MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2016-06-10 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\to$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\to$	4
		$54 \leq S_1 + 2S_2$	$\to$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\to$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\to$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\to$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\to$	B
		$61 \leq S_1 + 2S_2$	$\to$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the values of β for which the function f, defined by

$$f(x) = 4x^3 + 7\beta x^2 + 4\beta^2 x + 7\,,$$

has a local maximum at the point -2.

2. Evaluate the integral

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{(4 - x^2)^{3/2}}\,,$$

and write the result in as simple form as possible.

3. Solve the initial-value problem $\begin{cases} y' = x(y-1)(y-3), \\ y(0) = 2\,. \end{cases}$

4. Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2n)^3}{(4n+5)\sqrt{n^6+7}}$ convergent or divergent? Explain!

5. Find the length of the curve $\begin{cases} x = \frac{1}{2}t^2\,, \\ y = \frac{1}{3}t^3\,, \end{cases}$ $\sqrt{3} \leq t \leq 2\sqrt{2}\,.$

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2016-06-10 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm de värden på β för vilka funktionen f, definierad genom

$$f(x) = 4x^3 + 7\beta x^2 + 4\beta^2 x + 7\,,$$

har ett lokalt maximum i punkten -2.

2. Beräkna integralen

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{(4-x^2)^{3/2}}\,,$$

och skriv resultatet på en så enkel form som möjligt.

3. Lös begynnelsevärdesproblemet $\begin{cases} y' = x(y-1)(y-3), \\ y(0) = 2\,. \end{cases}$

4. Är serien $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2n)^3}{(4n+5)\sqrt{n^6+7}}$ konvergent eller divergent? Förklara!

5. Bestäm längden av kurvan $\begin{cases} x = \frac{1}{2}t^2\,, \\ y = \frac{1}{3}t^3\,, \end{cases}$ $\sqrt{3} \leq t \leq 2\sqrt{2}\,.$

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2016-06-10

1. $f(x) = 4x^3 + 7\beta x^2 + 4\beta^2 x + 7$

Since f is differentiable for all x in $\mathbb{R}$, the only possibility for f to have a local maximum at -2 is that the point is at least a stationary point of f. If also $f''(-2) < 0$, then we are sure that -2 is a local maximum point. If $f''(-2) = 0$, then some other method has to be applied for a final conclusion whether -2 is a local minimum point, a local maximum point or a terrace point.

Differentiation gives $f'(x) = 12x^2 + 14\beta x + 4\beta^2$, $f''(x) = 24x + 14\beta$

Necessary condition: $0 = f'(-2) = 48 - 28\beta + 4\beta^2 = 4(\beta-3)(\beta-4) \Leftrightarrow \begin{cases}\beta = 3 \\ \text{or} \\ \beta = 4\end{cases}$

Since $f''(-2) = -48 + 14\beta = \begin{cases}-6 & \text{if } \beta = 3 \\ 8 & \text{if } \beta = 4\end{cases}$ We conclude that only for $\beta = 3$ is -2 a local maximum point.

2. $$\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{(4-x^2)^{3/2}} \left[\begin{array}{l} x = 2\sin(\theta) \\ dx = 2\cos(\theta)\,d\theta \\ x_1 = \sqrt{2} \;—\; \theta_1 = \frac{\pi}{4} \\ x_2 = \sqrt{3} \;—\; \theta_2 = \frac{\pi}{3} \end{array}\right] = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\cos(\theta)\,d\theta}{\left[4(1-\sin^2(\theta))\right]^{3/2}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\cos(\theta)\,d\theta}{8\,|\cos(\theta)|^3}$$

$$\left(\begin{array}{l}\text{since } \cos(\theta) > 0 \\ \text{in the interval} \\ \text{of } \theta\end{array}\right) = \frac{1}{4}\int_{\pi/4}^{\pi/3} \frac{d\theta}{\cos^2(\theta)} = \frac{1}{4}\left[\tan(\theta)\right]_{\pi/4}^{\pi/3} = \frac{1}{4}(\sqrt{3} - 1)$$

3. $\begin{cases} \text{DE:} & y' = x(y-1)(y-3) \\ \text{IV:} & y(0) = 2 \end{cases}$

The DE is nonlinear but separable, and may (for $1 < y < 3$) be written as $\frac{1}{(y-1)(y-3)}\, y' = x$

$\Leftrightarrow \frac{1}{2}\left(\frac{1}{y-3} - \frac{1}{y-1}\right) y' = x \Leftrightarrow \left(\frac{1}{y-3} - \frac{1}{y-1}\right) y' = 2x$

Working out $\int dx$ on both sides gives

$\ln|y-3| - \ln|y-1| = x^2 + \tilde{C} \Leftrightarrow \ln\left|\frac{y-3}{y-1}\right| = x^2 + \tilde{C}$

$\Leftrightarrow \left|\frac{y-3}{y-1}\right| = e^{x^2 + \tilde{C}} = e^{x^2} \cdot e^{\tilde{C}} \Leftrightarrow \frac{y-3}{y-1} = e^{x^2} \cdot C$

where (applying the IV) $\frac{2-3}{2-1} = e^0 \cdot C$ i.e. $C = -1$

Thus $y - 3 = -e^{x^2}(y-1)$ i.e. $y(1 + e^{x^2}) = 3 + e^{x^2}$ i.e. $y = \frac{3 + e^{x^2}}{1 + e^{x^2}}$

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(4) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{(1+2n)^3}{(4n+5)\sqrt{n^6+7}} = \dfrac{(2n)^3\left(1+\frac{1}{2n}\right)^3}{4n\left(1+\frac{5}{4n}\right)n^3\sqrt{1+\frac{7}{n^6}}}$

$$= \frac{1}{n}\,\frac{2\left(1+\frac{1}{2n}\right)^3}{\left(1+\frac{5}{4n}\right)\sqrt{1+\frac{7}{n^6}}} = \frac{1}{n}\underbrace{B(n)}_{\text{bounded as } n\to\infty}$$

We have that $\displaystyle\lim_{n\to\infty}\frac{a_n}{1/n} = 2\,\frac{(1+0)^3}{(1+0)\sqrt{1+0}} = 2 > 0$

where $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent according to the integral test.
The comparison test (on limit form) then gives that even the series $\sum_{n=1}^{\infty} a_n$ diverges.

(5) $\gamma: \begin{cases} x = \frac{1}{2}t^2 \\ y = \frac{1}{3}t^3 \end{cases}, \quad \sqrt{3} \le t \le 2\sqrt{2}$

The length L of the curve γ is

$$\int_{\sqrt{3}}^{2\sqrt{2}} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_{\sqrt{3}}^{2\sqrt{2}} \sqrt{t^2+(t^2)^2}\,dt$$

$$= \int_{\sqrt{3}}^{2\sqrt{2}} t\sqrt{1+t^2}\,dt$$

$$= \left[\frac{2}{3}\,\frac{1}{2}\left(1+t^2\right)^{3/2}\right]_{\sqrt{3}}^{2\sqrt{2}}$$

$$= \frac{1}{3}\left[(1+8)^{3/2}-(1+3)^{3/2}\right]$$

$$= \frac{1}{3}\left(9\cdot 3 - 4\cdot 2\right)$$

$$= \frac{1}{3}(27-8) = \frac{19}{3}$$

Answer $\frac{19}{3}$ l.u.

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

Examination TEN2 – 2016-06-10 Maximum points for subparts of the problems in the final examination

1. $\beta = 3$

1p: Correctly concluded that for -2 to be a local maximum point of f it is necessary that -2 is a stationary point of f (since f is differentiable for all $x \in R$). Also correctly concluded that if $f''(-2) < 0$, then we are sure that -2 is a local maximum point. (If $f''(-2) = 0$ then some other method has to be applied for a final conclusion.)
1p: Correctly differentiated f twice, and correctly found the two possible β-values for which -2 has to be analyzed
2p: Correctly concluded that -2 is a local maximum point if $\beta = 3$, and a local minimum point if $\beta = 4$

2. $\frac{1}{4}(\sqrt{3} - 1)$

1p: Correctly by the substitution $x = 2\sin(\theta)$ translated the integrand and the limits of the integral
1p: Correctly simplified the integrand into $1/(2\cos(\theta))^2$
1p: Correctly found the antiderivative $\tan(\theta)/4$
1p: Correctly evaluated the antiderivative at the limits and by that correctly found the value of the integral

3. $y = \dfrac{3 + e^{x^2}}{1 + e^{x^2}}$

1p: Correctly identified the differential equation as nonlinear and separable, and correctly found the partial fractions of $[(y-1)(y-3)]^{-1}$
1p: Correctly found the antiderivatives of both sides of the separated differential equation
1p: Correctly adapted the solution to the initial value
1p: Correctly solved for y

4. The series is divergent

1p: Correctly found that the terms a_n of the series have the property of being equal to $n^{-1}B(n)$, where $B(n) \to 2$ as $n \to \infty$
1p: Correctly found that the comparison test is applicable and that the series $\Sigma\, n^{-1}$ is the one to compare with
1p: Correctly noted that the series $\Sigma\, n^{-1}$ is divergent according to the integral test
1p: Correctly concluded that the series is divergent since the series compared with, namely $\Sigma\, n^{-1}$, is divergent

5. $\dfrac{19}{3}$ l.u.

1p: Correctly formulated an integral (with explicit expressions of the derivatives dx/dt and dy/dt) whose value is the length of the curve
1p: Correctly rewrited the integrand into $t(1+t^2)^{1/2}$ in preparation for finding the antiderivative
1p: Correctly found an antiderivative of the integrand
1p: Correctly found the value of the integral, and by that the length of the curve γ