Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
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MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2016-08-18
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Låt T vara den linjära transformationen $T(x_1, x_2, x_3) = (1, -2, 1) \times (x_1, x_2, x_3)$. Bestäm standardmatrisen för T och $T \circ T$. (5p)

2 Låt $S_a = \{(1, 0, 1, -1), (1, 3a, 1, -1), (1, 3, 1 + a, -1)\}$, för $a \in \mathbb{R}$. Bestäm de värden på a sådana att S_a är linjärt beroende, och bestäm dimensionen av $\text{span}(S_a)$ för varje sådant a. (5p)

3 Den linjära transformationen T definieras av $T(\mathbf{x}) = A\mathbf{x}$ där

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

a. Bestäm en bas till värderummet till T. (3p)

b. Avgör om vektorn $(-1, 2, 1)$ tillhör T:s värderum. (2p)

4 Matrisen

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

har egenvärden $\lambda = 1$, $\lambda = -2$, och $\lambda = 4$.

a. Bestäm alla egenvektorer till A. (4p)

b. Avgör om A är diagonaliserbar. Motivera ditt svar. (1p)

5 Låt $S = \{(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), (0, 1, 0), (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\}$.

a. Visa att S är en ortonormal bas till $\mathbb{R}^3$. (3p)

b. Vektorn $\mathbf{v}$ har koordinater $(1, 2, 3)$ i standardbasen. Bestäm $\mathbf{v}$:s koordinater i basen S, alltså bestäm $(\mathbf{v})_S$. (2p)

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Examination Vector algebra
MAA150 - TEN2
Date: Aug 18, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Let T be the linear transformation $T(x_1, x_2, x_3) = (1, -2, 1) \times (x_1, x_2, x_3)$. Find the standard matrix for T and $T \circ T$. (5p)

2 Let $S_a = \{(1, 0, 1, -1), (1, 3a, 1, -1), (1, 3, 1 + a, -1)\}$, for $a \in \mathbb{R}$. Determine the values of a such that S_a is linearly dependent, and find the dimension of $\text{span}(S_a)$ for each such a. (5p)

3 The linear transformation T is defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

a. Find a basis for the range of T. (3p)

b. Determine if the vector $(-1, 2, 1)$ is in the range of T. (2p)

4 The matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

has eigenvalues $\lambda = 1$, $\lambda = -2$, and $\lambda = 4$.

a. Find all eigenvectors of A. (4p)

b. Determine if A is diagonalizable. Motivate your answer. (1p)

5 Let $S = \{(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), (0, 1, 0), (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\}$.

a. Show that S is an orthonormal basis for $\mathbb{R}^3$. (3p)

b. The vector $\mathbf{v}$ has coordinate vector $(1, 2, 3)$ in the standard basis. Find the coordinate vector of $\mathbf{v}$ relative to the basis S, that is find $(\mathbf{v})_S$. (2p)

MAA150 Vektoralgebra, vt-16.

Assessment criterias for TEN2 2016-08-18

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. [**5 points**]
   computing the cross product **(2p)**, finding the standard matrix for T **(2p)**, computing the standard matrix for $T \circ T$ **(1p)**

2. [**5 points**]
   the equation for linear dependence **(1p)**, relevant method that shows that they are linearly independent for $a \neq 0$ **(1p)**, correct motivation of linear dependence for $a = 0$ **(2p)**, correct dimension for $a = 0$ **(1p)**

3. [**5 points**]
   a. Relevant method with row operations **(2p)**, a correct basis **(1p)**
   b. correct motivation that the vector is in the range of T **(2p)**

4. [**5 points**]
   a. Relevant method for finding eigenvectors **(1p)**, each correct eigenvector 1p **(3p)**
   b. correct motivation **(1p)**

5. [**5 points**]
   a. checking that the vectors are orthonormal **(2p)**, checking they form a basis **(1p)**
   b. Finding $(\mathbf{v})_S$ **(2p)**

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1

$$T(x_1,x_2,x_3) = (1,-2,1) \times (x_1,x_2,x_3) =$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = (-2x_3 - x_2, -(x_3 - x_1), x_2 + 2x_1)$$

$$= (-2x_3 - x_2, -x_3 + x_1, x_2 + 2x_1) \quad (2p)$$

Standard matrix

$T(1,0,0) = (0,1,2)$, $T(0,1,0) = (-1,0,1)$

$T(0,0,1) = (-2,-1,0)$

So $[T] = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$. (2p)

and

$$[T \circ T] = [T]\cdot[T] = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \quad (1p)$$

$$= \begin{bmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{bmatrix}$$

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2. $S = \{(1,0,1,-1), (1,3a,1,-1), (1,3,1+a,-1)\}$

S is linearly dependent iff there there is a non-trivial solution to

$$k_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 3a \\ 1 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 3 \\ 1+a \\ -1 \end{bmatrix} = \bar{0}$$ (1p)

i.e at least one of the $k_i \neq 0$, where $i = 1,2,3$

1 −1 $\times\frac{1}{3}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3a & 3 \\ 1 & 1 & 1+a \\ -1 & -1 & -1 \end{bmatrix}}_{(*)}\left|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right. \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ (1p)

If $a \neq 0$ then $k_1 = k_2 = k_3 = 0$ is the only solution.

If $a = 0$ the row-echelon form of $(*)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has infinitely many solutions so S_0 is linearly dependent. (2p) The dimension is 2 since S_0 has 2 leading 1's. (1p)

Answer: S_a is linearly dependent for $a = 0$ and then $\dim(S_a) = 2$

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(3)

$T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

a) The range im(T) = col(A)

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{matrix} -2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{matrix} \\ 1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \times(-1) \\ \times(-\frac{1}{2}) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2p)

leading 1's

so $B = \{(1,2,0), (2,3,1), (1,1,-1)\}$
is a basis of col(A) = im(T)

(1p)

b) Since dim(im(T)) = 3 and $im(T) \subset \mathbb{R}^3$
we have $im(T) = \mathbb{R}^3$ so especially $(-1,2,1) \in im(T)$

(2p)

Answer: a) $B = \{(1,2,0), (2,3,1), (1,1,-1)\}$

b) Yes, $(-1,2,1)$ is in the range of T.

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(4) For Eigenvectors solve $(A-\lambda I)\bar{v}=\bar{0}$, $\bar{v}=(v_1,v_2,v_3)$

$$\lambda=1: \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3=t \Rightarrow v_2=t$; $v_1=-2t$, Taking $t=1$ gives

$\bar{u}_1=(-2,1,1)$ (2p)

$$\lambda=-2: \left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -8 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3=4t \Rightarrow v_2=-5t$, $v_1=10t-12t=-2t$

Taking $t=1$ gives $\bar{u}_2=(-2,-5,4)$ (1p)

$$\lambda=4: \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 1 & -4 & 3 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3=t \Rightarrow v_2=t$, $v_1=t$. Taking $t=1$ gives

$\bar{u}_3=(1,1,1)$ (1p)

Answer a) Eigenvalues and corresponding eigenvectors

(1) $\lambda_1=1$, $\bar{u}_1=(-2,1,1)$ (2) $\lambda_2=-2$, $\bar{u}_2=(-2,-5,4)$

(3) $\lambda_3=4$, $\bar{u}_3=(1,1,1)$

Answer b) A is digonalizeble since A have three distinct eigenvalues. (1p)

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(5) $S = \{ (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), (0,1,0), (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) \} = \{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \}$

$\bar{v} = (1,2,3)$

a) S is an orthonormal basis iff

i) S is an orthonormal set

ii) S is a basis

i)
$$\left.\begin{aligned}
&(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}) \bullet (0,1,0) = 0+0+0 = 0 \\
&(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}) \bullet (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}) = \tfrac{1}{2} + 0 - \tfrac{1}{2} = 0 \\
&(0,1,0) \bullet (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}) = 0+0+0 = 0
\end{aligned}\right\} \text{ ok!}$$

(2p)

$$\left.\begin{aligned}
&\|(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}})\| = \sqrt{(\tfrac{1}{\sqrt{2}})^2 + (-\tfrac{1}{\sqrt{2}})^2} = 1 \\
&\|(0,1,0)\| = \sqrt{1^2} = 1 \\
&\|(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}})\| = \sqrt{(\tfrac{1}{\sqrt{2}})^2 + (\tfrac{1}{\sqrt{2}})^2} = 1
\end{aligned}\right\} \text{ ok}$$

(Thm 6.3.1)

ii) S is a basis since, S is orthogonal $\Rightarrow$ S linearly indep. and 3 linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$ (Thm 4.5.4) (1p)

b) Since S is an orthonormal basis.

$$(\bar{v})_S \underset{\uparrow}{=} (\bar{v} \bullet \bar{v}_1, \bar{v} \bullet \bar{v}_2, \bar{v} \bullet \bar{v}_3) = \left(\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}, 2, \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} \right)$$

(Thm 6.3.2)

$$= (-\sqrt{2}, 2, 2\sqrt{2})$$ (2p)