MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2015-03-23 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Sketch the graph of the function f, defined by

$$f(x) = \frac{x^3}{9(x-2)}\,,$$

by utilizing the guidance given by asymptotes and stationary points.

2. Evaluate the integral

$$\int_1^{\sqrt{3}} x \arctan(x)\, dx\,,$$

and write the result in as simple form as possible.

3. Solve the initial-value problem $\begin{cases} y' = 1 - y^2, \\ y(0) = 0\,. \end{cases}$

4. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^{2n} + 3^n}{3n + \pi^n}$$

converges or not. Irrespective whether the answer is YES or NO, give an explanation of why!

5. Find the area of the surface generated by rotating the curve

$$y = \tfrac{1}{4}x^2, \quad 0 \leq x \leq \sqrt{12}\,,$$

about the y-axis.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2015-03-23 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Skissa grafen till funktionen f, definierad enligt

$$f(x) = \frac{x^3}{9(x-2)}\,,$$

genom att använda den vägledning som fås från asymptoter och stationära punkter.

2. Beräkna integralen

$$\int_1^{\sqrt{3}} x \arctan(x)\, dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

3. Lös begynnelsevärdesproblemet $\begin{cases} y' = 1 - y^2, \\ y(0) = 0\,. \end{cases}$

4. Avgör om serien

$$\sum_{n=1}^{\infty} \frac{2^{2n} + 3^n}{3n + \pi^n}$$

är konvergent eller ej. Oavsett om svaret är JA eller NEJ, ge en förklaring till varför!

5. Bestäm arean av den yta som genereras genom att rotera kurvan

$$y = \tfrac{1}{4}x^2, \quad 0 \leq x \leq \sqrt{12}\,,$$

kring y-axeln.

If you prefer the problems formulated in English, please turn the sheet.

MAA151 / Solutions to the final exam TEN2 2015-03-23

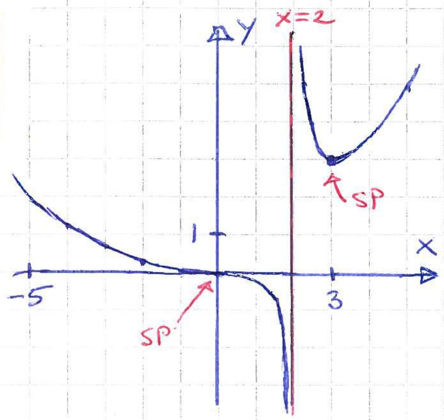
1 $f(x) = \dfrac{x^3}{9(x-2)}$

$$f'(x) = \frac{1}{9}\,\frac{3x^2(x-2) - x^3\cdot 1}{(x-2)^2} = \frac{1}{9}\,\frac{2x^3-6x^2}{(x-2)^2} = \frac{2x^2(x-3)}{9(x-2)^2}$$

First derivative test

x		0		2		3	
$f'(x)$	$-$	0	$-$	$\nexists$	$-$	0	$+$
$f(x)$	$\searrow$	ter.	$\searrow$	$\nexists$	$\searrow$	loc min	$\nearrow$

$\boxed{\begin{aligned} f(0)&=0 \\ f(3)&=3 \end{aligned}}$



$x=2$ is a double-sided vertical asymptote of the curve $y=f(x)$ (since $f(x) \xrightarrow{x\to 2^{\pm}} \pm\infty$). The are no non-vertical asymptotes (since degree$(x^3) > 1 +$ degree$(9(x-2))$).

2 $$\int_1^{\sqrt{3}} x\arctan(x)\,dx \overset{\text{Partial integration}}{=} \left[\frac{x^2}{2}\cdot\arctan(x)\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{x^2}{2}\,\frac{dx}{1+x^2}$$

$$= \left(\frac{3}{2}\frac{\pi}{3} - \frac{1}{2}\frac{\pi}{4}\right) - \frac{1}{2}\int_1^{\sqrt{3}} \frac{x^2+1-1}{1+x^2}\,dx = \left(\frac{\pi}{2}-\frac{\pi}{8}\right) - \frac{1}{2}\int_1^{\sqrt{3}}\left(1-\frac{1}{1+x^2}\right)dx$$

$$= \frac{(4-1)\pi}{8} - \frac{1}{2}\Big[x - \arctan(x)\Big]_1^{\sqrt{3}} = \frac{3\pi}{8} - \frac{1}{2}\left[\left(\sqrt{3}-\frac{\pi}{3}\right) - \left(1-\frac{\pi}{4}\right)\right]$$

$$= \frac{3\pi}{8} - \frac{1}{2}(\sqrt{3}-1) + \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{3\pi}{8} + \frac{\pi}{24} - \frac{1}{2}(\sqrt{3}-1) = \underline{\frac{1}{2}\left[\frac{5\pi}{6} - (\sqrt{3}-1)\right]}$$

(alt. order of eval. of the limits $\ldots = \frac{1}{2}\Big[(x^2+1)\arctan(x) - x\Big]_1^{\sqrt{3}} = \frac{1}{2}\left[\left(4\frac{\pi}{3} - 2\cdot\frac{\pi}{4}\right) - (\sqrt{3}-1)\right] = \frac{1}{2}\left[\frac{5\pi}{6} - (\sqrt{3}-1)\right]$)

3 $$\begin{cases} y' = 1-y^2 & \text{(DE)} \\ y(0) = 0 & \text{(IC)} \end{cases}$$

The DE is separable and may (for $-1<y<1$) be written as

$$\frac{1}{1-y^2}\frac{dy}{dx} = 1 \iff \frac{1}{2}\left(\frac{1}{1+y} + \frac{1}{1-y}\right)\frac{dy}{dx} = 1$$

Operating with $\int dx$ gives $\ln|1+y| - \ln|1-y| = 2x + C$

$\iff \ln\left(\dfrac{1+y}{1-y}\right) = 2x + C$ where the abs. bars are not needed since we have that $-1<y<1$

Adapting to the IC gives $\ln\left(\frac{1+0}{1-0}\right) = 2\cdot 0 + C$ i.e. $C = 0$

Thus $\dfrac{1+y}{1-y} = e^{2x} \iff 1+y = e^{2x}(1-y) \iff \underline{y = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{e^x - e^{-x}}{e^x+e^{-x}} = \tanh(x)}$

Cont. MAA151
TEN2 2015-03-23

4 $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2^{2n}+3^n}{3n+\pi^n} = \dfrac{4^n}{\pi^n}\dfrac{1+\left(\frac{3}{4}\right)^n}{1+\frac{3n}{\pi^n}}$

from which we note that $a_n \to \infty$ as $n \to \infty$

i.e. the series diverges.

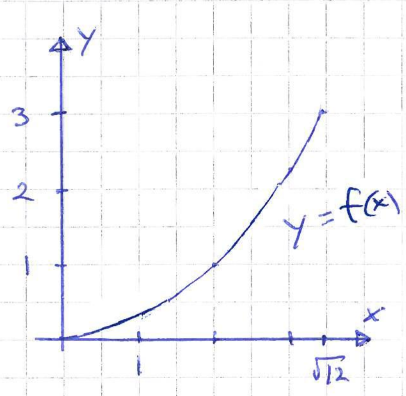
This may also be concluded from

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{4}{\pi}\right)^{n+1}\left(\frac{1+\left(\frac{3}{4}\right)^{n+1}}{1+\frac{3(n+1)}{\pi^{n+1}}}\right)\left(\frac{\pi}{4}\right)^n\left(\frac{1+\frac{3n}{\pi^n}}{1+\left(\frac{3}{4}\right)^n}\right)$$

$$= \frac{4}{\pi}\,\frac{1+0}{1+0}\,\frac{1+0}{1+0} = \frac{4}{\pi} > 1$$

from which the ratio test says that $\sum_{n=1}^{\infty} a_n$ diverges.

5



$$\begin{cases} f(x) = \frac{1}{4}x^2 \\ f'(x) = \frac{1}{2}x \end{cases}$$

$\sqrt{12} = \sqrt{4\cdot 3} = 2\sqrt{3}$

The area A_y of the surface generated by rotating, about the y-axis, the curve $y = \frac{1}{4}x^2$, $0 \le x \le \sqrt{12}$, is

$$A_y = \int_0^{\sqrt{12}} 2\pi x\sqrt{1+\left(\frac{d}{dx}\left(\frac{1}{4}x^2\right)\right)^2}\,dx$$

$$= 2\pi\int_0^{2\sqrt{3}} x\,dx\sqrt{1+\frac{1}{4}x^2} \qquad \left[\begin{aligned} 1+\tfrac{1}{4}x^2 &= u \\ \tfrac{1}{2}x\,dx &= du \end{aligned}\right]$$

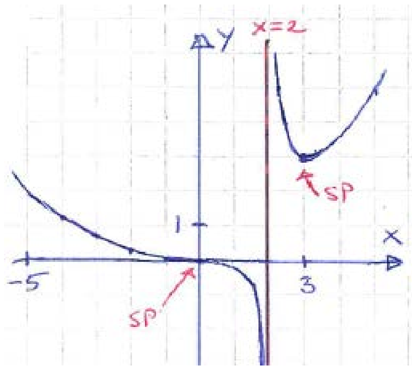
$$= 2\pi\int_1^4 2\,du\sqrt{u} = 4\pi\left[\frac{u^{3/2}}{3/2}\right]_1^4$$

$$= 4\pi\cdot\frac{2}{3}\left(4\sqrt{4}-1\sqrt{1}\right) \text{ a.u.}$$

$$= \frac{8\pi}{3}(8-1) \text{ a.u.} = \frac{56\pi}{3} \text{ a.u.}$$

Examination TEN2 – 2015-03-23 Maximum points for subparts of the problems in the final examination

1. The graph has an asymptote $x=2$, a local minimum at $(3,3)$ and a terrace point at $(0,0)$



1p: Correctly found the vertical asymptote of the graph and how the graph relates to the asymptote on its both sides
1p: Correctly found the local minimum point of the graph
1p: Correctly found the terrace point of the graph
1p: Correctly sketched the graph

2. $\frac{1}{2}\left(\frac{5\pi}{6}-\left(\sqrt{3}-1\right)\right)$

1p: Correctly integrated by parts as a first step in the evaluation of the integral
1p: Correctly worked out the polynomial division in the remaining integral to prepare for a final step
1p: Correctly found the antiderivative of the integrand in the remaining integral
1p: Correctly evaluated the limits of the integral

3. $y=\frac{e^{2x}-1}{e^{2x}+1}=\tanh(x)$

1p: Correctly identified the differential equation as nonlinear and separable, and correctly worked out the partial fractions of the expression in y
1p: Correctly found the antiderivatives of both sides of the separated differential equation
1p: Correctly adapted the equation to the initial value
1p: Correctly solved for y

4. The series is divergent

1p: Correctly identified that the terms of the series are dominated by exponentials 4^n in the numerator and by the exponentials π^n in the the denominator

----- **One scenario for the other three points** -------

1p: Correctly noted that the terms of the series does not have the limit zero (in fact, in this case goes to infinity)
2p: Correctly concluded that the series diverges

------------ **Another scenario for the other three points** -----------------------

2p: Correctly found that the limit of the test quantity in the ratio test equals $4/\pi$
1p: Correctly, from the fact that $4/\pi>1$ concluded that the series diverges

5. $\frac{56\pi}{3}$ a.u.

1p: Correctly formulated an integral expression for the area of the surface generated by the rotated curve about the y-axis
2p: Correctly changed variables by a suitable substitution which simplifies finding an antiderivative
1p: Correctly determined the integral