MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2016-09-29 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Evaluate the integral $\displaystyle\int_0^{\pi/2} \left|\sin(x - \tfrac{\pi}{6})\right|\, dx\,.$

2. Find out whether
$$\lim_{x\to 0} \frac{\cos(5x)}{x}$$
exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

3. Find an equation for the tangent line τ to the curve $\gamma : (x, y) = \left(e^{2t-2}, \frac{t}{t+1}\right)$ at the point P for which the y-coordinate is equal to $\frac{1}{2}$.

4. Let $f(x) = 1/\sqrt{x-1}$ and $g(x) = 1/x^2$. Find the function expression, the domain and the range of the composition $g \circ f$.

5. Find to the differential equation $y'' + 2y' + 2y = 0$ the solution that satisfies the initial conditions $y(0) = 1\,,\ y'(0) = -1$.

6. How large is the distance between the curve $\gamma : y = x^2,\ x \geq 0$ and the point $P : (0, \frac{5}{2})$, and which point on γ is closest to P?

7. Find the numerical sequence $\{a_n\}_{n=0}^{\infty}$ for which the power series $\sum_{n=0}^{\infty} a_n x^n$ has the sum $1/(3 - 5x)$. Also, find the interval of convergence of the power series.

8. Find the GENERAL antiderivative of $x \curvearrowright f(x) = \dfrac{\sqrt{x}}{x+1}\,.$

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{lcrcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Beräkna integralen $\displaystyle\int_0^{\pi/2} \left|\sin(x - \tfrac{\pi}{6})\right| dx$.

2. Utred om
$$\lim_{x\to 0} \frac{\cos(5x)}{x}$$
existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

3. Bestäm en ekvation för tangenten τ till kurvan $\gamma : (x, y) = \left(e^{2t-2}, \frac{t}{t+1}\right)$ i den punkt P för vilken y-koordinaten är lika med $\frac{1}{2}$.

4. Låt $f(x) = 1/\sqrt{x-1}$ och $g(x) = 1/x^2$. Bestäm funktionsuttrycket, definitionsmängden och värdemängden för sammansättningen $g \circ f$.

5. Bestäm till differentialekvationen $y'' + 2y' + 2y = 0$ den lösning som satisfierar begynnelsevillkoren $y(0) = 1$, $y'(0) = -1$.

6. Hur stort är avståndet mellan kurvan $\gamma : y = x^2,\ x \geq 0$ och punkten $P : (0, \frac{5}{2})$, och vilken punkt på γ ligger närmast P.

7. Bestäm den talföljd $\{a_n\}_{n=0}^{\infty}$ för vilken potensserien $\sum_{n=0}^{\infty} a_n x^n$ har summan $1/(3-5x)$. Bestäm även konvergensintervallet för potensserien.

8. Bestäm den GENERELLA primitiva funktionen till $x \curvearrowright f(x) = \dfrac{\sqrt{x}}{x+1}$.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2016-09-29

1. $$\int_0^{\pi/2} \left|\sin\left(x-\frac{\pi}{6}\right)\right| dx = \int_0^{\pi/6} \left[-\sin\left(x-\frac{\pi}{6}\right)\right] dx + \int_{\pi/6}^{\pi/2} \sin\left(x-\frac{\pi}{6}\right) dx$$

$$= \left[\cos\left(x-\frac{\pi}{6}\right)\right]_0^{\pi/6} + \left[-\cos\left(x-\frac{\pi}{6}\right)\right]_{\pi/6}^{\pi/2} = \left(\cos(0) - \cos\left(-\frac{\pi}{6}\right)\right)$$

$$+ \left(-\cos\left(\frac{\pi}{3}\right) + \cos(0)\right) = 1 - \frac{\sqrt{3}}{2} - \frac{1}{2} + 1 = \frac{3}{2} - \frac{\sqrt{3}}{2} = \underline{\frac{\sqrt{3}}{2}\left(\sqrt{3}-1\right)}$$

2. <u>The limit of $\frac{\cos(5x)}{x}$ does not exist as $x \to 0$</u>

since $\cos(5x) \to \cos(0) = 1$ as $x \to 0$, i.e. since

$$\left|\frac{\cos(5x)}{x}\right| \to +\infty \text{ as } x \to 0$$

3. $$\gamma : (x, y) = \left(e^{2t-2}, \frac{t}{t+1}\right) \equiv \left(f(t), g(t)\right)$$

$y_p = \frac{1}{2}$ gives $\frac{t_p}{t_p+1} = \frac{1}{2} \iff t_p = 1$ <u>Thus</u> $P : \left(1, \frac{1}{2}\right)$

<u>The slope of γ at P is</u> $k_p = \left.\frac{dy}{dx}\right|_p = \left.\left(\frac{dy}{dt}\frac{dt}{dx}\right)\right|_p$

<u>i.e.</u> $$k_p = \frac{g'(t_p)}{f'(t_p)} = \frac{\left(\frac{1}{(t_p+1)^2}\right)}{2e^{2t_p-2}} = \frac{\frac{1}{(1+1)^2}}{2e^0} = \frac{1}{8}$$

<u>Thus</u>, the tangent line τ to γ at P is given by the equation $\underline{y - \frac{1}{2} = \frac{1}{8}(x-1) \iff y = \frac{1}{8}(x+3)}$

4. $$f(x) = \frac{1}{\sqrt{x-1}}, \quad g(x) = \frac{1}{x^2}$$

$$\underline{g \circ f(x)} = g(f(x)) = g\left(\frac{1}{\sqrt{x-1}}\right) = \frac{1}{\left(\frac{1}{\sqrt{x-1}}\right)^2} = \underline{x-1}$$

$$\underline{D_{g\circ f}} = \{x : x \in D_f, f(x) \in D_g\}$$

$$= \left\{x : x \in (1, \infty), \frac{1}{\sqrt{x-1}} \neq 0\right\} = \underline{(1, \infty)}$$

$$\underline{R_{g\circ f}} = \{g\circ f(x) : x \in D_{g\circ f}\}$$

$$= \{x-1 : x \in (1,\infty)\} = \underline{(0, \infty)}$$

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(5) DE: $y''+2y'+2y=0$ IV: $y(0)=1$, $y'(0)=-1$

The auxiliary eq. is $0=r^2+2r+2=(r+1)^2+1=(r+1)^2-i^2=(r+1+i)(r+1-i)$

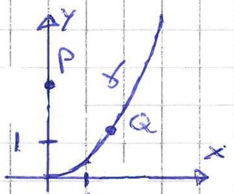
Thus, the general solution of the homogeneous linear 2nd order DE is $y=e^{-x}[C_1\cos(x)+C_2\sin(x)]$.

The IV:s give $\begin{cases}1=y(0)=1(C_1\cdot 1+C_2\cdot 0)\\ -1=y'(0)=-1(C_1\cdot 1+C_2\cdot 0)+1(-C_1\cdot 0+C_2\cdot 1)\end{cases} \Leftrightarrow \begin{cases}C_1=1\\ C_2=0\end{cases}$

Thus the solution of the IVP is $y=e^{-x}\cos(x)$

(6) γ: $y=x^2$, $x\geq 0$

P: $(0,\frac{5}{2})$



The distance $g(x)$ between a point $Q:(x,x^2)$, $x\geq 0$ on γ and the point P is $\sqrt{(x-0)^2+(x^2-\frac{5}{2})^2}$, $x\geq 0$. Then

$$g'(x)=\frac{1}{2}\frac{1}{g(x)}\left[2x+2\left(x^2-\frac{5}{2}\right)2x\right]=\frac{x}{g(x)}(1+2x^2-5)=\frac{2x}{g(x)}(x+\sqrt{2})(x-\sqrt{2})$$

A first deriv. test is

	EP		SP	
x	0		$\sqrt{2}$	
$g'(x)$		$-$	0	$+$
$g(x)$	loc. max	$\searrow$	loc. min	$\nearrow$

i.e. $g(x)\geq g(\sqrt{2})=\frac{3}{2}$

i.e. the distance between γ and P equals $\frac{3}{2}$ l.u., and the point with the coordinates $(\sqrt{2},2)$ is the point on γ closest to P.

(7) $$\frac{1}{3-5x}=\frac{1}{3}\frac{1}{1-\frac{5x}{3}}=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{5x}{3}\right)^n=\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{5}{3}\right)^n x^n$$

for $\left|\frac{5x}{3}\right|<1 \Leftrightarrow |x|<\frac{3}{5}$

i.e. $a_n=\frac{1}{3}\left(\frac{5}{3}\right)^n$, $n\geq 0$ are the terms of the sequence $\{a_n\}_{n=0}^{\infty}$, and the interval of convergence is $\left(-\frac{3}{5},\frac{3}{5}\right)$.

(8) $$\int f(x)\,dx=\int\frac{\sqrt{x}}{x+1}dx\left[\begin{array}{l}\sqrt{x}=u\\ x=u^2\\ dx=2u\,du\end{array}\right]=\int\frac{u\cdot 2u\,du}{u^2+1}$$

$$=2\int\frac{u^2+1-1}{u^2+1}du=2\int\left(1-\frac{1}{u^2+1}\right)du$$

$$=2(u-\arctan(u))+C=2\left(\sqrt{x}-\arctan(\sqrt{x})\right)+C$$

Examination TEN1 – 2016-09-29		Maximum points for subparts of the problems in the final examination
1.	$\frac{\sqrt{3}}{2}(\sqrt{3}-1)$	**2p**: Correctly taken account of the absolute value bars and correctly worked out the topical antiderivatives **1p**: Correctly evaluated the antiderivatives at the limits **Note**: The student who has not taken account of the absolute value bars obtains **0p**. The student who has found a negative value, and has not commented such a value as being unreasonable obtains **0p**. The student who has found a negative value for the integral, but at least has commented such a value as being unreasonable, can obtain **at most 1p**.
2.	The limit does not exist	**3p**: Correctly noted that the numerator of $\cos(5x)/x$ has a non-zero limit as $x \to 0$ in comparison with the denominator which has the limit zero, and from this correctly concluded that the limit does not exist
3.	$\tau: \; y = \frac{1}{8}(x+3)$	**1p**: Correctly found the parameter value t_P at the point P and also correctly found the first coordinate x_P **1p**: Correctly found the slope at the point P **1p**: Correctly formulated an equation for the tangent line τ to the curve γ at the point P
4.	$g \circ f(x) = x-1$ where $D_{g\circ f} = (1,\infty)$, $V_{g\circ f} = (0,\infty)$	**1p**: Correctly found the expression for $g \circ f(x)$ **1p**: Correctly found the domain of the composition $g \circ f$ **1p**: Correctly found the range of the composition $g \circ f$
5.	$y = e^{-x}\cos(x)$ **Note**: The student who, for the general solution, has found anything else but a linear combination of $e^{-x}\cos(x)$ and $e^{-x}\sin(x)$ obtains **0p**.	**1p**: Correctly found the general solution of the DE **1p**: Correctly differentiated the general solution in preparing for the adaption to the initial values **1p**: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP
6.	The distance between the curve γ and the point P equals $\frac{3}{2}$ l.u. The point with the coordinates $(\sqrt{2},2)$ is the point on γ closest to P.	**1p**: Correctly for the optimization problem formulated an explicit function of one variable measuring the distance between a point on the curve γ and the point P **1p**: Correctly, by completing the square in the argument of the square root function and by estimating the x-depending square to be ≥ 0, **OR** by a first derivative test, found the distance between the curve γ and the point P **1p**: Correctly found the coordinates of the point on γ closest to the point P
7.	$a_n = \frac{1}{3}(\frac{5}{3})^n$ for $n \geq 0$ The interval of convergence is $(-\frac{3}{5},\frac{3}{5})$	**1p**: Correctly expanded $1/(3-5x)$ in a power series in x **1p**: Correctly identified the coefficients of the power series **1p**: Correctly found the interval of convergence
8.	$\int f(x)\,dx = 2(\sqrt{x} - \arctan(\sqrt{x})) + C$	**1p**: Correctly applied a substitution which simplifies the finding of the general antiderivative of f **1p**: Correctly found an antiderivative of f **1p**: Correctly found the general antiderivative of f