Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2017-08-16
Hjälpmedel: penna,
sudd och linjal.

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1a. Lös ekvationssystemet (3p)

$$\begin{aligned} 3x_1 - 9\,x_2 + 6\,x_3 + 3\,x_4 &= 3 \\ -\,x_1 + 3\,x_2 + 2\,x_3 - x_4 &= -1 \end{aligned}$$

b. Bestäm ett villkor på konstanterna b_1, b_2, och b_3 för att ekvationssystemet nedan skall vara konsistent. (3p)

$$\begin{aligned} 2\,x + 4\,y - 6\,z &= b_1 \\ 3x + 4\,y - 4z &= b_2 \\ 2x + 2\,y - z &= b_3 \end{aligned}$$

2 För vilka reella tal c är matrisen nedan inverterbar? (4p)

$$\begin{bmatrix} 2c & -c & 1 \\ 1 & 1 & 2 \\ 3 & -2 & c \end{bmatrix}$$

3 Lös ekvationen $A^T X + B = 2I$ med hjälp av matrisoperationer och invers, där (5p)

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ och } B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

4 Givet linjen $l : 2x - y + 1 = 0$.

a. Finn vektorformen för linjen l. (2p)

b. Bestäm normen av projektionen av $\mathbf{u} = (1, 5)$ i ritningen av normalen till linjen l. (4p)

5 Visa att punkterna $A(1, 2, -1)$, $B(1, 3, 2)$, $C(-1, 1, -3)$ och $D(-1, 3, 3)$ tillhör ett plan i $\mathbb{R}^3$. (4p)

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Examination Vector algebra
MAA150 - TEN1
Date: August 16, 2017
Exam aids: pencil,
eraser, and ruler.

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1a. Solve the linear system (3p)

$$\begin{aligned} 3x_1 - 9\,x_2 + 6\,x_3 + 3\,x_4 &= 3 \\ -\,x_1 + 3\,x_2 + 2\,x_3 - x_4 &= -1 \end{aligned}$$

b. Find a condition for the constants b_1, b_2, and b_3 such that the linear system below is consistent. (3p)

$$\begin{aligned} 2\,x + 4\,y - 6\,z &= b_1 \\ 3x + 4\,y - \ 4z &= b_2 \\ 2x + 2\,y - \ z &= b_3 \end{aligned}$$

2 For which real numbers c is the matrix below invertible? (4p)

$$\begin{bmatrix} 2c & -c & 1 \\ 1 & 1 & 2 \\ 3 & -2 & c \end{bmatrix}$$

3 Solve the equation $A^T X + B = 2I$ by using matrix operations and invers, where (5p)

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

4 Given the line $l : 2x - y + 1 = 0$.

a. Find the vector form of the line l. (2p)

b. Find the norm of the projection of $\mathbf{u} = (1, 5)$ in the direction of the normal of the line l. (4p)

5 Show that the points $A(1, 2, -1)$, $B(1, 3, 2)$, $C(-1, 1, -3)$, and $D(-1, 3, 3)$ belongs to a plane in $\mathbb{R}^3$. (4p)

MAA150 Vektoralgebra, VT2017.

Assessment criteria for TEN1 2017-08-16

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Points may be deducted for erroneous mathematical statements, calculations, or failure to use proper mathematical notation.

Assessment problems

1. [**6 points**]
   a. Relevant row operations **(1p)**, setting free variables correctly **(1p)**, the correct solution **(1p)**.
   b. Solving the system/relevant row operations **(2p)**, a correct condition **(1p)**

2. [**4 points**]
   A condition for A being invertible **(1p)**, checking the condition, e.g. by evaluating the determinant **(2p)**, correct values of c **(1p)**

3. [**5 points**]
   Solving the system with matrix operations **(2p)**, finding the inverse of A^T including computations **(2p)**, correct solution **(1p)**

4. [**6 points**]
   a. Finding the vector form of the line **(2p)**
   b. Finding a normal to l **(1p)**, computing the projection **(2p)**, computing the norm of the projection **(1p)**

5. [**4 points**]
   A condition that the points belong to a plane **(1p)**, computing relevant vectors **(1p)**, checking the condition and correct conclusion **(2p)**

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1a
$$\left[\begin{array}{cccc|c} 3 & -9 & 6 & 3 & 3 \\ -1 & 3 & 2 & -1 & -1 \end{array}\right] \overset{3}{\leftarrow} \sim \left[\begin{array}{cccc|c} 0 & 0 & 12 & 0 & 0 \\ -1 & 3 & 2 & -1 & -1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} -1 & 3 & 2 & -1 & -1 \\ 0 & 0 & 12 & 0 & 0 \end{array}\right] \begin{array}{l} \times(-1) \\ \times(\frac{1}{12}) \end{array} \sim \left[\begin{array}{cccc|c} 1 & -3 & -2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \overset{2}{\leftarrow}$$ (1p)

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

free: x_2, x_4

$x_2 = s, x_4 = t \Rightarrow x_3 = 0$ and
$x_1 = 1 + 3s - t$ (1p)

<u>Answer a:</u> $x_1 = 1+3s-t, x_2 = s, x_3 = 0, x_4 = t$ where $s, t \in \mathbb{R}$ (1p)

1b
$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & b_1 \\ 3 & 4 & -4 & b_2 \\ 2 & 2 & -1 & b_3 \end{array}\right] \begin{array}{l} (-\frac{3}{2}) \; (-1) \end{array} \sim \left[\begin{array}{ccc|c} 2 & 4 & -6 & b_1 \\ 0 & -2 & 5 & b_2 - \frac{3}{2}b_1 \\ 0 & -2 & 5 & b_3 - b_1 \end{array}\right] (-1)$$

$$\sim \left[\begin{array}{ccc|c} 2 & 4 & -6 & b_1 \\ 0 & -2 & 5 & b_2 - \frac{3}{2}b_1 \\ 0 & 0 & 0 & b_3 - b_2 + \frac{1}{2}b_1 \end{array}\right] (3)$$

so equation (3) (2p)

will be $0 = b_3 - b_2 + \frac{1}{2}b_1$. If this is satisfied the system has z as a free variable and is consistent. Otherwise it is inconsistent. (1p)

<u>Answer b:</u> The linear system is consistent iff
$$0 = b_3 - b_2 + \frac{1}{2}b_1$$

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(2) The matrix A is invertible $\Leftrightarrow \det(A) \neq 0$ (1p)

$$\begin{array}{ccc} + & + & + \\ \left|\begin{matrix} 2c & -c & 1 \\ 1 & 1 & 2 \\ 3 & -2 & c \end{matrix}\right| \\ - & - & - \end{array} = 2c^2 - 6c - 2 - 3 + 8c + c^2 =$$

(2p)

$$= 3c^2 + 2c - 5$$

$$3c^2 + 2c - 5 = 0 \Leftrightarrow c^2 + \frac{2}{3}c - \frac{5}{3} = 0 \Leftrightarrow$$

$$\Leftrightarrow c = -\frac{1}{3} \pm \sqrt{\left(-\frac{1}{3}\right)^2 + \frac{5}{3}} = -\frac{1}{3} \pm \sqrt{\frac{16}{9}} = -\frac{1}{3} \pm \frac{4}{3}$$

So the matrix is invertible iff $c \neq 1$ and $c \neq -\frac{5}{3}$

Answer: $c \neq 1$ and $c \neq -\frac{5}{3}$ (1p)

③ $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$A^T X + B = 2I \iff A^T X = 2I - B \iff X = (A^T)^{-1} \cdot (2I - B)$

↑ if A^T is invertible (2p)

We find the inverse of $A^T = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-3} \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] \xrightarrow{-1} \sim \left[\begin{array}{cc|cc} 1 & 0 & 4 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}}_{= (A^T)^{-1}}$$

So

$$X = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \cdot \left(2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -8 \\ -1 & 7 \end{bmatrix}$$ (2p)

Answer: $X = \begin{bmatrix} 1 & -8 \\ -1 & 7 \end{bmatrix}$ (1p)

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(4) $\ell: 2x - y + 1 = 0$

(a) We find the vector form of ℓ by solving $2x-y+1=0$. Setting the free variable $y=t$ we have that

$$x = -\frac{1+y}{2} = -\frac{1}{2} + \frac{t}{2} \quad \text{so}$$

$$\ell : (x,y) = \left(-\frac{1}{2}+\frac{t}{2}, t\right) = \left(-\frac{1}{2},0\right) + t\cdot\left(\frac{1}{2},1\right) \qquad (2p)$$

(b) A normal to ℓ is $\bar{n} = (2,-1)$ so (1p)

$$\operatorname{proj}_{\bar{n}}\bar{u} = \frac{\bar{u}\circ\bar{n}}{\|\bar{n}\|^2}\cdot\bar{n} = \frac{(1,5)\circ(2,-1)}{2^2+(-1)^2}\cdot(2,-1) =$$

$$= \frac{2-5}{5}\cdot(2,-1) = -\frac{3}{5}\cdot(2,-1) \qquad (2p)$$

$$\|\operatorname{proj}_{\bar{n}}\bar{u}\| = \left\|-\frac{3}{5}\cdot(2,-1)\right\| = \left|-\frac{3}{5}\right|\cdot\|(2,-1)\| =$$

$$= \frac{3}{5}\cdot\sqrt{5} = \frac{3}{\sqrt{5}} \qquad (1p)$$

Answer a) $\ell: \left(-\frac{1}{2},0\right) + t\cdot\left(\frac{1}{2},1\right)$, where $t \in \mathbb{R}$

b) $\frac{3}{\sqrt{5}}$

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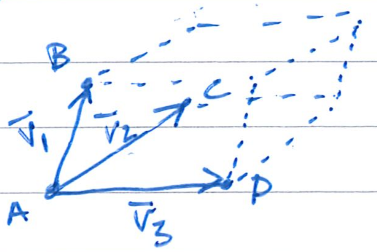

The four points belong to a plane iff the vectors $\bar{v}_1 = \overrightarrow{AB}$, $\bar{v}_2 = \overrightarrow{AC}$, $\bar{v}_3 = \overrightarrow{AD}$ do not span a volume, which is equivalent to

$$\underbrace{\det([\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3])}_{\pm \text{ Volume of the parallelepiped spanned by } \bar{v}_1, \bar{v}_2 \text{ and } \bar{v}_3.} = 0 \qquad (1p)$$

$\bar{v}_1 = \overrightarrow{AB} = (1,3,2) - (1,2,-1) = (0,1,3)$

$\bar{v}_2 = \overrightarrow{AC} = (-1,1,-3) - (1,2,-1) = (-2,-1,-2)$

$\bar{v}_3 = \overrightarrow{AD} = (-1,3,3) - (1,2,-1) = (-2,1,4)$ (1p)

Cofactor exp. column 1

$$\begin{vmatrix} 0 & -2 & -2 \\ 1 & -1 & 1 \\ 3 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -2 & -2 \\ -2 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} -2 & -2 \\ -1 & 1 \end{vmatrix}$$

$$= -(-8-4) + 3(-2-2) = 12 - 12 = 0$$

So the points belong to a plane. (2p)

Answer: The points belong to the same plane.