TEN1 2018-01-09
MMA130 Mathematical Logic for Computer Science
Duration: 3 hours
Tools: none
Passing grade requires 15 p or more

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Questions may be answered in English or Swedish. Solutions should be presented in such a way that reasoning and calculations are easy to follow. All answers must be supported by an argument, e.g., if you claim something has a particular property then you must also show that this is the case.

1 A basic board of traffic lights has one red, one yellow (amber, orange), and one green light. Let r, y, and g be propositional atoms denoting that the red, yellow, and green respectively lights are on in a particular group of lights.

a In *English signalling order*, the lights cycle through the states of red, red+yellow, green, and yellow before returning to red. Express as a propositional logic formula the claim that the lights are in one of these four states. (1 p)

b Express in linear time temporal logic the claim that all future states of the lights are among these four states. (1 p)

c Express in linear time temporal logic the claim that the lights will always eventually show green. (1 p)

d Express in linear time temporal logic the claim that at any time when the light is green, it will remain green until switching to yellow. (2 p)

e Interpret in natural language the claim $\mathrm{F}(y \to \mathrm{X}r \land \mathrm{XX}r \land \mathrm{XXX}r)$. (1 p)

2 Give a natural deduction proof of $p \to (q \to r) \vdash (q \land p) \to r$. Provide justifications of all steps. (6 p)

3 A natural deduction proof in the predicate calculus with equality has the steps

1. z_2
2. $\quad z_1 = z_2$
3. $\quad z_2 = z_2$
4. $\quad z_2 = z_1$
5. $\quad z_1 = z_2 \to z_2 = z_1$
6. $\quad \forall z_2\, (z_1 = z_2 \to z_2 = z_1)$

(in mathematics generally, one would typically go on to state symmetry of the $=$ relation as the closed formula $\forall z_1 \forall z_2\, (z_1 = z_2 \to z_2 = z_1)$, but that extra proof step would not exercise any new feature of the predicate calculus, so we don't bother with that here). **Provide detailed justifications for the steps in this proof**—explain for each step which earlier steps (if any) you make use of, which rule (in the accompanying list of rules) you use, what each metasyntactic variable (e.g. formula ϕ, term t, or variable x) in that rule comes out as in this particular application of the rule—and draw scope boxes for rules that have them. (6 p)

4 One approach to formalising the *Zebra puzzle* in predicate logic (as opposed to formalising it in propositional logic, as was done in one lecture) is to let variables range over house numbers, whereas properties of houses (or their inhabitants, as appropriate) are expressed using predicates.

a Construct a predicate logic language that allows you to express the following claims, and sketch a standard interpretation/model for that language:

> The Englishman lives in the red house.
> The Norwegian lives next to the blue house.
> Tea is drunk in the first house.
> The Englishman and the Norwegian do not live in the same house.
> There is exactly one blue house.

(The full puzzle would require more claims, and most likely also a larger language than the one you're constructing here, but this fragment at least tests the principle.) (4 p)

b Describe a nonstandard model which, when satisfying the claims above (as you encoded them), would have some property not expected in the standard solution; it could for example be that tea is drunk in more than one house. Then suggest an additional axiom which would rule out that model. (2 p)

5 Consider the Kripke model $M = (W, R, L)$ where $W = \{a, b, c\}$, $R = \{(a, b), (b, c), (a, c), (c, a)\}$, $L(a) = \{p\}$, $L(b) = \{p, q\}$, and $L(c) = \{r\}$.

a Draw a graph for M. (2 p)

b Determine the set of worlds where $\Box\Box p$ is satisfied. (1 p)

c Determine the set of worlds where $\Box(\Diamond p \to r)$ is satisfied. (1 p)

d Parse the formula $\mathrm{A}[p \mathrel{\mathrm{U}} r]$ (what logic is it in? What is its structure?) and determine the set of worlds/states in M where it is satisfied. (2 p)

Good luck!

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MMA130, Mathematical Logic for Computer Science

Collection of Formulas

Anna Fedyszak-Koszela

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1 Natural deduction - basic rules

1. $(\wedge\ i)$: $\dfrac{\phi \qquad \psi}{\phi \wedge \psi}$

2. $(\wedge\ e)$: $\dfrac{\phi \wedge \psi}{\phi}$, $\dfrac{\phi \wedge \psi}{\psi}$

3. $(\vee\ i)$: $\dfrac{\phi}{\phi \vee \psi}$, $\dfrac{\psi}{\phi \vee \psi}$

4. $(\vee\ e)$: $$\dfrac{\phi \vee \psi \quad \begin{bmatrix} \phi \\ \vdots \\ \chi \end{bmatrix} \quad \begin{bmatrix} \psi \\ \vdots \\ \chi \end{bmatrix}}{\chi}$$

5. $(\rightarrow i)$: $$\dfrac{\begin{bmatrix} \phi \\ \vdots \\ \psi \end{bmatrix}}{\phi \rightarrow \psi}$$

6. $(\rightarrow e)$: $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$

7. $(\neg\ i)$: $$\dfrac{\begin{bmatrix} \phi \\ \vdots \\ \perp \end{bmatrix}}{\neg\phi}$$

8. $(\neg\, e)$: $\dfrac{\phi \quad \neg\phi}{\bot}$

9. $(\bot\, e)$: $\dfrac{\bot}{\phi}$

10. $(\neg\neg\, e)$: $\dfrac{\neg\neg\phi}{\phi}$

11. (MT): $\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}$

12. $(\neg\neg\, i)$: $\dfrac{\phi}{\neg\neg\phi}$

13. (PBC): $\dfrac{\begin{bmatrix} \neg\phi \\ \vdots \\ \bot \end{bmatrix}}{\phi}$

14. (LEM): $\dfrac{}{\phi \vee \neg\phi}$

15. $(copy)$: $\dfrac{\phi}{\phi}$

16. $(=\, i)$: $\dfrac{}{t = t}$, for any term t

17. *Principle of Substitution* $(=\, e)$:

$$\frac{t_1 = t_2 \qquad \phi\,[t_1/x]}{\phi\,[t_2/x]},$$

for t_1 free for x in ϕ, and for t_2 free for x in ϕ; all occurrences of t_1 in $\phi\,[t_1/x]$ are replaced by t_2

18. $(\forall x\; i)$:

$$\frac{\begin{bmatrix} x_0 & \\ & \phi\,[x_0/x] \end{bmatrix}}{\forall x \phi},$$

for x_0 - new, doesn't occur anywhere outside its box,
for x_0 - not free in open P before its box

19. $(\forall x\ e)$: $\dfrac{\forall x \phi}{\phi[t/x]}$,

for any term t free for x in ϕ; all free occurrences of x in ϕ are replaced by t

20. $(\exists x\ i)$: $\dfrac{\phi[t/x]}{\exists x \phi}$,

for some term t free for x in ϕ

21. $(\exists x\ e)$:

$$\dfrac{\exists x \phi, \quad \begin{bmatrix} x_0 & \phi[x_0/x] \\ & \vdots \\ & \chi \end{bmatrix} \begin{matrix} 1,\ P \\ \\ 2 \end{matrix}}{\chi}$$

for x_0 - new, doesn't occur anywhere outside its box;

1: x_0 not free in open P before its box,

2: x_0 not free in χ

2 Modal logic

- Let $M = (W, R, L)$ be a Kripke model of basic modal logic, $x \in W$, and ϕ be a formula. We will define when formula ϕ is true in the world x. This is done via a satisfaction relation $x \Vdash \phi$ by structural induction on ϕ:

  1. $x \Vdash T$
  2. *not* $(x \Vdash \bot)$
  3. $x \Vdash p$ iff $p \in L(x)$
  4. $x \Vdash \neg\phi$ iff *not* $(x \Vdash \phi)$
  5. $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
  6. $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
  7. $x \Vdash \phi \rightarrow \psi$ iff $x \Vdash \psi$, whenever we have $x \Vdash \phi$
  8. $x \Vdash \phi \leftrightarrow \psi$ iff $(x \Vdash \phi$ iff $x \Vdash \psi)$
  9. $x \Vdash \Box\psi$ iff, for each $y \in W$ with $R(x, y)$, we have $y \Vdash \psi$
  10. $x \Vdash \Diamond\psi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \psi$

  When $x \Vdash \phi$ holds, we say 'x satisfies ϕ' or 'ϕ is true in world x'.

- A model $M = (W, R, L)$ of basic modal logic is said to satisfy a formula ϕ if every state (world) x in the model satisfies it.
  We write: $M \models \phi$ iff, for each $x \in W$, $x \Vdash \phi$.

3 Logic systems with multiple truth values

- Let L be a formal language without non-logical symbols and S_L - the set with all sentences in L. For every positive integer n we define the non-classical logic system $L_n = (L, V_n)$ with $n+1$ truth values and with the valuation $V_n : S_L \longrightarrow \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ by following 5 conditions:

  1. $V_n(\neg\phi) = 1 - V_n(\phi)$ for any sentence $\phi \in S_L$
  2. $V_n(\phi \wedge \psi) = \min\{V_n(\phi), V_n(\psi)\}$ for any two sentences $\phi, \psi \in S_L$
  3. $V_n(\phi \vee \psi) = \max\{V_n(\phi), V_n(\psi)\}$ for any two sentences $\phi, \psi \in S_L$
  4. $V_n(\phi \rightarrow \psi) = \min\{1, (1 - V_n(\phi) + V_n(\psi))\}$ for any two sentences $\phi, \psi \in S_L$
  5. $V_n(\phi \leftrightarrow \psi) = 1 - |V_n(\phi) - V_n(\psi)|$ for any two sentences $\phi, \psi \in S_L$

- L_2 is a non-classical logic with 3 truth values $\{0, \frac{1}{2}, 1\}$:

ϕ	φ	$\neg\phi$	$\phi \wedge \varphi$	$\phi \vee \varphi$	$\phi \rightarrow \varphi$
$x = V(\phi)$	$y = V(\varphi)$	$1 - x$	$min\{x, y\}$	$max\{x, y\}$	$min\{1, 1 - x + y\}$
1	1	0	1	1	1
1	1/2	0	1/2	1	1/2
1	0	0	0	1	0
1/2	1	1/2	1/2	1	1
1/2	1/2	1/2	1/2	1/2	1
1/2	0	1/2	0	1/2	1/2
0	1	1	0	1	1
0	1/2	1	0	1/2	1
0	0	1	0	0	1