MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2016-01-08 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Let $f(x) = \sqrt{2 - x^2}$ and $g(x) = x^2$. Find the function expression, the domain and the range of the composition $g \circ f$.

2. Find the function f such that $f'(x) = 4x\ln(x)$ and $f(1) = 2015$.

3. Determine whether

$$\lim_{x\to 0} \frac{\sin(x^2)}{x}$$

exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

4. Find the area of the bounded region precisely enclosed by the curves

$$y = e^{|x|} \quad \text{and} \quad y = e\,.$$

5. The sum of two non-negative numbers is 10. Which are the numbers if the square of the first number plus four times the square of the second is a minimum? Prove your conclusion.

6. Find an equation for the tangent line τ to the curve $\gamma : \begin{cases} x = \arcsin(t), \\ y = \sqrt{1-t^2}, \end{cases}$ at the point P for which the x-coordinate is equal to $\pi/3$.

7. Determine to the differential equation $4y'' + 4y' + y = 0$ the solution that satisfies the initial conditions $y(0) = 2\,,\ y'(0) = 0$.

8. Explain what a geometric series is. Then, give an example of each of a convergent and a divergent geometric series. Explain also the convergence and the divergence in the examples.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{lcrcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Låt $f(x) = \sqrt{2 - x^2}$ och $g(x) = x^2$. Bestäm funktionsuttrycket, definitionsmängden och värdemängden för sammansättningen $g \circ f$.

2. Bestäm funktionen f sådan att $f'(x) = 4x \ln(x)$ och $f(1) = 2015$.

3. Avgör om

$$\lim_{x \to 0} \frac{\sin(x^2)}{x}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

4. Bestäm arean av det begränsade område som precis innesluts av kurvorna

$$y = e^{|x|} \quad \text{och} \quad y = e\,.$$

5. Summan av två icke-negativa tal är 10. Vilka är talen om kvadraten av det ena talet plus fyra gånger kvadraten av det andra är minimal? Bevisa din slutsats!

6. Bestäm en ekvation för tangenten τ till kurvan $\gamma : \begin{cases} x = \arcsin(t), \\ y = \sqrt{1 - t^2}, \end{cases}$

i den punkt P för vilken x-koordinaten är lika med $\pi/3$.

7. Bestäm till differentialekvationen $4y'' + 4y' + y = 0$ den lösning som satisfierar begynnelsevillkoren $y(0) = 2\,,\ y'(0) = 0$.

8. Förklara vad en geometrisk serie är för något. Ge sedan ett exempel på vardera en konvergent respektive en divergent geometrisk serie. Förklara även konvergensen respektive divergensen i exemplen.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2016-01-08

① $f(x) = \sqrt{2-x^2}$, $g(x) = x^2$

We get $(g \circ f)(x) = g(f(x)) = g(\sqrt{2-x^2}) = (\sqrt{2-x^2})^2 = \underline{2-x^2}$

where
$$\begin{cases} D_{g\circ f} = \{x : x \in D_f, f(x) \in D_g\} = \{x : 2-x^2 \geq 0\} \\ \qquad = \{x : -\sqrt{2} \leq x \leq \sqrt{2}\} = \underline{[-\sqrt{2}, \sqrt{2}]} \\ V_{g\circ f} = \{2-x^2 : -\sqrt{2} \leq x \leq \sqrt{2}\} = \underline{[0, 2]} \end{cases}$$

② $f'(x) = 4x\ln(x)$ and $f(1) = 2015$

Integration by parts gives

$$\int 4x\ln(x)\,dx = 2x^2\ln(x) - \int 2x^2 \cdot \frac{1}{x}\,dx = 2x^2\ln(x) - x^2 + C$$

where $1^2[2\underbrace{\ln(1)}_{=0} - 1] + C = f(1) = 2015$ i.e. $C = 2016$

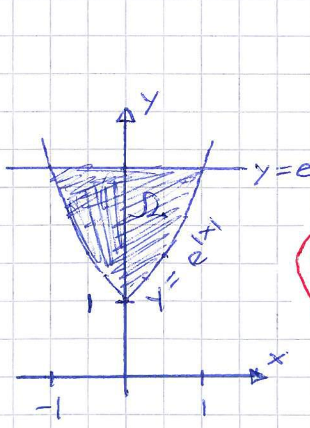
Thus $\underline{f(x) = x^2(2\ln(x) - 1) + 2016}$

product rule applicable since both limits exist

③ $$\lim_{x\to 0} \frac{\sin(x^2)}{x} = \left(\lim_{x\to 0} x\right) \cdot \left(\lim_{x\to 0} \frac{\sin(x^2)}{x^2} \begin{bmatrix} x^2 = u \\ u \to 0^+ \Rightarrow x = 0 \end{bmatrix}\right)$$

$$= \underbrace{\left(\lim_{x\to 0} x\right)}_{\text{obvious limit}} \underbrace{\left(\lim_{u\to 0^+} \frac{\sin(u)}{u}\right)}_{\text{standard limit}} = 0 \cdot 1 = \underline{0}$$

④



$$A(\Omega) = \int_{-1}^{1} (e - e^{|x|})\,dx$$

An even function integrated on an even interval

$$= 2\int_0^1 (e - e^x)\,dx$$

$$= 2\left[ex - e^x\right]_0^1$$

$$= 2\left[(e-e) - (0-1)\right] = 2$$

Answer: $\underline{2 \text{ a.u.}}$

Cont. MAA151
TEN1 2016-01-08

(5) Find $\min\limits_{\substack{\{x+y=10\} \\ \{x,y\geq 0\}}} (x^2+4y^2) = \min\limits_{0\leq x\leq 10} [x^2+4(10-x)^2]$

Let $f(x) = x^2+4(10-x)^2$ with $D_f = [0,10]$

Then $f'(x) = 2x + 4\cdot 2(10-x)(-1) = 2x+8(x-10) = 10(x-8)$

First derivative test

	EP		SP		EP
x	0		8		10
$f'(x)$		$-$	0	$+$	
$f(x)$	loc max	$\searrow$	loc min	$\nearrow$	loc max

i.e. f has a minimum at 8

Answer: The numbers are 8 and 2

(6) $\gamma: \begin{cases} x = f(t) \\ y = g(t) \end{cases}$ where $\begin{cases} f(t) = \arcsin(t) \\ g(t) = \sqrt{1-t^2} \end{cases}$

$x_p = \frac{\pi}{3} \Leftrightarrow \arcsin(t_p) = \frac{\pi}{3} \Leftrightarrow t_p = \frac{\sqrt{3}}{2}$

Thus $y_p = g(t_p) = \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

The slope of γ at P is $k_p = \left.\frac{dy}{dx}\right|_p = \left.\frac{dy}{dt}\right|_p \left.\frac{dt}{dx}\right|_p$

i.e. $k_p = \frac{g'(t_p)}{f'(t_p)} = \left.\frac{\frac{1}{2}\frac{1}{\sqrt{1-t^2}}(-2t)}{\frac{1}{\sqrt{1-t^2}}}\right|_{t=t_p} = -t_p = -\frac{\sqrt{3}}{2}$

Thus

The tangent line to γ at P is given by $y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$

(7) DE: $4y''+4y'+y=0$ IV: $y(0)=2$, $y'(0)=0$

The auxiliary equation is $0 = 4r^2+4r+1 = (2r+1)^2 = 4\left(r+\frac{1}{2}\right)^2$

Thus the general solution of the homogeneous linear 2nd order DE is $y = (C_0 + C_1 x)e^{-x/2}$.

The initial values give $\begin{cases} 2 = (C_0+0)e^0 = C_0 \\ 0 = (C_1 - \frac{1}{2}C_0 + 0)e^0 = C_1 - \frac{1}{2}C_0 \end{cases} \Leftrightarrow \begin{cases} C_0 = 2 \\ C_1 = 1 \end{cases}$

Thus the solution of the IVP is $y = (2+x)e^{-x/2}$

(8) A series of the form $\alpha \sum_{n=0}^{\infty} \beta^n = \alpha + \alpha\beta + \alpha\beta^2 + \dots$ is a geometric series, and is convergent iff $|\beta|<1$ (and thus divergent iff $|\beta| \geq 1$)

Examples $\begin{cases} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \text{ is a convergent geometric series} \\ \sum_{n=0}^{\infty} (-1)^n \text{ is a divergent } \text{—}\,\text{''}\,\text{—} \quad \text{—}\,\text{''}\,\text{—} \end{cases}$

Examination TEN1 – 2016-01-08 Maximum points for subparts of the problems in the final examination

1. $g \circ f(x) = 2 - x^2$

where $D_{g\circ f} = [-\sqrt{2}, \sqrt{2}]$, $V_{g\circ f} = [0,2]$

1p: Correctly found the expression for $g \circ f(x)$
1p: Correctly found the domain of the composition $g \circ f$
1p: Correctly found the range of the composition $g \circ f$

2. $f(x) = x^2(2\ln(x) - 1) + 2016$

1p: Correctly worked out the first progressive step in determining the antiderivative by parts
1p: Correctly worked out the second progressive step in determining the antiderivative by parts
1p: Correctly adapted the antiderivative to the value at 1

3. The limit exists and is equal to 0

Note: The student who have argued that the limit does not exist based on the fact that the fraction at the limit point is of the type “0/0” obtains **0p**. The student who have claimed that a fraction of the type “0/0” is equal to 0 obtains **0p**.

1p: Correctly extended the fraction with a factor x to be able to utilize the standard limit $\lim_{u\to 0} \sin(u)/u = 1$
2p: Correctly concluded that the limit exists and is equal to 0

4. 2 a.u.

1p: Correctly found the intersection of the two enclosing curves, and correctly formulated an integral for the area
1p: Correctly treated the argument $|x|$ of the exponential function
1p: Correctly found the area

5. 8 and 2

1p: Correctly for the optimization problem formulated a function of one variable including the specification of its domain
1p: Correctly determined and concluded about the local extreme points of the function
1p: Correctly found the numbers x and y which give the minimum value of the weighted sum of squares $x^2 + 4y^2$

6. $\tau : y - \frac{1}{2} = -\frac{\sqrt{3}}{2}(x - \frac{\pi}{3})$

Note: The student who has failed in finding more than one of t_P, y_P and k_P, but at least has formulated the equation $\tau : y - y_P = k_P(x - \pi/3)$, will still get the 3rd point.

1p: Correctly found the parameter value t_P at the point P and also the second coordinate y_P
1p: Correctly found the slope at the point P
1p: Correctly formulated an equation for the tangent line τ to the curve γ at the point P

7. $y = (2 + x)e^{-x/2}$

Note: The student who has stated that $y = Ae^{-x/2} + Be^{-x/2}$ is the general solution of the differential equation, and who has not found any explanation to the impossible conditions occuring when adapting to the initial values, obtains **0p**.

1p: Correctly found the general solution of the DE
1p: Correctly differentiated the general solution in preparing for the adaption to the initial values
1p: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP

8. A geometric series is of the form $\alpha\Sigma_{n=0}^{\infty}\beta^n$, and is convergent iff $|\beta| < 1$. $\Sigma_{n=0}^{\infty}(\frac{1}{2})^n$ and $\Sigma_{n=0}^{\infty}(-1)^n$ are examples of a convergent and a divergent series respectively.

1p: Correctly explained what a geometric series is
1p: Correctly stated and explained the convergence of a convergent geometric series
1p: Correctly stated and explained the divergence of a divergent geometric series