Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
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MÄLARDALENS HÖGSKOLA
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Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2016-08-16
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Bestäm ekvationen för skärningslinjen, l, mellan de två planen $\Pi_1 : 2x + y - 3z = 3$ och $\Pi_2 : x - 2y + z = -1$, och visa att punkten $(-1, -1, -2)$ tillhör l. (4p)

2 Bestäm konstanten b så att ekvationsystemet blir konsistent och bestäm sedan den allmänna lösningen för detta värde på b. (5p)

$$\begin{array}{rcrcrcl} x & - & y & + & & = & 1 \\ & & y & + & z & = & b \\ 2x & & & + & 2z & = & 1 \end{array}$$

3 Låt

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. Beräkna determinanten av A. (2p)

b. Bestäm inversen till A och beräkna $(AB^{-1})^{-1}$. (4p)

4 Låt Π vara planet som ges av ekvationen $x + 2y + 2z = -4$ och Q vara punkten $(2, 2, 1)$.

a. Bestäm avståndet från Q till planet Π. (3p)

b. Hitta den punkt i planet Π som ligger närmast Q. (3p)

5 Lös ekvationen $z^3 = -8i$. Svara på formen $a + bi$. (4p)

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Examination Vector algebra
MAA150 - TEN1
Date: August 16, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Find the equation of the line, l, of intersection between the two planes $\Pi_1 : 2x + y - 3z = 3$ and $\Pi_2 : x - 2y + z = -1$, and show that the point $(-1, -1, -2)$ belong to l. (4p)

2 Determine the constant b such that the linear system is consistent and then find the general solution for that value of b. (5p)

$$\begin{array}{rcrcrcr} x & - & y & + & & = & 1 \\ & & y & + & z & = & b \\ 2x & & & + & 2z & = & 1 \end{array}$$

3 Given

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. Evaluate the determinant of A. (2p)

b. Find the inverse of A and compute $(AB^{-1})^{-1}$. (4p)

4 Let Π be the plane given by the equation $x + 2y + 2z = -4$ and let Q be the point $(2, 2, 1)$.

a. Find the distance from Q to the plane Π. (3p)

b. Find the point belonging to the plane Π that is closest to Q. (3p)

5 Solve the equation $z^3 = -8i$. Give the answer in the form $a + bi$. (4p)

MAA150 Vektoralgebra, vt-16.

Assessment criterias for TEN1 2016-08-16

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. [**4 points**]
Relevant method, e.g. row operations on the augmented matrix **(2p)**, the correct line **(1p)**, correctly checking that the point belong to the line **(1p)**

2. [**5 points**]
Relevant method for finding b **(2p)**, the correct b **(1p)**, solving the equation correctly for any b **(1p)** and the correct solution **(1p)**

3. [**6 points**]
a. Computing the determinant **(2p)**
b. Relevant row operations **(2p)**, correct inverse **(1p)**, motivation that $(AB^{-1})^{-1}$ is undefined since B^{-1} is undefined **(1p)**

4. [**6 points**]
a. Relevant method maximum **(2p)**, correct answer **(1p)**
b. Relevant method maximum **(2p)**, correct answer **(1p)**

5. [**4 points**]
Finding polar form of $-8i$ **(1p)**, setting $z = r(\cos(\theta) + i\sin(\theta))$ and obtaining the equations for r and θ **(1p)**, finding the correct values of r and θ **(1p)**, the correct solutions in form $a + bi$ **(1p)**

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(1) $\Pi_1: 2x + y - 3z = 3$, $\Pi_2: x - 2y + z = -1$ } Solve for general solution

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 3 \\ 1 & -2 & 1 & -1 \end{array}\right] \updownarrow \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 2 & 1 & -3 & 3 \end{array}\right] \begin{array}{c} -2 \\ \end{array} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 5 & -5 & 5 \end{array}\right] \times \frac{1}{5} \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad (2p)$$

$\uparrow$ z free variable

Set $z = t \Rightarrow y = 1 + t$, $x = -1 + 2y - z = -1 + 2(1+t) - t$
$= 1 + t$

so $\ell: (x,y,z) = (1,1,0) + t \cdot (1,1,1)$, $t \in \mathbb{R}$ $(*)$ (1p)

The point $P = (-1,-1,2) \in \ell$ since taking $t = -2$ in $(*)$ gives $(1,1,0) - 2(1,1,1) = (-1,-1,-2)$ (1p)

Answer: $\ell: (x,y,z) = (1,1,0) + t \cdot (1,1,1)$, $t \in \mathbb{R}$

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(2)
$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & b \\ 2 & 0 & 2 & 1 \end{array}\right] \overset{-2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & b \\ 0 & 2 & 2 & -1 \end{array}\right] \overset{-2}{}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & -1-2b \end{array}\right] \Rightarrow \quad 0 = -1-2b \Leftrightarrow b = -\frac{1}{2}$$

The linear system is consistent if $b = -\frac{1}{2}$, (3p)
and then has infinitely many solutions.
With $b = -1/2$ we get

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad z = t \Rightarrow y = -\frac{1}{2} - t, \quad x = 1 + y = \frac{1}{2} - t, \quad t \in \mathbb{R}$$

z free variable

Answer: General solution for $b = -\frac{1}{2}$ is (2p)

$$\begin{cases} x = \frac{1}{2} - t \\ y = -\frac{1}{2} - t \\ z = t \end{cases}, \quad t \in \mathbb{R}$$

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3a
$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} =$$

$$= 3\cdot(-1) - 1\cdot 1 = -4$$

(2p)

3b
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] (-1) \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 3 & 1 & -1 & 1 & 0 \end{array}\right] (-3) \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 4 & -1 & 1 & -3 \end{array}\right] \times \frac{1}{4}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{array}\right] (1) \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{array}\right] (1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{array}\right] \quad \text{so } A^{-1} = \begin{bmatrix} 3/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & -3/4 \end{bmatrix}$$

(the right block is labelled A^{-1})

(3p)

$(AB^{-1})^{-1}$ is undefined since B does not have an invers due to $\det(B) = 0$.

(1p)

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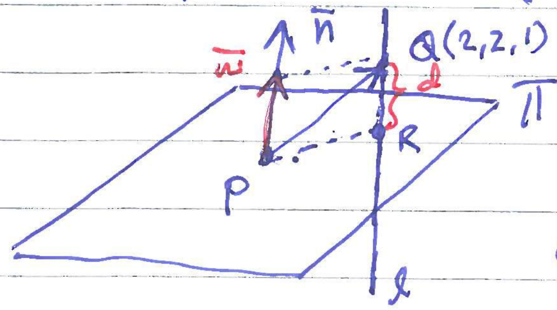
(4) $\Pi: x+2y+2z=-4$, $Q=(2,2,1)$

Take any $P\in\Pi$, e.g $x=y=0 \Rightarrow z=-2$, i.e $P=(0,0,-2)$



$\bar{n}=(1,2,2)$ is a normal to Π

then

$d=\|\bar{w}\|=\|\text{proj}_{\bar{n}}\overrightarrow{PQ}\|$

where $\overrightarrow{PQ}=(2,2,1)-(0,0,-2)$
$=(2,2,3)$ (1p)

$$\bar{w}=\text{proj}_{\bar{n}}(2,2,3)=\frac{(2,2,3)\bullet(1,2,2)}{\|(1,2,2)\|^2}\begin{bmatrix}1\\2\\2\end{bmatrix}=\frac{12}{9}\begin{bmatrix}1\\2\\2\end{bmatrix}=\frac{4}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

så $d=\|\bar{w}\|=\frac{4}{3}\cdot\|(1,2,2)\|=\frac{4}{3}\cdot 3=4$ (2p)

Answer 4a : 4 l.u.

$\ell: (x,y,z)=\overrightarrow{OQ}+t\cdot\bar{n}$, $t\in\mathbb{R}$ especially

$R=(2,2,1)+t\cdot(1,2,2)=(2+t,\ 2+2t,\ 1+2t)$

$R\in\Pi \iff (2+t)+2(2+2t)+2(1+2t)=-4$

$\iff 8+9t=-4 \iff t=-\frac{12}{9}=-\frac{4}{3}$

so $R=(2,2,1)-\frac{4}{3}(1,2,2)=\left(\frac{2}{3},-\frac{2}{3},-\frac{5}{3}\right)$ (3p)

Answer b: $\left(\frac{2}{3},-\frac{2}{3},-\frac{5}{3}\right)$

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5

(*) $z^3 = -8i = 8\cdot\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

↑ polar form

Set $z = r\cdot(\cos\theta + i\sin\theta)$, then $z^3 = r^3(\cos(3\theta) + i\sin(3\theta))$

$$\begin{cases} r^3 = 8 \\ 3\theta = -\frac{\pi}{2} + 2\pi n, \quad n \in \mathbb{Z} \end{cases}$$ (2p)

which gives

$$\begin{cases} r = 2 \\ \theta = -\frac{\pi}{6} + \frac{2\pi}{3}\cdot n, \quad n \in \mathbb{Z} \end{cases}$$ (1p)

We can take $n = 0, 1, 2$ since (*) has 3 solutions:

$\theta_0 = -\frac{\pi}{6}$, $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{7\pi}{6}$

$z_1 = 2\cdot\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$

(1p)

$z_2 = 2\cdot\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2\cdot i$

$z_3 = 2\cdot\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = 2\cdot\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

Answer : $z = \sqrt{3} - i$, $z = 2i$, or $z = -\sqrt{3} - i$