MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA150 Vector Algebra, TEN2
Date: 2018-03-22 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The PASS-marks 3, 4 and 5 require a minimum of 12, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 12, 13, 16, 20 and 24 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the marks for a completed course are determined according to

$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 47$	$\rightarrow$	3	$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 38$	$\rightarrow$	E
$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 62$	$\rightarrow$	4	$S_1, S_2 \geq 12$	AND	$39 \leq S_1 + 2S_2 \leq 47$	$\rightarrow$	D
		$63 \leq S_1 + 2S_2$	$\rightarrow$	5	$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 59$	$\rightarrow$	C
					$S_1, S_2 \geq 12$	AND	$60 \leq S_1 + 2S_2 \leq 71$	$\rightarrow$	B
							$72 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. The linear transformations $F : \mathbb{E}^2 \to \mathbb{E}^2$ and $G : \mathbb{E}^2 \to \mathbb{E}^2$ are such that F rotates vectors one eighth turn counterclockwise, while G by the factors $\sqrt{2}$ and $1/\sqrt{2}$ rescales the first and the second of the coordinates respectively of vectors. Find the standard matrices of the compositions $F \circ G$ and $G \circ F$.
($\mathbb{E}^2$ is the vector space $\mathbb{R}^2$ equipped with the standard inner product.)

2. Find, for each real value of β, the dimension of and a basis for the subspace

$$M_\beta = \operatorname{span}\{(-7, 5, 8), (6, -9, -4), (1, \beta, 16)\}$$

of $\mathbb{R}^3$. Also, find all values of β for which the vector $v = (15, -6, -20)$ belongs to the subspace, and state for these β the coordinates of v relative to the chosen bases respectively.

3. The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

relative to the standard basis. Find a basis for each eigenspace of F, and conclude whether F is diagonalizable or not.

4. Find the orthogonal projection of the vector $(5, -2, 3)$ on the vector space M defined according to

$$M = \{(x_1, x_2, x_3) \in \mathbb{E}^3 : x_1 + x_2 + x_3 = 0\},$$

where $\mathbb{E}^3$ is the vector space $\mathbb{R}^3$ equipped with the standard inner product.

5. Solve the equation

$$z^4 + 8 = 8\sqrt{3}\,i$$

and give in the complex plane an illustration of the solution set.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 12, 16 respektive 21 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av ett sammanfattningsbetyg på en slutförd kurs enligt

$S_1, S_2 \geq 12$	OCH	$S_1 + 2S_2 \leq 47$	$\rightarrow$	3
$S_1, S_2 \geq 12$	OCH	$48 \leq S_1 + 2S_2 \leq 62$	$\rightarrow$	4
		$63 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. De linjära avbildningarna $F : \mathbb{E}^2 \to \mathbb{E}^2$ och $G : \mathbb{E}^2 \to \mathbb{E}^2$ är sådana att F roterar vektorer en åttondels varv moturs, medan G med faktorerna $\sqrt{2}$ och $1/\sqrt{2}$ skalar om den första respektive den andra av koordinaterna för vektorer. Bestäm standardmatriserna för sammansättningarna $F \circ G$ och $G \circ F$.
($\mathbb{E}^2$ är vektorrummet $\mathbb{R}^2$ utrustat med standardskalärprodukten.)

2. Bestäm, för varje reellt värde på β, dimensionen av och en bas för delrummet

$$M_\beta = \text{span}\{(-7, 5, 8), (6, -9, -4), (1, \beta, 16)\}$$

till $\mathbb{R}^3$. Bestäm även alla värden på β för vilka vektorn $v = (15, -6, -20)$ tillhör delrummet, och ange för dessa β koordinaterna för v relativt respektive av de valda baserna.

3. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ har matrisen

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

relativt standardbasen. Bestäm en bas för varje egenrum till F, och avgör huruvida F är diagonaliserbar eller inte.

4. Bestäm den ortogonala projektionen av vektorn $(5, -2, 3)$ på vektorrummet M definierat enligt

$$M = \{(x_1, x_2, x_3) \in \mathbb{E}^3 : x_1 + x_2 + x_3 = 0\},$$

där $\mathbb{E}^3$ är vektorrummet $\mathbb{R}^3$ utrustat med standardskalärprodukten.

5. Lös ekvationen

$$z^4 + 8 = 8\sqrt{3}\, i$$

och ge i det komplexa talplanet en illustration av lösningsmängden.

If you prefer the problems formulated in English, please turn the page.

MAA150 / Solutions to the final exam. TEN2 2018-03-22

1 $F : R^2 \to R^2$ rotates vectors $\frac{1}{8}$ turn counterclockwise, i.e. rotates vectors an angle of $+\frac{\pi}{4}$.

This means especially that $\begin{cases} F((1,0)) = (\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4})) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \\ F((0,1)) = (\cos(\frac{3\pi}{4}), \sin(\frac{3\pi}{4})) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \end{cases}$

Thus, F has the standard matrix $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$G : R^2 \to R^2$ is such that $G((1,0)) = (\sqrt{2}, 0)$ and $G((0,1)) = (0, \frac{1}{\sqrt{2}})$

Thus, G has the standard matrix $B = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$

$\begin{cases} \text{The composition } F \circ G \text{ has the standard matrix } AB = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix} \\ \text{— ll —} \quad G \circ F \quad \text{— ll —} \quad BA = \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix} \end{cases}$

2 $M_\beta = \text{span}\{\overset{u_1}{(-7,5,8)}, \overset{u_2}{(6,-9,-4)}, \overset{u_3}{(1,\beta,16)}\}$, $\quad v = (15,-6,-20)$

The questions asked can be answered by a study of the condition $v \in M_\beta$ which is equivalent with the equation

$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = v \iff \underbrace{\begin{pmatrix} -7 & 6 & 1 \\ 5 & -9 & \beta \\ 8 & -4 & 16 \end{pmatrix}}_{A}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 15 \\ -6 \\ -20 \end{pmatrix}}_{B}$

where

$(A|B) \sim \left(\begin{array}{ccc|c} -7 & 6 & 1 & 15 \\ -2 & -3 & 14+\beta & 9 \\ 1 & 2 & 17 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 20 & 120 & -20 \\ 0 & 1 & 35+\beta & -1 \\ 1 & 2 & 17 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 17 & -5 \\ 0 & 1 & 6 & -1 \\ 0 & 1 & 35+\beta & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 5 & -3 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 29+\beta & 0 \end{array}\right)$

from which we conclude that: $\boxed{\dim(M_\beta) = \begin{cases} 2 & \text{if } \beta = -29 \\ 3 & \text{if } \beta \neq -29 \end{cases}}$

A basis for M_β is u_1, u_2 if $\beta = -29$ and u_1, u_2, u_3 if $\beta \neq -29$.

$v \in M_\beta$ for every $\beta \in R$ and $\begin{cases} \text{coord}_{u_1,u_2}(v) = (-3,-1) & \text{if } \beta = -29 \\ \text{coord}_{u_1,u_2,u_3}(v) = (-3,-1,0) & \text{if } \beta \neq -29 \end{cases}$

3 $F : R^3 \to R^3$, $\quad A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

Eigenvalues: $0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ 0 & 1 & 3-\lambda \end{pmatrix} = (1-\lambda)[(2-\lambda)(3-\lambda)-2]$

$= -(\lambda-1)(\lambda^2-5\lambda+6-2) = -(\lambda-1)(\lambda-1)(\lambda-4)$

$\begin{cases} \lambda_{1,2} = 1: & A - \lambda_{1,2} I = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad k_{1,2} = t_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \\ \lambda_3 = 4: & A - \lambda_3 I = \begin{pmatrix} -3 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -3 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}; \quad k_3 = t_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ t_3 \neq 0 \end{cases}$

where at least one of t_1 and t_2 is diff. from zero

Thus $\begin{cases} \text{e.g. } (1,0,0), (0,2,-1) \text{ is a basis for the eigenspace } E_{\lambda=1} \\ \text{e.g. } (1,1,1) \quad \text{— ll —} \quad \text{— ll —} \quad E_{\lambda=4} \end{cases}$

F is diagonalizable since the geometric multiplicities of the eigenvalues equal their algebraic multiplicities respectively.

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4 $M = \{(x_1, x_2, x_3) \in E^3 : x_1 + x_2 + x_3 = 0\}$

We notice that $(x_1, x_2, x_3)_M = (x_1, x_2, -x_1 - x_2) = x_1(1,0,-1) + x_2(0,1,-1)$
i.e. $\overset{u_1}{(1,0,-1)}, \overset{u_2}{(0,1,-1)}$ is a basis for M. However, in order to get a straightforward formula for orthogonal projections on M, we need an ON-basis for M. The Gram-Schmidt procedure on u_1, u_2 gives an ON-basis e_1, e_2 as follows:

$$\begin{cases} f_1 = u_1 \quad \text{and} \quad e_1 = \frac{1}{\|f_1\|} f_1 = \frac{1}{\sqrt{2}}(1,0,-1) \\ f_2 = u_2 - (u_2 \cdot e_1)e_1 = (0,1,-1) - \frac{1}{2}(0+0+1)(1,0,-1) = \frac{1}{2}(-1,2,-1) \\ \quad \text{and} \quad e_2 = \frac{1}{\sqrt{1+4+1}}(-1,2,-1) = \frac{1}{\sqrt{6}}(-1,2,-1) \end{cases}$$

Then the orthogonal projection of $(5,-2,3)$ on M is

$$\begin{aligned} \text{proj}_M((5,-2,3)) &= [(5,-2,3) \cdot e_1]e_1 + [(5,-2,3) \cdot e_2]e_2 \\ &= \tfrac{1}{2}(5+0-3)(1,0,-1) + \tfrac{1}{6}(-5-4-3)(-1,2,-1) \\ &= (1,0,-1) - 2(-1,2,-1) = \underline{(3,-4,1)} \end{aligned}$$

5 $z^4 + 8 = 8\sqrt{3}i \iff z^4 = 8(-1+\sqrt{3}i) = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$$= 2^4\left[\cos\left(\tfrac{2\pi}{3}\right) + \sin\left(\tfrac{2\pi}{3}\right)i\right] = 2^4 e^{\frac{2\pi}{3}i}$$

Let $z = re^{i\theta}$ where r is nonnegative and $\theta \in R$

Then $z^4 + 8 = 8\sqrt{3}i \iff r^4 e^{4i\theta} = 2^4 e^{\frac{2\pi}{3}i}$

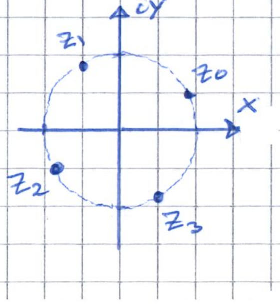
Working out the absolute values of the LHS and the RHS gives $r^4 = 2^4$ and then $e^{4i\theta} = e^{\frac{2\pi i}{3}}$ remains, i.e. the eq. is equivalent with $r = 2$ and $4\theta = \frac{2\pi}{3} + n \cdot 2\pi$ (n integer)

Thus the solutions of the eq. $z^4 + 8 = 8\sqrt{3}i$ are

$$\underline{z_n = 2e^{i\left(\frac{\pi}{6} + n\cdot\frac{\pi}{2}\right)}}, \quad n = 0,1,2,3$$
(is enough since e.g. $\ldots = z_{-4} = z_0 = z_4 = \ldots$)

i.e.
$$\begin{cases} z_0 = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sqrt{3} + i \\ z_1 = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i \\ z_2 = 2\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -\sqrt{3} - i \\ z_3 = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i \end{cases}$$

The roots of the eq. lie on a circle of radius 2 and centered at origin. The angle between two consecutive "radii" is $\pi/2$.



Final examination TEN2 – 2018-03-22 | Maximum points for subparts of the problems in the final examination

1. The standard matrix of $F \circ G$ is $\begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$ and that of $G \circ F$ is $\begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

2p: Correctly found the standard matrix of F
1p: Correctly found the standard matrix of G
1p: Correctly found the standard matrix of $F \circ G$
1p: Correctly found the standard matrix of $G \circ F$

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2. $\dim(M_\beta) = \begin{cases} 2 & \text{if } \beta = -29 \\ 3 & \text{if } \beta \neq -29 \end{cases}$

A basis for M_β is e.g. u_1, u_2 if $\beta = -29$ and u_1, u_2, u_3 if $\beta \neq -29$, where the vectors are those listed in the span-definition of M_β.The coordinates of v relative to the chosen bases are $\begin{cases} -3, -1 & \text{if } \beta = -29 \\ -3, -1, 0 & \text{if } \beta \neq -29 \end{cases}$

1p: Correctly found the dimension of M_β for each value of β
1p: Correctly found a basis for M_β for each value of β
1p: Correctly found that $v \in M_\beta$ for every value of β
1p: Correctly found the coordinates of v relative the chosen basis as $\beta = -29$
1p: Correctly found the coordinates of v relative the chosen basis as $\beta \neq -29$

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3. Bases for the eigenspaces of F are e.g. $(1,0,0)$, $(0,2,-1)$ corresponding to the eigenvalue 1, and e.g. $(1,1,1)$ corresponding to the eigenvalue 4.

F is diagonalizable since the geometric multiplicities of the eigenvalues equals their algebraic multiplicies respectively

1p: Correctly found the eigenvalues of F
2p: Correctly found a basis for the eigenspace corresponding to the eigenvalue 1
1p: Correctly found a basis for the eigenspace corresponding to the eigenvalue 4
1p: Correctly concluded that F is diagonalizable

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4. $\text{proj}_M\big((5,-2,3)\big) = (3,-4,1)$

1p: Correctly found a basis for M
2p: Correctly orthogonalized the chosen basis for M as a proper preparation for finding of the orthogonal projection of the vector $(5,-2,3)$ on M
2p: Correctly found the orthogonal projection asked for

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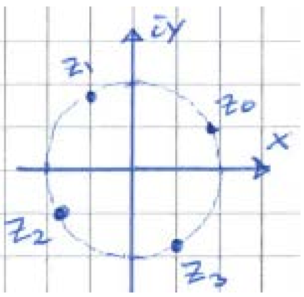
5. The roots of the equation are

$z_n = 2e^{i\left(\frac{\pi}{6} + n\frac{\pi}{2}\right)}$, $n = 0,1,2,3$



1p: Correctly reformulated the equation as $z^4 = 16e^{2\pi i/3}$
1p: Correctly substituted z by $re^{i\theta}$ for solving the equation by the use of polar coordinates, and correctly found (by taking the absolute values of the LH and the RH sides of the equation) that $r = 2$
1p: Correctly from the remaining equation $e^{4i\theta} = e^{2\pi i/3}$ concluded that $4\theta = \frac{2\pi}{3} + n2\pi$, where n is an integer
1p: Correctly summarized the four roots (irrespective of form)
1p: Correctly illustrated the solution set
