MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA153 Linear Algebra
Date: 2018-01-09 *Write time:* 5 hours
Aid: Writing materials, ruler

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. Find for each real value of a, the dimension of and a basis for the subspace $\text{span}\{(3,1,-2,4),(7,2,0,9),(-14,-4,1,a-15),(4a,a,3a,6a)\}$ of $\mathbb{R}^4$. Also, find all values of a for which the vector $v=(7a+10,2a+3,-4,6a+5)$ belong to the subspace, and find for those a the coordinates of v relative to the chosen bases respectively.

2. Let $\mathcal{P}_2$ be the vector space spanned by the polynomial functions p_0, p_1, p_2 where $p_n(x)=x^n$, and define by $F(p)(x)=p(x+2)+xp'(x+2)$ a linear operator on $\mathcal{P}_2$. Prove that F has an inverse and find $F^{-1}(3p_0+2p_1-p_2)$.

3. Find the geometric meaning of the equation

$$c=(2x+y-2z)^2+(x+3y-3z)^2-(x-2y+z)^2$$

for every real number c.

4. The linear operator $F:\mathbb{R}^4\to\mathbb{R}^4$ is with $u=(x_1,x_2,x_3,x_4)$ defined according to

$$F(u)=(2x_1-x_2+3x_3-x_4,x_1+x_2-2x_4,-x_1+3x_2-4x_3-2x_4,x_1-2x_2+3x_3+x_4).$$

Find the kernel and the image of F, and find out whether F is bijective or not.

5. Find an ON-basis for $M=\text{span}\{(1,0,1,0),(0,1,0,1)\}\subset E$, where E is the vector space $\mathbb{R}^4$ equipped with the inner product $\langle\ |\ \rangle$ given by

$$\begin{aligned}\langle u|v\rangle &= x_1y_1+x_2y_2+18x_3y_3+10x_4y_4+(x_1y_3+x_3y_1)\\ &\quad -3(x_2y_3+x_3y_2)-2(x_2y_4+x_4y_2)+12(x_3y_4+x_4y_3).\end{aligned}$$

where $u=(x_1,x_2,x_3,x_4)$ and $v=(y_1,y_2,y_3,y_4)$.

6. Let $m_1=\begin{pmatrix}1&2\\2&-3\end{pmatrix}$, $m_2=\begin{pmatrix}-2&-1\\-1&1\end{pmatrix}$, $m_3=\begin{pmatrix}3&1\\1&4\end{pmatrix}$. Prove that m_1,m_2,m_3 is a basis for the vector space of all symmetric 2×2-matrices with real-valued entries. Also, find the coordinates of $\begin{pmatrix}2&5\\5&-3\end{pmatrix}$ relative to the (ordered) basis m_1,m_2,m_3.

7. The vectors $u+v$, $u+3v$ and $2u-v$ have the lengths 1, $\sqrt{17}$ and $\sqrt{19}$ respectively in an Euclidean space E. Find the length of the orthogonal projection of the vector $F(u)$ on the vector $F(v)$, where F is an isometric linear operator $F:E\to E$.

8. The linear operator $F:\mathbb{R}^3\to\mathbb{R}^3$ has, relative to the standard basis, the matrix

$$\begin{pmatrix}3&0&1\\2&1&1\\1-\beta&1+\beta&2\end{pmatrix}$$

where $\beta\in\mathbb{R}$. Find the numbers β for which the operator är diagonalizable, and state a basis of eigenvectors for each of these β.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Bestäm för varje reellt värde på a dimensionen av och en bas för delrummet $\operatorname{span}\{(3,1,-2,4),(7,2,0,9),(-14,-4,1,a-15),(4a,a,3a,6a)\}$ till $\mathbb{R}^4$. Bestäm även alla värden på a för vilka vektorn $v=(7a+10,2a+3,-4,6a+5)$ tillhör delrummet, och ange för dessa a koordinaterna för v relativt respektive av de valda baserna.

2. Låt $\mathcal{P}_2$ vara det linjära rummet som spänns upp av polynomfunktionerna p_0, p_1, p_2 där $p_n(x)=x^n$, och definiera genom $F(p)(x)=p(x+2)+xp'(x+2)$ en linjär operator på $\mathcal{P}_2$. Bevisa att F har en invers och bestäm $F^{-1}(3p_0+2p_1-p_2)$.

3. Bestäm den geometriska innebörden av ekvationen

$$c=(2x+y-2z)^2+(x+3y-3z)^2-(x-2y+z)^2$$

för varje reellt tal c.

4. Den linjära operatorn $F:\mathbb{R}^4\to\mathbb{R}^4$ är med $u=(x_1,x_2,x_3,x_4)$ definierad enligt

$$F(u)=(2x_1-x_2+3x_3-x_4,x_1+x_2-2x_4,-x_1+3x_2-4x_3-2x_4,x_1-2x_2+3x_3+x_4).$$

Bestäm F:s nollrum respektive värderum, och avgör om F är bijektiv eller inte.

5. Bestäm en ON-bas för $M=\operatorname{span}\{(1,0,1,0),(0,1,0,1)\}\subset E$, där E är vektorrummet $\mathbb{R}^4$ utrustat med skalärprodukten $\langle\ |\ \rangle$ given av

$$\begin{aligned}\langle u|v\rangle &= x_1y_1+x_2y_2+18x_3y_3+10x_4y_4+(x_1y_3+x_3y_1)\\ &\quad -3(x_2y_3+x_3y_2)-2(x_2y_4+x_4y_2)+12(x_3y_4+x_4y_3).\end{aligned}$$

där $u=(x_1,x_2,x_3,x_4)$ and $v=(y_1,y_2,y_3,y_4)$.

6. Låt $m_1=\begin{pmatrix}1&2\\2&-3\end{pmatrix}$, $m_2=\begin{pmatrix}-2&-1\\-1&1\end{pmatrix}$, $m_3=\begin{pmatrix}3&1\\1&4\end{pmatrix}$. Visa att m_1,m_2,m_3 är en bas för vektorrummet av alla symmetriska 2×2-matriser med reellvärda element. Bestäm även koordinaterna för $\begin{pmatrix}2&5\\5&-3\end{pmatrix}$ relativt (den ordnade) basen m_1,m_2,m_3.

7. Vektorerna $u+v$, $u+3v$ och $2u-v$ har längderna 1, $\sqrt{17}$ respektive $\sqrt{19}$ i ett euklidiskt rum E. Bestäm längden av den ortogonala projektionen av vektorn $F(u)$ på vektorn $F(v)$, där F är en isometrisk linjär operator $F:E\to E$.

8. Den linjära operatorn $F:\mathbb{R}^3\to\mathbb{R}^3$ har relativt standardbasen matrisen

$$\begin{pmatrix}3&0&1\\2&1&1\\1-\beta&1+\beta&2\end{pmatrix}$$

där $\beta\in\mathbb{R}$. Bestäm de tal β för vilka operatorn är diagonaliserbar, och ange en bas av egenvektorer till F för var och en av dessa β.

If you prefer the problems formulated in English, please turn the page.

MAA153 / Solutions to the final exam. 2018-01-09

① $\begin{cases} U = \operatorname{span}\{\overset{u_1}{(3,1,-2,4)}, \overset{u_2}{(7,2,0,9)}, \overset{u_3}{(-14,-4,1,a-15)}, \overset{u_4}{(4a,a,3a,6a)}\} \subset R^4 \\ v = (7a+10,\ 2a+3,\ -4,\ 6a+5) \end{cases}$

A test equation for linear dependence of u_1, u_2, u_3, u_4 and for the question whether $v \in U$ or not is $\sum_{i=1}^{4} \lambda_i u_i = v$,

i.e. $\begin{pmatrix} 3 & 7 & -14 & 4a \\ 1 & 2 & -4 & a \\ -2 & 0 & 1 & 3a \\ 4 & 9 & a-15 & 6a \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 7a+10 \\ 2a+3 \\ -4 \\ 6a+5 \end{pmatrix}$ i.e. $C\lambda = B$

where $(C|B) \sim \left(\begin{array}{cccc|c} 0 & 1 & -2 & a & a+1 \\ 1 & 2 & -4 & a & 2a+3 \\ 0 & 4 & -7 & 5a & 4a+2 \\ 0 & 1 & a+1 & 2a & -2a-7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -a & 1 \\ 0 & 1 & -2 & a & a+1 \\ 0 & 0 & 1 & a & -2 \\ 0 & 0 & a+3 & a & -3a-8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -a & 1 \\ 0 & 1 & 0 & 3a & a-3 \\ 0 & 0 & 1 & a & -2 \\ 0 & 0 & 0 & -a(a+2) & -(a+2) \end{array}\right)$

If $a=-2$ then $(C|B) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -6 & -5 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ i.e. e.g. u_1, u_2, u_3 is a basis for U, and $\operatorname{coord}_{u_1,u_2,u_3}(v) = (1,-5,-2)$

If $a=0$ then $(C|B) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right)$ i.e. e.g. u_1, u_2, u_3 is a basis for U, and $v \notin U$

If $a \neq -2, 0$ then $(C|B) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & a-6 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1/a \end{array}\right)$ i.e. e.g. u_1, u_2, u_3, u_4 is a basis for U, and $\operatorname{coord}_{u_1,u_2,u_3,u_4}(v) = (2, a-6, -3, \frac{1}{a})$

In this case even the standard basis works. The coordinates are then $7a+10, 2a+3, -4, 6a+5$

② $\begin{cases} P_2 = \operatorname{span}\{p_0, p_1, p_2\} \quad \text{where} \quad p_n(x) = x^n \\ F: P_2 \to P_2 \quad \text{where} \quad F(p)(x) = p(x+2) + x\,p'(x+2) \end{cases}$

The matrix A of F relative to the basis p_0, p_1, p_2 is given by the coordinates of the images $F(p_0)$, $F(p_1)$ and $F(p_2)$.

We have that

$$\begin{cases} F(p_0)(x) = p_0(x+2) + x(p_0)'(x+2) = 1 + x\cdot 0 = p_0(x) \\ F(p_1)(x) = p_1(x+2) + x(p_1)'(x+2) = (x+2) + x\cdot 1 = 2p_0(x) + 2p_1(x) \\ F(p_2)(x) = p_2(x+2) + x(p_2)'(x+2) = (x+2)^2 + x\,2(x+2) = 4p_0(x) + 8p_1(x) + 3p_2(x) \end{cases}$$

Whereby $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{pmatrix}$

Since $\det(A) = 6 \neq 0$, we know that $A^{-1}\,\exists$ and therefore $F^{-1}\,\exists$

q.e.d.

We get

$$F^{-1}(3p_0 + 2p_1 - p_2) = (p_0\ p_1\ p_2)\, A^{-1}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = (p_0\ p_1\ p_2)\,\frac{1}{6}\begin{pmatrix} 6 & -6 & 8 \\ 0 & 3 & -8 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$= (p_0\ p_1\ p_2)\,\frac{1}{6}\begin{pmatrix} -2 \\ 14 \\ -2 \end{pmatrix} = -\frac{1}{3}p_0 + \frac{7}{3}p_1 - \frac{1}{3}p_2$$

Cont. MAA153
2018-01-09

(3) $c = (2x+y-2z)^2 + (x+3y-3z)^2 - (x-2y+z)^2$

$= 4x^2 + 6y^2 + 12z^2 + 14xy - 18yz - 16zx$

$= \left[(2x + \frac{7}{2}y - 4z)^2 - \frac{49}{4}y^2 - 16z^2 + 28yz\right] + 6y^2 + 12z^2 - 18yz$

$= (2x + \frac{7}{2}y - 4z)^2 - \frac{25}{4}y^2 - 4z^2 + 10yz = (2x + \frac{7}{2}y - 4z)^2 - (\frac{5}{2}y - 2z)^2$

Let $\begin{cases} 2x + \frac{7}{2}y - 4z = \tilde{x} \\ \frac{5}{2}y - 2z = \tilde{y} \\ z = \tilde{z} \end{cases}$ i.e. $\begin{pmatrix} 2 & \frac{7}{2} & -4 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$

Since the matrix $\begin{pmatrix} 2 & 7/2 & -4 \\ 0 & 5/2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ is invertible, the numbers $\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of a vector u relative to a basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ (where x, y, z are the coordinates of u relative to a original basis e_1, e_2, e_3). Thus, we can conclude that the signature of the quadratic form $\underbrace{(2x+y-2z)^2 + (x+3y-3z)^2 - (x-2y+z)^2}_{q(u)}$ is $(1, -1, 0)$.

Then the equation $c = q(u)$ means a hyperbolic cylinder if $c \neq 0$ and the union of two planes if $c = 0$.

(4) $\begin{cases} F: R^4 \to R^4 \quad \text{where} \\ F(u) = (2x_1 - x_2 + 3x_3 - x_4,\ x_1 + x_2 - 2x_4,\ -x_1 + 3x_2 - 4x_3 - 2x_4,\ x_1 - 2x_2 + 3x_3 + x_4) \end{cases}$

The matrix A of F relative to the standard basis for R^4 is

$$A = \begin{pmatrix} 2 & -1 & 3 & -1 \\ 1 & 1 & 0 & -2 \\ -1 & 3 & -4 & -2 \\ 1 & -2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -3 & 3 & 3 \\ 1 & 1 & 0 & -2 \\ 0 & 4 & -4 & -4 \\ 0 & -3 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_{rref}$$

We conclude from A_{rref} that

$\mathrm{im}(F) = \mathrm{span}\{(2,1,-1,1), (-1,1,3,-2)\}$

$\ker(F) = \mathrm{span}\{(-1,1,1,0), (1,1,0,1)\}$

Since $\mathrm{im}(F)$ is a proper subset of R^4, F is not surjective and therefore also not bijective.

(A conclusion that F is not bijective also follows from the fact that $\dim(\ker(F)) > 0$, indicating that F is not injective and therefore also not bijective.)

Cont. MAA153
2018-01-09

(5)
$$\begin{cases} M = \operatorname{span}\{\overset{u_1}{(1,0,1,0)}, \overset{u_2}{(0,1,0,1)}\} \subset E \quad \underline{\text{where}} \\ \langle u|v\rangle = x_1y_1 + x_2y_2 + 18x_3y_3 + 10x_4y_4 + (x_1y_3 + x_3y_1) \\ \qquad -3(x_2y_3+x_3y_2) - 2(x_2y_4+x_4y_2) + 12(x_3y_4+x_4y_3) \end{cases}$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 1 & -3 & 18 & 12 \\ 0 & -2 & 12 & 10 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = X^T A Y$$

The Gram-Schmidt orthogonalization procedure gives an ON-basis e_1, e_2 for M as:

Step 1: $f_1 = u_1$ where $\|u_1\|^2 = \langle u_1|u_1\rangle = (1\,0\,1\,0)A\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = (1\,0\,1\,0)\begin{pmatrix}2\\-3\\19\\12\end{pmatrix} = 21$

whereby $e_1 = \frac{1}{\|f_1\|} f_1 = \frac{1}{\sqrt{21}}(1,0,1,0)$

Step 2: $f_2 = u_2 - \langle u_2|e_1\rangle e_1 = (0,1,0,1) - \frac{1}{21}(0\,1\,0\,1)\begin{pmatrix}2\\-3\\19\\12\end{pmatrix}(1,0,1,0)$

$= (0,1,0,1) - \frac{1}{21}\cdot 9\,(1,0,1,0) = \frac{1}{7}[(0,7,0,7)-(3,0,3,0)] = \frac{1}{7}(-3,7,-3,7)$

where $\|(-3,7,-3,7)\|^2 = (-3\,7\,-3\,7)A\begin{pmatrix}-3\\7\\-3\\7\end{pmatrix} = (-3\,7\,-3\,7)\begin{pmatrix}-6\\2\\6\\20\end{pmatrix} = 154$

whereby $e_2 = \frac{1}{\sqrt{154}}(-3,7,-3,7)$

i.e. $\frac{1}{\sqrt{21}}(1,0,1,0),\ \frac{1}{\sqrt{154}}(-3,7,-3,7)$ is an ON-basis for M

(6) $m_1 = \begin{pmatrix}1 & 2\\2 & -3\end{pmatrix}$, $m_2 = \begin{pmatrix}-2 & -1\\-1 & 1\end{pmatrix}$, $m_3 = \begin{pmatrix}3 & 1\\1 & 4\end{pmatrix}$, $m = \begin{pmatrix}2 & 5\\5 & -3\end{pmatrix}$

A test eq. for m_1, m_2, m_3 is

$$\begin{pmatrix}0&0\\0&0\end{pmatrix} = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 \Leftrightarrow \begin{pmatrix}0&0\\0&0\end{pmatrix} = \begin{pmatrix}\lambda_1 - 2\lambda_2 + 3\lambda_3 & 2\lambda_1 - \lambda_2 + \lambda_3 \\ 2\lambda_1 - \lambda_2 + \lambda_3 & -3\lambda_1 + \lambda_2 + 4\lambda_3\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}1&-2&3\\2&-1&1\\-3&1&4\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}1&-2&3\\0&3&-5\\0&-5&13\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}3&0&-1\\0&3&-5\\0&0&14\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$ i.e. m_1, m_2, m_3 is a linearly indep. set of vectors and is therefore a basis for the 3-dimensional vector space of all symmetric 2×2-matrices with real-valued entries.

q.e.d.

The coordinates x_1, x_2, x_3 of m relative to the basis m_1, m_2, m_3 are given by the eq. $x_1m_1 + x_2m_2 + x_3m_3 = m$

i.e. $\begin{pmatrix}1&-2&3\\2&-1&1\\-3&1&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2\\5\\-3\end{pmatrix} \Leftrightarrow \begin{pmatrix}1&-2&3\\0&3&-5\\0&-5&13\end{pmatrix}X = \begin{pmatrix}2\\1\\3\end{pmatrix} \Leftrightarrow \begin{pmatrix}3&0&-1\\0&3&-5\\0&0&14\end{pmatrix}X = \begin{pmatrix}8\\1\\14\end{pmatrix}$

$\Leftrightarrow \begin{pmatrix}3&0&0\\0&3&0\\0&0&1\end{pmatrix}X = \begin{pmatrix}9\\6\\1\end{pmatrix} \Leftrightarrow X = \begin{pmatrix}3\\2\\1\end{pmatrix}$ i.e. $\operatorname{coord}_{m_1,m_2,m_3}(m) = (3,2,1)$

Cont. MAA153
2018-01-09

7 We know that $\|u+v\|=1$, $\|u+3v\|=\sqrt{17}$, $\|2u-v\|=\sqrt{19}$

i.e. that
$$\begin{cases} 1^2=\|u+v\|^2=\langle u+v|u+v\rangle=\|u\|^2+2\langle u|v\rangle+\|v\|^2\\ (\sqrt{17})^2=\|u+3v\|^2=\|u\|^2+6\langle u|v\rangle+9\|v\|^2\\ (\sqrt{19})^2=\|2u-v\|^2=4\|u\|^2-4\langle u|v\rangle+\|v\|^2\end{cases}$$

i.e. $\begin{pmatrix}1&2&1\\1&6&9\\4&-4&1\end{pmatrix}\begin{pmatrix}\langle u|u\rangle\\\langle u|v\rangle\\\langle v|v\rangle\end{pmatrix}=\begin{pmatrix}1\\17\\19\end{pmatrix}$ i.e. $A\begin{pmatrix}\langle u|u\rangle\\\langle u|v\rangle\\\langle v|v\rangle\end{pmatrix}=B$

where $(A|B)\sim\left(\begin{array}{ccc|c}1&2&1&1\\0&4&8&16\\0&-12&-3&15\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&-3&-7\\0&1&2&4\\0&0&21&63\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&0&2\\0&1&0&-2\\0&0&1&3\end{array}\right)$ i.e. $\begin{cases}\langle u|u\rangle=2\\\langle u|v\rangle=-2\\\langle v|v\rangle=3\end{cases}$

Then $\|\text{proj}_{F(v)}(F(u))\|=\left\|\dfrac{\langle F(u)|F(v)\rangle}{\|F(v)\|^2}F(v)\right\|=\dfrac{|\langle F(u)|F(v)\rangle|}{\|F(v)\|}$

since $\langle F(u)|F(v)\rangle=\langle u|v\rangle$ for an isometric linear operator F

$$=\frac{|\langle u|v\rangle|}{\|v\|}=\frac{|-2|}{\sqrt3}=\frac{2}{\sqrt3}$$

8 $F:R^3\to R^3$ has the matrix $A=\begin{pmatrix}3&0&1\\2&1&1\\1-\beta&1+\beta&2\end{pmatrix}$ relative to the standard basis for R^3. $(\beta\in R)$

Eigenvalues: $0=\det(A-\lambda I)=\det\begin{pmatrix}3-\lambda&0&1\\2&1-\lambda&1\\1-\beta&1+\beta&2-\lambda\end{pmatrix}\overset{(-1)}{=}\det\begin{pmatrix}3-\lambda&0&1\\\lambda-1&1-\lambda&0\\1-\beta&1+\beta&2-\lambda\end{pmatrix}$ $(+1)$

$=\det\begin{pmatrix}3-\lambda&0&1\\0&1-\lambda&0\\2&1+\beta&2-\lambda\end{pmatrix}=0+(1-\lambda)[(3-\lambda)(2-\lambda)-2]+0$

$=-(\lambda-1)(\lambda^2-5\lambda+6-2)=-(\lambda-1)(\lambda-1)(\lambda-4)$

$\lambda_{1,2}=1$: $A-\lambda_{1,2}I=\begin{pmatrix}2&0&1\\2&0&1\\1-\beta&1+\beta&1\end{pmatrix}\sim\begin{pmatrix}2&0&1\\-(1+\beta)&1+\beta&0\\0&0&0\end{pmatrix}$ $\begin{cases}\overset{\text{if } 1+\beta=0}{\sim}\begin{pmatrix}2&0&1\\0&0&0\\0&0&0\end{pmatrix}\\ \overset{\text{if } 1+\beta\neq0}{\sim}\begin{pmatrix}2&0&1\\-1&1&0\\0&0&0\end{pmatrix}\sim\begin{pmatrix}2&0&1\\0&2&1\\0&0&0\end{pmatrix}\end{cases}$

from which it can be concluded that F is diagonalizable if and only if $\beta=-1$ (since only then $\dim(\lambda_{1,2}\text{-eigenspace})=2=\text{multiplicity}(\lambda_{1,2})$)

$\lambda_3=4$: $A-\lambda_3 I=\begin{pmatrix}-1&0&1\\2&-3&1\\1-\beta&1+\beta&-2\end{pmatrix}\sim\begin{pmatrix}1&0&-1\\0&-3&3\\0&1+\beta&-1-\beta\end{pmatrix}\sim\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}$ (indep. of β)

Answer: F is diagonalizable iff $\beta=-1$. A basis of eigenvectors is then e.g. $(1,0,-2)$, $(0,1,0)$, $(1,1,1)$.

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EXAMINATION IN MATHEMATICS
MAA153 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2017/18

Examination 2018-01-09 — Maximum points for subparts of the problems in the final examination

1.	$\dim(U)=\begin{cases}3 & \text{if } (a=-2)\vee(a=0)\\ 4 & \text{if } a\neq -2,0\end{cases}$ where U denotes the subspace of R^4. A basis for U is e.g. $(3,1,-2,4)$, $(7,2,0,9)$, $(-14,-4,1,a-15)$ if $(a=-2)\vee(a=0)$, and e.g. the standard basis of R^4 if $a\neq -2,0$. $v\in U$ iff $a\neq 0$. The coordinates of v relative to the indicated bases are $1,-5,-2$ and $7a+10, 2a+3, -4, 6a+5$ respectively. If, in the latter case, the vectors spanning U are chosen as a basis then the coordinates of v are $2, a-6, -3, 1/a$.	**1p**: Correctly found the dimension of and a basis for the subspace if $(a=-2)\vee(a=0)$ **1p**: Correctly found the dimension of and a basis for the subspace if $a\neq -2,0$ **1p**: Correctly concluded that the vector v belong to the subspace if $a\neq 0$ **1p**: Correctly for the case $a=-2$ found the coordinates of v relative to the chosen basis for the subspace U **1p**: Correctly for the case $a\neq -2,0$ found the coordinates of v relative to the chosen basis for the subspace U
2.	Proof $F^{-1}(3p_0+2p_1-p_2)$ $=-\frac{1}{3}p_0+\frac{7}{3}p_1-\frac{1}{3}p_2$	**2p**: Correctly found the matrix A of F relative to the ordered basis p_0, p_1, p_2 **1p**: Correctly concluded that F has an inverse (due to the fact that the matrix A is invertible) **1p**: Correctly concluded that $F^{-1}(3p_0+2p_1-p_2)$ is equal to $(p_0\ p_1\ p_2)\mathsf{A}^{-1}(3\ \ 2\ \ -1)^T$ **1p**: Correctly evaluated $(p_0\ p_1\ p_2)\mathsf{A}^{-1}(3\ \ 2\ \ -1)^T$
3.	The equation describes an hyperbolic cylinder if $c\neq 0$, and the union of two (intersecting) planes if $c=0$	**2p**: Correctly proved that the right hand side of the equation is a quadratic form with the signature $(1,-1,0)$ **1p**: Correctly concluded that the geometric meaning of the equation is an hyperbolic cylinder if $c>0$ **1p**: Correctly concluded that the geometric meaning of the equation is the union of two planes if $c=0$ **1p**: Correctly concluded that the geometric meaning of the equation is an hyperbolic cylinder if $c<0$
4.	$\mathrm{im}(F)=\mathrm{span}\{(2,1,-1,1),(-1,1,3,-2)\}$ $\ker(F)=\mathrm{span}\{(-1,1,1,0),(1,1,0,1)\}$ F is not bijective	**1p**: Correctly identified the matrix of F relative to e.g. the standard basis for R^4. **1p**: Correctly found the reduced row echelon form of the standard matrix in preparation for the conclusions about the image of F, the kernel of F and whether F is bijective or not **1p**: Correctly found the image of F **1p**: Correctly found the kernel of F **1p**: Correctly found that F is not bijective

5. An ON-basis for M is e.g.

$\frac{1}{\sqrt{21}}(1,0,1,0), \frac{1}{\sqrt{154}}(-3,7,-3,7)$

or e.g.

$\frac{1}{\sqrt{7}}(0,1,0,1), \frac{1}{\sqrt{462}}(7,-9,7,-9)$

or e.g. "$\pm$ versions" of the above ex:s

2p: Correctly initiated a Gram-Schmidt process to find an ON-basis e_1, e_2, and correctly as a first step in the process normed the first vector to e_1

2p: Correctly in the G-S process found (defined and evaluated) a nonzero vector orthogonal to e_1

1p: Correctly normed the vector orthogonal to e_1, and correctly summarized an ON-basis for M

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6. Proof

The coordinates of $\begin{pmatrix} 2 & 5 \\ 5 & -3 \end{pmatrix}$ relative to the basis m_1, m_2, m_3 are $(3,2,1)$

2p: Correctly proved that the set of vectors m_1, m_2, m_3 is a basis for the topical vector space

3p: Correctly found the coordinates of the matrix $\begin{pmatrix} 2 & 5 \\ 5 & -3 \end{pmatrix}$ relative to the basis m_1, m_2, m_3

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7. $\text{proj}_{F(v)}(F(u)) = \text{proj}_v(u) = \dfrac{2}{\sqrt{3}}$

2p: Correctly, for a reformulation of the expression for the orthogonal projection of $F(u)$ on $F(v)$, used the fact that for an isometric linear operator F on E, the inner product $\langle F(u) | F(v) \rangle$ equals $\langle u | v \rangle$ for every $u, v \in E$

1p: Correctly found the value of $\langle u | v \rangle$

1p: Correctly found the value of $\| v \|$

1p: Correctly combined the values of $\langle u | v \rangle$ and $\| v \|$ for the value of the orthogonal projection of $F(u)$ on $F(v)$

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8. The linear operator F is diagonalizable iff $\beta = -1$. A basis of eigenvectors is then e.g. $(1,0,-2), (0,1,0), (1,1,1)$.

2p: Correctly found that the linear operator F is diagonalizable iff $\beta = -1$

3p: Correctly for $\beta = -1$ found a basis of eigenvectors
