MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2018-01-04 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. The sum of two non-negative numbers x and y equals 4. Which is the smallest interval that surely contains the number $x^3 + 3y^2$?

2. For which x is the series

$$\ln(x) + \ln^2(x) + \ln^3(x) + \ldots$$

convergent? Find the sum of the series for these x.

3. Find the inverse of the function f defined by $f(x) = \dfrac{1}{\sqrt{x+2}}$. Especially, specify the domain and the range of the inverse.

4. Find out whether

$$\lim_{x \to +\infty} \left[\frac{x^3}{(x-1)^2} - \frac{x^3}{(x+1)^2} \right]$$

exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

5. Find the area of the bounded region which is precisely enclosed by the curves

$$\gamma_1 : y = -|x| \quad \text{and} \quad \gamma_2 : y = 2 - x^2.$$

6. Find to the differential equation $y'' - 8y' + 16y = 0$ the solution that satisfies the initial conditions $y(0) = 1$, $y'(0) = 7$.

7. Find to the function $x \curvearrowright f(x) = \dfrac{6}{x^2 - 9}$ the antiderivative F that have the value 0 at the point 0.

8. Prove that the function f defined by $f(x) = x\sqrt{\ln(x)}$ is invertible, and find the derivative of the inverse at the point $2e^4$.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN1
Datum: 2018-01-04 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{lcrcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Summan av två icke-negativa tal x och y är lika med 4. Vilket är det minsta intervall som garanterat innehåller talet $x^3 + 3y^2$?

2. För vilka x är serien

$$\ln(x) + \ln^2(x) + \ln^3(x) + \ldots$$

konvergent? Bestäm seriens summa för dessa x.

3. Bestäm inversen till funktionen f definierad enligt $f(x) = \dfrac{1}{\sqrt{x+2}}$. Specificera särskilt inversens definitionsmängd och värdemängd.

4. Utred om

$$\lim_{x\to+\infty} \left[\frac{x^3}{(x-1)^2} - \frac{x^3}{(x+1)^2} \right]$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

5. Bestäm arean av det begränsade område som precis innesluts av kurvorna

$$\gamma_1 : y = -|x| \quad \text{och} \quad \gamma_2 : y = 2 - x^2.$$

6. Bestäm till differentialekvationen $y'' - 8y' + 16y = 0$ den lösning som satisfierar begynnelsevillkoren $y(0) = 1$, $y'(0) = 7$.

7. Bestäm till funktionen $x \curvearrowright f(x) = \dfrac{6}{x^2 - 9}$ den primitiv F som har värdet 0 i punkten 0 .

8. Bevisa att funktionen f definierad enligt $f(x) = x\sqrt{\ln(x)}$ är inverterbar, och bestäm derivatan till inversen i punkten $2e^4$.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam. TEN1 2018-01-04

1 Define $f(x) = x^3+3y^2\Big|_{y=4-x} = x^3+3(4-x)^2$ with $D_f=[0,4]$

Then the range of f is the shortest interval that for sure contains the number x^3+3y^2 where $x,y\geq 0$ and $x+y=4$.

Diff. gives $f'(x) = 3x^2+6(4-x)(-1) = 3(x^2+2x-8) = 3(x+4)(x-2)$

Thus

x	0 (EP)		2 (SP)		4 (EP)
$f'(x)$		$-$	0	$+$	
$f(x)$	loc max	$\searrow$	loc min	$\nearrow$	loc max

and
$$\begin{cases} f_{min} = f(2) = 8+3\cdot 2^2 = 20 \\ f_{max} = \max(f(0), f(4)) = \max(48, 64) = 64 \end{cases}$$

gives that $R_f=[20,64]$

Answer: $[20,64]$

2 $\ln(x)+\ln^2(x)+\ln^3(x)+\dots$ is a geometric series

and converges iff $|\ln(x)|<1 \iff e^{-1}<x<e$.

The sum of the series is then $\ln(x)\,\dfrac{1}{1-\ln(x)}$

3 $f(x) = \dfrac{1}{\sqrt{x+2}}$ where $D_f = (-2,\infty)$ and $R_f=(0,\infty)$

Then
$$\begin{cases} f(x)=y \\ \dfrac{1}{\sqrt{x+2}} = y \end{cases} \iff \begin{cases} f(x)=y \\ \sqrt{x+2} = \dfrac{1}{y} \end{cases} \iff \begin{cases} x = f^{-1}(y) \\ x = \dfrac{1}{y^2}-2,\ y>0 \end{cases}$$

Thus $f^{-1}(x) = \dfrac{1}{x^2}-2$ where $\begin{cases} D_{f^{-1}} = R_f = (0,\infty) \\ R_{f^{-1}} = D_f = (-2,\infty) \end{cases}$

4
$$\frac{x^3}{(x-1)^2}-\frac{x^3}{(x+1)^2} = x^3\cdot\frac{(x+1)^2-(x-1)^2}{(x-1)^2(x+1)^2} = x^3\,\frac{(x+1+x-1)(x+1-(x-1))}{(x^2-1)^2}$$
$$= x^3\,\frac{2x\cdot 2}{(x^2-1)^2} = \frac{4x^4}{(x^2-1)^2}$$

Thus
$$\lim_{x\to+\infty}\left(\frac{x^3}{(x-1)^2}-\frac{x^3}{(x+1)^2}\right) = \lim_{x\to+\infty}\frac{4x^4}{x^4\left(1-\frac{1}{x^2}\right)^2}$$
$$= 4\cdot 1\,\frac{1}{(1-0)^2} = 4$$

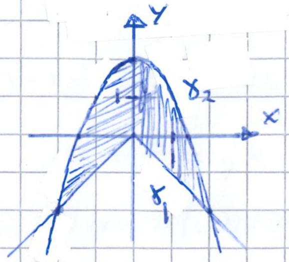
Cont. MAA151
TEN1 2018-01-04

5

$$\begin{cases}\gamma_1: \; y = -|x| \\ \gamma_2: \; y = 2 - x^2\end{cases}$$

$$\gamma_1 \cap \gamma_2: \begin{cases} y = -|x| \\ y = 2 - x^2 \end{cases} \Leftrightarrow \begin{cases} x \geq 0,\ y = -x,\ x^2 - x - 2 = 0 \\ \text{OR} \\ x < 0,\ y = x,\ x^2 + x - 2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x \geq 0,\ y = -x,\ (x+1)(x-2) = 0 \\ \text{OR} \\ x < 0,\ y = x,\ (x+2)(x-1) = 0 \end{cases} \Leftrightarrow \begin{cases} (x,y) = (2,-2) \\ \text{OR} \\ (x,y) = (-2,-2) \end{cases}$$



$$A = \int_{-2}^{2} \left[2 - x^2 - (-|x|)\right] dx = 2\int_0^2 (2 - x^2 + x)\, dx$$

$$= 2\left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_0^2 = 2\left[\left(4 - \frac{8}{3} + 2\right) - (0 - 0 + 0)\right] = \frac{20}{3}$$

Answer: $\frac{20}{3}$ a.u.

6

DE: $y'' - 8y' + 16y = 0$ IV:s $y(0) = 1,\ y'(0) = 7$

The auxiliary eq. is $0 = r^2 - 8r + 16 = (r-4)^2$

Thus the general solution of the DE is $y = (A + Bx)e^{4x}$

The IV:s give $\begin{cases} 1 = y(0) = (A + B\cdot 0)e^0 = A \\ 7 = y'(0) = (4A + B + 4B\cdot 0)e^0 = 4A + B \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 3 \end{cases}$

Thus $y = (3x+1)e^{4x}$ solves the IVP.

7

We know that $F'(x) = \frac{6}{x^2 - 9}$ and $F(0) = 0$. Then

$$F(x) - 0 = F(x) - F(0) = \int_0^x \frac{6\,dt}{t^2 - 9} = \int_0^x \left(\frac{1}{t-3} - \frac{1}{t+3}\right) dt$$

$$= \left[\ln\left|\frac{t-3}{t+3}\right|\right]_0^x = \ln\left|\frac{x-3}{x+3}\right| - \ln(1)$$

Since $\frac{3-0}{3+0} > 0$, we finally get that $F(x) = \ln\left(\frac{3-x}{3+x}\right)$

(where the interval of existence for F is $(-3,3)$)

8

$f(x) = x\sqrt{\ln(x)}$ Diff. gives $f'(x) = 1\cdot\sqrt{\ln(x)} + x\,\frac{1}{2}\,\frac{\frac{1}{x}}{\sqrt{\ln(x)}} = \frac{\ln(x) + \frac{1}{2}}{\sqrt{\ln(x)}}$

Since $f'(x) > 0$ in $(1,\infty)$ where $D_f = [1,\infty)$, we get that f is increasing in D_f and therefore is invertible q.e.d.

We note that $f(e^4) = e^4\sqrt{\ln(e^4)} = e^4\sqrt{4} = 2e^4$

and thus that $f^{-1}(2e^4) = e^4$. Since $f'(e^4) = \frac{4 + \frac{1}{2}}{\sqrt{4}} = \frac{9}{4} \neq 0$

and since f^{-1} is continuous (due to that f is cont. and monotonic)

we get that $(f^{-1})'(2e^4) = \frac{1}{f'(e^4)} = \frac{1}{9/4} = \frac{4}{9}$

Examination TEN1 – 2018-01-04 | Maximum points for subparts of the problems in the final examination

1. $x^3+3y^2 \in [20,64]$

1p: Correctly for the problem formulated a function of one variable including the specification of its domain
1p: Correctly found and concluded about the local extreme points of the function
1p: Correctly found the range of the function, and thereby the interval in which the number x^3+3y^2 lies

2. The series converges iff $x \in (e^{-1}, e)$.

The sum of the series equals $\dfrac{\ln(x)}{1-\ln(x)}$

1p: Correctly noted that the series is geometric, and correctly found the upper (non-included) limit of the interval of convergence
1p: Correctly found the lower (non-included) limit of the interval of convergence
1p: Correctly found the sum of the series

3. $f^{-1}(x) = -2 + x^{-2}$
$D_{f^{-1}} = (0,\infty)$, $R_{f^{-1}} = (-2,\infty)$

1p: Correctly found the expression of $f^{-1}(x)$
1p: Correctly found the domain of f^{-1}
1p: Correctly found the the range of f^{-1}

4. The limit exists and is equal to 4

1p: Correctly brought the terms together with a least common denominator and correctly simplified the numerator
1p: Correctly identified the dominating factors as $x \to +\infty$
1p: Correctly found the limit

5. $20/3$ a.u.

1p: Correctly found the intersection of the two enclosing curves, and correctly formulated an integral for the area
1p: Correctly found the needed antiderivatives
1p: Correctly found the limits in the integral and the area

6. $y = (3x+1)e^{4x}$

Note: The student who has stated that $y = Ae^{4x} + Be^{4x}$ is the general solution of the differential equation, and who has not found any explanation to the impossible conditions occuring when adapting to the initial values, can not obtain any other sum of points than **0p**.

1p: Correctly found one solution of the DE
1p: Correctly found the general solution of the DE
1p: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP

7. $F(x) = \ln\left(\dfrac{3-x}{3+x}\right)$

1p: Correctly found the partial fractions of $f(x)$
1p: Correctly found the general antiderivative of f
1p: Correctly adapted the antiderivative to the value at 0

8. f is invertible since $f'(x) > 0$ on D_f

$(f^{-1})'(2e^4) = \dfrac{1}{f'(e^4)} = 4/9$

1p: Correctly proved that f is invertible
1p: Correctly found that $f^{-1}(2e^4) = e^4$
1p: Correctly determined the value of $(f^{-1})'(2e^4)$