Divsion of Mathematics
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Examination Vector algebra
MAA150 - TEN1
Date: Februari 15, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Solve the linear system (3p)

$$\begin{aligned} -x + 2y + z &= 3 \\ 2x - 4y + z &= -1 \end{aligned}$$

2 Given that the augmented matrix for a linear system is (3p)

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & a & -1 & 0 \\ 0 & 0 & 1-a & 1-a \end{array}\right],$$

determine the values of a for which the linear system has no solution, exactly one solution, or infinitely many solutions. Motivate your answer.

3 Given the matrices

$$B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix},$$

a. Use the determinant to show that B is invertible. (3p)

b. Find B^{-1} and use it to solve the equation $XB = C$. (5p)

4 The line l passes through the points $P : (1, 2, -1)$ and $Q : (1, 0, 3)$ intersects the plane $\Pi : 2x - y + z = 4$ at the point R.

a. Find the vector form of the line l and the coordinates of R. (3p)

b. Find the angle between the line and the plane. Include an illustrative figure in which the angle is clearly marked. (3p)

5 Find all solutions of the equation $z^3 = -27i$. Give the solutions in the form $a + bi$ and mark the solutions in the complex plane. (5p)

Avdelningen för Matematik
och tillämpad matematik
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Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2016-02-15
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Lös det linjära ekvationssystemet (3p)

$$\begin{aligned} -x+2y+z &= 3 \\ 2x-4y+z &= -1 \end{aligned}$$

2 Givet att den utvidgade matrisen för ett linjärt ekvationssystem är (3p)

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & a & -1 & 0 \\ 0 & 0 & 1-a & 1-a \end{array}\right],$$

avgör för vilka värden på a som ekvationssytemet saknar lösning, har exakt en lösning eller oändligt många lösningar. Motivera ditt svar.

3 Givet matriserna

$$B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ och } C = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix},$$

a. Använd determinanten för att visa att B är inverterbar. (3p)

b. Bestäm B^{-1} och använd den för att lösa ekvationen $XB = C$. (5p)

4 Linjen l går genom punkterna $P : (1, 2, -1)$ och $Q : (1, 0, 3)$, skär planet $\Pi : 2x - y + z = 4$ i punkten R.

a. Bestäm på vektorform ekvationen för linjen l och koordinaterna för R. (3p)

b. Bestäm vinkeln mellan linjen planet. Inkludera en illustrativ figur i vilken vinkeln är tydligt markerad. (3p)

5 Bestäm alla lösningar till ekvationen $z^3 = -27i$. Ange lösningarna på formen $a + bi$ och markera lösningarna i det komplexa talplanet. (5p)

MAA150 Vektoralgebra, vt-16.

Assessment criterias for TEN1 2016-02-15

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. Complete solution gives 3 points.

- Relevant row operations on the augmented matrix **(1p)**
- Setting the free variable to a parameter **(1p)**
- Correct answer **(1p)**

2. Complete solution gives 3 points.

- Correctly determining that there is a unique solution if $a \neq 0$ and $a \neq 1$ **(1p)**
- Correctly determining that there is no solution if $a = 0$ **(1p)**
- Correctly determining that there is infinitely many solutions if $a = 1$ **(1p)**

3. **a.** Complete solution gives 3 points.

- Stating a condition for when A is invertible **(1p)**
- Computing the determinant **(2p)**

b. Complete solution gives 5 points.

- Correct method and relevant row operations to find B^{-1} **(2p)**
- Correct B^{-1} **(1p)**
- Solving $X = CB^{-1}$ correctly **(1p)**
- Computing the matrix multiplication correctly **(1p)**

4. **a.** Complete solution gives 3 points.

- Calculating a vector in the direction of the line **(1p)**
- Giving the vector form of the line l **(1p)**
- Finding the coordinates of R **(1p)**

b. Complete solution 3 points

- Finding $\cos(\theta)$ for the angle θ between the normal vector and a vector in the direction of the line **(2p)**
- A correct and illustrative figure including relevant angles **(1p)**

5. Complete solution gives 5 points.

- Finding polar form of $-27i$ **(1p)**
- Setting $z = r(\cos(\theta) + i\sin(\theta))$ and obtaining the equations for r and θ **(1p)**
- Finding the correct values of r and θ **(1p)**
- Finding the solutions in form $a + bi$ **(1p)**
- Marking the correct solutions in the complex plane **(1p)**

(1)
$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 3 \\ 2 & -4 & 1 & -1 \end{array}\right] \underset{R_2 = R_2 + 2R_1}{\sim} \left[\begin{array}{ccc|c} -1 & 2 & 1 & 3 \\ 0 & 0 & 3 & 5 \end{array}\right] \begin{array}{l} \times(-1) \\ \times \frac{1}{3} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 5/3 \end{array}\right] \text{(1p)} \qquad y = t \Rightarrow z = 5/3, \; x = -3 + \frac{5}{3} + 2t = -\frac{4}{3} + 2t$$

(1p)

x, z leading variables, y free

Answer a: $x = -\frac{4}{3} + 2t$, $y = t$, $z = 5/3$ where $t \in \mathbb{R}$. (1p)

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(2) If $a \neq 0$ and $a \neq 1$ the equations becomes

$$\begin{cases} x + a \cdot y + z = 1 \\ \quad y - \frac{1}{a} z = 0 \\ \quad\quad z = 1 \end{cases}$$

which has a unique solution

(1p)

If $a = 0$ the equations becomes

$$\begin{cases} x \quad + z = 1 \\ \quad -z = 0 \\ \quad\; z = 1 \end{cases}$$

This is a contradiction, so there is no solution. (1p)

If $a = 1$ the equations becomes

$$\begin{cases} x + y + z = 1 \\ \quad y - z = 0 \end{cases}$$

free variable

which has infinitely many solutions, since y is a free variable.

(1p)

Answer b: The system has $\begin{cases} \text{exactly one sol. if } a \neq 0 \text{ and } a \neq 1 \\ \text{no sol. if } a = 0 \\ \text{infinitely many sol. if } a = 1 \end{cases}$

③ (a) B is invertible $\Leftrightarrow \det(B) \neq 0$ (1p)

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \underset{R_2=R_2+R_1}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = [\text{cofactor}$$

$$\text{expansion column } 1] = 1 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 \cdot (1\cdot 1 - (-1)\cdot 1) = 2 \quad (2p)$$

Answer a: B is invertible since $\det(B) \neq 0$

(b) $X \cdot B = C \Leftrightarrow X \cdot B \cdot B^{-1} = C \cdot B^{-1} \Leftrightarrow X = C \cdot B^{-1}$ (1p)

Find B^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_2=R_2+R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_3 = R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \underset{R_3=\frac{R_3}{2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right] R_2 = R_2 - R_3$$

(*)

(*) = (2p)

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right] \underset{R_1=R_1-2R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 0 & -1 & 1 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{matrix}}_{=B^{-1}} \quad (1p)$$

$$X = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -5/2 & 5/2 \\ -1/2 & -3/2 & -1/2 \end{bmatrix} \quad (1p)$$

(4)

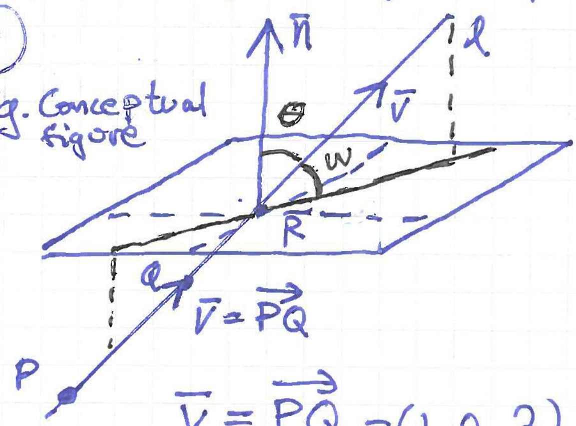

Fig. Conceptual figure

$\Pi: 2x - y + z = 4$

$\vec{v} = \overrightarrow{PQ} = (1,0,3) - (1,2,-1) = (0,-2,4)$ (1p)

$l: (x,y,z) = (1,2,-1) + t(0,-2,4) = (1, 2-2t, -1+4t)$ (1p)

The point of intersection R ($R \in l$ and $R \in \Pi$)

Inserting $(x,y,z) = (1, 2-2t, -1+4t)$ in the equation of Π: $2 - (2-2t) + (-1+4t) = 4 \Leftrightarrow t = \frac{5}{6}$

$\Rightarrow R = \left(1, 2 - \frac{10}{6}, -1 + \frac{20}{6}\right) = \left(1, \frac{1}{3}, \frac{7}{3}\right)$ (1p)

The plane has normal $\vec{n} = (2,-1,1)$ and θ satisfies

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{n}}{\|\vec{v}\| \cdot \|\vec{n}\|} = \frac{(0,-2,4) \cdot (2,-1,1)}{\sqrt{(-2)^2+4^2} \cdot \sqrt{2^2+(-1)^2+1^2}} = \frac{6}{\sqrt{20} \cdot \sqrt{6}} \quad \text{(2p)}$$

$$= \sqrt{\frac{6}{20}} = \sqrt{\frac{3}{10}}$$

$\Rightarrow \theta = \cos^{-1}\left(\sqrt{\frac{3}{10}}\right)$ so $w = \frac{\pi}{2} - \theta$ (+fig: 1p)

Answer (a) $R = (1, 1/3, 7/3)$ (b) $\frac{\pi}{2} - \cos^{-1}\left(\sqrt{\frac{3}{10}}\right)$

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(5) $z^3 = -27i$. Write $-27i$ on polar form

$-27i = 27\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$ (1p)

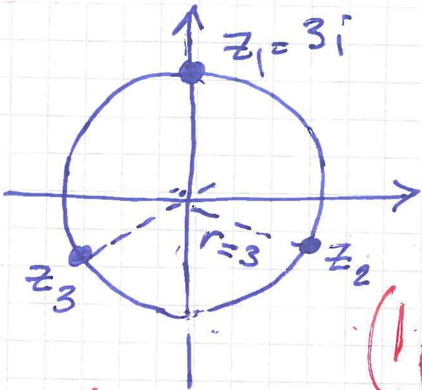
Set $z = r(\cos(\theta) + i\sin(\theta))$ so $z^3 = r^3 \cdot (\cos(3\theta) + i\sin(3\theta))$

This gives

$$\begin{cases} r^3 = 27 \\ 3\theta = \frac{3\pi}{2} + 2\pi n, \quad n = -1, 0, 1 \end{cases} \quad \text{(1p)}$$

$$\Rightarrow \begin{cases} r = 3 \\ \theta = \frac{\pi}{2} + \frac{2\pi}{3} n \quad , n = -1, 0, 1. \end{cases}$$

$\theta_1 = \frac{\pi}{2}$, $\theta_2 = -\frac{\pi}{6}$, $\theta_3 = \frac{7\pi}{6}$ (1p)



(1p)

$n=0$: $z_1 = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 3i$

$n=-1$: $z_2 = 3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$n=1$: $z_3 = 3\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ (1p)

Answer : $z_1 = 3i$, $z_2 = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$, $z_3 = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$
