Divsion of Mathematics
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Examination Vector algebra
MAA150 - TEN1
Date: 2015-11-30
Exam aids: not any

**All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.**

**1** Given the linear system

$$\begin{aligned} x+y+az &= 1 \\ x-y-z &= 0 \\ x-y+az &= -1 \end{aligned}$$

**a.** Find all solutions of the linear system for $a = 2$. (3p)

**b.** For what values of $a$ is the linear system inconsistent? (3p)

**2** Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$$

**a.** Evaluate the determinant of $A$. (3p)

**b.** Find the inverse of $A$ and use the inverse to solve the equation $AX + B = 0$. (4p)

**3** Given the planes

$$\Pi_1 : x+y-z=2 \text{ and } \Pi_2 : -x-y+z=-3$$

**a.** Show that the planes are parallel. (2p)

**b.** Find the distance between $\Pi_1$ and $\Pi_2$. (4p)

**4** Find all solutions to the equation $z^2 + 4iz - 4 - 8i = 0$. Give the answer in the form $a + bi$. (6p)

Avdelningen för Matematik
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Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2015-11-30
Hjälpmedel: inga

**Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar!**

**1** Givet ekvationssystemet

$$\begin{aligned} x + y + az &= 1 \\ x - y - z &= 0 \\ x - y + az &= -1 \end{aligned}$$

**a.** Bestäm alla lösningar till ekvationssystemet för $a = 2$. (3p)

**b.** För vilka värden på $a$ är ekvationssystemet inkonsistent? (3p)

**2** Låt

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ och } B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$$

**a.** Beräkna determinanten av $A$. (3p)

**b.** Bestäm inversen till $A$ och använd inversen för att lösa ekvationen $AX + B = 0$. (4p)

**3** Givet planen

$$\Pi_1 : x + y - z = 2 \text{ och } \Pi_2 : -x - y + z = -3$$

**a.** Visa att planen är parallella. (2p)

**b.** Bestäm avståndet mellan $\Pi_1$ och $\Pi_2$. (4p)

**4** Bestäm alla lösningar till ekvationen $z^2 + 4iz - 4 - 8i = 0$. Svara på formen $a + bi$. (6p)

# MAA150 Vektoralgebra, ht-15.

## Assessment criterias for TEN1 2015-11-30

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### General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

### Assessment problems

**1.** a. Complete solution gives 3 points. Relevant row operations gives maximum 2 points. Correct answer gives 1 point.

b. Complete solution gives 3 points. Relevant row operations gives 1 point. Correctly finding a row for determining consistency and deducing the correct answer gives 2 points.

**2.** a. Compting $\det(A)$ correct gives 3 points. Relevant cofactor expansion or row reduction gives maximum 2 points. Correct answer gives 1 point.

b. Complete solution gives 4 points. Determining the inverse gives 3 points. Correct method with relevant row operations gives 2 points. Correct answer gives 1 point. Finding $X$ gives 1 point.

**3.** a. Stating a correct criteria that two planes are parallel gives 1 point. Showing that it is satisfied gives 1 point.

b. Complete solution gives 4 points. Correct method gives maximum 2 points. The computations involved gives maximum 2 points.

**4.** Complete solution gives 6 points. The correct method gives maximum 2 points, where setting $z = x + yi$ gives 1 point and finding the equation system for $x$ and $y$ gives 1 point. Solving for $x$ and $y$ correctly gives 2 points. The correct answer gives 1 point for each root.

# Solutions Vector algebra TEN1 2015-11-30

①a
$$\begin{cases} x+y+2z=1\\ x-y-z=0\\ x-y+2z=-1\end{cases}\qquad \left[\begin{array}{ccc|c}1&1&2&1\\1&-1&-1&0\\1&-1&2&-1\end{array}\right]\begin{matrix}\textcircled{-1}\ \textcircled{-1}\\ \leftarrow\\ \leftarrow\end{matrix}\sim$$

$$\sim\left[\begin{array}{ccc|c}1&1&2&1\\0&-2&-3&-1\\0&-2&0&-2\end{array}\right]\begin{matrix}\\ \leftarrow\\ \textcircled{-1}\end{matrix}\sim\left[\begin{array}{ccc|c}1&1&2&1\\0&0&-3&1\\0&-2&0&-2\end{array}\right]\begin{matrix}\\ \updownarrow\\ \end{matrix}\sim\left[\begin{array}{ccc|c}1&1&2&1\\0&1&0&1\\0&0&1&-1/3\end{array}\right]\begin{matrix}\leftarrow\ \leftarrow\\ \textcircled{-1}\\ \textcircled{-2}\end{matrix}$$

$$\sim\left[\begin{array}{ccc|c}1&0&0&2/3\\0&1&0&1\\0&0&1&-1/3\end{array}\right]$$

**Answer**: $x=2/3$, $y=1$, $z=-1/3$

①b
$$\left[\begin{array}{ccc|c}1&1&a&1\\1&-1&-1&0\\1&-1&a&-1\end{array}\right]\begin{matrix}\textcircled{-1}\\ \leftarrow\\ \leftarrow\end{matrix}\sim\left[\begin{array}{ccc|c}1&1&a&1\\0&-2&-1-a&-1\\0&-2&0&-2\end{array}\right]\begin{matrix}\\ \leftarrow\\ \textcircled{-1}\end{matrix}\sim\left[\begin{array}{ccc|c}1&1&a&1\\0&0&-1-a&1\\0&-2&0&-2\end{array}\right]\times\left(-\tfrac{1}{2}\right)$$

$$\sim\left[\begin{array}{ccc|c}1&1&a&1\\0&0&-1-a&1\\0&1&0&1\end{array}\right]\begin{matrix}\\ \updownarrow\\ \end{matrix}\sim\left[\begin{array}{ccc|c}1&1&a&1\\0&1&0&1\\0&0&-1-a&1\end{array}\right]\qquad\begin{cases}x+y+az=1\\ y=1\\ (-1-a)z=1 \quad ③\end{cases}$$

if $a=-1$ ③ is $0=1$, so the system is inconsistent. For all other values of $a$, the system is consistent.

**Answer**: The system is inconsistent if $a=-1$.

## 2a

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \begin{matrix} \textcircled{1} \\ \leftarrow \\ \phantom{0} \end{matrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \begin{matrix} \phantom{0} \\ \textcircled{1} \\ \leftarrow \end{matrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} =$$

$$\underset{\substack{\uparrow \\ \text{upper triangular matrix}}}{=} 1 \cdot 1 \cdot 2 = 2$$

Answer : $\det(A) = 2$

## 2b

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \textcircled{1} \\ \leftarrow \\ \phantom{0} \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \phantom{0} \\ \textcircled{1} \\ \leftarrow \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \begin{matrix} \phantom{0} \\ \phantom{0} \\ \times(\frac{1}{2}) \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right] \begin{matrix} \phantom{0} \\ \leftarrow \\ \textcircled{-1} \end{matrix} \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right] \begin{matrix} \leftarrow \\ \textcircled{-2} \\ \phantom{0} \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & \underbrace{1/2 \quad 1/2 \quad 1/2}_{A^{-1}} \end{array}\right]$$

$$AX + B = 0 \iff AX = -B \iff X = A^{-1}(-B) \iff$$

$$X = -A^{-1}B = -\begin{bmatrix} 0 & -1 & 1 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & -2 \end{bmatrix} = -\begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & -2 \\ -2 & 0 \end{bmatrix}$$

Answer : $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 3 \\ -2 & -2 \\ -2 & 0 \end{bmatrix}$

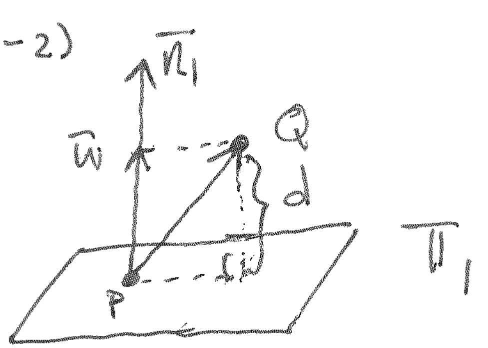
(3a) $\begin{cases} \Pi_1 : x + y - z = 2 & ① \\ \Pi_2 : -x - y + z = -3 & ② \end{cases}$

$\left.\begin{array}{l} \text{Normal to } \Pi_1 : \overline{n}_1 = (1,1,-1) \\ \text{Normal to } \Pi_2 : \overline{n}_2 = (-1,-1,1) \end{array}\right\}$ $\overline{n}_1 = -\overline{n}_2$, so the planes are parallel

(3b) Take any $Q \in \Pi_2$, e.g. let $x = y = 0 \Rightarrow z = -3$ by ②, so $Q = (0,0,-3)$

Take any $P \in \Pi_1$, e.g. let $x = y = 0 \Rightarrow z = -2$

i.e. $P = (0,0,-2)$



Let $\overline{w} = \text{proj}_{\overline{n}_1}\overrightarrow{PQ}$, then $d = \|\overline{w}\|$

$$\overrightarrow{PQ} = (0,0,-3) - (0,0,-2) = (0,0,-1)$$

$$\overline{w} = \text{proj}_{\overline{n}_1}\overrightarrow{PQ} = \frac{(0,0,-1) \bullet (1,1,-1)}{\|(1,1,-1)\|^2}(1,1,-1)$$

$$= \frac{1}{\left(\sqrt{1^2 + 1^2 + (-1)^2}\right)^2} \cdot (1,1,-1) = \frac{1}{3}(1,1,-1)$$

$$d = \|\overline{w}\| = \frac{1}{3} \cdot \|(1,1,-1)\| = \frac{1}{3} \cdot \sqrt{3} = \frac{1}{\sqrt{3}}$$

<u>Answer</u>: The distance is $\frac{1}{\sqrt{3}}$.

④ $z^2+4iz-4-8i=0$

$(z+2i)^2-(2i)^2-4-8i=0$

$(z+2i)^2-8i=0$

$(z+2i)^2=8i \underset{w=z+2i}{\iff} w^2=8i$

Let $w=x+yi$, then $w^2=x^2+2xyi-y^2=8i$ so

$$\begin{cases} x^2-y^2=0 \\ 2xy=8 \end{cases} \iff \begin{cases} x^2-y^2=0 & \text{①} \\ xy=4 & \text{②} \end{cases}$$

$|w^2|=|8i|=8$, $|w|^2=x^2+y^2$ gives

③ $x^2+y^2=8$

①+③ gives $2x^2=8 \iff x^2=4 \iff x=\pm 2$

② gives $x=2 \Rightarrow y=2$

$x=-2 \Rightarrow y=-2$

Then

$w_1=z_1+2i=2+2i \iff z_1=2$

$w_2=z_2+2i=-2-2i \iff z_2=-2-4i$

Answer: $z=2$ or $z=-2-4i$