Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2017-01-02
Hjälpmedel: penna,
sudd och linjal.

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Finn alla lösningar till ekvationssystemet $A\mathbf{x} = \mathbf{b}$, där (4p)

$$A = \begin{bmatrix} 2 & -1 & 2 & 1 \\ -2 & 1 & 1 & -2 \\ 4 & -2 & 1 & -1 \end{bmatrix} \text{ och } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

2 Låt

$$B = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

a. Bestäm inversen till B. (3p)

b. Beräkna $\det(B)$. (3p)

3 Givet vektorerna $\mathbf{u} = (-1, 0, -1)$ och $\mathbf{v} = (2, -2, 1)$.

a. Bestäm vinkeln mellan $\mathbf{u}$ och $\mathbf{v}$. (2p)

b. Bestäm en enhetsvektor $\mathbf{w} \in \mathbb{R}^3$ som är ortogonal mot både $\mathbf{u}$ och $\mathbf{v}$. (3p)

4 Låt l vara linjen som på parameterform ges av $x = 1 + t$, $y = 1 - t$, $z = 2 + t$. Bestäm ekvationen för planet som går genom punkten $P(1, -3, 5)$ och innehåller linjen l. Ange svaret på punkt-normal form och inkludera en relevant figur i lösningen. (5p)

5 Bestäm det minsta avståndet från punkten $Q(3, 1, 0)$ till linjen som går genom punkterna $A(1, 1, -2)$ och $B(-1, 2, -1)$. (5p)

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Examination Vector algebra
MAA150 - TEN1
Date: January 2, 2017
Exam aids: pencil,
eraser, and ruler.

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Find all solutions to the linear system $A\mathbf{x} = \mathbf{b}$, where (4p)

$$A = \begin{bmatrix} 2 & -1 & 2 & 1 \\ -2 & 1 & 1 & -2 \\ 4 & -2 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

2 Let

$$B = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

a. Find the inverse of B. (3p)

b. Evaluate $\det(B)$. (3p)

3 Given the vectors $\mathbf{u} = (-1, 0, -1)$ and $\mathbf{v} = (2, -2, 1)$.

a. Find the angle between $\mathbf{u}$ and $\mathbf{v}$. (2p)

b. Find a unit vector $\mathbf{w} \in \mathbb{R}^3$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. (3p)

4 Let l be the line that has parameter form $x = 1 + t$, $y = 1 - t$, $z = 2 + t$. Find the equation of the plane that passes through the point $P(1, -3, 5)$ and contains the line l. Give the answer in point-normal form and include a relevant figure in the solution. (5p)

5 Determine the smallest distance from the point $Q(3, 1, 0)$ to the line that passes through the points $A(1, 1, -2)$ and $B(-1, 2, -1)$. (5p)

MAA150 Vektoralgebra, HT2016.

Assessment criteria for TEN1 2017-01-02

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Points may be deducted for erroneous mathematical statements, calculations, or failure to use proper mathematical notation.

Assessment problems

1. [**4 points**]
Relevant row operations on the augmented matrix **(2p)**, solving including setting a parameter (no point if you find that t is a value) **(1p)**, the correct solution **(1p)**.

2. [**6 points**]
a. Relevant method and row operations **(2p)**, correct inverse **(1p)**
b. Evaluating the determinant with a valid method **(2p)**, correct value **(1p)**.

3. **5 points**]
a. Relevant method/formula **(1p)**, finding the angle **(1p)**
b. Finding an orthogonal vector (e.g. using the cross product) **(2p)**, normalizing the vector **(1p)**

4. [**5 points**]
Finding two vectors in the plane **(1p)**, relevant figure **(1p)**, finding a normal (e.g. by cross product) **(1p)**, finding the equation of the plane **(2p)**

5. [**5 points**]
Method; such as relevant vectors and explaining figure **(2p)**, computing relevant projection **(2p)**, computing the distance **(1p)**

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(1)

$$\left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ -2 & 1 & 1 & -2 & 1 \\ 4 & -2 & 1 & -1 & -5 \end{array}\right] \begin{array}{c} 1 \quad -2 \end{array} \sim \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & -3 & -5 \end{array}\right] \begin{array}{c} 1 \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right] \times\left(-\tfrac{1}{4}\right) \sim \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 1 \quad -1 \end{array} \sim$$

(2p)

$$\sim \left[\begin{array}{cccc|c} 2 & -1 & 2 & 0 & -1 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \text{so} \begin{cases} 2x_1 - x_2 + 2x_3 = -1 \\ 3x_3 = 2 \\ x_4 = 1 \end{cases} \quad \begin{array}{l} x_3 = \frac{2}{3} \\ x_4 = 1 \end{array}$$

x_2 free variable (1p)

Set $x_2 = t$, then $x_1 = \dfrac{-1 + x_2 - 2x_3}{2} = \dfrac{-1 + t - \frac{4}{3}}{2} =$

$= -\frac{1}{2} + \frac{1}{2}t - \frac{2}{3} = -\frac{7}{6} + \frac{1}{2}t$

Answer: $\bar{x} = \left(-\frac{7}{6} + \frac{1}{2}t, t, \frac{2}{3}, 1\right)$ where $t \in \mathbb{R}$. (1p)

Check:

$2\left(-\frac{7}{6} + \frac{1}{2}t\right) - t + 2\cdot\frac{2}{3} + 1 = -\frac{7}{3} + t - t + \frac{4}{3} + \frac{3}{3} = 0$ ok

$-2\left(-\frac{7}{6} + \frac{1}{2}t\right) + t + \frac{2}{3} - 2\cdot 1 = \frac{7}{3} - t + t + \frac{2}{3} - \frac{6}{3} = \frac{3}{3} = 1$ ok.

$4\left(-\frac{7}{6} + \frac{1}{2}t\right) - 2t + \frac{2}{3} - 1 = -\frac{14}{3} + 2t - 2t + \frac{2}{3} - \frac{3}{3} = -\frac{15}{3} = -5$ ok.

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(2) Method $[B \mid I] \sim [I \mid B^{-1}]$

a)
$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \; 1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \; 1 \quad \times(-1) \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 \end{array}\right] \; -1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 \end{array}\right] \times\tfrac{1}{2} \sim$$ (2p)

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & \tfrac{1}{2} & \tfrac{1}{2} & 1 \end{array}\right] \; 1 \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & 0 & 1 \\ \tfrac{1}{2} & -\tfrac{1}{2} & 1 \\ \tfrac{1}{2} & \tfrac{1}{2} & 1 \end{matrix}}_{= B^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$ (1p)

b)
$$\det(B) = \begin{vmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \; \tfrac{1}{2} = \begin{vmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \end{vmatrix} \; \tfrac{1}{2} = \begin{vmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$ (2p)

(diagonal matrix) $= 2 \cdot (-1) \cdot 1 = -2$ (1p)

Answer a) $B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix}$

b) $\det(B) = -2$

Check.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1/2 & -1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ok!

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(3) $\bar{u} = (-1, 0, -1)$, $\bar{v} = (2, -2, 1)$

a) $\bar{u} \bullet \bar{v} = \|\bar{u}\| \cdot \|\bar{v}\| \cdot \cos(\theta)$ where θ is the angle between $\bar{u}$ and $\bar{v}$, so

$$\theta = \cos^{-1}\left(\frac{(-1,0,-1)\bullet(2,-2,1)}{\sqrt{(-1)^2+(-1)^2}\cdot\sqrt{2^2+(-2)^2+1^2}}\right) \overset{(1p)}{=} \cos^{-1}\left(\frac{-2-1}{\sqrt{2}\cdot\sqrt{9}}\right)$$

$$= \cos^{-1}\left(\frac{-3}{\sqrt{2}\cdot 3}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4} \quad (1p)$$

b) $\bar{u} \times \bar{v}$ is orthogonal to both $\bar{u}$ and $\bar{v}$

$$\bar{u} \times \bar{v} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \quad (2p)$$

So $\bar{w} = \frac{1}{\|\bar{u} \times \bar{v}\|} \cdot \bar{u} \times \bar{v} = \frac{1}{\sqrt{(-2)^2+(-1)^2+2^2}}(-2,-1,2)$

$$= \frac{1}{3}(-2,-1,2) = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) \quad (1p)$$

is a unit vector orthogonal to both $\bar{u}$ and $\bar{v}$

Answer a) $\frac{3\pi}{4}$

b) $\bar{w} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

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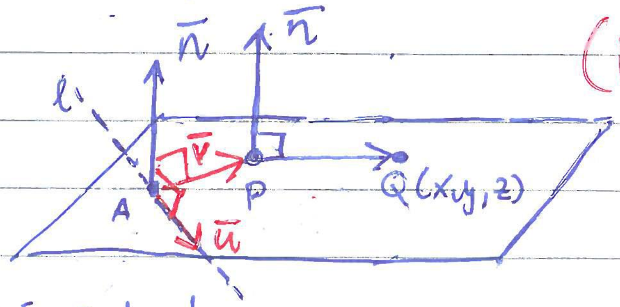
(4) $\ell : \begin{cases} x = 1+t \\ y = 1-t \\ z = 2+t \end{cases}$, $t \in \mathbb{R}$. $P(1,-3,5)$. Vector form of ℓ: $(x,y,z) = (1+t, 1-t, 2+t) =$

$$= (1,1,2) + t\cdot(1,-1,1), \quad t \in \mathbb{R}$$

We need two vectors in the plane, and can take one in the direction of ℓ, e.g. $\bar{u} = (1,-1,1)$. The second can be $\overrightarrow{AP}$ where A is $(1,1,2)$, i.e. $\bar{v} = (1,-3,5) - (1,1,2) = (0,-4,3)$. Then a normal to the plane is $\bar{n} = \bar{u} \times \bar{v}$ (1p)

Illustrative figure.



(1p)

$\overrightarrow{PQ} = (x-1, y+3, z-5)$

$$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & -4 & 3 \end{vmatrix} = i\cdot\begin{vmatrix} -1 & 1 \\ -4 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 0 & -4 \end{vmatrix}$$

$$= i - 3j - 4k = (1,-3,-4)$$ (1p)

The equation of the plane: $\bar{n} \bullet \overrightarrow{PQ} = 0$

$$(1,-3,-4) \bullet (x-1, y+3, z-5) = 0$$

$$\Updownarrow$$

$$(x-1) - 3(y+3) - 4(z-5) = 0$$ (2p)

Answer: $(x-1) - 3(y+3) - 4\cdot(z-5) = 0$

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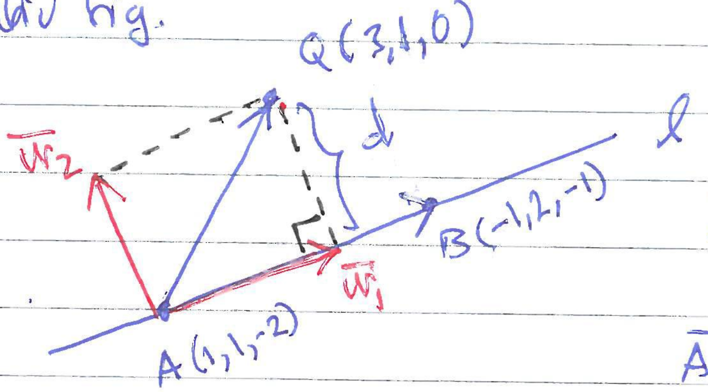
(5) Illustrative fig.



(1p)

Direction of l: $\overrightarrow{AB}$

$\overrightarrow{AB} = (-1,2,-1)-(1,1,-2)$
$= (-2,1,1)$

$\overrightarrow{AQ} = (3,1,0)-(1,1,-2)$
$= (2,0,2)$

Use orthogonal projection to find $\overline{w}_1$ and $\overline{w}_2$ such that $\overline{u} = \overline{w}_1 + \overline{w}_2$, $\overline{w}_1$ in the direction of l and $\overline{w}_1 \bullet \overline{w}_2 = 0$. Then $d = \|\overline{w}_2\|$. (1p)

$$\overline{w}_1 = \text{proj}_{\overrightarrow{AB}} \overrightarrow{AQ} = \frac{(2,0,2)\bullet(-2,1,1)}{\|(-2,1,1)\|^2}\cdot(-2,1,1) =$$

$$= \frac{-4+2}{(-2)^2+1^2+1^2} = -\frac{1}{3}(-2,1,1) = \left(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\right) \quad (2p)$$

$$\overline{w}_2 = (2,0,2)-\left(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\right) = \left(\frac{4}{3},\frac{1}{3},\frac{7}{3}\right)$$

so $$d = \|\overline{w}_2\| = \sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{7}{3}\right)^2} = \sqrt{\frac{16+1+49}{9}} = \sqrt{\frac{22}{3}} \quad (1p)$$

Answer: $\sqrt{\frac{22}{3}}$ l.u.