MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2015-09-28 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find to the curve $\gamma : y = \ln(2e^2 - x^2)$ an equation for the tangent line τ which is parallel to the straight line $\lambda : 2x + ey = 0$.

2. Find the inverse of the function $x \curvearrowright f(x) = \sqrt{x-1}$, and specify its domain and range. Also sketch the graphs of f and f^{-1} in the same coordinate system.
Note that the word '*its*' refers to '*the inverse of the function*'.

3. Evaluate the integral $\displaystyle\int_0^1 \left|2x - 3x^2\right| \, dx$.

4. Find the range of the function $x \curvearrowright f(x) = \left(x^2 - 6x\right)^2,\ D_f = [1, 4]$.

5. Find the real numbers x for which

$$7x^2 + 20 + \frac{100}{7} + \ldots$$

is a geometric series. Then, determine for each geometric series whether it is convergent or not, and find in each case of convergence the sum of the series.

6. Determine whether

$$\lim_{x\to\infty} \left(\frac{3e^x + e^{2x}}{e^x + 1} - e^x \right)$$

exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

7. Find the function f whose derivative is equal to the function $x \curvearrowright x\cos(3x)$ and whose value at the point π is equal to 0.

8. Solve for $x > 0$ the initial-value problem $xy' + 2y = x^2$, $y(2) = 2$.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN1
Datum: 2015-09-28 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm till kurvan $\gamma : y = \ln(2e^2 - x^2)$ en ekvation för den tangent τ som är parallell med den räta linjen $\lambda : 2x + ey = 0$.

2. Bestäm inversen till funktionen $x \curvearrowright f(x) = \sqrt{x-1}$, och specificera dess definitionsmängd och värdemängd. Skissa även i ett och samma koordinatsystem graferna till f och f^{-1}. Notera att ordet '*dess*' syftar på '*inversen till funktionen*'.

3. Beräkna integralen $\displaystyle\int_0^1 \left|2x - 3x^2\right| \, dx$.

4. Bestäm värdemängden för funktionen $x \curvearrowright f(x) = \left(x^2 - 6x\right)^2,\ D_f = [1,4]$.

5. Bestäm de reella tal x för vilka

$$7x^2 + 20 + \frac{100}{7} + \ldots$$

är en geometrisk serie. Avgör sedan för varje geometrisk serie om den är konvergent eller ej, och bestäm i varje fall av konvergens seriens summa.

6. Avgör om

$$\lim_{x\to\infty} \left(\frac{3e^x + e^{2x}}{e^x + 1} - e^x \right)$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

7. Bestäm den funktion f vars derivata är lika med funktionen $x \curvearrowright x\cos(3x)$ och vars värde i punkten π är lika med 0.

8. Lös för $x > 0$ begynnelsevärdesproblemet $xy' + 2y = x^2$, $y(2) = 2$.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2015-09-28

1 $\gamma:\ y=\underbrace{\ln(2e^2-x^2)}_{f(x)}$, $\lambda:\ 2x+ey=0$, $\tau \,//\, \lambda$

Differentiating f gives: $f'(x)=\dfrac{-2x}{2e^2-x^2}$

The slope of the tangent line is equal to the slope of λ, i.e. $f'(x_P)=-\frac{2}{e}$, where P is the point where the graph of f intersects γ. The equation for x_P is equivalent to

$$\begin{cases}\dfrac{-2x_P}{2e^2-x_P^2}=-\dfrac{2}{e}\\ x_P\in D_f\end{cases} \iff \begin{cases}0=x_P^2+ex_P-2e^2=(x_P+2e)(x_P-e)\\ x_P\in D_f\end{cases} \iff x_P=e$$

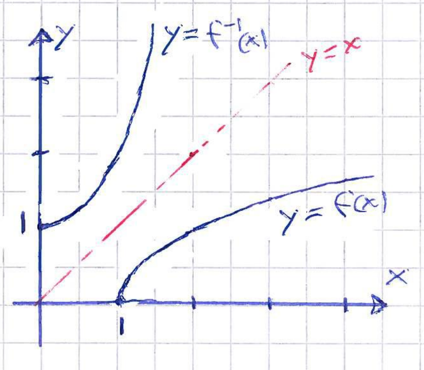
Thus $\tau:\ y-\underbrace{f(e)}_{2}=-\frac{2}{e}(x-e) \iff y=4-\frac{2}{e}x$

$\iff \underline{2x+ey=4e}$

2 $f(x)=\sqrt{x-1}$, $\begin{cases}D_f=[1,\infty)\\ V_f=[0,\infty)\end{cases}$

We have $\begin{cases}f(x)=y\\ \sqrt{x-1}=y\end{cases} \iff \begin{cases}x=f^{-1}(y)\\ x=1+y^2,\ y\geq 0\end{cases}$

i.e. $\underline{f^{-1}(x)=1+x^2}$ $\begin{cases}D_{f^{-1}}=V_f=[0,\infty)\\ V_{f^{-1}}=D_f=[1,\infty)\end{cases}$



3 $$\int_0^1|2x-3x^2|\,dx=\int_0^1\left|\left(\tfrac{2}{3}-x\right)3x\right|dx=\int_0^{2/3}(2x-3x^2)\,dx+\int_{2/3}^1(3x^2-2x)\,dx$$

$$=\left[x^2-x^3\right]_0^{2/3}+\left[x^3-x^2\right]_{2/3}^1=\left[\left(\tfrac{4}{9}-\tfrac{8}{27}\right)-(0-0)\right]+\left[(1-1)-\left(\tfrac{8}{27}-\tfrac{4}{9}\right)\right]$$

$$=\frac{4}{27}-0+0-\left(-\frac{4}{27}\right)=\underline{\frac{8}{27}=\left(\frac{2}{3}\right)^3}$$

4 $f(x)=(x^2-6x)^2$, $D_f=[1,4]$

We have $f'(x)=2(x^2-6x)(2x-6)=4x(x-3)(x-6)$

1st deriv. test

	EP	SP	EP
x	1	3	4
$f'(x)$	+	0	−
$f(x)$	loc. min. ↗	loc. max	↘ loc. min.

$$\begin{cases}f_{min}=\min(f(1),f(4))\\ \qquad=\min(25,64)=25\\ f_{max}=f(3)=81\end{cases}$$

Since f is a continuous function in an closed and bounded interval it attains all values in the interval $[f_{min},f_{max}]$, i.e. $\underline{V_f=[25,81]}$

Cont. MAA151
TEN1, 2015-09-28

(5) $$7x^2 + 20 + \frac{100}{7} + \dots = \sum_{n=0}^{\infty} a_n$$

The series is geometric if $\frac{a_3}{a_2} = \frac{a_2}{a_1} \Leftrightarrow \frac{\left(\frac{100}{7}\right)}{20} = \frac{20}{7x^2}$

$$\Leftrightarrow 1 = \left(\frac{2}{x}\right)^2 \Leftrightarrow x = -2 \vee x = 2$$

The ratio is in both cases $\frac{5}{7}$ and since (the absolute value of) that number is less than one, both series converge.

The common sum of the series is $7 \cdot 4 \frac{1}{1-\frac{5}{7}} = \frac{28}{2/7} = 98$

(6) $$\lim_{x\to\infty}\left(\frac{3e^x + e^{2x}}{e^x+1} - e^x\right) = \lim_{x\to\infty}\left[\frac{3e^x + e^{2x} - e^x(e^x+1)}{e^x+1}\right] = \lim_{x\to\infty}\frac{2e^x}{e^x+1}$$

$$= \lim_{x\to\infty}\frac{2}{1+e^{-x}} = \frac{2}{1+0} = 2$$

(7) Given $f'(x) = x\cos(3x)$ and $f(\pi) = 0$

We have $f(x) = \int dx\, x\cos(3x) = x \cdot \frac{1}{3}\sin(3x) - \int dx\, 1 \cdot \frac{1}{3}\sin(3x)$

$$= \frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C$$ ← a constant

where $0 = f(\pi) = \frac{\pi}{3}\sin(3\pi) + \frac{1}{9}\cos(3\pi) + C$

$$= \frac{\pi}{3}\cdot 0 + \frac{1}{9}(-1) + C \quad \text{i.e.} \quad C = \frac{1}{9}$$

Thus $f(x) = \frac{1}{9}\left[1 + \cos(3x) + 3x\sin(3x)\right]$

(8) $$xy' + 2y = x^2, \quad x > 0, \quad y(2) = 2$$

Mult. with x gives $\underbrace{x^2 \cdot y' + 2x \cdot y}_{\frac{d}{dx}(x^2 y)} = x^3$

and then $x^2 y = \int dx\, x^3 = \frac{1}{4}x^4 + C$

i.e. $y = \left(\frac{x}{2}\right)^2 + \frac{C}{x^2}$

where $2 = y(2) = \left(\frac{2}{2}\right)^2 + \frac{C}{2^2}$ i.e. $C = 4$

Thus $y = \left(\frac{x}{2}\right)^2 + \left(\frac{2}{x}\right)^2, \quad x > 0$

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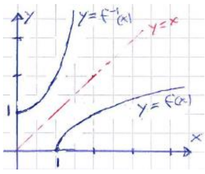
EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

Examination TEN1 – 2015-09-28 Maximum points for subparts of the problems in the final examination

1. $2x + ey = 4e$

1p: Correctly formulated an equation equalizing the explicit derivative of the function of the curve with the explicit slope of the straight line
1p: Correctly solved the equation for the intersection $\tau \cap \gamma$ respecting the domain of the logarithmic function
1p: Correctly formulated an equation for the topical tangent line τ to the curve γ

2. $f^{-1}(x) = 1 + x^2$
$D_{f^{-1}} = [0, \infty)$
$V_{f^{-1}} = [1, \infty)$



1p: Correctly determined the expression for, and the domain of, f^{-1}
1p: Correctly determined the the range of the function f^{-1}
1p: Correctly sketched the graphs of f and f^{-1}

3. $(2/3)^3 = 8/27$

Note: The student who have not taken account of the absolute value bars can not obtain more than **0p**.

1p: Correctly divided the integral in two integrals, each in which the absolute value bar can be removed
1p: Provided that the division in two integrals is proper, correctly determined an antiderivative for each integrand
1p: Correctly determined the value of the integral

4. $V_f = [25, 81]$

Note: It is not necessary to have referred properly to the theorems (of intermediate values and extreme values) supporting a correctly given answer. It is enough to have correctly conducted a first derivative test, and to have drawn the right conclusions thereof.

1p: Correctly found the stationary point of the function
1p: Correctly with e.g. a 1st-derivative test found the local extreme points of the function
1p: Correctly found the extreme values of the function, and correctly determined the range of f

5. The series is geometric if $(x = -2) \vee (x = 2)$. In both cases, the series converges (the ratio equals $5/7$) and the sum of the series equals 98.

1p: Correctly determined the x for which the series is a geometric series
1p: Correctly explained why both the series are convergent
1p: Correctly determined the common sum of the two series

6. The limit exists and is equal to 2

1p: Correctly brought the terms together with a least common denominator and correctly simplified the numerator
1p: Correctly identified the dominating factors
1p: Correctly determined the limit

7. $F(x) = \frac{1}{9}\left[1 + \cos(3x) + 3x\sin(3x)\right]$

1p: Correctly worked out the first progressive step in determining the antiderivative by parts
1p: Correctly worked out the second progressive step in determining the antiderivative by parts
1p: Correctly adapted the antiderivative to the value at π

8. $y = \left(\frac{x}{2}\right)^2 - \left(\frac{2}{x}\right)^2$

1p: Correctly written the DE in standard form, correctly determined an integrating factor, and correctly reformulated the left-hand-side of the DE into an exact derivative
1p: Correctly found the general solution of the DE
1p: Correctly adapted the general solution to the initial value, and correctly summarized the solution of the IVP