MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2016-01-15 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Solve the initial-value problem $\begin{cases} y' = xy^2\cos(x), \\ y(0) = \frac{1}{3}. \end{cases}$

2. Sketch the graph of the function f, defined by

$$f(x) = \frac{x^2+4x+6}{x^2-4},$$

by utilizing the guidance given by asymptotes and stationary points.

3. Evaluate the generalized integral

$$\int_1^\infty \frac{2\,dx}{x^3+x},$$

and write the result in as simple form as possible.

4. Find the area of the surface generated by rotating the curve

$$y = 2\sqrt{x}, \quad 3 \leq x \leq 8,$$

about the x-axis.

5. Is the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1+2\sqrt{n}}$$

absolutely convergent, conditionally convergent or divergent?

NOTE: Do not forget that an answer must be accompanied by a relevant justification.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA	**TENTAMEN I MATEMATIK**
Akademin för utbildning, kultur och kommunikation	MAA151 Envariabelkalkyl, TEN2
Avdelningen för tillämpad matematik	*Datum:* 2016-01-15 *Skrivtid:* 3 timmar
Examinator: Lars-Göran Larsson	*Hjälpmedel:* Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Lös begynnelsevärdesproblemet $\begin{cases} y' = xy^2\cos(x), \\ y(0) = \frac{1}{3}. \end{cases}$

2. Skissa grafen till funktionen f, definierad enligt

$$f(x) = \frac{x^2 + 4x + 6}{x^2 - 4},$$

genom att använda den vägledning som fås från asymptoter och stationära punkter.

3. Beräkna den generaliserade integralen

$$\int_1^\infty \frac{2\,dx}{x^3 + x},$$

och skriv resultatet på en så enkel form som möjligt.

4. Bestäm arean av den yta som genereras genom att rotera kurvan

$$y = 2\sqrt{x}, \quad 3 \leq x \leq 8,$$

kring x-axeln.

5. Är serien

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1 + 2\sqrt{n}}$$

absolut konvergent, betingat konvergent eller divergent?

NOTERA: Glöm inte att ett svar måste åtföljas av en relevant motivering.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2016-01-15

1 DE: $y' = xy^2\cos(x)$, IV: $y(0) = \frac{1}{3}$

The DE is separable and may (for $y>0$) be written as $\frac{1}{y^2}\frac{dy}{dx} = x\cos(x)$. Taking $\int dx$ on both sides gives

$$-\frac{1}{y} = \int dx\; x\cos(x) = x\cdot\sin(x) - \int dx\; 1\cdot\sin(x)$$
$$= x\sin(x) + \cos(x) + C$$

where the IV gives $-\frac{1}{1/3} = 0 + 1 + C \Leftrightarrow C = -4$

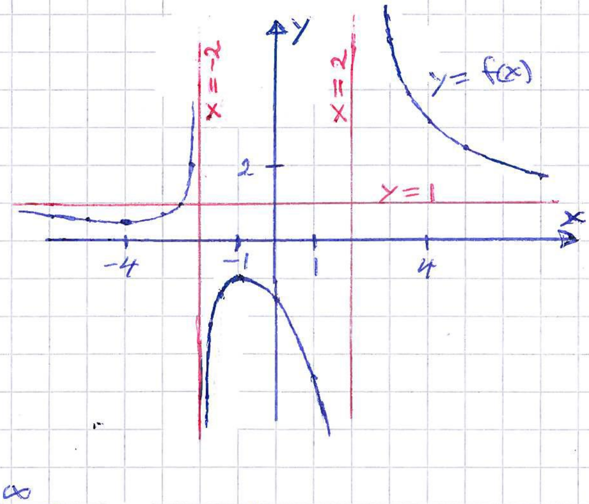
Thus $y = \dfrac{-1}{x\sin(x)+\cos(x)-4}$ i.e. $y = \dfrac{1}{4-\cos(x)-x\sin(x)}$

2 $f(x) = \dfrac{x^2+4x+6}{x^2-4}$ Differentiation gives:

$$f'(x) = \frac{(2x+4)(x^2-4)-(x^2+4x+6)2x}{(x^2-4)^2} = \frac{\cancel{2x^3}+4x^2-8x-16-\cancel{2x^3}-8x^2-12x}{[(x+2)(x-2)]^2}$$
$$= \frac{-4x^2-20x-16}{(x+2)^2(x-2)^2} = -4\,\frac{x^2+5x+4}{(x+2)^2(x-2)^2} = -4\,\frac{(x+4)(x+1)}{(x+2)^2(x-2)^2}$$

A first derivative test $\quad f(-4) = \frac{1}{2},\ f(-1) = -1$

		SP		Asym.		SP		Asym.	
x		-4		-2		-1		2	
$f'(x)$	$-$	0	$+$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
$f(x)$	$\searrow$	loc. min	$\nearrow$	$\nexists$	$\nearrow$	loc. max	$\searrow$	$\nexists$	$\searrow$



$x=-2$ and $x=2$ are two-sided vertical asymptotes of the curve $y=f(x)$ (since $f(x) \xrightarrow{|x|\to 2^{\mp}} \mp\infty$).

$y=1$ is a two-sided non-vertical asymptote (since $f(x) \xrightarrow{x\to\pm\infty} 1$).

3
$$\int_1^\infty \frac{2\,dx}{x^3+x} = \int_1^\infty \frac{2\,dx}{x(x^2+1)} = \int_1^\infty \left(\frac{2}{x} - \frac{2x}{x^2+1}\right)dx$$

decomposition into partial fractions

$$= \lim_{A\to+\infty}\Big[2\ln|x| - \ln(x^2+1)\Big]_1^A = \lim_{A\to+\infty}\left[\ln\left(\frac{x^2}{x^2+1}\right)\right]_1^A$$

ln is continuous

$$= \lim_{A\to+\infty}\left[\ln\left(\frac{A^2}{A^2+1}\right) - \ln\left(\frac{1}{1+1}\right)\right] = \ln\left(\lim_{A\to+\infty}\left(\frac{A^2}{A^2+1}\right)\right) - \ln\left(\frac{1}{2}\right)$$

$$= \ln(1) - (-\ln(2)) = 0 + \ln(2) = \ln(2)$$

Cont. MAA151
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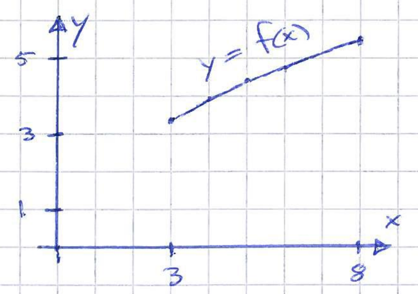
(4)



$$\begin{cases} f(x) = 2\sqrt{x}, & 3 \le x \le 8 \\ f'(x) = \frac{1}{\sqrt{x}} \end{cases}$$

The area A_x of the surface generated by rotating, about the x-axis, the curve $y = 2\sqrt{x}$, $3 \le x \le 8$, is

$$A_x = \int_3^8 2\pi |f(x)| \cdot \sqrt{1 + (f'(x))^2}\, dx = 2\pi \int_3^8 2\sqrt{x}\sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx$$

$$= 4\pi \int_3^8 \sqrt{x+1}\, dx = 4\pi \left[\frac{2}{3}(x+1)^{3/2}\right]_3^8 = \frac{8\pi}{3}\left(9^{3/2} - 4^{3/2}\right)$$

$$= \frac{8\pi}{3}(9 \cdot 3 - 4 \cdot 2) = \frac{8\pi}{3} \cdot 19 = \frac{152\pi}{3}$$ Answer: $\frac{152\pi}{3}$ a.u.

(5) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1 + 2\sqrt{n}}$ where $\cos(n\pi) = (-1)^n$

Let $\frac{(-1)^n}{1+2\sqrt{n}} = a_n$ Then $|a_n| = \frac{1}{1+2\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\frac{1}{2 + \frac{1}{\sqrt{n}}}\right)$

a) Since $\lim_{n\to\infty} \frac{|a_n|}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{2 + \frac{1}{\sqrt{n}}} = \frac{1}{2+0} = \frac{1}{2} > 0$

and since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent according to the integral criteria, we get according to the comparison test that the series $\sum_{n=1}^{\infty} |a_n|$ is divergent, i.e. the series $\sum_{n=1}^{\infty} a_n$ is definitely not abs. convergent.

b) Furthermore, since
- I. $a_n \cdot a_{n+1} < 0$ for $n \ge 1$, the terms are alternating in sign.
- II. $\frac{1}{1+2\sqrt{n+1}} \le \frac{1}{1+2\sqrt{n}}$ for $n \ge 1$, i.e. the number sequence $\{|a_n|\}$ is non-increasing.
- III. $\lim_{n\to\infty} \frac{(-1)^n}{1+2\sqrt{n}} = 0$

the Leibniz' criteria says that the series $\sum_{n=1}^{\infty} a_n$ is convergent. To summarize, a) and b) give that the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{1+2\sqrt{n}}$ is conditionally convergent.

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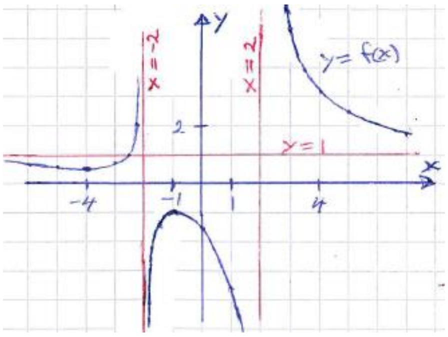
EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

Examination TEN2 – 2016-01-15 Maximum points for subparts of the problems in the final examination

1. $y = \dfrac{1}{4 - \cos(x) - x\sin(x)}$

1p: Correctly identified the differential equation as nonlinear and separable, and correctly found the antiderivative of $y^{-2}y'$
1p: Correctly found the antiderivative of the other side of the separated differential equation, i.e. of $x\cos(x)$
1p: Correctly adapted the equation to the initial value
1p: Correctly solved for y

2. The graph has the two-sided asymptotes $x = -2$, $x = 2$ and $y = 1$, has a local minimum at $(-4, \frac{1}{2})$ and has a local maximum at $(-1, -1)$



1p: Correctly found the asymptotes of the graph
1p: Correctly classified the local extreme points of the graph
1p: Correctly sketched the graph according to how the graph relates to the asymptotes on their both sides respectively
1p: Correctly completed the sketch of the graph

3. $\ln(2)$

------------------------------------ **One scenario** ------------------------------------
1p: Correctly decomposed the integrand into partial fractions
1p: Correctly found an antiderivative of the integrand
2p: Correctly evaluated the found antiderivative at the limits
-------------------------------- **Another scenario** --------------------------------
(1+1)p: Correctly, by the substitution $x = \tan(\theta)$, translated the integrand into $2\cot(\theta)$ (**1p**) and interval into $[\pi/4, \pi/2]$ (**1p**)
(1+1)p: Correctly found an antiderivative of the new integrand (**1p**) and finally determined the value of the integral (**1p**)

4. $\dfrac{152\pi}{3}$ a.u.

1p: Correctly formulated an explicit integral expression for the area of the surface generated by the curve rotated about the *x*-axis
1p: Correctly merged the two *x*-dependent factors of the integrand into the factor $\sqrt{1+x}$
1p: Correctly found an antiderivative of the integrand
1p: Correctly, at the limits, evaluated the found antiderivative

5. The series is conditionally convergent

2p: Correctly found that the series satisfies the three conditions for being convergent according to Leibniz' criteria
1p: Correctly identified that the absolute value of the terms of the series are equal to $n^{-1/2}B(n)$, where $B(n) \to 1/2$ if $n \to \infty$, and from this correctly by the integral criteria and the comparison test found that the series **is not** absolutely convergent
1p: Correctly concluded that the series is conditionally conv.