Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2017-02-16
Hjälpmedel: penna,
sudd och linjal.

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Låt

$$A = \begin{bmatrix} -1 & 3 & -1 \\ 2 & -6 & 5 \\ 3 & -9 & 5 \end{bmatrix} \text{ och } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

a. Bestäm den allmänna lösningen till $A\mathbf{x} = \mathbf{0}$. Ange lösningen på parameterform. (3p)

b. För vilket reellt tal c är ekvationssystemet $A\mathbf{x} = \mathbf{b}$ konsistent, då $\mathbf{b} = \begin{bmatrix} -7 \\ 2 \\ c \end{bmatrix}$? (3p)

2 För vilka reella tal k är matrisen nedan inverterbar? (4p)

$$\begin{bmatrix} 2 & -2 & 1 \\ 1 & k & 2 \\ 3 & -2k & k \end{bmatrix}$$

3 Lös ekvationen $A + XB = 0$, där (5p)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ och } B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

4 Punkterna $A(1, 2, -1)$, $B(1, -2, 2)$ och $C(-1, 1, 1)$ bildar en triangel i $\mathbb{R}^3$.

a. Bestäm avståndet mellan punkterna A och B. (2p)

b. Bestäm arean av triangeln med hörn i A, B och C. (3p)

5 Bestäm det minsta avståndet mellan de parallella linjerna l_1 och l_2, vilka på parameterform ges av: (5p)

$$l_1 : \begin{cases} x = & 1 - 2t \\ y = & 2 + t \end{cases} \text{ och } l_2 : \begin{cases} x = & -1 + 4t \\ y = & 1 - 2t \end{cases}.$$

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Examination Vector algebra
MAA150 - TEN1
Date: Februari 16, 2017
Exam aids: pencil,
eraser, and ruler.

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Let

$$A = \begin{bmatrix} -1 & 3 & -1 \\ 2 & -6 & 5 \\ 3 & -9 & 5 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

a. Find the general solution to $A\mathbf{x} = \mathbf{0}$. Give the solution in parametric form. (3p)

b. For what real value c is the linear system consistent when $\mathbf{b} = \begin{bmatrix} -7 \\ 2 \\ c \end{bmatrix}$? (3p)

2 For what real values k is the matrix below invertible? (4p)

$$\begin{bmatrix} 2 & -2 & 1 \\ 1 & k & 2 \\ 3 & -2k & k \end{bmatrix}$$

3 Solve the equation $A + XB = 0$, where (5p)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

4 The points $A(1, 2, -1)$, $B(1, -2, 2)$, and $C(-1, 1, 1)$, form a triangle in $\mathbb{R}^3$.

a. Find the distance between the points A and B. (2p)

b. Find the area of the triangle with vertices in A, B, and C. (3p)

5 Find the shortest distance between the parallel lines l_1 and l_2, which have the parametric forms: (5p)

$$l_1 : \begin{cases} x = & 1 - 2t \\ y = & 2 + t \end{cases} \text{ and } l_2 : \begin{cases} x = & -1 + 4t \\ y = & 1 - 2t \end{cases}.$$

MAA150 Vektoralgebra, VT2017.

Assessment criteria for TEN1 2017-02-16

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Points may be deducted for erroneous mathematical statements, calculations, or failure to use proper mathematical notation.

Assessment problems

1. [**6 points**]
a. Method; e.g. relevant row operations **(1p)**, correctly setting free parameter (no point if you find that t is a value) **(1p)**, the correct solution **(1p)**.
b. Method; e.g.relevant row operations **(2p)**, finding c **(1p)**

2. [**4 points**]
A condition for invertability **(1p)**, checking the condition, e.g computing the determinant **(2p)**, finding the values for which the matrix is invertible **(1p)**

3. **5 points**]
Solving for X algebraically using matrix operations **(1p)**, method for finding B^{-1} **(2p)**, correct B^{-1} **(1p)**, correct X **(1p)**

4. [**5 points**]
a. The correct vector $\overrightarrow{AB}$ **(1p)**, computing the distance **(1p)**
b. Computing the crossproduct of two relevant vectors **(2p)**, computing the area **(1p)**

5. [**5 points**]
Method; such as relevant vectors and figure **(2p)**, finding the distance, e.g. by relevant projection **(3p)**

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ⓘa

$$\left[\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 2 & -6 & 5 & 0 \\ 3 & -9 & 5 & 0 \end{array}\right] \begin{matrix} ② \; ③ \\ \leftarrow \\ \leftarrow \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \begin{matrix} \\ \boxed{-2/3} \\ \leftarrow \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \times(-1) \\ \times(\frac{1}{3}) \\ \end{matrix}$$

(1p)

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \underset{\uparrow}{0} & 0 & 0 \end{array}\right]$$

x_2 free variable

Set $x_2 = t$, then $x_3 = 0$ (1p)

and $x_1 = 3x_2 - x_3 = 3t$

Hence $$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ 0 \end{bmatrix} = t \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Answer a: $x_1 = 3t$, $x_2 = t$, $x_3 = 0$, where $t \in \mathbb{R}$. (1p)

①b

$$\left[\begin{array}{ccc|c} -1 & 3 & -1 & -7 \\ 2 & -6 & 5 & 2 \\ 3 & -9 & 5 & c \end{array}\right] \begin{matrix} ② \; ③ \\ \leftarrow \\ \leftarrow \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & -1 & -7 \\ 0 & 0 & 3 & -12 \\ 0 & 0 & 2 & c-21 \end{array}\right] \begin{matrix} \\ \boxed{-2/3} \\ \leftarrow \end{matrix} \sim \left[\begin{array}{ccc|c} -1 & 3 & -1 & -7 \\ 0 & 0 & 3 & -12 \\ 0 & 0 & 0 & c-13 \end{array}\right] (*)$$

(2p)

Then (*) yields $0 = c - 13$, so there is no solution if $c \neq 13$. If $c = 13$ then the equation system is

$$\begin{cases} -x_1 + 3x_2 - x_3 = -7 \\ \qquad\qquad 3x_3 = -12 \end{cases}$$

which has x_2 as free variable, and therefore has infinitely many solutions, i.e. consistent (1p)

Answer b: The system $A\bar{x} = \bar{b}$ is consistent iff $c = 13$.

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② Let $A = \begin{bmatrix} 2 & -2 & 1 \\ 1 & k & 2 \\ 3 & -2k & k \end{bmatrix}$.

A is inverteble $\Longleftrightarrow$ $\det(A) \neq 0$ (1p)

$$\det(A) = \begin{vmatrix} 2 & -2 & 1 \\ 1 & k & 2 \\ 3 & -2k & k \end{vmatrix} \updownarrow = - \begin{vmatrix} 1 & k & 2 \\ 2 & -2 & 1 \\ 3 & -2k & k \end{vmatrix} \underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}}{=} - \begin{vmatrix} 1 & k & 2 \\ 0 & -2-2k & -3 \\ 0 & -5k & k-6 \end{vmatrix} =$$

Cofactor exp. Column 1

$$= - \begin{vmatrix} -2-2k & -3 \\ -5k & k-6 \end{vmatrix} = -\left(-(2+2k)(k-6) - 15k\right) = (2+2k)(k-6) + 15k =$$

$$= 2k - 12 + 2k^2 - 12k + 15k = 2k^2 + 5k - 12$$ (2p)

$$2k^2 + 5k - 12 = 0 \Longleftrightarrow k^2 + \frac{5}{2}k - 6 = 0 \Longleftrightarrow k = -\frac{5}{4} \pm \sqrt{\frac{25}{16} + \frac{96}{16}}$$

$$\Longleftrightarrow k = -\frac{5}{4} \pm \sqrt{\frac{121}{16}} = -\frac{5}{4} \pm \frac{11}{4} \Longleftrightarrow k = -4 \text{ or } k = \frac{3}{2}$$

Answer: A is invertible iff $k \neq -4$ and $k \neq \frac{3}{2}$ (1p)

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③ $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$A + X \cdot B = 0 \Leftrightarrow X \cdot B = -A \underset{\uparrow}{\Leftrightarrow} X = (-A) \cdot B^{-1}$ (4p)

if B has an inverse

Finding B^{-1}: $[B \mid I] \sim [I \mid B^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\textcircled{-1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right] \overset{\textcircled{1}}{\sim}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \overset{\textcircled{-1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & -2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \times(-\frac{1}{2}) \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \quad \text{so } B^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}$$

$= B^{-1}$ (3p)

Hence $X = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{7}{2} & -2 & -\frac{5}{2} \\ -\frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$ (1p)

Answer: $X = \begin{bmatrix} -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{7}{2} & -2 & -\frac{5}{2} \\ -\frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$

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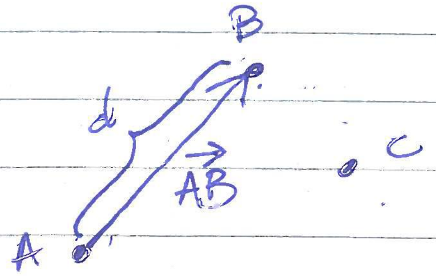
④a $A(1,2,-1)$, $B(1,-2,2)$, $C(-1,1,1)$



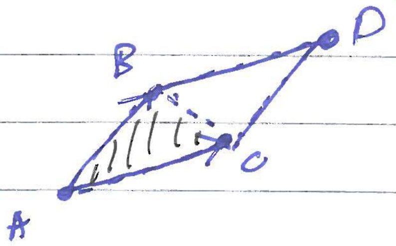
$d = \| \vec{AB} \|$

$\vec{AB} = (1,-2,2) - (1,2,-1)$
$= (0,-4,3)$ (1p)

$\| \vec{AB} \| = \sqrt{0^2+(-4)^2+3^2} = \sqrt{25} = 5$ (1p)

Answer a: 5 length units

④b



Area ABCD $= \| \vec{AB} \times \vec{AC} \|$ so

Area ABC $= \dfrac{\| \vec{AB} \times \vec{AC} \|}{2}$

$\vec{AB} = (1,-2,2) - (1,2,-1) = (0,-4,3)$

$\vec{AC} = (-1,1,1) - (1,2,-1) = (-2,-1,2)$

$$\vec{AB} \times \vec{AC} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \times \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} -4 & -1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & -2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ -4 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \\ -8 \end{bmatrix}$$ (2p)

$\| \vec{AB} \times \vec{AC} \| = \sqrt{(-5)^2+(-6)^2+(-8)^2} = \sqrt{25+36+64} = \sqrt{125} = 5 \cdot \sqrt{5}$

Answer b: $\dfrac{5 \cdot \sqrt{5}}{2}$ area units. (1p)

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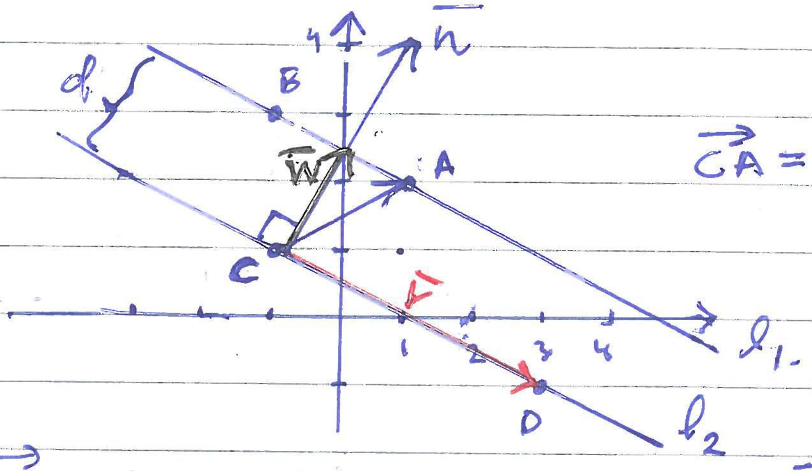
⑤ $l_1: \begin{cases} x = 1-2t \\ y = 2+t \end{cases}$, $l_2: \begin{cases} x = -1+4t \\ y = 1-2t \end{cases}$

l_1) $t=0 \Rightarrow (1,2) \in l_1$ l_2) $t=0 \Rightarrow (-1,1) \in l_2$

$t=1 \Rightarrow (-1,3) \in l_2$ $t=1 \Rightarrow (3,-1) \in l_2$

Let $A(1,2)$, $B(-1,3)$. and $C(-1,1)$, $D(3,-1)$



$\vec{CA} = (1,2)-(-1,1) = (2,1)$

(2p)

Let $\bar{v} = \vec{CD} = (3,-1)-(-1,1) = (4,-2)$, then $\bar{n} = (2,4)$ is a normal to l_2 since $\bar{v} \cdot \bar{n} = 4 \cdot 2 - 2 \cdot 4 = 0$. (1p)

The distance, d, between l_1 and l_2 equals

$$d = \|\bar{w}\| = \| \text{proj}_{\bar{n}} \vec{CA} \| = \left\| \frac{(2,1)\cdot(2,4)}{\|(2,4)\|^2} \cdot (2,4) \right\|$$

$$= \frac{|(2,1)\cdot(2,4)|}{\|(2,4)\|} = \frac{2\cdot 2 + 1 \cdot 4}{\sqrt{2^2+4^2}} = \frac{8}{\sqrt{20}} = \frac{4}{\sqrt{5}}$$ (2p)

Answer: $\frac{4}{\sqrt{5}}$ length units