Divsion of Mathematics
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Examination Vector algebra
MAA150 - TEN2
Date: June 9, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Determine if the points $A(1,1,2)$, $B(1,0,1)$, $C(-1,2,1)$, and $D(0,-1,1)$ belong to a plane. (4p)

2 Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection on the xy-plane, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ a contraction by 1/2. Find the standard matrices for T_1, T_2, and $T_1 \circ T_2$. Motivate your answer. (5p)

3 Given that $B_1 = \{(1,-1),(3,-1)\}$ and $B_2 = \{(1,0),(-1,2)\}$, and that the coordinate vector of $\mathbf{v}$ relative to B_1 is $(\mathbf{v})_{B_1} = (1,2)_{B_1}$

a. Find the transition matrix $P_{B_1 \to B_2}$ from the basis B_1 to the basis B_2. (3p)

b. Find the coordinate vector of $\mathbf{v}$ relative to B_2. (2p)

4 For the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

a. Find all eigenvalues of A. (4p)

b. Motivate why A is diagonalizable. (1p)

5 Let $S = \{(1,0,0),(1,1,1),(4,1,-1),(0,1,1)\}$ and $W = \operatorname{span}(S)$.

a. Find a basis for W consisting of vectors from S. (2p)

b. Construct an orthonormal basis for W. (4p)

Avdelningen för Matematik
och tillämpad matematik
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Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2016-06-09
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Avgör om punkterna $A(1,1,2)$, $B(1,0,1)$, $C(-1,2,1)$, och $D(0,-1,1)$ ligger i ett plan. (4p)

2 Låt $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ vara projektionen på xy-planet, och $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ en kontraktion med kontraktionsfaktor 1/2. Bestäm standardmatrisen för T_1, T_2, och $T_1 \circ T_2$. Motivera ditt svar. (5p)

3 Givet att $B_1 = \{(1,-1),(3,-1)\}$ och $B_2 = \{(1,0),(-1,2)\}$, och koordinaterna för vektorn $\mathbf{v}$ i basen B_1 är $(\mathbf{v})_{B_1} = (1,2)_{B_1}$

a. Bestäm övergångsmatrisen $P_{B_1 \to B_2}$ från basen B_1 till basen B_2. (3p)

b. Bestäm koordinaterna för $\mathbf{v}$ i basen B_2. (2p)

4 För matrisen

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

a. Bestäm alla egenvärden till A. (4p)

b. Motivera varför A är diagonaliserbar. (1p)

5 Låt $S = \{(1,0,0),(1,1,1),(4,1,-1),(0,1,1)\}$ och $W = \text{span}(S)$.

a. Bestäm en bas för W bestående av vektorer ur S. (2p)

b. Konstruera en ortonormal bas för W. (4p)

MAA150 Vektoralgebra, vt-16.

Assessment criterias for TEN2 2016-06-09

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. [**4 points**]
Finding relevant vectors **(2p)**, motivating why the points do not belong to a plane **(2p)**

2. [**5 points**]
Figures for projection and standard matrices for T_1 and T_2 **(3p)**, standard matrix for $T_1 \circ T_2$ **(2p)**

3. [**5 points**]
a. Relevant method with row operations **(2p)**, correct matrix **(1p)**
b. computing $(\mathbf{v})_{\mathbf{B_2}}$**(2p)**

4. [**5 points**]
a. Correct $A - \lambda I$ **(1p)**, characteristic equation **(1p)**, finding the roots **(3p)**: 1 point for each root
b. correct motivation **(1p)**

5. [**6 points**]
a. Relevant row operations **(1p)**, a correct basis **(1p)**
b. Finding an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ **(3p)**: 1 point for each vector $\mathbf{v_i}$, normalizing the vectors **(1p)**

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① $\bar{v}_1 = \overrightarrow{AD} = (0,-1,1)-(1,1,2) = (-1,-2,-1)$
$\bar{v}_2 = \overrightarrow{AC} = (-1,2,1)-(1,1,2) = (-2,1,-1)$
$\bar{v}_3 = \overrightarrow{AB} = (1,0,1)-(1,1,2) = (0,-1,-1)$

A, B, C, and D belongs to the same plane iff

$$\underbrace{\det([\bar{v}_1\ \bar{v}_2\ \bar{v}_3])}_{\pm \text{ Volume of the parallelepiped spanned by } \bar{v}_1, \bar{v}_2 \text{ and } \bar{v}_3} = 0$$

$$\begin{vmatrix} -1 & -2 & -1 \\ -2 & 1 & -1 \\ 0 & -1 & -1 \end{vmatrix} \xleftarrow{-2} = \begin{vmatrix} -1 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & -1 & -1 \end{vmatrix} =$$

$$= (-1)\cdot \begin{vmatrix} 5 & 1 \\ -1 & -1 \end{vmatrix} = -(-5+1) = 4 \neq 0$$

__Answer__: They do not belong to a plane

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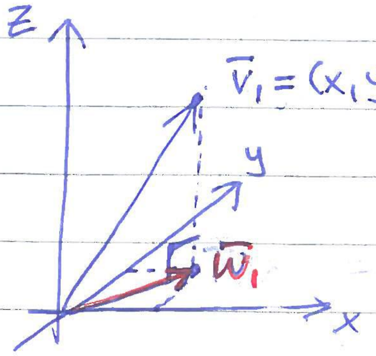
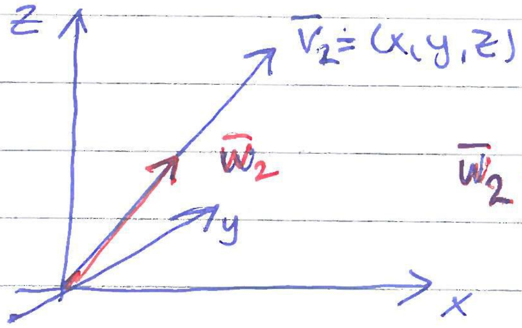
(2) Draw a sketch to find T_1 and T_2

T_1:



$\bar{w}_1 = T_1(\bar{v}_1) = (x, y, 0)$

T_2



$\bar{w}_2 = T_2(\bar{v}_2) = \frac{1}{2} \cdot \bar{v}_1 = \frac{1}{2}(x, y, z)$
$= \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$

Find the standard matrix $[T_1]$ of T_1

$T_1(1,0,0) = (1,0,0)$, $T_1(0,1,0) = (0,1,0)$, $T_1(0,0,1) = (0,0,0)$

so $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Standard matrix $[T_2]$ of T_2

$T_2(1,0,0) = (\frac{1}{2}, 0, 0)$, $T_2(0,1,0) = (0, \frac{1}{2}, 0)$, $T_2(0,0,1) = (0, 0, \frac{1}{2})$

so $[T_2] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Then the standard matrix

of $T_1 \circ T_2$ is $[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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3a $B_1 = \{(1,-1),(3,-1)\}$, $B_2 = \{(1,0),(-1,2)\}$

$(\bar{v})_{B_1} = (1,2)_{B_1}$

Transitionmatrix

$$\left[\text{ny bas} \,\middle|\, \text{gammal bas}\right] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 3 \\ 0 & 2 & -1 & -1 \end{array}\right] \sim \times \tfrac{1}{2}$$

$$\sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 3 \\ 0 & 1 & -1/2 & -1/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 5/2 \\ 0 & 1 & -1/2 & -1/2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1/2 & 5/2 \\ -1/2 & -1/2 \end{matrix}}_{= P_{B_1 \to B_2}}$$

Answer 3a: $P_{B_1 \to B_2} = \begin{bmatrix} 1/2 & 5/2 \\ -1/2 & -1/2 \end{bmatrix}$

3b $(\bar{v})_{B_2} = P_{B_1 \to B_2} (\bar{v})_{B_1} = \begin{bmatrix} 1/2 & 5/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$

$$= \begin{bmatrix} \frac{1}{2} \cdot 1 + \frac{5}{2} \cdot 2 \\ -\frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -3/2 \end{bmatrix}$$

Answer 3b: $(\bar{v})_{B_2} = \left(\frac{11}{2}, -\frac{3}{2}\right)$

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4a

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A - \lambda \cdot I = \begin{bmatrix} 2-\lambda & 0 & 2 \\ 1 & -\lambda & 3 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

(KE) $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 2 \\ 1 & -\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (2-\lambda)\cdot(-\lambda)\cdot(1-\lambda) + 0 + 4 + 2\lambda - 6(2-\lambda) - 0$$

$$= -\lambda(2-\lambda)\cdot(1-\lambda) - 12 + 8\lambda = (-2\lambda + \lambda^2)(1-\lambda) - 8 + 8\lambda =$$
$$= -2\lambda + \underline{2\lambda^2} + \underline{\lambda^2} - \lambda^3 - 8 + 8\lambda = -\lambda^3 + 3\lambda^2 + 6\lambda - 8$$

(KE) $\lambda^3 - 3\lambda^2 - 6\cdot\lambda + 8 = 0$

$\lambda = 1$ is a root to LHS since $1^3 - 3\cdot 1^2 - 6\cdot 1 + 8 = 0$

Long division gives

$$\begin{array}{l|l} \multicolumn{1}{l}{\lambda^2 - 2\lambda - 8} & \\ \hline \lambda^3 - 3\lambda^2 - 6\lambda + 8 & \lambda - 1 \\ -(\lambda^3 - \lambda^2) & \\ \quad -2\lambda^2 - 6\lambda + 8 & \\ \quad -(-2\lambda^2 + 2\lambda) & \\ \qquad -8\lambda + 8 & \\ \qquad -(-8\lambda + 8) & \\ \qquad\qquad 0 & \end{array}$$

so

(KE) $(\lambda - 1)(\lambda^2 - 2\lambda - 8) = 0$

$$\lambda^2 - 2\lambda - 8 = 0$$
$$\Updownarrow$$
$$\lambda = 1 \pm \sqrt{9} = 1 \pm 3$$
$$\Updownarrow$$
$$\lambda = -2 \text{ or } \lambda = 4$$

So eigenvalues are $\lambda = 1$, $\lambda = -2$, and $\lambda = 4$

4b Since the eigenvalues are distinct, A is diagonalizable

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5a
$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\uparrow \ \uparrow \ \uparrow$ leading 1:s

So $B = \{(1,0,0), (1,1,1), (4,1,-1)\}$ is a basis for $W = \text{span}(S)$

5b Gram-Schmidt.

$\bar{v}_1 = (1,0,0)$

$$\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(1,0,0)\bullet(1,1,1)}{\|(1,0,0)\|^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{v}_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} - \frac{(1,0,0)\bullet(4,1,-1)}{\|(1,0,0)\|^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{(0,1,1)\bullet(4,1,-1)}{\|(0,1,1)\|^2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} - \frac{4}{1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Normalize : $\bar{u}_1 = \frac{1}{\|\bar{v}_1\|} \bar{v}_1 = (1,0,0)$ $\quad \bar{u}_2 = \frac{\bar{v}_2}{\|\bar{v}_2\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\bar{u}_3 = \frac{\bar{v}_3}{\|\bar{v}_3\|} = \frac{1}{\sqrt{1^2+(-1)^2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Answer: An orthonormal basis is $S = \{(1,0,0), (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$