EXAMINATION IN MATHEMATICS
MAA150 Vector Algebra, TEN1
Date: 2017-12-01 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The PASS-marks 3, 4 and 5 require a minimum of 12, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 12, 13, 16, 20 and 24 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN1 S_2, the marks for a completed course are determined according to

$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 47$	$\rightarrow$	3
$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 62$	$\rightarrow$	4
		$63 \leq S_1 + 2S_2$	$\rightarrow$	5

$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 38$	$\rightarrow$	E
$S_1, S_2 \geq 12$	AND	$39 \leq S_1 + 2S_2 \leq 47$	$\rightarrow$	D
$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 59$	$\rightarrow$	C
$S_1, S_2 \geq 12$	AND	$60 \leq S_1 + 2S_2 \leq 71$	$\rightarrow$	B
		$72 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find out whether the planes π_1 och π_2 defined by

$$\begin{cases} \pi_1 : (x, y, z) = (1 + r + 3s, 2 - 2r - s, 3 + r + 2s), \quad r, s \in R \\ \pi_2 : 3x - y - 5z + 28 = 0 \end{cases}$$

intersect or not. If the answer is YES: Find the angle between the planes. If the answer is NO: Find the distance between the planes. It is assumed that the standard basis is a right-handed ON-basis.

2. Compute the determinant of

$$\frac{1}{5}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^T \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^{-1}.$$

3. Sketch the region $\Omega = \{z \in \mathbb{C} : \text{Im}(z) \geq 2, \ |z| \leq 4\}$ where $\mathbb{C}$ denotes the set of all complex numbers. Then find on both rectangular and polar form the complex number which belongs to Ω and has smallest possible real part.

4. Find, for every real value of the parameter β, the triples (x, y, z) that satisfies the system of linear equations

$$\begin{cases} 2x + y - z = -3\,, \\ 4x + 5y + z = -3\,, \\ \beta x + y + 2z = 3\,. \end{cases}$$

5. Find an equation for the line λ which includes the point $P_1 : (3, -5, 4)$, is parallel with the plane $\pi_2 : x + 2y + 3z + 4 = 0$, and is perpendicular to vectors parallel with the line $\lambda_3 : (x, y, z) = (-7 + 2t, 5 - 2t, 3 - t), \ t \in R$. It is assumed that the standard basis is a right-handed ON-basis.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA150 Vektoralgebra, TEN1
Datum: 2017-12-01 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 12, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av ett sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{lccc} S_1, S_2 \geq 12 & \text{OCH} & S_1 + 2S_2 \leq 47 & \rightarrow & 3 \\ S_1, S_2 \geq 12 & \text{OCH} & 48 \leq S_1 + 2S_2 \leq 62 & \rightarrow & 4 \\ & & 63 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Utred om planen π_1 och π_2 definierade genom

$$\begin{cases} \pi_1 : (x, y, z) = (1 + r + 3s, 2 - 2r - s, 3 + r + 2s), \quad r, s \in R \\ \pi_2 : 3x - y - 5z + 28 = 0 \end{cases}$$

skär varandra eller ej. Om svaret är JA: Bestäm vinkeln mellan planen. Om svaret är NEJ: Bestäm avståndet mellan planen. Det antages att standardbasen är en högerorienterad ON-bas.

2. Beräkna determinanten av

$$\frac{1}{5}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^T \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^{-1}.$$

3. Skissa området $\Omega = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 2, \ |z| \leq 4\}$ där $\mathbb{C}$ betecknar mängden av alla komplexa tal. Bestäm sedan på både rektangulär och polär form det komplexa tal som tillhör Ω och har minsta möjliga realdel.

4. Bestäm, för varje reellt värde på parametern β, de taltriplar (x, y, z) som satisfierar det linjära ekvationssystemet

$$\begin{cases} 2x + y - z = -3\,, \\ 4x + 5y + z = -3\,, \\ \beta x + y + 2z = 3\,. \end{cases}$$

5. Bestäm en ekvation för den linje λ som inkluderar punkten $P_1 : (3, -5, 4)$, är parallell med planet $\pi_2 : x + 2y + 3z + 4 = 0$, och är vinkelrät mot vektorer som är parallella med linjen $\lambda_3 : (x, y, z) = (-7 + 2t, 5 - 2t, 3 - t),\ t \in R$. Det antages att standardbasen är en högerorienterad ON-bas.

If you prefer the problems formulated in English, please turn the page.

MAA150 / Solutions to the final exam. TEN1 2017-12-01

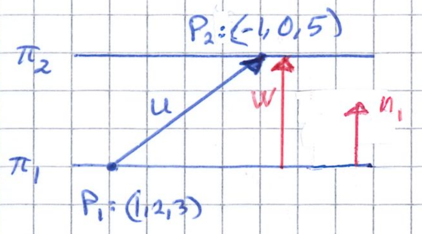
1

$$\begin{cases} \pi_1 : (x,y,z) = (1,2,3) + r(1,-2,1) + s(3,-1,2), & r,s \in \mathbb{R} \\ \pi_2 : 3x - y - 5z + 28 = 0 \end{cases}$$

The equations for π_1 and π_2 tell us e.g. that $v_{1a} = (1,-2,1)$ and $v_{1b} = (3,-1,2)$ are two (non-proportional) vectors parallel with π_1 and that $n_2 = (3,-1,-5)$ is a vector orthogonal to π_2. Since

$$n_1 = v_{1a} \times v_{1b} = (1,-2,1) \times (3,-1,2) = (-4-(-1), 3-2, -1-(-6)) = (-3,1,5)$$

is a vector orthogonal to π_1 and $n_1 = -n_2$, we conclude that the planes π_1 and π_2 are parallel. The distance



between the planes equals, according to the illustration to the left, the length of the vector w which is nothing but the orthogonal projection of the vector $u = u_{\overrightarrow{P_1P_2}}$ on n_1 where P_1 is a point in π_1 and P_2 a point in π_2.

Thus

$$\text{distance}(\pi_1,\pi_2) = \|\text{proj}_{n_1}(u)\| = \left\| \frac{u \cdot n_1}{\|n_1\|^2} n_1 \right\| = \frac{|u \cdot n_1|}{\|n_1\|}$$

$$= \frac{|(-2,-2,2)\cdot(-3,1,5)|}{\|(-3,1,5)\|} = \frac{|6-2+10|}{\sqrt{9+1+25}} = \frac{14}{\sqrt{35}} = 2\sqrt{\frac{7}{5}}$$

Answer: $2\sqrt{\frac{7}{5}}$ l.u.

2 Let $\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix} = A$ Then

$$\det\left[\frac{1}{5}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}^T\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix}\right] = \det\left(\frac{1}{5} A A^T A^{-1}\right)$$

The product rule for determinants is used

The factor 1/5 is associated with a copy of the identity matrix

$$= \det\left(\left(\frac{1}{5}I\right) A A^T A^{-1}\right) = \det\begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \det(A) \det(A^T) \det(A^{-1})$$

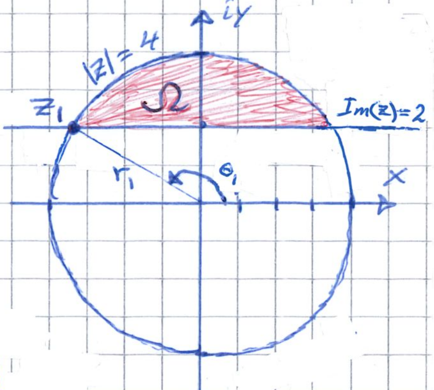
The general rules for determinants of transposed and inverse matrices respectively are used

$$= \left(\frac{1}{5}\right)^3 \det(A) \det(A) \frac{1}{\det(A)} = \frac{1}{125} \det(A)$$

$$= \frac{1}{125} \det\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 8 & 2 \end{pmatrix} \begin{matrix} \\ -2 \quad -4 \\ \\ \end{matrix} = \frac{1}{125} \det\begin{pmatrix} 0 & -1 & -4 \\ 1 & 2 & 3 \\ 0 & 0 & -10 \end{pmatrix}$$

$$= \frac{1}{125}\left[0 + 0 - 10(-1)^{3+3} \det\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\right] = \frac{1}{125}(-10)[0-(-1)] = -\frac{2}{25}$$

Expansion along the third row



3 $\Omega = \{ z \in \mathbb{C} : Im(z) \geq 2,\ |z| \leq 4 \}$

i.e. Ω consists of the complex numbers which are on or above the line $Im(z) = 2$ but not more than 4 length units from the origin (the redmarked region including its boundary). The complex number $z_1 \in \Omega$ having the smallest possible real part is given by

$$\begin{cases} x_1 = r_1 \cos(\theta_1) \\ y_1 = r_1 \sin(\theta_1) \end{cases} \text{ where } \begin{cases} r_1 = 4 \\ y_1 = 2 \end{cases} \text{ giving } \begin{cases} \sin(\theta_1) = \frac{2}{4} = \sin(\pi/6) = \sin\left(\frac{5\pi}{6}\right) \\ x_1 = 4\cos\left(\frac{5\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} \end{cases}$$

i.e. $$\underbrace{z_1 = -2\sqrt{3} + 2i}_{\text{rectangular form}} = \underbrace{4\left[\cos(5\pi/6) + i\sin(5\pi/6)\right]}_{\text{polar form}}$$

4 $-\beta$ $\quad -2$

$$\begin{cases} 2x + y - z = -3 \\ 4x + 5y + z = -3 \\ \beta x + y + 2z = 3 \end{cases} \Leftrightarrow \begin{cases} 2x + y - z = -3 \\ 3y + 3z = 3 \\ (2-\beta)y + (4+\beta)z = 6 + 3\beta \end{cases}$$

$\beta - 2$ $\quad -1$

$$\Leftrightarrow \begin{cases} 2x + y - z = -3 \\ y + z = 1 \\ (2-\beta)y + (4+\beta)z = 6 + 3\beta \end{cases} \Leftrightarrow \begin{cases} 2x - 2z = -4 \\ y + z = 1 \\ (2\beta + 2)z = 4\beta + 4 \end{cases} \Leftrightarrow \begin{cases} x - z = -2 \\ y + z = 1 \\ (\beta + 1)z = 2(\beta + 1) \end{cases}$$

If $\beta = -1$ then $\begin{cases} x = z - 2 \\ y = -z + 1 \\ 0 = 0 \end{cases}$ Let $z = t \in \mathbb{R}$ then

$(x, y, z)_{\beta = -1} = (-2 + t, 1 - t, t) = (-1 - s, s, 1 - s) = (r, -1 - r, 2 + r)$, $r, s, t \in \mathbb{R}$ (alternative parametrizations)

If $\beta \neq -1$ then $\begin{cases} x = z - 2 \\ y = -z + 1 \\ z = 2 \end{cases} \Leftrightarrow (x, y, z)_{\beta \neq -1} = (0, -1, 2)$

Answer: $$\boxed{(x, y, z) = \begin{cases} (0, -1, 2) & \text{if } \beta \neq -1 \\ (0, -1, 2) + s(1, -1, 1) & \text{if } \beta = -1 \end{cases}}$$

5 $P_1 : (3, -5, 4)$, $\quad \pi_2 : x + 2y + 3z + 4 = 0$, $\quad \lambda_3 : (x, y, z) = (-7 + 2t, 5 - 2t, 3 - t)$, $t \in \mathbb{R}$

We know that
- a) $\lambda \supset P_1$
- b) $\lambda \parallel \pi_2$ and therefore $\lambda \perp n_2$ where $n_2 = (1, 2, 3)$ is a vector orthogonal to π_2
- c) $\lambda \perp v_3$ where $v_3 = (2, -2, -1)$ is a vector parallel with λ_3

Combining (b) and (c) gives that the line λ is parallel with the vector $v = n_2 \times v_3 = (1, 2, 3) \times (2, -2, -1) = (4, 7, -6)$, and we finally get that $\lambda : (x, y, z) = (3, -5, 4) + r(4, 7, -6)$, $r \in \mathbb{R}$ is an equation for the line λ.

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EXAMINATION IN MATHEMATICS
MAA150 Vector algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2017/18

Final examination TEN1 – 2017-12-01 Maximum points for subparts of the problems in the final examination

1. The planes π_1 and π_2 are parallel, and the distance between them is $2\sqrt{\frac{7}{5}}$ l.u.

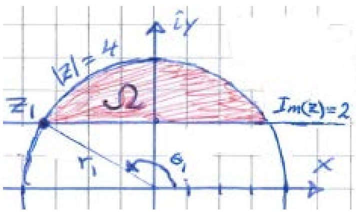
2p: Correctly proved the planes π_1 and π_2 are parallel based on the fact that their normal vectors are parallel
1p: Correctly concluded that the distance between the planes equals the length of the orthogonal projection of a vector u on a vector n, where u is represented by the directed line segment $\overrightarrow{P_1P_2}$ from a point P_1 in π_1 to a point P_2 in π_2 (both explicitly found), and where n is a normal vector to the planes
1p: Correctly found the (evaluated) expression for the orthogonal projection of u on n
1p: Correctly found the distance between the planes as the length of the orthogonal projection of u on n

2. $-\frac{2}{25}$

-------------------- **Another scenario** ----------------
1p: Correctly carried out the matrix product AA^T
1p: Correctly found the inverse of the matrix A, and correctly carried out the matrix product $(\mathsf{AA}^T)\mathsf{A}^{-1}$
3p: Correctly found the value of the determinant of $\frac{1}{5}\mathsf{AA}^T\mathsf{A}^{-1}$, where a correct treatment of the factor $1/5$ gives one (**1**) of the total of three points

------------------------------ **One scenario** ------------------------------
1p: Correctly applied the product rule for determinants giving that the determinant equals $\det(\frac{1}{5}\mathsf{A})\det(\mathsf{A}^T)\det(\mathsf{A}^{-1})$
1p: Correctly treated the matrix factor $1/5$
1p: Correctly treated the determinant factor $\det(\mathsf{A}^T)$
1p: Correctly treated the determinant factor $\det(\mathsf{A}^{-1})$
1p: Correctly found the value of the determinant

3. $-2\sqrt{3}+2i = 4\,[\cos(\frac{5\pi}{6}) + i\cos(\frac{5\pi}{6})]$



1p: Correctly interpreted the inequality $|z| \le 4$ geometrically
1p: Correctly interpreted the inequality $\mathrm{Im}(z) \ge 2$ geometrically, and correctly sketched the region Ω
1p: Correctly concluded about the imaginary part and the absolute value of the complex number z_1 wich has the smallest real part in Ω
1p: Correctly found a representative argument of z_1
1p: Correctly found the real part of z_1, and correctly summarized the complex number on rectangle and polar forms

4. $(x,y,z)_{\beta\ne-1} = (0,-1,2),\quad t \in R$

$$(x,y,z)_{\beta=-1} = (r,-1-r,2+r),\quad r\in R$$
$$= (-1-s,s,1-s),\quad s\in R$$
$$= (-2+t,1-t,t),\quad t\in R$$

1p: Correctly concluded that the solving of the system of linear equations has to be divided into two cases, namely $\beta \ne -1$ and $\beta = -1$, for which different solution scenarios are found
2p: Correctly found the (unique) triple if $\beta \ne -1$
2p: Correctly found the (parametric) triples if $\beta = -1$

5. $\lambda : (x,y,z) = (3,-5,4) + r(4,7,-6)$
$r \in R$

2p: Correctly concluded that a vector v parallel with the line λ is found as the vector product of n_2 and v_3, where n_2 is a normal vector of the plane π_2 (e.g. $(1,2,3)$) and v_3 is a vector parallel with the line λ_3 (e.g. $(2,-2,-1)$)
1p: Correctly calculated a vector v parallel with the line λ
2p: Correctly formulated an equation for the line λ which includes the point $P_1 : (3,-5,4)$ and is parallel with the vector v