MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2017-11-09 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\to$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\to$	4
		$54 \leq S_1 + 2S_2$	$\to$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\to$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\to$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\to$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\to$	B
		$61 \leq S_1 + 2S_2$	$\to$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Evaluate the integral

$$\int_e^{e^e} \frac{\ln(\ln(x))}{x}\,dx\,,$$

and write the result in as simple form as possible.

2. Sketch the graph of the function f, defined by

$$f(x) = (x^2-1)\,e^{-\frac{1}{2}(x^2-2)}\,,$$

by utilizing the guidance given by asymptotes and stationary points. When sketching, the approximations $e \approx 2.72$, $\sqrt{e} \approx 1.65$, $1/\sqrt{e} \approx 0.607$ and $1/e \approx 0.369$ might be worth knowing.

3. Find the volume of the solid generated by rotating about the x-axis the bounded region that in the first quadrant is precisely enclosed by the curves $x + y = 1$ and $\sqrt{x} + y = 1$.

4. Find the real numbers x for which the power series

$$\sum_{n=1}^{\infty} \frac{n(5x-1)^n}{n+1}$$

is convergent. Are there any of these x for which the series is not absolutely convergent, i.e. is (only) conditionally convergent?

5. Solve the initial-value problem

$$y'' + 9y = \cos(x)\,, \quad y(0) = y'(0) = 0\,.$$

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2017-11-09 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Beräkna integralen

$$\int_e^{e^e} \frac{\ln(\ln(x))}{x}\,dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

2. Skissa grafen till funktionen f, definierad enligt

$$f(x) = (x^2-1)\,e^{-\frac{1}{2}(x^2-2)}\,,$$

genom att använda den vägledning som ges av asymptoter och stationära punkter. Vid skissning kan approximationerna $e \approx 2.72$, $\sqrt{e} \approx 1.65$, $1/\sqrt{e} \approx 0.607$ och $1/e \approx 0.369$ tänkas vara av värde att känna till.

3. Bestäm volymen av den kropp som genereras genom att kring x-axeln rotera det begränsade område som i den första kvadranten precis är inneslutet av kurvorna $x + y = 1$ och $\sqrt{x} + y = 1$.

4. Bestäm de reella tal x för vilka potensserien

$$\sum_{n=1}^{\infty} \frac{n(5x-1)^n}{n+1}$$

är konvergent. Är det några av dessa x för vilka serien inte är absolutkonvergent, dvs. är (endast) betingat konvergent?

5. Lös begynnelsevärdesproblemet

$$y'' + 9y = \cos(x)\,, \quad y(0) = y'(0) = 0\,.$$

If you prefer the problems formulated in English, please turn the page.

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① $$\int_e^{e^e} \frac{\ln(\ln(x))}{x}\,dx \begin{bmatrix}\ln(x)=u\\ \frac{1}{x}dx=du\end{bmatrix} = \int_1^e \ln(u)\,du = \Big[u\ln(u)\Big]_1^e - \int_1^e u\cdot\frac{1}{u}\,du$$

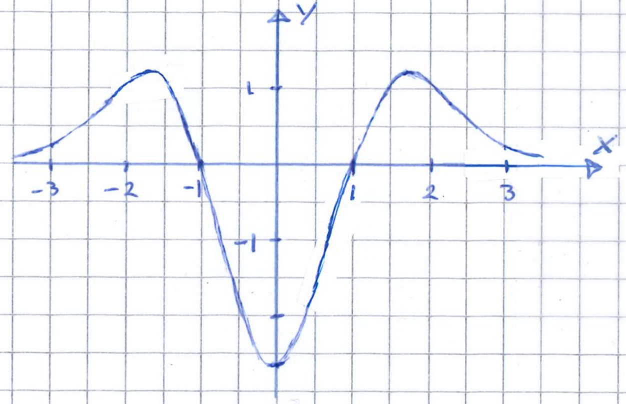
$$= \Big[u(\ln(u)-1)\Big]_1^e = e(1-1)-1(0-1) = \underline{1}$$

② $f(x)=(x^2-1)e^{-\frac{1}{2}(x^2-2)}$ Differentiation gives

$$f'(x) = 2xe^{-\frac{1}{2}(x^2-2)} + (x^2-1)e^{-\frac{1}{2}(x^2-2)}\cdot(-x) = -xe^{-\frac{1}{2}(x^2-2)}(-2+x^2-1) = -(x+\sqrt{3})x(x-\sqrt{3})e^{-\frac{1}{2}(x^2-2)}$$

A first derivative test is

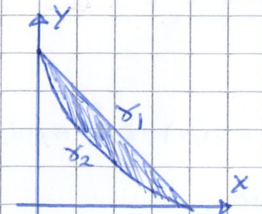
x		$-\sqrt{3}$		0		$\sqrt{3}$	
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$	$\nearrow$	loc. max	$\searrow$	loc. min	$\nearrow$	loc max	$\searrow$



$$\begin{cases} f(0)=(0-1)e^{-\frac{1}{2}(0-2)} = -e \\ f(\pm\sqrt{3})=(3-1)e^{-\frac{1}{2}(3-2)} = 2\frac{1}{\sqrt{e}} \end{cases}$$

The are no vertical asymptotes but a two-sided non-vertical one namely $y=0$ (since $\lim_{x\to\pm\infty} f(x)=0$).

③



$$\begin{cases} \gamma_1: \; x+y=1, \quad 0\le x\le 1 \\ \gamma_2: \; \sqrt{x}+y=1, \quad 0\le x\le 1 \end{cases}$$

Let V_x denote the volume of the solid generated by rotating about the x-axis the region precisely enclosed by γ_1 and γ_2.

By the method of slicing, we get

$$V_x = \int_0^1 \pi(1-x)^2\,dx - \int_0^1 \pi(1-\sqrt{x})^2\,dx = \pi\int_0^1 (1-2x+x^2-1+2\sqrt{x}-x)\,dx$$

$$= \pi\left[\frac{x^3}{3}-\frac{3}{2}x^2+\frac{4}{3}x\sqrt{x}\right]_0^1 = \pi\left[\left(\frac{1}{3}-\frac{3}{2}+\frac{4}{3}\right)-(0-0+0)\right] = \pi\left(\frac{5}{3}-\frac{3}{2}\right) = \frac{\pi}{6}$$

By the method of cylindrical shells, we get

$$V_x = \int_0^1 2\pi y\left[(1-y)-(1-y)^2\right]dy = 2\pi\int_0^1 y(1-y)\left[1-(1-y)\right]dy = 2\pi\int_0^1 y^2(1-y)\,dy$$

$$= 2\pi\left[\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1 = 2\pi\left[\left(\frac{1}{3}-\frac{1}{4}\right)-(0-0)\right] = \frac{\pi}{6}$$

Answer: $\underline{\frac{\pi}{6}}$ v.u.

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(4) $\sum_{n=1}^{\infty} \frac{n(5x-1)^n}{n+1}$ Define $a_n(x) = \frac{n(5x-1)^n}{n+1}$

Let $A(x) = \begin{cases} \lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty} \frac{n+1}{n+2}\frac{n+1}{n}|5x-1| = 5\left|x-\frac{1}{5}\right| & \text{if } x \neq \frac{1}{5} \\ 0 & \text{if } x = \frac{1}{5} \end{cases}$

The series $\sum_{n=1}^{\infty} a_n(x)$ converges absolutely if $A(x) < 1 \Leftrightarrow 0 < x < \frac{2}{5}$

— " — — " — diverges if $A(x) > 1 \Leftrightarrow x < 0 \vee x > \frac{2}{5}$

For $x = 0$, we have the series $\sum_{n=1}^{\infty} \frac{n}{n+1}(-1)^n$ which diverges since the terms of the series does not have the limit 0.

For $x = \frac{2}{5}$, — " — $\sum_{n=1}^{\infty} \frac{n}{n+1}$ — " —

Thus, the series $\sum_{n=1}^{\infty} a_n(x)$ converges in the interval $(0, \frac{2}{5})$. Especially, the convergence is absolutely for all points in the interval.

(5) DE: $y'' + 9y = \cos(x)$ IV:s $y(0) = y'(0) = 0$

The auxiliary equation of the corresponding homogeneous DE is $r^2 + 9 = 0 \Leftrightarrow (r+3i)(r-3i) = 0$.

Thus $y_H = A\cos(3x) + B\sin(3x)$

For a particular solution, $y_p = a\cos(x) + b\sin(x)$ works. Substitution of y_p into the DE gives

$[-a\cos(x) - b\sin(x)] + 9[a\cos(x) + b\sin(x)] = \cos(x)$ (in an interval)

$\Leftrightarrow (8a-1)\cos(x) + 8b\sin(x) = 0 \Leftrightarrow \begin{cases} a = \frac{1}{8} \\ b = 0 \end{cases}$

Thus $y = A\cos(3x) + B\sin(3x) + \frac{1}{8}\cos(x)$ is the general solution of the non-homogeneous DE.

An adaption to the IV:s gives $\begin{cases} A\cdot 1 + B\cdot 0 + \frac{1}{8}\cdot 1 = 0 \\ -3A\cdot 0 + 3B\cdot 1 - \frac{1}{8}\cdot 0 = 0 \end{cases} \Leftrightarrow \begin{cases} A = -\frac{1}{8} \\ B = 0 \end{cases}$

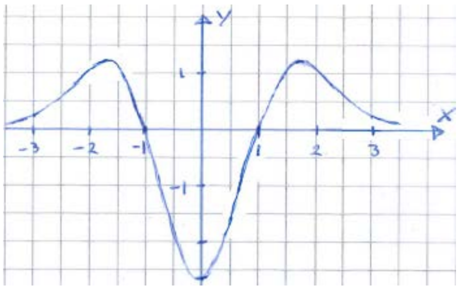
The solution of the IVP is thus $y = \frac{1}{8}[\cos(x) - \cos(3x)]$

Examination TEN2 – 2017-11-09 Maximum points for subparts of the problems in the final examination

1. 1

2p: Correctly by the substitution $\ln(x) = u$ translated the integral into $\int_1^e \ln(u)\,du$
1p: Correctly found an antiderivative of the integrand
1p: Correctly evaluated the antiderivative at the limits

2.



2p: Correctly found and classified the stationary points of the function
1p: Correctly illustrated the stationary points of the function
1p: Correctly found the asymptote of the graph, correctly sketched how the graph relates to the asymptote, and correctly completed the sketch of the graph

3. $\frac{\pi}{6}$ v.u.

2p: Correctly formulated an integral for the volume of the solid obtained by rotating the region about the x-axis (irrespective whether the method of slicing or the method of cylindrical shells have been applied)
1p: Correctly found an antiderivative of the integrand
1p: Correctly evaluated the antiderivative at the limits

4. The series is convergent for $0 < x < \frac{2}{5}$, and the convergence is absolute in the whole interval of convergence

1p: Correctly, by e.g. the ratio test, found that the series is absolutely convergent for $|5x - 1| < 1$, i.e. for $0 < x < \frac{2}{5}$, and hopefully correctly mentioned that the series definitely is divergent for $|x - \frac{1}{5}| > \frac{1}{5}$
1p: Correctly found that the series is divergent for $x = 0$
1p: Correctly found that the series is divergent for $x = \frac{2}{5}$
1p: Correctly stressed that the series is absolutely convergent in the whole interval of convergence

5. $y = \frac{1}{8}\big(\cos(x) - \cos(3x)\big)$

1p: Correctly identified the differential equation as a non-homogeneous linear DE of second order, and correctly found the general solution y_h of the corr. homog. DE
1p: Correctly proposed a formula for a part. sol. of the DE
1p: Correctly found a particular solution of the DE, and correctly summarized the general solution of the DE
1p: Correctly adapted the general solution of the DE to the IV