MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2017-11-06 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Let $f(x) = \dfrac{1}{|\ln(x)| + 3}$. State (and explain) the domain and the range of f.

2. Find to the differential equation $x^2 \dfrac{dy}{dx} + 5xy = 8\,,\ x > 0\,,$ the unique solution which satisfies $y(2) = 2\,.$

3. Evaluate the integral

$$\int_{-1}^{1} \left[(x+1)^{2016} - (x-1)^{2016}\right]\, dx$$

and write the result in as simple form as possible.

4. Is the series

$$\sum_{n=1}^{\infty} 3^n 4^{1-n}$$

convergent or not? If your answer is YES: Give an explanation of why and find the sum of the series! If your answer is NO: Give an explanation of why!

5. The function f is differentiable, and it is known that

$$\begin{array}{llll} f(1) = 2\,, & f(2) = -4\,, & f(2\sqrt{2}) = -7\,, & f(2\sqrt{3}) = 5\,, \\ f'(1) = \sqrt{5}\,, & f'(2) = \sqrt{3}\,, & f'(2\sqrt{2}) = 6\,, & f'(2\sqrt{3}) = \sqrt{2}\,. \end{array}$$

Find an equation for the tangent line τ to the curve $\gamma : y = f(4\sin(x\pi/6))$ at the point P whose x-coordinate is equal to 1.

6. Find the GENERAL antiderivative of the function $x \curvearrowright f(x) = (2x+1)e^{x/3}\,.$

7. Find out whether

$$\lim_{x\to+\infty} x\sin(1/x)$$

exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

8. Find the range of the function $x \curvearrowright f(x) = x/(x^2+4),\ D_f = [1, 3]$.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN1
Datum: 2017-11-06 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Låt $f(x) = \dfrac{1}{|\ln(x)| + 3}$. Ange (och förklara) definitionsmängden och värdemängden för f.

2. Bestäm till differentialekvationen $x^2 \dfrac{dy}{dx} + 5xy = 8\,,\ x > 0\,,$ den unika lösning som satisfierar $y(2) = 2\,.$

3. Beräkna integralen

$$\int_{-1}^{1} \left[(x+1)^{2016} - (x-1)^{2016}\right]\, dx$$

och skriv resultatet på en så enkel form som möjligt.

4. Är serien

$$\sum_{n=1}^{\infty} 3^n 4^{1-n}$$

konvergent eller ej? Om ditt svar är JA: Ge en förklaring till varför och bestäm summan av serien! Om ditt svar är NEJ: Ge en förklaring till varför!

5. Funktionen f är deriverbar, och det är bekant att

$$\begin{array}{llll} f(1) = 2\,, & f(2) = -4\,, & f(2\sqrt{2}) = -7\,, & f(2\sqrt{3}) = 5\,, \\ f'(1) = \sqrt{5}\,, & f'(2) = \sqrt{3}\,, & f'(2\sqrt{2}) = 6\,, & f'(2\sqrt{3}) = \sqrt{2}\,. \end{array}$$

Bestäm en ekvation för tangenten τ till kurvan $\gamma : y = f(4\sin(x\pi/6))$ i punkten P vars x-koordinat är lika med 1.

6. Bestäm den GENERELLA primitiven till funktionen $x \curvearrowright f(x) = (2x+1)e^{x/3}\,.$

7. Utred om

$$\lim_{x \to +\infty} x \sin(1/x)$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

8. Bestäm värdemängden för funktionen $x \curvearrowright f(x) = x/(x^2+4),\ D_f = [1,3].$

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2017-11-06

1 $f(x) = \dfrac{1}{|\ln(x)| + 3}$

We note immediately that, since the denominator is positive for all $x \in D_{\ln}$, $D_f = D_{\ln} = (0, \infty)$. Furthermore, since $0 < f(x) \leq \dfrac{1}{|\ln(1)|+3} = \dfrac{1}{3}$ and f is continuous and $\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to +\infty}$, we conclude that $R_f = (0, \frac{1}{3}]$

2 DE: $x^2 \frac{dy}{dx} + 5xy = 8$, $x > 0$ IV: $y(2) = 2$

The DE in standard form is $\dfrac{dy}{dx} + \dfrac{5}{x} y = \dfrac{8}{x^2}$

Mult. with an integrating factor x^5 gives $\dfrac{d}{dx}(x^5 y) = x^5 \cdot \dfrac{8}{x^2}$

and thus $y = \dfrac{1}{x^5}(2x^4 + c)$

The IV gives $2 = y(2) = \frac{1}{32}(2 \cdot 16 + c) \Leftrightarrow c = 32$

The solution of the IVP is thus $y = \dfrac{2}{x}\left[1 + \left(\dfrac{2}{x}\right)^4\right]$, $x > 0$

3 $\displaystyle\int_{-1}^{1} \underbrace{\left[(x+1)^{2016} - (x-1)^{2016}\right]}_{f(x)} dx = 0$ since the integrand is an odd function (i.e. since $f(-x) = -f(x)\ \forall\ x \in [-1, 1]$)

$$f(-x) = (-x+1)^{2016} - (-x-1)^{2016} = (-1)^{2016}(x-1)^{2016} - (-1)^{2016}(x+1)^{2016}$$
$$= (x-1)^{2016} - (x+1)^{2016} = -\left[(x+1)^{2016} - (x-1)^{2016}\right] = -f(x)$$

(Alt. eval. $\displaystyle\int_{-1}^{1} f(x)\,dx = \left[\frac{(x+1)^{2017}}{2017} - \frac{(x-1)^{2017}}{2017}\right]_{-1}^{1} = \left(\frac{2^{2017}}{2017} - 0\right) - \left(0 - \frac{(-2)^{2017}}{2017}\right)$

$$= \frac{2^{2017}}{2017} + (-1)^{2017}\frac{2^{2017}}{2017} = \frac{2^{2017}}{2017}(1 - 1) = 0$$)

4 $\displaystyle\sum_{n=1}^{\infty} 3^n 4^{1-n} = 4\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = 4\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k+1} = 3\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k$

is a geometric series with the ratio $\frac{3}{4}$.

Since the absolute value of the ratio is less than one, the series is convergent. The sum of the series is $3 \cdot \dfrac{1}{1 - \frac{3}{4}} = 3 \cdot \dfrac{1}{1/4} = 12$.

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from the table

(5) $\gamma: \quad y = \underbrace{f(4\sin(x\pi/6))}_{g(x)} , \qquad P: (1, g(1)) = (1, f(2)) = (1, -4)$

Differentiation gives $g'(x) = f'(4\sin(\frac{x\pi}{6}))\cdot 4\cos(\frac{x\pi}{6})\cdot\frac{\pi}{6}$

from the table

whereby $g'(1) = f'(2)\cdot 4\cdot\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6} = \sqrt{3}\cdot\frac{\sqrt{3}\pi}{3} = \pi$

Thus, the tangent line τ to γ at P is given by the equation $y - g(1) = g'(1)(x-1) \iff y+4 = \pi(x-1)$

(6) $\int f(x)dx = \int (2x+1)e^{x/3}dx = (2x+1)(3e^{x/3}) - \int 2\cdot(3e^{x/3})dx$

$= (2x+1)3e^{x/3} - 6(3e^{x/3}) + C = 3(2x-5)e^{x/3} + C$

(7) $\lim_{x\to+\infty} x\sin(1/x) \left[\begin{array}{l} \frac{1}{x} = u \\ u\to 0^+ \Rightarrow x\to+\infty\end{array}\right] = \underbrace{\lim_{u\to 0^+} \frac{1}{u}\sin(u) = 1}_{\text{a standard limit}}$

(8) $f(x) = \frac{x}{x^2+4}, \qquad D_f = [1,3]$

Differentiation gives $f'(x) = \frac{1\cdot(x^2+4) - x\cdot 2x}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$

$= -\frac{(x+2)(x-2)}{(x^2+4)^2}$

A first derivative test is

	EP		SP		EP
x	1		2		3
$f'(x)$		+	0	−	
$f(x)$	loc. min	↗	loc. max	↘	loc. min

Since f is continuous in a closed and bounded D_f, it has a minimum f_{min} and a maximum f_{max}, and takes all values between these extremes.

We get $\begin{cases} f_{min} = \min(f(1), f(3)) = \min(\frac{1}{5}, \frac{3}{13}) = \min(\frac{13}{65}, \frac{15}{65}) = \frac{1}{5} \\ f_{max} = f(2) = \frac{2}{8} = \frac{1}{4}\end{cases}$

Thus $R_f = [\frac{1}{5}, \frac{1}{4}]$

Examination TEN1 – 2017-11-06	Maximum points for subparts of the problems in the final examination
1. $D_f=(0,\infty)$, $R_f=(0,\frac{1}{3}]$	**1p**: Correctly found and explained the domain of f **1p**: Correctly found an explained the lower limit of the range **1p**: Correctly found an explained the upper limit of the range, and correctly summarized the range
2. $y=\frac{2}{x}\left(1+\left(\frac{2}{x}\right)^4\right)$, $x>0$	**1p**: Correctly found and multiplied with an integrating factor, and correctly rewritten the LHS of the DE as an exact derivative **1p**: Correctly found the general solution of the DE **1p**: Correctly adapted the general solution to the initial value, and correctly summarized the solution of the IVP
3. 0	----------------------------------- **One scenario** ----------------------------------- **1p**: Correctly proved that the integrand is an odd function **2p**: Correctly concluded that the integral equals zero since an integral of an odd function over an interval which is symmetric about zero equals zero -------------------------------- **Another scenario** -------------------------------- **1p**: Correctly found an antiderivative of the integrand **2p**: Correctly evaluated the antiderivative at the limits
4. The series is a convergent. The sum of the series equals 12	**1p**: Correctly concluded that the series is geometric **1p**: Correctly concluded that the series is convergent **1p**: Correctly found the sum of the series
5. $\tau:\quad y+4=\pi(x-1)$	**1p**: Correctly found the derivative of the function of the graph with the purpose of finding the slope at P **1p**: Correctly found the slope of the curve γ at P **1p**: Correctly evaluated the function at P, and correctly found an equation for the tangent line to the curve γ at P
6. $\int f(x)\,dx=3(2x-5)e^{x/3}+C$ where C is a constant	**1p**: Correctly worked out the first progressive step to find an antiderivative by parts **1p**: Correctly worked out the second progressive step to find an antiderivative by parts **1p**: Correctly included a constant in an otherwise correctly found antiderivative
7. The limit exists and equals 1 **Note**: The student who have argued that the limit does not exist based on the fact that the product at the limit point is of the type "$\infty\cdot 0$" obtains **0p**. The student who have claimed that a product of the type "$\infty\cdot 0$" is equal to 0 obtains **0p**, especially if the succeeding conclusion is of the kind "*the limit does not exist since the value is equal to 0*".	**2p**: Correctly by a substitution reformulated the limit as $\lim\limits_{u\to 0^+}\frac{\sin(u)}{u}$ **1p**: Correctly identified the limit as a standard limit with the value 1
8. $R_f=[\frac{1}{5},\frac{1}{4}]$ **Note**: To get full marks, a student should either correctly have concluded from a properly done first derivative test, or alternatively exhaustively have referred to the two theorems about intermediate values and extreme values respectively.	**1p**: Correctly differentiated the function f, and correctly concluded about the local extreme points of the function **1p**: Correctly found the minimum of f **1p**: Correctly found the maximum of f, and correctly stated the range of f