Divsion of Mathematics
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Mälardalen University
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MÄLARDALENS HÖGSKOLA
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Examination Vector algebra
MAA150 - TEN1
Date: June 3, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Find all solutions to the linear system (3p)

$$\begin{aligned} -x + 3y + z &= 2 \\ 3x - 9y - 2z &= -1 \end{aligned}$$

2 Determine the values of a for which the linear system has no solution, exactly one solution, or infinitely many solutions. (5p)

$$\begin{array}{rcrcrcl} x & + & a\,y & + & z & = & 1 \\ 2\,x & + & 3a\,y & + & z & = & 2 \\ a\,x & & & + & (a+1)\,z & = & 0 \end{array}$$

3 Given

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 1 \end{bmatrix}$$

.

a. Evaluate $\det(A)$. Is A invertible? Motivate your answer. (4p)

b. Evaluate $\det(2B^{-5})$ if B is a 4×4 matrix such that $\det(B) = -2$. (2p)

4 The plane Π passes through the point $P = (1, 2, 3)$ and has a normal parallel to the line l having parametric form $x = 1 - t$, $y = -1 + 2t$, and $z = 2 + 2t$, $t \in \mathbb{R}$.

a. Sketch an figure that illustrates the problem in a relevant way. The figure should illustrate the plane Π, the line l, and the point P. (2p)

b. Find the equation of the plane Π. Give the answer in point-normal form. (3p)

5 Find all solutions to the equation $z^2 - iz + 1 - 3i = 0$. Give the answer in the form $a + bi$. (6p)

Avdelningen för Matematik
och tillämpad matematik
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Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2016-06-03
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Bestäm alla lösningar till ekvationssystemet (3p)

$$\begin{aligned} -x+3y+z &= 2 \\ 3x-9y-2z &= -1 \end{aligned}$$

2 Avgör för vilka värden på a som ekvationssystemet saknar lösning, har exakt en lösning, eller har oändligt många lösningar. (5p)

$$\begin{array}{rcrcrcl} x & + & ay & + & z & = & 1 \\ 2x & + & 3ay & + & z & = & 2 \\ ax & & & + & (a+1)z & = & 0 \end{array}$$

3 Givet

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 1 \end{bmatrix}$$

.

a. Beräkna $\det(A)$. Är A inverterbar? Motivera ditt svar. (4p)

b. Beräkna $\det(2B^{-5})$ om B är en 4×4 matris med $\det(B) = -2$. (2p)

4 Planet Π går genom punkten $P = (1, 2, 3)$ och har en normal parallell med linjen l som har parametrisk form $x = 1-t$, $y = -1+2t$, och $z = 2+2t$, $t \in \mathbb{R}$.

a. Rita en figur som illustrerar problemet på ett relevant sätt. Figuren skall illustrera planet Π, linjen l och punkten P. (2p)

b. Bestäm ekvationen för planet Π. Ange ekvationen på punkt-normal form. (3p)

5 Bestäm alla lösningar till ekvationen $z^2 - iz + 1 - 3i = 0$. Svara på formen $a + bi$. (6p)

MAA150 Vektoralgebra, vt-16.

Assessment criterias for TEN1 2016-02-15

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. [**3 points**]
Relevant row operations on the augmented matrix **(1p)**, setting the free variable to a parameter **(1p)**, correct answer **(1p)**

2. [**5 points**]
Correctly determining that there is a unique solution if $a \neq 0$ and $a \neq 1$ **(3p)**, correctly determining that there is no solution if $a = 0$ **(1p)**, correctly determining that there is infinitely many solutions if $a = 1$ **(1p)**

3. [**6 points**]
a. Computing the determinant **(3p)**, correctly determining that A is invertible **(1p)**
b. using relevant matrix arithmetic rules **(1p)**, correct answer **(1p)**

4. [**5 points**]
a. A general sketch containing all relevant information **(2p)**
b. finding a vector in the direction of the line **(1p)**, using a normal and finding the equation of the plane **(2p)**

5. [**6 points**] The correct method gives maximum 2 points, where setting $z = x + yi$ gives **(1p)** and finding the equation system for x and y gives **(1p)**. Solving for x and y correctly **(2p)**. The correct answer gives **(2p)** with 1p for each root with possible partial points for checked the answer

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① $\begin{cases} -x+3y+z=2 \\ 3x-9y-2z=-1 \end{cases}$

Augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 2 \\ 3 & -9 & -2 & -1 \end{array}\right] \begin{array}{c} ③ \\ \end{array} \sim \left[\begin{array}{ccc|c} -1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{c} ① \\ ② \end{array} \quad (1p)$$

y is a free variable

Let $y=t$ then (1p)

② $z=5$

① $x = 3y+z-2 = 3t+5-2 = 3t+3$

Answer: $\begin{cases} x = 3+3t \\ y = t \\ z = 5 \end{cases}, \quad t \in \mathbb{R}$ (1p)

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(2) Let $A = \begin{bmatrix} 1 & a & 1 \\ 2 & 3a & 1 \\ a & 0 & a+1 \end{bmatrix}$, then the linear system has exactly one solution if $\det(A) \neq 0$

$$\begin{vmatrix} 1 & a & 1 \\ 2 & 3a & 1 \\ a & 0 & a+1 \end{vmatrix} \underset{\text{rule}}{=} 3a(a+1) + 0 + a^2 - 3a^2 - 2a(a+1) - 0 =$$

(2p)

$$= \cancel{3a^2} + 3a + a^2 - \cancel{3a^2} - 2a^2 - 2a = a - a^2 = a(a-1)$$

$$\det(A) = a(a-1) = 0 \iff a = 0 \text{ or } a = 1$$

Exactly one solution $\iff \det(A) \neq 0 \iff a \neq 0$ and $a \neq 1$ (1p)

For $a=0$: we get the system $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow z = 0 \Big\} \Rightarrow 2x = 2 \Rightarrow x = 1$

$y = t$

y free variable

$$\begin{cases} x = 1 \\ y = t \\ z = 0 \end{cases}, \ t \in \mathbb{R}$$

So there are infinitely many solutions for $a=0$. (1p)

For $a=1$: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & 0 & 2 & 0 \end{array}\right] \begin{matrix} (-2) & (-1) \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \end{array}\right] (1)$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \Rightarrow 0 = -1 \ \lightning \text{ contradiction}$$

so no solution (1p)

Answer:
$$\begin{cases} a = 1 & \text{no solution} \\ a = 0 & \text{infinitely many solutions.} \\ a \neq 1 \text{ and } a \neq 0 & \text{exactly one solution} \end{cases}$$

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3a

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 1 \end{vmatrix} \overset{(-1)}{=} \begin{vmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & -1 & 1 \end{vmatrix} = \text{cofactor exp.}$$

$$= 0 - (-2) \cdot \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{vmatrix} + 0 + 0 = 2 \cdot \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{vmatrix} \begin{matrix} 1 & 0 \\ -1 & 2 \\ 2 & -1 \end{matrix}$$

$$= 2 \cdot \left(1 \cdot 2 \cdot 1 + 0 + 1 \cdot (-1)^2 - 2^2 \cdot 1 - 0 - 0\right) =$$

$$= 2 \cdot (2 + 1 - 4) = -2$$ (3p)

A is invertible $\iff \det(A) \neq 0$ (1p)

Answer 3b : $\det(A) = -2 \neq 0$ so A is invertible.

3b If $\det(A) = -2$ and B is a 4x4 matrix

$$\det(2 \cdot B^{-5}) = 2^4 \cdot \det(B^{-5}) = 2^4 (\det(B))^{-5} =$$

$$= 2^4 \cdot (-2)^{-5} = -2^4 \cdot 2^{-5} = -\frac{1}{2}$$

(2p)

Answer 3b : $\det(2 \cdot B^{-5}) = -\frac{1}{2}$

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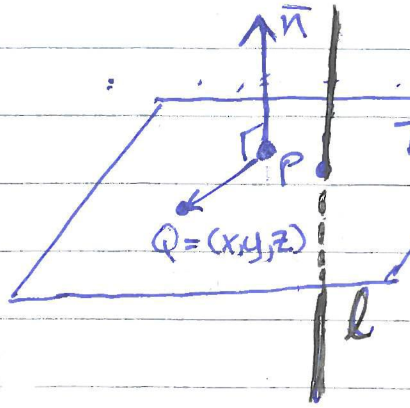
④ $P=(1,2,3)\in\Pi$, $\bar{n}\,/\!/\,\ell$ where



(2p)

$\ell:(x,y,z)=(1-t,-1+2t,2+2t)$
$=(1,-1,2)+t\underbrace{(-1,2,2)}_{\text{Direction of }\ell}, \; t\in\mathbb{R}$

so we can chose
$\bar{n}=(-1,2,2)$ (1p)

Then the equation of Π is then given by

$\vec{PQ}\bullet\bar{n}=0 \iff (x-1,y-1,z-3)\bullet(-1,2,2)=0$ (1p)

$\iff -(x-1)+2(y-1)+2(z-3)=0$

Answer: $-(x-1)+2(y-1)+2(z-3)=0$ (1p)

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(5) $z^2 - iz + 1 - 3i = 0$

Complete the square: $\left(z-\frac{i}{2}\right)^2 - \left(\frac{i}{2}\right)^2 + 1 - 3i = 0$

$\Leftrightarrow \left(z-\frac{i}{2}\right)^2 = -\frac{5}{4} + 3i$. Let $w = z - \frac{i}{2}$

so $w^2 = -\frac{5}{4} + 3i$

Set $w = x + iy$, then (1p)

$x^2 - y^2 + 2xyi = -\frac{5}{4} + 3i$ which gives

$$\begin{cases} x^2 - y^2 = -\frac{5}{4} & (1) \\ 2xy = 3 & (2) \end{cases}$$ (1p)

(3) $|w|^2 = x^2 + y^2 = \left|-\frac{5}{4} + 3i\right| = \sqrt{\frac{25}{16} + \frac{9 \cdot 16}{16}} = \frac{13}{4}$

(1)+(3) gives $2x^2 = \frac{8}{4} = 2 \Rightarrow x = \pm 1$

(2) gives $\begin{cases} x = 1 \Rightarrow y = \frac{3}{2} \\ x = -1 \Rightarrow y = -\frac{3}{2} \end{cases} \Rightarrow \begin{array}{l} z - \frac{i}{2} = 1 + \frac{3}{2}i \\ z - \frac{i}{2} = -1 - \frac{3}{2}i \end{array}$ (2p)

So $z_1 = 1 + 2i$, $z_2 = -1 - i$

Answer: $z = 1 + 2i$ or $z = -1 - i$ (2p)