MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA150 Vector Algebra, TEN2
Date: 2018-01-08 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The PASS-marks 3, 4 and 5 require a minimum of 12, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 12, 13, 16, 20 and 24 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the marks for a completed course are determined according to

$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 47$	$\to$	3	$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 38$	$\to$	E
$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 62$	$\to$	4	$S_1, S_2 \geq 12$	AND	$39 \leq S_1 + 2S_2 \leq 47$	$\to$	D
		$63 \leq S_1 + 2S_2$	$\to$	5	$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 59$	$\to$	C
					$S_1, S_2 \geq 12$	AND	$60 \leq S_1 + 2S_2 \leq 71$	$\to$	B
							$72 \leq S_1 + 2S_2$	$\to$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find an ON-basis for the subspace

$$\operatorname{span}\{(2,-1,1),(3,0,2),(1,1,1),(1,-2,0)\}$$

of $\mathbb{E}^3$, where the latter is the vector space $\mathbb{R}^3$ equipped with the standard inner product.

2. The linear transformation $F : \mathbb{R}^4 \to \mathbb{R}^3$ is defined by

$$F(u) = (-3x_1 + 2x_2 + 7x_3 - 7x_4, 2x_1 + 3x_2 + 4x_3 + 9x_4, x_1 + x_2 + x_3 + 4x_4),$$

where $u = (x_1, x_2, x_3, x_4)$. Find the standard matrix of F, i.e. the matrix of F relative to the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$. Also, find a basis for the null space (synonymously *kernel*) of F.

3. The polynomial $2z^4 - 16z^3 + 76z^2 - 112z + 50$ has the zero $3 - 4i$. Write the polynomial in a factorized form where all the z-dependent factors have a degree of one.

4. Let e_1, e_2, e_3 be a basis for the vector space $\mathbb{V}$, and introduce the vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ according to

$$\begin{cases} \tilde{e}_1 = e_2 - e_3\,, \\ \tilde{e}_2 = -e_1 + 2e_2 - 2e_3\,, \\ \tilde{e}_3 = 2e_1 - 6e_2 + 7e_3\,. \end{cases}$$

Prove that even $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a basis for $\mathbb{V}$, and find the coordinates of the vector $3e_1 - 2e_2 + e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

5. The linear operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ has the matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 7 \\ 6 & 4 \end{pmatrix}$$

relative to the standard basis. Prove that F is diagonalizable by finding its diagonal matrix D relative to a basis of eigenvectors. State especially the change-of-basis matrix S which according to $\mathsf{D} = \mathsf{S}^{-1}\mathsf{AS}$ diagonalizes F.

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 12, 16 respektive 21 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av ett sammanfattningsbetyg på en slutförd kurs enligt

$S_1, S_2 \geq 12$	OCH	$S_1 + 2S_2 \leq 47$	$\rightarrow$	3
$S_1, S_2 \geq 12$	OCH	$48 \leq S_1 + 2S_2 \leq 62$	$\rightarrow$	4
		$63 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm en ON-bas för delrummet

$$\text{span}\{(2,-1,1),(3,0,2),(1,1,1),(1,-2,0)\}$$

till $\mathbb{E}^3$, där det senare är vektorrummet $\mathbb{R}^3$ utrustat med standardskalärprodukten.

2. Den linjära avbildningen $F : \mathbb{R}^4 \to \mathbb{R}^3$ är definierad genom

$$F(u) = (-3x_1 + 2x_2 + 7x_3 - 7x_4, 2x_1 + 3x_2 + 4x_3 + 9x_4, x_1 + x_2 + x_3 + 4x_4),$$

där $u = (x_1, x_2, x_3, x_4)$. Bestäm standardmatrisen för F, dvs avbildningsmatrisen för F relativt standardbaserna för $\mathbb{R}^4$ och $\mathbb{R}^3$. Bestäm även en bas för F:s nollrum (synonymt *kärna*).

3. Polynomet $2z^4 - 16z^3 + 76z^2 - 112z + 50$ har nollstället $3 - 4i$. Skriv polynomet på en faktoriserad form där alla z-beroende faktorer har graden ett.

4. Låt e_1, e_2, e_3 vara en bas för vektorrummet $\mathbb{V}$, och introducera vektorerna $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ enligt

$$\begin{cases} \tilde{e}_1 = e_2 - e_3\,, \\ \tilde{e}_2 = -e_1 + 2e_2 - 2e_3\,, \\ \tilde{e}_3 = 2e_1 - 6e_2 + 7e_3\,. \end{cases}$$

Bevisa att även $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ är en bas för $\mathbb{V}$, och bestäm koordinaterna för vektorn $3e_1 - 2e_2 + e_3$ relativt basen $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

5. Den linjära operatorn $F : \mathbb{R}^2 \to \mathbb{R}^2$ har matrisen

$$\mathsf{A} = \begin{pmatrix} 3 & 7 \\ 6 & 4 \end{pmatrix}$$

relativt standardbasen. Bevisa att F diagonaliserbar genom att bestämma dess diagonala matris D relativt en bas av egenvektorer. Ange speciellt den basbytesmatris S som enligt $\mathsf{D} = \mathsf{S}^{-1}\mathsf{AS}$ diagonaliserar F.

If you prefer the problems formulated in English, please turn the page.

MAA150/ Solutions to the final exam. TEN2 2018-01-08

① Let $\mathcal{U} = \text{span}\{\overset{u_1}{(2,-1,1)}, \overset{u_2}{(3,0,2)}, \overset{u_3}{(1,1,1)}, \overset{u_4}{(1,-2,0)}\} \subset E^3$.

The coordinate matrix C of the vectors u_1, u_2, u_3, u_4 relative to the standard basis is

$$C = \begin{pmatrix} 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -2 \\ 1 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 1 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = C_{rref}$$

From C_{rref} we conclude that e.g. u_1, u_2 is a basis for the subspace $\mathcal{U}$. The Gram-Schmidt procedure the gives

$$\begin{cases} f_1 = u_1 = (2,-1,1), \quad e_1 = \frac{1}{\|f_1\|} f_1 = \frac{1}{\sqrt{4+1+1}} f_1 = \frac{1}{\sqrt{6}}(2,-1,1) \\ f_2 = u_2 - (u_2 \cdot e_1) e_1 = (3,0,2) - \left[(3,0,2) \cdot \frac{1}{\sqrt{6}}(2,-1,1)\right] \frac{1}{\sqrt{6}}(2,-1,1) \\ \quad = (3,0,2) - \frac{1}{6}(6+0+2)(2,-1,1) = \frac{1}{3}\left[(9,0,6) - (8,-4,4)\right] = \frac{1}{3}(1,4,2) \\ e_2 = \frac{1}{\|f_2\|} f_2 = \frac{1}{\|(1,4,2)\|}(1,4,2) = \frac{1}{\sqrt{1+16+4}}(1,4,2) = \frac{1}{\sqrt{21}}(1,4,2) \end{cases}$$

Thus $\frac{1}{\sqrt{6}}(2,-1,1), \frac{1}{\sqrt{21}}(1,4,2)$ is an ON-basis for $\mathcal{U}$

② $F: R^4 \to R^3$ is a linear transformation defined by

$$F((x_1,x_2,x_3,x_4)) = (-3x_1+2x_2+7x_3-7x_4,\ 2x_1+3x_2+4x_3+9x_4,\ x_1+x_2+x_3+4x_4)$$

Let e_1, e_2, e_3, e_4 and f_1, f_2, f_3 denote the standard bases for R^4 and R^3 respectively. Then

$$\begin{cases} F(e_1) = F((1,0,0,0)) = (-3,2,1) = -3f_1 + 2f_2 + f_3 \\ F(e_2) = F((0,1,0,0)) = (2,3,1) = 2f_1 + 3f_2 + f_3 \\ F(e_3) = F((0,0,1,0)) = (7,4,1) = 7f_1 + 4f_2 + f_3 \\ F(e_4) = F((0,0,0,1)) = (-7,9,4) = -7f_1 + 9f_2 + 4f_3 \end{cases}$$

Thus, the standard matrix A of F equals $\begin{pmatrix} -3 & 2 & 7 & -7 \\ 2 & 3 & 4 & 9 \\ 1 & 1 & 1 & 4 \end{pmatrix}$

Since $A \sim \begin{pmatrix} 0 & 5 & 10 & 5 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_{rref}$

we get that null(F): $AX = 0 \iff A_{rref} X = 0$

$$\iff \begin{cases} x_1 - x_3 + 3x_4 = 0 \\ x_2 + 2x_3 + x_4 = 0 \end{cases} \iff (x_1,x_2,x_3,x_4) \overset{x_3 = r,\ x_4 = s,\ r,s \in R}{=} (r-3s, -2r-s, r, s)$$
$$= r(1,-2,1,0) + s(-3,-1,0,1)$$

i.e. a basis for null(F) is e.g. $(1,-2,1,0), (-3,-1,0,1)$.

Cont. MAA150
TEN2, 2018-01-08

(3) $2z^4-16z^3+76z^2-112z+50 = 2(z^4-8z^3+38z^2-56z+25)$

$\overset{*}{=} 2[z-(3-4i)][z-(3+4i)][z^2+\alpha z+\beta]$

$= 2[(z-3)^2+16][z^2+\alpha z+\beta] = 2(z^2-6z+25)(z^2-2z+1)$

$= 2[z-(3-4i)][z-(3+4i)](z-1)(z-1)$

* (Since the coefficients of the polynomial are real-valued, $3+4i$ is a zero if $3-4i$ is a zero.)

(4) $\begin{cases} \tilde{e}_1 = e_2-e_3 \\ \tilde{e}_2 = -e_1+2e_2-2e_3 \\ \tilde{e}_3 = 2e_1-6e_2+7e_3 \end{cases} \Leftrightarrow \underbrace{(\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)}_{\text{A row matrix } \tilde{e} \text{ of vectors as entries}} = (e_1\ e_2\ e_3)\underbrace{\begin{pmatrix} 0 & -1 & 2 \\ 1 & 2 & -6 \\ -1 & -2 & 7 \end{pmatrix}}_{\text{The change-of-basis matrix } S \text{ from the basis } e_1,e_2,e_3 \text{ to the basis } \tilde{e}_1,\tilde{e}_2,\tilde{e}_3}$

i.e. $\tilde{e} = eS$

where $S \sim \begin{pmatrix} 0 & -1 & 2 \\ 1 & 2 & -6 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ i.e. $\operatorname{rank}(S) = 3$

i.e. $S^{-1}\exists$ i.e. $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a linearly independent set of vectors i.e. $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a basis for the (3-dimensional) vector space V.

q.e.d.

Furthermore

$3e_1-2e_2+e_3 = \overbrace{(e_1 e_2 e_3)\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}}^{eX} = \overbrace{(e_1 e_2 e_3)S}^{\tilde{e}}\ \overbrace{S^{-1}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}}^{\tilde{X}} = \tilde{e}\tilde{X}$

i.e. $S\tilde{X} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ where $\tilde{X}$ is the coordinate matrix of $3e_1-2e_2+e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

Since

$(S|X) = \left(\begin{array}{ccc|c} 0 & -1 & 2 & 3 \\ 1 & 2 & -6 & -2 \\ -1 & -2 & 7 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -1 & 2 & 3 \\ 1 & 2 & -6 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -1 & 0 & 5 \\ 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right)$

we conclude that $\tilde{X} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and thus $\operatorname{coord}_{\tilde{e}}(3e_1-2e_2+e_3) = (2,-5,-1)$

(5) $F: \mathbb{R}^2 \to \mathbb{R}^2$ has the matrix $A = \begin{pmatrix} 3 & 7 \\ 6 & 4 \end{pmatrix}$ relative to the standard basis.

Eigenvalues: $0 = \det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 7 \\ 6 & 4-\lambda \end{pmatrix} = (\lambda-3)(\lambda-4)-42$

$= \lambda^2-7\lambda-30 = (\lambda+3)(\lambda-10) \Leftrightarrow \lambda=-3 \vee \lambda=10$

Since the eigenvalues are distinct, the linear operator is diagonalizable.

$\begin{cases} \lambda_1=-3: & A-\lambda_1 I = \begin{pmatrix} 6 & 7 \\ 6 & 7 \end{pmatrix} \sim \begin{pmatrix} 6 & 7 \\ 0 & 0 \end{pmatrix}; & t_1(7,-6),\ t_1 \neq 0 \text{ are the eigenvectors} \\ \lambda_2=10: & A-\lambda_2 I = \begin{pmatrix} -7 & 7 \\ 6 & -6 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}; & t_2(1,1),\ t_2 \neq 0 \quad \text{———} // \text{———} \end{cases}$

Thus a basis of eigenvectors is e.g. $(7,-6), (1,1)$.

The matrix D of F relative to the (ordered) basis $(7,-6), (1,1)$ is $\begin{pmatrix} -3 & 0 \\ 0 & 10 \end{pmatrix}$ and is given by $D = S^{-1}AS$, where $S = \begin{pmatrix} 7 & 1 \\ -6 & 1 \end{pmatrix}$ is the change-of-basis matrix from the standard basis to $(7,-6), (1,1)$.

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA150 Vector algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2017/18

Final examination TEN2 – 2018-01-08 Maximum points for subparts of the problems in the final examination

1. An ON-basis for the subspace is e.g.

$\frac{1}{\sqrt{6}}(2,-1,1)$, $\frac{1}{\sqrt{21}}(1,4,2)$

2p: Correctly found a basis for the subspace
2p: Correctly orthogonalized the two vectors of the basis
1p: Correctly normed the two vectors of the basis

---

2. The standard matrix of F is

$$\begin{pmatrix} -3 & 2 & 7 & -7 \\ 2 & 3 & 4 & 9 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

A basis for the null space of F is e.g. $(1,-2,1,0)$, $(-3,-1,0,1)$

2p: Correctly found the standard matrix of F
1p: Correctly found the reduced row echelon form of the standard matrix in preparation for finding the null space of F
2p: Correctly found/identified a basis for the null space of F (**1p** for each of two basis vectors)

---

3. The factorized polynomial is

$2(z-3+4i)(z-3-4i)(z-1)(z-1)$

1p: Correctly concluded that the complex conjugate of the given zero is also a zero of the polynomial
1p: Correctly worked out the product $(z-3+4i)(z-3-4i)$ in preparation for finding the other two zeros
2p: Correctly found the polynomial factor (z^2-2z+1) besides the factor $2(z^2-6z+25)$
1p: Correctly worked out the remaining factorization

---

4. Proof

The coordinates of $3e_1-2e_2+e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are $2,-5,-1$

2p: Correctly proved that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a basis for V
1p: Correctly found an equation for the coordinate matrix of the vector $3e_1-2e_2+e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$
2p: Correctly found the coordinates of $3e_1-2e_2+e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$

---

5. If D is chosen as $\begin{pmatrix} -3 & 0 \\ 0 & 10 \end{pmatrix}$ then

$S = \begin{pmatrix} 7r & s \\ -6r & s \end{pmatrix}$ where $r, s \neq 0$.

If D is chosen as $\begin{pmatrix} 10 & 0 \\ 0 & -3 \end{pmatrix}$ then

$S = \begin{pmatrix} s & 7r \\ s & -6r \end{pmatrix}$ where $r, s \neq 0$.

1p: Correctly found the eigenvalues of F
1p: Correctly found an eigenvector corresponding to the eigenvalue -3, and an eigenvector corresponding to the eigenvalue 10
1p: Based on the eigenvalues of F, correctly formulated one of the two possible diagonal matrices
2p: Correctly constructed a change-of-basis matrix S which according to $\mathrm{D} = \mathrm{S}^{-1}\mathrm{AS}$ diagonalizes F to the chosen diagonal matrix D
