MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2017-06-08 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Sketch the graph of the function f, defined by

$$f(x) = \left(\frac{x+2}{x-1}\right)^2,$$

by utilizing the guidance given by asymptotes and stationary points.

2. Evaluate the integral

$$\int_0^{\pi/2} \cos(x)\sin^3(x)\, e^{-\sin^2(x)}\, dx\,,$$

and write the result in as simple form as possible.

3. Prove that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^{n-\frac{1}{2}}}$$

is convergent. Then, find the sum of the series.

4. Find the Taylor polynomial of order 2 about the point 1 for the function f whose function curve $y = f(x)$ with $f(1) = e$ is the solution of the equation

$$x\big(y + e\ln(y)\big) = 2e$$

in a neighbourhood of $P : (1, e)$.

5. Find the area of the surface generated by rotating the curve

$$y = \sqrt{x} - \tfrac{1}{3}x\sqrt{x}, \quad 0 \leq x \leq 3\,,$$

about the y-axis.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2017-06-08 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Skissa grafen till funktionen f, definierad enligt

$$f(x) = \left(\frac{x+2}{x-1}\right)^2,$$

genom att använda den vägledning som fås från asymptoter och stationära punkter.

2. Beräkna integralen

$$\int_0^{\pi/2} \cos(x)\sin^3(x)\, e^{-\sin^2(x)}\, dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

3. Bevisa att serien

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^{n-\frac{1}{2}}}$$

är konvergent. Bestäm sedan summan av serien.

4. Bestäm Taylorpolynomet av ordning 2 kring punkten 1 för den funktion f vars funktionskurva $y = f(x)$ med $f(1) = e$ är lösningen till ekvationen

$$x\big(y + e\ln(y)\big) = 2e$$

i en omgivning till $P : (1, e)$.

5. Bestäm arean av den yta som genereras genom att rotera kurvan

$$y = \sqrt{x} - \tfrac{1}{3}x\sqrt{x}, \quad 0 \leq x \leq 3\,,$$

kring y-axeln.

If you prefer the problems formulated in English, please turn the page.

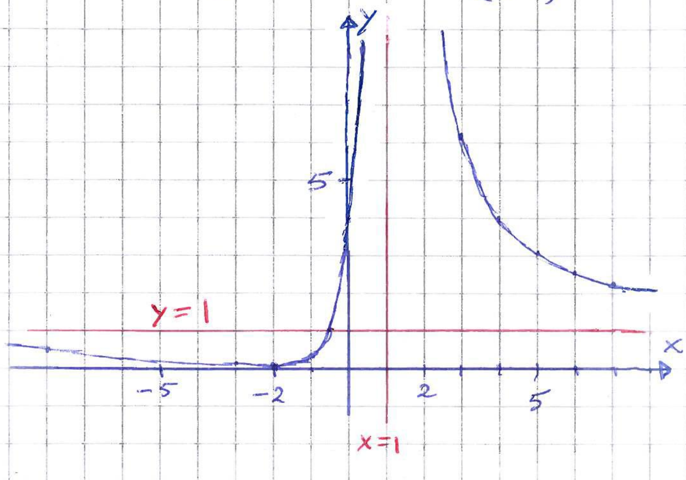
MAA151 / Solutions to the final exam TEN2 2017-06-08

1 $f(x) = \left(\frac{x+2}{x-1}\right)^2$

Differentiation gives $f'(x) = 2\left(\frac{x+2}{x-1}\right)\frac{1(x-1)-(x+2)\cdot 1}{(x-1)^2} = -\frac{6(x+2)}{(x-1)^3}$

A first-derivative test is

x		SP -2		As 1	
$f'(x)$	$-$	0	$+$	$\#$	$-$
$f(x)$	$\searrow$	loc. min.	$\nearrow$	$\#$	$\searrow$



$x = -2$ is a vertical asymptote (since f is unbounded in every punctured neighbourhood of 1)

$y = 1$ is a nonvertical asymptote as $x \to \pm\infty$

2 $\int_0^{\pi/2} \cos(x)\sin^3(x) e^{-\sin^2(x)}\,dx = \int_0^{\pi/2} \sin^2(x) e^{-\sin^2(x)} \sin(x)\cos(x)\,dx$ $\left[\begin{array}{l}\sin^2(x) = u \\ 2\sin(x)\cos(x)\,dx = du\end{array}\right]$

$$= \int_0^1 u e^{-u} \tfrac{1}{2}\,du = \tfrac{1}{2}\left(\left[u(-e^{-u})\right]_0^1 - \int_0^1 1\cdot(-e^{-u})\,du\right)$$

$$= \tfrac{1}{2}\left[u(-e^{-u}) - e^{-u}\right]_0^1 = \tfrac{1}{2}\left[(-e^{-1}-e^{-1}) - (0-1)\right] = \underline{\tfrac{1}{2}\left(1-\tfrac{2}{e}\right)}$$

3 $\sum_{n=0}^{\infty} a_n$ where $a_n = \frac{(-1)^n}{(2n+1)\,3^{n-\frac{1}{2}}}$

According to the ratio test, the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+1)\,3^{n-\frac{1}{2}}}{(2(n+1)+1)\,3^{n+1-\frac{1}{2}}} = \lim_{n\to\infty} \frac{2n(1+\frac{1}{2n})}{2n(1+\frac{3}{2n})}\frac{1}{3}$$

$$= 1\cdot\frac{1+0}{1+0}\cdot\frac{1}{3} = \frac{1}{3} \text{ which is less than } 1$$

tell us that the series $\sum_{n=0}^{\infty} a_n$ is (absolutely) convergent. q.e.d.

The sum of the series is given by an identification with the Maclaurin series of the arctangent function:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^{n-\frac{1}{2}}} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{\sqrt{3}}\right)^{2n-1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}\left(\frac{1}{\sqrt{3}}\right)^{-2}$$

$$= 3\sum_{n=0}^{\infty}(-1)^n\frac{(1/\sqrt{3})^{2n+1}}{2n+1} = 3\arctan\left(\frac{1}{\sqrt{3}}\right) = 3\cdot\frac{\pi}{6} = \frac{\pi}{2}$$

i.e. the sum of the series $\sum_{n=0}^{\infty} a_n$ is $\frac{\pi}{2}$.

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(4) The equation $y=f(x)$ expresses the solution of the equation $x(y+e\ln(y))=2e$ in a neighbourhood of the point $P:(1,e)$. In order to find at least Taylor polynomials of f, we differentiate the equation implicitly with respect to x and up to the order of the Taylor polynomial asked for.

Diff. with respect to x gives $1\cdot(y+e\ln(y))+x(y'+e\frac{1}{y}y')=0$

At the point P (with $\frac{dy}{dx}|_P=k$), we get $e+e\cdot 1+1(k+\frac{e}{e}k)=0$

$\Leftrightarrow \quad 2e+2k=0 \quad \Leftrightarrow \quad k=-e$

One more diff. gives

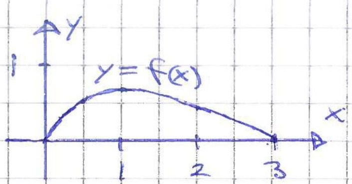
$$(y'+e\frac{1}{y}y')+1\cdot(y'+e\frac{1}{y}y')+x\left(y''+e(-\frac{1}{y^2}y')y'+e\frac{1}{y}y''\right)=0$$

At P (with $\frac{d^2y}{dx^2}|_P=\alpha$), we get $2(k+\frac{e}{e}k)+1(\alpha-\frac{e}{e^2}k^2+\frac{e}{e}\alpha)=0$

$\Leftrightarrow \quad 4k+2\alpha-\frac{1}{e}k^2=0$ Thus $\alpha=\frac{1}{2}\left(\frac{k^2}{e}-4k\right)=\frac{1}{2}(e+4e)=\frac{5e}{2}$

The Taylor polynomial of order 2 for f about the point 1 is then $f(1)+f'(1)(x-1)+\frac{1}{2}f''(1)(x-1)^2=e-e(x-1)+\frac{5e}{4}(x-1)^2$

(5)



$f(x)=\sqrt{x}-\frac{1}{3}x\sqrt{x}=(1-\frac{x}{3})\sqrt{x}, \quad 0\le x\le 3$

The area A_y of the surface generated by rotating the curve $y=f(x)$ about the y-axis is

$$A_y=\int_0^3 2\pi|x|\sqrt{1+(f'(x))^2}\,dx=\int_0^3 2\pi x\sqrt{1+\left(\frac{1}{2\sqrt{x}}-\frac{1}{3}\frac{3}{2}\sqrt{x}\right)^2}\,dx$$

$$=2\pi\int_0^3 x\sqrt{1+\left(\frac{1}{4x}-\frac{1}{2}+\frac{x}{4}\right)}\,dx=2\pi\int_0^3 x\sqrt{\left(\frac{1}{2\sqrt{x}}+\frac{\sqrt{x}}{2}\right)^2}\,dx$$

$$=2\pi\int_0^3 x\left(\frac{1}{2\sqrt{x}}+\frac{1}{2}\sqrt{x}\right)dx=\pi\int_0^3(\sqrt{x}+x\sqrt{x})\,dx$$

$$=\pi\left[\frac{2}{3}x^{3/2}+\frac{2}{5}x^{5/2}\right]_0^3=2\pi\left(\frac{1}{3}3\sqrt{3}+\frac{1}{5}9\sqrt{3}\right)=2\pi\left(1+\frac{9}{5}\right)\sqrt{3}=\frac{28\sqrt{3}\pi}{5}$$

Answer: $\frac{28\sqrt{3}\pi}{5}$ a.u.

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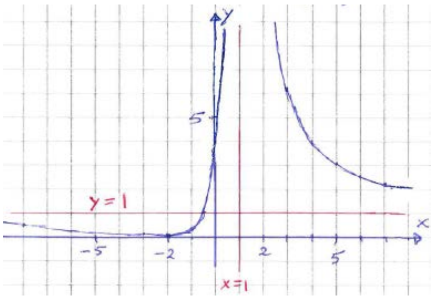
EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN2 – 2017-06-08 — Maximum points for subparts of the problems in the final examination

1. The graph has a local minimum at $P:(-2,0)$, and has the asymptotes $x=1,\ y=1$



1p: Correctly found and classified the local (and also global) minimum point of the graph
1p: Correctly found the vertical asymptote of the graph, and correctly sketched the graph according to how it relates to the vertical asymptote
1p: Correctly found the non-vertical asymptote of the graph, and correctly sketched the graph according to how it relates to the non-vertical asymptote
1p: Correctly completed the sketch of the graph

2. $y=\frac{1}{2}\left(1-2e^{-1}\right)$

1p: Correctly by the substitution $\sin^2(x)=u$ translated the integral into $\int_0^1 \frac{1}{2}ue^{-u}\,du$, or alternatively by the substitution $-\sin^2(x)=v$ translated the integral into $\int_0^{-1}\frac{1}{2}ve^{v}\,dv$
1p: Correctly worked out the first of the two progressive step of a partial integration
1p: Correctly worked out the second progressive step of a partial integration
1p: Correctly evaluated the antiderivative at the limits of the integral

3. The series is convergent, and the sum of the series is $\pi/2$

2p: Correctly found, <u>either by</u> the ratio test combined with the fact that absolute convergence imply convergence or <u>by</u> the Leibniz's criteria, that the series is convergent
1p: Correctly identified the series as, except for a factor 3, the Maclaurin series of the arctangent function at the point $1/\sqrt{3}$
1p: Correctly found the sum of the series

4. $e-e(x-1)+\frac{5}{4}e(x-1)^2$

Note: A student who has calculated one or both of the derivatives wrongly, but who correctly from the values obtained write a second order Taylor polynomial about the point 1, obtains the last point.

1p: Correctly differentiated with respect to x in the LHS and the RHS of the equation
1p: Correctly found the value of $f'(1)$
1p: Correctly differentiated once more with respect to x in the LHS and the RHS of the equation, <u>and</u> correctly found the value of $f''(1)$
1p: Correctly formulated the explicit Taylor polynomial of order 2 for f about the point 1

5. $\dfrac{28\sqrt{3}\pi}{5}$ a.u.

1p: Correctly formulated an explicit integral expression for the area of the surface obtained by rotating the curve about the y-axis
1p: Correctly rewrited the square root factor of the integrand into a sum of two powers of x
1p: Correctly found an antiderivative of the integrand
1p: Correctly evaluated the limits of the integration