MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Erik Darpö

TENTAMEN I MATEMATIK
MAA150 Vektoralgebra — TEN1
Datum: 9 januari 2015 — *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

*Denna tentamen TEN1 består av 7 uppgifter, med en sammanlagd poängsumma om 25 poäng. För betyget **3** krävs en erhållen poängsumma om minst 12 poäng, för betyget **4** krävs 16 poäng, och för betyget **5** krävs 20 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar (undantaget uppgift 1, där endast svar krävs). Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.*

1. Ange

a) ekvationen för den linje i rummet som innehåller punkterna $\begin{pmatrix}1\\1\\1\end{pmatrix}$ och $\begin{pmatrix}7\\5\\-3\end{pmatrix}$;

b) absolutbeloppet $|z|$ av det komplexa talet $z = \sqrt{6} - \sqrt{2}i$;

c) den polära formen av talet z från föregående deluppgift.

Endast svar krävs på uppgift 1. *(3p)*

2. Bestäm vektorprodukten $u \times v$ av vektorerna $u = \begin{pmatrix}1\\1\\0\end{pmatrix}$ och $v = \begin{pmatrix}0\\-1\\2\end{pmatrix}$. *(2p)*

3. Lös det linjära ekvationssystemet
$$\begin{cases} x_1 + x_2 - x_3 = 1, \\ x_1 + 2x_2 + x_4 = 0, \\ -x_1 + 2x_3 + x_4 = -2. \end{cases}$$
(4p)

4. Bestäm vinkeln mellan vektorerna u och v, givet att $\|u\| = \|v\| = \|u + v\| = 1$.

Not: $\|u\|$ *betecknar längden av vektorn* u *(skrivs ibland även som* $|u|$*).* *(4p)*

5. Vilka komplexa tal z uppfyller andragradsekvationen $z^2 + iz + \frac{7}{4} = 0$? *(4p)*

6. Avgör vilka av följande avbildningar som är linjära, och ange i förekommande fall deras matriser:

a) $F : \mathbb{R}^3 \to \mathbb{R}^2, \quad F\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-x_3\\x_1 - 2x_3\end{pmatrix};$

b) $G : \mathbb{R}^2 \to \mathbb{R}^3, \quad G\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 + x_2\\x_1x_2\\x_2 - x_1\end{pmatrix}.$

Glöm inte att förklara varför avbildningarna är, eller inte är, linjära. *(4p)*

7. Avgör om matrisen $A = \begin{pmatrix}1 & 0 & 2\\1 & 2 & 1\\0 & 1 & -1\end{pmatrix}$ är inverterbar, och bestäm i så fall dess invers.

(4p)

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Division of Applied Mathematics
Examiner: Erik Darpö

EXAMINATION IN MATHEMATICS
MAA150 Vector algebra TEN1
Date: 9th January 2015 *Time:* 3 hours
Materials allowed: Writing material only

This exam TEN1 consists of 7 problems, with a total score of 25 points. To obtain the grades **3**, **4** *and* **5**, *scores of at least 12, 16 respectively 20 points are required.*
The solutions to the problem 2–7 are to include motivations and clear answers to the questions asked. To problem 1, only correct answers are required.

1. Determine

a) the equation of the straight line in space that contains the points $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}7\\5\\-3\end{pmatrix}$;

b) the modulus $|z|$ of the complex number $z = \sqrt{6} - \sqrt{2}i$;

c) the polar form of the number z in the problem above.

Only answers are required to problem 1. *(3p)*

2. Determine the cross product $u \times v$ of the vectors $u = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $v = \begin{pmatrix}0\\-1\\2\end{pmatrix}$. *(2p)*

3. Solve the linear system of equations $\begin{cases} x_1 + x_2 - x_3 = 1, \\ x_1 + 2x_2 + x_4 = 0, \\ -x_1 + 2x_3 + x_4 = -2. \end{cases}$ *(4p)*

4. Determine the angle between the vectors u and v, given that $\|u\| = \|v\| = \|u+v\| = 1$.
Note: $\|u\|$ *denotes the length of the vector u (sometime also written as $|u|$).* *(4p)*

5. Which complex numbers z satisfy the quadratic equation $z^2 + iz + \frac{7}{4} = 0$? *(4p)*

6. Determine which of the following maps are linear, and find the matrices of those that are:

a) $F : \mathbb{R}^3 \to \mathbb{R}^2, \quad F\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-x_3\\x_1 - 2x_3\end{pmatrix}$;

b) $G : \mathbb{R}^2 \to \mathbb{R}^3, \quad G\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 + x_2\\x_1x_2\\x_2 - x_1\end{pmatrix}$.

Remember to explain why the maps are, or are not, linear. *(4p)*

7. Determine whether the matrix $A = \begin{pmatrix}1 & 0 & 2\\1 & 2 & 1\\0 & 1 & -1\end{pmatrix}$ is invertible and, in case it is, determine its inverse. *(4p)*

MAA150 Vector Algebra

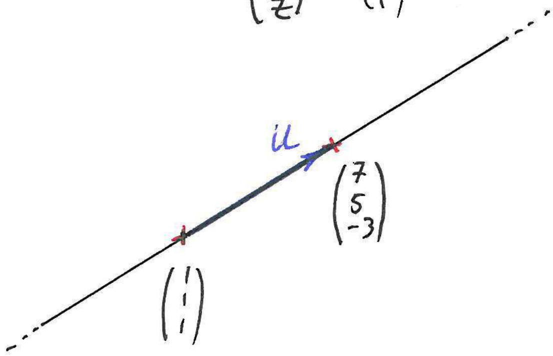
Solutions to the exam TEN1 9/1/2015

1.a) $u = \begin{pmatrix} 7 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix}$ is a vector parallel to the line.

Every point on the line can be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + tu$

for some $t \in \mathbb{R}$, i.e.,

$$\underline{\underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix}}}, \quad t \in \mathbb{R}$$



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b) $z = \sqrt{6} - \sqrt{2}i$

$$|z| = \sqrt{\sqrt{6}^2 + (-\sqrt{2})^2} = \sqrt{6+2} = \sqrt{8} = \underline{\underline{2\sqrt{2}}}$$

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c) $z = \sqrt{6} - \sqrt{2}i = 2\sqrt{2}\left(\frac{\sqrt{6}}{2\sqrt{2}} - \frac{\sqrt{2}}{2\sqrt{2}}i\right) = 2\sqrt{2}\cdot\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$$\underline{\underline{= 2\sqrt{2}\cdot\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}}$$

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2) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 2e_1 + 0e_2 - e_3 - 0e_3 - 0e_1 - 2e_2$

$$= 2e_1 - 2e_2 - e_3 = \underline{\underline{\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}}}$$

3) $$\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_1 + 2x_2 + x_4 = 0 \\ -x_1 + 2x_3 + x_4 = -2 \end{cases}$$

Augmented matrix:

pivot elements

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 2 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - 2x_3 - x_4 = 2 \\ x_2 + x_3 + x_4 = -1 \end{cases}$$

Set $x_3 = r$, $x_4 = t$

$$\begin{cases} x_1 = 2 + 2r + t \\ x_2 = -1 - r - t \\ x_3 = r \\ x_4 = t \end{cases} \qquad r, t \in \mathbb{R}$$

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4) $\|u\| = \|v\| = \|u+v\| = 1$

$$1 = \|u+v\|^2 = (u+v)\cdot(u+v) = u\cdot u + u\cdot v + v\cdot u + v\cdot v$$

$$= \underbrace{\|u\|^2}_{=1} + 2u\cdot v + \underbrace{\|v\|^2}_{=1} = 2 + 2u\cdot v$$

$$\Longrightarrow u\cdot v = \frac{1-2}{2} = -\frac{1}{2}.$$

By definition, $u\cdot v = \|u\|\|v\|\cos\theta$, where θ is the angle between u and v.

$\Longrightarrow u\cdot v = \cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$ (because, by convention, $0 \le \theta \le \pi$)

The angle between u and v is $\frac{2\pi}{3}$.

5) $z^2 + iz + \frac{7}{4} = 0$

$$\left(z+\frac{i}{2}\right)^2 - \left(\frac{i}{2}\right)^2 + \frac{7}{4} = 0$$

$$\left(z+\frac{i}{2}\right)^2 + \frac{1}{4} + \frac{7}{4} = 0$$

$$\left(z+\frac{i}{2}\right)^2 = -2$$

Set $w = z + \frac{i}{2}$, then $w^2 = -2$

$$w = \pm\sqrt{2}\, i$$

$$z = w - \frac{i}{2} = \pm\sqrt{2}\, i - \frac{i}{2}$$

So $z = \left(-\sqrt{2} - \frac{1}{2}\right) i$ or $z = \left(\sqrt{2} - \frac{1}{2}\right) i$

6a) $F: \mathbb{R}^3 \to \mathbb{R}^2$, $F(x) = \begin{pmatrix} -x_3 \\ x_1 - 2x_3 \end{pmatrix}$

Set $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}$. Then $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_1 - 2x_3 \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since F is given by multiplication with a matrix ($F(x) = Ax$), F is a linear map.

b) $G: \mathbb{R}^2 \to \mathbb{R}^3$, $G(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \\ x_2 - x_1 \end{pmatrix}$

$$G\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 1\cdot 0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad G\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 0\cdot 1 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow G\begin{pmatrix} 1 \\ 0 \end{pmatrix} + G\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

But $G\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1\cdot 1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \neq G\begin{pmatrix} 1 \\ 0 \end{pmatrix} + G\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Since every linear map must satisfy $G(x+y) = G(x) + G(y)$ for all x, y, it follows that G is not linear.

7) $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

A is invertible $\iff$ rank $A = 3 \iff \operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In this case, the same row operations that transform A into the identity matrix I_3 will transform I_3 into A^{-1}:

$$(A \mid I_3) = \begin{array}{c} -1 \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} \\ \\ -1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \begin{array}{c} \\ -1 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} \\ \\ 2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 2 \end{array}\right)$$

$$\sim \begin{array}{c} \\ \\ -1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 4 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 2 \end{array}\right) \sim \left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I_3} \middle| \underbrace{\begin{array}{ccc} 3 & -2 & 4 \\ -1 & 1 & -1 \\ -1 & 1 & -2 \end{array}}_{A^{-1}}\right)$$

So A is invertible, and

$$A^{-1} = \begin{pmatrix} 3 & -2 & 4 \\ -1 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$$

Answers to examination TEN1 in MAA150 Vector algebra 9th January 2015

1. a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix}, \quad t \in \mathbb{R}.$

b) $|z| = 2\sqrt{2} = \sqrt{8}.$

c) $z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$

2. $u \times v = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

3. $\begin{cases} x_1 = & 2 + 2r + t, \\ x_2 = & -1 - r - t, \\ x_3 = & r, \\ x_4 = & t, \end{cases} \qquad \text{where } r, t \in \mathbb{R}.$

4. The angle is $2\pi/3$.

5. $z = -i\left(\frac{1}{2} \pm \sqrt{2}\right)$

6. The map F is linear, and has the matrix $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \end{pmatrix}$. The map G is not linear.

7. The matrix A is invertible, and $A^{-1} = \begin{pmatrix} 3 & -2 & 4 \\ -1 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$.

MAA150 Vector algebra autumn term 2014

Assessment criteria for TEN1 9/1/2015

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1. One point each per correct answer.

2. Two points for complete solution. An incomplete solution that shows some degree of understanding of how to calculate the vector product may give one point, a correct answer with insufficent motivation may also give one point. A correct answer alone does not merit any points at all.

3. A total of three points is given for the reduction to row-reduced echelon form, one point for correct interpretation of the same. A minimum of one point is given if at least one relevant row operation is performed. Only writing down the augmented matrix of the system gives no points.

4. Two points for determining $u \bullet v$, two points for finding the angle using the formula $u \bullet v = \|u\|\|v\| \cos\theta$.

5. One point for rewriting the equation as $(z + \frac{i}{2})^2 + 2 = 0$ (or similar formulation), two points for solving the equation $w^2 = -2$, and another point for determining the solution of the original equation.

6. Two points each. For full score, it is necessary to prove the statements made about F and G. Correct answer with no or insufficient motivation gives one point on each part.

7. In principle, two points for detecting that A is invertible, and two points for finding the inverse. Writing up the matrix $(A \mid \mathrm{I}_3)$ and starting to do row operations on it gives one point. Repeated miscalculations, or mistakes of more principal character, may result in loss of points.