MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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**EXAMINATION IN MATHEMATICS**
MAA151 Single Variable Calculus, TEN2
*Date:* 2016-11-03 *Write time:* 3 hours
*Aid:* Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted $S_2$, and that obtained at examination TEN1 $S_1$, the mark for a completed course is according to the following:

| | | | | |
|---|---|---|---|---|
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $S_1 + 2S_2 \leq 41$ | $\rightarrow$ | 3 |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $42 \leq S_1 + 2S_2 \leq 53$ | $\rightarrow$ | 4 |
| | | $54 \leq S_1 + 2S_2$ | $\rightarrow$ | 5 |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $S_1 + 2S_2 \leq 32$ | $\rightarrow$ | E |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $33 \leq S_1 + 2S_2 \leq 41$ | $\rightarrow$ | D |
| $S_1 \geq 11,\ S_2 \geq 9$ | AND | $42 \leq S_1 + 2S_2 \leq 51$ | $\rightarrow$ | C |
| | | $52 \leq S_1 + 2S_2 \leq 60$ | $\rightarrow$ | B |
| | | $61 \leq S_1 + 2S_2$ | $\rightarrow$ | A |

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

**1.** Evaluate the integral
$$\int_{2/3}^{7/6} \sqrt{5 + 3x(4 - 3x)}\, dx\,,$$
and write the result in as simple form as possible.

**2.** Sketch the graph of the function $f$ defined by
$$f(x) = \frac{2(x+1)}{x^2+3}\,.$$
Also, specify the range of the function.

**3.** Find the area of the surface generated by rotating the curve
$$y = x^3, \quad 0 \leq x \leq 1\,,$$
about the $x$-axis.

**4.** Prove that the series
$$\sum_{n=2}^{\infty} \frac{\pi^{2n}}{(-9)^n (2n)!}$$
is convergent. Then, find the sum of the series.

**5.** Find an equation for the tangent line $\tau$ to the curve
$$\gamma:\ xy\arctan(xy) + \ln(2) = \frac{\pi}{4} + \ln(1 + x^2y^2)$$
at the point $P : (\frac{1}{\sqrt{3}}, \sqrt{3})$.

**Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.**

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns $S_2$, och den vid tentamen TEN1 erhållna $S_1$, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$$\begin{array}{lcrcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

**1.** Beräkna integralen

$$\int_{2/3}^{7/6} \sqrt{5 + 3x(4 - 3x)}\, dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

**2.** Skissa grafen till funktionen $f$ definierad genom

$$f(x) = \frac{2(x+1)}{x^2+3}\,.$$

Specificera även funktionens värdemängd.

**3.** Bestäm arean av den yta som genereras genom att rotera kurvan

$$y = x^3, \quad 0 \leq x \leq 1\,,$$

kring $x$-axeln.

**4.** Bevisa att serien

$$\sum_{n=2}^{\infty} \frac{\pi^{2n}}{(-9)^n(2n)!}$$

är konvergent. Bestäm därefter summan av serien.

**5.** Bestäm en ekvation för tangenten $\tau$ till kurvan

$$\gamma:\ xy\arctan(xy) + \ln(2) = \frac{\pi}{4} + \ln(1 + x^2y^2)$$

i punkten $P : (\frac{1}{\sqrt{3}}, \sqrt{3})$.

**If you prefer the problems formulated in English, please turn the page.**

# MAA151 / Solutions to the final exam TEN2 2016-11-03

(1) $$\int_{2/3}^{7/6} \sqrt{5+3x(4-3x)}\,dx = \int_{2/3}^{7/6} \sqrt{9-(3x-2)^2}\,dx \quad \begin{bmatrix} 3x-2 = 3\sin(\theta) \\ 3dx = 3\cos(\theta)d\theta \end{bmatrix}$$

$$= \int_0^{\pi/6} \sqrt{9(1-\sin^2(\theta))}\cos(\theta)\,d\theta = 3\int_0^{\pi/6} |\cos(\theta)|\cos(\theta)\,d\theta \overset{\substack{\cos(\theta)\geq 0 \text{ in} \\ \text{'}\sin(\theta)\text{-substitutions'}}}{=} 3\int_0^{\pi/6} \cos^2(\theta)\,d\theta$$

$$= 3\int_0^{\pi/6} \frac{1}{2}(1+\cos(2\theta))\,d\theta = \frac{3}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6} = \frac{3}{2}\left[\left(\frac{\pi}{6}+\frac{1}{2}\frac{\sqrt{3}}{2}\right)-(0+0)\right]$$

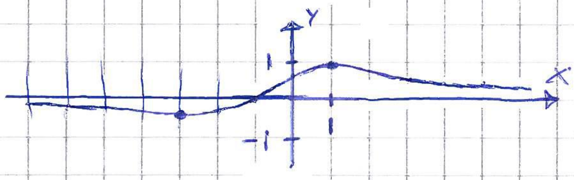
$$= \frac{3}{2}\frac{1}{12}(2\pi+3\sqrt{3}) = \underline{\frac{1}{8}(2\pi+3\sqrt{3})}$$

(2) $$f(x) = \frac{2(x+1)}{x^2+3}$$

Diff. gives $$f'(x) = 2\frac{1\cdot(x^2+3)-(x+1)\cdot 2x}{(x^2+3)^2} = -2\cdot\frac{x^2+2x-3}{(x^2+3)^2} = -2\frac{(x+3)(x-1)}{(x^2+3)^2}$$

A first-deriv. test:

| $x$ | | $-3$ | | $1$ | |
|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ | $0$ | $-$ |
| $f(x)$ | $\searrow$ | loc. min | $\nearrow$ | loc. max | $\searrow$ |

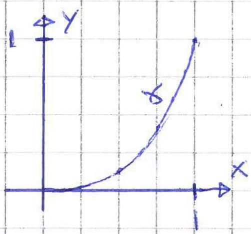



There are no vertical asymptotes
At "$\pm\infty$", we have the asymptote
$y=0$ (since $\lim_{x\to\pm\infty} f(x) = 0$)

$$f(-3) = \frac{2(-3+1)}{(-3)^2+3} = -\frac{1}{3}, \quad f(1) = \frac{2(1+1)}{1^2+3} = 1$$

$$\underline{R_f = [f(-3), f(1)] = \left[-\frac{1}{3}, 1\right]}$$

(3)




$\gamma$: $y = \underset{\substack{\uparrow \\ f(x)}}{x^3}$, $0 \leq x \leq 1$

The area $A_x$ of the surface generated by rotating the curve $y=f(x)$ about the x-axis is

$$A_x = \int_0^1 2\pi|f(x)|\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx \quad \begin{bmatrix} 1+9x^4 = u \\ 36x^3dx = du \end{bmatrix}$$

$$= 2\pi\int_1^{10} \frac{1}{36}\sqrt{u}\,du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\pi}{27}(10\sqrt{10}-1)$$

Answer: $\underline{\frac{\pi}{27}(10\sqrt{10}-1) \text{ a.u.}}$

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(4) $\sum_{n=2}^{\infty} \frac{\pi^{2n}}{(-9)^n (2n)!}$ Let $\frac{\pi^{2n}}{(-9)^n (2n)!} = a_n$

Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\pi^{2(n+1)}}{9^{n+1}(2(n+1))!} \frac{9^n (2n)!}{\pi^{2n}}$

$$= \lim_{n\to\infty} \frac{\pi^2}{9} \frac{1}{(2n+1)(2n+2)} = 0 < 1$$

i.e. the series $\sum_{n=2}^{\infty} a_n$ is (absolutely) convergent according to the ratio test. q.e.d.

Furthermore $\sum_{n=2}^{\infty} \frac{\pi^{2n}}{(-9)^n (2n)!} = \sum_{n=2}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{3}\right)^{2n}}{(2n)!}\right) - \left[(-1)^0 \frac{\left(\frac{\pi}{3}\right)^0}{0!} + (-1)^1 \frac{\left(\frac{\pi}{3}\right)^2}{2!}\right]$$

$$= \cos\left(\frac{\pi}{3}\right) - \left(1 - \frac{\pi^2}{18}\right) = \frac{1}{2} - 1 + \frac{\pi^2}{18} = \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - 1\right]$$

i.e. the sum of the series $\sum_{n=2}^{\infty} a_n$ is $\frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - 1\right]$

(5) $\gamma$: $xy\arctan(xy) + \ln(2) = \frac{\pi}{4} + \ln(1 + x^2y^2)$, $P: \left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)$

Differentiation with respect of $x$ gives

$$\left[1\cdot\arctan(xy) + xy\frac{1}{1+x^2y^2}\right](1\cdot y + xy') + 0 = 0 + \frac{1}{1+x^2y^2} 2(xy)(1\cdot y + xy')$$

Let $y'|_P = k$. Then at $P$, we get

$$\left(\arctan(1) + 1\cdot\frac{1}{1+1^2} - \frac{1}{1+1^2}2\cdot 1\right)\left(\sqrt{3} + \frac{1}{\sqrt{3}}k\right) = 0$$

i.e. $\underbrace{\left(\frac{\pi}{4} - \frac{1}{2}\right)}_{\neq 0} \frac{1}{\sqrt{3}}(3 + k) = 0$ i.e. $k = -3$

Thus an equation for the tangent line $\tau$ to $\gamma$ at $P$ is

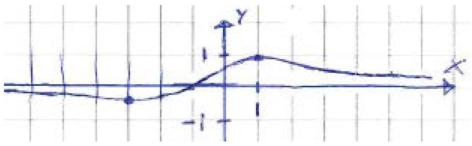
$$y - \sqrt{3} = -3\left(x - \frac{1}{\sqrt{3}}\right) \Leftrightarrow y = -3x + 2\sqrt{3}$$

**Examination TEN2 – 2016-11-03** Maximum points for subparts of the problems in the final examination

**1.** $\dfrac{2\pi+3\sqrt{3}}{8}$

**1p**: Correctly rewrited the integrand into $\sqrt{9-(3x-2)^2}$, and correctly by the substitution $3x-2=3\sin(\theta)$ translated $\sqrt{9-(3x-2)^2}\,dx$ into $3\cos^2(\theta)\,d\theta$
**1p**: Correctly translated the limits of the integral in connection with the substitution $3x-2=3\sin(\theta)$
**1p**: Correctly rewrited the integrand according to $\cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$, and correctly found an antiderivative
**1p**: Correctly evaluated the antiderivative at the limits and by that correctly found the value of the integral

**2.** The graph has a global minimum at $P_1:(-3,-\frac{1}{3})$ and global maximum at $P_2:(1,1)$. The only asymptote is $y=0$, and the range is $[-\frac{1}{3},1]$.




**1p**: Correctly found the stationary points and the asymptote
**1p**: Correctly classified the stationary points
**1p**: Correctly sketched the graph according to how the graph relates to the extreme points and the asymptote
**1p**: Correctlys found the range

**3.** $\dfrac{\pi}{27}(10\sqrt{10}-1)$ a.u.

**1p**: Correctly formulated an explicit integral expression for the area of the surface generated by the curve rotated about the *x*-axis
**2p**: Correctly by e.g. a substitution $1+9x^4=u$ translated the integral into a standard integral with the integrand $\sqrt{u}$
**1p**: Correctly found an antiderivative of the integrand, and correctly evaluated the antiderivative at the limits

**4.** The series is convergent, and the sum of the series is $\frac{1}{2}[(\frac{\pi}{3})^2-1]$

**2p**: Correctly, by the ratio test, found that the series is (absolutely) convergent
**1p**: Correctly identified the series as, except for the first two terms, the Maclaurin series for the cosine function at the point $\pi/3$
**1p**: Correctly found the sum of the series

**5.** $\tau: 3x+y=2\sqrt{3}$

**2p**: Correctly, with the purpose of finding the slope at the point $P$, implicitly differentiated the equation with respect to $x$
**1p**: Correctly determined the slope $dy/dx|_P$ at the point $P$
**1p**: Correctly found an equation for the tangent line $\tau$