EXAMINATION IN MATHEMATICS
MAA150 Vector Algebra, TEN1
Date: 2018-02-15 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The PASS-marks 3, 4 and 5 require a minimum of 12, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 12, 13, 16, 20 and 24 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN1 S_2, the marks for a completed course are determined according to

$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 47$	$\rightarrow$	3
$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 62$	$\rightarrow$	4
		$63 \leq S_1 + 2S_2$	$\rightarrow$	5

$S_1, S_2 \geq 12$	AND	$S_1 + 2S_2 \leq 38$	$\rightarrow$	E
$S_1, S_2 \geq 12$	AND	$39 \leq S_1 + 2S_2 \leq 47$	$\rightarrow$	D
$S_1, S_2 \geq 12$	AND	$48 \leq S_1 + 2S_2 \leq 59$	$\rightarrow$	C
$S_1, S_2 \geq 12$	AND	$60 \leq S_1 + 2S_2 \leq 71$	$\rightarrow$	B
		$72 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the 4-tuples (a, b, c, d) that solves the system of linear equations

$$\begin{cases} 5a - 2b - 7c + 8d = 0\,, \\ 2a + 3b + c + 7d = 0\,, \\ 3a - 5b - 8c + d = 0\,, \\ 4a + b - 3c + 9d = 0\,. \end{cases}$$

2. Find an equation on parameter-free form for the plane π which contains the point $P_1 : (3, -4, 2)$, is parallel with the line $\lambda_2 : (x, y, z) = (3 + 5t, 2t, 4 - 3t)$, and is orthogonal to the plane $\pi_3 : x + 3y + z + 2 = 0$. It is assumed that the standard basis is a right-handed ON-basis.

3. Find the matrix A that solves the equation $\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}^T = \mathsf{A} + \mathsf{A}\begin{pmatrix} 2 & 7 \\ 4 & 8 \end{pmatrix}$.

4. Find the coordinates of the point Q which belongs to the line

$$\lambda : (x, y, z) = (1 - 2t, 2 - t, 3 + t)$$

and which is closest to the point $P : (2, -1, 5)$. It is assumed that the standard basis is an ON-basis.

5. Find, in polar form, the complex number w which solves the equation

$$\frac{i + \sqrt{3}}{i + 3} = \frac{1}{\overline{w}\,(3i - 1)}\,,$$

where $\overline{w}$ denotes the complex conjugate of w.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 12, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av ett sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{lcll} S_1, S_2 \geq 12 & \text{OCH} & S_1 + 2S_2 \leq 47 & \rightarrow \ 3 \\ S_1, S_2 \geq 12 & \text{OCH} & 48 \leq S_1 + 2S_2 \leq 62 & \rightarrow \ 4 \\ & & 63 \leq S_1 + 2S_2 & \rightarrow \ 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm de 4-tipplar (a, b, c, d) som löser det linjära ekvationssystemet

$$\begin{cases} 5a - 2b - 7c + 8d = 0\,, \\ 2a + 3b + c + 7d = 0\,, \\ 3a - 5b - 8c + d = 0\,, \\ 4a + b - 3c + 9d = 0\,. \end{cases}$$

2. Bestäm en ekvation på parameterfri form för det plan π som innehåller punkten $P_1 : (3, -4, 2)$, är parallellt med linjen $\lambda_2 : (x, y, z) = (3 + 5t, 2t, 4 - 3t)$, och är ortogonalt mot planet $\pi_3 : x + 3y + z + 2 = 0$. Det antages att standardbasen är en högerorienterad ON-bas.

3. Bestäm den matris A som löser ekvationen $\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}^T = \mathsf{A} + \mathsf{A}\begin{pmatrix} 2 & 7 \\ 4 & 8 \end{pmatrix}$.

4. Bestäm koordinaterna för den punkt Q som tillhör linjen

$$\lambda : (x, y, z) = (1 - 2t, 2 - t, 3 + t)$$

och som ligger närmast punkten $P : (2, -1, 5)$. Det antages att standardbasen är en ON-bas.

5. Bestäm, på polär form, det komplexa tal w som löser ekvationen

$$\frac{i + \sqrt{3}}{i + 3} = \frac{1}{\overline{w}\,(3i - 1)}\,,$$

där $\overline{w}$ betecknar komplexkonjugatet till w.

If you prefer the problems formulated in English, please turn the page.

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$$\begin{cases} 5a-2b-7c+8d=0 \\ 2a+3b+c+7d=0 \\ 3a-5b-8c+d=0 \\ 4a+b-3c+9d=0 \end{cases} \iff A\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad \text{where}$$

$$A=\begin{pmatrix} 5 & -2 & -7 & 8 \\ 2 & 3 & 1 & 7 \\ 3 & -5 & -8 & 1 \\ 4 & 1 & -3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & -8 & -9 & -6 \\ 2 & 3 & 1 & 7 \\ 1 & -8 & -9 & -6 \\ 0 & -5 & -5 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -8 & -9 & -6 \\ 0 & 19 & 19 & 19 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_{rref}$$

Thus $A\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \iff \begin{cases} a-c+2d=0 \\ b+c+d=0 \end{cases}$ Let $c=r,\ d=s \quad r,s\in R$

Then $a=r-2s,\ b=-r-s$

Thus $(a,b,c,d)=(r-2s,-r-s,r,s),\ r,s\in R$ are the 4-tuples that solves the system of linear equations

2 $P_1: (3,-4,2)$, $\lambda_2: (x,y,z)=(3+5t, 2t, 4-3t)$, $\pi_3: x+3y+z+2=0$

We know that $\pi \ni P_1$, $\pi \parallel \lambda_2$ and $\pi \perp \pi_3$ where the 2^{nd} condition means that π is parallel with the vector $v_2=(5,2,-3)$, and the 3^{rd} condition means that π is also parallel with the normal vector $n_3=(1,3,1)$ to π_3.

Thus, a normal vector n to the plane π equals $v_2\times n_3$, i.e.

$$n=v_2\times n_3=(5,2,-3)\times(1,3,1)=(2\cdot1-(-3)\cdot3, (-3)\cdot1-5\cdot1, 5\cdot3-2\cdot1)=(11,-8,13)$$

An eq. for π is then $n\cdot(x-x_1, y-y_1, z-z_1)=0$, i.e.

$$\pi:\ 11(x-3)-8(y+4)+13(z-2)=0 \iff 11x-8y+13z-91=0$$

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$$\begin{pmatrix} 2 & -1 \\ 5 & 3\end{pmatrix}^T = A + A\begin{pmatrix} 2 & 7 \\ 4 & 8\end{pmatrix} \iff \begin{pmatrix} 2 & 5 \\ -1 & 3\end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + A\begin{pmatrix} 2 & 7 \\ 4 & 8\end{pmatrix}$$

$$\iff \begin{pmatrix} 2 & 5 \\ -1 & 3\end{pmatrix} = A\begin{pmatrix} 3 & 7 \\ 4 & 9\end{pmatrix}$$

$$\iff A=\begin{pmatrix} 2 & 5 \\ -1 & 3\end{pmatrix}\begin{pmatrix} 3 & 7 \\ 4 & 9\end{pmatrix}^{-1} = \begin{pmatrix} 2 & 5 \\ -1 & 3\end{pmatrix}\frac{1}{27-28}\begin{pmatrix} 9 & -7 \\ -4 & 3\end{pmatrix}$$

$$=\begin{pmatrix} 2 & 5 \\ -1 & 3\end{pmatrix}\begin{pmatrix} -9 & 7 \\ 4 & -3\end{pmatrix} = \begin{pmatrix} -18+20 & 14-15 \\ 9+12 & -7-9\end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 21 & -16\end{pmatrix}$$

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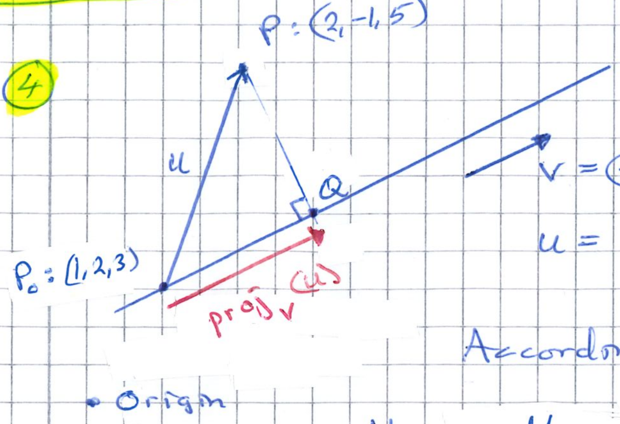

$\lambda : (x,y,z) = (1-2t,\ 2-t,\ 3+t)$

$v = (-2,-1,1) \parallel \lambda$

$u = u_{\overrightarrow{OP}} - u_{\overrightarrow{OP_0}} = (2,-1,5) - (1,2,3) = (1,-3,2)$

According to the illustration, we get that

$$u_{\overrightarrow{OQ}} = u_{\overrightarrow{OP_0}} + u_{\overrightarrow{P_0Q}} = (1,2,3) + proj_v(u)$$

$$= (1,2,3) + \frac{(1,-3,2)\cdot(-2,-1,1)}{\|(-2,-1,1)\|^2}(-2,-1,1)$$

$$= (1,2,3) + \frac{-2+3+2}{4+1+1}(-2,-1,1)$$

$$= (1,2,3) + \frac{1}{2}(-2,-1,1) = \left(0, \frac{3}{2}, \frac{7}{2}\right)$$

Thus $Q : \left(0, \frac{3}{2}, \frac{7}{2}\right)$ is the point in λ closest to P.

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$$\frac{i+\sqrt{3}}{i+3} = \frac{1}{\overline{w}\,(3i-1)} \iff \overline{w} = \frac{1}{3i-1}\cdot\frac{i+3}{i+\sqrt{3}} = \frac{1}{-1+3i}\cdot\frac{3+i}{\sqrt{3}+i}$$

$$\iff w = \frac{1}{-1-3i}\,\frac{3-i}{\sqrt{3}-i} = \frac{-1+3i}{1+9}(3-i)\,\frac{\sqrt{3}+i}{3+1}$$

$$= \frac{(-3+3+i+9i)}{10\cdot 4}(\sqrt{3}+i) = \frac{i}{4}(\sqrt{3}+i) = \frac{1}{2}\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = \frac{1}{2}e^{i2\pi/3}$$

(Alt. eval. (doing the complex conjugation at the end)

$$\iff \overline{w} = \frac{1}{-1+3i}\,\frac{3+i}{\sqrt{3}+i} = \frac{(-1-3i)}{1+9}(3+i)\,\frac{\sqrt{3}-i}{3+1}$$

$$= \frac{-3+3-i-9i}{10\cdot 4}(\sqrt{3}-i) = -\frac{i}{4}(\sqrt{3}-i) = \frac{1}{2}\,\frac{-1-i\sqrt{3}}{2}$$

$$\iff w = \frac{1}{2}\,\frac{-1+i\sqrt{3}}{2} = \frac{1}{2}e^{i2\pi/3}$$
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EXAMINATION IN MATHEMATICS
MAA150 Vector algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2017/18

Final examination TEN1 – 2018-02-15	Maximum points for subparts of the problems in the final examination
1. $(a,b,c,d)=(r-2s,-r-s,r,s)$ where $r,s\in R$	**2p**: Correctly found at least the echelon form of the coefficient matrix in preparation for finding the 4-tuples **1p**: Correctly, from the (row-reduced) echelon form, identified the correponding system of linear equations **2p**: Correctly found the general 4-tuple solving the system of linear equations
2. $\pi\,:\;11x-8y+13z-91=0$	**1p**: Correctly concluded that a normal of the plane π is a vector n which is orthogonal to the line λ_2 and to a normal of the plane π_3 **2p**: Correctly identified a vector v_2 parallel with λ_2, and a vector n_3 normal to π_3, and correctly worked out a vector n orthogonal to the vectors v_2 and n_3 **2p**: Correctly formulated an equation for the plane π on parameter-free form
3. $\begin{pmatrix}2 & -1\\ 21 & -16\end{pmatrix}$	------------------------------ **One scenario** -- **2p**: Correctly factorized the RHS of the equation **1p**: Correctly transposed the matrix of the LHS of the equation, and from the resulting equation $\mathrm{C}=\mathrm{AB}$ correctly found that $\mathrm{A}=\mathrm{CB}^{-1}$ **1p**: Correctly found (the entries of) the matrix B^{-1} **1p**: Correctly found (the entries of) the matrix A ----------------------------**Another scenario** --------------------------------------- **3p**: Correctly proposed the entries of the matrix A and correctly worked out the matrix operations of the LHS and the RHS of the equation **2p**: Correctly solved the system of (four) linear equations, and correctly put together the matrix A
4. $Q:\left(0\,,\,\frac{3}{2}\,,\,\frac{7}{2}\right)$	**2p**: Correctly concluded that the point Q can be addressed by adding to the coordinates of a point P_0 of the line, the coordinates of the ortogonal projection of a vector u, represented by the directed line segment from the point P_0 to the point P, on a vector v parallell with the line **1p**: Correctly identified vectors u and v **1p**: Correctly found the orthogonal projection of u on v **1p**: Correctly found the coordinates of the point Q
5. $w=\frac{1}{2}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$ $=\frac{1}{2}e^{i2\pi/3}$	**3p**: Correctly found w on the form $a+bi,\;a,b\in R$, where a correct treatment of the involved complex conjugation is worth one (**1**) of the three points **1p**: Correctly found a representative argument of w **1p**: Correctly found the absolute value of w, and correctly written w in polar form