Divsion of Mathematics
and applied mathematics
Mälardalen University
Examiner: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Exam Flervariabelkalkyl
MAA152 - TEN1
Date: May 3, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 A particle moves along a path where its position vector at time t is given by

$$\mathbf{r}(t) = e^t\,\mathbf{i} + t\sqrt{2}\,\mathbf{j} + e^{-t}\,\mathbf{k}.$$

Time is given in seconds and distance in meters.

a. Find the velocity $\mathbf{v}(t)$ and the acceleration $\mathbf{a}(t)$ of the particle. (2p)

b. Find the distance traveled by the particle between the time $t = 0$ and $t = 3$. (2p)
Hint: $e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$

2 For the function

$$f(x, y) = \frac{y^2}{x^2 + xy}$$

a. Find the domain of $f(x, y)$ and mark the points in the xy-plane where f is not defined. (2p)

b. Find the directional derivative of f at $(1, 1)$ in the direction of $(-3, 4)$. (2p)

c. Show that $f(x, y)$ does not have a limit as $(x, y) \to (0, 0)$. (3p)

3 Find the equation of the tangent plane to the surface $\cos(xz) + yz^2 = 1$ at the point $(\pi, 2, -1)$. (5p)

4 Find the maximum and minimum value that the function $f(x, y) = x^2 + y^2 - x - y$ assumes in the region $D = \{(x, y) : x^2 + y^2 \leq 1,\ y \geq 0\}$. (5p)

5 Prove that if $f(x, y)$ is differentiable at (a, b) and $\nabla f(a, b) \neq \bar{\mathbf{0}}$, then $\nabla f(a, b)$ is a normal to the level curve of f that passes through the point (a, b). (4p)

Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Flervariabelkalkyl
MAA152 - TEN1
Datum: 2016-05-03
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 En partikel rör sig längs en bana så att partikelns position vid tiden t ges av ortsvektorn

$$\mathbf{r}(t) = e^t\,\mathbf{i} + t\sqrt{2}\,\mathbf{j} + e^{-t}\,\mathbf{k}.$$

Tiden anges i sekunder och längdenheten är meter.

a. Bestäm partikelns hastighet $\mathbf{v}(t)$ och acceleration $\mathbf{a}(t)$. (2p)

b. Bestäm sträckan som partikeln färdas mellan tiden $t = 0$ till $t = 3$. (2p)
Ledning: $e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$

2 För funktionen

$$f(x, y) = \frac{y^2}{x^2 + xy}$$

a. Bestäm domänet till $f(x, y)$ och markera de punkter i xy-planet där f inte är definierad. (2p)

b. Bestäm riktningsderivatan till f i punkten $(1, 1)$ i riktningen $(-3, 4)$. (2p)

c. Visa att $f(x, y)$ saknar gränsvärde då $(x, y) \to (0, 0)$. (3p)

3 Bestäm ekvationen för tangentplanet till ytan $\cos(xz) + yz^2 = 1$ i punkten $(\pi, 2, -1)$. (5p)

4 Bestäm det största och minsta värdet som funktionen $f(x, y) = x^2 + y^2 - x - y$ antar på området $D = \{(x, y) : x^2 + y^2 \leq 1,\ y \geq 0\}$. (5p)

5 Bevisa att om funktionen $f(x, y)$ är differentierbar i (a, b) och $\nabla f(a, b) \neq \bar{\mathbf{0}}$, så är $\nabla f(a, b)$ en normal till nivåkurvan till f som går genom punkten (a, b). (4p)

MAA152 Flervariabelkalkyl, VT16.

Assessment criterias for TEN1 2016-05-03

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1.
- a) Correct expression for $\mathbf{a}(\mathbf{t})$ **(1p)** and $\mathbf{v}(\mathbf{t})$ **(1p)**
- b) Calculating $v(t)$ **(1p)**, and finding the length of the curve **(1p)**

2.
- a) Equations for the set where f is undefined **(1p)**, correctly expressing D or correct figure **(1p)**
- b) Finding $\nabla f(1,1)$ **(1p)**, directional derivative **(1p)**
- c) Computing relevant limits **(2p)**, correct conclusion including motivation **(1p)**

3.
- using a relevant method **(2p)**
- relevant computations **(2p)**
- equation of the tangent plane in any form **(1p)**

4.
- finding the critical point and sketching the region D **(1p)**
- parametrizing the boundary **(2p)**
- finding relevant points on the boundary and finding max/min **(2p)**

5.
- relevant method **(1p)**, correct motivation **(2p)**, presentation of proof **(1p)**

TEN1 : MAA152 2015-05-03

1. $\bar{r}(t) = e^t\,\bar{i} + t\sqrt{2}\,\bar{j} + e^{-t}\,\bar{k}$

a) $\bar{v}(t) = \bar{r}'(t) = e^t\,\bar{i} + \sqrt{2}\,\bar{j} - e^{-t}\,\bar{k}$ (1p)

$\bar{a}(t) = \bar{v}'(t) = e^t\,\bar{i} + 0\cdot\bar{j} + e^{-t}\,\bar{k} = e^t\,\bar{i} + e^{-t}\,\bar{k}$ (1p)

b) $v(t) = |\bar{v}(t)| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (-e^{-t})^2} = \sqrt{e^{2t} + 2 + e^{-2t}}$

$= \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$

$$s = \int_0^3 v(t)\,dt = \int_0^3 (e^t + e^{-t})\,dt = \left[e^t - e^{-t} \right]_0^3 =$$

$$= e^3 - e^{-3} = 2\sinh(3)$$ (2p)

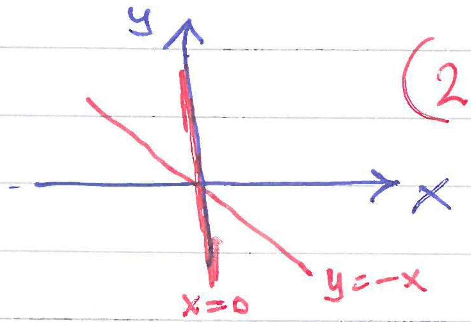
Svar: $2\sinh(3)$ l.e

TEN1 : MAA152 2016-05-03

(2) $f(x,y) = \dfrac{y^2}{x^2+xy} = \dfrac{y^2}{x(x+y)} = y^2(x^2+xy)^{-1}$

a) f är definierad utom då $x(x+y)=0 \Leftrightarrow$

$\Leftrightarrow$ $x=0$ eller $y=-x$, dvs



(2p)

Svar a) $D_f = \{(x,y) : x \neq 0, y \neq -x\}$

b) $\nabla f(x,y) = \dfrac{-y^2(2x+y)}{(x^2+xy)^2}\bar{i} + \dfrac{2y(x^2+xy)-y^2 \cdot x}{(x^2+xy)^2}\bar{j}$

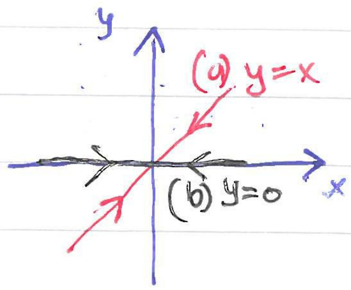
$\nabla f(1,1) = -\dfrac{3}{4}\bar{i} + \dfrac{3}{4}\bar{j} = \dfrac{1}{4}(-3,3)$ (1p)

$|(-3,4)| = 5$ så $\bar{u} = \frac{1}{5}(-3,4)$ är enhetsvektorn i riktning $(-3,4)$.

$D_{\bar{u}}f(1,1) = \bar{u} \bullet \nabla f(1,1) = \frac{1}{5}(-3,4) \bullet \frac{1}{4}(-3,3) = \frac{21}{20}$ (1p)

Svar b) 21/20

c)



(a) $\lim\limits_{(x,x)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

(b) $\lim\limits_{(x,0)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} 0 = 0$

(2p)

Svar: (a) och (b) ej lika så gränsvärdet existerar ej. (1p)

TEN 1 : MAA152 · 2016-05-03

③ $S: \cos(xz) + yz^2 = 1 \qquad P_0 = (\pi, 2, -1)$

Om $g(x,y,z) := \cos(xz) + yz^2 - 1 - \pi$ blir S nivåytan till $g(x,y,z) = 0$ som har en normal $\nabla g(P_0)$. om $\nabla g(P_0) \neq \bar{0}$. (2p)

$$\nabla g(x,y,z) = -\sin(xz)\cdot z\,\bar{i} + z^2\,\bar{j} + (-\sin(xz)\cdot x + 2yz)\,\bar{k}$$

$$\nabla g(\pi,2,-1) = -\sin(-\pi)(-1)\,\bar{i} + (-1)^2\,\bar{j} + (-\sin(-\tfrac{\pi}{2})\cdot\pi - 4)\,\bar{k}$$
$$= \bar{j} - 4\bar{k} \doteq (0,1,-4) \qquad (2p)$$

Om $P = (x,y,z)$ ges tangentplanet av ekv.

$$\nabla g(P_0) \bullet \overrightarrow{P_0P} = 0 \iff (0,1,-4)\bullet(x-\pi, y-2, z+1) = 0$$

$$\iff (y-2) - 4(z+1) = 0 \iff y - 4z = 6 \qquad (1p)$$

Svar: $y - 4z = 6$

Alt. lösning: Implicit derivering av $z = z(x,y)$

$\frac{\partial}{\partial x}$: $-\sin(xz)\cdot(x\frac{\partial z}{\partial x} + z) + 2yz\cdot\frac{\partial z}{\partial x} = 0 \Big|_{P_0} \Rightarrow -4\frac{\partial z}{\partial x}\Big|_{P_0} = 0$

så $z_1(\pi,2,-1) = 0$ (2p)

$\frac{\partial}{\partial y}$: $-\sin(xz)\cdot x\frac{\partial z}{\partial y} + z^2 + 2yz\cdot\frac{\partial z}{\partial y} = 0 \Big|_{P_0} \Rightarrow 1 - 4\frac{\partial z}{\partial y} = 0$

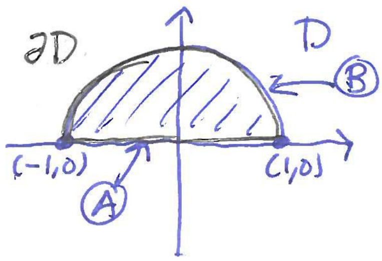
$z_2(\pi,2,-1) = \frac{1}{4}$ (2p)

så $z = -1 + \frac{1}{4}(y-2) \iff 4z = -4 + y - 2$ (1p)

$\iff y - 4z = 6$

TEN 2: MAA 152 2015-05-03

(4) $f(x,y) = x^2 + y^2 - x - y$



$D = \{(x,y) : x^2 + y^2 \leq 1,\ y \geq 0\}$

f är kont. på D och har kont. partiella derivator så $f_{min/max}$ existerar och antas i CP eller på randen, ∂D.

CP: $\begin{cases} f_1(x,y) = 2x - 1 \\ f_2(x,y) = 2y - 1 \end{cases} \Rightarrow \begin{matrix} x = \frac{1}{2} \\ y = \frac{1}{2} \end{matrix}$ dvs $(\frac{1}{2}, \frac{1}{2}) \in D$

$f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2}$ min

(2p) ∂D: (A) $y = 0$: $g_A(x) := f(x,0) = x^2 - x$, $-1 \leq x \leq 1$

$g_A'(x) = 2x - 1 \Rightarrow x = \frac{1}{2}$

g_A antar max/min i $x = -1$, $x = \frac{1}{2}$, eller $x = 1$

$g_A(-1) = f(-1,0) = 2$, $g_A(\frac{1}{2}) = f(\frac{1}{2}, 0) = -\frac{1}{4}$, $g_A(1) = f(1,0) = 0$

max

(2p) (B) Parametrisera halvcirkeln: $\begin{matrix} x = \cos(\theta) \\ y = \sin(\theta) \end{matrix}$, $0 \leq \theta \leq \pi$

$g_B(\theta) := f(\cos(\theta), \sin(\theta)) = \cos^2\theta + \sin^2(\theta) - \cos(\theta) - \sin(\theta)$
$= 1 - \cos(\theta) - \sin(\theta)$

$g_B'(\theta) = \sin(\theta) - \cos(\theta) = 0 \Rightarrow \tan(\theta) = 1 \Rightarrow \theta = \frac{\pi}{4}$

g_B antar max/min i $\theta = 0$, $\theta = \frac{\pi}{4}$, eller $\theta = \pi$

max

$g_B(0) = f(1,0) = 0$, $g_B(\frac{\pi}{4}) = f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 1 - \sqrt{2}$, $g_B(\pi) = f(-1,0) = 2$

$1 - \sqrt{2} \approx -0{,}41 > -\frac{1}{2}$

$\sqrt{2} < \frac{3}{2}$ så

$1 - \sqrt{2} > 1 - \frac{3}{2} = -\frac{1}{2}$

Svar: $f_{max} = 2$ och $f_{min} = -\frac{1}{2}$