Mälardalen University
School of Education, Culture and Communication
Division of Applied Mathematics

EXAM TEN1 MAA153
LINEAR ALGEBRA
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Suggested solutions

1

Let T be the linear map in R^3 that maps the vectors $(1, 0, -1)$, $(0, 1, 0)$ and $(0, 1, 1)$, to the vectors $(1, 1, 1)$, $(1, -1, 1)$, and $(1, 0, 1)$, respectively. Denote by v the vector $(1, 2, 3)$.

(i). *2 points.* Determine whether v belongs to the range of T.

(ii). *2 points.* Determine whether v belongs to the kernel of T.

(iii). *2 points.* Show that the intersection of the range of T and the kernel of T equals the image under T of the kernel of T composed with itself: $T(V) \cap \ker T = T(\ker T^2)$.

(iv). *4 points.* Determine the vector w, if it exists, that, among all the vectors belonging to both the range and the kernel of T, lies closest to v in the standard Euclidean metric.

Solution.

(i). No. The range of T is the image under T of the domain of T, which, the first set of vectors being linearly independent (check!), equals R^3. The vector v thus lies in the range of T if and only if it is in the span of the second set of vectors, and this it is not (check!).

(ii). No. The vector v is the linear combination of the first (ordered) set of vectors with coefficients $(1, -2, 4)$ (check!), and $T(v)$ is the the linear combination, with the same coefficients, of the second set of vectors, but the latter combination is not zero (check!).

(iii). A vector y is in the kernel of T if and only if $T(y) = 0$, and y is in the range of T if and only if $y = T(x)$ for some x in the domain of T, so y is both in the range and in the kernel if and only if $y = T(x)$ for some x satisfying $T(T(x)) = 0$, i.e. if and only if y is in the image under T of the kernel of T^2.

(iv). A basis B for $\ker T^2$ is found by solving $M^2 x = 0$, with M the matrix for T in the standard basis. The orthogonal projection of v onto the span of $T(B)$ is the sought vector w, and the distance from v to the intersection of the range and the kernel of T is $\|v - w\|$. Presently, B consists of a non-zero multiple of $(1, -1, -3)$, which T maps to zero, so $w = 0$ and the sought distance equals $\|v\|$, which equals $\sqrt{14}$.

2

Let V be the vector space of all real polynomial functions of degree at most two, in the interval $(-1,1)$, equipped with the scalar product $(f|g) = \int_{-1}^{1} f(t)g(t)\,t^2\,dt$. Define $\delta : V \to R$ by $\delta(f) = f(0)$.

(i). *1 point.* Show that δ is linear.

(ii). *4 points.* Find an orthonormal basis $\mathcal{B}$ for the kernel of δ.

(iii). *3 points.* Extend the set $\mathcal{B}$ to an orthonormal basis $\mathcal{C}$ in V.

(iv). *2 points.* Determine the matrix for δ in the basis $\mathcal{C}$.

Solution.

(i). The vector space operations in the function space V are defined 'point-wise', $\delta(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$, $x \in (-1,1)$, $\alpha, \beta \in R$, $f, g \in V$, and putting $x = 0$ proves the claim.

(ii). The general element $p \in V$ has the form $p(t) = at^2 + bt + c$, and p is in the kernel of δ if and only if $\delta(p) = p(0) = c = 0$, so a basis in the kernel of δ is given by the monomials t and t^2, which are orthogonal (the integral of an odd function over (-1.1) is zero), duly normalised.

(iii). The monomial 1, which is not in the kernel of δ, is orthogonal to t, but not to t^2. Subtract then from 1 its orthogonal projection on (the line spanned by) t^2 to get a vector f orthogonal to the kernel.

(iv). The matrix for δ is $(\delta(t/\|t\|), \delta(t^2/\|t^2\|), \delta(f/\|f\|) = (0, 0, 1/\|f\|)$.

3

Let $\mathcal{L}$ be the vector space of the linear transformation of the standard Euclidean plane R^2. Write E for the identity map and write I for the counter-clockwise rotation of the plane by the angle $\pi/2$. Show:

(i). *2 points.* The set $\mathcal{C}$ of the transformations in $\mathcal{L}$ that commute with I is a vector subspace of $\mathcal{L}$.

(ii). *3 points.* The pair (E, I) is an (ordered) basis for $\mathcal{C}$.

(iii). *2 points.* The maps in $\mathcal{C}$ commute: $AB - BA = 0$ for $A, B \in \mathcal{C}$.

(iv). *3 points.* For $A, B \in \mathcal{C}$ the following hold:

$$x(AB) = x(A)x(B) - y(A)y(B)$$

and

$$y(AB) = x(A)y(B) + y(A)x(B),$$

with $(x(T), y(T))$ denoting the coordinates of $T \in \mathcal{C}$ in the basis (E, I).

Solution.

(i). The set $\mathcal{C}$ is the kernel of the linear map $T \mapsto TI - IT$ in $\mathcal{L}$.

(ii). The set $\{E, I\}$ is linearly independent (since so is $\{E(u), I(u)\}$ for any $v \neq 0$) and the space $\mathcal{C}$ is of dimension two. Indeed, one easily finds two linearly independent maps in the range $\mathcal{R}$ of $T \mapsto TI - IT$ in $\mathcal{L}$, and then $2 \leq \dim \mathcal{C} = 4 - \dim \mathcal{R} \leq 2$.

(iii). Linear combinations of pair-wise commuting maps commute.

(iv). Compose $A = x(A)E + y(A)I$ with $B = x(B)E + y(B)I$ and collect the terms, remembering that $E^2 = -I$.

4

Let T be the linear map in R^3 that maps $(1,0,0), (0,1,0), (0,0,1)$ to $(1,1,0), (-1,-2,1), (-1,1,-3)$, respectively.

(i). *2 points.* Show that there exists a non-zero polynomial p of degree at most three, such that $p(T)$ maps the vector $(1,0,0)$ to zero.

(ii). *3 points.* Show that $p(T)$ then maps every vector to zero.

(iii). *5 points.* Determine all such polynomials p.

Solution.

(i). Denote $(1,0,0)$ by e. The four vectors $T^k(e)$, $k = 0,1,2,3$, in R^3 are not linearly independent (for R^3 is of dimension three), so there are real numbers a_k, $k = 0,1,2,3$, not all zero, such that $\sum_k a_k T^k(e) = p(T)(e) = 0$, with $p(x) = \sum_k a_k x^k$.

(ii). The vectors $T^k(e)$, $k = 0,1,2$, are linearly independent (check!), hence a basis in R^3, and $p(T)(T^k(e)) = T^k(p(T)(e)) = 0$, $k = 0,1,2$.

(iii). Let M be the matrix for T in the standard basis (determine this matrix!), compute $v_k := T^k(e) = M^k e$, $k = 0,1,2,3$, (recursively, since $M^k e = M(M^{k-1}e)$), and solve the homogeneous system $\sum_k a_k v_k = 0$.

5

Consider R^3 with Euclidean metric. Call T the linear map that maps $(1,0,0), (1,1,0), (1,1,1)$ to $(1,0,0), (0,-1,1), (-1,0,-2)$, respectively. Put $y = (1,2,3)$.

10 points. Find all the unit eigenvectors, and the eigenvalues, of T, and, using these, solve the equation $Tx = y$.

Solution.

v is eigenvector for T with eigenvalue λ iff $v \neq 0$ and $Tv - \lambda v = (T - \lambda E)v = 0$, and λ, v exist iff $\det(M - \lambda E) = 0$, M due matrix for T. This has $\lambda_1 = 1$ as root (with eigenvector $v_1 = (1,0,0)$), and the roots λ_2, λ_3, are easily seen to be $-2 - \sqrt{2}, -2 + \sqrt{2}$, respectively, with unit eigenvectors v_2, v_3, obtained by solving corresponding homogeneous systems of equations; $v_1 = (3 - \sqrt{2}, 7\sqrt{2}/2, 7 + 7\sqrt{2}/2) \cdot \sqrt{109 + 43\sqrt{2}})$, and similarly for v_2. With $y = (y_1, y_2, y_3)$ in basis (v_1, v_2, v_3) solves $Tx = y$ in form $x = (y_1/\lambda_1, y_2/\lambda_2, y_3/\lambda_3)$ (the computation of y_1, y_2, y_3 is cumbersome and would not be required for full credit).