MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA153 Linear Algebra
Date: 2016-08-17 *Write time:* 5 hours
Aid: Writing materials, ruler

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. Find an orthonormal basis for the three-dimensional Euclidean space $\mathbb{E}$ for which the inner product is fixed as

$$\langle \mathbf{u}|\mathbf{v}\rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3 - (x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2)\,,$$

where x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of $\mathbf{u}$ and $\mathbf{v}$ respectively in the (ordered) basis $(1,0,0), (0,1,0), (0,0,1)$.

2. A linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ projects vectors along the vector $(1,-2,1)$ on the subspace $\{(a,b,c) \in \mathbb{R}^3 : 2a + 2b + c = 0\}$ of $\mathbb{R}^3$. Find the matrix of F with respect to the standard basis.

3. Explain which type of surface that is described by the equation

$$2x^2 - 3y^2 + 4yz = 1,$$

where x, y, z denotes the coordinates of a point in an orthonormal system. Find also the distance between the surface and the origin, and check whether any rotational symmetry exists. Find if so for the latter an equation (expressed in x, y, z) for the rotational axis.

4. The linear transformation $F : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$F((a,b,c,d)) = (4a - 3b + 2d, 2a - 5b + c + 3d, 3a - 7b + 2c + 4d, -a + 4b + c - 3d).$$

Find a basis for each of the image of F and the kernel of F.

5. Let $\mathcal{M}$ denote the linear space of all real-valued, anti-symmetric (skew-symmetric) 3×3-matrices, i.e. real-valued matrices M which satisfy $M^T = -M$. Prove att

$$M_1 = \begin{pmatrix} 0 & -6 & 2 \\ 6 & 0 & 7 \\ -2 & -7 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

is a basis for $\mathcal{M}$. Also, find the coordinates of the matrix $M = \begin{pmatrix} 0 & 3 & -1 \\ -3 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ with respect to the basis M_1, M_2, M_3.

6. Let $\mathcal{P}$ be the linear space which is spanned by the real-valued polynomial functions p_n, $n = 0, 1, 2, \ldots$ where $p_n(x) = x^n$ in the interval $[0,1]$, and which is equipped with the inner product $\langle p|q\rangle = \int_0^1 x^2 p(x)q(x)\,dx$. Find the orthogonal projection of $p_0 + p_2$ on the subspace spanned by p_1. Also, find its length.

7. The subspace $\mathbb{U}$ of $\mathbb{R}^4$ is spanned by the vectors $(1,2,\beta,1)$, $(2,\beta,1,1)$, $(3,3,1,2)$ and $(4,5,6,3)$. Find, for each value of β, the dimension of $\mathbb{U}$ and a basis for $\mathbb{U}$.

8. The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F((x_1, x_2, x_3)) = (x_1 + 6x_2, 6x_1 + x_2, -5x_3).$$

Prove that F is diagonalizable and find a basis of eigenvectors of F. Also, specify the matrix of F with respect to the chosen basis of eigenvectors.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Bestäm en ortonormerad bas för det tredimensionella euklidiska rum $\mathbb{E}$ för vilket skalärprodukten är fixerad till

$$\langle \mathbf{u}|\mathbf{v}\rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3 - (x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2)\,,$$

där x_1, x_2, x_3 och y_1, y_2, y_3 är koordinaterna för $\mathbf{u}$ respektive $\mathbf{v}$ i (den ordnade) basen $(1,0,0), (0,1,0), (0,0,1)$.

2. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ projicerar vektorer längs vektorn $(1,-2,1)$ på underrummet $\{(a,b,c) \in \mathbb{R}^3 : 2a+2b+c = 0\}$ till $\mathbb{R}^3$. Bestäm avbildningsmatrisen för F med avseende på standardbasen.

3. Förklara vilken typ av yta som beskrivs av ekvationen

$$2x^2 - 3y^2 + 4yz = 1,$$

där x, y, z betecknar en punkts koordinater i ett ON-system. Bestäm även avståndet mellan ytan och origo, och undersök om någon rotationssymmetri föreligger. Bestäm i så fall för det senare en ekvation (uttryckt i x, y, z) för rotationsaxeln.

4. Den linjära avbildningen $F : \mathbb{R}^4 \to \mathbb{R}^4$ är definierad enligt

$$F((a,b,c,d)) = (4a-3b+2d, 2a-5b+c+3d, 3a-7b+2c+4d, -a+4b+c-3d).$$

Bestäm en bas för vart och ett av F:s värderum och F:s nollrum.

5. Låt $\mathcal{M}$ beteckna det linjära rummet av alla reellvärda, antisymmetriska (skevsymmetriska) 3×3-matriser, dvs reellvärda matriser M som uppfyller $M^T = -M$. Bevisa att

$$M_1 = \begin{pmatrix} 0 & -6 & 2 \\ 6 & 0 & 7 \\ -2 & -7 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

är en bas för $\mathcal{M}$. Bestäm även koordinaterna för matrisen $M = \begin{pmatrix} 0 & 3 & -1 \\ -3 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ med avseende på basen M_1, M_2, M_3.

6. Låt $\mathcal{P}$ vara det linjära rum som spänns upp av de reellvärda polynomfunktionerna p_n, $n = 0, 1, 2, \ldots$ där $p_n(x) = x^n$ i intervallet $[0,1]$, och som är utrustat med skalärprodukten $\langle p|q\rangle = \int_0^1 x^2 p(x)q(x)\,dx$. Bestäm den ortogonala projektionen av $p_0 + p_2$ på underrummet som spänns upp av p_1. Bestäm även dess längd.

7. Underrummet $\mathbb{U}$ till $\mathbb{R}^4$ spänns upp av vektorerna $(1,2,\beta,1)$, $(2,\beta,1,1)$, $(3,3,1,2)$ och $(4,5,6,3)$. Bestäm, för varje värde på β, dimensionen av $\mathbb{U}$ och en bas för $\mathbb{U}$.

8. Den linjära operatorn $F : \mathbb{R}^3 \to \mathbb{R}^3$ ges av

$$F((x_1,x_2,x_3)) = (x_1 + 6x_2, 6x_1 + x_2, -5x_3).$$

Bevisa att F är diagonaliserbar och bestäm en bas av egenvektorer till F. Specificera även F:s matris med avseende på den valda basen av egenvektorer.

If you prefer the problems formulated in English, please turn the page.

MAA153 & MMA129 / Solutions to the final exam 2016-08-17

① $\langle u|v\rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3 - (x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2)$

$$= (x_1\ x_2\ x_3)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where the coordinates are given in the standard basis $(1,0,0), (0,1,0), (0,0,1)$, here denoted by e_1, e_2, e_3.

Let $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ be the ON-basis formed by the Gram-Schmidt procedure on e_1, e_2, e_3. We then get

$$\|e_1\|^2 = (1\,0\,0)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 \qquad \text{Thus } \tilde{e}_1 = \frac{1}{\sqrt{2}}(1,0,0)$$

$$f_2 = e_2 - \langle e_2|\tilde{e}_1\rangle\tilde{e}_1 = e_2 - \frac{1}{2}(0\,1\,0)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e_1 = e_2 - \frac{1}{2}(-1)e_1$$

$$= (0,1,0) + \frac{1}{2}(1,0,0) = \frac{1}{2}(1,2,0)$$

where $\|(1,2,0)\|^2 = (1\,2\,0)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = (1\,2\,0)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = 2$ Thus $\tilde{e}_2 = \frac{1}{\sqrt{2}}(1,2,0)$

$$f_3 = e_3 - \langle e_3|\tilde{e}_1\rangle\tilde{e}_1 - \langle e_3|\tilde{e}_2\rangle\tilde{e}_2 = e_3 - \frac{1}{2}(0\,0\,1)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}e_1 - \frac{1}{2}(0\,0\,1)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}(1,2,0)$$

$$= (0,0,1) - 0(1,0,0) + (1,2,0) = (1,2,1)$$

where $\|(1,2,1)\|^2 = (1\,2\,1)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (1\,2\,1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$ Thus $\tilde{e}_3 = (1,2,1)$

Answer An ON-basis for E is e.g. $\frac{1}{\sqrt{2}}(1,0,0), \frac{1}{\sqrt{2}}(1,2,0), (1,2,1)$

② $F: R^3 \to R^3$ projects vectors along the vector $(1,-2,1)$ on the linear space $\{(a,b,c) \in R^3 : 2a+2b+c=0\} = M$

We notice that $F((1,-2,1)) = (0,0,0)$ and that $F(u) = u$ for all $u \in M$. Since $u_M = (a, b, -2a-2b) = a(1,0,-2) + b(0,1,-2)$ we have that $(1,0,-2), (0,1,-2)$ is a basis for M.

Thus the matrix A of F in the standard basis is given by $A\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

i.e. by $A\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix}$

We get $$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix}\frac{1}{(0+1+0)-(0-2+4)}\begin{pmatrix} 2 & 2 & 1 \\ -3 & -2 & -1 \\ 4 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 3 & 2 & 1 \\ -4 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ -4 & -3 & -2 \\ 2 & 2 & 2 \end{pmatrix}$$

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(3) $1 = 2x^2 - 3y^2 + 4yz = X^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 0 \end{pmatrix} X = X^T G X$

where x, y, z are the coordinates in an ON-system.

The eigenvalues of the symmetric operator which has the matrix G are given by

$$0 = \det(G - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & -3-\lambda & 2 \\ 0 & 2 & -\lambda \end{pmatrix} = -(\lambda-2)[(\lambda+3)\lambda - 4] = -(\lambda+4)(\lambda-1)(\lambda-2)$$

$$\begin{cases} \lambda_1 = -4: & G - \lambda_1 I = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}; \quad k_1 = t_1 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad t_1 \neq 0 \\ \lambda_2 = 1: & G - \lambda_2 I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}; \quad k_2 = t_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad t_2 \neq 0 \\ \lambda_3 = 2: & G - \lambda_3 I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad k_3 = t_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad t_3 \neq 0 \end{cases}$$

The orthogonal matrix $S = \begin{pmatrix} 0 & 0 & 1 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$ is a change-of-basis matrix from the given ON-basis to an ON-basis of eigenvectors (of the symmetric operator which has the matrix G in the original ON-basis). Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the coordinates in the ON-basis of eigenvectors. Then, the equation $1 = X^T G X$ becomes $1 = \tilde{X}^T S^T S \tilde{G} S^{-1} S \tilde{X} = \tilde{X}^T \tilde{G} \tilde{X} = -4\tilde{x}^2 + \tilde{y}^2 + 2\tilde{z}^2$ from which we conclude that the surface is a one-sheeted hyperboloid for which $\tilde{y}^2 + 2\tilde{z}^2 = 1 + 4\tilde{x}^2 \geq 1 + 0$ implies that the distance(surface, origin) $= \min\left(1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ (the minor half-axis-lengt of the ellipse $\left(\frac{\tilde{y}}{1}\right)^2 + \left(\frac{\tilde{z}}{1/\sqrt{2}}\right)^2 = 1$). Since the half-axes of the ellipse $\tilde{y}^2 + 2\tilde{z}^2 = 1$ are different, there is no rotational symmetry.

(4) $F: R^4 \to R^4$, $F((a,b,c,d)) = (4a-3b+2d, 2a-5b+c+3d, 3a-7b+2c+4d, -a+4b+c-3d)$.

The matrix A of F in the standard basis is

$$A = \begin{pmatrix} 4 & -3 & 0 & 2 \\ 2 & -5 & 1 & 3 \\ 3 & -7 & 2 & 4 \\ -1 & 4 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 13 & 4 & -10 \\ 0 & 3 & 3 & -3 \\ 0 & 5 & 5 & -5 \\ -1 & 4 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & -9 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which gives that a basis for $\mathrm{im}(F)$ is e.g. $(4,2,3,-1), (-3,-5,-7,4), (0,1,2,1)$ and that vectors (x_1, x_2, x_3, x_4) in $\ker(F)$ are given by

$$\begin{cases} x_1 = 0 \\ 3x_2 - 2x_4 = 0 \\ 3x_3 - x_4 = 0 \end{cases} \quad \text{i.e.} \quad u_{\ker(F)} = \left(0, \tfrac{2}{3}x_4, \tfrac{1}{3}x_4, x_4\right) = \tfrac{1}{3}x_4 (0, 2, 1, 3)$$

i.e. a basis for $\ker(F)$ is e.g. $(0,2,1,3)$.

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⑤ A general vector of $\mathcal{M}$ is $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ and is equal to
$a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = ae_1 + be_2 + ce_3$
where e_1, e_2, e_3 are linearly independent, i.e. the vectors e_1, e_2, e_3 constitutes a basis for $\mathcal{M}$. In that basis, the matrices M_1, M_2, M_3 can be expressed as

$$\begin{cases} M_1 = -6e_1 + 2e_2 + 7e_3 \\ M_2 = 2e_1 - e_2 - 2e_3 \\ M_3 = e_1 - e_3 \end{cases} \quad \text{i.e. } (M_1\ M_2\ M_3) = (e_1\ e_2\ e_3)\underbrace{\begin{pmatrix} -6 & 2 & 1 \\ 2 & -1 & 0 \\ 7 & -2 & -1 \end{pmatrix}}_{S}$$

where $S \sim \begin{pmatrix} 0 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

i.e. $\operatorname{rank}(S) = 3$ i.e. S is a change-of-basis matrix
i.e. M_1, M_2, M_3 is a basis since e_1, e_2, e_3 is a basis q.e.d.

Furthermore, $M = \begin{pmatrix} 0 & 3 & -1 \\ -3 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} = 3e_1 - e_2 - 2e_3 = (e_1\ e_2\ e_3)\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

$$= (M_1\ M_2\ M_3)\, S^{-1}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = (M_1\ M_2\ M_3)\frac{1}{(-6+0-4)-(-7+0-4)}\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$= (M_1\ M_2\ M_3)\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} = M_1 + 3M_2 + 3M_3$$ i.e. the coordinates of M with respect to the basis M_1, M_2, M_3 are $1, 3, 3$.

⑥ $\begin{cases} \mathcal{P} = \operatorname{span}\{p_0, p_1, p_2\} \text{ where } p_n(x) = x^n \text{ in } [0,1] \\ \mathcal{P} \text{ is equipped with the inner product } \langle p | q \rangle = \int_0^1 x^2 p(x) q(x)\, dx \end{cases}$

The orthogonal projection of $p_0 + p_2$ on $\operatorname{span}\{p_1\}$ is

$$\frac{\langle p_0 + p_2 | p_1 \rangle}{\langle p_1 | p_1 \rangle} p_1 = \frac{\int_0^1 x^2(1+x^2)\, x\, dx}{\int_0^1 x^2 \cdot x \cdot x\, dx} p_1 = \frac{\left[\frac{1}{4}x^4 + \frac{1}{6}x^6\right]_0^1}{\left[\frac{1}{5}x^5\right]_0^1} p_1$$

$$= \frac{\frac{1}{4} + \frac{1}{6}}{\frac{1}{5}} p_1 = 5 \cdot \frac{6+4}{24} p_1 = \frac{25}{12} p_1$$

The length of orthogonal projection is

$$\left\| \frac{25}{12} p_1 \right\| = \frac{25}{12} \|p_1\| = \frac{25}{12}\sqrt{\langle p_1 | p_1 \rangle} \underset{\uparrow\ \text{from above}}{=} \frac{25}{12}\sqrt{\frac{1}{5}} = \frac{5\sqrt{5}}{12}$$

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(7) $U = \text{span}\{\underbrace{(1,2,\beta,1)}_{u_1}, \underbrace{(2,\beta,1,1)}_{u_2}, \underbrace{(3,3,1,2)}_{u_3}, \underbrace{(4,5,6,3)}_{u_4}\}$

The coordinate matrix for u_1, u_2, u_3, u_4 is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \beta & 3 & 5 \\ \beta & 1 & 1 & 6 \\ 1 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & \beta-4 & -3 & -3 \\ 0 & 1-2\beta & 1-3\beta & 6-4\beta \\ 0 & -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\beta & 5-2\beta \\ 0 & 0 & 1-\beta & 1-\beta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\beta & 5-2\beta \\ 0 & 0 & 1 & \beta-4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \beta-4 \\ 0 & 0 & 0 & (\beta-1)(\beta-5) \end{pmatrix}. \quad \text{i.e. } \dim(U) = \begin{cases} 3 & \text{if } \beta=1 \vee \beta=5 \\ 4 & \text{if } \beta \neq 1,5 \end{cases}$$

A basis for U is e.g. u_1, u_2, u_3 if $\beta=1 \vee \beta=5$, and e.g. the standard basis if $\beta \neq 1,5$.

(8) $F: R^3 \to R^3$ where $F((x_1,x_2,x_3)) = (x_1+6x_2,\ 6x_1+x_2,\ -5x_3)$

The matrix A of F in the standard basis is $\begin{pmatrix} 1 & 6 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix}$

Eigenvalues of F: $0 = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 6 & 0 \\ 6 & 1-\lambda & 0 \\ 0 & 0 & -5-\lambda \end{pmatrix}$

$= -(\lambda+5)[(\lambda-1)^2-6^2] = -(\lambda+5)^2(\lambda-7)$

$\lambda_{1,2} = -5$: $A-\lambda_{1,2}I = \begin{pmatrix} 6 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $k_{1,2} = t_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

where at least one of t_1, t_2 is nonzero.

Since the dimension of the eigenspace for the repeated eigenvalue equals 2, the operator F is diagonalizable. q.e.d.

$\lambda_3 = 7$: $A-\lambda_3 I = \begin{pmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \\ 0 & 0 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; $k_3 = t_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $t_3 \neq 0$

A basis of eigenvectors is e.g. $(0,0,1), (1,-1,0), (1,1,0)$.

The matrix $\tilde{A}$ of F in the chosen (ordered) basis of eigenvectors is equal to $\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

i.e. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

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EXAMINATION IN MATHEMATICS
MAA153 Linear algebra & MMA129 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2015/16

Examination 2016-08-17	Maximum points for subparts of the problems in the final examination

1. An ON-basis for the Euclidean space E is e.g. $\frac{1}{\sqrt{2}}(1,0,0)$, $\frac{1}{\sqrt{2}}(1,2,0)$, $(1,2,1)$

1p: Correctly interpreted the given inner product (in terms of at least one explicit evaluation)
1p: Correctly applied the Gram-Schmidt procedure for finding an ON-basis from the standard basis
3p: Correctly determined an ON-basis (**1p** for each vector)

2. $\begin{pmatrix} 3 & 2 & 1 \\ -4 & -3 & -2 \\ 2 & 2 & 2 \end{pmatrix}$

1p: Correctly noted that F maps vectors in the subspace on themselves, i.e. $F(u)=u$ for each u in the subspace
1p: Correctly from the condition for the given subspace identified two linearly independent vectors which span the subspace
1p: Correctly noted that F maps vectors in the span of $(1,-2,1)$ on the zero vector, i.e. $F(v)=0$ for each v in the span of $(1,-2,1)$
1p: Correctly on the form CB^{-1}, and in the standard basis, found the matrix A of the linear transformation F
1p: Correctly found the explicit expression for the matrix A

3. By diagonalization, the equation may be reformulated as $1=-4\tilde{x}^2+\tilde{y}^2+2\tilde{z}^2$ which, based on the fact that $\tilde{x},\tilde{y},\tilde{z}$ denotes the coordinates of a point in an ON-system, describes a one-sheeted hyperboloid without any rotational symmetry and with a distance to the origin equal to $\frac{1}{\sqrt{2}}$ l.u.

2p: Correctly found that $2x^2-3y^2+4yz=1$ describes a one-sheeted hyperboloid
1p: Correctly concluded that there is no rotational symmetry of the surface
2p: Correctly found the distance between the surface and the origin

4. A basis for the kernel of F is e.g. $(0,2,1,3)$

A basis for the image of F is e.g. $(4,2,3,-1)$, $(-3,-5,-7,4)$, $(0,1,2,1)$

2p: Correctly found a basis for the kernel of F
3p: Correctly found a basis for the image of F

5. A basis for $\mathcal{M}$ is e.g. e_1,e_2,e_3 where

$$e_1=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad e_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad e_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The coordinates of the matrix M with respect to the basis M_1, M_2, M_3 are $1,3,3$.

2p: Correctly proven that M_1, M_2, M_3 is a basis for $\mathcal{M}$
3p: Correctly found the coordinates of the matrix M relative to the basis M_1, M_2, M_3

6.	$(p_0+p_2)_{p_1} = \frac{25}{12}p_1$ $\left\|(p_0+p_2)_{p_1}\right\| = \frac{5\sqrt{5}}{12}$	**1p**: Correctly stated an expression for the orthogonal projection of p_0+p_2 on p_1 **3p**: Correctly evaluated the orthogonal projection **1p**: Correctly found the length of the orthogonal projection
7.	$\dim(U) = \begin{cases} 3 & \text{if } (\beta=1)\vee(\beta=5) \\ 4 & \text{if } \beta\neq 1,5 \end{cases}$ $(\beta=1)\vee(\beta=5)$: A basis for U is e.g. $(1,2,\beta,1),(2,\beta,1,1),(3,3,1,2)$ $\beta\neq 1,5$: A basis for the U is e.g. the standard basis in $\mathcal{R}^4$	**1p**: Correctly initiated an analysis of the vectors spanning the subspace U, and correctly determined the reduced row-echelon form of the coordinate matrix of the vectors **1p**: Correctly concluded that there are two cases, namely $(\beta=1)\vee(\beta=5)$ and $\beta\neq 1,5$ respectively **2p**: Correctly in the case $(\beta=1)\vee(\beta=5)$ found the dimension of and a basis for the subspace U **1p**: Correctly in the case $\beta\neq 1,5$ found the dimension of and a basis for the subspace U
8.	The linear operator is diagonalizable since the eigenspace for the repeated eigenvalue is two-dimensional. A basis of eigenvectors is e.g. $(0,0,1)$, $(1,-1,0)$, $(1,1,0)$. The matrix of F with respect to the chosen (ordered) basis of eigenvectors equals $\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$	**1p**: Correctly proven that F is diagonalizable **1p**: Correctly found a two-dimensional basis for the eigen-space spanned by the eigenvectors with the repeated eigenvalue -5 **1p**: Correctly found a basis for the the eigenspace spanned by the eigenvectors with the eigenvalue 7, and correctly stated a basis of eigenvectors **2p**: Correctly found the matrix of F with respect to the chosen (ordered) basis of eigenvectors