Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
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Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2016-09-29
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Bestäm alla lösningar till ekvationssystemet (4p)

$$\begin{array}{rcrcrcrcl} 2x & + & 2y & + & 2z & + & 2w & = & 1 \\ -x & - & y & + & z & + & w & = & 2 \end{array}$$

2 Givet

$$A = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

a. Beräkna determinanten av A. (3p)

b. Bestäm inversen till A. (3p)

3 Låt Π vara planet som ges av ekvationen $x+2y+2z=-4$ och l linjen som på parameterform ges av $x=1-2t$, $y=2+t$, $z=3$.

a. Visa att linjen l löper parallellt med planet Π. (2p)

b. Bestäm avståndet mellan linjen och planet genom att använda lämplig projektion. (4p)

4 Bestäm ekvationen för planet som innehåller punkterna $A(3,-1,-2)$, $B(1,-1,0)$ och $C(1,0,1)$. Svara på punkt-normal form. (5p)

5 Avgör om påståendet är sant eller falskt genom att bevisa påståendet om det är sant, eller ge ett motexempel om det är falskt. (4p)

a. Om $\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathrm{proj}_{\mathbf{a}}\mathbf{v}$ gäller för något $\mathbf{a} \neq \mathbf{0}$ är $\mathbf{u} = \mathbf{v}$.

b. Om $\mathbf{u}$ och $\mathbf{v}$ är ortgonala vektorer i $\mathbb{R}^n$ gäller att $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

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Examination Vector algebra
MAA150 - TEN1
Date: September 29, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Find all solutions to the linear system (4p)

$$\begin{array}{rcrcrcrcl} 2x & + & 2y & + & 2z & + & 2w & = & 1 \\ -x & - & y & + & z & + & w & = & 2 \end{array}$$

2 Given

$$A = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

a. Evaluate the determinant of A. (3p)

b. Find the inverse of A. (3p)

3 Let Π be the plane given by $x + 2y + 2z = -4$ and l the line that has parametric equations $x = 1 - 2t$, $y = 2 + t$, $z = 3$.

a. Show that the line l runs parallel to the plane Π. (2p)

b. Determine the distance between the line and the plane by using appropriate projection. (4p)

4 Find the equation of the plane the contains the points $A(3, -1, -2)$, $B(1, -1, 0)$, and $C(1, 0, 1)$. Give the answer in point-normal form. (5p)

5 Determine if the statement is true or false by proving the statement if it is true, or giving a counter-example if it is false. (4p)

a. If $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{a}}\mathbf{v}$ holds for some $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u} = \mathbf{v}$.

b. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$, then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

MAA150 Vektoralgebra, HT2016.

Assessment criterias for TEN1 2016-09-29

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. [**4 points**]
   Relevant method, e.g. row operations on the augmented matrix **(2p)**, setting free variables to parameters **(1p)**, correct solution **(1p)**

2. [**6 points**]
   a. Computing the determinant using a relevant method **(2p)**, correct value **(1p)**
   b. Relevant method and row operations **(2p)**, correct inverse **(1p)**

3. [**6 points**]
   a. Correct condition for the line being parallel to the plane **(1p)**, checking that the condition is satisfied **(1p)**
   b. finding relevant vectors and figure maximum **(2p)**, computing the norm of a relevant projection **(2p)**

4. [**5 points**]
   Relevant method and vectors **(1p)**,computing the cross product **(2p)**, finding the point-normal form **(2p)**

5. [**4 points**]
   a. correct counterexample **(1p)**, properly explained **(1p)**
   b. proof **(2p)**

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(1)
$$(*)\begin{cases} 2x+2y+2z+2w=1 \\ -x-y+z+w=2 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 1 \\ -1 & -1 & 1 & 1 & 2 \end{array}\right] (\times \tfrac{1}{2}) \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1/2 \\ -1 & -1 & 1 & 1 & 2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1/2 \\ 0 & 0 & 2 & 2 & 5/2 \end{array}\right] (\times \tfrac{1}{2}) \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 & 5/4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -3/4 \\ 0 & 0 & 1 & 1 & 5/4 \end{array}\right] \qquad \begin{array}{l} x+y = -3/4 \\ z+w = 5/4 \end{array}$$ (2p)

Free variables

Set $y=s$ and $w=t$, then (1p)

$x = -\frac{3}{4} - s$ and $z = 5/4 - t$

Answer: $\begin{cases} x = -3/4 - s \\ y = s \\ z = 5/4 - t \\ w = t \end{cases}$ where $s,t \in \mathbb{R}$ (1p)

Check (*): $2\cdot(-\frac{3}{4}-s)+2s+2(\frac{5}{4}-t)+2\cdot t =$

$= -\frac{6}{4} \cancel{-2s} \cancel{+2s} + \frac{10}{4} \cancel{-2t} \cancel{+2t} = \frac{4}{4} = 1$ ok!

$-(-\frac{3}{4}-s)-s+(\frac{5}{4}-t)+t = \frac{3}{4} \cancel{+s} \cancel{-s} + \frac{5}{4} \cancel{-t} \cancel{+t}$

$= \frac{8}{4} = 2$ ok!

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2a

$$\begin{vmatrix} -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{vmatrix} \underset{\text{exp. Row3}}{\overset{\text{Cofactor}}{=}} 2 \cdot \begin{vmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = \quad (2p)$$

$$= 2 \cdot \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = (\text{diagonal matrix}) = 2 \cdot (-1) \cdot (-1) \cdot 1 = 2$$

Answer a: $\det(A) = 2$ (1p)

2b

$$\left[\begin{array}{cccc|cccc} -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ \times\frac{1}{2} \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|cccc} -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \times(-1) \\ \times(-1) \\ \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \quad \text{Answer b} \quad (2p)$$

Answer b: $$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & -1 \\ -1 & -1 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$ (1p)

Check: $$\begin{bmatrix} 0 & 0 & \frac{1}{2} & -1 \\ -1 & -1 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ok!

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(3a) $\Pi: x+2y+2z=-4$

$\ell: (x,y,z) = (1-2t, 2+t, 3) = (1,2,3) + t\cdot(-2,1,0)$

The line ℓ is parallel to the plane Π if the line does not intersect the plan. Inserting the parametric form in the equation of the plane gives

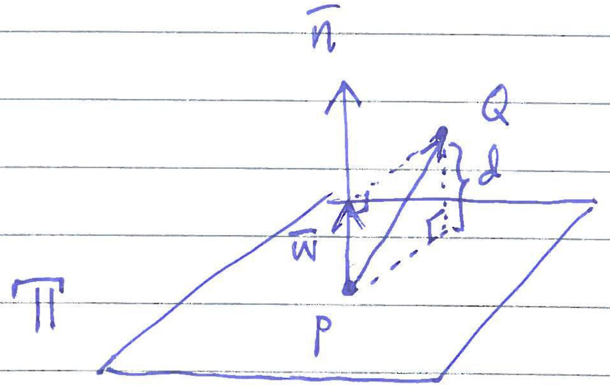
$$1-2t+2\cdot(2+t)+2\cdot 3 = 11 \neq -4$$

for any t, so ℓ does not intersect Π. (2p)

(3b) Pick any $Q\in \ell$, e.g. $t=0$ gives $Q(1,2,3)$.

Take any $P \in \Pi$, e.e. $x=y=0 \Rightarrow z=-2$

; i.e $P(0,0,-2)\in\Pi$.



$\bar{w} = \text{proj}_{\bar{n}} \overrightarrow{PQ}$ and $d = \|\bar{w}\|$

$\bar{n} = (1,2,2) \Rightarrow \|\bar{n}\| = \sqrt{9} = 3$ (2p)

$\overrightarrow{PQ} = (1,2,3)-(0,0,-2) = (1,2,5)$

$$d = \|\text{proj}_{\bar{n}} \overrightarrow{PQ}\| = \left\| \frac{(1,2,5)\bullet(1,2,2)}{\|(1,2,2)\|^2}\cdot(1,2,2)\right\| = \frac{|(1,2,5)\bullet(1,2,2)|}{\|(1,2,2)\|} =$$

$$= \frac{1+4+10}{3} = 5$$ (2p)

<u>Answer</u> : 5. length units.

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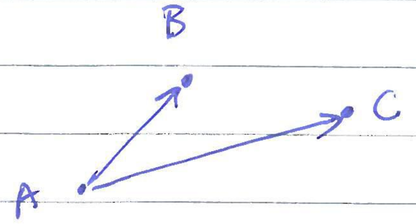
(4) $A(3,-1,-2)$, $B(1,-1,0)$, $C(1,0,1)$



$\vec{AB} = (1,-1,0)-(3,-1,-2)$
$= (-2,0,2)$
$\vec{AC} = (1,0,1)-(3,-1,-2)$ (1p)
$= (-2,1,3)$

$\vec{AB}$ and $\vec{AC}$ are not colinear so the normal can be

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0\cdot 3 - 2\cdot 1 \\ -((-2)\cdot 3 - 2\cdot(-2)) \\ (-2)\cdot 1 - 2\cdot(-2) \end{bmatrix} =$$
$$= \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$$ (2p)

Let $P(x,y,z)$ be any point in the plane, then the equation of the plane is given by

$$\vec{n} \bullet \vec{AP} = 0 \iff (-2,2,-2)\bullet(x-3, y+1, z+2) = 0$$

$$\iff -2(x-3) + 2(y+1) - 2\cdot(z+2) = 0$$
$$\iff -(x-3) + (y+1) - (z+2) = 0$$ (2p)

Answer: $-(x-3)+(y+1)-(z+2) = 0$

Check: Does the points belong to the plane?

A: $-(3-3)+(-1+1)-(-2+2)=0$ ok!
B: $-(1-3)+(-1+1)-(0+2) = 2-2=0$ ok!
C: $-(1-3)+(0+1)-(1+2) = 2+1-3=0$ ok!

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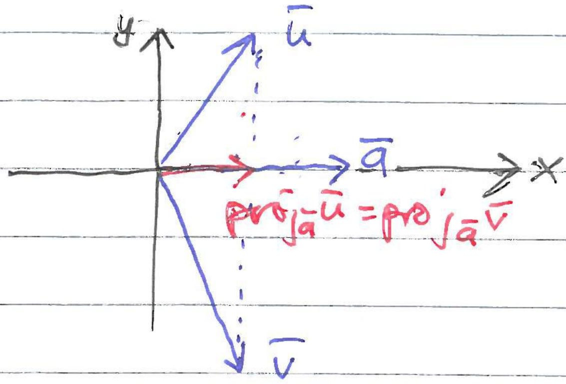
5 (a) is false, e.g. take $\bar{u} = (1,2)$, $\bar{v} = (1,-3)$ and $\bar{a} = (2,0)$, then



$$\text{proj}_{\bar{a}}\bar{u} = (1,0)$$

$$\text{proj}_{\bar{a}}\bar{v} = (1,0)$$

(2p)

but $\bar{u} \neq \bar{v}$

(b) is true: if $\bar{u}$ and $\bar{v}$ are orthogonal i.e. $\bar{u}\cdot\bar{v} = 0$, then

$$\|\bar{u}+\bar{v}\|^2 = (\bar{u}+\bar{v})\cdot(\bar{u}+\bar{v}) = \bar{u}\cdot\bar{u} + \bar{v}\cdot\bar{u} + \bar{v}\cdot\bar{u} + \bar{v}\cdot\bar{v}$$

$$= \|\bar{u}\|^2 + \underbrace{2(\bar{u}\cdot\bar{v})}_{=0} + \|\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$$

so $\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$ $\square$

(2p)