Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2017-03-20
Hjälpmedel: penna,
sudd och linjal

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Faktorisera polynomet $p(z) = 2z^4 - 2z^2 - 4z + 4$ i linjära faktorer, givet att $z = 1$ är en dubbelrot till $p(z)$. (5p)

2 Låt $W = \text{span}(S)$, där S består av vektorerna

$$\mathbf{v}_1 = (3, 1, 2, -1), \mathbf{v}_2 = (1, -1, 0, 2) \text{ och } \mathbf{v}_3 = (-1, -3, -2, 5).$$

a. Bestäm en bas för W som består av vektorer tillhörande S. (3p)

b. Bestäm den ortogonala projektionen av $\mathbf{u} = (1, 1, 1, 1)$ på W. (3p)

3 Bestäm standardmatrisen för den sammansatta transformationen $T : \mathbb{R}^2 \to \mathbb{R}^2$ som ges av först en rotation 45° moturs, och sedan den ortogonala projektionen på y-axeln. (5p)

4 För matrisen (5p)

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

a. Bestäm samtliga egenvärden till A. (2p)

b. Bestäm baser för egenrummen till A. (3p)

5 Bevisa att om $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ är en bas för vektorrummet V, då kan varje vektor $\mathbf{v}$ i V skrivas som $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$ på exakt ett sätt. (4p)

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Examination Vector algebra
MAA150 - TEN2
Date: March 20, 2017
Exam aids: pencil,
eraser and ruler

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Factor the polynomial $p(z) = 2z^4 - 2z^2 - 4z + 4$ into linear factors, given that $z = 1$ is a double root of $p(z)$. (5p)

2 Let $W = \text{span}(S)$, where S consist of the vectors

$$\mathbf{v}_1 = (3, 1, 2, -1), \mathbf{v}_2 = (1, -1, 0, 2), \text{ and } \mathbf{v}_3 = (-1, -3, -2, 5).$$

a. Find a basis for W consisting of vectors from S. (3p)

b. Find the orthogonal projection of $\mathbf{u} = (1, 1, 1, 1)$ onto W. (3p)

3 Find the standard matrix for the composite transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by first a rotation of 45° counterclockwise, followed by the orthogonal projection on the y-axis. (5p)

4 For the matrix (5p)

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

a. Find the eigenvalues of A. (2p)

b. Find bases for the eigenspaces of A. (3p)

5 Prove that if $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n$ in exactly one way. (4p)

MAA150 Vektoralgebra, HT2017

Assessment criteria for TEN2 2017-02-20

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Points may be deducted for erroneous mathematical statements, calculations, or failure to use proper mathematical notation.

Assessment problems

1. [**5 points**]
Finding the factor $(z^2 + 2z + 2)$ by long-division **(2p)**, finding the roots $-1 \pm i$ **(2p)**, correct factorization **(1p)**

2. [**6 points**]
a. Correct method for finding the basis **(2p)**, a correct basis **(1p)**
b. Checking that the basis is orthogonal **(1p)**, computing the projection **(2p)**

3. [**5 points**]
Characteristic equation **(1p)**, correct eigenvalues **(1p)**, finding the eigenvectors **(2p)**, correct bases for each eigenspace **(1p)**

4. [**5 points**]
Correct standard matrix for the rotation **(1p)**, the correct standard matrix for the projection with motivation **(2p)**, standard matrix for the composition **(2p)**

5. [**4 points**]
Valid method and presentation **(2p)**, correct arguments **(2p)**

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(1) $p(z) = 2z^4 - 2z^2 - 4z + 4$.

Since $z=1$ is a double root of $p(z)$, $(z-1)^2 = z^2 - 2z + 1$ is a factor of $p(z)$.

$$
\begin{array}{rl}
2z^2 + 4z + 4 & \\
\hline
2z^4 - 2z^2 - 4z + 4 & \big| \; z^2 - 2z + 1 \\
-(2z^4 - 4z^3 + 2z^2) & \\
\hline
4z^3 - 4z^2 - 4z + 4 & \\
-(4z^3 - 8z^2 + 4z) & \\
\hline
4z^2 - 8z + 4 & \\
-(4z^2 - 8z + 4) & \\
\hline
0 &
\end{array}
$$

(2p)

Therefore $p(z) = (z-1)^2 \cdot (2z^2 + 4z + 4) = 2 \cdot (z-1)^2 \cdot (z^2 + 2z + 2)$

To find the remaining roots we find the roots of $z^2 + 2z + 2$.

$$z^2 + 2z + 2 = 0 \iff (z+1)^2 - 1 + 2 = 0 \iff (z+1)^2 = -1$$

So $z + 1 = \pm i$ or $z = -1 \pm i$ (2p)

Answer: $p(z) = 2 \cdot (z-1)^2 \cdot (z - (-1+i)) \cdot (z - (-1-i))$ (1p)

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(2) $S=\{\bar{v}_1,\bar{v}_2,\bar{v}_3\}$ where $W=\text{span}(S)$ and
$\bar{v}_1=(3,1,2,-1)$, $\bar{v}_2=(1,-1,0,2)$, and $\bar{v}_3=(-1,-3,-2,5)$

(a)

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & -3 \\ 2 & 0 & -2 \\ -1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 3 & 1 & -1 \\ 2 & 0 & -2 \\ -1 & 2 & 5 \end{bmatrix} \begin{matrix} -3 & -2 & 1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 4 & 8 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix} \begin{matrix} -2 & -4 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2p)$$

Leading 1 in column 1 and 2, so $B=\{\bar{v}_1,\bar{v}_2\}$ is a basis for V. (1p)

Answer a: $B=\{\bar{v}_1,\bar{v}_2\}$ is a basis for W.

$$v_1 \bullet v_2 = (3,1,2,-1)\bullet(1,-1,0,2) = 3-1+0-2=0$$

so B is an orthogonal basis. (1p)

Therfore, $\bar{u}=(1,1,1,1)$, we hae that

$$\text{proj}_W u = \frac{\bar{u}\bullet\bar{v}_1}{\|\bar{v}_1\|^2}\bar{v}_1 + \frac{\bar{u}\bullet\bar{v}_2}{\|\bar{v}_2\|^2}\bar{v}_2 =$$

$$= \frac{(1,1,1,1)\bullet(3,1,2,-1)}{(\sqrt{3^2+1^2+2^2+(-1)^2})^2}(3,1,2,-1) + \frac{(1,1,1,1)\bullet(1,-1,0,2)}{(\sqrt{1^2+(-1)^2+0^2+2^2})^2}(1,-1,0,2) =$$

$$= \frac{5}{15}(3,1,2,-1) + \frac{2}{6}(1,-1,0,2) = (1,\tfrac{1}{3},\tfrac{2}{3},-\tfrac{1}{3}) + (\tfrac{1}{3},-\tfrac{1}{3},0,\tfrac{2}{3}) =$$

$$= (\tfrac{4}{3},0,\tfrac{2}{3},\tfrac{1}{3})$$

Answer b: $(\tfrac{4}{3},0,\tfrac{2}{3},\tfrac{1}{3})$ (2p)

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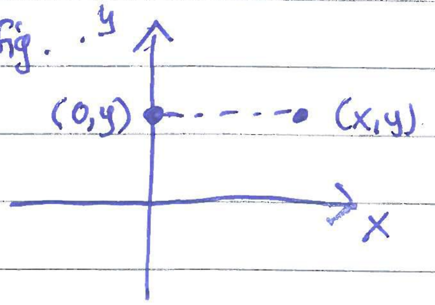
③ Standard matrix for a rotation by 45° is

$$[T_1] = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$ (1p)

Standard matrix for the projection onto the y-axis:

① Fig.



② $T_2(x,y) = (0,y)$

③ $[T_2] = [T_2(\bar{e}_1)\ T_2(\bar{e}_2)]$

$T(\bar{e}_1) = T(1,0) = (0,0)$

$T(\bar{e}_2) = T(0,1) = (0,1)$

so $[T] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (2p)

Standard matrix for $T = T_2 \circ T_1$ is $[T] = [T_2] \cdot [T_1] =$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$ (2p)

Answer: $[T] = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

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(4) $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow A - \lambda \cdot I = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ -2 & 0 & 1-\lambda \end{bmatrix}$

(a)

Cofactor exp. row 1

$$\det(A - \lambda \cdot I) = \begin{vmatrix} \overset{+}{3-\lambda} & \overset{-}{0} & \overset{+}{0} \\ 1 & 1-\lambda & 2 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda) \cdot \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (3-\lambda) \cdot (1-\lambda)^2$$ (1p)

(CE) $(3-\lambda) \cdot (1-\lambda)^2 = 0 \Leftrightarrow \lambda = 1$ or $\lambda = 3$ (1p)

Answer a: Eigenvalues are $\lambda = 1$ and $\lambda = 3$

(b) Eigenvectors: Solve $(A - \lambda \cdot I) \cdot \bar{v} = \bar{0}$ where $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

$\boxed{\lambda = 1}$ $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \overset{(1)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{(-2)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

v_2 free

$v_2 = t \Rightarrow v_1 = 0$ and $v_3 = 0$ so $\bar{v} = t \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (1p)

$\boxed{\lambda = 3}$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \overset{(2)}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

v_3 free

$v_3 = 2t \Rightarrow v_2 = t$ and $v_1 = -2t$ so $\bar{v} = t \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ (1p)

Answer b: A basis for the eigenspace to $\lambda = 1$ is $B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

— " — $\lambda = 3$ is $B_3 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$

(1p)

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(5) <u>Proof</u>. Every $\bar{v} \in V$ can be expressed as a linear combination of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ since B spans the vectorspace V. Assume that $\bar{v}$ can be expressed as $v = a_1\bar{v}_1 + a_2\bar{v}_2 + \ldots + a_n\bar{v}_n$ and also as $\bar{v} = b_1\bar{v}_1 + b_2\bar{v}_2 + \ldots + b_n\bar{v}_n$.

If we can show that $a_i = b_i$ for $i = 1, 2, \ldots, n$, the proof is complete. But we have that

$$\bar{0} = \bar{v} - \bar{v} = (a_1 - b_1)\bar{v}_1 + (a_2 - b_2)\bar{v}_2 + \ldots + (a_n - b_n)\bar{v}_n$$

Since B is linearly independent the equation

$$\bar{0} = k_1\bar{v}_1 + k_2\bar{v}_2 + \ldots + k_n\bar{v}_n$$

has only the trivial solution $k_1 = k_2 = \ldots = k_n = 0$ so $a_i - b_i = 0$ for $i = 1, 2, \ldots, n$. That is;

$$a_1 = b_1,\ a_2 = b_2, \ldots, a_n = b_n$$ so the

two expressions for $\bar{v}$ are the same $\square$ (4p)

(See Theorem 4.4.1 (p. 198) in the course book.)