MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2015-11-05 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Evaluate the integral

$$\int_1^{\sqrt{3}} \frac{dx}{(1+x^2)^2}\,,$$

and write the result in as simple form as possible.

2. Find the solution of the initial-value problem

$$y'' + 2y' + y = 4xe^x, \quad y(0) = -2\,, \quad y'(0) = 2\,.$$

3. Find the real numbers x for which the power series

$$\sum_{n=1}^{\infty} \frac{(2-3x)^n}{n^2}.$$

is convergent. Are there any of these x for which the series is only convergent but not absolutely convergent, i.e. is conditionally convergent?

4. Sketch the graph of the function f, defined by

$$f(x) = \frac{x}{\sqrt{x^4+1}}\,,$$

by utilizing the guidance given by asymptotes and stationary points. Also, specify the range of the function.

5. Find the volume of the solid generated by rotating about the y-axis the bounded region precisely enclosed by the curves $y = 1/x$, $x = 4$ and $y = \sqrt{x}$.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2015-11-05 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Beräkna integralen

$$\int_1^{\sqrt{3}} \frac{dx}{(1+x^2)^2}\,,$$

och skriv resultatet på en så enkel form som möjligt.

2. Bestäm lösningen till begynnelsevärdesproblemet

$$y'' + 2y' + y = 4xe^x, \quad y(0) = -2\,, \quad y'(0) = 2\,.$$

3. Bestäm de reella tal x för vilka potensserien

$$\sum_{n=1}^{\infty} \frac{(2-3x)^n}{n^2}.$$

är konvergent. Är det några av dessa x för vilka serien endast är konvergent men inte absolutkonvergent, dvs. är betingat konvergent?

4. Skissa grafen till funktionen f, definierad enligt

$$f(x) = \frac{x}{\sqrt{x^4+1}}\,,$$

genom att använda den vägledning som fås från asymptoter och stationära punkter. Specificera även funktionens värdemängd.

5. Bestäm volymen av den kropp som genereras genom att kring y-axeln rotera det begränsade område som precis innesluts av kurvorna $y = 1/x$, $x = 4$ och $y = \sqrt{x}$.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2015-11-05

(1) $$\int_1^{\sqrt{3}} \frac{dx}{(1+x^2)^2} \left[\begin{aligned} x &= \tan(\theta) \\ dx &= \frac{1}{\cos^2\theta}d\theta \end{aligned}\right] = \int_{\pi/4}^{\pi/3} \frac{d\theta}{\cos^2(\theta)} \frac{1}{(1+\tan^2(\theta))^2}$$

$$= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\cos^2(\theta)} \frac{1}{(1/\cos^2(\theta))^2} = \int_{\pi/4}^{\pi/3} d\theta \cos^2(\theta) = \int_{\pi/4}^{\pi/3} d\theta \, \frac{1}{2}(1+\cos(2\theta))$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\pi/4}^{\pi/3} = \frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{1}{2}\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{4}+\frac{1}{2}\cdot 1\right)\right] = \underline{\frac{\pi}{24} - \frac{2-\sqrt{3}}{8}}$$

(2) DE: $y'' + 2y' + y = 4xe^x$ IV: $y(0) = -2$, $y'(0) = 2$

The general solution of the DE is $y_h + y_p$, where y_h is the general solution of the corresponding homogeneous eq. $y''+2y'+y=0$ and y_p is a particular solution of the DE.

The auxiliary eq. of the hom. DE is $0 = r^2+2r+1 = (r+1)^2$ implying that $y_h = (C_0+C_1x)e^{-x}$. For y_p, we seek a solution of the form $x^0(a_0+a_1x)e^x = (a_0+a_1x)e^x$.

Differentiating y_p and substituting into the DE give

$$\begin{cases} y_p' = (a_1+a_0+a_1x)e^x \\ y_p'' = (2a_1+a_0+a_1x)e^x \end{cases} \quad \left[(2a_1+a_0+a_1x)+2(a_1+a_0+a_1x)+(a_0+a_1x)\right]e^x = 4xe^x$$

$$\Leftrightarrow 4(a_0+a_1)+4a_1x = 4x \Leftrightarrow \begin{cases} 4(a_0+a_1) = 0 \\ 4a_1 = 4 \end{cases}$$

Thus, the general solution of the DE is $y = (C_0+C_1x)e^{-x} + (x-1)e^x$

The IV:s give $$\begin{cases} -2 = y(0) = (C_0+0)\cdot 1 + (0-1)\cdot 1 = C_0 - 1 \\ 2 = y'(0) = \left[(C_1-C_0-C_1x)e^{-x} + (1+x-1)e^x\right]_{x=0} = C_1 - C_0 \end{cases} \text{ i.e. } \begin{cases} C_0 = -1 \\ C_1 = 1 \end{cases}$$

The (unique) solution of the IVP is thus $\underline{y = (x-1)(e^x+e^{-x}) = 2(x-1)\cosh(x)}$

(3) $\sum_{n=1}^{\infty} a_n(x)$ where $a_n(x) = \frac{(2-3x)^n}{n^2}$

Let $A(x) = \lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty}\left|\frac{(2-3x)^{n+1}}{(n+1)^2}\frac{n^2}{(2-3x)^n}\right| = |2-3x|\cdot 1 = 3\left|x-\frac{2}{3}\right|$

* The series diverges if $A(x) > 1 \Leftrightarrow \left|x-\frac{2}{3}\right| > \frac{1}{3}$.
* — " — converges absolutely if $A(x) < 1 \Leftrightarrow \left|x-\frac{2}{3}\right| < \frac{1}{3}$ (i.e. $\frac{1}{3} < x < 1$)
* For $x = \frac{1}{3}$: $\sum_{n=1}^{\infty}\frac{(2-3\cdot\frac{1}{3})^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which is absolutely convergent according to the integral criteria.
* For $x=1$: $\sum_{n=1}^{\infty}\frac{(2-3\cdot 1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ — " —

Thus The series is convergent for $\frac{1}{3} \le x \le 1$. The convergence is absolutely for all those x.

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$$f(x)=\frac{x}{\sqrt{x^4+1}},\quad f'(x)=\frac{1\cdot\sqrt{x^4+1}-x\,\frac{1}{2}\frac{4x^3}{\sqrt{x^4+1}}}{\left(\sqrt{x^4+1}\right)^2}=\frac{(x^4+1)-2x^4}{(x^4+1)^{3/2}}$$

$$=-\frac{x^4-1}{(x^4+1)^{3/2}}=-\frac{(x^2+1)(x+1)(x-1)}{(x^4+1)^{3/2}}$$

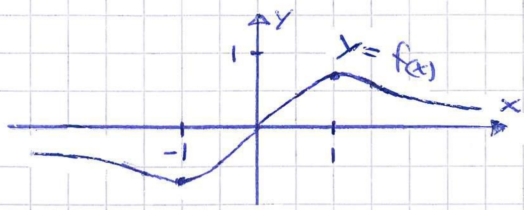
First derivative test

x		-1		1	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	loc. min	$\nearrow$	loc. max	$\searrow$

* There are no vertical asymptotes.

* $y=0$ is an asymptote at "$\pm\infty$" since $f(x)\to 0$ as $|x|\to\infty$

Since $f(-1)<\lim_{x\to\infty} f(x)=0$ the local min.p. -1 is also a min.p. Correspondingly, since $f(1)>\lim_{x\to-\infty} f(x)=0$, 1 is a max.p.



$$\begin{cases} f(-1)=-\frac{1}{\sqrt{2}} \\ f(1)=\frac{1}{\sqrt{2}}\end{cases}$$

f continuous then gives that $V_f=\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$.

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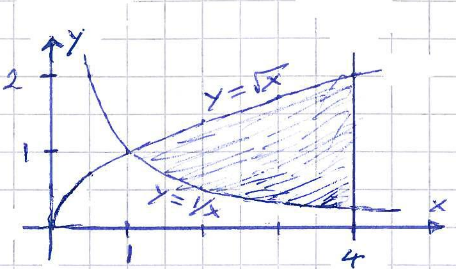

The bounded region precisely enclosed by the curves $y=\frac{1}{x}$, $x=4$ and $y=\sqrt{x}$ is rotated about the y-axis giving a solid. The volume of that solid is denoted V_y.

By the method of cylindrical shells, we get

$$V_y=\int_1^4 2\pi x\left(\sqrt{x}-\frac{1}{x}\right)dx=2\pi\left[\frac{2}{5}x^{5/2}-x\right]_1^4$$

$$=2\pi\left[\left(\frac{2}{5}4\cdot4\cdot2-4\right)-\left(\frac{2}{5}\cdot1-1\right)\right]=2\pi\left[\frac{2}{5}(32-1)-(4-1)\right]$$

$$=\frac{2\pi}{5}(2\cdot31-3\cdot5)=\frac{2\pi}{5}\cdot47=\frac{94\pi}{5}$$

We may also get the result by the method of slicing

$$V_y=\int_{1/4}^{1}\pi\left(4^2-\left(\frac{1}{y}\right)^2\right)dy+\int_1^2\pi\left(4^2-(y^2)^2\right)dy$$

$$=\pi\left[16y+\frac{1}{y}\right]_{1/4}^{1}+\pi\left[16y-\frac{y^5}{5}\right]_1^2$$

$$=\pi\left[(16+1)-(4+4)\right]+\pi\left[\left(32-\frac{32}{5}\right)-\left(16-\frac{1}{5}\right)\right]$$

$$=\pi\left[9+16-\frac{31}{5}\right]=\pi\left(\frac{25\cdot5-31}{5}\right)=\frac{94\pi}{5}$$

Answer: $\frac{94\pi}{5}$ v.u.

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

Examination TEN2 – 2015-11-05 Maximum points for subparts of the problems in the final examination

1. $\frac{1}{8}\left(\frac{\pi}{3}-(2-\sqrt{3})\right)$

----- One scenario for the first three points -------

1p: Correctly rewrited the integrand as $(1+x^2)^{-1}-x^2(1+x^2)^{-2}$
1p: Correctly worked out the first step in determining the antiderivative by parts in the second term $x\cdot(1+x^2)^{-2}x$
1p: Correctly found the antiderivative of the integrand

------ Another (the standard) scenario for the first three points -----------

1p: Correctly, by the substitution $x=\tan(\theta)$, translated the integrand of the integral
1p: Correctly, by the substitution $x=\tan(\theta)$, translated the limits of the integral
1p: Correctly rewrited the integrand according to $\cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$, and correctly found an antiderivative

-------------------- The scenario for the fourth point -------------------------

1p: Correctly evaluated the antiderivative at the limits, i.e. correctly determined the value of the integral

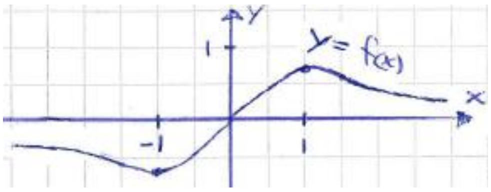
2. $y=(x-1)(e^x+e^{-x})$
$=2(x-1)\cosh(x)$

1p: Correctly identified the differential equation as a non-homogeneous linear DE of second order, and correctly found the general solution y_h of the corr. homog. DE
1p: Correctly determined a particular solution of the DE
1p: Correctly formulated the general solution of the DE
1p: Correctly adapted the general solution of the DE to the IV

3. The series is convergent for $\frac{1}{3}\le x\le 1$, and the convergence is absolutely for all those x

1p: Correctly, by e.g. the ratio test, found that the series is absolutely convergent for $|2-3x|<1$, i.e. for $\frac{1}{3}<x<1$, and hopefully correctly mentioned that the series definitely is divergent for $|x-\frac{2}{3}|>\frac{1}{3} \Leftrightarrow (x<\frac{1}{3})\vee(x>1)$
1p: Correctly found that the series is convergent for $x=\frac{1}{3}$
1p: Correctly found that the series is convergent for $x=1$
1p: Correctly found that the series is also absolutely convergent at the endpoints of the interval of convergence, and by this stressed that there are no x for which the series is (only) conditionally convergent

4. The graph has the asymptote $y=0$ at "$\pm\infty$", a minimum at $(-1,-\frac{1}{\sqrt{2}})$ and a maximum at $(1,\frac{1}{\sqrt{2}})$. $V_f=[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]$



1p: Correctly determined the equation for the one and only (horizontal) asymptote of the graph
1p: Correctly found the stationary points of the function and correctly classified them
1p: Correctly found that the local extreme points also are (absolute) extreme points, and by this together with the fact that the function is continuous correctly specified the range of the function
1p: Correctly sketched the graph

5. $\frac{94\pi}{5}$ v.u.

2p: Correctly formulated an integral (or a sum of integrals) for the volume obtained by rotating the region about the y-axis (irrespective whether the method of cylindrical shells or the method of slicing have been applied)
1p: Correctly found the antiderivative(s) of the integrand(s)
1p: Correctly evaluated the limits of the integral