MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2016-12-02 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Prove that the explicitly given terms of the series

$$8x + \frac{2}{x} + \frac{1}{2x^3} + \ldots$$

are the first three in a geometric series. Then assume that the symbol "..." denotes all the other terms of precisely that geometric series. For which x converges the series? Find the sum of the series for these x.

2. Find the area of the region $\big\{(x,y) \in R^2 : \sqrt{x} \leq y \leq 1/\sqrt{x}\big\}$.

3. Find out whether
$$\lim_{x\to+\infty} \frac{6e^{-x} + 5x^3}{\big(4x\sqrt{x} + 3\ln(x)\big)^2}$$
exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

4. The product of two positive real numbers x and y equals 10. Which are the numbers if the weighted sum $5x + 2y$ is a minimum? Prove your answer!

5. Find the function f such that $f'(x) = 4x\cos(2x)$ and $f(\frac{\pi}{6}) = \frac{1}{2}(1 + \frac{\pi}{\sqrt{3}})$.

6. Let $f(x) = \ln\big(e - |x|\big)$. State (and explain) the domain and the range of f. For the latter, a decent sketch of the graph of the function could be a way to give an explanation.

7. Prove that the function f defined by $f(x) = x^3 + 3x^2 + 6x$ is invertible, and find the derivative of f^{-1} at the point -4.

8. Solve for $x > 0$ the initial-value problem $x^2y' - y = 3\,,\ y(1) = 0\,.$

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{llll} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Visa att de explicit utskrivna termerna i serien

$$8x + \frac{2}{x} + \frac{1}{2x^3} + \dots$$

är de tre första i en geometrisk serie. Antag sedan att symbolen "..." betecknar övriga termer i just denna geometriska serie. För vilka x konvergerar serien? Bestäm seriens summa för dessa x.

2. Bestäm arean av området $\big\{(x,y) \in R^2 : \sqrt{x} \leq y \leq 1/\sqrt{x}\big\}$.

3. Utred om

$$\lim_{x\to+\infty} \frac{6e^{-x} + 5x^3}{\big(4x\sqrt{x} + 3\ln(x)\big)^2}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

4. Produkten av två positiva reella tal x och y är lika med 10. Vilka är talen om den viktade summan $5x + 2y$ är minimal? Bevisa ditt svar!

5. Bestäm funktionen f så att $f'(x) = 4x\cos(2x)$ och $f(\frac{\pi}{6}) = \frac{1}{2}(1 + \frac{\pi}{\sqrt{3}})$.

6. Låt $f(x) = \ln\big(e - |x|\big)$. Ange (och förklara) definitionsmängden och värdemängden för f. För den senare skulle en hyfsad skiss av grafen till funktionen kunna vara ett sätt att ge en förklaring.

7. Bevisa att funktionen f definierad enligt $f(x) = x^3 + 3x^2 + 6x$ är inverterbar, och bestäm derivatan till f^{-1} i punkten -4.

8. Lös för $x > 0$ begynnelsevärdesproblemet $x^2y' - y = 3\,,\ \ y(1) = 0\,.$

MAA151 / Solutions to the final exam TEN1 2016-12-02

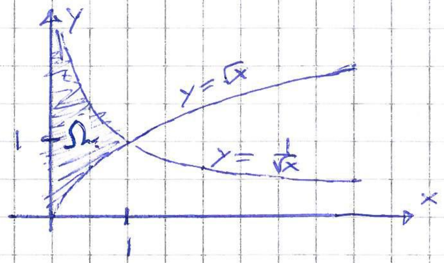
1

$$8x + \frac{2}{x} + \frac{1}{2x^3} + \dots \equiv a_0(x) + a_1(x) + a_2(x) + \dots$$

Since $a_1(x) = \frac{1}{4x^2} a_0(x)$ and $a_2(x) = \frac{1}{4x^2} a_1(x)$, the three first terms are the three first terms of a geometric series with the ratio $\frac{1}{4x^2}$. q.e.d. The geometric series converges iff $\left|\frac{1}{4x^2}\right| < 1 \Leftrightarrow |x| > \frac{1}{2} \Leftrightarrow x < -\frac{1}{2} \lor x > \frac{1}{2}$

The sum of the series for those x is $8x\left(\frac{1}{1-\frac{1}{4x^2}}\right) = \frac{32x^3}{4x^2-1}$.

2



$$A(\Omega) = A\left(\left\{(x,y) \in \mathbb{R}^2 : \sqrt{x} \le y \le \frac{1}{\sqrt{x}}\right\}\right)$$

$$= \int_0^1 \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) dx = \left[2\sqrt{x} - \frac{2}{3}x\sqrt{x}\right]_0^1$$

$$= \left[\left(2 - \frac{2}{3}\right) - (0 - 0)\right] = \frac{4}{3}$$

$\boxed{\text{Answer } \frac{4}{3} \text{ a.u.}}$

3

$$\lim_{x \to +\infty} \frac{6e^{-x} + 5x^3}{(4x\sqrt{x} + 3\ln(x))^2} = \lim_{x \to +\infty} \frac{x^3\left(\frac{6}{x^3e^x} + 5\right)}{x^3\left(4 + 3\frac{\ln(x)}{x\sqrt{x}}\right)^2}$$

$$= \frac{\left(\lim_{x \to +\infty} \frac{6}{x^3e^x}\right) + 5}{\left(4 + 3\lim_{x \to +\infty} \frac{\ln(x)}{x^{3/2}}\right)^2} = \frac{0 + 5}{(4 + 3 \cdot 0)^2} = \frac{5}{16}$$

where
$$\begin{cases} \frac{6}{x^3e^x} \to 0 \text{ as } x \to +\infty & \text{(obvious)} \\ \frac{\ln(x)}{x^{3/2}} \to 0 \text{ as } x \to +\infty & \text{(since positive powers of } x \text{ are "stronger" than logarithms as } x \to +\infty) \end{cases}$$

4

Find minimum of $5x + 2y$ as $x, y > 0$ and $xy = 10$.

Let $f(x) = 5x + 2y\Big|_{\substack{xy=10 \\ x,y>0}} = 5x + \frac{20}{x}$ for $x > 0$

Then $f'(x) = 5 - \frac{20}{x^2} = \frac{5}{x^2}(x^2 - 4) = \frac{5}{x^2}(x+2)(x-2)$

$f''(x) = \frac{40}{x^3}$ and $f''(2) = \frac{40}{8} = 5 > 0$

Thus: There is only one stationary point and no boundary nor singular points. Since the SP 2 is a local minimum point (due to the fact that $f''(2) > 0$) and the function f is continuous, the SP is also a global minimum.

Answer The numbers x and y are 2 and 5 respectively.

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5 $f'(x) = 4x\cos(2x)$ and $f\left(\frac{\pi}{6}\right) = \frac{1}{2}\left(1+\frac{\pi}{\sqrt{3}}\right)$

We get $f(x) - \frac{1}{2}\left(1+\frac{\pi}{\sqrt{3}}\right) = f(x) - f\left(\frac{\pi}{6}\right) = \int_{\pi/6}^{x} f'(t)\,dt = \int_{\pi/6}^{x} 4t\cos(2t)\,dt$

partial integr.

$= \left[2t\sin(2t)\right]_{\pi/6}^{x} - \int_{\pi/6}^{x} 2\sin(2t)\,dt = \left[2t\sin(2t) + \cos(2t)\right]_{\pi/6}^{x}$

$= \left(2x\sin(2x) + \cos(2x)\right) - \left(\frac{\pi}{3}\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$ i.e. $f(x) = 2x\sin(2x) + \cos(2x)$

6 $f(x) = \ln(e - |x|)$

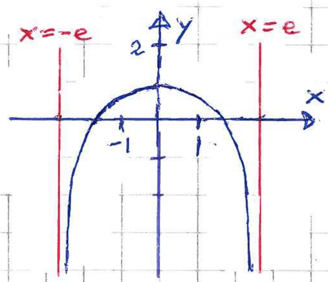
the argument of ln has to be > 0

$D_f = \{x : e - |x| > 0\} = \{x : |x| < e\} = \{x : -e < x < e\} = (-e, e)$

$R_f = \{y : y = f(x), x \in D_f\} = (-\infty, 1]$

(Xtra: $f'(x) = \begin{cases} \frac{1}{e+x}, & -e < x < 0 \\ -\frac{1}{e-x}, & 0 < x < e \end{cases}$

(f' ∄ at 0))



7 $f(x) = x^3 + 3x^2 + 6x$

Since $f'(x) = 3x^2 + 6x + 6 = 3(x^2 + 2x + 2) = 3\left[(x+1)^2 + 1\right] \geq 3 > 0$,
f is increasing in $D_f = \mathbb{R}$ and is therefore invertible. q.e.d.

We note that $f(-1) = -1 + 3 - 6 = -4$ and thus $-1 = f^{-1}(-4)$

Since $f'(-1) \neq 0$ and since f^{-1} is continuous (due to that f is cont. and monotonic)

we get that $(f^{-1})'(-4) = \frac{1}{f'(-1)} = \frac{1}{3}$

8 $x^2y' - y = 3$ for $x > 0$ and $y(1) = 0$

a linear, 1st order DE

DE on standard form $y' - \frac{1}{x^2}y = \frac{3}{x^2}$ Int. factor = choose $e^{\frac{1}{x}}$

Thus $e^{\frac{1}{x}} \cdot y' + e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) \cdot y = e^{\frac{1}{x}} \cdot \frac{3}{x^2}$ giving

$e^{\frac{1}{x}}y = \int e^{\frac{1}{x}} \cdot \frac{3}{x^2}\,dx = -3e^{\frac{1}{x}} + C$; $y = Ce^{-\frac{1}{x}} - 3$

where $0 = y(1) = Ce^{-1} - 3 \Leftrightarrow C = 3e$ Thus $y = 3\left(e^{\frac{x-1}{x}} - 1\right)$

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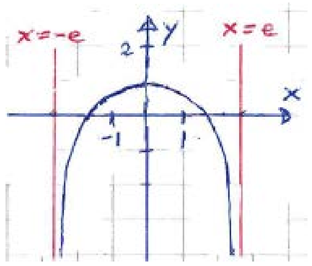
EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN1 – 2016-12-02 Maximum points for subparts of the problems in the final examination

1.	The series is a geometric series since it has a quotient. The series converges for $x\in(-\infty,-\frac{1}{2})\cup(\frac{1}{2},\infty)$. For those x, the sum of the series is $32x^3(4x^2-1)^{-1}$	**1p**: Correctly proved that the series is a geometric series **1p**: Correctly found the interval of convergence **1p**: Correctly found the sum of the series
2.	$4/3$ a.u.	**1p**: Correctly formulated an integral for the area **1p**: Correctly found the antiderivative of the integrand **1p**: Correctly found the area of the region
3.	The limit exists and is equal to $5/16$	**1p**: Correctly brought out the dominating factors from the numerator and denominator respectively **1p**: Correctly concluded that the limit exists **1p**: Correctly found the limit
4.	$x=2$ and $y=5$	**1p**: Correctly for the optimization problem formulated a function of one variable, and correctly found the factorized derivative of the function **1p**: Correctly worked out a test for a minimum **1p**: Correctly found the numbers x and y
5.	$f(x)=\cos(2x)+2x\sin(2x)$	**1p**: Correctly worked out the first progressive step in finding the antiderivative by parts **1p**: Correctly worked out the second progressive step in finding the antiderivative by parts **1p**: Correctly adapted the antiderivative to the value at $\frac{\pi}{6}$
6.	$D_f=(-e,e)$, $R_f=(-\infty,1]$ 	**1p**: Correctly deduced the domain of the function f **2p**: Correctly deduced the range of the function f
7.	f is invertible since $f'(x)>0$ on D_f $(f^{-1})'(-4)=1/3$	**1p**: Correctly proved that f is invertible **1p**: Correctly found that $f^{-1}(-4)=-1$ **1p**: Correctly determined the value of $(f^{-1})'(-4)$
8.	$y=3(e^{(x-1)/x}-1)$	**1p**: Correctly found and multiplied with an integrating factor, and correctly rewritten the LHS of the DE as an exact derivative **1p**: Correctly found the general solution of the DE **1p**: Correctly adapted the general solution to the initial value, and correctly summarized the solution of the IVP