MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
*Examinator:* Erik Darpö

**TENTAMEN I MATEMATIK**
MAA150 Vektoralgebra — TEN2
*Datum:* 12 januari 2015 — *Skrivtid:* 3 timmar
*Hjälpmedel:* Skrivdon

*Denna tentamen TEN2 består av 6 uppgifter, med en sammanlagd poängsumma om 25 poäng. För betyget **3** krävs en erhållen poängsumma om minst 12 poäng, för betyget **4** krävs 16 poäng, och för betyget **5** krävs 20 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.*

**1.** Ange en matris $A \in \mathbb{R}^{3\times 3}$ som uppfyller att

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \ker A \qquad \text{och} \qquad \operatorname{im} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

*Glöm inte att visa varför $A$ har dessa egenskaper.* *(4p)*

**2.** Låt $u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $u_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ och $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^4$.

Bestäm en bas för $U$, och ange koordinaterna för var och en av vektorerna $u_1, u_2, u_3, u_4$ i denna bas. *(5p)*

**3.** Bestäm en ortonormerad bas i underrummet $V = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$. *(4p)*

**4.** Låt $A = \dfrac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -1 & \sqrt{2} \\ \sqrt{3} & 1 & -\sqrt{2} \\ 0 & 2 & \sqrt{2} \end{pmatrix}$ och $B = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

Avgör vilka av matriserna $A$, $B$, $AB$, $A^{-1}$ som är ortogonala. *(4p)*

**5.** Beräkna determinanten $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & 2 \\ 1 & 4 & 5 & 1 \\ 0 & 5 & 6 & 0 \end{vmatrix}$. *(4p)*

**6.** Låt $A = \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$, där $a \in \mathbb{R}$ är en godtycklig konstant.

a) Bestäm egenvärdena till matrisen $A$, för varje värde på konstanten $a$.

b) För vilka värden på $a$ är matrisen $A$ diagonaliserbar?

*(4p)*

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Division of Applied Mathematics
*Examiner:* Erik Darpö

**EXAMINATION IN MATHEMATICS**
MAA150 Vector algebra TEN2
*Date:* 12 January 2015 *Time:* 3 hours
*Materials allowed:* Writing material only

*This exam TEN2 consists of 6 problems, with a total score of 25 points. To obtain the grades **3**, **4** and **5**, scores of at least 12, 16 respectively 20 points are required.*
*All solutions are to include motivations and clear answers to the questions asked.*

**1.** Find a matrix $A \in \mathbb{R}^{3\times 3}$ that satisfies

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \in \ker A \qquad \text{and} \qquad \operatorname{im} A = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

*Remember to explain* <u>*why*</u> *A has these properties.* *(4p)*

**2.** Let $u_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, $u_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $u_3 = \begin{pmatrix}-1\\0\\2\end{pmatrix}$, $u_4 = \begin{pmatrix}0\\1\\1\end{pmatrix}$ and $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^4$.

Determine a basis of $U$, and give the coordinates of each of the vectors $u_1, u_2, u_3, u_4$ in this basis. *(5p)*

**3.** Determine an orthonormal basis of the subspace $V = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$.

*(4p)*

**4.** Let $A = \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3} & -1 & \sqrt{2}\\ \sqrt{3} & 1 & -\sqrt{2}\\ 0 & 2 & \sqrt{2}\end{pmatrix}$ and $B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\\ -1 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$.

Determine which of the matrices $A$, $B$, $AB$, $A^{-1}$ are orthogonal. *(4p)*

**5.** Compute the determinant $\begin{vmatrix}1 & 2 & 3 & 0\\ 1 & 3 & 4 & 2\\ 1 & 4 & 5 & 1\\ 0 & 5 & 6 & 0\end{vmatrix}$. *(4p)*

**6.** Let $A = \begin{pmatrix}1 & 1\\ 0 & a\end{pmatrix}$, where $a \in \mathbb{R}$ is an arbitrary constant.

a) Determine the eigenvalues of the matrix $A$, for all values of the constant $a$.

b) For which values of $a$ is the matrix $A$ diagonalisable?

*(4p)*

# MAA150 Vector algebra

## Solutions to the exam TEN2 12/1/2015

1) Take for example $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

$Ae_3$ is the third column in $A$, so $e_3 \in \ker A$ means that the third column must be $0$.

In other words: $A = \begin{pmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ u_3 & v_3 & 0 \end{pmatrix}$, for some $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$.

Now $\operatorname{im} A = \operatorname{span}\left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\}$

So $\operatorname{im} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ means that

$$\operatorname{span}\left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

This is satisfied if (for example) $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. In this case, we have $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ as above.

2) A basis of $U = \operatorname{span}\{u_1, u_2, u_3, u_4\}$ can be found by removing vectors $u_i$ which can be written as linear combinations of the remaining ones.

Find linear relations among $u_1, u_2, u_3, u_4$:

(*) $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0$ (where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$)

$$\begin{cases} \lambda_1 + \lambda_2 - \lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + \lambda_4 = 0 \\ -\lambda_1 + 2\lambda_3 + \lambda_4 = 0 \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} | & | & | & | & 0 \\ u_1 & u_2 & u_3 & u_4 & 0 \\ | & | & | & | & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

pivot elements

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} \lambda_1 = 2r + t \\ \lambda_2 = -r - t \\ \lambda_3 = r \\ \lambda_4 = t \end{cases} \text{ where } r, t \in \mathbb{R}. \quad (**)$$

This means that $u_1, u_2$ form a basis of $U$. Write $\underline{u} = (u_1, u_2)$

$[u_1]_{\underline{u}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $[u_2]_{\underline{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The solution (**) to the equation (*) means that

(#) $(2r+t)u_1 + (-r-t)u_2 + ru_3 + tu_4 = 0$ for all $r, t \in \mathbb{R}$

<u>Insert $r=1, t=0$</u> into eq. (#): $2u_1 - u_2 + u_3 = 0$

$\Rightarrow u_3 = -2u_1 + u_2$, that is, $[u_3]_{\underline{u}} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Insert $r=0, t=1$ into eq. (#): $u_1 - u_2 + u_4 = 0$

$\Rightarrow u_4 = -u_1 + u_2$, that is $[u_4]_{\underline{u}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\boxed{\text{Answer: } \underline{u} = (u_1, u_2) \text{ is a basis of } U, \text{ and } [u_1]_{\underline{u}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, [u_2]_{\underline{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, [u_3]_{\underline{u}} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, [u_4]_{\underline{u}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

3) $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V \iff x_1 + x_2 + x_3 = 0$

Set $x_2 = r$, $x_3 = t$.

$$x = \begin{pmatrix} -r-t \\ r \\ t \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad r,t \in \mathbb{R}$$

So $V = \text{span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{= u_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{= u_2} \right\}$

Applying the Gram-Schmidt algorithm on $u_1, u_2$ gives an orthonormal basis of $V$:

$$f_1 = \hat{u}_1 = \frac{1}{\|u_1\|} u_1 = \frac{1}{\sqrt{(-1)^2 + 1^2 + 0^2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

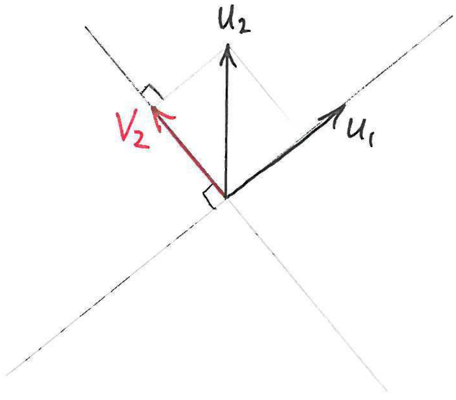
$$v_2 = u_2 - P_{f_1}(u_2) = u_2 - (u_2 \cdot f_1) f_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left( \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left( (-1)^2 + 0 \cdot 1 + 1 \cdot 0 \right) \cdot \frac{1}{\sqrt{2}^2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$f_2 = \hat{v}_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 1^2}} \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{\frac{3}{2}}} \cdot \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

The vectors $f_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $f_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ constitute an orthonormal basis of $V$.



4) • The matrix $A$ is orthogonal, since its columns form an orthonormal basis of $\mathbb{R}^3$:

$$\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3}\\\sqrt{3}\\0\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}=\frac{1}{\sqrt{6}^2}\left(\sqrt{3}\cdot(-1)+\sqrt{3}\cdot 1+0\cdot 2\right)=0$$

$$\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3}\\\sqrt{3}\\0\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{6}^2}\left(\sqrt{3}\cdot\sqrt{2}+\sqrt{3}\cdot(-\sqrt{2})+0\cdot\sqrt{2}\right)=0$$

$$\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{6}^2}\left((-1)\sqrt{2}+1\cdot(-\sqrt{2})+2\sqrt{2}\right)=0$$

$$\left\|\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3}\\\sqrt{3}\\0\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\left\|\begin{pmatrix}\sqrt{3}\\\sqrt{3}\\0\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\sqrt{\sqrt{3}^2+\sqrt{3}^2+0}=1$$

$$\left\|\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\left\|\begin{pmatrix}-1\\1\\2\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\sqrt{(-1)^2+1^2+2^2}=1$$

$$\left\|\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\left\|\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\\sqrt{2}\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\sqrt{\sqrt{2}^2+(-\sqrt{2})^2+\sqrt{2}^2}=1$$

• The matrix $B$ is not orthogonal, since its columns do not form an orthonormal basis of $\mathbb{R}^3$. For example:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}=\frac{1}{\sqrt{2}^2}\left(1\cdot 0+(-1)^2+0\cdot 1\right)=\frac{1}{2}\neq 0.$$

• The inverse of an orthogonal matrix is orthogonal, so $A^{-1}$ is orthogonal.

The product of two orthogonal matrices is orthogonal. If $AB$ were orthogonal, this would therefore imply that $A^{-1}\cdot(AB) = A^{-1}AB = IB = B$ is orthogonal, which is not true.

- Hence, $AB$ is not orthogonal.

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5) $$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & 2 \\ 1 & 4 & 5 & 1 \\ 0 & 5 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 5 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 5 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ -1 & 6 & 0 \end{vmatrix} = -1\cdot\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= -1\cdot(1\cdot 1 - 2\cdot 2) = \underline{\underline{3}}$$

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6) The eigenvalues are the solutions $\lambda$ to the equation $\det(A-\lambda I_2)=0$.

$$\det(A-\lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & a-\lambda \end{vmatrix} = (1-\lambda)(a-\lambda)$$

$\Rightarrow$ The eigenvalues are $\lambda=1$ and $\lambda=a$.

- If $a\neq 1$ then $A$ has two distinct eigenvalues, and is therefore diagonalisable.
- If $a=1$: $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\lambda=1$ is the only eigenvalue.

$$(A - 1\cdot I_2)x = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0 \iff x_2 = 0$$

(here $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$) $\iff x = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ t \in \mathbb{R}$.

Hence, every eigenvector of $A$ belongs to $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, so no basis consisting of eigenvectors can exist.

$\Rightarrow$ A is not diagonalisable.

Conclusion: The eigenvalues of $A$ are $1$ and $a$.
$A$ is diagonalisable if and only if $a \neq 1$.

**Answers to examination TEN2 in MAA150 Vector algebra 12 January 2015**

**1.** For example $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

**2.** A basis of $U$ is $\underline{u} = (u_1, u_2)$, and

$$[u_1]_{\underline{u}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ [u_2]_{\underline{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ [u_3]_{\underline{u}} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ [u_4]_{\underline{u}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**3.** For example $\dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

**4.** The matrices $A$ and $A^{-1}$ are orthogonal, $B$ and $AB$ are not.

**5.** The value of the determinant is 3.

**6.** The eigenvalues are 1 and $a$. The matrix $A$ is diagonalisable if, and only if, $a \neq 1$.

# MAA150 Vector algebra autumn term 2014

## Assessment criteria for TEN2 12/1/2015

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**1.** Two points each for the kernel and the image. Insufficient motivation results in deduction of one point each for the two parts.

**2.** One point each of the following:
1. writing up the equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0$,
2. solving it,
3. interpreting the solution to find a basis.

Two points are given for correctly determining the coordinates of the vectors $u_i$ in the given basis.

**3.** One point for finding a basis of $V$, three points for making this basis orthogonal (e.g., using the Gram–Schmidt algorithm).
In the Gram–Schmidt algorithm, one point is given for computing the first basis vector, one point for applying the correct formulae for computing the second, and a third point for a complete and correct solution.

**4.** In principle, one point per matrix.

**5.** Computation of the determinant using a valid algorithm gives full score, with mistakes along the way resulting in loss of points. The basic rule is that two minor mistakes, or one more important, results in a deduction of one point. A calculation that is too brief to follow may result in deduction of one or several points.

**6.** Two points each for a) and b).