MÄLARDALEN UNIVERSITY
School of education, culture and communication
Division of applied mathematics
*Examiner:* Erik Darpö

**Examination in Mathematics**
MAA150 Vector algebra TEN1
*Date:* 5th December 2014 *Time:* 3 hours
*Materials allowed:* Writing material only.

This exam TEN1 consists of 6 problems, with a total score of 25 points. To obtain the grades **3**, **4** and **5**, scores of at least 12, 16 respectively 20 points are required.
The solutions to the problem 2–5 are to include motivations and clear answers to the questions asked. To problem 1, only correct answers are required.

**1.** Determine

a) the equation (on normal form) of the plane in $\mathbb{R}^3$ that is perpendicular to the vector $n = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and contains the point $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

b) the modulus $|z|$ of the complex number $z = -1 - i$;

c) the argument $\arg(z)$ of the complex number $z = -1 - i$;

d) the matrix of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_1 \end{pmatrix}$.

*Only answers are required to Problem 1.* *(4p)*

**2.** Let $a \in \mathbb{R}$ be an arbitrary constant, and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & a & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Solve the equation $Ax = b$ for all possible values of $a$. *(4p)*

**3.** Write the complex number $z = \dfrac{(\sqrt{3} + i)^{60}}{2^{55}}$ on the form $z = a + bi$
(where $a$ and $b$ are real numbers). *(4p)*

**4.** Let $\ell \subset \mathbb{R}^2$ be the line through the origin that is parallel to the vector $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The linear map $P = P_u : \mathbb{R}^2 \to \mathbb{R}^2$ is given as orthogonal projection onto the line $\ell$.

a) Determine $P\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) Find the matrix of $P$.

*(5p)*

**5.** Determine the area of the triangle in $\mathbb{R}^3$ with corners in the points $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. *(4p)*

**6.** Solve the matrix equation $AX = B$, where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}, \quad \text{and} \quad X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}.$$

*(4p)*

# MAA150: Solutions to the exam TEN1 5 December 2014

1. a) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = 0$

$1 \cdot (x-1) + 0 \cdot (y-0) + (-1) \cdot (z-0) = 0$

$\underline{\underline{x - z = 1}}$ – the equation of the plane.

b) $|z| = |-1-i| = \sqrt{(-1)^2 + (-1)^2} = \underline{\underline{\sqrt{2}}}$

c) $z = \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) = \sqrt{2} \cdot \left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right)$

$\frac{5\pi}{4}$ is an argument of $z = -1 - i$

d) $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ – the matrix of $T$

2) The augmented matrix of $Ax=b$ is

$$-1\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & a & 1 & 1 \end{array}\right) \sim -a\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & a & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1-a & 1 \end{array}\right)$$

If $1-a \neq 0$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{1-a} \end{array}\right) -1 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{a-1} \\ 0 & 0 & 1 & \frac{1}{1-a} \end{array}\right) \quad \underline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{a-1} \\ \frac{1}{1-a} \end{pmatrix}}$$

$\underline{\text{If } 1-a=0}$:
$(a=1)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1-a & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{cases} x_1 = 1 \\ x_2 + x_3 = 0 \\ 0 = 1 \end{cases}$$

<u>No solutions</u>

The equation $Ax=b$ has solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{a-1} \\ \frac{1}{1-a} \end{pmatrix}$$ if $a \neq 1$, and no solution if $a=1$.

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3) $$|\sqrt{3}+i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\sqrt{3}+i = 2\cdot\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$(\sqrt{3}+i)^{60} = \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^{60} = 2^{60}\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{60}$$

$$= 2^{60}\left(\cos\left(60\cdot\frac{\pi}{6}\right) + i\sin\left(60\cdot\frac{\pi}{6}\right)\right) = 2^{60}\left(\underbrace{\cos 10\pi}_{1} + i\underbrace{\sin 10\pi}_{0}\right)$$

$$= 2^{60}$$

$$\frac{(\sqrt{3}+i)^{60}}{2^{55}} = \frac{2^{60}}{2^{55}} = 2^5 = \underline{\underline{32}}$$

4) $P(w) = P_u(w) = \frac{w \cdot u}{u \cdot u}\, u$ for all $w \in \mathbb{R}^2$.

$$u \cdot u = \begin{pmatrix}1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\end{pmatrix} = 1^2 + 2^2 = 5$$

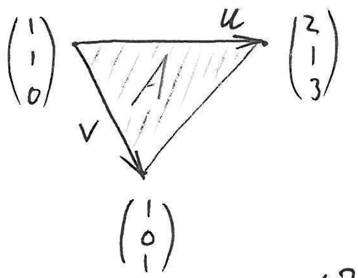
a) $$P\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{5}\left(\begin{pmatrix}1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\2\end{pmatrix}\right)\begin{pmatrix}1\\2\end{pmatrix} = \underline{\underline{\frac{1}{5}\begin{pmatrix}1\\2\end{pmatrix}}}$$

$$P\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{5}\left(\begin{pmatrix}0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\end{pmatrix}\right)\begin{pmatrix}1\\2\end{pmatrix} = \frac{2}{5}\begin{pmatrix}1\\2\end{pmatrix} = \underline{\underline{\frac{1}{5}\begin{pmatrix}2\\4\end{pmatrix}}}$$

b) The matrix $A$ of $P$ is

$$A = \begin{pmatrix} | & | \\ P\begin{pmatrix}1\\0\end{pmatrix} & P\begin{pmatrix}0\\1\end{pmatrix} \\ | & | \end{pmatrix} = \underline{\underline{\frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}}}$$

5) $$u = \begin{pmatrix}2\\1\\3\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\3\end{pmatrix}$$




$$v = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 3 \\ 0 & -1 & 1 \end{vmatrix} = -e_3 + 3e_1 - e_2 = 3e_1 - e_2 - e_3 = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$$

$$\|u \times v\| = \sqrt{3^2 + (-1)^2 + (-1)^2} = \sqrt{11}$$

The area of the triangle: $A = \frac{\|u \times v\|}{2} = \underline{\underline{\frac{\sqrt{11}}{2}}}$

6) If the matrix $A$ is invertible, then the equation has a unique solution $X = A^{-1}B$.

Try to invert $A$:

$$(A|I_2) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

rank $=2$, so $A$ is invertible.

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$A^{-1}$

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$X = A^{-1}B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The equation has a unique solution $X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Alternative solution:

$$A\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \iff \begin{cases} A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{augm. matr.: } \left(A \,\middle|\, \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\ A\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} & \text{augm. matr. } \left(A \,\middle|\, \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) \end{cases}$$

We solve these two systems simultaneously:

$$\left(A \,\middle|\, \begin{matrix} 0 & 2 \\ 1 & 3 \end{matrix}\right) = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right). \text{ So } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

# MAA150 Vector algebra autumn term 2014

## Assessment criteria for TEN1 2014-12-05

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**1.** One point for correct answer to each of a – d.

**2.** Interpretation of the equation $Ax = b$ as a linear system: *1p*
Reduction of the system to upper triangular (or some other essentialy equivalent) form: *1p*
One point each for solving the two cases $a \neq 0$ and $a = 0$ respectively.

**3.** Two points for writing the number $\sqrt{3} + i$ on polar form, one point for applying de Moivre's formula, and one additional point for correctly answering the question.

**4.** Identfying two vectors that span the triangle: *1p*
Taking the vector product of the two vectors: *1p*
Correctly computing the vector product: *1p*
Computing the area of the triangle from the length of the vector product: *1p*

**5.** Identifying that $X = A^{-1}B$: *1p*
Explicitly computing the inverse of the matrix $A$: *2p*
Computing $X$: *1p*

*Alternatively:*
Interpreting the equation $AX = B$ as a linear system of equations: *1p*
Solving the system: *2p*
Interpreting the solution of the system to find the matrix $X$: *1p*