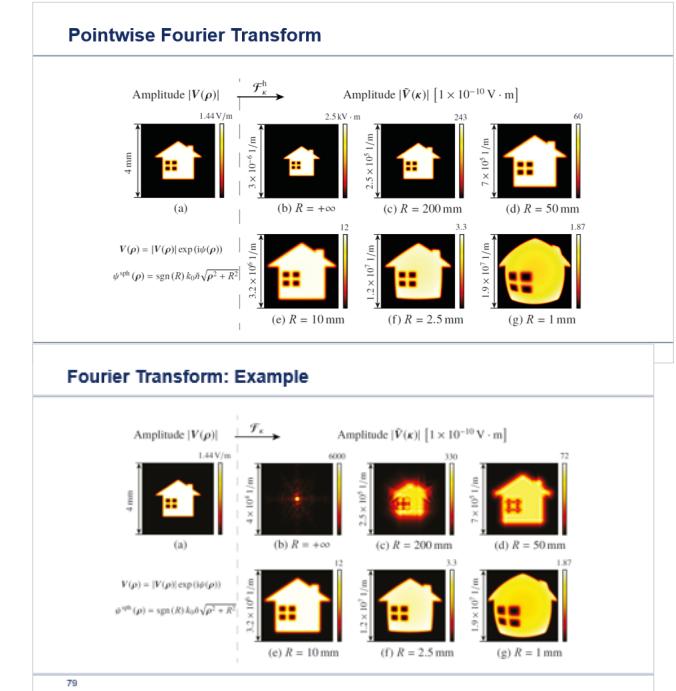
# Pointwise Fourier Transform

the stationary phase integral method [4].

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[1] Zengnian Wang, Oleg Bespalov, Dario, Christian Hirsch, and Peter W. Smith, "A pointwise Fourier transform for the computation of wavefronts in diffraction theory," <i>Opt. Express</i> , vol. 18, pp. 108–120, 2010.			
[2] Z. Wang, "A pointwise Fourier transform for the computation of wavefronts in diffraction theory," <i>J. Opt. Soc. Am. A</i> , vol. 27, pp. 104–114, 2010.			
[3] Z. Wang, "A pointwise Fourier transform for the computation of wavefronts in diffraction theory," <i>Computing in Science and Engineering</i> , vol. 12, pp. 10–16, 2010.			
[4] Z. Wang, "A pointwise Fourier transform for the computation of wavefronts in diffraction theory," <i>Computing in Science and Engineering</i> , vol. 12, pp. 10–16, 2010.			

$\partial_{xx}\psi(\rho), \partial_{yy}\psi(\rho)$	$\partial_{xy}\psi(\rho)$	$ \mathbf{J}_\kappa(\rho) $	factor $\sigma(\rho)$
same sign ( $\pm$ )	small	$> 0$	$\pm i$
one/both = 0	$\neq 0$	$< 0$	1
different sign	any	$< 0$	1
same sign	large	$< 0$	1



# Pointwise Fourier Transform

version as in Eq. (96).

- To this end, analogously to Eq. (97), we write

$$dk_x dk_y = |\mathbf{J}_\kappa(\rho)| dx dy, \quad (100)$$

with

$$\mathbf{J}_\kappa(\rho) = \begin{pmatrix} \partial_x k_x(\rho) & \partial_x k_y(\rho) \\ \partial_y k_x(\rho) & \partial_y k_y(\rho) \end{pmatrix} = \begin{pmatrix} \partial_{xx}\psi(\rho) & \partial_{xy}\psi(\rho) \\ \partial_{yx}\psi(\rho) & \partial_{yy}\psi(\rho) \end{pmatrix}, \quad (101)$$

where we used the mapping Eq. (92) on the right side of Eq. (101).

- Comparison of Eq. (97) and Eq. (100) reveals

$$|\mathbf{J}_\rho(\kappa)| = |\mathbf{J}_\kappa(\rho)|^{-1} [\rho \leftarrow \rho(\kappa)]. \quad (102)$$

## Pointwise Fourier Transform

- If we assume a pointwise connection according to Eq. (92), then we may substitute  $\rho$  in the integrand on the left side by  $\rho(\kappa)$  and obtain

$$(|V(\rho)|^2 dx dy)[\rho \leftarrow \rho(\kappa)] = (|V(\rho)|^2)[\rho \leftarrow \rho(\kappa)] |\mathbf{J}_\rho(\kappa)| dk_x dk_y \quad (95)$$

with the determinant of the Jacobian matrix

$$\mathbf{J}_\rho(\kappa) = \begin{pmatrix} \partial_{x_\rho} x(\kappa) & \partial_{x_\rho} y(\kappa) \\ \partial_{y_\rho} x(\kappa) & \partial_{y_\rho} y(\kappa) \end{pmatrix} \quad (96)$$

and the partial derivative notation  $\partial f / \partial x = \partial_x f$ .

- Here we used the identity

$$dx dy = |\mathbf{J}_\rho(\kappa)| dk_x dk_y. \quad (97)$$



Pointwise Fourier Transform

# Pointwise Fourier Transform

- Then we reformulate Eq. (99) and find

$$\begin{aligned}\tilde{V}(\kappa) &= \left( V(\rho)\sigma(\rho) \exp(-i\nabla\psi(\rho)\cdot\rho) |\mathbf{J}_\kappa(\rho)|^{-1/2} \right) [\rho \leftarrow \rho(\kappa)] \\ &= (V(\rho)f(\rho))[\rho \leftarrow \rho(\kappa)]\end{aligned}\quad (103)$$

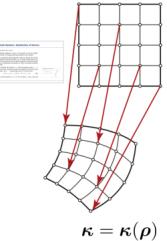
in accordance with the formal definition of the pointwise Fourier transform in Eq. (87).

- The determinant of  $\mathbf{J}_\kappa(\rho)$  follows directly from Eq. (101) and by that from the wavefront phase  $\psi(\rho)$ .
- In analogy to the derivation of Eqs. (103) and (99) the equations of the inverse PFT can be derived.
- Starting from Eq. (81) and inserting into Eq. (74) leads via the same arguments as discussed before to the mapping equation

$$\rho(\kappa) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \kappa \mapsto \rho(\kappa) = -\nabla_k \tilde{\phi}(\kappa), \quad (104)$$

## Pointwise Fourier Transform

- In a compact form we write  
 $\tilde{V}(\kappa) = (\mathcal{F}_k^{1:1}V)(\kappa) := (f(\rho)V(\rho)[\rho \leftarrow \rho(\kappa)]) =: \tilde{f}(\kappa)\tilde{V}(\kappa). \quad (87)$
- Here we use the notation for substitution and the change of the function symbol, e.g. from  $f$  to  $\tilde{f}$ , which we also applied in Eq. (50).
- In words Eq. (87) means: The field values  $V$  at some position  $\rho$  are mapped to a specific position  $\kappa(\rho)$  and weighted by  $f(\rho)$  before the mapping or with  $f(\kappa) : \mathbb{R}^2 \rightarrow \mathbb{C}$  afterwards. The resulting value at  $\kappa(\rho)$  is the value of the Fourier transformed function at  $\kappa$ , that is  $\tilde{V}(\kappa)$ .
- The inverse pointwise Fourier transform in compact notation is given by  
 $V(\rho) = ((\mathcal{F}_k^{1:1})^{-1}\tilde{V})(\rho) := (\tilde{f}(\kappa)\tilde{V}(\kappa))[\kappa \leftarrow \kappa(\rho)] =: \tilde{F}(\rho)\tilde{V}(\rho). \quad (88)$



76

# Pointwise Fourier Transform

with  $\nabla_k = (\partial_{k_x}, \partial_{k_y})$ .

- Via Parseval's theorem Eq. (94) and suitable substitutions follows

$$\begin{aligned} V(\rho) &= \left( (\tilde{V}(\kappa)\tilde{\sigma}(\kappa) \exp(-i\nabla_k \tilde{\phi}(\kappa) \cdot \kappa)) [\kappa \leftarrow \kappa(\rho)] \right) \sqrt{|\mathbf{J}_\kappa(\rho)|} \\ &= \tilde{V}(\rho) \tilde{F}(\rho) \end{aligned} \quad (105)$$

with the notation introduced in Eq. (88).

- The factor  $\tilde{\sigma}(\kappa)$  follows the same rules as in Tab. 3 when  $\psi(\rho)$  is replaced by  $\tilde{\phi}(\kappa)$  together with the derivatives.
- As alternative to Eq. (105) we conclude

$$\begin{aligned} V(\rho) &= \left( \tilde{V}(\kappa)\tilde{\sigma}(\kappa) \exp \left( -i\nabla_k \tilde{\phi}(\kappa) \cdot \kappa \right) |\mathbf{J}_\rho(\kappa)|^{-1/2} \right) [\kappa \leftarrow \kappa(\rho)] \\ &= (\tilde{V}(\kappa)\tilde{F}(\kappa)) [\kappa \leftarrow \kappa(\rho)] \end{aligned} \quad (106)$$

$\partial_{xx}\psi(\rho), \partial_{yy}\psi(\rho)$	$\partial_{xy}\psi(\rho)$	$ \mathbf{J}_\kappa(\rho) $	factor $\sigma(\rho)$
same sign ( $\pm$ )	small	$> 0$	$\pm i$
one/both = 0	$\neq 0$	$< 0$	1
different sign	any	$< 0$	1
same sign	large	$< 0$	1

# Pointwise Fourier Transform

with

$$\mathbf{J}_\rho(\boldsymbol{\kappa}) = \begin{pmatrix} \partial_{k_x} x(\boldsymbol{\kappa}) & \partial_{k_x} y(\boldsymbol{\kappa}) \\ \partial_{k_y} x(\boldsymbol{\kappa}) & \partial_{k_y} y(\boldsymbol{\kappa}) \end{pmatrix} = - \begin{pmatrix} \partial_{k_x k_x} \tilde{\phi}(\boldsymbol{\kappa}) & \partial_{k_x k_y} \tilde{\phi}(\boldsymbol{\kappa}) \\ \partial_{k_y k_x} \tilde{\phi}(\boldsymbol{\kappa}) & \partial_{k_y k_y} \tilde{\phi}(\boldsymbol{\kappa}) \end{pmatrix}, \quad (107)$$

where we used the mapping Eq. (104) on the right side of Eq. (107).

$$\rho(\boldsymbol{\kappa}) = -\nabla_k \tilde{\phi}(\boldsymbol{\kappa})$$

### 3.3.3 Pointwise Fourier transform: algorithm

- The equations on the PFT and the inverse PFT can be applied directly on the complex amplitudes  $V$  and  $\tilde{V}$  respectively. Some comments on the algorithm of the PFT should be given next.
- Each component comes with a wavefront phase in both domains. It is best, if operations on the field components provide the wavefront phase in the output as well.
- In general a Fourier transform cannot deliver the wavefront phase

# Pointwise Fourier Transform

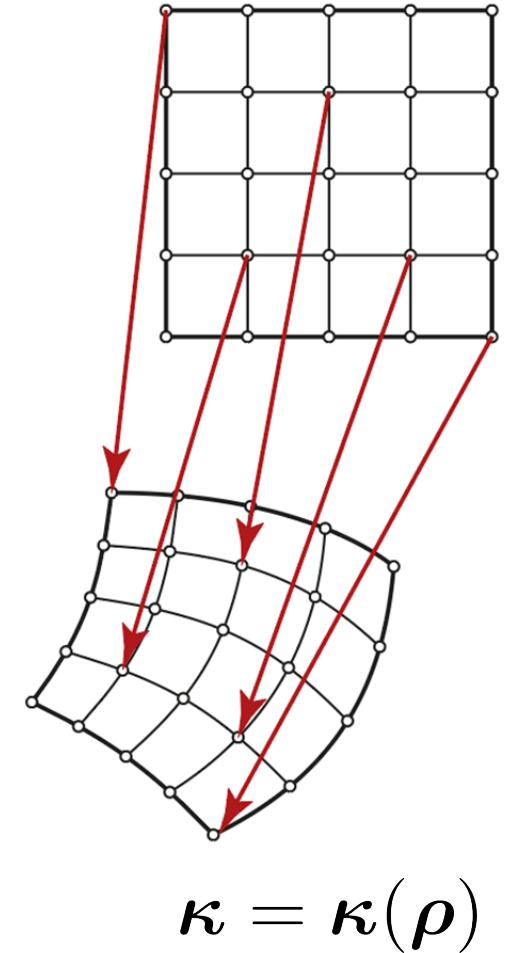
in its output but the PFT can. From the formulas we directly obtain

$$\tilde{\phi}(\kappa) = (\psi(\rho) - \nabla\psi(\rho)\cdot\rho)[\rho \leftarrow \rho(\kappa)] \quad (108)$$

and

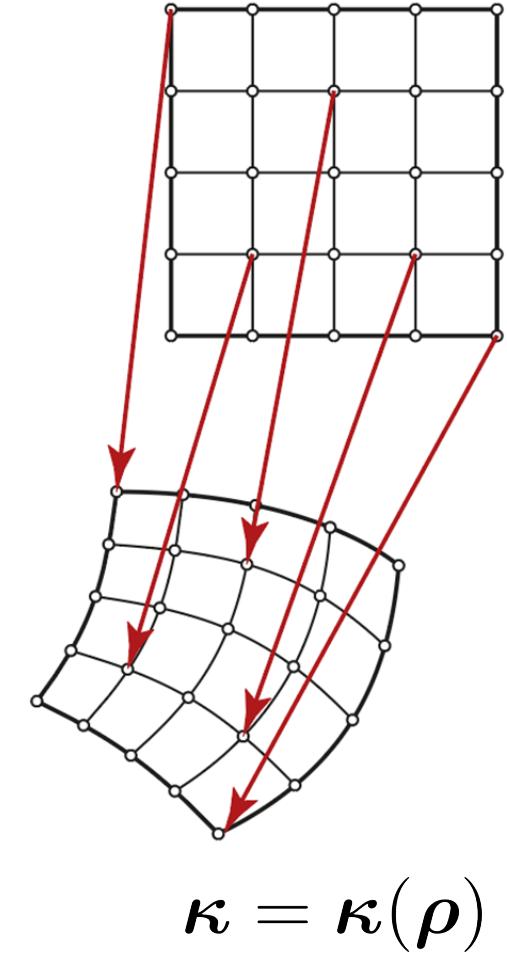
$$\psi(\rho) = (\tilde{\phi}(\kappa) - \nabla_k \tilde{\phi}(\kappa)\cdot\kappa)[\kappa \leftarrow \kappa(\rho)]. \quad (109)$$

- The transformations of the wavefront phases constitute Legendre transformations.
- The mappings according to equations Eq. (92) and Eq. (104) can be arbitrary continuous functions dependent of the wavefront phase.
- In the PFT algorithms the coordinates in both domains are connected and represented by meshes. In order to avoid overlapping of some parts of the mesh with others, the mappings should be bijective which also means the coordinates in both domains are connected by a homeomorphism.



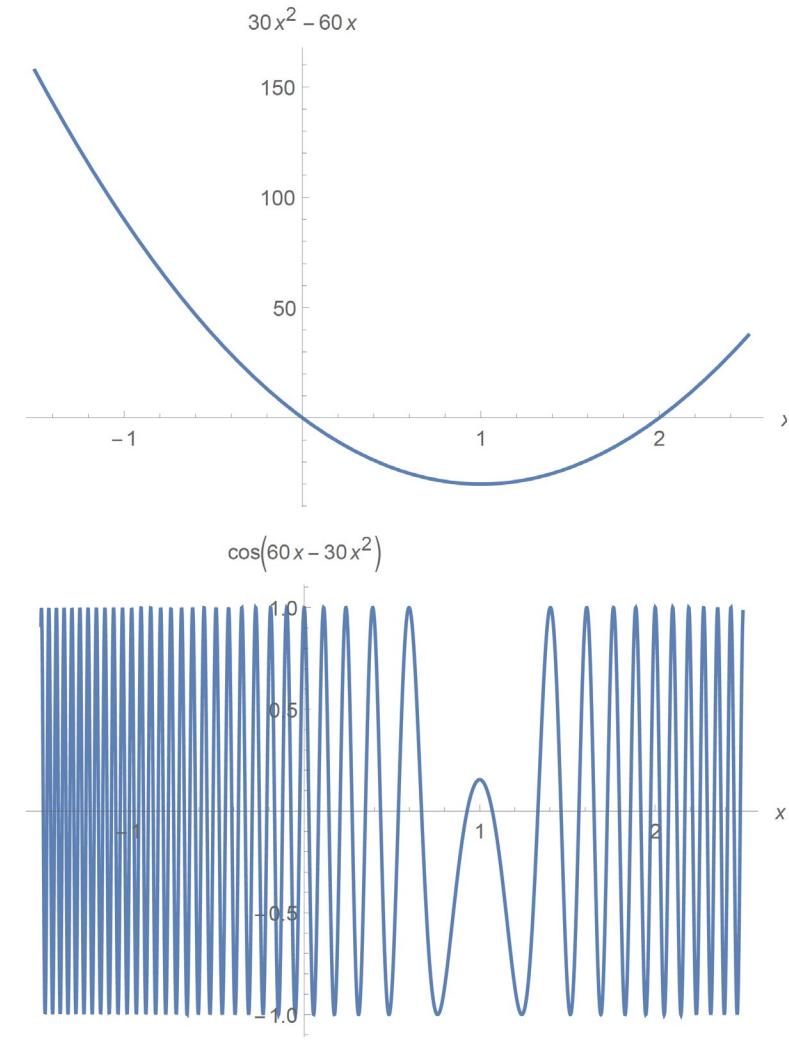
# Pointwise Fourier Transform

- In this case, we refer to the PFT also to as the homeomorphic Fourier transform (HFT).
- The accuracy of the HFT is higher than the one of a non-bijective PFT. Nevertheless, we also use the PFT in case of a non-bijective mapping between the coordinates.
- In general, we must care about the accuracy of the PFT before its application is justified in modeling. This is a mathematical decision and we have a rigorous reference with the rigorous Fourier transform. In practice we need accuracy criteria and numerical procedures to evaluate the accuracy of the PFT. In what follows we assume, that such concepts are available and applied when needed.
- The PFT is a pointwise Fourier transform and thus obviously linear in the number  $N^{\text{PFT}}$  of sampling values, e.g.  $V(\rho_n)$ , which are needed to perform a PFT.



# Pointwise Fourier Transform

- Because of the Fast Fourier Transform (FFT) also the rigorous Fourier transform can be implemented being linear in the number  $N^{\text{FFT}}$  of sampling values  $V(m_x\delta x, m_y\delta y)$ .
- However, (1) for the FFT the sampling must be equidistantly on a grid with period  $(\delta x, \delta y)$  whereas the PFT works also with gridless data samples, and (2) typically  $N^{\text{FFT}} \gg N^{\text{PFT}}$  because the FFT requires a sampling of the complex amplitude including the cosine fluctuations (see figure before), whereas the PFT needs to sample the smooth function  $\psi$  only.
- It should be mentioned, that the Jacobian determinants in the PFT equations represent always, according to Eq. (97) and Eq. (100), the change of the size of a small area in one domain under mapping into the other one.  
$$dk_x dk_y = |\mathbf{J}_\kappa(\boldsymbol{\rho})| dx dy$$



# Pointwise Fourier Transform

## 3.3.4 Pointwise Fourier transform: spherical phase

- The spherical wavefront phase of Eq. (90) is of special interest in optics. Thus we like to explicitly provide the PFT for this case. From Eq. (92) follows

$$\kappa(\rho) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \rho \mapsto \frac{Rk_0n}{|R|} \frac{\rho}{\sqrt{\rho^2 + R^2}} = \frac{Rk_0n}{|R|} \frac{\rho}{r(\rho)} = \frac{Rk_0n}{|R|} \hat{s}_\perp(\rho), \quad (110)$$

with the scalar product  $\rho^2 := \rho \cdot \rho$  and the unit direction vector  $\hat{s}_\perp := \rho/r$ .

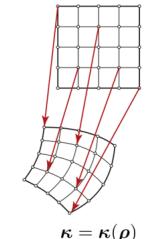
- This mapping can be inverted. By solving for  $\rho^2$  and using that the directions of  $\rho$  and  $\kappa$  are parallel, which obviously follows from

### Pointwise Fourier Transform

a spherical wavefront phase  $\psi^{\text{sph}}(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r}$ , which becomes in a plane at distance  $z = R$

$$\psi^{\text{sph}}(\rho) := \frac{Rk_0n}{|R|} \sqrt{x^2 + y^2 + R^2} = \frac{Rk_0n}{|R|} r(\rho). \quad (90)$$

- The term  $R/|R|$  cares about the sign of  $R$ . A negative radius of curvature  $R$ , i.e.  $R < 0$ , indicates a convergent spherical wave and  $R > 0$  indicates a divergent wave.
- Figure 1 shows, that for small radius of curvature the Fourier transformed function  $|V(\kappa)|$  looks mainly like a geometrically distorted version of the house-shaped function in (a).
- A geometric distortion is mainly defined by a mapping between the coordinates in  $x$ - and  $k$ -domain. That means for small enough  $R$  we obtain a situation as we formulated in Eq. (87) mathematically.
- What can govern the mapping between the coordinates, that means



# Pointwise Fourier Transform

Eq. (110), we find

$$\rho(\kappa) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \kappa \mapsto R \frac{\kappa}{\sqrt{k_0^2 n^2 - \kappa^2}} = R \frac{\kappa}{k_z(\kappa)}. \quad (111)$$

- Plugging  $\rho$  from Eq. (111) into  $r = \sqrt{\rho^2 + R^2}$  leads to the helpful relationship

$$r = \frac{k_0 n |R|}{k_z}. \quad (112)$$

- According to Eq. (111)  $\rho \in \mathbb{R}^2$  requires that the PFT for a spherical wavefront phase is applied to  $\kappa^2 < k_0^2 n^2$  only.
- From a physical optics point of view that means the PFT does not include evanescent plane waves.
- Next we turn to the determination of  $\sqrt{|\mathbf{J}_\rho(\kappa)|}$  of Eq. (99) by direct evaluation of Eq. (96) for  $\rho(\kappa)$  of Eq. (111).

## Pointwise Fourier Transform

- If we assume a pointwise connection according to Eq. (92), then we may substitute  $\rho$  in the integrand on the left side by  $\rho(\kappa)$  and obtain

$$(|V(\rho)|^2 dx dy)[\rho \leftarrow \rho(\kappa)] = (|V(\rho)|^2)[\rho \leftarrow \rho(\kappa)] |\mathbf{J}_\rho(\kappa)| dk_x dk_y \quad (95)$$

with the determinant of the Jacobian matrix

$$\mathbf{J}_\rho(\kappa) = \begin{pmatrix} \partial_{k_x} x(\kappa) & \partial_{k_x} y(\kappa) \\ \partial_{k_y} x(\kappa) & \partial_{k_y} y(\kappa) \end{pmatrix} \quad (96)$$

and the partial derivative notation  $\partial f / \partial x = \partial_x f$ .

- Here we used the identity

$$dx dy = |\mathbf{J}_\rho(\kappa)| dk_x dk_y. \quad (97)$$

86



## Pointwise Fourier Transform

- Plugging Eq. (95) into the left side of Eq. (94) reveals

$$|\tilde{V}(\kappa)| = \left( |V(\rho)| [\rho \leftarrow \rho(\kappa)] \right) \sqrt{|\mathbf{J}_\rho(\kappa)|}. \quad (98)$$

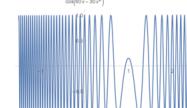
- Combining this result with Eq. (93) leads to the pointwise Fourier transform

$$\tilde{V}(\kappa) = \left( (V(\rho) \sigma(\rho) \exp(-i \nabla \psi(\rho) \cdot \rho)) [\rho \leftarrow \rho(\kappa)] \right) \sqrt{|\mathbf{J}_\rho(\kappa)|} = V(\kappa) f(\kappa) \quad (99)$$

with the notation of Eq. (87).

- The factor  $\sigma(\rho)$  takes into account, that by Parseval's theorem we do not get information about an additional possible phase factor  $\sigma(\rho) = \exp(i\alpha(\rho))$ .

- However, that can be obtained by a more elaborate evaluation of the integral around the critical values. That approach is known as



# Pointwise Fourier Transform

- That yields after a straightforward derivation to

$$\sqrt{|\mathbf{J}_\rho(\kappa)|} = \frac{k_0 n |R|}{k_z^2(\kappa)}. \quad (113)$$

- By evaluating the second derivative of  $\psi^{\text{sph}}(\rho)$  Tab. 3 provides  $\sigma = iR/|R|$ .

- The wavefront phase in k-domain results via Eq. (108) and we get

$$\tilde{\phi}(\kappa) = k_z(\kappa)R \quad (114)$$

by applying  $\psi^{\text{sph}}(\rho)[\rho \leftarrow \rho(\kappa)] = k_0^2 n^2 R / k_z(\kappa)$  (see Eq. (112)).

- By combining all results we obtain from Eq. (99)

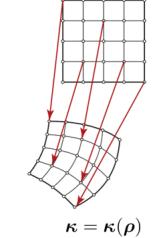
$$\begin{aligned} \tilde{V}(\kappa) &= \mathcal{F}_k^{1:1} \left( \rho \mapsto U(\rho) \exp(i\psi^{\text{sph}}(\rho)) \right)(\kappa) \\ &= i \frac{k_0 n R}{k_z(\kappa)} \frac{\exp(ik_z(\kappa)R)}{k_z(\kappa)} \left( U(\rho)[\rho \leftarrow R \frac{\kappa}{k_z(\kappa)}] \right). \end{aligned} \quad (115)$$

$\partial_{xx}\psi(\rho), \partial_{yy}\psi(\rho)$	$\partial_{xy}\psi(\rho)$	$ \mathbf{J}_\kappa(\rho) $	factor $\sigma(\rho)$
same sign ( $\pm$ )	small	$> 0$	$\pm i$
one/both = 0	$\neq 0$	$< 0$	1
different sign	any	$< 0$	1
same sign	large	$< 0$	1

**Pointwise Fourier Transform**

in its output but the PFT can. From the formulas we directly obtain  
 $\tilde{\phi}(\kappa) = (\psi(\rho) - \nabla \psi(\rho) \cdot \rho)[\rho \leftarrow \rho(\kappa)] \quad (108)$   
and  
 $\psi(\rho) = (\tilde{\phi}(\kappa) - \nabla_\kappa \tilde{\phi}(\kappa) \cdot \kappa)[\kappa \leftarrow \kappa(\rho)]. \quad (109)$

- The transformations of the wavefront phases constitute Legendre transformations.
- The mappings according to equations Eq. (92) and Eq. (104) can be arbitrary continuous functions dependent of the wavefront phase.
- In the PFT algorithms the coordinates in both domains are connected and represented by meshes. In order to avoid overlapping of some parts of the mesh with others, the mappings should be bijective which also means the coordinates in both domains are connected by a homeomorphism.



$\kappa = \kappa(\rho)$

# Pointwise Fourier Transform

## 3.3.5 Inverse pointwise Fourier transform: $k_z z$ -phase

- The phase  $\tilde{\phi}(\boldsymbol{\kappa}) = k_z(\boldsymbol{\kappa})z$  with  $k_z(\boldsymbol{\kappa}) = \sqrt{k_0^2 n^2 - \boldsymbol{\kappa}^2}$  is of special importance for the inverse PFT.
- With respect to Eq. (114) we may understand that also as the inverse PFT to the case discussed in Sec. 3.3.4 for  $R = z$ .
- Analogously to the discussion in the previous section we derive from Eq. (104) the mapping equation and its inverse version.
- The results are identical with Eq. (110) and Eq. (111) when we replace  $R$  by  $z$ . That does not come as a surprise.
- Next we derive  $\sqrt{|\mathbf{J}_\kappa(\boldsymbol{\rho})|}$  by combining Eq. (101) and Eq. (111). Then we receive after straightforward calculation

$$\sqrt{|\mathbf{J}_\kappa(\boldsymbol{\rho})|} = \frac{k_0 n |z|}{\boldsymbol{\rho}^2 + z^2} = \frac{k_0 n |z|}{r^2(\boldsymbol{\rho})}. \quad (116)$$

$$\tilde{\phi}(\boldsymbol{\kappa}) = k_z(\boldsymbol{\kappa})R$$

### Pointwise Fourier Transform

- Then we reformulate Eq. (99) and find
$$\begin{aligned}\tilde{V}(\boldsymbol{\kappa}) &= \left( V(\boldsymbol{\rho}) \sigma(\boldsymbol{\rho}) \exp(-i \nabla \psi(\boldsymbol{\rho}) \cdot \boldsymbol{\rho}) |\mathbf{J}_\kappa(\boldsymbol{\rho})|^{-1/2} \right) (\boldsymbol{\rho} \leftarrow \boldsymbol{\rho}(\boldsymbol{\kappa})) \\ &= (V(\boldsymbol{\rho}) f(\boldsymbol{\rho})) (\boldsymbol{\rho} \leftarrow \boldsymbol{\rho}(\boldsymbol{\kappa}))\end{aligned}\quad (103)$$
in accordance with the formal definition of the pointwise Fourier transform in Eq. (67).
- The determinant of  $\mathbf{J}_\kappa(\boldsymbol{\rho})$  follows directly from Eq. (101) and by that from the wavefront phase  $\psi(\boldsymbol{\rho})$ .
- In analogy to the derivation of Eqs. (103) and (99) the equations of the inverse PFT can be derived.
- Starting from Eq. (81) and inserting into Eq. (74) leads via the same arguments as discussed before to the mapping equation

$$\boldsymbol{\rho}(\boldsymbol{\kappa}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \boldsymbol{\kappa} \mapsto \boldsymbol{\rho}(\boldsymbol{\kappa}) = -\nabla \tilde{\phi}(\boldsymbol{\kappa}). \quad (104)$$



# Pointwise Fourier Transform

- With Eq. (112) we easily see, that the Jacobian in Eq. (116) and Eq. (113) are reciprocal to each other in accordance with Eq. (102).
- According to Tab. 3 for k-domain  $\tilde{\sigma} = -iz/|z|$  follows from e.g.  $\operatorname{sgn}(\partial_{k_x}^2 \tilde{\phi}(\boldsymbol{\kappa})) = -z/|z|$ .
- The wavefront phase Eq. (109) leads to  $\psi^{\text{sph}}(\boldsymbol{\rho})$  of Eq. (90) with  $R = z$ .
- By combining all results we obtain from Eq. (105)

$$\begin{aligned} V(\boldsymbol{\rho}) &= \left( \mathcal{F}_k^{1:1} \right)^{-1} \left( \boldsymbol{\kappa} \mapsto \tilde{A}(\boldsymbol{\kappa}) \exp(i k_z(\boldsymbol{\kappa}) z) \right)(\boldsymbol{\rho}) \\ &= -i \frac{k_0 n z \exp(i \frac{z k_0 n}{|z|} r(\boldsymbol{\rho}))}{r(\boldsymbol{\rho})} \left( \tilde{A}(\boldsymbol{\kappa}) [\boldsymbol{\kappa} \leftarrow \frac{k_0 n z}{|z|} \frac{\boldsymbol{\rho}}{r(\boldsymbol{\rho})}] \right). \quad (117) \end{aligned}$$

- Now we may perform the test of the consistency of Eqs. (117) and (115) by assuming  $R = z$  and the result of (115) for  $\tilde{A}$  in

$\partial_{xx}\psi(\boldsymbol{\rho}), \partial_{yy}\psi(\boldsymbol{\rho})$	$\partial_{xy}\psi(\boldsymbol{\rho})$	$ \mathbf{J}_{\boldsymbol{\kappa}}(\boldsymbol{\rho}) $	factor $\sigma(\boldsymbol{\rho})$
same sign ( $\pm$ )	small	$> 0$	$\pm i$
one/both = 0	$\neq 0$	$< 0$	1
different sign	any	$< 0$	1
same sign	large	$< 0$	1

## Pointwise Fourier Transform

with  $\nabla_k = (\partial_{k_x}, \partial_{k_y})$ .

- Via Parseval's theorem Eq. (94) and suitable substitutions follows

$$\begin{aligned} V(\boldsymbol{\rho}) &= \left( (\hat{V}(\boldsymbol{\kappa}) \tilde{\sigma}(\boldsymbol{\kappa}) \exp(-i \nabla_k \tilde{\phi}(\boldsymbol{\kappa}) \cdot \boldsymbol{\kappa}) ) [\boldsymbol{\kappa} \leftarrow \boldsymbol{\kappa}(\boldsymbol{\rho})] \right) \sqrt{|\mathbf{J}_{\boldsymbol{\kappa}}(\boldsymbol{\rho})|} \\ &= \hat{V}(\boldsymbol{\rho}) \tilde{F}(\boldsymbol{\rho}) \quad (105) \end{aligned}$$

with the notation introduced in Eq. (88).

- The factor  $\tilde{\sigma}(\boldsymbol{\kappa})$  follows the same rules as in Tab. 3 when  $\psi(\boldsymbol{\rho})$  is replaced by  $\tilde{\phi}(\boldsymbol{\kappa})$  together with the derivatives.

- As alternative to Eq. (105) we conclude

$$\begin{aligned} V(\boldsymbol{\rho}) &= \left( (\hat{V}(\boldsymbol{\kappa}) \tilde{\sigma}(\boldsymbol{\kappa}) \exp(-i \nabla_k \tilde{\phi}(\boldsymbol{\kappa}) \cdot \boldsymbol{\kappa}) |\mathbf{J}_{\boldsymbol{\rho}}(\boldsymbol{\rho})|^{-1/2} ) [\boldsymbol{\kappa} \leftarrow \boldsymbol{\kappa}(\boldsymbol{\rho})] \right) [\boldsymbol{\kappa} \leftarrow \boldsymbol{\kappa}(\boldsymbol{\rho})] \\ &= (\hat{V}(\boldsymbol{\kappa}) \tilde{F}(\boldsymbol{\kappa})) [\boldsymbol{\kappa} \leftarrow \boldsymbol{\kappa}(\boldsymbol{\rho})] \quad (106) \end{aligned}$$

$\partial_{xx}\psi(\boldsymbol{\rho}), \partial_{yy}\psi(\boldsymbol{\rho})$	$\partial_{xy}\psi(\boldsymbol{\rho})$	$ \mathbf{J}_{\boldsymbol{\kappa}}(\boldsymbol{\rho}) $	factor $\sigma(\boldsymbol{\rho})$
same sign ( $\pm$ )	small	$> 0$	$\pm i$
one/both = 0	$\neq 0$	$< 0$	1
different sign	any	$< 0$	1
same sign	large	$< 0$	1

93

$$\begin{aligned} \tilde{V}(\boldsymbol{\kappa}) &= \mathcal{F}_k^{1:1} \left( \boldsymbol{\rho} \mapsto U(\boldsymbol{\rho}) \exp(i \psi^{\text{sph}}(\boldsymbol{\rho})) \right)(\boldsymbol{\kappa}) \\ &= i \frac{k_0 n R}{k_z(\boldsymbol{\kappa})} \frac{\exp(i k_z(\boldsymbol{\kappa}) R)}{k_z(\boldsymbol{\kappa})} \left( U(\boldsymbol{\rho}) [\boldsymbol{\rho} \leftarrow R \frac{\boldsymbol{\kappa}}{k_z(\boldsymbol{\kappa})}] \right) \end{aligned}$$

# Pointwise Fourier Transform

Eqs. (117) , that means  $\tilde{A}(\kappa) = ik_0nR/k_z^2(\kappa) U(\rho)[\rho \leftarrow R\kappa/k_z(\kappa)]$ .

Plugging into (117) and applying Eq. (112) leads to the expected result  $V(\rho) = U(\rho) \exp(i\frac{zk_0n}{|z|}r(\rho))$ .

Name	Notation	
	x-domain	k-domain
Fourier operator (73),(74)	$\mathcal{F}_k$	$\mathcal{F}_k^{-1}$
Pointwise Fourier transform	$\mathcal{F}_k^{1:1}$	$(\mathcal{F}_k^{1:1})^{-1}$
weight functions	$f(\rho)$	$\tilde{F}(\kappa)$
coordinate map	$\rho \mapsto \kappa(\rho)$	$\kappa \mapsto \rho(\kappa)$
coordinate substitution	$[\rho \leftarrow \rho(\kappa)]$	$[\kappa \leftarrow \kappa(\rho)]$

Table 2: Notation and terminology: Fourier transform

## **4. Electromagnetic fields in isotropic homogeneous media**

# Electromagnetic Fields in Isotropic Homogeneous Media

## 4 Electromagnetic fields in isotropic homogeneous media

- In this chapter we discuss fields in homogeneous and isotropic media. After considering the general solution in Sec. 4.1, we turn to a detailed discussion of the propagation operator concept in Sec. 4.3.
- In physical-optics modeling we typically implement propagation operators via Fourier transform in the k-domain.
- In the x-domain these techniques are integral operators with some special cases are discussed in Sec. 4.4.

$$\begin{aligned}\partial_y E_z(\mathbf{r}) - \partial_z E_y(\mathbf{r}) &= ik_0 H_x(\mathbf{r}) \\ \partial_z E_x(\mathbf{r}) - \partial_x E_z(\mathbf{r}) &= ik_0 H_y(\mathbf{r}) \\ \partial_x E_y(\mathbf{r}) - \partial_y E_x(\mathbf{r}) &= ik_0 H_z(\mathbf{r}) \\ \partial_y H_z(\mathbf{r}) - \partial_z H_y(\mathbf{r}) &= -ik_0 \check{\epsilon}_r E_x(\mathbf{r}) \\ \partial_z H_x(\mathbf{r}) - \partial_x H_z(\mathbf{r}) &= -ik_0 \check{\epsilon}_r E_y(\mathbf{r}) \\ \partial_x H_y(\mathbf{r}) - \partial_y H_x(\mathbf{r}) &= -ik_0 \check{\epsilon}_r E_z(\mathbf{r})\end{aligned}$$

## 4.1. General Solution

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### 4.1 Solution of Maxwell's equations in isotropic homogeneous media

- Electromagnetic fields in homogeneous media are governed by the curl Maxwell's equations

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega\mu_0 \breve{\mathbf{H}}(\mathbf{r}, \omega) \quad (118)$$

$$\nabla \times \breve{\mathbf{H}}(\mathbf{r}, \omega) = -i\omega\epsilon_0\check{\epsilon}_r(\omega)\mathbf{E}(\mathbf{r}, \omega) \quad (119)$$

with the generalized permittivity  $\check{\epsilon}_r$  of Eq. (1), the vacuum permittivity

$$\epsilon_0 = 8.854\ 187\ 817\ 6 \times 10^{-12} \text{ N}^{-1} \text{ m}^{-2} \text{ C}^2. \quad (120)$$

and the vacuum permeability

$$\mu_0 = 1.256\ 637\ 061\ 4 \times 10^{-6} \text{ m kg C}^{-2}. \quad (121)$$

## 4.1. General Solution

---

- The notation  $\check{\mathbf{H}}$  for the magnetic field is introduced, because we like to replace the magnetic field  $\check{\mathbf{H}}$  by

$$\mathbf{H} := \eta_0 \check{\mathbf{H}} \text{ with } \eta_0 = \sqrt{\epsilon_0 / \mu_0}. \quad (122)$$

- As shown below this substitution gives a more symmetric formulation of Maxwell's equations.
- More importantly, because of the  $\mu_0 \approx 10^5 \times \epsilon_0$  the magnetic field is always about this factor smaller than the electric field. For numerical algorithms this is not a favorite situation.
- By replacing  $\check{\mathbf{H}}$  by  $\mathbf{H}$  according to Eq. (122) we obtain the equations

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = ik_0(\omega) \mathbf{H}(\mathbf{r}, \omega) \quad (123)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -ik_0(\omega) \check{\epsilon}_r(\omega) \mathbf{E}(\mathbf{r}, \omega) \quad (124)$$

using  $1/c = \sqrt{\epsilon_0 \mu_0}$ ,  $k_0 = \omega/c$ ,  $\eta_0 = \mu_0 \omega/k_0$ , and  $\eta_0 = k_0/(\epsilon_0 \omega)$ .

## 4.1. General Solution

---

- In what follows we work with the slightly modified magnetic field and must remember to replace it when we deal with expressions including the original magnetic field.
- We like to emphasize, that the equations Eq. (123) and Eq. (124) formulate six equations for the six field components and thus they are sufficient to discuss fields in homogeneous media.
- For the further discussion we consider the six equations explicitly, that means (skipping the  $\omega$ -dependency)

$$\begin{aligned}\partial_y E_z(\mathbf{r}) - \partial_z E_y(\mathbf{r}) &= ik_0 H_x(\mathbf{r}) \\ \partial_z E_x(\mathbf{r}) - \partial_x E_z(\mathbf{r}) &= ik_0 H_y(\mathbf{r}) \\ \partial_x E_y(\mathbf{r}) - \partial_y E_x(\mathbf{r}) &= ik_0 H_z(\mathbf{r}) \\ \partial_y H_z(\mathbf{r}) - \partial_z H_y(\mathbf{r}) &= -ik_0 \check{\epsilon}_r E_x(\mathbf{r}) \\ \partial_z H_x(\mathbf{r}) - \partial_x H_z(\mathbf{r}) &= -ik_0 \check{\epsilon}_r E_y(\mathbf{r}) \\ \partial_x H_y(\mathbf{r}) - \partial_y H_x(\mathbf{r}) &= -ik_0 \check{\epsilon}_r E_z(\mathbf{r})\end{aligned}\tag{125}$$

## 4.1. General Solution

with the partial derivative, e.g.  $\partial_x$ .

- In each  $z$ -position we may perform a Fourier transform  $\mathcal{F}_k$  according to Eq. (72) and obtain

$$\begin{aligned} ik_y \tilde{E}_z(\boldsymbol{\kappa}, z) - \partial_z \tilde{E}_y(\boldsymbol{\kappa}, z) &= ik_0 \tilde{H}_x(\boldsymbol{\kappa}, z) \\ \partial_z \tilde{E}_x(\boldsymbol{\kappa}, z) - ik_x \tilde{E}_z(\boldsymbol{\kappa}, z) &= ik_0 \tilde{H}_y(\boldsymbol{\kappa}, z) \\ ik_x \tilde{E}_y(\boldsymbol{\kappa}, z) - ik_y \tilde{E}_x(\boldsymbol{\kappa}, z) &= ik_0 \tilde{H}_z(\boldsymbol{\kappa}, z) \\ ik_y \tilde{H}_z(\boldsymbol{\kappa}, z) - \partial_z \tilde{H}_y(\boldsymbol{\kappa}, z) &= -ik_0 \check{\epsilon}_r \tilde{E}_x(\boldsymbol{\kappa}, z) \\ \partial_z \tilde{H}_x(\boldsymbol{\kappa}, z) - ik_x \tilde{H}_z(\boldsymbol{\kappa}, z) &= -ik_0 \check{\epsilon}_r \tilde{E}_y(\boldsymbol{\kappa}, z) \\ ik_x \tilde{H}_y(\boldsymbol{\kappa}, z) - ik_y \tilde{H}_x(\boldsymbol{\kappa}, z) &= -ik_0 \check{\epsilon}_r \tilde{E}_z(\boldsymbol{\kappa}, z) \end{aligned} \tag{126}$$

utilizing  $\partial_x \rightarrow ik_x$  and for  $y$  accordingly.

- Equations (126) comprise four differential and two algebraic equations, namely the  $\tilde{H}_z$  and  $\tilde{E}_z$  equations. We can use the algebraic equations to eliminate  $\tilde{H}_z$  and  $\tilde{E}_z$  in the differential equations by

## 4.1. General Solution

substitution.

- From Eq. (126) we conclude

$$\tilde{H}_z(\boldsymbol{\kappa}, z) = \frac{k_x \tilde{E}_y(\boldsymbol{\kappa}, z) - k_y \tilde{E}_x(\boldsymbol{\kappa}, z)}{k_0}, \quad (127)$$

$$\tilde{E}_z(\boldsymbol{\kappa}, z) = -\frac{k_x \tilde{H}_y(\boldsymbol{\kappa}, z) - k_y \tilde{H}_x(\boldsymbol{\kappa}, z)}{k_0 \check{\epsilon}_r}, \quad (128)$$

and after suitable sorting we obtain

$$\frac{d}{dz} \begin{pmatrix} \tilde{E}_x(\boldsymbol{\kappa}, z) \\ \tilde{E}_y(\boldsymbol{\kappa}, z) \\ \tilde{H}_x(\boldsymbol{\kappa}, z) \\ \tilde{H}_y(\boldsymbol{\kappa}, z) \end{pmatrix} = \begin{pmatrix} \tilde{\Omega}_{11}(\boldsymbol{\kappa}) & \tilde{\Omega}_{12}(\boldsymbol{\kappa}) & \tilde{\Omega}_{13}(\boldsymbol{\kappa}) & \tilde{\Omega}_{14}(\boldsymbol{\kappa}) \\ \tilde{\Omega}_{21}(\boldsymbol{\kappa}) & \tilde{\Omega}_{22}(\boldsymbol{\kappa}) & \tilde{\Omega}_{23}(\boldsymbol{\kappa}) & \tilde{\Omega}_{24}(\boldsymbol{\kappa}) \\ \tilde{\Omega}_{31}(\boldsymbol{\kappa}) & \tilde{\Omega}_{32}(\boldsymbol{\kappa}) & \tilde{\Omega}_{33}(\boldsymbol{\kappa}) & \tilde{\Omega}_{34}(\boldsymbol{\kappa}) \\ \tilde{\Omega}_{41}(\boldsymbol{\kappa}) & \tilde{\Omega}_{42}(\boldsymbol{\kappa}) & \tilde{\Omega}_{43}(\boldsymbol{\kappa}) & \tilde{\Omega}_{44}(\boldsymbol{\kappa}) \end{pmatrix} \begin{pmatrix} \tilde{E}_x(\boldsymbol{\kappa}, z) \\ \tilde{E}_y(\boldsymbol{\kappa}, z) \\ \tilde{H}_x(\boldsymbol{\kappa}, z) \\ \tilde{H}_y(\boldsymbol{\kappa}, z) \end{pmatrix}. \quad (129)$$

## 4.1. General Solution

- In compact form

$$\frac{d}{dz} \tilde{\mathbf{V}}_{\perp}(\boldsymbol{\kappa}, z) = \tilde{\boldsymbol{\Omega}}(\boldsymbol{\kappa}) \tilde{\mathbf{V}}_{\perp}(\boldsymbol{\kappa}, z) \quad (130)$$

with the vector  $\tilde{\mathbf{V}} = \mathcal{F}\mathbf{V}$  includes all transversal field components.

- The matrix  $\tilde{\boldsymbol{\Omega}}$  is explicitly given by

$$\tilde{\boldsymbol{\Omega}}(\boldsymbol{\kappa}) = \begin{pmatrix} 0 & 0 & i \frac{k_x k_y}{k_0} \frac{1}{\check{\epsilon}_r} & ik_0 - i \frac{k_x^2}{k_0} \frac{1}{\check{\epsilon}_r} \\ 0 & 0 & -ik_0 + i \frac{k_y^2}{k_0} \frac{1}{\check{\epsilon}_r} & -i \frac{k_x k_y}{k_0} \frac{1}{\check{\epsilon}_r} \\ -i \frac{k_x k_y}{k_0} & -ik_0 \check{\epsilon}_r + i \frac{k_x^2}{k_0} & 0 & 0 \\ ik_0 \check{\epsilon}_r - i \frac{k_y^2}{k_0} & i \frac{k_x k_y}{k_0} & 0 & 0 \end{pmatrix}. \quad (131)$$

## 4.1. General Solution

- The structure of the matrix becomes more obvious by

$$\tilde{\Omega} = \begin{pmatrix} 0 & 0 & iA/\check{\epsilon}_r & iB_x/\check{\epsilon}_r \\ 0 & 0 & -iB_y/\check{\epsilon}_r & -iA/\check{\epsilon}_r \\ -iA & -iB_x & 0 & 0 \\ iB_y & iA & 0 & 0 \end{pmatrix}, \quad (132)$$

with

$$\begin{aligned} A &:= \frac{k_x k_y}{k_0} \\ B_x &:= \frac{1}{k_0} (k_0^2 \check{\epsilon}_r - k_x^2) = \frac{1}{k_0} (k_z^2 + k_y^2) \\ B_y &:= \frac{1}{k_0} (k_0^2 \check{\epsilon}_r - k_y^2) = \frac{1}{k_0} (k_z^2 + k_x^2) \end{aligned} \quad (133)$$

- For the second step in the “B”-equations we used  $k_z$  from Eq. (137).

## 4.1. General Solution

---

- Equation (130) is in form of a first-order linear homogeneous differential equation system. Mathematics provide a full concept to deal with such systems. We like to follow the basic steps of this mathematical concept.
- **Step 1:** We search for solutions of the form

$$\tilde{V}_\perp(z) = \mathbf{v} e^{\alpha z} \quad (134)$$

with a vector  $\mathbf{v} \in \mathbb{C}^4$  and  $\alpha \in \mathbb{C}$ . Both quantities are not function of  $z$ .

- The explicit  $\kappa$  dependency of all quantities is skipped here. Insertion of Eq. (134) into Eq. (130) yields after division of both sides through  $e^{\alpha z}$  the algebraic equation

$$\tilde{\Omega} \mathbf{v} = \alpha \mathbf{v} \iff (\tilde{\Omega} - \alpha \mathbf{I}) \mathbf{v} = 0 \quad (135)$$

with the identity matrix  $\mathbf{I}$ .

## 4.1. General Solution

---

- **Step 2:** Equation (135) constitutes an eigenvalue equation of the matrix  $\tilde{\Omega}$ , the eigenvalues  $\alpha$ , and the eigenvectors  $v$ .
- $(\tilde{\Omega} - \alpha I)v = 0$  has nonzero solutions  $v$  only if the determinant of  $(\tilde{\Omega} - \alpha I)$  is zero.
- Hence we look for solutions of

$$|\tilde{\Omega} - \alpha I| = 0, \quad (136)$$

which provides the characteristic polynomial of  $\tilde{\Omega}$ . This polynomial is of degree  $J = 4$ .

- A straightforward but lengthy evaluation of Eq. (136) leads to two distinct eigenvalues

$$\alpha_{\pm}(\kappa) = \pm i \sqrt{\check{\epsilon}_r k_0^2 - \kappa^2} =: \pm i k_z(\kappa), \quad (137)$$

with  $\kappa^2 = \kappa \cdot \kappa = k_x^2 + k_y^2$ .

## 4.1. General Solution

---

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## 4.1. General Solution

---

- On the right side of Eq. (137) we defined  $\alpha =: ik_z$ , which implies the dispersion relation

$$k^2(\omega) = \mathbf{k}(\omega) \cdot \mathbf{k}(\omega) = k_x^2(\omega) + k_y^2(\omega) + k_z^2(\omega) = \epsilon_r(\omega)k_0^2, \quad (138)$$

with explicit emphasis of the  $\omega$ -dependency and .

- **Step 3:** Next we calculate the eigenvectors per eigenvalue by solving Eq. (135) per eigenvalue.
- In practice the solution of eigenvalues and eigenvectors can be performed numerically per  $\kappa$  and  $\omega$ . But in the present case the problem can be solved analytically.
- Before we do that, we like to discuss the consequences of having the eigenvectors.

## 4.1. General Solution

---

- **Step 4:** Assuming we have the eigenvectors we next turn to the construction of our general solution of Maxwell's equations in homogeneous and isotropic media as formulated by Eq. (130) in k-domain.
- With the  $J$  eigenvectors  $\mathbf{v}_j$ , we can define the matrix

$$\mathbf{P} := (\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_J) \quad (139)$$

with the eigenvectors as columns.

- That allows us to define any linear combination of the eigenvectors, which constitutes a base of our solution space, by

$$\mathbf{v} = \sum_{j=1}^J \tilde{C}_j \mathbf{v}_j = \mathbf{P} \tilde{\mathbf{C}} \quad (140)$$

with  $\tilde{\mathbf{C}} = (\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_J)^T$ .

## 4.1. General Solution

- Any vector  $v$  of Eq. (140) satisfies Eq. (135). This can be nicely written by

$$\tilde{\Omega} \mathbf{P} \tilde{C} = \mathbf{P} \mathbf{D}_\alpha \tilde{C}, \quad (141)$$

with the diagonal matrix

$$\mathbf{D}_\alpha := \begin{pmatrix} ik_z & 0 & 0 & 0 \\ 0 & ik_z & 0 & 0 \\ 0 & 0 & -ik_z & 0 \\ 0 & 0 & 0 & -ik_z \end{pmatrix} \quad (142)$$

with the eigenvalues in the diagonal for  $J = 4$  in our case.

## 4.1. General Solution

- In order to formulate the general solution we also define

$$\mathbf{D}_e(z) := \begin{pmatrix} e^{ik_z z} & 0 & 0 & 0 \\ 0 & e^{ik_z z} & 0 & 0 \\ 0 & 0 & e^{-ik_z z} & 0 \\ 0 & 0 & 0 & e^{-ik_z z} \end{pmatrix}. \quad (143)$$

- Then we multiply both sides of Eq. (141) with  $\mathbf{D}_e$  and obtain

$$\tilde{\Omega} \mathbf{P} \mathbf{D}_e(z) \tilde{\mathbf{C}} = \mathbf{P} \mathbf{D}_\alpha \mathbf{D}_e(z) \tilde{\mathbf{C}}, \quad (144)$$

in which we can now replace  $\mathbf{D}_\alpha$  by the differential  $d/dz$  and

$$\tilde{\Omega} \mathbf{P} \mathbf{D}_e(z) \tilde{\mathbf{C}} = \frac{d}{dz} \mathbf{P} \mathbf{D}_e(z) \tilde{\mathbf{C}} \quad (145)$$

results.

## 4.1. General Solution

---

- From that we finally conclude the general solution of Maxwell's equations in k-domain for homogeneous and isotropic media by

$$\tilde{\mathbf{V}}_{\perp}(\kappa, z, \omega) = \mathbf{P}(\kappa, \omega) \mathbf{D}_e((\kappa, z, \omega)) \tilde{\mathbf{C}}(\kappa, \omega), \quad (146)$$

with all dependencies explicitly noted.

- What does this equation express?
- The two matrices follow from the eigenvalue problem stated in Eq. (135). They must be solved per  $\omega$  and  $\kappa$ .
- The coefficient vector  $\tilde{\mathbf{C}}(\kappa, \omega)$  can be freely selected to construct a solution.
- Typically it is further specified by additional initial values or boundary values. This is very typical for tasks with differential equations involved.

## 4.1. General Solution

---

- We also see from Eq. (143) that two coefficients belong to forward propagating fields and two belong to fields which propagate into the negative direction. The general solution naturally includes both directions.
- Observe that per direction the general field is fully specified by two coefficients of  $\tilde{C}$  respectively!
- Via Eq. (146) the transversal field components can be determined and then the z-components follow from Eq. (128) and Eq. (127).
- The meaning of the coefficients depends on the eigenvector selection which is not fully fixed in homogeneous and isotropic media. That is to be considered next. So we proceed with the details of Step 3.

## 4.1. General Solution

---

- Details of **Step 3**: The eigenvectors are defined by solutions of the Eq. (135) per eigenvalue.
- There are different ways to calculate the eigenvectors. We search for the eigenvectors by constructing the matrix  $\mathbf{P}$  using Eq. (141) without  $\tilde{\mathbf{C}}$  in form of

$$\tilde{\Omega}\mathbf{P} = \mathbf{P}\mathbf{D}_\alpha \quad (147)$$

with  $\tilde{\Omega}$  of Eq. (132) and  $\mathbf{D}_\alpha$  of Eq. (142).

- What do we know about the eigenvectors *a priori*?
- From Eq. (129) together with Eq. (132) we conclude, that  $\tilde{\mathbf{E}}_\perp$  follows from  $\tilde{\mathbf{H}}_\perp$  and vice versa. The z-components then follow by Eq. (127) and Eq. (128) respectively.
- Thus, two components are independent, whereas all other components follow. That should be reflected by the general solution

## 4.1. General Solution

Eq. (146) and first search for a  $\mathbf{P}$  of the form

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}, \quad (148)$$

which results according to Eq. (146) for example in  $\tilde{E}_x(\kappa, z) = \tilde{C}_1(\kappa)e^{ik_z z} + \tilde{C}_3(\kappa)e^{-ik_z z}$ .

- This expresses, that  $\tilde{E}_x$  is chosen to be the independent component. The same argument holds for  $\tilde{E}_y$ .
- Thus, we refer to a solution with the structure of Eq. (148) as the electric-field-type solution.
- Inserting Eq. (148) into Eq. (147) leads after straightforward calculations to

## 4.1. General Solution

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -A/k_z & -B_x/k_z & A/k_z & B_x/k_z \\ B_y/k_z & A/k_z & -B_y/k_z & -A/k_z \end{pmatrix}, \quad (149)$$

with the definitions of Eq. (133) and Eq. (137).

- For the electric-field-type solution we rename the components of  $\tilde{\mathbf{C}}$  and obtain

$$\tilde{\mathbf{C}} = (\tilde{C}_x^+, \tilde{C}_y^+, \tilde{C}_x^- +, \tilde{C}_y^-)^T \quad (150)$$

according to their physical meaning to weight the forward and backward propagating x- and y-components of the electric field.

## 4.1. General Solution

- It should be noted, that also by a straightforward mathematical solution of the eigenvalue problem we would find a solution of the same style as in Eq. (149).
- Obviously four different versions are possible by shifting the  $(1, 0, 1, 0)$  and  $(0, 1, 0, 1)$  rows to other positions.
- We like to consider also the solution

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -A/B_x & -k_z/B_x & -A/B_x & k_z/B_x \\ 0 & 1 & 0 & 1 \\ k_z\epsilon_r/B_x & -A/B_x & -k_z\epsilon_r/B_x & -A/k_z \end{pmatrix}, \quad (151)$$

## 4.1. General Solution

---

for which we rename the components of  $\tilde{C}$  by

$$\tilde{C} = (\tilde{C}_+^{\text{TE}}, \tilde{C}_+^{\text{TM}}, \tilde{C}_-^{\text{TE}}, \tilde{C}_-^{\text{TM}})^T. \quad (152)$$

- This solution is referred as the TE-TM-type solution whereby TE and TM indicate transversal electric and transversal magnetic respectively.
- This naming has historical roots and does not mean, that there would be no z-component of the fields in general. It emphasizes the fact, that the selected independent components are the transversal electric field component  $\tilde{E}_x$  and the transversal magnetic field component  $\tilde{H}_x$ .
- We may insert either  $\mathbf{P}$  of Eq. (149) or Eq. (151) into Eq. (146) and obtain a full electromagnetic solution of Maxwell's equations in homogeneous and isotropic media.

## 4.2. Vectorial Field Components in Plane

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- The vector  $\tilde{C}(\kappa, \omega)$  is the remaining freedom to be selected. Any selection constructs a solution! Let us consider that a bit more.

### 4.2 Electromagnetic field components in a plane

- Let us face the following practical task in optical modeling: We know the x- and y-component of the electric field in some plane  $P$  through  $z_0$  and denote the field components by  $E_{\perp}(\rho, z_0) = (E_x(\rho, z_0), E_y(\rho, z_0))^T$  and in k-domain  $\tilde{E}_{\perp}(\kappa, z_0) = (\tilde{E}_x(\kappa, z_0), \tilde{E}_y(\kappa, z_0))^T$ .
- How can we calculate the missing four components?
- The answer must follow from the general electric-field-type solution given by Eq. (146) and the corresponding matrix in Eq. (149) and vector in Eq. (150).
- Assume the field components given belong to a field propagating into positive z-direction. Then we know  $\tilde{C}_x^- = \tilde{C}_y^- = 0$  and we may

## 4.2. Vectorial Field Components in Plane

reduce the general solution of Eq. (146) to

$$\begin{pmatrix} \tilde{E}_x(\boldsymbol{\kappa}, z) \\ \tilde{E}_y(\boldsymbol{\kappa}, z) \\ \tilde{H}_x(\boldsymbol{\kappa}, z) \\ \tilde{H}_y(\boldsymbol{\kappa}, z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -A(\boldsymbol{\kappa})/k_z(\boldsymbol{\kappa}) & -B_x(\boldsymbol{\kappa})/k_z(\boldsymbol{\kappa}) \\ B_y(\boldsymbol{\kappa})/k_z(\boldsymbol{\kappa}) & A/k_z(\boldsymbol{\kappa}) \end{pmatrix} \begin{pmatrix} \tilde{C}_x^+(\boldsymbol{\kappa}) \\ \tilde{C}_y^+(\boldsymbol{\kappa}) \end{pmatrix} e^{ik_z(\boldsymbol{\kappa})z} \quad (153)$$

with all dependencies on  $z$  and  $\boldsymbol{\kappa}$  but skipping  $\omega$ .

- For  $z = z_0$  we conclude from the initial value condition  $\tilde{\mathbf{E}}_\perp(\boldsymbol{\kappa}, z_0) = (\tilde{E}_x(\boldsymbol{\kappa}, z_0), \tilde{E}_y(\boldsymbol{\kappa}, z_0))^T$  the identity

$$\begin{pmatrix} \tilde{C}_x^+(\boldsymbol{\kappa}) \\ \tilde{C}_y^+(\boldsymbol{\kappa}) \end{pmatrix} = \begin{pmatrix} \tilde{E}_x(\boldsymbol{\kappa}, z_0) \\ \tilde{E}_y(\boldsymbol{\kappa}, z_0) \end{pmatrix} e^{-ik_z(\boldsymbol{\kappa})z_0} \quad (154)$$

and that leads in  $z = z_0$  to

## 4.2. Vectorial Field Components in Plane

$$\begin{pmatrix} \tilde{H}_x(\kappa, z_0) \\ \tilde{H}_y(\kappa, z_0) \end{pmatrix} = \begin{pmatrix} -A(\kappa)/k_z(\kappa) & -B_x(\kappa)/k_z(\kappa) \\ B_y(\kappa)/k_z(\kappa) & A/k_z(\kappa) \end{pmatrix} \begin{pmatrix} \tilde{E}_x(\kappa, z_0) \\ \tilde{E}_y(\kappa, z_0) \end{pmatrix}, \quad (155)$$

where we skipped first two rows because they simply state identities.

- Equation (155) gives us the answer to the question on how to calculate  $\tilde{\mathbf{H}}_{\perp}$  from  $\tilde{\mathbf{E}}_{\perp}$  in the plane  $z_0$ .
- With  $\tilde{\mathbf{E}}_{\perp}$  we may also calculate  $\tilde{H}_z$  via Eq. (127) which means

$$\tilde{H}_z(\kappa, z_0) = \frac{k_x \tilde{E}_y(\kappa, z_0) - k_y \tilde{E}_x(\kappa, z_0)}{k_0}. \quad (156)$$

## 4.2. Vectorial Field Components in Plane

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- Inserting Eq. (155) into Eq. (128) yields by applying Eq. (133)

$$\tilde{E}_z(\kappa, z_0) = -\frac{k_x \tilde{E}_x(\kappa, z_0) + k_y \tilde{E}_y(\kappa, z_0)}{k_z(\kappa)}. \quad (157)$$

- This result does not come as a surprise since it shows the  $\nabla \cdot \mathbf{E}(\rho, z) = 0$  equation in the k-domain for  $z = z_0$ . This explicitly shows, that Maxwell's div-equations are included in the rot-equations.
- By Eqs. (155), Eq. (156), and Eq. (157) we obtain the missing four components in the k-domain and inverse Fourier transform provides the full field in the space domain and the stated task of this section is fully solved.
- We introduce some compact matrix notation to summarize the dependency of all components from the transversal electric field components.

## 4.2. Vectorial Field Components in Plane

- Hence, we define

$$\tilde{\mathbf{V}}(\kappa, z_0) = \mathbf{P}^{\text{all}}(\kappa) \tilde{\mathbf{E}}_{\perp}(\kappa, z_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -k_x/k_z(\kappa) & -k_y/k_z(\kappa) \\ -A(\kappa)/k_z(\kappa) & -B_x(\kappa)/k_z(\kappa) \\ B_y(\kappa)/k_z(\kappa) & A/k_z(\kappa) \\ -k_y/k_0 & k_x/k_0 \end{pmatrix} \begin{pmatrix} \tilde{E}_x(\kappa, z_0) \\ \tilde{E}_y(\kappa, z_0) \end{pmatrix} \quad (158)$$

with  $\tilde{\mathbf{V}} := (\tilde{E}_x, \tilde{E}_y, \tilde{E}_z, \tilde{H}_x, \tilde{H}_y, \tilde{H}_z)^T$ .

- A closer look into the component matrix  $\mathbf{P}^{\text{all}}(\kappa)$  reveals a problem.

## 4.3. Propagation Between Parallel Planes

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- The division through  $k_z(\kappa) = \sqrt{\check{\epsilon}_r k_0^2 - \kappa^2}$  is not defined for  $k_z = 0$  and can provide huge values for the resulting components for  $k_z \approx 0$ .
- That happens for  $k^2 \approx k_0^2 \Re \check{\epsilon}_r$  in combination with non-vanishing values  $\tilde{E}_\perp(\kappa, z_0)$  for that  $\kappa$  values.
- What is the physical reason for this singularity and how can we deal with it? We leave this to a special discussion in a later section. Here we just mention, that in practice we face this problem not often.

### 4.3 Field propagation between parallel planes

- After discussion of the construction of the electromagnetic field in the plane  $z_0$  from the initial components  $\tilde{E}_\perp(\kappa, z_0)$  we ask for the field for all  $z$  assuming the initial components belong to the field propagating into the positive  $z$ -direction.

## 4.3. Propagation Between Parallel Planes

- The field which propagates into the opposite direction can dealt with fully analogously simply by changing the sign of the terms in the 4th and 5th rows in Eq. (158) according to the second half of the matrix in Eq. (149).
- Let us use the notation from Eq. (158) to formulate the full field  $\tilde{V}$  in k-domain by combining Eq. (153), Eq. (154), Eq. (156), and Eq. (157).
- Then we obtain the compact result

$$\tilde{V}(\kappa, z) = \mathbf{P}^{\text{all}}(\kappa) \tilde{E}_{\perp}(\kappa, z_0) e^{ik_z(\kappa)\Delta z} \quad (159)$$

with  $\Delta z = z - z_0$ .

- This is an impressive result which states: If you know the transversal electric field components in some plane  $z_0$  for the field propagating along the z-axis, then you know all field components for any  $z$  for that field, that means in any other z-plane of interest!

## 4.3. Propagation Between Parallel Planes

- Indeed, also  $\Delta z < 0$  is permitted, but we must understand the limitations of an inverse propagation. It means to go back into the “propagation history” of the field.
- However, we know that for  $\kappa$  with imaginary parts  $k''$  of  $k_z(\kappa)$  propagating means often a fast attenuation of field components by  $e^{-k''_z(\kappa)\Delta z}$ . As a consequence they are lost and can not be reconstructed by inverse propagation in practice.
- Equation (159) can be read in two ways:
- (1) By

$$\tilde{E}_\perp(\kappa, z) = \tilde{E}_\perp(\kappa, z_0) e^{ik_z(\kappa)\Delta z} \quad (160)$$

we may first obtain the transversal electric field components in the plane  $z$ .

## 4.3. Propagation Between Parallel Planes

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- In this plane we then apply the matrix multiplication to calculate the remaining components which we write by

$$\tilde{\mathbf{V}}(\kappa, z) = \mathbf{P}^{\text{all}}(\kappa) \tilde{\mathbf{E}}_{\perp}(\kappa, z). \quad (161)$$

- (2) We evaluate  $\tilde{\mathbf{V}}(\kappa, z_0) = \mathbf{P}^{\text{all}}(\kappa) \tilde{\mathbf{E}}_{\perp}(\kappa, z_0)$  first and then obtain the field in any  $z$  by

$$\tilde{\mathbf{V}}(\kappa, z) = \tilde{\mathbf{V}}(\kappa, z_0) e^{ik_z(\kappa) \Delta z}. \quad (162)$$

- Also this result should come without a surprise. It means, that the six field components, as soon as they are known in  $z_0$ , can be obtained in any  $z$  independently!
- This decoupling of the  $z$ -dependency is typically discussed in the context of Helmholtz' equation. Then it is sometimes connected to scalar modeling and even understood as an approximation.

## 4.3. Propagation Between Parallel Planes

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- From our discussion we understand, that the component decoupling is related to the  $z$ -dependency only. The general mutual component dependency, which is fully expressed by  $\mathbf{P}^{\text{all}}$  of Eq. (158) is not affected by the specific  $z$ -decoupling.
- The difference between understanding (1) and (2) manifests in the implementation of an algorithm to provide access to the field components, though it is a minor difference only.
- In (1) we perform Eq. (160) and can calculate the remaining components in  $z$  on demand via Eq. (161).
- In fact that replaces the initial values from  $z_0$  to  $z$  only and considers then the task discussed in Sec. 4.2.
- In (2) we stay with the storage of the original field components in  $z_0$  and calculate all components on demand.

## 4.3. Propagation Between Parallel Planes

- In VirtualLab Fusion we currently use approach (1), since that fits best to our current understanding and implementation of operator sequences.
- Equations (162) and (160) constitute propagation equations, since they provide the components in plane  $P'$  from the knowledge of the components in an initial plane  $P$ .
- When both planes are parallel and we use the same coordinate system the planes are simply expressed by different  $z$  and we end up with the equations (162) and (160). We express this also in an operator notation by

$$\tilde{E}_\perp(\kappa, z) = (\tilde{\mathcal{P}}_\parallel \tilde{E}_\perp(z_0))(\kappa) \quad (163)$$

with the pointwise operator

$$\tilde{\mathcal{P}}_\parallel : \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}) \rightarrow \mathbb{L}_2^{\mathbb{C}}(\mathbb{R}), \tilde{V} \mapsto (\kappa \mapsto \tilde{V}(\kappa) e^{ik_z(\kappa)\Delta z}). \quad (164)$$

## 4.3. Propagation Between Parallel Planes

- Analogously we have the equation

$$\tilde{\mathbf{V}}(\boldsymbol{\kappa}, z) = (\tilde{\mathcal{P}}_{\parallel} \tilde{\mathbf{V}}(z_0))(\boldsymbol{\kappa}) \quad (165)$$

for the full field vector.

- As discussed in Sec. 2.3 we may also write  $\tilde{\mathbf{V}}(P') = \tilde{\mathcal{P}}_{\parallel} \tilde{\mathbf{V}}(P)$  or  $\tilde{\mathbf{V}}(\boldsymbol{\kappa}') = (\tilde{\mathcal{P}}_{\parallel} \tilde{\mathbf{V}}(P))(\boldsymbol{\kappa}')$ .
- Next we express the full propagation in x-domain by

$$\mathbf{V}(\boldsymbol{\rho}, z) = \mathcal{F}_k^{-1}(\boldsymbol{\kappa} \mapsto \mathbf{P}^{\text{all}}(\boldsymbol{\kappa}) \tilde{\mathbf{E}}_{\perp}(\boldsymbol{\kappa}, z_0) e^{ik_z(\boldsymbol{\kappa})\Delta z})(\boldsymbol{\rho}, z) \quad (166)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \mathbf{P}^{\text{all}}(\boldsymbol{\kappa}) \tilde{\mathbf{E}}_{\perp}(\boldsymbol{\kappa}, z_0) e^{ik_z(\boldsymbol{\kappa})z_0} \right) e^{i\mathbf{k}(\boldsymbol{\kappa}) \cdot \mathbf{r}} dk_x dk_y \quad (167)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \tilde{\mathbf{V}}(\boldsymbol{\kappa}, z_0) e^{ik_z(\boldsymbol{\kappa})z_0} \right) e^{i\mathbf{k}(\boldsymbol{\kappa}) \cdot \mathbf{r}} dk_x dk_y \quad (168)$$

## 4.3. Propagation Between Parallel Planes

using Eq. (74) and, via Eq. (137), with

$$\mathbf{k} := (k_x, k_y, k_z(\kappa)) \wedge k_z(\kappa) = \sqrt{\check{\epsilon}_r k_0^2 - \kappa^2} \quad (169)$$

and  $\mathbf{r} := (x, y, z)$ .

- Per  $\kappa$  the integrand is in form of a electromagnetic plane wave. That is the reason why the expression in Eq. (168) is also referred to as the Angular Spectrum of Plane Waves (SPW) decomposition of the field in x-domain - typically discussed for one component only.
- This is a useful interpretation and our discussions shows, that it follows from a thorough investigation of the solutions of Maxwell's equations in k-domain.
- Because of the decoupling of the components for the propagation as it is explicitly expressed in Eq. (168) we next concentrate for a

## 4.3. Propagation Between Parallel Planes

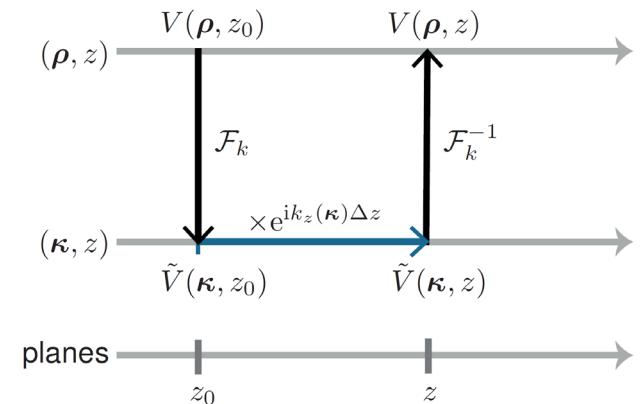
deeper discussion on

$$V(\rho, z) = \mathcal{F}_k^{-1} \left( \kappa \mapsto e^{ik_z(\kappa)\Delta z} (\mathcal{F}_k V(z_0))(\kappa) \right)(\rho, z) \quad (170)$$

$$=: (\mathcal{P}_{\parallel} V(z_0))(\rho, z), \quad (171)$$

which constitutes the basic equation for propagating arbitrary field components between parallel planes in the  $x$ -domain.

- Obviously it includes two Fourier transforms. These Fourier transforms are in general integral operators but, according to our discussion in Sec. 3.3 and dependent on the wavefront phases in both domains for a specific situation, the pointwise Fourier transform might be a very good approximation as well for one or both of them.
- This understanding has far-reaching consequences for fast physical-optics modeling. In the next section 4.4 we will study this in more detail.



# 4.4. Diffraction Integrals and Ray Tracing

## 4.4 Diffraction integrals and ray tracing

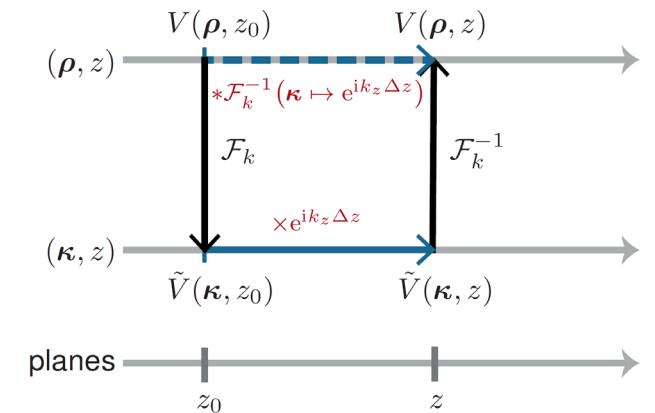
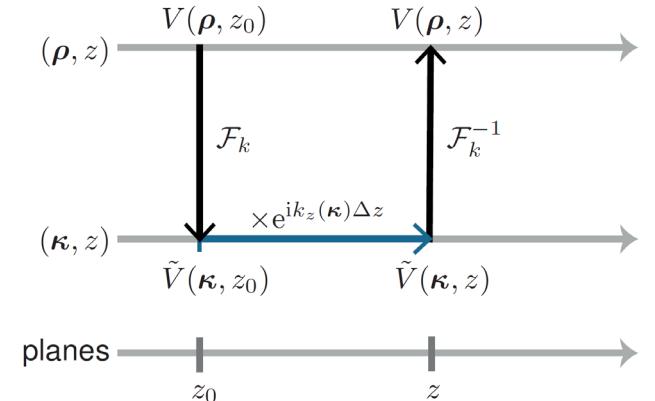
### 4.4.1 Rayleigh integrals

- Equation (171) is illustrated in Fig. 4 in a field tracing diagram.
- According to the convolution theorem a product in the k-domain results in a convolution in the x-domain. This is illustrated in Fig. 5.

- As a consequence we can reformulate Eq. (171) and obtain

$$V(\rho, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\rho', z_0) \left( \mathcal{F}_k^{-1}(\kappa \mapsto e^{ik_z(\kappa)\Delta z})(\rho - \rho') \right) dx' dy'. \quad (172)$$

- For further evaluation we need to derive an expression for



## 4.4.1 Rayleigh Integrals

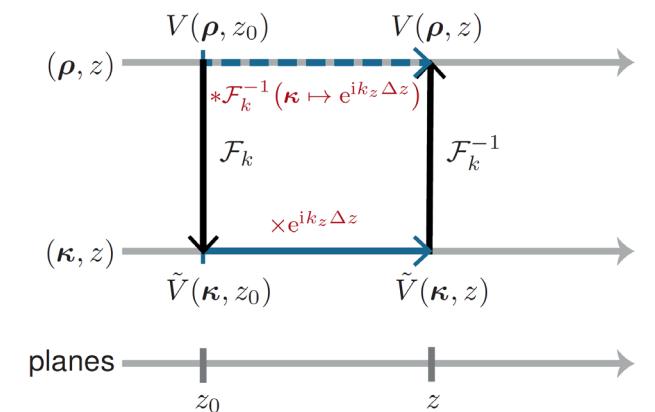
$\mathcal{F}_k^{-1}(\kappa \mapsto e^{ik_z(\kappa)\Delta z})$ . We know (see Introduction lecture Eq. (1.78))

$$\frac{e^{ikr}}{r} = \mathcal{F}_k^{-1} \left( \kappa \mapsto i \frac{e^{ik_z(\kappa)z}}{k_z(\kappa)} \right) (\rho, z) \quad (173)$$

- This is close to what we look for, but we need to deal with the denominator.
- To this end both sides are differentiated with respect to  $z$  and we obtain by utilizing the independence of  $\mathcal{F}_k$  from  $z$

$$\frac{\partial}{\partial z} \frac{e^{ikr}}{r} = -\mathcal{F}_k^{-1} \left( \kappa \mapsto e^{ik_z(\kappa)z} \right) (\rho, z), \quad (174)$$

which is the expression we look for in Eq. (172) when we replace  $z$  by  $\Delta z$  which also means replacing  $r = (x, y, z)$  by  $r = (x, y, \Delta z)$  and  $r = \|r\|$ .



## 4.4.1 Rayleigh Integrals

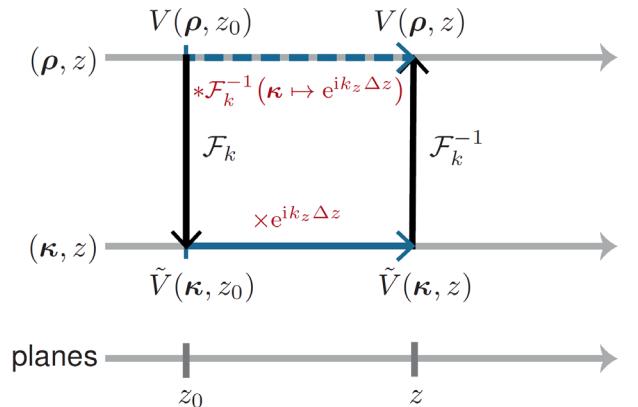
- As a result we obtain

$$V(\rho, z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\rho', z_0) \frac{\partial}{\partial z} \frac{e^{ikR}}{R} dx' dy', \quad (175)$$

with  $k = k_0\sqrt{\epsilon_r}$  and  $\mathbf{R} = \mathbf{r} - \mathbf{\rho}'$  and

$$R = \|\mathbf{R}\| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z_0)^2}.$$

- This integral is called Rayleigh diffraction or propagation integral of first kind. We may say according to Eq. (175) that at each point  $\rho$  a wave with function  $\partial_z e^{ikr}/r$  emerges with amplitude  $V(\rho, z_0)$ .



## 4.4.1 Rayleigh Integrals

- Let us have a closer look into this special term by applying the chain rule and we obtain

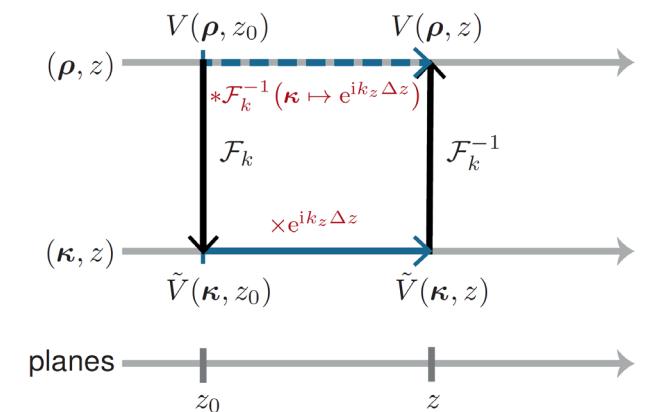
$$\frac{\partial}{\partial z} \frac{e^{ikr}}{r} = \frac{e^{ikr}}{r} \frac{z}{r} \left( ik - \frac{1}{r} \right) \quad (176)$$

$$= \frac{e^{ikr}}{r} \cos \theta \left( ik - \frac{1}{r} \right) \quad (177)$$

$$\approx ik \cos \theta \frac{e^{ikr}}{r} \text{ if } r \gg \lambda , \quad (178)$$

whereby in the last line we assumed a propagation distance larger than a few wavelengths.

- The cosine term is called inclination factor.
- We see that in Eq. (178) the convolution kernel is mainly what we understand as a scalar spherical wave. In other words, Eq. (175) states the Huygens' principle as a conclusion from Maxwell's equations!

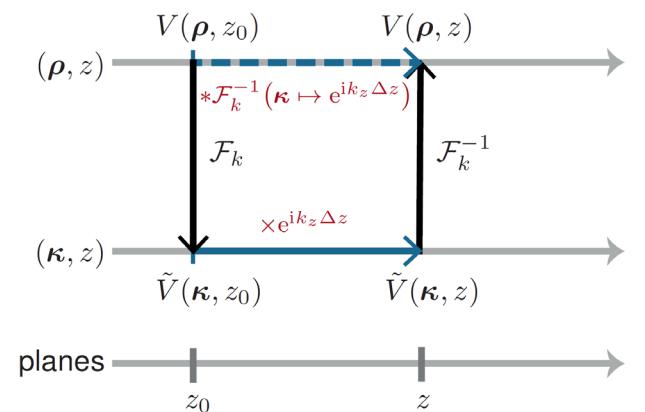


## 4.4.1 Rayleigh Integrals

- According to Eq. (164) we can apply Eq. (175) to all components, when we have calculated the matrix multiplication in Eq. (158) before to obtain  $\mathbf{V}(\boldsymbol{\kappa}, z_0)$ .
- Nevertheless, we like to show at the example of  $E_z$ , that we can also formulate an integral, which uses  $E_x$  and  $E_y$  in the space domain.
- Here we proceed analogously to Eq. (172), but now we need to perform the convolution according to the inverse Fourier transform of Eq. (157) with  $z$  instead of  $z_0$ .
- That means we need to know the inverse Fourier transforms

$$\mathcal{F}_k^{-1}\left(\boldsymbol{\kappa} \mapsto k_x \frac{e^{ik_z(\boldsymbol{\kappa})\Delta z}}{k_z(\boldsymbol{\kappa})}\right)(\rho) \quad (179)$$

$$\mathcal{F}_k^{-1}\left(\boldsymbol{\kappa} \mapsto k_y \frac{e^{ik_z(\boldsymbol{\kappa})\Delta z}}{k_z(\boldsymbol{\kappa})}\right)(\rho). \quad (180)$$



## 4.4.1 Rayleigh Integrals

- We start again from Eq. (173) and obtain by differentiating the Fourier kernel with respect to  $x$

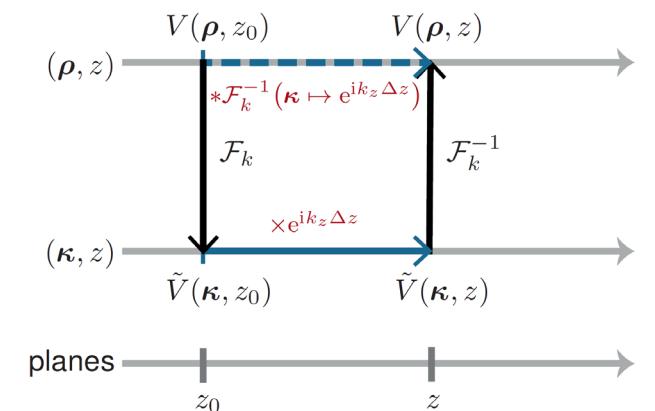
$$\frac{\partial}{\partial x} \frac{e^{ikr}}{r} = \mathcal{F}_k^{-1} \left( \boldsymbol{\kappa} \mapsto i \frac{e^{ik_z(\boldsymbol{\kappa})z}}{k_z(\boldsymbol{\kappa})} \right) (\boldsymbol{\rho}, z) \quad (181)$$

$$= -\mathcal{F}_k^{-1} \left( \boldsymbol{\kappa} \mapsto k_x \frac{e^{ik_z(\boldsymbol{\kappa})z}}{k_z(\boldsymbol{\kappa})} \right) (\boldsymbol{\rho}). \quad (182)$$

- That is exactly what we look for and it is valid for  $y$  as well. Combining it all leads to

$$E_z(\boldsymbol{\rho}, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( E_x(\boldsymbol{\rho}', z_0) \frac{\partial}{\partial x} \frac{e^{ikR}}{R} + E_y(\boldsymbol{\rho}', z_0) \frac{\partial}{\partial y} \frac{e^{ikR}}{R} \right) dx' dy'. \quad (183)$$

- The derivatives look identical to Eqs. (176) to (178) when we replace  $z/r$  by  $x/r$  or  $y/r$ .



## 4.4.1 Rayleigh Integrals

- Next we like to discuss a second type of Rayleigh integral, by replacing the field component  $V(\rho, z_0)$  by its derivative with respect to  $z$ .
- To this end we use the component representation in Eq. (171). Let us differentiate both sides with respect to  $z$ .

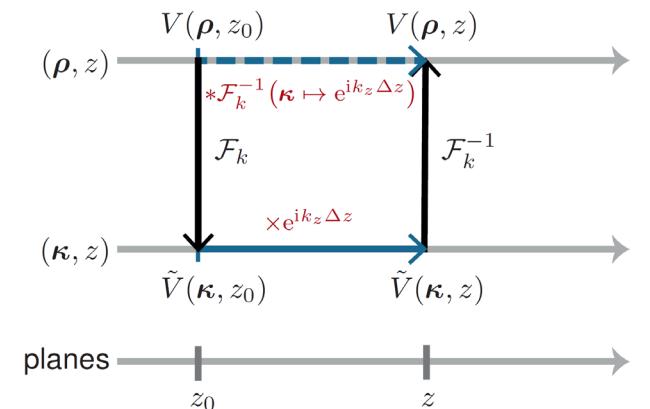
- Then we obtain

$$\frac{\partial}{\partial z} V(\rho, z) = \mathcal{F}_k^{-1} \left( \kappa \mapsto i k_z(\kappa) e^{ik_z(\kappa)\Delta z} \tilde{V}(\kappa, z_0) \right)(\rho, z). \quad (184)$$

- Selecting  $z = z_0$ , performing the Fourier transform on both sides and resolving for  $\tilde{V}(\kappa, z_0)$  results in

$$\tilde{V}(\kappa, z_0) = -i \frac{1}{k_z(\kappa)} \mathcal{F}_k \left( \rho \mapsto (\partial_z V(\rho, z))_{z_0} \right), \quad (185)$$

with  $(\partial_z V(\rho, z))_{z_0}$  indicates the  $z$  derivative at  $z = z_0$ .



## 4.4.1 Rayleigh Integrals

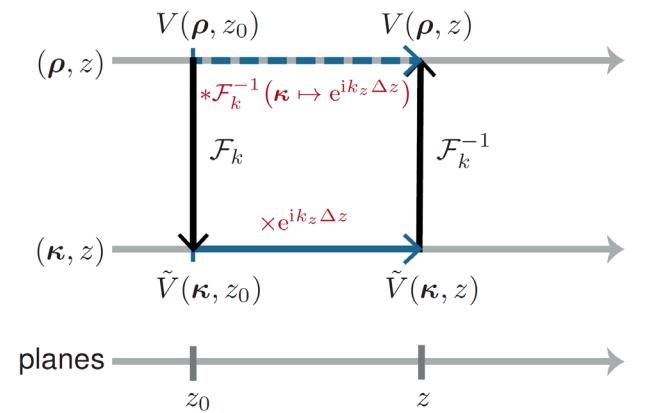
- Inserting this result into Eq. (171) yields the convolution integral

$$V(\rho, z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\partial_z V(\rho, z))_{z_0} \left( \mathcal{F}_k^{-1} \left( \kappa \mapsto i \frac{e^{ik_z(\kappa)\Delta z}}{k_z(\kappa)} \right) (\rho - \rho') \right) dx' dy'. \quad (186)$$

- The Fourier transform of the term in parentheses is known from Eq. (173) and we finally obtain Rayleigh's integral of second type

$$V(\rho, z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\partial_z V(\rho, z))_{z_0} \frac{e^{ikR}}{R} dx' dy'. \quad (187)$$

- Comparison with Eq. (175) shows, that the z-derivative has changed its position from convolution kernel to field component.
- Equation (175) is remarkable. It shows, that we can calculate the z-derivative of a field component via its Fourier transform and vice versa.



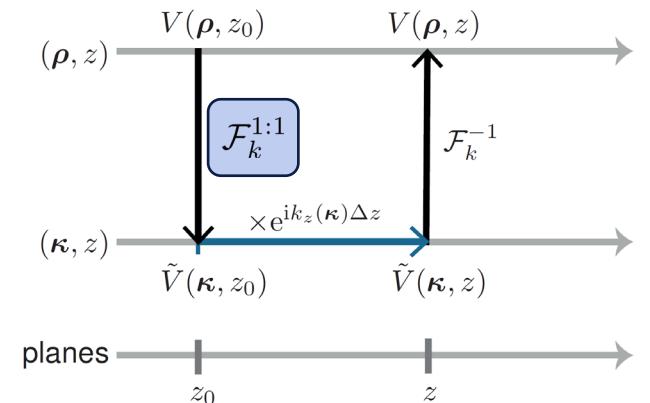
## 4.4.2 Debye Integral

### 4.4.2 Debye integral

- Next we like to consider an assumption about the first Fourier transform in Eq. (171), namely a pointwise FT.
- Then we write with Eq. (103)

$$\begin{aligned} V(\rho, z) &= \mathcal{F}_k^{-1} \left( \kappa \mapsto e^{ik_z(\kappa)\Delta z} (\mathcal{F}_k^{1:1} V(z_0))(\kappa) \right)(\rho, z) \\ &= -i \mathcal{F}_k^{-1} \left( \kappa \mapsto e^{ik_z(\kappa)\Delta z} \times \right. \\ &\quad \left. \times \left( V(\rho, z_0) \exp(-i\nabla\psi(\rho, z_0) \cdot \rho) |\mathbf{J}_\kappa(\rho, z_0)|^{-1/2} \right) [\rho \leftarrow \rho(\kappa)] \right). \end{aligned} \quad (188)$$

- We assumed  $\sigma(\rho) = -i$  since the situation is typically related to convergent fields (see Table 3).
- The most typical application scenario is a convergent field behind a lens system, e.g. in the exit pupil, which should be propagated into the focal region.

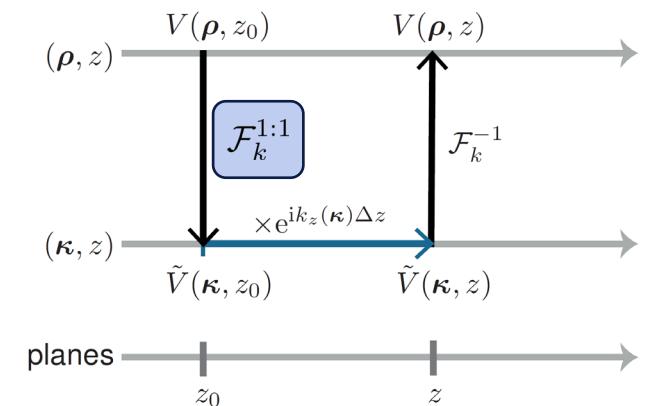


## 4.4.2 Debye Integral

- Then, the wavefront behind the lens is often strong enough to approximate the Fourier transform by a PFT.
- In a practical situation that must be tested of course. If the lens system has a low numerical aperture (NA), for example in case of some laser beam lens systems, the assumption might be not justified and Eq. (171) is to be used with rigorous FT's.
- Equation (188) in general requires a numerical evaluation for PFT and inverse FT.
- However, there is a situation of special concern, namely a spherical wavefront phase  $\psi^{\text{sph}}$  of Eq. (90) with negative  $R$  which means

$$\psi^{\text{sph}}(\rho, z_0) := -k_0 n \sqrt{x^2 + y^2 + R^2} = -k_0 n r(\rho). \quad (189)$$

- In lens systems we ideally like to obtain a spherical wavefront with a few aberrations only. These aberrations are to be assumed in the phase of the  $U$ -field.



## 4.4.2 Debye Integral

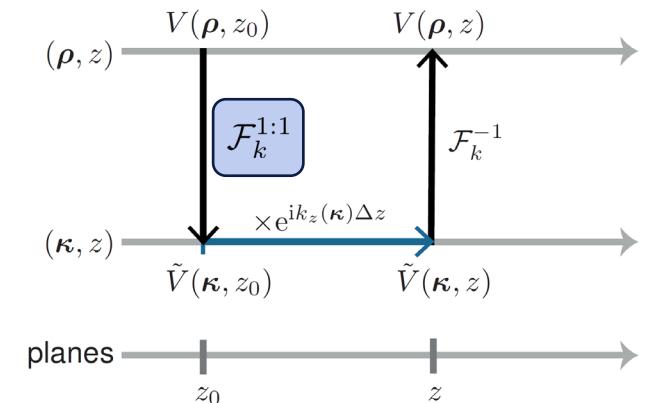
- That means we decompose the field component according to  $V(\rho, z_0)$   
 $U(\rho, z_0)e^{i\psi^{\text{sph}}(\rho, z_0)}$ .
- For a spherical wavefront phase we know the PFT from Eq. (115) and Eq. (188) changes to

$$\begin{aligned} V(\rho, z) &= \mathcal{F}_k^{-1} \left( \kappa \mapsto e^{ik_z(\kappa)\Delta z} (\mathcal{F}_k^{1:1} V(z_0))(\kappa) \right)(\rho, z) \\ &= i \mathcal{F}_k^{-1} \left[ \kappa \mapsto \frac{k_0 n R}{k_z^2(\kappa)} e^{ik_z(\kappa)(R+\Delta z)} \left( U(\rho, z_0) [\rho \leftarrow R \frac{\kappa}{k_z(\kappa)}] \right) \right](\rho, z) \end{aligned} \quad (190)$$

- With the interpretation of a convergent, aberrant spherical wave it is consequent to reformulate  $U$  by

$$U(\rho, z_0) = \frac{t(\rho, z_0)}{r} \quad (191)$$

with  $t(\rho)$  includes the aberrations and apodization effects.



## 4.4.2 Debye Integral

- With Eq. (112) we can directly perform the substitution and obtain

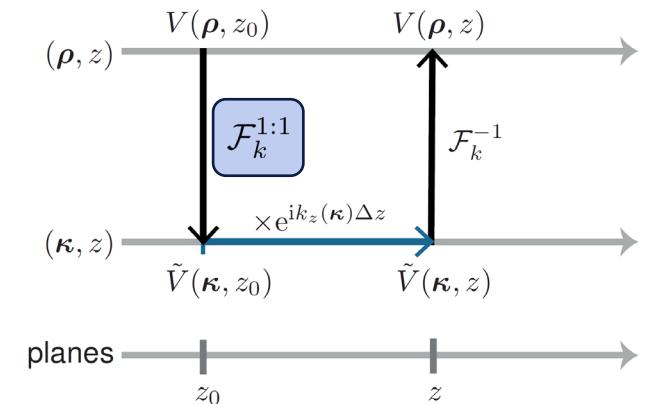
$$U(\rho, z_0)[\rho \leftarrow R \frac{\kappa}{k_z(\kappa)}] = \frac{k_z}{k_0 n |R|} \left( t(\rho, z_0)[\rho \leftarrow R \frac{\kappa}{k_z(\kappa)}] \right). \quad (192)$$

- Plugging it into Eq. (190) and assuming  $R < 0$  reveals the Debye integral

$$V(\rho, z) = -i \mathcal{F}_k^{-1} \left[ \kappa \mapsto \frac{1}{k_z(\kappa)} \left( t(\rho, z_0)[\rho \leftarrow R \frac{\kappa}{k_z(\kappa)}] \right) e^{ik_z(\kappa)(\Delta z - |R|)} \right] (\rho, z) \quad (193)$$

$$= -\frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\check{t}(\kappa)}{k_z(\kappa)} e^{ik_z(\kappa)(\Delta z - |R|)} e^{i\kappa \cdot \rho} dk_x dk_y \quad (194)$$

with Eq. (50) for changing function notation after substitution.

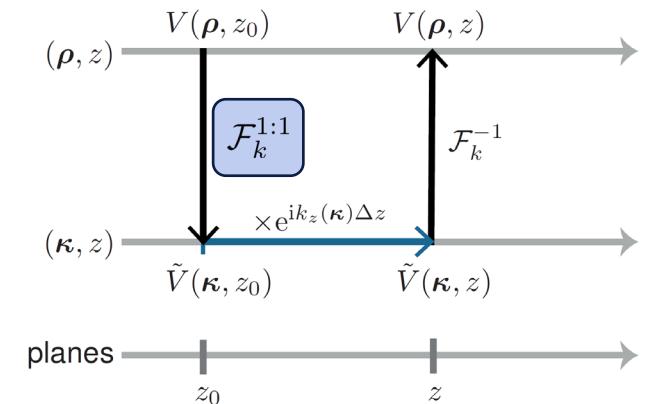


## 4.4.2 Debye Integral

- In the focal plane, that is  $\Delta z = |R|$ , we simply get

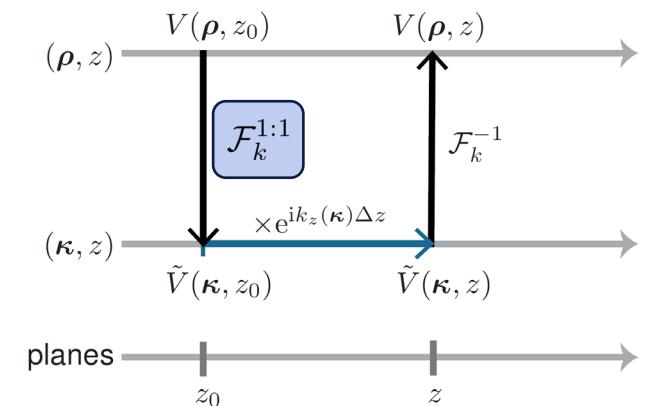
$$V(\rho, z_0 + |R|) = -i \mathcal{F}_k^{-1} \left( \kappa \mapsto \frac{\check{t}(\kappa)}{k_z(\kappa)} \right) (\rho, z_0 + |R|) \quad (195)$$

- That is the inverse Fourier transform of the exit pupil function  $t$  which includes aberrations via  $\arg(t)$ , the stop, apodization and vignette effects via  $|t|$  after coordinate transformation and division through  $k_z$ .
- The effects of the substitution deforms for significantly non-paraxial situations only, since otherwise  $k_z \approx k_0 n$  and besides scaling and constants the Debye integral results in a Fourier transform of the exit pupil function.
- To assume the focal spot is mainly just the Fourier transform of the exit pupil function is a popular simplification of the correct result.



## 4.4.2 Debye Integral

- Let us also keep in mind, that the Debye integral is an approximation itself!
- However, it is a fully vectorial solution, since we just replace the first Fourier transform by a PFT and that can also be done in the full vectorial solution of Eq. (166).
- In VirtualLab Fusion the first Fourier transform is replaced by a PFT when the numerical accuracy is sufficient. This is a pure numerical decision.
- In this case it leads to the propagation technique according to Eq. (188), since in general we know the wavefront phase together with aberrations.
- The Debye integral can be enforced in the Fourier transform settings by restricting the PFT to use the spherical part of the wavefront only.



## 4.4.3 Far-Field Integral

### 4.4.3 Far-field integral

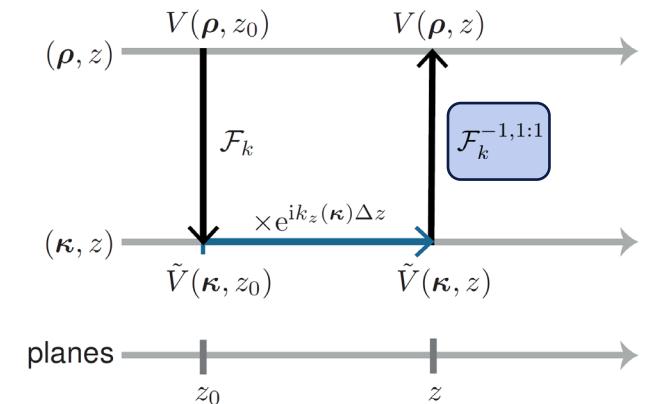
- Next we discuss the case, in which the inverse Fourier transform in Eq. (171) is replaced by a pointwise Fourier transform.
- To this end we define via Eq. (81)

$$\tilde{\phi}(\kappa, \Delta z) := \tilde{\phi}(\kappa, z_0) + k_z(\kappa) \Delta z. \quad (196)$$

and obtain via Eq. (106)

$$V(\rho, z) = i \left( \tilde{V}(\kappa, z_0) \exp \left( -i \nabla_k \tilde{\phi}(\kappa, \Delta z) \cdot \kappa \right) |\mathbf{J}_{\rho, z_0}(\kappa)|^{-1/2} \right) [\kappa \leftarrow \kappa(\rho)]. \quad (197)$$

with  $\tilde{\sigma}(\kappa) = i$  according to Table 3) since we assume here divergent fields, which are propagated far enough to develop a strong divergent wavefront phase  $\tilde{\phi}(\kappa, \Delta z)$ .



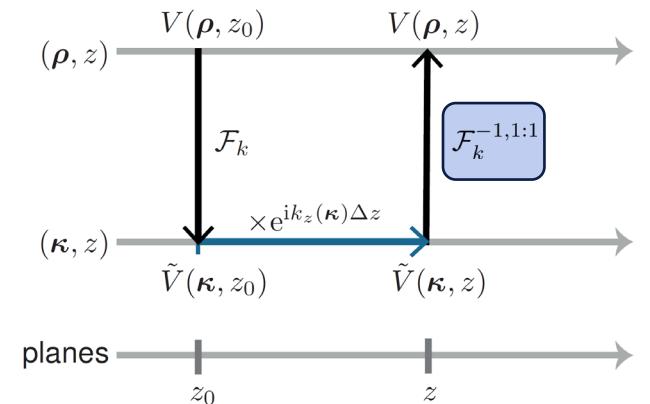
### 4.4.3 Far-Field Integral

- Equation (197) expresses the fact, that in a far field we obtain the field as Fourier transform of the input field and additional weight functions combined with a subsequent coordinate transformation.
- Like in Sec. 4.4.2 the discussion of a spherical wavefront phase provides us with a more compact result, because we know the PFT analytically.
- To this end we define

$$\tilde{\phi}^{\text{sph}}(\kappa, z) := k_z(\kappa)z_0 + k_z(\kappa)\Delta z = k_z(\kappa)z, \quad (198)$$

that means we include the spherical phase part which might come with the field in the plane  $z_0$ .

- That implies, that  $\tilde{A}(\kappa, z_0) = \tilde{A}(\kappa, 0) =: \tilde{A}(\kappa)$ , since propagation in  $k$ -domain only provides the additional  $k_z(\kappa)\Delta z$ -phase.



### 4.4.3 Far-Field Integral

- The inverse Fourier transform for the wavefront phase of Eq. (198) is known from Eq. (117) and we obtain directly

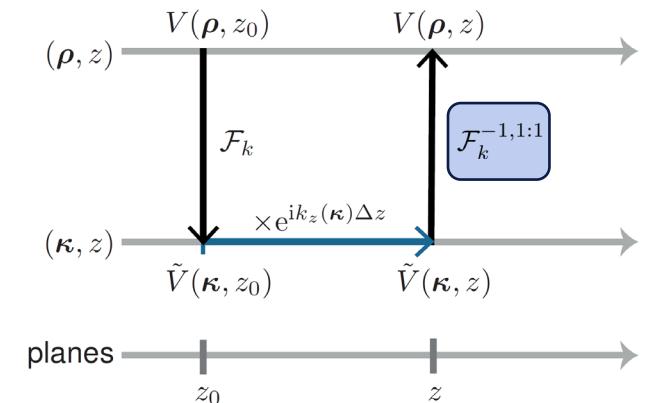
$$V(\rho, z) = -i \frac{k_0 n z \exp(ik_0 n r)}{r} \left( \tilde{A}(\kappa) [\kappa \leftarrow k_0 n \frac{\rho}{r}] \right) \quad (199)$$

$$= -i k_0 n \cos \theta \tilde{A}(k_0 n \hat{s}_\perp) \frac{\exp(ik_0 n r)}{r} \quad (200)$$

$$= V(r \hat{s}). \quad (201)$$

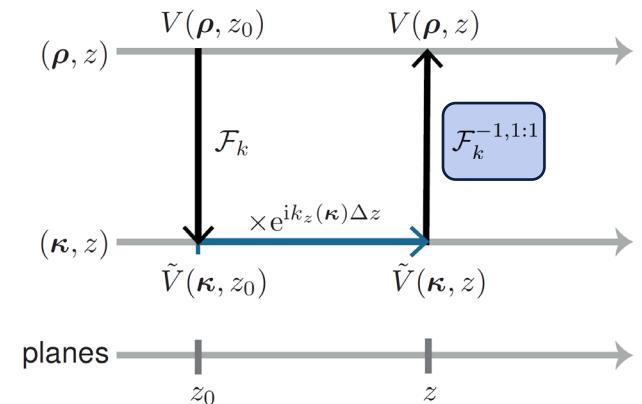
with  $r = \sqrt{x^2 + y^2 + z^2}$  and  $\tilde{A}(\kappa)$  the Fourier transform of the incident field component with extracted spherical wavefront phase, which provides  $z_0$  for Eq. (198).

- In Eq. (200) and Eq. (201) we used  $\hat{s} = (\hat{s}_\perp, \hat{s}_z) = \mathbf{r}/r$  and  $\hat{s}_z = \cos \theta$ .



### 4.4.3 Far-Field Integral

- According to his result, which is referred to as far field integral, a field component in a far field constitutes a spherical wave component which is modulated by the Fourier transform in the input field and weighted by the inclination factor  $\cos \theta$ .
- As in case of the Debye integral the solution of the far field integral can be applied to all components according to the full vectorial equation Eq. (166).
- In VirtualLab Fusion the second Fourier transform is replaced by a PFT on numerical accuracy reasons only.
- If accurate enough, the inverse PFT leads to the propagation technique according to Eq. (197), since in general we know the wavefront phase together with aberrations also in the far field.
- The classical far field integral can be enforced in the Fourier transform settings by restricting the inverse PFT to use the spherical part of the wavefront only.



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## References

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