TEN1 2018-01-09 Solutions MMA130 Math. Logic

1. A basic board of traffic lights has one red, one yellow (amber, orange), and one green light. Let r, y, and g be propositional atoms denoting that the red, yellow, and green respectively lights are on in a particular group of lights.

a) In English signalling order, the lights cycle through the states of red, red+yellow, green, and yellow before returning to red. Express as a propositional logic formula the claim that the lights are in one of these four states.

<u>Answer</u>: $(r \wedge \neg g) \vee (\neg r \wedge \neg y \wedge g) \vee (\neg r \wedge y \wedge \neg g)$

b) Express in linear time temporal logic the claim that all future states of the lights are among these four states.

<u>Answer</u>: G of the answer to (a), i.e.
$G((r \wedge \neg g) \vee (\neg r \wedge \neg y \wedge g) \vee (\neg r \wedge y \wedge \neg g))$

c) Express in linear time temporal logic the claim that the lights will always eventually show green.

To show green is not just g, but $\neg r \wedge \neg y \wedge g$.
To eventually show green is F of that, but this could be satisfied by showing green now and never again; "always" additionally requires a G.
Answer: $GF(\neg r \wedge \neg y \wedge g)$

d) Express in linear time temporal logic the claim that at any time when the light is green, it will remain green until switching to yellow.

"Green until yellow" is $(\neg r \wedge \neg y \wedge g)\, U\, (\neg r \wedge y \wedge \neg g)$.
"At any time when" is G of an implication. Thus we get

<u>Answer</u>: $G\left((\neg r \wedge \neg y \wedge g) \rightarrow (\neg r \wedge \neg y \wedge g)\, U\, (\neg r \wedge y \wedge \neg g)\right)$

e) Interpret in natural language the claim $F(y \rightarrow Xr \wedge XXr \wedge XXXr)$.

The $Xr \wedge XXr \wedge XXXr$ part means "during the next, second next, and third next time steps, the red light will be on"; let's call this t for short. Then $y \rightarrow t$ is "if the yellow light is on, then t". But it gets strange when we get to the F, since literally the meaning of the whole thing is

<u>Answer</u>: There is some future time with the property that if the yellow light is on then, the red light will be on during the following three time steps.

(What's strange is that this is trivially satisfied if there is even a single time step at which the yellow light is not on. But not all formulae one encounters do necessarily mean what the author intended.)

2 Give a natural deduction proof of
$p \to (q \to r) \vdash (q \wedge p) \to r$
Provide justifications of all steps.

Solution: Since the sought conclusion $(q \wedge p) \to r$ is an implication, the last step will most likely be by $(\to i)$, and we will have $q \wedge p$ as an hypothesis. Then the proof pretty much writes itself (since there aren't many choices of things to do that wouldn't advance the proof):

1.	$p \to (q \to r)$	Premise	
2.	$q \wedge p$	Hypothesis	
3.	p	$\wedge e$ on line 2	(rule 2b)
4.	$q \to r$	$\to e$ on lines 1, 3	(rule 6)
5.	q	$\wedge e$ on line 2	(rule 2a)
6.	r	$\to e$ on lines 4, 5	(rule 6)
7.	$(q \wedge p) \to r$	$\to i$ on lines 2–6.	(rule 5)

3. A natural deduction proof in the predicate calculus with equality has the steps

1. z_2
2. $z_1 = z_2$ (Typo in exam!)
3. $z_1 = z_1$
4. $z_2 = z_1$
5. $z_1 = z_2 \to z_2 = z_1$
6. $\forall z_2\,(z_1 = z_2 \to z_2 = z_1)$

Provide detailed justifications for the steps in this proof — explain for each step which earlier steps (if any) you make use of, which rule (in the accompanying list of rules) you use, what each metasyntactic variable (e.g. formula ϕ, term t, or variable x) in that rule comes out as in this particular application of the rule — and draw scope boxes for rules that have them.

Solution: First, some planning. Step 6 is just step 5 with an extra $\forall z_2$, so it was most likely formed by $\forall i$ (rule 18). This explains the z_2 on line 1 as a fresh variable declaration (the x_0 in rule 18 is z_2, as is the x). Step 5 has the form (step 2) $\to$ (step 4), so this is likely an $\to i$ (rule 5) on lines 2–4, which would make line 2 a hypothesis. This leaves lines 3 and 4 to explain.

Since this was explicitly stated to be a proof in the predicate calculus with equality, we should probably make use of the rules for equality (rules 16 and 17). Rule 16 ($=i$) can indeed handily explain $z_1 = z_1$ (line 3), so is it then up to rule 17 (=e) to explain line 4? As it turns out, that is indeed the case, although we probably need to spell out the metasyntactic variable values before it becomes apparent. The fully annotated proof is

1.	z_2	Fresh variable declaration
2.	$z_1 = z_2$	Hypothesis
3.	$z_1 = z_1$	Rule 16: =i t is z_1
4.	$z_2 = z_1$	Rule 17: =e t_1 is z_1, t_2 is z_2, so that line 2 is $t_1 = t_2$ ϕ is $x = z_1$, so that $\phi[t_2/x]$ is line 4 $\phi[t_1/x]$ is line 3
5.	$z_1 = z_2 \to z_2 = z_1$	Rule 5: →i ϕ is $z_1 = z_2$ (line 2) ψ is $z_2 = z_1$ (line 4)
6.	$\forall z_2 (z_1 = z_2 \to z_2 = z_1)$	Rule 18: ∀i x_0 and x are both z_2, fresh by line 1. ϕ is line 5.

4 One approach to formalising the Zebra puzzle in predicate logic (as opposed to formalising it in propositional logic, as was done in one lecture) is to let variables range over house numbers, whereas properties of houses (or their inhabitants, as appropriate) are expressed using predicates.

a) Construct a predicate logic language that allows you to express the following claims, and sketch a standard interpretation/model for that language:

1. The Englishman lives in the red house.
2. The Norwegian lives next to the blue house.
3. Tea is drunk in the first house.
4. The Englishman and the Norwegian do not live in the same house.
5. There is exactly one blue house.

(The full puzzle would require more claims, and most likely also a larger language than the one you're constructing here, but this fragment at least tests the principle.)

Solution: We can do the above with seven predicates, with intended interpretations as follows:

$B(x)$: House x is blue.

$E(x)$: An Englishman lives in house x.

$N(x)$: A Norwegian lives in house x.

$R(x)$: House x is red.

$T(x)$: Tea is drunk in house x.

$S(x,y)$: House x is is the left neighbour of house y.

$x = y$: x is equal to y.

The five claims can then be formalised as follows:

1. $\exists x: (E(x) \wedge R(x))$

(Several variations, taking different approaches to the uniqueness of the Englishman and the red house are possible.)

2. $\exists x: \exists y: (N(x) \wedge B(y) \wedge (S(x,y) \vee S(y,x)))$
3. $\exists x: (T(x) \wedge \forall y: \neg S(y,x))$
4. $\neg \exists x: (E(x) \wedge N(x))$
5. $\exists x: B(x) \wedge \forall x: \forall y: ((B(x) \wedge B(y)) \rightarrow x = y)$

The domain A of the standard interpretation moreover has $A = \{1,2,3,4,5\}$.

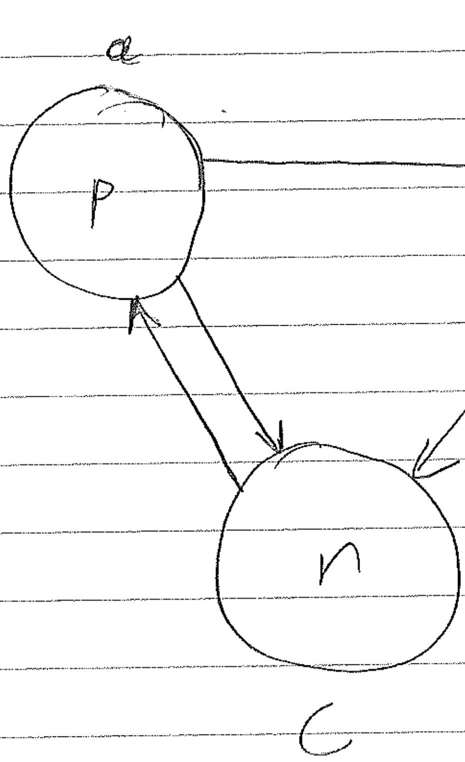
b) Describe a nonstandard model which, when satisfying the claims above (as you encoded them), would have some property not expected in the standard solution; it could for example be that tea is drunk in more than one house. Then suggest an additional axiom which would rule out that model.

If we take $\{1,2,3,5,6,7\}$ as domain and define $S(x,y)$ as $x+1=y$, then we do indeed get the consequence that tea is drunk both in house 1 and house 5, because these are both "first" houses on account of not having an immediate predecessor.
To eliminate that possibility, we add an axiom stating that all houses without predecessor (i.e., first houses) are equal:

$$\forall x: \forall y: ((\neg \exists z: S(z,x) \wedge \neg \exists z: S(z,y)) \rightarrow x = y)$$

5. Consider the Kripke model $M = (W, R, L)$ where $W = \{a, b, c\}$, $R = \{(a,b), (b,c), (a,c), (c,a)\}$, $L(a) = \{p\}$, $L(b) = \{p, q\}$ and $L(c) = \{r\}$.

a) Draw a graph for M.



b) Determine the set of worlds where $\Box\Box p$ is satisfied.

Solution: A table of relevant subformulae and their truth values in different worlds are

	a	b	c
p	$\top$	$\top$	$\bot$
$\Box p$	$\bot$	$\bot$	$\top$
$\Box\Box p$	$\bot$	$\top$	$\bot$

Answer: $\Box\Box p$ is satisfied in $\{b\}$.

c) Determine the set of worlds where $\Box(\Diamond p \to r)$ is satisfied.

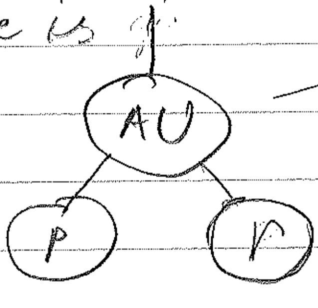
Table again:

	a	b	c
p	$\top$	$\top$	$\bot$
$\Diamond p$	$\top$	$\bot$	$\top$
r	$\bot$	$\bot$	$\top$
$\Diamond p \to r$	$\bot$	$\top$	$\top$
$\Box(\Diamond p \to r)$	$\top$	$\top$	$\bot$

<u>Answer:</u> $\Box(\Diamond p \to r)$ is satisfied in $\{a, b\}$.

d) Parse the formula $A[p\,U\,r]$ (what logic is it in? What is its structure?) and determine the set of worlds/states in M where it is satisfied.

This is Computation Tree Logic (CTL), and the formula structure is



— in all futures, until

The possible paths starting in a either go $a \to b \to c$ or $a \to c$. In c, r is satisfied, and in a, b both p is satisfied, so indeed we have p until r in all futures. Possible paths from b all start $b \to c$, with the same conclusion. In c, r is true immediately. Thus $A[p\,U\,r]$ is satisfied in all worlds of this model.