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Examination Vector algebra
MAA150 - TEN2
Date: 2015-03-21
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Find the area of the parallelogram with vertices $A(-2,-2)$, $B(1,-1)$, $C(4,3)$, and $D(1,2)$. (4p)

2 Let T_1 and T_2 be the linear transformations: $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation $-60°$ (i.e. $60°$ clockwise), and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ a reflection in the y-axis. Find the standard matrices for T_1, T_2, and $T_1 \circ T_2$. Motivate your answer. (5p)

3 Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 4 & 2 \end{bmatrix}.$$

a. Show that $\mathbf{v}_1 = (0,-2,2)$ is an eigenvector of A and find the corresponding eigenvalue λ_1. (3p)
b. Find all eigenvalues of A and determine if A is diagonalizable. Hint: use λ_1 from part (a). (3p)

4 Let $S = \{(2,1,-4),(-1,1,3),(1,2,-1)\}$ and $V = \mathrm{span}(S)$.
a. Find a basis for V. (3p)
b. Find a basis for $V^{\perp}$. (2p)

5 Let W be the vector space spanned by the vectors $\mathbf{v}_1 = (1,0,-1,-1)$, $\mathbf{v}_2 = (2,1,1,1)$, and $\mathbf{v}_3 = (0,0,1,-1)$.
a. Show that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is an ortogonal set. (2p)
b. Find the orthogonal projection of $\mathbf{u} = (-1,1,0,1)$ on W. (3p)

Avdelningen för Matematik
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Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2016-03-21
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Bestäm arean av parallellogrammet med hörn i $A(-2,-2)$, $B(1,-1)$, $C(4,3)$, och $D(1,2)$. (4p)

2 Låt T_1 och T_2 vara de linjära avbildningarna: $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ är en rotation $-60°$ (dvs. $60°$ medurs), och $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ en spegling i y-axeln. Bestäm standardmatriserna till T_1, T_2, och $T_1 \circ T_2$. Motivera ditt svar. (5p)

3 Låt A vara matrisen

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 4 & 2 \end{bmatrix}.$$

a. Visa att $\mathbf{v}_1 = (0,-2,2)$ är en egenvektor till A och bestäm tillhörande egenvärde λ_1. (3p)

b. Bestäm alla egenvärden till A och avgör om A är diagonaliserbar. Tips: utnyttja λ_1 från (a). (3p)

4 Låt $S = \{(2,1,-4),(-1,1,3),(1,2,-1)\}$ och $V = \operatorname{span}(S)$.

a. Bestäm en bas för V. (3p)

b. Bestäm en bas för $V^\perp$. (2p)

5 Låt W vara vektorrummet som spänns upp av vektorerna $\mathbf{v}_1 = (1,0,-1,-1)$, $\mathbf{v}_2 = (2,1,1,1)$, och $\mathbf{v}_3 = (0,0,1,-1)$.

a. Visa att $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ är en ortogonal mängd vektorer. (2p)

b. Bestäm den ortogonala projektionen av $\mathbf{u} = (-1,1,0,1)$ på W. (3p)

MAA150 Vektoralgebra, vt-16.

Assessment criterias for TEN2 2016-03-21

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1.
- Correct and relevant formula for the area relevant for the problem **(1p)**
- Finding relevant vectors correctly **(2p)**
- Computing the area **(1p)**

Max 2p if you misuse mathematical notation when calculating the area, i.e. using cross product for vectors in $\mathbb{R}^2$, setting matrices equal to scalars, taking determinants of 2×3 matrices, setting the result of the cross product equal to a scalar or the result of the scalar product equal to a vector, using point wise multiplication of vectors etc.

2.
- Stating the standard matrix for T_1 for $-60°$ (no motivation needed) **(1p)**
- Finding the correct standard matrix for T_2 **(1p)** and proper motivation **(1p)**
- Finding the standard matrix for $T_1 \circ T_2$: The step $[T1 \circ T2] = [T1] \cdot [T2]$ is **(1p)** and doing the multiplication correct is 1p. **(1p)**

3. **a.**

- Stating or using the definition of eigenvalue and eigenvector **(1p)**
- Checking the condition **(1p)** and finding the correct eigenvalue **(1p)**

b.

- Finding the remaining two eigenvalues: **(1p)** for calculating the characteristic polynomial and **(1p)** for finding the other two eigenvalues.
- Correct motivation that A is diagonalizable **(1p)**

4. **a.**

- Method and setting up the relevant matrix **(1p)**
- Relevant row reductions**(1p)**
- Finding a basis **(1p)**

b.

- Correct method for finding the basis **(1p)**
- Computing the basis **(1p)**

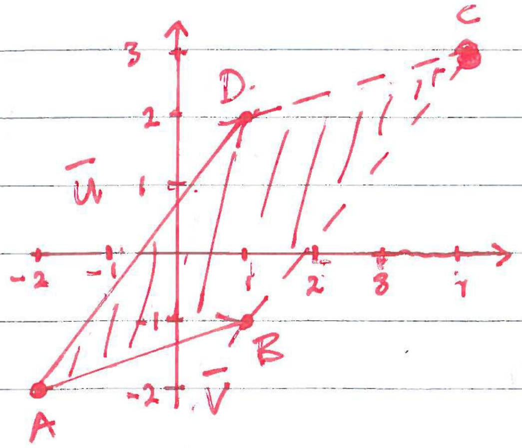
5. **a.**

- Checking that one pair of vector is orthogonal **(1p)**
- Checking that the remaining pair of vectors are pairwise orthogonal gives **(1p)**

b.

- Stating the correct formula for orthogonal projection **(1p)**
- Computing the projection correctly **(2p)**

(1)



$\bar{v} = \overrightarrow{AB} = (1,-1) - (-2,-2) =$
$= (3,1)$ (1p)

$\bar{u} = \overrightarrow{AD} = (1,2) - (-2,-2) =$
$= (3,4)$ (1p)

$$\text{Area } ABCD = \left|\det\left(\begin{bmatrix}\bar{u} & \bar{v}\end{bmatrix}\right)\right| = \left|\det\left(\begin{bmatrix}3 & 3\\4 & 1\end{bmatrix}\right)\right| =$$

$$= |3\cdot 1 - 3\cdot 4| = |-9| = 9$$ (2p)

Answer: 9 area units.

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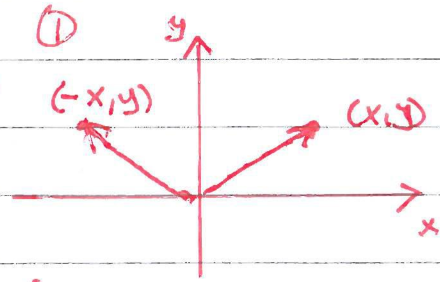
(2)

T_1) $[T_1] = \begin{bmatrix}\cos(-60^\circ) & -\sin(-60^\circ)\\ \sin(-60^\circ) & \cos(-60^\circ)\end{bmatrix} = \begin{bmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}$ (1p)

T_2) (1)



(2) $T_2(x,y) = (-x,y)$

(3) $T_2(\bar{e}_1) = T_2(1,0) = (-1,0)$
$T_2(\bar{e}_2) = T_2(0,1) = (0,1)$

(1p)

Therefore

$[T_2] = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ (1p) so

$[T_1 \circ T_2] = [T_1]\cdot[T_2] = \begin{bmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}\cdot\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}$ (2p)

Answer: $[T_1] = \begin{bmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}$, $[T_2] = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$, $[T_1 \circ T_2] = \begin{bmatrix}-1/2 & \sqrt{3}/2\\ \frac{\sqrt{3}}{2} & 1/2\end{bmatrix}$

(3a) λ is an eigenvalue with eigenvector $\bar{v}$ iff $A\bar{v}=\lambda\cdot\bar{v}$

$$A\cdot\bar{v}=\begin{bmatrix}1&1&1\\0&-1&1\\1&4&2\end{bmatrix}\cdot\begin{bmatrix}0\\-2\\2\end{bmatrix}=\begin{bmatrix}0\\4\\-4\end{bmatrix}=(-2)\cdot\begin{bmatrix}0\\-2\\2\end{bmatrix}=(-2)\cdot\bar{v} \quad (2p)$$

Therefore $\bar{v}$ is an eigenvector for eigenvalue $\lambda=-2$. (1p)

Answer a: $\lambda=-2$.

(3b) (CE) $\det(A-\lambda\cdot I)=0$ which here is

$$p(\lambda)=\begin{vmatrix}1-\lambda&1&1\\0&-1-\lambda&1\\1&4&2-\lambda\end{vmatrix}=(-1-\lambda)\cdot\begin{vmatrix}1-\lambda&1\\1&2-\lambda\end{vmatrix}-\begin{vmatrix}1-\lambda&1\\1&4\end{vmatrix}=$$

cofactor exp. row 2

$$=(-1-\lambda)\left((1-\lambda)(2-\lambda)-1\right)-\left(4(1-\lambda)-1\right)=\ldots=\boxed{-\lambda^3+2\lambda^2+6\lambda-4=0}$$

(CE) (1p)

$\lambda=-2$ is an eigenvalue so $p(-2)=0$, Long division gives

$$\begin{array}{l}
\quad -\lambda^2+4\lambda-2\\
\hline
-\lambda^3+2\lambda^2+6\lambda-4 \;\big|\; \lambda+2\\
-(-\lambda^3-2\lambda^2)\\
\hline
\qquad 4\lambda^2+6\lambda-4\\
\quad -(4\lambda^2+8\lambda)\\
\hline
\qquad\quad -2\lambda-4\\
\qquad -(-2\lambda-4)\\
\hline
\qquad\qquad 0
\end{array}$$

Find the remaining roots to $p(\lambda)$

$-\lambda^2+4\lambda-2=0 \Rightarrow \lambda=2\pm\sqrt{2}$ (1p)

Eigenvalues: $\lambda=-2$, $\lambda=2\pm\sqrt{2}$

Answer b: $\lambda=-2$, $\lambda=2\pm\sqrt{2}$, A is diagonalizable because A has 3 distinct eigenvalues. (1p)

4a

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ -4 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ -4 & 3 & -1 \end{bmatrix} \begin{matrix} (-2) & (4) \\ & \\ & \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (2p)$$

leading 1:s

Then $B_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for V. (1p)

4b Since V is a subspace of $\mathbb{R}^3$ and $\dim(V) = 2$, V is a plane. This means $V^\perp$ is a line through the origin in the direction of the normal. (1p)

$$\bar{n} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

so $B_2 = \left\{ \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \right\}$ is a basis for $V^\perp$ (1p)

5a

$$\left.\begin{aligned} \bar{v}_1 \bullet \bar{v}_2 &= (1,0,-1,-1) \bullet (2,1,1,1) = 0 \\ \bar{v}_1 \bullet \bar{v}_3 &= (1,0,-1,-1) \bullet (0,0,1,-1) = 0 \\ \bar{v}_2 \bullet \bar{v}_3 &= (2,1,1,1) \bullet (0,0,1,-1) = 0 \end{aligned}\right\} \text{ orthogonal set!}$$

(2p)

5b

$$\operatorname{proj}_W \bar{u} = \frac{\bar{u} \bullet \bar{v}_1}{\|\bar{v}_1\|^2} \cdot \bar{v}_1 + \frac{\bar{u} \bullet \bar{v}_2}{\|\bar{v}_2\|} \cdot \bar{v}_2 + \frac{\bar{u} \bullet \bar{v}_3}{\|\bar{v}_3\|^2} \bar{v}_3 = \quad (1p)$$

$$= \frac{(-1,1,0,1) \bullet (1,0,-1,-1)}{3} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} + \frac{(-1,1,0,1) \bullet (2,1,1,1)}{7} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{(-1,1,0,1) \bullet (0,0,1,-1)}{2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} =$$

$$= -\frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 0 \\ 1/6 \\ 7/6 \end{bmatrix} \quad (2p)$$