MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2015-02-20 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the real numbers x for which
$$\frac{5}{2} + x + \frac{8}{125} + \ldots$$
is a geometric series. Then, determine for each of the geometric series whether it is convergent or not, and find in case of convergence its sum.

2. Find the solution of the differential equation $y'' + 8y' + 15y = 0$, satisfying the initial conditions $y(0) = 5$, $y'(0) = -11$.

3. Let
$$f(x) = \frac{\cos(x)}{1 + \sin(x)}.$$
Find the area of the triangle region which lies in the first quadrant, and which is precisely enclosed by the positive coordinate axes and the tangent line to the curve $y = f(x)$ at the point P whose x-coordinate is equal to $\pi/6$.

4. Which antiderivative G of the function $x \curvearrowright g(x) = \dfrac{1}{(\sqrt{x}-1)\sqrt{x}}$ has the value -2 at the point 4.

5. Determine whether $\displaystyle\lim_{n\to\infty}\left(\frac{n^2}{n+3} - \frac{n^3}{n^2+2}\right)$ exists or not.

If the answer is NO: Give an explanation of why!

If the answer is YES: Give an explanation of why and find the limit!

6. Evaluate the integral $\displaystyle\int_0^2 |x^2 - 3x + 2|\,dx$.

7. Let $f(x) = \sqrt{x}$ and $g(x) = \ln(x)$. Find the domain of the composition $f \circ g \circ g$.

8. Does the function f, defined as $f(x) = (2x+3)e^{-x}$, has any maximum value or/and any minimum value? Determine if so, this/these value(s).

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm de reella tal x för vilka

$$\frac{5}{2} + x + \frac{8}{125} + \ldots$$

är en geometrisk serie. Avgör sedan för var och en av de geometriska serierna om den är konvergent eller ej, och bestäm i händelse av konvergens dess summa.

2. Bestäm till differentialekvationen $y'' + 8y' + 15y = 0$ den lösning som uppfyller begynnelsevillkoren $y(0) = 5$, $y'(0) = -11$.

3. Låt

$$f(x) = \frac{\cos(x)}{1 + \sin(x)}.$$

Bestäm arean av det triangelområde som ligger i den första kvadranten, och som precis innesluts av de positiva koordinataxlarna och tangenten till kurvan $y = f(x)$ i punkten P vars x-koordinat är lika med $\pi/6$.

4. Vilken primitiv funktion G till funktionen $x \curvearrowright g(x) = \dfrac{1}{(\sqrt{x} - 1)\sqrt{x}}$ har värdet -2 i punkten 4?

5. Avgör om $\displaystyle\lim_{n\to\infty} \left(\frac{n^2}{n+3} - \frac{n^3}{n^2+2} \right)$ existerar eller ej.

Om svaret är NEJ: Ge en förklaring till varför!

Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

6. Beräkna integralen $\displaystyle\int_0^2 |x^2 - 3x + 2|\, dx$.

7. Låt $f(x) = \sqrt{x}$ och $g(x) = \ln(x)$. Bestäm definitionsmängden för sammansättningen

$$f \circ g \circ g.$$

8. Har funktionen f, definierad enligt $f(x) = (2x + 3)e^{-x}$, något största värde eller/och något minsta värde? Bestäm i så fall detta/dessa värde(n).

If you prefer the problems formulated in English, please turn the sheet.

MAA151 / Solutions to the final exam TEN1 2015-02-20

1 $\frac{5}{2}+x+\frac{8}{125}+\ldots = \frac{5}{2}\left(1+\frac{2x}{5}+\frac{16}{625}+\ldots\right)$

The series is geometric iff $\left(\frac{2x}{5}\right)^2=\frac{16}{625} \Leftrightarrow x^2=\frac{4}{25}$

$\Leftrightarrow\ x=\frac{2}{5}\ \vee\ x=-\frac{2}{5}$

The series converges for these x-values since

$\left|\frac{2x}{5}\right| = \left|\frac{2}{5}\left(\pm\frac{2}{5}\right)\right| = \frac{4}{25} < 1$

The sums of the series are $\frac{5}{2}\frac{1}{1-\frac{4}{25}}$ and $\frac{5}{2}\frac{1}{1-\left(-\frac{4}{25}\right)}$

respectively, i.e. $\frac{125}{42}$ and $\frac{125}{58}$ respectively.

2 DE: $y''+8y'+15y=0$ IV: $y(0)=5$, $y'(0)=-11$

The auxiliary equation of the DE is

$0=r^2+8r+15=(r+5)(r+3) \Leftrightarrow r=-5\ \vee\ r=-3$

The general solution of the DE is then $y=C_1e^{-5x}+C_2e^{-3x}$

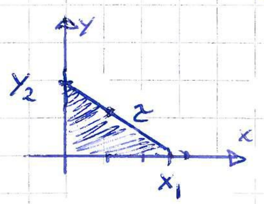
The IV:s give $\begin{cases}5=y(0)=C_1+C_2\\-11=y'(0)=-5C_1-3C_2\end{cases} \Leftrightarrow \begin{cases}C_1+C_2=5\\2C_2=14\end{cases} \Leftrightarrow \begin{cases}C_1=-2\\C_2=7\end{cases}$

The solution of the IVP is then $y=7e^{-3x}-2e^{-5x}$

3 $f(x)=\frac{\cos(x)}{1+\sin(x)}$

use of the Pythagorean trigonometric identity

$f'(x)=\frac{-\sin(x)(1+\sin(x))-\cos(x)\cos(x)}{(1+\sin(x))^2} = \frac{-\sin(x)-1}{(1+\sin(x))^2} = -\frac{1}{1+\sin(x)}$



At the point P, we have: $\begin{cases}f\left(\frac{\pi}{6}\right)=\frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}}=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}\\ f'\left(\frac{\pi}{6}\right)=-\frac{1}{1+\frac{1}{2}}=-\frac{2}{3}\end{cases}$

An eq. for the tangent line z at P is

$y-\frac{1}{\sqrt{3}}=-\frac{2}{3}\left(x-\frac{\pi}{6}\right)$ where $\begin{cases}\text{if } y_1=0 \text{ then } x_1=\frac{\pi}{6}+\frac{\sqrt{3}}{2}\\ \text{if } x_2=0 \text{ then } y_2=\frac{\pi}{9}+\frac{1}{\sqrt{3}}\end{cases}$

The area A of the triangle region is

$A=\frac{1}{2}x_1\cdot y_2=\frac{1}{2}\left(\frac{\pi}{6}+\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{9}+\frac{1}{\sqrt{3}}\right)\text{ a.u.}=\frac{1}{2}\frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\left(1+\frac{\pi}{3\sqrt{3}}\right)^2\text{a.u.}=\frac{1}{4}\left(1+\frac{\pi}{3\sqrt{3}}\right)^2\text{ a.u.}$

4 $G(x)=\int\frac{dx}{(\sqrt{x}-1)\sqrt{x}}\ \begin{bmatrix}\sqrt{x}-1=u\\ \frac{1}{2}\frac{1}{\sqrt{x}}dx=du\end{bmatrix}=\int\frac{2\,du}{u}=2\ln|\sqrt{x}-1|+C$

where $-2=G(4)=2\ln(\sqrt{4}-1)+C=2\ln(1)+C=2\cdot 0+C$ i.e. $C=-2$

Thus $G(x)=2\left[\ln(\sqrt{x}-1)-1\right]$ (where the absolute value bars are not needed since from the IV G is defined for $\sqrt{x}>1$)

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$$\lim_{n\to\infty}\left(\frac{n^2}{n+3}-\frac{n^3}{n^2+2}\right)=\lim_{n\to\infty}\frac{n^2(n^2+2)-n^3(n+3)}{(n+3)(n^2+2)}=\lim_{n\to\infty}\frac{2n^2-3n^3}{(n+3)(n^2+2)}$$

$$=\lim_{n\to\infty}\frac{n^3(-3+2/n)}{n^3(1+3/n)(1+2/n^2)}=\lim_{n\to\infty}\frac{n^3}{n^3}\left(\frac{\lim\limits_{n\to\infty}(-3+\frac{2}{n})}{\lim\limits_{n\to\infty}(1+\frac{3}{n})\lim\limits_{n\to\infty}(1+\frac{2}{n^2})}\right)$$

$$=1\cdot\frac{-3+0}{(1+0)(1+0)}=\underline{-3}$$

6

$$\int_0^2|x^2-3x+2|\,dx=\int_0^2|(x-1)(x-2)|\,dx=\int_0^1(x-1)(x-2)\,dx+\int_1^2\underbrace{[-(x-1)(x-2)]}_{\text{since }(x-1)(x-2)<0}dx$$

$$=\left[\frac{x^3}{3}-3\frac{x^2}{2}+2x\right]_0^1-\left[\frac{x^3}{3}-\frac{3}{2}x^2+2x\right]_1^2=\left(\frac{1}{3}-\frac{3}{2}+2\right)-(0)-\left[\left(\frac{8}{3}-6+4\right)-\left(\frac{1}{3}-\frac{3}{2}+2\right)\right]$$

$$=2\left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{8}{3}-2\right)=2\cdot\frac{5}{6}-\frac{2}{3}=\frac{5}{3}-\frac{2}{3}=\underline{1}$$

7

$$\begin{cases} f(x)=\sqrt{x} & \underline{\text{where}}\ D_f=[0,\infty) \\ g(x)=\ln(x) & \underline{\text{where}}\ D_g=(0,\infty) \end{cases}$$

__We get__ $f\circ g\circ g(x)=\sqrt{\ln(\ln(x))}$ __where__

$$D_{f\circ g\circ g}=\{x:\underbrace{\ln(\ln(x))\geq 0}_{\text{since } D_{\sqrt{\ }}=[0,\infty)}\}=\{x:\underbrace{\ln(x)\geq 1}_{\text{since }\ln(u)\geq 0\iff u\geq 1}\}=\{x:\underbrace{x\geq e}_{\text{since }\ln(x)\geq 1\iff x\geq e}\}$$

$$=\underline{[e,\infty)}$$

8

$$\begin{cases} f(x)=(2x+3)e^{-x} \\ f'(x)=2e^{-x}+(2x+3)e^{-x}(-1) \end{cases}$$

$$=-(2x+3-2)e^{-x}=-2\left(x+\tfrac{1}{2}\right)e^{-x}$$

__First derivative test__

x		$-\frac{1}{2}$	
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	loc. max.	$\searrow$

__and__ $f(-\frac{1}{2})=[2(-\frac{1}{2})+3]e^{-(-\frac{1}{2})}=2\sqrt{e}$

From the first derivative test, we conclude that f has a maximum at $-\frac{1}{2}$, but has no minimum.

The maximum value is $2\sqrt{e}$.

Examination TEN1 – 2015-02-20	Maximum points for subparts of the problems in the final examination
1. The series is geometric iff $(x=\frac{2}{5})\vee(x=-\frac{2}{5})$ In both cases, the series converges. The sum of the series is equal to $\frac{125}{42}$ as $x=\frac{2}{5}$ and $\frac{125}{58}$ as $x=-\frac{2}{5}$	**1p**: Correctly found that the series is geometric iff $(x=\frac{2}{5})\vee(x=-\frac{2}{5})$ **1p**: Correctly determined that the series converges in both of the geometric cases **1p**: Correctly determined the sums of the two series **The student** who has found only one of the two x-values, but at least correctly has solved the problem for that x, may obtain a total of **2p**.
2. $y=7e^{-3x}-2e^{-5x}$	**1p**: Correctly found one solution of the DE **1p**: Correctly found the general solution of the DE **1p**: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP **The student** who wrongly has claimed that the general solution is either of $y=Ae^{-3x}+Bxe^{-5x}$ or $y=Axe^{-3x}+Be^{-5x}$, but then at least correctly has adapted the proposed solution to the initial values, may obtain a total of **1p**.
3. $\frac{1}{4}(1+\frac{\pi}{3\sqrt{3}})^2$ a.u.	**1p**: Correctly determined the derivative of the the function f **1p**: Correctly determined an equation for the tangent line to the curve $y=f(x)$ at the point P **1p**: Correctly determined the area of the triangle region
4. $G(x)=2\left(\ln(\sqrt{x}-1)-1\right)$	**1p**: Correctly applied a substitution which simplifies the determination of the general antiderivative of g **1p**: Correctly determined the general antiderivative of g **1p**: Correctly adapted the general antiderivative to the initial value, and correctly summarized the solution of the IVP
5. The limit exists and is equal to -3	**1p**: Correctly brought the rational expressions together with a least common denominator **1p**: Correctly found that the limit exists **1p**: Correctly determined the limit
6. 1	**1p**: Correctly divided the integral in two integrals, each in which the absolute value bars can be removed **1p**: Correctly determined an antiderivative for each integrand **1p**: Correctly determined the value of the integral
7. $D_{f\circ g\circ g}=[e,\infty)$	**1p**: Correctly noted that $x\in D_{f\circ g\circ g}\Leftrightarrow \ln(\ln(x))\geq 0$ **1p**: Correctly noted that $\ln(\ln(x))\geq 0\Leftrightarrow \ln(x)\geq 1$ **1p**: Correctly noted that $\ln(x)\geq 1\Leftrightarrow x\geq e$, and by this finally concluded that $D_{f\circ g\circ g}=[e,\infty)$
8. f has no minimum value f has the maximum value $2\sqrt{e}$	**1p**: Correctly found that the function has no minimum value **1p**: Correctly found that the function has a maximum value **1p**: Correctly determined the maximum value