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Examination Vector algebra
MAA150 - TEN2
Date: 2015-01-13
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Find the equation of the plane that contains the line $l : (x, y.z) = (1, 3, 0) + t(1, -1, 2)$ and the point $P = (-1, 0, 2)$. State the equation in general form. (5p)

2 Let the linear transformation T be given by

$$T(x_1, x_2, x_3, x_4) = (2x_1 + x_4, x_1 + x_2 - 2x_4, 4x_1 + 4x_3).$$

a. Find the standard matrix of T. (3p)

b. Determine if $\mathbf{v} = (1, 0, 3)$ is in the range of T. (2p)

3 The vector $\mathbf{v}$ has coordinate vector $(5, -2, 3)$ relative the standard basis. Find the coordinate vector of $\mathbf{v}$ relative the basis B, where $B = \{(1, 0, -1), (1, 0, 1), (1, 2, -1)\}$. (4p)

4 Determine if the matrix A is diagonalizable, where (5p)

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 3 \end{bmatrix}.$$

5 Let $S = \{(0, 1, 1, 1), (1, 1, 1, 1), (4, 1, -1, 2), (-1, 1, 3, 0)\}$ and $W = \mathrm{span}(S)$.

a. Find a basis for W consisting of vectors from S. (3p)

b. Construct an orthogonal basis for W. (3p)

Avdelningen för Matematik
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Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2016-01-13
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Bestäm ekvationen för planet som innehåller linjen $l : (x, y.z) = (1, 3, 0) + t(1, -1, 2)$ och punkten $P = (-1, 0, 2)$. Ange ekvationen på allmän form. (5p)

2 Givet att den linjära avbildningen T ges av

$$T(x_1, x_2, x_3, x_4) = (2x_1 + x_4, x_1 + x_2 - 2x_4, 4x_1 + 4x_3).$$

a. Bestäm standardmatrisen för T. (3p)

b. Avgör om $\mathbf{v} = (1, 0, 3)$ tillhör T:s värderum. (2p)

3 Vektorn $\mathbf{v}$ har koordinatvektor $(5, -2, 3)$ i standardbasen. Bestäm koordinatvektorn för $\mathbf{v}$ i basen B, då $B = \{(1, 0, -1), (1, 0, 1), (1, 2, -1)\}$. (4p)

4 Avgör om matrisen A är diagonaliserbar, då (5p)

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 3 \end{bmatrix}.$$

5 Låt $S = \{(0, 1, 1, 1), (1, 1, 1, 1), (4, 1, -1, 2), (-1, 1, 3, 0)\}$ och $W = \text{span}(S)$.

a. Bestäm en bas för W bestående av vektorer från S. (3p)

b. Konstruera en ortogonal bas för W. (3p)

MAA150 Vektoralgebra, ht-15.

Assessment criterias for TEN2 2016-01-13

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. Complete solution gives 5 points. Determining two vectors that lie in the plan to find the normal gives 2 points, Using the crossproduct to compute a normal gives 2 points, 1 point for the equation of the plane.

2. a. Complete solution gives 3 points. 2 point for the correct matrix and 1 point for a proper motivation.

b. Complete solution gives 2 points. Setting up the correct system of equations gives 1 point. Showing the it has a solution gives 1 point.

3. Complete solution gives 4 points. Correct method gives maximum 2 points; either by finding the transition matrix or solving an equation system. Computations with relevant row operations 1 point. The correct answer 1 point.

4. Complete solution gives 5 points. Stating a correct criteria for A being diagonalizable gives 1 point. Computations for checking the criteria gives maximum 3 points; typically finding eigenvalues 1 point, eigenvectors 2 points. The correct conclusion gives 1 point.

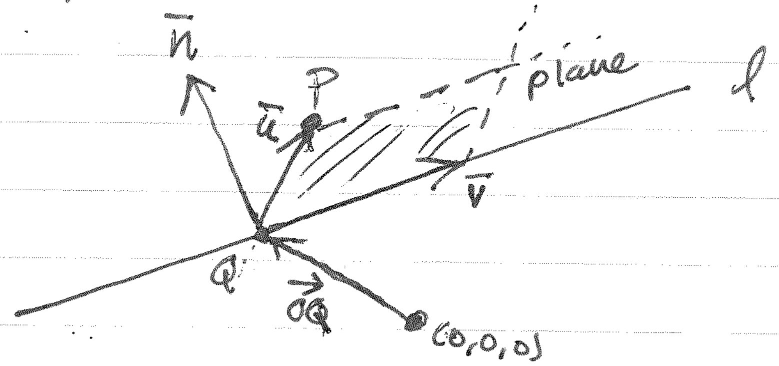
5. a. Complete solution gives 3 points. Correct method with relevant row operations gives 2 points. Selecting the the basis correctly from the original matrix based on the row-reduced matrix gives 1 point.

b. Complete solution gives 3 points. Using Gram-Schmidt method gives maximum 2 points. Finding an orthogonal basis 1 point.

① $\ell : \underbrace{(1,3,0)}_{Q} + t\,\underbrace{(1,-1,2)}_{\bar{v}}$, $P = (-1,0,2)$

To find a normal take two vectors that lie in the plane and use crossproduct.

Fig. Conceptual illustration



we can e.g. take $\bar{v} = (1,-1,2)$ and

$$\bar{u} = \overrightarrow{QP} = (-1,0,2) - (1,3,0) = (-2,-3,2)$$

Crossproduct gives

$$\bar{n} = \bar{v} \times \bar{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} -1 & -3 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ -1 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix}$$

Equation of plane: $\bar{n} \bullet ((x,y,z) - (1,3,0)) = 0$

$$\Leftrightarrow (4,-6,-5) \bullet (x-1, y-3, z-0) = 0$$
$$\Leftrightarrow \underline{4x - 6y - 5z + 14 = 0}$$

Answer: $4x - 6y - 5z + 14 = 0$

② ⓐ $T(x_1, x_2, x_3, x_4) = \begin{pmatrix} 2x_1 + x_4 \\ x_1 + x_2 - 2x_4 \\ 4x_1 + 4x_3 \end{pmatrix}$

$T(\bar{e}_1) = (2,1,4)$, $T(\bar{e}_2) = (0,1,0)$

$T(\bar{e}_3) = (0,0,4)$, $T(\bar{e}_4) = (1,-2,0)$

Standard matrix is

$$[T] = \big[\, T(\bar{e}_1)\ T(\bar{e}_2)\ T(\bar{e}_3)\ T(\bar{e}_4) \,\big] = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 \\ 4 & 0 & 4 & 1 \end{bmatrix}$$

ⓑ
$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ 4 & 0 & 4 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 4 & 0 & 4 & 0 & 3 \end{array}\right] \begin{array}{c} (-2)\ (-4) \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 5 & 1 \\ 0 & -4 & 4 & 8 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 5 & 1 \\ 0 & 0 & 4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -5/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/4 \end{array}\right]$$

free var.

This has infinitely many solutions. so $(1,0,3) \in \operatorname{im}(T)$

Answer b: Yes!

③ $\bar{v} = (5,-2,3)$, $B = \{(1,0,-1), (1,0,1), (1,2,-1)\} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$

$\bar{v} = k_1\bar{u}_1 + k_2\bar{u}_2 + k_3\bar{u}_3 \Leftrightarrow (\bar{v})_B = (k_1, k_2, k_3)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 0 & 2 & -2 \\ -1 & 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 0 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Answer: $(\bar{v})_B = (2,4,-1)_B$

(4) $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 3 \end{bmatrix}$

algebraic mult. of $\lambda = 3$ $= 2$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & -2 & 3-\lambda \end{vmatrix} = \underbrace{(3-\lambda)^2}_{} (2-\lambda) = 0$$

Eigenvalues : $\lambda = 3$, $\lambda = 2$

$\boxed{\lambda = 3}$ $(A - 3I)\bar{v} = \bar{0}$ gives $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{array}\right] \sim$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $v_3 = t$, $v_2 = 0$, $v_1 = 0$ $t = 1$ gives

$\bar{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ eigenvector to $\lambda = 3$.

geometric mult. of $\lambda = 3$ is 1, while algebraic mult. is 2. Therefore

<u>A is not diagonalizable.</u>

5 a
$$\begin{bmatrix} 0 & 1 & 4 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ 1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ 0 & 1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B}_1 = \left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\bar{u}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\bar{u}_2}, \underbrace{\begin{bmatrix} 4 \\ 1 \\ -1 \\ 2 \end{bmatrix}}_{\bar{u}_3} \right\}$$

is a basis for W.

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b
$$\bar{v}_1 = \bar{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{v}_2 = \bar{u}_2 - \frac{\bar{u}_2 \bullet \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(1,1,1,1) \bullet (0,1,1,1)}{\sqrt{0^2+1^2+1^2+1^2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{v}_3 = \bar{u}_3 - \frac{\bar{u}_3 \bullet \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1 - \frac{\bar{u}_3 \bullet \bar{v}_2}{\|\bar{v}_2\|^2} \bar{v}_2 =$$

$$= \begin{bmatrix} 4 \\ 1 \\ -1 \\ 2 \end{bmatrix} - \frac{(4,1,-1,2) \bullet (0,1,1,1)}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(4,1,-1,2) \bullet (1,0,0,0)}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ -1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ -5/3 \\ 4/3 \end{bmatrix}$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/3 \\ -5/3 \\ 4/3 \end{bmatrix} \right\}$$ is an orthogonal basis for W.