Exam

March 25th, 2019
Västerås

ELA407 – Control Theory

(Till tentamensvakten: engelsk information behövs)

Teacher	Alessandro Papadopoulos, tel: 021-1073 23	
Exam duration	14:10-19:30	
Help allowed	Non Programmable Calculator, language dictionary, ruler, and APPENDIX attached to this exam.	
Points	60 p	
Grading	Swedish grades:	ECTS grades
	< 30p $\rightarrow$ failed	$< 30 \rightarrow$ failed
	$30 - 41$p $\rightarrow$ grade 3	$30 - 34$p $\rightarrow$ D
	$42 - 53$p $\rightarrow$ grade 4	$35 - 44$p $\rightarrow$ C
	$54 - 60$p $\rightarrow$ grade 5	$45 - 54$p $\rightarrow$ B
		$55 - 60$p $\rightarrow$ A

Instructions

- Answers MUST be written in **English**.
- **Short and precise** answers are preferred. Do not write more than necessary. The clarity and the order of the answers will be considered in the evaluation.
- Use a **new sheet** for each of the assignments.
- If some assumptions are missing, or if you think the assumptions are unclear, **write down what do you assume** to solve the problem.
- Write **clearly**. If I cannot read it, you get zero points.
- During the exam you are not allowed to consult books or any kind of notes.

Good luck!!!

Turn the page

EXERCISE 1 (STABILITY AND TRANSFER FUNCTION) **10 POINTS**

Consider the Linear Time-Invariant (LTI) system with input $u(t)$ and output $y(t)$ described by the following equations:

$$\begin{cases} \dot{x}_1(t) = -\sqrt{3}x_1(t) - \sqrt{2}\alpha x_2(t) + u(t) \\ \dot{x}_2(t) = -4x_2(t) + 2u(t) \\ y(t) = x_2(t) \end{cases}$$

2/3

where $\alpha \in \mathbb{R}$ is a parameter of the system.
Respond to the following questions:

1. Determine what are the values of α that make the system asymptotically stable.

2. Compute the transfer function of the system with input $u(t)$ and output $y(t)$.

3. Setting $\alpha = 0$, determine the analytical expression of the output response of the system for initial conditions $x_1(0) = 1$ and $x_2(0) = 1$, and input $u(t) = 1$, $t \geq 0$.

EXERCISE 2 (TRANSFER FUNCTIONS AND STEP RESPONSE) **10 POINTS**

Given the LTI system with input $u(t)$ and output $y(t)$ described by the following equations

2/2

$$\begin{cases} \dot{x}_1(t) = 3x_2(t) \\ \dot{x}_2(t) = -10x_1(t) - 11x_2(t) + u(t) \\ y(t) = x_1(t) - 2x_2(t) \end{cases}$$

1. Compute the transfer function of the system with input $u(t)$ and output $y(t)$.
2. Sketch the output response of the system when $u(t) = \text{step}(t)$, indicating the initial value, the final value (if possible), the initial derivative, and the settling time.

EXERCISE 3 (FREQUENCY RESPONSE) **10 POINTS**

Consider the transfer function of a second-order system

2/3

$$G(s) = 10\frac{1+s}{(1-s)(1+10s)}$$

1. Determine the zeros, poles, type and gain of the transfer function.
2. Determine if the system is asymptotically stable.
3. Draw the asymptotic Bode diagrams of the frequency response of the system with transfer function $G(s)$ (you can use the semi-logarithmic axes at the end of the Appendix).

1/2 ~ 2/2

EXERCISE 4 (CLOSED-LOOP STABILITY) **10 POINTS**

Consider a third-order LTI system that has a transfer function $L(s)$, whose Bode diagrams are presented in Figure 1.

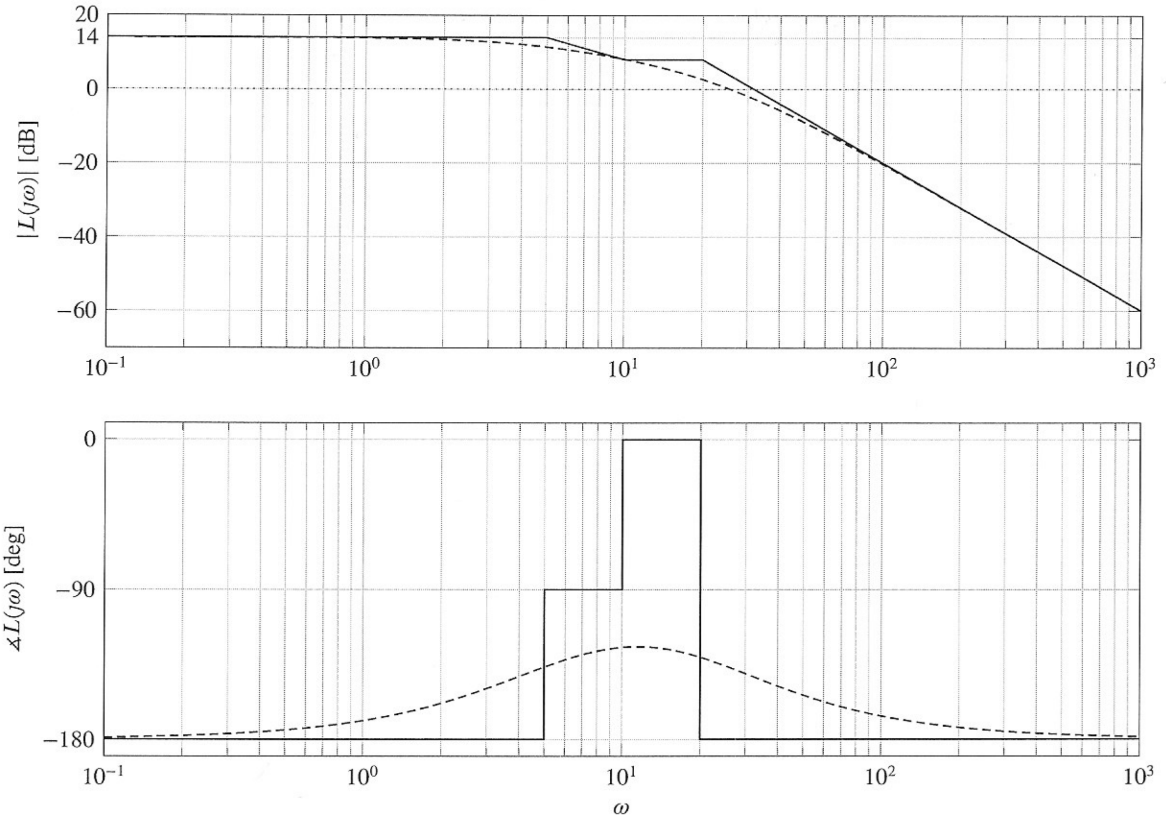

Figure 1: Asymptotic Bode plots (solid line) and exact (dashed line) of the frequency response associated with the transfer function $L(s)$.

1. Derive from the Bode diagrams the expression of $L(s)$, assuming that all the singularities are real.
2. Evaluate the stability of the closed-loop system of Figure 2.

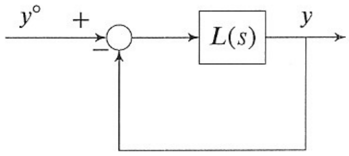

Figure 2: Closed-loop system.

C

EXERCISE 5 (CONTROL DESIGN) **10 POINTS**

Consider the control system in Figure 3 where

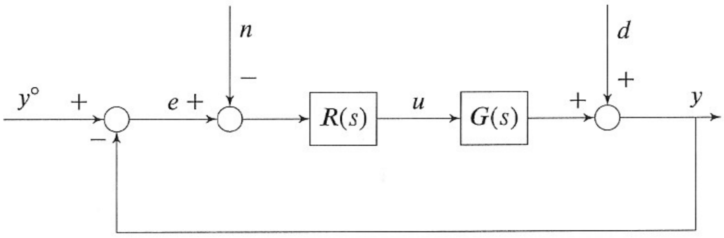

Figure 3: Control scheme.

$$G(s) = \frac{10}{s(1+s)^2}$$

Design a controller $R(s)$ such that:

1. $|e_{\infty,y^\circ}| = 0$ when $y^\circ = \text{step}(t)$,
2. $|e_{\infty,n}| < 0.1$ when $n(t) = \sin(\omega_n t)$, with $\omega_n > 10$rad/s
3. $|e_{\infty,d}| < 0.1$ when $d(t) = \sin(\omega_d t)$, with $\omega_d < 0.01$rad/s
4. $\omega_c \geq 1$rad/s, and
5. $\varphi_m \geq 60^\circ$.

You can use the semi-logarithmic axes attached in the Appendix to support the control design.

1/2

EXERCISE 6 (DISCRETE-TIME SYSTEMS) **10 POINTS**

1. Discuss what are the necessary and sufficient conditions for the asymptotic stability of Linear Time-Invariant system in continuous-time, and in discrete-time system, and make an example of an asymptotically stable LTI system of order 2 for both continuous-time and discrete-time.
2. Discuss what are the main criteria for the stability of a closed-loop system in continuous-time, clearly stating the main hypotheses and the criteria.

A Formulas

Continuous-time systems

- Diagonalization of a matrix. Assume that $\boldsymbol{A}$ is a diagonalizable matrix

$$\exists \boldsymbol{T}, \det(\boldsymbol{T}) \neq 0 : \hat{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

  and $\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$, where $\boldsymbol{v}_i$, $i = 1, \dots, n$ are the eigenvector of the matrix $\boldsymbol{A}$.

- Matrix exponential: $e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty} \dfrac{(\boldsymbol{A}t)^k}{k!} = \boldsymbol{I} + \boldsymbol{A}t + \dfrac{\boldsymbol{A}^2 t^2}{2!} + \dfrac{\boldsymbol{A}^3 t^3}{3!} + \dots$
  the matrix exponential can also be computed as:

$$e^{\boldsymbol{A}t} = \boldsymbol{T}^{-1} \text{diag}\left(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}\right) \boldsymbol{T}$$

- Lagrange equation: $e^{\phi \jmath} = \cos(\phi) + \jmath \sin(\phi)$

- $\text{tr}(\boldsymbol{A}) = \sum_i^n \lambda_i$, $\det(\boldsymbol{A}) = \prod_i^n \lambda_i$

- Routh-Hurwitz table. Given a polynomial $\chi(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n$ the Routh-Hurwitz table is constructed as:

$$\begin{array}{cccc} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ h_1 & h_2 & h_3 & \cdots \\ k_1 & k_2 & k_3 & \cdots \\ l_1 & l_2 & l_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

  and the entries in a generic row are computed with the following rule

$$l_i = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_{i+1} \\ k_1 & k_{i+1} \end{bmatrix}$$

Laplace and transform

- Laplace transform of canonical signals:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
$\text{imp}(t)$	1
$\text{step}(t)$	$\dfrac{1}{s}$
t^k	$\dfrac{k!}{s^{k+1}}$
$\cos(\omega t)$	$\dfrac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2 + \omega^2}$
e^{at}	$\dfrac{1}{s - a}$
$e^{at} t^k$	$\dfrac{k!}{(s-a)^{k+1}}$
$e^{at} \cos(\omega t)$	$\dfrac{s - a}{(s-a)^2 + \omega^2}$
$e^{at} \sin(\omega t)$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

Frequency response

- Given a generic transfer function

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + s\tau_i)}{\prod_j (1 + sT_j)}$$

$$|G(\jmath\omega)| = \frac{|\mu|}{|\jmath\omega|^g} \frac{\prod_i |1 + \jmath\omega\tau_i|}{\prod_j |1 + \jmath\omega T_j|} = \frac{|\mu|}{\omega^g} \frac{\prod_i \sqrt{1 + (\omega\tau_i)^2}}{\prod_j \sqrt{1 + (\omega T_j)^2}}$$

$$\measuredangle G(\jmath\omega) = \measuredangle\mu - 90^\circ g + \sum_i \text{atan}(\omega\tau_i) - \sum_j \text{atan}(\omega T_j)$$

- Conversion in decibel of a number x

$$|x|_{\text{dB}} = 20 \log_{10} |x|$$

- Rules for Magnitude Bode plot:

  1. For $\omega \to 0$, the plot starts having a slope $-g$ and going through point $\omega = 1$rad/s, $|G(\jmath 1)|_{\text{dB}} = |\mu|_{\text{dB}}$.
  2. At every frequency corresponding to p real poles (zeros), slope decreases (increases) of p units.
  3. At every frequency corresponding to the natural frequency of p complex poles (zeroes), slope decreases (increases) of $2p$ units.
  4. For $\omega \to \infty$, the slope of the plot equals the difference between the number of zeros and the number of poles of the transfer function.

- Rules for the Phase Bode plot:

  1. For $\omega \to 0$, the plot starts having a line parallel to the x-axis and crossing the y-axis at $\measuredangle\mu - g90^\circ$.
  2. At every frequency corresponding to p real zeros in the left half plane or p real poles in the right half plane, the phase increases of $p90^\circ$.
  3. At every frequency corresponding to p real zeros in the right half plane or p real poles in the left half plane, the phase decreases of $p90^\circ$.
  4. At every frequency corresponding to p complex zeros in the left half plane or p complex poles in the right half plane, the phase increases of $p180^\circ$.
  5. At every frequency corresponding to p complex zeros in the right half plane or p complex poles in the left half plane, the phase decreases of $p180^\circ$.

Root locus

Considering the transfer function expressed in the form:

$$L(s) = \rho \frac{\prod_i (s + z_i)}{\prod_j (s + p_j)}$$

the root locus can be plotted according to the following rules:

R1 The root locus has $2n$ branches (closed-loop poles migrating in the complex plane): n belong to the "direct" locus, n to the "inverse" one.

R2 The root locus is symmetric with respect to the real axis.

R3 Branches begin ($|\rho| \to 0$) at the poles of $L(s)$.

R4 When $|\rho| \to \infty$, m branches end at the zeros of $L(s)$, while ν branches go to infinity approaching an asymptote.

R5 The root locus lies on the real axis to the left of an

- odd number (direct locus)
- even number (inverse locus)

of singularities (poles and zeros)

R6 Asymptotes intersect at a point on the real axis, whose coordinate x_a and whose angle with the real axis θ_a are given by

$$x_a = \frac{1}{\nu}\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right) \qquad \theta_a = \begin{cases} \dfrac{180^\circ + k360^\circ}{\nu} & k = 0, 1, \ldots, \nu - 1 \text{ (direct locus)} \\ \dfrac{k360^\circ}{\nu} & k = 0, 1, \ldots, \nu - 1 \text{ (inverse locus)} \end{cases}$$

R7 When $\nu \geq 2$, the center of gravity is constant, and lies at a point on the real axis, whose coordinate is given by

$$\text{centroid} = -\frac{1}{n}\sum_{i=1}^{n} p_i$$

R8 At any point $\bar{s}$ on the locus, the absolute value of the gain can be calculated as the product of distances from the point to the poles divided by the product of distances from the point to the zeros (if there are no zeros, the denominator is 1)

$$|\rho| = \frac{\prod_{i=1}^{n} |\bar{s} + p_i|}{\prod_{i=1}^{m} |\bar{s} + z_i|}$$

B Semi-logarithmic axes