MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
*Examiner:* Lars-Göran Larsson

**EXAMINATION IN MATHEMATICS**
MAA151 Single Variable Calculus, TEN2
*Date:* 2016-08-19 *Write time:* 3 hours
*Aid:* Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted $S_2$, and that obtained at examination TEN1 $S_1$, the mark for a completed course is according to the following:

| | | | | |
|---|---|---|---|---|
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $S_1 + 2S_2 \leq 41$ | $\rightarrow$ | 3 |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $42 \leq S_1 + 2S_2 \leq 53$ | $\rightarrow$ | 4 |
| | | $54 \leq S_1 + 2S_2$ | $\rightarrow$ | 5 |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $S_1 + 2S_2 \leq 32$ | $\rightarrow$ | E |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $33 \leq S_1 + 2S_2 \leq 41$ | $\rightarrow$ | D |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $42 \leq S_1 + 2S_2 \leq 51$ | $\rightarrow$ | C |
| | | $52 \leq S_1 + 2S_2 \leq 60$ | $\rightarrow$ | B |
| | | $61 \leq S_1 + 2S_2$ | $\rightarrow$ | A |

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

**1.** Find the solution of the initial-value problem

$$y'' + 4y = 5e^{-4x}, \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

**2.** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{7^n + 8^n}{3^{2n}}$$

converges or not. Irrespective whether the answer is YES or NO, give an explanation of why!

**3.** Find the area of the surface generated by rotating about the $y$-axis the triangle which has its vertices at the points $P : (1,1)$, $Q : (2,2)$ and $R : (1,2)$.

**4.** Evaluate the integral

$$\int_1^e \sqrt{x}\, \ln^2(x)\, dx\,,$$

and write the result in as simple form as possible.

**5.** Sketch the graph of the function $f$, defined by

$$f(x) = \frac{1}{x+1} + \arctan(x)\,,$$

by utilizing the guidance given by asymptotes and stationary points.

**Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.**

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns $S_2$, och den vid tentamen TEN1 erhållna $S_1$, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

| | | | | |
|---|---|---|---|---|
| $S_1 \geq 11,\ S_2 \geq 9$ | OCH | $S_1 + 2S_2 \leq 41$ | $\rightarrow$ | 3 |
| $S_1 \geq 11,\ S_2 \geq 9$ | OCH | $42 \leq S_1 + 2S_2 \leq 53$ | $\rightarrow$ | 4 |
| | | $54 \leq S_1 + 2S_2$ | $\rightarrow$ | 5 |

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

**1.** Bestäm lösningen till begynnelsevärdesproblemet

$$y'' + 4y = 5e^{-4x}, \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

**2.** Avgör om serien

$$\sum_{n=1}^{\infty} \frac{7^n + 8^n}{3^{2n}}$$

är konvergent eller ej. Oavsett om svaret är JA eller NEJ, ge en förklaring till varför!

**3.** Bestäm arean av den yta som genereras genom att kring $y$-axeln rotera den triangel som har sina hörn i punkterna $P : (1,1)$, $Q : (2,2)$ och $R : (1,2)$.

**4.** Beräkna integralen

$$\int_1^e \sqrt{x}\, \ln^2(x)\, dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

**5.** Skissa grafen till funktionen $f$, definierad enligt

$$f(x) = \frac{1}{x+1} + \arctan(x)\,,$$

genom att använda den vägledning som fås från asymptoter och stationära punkter.

**If you prefer the problems formulated in English, please turn the page.**

# MAA151 / Solutions to the final exam TEN2 2016-08-19

**1** DE: $y''+4y=5e^{-4x}$ IV: $y(0)=1$, $y'(0)=0$

The general solution of the DE is $y=y_H+y_p$ where $y_H$ is the general solution of the corresponding homogeneous equation $y''+4y=0$ and $y_p$ is a particular solution of the DE.

The auxiliary equation of $y''+4y=0$ is $0=r^2+4=(r+2i)(r-2i)$ which imply that $y_H=C_1\cos(2x)+C_2\sin(2x)$.

For $y_p$, the form $x^0Ae^{-4x}$ is appropriate, where $Ae^{-4x}$ is a generalization of the RHS of the DE and where the zero in $x^0$ equals the multiplicity of the number $-4$ (in $e^{-4x}$) among the roots of the auxiliary equation.

Differentiating $y_p$ and substituting into the DE give

$$[(-4)^2+4]Ae^{-4x}=5e^{-4x} \Leftrightarrow A=\frac{5}{20}=\frac{1}{4}$$

Thus, the general solution of the DE is $y=C_1\cos(2x)+C_2\sin(2x)+\frac{1}{4}e^{-4x}$

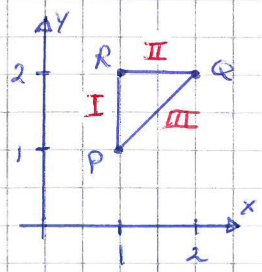
An adaption to the IV:s gives $\begin{cases}1=y(0)=C_1\cdot 1+C_2\cdot 0+\frac{1}{4}e^0\\ 0=y'(0)=-2C_1\cdot 0+2C_2\cdot 1-e^0\end{cases} \Leftrightarrow \begin{cases}C_1=\frac{3}{4}\\ C_2=\frac{1}{2}\end{cases}$

The (unique) solution of the IVP is thus $y=\frac{1}{4}\left(3\cos(2x)+2\sin(2x)+e^{-4x}\right)$

**2** $\sum_{n=1}^{\infty}\frac{7^n+8^n}{3^{2n}}$ Let $\frac{7^n+8^n}{3^{2n}}=a_n$

Then $a_n=\frac{7^n+8^n}{(3^2)^n}=\frac{7^n+8^n}{9^n}<\frac{8^n+8^n}{9^n}=2\frac{8^n}{9^n}=2\left(\frac{8}{9}\right)^n=b_n$

Since the series $\sum_{n=1}^{\infty}b_n$ is convergent (it is a geometric series with a quotient $=\frac{8}{9}<1$, i.e. a convergent geometric series), we get by the comparision test with $a_n<b_n$ that also $\sum_{n=1}^{\infty}a_n$ is convergent.

**3** 


We note that the surface generated by curve$_I$ is a circular cylinder, that by curve$_{II}$ is an annulus, and that by curve$_{III}$ is a part of a cirkular cone. We get

$(A_y)_I=2\pi\cdot 1(2-1)=2\pi$, $(A_y)_{II}=\pi(2^2-1^2)=3\pi$

$$(A_y)_{III}=\int_1^2 2\pi x\sqrt{1+1^2}\,dx=\pi\sqrt{2}\left[x^2\right]_1^2=\pi\sqrt{2}(4-1)=3\sqrt{2}\pi$$

Thus $A_y=2\pi+3\pi+3\sqrt{2}\pi=(5+3\sqrt{2})\pi$ Answer: $(5+3\sqrt{2})\pi$ a.u.

Cont. MAA151
2016-08-19

(4)

$$\int_1^e \sqrt{x}\,\ln^2(x)\,dx = \left[\tfrac{2}{3}x^{3/2}\ln^2(x)\right]_1^e - \int_1^e \tfrac{2}{3}x^{3/2}\cdot 2\ln(x)\cdot\tfrac{1}{x}\,dx$$

$$= \tfrac{2}{3}\left(e^{3/2}\cdot 1^2 - 1^{3/2}\cdot 0^2\right) - \tfrac{4}{3}\int_1^e x^{1/2}\ln(x)\,dx$$

$$= \tfrac{2}{3}e^{3/2} - \tfrac{4}{3}\left(\left[\tfrac{2}{3}x^{3/2}\ln(x)\right]_1^e - \int_1^e \tfrac{2}{3}x^{3/2}\tfrac{1}{x}\,dx\right)$$

$$= \tfrac{2}{3}e^{3/2} - \tfrac{8}{9}\left(e^{3/2}\cdot 1 - 1^{3/2}\cdot 0\right) + \tfrac{8}{9}\left[\tfrac{2}{3}x^{3/2}\right]_1^e$$

$$= \tfrac{2}{27}\;9e^{3/2} - 12e^{3/2} + 8\left(e^{3/2} - 1^{3/2}\right) = \underline{\tfrac{2}{27}\left(5e\sqrt{e} - 8\right)}$$

(5) $f(x) = \dfrac{1}{x+1} + \arctan(x)$

Differentiation gives: $f'(x) = -\dfrac{1}{(x+1)^2} + \dfrac{1}{1+x^2} = \dfrac{-(1+x^2) + (x^2+2x+1)}{(x+1)^2(1+x^2)}$

$$= \frac{2x}{(x+1)^2(1+x^2)}$$

A first derivative test is

| | | As | | SP | |
|---|---|---|---|---|---|
| $x$ | | $-1$ | | $0$ | |
| $f'(x)$ | $-$ | # | $-$ | $0$ | $+$ |
| $f(x)$ | ↘ | # | ↘ | loc. min. | ↗ |

$\underline{x=-1}$ is a double-sided vertical asymptote of the curve $y = f(x)$ since $f(x) \to \pm\infty$ as $x \to (-1)^{\pm}$.

$\underline{y = -\frac{\pi}{2}}$ is a nonvertical asymptote "at $-\infty$" and $\underline{y = \frac{\pi}{2}}$ "at $\infty$"

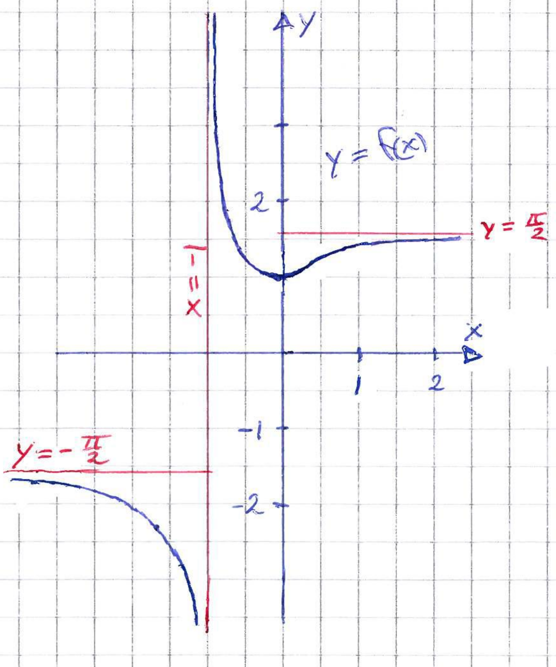
(since $\lim\limits_{x\to\pm\infty} f(x) = 0 \pm \frac{\pi}{2} = \pm\frac{\pi}{2}$)

(Note For all pairs of '$\pm$' concerning the asymptotes, the signs are synchronized.)

Also $f(0) = \dfrac{1}{0+1} + \arctan(0)$

$= 1 + 0 = 1$

i.e. the coordinates of the local min. point are $(0, 1)$.



**Examination TEN2 – 2016-08-19** | Maximum points for subparts of the problems in the final examination

**1.** $y = \frac{1}{4}\left(3\cos(2x) + 2\sin(2x) + e^{-4x}\right)$

**1p**: Correctly identified the differential equation as a non-homogeneous linear DE of second order, and correctly found the general solution $y_h$ of the corr. homog. DE
**1p**: Correctly proposed a formula for a part. sol. of the DE
**1p**: Correctly found a particular solution of the DE, and correctly summarized the general solution of the DE
**1p**: Correctly adapted the general solution of the DE to the IV

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**2.** The series converges

**1p**: Correctly identified that the terms of the series are dominated by exponentials $8^n$ in the numerator and by the exponentials $9^n$ in the the denominator

----- **One scenario for the other three points** -------
**1p**: Correctly found that the limit of the test quantity in the ratio test equals $8/9$
**2p**: Correctly, from the fact that $8/9 < 1$, concluded that the series converges according to the ratio test

------------ **Another scenario for the other three points** ----------------------
**1p**: Correctly noted that the terms of the series are less than e.g. $2(8/9)^n$
**2p**: Correctly concluded that the series converges according to the comparison test (with a comparion with the convergent geometric series $\sum_{n=1}^{\infty} 2(8/9)^n$ )

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**3.** $(5 + 3\sqrt{2})\pi$ a.u.

**1p**: Correctly realized that the surface considered consists of three parts which most easily are described separately concerning matters of finding the total area, and correctly found the area of the circular cylinder surface of radius 2
**1p**: Correctly found the area of the annulus part of the surface
**2p**: Correctly found the area of the conic part of the surface, and correctly added the three contributions to the area

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**4.** $\dfrac{2}{27}(5e\sqrt{e} - 8)$

----------------------------------- **One scenario** -----------------------------------
**2p**: Correctly worked out the first step of a partial integration
**1p**: Correctly worked out the second step of a partial integration
**1p**: Correctly worked out the third step of a partial integration
-------------------------------- **Another scenario** --------------------------------
**1p**: Correctly, by the substitution $\ln(x) = u$, translated the integral into into $\int_0^1 u^2 e^{3u/2}\,du$
**3p**: Correctly worked out the three steps of a partial integration in $u$

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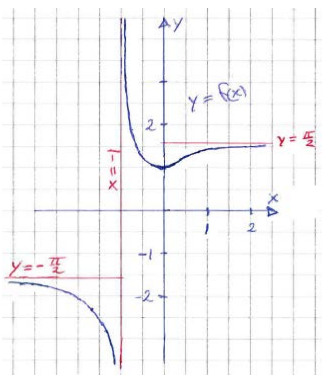
**5.** The graph has a local minimum at $P:(0,1)$, and has the asymptotes $x = -1$, $y = -\pi/2$ and $y = \pi/2$




**1p**: Correctly found the asymptotes of the graph
**1p**: Correctly classified the local extreme point of the graph
**1p**: Correctly sketched the graph according to how the graph relates to the asymptotes on their both sides respectively
**1p**: Correctly completed the sketch of the graph

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