MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2015-06-12 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$$\begin{array}{lcrcl}
S_1 \geq 11,\ S_2 \geq 9 & \text{AND} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\
S_1 \geq 11,\ S_2 \geq 9 & \text{AND} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\
 & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \\
 & & & & \\
S_1 \geq 11,\ S_2 \geq 9 & \text{AND} & S_1 + 2S_2 \leq 32 & \rightarrow & \text{E} \\
S_1 \geq 11,\ S_2 \geq 9 & \text{AND} & 33 \leq S_1 + 2S_2 \leq 41 & \rightarrow & \text{D} \\
S_1 \geq 11,\ S_2 \geq 9 & \text{AND} & 42 \leq S_1 + 2S_2 \leq 51 & \rightarrow & \text{C} \\
 & & 52 \leq S_1 + 2S_2 \leq 60 & \rightarrow & \text{B} \\
 & & 61 \leq S_1 + 2S_2 & \rightarrow & \text{A}
\end{array}$$

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Evaluate the integral

$$\int_{-5/2}^{-1} \frac{dx}{\sqrt{-3-4x-x^2}}\,,$$

and write the result in as simple form as possible.

2. Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(3x-6)^n}{\sqrt{n}}.$$

3. Find the volume of the solid generated by rotating about the x-axis the bounded region precisely enclosed by the curves $y-1=\sqrt{x-1}$ and $y=x$.

4. Solve the initial-value problem

$$y'' + 3y' + 2y = e^{-2x}, \quad y(0)=5\,, \quad y'(0)=-9\,.$$

5. Find all local extremes values of the function f, defined by

$$f(x) = \big[x\ln(x)\big]^2\,,$$

and determine whether there is any maximum value and/or any minimum value.

Om du föredrar frågorna skrivna på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2015-06-12 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Beräkna integralen

$$\int_{-5/2}^{-1} \frac{dx}{\sqrt{-3-4x-x^2}}\,,$$

och skriv resultatet på en så enkel form som möjligt.

2. Bestäm konvergensintervallet till potensserien

$$\sum_{n=1}^{\infty} \frac{(3x-6)^n}{\sqrt{n}}.$$

3. Bestäm volymen av den kropp som genereras genom att kring x-axeln rotera det begränsade område som precis innesluts av kurvorna $y-1=\sqrt{x-1}$ och $y=x$.

4. Lös begynnelsevärdesproblemet

$$y''+3y'+2y=e^{-2x}, \quad y(0)=5\,, \quad y'(0)=-9\,.$$

5. Bestäm alla lokala extremvärden för funktionen f, definierad enligt

$$f(x)=\big[x\ln(x)\big]^2\,,$$

och avgör om det finns något största värde och/eller något minsta värde.

If you prefer the questions written in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2015-06-12

(1)

$$\int_{-5/2}^{-1} \frac{dx}{\sqrt{-3-4x-x^2}} = \int_{-5/2}^{-1} \frac{dx}{\sqrt{1-(x+2)^2}} \begin{bmatrix} x+2 = \sin(\theta) \\ dx = \cos(\theta)\,d\theta \end{bmatrix} = \int_{-\pi/6}^{\pi/2} \frac{\cos(\theta)\,d\theta}{\sqrt{1-\sin^2(\theta)}}$$

$$= \int_{-\pi/6}^{\pi/2} \frac{\cos(\theta)\,d\theta}{\sqrt{\cos^2(\theta)}} = \int_{-\pi/6}^{\pi/2} \frac{\cos(\theta)\,d\theta}{|\cos(\theta)|} \overset{\cos(\theta)>0}{=} \int_{-\pi/6}^{\pi/2} d\theta = \frac{\pi}{2} - \left(-\frac{\pi}{6}\right) = \frac{2\pi}{3}$$

(2)

$$\sum_{n=1}^{\infty} \frac{(3x-6)^n}{\sqrt{n}} \qquad \text{Let } \frac{(3x-6)^n}{\sqrt{n}} = a_n(x)$$

- The series converges absolutely if

$$1 > \lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(3x-6)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(3x-6)^n} \right|$$

$$= 3|x-2| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 3|x-2|$$

i.e. if $|x-2| < \frac{1}{3} \iff -\frac{1}{3} < x-2 < \frac{1}{3} \iff \frac{5}{3} < x < \frac{7}{3}$

- The series diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| > 1$

i.e. if $|x-2| > \frac{1}{3} \iff x < \frac{5}{3} \lor x > \frac{7}{3}$

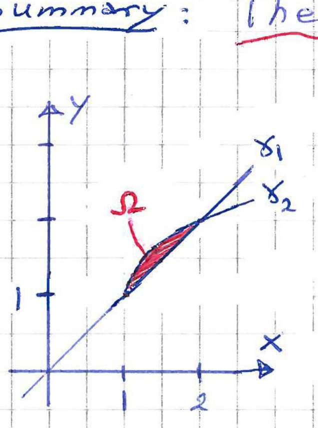
- For the remaining x, we have that

$x = \frac{5}{3}$: $\sum_{n=1}^{\infty} \frac{(5-6)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges according to the Leibniz' theorem (since ① a_n is alternating in sign, ② $|a_{n+1}| < |a_n|$ for all n, and ③ $a_n \to 0$ when $n \to \infty$).

$x = \frac{7}{3}$: $\sum_{n=1}^{\infty} \frac{(7-6)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges according to the integral theorem.

Summary: The interval of convergence is $\left[\frac{5}{3}, \frac{7}{3}\right)$

(3)



$\gamma_1: \; y = x \qquad \gamma_2: \; y - 1 = \sqrt{x-1}$

$$\gamma_1 \cap \gamma_2 : \begin{cases} y = x \\ x = 1 + \sqrt{x-1} \end{cases} \iff \begin{cases} y = x \\ \sqrt{x-1}\left(\sqrt{x-1} - 1\right) = 0 \end{cases}$$

$$\iff (x,y) = (1,1) \lor (x,y) = (2,2)$$

The volume of the solid generated by rotating about the x-axis the region Ω is

$$V_x = \pi \int_1^2 \left[\left(1+\sqrt{x-1}\right)^2 - x^2\right] dx \begin{bmatrix} x-1 = u \\ dx = du \end{bmatrix} = \pi \int_0^1 \left[\left(1+\sqrt{u}\right)^2 - (1+u)^2\right] du$$

$$= \pi \int_0^1 \left(2\sqrt{u} - u - u^2\right) du = \pi \left[\frac{4}{3}u^{3/2} - \frac{1}{2}u^2 - \frac{1}{3}u^3\right]_0^1 = \pi\left(\frac{4}{3} - \frac{1}{2} - \frac{1}{3}\right) \text{ v.u.} = \frac{\pi}{2} \text{ v.u.}$$

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④ <u>DE</u>: $y'' + 3y' + 2y = e^{-2x}$ IV: $\begin{cases} y(0) = 5 \\ y'(0) = -9 \end{cases}$

The general solution of the DE is $y_h + y_p$, <u>where</u> y_h is the general solution of the corresponding homogeneous eq. $y'' + 3y' + 2y = 0$ and y_p is a particular solution of the DE.

The auxiliary equation of $y'' + 3y' + 2y = 0$ is $0 = r^2 + 3r + 2 = (r+2)(r+1)$ <u>i.e.</u> $y_h = C_1 e^{-2x} + C_2 e^{-x}$.

For y_p, we seek a solution of the form $x^1 a_0 e^{-2x}$, where the exponent of x is the multiplicity of -2 in e^{-2x} among the roots of the auxiliary equation and where $a_0 e^{-2x}$ is a generalization of the RHS of the DE.

<u>Differentiating y_p and substituting into the DE give</u>

$$\begin{cases} y_p' = a_0(1-2x)e^{-2x} \\ y_p'' = a_0(-4+4x)e^{-2x} \end{cases} \qquad [(\cancel{4x}-4) + 3(1-\cancel{2x}) + \cancel{2x}]\,a_0 e^{-2x} = e^{-2x}$$
i.e. $a_0 = -1$

<u>Thus</u> $y = y_h + y_p = C_1 e^{-2x} + C_2 e^{-x} - x e^{-2x}$ is the general solution

<u>The IV:s give</u>: $\begin{cases} 5 = C_1 + C_2 \\ -9 = -2C_1 - C_2 - 1 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = 5 \\ 2C_1 + C_2 = 8 \end{cases} \Leftrightarrow \begin{cases} C_1 = 3 \\ C_2 = 2 \end{cases}$

The (unique) solution of the IVP is thus $\underline{y = (3-x)e^{-2x} + 2e^{-x}}$

⑤ $f(x) = (x\ln(x))^2$ <u>Then</u> $f'(x) = 2[x\ln(x)]\left(1\cdot\ln(x) + x\cdot\frac{1}{x}\right)$
$= 2x\ln(x)(\ln(x)+1)$

<u>First derivative test</u>

x		e^{-1}		1	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	loc max	$\searrow$	loc min	$\nearrow$

<u>Also</u>

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} [x\ln(x)]^2 \overset{\text{rule}}{=} \left(\lim_{x\to 0^+} x\ln(x)\right)^2$

subst. and $\ln(\frac{1}{t}) = -\ln(t)$

$= \left(\lim_{t\to\infty} -\frac{\ln(t)}{t}\right)^2 = (-0)^2 = 0$

power of t dominates over $\ln(t)$

$f(x) \to \infty$ when $x \to \infty$

<u>i.e.</u> $\begin{cases} f(e^{-1}) = (e^{-1}\ln(e^{-1}))^2 = (-e^{-1})^2 = \underline{e^{-2} \text{ is a loc. max of } f} \\ f(1) = (1\cdot\ln(1))^2 = 0^2 = \underline{0 \text{ is a loc. min of } f} \end{cases}$

<u>There is no maximum value of f,</u>
<u>but 0 is the minimum value.</u>

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2014/15

Examination TEN2 – 2015-06-12 Maximum points for subparts of the problems in the final examination

1. $\dfrac{2\pi}{3}$

1p: Correctly completed the square in the argument of the root function and found the expression $1-(x+2)^2$

----- **One scenario for the other three points** -------

2p: Correctly found that $\arcsin(x+2)$ is an antiderivative of the integrand

1p: Correctly evaluated the antiderivative at the limits

------------ **Another scenario for the other three points** ----------------------

1p: Correctly, in a substitution $x+2=\sin(\theta)$, translated the integrand of the integral

1p: Correctly, in a substitution $x+2=\sin(\theta)$, translated the limits of the integral

1p: Correctly, in a substitution $x+2=\sin(\theta)$, simplified the integrand into the constant 1, and finally correctly determined the value of the integral

2. $[\frac{5}{3},\frac{7}{3})$

1p: Correctly, by e.g. the ratio test, found that the series is (absolutely) convergent for $|3x-6|<1$

1p: Correctly translated the $|3x-6|<1$ into $\frac{5}{3}<x<\frac{7}{3}$, and hopefully correctly mentioned that the series definitely is divergent for $|x-2|>\frac{1}{3} \Leftrightarrow (x<\frac{5}{3}) \vee (x>\frac{7}{3})$

1p: Correctly, by the Leibniz' theorem, found that the series is convergent for $x=\frac{5}{3}$ (by the integral test it also follows that the convergence is conditionally)

1p: Correctly, by e.g. the integral test, found that the series is divergent for $x=\frac{7}{3}$

3. $\dfrac{\pi}{2}$ v.u.

1p: Correctly determined the intersection of the two enclosing curves, and by this also the integration interval (irrespective whether the integration is performed with respect to x or y)

1p: Correctly formulated an integral for the volume obtained by rotating the region about the x-axis

1p: Correctly found the antiderivative of the integrand

1p: Correctly evaluated the limits of the integral

4. $y=(3-x)e^{-2x}+2e^{-x}$

1p: Correctly identified the differential equation a nonhomogeneous linear DE of second order, and correctly found the general solution y_h of the corresponding hom. DE

1p: Correctly proposed a working function form for a particular solution y_p of the DE

1p: Correctly differentiated the assumed y_p and correctly determined the exact form of the solution

1p: Correctly adapted the general solution to the initial values

5. 0 is a local minimum of f

e^{-2} is a local maximum of f

0 is also a minimum value of f

There is no maximum value of f

1p: Correctly found the local minimum of f

1p: Correctly found the local maximum of f

1p: Correctly concluded that the local minimum of f is also a (global) minimum

1p: Correctly concluded that there is no maximum of f