MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2017-06-02 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the explicit function expression for f defined by

$$f(x) = \int_{\frac{1}{2}}^{x} u\cos(\pi u)\,du\,.$$

2. Classify all local extreme points of the function f defined by

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 7\,.$$

Is there any absolute (global) extreme point of f?

3. Find the numerical sequence $\{c_n\}_{n=0}^{\infty}$ for which the power series $\sum_{n=0}^{\infty} c_n x^n$ has the sum $x/(2-3x)$. Also, find the interval of convergence of the power series.

4. Let f be an odd function in the interval $[-a, a]$ where $a > 0$. Prove that

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx\,, \quad \text{i.e. prove that} \quad \int_{-a}^{a} f(x)\,dx = 0\,.$$

5. Find out whether $$\lim_{x\to 0} \frac{x}{|3x-4| - |3x+4|}$$ exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

6. Find an equation for the tangent line τ to the curve $\gamma : (x, y) = \left(e^3/t,\ \sqrt{\ln(t)}\right)$ at the point P where the parameter t is equal to e^4.

7. Solve the initial-value problem $2y' + 8xy = x,\ \ y(0) = 0\,.$

8. Find the function expression for the inverse of the function f defined by

$$f(x) = \frac{x}{x-2}.$$

Also, specify the domain and the range of the inverse.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{llrcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm det explicita funktionsuttrycket för f definierad enligt

$$f(x) = \int_{\frac{1}{2}}^{x} u\cos(\pi u)\,du\,.$$

2. Klassificera alla lokala extrempunkter för funktionen f definierad av

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 7\,.$$

Finns det någon absolut (global) extrempunkt för f?

3. Bestäm den talföljd $\{c_n\}_{n=0}^{\infty}$ för vilken potensserien $\sum_{n=0}^{\infty} c_n x^n$ har summan $x/(2-3x)$. Bestäm även konvergensintervallet för potensserien.

4. Låt f vara en udda funktion i intervallet $[-a, a]$ där $a > 0$. Bevisa att

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx\,, \quad \text{dvs bevisa att} \quad \int_{-a}^{a} f(x)\,dx = 0\,.$$

5. Utred om

$$\lim_{x\to 0} \frac{x}{|3x-4| - |3x+4|}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

6. Bestäm en ekvation för tangenten τ till kurvan $\gamma : (x, y) = \left(e^3/t, \sqrt{\ln(t)}\right)$ i den punkt P där parametern t är lika med e^4.

7. Lös begynnelsevärdesproblemet $2y' + 8xy = x,\ \ y(0) = 0\,.$

8. Bestäm funktionsuttrycket för inversen till funktionen f definierad enligt

$$f(x) = \frac{x}{x-2}.$$

Specificera även inversens definitionsmängd och värdemängd.

If you prefer the problems formulated in English, please turn the page.

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① $$f(x) = \int_{1/2}^{x} u\cos(\pi u)\,du = \left[u\cdot\frac{1}{\pi}\sin(\pi u)\right]_{1/2}^{x} - \int_{1/2}^{x} 1\cdot\frac{1}{\pi}\sin(\pi u)\,du$$

$$= \left[\frac{u}{\pi}\sin(\pi u) + \frac{1}{\pi^2}\cos(\pi u)\right]_{1/2}^{x} = \left(\frac{x}{\pi}\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x)\right)$$

$$- \left(\frac{1}{2\pi}\sin\left(\frac{\pi}{2}\right) + \frac{1}{\pi^2}\cos\left(\frac{\pi}{2}\right)\right) = \frac{1}{\pi^2}\left[\cos(\pi x) + \pi x\sin(\pi x) - \frac{\pi}{2}\right]$$

② $f(x) = 3x^4 - 4x^3 - 12x^2 + 7$

Differentiation gives $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2)$

A first deriv. test

x		SP -1		SP 0		SP 2	
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	loc. min.	$\nearrow$	loc. max.	$\searrow$	loc. min	$\nearrow$

We conclude that -1 and 2 are local minimum points of f, and that 0 is a local maximum point of f. Since f is continuous in $\mathbb{R}$ and $f(x) \to +\infty$ as $|x| \to +\infty$, we also conclude that f has a global minimum but not a global maximum.

2 is the global minimum point of f since $-25 = f(2) < f(-1) = 2$.

③ $$\frac{x}{2-3x} = \frac{x}{2}\,\frac{1}{1-\frac{3}{2}x} = \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{3}{2}x\right)^n = \sum_{n=0}^{\infty}\frac{3^n}{2^{n+1}}x^{n+1} = \sum_{k=1}^{\infty}\frac{3^{k-1}}{2^k}x^k$$

iff $\left|\frac{3}{2}x\right| < 1 \iff |x| < \frac{2}{3}$

Compare with general form of a power series in x

$$= \sum_{k=0}^{\infty} c_k x^k$$

Thus the numerical sequence asked for is $\{c_k\}_{k=0}^{\infty}$ where $\boxed{\begin{cases} c_0 = 0 \\ c_k = \frac{3^{k-1}}{2^k}, & k \geq 1 \end{cases}}$

The interval of convergence is $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

④ f is odd in $[-a, a]$ where $a > 0$, i.e. $f(-x) = -f(x)$ for every $x \in [-a, a]$

Thus

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx \begin{bmatrix} x = -u \\ dx = -du \end{bmatrix} + \int_{0}^{a} f(x)\,dx$$

$$= \int_{a}^{0} f(-u)(-du) + \int_{0}^{a} f(x)\,dx = \int_{a}^{0} (-f(u))(-du) + \int_{0}^{a} f(x)\,dx$$

$f(-u) = -f(u)$

$$= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0 \qquad \text{q.e.d.}$$

changing the order of the limits of the first integral

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$$\lim_{x\to 0} \frac{x}{|3x-4|-|3x+4|} \underset{\text{Since } (3x-4)<0 \text{ and } (3x+4)>0 \text{ in a neighbourhood of } 0}{=} \lim_{x\to 0} \frac{x}{(4-3x)-(3x+4)} = \lim_{x\to 0} \frac{x}{-6x} = -\frac{1}{6}$$

i.e. the limit exists and equals $-\frac{1}{6}$

6 $\gamma: (x,y) = \left(\frac{e^3}{t}, \sqrt{\ln(t)}\right) \equiv (f(t), g(t))$, $t_p = e^4$

We get $\begin{cases} x_p = f(t_p) = \frac{e^3}{e^4} = \frac{1}{e} \\ y_p = g(t_p) = \sqrt{\ln(e^4)} = \sqrt{4} = 2 \end{cases}$

The slope k_p at P equals $\left(\frac{dy}{dx}\right)_p = \left(\frac{dy}{dt}\frac{dt}{dx}\right)_p = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)_p = \frac{g'(t_p)}{f'(t_p)}$

i.e. $k_p = \left(\frac{\frac{1}{2}\frac{1}{\sqrt{\ln(t)}}\cdot\frac{1}{t}}{-\frac{e^3}{t^2}}\right)_p = -\frac{t_p}{2e^3\sqrt{\ln(t_p)}} = -\frac{e^4}{2e^3\cdot 2} = -\frac{e}{4}$

Thus, an equation for the tangent line z to γ at P is

$$y - y_p = k_p(x - x_p) \iff y - 2 = -\frac{e}{4}\left(x - \frac{1}{e}\right) = -\frac{e}{4}x + \frac{1}{4}$$

$$\iff y = \frac{1}{4}(9 - ex)$$

7 DE: $2y' + 8xy = x$, IV: $y(0) = 0$

The DE in standard form is $y' + 4xy = \frac{1}{2}x$

Mult. with an integrating factor e^{2x^2} gives $\left(e^{2x^2}\cdot y\right)' = e^{2x^2}\cdot\frac{1}{2}x$

and thus $y = e^{-2x^2}\left(\frac{1}{8}e^{2x^2} + C\right) = \frac{1}{8} + Ce^{-2x^2}$.

The IV gives $0 = y(0) = \frac{1}{8} + C\cdot e^0$ i.e. $C = -\frac{1}{8}$

The solution of the IVP is then $y = \frac{1}{8}\left(1 - e^{-2x^2}\right)$

8 $f(x) = \frac{x}{x-2}$ where $D_f = \mathbb{R}\setminus\{2\}$ and $R_f = \mathbb{R}\setminus\{1\}$

(R_f since $f(x) = 1 + \frac{2}{x-2}$ and $\frac{2}{x-2}$ can attain all real values but 0)

We have $\begin{cases} f(x) = y \\ \frac{x}{x-2} = y \end{cases} \iff \begin{cases} f(x) = y \\ x = y(x-2) \end{cases} \iff \begin{cases} f(x) = y \\ x(1-y) = -2y \end{cases} \iff \begin{cases} x = f^{-1}(y) \\ x = \frac{2y}{y-1} \end{cases}$

where the existence of f^{-1} is proved by the possibility to solve for x uniquely from $f(x) = y$.

Thus $f^{-1}(x) = \frac{2x}{x-1}$ and $D_{f^{-1}} = R_f = \mathbb{R}\setminus\{1\}$ and $R_{f^{-1}} = D_f = \mathbb{R}\setminus\{2\}$

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN1 – 2017-06-02 — Maximum points for subparts of the problems in the final examination

1.	$f(x)=\frac{1}{\pi^2}\left(\cos(\pi x)+\pi x\sin(\pi x)-\frac{\pi}{2}\right)$	**1p**: Correctly worked out the first progressive step in finding an antiderivative by parts **1p**: Correctly worked out the second progressive step in finding an antiderivative by parts **1p**: Correctly worked out the limits of the integral
2.	-1 is a local minimum point of f 0 is a local maximum point of f 2 is an absolute minimum point of f	**1p**: Correctly found the stationary points of the function **1p**: Correctly classified the local extreme points of f **1p**: Correctly classified the absolute minimum point of f
3.	$\frac{x}{2-3x}=\sum_{n=0}^{\infty}c_n x^n$ where $c_0=0$ and $c_n=\frac{1}{3}(\frac{3}{2})^n$, $n\geq 1$ The interval of convergence is $(-\frac{2}{3},\frac{2}{3})$	**1p**: Correctly expanded $x/(2-3x)$ in a power series in x **1p**: Correctly identified the coefficients of the power series **1p**: Correctly found the interval of convergence
4.	Proof	**3p**: Correctly proved that the integral from $-a$ to 0 is equal to minus the integral from 0 to a, and by this proved that the integral from $-a$ to a equals zero
5.	The limit exists and is equal to $-1/6$ **Note**: A student who wrongly has interpreted the absolute value bars obtains at most **1p** and this if and only if the resulting limit is correctly evaluated.	**1p**: Correctly, in the neighbourhood of zero, taken account of the absolute value bars **1p**: Correctly simplified the form of the denominator **1p**: Correctly found the limit
6.	$\tau: y-2=-\frac{e}{4}(x-\frac{1}{e}) \Leftrightarrow y=\frac{1}{4}(9-ex)$	**1p**: Correctly found the slope of the curve at a general point **1p**: Correctly found the coordinates of, and the slope at, the point P **1p**: Correctly formulated an equation for the tangent line τ to the curve γ at the point P
7.	$y=\frac{1}{8}(1-e^{-2x^2})$	**1p**: Correctly found and multiplied with an integrating factor, and correctly rewritten the LHS of the DE as an exact derivative **1p**: Correctly found the general solution of the DE **1p**: Correctly adapted the general solution to the initial value, and correctly summarized the solution of the IVP
8.	$f^{-1}(x)=\frac{2x}{x-1}$ $D_{f^{-1}}=R\setminus\{1\}$, $R_{f^{-1}}=R\setminus\{2\}$	**1p**: Correctly found the function expression for the inverse **1p**: Correctly found the domain of the inverse **1p**: Correctly found the range of the inverse