MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2017-08-18 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the volume of the solid generated by rotating the circular disc

$$x^2 + (y-1)^2 \leq 1$$

about the x-axis.

2. Is the series

$$\sum_{n=0}^{\infty} \frac{\sin((2n+1)\frac{\pi}{2})}{2n+1}$$

absolutely convergent, conditionally convergent or divergent?

NOTE: Do not forget that an answer must be accompanied by a relevant justification.

3. Solve the initial-value problem $\begin{cases} y'\sin(x) = y^2\cos(x), \\ y(\frac{\pi}{2}) = 1\,. \end{cases}$

4. Evaluate the integral

$$\int_2^3 \frac{dx}{x^2(x-1)}\,,$$

and write the result in as simple form as possible.

5. Let the function f_β be defined by

$$f_\beta(x) = \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2 + \beta\,, \quad x \in R\,,$$

where β is a real-valued parameter. Find the number of zeroes of f_β for every value of β.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2017-08-18 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm volymen av den kropp som genereras genom att rotera den cirkulära skivan

$$x^2 + (y-1)^2 \leq 1$$

kring x-axeln.

2. Är serien

$$\sum_{n=0}^{\infty} \frac{\sin((2n+1)\frac{\pi}{2})}{2n+1}$$

absolut konvergent, betingat konvergent eller divergent?

NOTERA: Glöm inte att ett svar måste åtföljas av en relevant motivering.

3. Lös begynnelsevärdesproblemet $\begin{cases} y'\sin(x) = y^2\cos(x), \\ y(\frac{\pi}{2}) = 1\,. \end{cases}$

4. Beräkna integralen

$$\int_2^3 \frac{dx}{x^2(x-1)}\,,$$

och skriv resultatet på en så enkel form som möjligt.

5. Låt funktionen f_β vara definierad genom

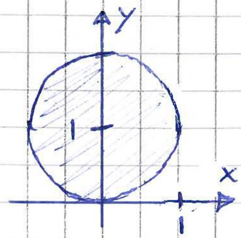
$$f_\beta(x) = \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2 + \beta\,, \quad x \in R\,,$$

där β är en reellvärd parameter. Bestäm antalet nollställen till f_β för varje värde på β.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam. TEN2 2017-08-18

(1)



The upper half of the circle is given by $y-1=\sqrt{1-x^2}$ and the lower half by $y-1=-\sqrt{1-x^2}$.
The volume V_x of the solid generated by rotating the circular disc $x^2+(y-1)^2\le 1$ about the x-axis is by the method of slicing given by

$$V_x=\int_{-1}^{1}\pi\left[\left(1+\sqrt{1-x^2}\right)^2-\left(1-\sqrt{1-x^2}\right)^2\right]dx=2\int_0^1\pi\, 4\sqrt{1-x^2}\,dx$$

$$=8\pi\underbrace{\int_0^1\sqrt{1-x^2}\,dx}_{\text{area measure of a }\frac14\text{ circle disc of radius 1}}=8\pi\left(\tfrac14\pi\cdot 1^2\right)=2\pi^2$$

Answer: $2\pi^2$ v.u.

(2) $\sum_{n=0}^{\infty}a_n$ where $a_n=\dfrac{\sin\left((2n+1)\frac{\pi}{2}\right)}{2n+1}=\dfrac{\sin\left(n\pi+\frac{\pi}{2}\right)}{2n+1}=\dfrac{(-1)^n}{2n+1}$

The series is not absolutely convergent since $\sum_{n=0}^{\infty}\frac{1}{2n+1}$ is divergent according to the comparison test (in a comparison with e.g. $\sum_{n=0}^{\infty}\frac{1}{n+1}=\sum_{m=1}^{\infty}\frac{1}{m}$ which is divergent according to the integral test).
However, the series $\sum_{n=0}^{\infty}a_n$ converges according to the alternating series test

all three conditions are satisfied
1) $a_n\cdot a_{n+1}<0$ i.e. altern. signs
2) $|a_{n+1}|\le|a_n|$ i.e. the terms are non-increasing
3) $a_n\to 0$ as $n\to\infty$

Thus The series $\sum_{n=0}^{\infty}a_n$ is conditionally convergent

(3) DE: $y'\sin(x)=y^2\cos(x)$ IV: $y\left(\frac{\pi}{2}\right)=1$

The DE is nonlinear but separable. For $y>0$ and $0<x<\pi$, the DE is $\frac{1}{y^2}y'=\frac{\cos(x)}{\sin(x)}$. Working out $\int dx$ on both sides gives $-\frac{1}{y}=\ln|\sin(x)|+C$ for which the IV gives $-\frac{1}{1}=\ln\left(\sin\left(\frac{\pi}{2}\right)\right)+C \Leftrightarrow C=-1$, where the absolute value bars obviously may be removed without any compensating sign (since $\sin(x)$ is to be treated in a neighbourhood of $\frac{\pi}{2}$ where $\sin(x)>0$). We get finally that

$y=\dfrac{1}{1-\ln(\sin(x))}$ solves the IVP in the interval $(0,\pi)$.

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$$\int_2^3 \frac{dx}{x^2(x-1)} = \int_2^3 dx\, \frac{1}{x}\left(\frac{1}{x-1} - \frac{1}{x}\right) = \int_2^3 dx \left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\right)$$

$$= \Big[\ln(x-1) - \ln(x) + \frac{1}{x^2}\Big]_2^3 = \Big[\ln\left(\frac{x-1}{x}\right) + \frac{1}{x}\Big]_2^3$$

$$= \left(\ln\left(\frac{2}{3}\right) + \frac{1}{3}\right) - \left(\ln\left(\frac{1}{2}\right) + \frac{1}{2}\right)$$

$$= \ln\left(\frac{2}{3}\right) + \ln(2) + \frac{1}{3} - \frac{1}{2} = \underline{\ln\left(\frac{4}{3}\right) - \frac{1}{6}}$$

5 $f_\beta(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 + \beta, \quad x \in R$

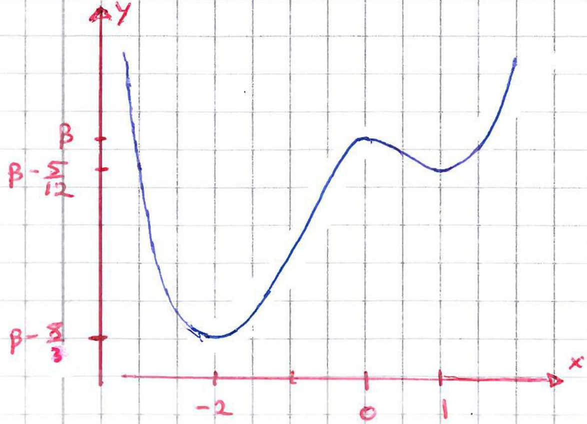
The function f_β has at most four zeroes (since it is a fourth-order polynomial function). Since $f_\beta(x) \to +\infty$ as $x \to \pm\infty$, the number of zeroes depends only on the locations of the local extreme values of f_β. Therefore, we work out a first-derivative test to get an overview of the values of the (continuous) function f_β.

Diff. of f_β gives: $f_\beta'(x) = x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1)$

x		-2		0		1	
$f_\beta'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f_\beta(x)$	$\searrow$	loc. min.	$\nearrow$	loc. max.	$\searrow$	loc. min.	$\nearrow$

$$\begin{cases} f_\beta(-2) = 4 - \frac{8}{3} - 4 + \beta = \beta - \frac{8}{3} \\ f_\beta(0) = 0 + 0 - 0 + \beta = \beta \\ f_\beta(1) = \frac{1}{4} + \frac{1}{3} - 1 + \beta = \beta - \frac{5}{12} \end{cases}$$

Irrespective of the location in the y-direction, the graph of f_β has the following appearance



Case	The number of zeroes of f_β
$\beta > \frac{8}{3}$	0
$\beta = \frac{8}{3}$	1
$\frac{5}{12} < \beta < \frac{8}{3}$	2
$\beta = \frac{5}{12}$	3
$0 < \beta < \frac{5}{12}$	4
$\beta = 0$	3
$\beta < 0$	2

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN2 – 2017-08-18 Maximum points for subparts of the problems in the final examination

1.	$2\pi^2$ v.u.	**2p**: Correctly formulated an integral for the volume of the solid obtained by rotating the region about the *x*-axis (irrespective whether the method of slicing or the method of cylindrical shells have been applied) **2p**: Correctly found the value of the integral
2.	The series is conditionally convergent	**2p**: Correctly found that the series satisfies the three conditions for being convergent according to Leibniz' criteria **2p**: Correctly found that the series is not absolutely convergent
3.	$y = \dfrac{1}{1-\ln(\sin(x))}$ for $x \in (0,\pi)$	**1p**: Correctly identified the differential equation as nonlinear and separable, and correctly in the separated differential equation found an antiderivative of $y^{-2}\, dy/dx$ **1p**: Correctly found the general antiderivative of the other side of the separated differential equation **1p**: Correctly adapted the equation of antiderivatives to the initial value **1p**: Correctly solved for y (including a correct motivation for the final choice of sign in the argument of the logarithmic function, all in order to adapt the solution to the initial value given and the implication thereof)
4.	$\ln(\frac{4}{3}) - \frac{1}{6}$	**1p**: Correctly decomposed the integrand into partial fractions **2p**: Correctly found an antiderivative of the integrand **1p**: Correctly evaluated the antiderivative at the limits
5.	The number of zeroes of f_β is equal to $\begin{cases} 2 & \text{if} \quad \beta < 0 \\ 3 & \text{if} \quad \beta = 0 \\ 4 & \text{if} \quad 0 < \beta < \frac{5}{12} \\ 3 & \text{if} \quad \beta = \frac{5}{12} \\ 2 & \text{if} \quad \frac{5}{12} < \beta < \frac{8}{3} \\ 1 & \text{if} \quad \beta = \frac{8}{3} \\ 0 & \text{if} \quad \beta > \frac{8}{3} \end{cases}$	**1p**: Correctly found and classified the local extreme points of the function f_β, all with the purpose of finding how the number of zeroes of f_β depends on β **1p**: Correctly, in two of the seven cases, found the number of zeroes of f_β **1p**: Correctly, in two more of the seven cases, found the number of zeroes of f_β **1p**: Correctly, in the three last of the seven cases, found the number of zeroes of f_β