MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2017-03-24 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the real numbers x for which the power series

$$\sum_{n=1}^{\infty} \frac{(4x-12)^n}{n}.$$

is convergent. Are there any of these x for which the series is not absolutely convergent, i.e. is (only) conditionally convergent?

2. Solve the initial-value problem $\begin{cases} y' = (xy)^3, \\ y(1) = -\frac{1}{2}\,. \end{cases}$

3. A closed (with bottom and lock) cylindrical tin is to hold π liters. Find the radius and height of the can so that the consumption of material (the number of square decimeter thin sheet) becomes a minimum.

4. Find the length of the curve $\begin{cases} x = \frac{1}{2}\cos^2(\theta)\,, \\ y = \frac{1}{2}\sin^2(\theta)\,, \end{cases} \quad 0 \leq \theta \leq \pi\,.$

5. Evaluate the generalized integral

$$\int_{9/2}^{\infty} \frac{dx}{4x(x-6)+45}\,,$$

and write the result in as simple form as possible.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2017-03-24 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm de reella tal x för vilka potensserien

$$\sum_{n=1}^{\infty} \frac{(4x-12)^n}{n}.$$

är konvergent. Är det några av dessa x för vilka serien inte är absolutkonvergent, dvs. är (endast) betingat konvergent?

2. Lös begynnelsevärdesproblemet $\begin{cases} y' = (xy)^3, \\ y(1) = -\frac{1}{2}\,. \end{cases}$

3. En sluten (med botten och lock) cylinderformad konservburk ska rymma π liter. Bestäm burkens radie och höjd så att åtgången av material (antalet kvadrat-decimeter tunn plåt) blir minimal.

4. Bestäm längden av kurvan $\begin{cases} x = \frac{1}{2}\cos^2(\theta)\,, \\ y = \frac{1}{2}\sin^2(\theta)\,, \end{cases} \quad 0 \leq \theta \leq \pi\,.$

5. Beräkna den generaliserade integralen

$$\int_{9/2}^{\infty} \frac{dx}{4x(x-6)+45}\,,$$

och skriv resultatet på en så enkel form som möjligt.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2017-03-24

1 $\sum_{n=1}^{\infty} a_n(x)$ where $a_n(x) = \frac{(4x-12)^n}{n} = \frac{4^n(x-3)^n}{n}$

Let $A(x) = \begin{cases} \lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty} \left|\frac{4^{n+1}(x-3)^{n+1}\, n}{(n+1)\, 4^n (x-3)^n}\right| = 4|x-3| & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$

The series converges absolutely if $A(x) < 1 \iff \frac{11}{4} < x < \frac{13}{4}$

— " — diverges if $A(x) > 1 \iff x < \frac{11}{4} \vee x > \frac{13}{4}$

For $x = \frac{11}{4}$, we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent accord. to the alternating series test, but not absolutely convergent accord. to the integral test.

For $x = \frac{13}{4}$, we have the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent accord. to the integral test.

Thus $\sum_{n=1}^{\infty} a_n(x)$ converges in the interval $[\frac{11}{4}, \frac{13}{4})$. The convergence is absolutely except at the point $11/4$ where it is only conditionally convergent.

2 DE: $y' = (xy)^3$, IV: $y(1) = -\frac{1}{2}$

The DE is nonlinear but separable. For $y < 0$, the DE may be formulated as $\frac{1}{y^3} y' = x^3$. Working out the operation $\int dx$ on both sides gives $-\frac{1}{2y^2} = \frac{1}{4}x^4 + C$.

The IV gives $-\frac{1}{2(-\frac{1}{2})^2} = \frac{1}{4}\cdot 1^4 + C \iff C = -2 - \frac{1}{4} = -\frac{9}{4}$

Then $\frac{1}{y^2} = -\frac{1}{2}x^4 + \frac{9}{2} = \frac{9 - x^4}{2}$

Solving for y (involving the IV again) gives

$$y = -\sqrt{\frac{2}{9 - x^4}}$$

Extra: (The possible x for the solution of the IVP are given by the necessary condition $9 - x^4 > 0$ $\iff -\sqrt{3} < x < \sqrt{3}$. Thus the so called interval of existence equals $(-\sqrt{3}, \sqrt{3})$.)

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3 Minimize the material area of a closed cylindrical tin which holds π liters (π dm^3). Let r be the radius of the cylinder and h the height. Then the volume π of the cylinder equals $\pi r^2 \cdot h$, where r and h are understood to be given in 'dm', and the expression to be minimized is the area $\pi r^2 + 2\pi h r + \pi r^2$. Using the condition $\pi = \pi r^2 h$ in the expression for the area gives the problem of minimizing the function f defined as $f(r) = 2\pi r^2 + 2\pi \frac{1}{r^2} r$ for $r > 0$.

We get $f'(r) = 4\pi r - \frac{2\pi}{r^2} = \frac{4\pi}{r^2}\left(r^3 - \frac{1}{2}\right)$ which in a first-derivative test

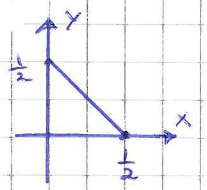
r		$\frac{1}{\sqrt[3]{2}}$	
$f'(r)$	$-$	0	$+$
$f(r)$	$\searrow$	loc. min.	$\nearrow$

gives that $\frac{1}{\sqrt[3]{2}}$ not only is a local minimum point, but also a global one. Thus a radius of $\frac{1}{\sqrt[3]{2}}$ dm and a height of $\sqrt[3]{4}$ dm minimizes the consumption of material.

$\left(f_{min} = 2\pi\left(\frac{1}{\sqrt[3]{4}} + \sqrt[3]{2}\right) = \frac{6\pi}{\sqrt[3]{4}}\right)$

4 Curve: $\begin{cases} x = \frac{1}{2}\cos^2(\theta) \\ y = \frac{1}{2}\sin^2(\theta) \end{cases}, \quad 0 \le \theta \le \pi$

Alt. 1 The curve is the straight line $x + y = \frac{1}{2}$ back and forth between the points $P: (\frac{1}{2}, 0)$ and $Q: (0, \frac{1}{2})$, and therefore has the length $2 \cdot \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$ l.u. $= \sqrt{2}$ l.u. However, even $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$ l.u. $= \frac{1}{\sqrt{2}}$ l.u. for the length of the curve $x + y = \frac{1}{2}$ between the points P and Q is accepted as an answer to the question.



Alt. 2 The standard evaluation is: $\text{Length(curve)} = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{\pi} \sqrt{[\cos(\theta)(-\sin(\theta))]^2 + [\sin(\theta)\cos(\theta)]^2}\, d\theta = \sqrt{2}\int_0^{\pi} |\cos(\theta)\sin(\theta)|\, d\theta$$

$$= \sqrt{2}\left[\int_0^{\pi/2} \cos(\theta)\sin(\theta)\, d\theta + \int_{\pi/2}^{\pi} (-\cos(\theta)\sin(\theta))\, d\theta\right] = \sqrt{2}\left(\left[\tfrac{1}{2}\sin^2(\theta)\right]_0^{\pi/2} - \left[\tfrac{1}{2}\sin^2(\theta)\right]_{\pi/2}^{\pi}\right)$$

$$= \sqrt{2}\,\tfrac{1}{2}[(1-0) - (0-1)] = \sqrt{2}$$

5
$$\int_{9/2}^{\infty} \frac{dx}{4x(x-6)+45} = \int_{9/2}^{\infty} \frac{dx}{(2x-6)^2+9} = \frac{1}{9}\int_{9/2}^{\infty} \frac{dx}{\left(\frac{2x-6}{3}\right)^2+1} \quad \left[\begin{array}{l} \frac{2(x-3)}{3} = u \\ \frac{2}{3}dx = du \end{array}\right]$$

$$= \frac{1}{9}\int_1^{\infty} \frac{\frac{3}{2}du}{u^2+1} = \frac{1}{6}\left[\arctan(u)\right]_1^{\infty}$$

$$= \frac{1}{6}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{6}\,\frac{\pi}{4} = \frac{\pi}{24}$$

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN2 – 2017-03-24 Maximum points for subparts of the problems in the final examination

1. The series is convergent for $\frac{11}{4} \le x < \frac{13}{4}$. Of those x, it is only conditionally convergent for $x = \frac{11}{4}$.

1p: Correctly, by e.g. the ratio test, found that the series is absolutely convergent for $|4(x-3)| < 1$, i.e. for $\frac{11}{4} < x < \frac{13}{4}$, and hopefully correctly mentioned that the series definitely is divergent for $|x-3| > \frac{1}{4}$
1p: Correctly found that the series is convergent for $x = \frac{11}{4}$
1p: Correctly found that the series is divergent for $x = \frac{13}{4}$
1p: Correctly stressed that the series is only conditionally convergent for $x = \frac{11}{4}$

2. $y = -\sqrt{\dfrac{2}{9-x^4}}$

1p: Correctly identified the differential equation as nonlinear and separable, and correctly found general antiderivatives of both sides of the separated differential equation
1p: Correctly adapted the integrated equation to the IV
1p: Correctly solved for y^2
1p: Correctly solved for y (the IV had to be used once more)

3. radius $= \dfrac{1}{\sqrt[3]{2}}$ dm

height $= \dfrac{2}{\sqrt[3]{2}}$ dm $= \sqrt[3]{4}$ dm

1p: Correctly for the optimization problem formulated a function of one variable
2p: Correctly found and concluded about the local extreme points of the function
1p: Correctly stated the radius and height which minimizes the consumption of material

4. $\sqrt{2}$ l.u. if the curve has been interpreted as *the path back and forth between the points* $P:(\frac{1}{2},0)$ and $Q:(0,\frac{1}{2})$

$1/\sqrt{2}$ l.u. if the curve has been interpreted as *the path between the points* $P:(\frac{1}{2},0)$ and $Q:(0,\frac{1}{2})$

<u>Notice</u>: Both interpretations/answers are accepted provided they have been justified.

------------------------------------ **One scenario** ------------------------------------

1p: Correctly formulated an integral for the length of the curve (an integral with explicit expressions for the derivatives $dx/d\theta$ and $dy/d\theta$)
1p: Correctly rewrited the integrand into $\sqrt{2}|\cos(\theta)\sin(\theta)|$
1p: Correctly treated the absolute values when integrating
1p: Correctly found an antiderivative of the integrand, and correctly found the length of the curve

--------------------------------- **Another scenario** ---------------------------------

2p: Correctly noticed that the curve is the straight line $x + y = \frac{1}{2}$ back and forth between two points, P and Q
1p: Correctly found the length of the straight line between the points P and Q
1p: Correctly found the length of the path back and forth between the points P and Q

5. $\dfrac{\pi}{24}$

1p: Correctly rewrited the denominator of the integrand into $(2x-6)^2 + 9$, and correctly by the substitution $2(x-3) = 3u$ translated the integrand into $1/(6(1+u^2))$
1p: Correctly translated the limits of the integral in connection with the substitution $2(x-3) = 3u$
1p: Correctly found an antiderivative of the integrand
1p: Correctly evaluated the antiderivative at the limits, and by that also correctly found the value of the integral