MMA130 TEN1 2018-08-14 Solutions

1. Find appropriate predicates and their specifications, then use them to translate the following into predicate logic.

   a) All green things with spirals belong to Pete.

   b) Pete has exactly one blue thing without spirals.

<u>Solution:</u> A scan of the sentences suggest the following predicates:

$G(x)$ = x is green

$T(x)$ = x is a thing

$S(x)$ = x has spirals

$P(x)$ = x belongs to Pete

$B(x)$ = x is blue

$x = y$: equality (needed for stating uniqueness).

a) is then: $\forall x: \big(G(x) \wedge T(x) \wedge S(x) \rightarrow P(x) \big)$

b) is then: $\exists x: \big(P(x) \wedge B(x) \wedge T(x) \wedge \neg S(x) \big) \wedge$

$$\forall x : \forall y : \big(P(x) \wedge B(x) \wedge T(x) \wedge \neg S(x) \wedge P(y) \wedge B(y) \wedge T(y) \wedge \neg S(y) \rightarrow x = y \big)$$

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2. Give a natural deduction proof of $\neg p \to \neg q \vdash (\neg p \to q) \to p$. Provide justifications of all steps.

Solution: We likely want to assume $\neg p \to \neg q$, seeking to prove p, but how to do that? One way to get anywhere is to then assume $\neg p$, which quickly leads to contradiction $\bot$, so we get a proof by contradiction.

1.	$\neg p \to \neg q$	Premise
2.	Assume $\neg p \to q$	Hypothesis
3.	Assume $\neg p$	Hypothesis
4.	q	by $\to e$ on 3,2
5.	$\neg q$	by $\to e$ on 3,1
6.	$\bot$	by $\neg e$ on 4,5
7.	p	by PBC on 3–6
8.	$(\neg p \to q) \to p$	by $\to i$ on 2–7.

QED

3. A basic board of traffic lights has one red, one yellow (amber, orange), and one green light. Let r, y and g be propositional atoms denoting that the red, yellow, and green respectively lights are on in a particular group of lights.

a) In the old Swedish signalling order, the lights cycle through the states of red, red+yellow, green, and green+yellow before returning to red. Express as a propositional logic formula the claim that the lights are in one of these four states.

Solution 1: Basic disjunctive normal form; essentially list the allowed states:

$$r \wedge \neg y \wedge \neg g \ \vee\ r \wedge y \wedge \neg g \ \vee\ \neg r \wedge \neg y \wedge g \ \vee\ \neg r \wedge y \wedge g$$

Solution 2: The state of y does not matter, so the above can be simplified to

$$r \wedge \neg g \ \vee\ \neg r \wedge g.$$

Solution 3: That can in turn be simplified using $\Leftrightarrow$ instead of $\wedge$ and $\vee$:

$$r \Leftrightarrow \neg g$$

(Red is on iff green is off.)

b) Express in linear time temporal logic the claim that all future states of the lights are in one of these four states.

Answer: $G(r \Leftrightarrow \neg g)$

Alternative answer: $XG(r \Leftrightarrow \neg g)$ if you want to exclude now from "the future".

c) Express in linear time logic the claim that the state after red+yellow is green.

Answer: $G(r \wedge y \wedge \neg g \rightarrow X(\neg r \wedge \neg y \wedge g))$

d) Interpret in natural language the claim $G(r \wedge \neg y \wedge \neg g \to Xr)$.

<u>Answer:</u> Whenever the traffic lights are in the red state, the red light will be on during the following time step.

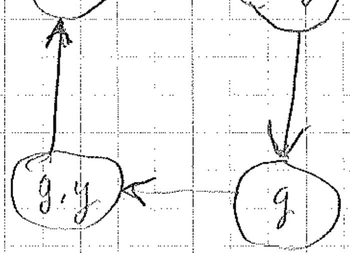
e) Express in linear time logic the claim that the red light, whenever it lights up, will remain lit for at least three time steps.

<u>Solution:</u> "lights up" must mean "goes from off to on", so there is an implication with condition $\neg r \wedge Xr$. That already has r on for one time step, so the conclusion needs to have it on for ~~two~~ additional time steps. This can be written as $XX(r \wedge Xr)$, so the whole thing becomes (for example):

$$G(\neg r \wedge Xr \to XX(r \wedge Xr))$$

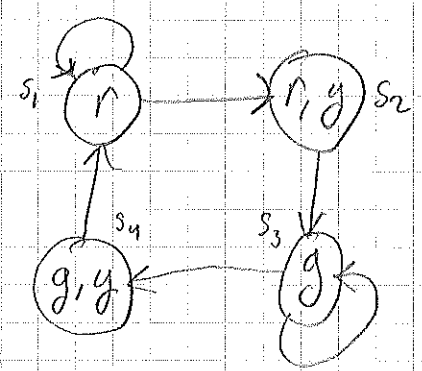
f) What is the minimal number of states in a transition system (worlds in a Kripke logic) that can reproduce the old Swedish signalling order?

<u>Answer:</u> 4, since there are 4 different combinations of lights in that signalling order. Attained by

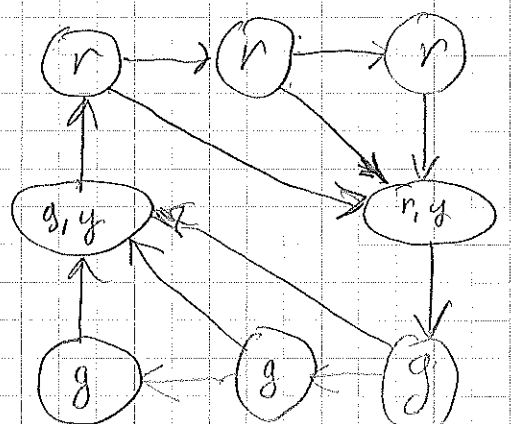


g) Draw a transition system which attains the minimum in (f), which additionally satisfies (i) the states with yellow only last for one time step whereas (ii) the states without yellow can last for more than one time step.

<u>Answer</u>:



<u>Remark</u>: If we have more states, we can attain that minimum without loops (only the big cycle), e.g. as



This would have different answers for some later questions.

h) Is the linear time temporal logic formula GFg true for the model given by that transition system?

<u>Answer</u>: No, because it allows paths that loop forever at the r state! Such paths do not satisfy GFg.

(The non-minimal-states transition system would satisfy that formula though.)

i) Which states of the transition system/model satisfy $\Diamond\Box r$?

<u>Solution:</u> r is satisfied by s_1 and s_2.

Thus $\Box r$ is satisfied by those states where <u>all</u> successors are in $\{s_1, s_2\}$, namely s_1 and s_4.

Finally $\Diamond\Box r$ is satisfied by those states where <u>some</u> successor is in $\{s_1, s_4\}$, which are s_1, s_4 and s_3.

<u>Answer:</u> s_1, s_3 and s_4 (the red, green, and green+yellow states).

j) Which states of the transition system/model satisfy $E[y \cup g]$?

<u>Solution:</u> This is a formula in Computation Tree Logic: there Exists a path where y Until g. Let's check the states one by one:

s_1: $\neg y$ and $\neg g$, so no such path exists (no first step).

s_2: Any path $s_2 \to s_3 \to$ whatever satisfies $y \cup g$, so yes.

s_3: This has g, so any path beginning here satisfies $y \cup g$.

s_4: Ditto. It makes no difference that y is true also at start.

<u>Answer:</u> s_2, s_3, and s_4 (the red+yellow, green, and green+yellow states)

4. Prove the validity of

$$\forall x\,(P(x,x) \to Q(x)) \vdash \exists x \forall y\, P(x,y) \to \exists x\, Q(x)$$

where P is a predicate symbol of two arguments and Q is a predicate symbol of one argument. Provide justifications of the steps in your proof.

<u>Solution</u>: Informally, this says "$Q(x)$ if $P(x,x)$" and "there exists an x such that for all y we have $P(x,y)$". Taking that y equal to this x yields $P(x,x)$, from which follows $Q(x)$ – this is the plan. Now for the minutae.

1.	$\forall x\,(P(x,x) \to Q(x))$	Premise
2.	Assume $\exists x \forall y\, P(x,y)$	Hypothesis (for $\to i$)
3.	z_0 $\forall y\, P(z_0,y)$	Hypothesis for $\exists e$
4.	$P(z_0,z_0)$	$\forall e$ on 3 (z_0/y)
5.	$P(z_0,z_0) \to Q(z_0)$	$\forall e$ on 1 (z_0/x)
6.	$Q(z_0)$	$\to e$ on 4,5
7.	$\exists x\, Q(x)$	$\exists i$ on 6 (z_0/x)
8.	$\exists x\, Q(x)$	$\exists e$ on 2, 3–7
9.	$\exists x{:} \forall y{:} P(x,y) \to \exists x{:} Q(x)$	$\to i$ on 2–8

QED