EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2016-06-07 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

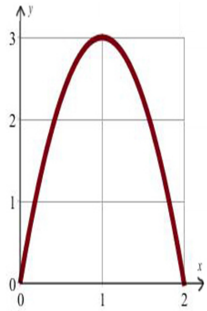
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the range of the function $x \curvearrowright f(x) = x - 2\arctan(x),\ D_f = [0, \sqrt{3}\,]$.

2. Find to the differential equation $y'' - 4y = 0$, the solution that satisfies the initial conditions $y(0) = 1,\ y'(0) = 0$.

3. To the right can be seen a sketch of the graph of the function f. Explain and make decent sketches of the graphs given by the equations $2y = f(x/2)$ and $y + 1 = f(x - 2)$.



4. Find the GENERAL antiderivative of $x \curvearrowright f(x) = \dfrac{x}{x^2 - 3x + 2}$.

5. Let
$$f(x) = 2 - \frac{3x}{x^2 + 2}.$$
Find the area of the triangle region Ω which lies in the first quadrant, and which is precisely enclosed by the positive coordinate axes and the tangent line τ to the curve $\gamma : y = f(x)$ at the point $P : (1, 1)$.

6. Find the numerical sequence $\{c_n\}_{n=0}^{\infty}$ for which the power series $\sum_{n=0}^{\infty} c_n x^n$ has the sum $x/(x + 2)$. Also, determine the interval of convergence of the power series.

7. Determine whether
$$\lim_{x \to 3^-} \frac{x^2 - 4x + 3}{|x^2 + x - 12|}$$
exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

8. Evaluate the integral $\displaystyle\int_{-4}^{0} \sqrt{16 - x^2}\, dx$ by interpreting it as a certain area measure.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN1
Datum: 2016-06-07 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

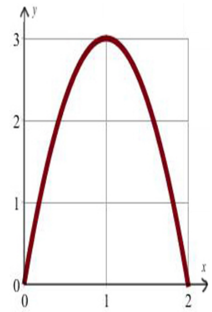
$$\begin{array}{lcrcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm värdemängden för funktionen

$$x \curvearrowright f(x) = x - 2\arctan(x), \qquad D_f = [0, \sqrt{3}\,].$$

2. Bestäm till differentialekvationen $y'' - 4y = 0$ den lösning som uppfyller begynnelsevillkoren $y(0) = 1$, $y'(0) = 0$.



3. Till höger i bild syns en skiss av grafen till funktionen f. Förklara och gör hyfsade skisser av de grafer som ges av ekvationerna $2y = f(x/2)$ och $y + 1 = f(x - 2)$.

4. Bestäm den GENERELLA primitiva funktionen till $x \curvearrowright f(x) = \dfrac{x}{x^2 - 3x + 2}$.

5. Låt

$$f(x) = 2 - \frac{3x}{x^2 + 2}.$$

Bestäm arean av det triangelområde Ω som ligger i den första kvadranten, och som precis innesluts av de positiva koordinataxlarna och tangenten τ till kurvan $\gamma : y = f(x)$ i punkten $P : (1, 1)$.

6. Bestäm den talföljd $\{c_n\}_{n=0}^{\infty}$ för vilken potensserien $\sum_{n=0}^{\infty} c_n x^n$ har summan $x/(x + 2)$. Bestäm även konvergensintervallet för potensserien.

7. Avgör om

$$\lim_{x \to 3^-} \frac{x^2 - 4x + 3}{|x^2 + x - 12|}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

8. Beräkna integralen $\displaystyle\int_{-4}^{0} \sqrt{16 - x^2}\, dx$ genom att tolka den som ett visst areamått.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2016-06-07

1 $f(x) = x - 2\arctan(x)$, $D_f = [0, \sqrt{3}]$

We begin by noting that f is a continuous function on a closed and finite domain, and therefore has a minimum, a maximum and attains all values between the minimum and the maximum. Differentiation gives

$$f'(x) = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} = \frac{(x+1)(x-1)}{x^2+1}$$ and by this we have

x	0		1		$\sqrt{3}$
$f'(x)$		$-$	0	$+$	
$f(x)$	loc. max	$\searrow$	loc. min	$\nearrow$	loc. max

i.e. $$\begin{cases} f_{min} = f(1) = 1 - 2\arctan(1) = 1 - 2\frac{\pi}{4} = 1 - \frac{\pi}{2} \\ f_{max} = \max\left(f(0), f(\sqrt{3})\right) = \max\left(0-0, \underbrace{\sqrt{3} - 2\frac{\pi}{3}}_{<0}\right) = 0 \end{cases}$$

i.e. $R_f = \left[-\left(\frac{\pi}{2} - 1\right), 0\right]$

2 DE: $y'' - 4y = 0$ IV: $y(0) = 1$, $y'(0) = 0$

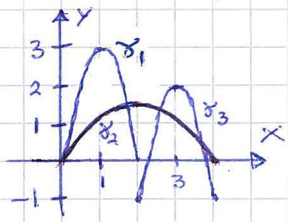
The auxiliary eq. is $0 = r^2 - 4 = (r+2)(r-2)$

Thus the general solution of the homogeneous linear 2nd order DE is $y = C_1 e^{-2x} + C_2 e^{2x}$. The initial values give

$$\begin{cases} 1 = y(0) = C_1 + C_2 \\ 0 = y'(0) = -2C_1 + 2C_2 \end{cases} \Leftrightarrow C_1 = C_2 = \frac{1}{2}$$

Thus the solution of the IVP is $y = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \cosh(2x)$

3 $$\begin{cases} \gamma_1: \ y = f(x) \\ \gamma_2: \ y = \frac{1}{2} f\left(\frac{1}{2}x\right) \\ \gamma_3: \ y + 1 = f(x-2) \end{cases}$$



The curve γ_2 is γ_1 but expanded horizontally by a factor 2 and compressed vertically by a factor 2. The curve γ_3 is γ_1 but shifted 2 units horizontally and -1 unit vertically.

4 $$f(x) = \frac{x}{x^2 - 3x + 2} = \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{(A+B)x - (2A+B)}{(x-1)(x-2)}$$

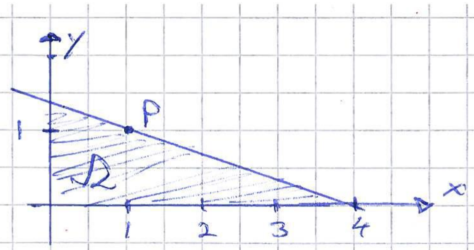
↑ partial fractions

i.e. $$\begin{cases} A + B = 1 \\ -2A - B = 0 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$$

Thus, the general antiderivative of f is

$$\int f(x)\,dx = \int \left(\frac{-1}{x-1} + \frac{2}{x-2}\right) dx = -\ln|x-1| + 2\ln|x-2| + C$$
$$= \ln\left(\frac{(x-2)^2}{|x-1|}\right) + C$$ where C is a constant

Cont. MAA151
TEN1 2016-06-07



(5) $f(x) = 2 - \frac{3x}{x^2+2}$

$$f'(x) = 0 - 3\,\frac{1\cdot(x^2+2) - x\cdot 2x}{(x^2+2)^2} = 3\,\frac{x^2-2}{(x^2+2)^2}$$

The tangent line τ to $y = f(x)$ at $P = (1,1)$ is given by $y - 1 = f'(1)(x-1)$ where $f'(1) = 3\,\frac{1-2}{(1+2)^2} = -\frac{1}{3}$

i.e. $\tau:\ y = 1 - \frac{1}{3}(x-1) = \frac{4}{3} - \frac{1}{3}x$

Thus $A(\Omega) = \frac{1}{2}\,4\cdot\frac{4}{3}$ a.u. $= \frac{8}{3}$ a.u.

(6) $$\frac{x}{x+2} = \frac{x}{2}\,\frac{1}{1+\frac{x}{2}} = \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{1}{2}\right)^n x^{n+1} \quad [n+1=k]$$

iff $\left|-\frac{x}{2}\right| < 1$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{2}\right)^k x^k = \sum_{k=0}^{\infty} c_k x^k$$

i.e. $\boxed{\begin{cases} c_0 = 0 \\ c_k = \frac{(-1)^{k-1}}{2^k}, & k \geq 1 \end{cases}}$ and the interval of convergence is equal to $(-2,2)$.

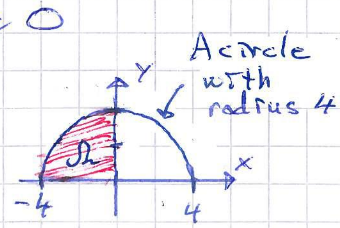
(7) $$\lim_{x\to 3^-}\frac{x^2-4x+3}{|x^2+x-12|} = \lim_{x\to 3^-}\frac{(x-1)(x-3)}{|(x+4)(x-3)|} = \lim_{x\to 3^-}\left(\frac{(x-1)(x-3)}{-(x+4)(x-3)}\right)$$

$$= -\left(\lim_{x\to 3^-}\frac{x-1}{x+4}\right)\left(\lim_{x\to 3^-}\frac{x-3}{x-3}\right) = -\left(\frac{3-1}{3+4}\right)\cdot 1 = -\frac{2}{7}$$

(since each of the two separate limits exists)

(since the function $x \to \frac{x-1}{x+4}$ is continuous at 3)

(8) $\sqrt{16-x^2} = y \iff x^2 + y^2 = 4^2 \wedge y \geq 0$



Thus $$\int_{-4}^{0}\sqrt{16-x^2}\,dx = A(\Omega), \text{ where}$$

$$= \frac{1}{4}\pi\cdot 4^2$$

$$= 4\pi$$

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

Examination TEN1 – 2016-06-07 — Maximum points for subparts of the problems in the final examination

1. $R_f = [-(\frac{\pi}{2}-1), 0]$

Note: To get full marks, it is not necessary to explicitly invoke the theorems that support a correct answer (i.e. the theorem about existence of extreme values and the intermediate-value theorem). It is sufficient to have properly conducted a first derivative test and made correct conclusions thereof, or alternatively, to exhaustively have applied the theorems indicated above.

1p: Correctly differentiated the function f, and correctly concluded about the local extreme points of the function
1p: Correctly found the maximum of f
1p: Correctly found the minimum of f, and correctly stated the range of f

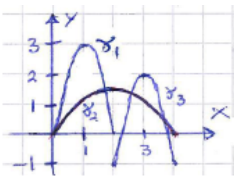
2. $y = \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh(2x)$

1p: Correctly found one solution of the DE
1p: Correctly found the general solution of the DE
1p: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP

3. The graph γ_2 given by the equation $2y = f(x/2)$ is the graph of f expanded horizontally by a factor of 2 and compressed vertically by a factor of 2. The graph γ_3 given by the equation $y+1 = f(x-2)$ is the graph of f shifted 2 units horizontally and -1 unit vertically.

2p: Correctly explained and sketched the graph γ_2 given by the equation $2y = f(x/2)$
1p: Correctly explained and sketched the graph γ_3 given by the equation $y+1 = f(x-2)$

Note: A clear and instructive *illustration* of a graph may be accounted for as also being an appropriate *explanation*. However, the student who have sketched the two graphs without any comments at all supporting the sketches, can obtain **at most 1p**.



4. $\int f(x)\,dx = 2\ln|x-2| - \ln|x-1| + C$

1p: Correctly found the partial fractions of $f(x)$
1p: Correctly found an antiderivative of f
1p: Correctly found the general antiderivative of f

5. $\frac{8}{3}$ a.u.

1p: Correctly found the derivative of the function f
1p: Correctly found an equation for the tangent line τ to the curve γ at the point P
1p: Correctly found the area of the triangle region Ω

6. $c_0 = 0$ and $c_n = (-1)^{n-1}(\frac{1}{2})^n$ for $n \geq 1$
The interval of convergence is $(-2, 2)$

1p: Correctly expanded $x/(x+2)$ in a power series in x
1p: Correctly identified the coefficients of the power series
1p: Correctly found the interval of convergence

7. The limit exists and is equal to $-2/7$

Note: The student who have argued that the limit does not exist based on the fact that the fraction at the limit point is of the type "0/0" obtains **0p**. The student who have claimed that a fraction of the type "1/0" or "0/0" is equal to 0 obtains **0p**, especially if the succeeding conclusion is of the kind "*the limit does not exist since the value is 0*".

1p: Correctly factorized the expression
1p: Correctly taken account of the absolute value bars
1p: Correctly concluded that the limit exists, and correctly found the limit

8. 4π

1p: Correctly identified the integrand as the function whose curve is the upper half of a circle with the centre at the origin and a radius equal to 4
1p: Correctly interpreted the integral as a measure of the area of a quarter-cricle disk
1p: Correctly evaluated the integral