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EXAMINATION IN MATHEMATICS
MAA153 Linear Algebra
Date: 2016-01-13 *Write time:* 5 hours
Aid: Writing materials

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F((x_1, x_2, x_3)) = (2x_1 - x_3, x_1 - x_2 + x_3, x_2 + 2x_3).$$

Find the matrix of F relative to the ordered basis $(2, -1, 0), (3, 2, 1), (2, 2, 1)$, and determine in that basis the coordinates of the vector $F((1, 1, 1))$.

2. Let $\mathcal{G}$ denote the vector space spanned by the half-integer power functions g_1 and g_2 definied according to $g_1(x) = \sqrt{x}$ and $g_2(x) = x\sqrt{x}$ in the interval $[0, 1]$. Determine an ON-basis for $\mathcal{G}$ which has been equipped with the inner product $\langle p|q\rangle = \int_0^1 p(t)q(t)\,dt$.

3. Find, relative to the standard basis, the matrix of the linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ which has the kernel $\operatorname{span}\{(2, -1, -1)\}$, which has the eigenvalue -1 with the eigenspace $\operatorname{span}\{(1, 3, 2)\}$, and which transforms the vector $(1, 2, 1)$ to the vector $(4, 1, 3)$.

4. Explain which type of surface that is described by the equation

$$5 = 2(x^2 + y^2 + z^2) + 6(xy + yz + zx)$$

where x, y, z denotes the coordinates of a point in an orthonormal system. Determine also the distance between the surface and the origin, and check whether any rotational symmetry exists. Determine if so for the latter an equation for the rotational axis.

5. Find a basis for the subspace $\mathbb{U} = \operatorname{span}\left\{\begin{pmatrix}-1 & 2\\-3 & 2\end{pmatrix}, \begin{pmatrix}5 & -3\\3 & -5\end{pmatrix}, \begin{pmatrix}2 & 3\\-6 & 1\end{pmatrix}, \begin{pmatrix}1 & 12\\-21 & 8\end{pmatrix}\right\}$ of the vector space of all real-valued 2×2-matrices for which the sum of the matrix elements is equal to zero. Also, find all numbers a for which $\begin{pmatrix}4a+3 & -7a-11\\a-1 & 2a+9\end{pmatrix} \in \mathbb{U}$, and determine for those numbers the coordinates of the vector in the chosen basis.

6. The linear transformation $F : \mathbb{R}^6 \to \mathbb{R}^3$ has in the standard bases the matrix

$$\begin{pmatrix} 3 & 2 & -1 & -5 & 17 & 2\\ 1 & 2 & -1 & 1 & 5 & 2\\ -2 & 1 & 2 & 3 & -5 & 1 \end{pmatrix}.$$

Find a basis for the kernel of F and a basis for the image of F.

7. The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ has relative to the standard basis the matrix

$$\begin{pmatrix} 1 & 1 & 2\\ a & 2 & 2-a\\ 0 & 1 & 3 \end{pmatrix}$$

where $a \in \mathbb{R}$. Find the values of a for which the operator är diagonalizable, and state a basis of eigenvectors for each of these a.

8. Determine the length of the orthogonal projection of the vector $2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$ on the vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ in the Euclidean space E for which the inner product is fixed as $\langle \mathbf{u}|\mathbf{v}\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2$, where x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of $\mathbf{u}$ and $\mathbf{v}$ respectively in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ ges av

$$F((x_1, x_2, x_3)) = (2x_1 - x_3, x_1 - x_2 + x_3, x_2 + 2x_3).$$

Bestäm F:s matris i den ordnade basen $(2, -1, 0), (3, 2, 1), (2, 2, 1)$, och bestäm i den basen koordinaterna för vektorn $F((1, 1, 1))$.

2. Låt $\mathcal{G}$ beteckna det linjära rum som spänns upp av de halvtaliga potensfunktionerna g_1 och g_2 definierade enligt $g_1(x) = \sqrt{x}$ och $g_2(x) = x\sqrt{x}$ på intervallet $[0, 1]$. Bestäm en ON-bas för $\mathcal{G}$ som har utrustats med skalärprodukten $\langle p|q\rangle = \int_0^1 p(t)q(t)\,dt$.

3. Bestäm, i standardbasen, matrisen för den linjära operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ som har nollrummet $\text{span}\{(2, -1, -1)\}$, som har egenvärdet -1 med egenrummet $\text{span}\{(1, 3, 2)\}$, och som avbildar vektorn $(1, 2, 1)$ på vektorn $(4, 1, 3)$.

4. Förklara vilken typ av yta som beskrivs av ekvationen

$$5 = 2(x^2 + y^2 + z^2) + 6(xy + yz + zx)$$

där x, y, z betecknar en punkts koordinater i ett ON-system. Bestäm även avståndet mellan ytan och origo, samt fastställ om någon rotationssymmetri föreligger. Bestäm i så fall för det senare en ekvation för rotationsaxeln.

5. Bestäm en bas för underrummet $\mathbb{U} = \text{span}\left\{\begin{pmatrix}-1 & 2\\-3 & 2\end{pmatrix}, \begin{pmatrix}5 & -3\\3 & -5\end{pmatrix}, \begin{pmatrix}2 & 3\\-6 & 1\end{pmatrix}, \begin{pmatrix}1 & 12\\-21 & 8\end{pmatrix}\right\}$ till det linjära rummet av alla reellvärda 2×2-matriser för vilka summan av matriselement är lika med noll. Bestäm även alla tal a för vilka $\begin{pmatrix}4a+3 & -7a-11\\a-1 & 2a+9\end{pmatrix} \in \mathbb{U}$, och bestäm för dessa tal koordinaterna för vektorn i den valda basen.

6. Den linjära avbildningen $F : \mathbb{R}^6 \to \mathbb{R}^3$ har i berörda standardbaser matrisen

$$\begin{pmatrix} 3 & 2 & -1 & -5 & 17 & 2\\ 1 & 2 & -1 & 1 & 5 & 2\\ -2 & 1 & 2 & 3 & -5 & 1 \end{pmatrix}.$$

Bestäm en bas för F:s nollrum och en bas för F:s värderum.

7. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ har i standardbasen matrisen

$$\begin{pmatrix} 1 & 1 & 2\\ a & 2 & 2-a\\ 0 & 1 & 3 \end{pmatrix}$$

där $a \in \mathbb{R}$. Bestäm de värden på a för vilka operatorn är diagonaliserbar, och ange en bas av egenvektorer till F för var och en av dessa a.

8. Bestäm längden av den ortogonala projektionen av vektorn $2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$ på vektorn $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ i det euklidiska rum E för vilket skalärprodukten är fixerad till $\langle \mathbf{u}|\mathbf{v}\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2$, där x_1, x_2, x_3 och y_1, y_2, y_3 är koordinaterna för $\mathbf{u}$ respektive $\mathbf{v}$ i basen $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

If you prefer the problems formulated in English, please turn the page.

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① $F: R^3 \to R^3$ is given by $F((x_1,x_2,x_3)) = (2x_1 - x_3, x_1 - x_2 + x_3, x_2 + 2x_3)$

i.e. $\begin{cases} F((1,0,0)) = (2,1,0) \\ F((0,1,0)) = (0,-1,1) \\ F((0,0,1)) = (-1,1,2) \end{cases}$ i.e. F has the matrix $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ in the standard basis

Let e_1, e_2, e_3 denote the standard basis, and $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ the other (ordered) basis $(2,-1,0), (3,2,1), (2,2,1)$. Then

$(\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3) = (e_1\ e_2\ e_3)\begin{pmatrix} 2 & 3 & 2 \\ -1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, i.e. $\tilde{e} = eS$ where

S is the change-of-basis matrix from e_1, e_2, e_3 to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

The matrix $\tilde{A}$ of F relative the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is then

$$\tilde{A} = S^{-1}AS = \frac{1}{(4+0-2)-(0+4-3)}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 2 & -6 \\ -1 & -2 & 7 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{1}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 2 & -6 \\ -1 & -2 & 7 \end{pmatrix}\begin{pmatrix} 4 & 5 & 3 \\ 3 & 2 & 1 \\ -1 & 4 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 6 & 7 \\ 16 & -15 & -19 \\ -17 & 19 & 23 \end{pmatrix}$$

Furthermore

$$F((1,1,1)) = F(e_1+e_2+e_3) = (F(e_1)\ F(e_2)\ F(e_3))\begin{pmatrix}1\\1\\1\end{pmatrix} = (e_1\ e_2\ e_3)\,A\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\,S^{-1}\begin{pmatrix}1\\1\\3\end{pmatrix} = (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\begin{pmatrix}5\\-15\\18\end{pmatrix} = 5\tilde{e}_1 - 15\tilde{e}_2 + 18\tilde{e}_3$$

i.e. the coordinates of $F((1,1,1))$ are $5, -15, 18$ in the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$

② $G = \text{span}\{g_1, g_2\}$ where $\begin{cases} g_1(x) = \sqrt{x} \\ g_2(x) = x\sqrt{x} \end{cases}$ in $[0,1]$.

is equipped with the inner product $\langle p|q\rangle = \int_0^1 p(t)q(t)\,dt$.

The Gram-Schmidt procedure gives

$$\begin{cases} f_1 = g_1 \text{ and } \|f_1\|^2 = \int_0^1 \sqrt{t}\sqrt{t}\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2} \\ e_1 = \frac{1}{\|f_1\|}f_1 = \sqrt{2}\,g_1 \end{cases}$$

$$\begin{cases} f_2 = g_2 - e_1\langle e_1|g_2\rangle = g_2 - e_1\int_0^1 \sqrt{2}\sqrt{t}\,t\sqrt{t}\,dt \\ \quad = g_2 - (\sqrt{2}\,g_1)\sqrt{2}\left[\frac{t^3}{3}\right]_0^1 = g_2 - \frac{2}{3}g_1 \\ \|f_2\|^2 = \int_0^1 \left(t\sqrt{t} - \frac{2}{3}\sqrt{t}\right)^2 dt = \int_0^1 \left(t^3 - \frac{4}{3}t^2 + \frac{4}{9}t\right)dt = \frac{1}{4} - \frac{4}{3}\frac{1}{3} + \frac{4}{9}\cdot\frac{1}{2} = \frac{1}{36} \\ e_2 = \frac{1}{\|f_2\|}f_2 = 6\left(g_2 - \frac{2}{3}g_1\right) = 6g_2 - 4g_1 \end{cases}$$

i.e. an ON-basis for G is e.g. $\sqrt{2}\,g_1,\ 2(3g_2 - 2g_1)$

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3 $F: R^3 \to R^3$ where $\begin{cases} \ker(F) = \text{span}\{(2,-1,-1)\} \\ E_{-1} = \text{span}\{(1,3,2)\} \quad \text{(eigenspace)} \\ F((1,2,1)) = (4,1,3) \end{cases}$

The matrix A of F relative to the standard basis is given by $A\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $A\begin{pmatrix}1\\3\\2\end{pmatrix} = -\begin{pmatrix}1\\3\\2\end{pmatrix}$, $A\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}4\\1\\3\end{pmatrix}$

i.e. by $A\begin{pmatrix}2&1&1\\-1&3&2\\-1&2&1\end{pmatrix} = \begin{pmatrix}0&-1&4\\0&-3&1\\0&-2&3\end{pmatrix}$ i.e. $A = \begin{pmatrix}0&-1&4\\0&-3&1\\0&-2&3\end{pmatrix}\begin{pmatrix}2&1&1\\-1&3&2\\-1&2&1\end{pmatrix}^{-1}$

i.e. $A = \begin{pmatrix}0&-1&4\\0&-3&1\\0&-2&3\end{pmatrix}\dfrac{1}{(6-2-2)-(-3+8-1)}\begin{pmatrix}-1&1&-1\\-1&3&-5\\1&-5&7\end{pmatrix} = \dfrac{1}{2}\begin{pmatrix}-5&23&-33\\-4&14&-22\\-5&21&-31\end{pmatrix}$

4 $5 = 2(x^2+y^2+z^2) + 6(xy+yz+zx) = X^T\begin{pmatrix}2&3&3\\3&2&3\\3&3&2\end{pmatrix}X = X^T G X$

where x, y, z are coordinates in an ON-system.

The eigenvalues of the (symmetric) operator which has the matrix G are given by

$0 = \det(G-\lambda I) = \det\begin{pmatrix}2-\lambda&3&3\\3&2-\lambda&3\\3&3&2-\lambda\end{pmatrix} = \det\begin{pmatrix}2-\lambda&3&3\\\lambda+1&-\lambda-1&0\\\lambda+1&0&-\lambda-1\end{pmatrix}$

$= (\lambda+1)^2\det\begin{pmatrix}2-\lambda&3&3\\1&-1&0\\1&0&-1\end{pmatrix} = (\lambda+1)^2\det\begin{pmatrix}5-\lambda&3&0\\1&-1&0\\1&0&-1\end{pmatrix} = -(\lambda+1)^2(\lambda-8)$

$\begin{cases} \lambda_{1,2} = -1: \quad G-\lambda_{1,2}I = \begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix} \sim \begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}; \quad k_{1,2} = t_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + t_2\begin{pmatrix}1\\-2\\1\end{pmatrix} \\ \text{(where at least one of } t_1 \text{ and } t_2 \text{ is nonzero)} \\ \lambda_3 = 8: \quad G-\lambda_3 I = \begin{pmatrix}-6&3&3\\3&-6&3\\3&3&-6\end{pmatrix} \sim \begin{pmatrix}0&9&-9\\0&-9&9\\3&3&-6\end{pmatrix} \sim \begin{pmatrix}1&1&-2\\0&1&-1\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}; \quad k_3 = t_3\begin{pmatrix}1\\1\\1\end{pmatrix} \end{cases}$

An ON-basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of eigenvectors is then e.g. $\begin{cases}\tilde{e}_1 = \frac{1}{\sqrt{2}}(e_1 - e_3) \\ \tilde{e}_2 = \frac{1}{\sqrt{6}}(e_1 - 2e_2 + e_3) \\ \tilde{e}_3 = \frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)\end{cases}$

The corresponding change-of-basis matrix S from e_1, e_2, e_3 to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is by this orthogonal, i.e. $S^{-1} = S^T$.

Thus $5 = X^T G X = (S\tilde{X})^T S\tilde{G}S^{-1}(S\tilde{X}) = \tilde{X}^T S^T S \tilde{G} S^{-1} S \tilde{X}$

$= \tilde{X}^T I \tilde{G} I \tilde{X} = \tilde{X}^T \tilde{G} \tilde{X} = -\tilde{x}^2 - \tilde{y}^2 + 8\tilde{z}^2$

since S is orthogonal

i.e. the equation describes a two-sheeted hyperboloid.

Since $8\tilde{z}^2 = \tilde{x}^2 + \tilde{y}^2 + 5 \geq 0 + 0 + 5$, the distance between the surface and the origin is equal to $\sqrt{\frac{5}{8}}$ l.u. $= \frac{1}{2}\sqrt{\frac{5}{2}}$ l.u.

Since $\lambda_1 = \lambda_2$, the surface has a rotational symmetry where the $\tilde{z}$-axis, i.e. $(x,y,z) = t(1,1,1)$, $t \in R$, is the symmetry axis.

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(5) $\mathcal{U} = \text{span}\left\{ \underbrace{\begin{pmatrix} -1 & 2 \\ -3 & 2 \end{pmatrix}}_{m_1}, \underbrace{\begin{pmatrix} 5 & -3 \\ 3 & -5 \end{pmatrix}}_{m_2}, \underbrace{\begin{pmatrix} 2 & 3 \\ -6 & 1 \end{pmatrix}}_{m_3}, \underbrace{\begin{pmatrix} 1 & 12 \\ -21 & 8 \end{pmatrix}}_{m_4} \right\}$, $\quad m = \begin{pmatrix} 4a+3 & -7a-11 \\ a-1 & 2a+9 \end{pmatrix}$

We know that $\mathcal{U}$ is a subspace of $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+c+d=0,\ a,b,c,d \in \mathbb{R} \right\}$ which in turn is a subspace of $M_{2,2}(\mathbb{R})$ which is spanned by e.g. $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_4}$.

We have that
$$\begin{cases} m_1 = -e_1 - 3e_2 + 2e_3 + 2e_4 \\ m_2 = 5e_1 + 3e_2 - 3e_3 - 5e_4 \\ m_3 = 2e_1 - 6e_2 + 3e_3 + e_4 \\ m_4 = e_1 - 21e_2 + 12e_3 + 8e_4 \end{cases}$$
and $m = (4a+3)e_1 + (a-1)e_2 + (-7a-11)e_3 + (2a+9)e_4$

A test equation for m_1, m_2, m_3, m_4, m is $\lambda_1 m_1 + \dots + \lambda_4 m_4 + \lambda m = 0$

i.e. $\begin{pmatrix} -1 & 5 & 2 & 1 & 4a+3 \\ -3 & 3 & -6 & -21 & a-1 \\ 2 & -3 & 3 & 12 & -7a-11 \\ 2 & -5 & 1 & 8 & 2a+9 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ i.e. $M\lambda = 0$ where

$$M \sim \begin{pmatrix} -1 & 5 & 2 & 1 & 4a+3 \\ 0 & -12 & -12 & -24 & -11a-10 \\ 0 & 7 & 7 & 14 & a-5 \\ 0 & 5 & 5 & 10 & 10a+15 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & -2 & -1 & -4a-3 \\ 0 & 1 & 1 & 2 & 2a+3 \\ 0 & -12 & -12 & -24 & -11a-10 \\ 0 & 7 & 7 & 14 & a-5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 9 & 6a+12 \\ 0 & 1 & 1 & 2 & 2a+3 \\ 0 & 0 & 0 & 0 & 13a+26 \\ 0 & 0 & 0 & 0 & -13a-26 \end{pmatrix}$$

We see that a basis for $\mathcal{U}$ is e.g. m_1, m_2, and that $m \in \mathcal{U}$ iff $a=-2$. In the latter case, $m = 0m_1 - 1m_2$ i.e. the coordinates of m are $0, -1$ in the chosen basis.

(6) $F: \mathbb{R}^6 \to \mathbb{R}^3$ has in the standard bases the matrix

$$\begin{pmatrix} 3 & 2 & -1 & -5 & 17 & 2 \\ 1 & 2 & -1 & 1 & 5 & 2 \\ -2 & 1 & 2 & 3 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 1 & 5 & 2 \\ 0 & -4 & 2 & -8 & 2 & -4 \\ 0 & 5 & 0 & 5 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 1 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 & 1 & -2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & -1 & -1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -3 & 6 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{pmatrix}$$

which gives that a basis for $\text{im}(F)$ is e.g. $(3,1,-2), (2,2,1), (-1,-1,2)$

and that $\ker(F) = \left\{ (x_1, \dots, x_6) \in \mathbb{R}^6 : \begin{matrix} x_1 - 3x_4 + 6x_5 = 0 \\ x_2 + x_4 + x_5 + x_6 = 0 \\ x_3 - 2x_4 + 3x_5 = 0 \end{matrix} \right\}$

i.e. $(x_1, \dots, x_6)_{\ker(F)} = (3x_4 - 6x_5, -x_4 - x_5 - x_6, 2x_4 - 3x_5, x_4, x_5, x_6)$

$= x_4(3,-1,2,1,0,0) + x_5(-6,-1,-3,0,1,0) + x_6(0,-1,0,0,0,1)$

i.e. $(3,-1,2,1,0,0), (-6,-1,-3,0,1,0), (0,-1,0,0,0,1)$ is e.g. a basis for $\ker(F)$

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7 $F: \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix $\begin{pmatrix} 1 & 1 & 2 \\ a & 2 & 2-a \\ 0 & 1 & 3 \end{pmatrix} = A$, $a \in \mathbb{R}$, relative to the standard basis.

Eigenvalues $\quad 0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 2 \\ a & 2-\lambda & 2-a \\ 0 & 1 & 3-\lambda \end{pmatrix} = \det\begin{pmatrix} 1-\lambda & 0 & \lambda-1 \\ a & 2-\lambda & 2-a \\ 0 & 1 & 3-\lambda \end{pmatrix}$

$$= -(\lambda-1)\det\begin{pmatrix} 1 & 0 & -1 \\ a & 2-\lambda & 2-a \\ 0 & 1 & 3-\lambda \end{pmatrix} = -(\lambda-1)\det\begin{pmatrix} 1 & 0 & 0 \\ a & 2-\lambda & 2 \\ 0 & 1 & 3-\lambda \end{pmatrix}$$

$$= -(\lambda-1)\cdot\left(1[(\lambda-2)(\lambda-3)-2] + 0 + 0\right) = -(\lambda-1)(\lambda^2-5\lambda+4) = -(\lambda-1)^2(\lambda-4)$$

$\lambda_{1,2} = 1$: $\quad A - \lambda_{1,2} I = \begin{pmatrix} 0 & 1 & 2 \\ a & 1 & 2-a \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} a & 0 & -a \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

F is diagonalizable iff the $\lambda_{1,2}$-eigenspace is two-dimensional i.e. iff $a = 0$. Then $k_{1,2} = t_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\lambda_3 = 4$: $\quad A - \lambda_3 I = \begin{pmatrix} -3 & 1 & 2 \\ a & -2 & 2-a \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -3 & 0 & 3 \\ a & 0 & -a \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$; $k_3 = t_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(independent of a)

i.e. F is diag. iff $a = 0$. $\quad$ A basis of eigenvectors is then e.g. $(1,0,0), (0,2,-1), (1,1,1)$.

8 $\langle u | v \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2$

$$= (x_1 \; x_2 \; x_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $\quad x_1, x_2, x_3$ and y_1, y_2, y_3 are the coordinates of u and v respectively in the basis e_1, e_2, e_3.

The orthogonal projection of $2e_1 - 3e_2 + e_3$ on $e_1 + e_2 + e_3$ is $(2e_1 - 3e_2 + e_3)_{e_1+e_2+e_3} = \dfrac{\langle 2e_1 - 3e_2 + e_3 | e_1 + e_2 + e_3 \rangle}{\| e_1 + e_2 + e_3 \|^2}(e_1 + e_2 + e_3)$

The length of the projection is

$$\| (2e_1 - 3e_2 + e_3)_{e_1+e_2+e_3} \| = \frac{|\langle 2e_1 - 3e_2 + e_3 | e_1 + e_2 + e_3 \rangle|}{\| e_1 + e_2 + e_3 \|}$$

$$= \frac{\left|(2 \; -3 \; 1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|}{\sqrt{(1 \; 1 \; 1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}} = \frac{\left|(2 \; -3 \; 1)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\right|}{\sqrt{(1 \; 1 \; 1)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}}} = \frac{|6-12+5|}{\sqrt{3+4+5}} = \frac{1}{\sqrt{12}}$$

$$= \frac{1}{2\sqrt{3}}$$

Examination 2016-01-13	Maximum points for subparts of the problems in the final examination
1. The matrix of F relative to the ordered basis $(2,-1,0),(3,2,1),(2,2,1)$ is equal to $$\begin{pmatrix} -5 & 6 & 7 \\ 16 & -15 & -19 \\ -17 & 19 & 23 \end{pmatrix}.$$ The coordinates of $F((1,1,1))$ relative to the ordered basis $(2,-1,0),(3,2,1),(2,2,1)$ are $5,-15,18$.	**1p**: Correctly identified the matrix of F relative to the standard basis and the change-of-basis matrix S from the standard basis to the basis $(2,-1,0),(3,2,1),(2,2,1)$ **2p**: Correctly found the matrix of F relative to the ordered basis $(2,-1,0),(3,2,1),(2,2,1)$ **2p**: Correctly found the coordinates of the vector $F((1,1,1))$ relative to the ordered basis $(2,-1,0),(3,2,1),(2,2,1)$
2. An ON-basis for the vector space G is e.g. $$\sqrt{2}g_1,\ 2(3g_2-2g_1)$$	**1p**: Correctly normalized one of the two functions g_1 and g_2 to become e_1 **1p**: Correctly formulated a linear combination of the two functions g_1 and g_2 which is not equal to the zero function and which is orthogonal to the first selected function, e.g. the function $g_2-e_1\langle e_1\vert g_2\rangle$, **1p**: Correctly evaluated the inner product $\langle e_1\vert g_2\rangle$ **2p**: Correctly normalized $g_2-e_1\langle e_1\vert g_2\rangle$ to become e_2, and correctly stated e_1,e_2 as an ON-basis for $\mathcal{G}$
3. F has the matrix $$\frac{1}{2}\begin{pmatrix} -5 & 23 & -33 \\ -4 & 14 & -22 \\ -5 & 21 & -31 \end{pmatrix}$$ relative to the standard basis	**1p**: Correctly interpreted the vector spanning the kernel, i.e. that $\mathsf{A}(2\ \ -1\ \ -1)^T=(0\ \ 0\ \ 0)^T$ for the matrix A of F **1p**: Correctly interpreted the vector spanning the eigenspace, i.e. that $\mathsf{A}(1\ \ 3\ \ 2)^T=-1(1\ \ 3\ \ 2)^T$ for the matrix A of F **1p**: Correctly, based on the three given conditions, formulated an equation for the matrix A **2p**: Correctly solved the equation for the matrix A
4. By diagonalization, the equation may be reformulated as $5=-\tilde{x}^2-\tilde{y}^2+9\tilde{z}^2$ which, based on the fact that $\tilde{x},\tilde{y},\tilde{z}$ denotes the coordinates of a point in an ON-system, describes a two-sheeted hyperboloid with an rotational symmetry axis along the $\tilde{z}$-axis $(x,y,z)=t(1,1,1)$, $t\in R$, and with a distance to the origin equal to $\frac{1}{2}\sqrt{\frac{5}{2}}$ l.u.	**2p**: Correctly found that $5=2(x^2+y^2+z^2)+6(xy+yz+zx)$ describes a two-sheeted hyperboloid **1p**: Correctly determined the distance between the origin and the surface **2p**: Correctly determined the rotational symmetry axis

5. A basis for $\mathcal{U}$ is e.g. m_1, m_2 where

$$m_1 = \begin{pmatrix} -1 & 2 \\ -3 & 2 \end{pmatrix},\ m_2 = \begin{pmatrix} 5 & -3 \\ 3 & -5 \end{pmatrix}$$

$$m = \begin{pmatrix} 4a+3 & -7a-11 \\ a-1 & 2a+9 \end{pmatrix} \in \mathcal{U} \text{ iff } a=-2$$

The coordinates of m in the chosen basis m_1, m_2 are $0, -1$, i.e. $m = 0m_1 - 1m_2$

2p: Correctly found a basis for $\mathcal{U}$
1p: Correctly determined that the fifth vector belong to $\mathcal{U}$ iff $a=-2$
2p: Correctly found the coordinates of the fifth vector relative to the chosen basis

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6. A basis for the kernel of F is e.g. $(3,-1,2,1,0,0)$, $(-6,-1,-3,0,1,0)$, $(0,-1,0,0,0,1)$.

A basis for the image of F is e.g. $(3,1,-2)$, $(2,2,1)$, $(-1,-1,2)$.

3p: Correctly determined a basis for the kernel of F
2p: Correctly determined a basis for the image of F

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7. The linear operator is diagonalizable iff $a=0$, and a basis of eigenvectors is then e.g. $(1,0,0)$, $(0,2,-1)$, $(1,1,1)$.

3p: Correctly found that the linear operator is diagonalizable iff $a=0$
2p: Correctly for $a=0$ found a basis of eigenvectors

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8. $\left\| (2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3)_{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} \right\| = \dfrac{1}{2\sqrt{3}}$

1p: Correctly stated an expression for the orthogonal projection of $2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$ on $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
1p: Correctly interpreted how the given inner product is applied
1p: Correctly determined the inner product of the vectors $2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
1p: Correctly determined the length of the vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
1p: Correctly determined the length of the orthogonal projection

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