Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2016-11-04
Hjälpmedel: penna,
sudd och linjal

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Lös ekvationen $z^2 - 3iz - (3+i) = 0$. Ange lösningarna på formen $a + bi$. (5p)

2 Låt $T : \mathbb{R}^3 \to \mathbb{R}^3$ vara den linjära transformationen som ges av

$$T(\mathbf{x}) = \begin{bmatrix} x_3 - x_2 \\ 2x_1 - x_2 + 4x_3 \\ 2x_1 + 3x_3 \end{bmatrix}, \text{ där } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

a. Bestäm en matris A så att $T(\mathbf{x}) = A\,\mathbf{x}$. Motivering krävs. (2p)

b. Avgör huruvida T är $1-1$. (2p)

3 Låt V vara det linjära höljet till S, dvs $V = \text{span}(S)$, där S består av vektorerna

$$\mathbf{v}_1 = (1, 2, -1), \mathbf{v}_2 = (-2, -4, 2), \mathbf{v}_3 = (1, 4, 3), \text{ och } \mathbf{v}_4 = (1, 1, -3).$$

a. Bestäm en bas för V som består av vektorer ur S. (3p)

b. Bestäm en ortogonal bas till V. (3p)

4 Bestäm alla egenvärden och tre linjärt oberoende egenvektorer till matrisen (5p)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 4 & -6 \\ 9 & 3 & -5 \end{bmatrix}.$$

5 Låt V vara underrummet till $\mathbb{R}^3$ som ges av planet $x - 2y - 3z = 0$. Bestäm en bas B för V sådan att koordinatvektorn för $\mathbf{u} = (-4, 1, -2) \in V$ i basen B blir $(1, 2)$, dvs $(\mathbf{u})_B = (1, 2)_B$. (5p)

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Examination Vector algebra
MAA150 - TEN2
Date: Nov 4, 2016
Exam aids: pencil,
eraser and ruler

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Solve the equation $z^2 - 3iz - (3+i) = 0$. Give the solutions in the form $a + bi$. (5p)

2 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{x}) = \begin{bmatrix} x_3 - x_2 \\ 2x_1 - x_2 + 4x_3 \\ 2x_1 + 3x_3 \end{bmatrix}, \text{ where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

a. Find a matrix A such that $T(\mathbf{x}) = A\,\mathbf{x}$. Motivation is required. (2p)
b. Determine whether or not T is $1-1$. (2p)

3 Let V be the span of S, i.e. $V = \operatorname{span}(S)$, where S consist of the vectors

$$\mathbf{v}_1 = (1,2,-1), \mathbf{v}_2 = (-2,-4,2), \mathbf{v}_3 = (1,4,3), \text{ and } \mathbf{v}_4 = (1,1,-3).$$

a. Find a basis for V consisting of vectors from S. (3p)
b. Find an orthogonal basis for V. (3p)

4 Find all eigenvalues and three linearly independent eigenvectors of the matrix (5p)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 4 & -6 \\ 9 & 3 & -5 \end{bmatrix}.$$

5 Let V be the subspace of $\mathbb{R}^3$ given by the plane $x - 2y - 3z = 0$. Find a basis B for V such that the coordinate vector of $\mathbf{u} = (-4,1,-2) \in V$ relative to B is $(1,2)$, i.e $(\mathbf{u})_B = (1,2)_B$. (5p)

MAA150 Vektoralgebra, HT2016

Assessment criteria for TEN2 2016-11-04

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. [**5 points**]
Correct method gives maximum 2 points, where setting $z = x + yi$ **(1p)** and finding the equation system for x and y **(1p)**. Solving for x and y **(2p)**. The correct solutions **(1p)**

2. [**4 points**]
a. Correct method with motivation **(1p)**, the correct standard matrix **(1p)**
b. A condition for T being $1 - 1$ **(1p)**, checking the condition **(1p)**

3. [**6 points**]
a. Correct method for finding the basis **(2p)**, a correct basis **(1p)**
b. Correct method for constructing an orthogonal basis **(2p)**, correct and relevant computations **(1p)**

4. [**5 points**]
Characteristic equation **(1p)**, correct eigenvalues **(1p)**, method of finding the eigenvectors **(2p)**, correct eigenvectors **(1p)**

5. [**5 points**]
finding the basis gives maximum 3p, where the proper form of vectors in V gives **(1p)**, an equation for the coordinate vector being $(1, 2)$ **(1p)**, two correct vectors **(1p)**, motivation that the vectors form a basis **(2p)**

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(1) $\boxed{z^2 - 3iz - (3+i) = 0}$ (*)

Completing the square: $\left(z - \frac{3i}{2}\right)^2 - \left(\frac{3i}{2}\right)^2 - (3+i) =$

$= \left(z - \frac{3i}{2}\right)^2 + \frac{9}{4} - 3 - i = \left(z - \frac{3i}{2}\right)^2 - \frac{3}{4} - i$. Then (*) can be written

$\boxed{\left(z - \frac{3i}{2}\right)^2 = \frac{3}{4} + i}$. Setting $w = z - \frac{3i}{2} = x + i \cdot y$ yields

$w^2 = (x+iy)^2 = x^2 + 2xyi - y^2 = \frac{3}{4} + i$ (1p)

Identifying real and imaginary parts gives

$$\begin{cases} x^2 - y^2 = \frac{3}{4} & (1) \\ 2xy = 1 & (2) \end{cases}$$ (1p)

Extra equation: $|w^2| = x^2 + y^2 = \left|\frac{3}{4} + i\right| = \sqrt{\frac{9}{16} + 1} = \frac{5}{4}$

$$\begin{cases} x^2 - y^2 = \frac{3}{4} \\ x^2 + y^2 = \frac{5}{4} \end{cases} \Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1$$

$x = \pm 1 \underset{(2)}{\Rightarrow} y = \pm\frac{1}{2}$ so $\begin{cases} z - \frac{3i}{2} = 1 + \frac{i}{2} \\ z - \frac{3i}{2} = -1 - \frac{i}{2} \end{cases}$ (2p)

__Answer:__ $z = 1 + 2i$ or $z = -1 + i$ (1p)

Check: $(1+2i)^2 - 3i(1+2i) - (3+i) = \cancel{1} + \cancel{4i} - \cancel{4} - \cancel{3i} + \cancel{6} - \cancel{3} - \cancel{i} = 0$ ok!

$(-1+i)^2 - 3i(-1+i) - (3+i) = \cancel{1} - \cancel{2i} - \cancel{1} + \cancel{3i} + \cancel{3} - \cancel{3} - \cancel{i} = 0$ ok!

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(2) $T(x) = \begin{bmatrix} x_3 - x_2 \\ 2x_1 - x_2 + 4x_3 \\ 2x_1 + 3x_3 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

a) Find A such that $T(\bar{x}) = A \cdot \bar{x}$, i.e find the standard matrix $[T]$. Since

$T(\bar{e}_1) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $T(\bar{e}_2) = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, $T(\bar{e}_3) = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ we get

$$A = [T] = \left[T(\bar{e}_1)\ T(\bar{e}_2)\ T(\bar{e}_3) \right] = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -1 & 4 \\ 2 & 0 & 3 \end{bmatrix}$$ (2p)

b) T is 1-1 $\Leftrightarrow$ $[T]$ is invertible $\Leftrightarrow$ $\det([T]) \neq 0$ (1p)

$$\begin{vmatrix} 0 & -1 & 1 \\ 2 & -1 & 4 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} = 2(-1+1) = 0$$

Cofactor exp.

Therefore T is not 1-1. (1p)

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(3) $S=\{\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4\}$, $\bar{v}_1=(1,2,-1)$, $\bar{v}_2=(-2,-4,2)$, $\bar{v}_3=(1,4,3)$ and $\bar{v}_4=(1,1,-3)$.

a) $$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & -4 & 4 & 1 \\ -1 & 2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (2p)

basis correspond to leading 1:s.

So $B=\{\bar{v}_1,\bar{v}_3\}=\{(1,2,-1),(1,4,3)\}$ is a basis for V. (1p)

b) Orthogonalize B using Gram-Schmidt

Set $\bar{u}_1=\bar{v}_1=\begin{bmatrix}1\\2\\-1\end{bmatrix}$ and

$$\bar{u}_2=\bar{v}_3-\text{proj}_{\bar{u}_1}\bar{v}_3=\begin{bmatrix}1\\4\\3\end{bmatrix}-\frac{(1,4,3)\cdot(1,2,-1)}{\|(1,2,-1)\|^2}\begin{bmatrix}1\\2\\-1\end{bmatrix}=$$

$$=\begin{bmatrix}1\\4\\3\end{bmatrix}-\frac{6}{6}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}0\\2\\4\end{bmatrix}$$ (2p)

Then $B=\{(1,2,-1),(0,2,4)\}$ is an orthogonal basis for V. (1p)

Check: orthogonal: $(1,2,-1)\cdot(0,2,4)=0+4-4=0$ ok.

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(4) $A=\begin{bmatrix}1&0&0\\9&4&-6\\9&3&-5\end{bmatrix}$

Cofactor expansion

$$\det(A-\lambda\cdot I)=\begin{vmatrix}1-\lambda&0&0\\9&4-\lambda&-6\\9&3&-5-\lambda\end{vmatrix}=(1-\lambda)\cdot\begin{vmatrix}4-\lambda&-6\\3&-5-\lambda\end{vmatrix}=$$

$$=(1-\lambda)((4-\lambda)\cdot(-5-\lambda)+18)=(1-\lambda)\cdot(-20+\lambda+\lambda^2+18)$$

$\lambda^2+\lambda-2=0 \Rightarrow \lambda=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{8}{4}}=-\frac{1}{2}\pm\frac{3}{2}$ so

(CE) $\det(A-\lambda\cdot I)=0 \Leftrightarrow \boxed{(1-\lambda)\cdot(\lambda+2)\cdot(\lambda-1)=0}$

Therefore the eigenvalues are $\lambda=1$ and $\lambda=-2$ (2p)

Eigenvectors

$\boxed{\lambda=1}$: $(A-I)\bar{v}=\bar{0}$ gives $\left[\begin{array}{ccc|c}0&0&0&0\\9&3&-6&0\\9&3&-6&0\end{array}\right]\sim\left[\begin{array}{ccc|c}3&1&-2&0\\0&0&0&0\\0&0&0&0\end{array}\right]$

Free variables

$v_2=s,\ v_3=t \Rightarrow v_1=\frac{2t-s}{3}$ so $\bar{v}=\begin{bmatrix}\frac{2t-s}{3}\\s\\t\end{bmatrix}=s\cdot\begin{bmatrix}-1/3\\1\\0\end{bmatrix}+t\cdot\begin{bmatrix}2/3\\0\\1\end{bmatrix}$

Hence $\begin{bmatrix}-1/3\\1\\0\end{bmatrix}$ and $\begin{bmatrix}2/3\\0\\1\end{bmatrix}$ are eigenvectors to $\lambda=1$.

$\boxed{\lambda=-2}$: $(A+2I)\bar{v}=\bar{0}$ gives $\left[\begin{array}{ccc|c}3&0&0&0\\9&6&-6&0\\9&3&-3&0\end{array}\right]\sim\left[\begin{array}{ccc|c}3&0&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$

free

$v_3=t \Rightarrow v_2=t, v_1=0$ so $\bar{v}=t\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ is an eigenvector to $\lambda=-2$.

(3p)

Answer: $\lambda=1$ has eigenvectors $\begin{bmatrix}-1/3\\1\\0\end{bmatrix}$ and $\begin{bmatrix}2/3\\0\\1\end{bmatrix}$

$\lambda=-2$ has eigenvector $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

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(5) $V = \{(x,y,z) \in \mathbb{R}^3 : x - 2y - 3z = 0\}$

Find a basis for V by solving $x - 2\underset{\uparrow}{y} - 3\underset{\uparrow}{z} = 0$ ← free variables.

Set $y = s$ and $z = t \Rightarrow x = 2s + 3t$, so every vektor in V is on the form $(x,y,z) = (2s+3t, s, t)$, where $s,t \in \mathbb{R}$ (*) (1p)

Take $\bar{v}_1 \in V$ not colinear to $(-4,1,-2)$, e.g. ($s=0, t=1$ in (*)) $\bar{v}_1 = (3,0,1)$.
We wish to find $\bar{v}_2 = (2s+3t, s, t)$ such that

$$[(-4,1,-2)]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_B \Leftrightarrow 1 \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} \quad (1p)$$

$$\Leftrightarrow \begin{cases} 4s + 6t = -7 \\ 2s = 1 \\ 2t = -3 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 4 & 6 & -7 \\ 2 & 0 & 1 \\ 0 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \end{array}\right] \quad \begin{array}{l} s = 1/2 \\ t = -3/2 \end{array}$$

so $\bar{v}_2 = \left(2 \cdot \frac{1}{2} + 3 \cdot \left(\frac{-3}{2}\right), 1/2, -3/2\right) = \left(-\frac{7}{2}, \frac{1}{2}, -\frac{3}{2}\right)$ (1)p

$B = \{\bar{v}_1, \bar{v}_2\}$ is linearly independent since they differ in the third component. $W = \text{span}(B) \subset V$, but since $\dim(W) = 2$ and $\dim(V) = 2$ (V is a plane $\mathbb{R}^3$) we must have $W = V$ so B is a basis for V with $[(-4,1,-2)]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_B$. (2p)

Check: $1 \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -7/2 \\ 1/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}$ ok!

$\bar{v}_1 \in V$: $3 - 2 \cdot 0 - 3 \cdot 1 = 0$ ok!

$\bar{v}_2 \in V$: $-\frac{7}{2} - 2 \cdot \frac{1}{2} - 3 \cdot \left(\frac{-3}{2}\right) = 0$ ok!