MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2017-02-17 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. The function g is twice differentiable, and it is known that

$$\begin{array}{lllll} g(0) = 7\,, & g(\frac{1}{2}) = -4\,, & g(\frac{1}{\sqrt{2}}) = 5\,, & g(\frac{\sqrt{3}}{2}) = 3\,, & g(1) = 0\,, \\ g'(0) = -2\,, & g'(\frac{1}{2}) = 6\,, & g'(\frac{1}{\sqrt{2}}) = 9\,, & g'(\frac{\sqrt{3}}{2}) = 8\,, & g'(1) = 2\,, \\ g''(0) = 3\,, & g''(\frac{1}{2}) = 7\,, & g''(\frac{1}{\sqrt{2}}) = 4\,, & g''(\frac{\sqrt{3}}{2}) = 6\,, & g''(1) = 7\,. \end{array}$$

Find the second derivative of the composite function $g \circ \cos$ at the point $\pi/3$.

2. Let $f(x) = e^{-x}|\sin(x)|$. Is the function f bounded, bounded above, bounded below, or unbounded both above and below? Explain!

3. Find the function f whose derivative is equal to the function $x \curvearrowright \dfrac{2}{x^2-1}$, and whose value at the point 3 is equal to 0.

4. Find to the differential equation $y'' + y' - 20y = 0$ the solution that satisfies the initial conditions $y(0) = 4\,,\ y'(0) = -2$.

5. Find the convergent geometric series whose second term is equal to -3 and whose sum is equal to 4.

6. Evaluate the integral $\displaystyle\int_0^1 |3 - 4x|\ dx\,.$

7. Find the range of the function $x \curvearrowright f(x) = x^2\sqrt{6 - x^2}$, $D_f = [1, \sqrt{5}\,]$.

8. Find out whether
$$\lim_{x \to -2} \frac{12 + 4x - x^2}{4 + 4x + x^2} \cdot \frac{3x + 6}{3x + 5}$$
exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$$\begin{array}{lclcl} S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & S_1 + 2S_2 \leq 41 & \rightarrow & 3 \\ S_1 \geq 11,\ S_2 \geq 9 & \text{OCH} & 42 \leq S_1 + 2S_2 \leq 53 & \rightarrow & 4 \\ & & 54 \leq S_1 + 2S_2 & \rightarrow & 5 \end{array}$$

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Funktionen g är två gånger deriverbar, och det är bekant att

$$\begin{array}{lllll} g(0) = 7\,, & g(\tfrac{1}{2}) = -4\,, & g(\tfrac{1}{\sqrt{2}}) = 5\,, & g(\tfrac{\sqrt{3}}{2}) = 3\,, & g(1) = 0\,, \\ g'(0) = -2\,, & g'(\tfrac{1}{2}) = 6\,, & g'(\tfrac{1}{\sqrt{2}}) = 9\,, & g'(\tfrac{\sqrt{3}}{2}) = 8\,, & g'(1) = 2\,, \\ g''(0) = 3\,, & g''(\tfrac{1}{2}) = 7\,, & g''(\tfrac{1}{\sqrt{2}}) = 4\,, & g''(\tfrac{\sqrt{3}}{2}) = 6\,, & g''(1) = 7\,. \end{array}$$

Bestäm andraderivatan av den sammansatta funktionen $g \circ \cos$ i punkten $\pi/3$.

2. Låt $f(x) = e^{-x}|\sin(x)|$. Är funktionen f begränsad, uppåt begränsad, nedåt begränsad, eller obegränsad både uppåt och nedåt? Förklara!

3. Bestäm den funktion f vars derivata är lika med funktionen $x \curvearrowright \dfrac{2}{x^2-1}$, och vars värde i punkten 3 är lika med 0.

4. Bestäm till differentialekvationen $y'' + y' - 20y = 0$ den lösning som satisfierar begynnelsevillkoren $y(0) = 4$, $y'(0) = -2$.

5. Bestäm den konvergenta geometriska serie vars andra term är lika med -3 och vars summa är lika med 4.

6. Beräkna integralen $\displaystyle\int_0^1 |3-4x|\,dx$.

7. Bestäm värdemängden för funktionen $x \curvearrowright f(x) = x^2\sqrt{6-x^2}$, $D_f = [1, \sqrt{5}\,]$.

8. Utred om

$$\lim_{x\to -2} \frac{12+4x-x^2}{4+4x+x^2} \cdot \frac{3x+6}{3x+5}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2017-02-17

1

$$\frac{d^2}{dx^2}(g\circ\cos)(x) = \frac{d^2}{dx^2}\, g(\cos(x)) \overset{\text{the chain rule}}{=} \frac{d}{dx}\left[g'(\cos(x))\cdot(-\sin(x))\right]$$

the product rule and the chain rule

$$= \left[g''(\cos(x))\cdot(-\sin(x))\right][-\sin(x)] + [g'(\cos(x))]\cdot[-\cos(x)]$$

$$= g''(\cos(x))\cdot\sin^2(x) - g'(\cos(x))\cdot\cos(x)$$

trig. values; from the given table of values

Thus $$\left.\frac{d^2}{dx^2}(g\circ\cos)(x)\right|_{x=\frac{\pi}{3}} = g''\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{\sqrt{3}}{2}\right)^2 - g'\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{2} = 7\cdot\tfrac{3}{4} - 6\cdot\tfrac{1}{2} = \underline{\tfrac{9}{4}}$$

2

$$f(x) = e^{-x}\cdot|\sin(x)|$$

especially $e^{-x} > 0$ for all $x \in R$

$\underline{f \text{ is bounded below}}$ since
$$\begin{cases} * \; e^{-x}\cdot|\sin(x)| \geq 0 \text{ for all } x \in D_f = R \\ * \; e^{-x}|\sin(x)| \to +\infty \text{ as } \begin{cases} x \to -\infty \\ x \neq n\pi \end{cases} \end{cases}$$

n integer

3

Given: $f'(x) = \frac{2}{x^2-1}$ and $f(3) = 0$

We get $$f(x) = \int \frac{2\,dx}{x^2-1} = \int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$$
$$= \ln|x-1| - \ln|x+1| + C = \ln\left|\frac{x-1}{x+1}\right| + C$$

where $$0 = f(3) = \ln\left(\frac{3-1}{3+1}\right) + C = \ln\left(\tfrac{1}{2}\right) + C = -\ln(2) + C$$
i.e. $C = \ln(2)$

Thus $$\underline{f(x)} = \ln\left(\frac{x-1}{x+1}\right) + \ln(2) = \underline{\ln\left(2\left(\frac{x-1}{x+1}\right)\right)}$$

where the absolute value signs have been interpreted for the interval of existence I_E about the point 2, i.e. $I_E = (1, \infty)$.

4

DE: $y'' + y' - 20y = 0$ IV:s: $y(0) = 4$, $y'(0) = -2$

The auxiliary equation is $0 = r^2 + r - 20 = (r+5)(r-4)$

Thus the general solution of the DE is $y = Ae^{-5x} + Be^{4x}$

The IV:s give $$\begin{cases} 4 = y(0) = A + B \\ -2 = y'(0) = -5A + 4B \end{cases} \Leftrightarrow \begin{cases} A + B = 4 \\ 9B = 18 \end{cases} \Leftrightarrow \begin{cases} A = 2 \\ B = 2 \end{cases}$$

Thus $\underline{y = 2(e^{-5x} + e^{4x})}$ is the solution of the IVP.

5 A geometric series is of the form $\alpha + \alpha\beta + \alpha\beta^2 + \ldots$ and has the sum $\frac{\alpha}{1-\beta}$ iff $|\beta|<1$.

We have: $\begin{cases} \alpha\beta = -3 \\ |\beta| < 1 \\ \frac{\alpha}{1-\beta} = 4 \end{cases} \Leftrightarrow \begin{cases} \alpha\beta = -3 \\ -1<\beta<1 \\ -3 = 4\beta(1-\beta) \end{cases} \Leftrightarrow \begin{cases} \alpha\beta = -3 \\ -1<\beta<1 \\ (2\beta-1)^2 = 4 \end{cases} \Leftrightarrow \begin{cases} \alpha\beta = -3 \\ -1<\beta<1 \\ (2\beta+1)(2\beta-3)=0 \end{cases}$

$\Leftrightarrow \begin{cases} \beta = -\frac{1}{2} \\ \alpha = 6 \end{cases}$ Thus the convergent geometric series is $\underline{6 - 3 + \frac{3}{2} - \ldots}$

6 $$\int_0^1 |3-4x|\,dx = \int_0^{3/4}(3-4x)\,dx + \int_{3/4}^1 [-(3-4x)]\,dx$$

$$= \left[3x-2x^2\right]_0^{3/4} - \left[3x-2x^2\right]_{3/4}^1 = \left(\frac{9}{4}-\frac{9}{8}\right)-(0-0)-\left[(3-2)-\left(\frac{9}{4}-\frac{9}{8}\right)\right]$$

$$= \frac{9}{8} - 1 + \frac{9}{8} = \frac{9}{4} - 1 = \underline{\frac{5}{4}}$$

7 $f(x) = x^2\sqrt{6-x^2}$, $D_f = [1, \sqrt{5}]$

Differentiation gives $f'(x) = 2x\sqrt{6-x^2} + x^2 \cdot \frac{1}{2}\frac{-2x}{\sqrt{6-x^2}} = \frac{2x}{\sqrt{6-x^2}}\left(6-x^2-\frac{1}{2}x^2\right)$

$$= \frac{2x}{\sqrt{6-x^2}}\left(-\frac{3}{2}\right)(x^2-4) = -\frac{3x}{\sqrt{6-x^2}}(x+2)(x-2)$$

A first derivative test is

	EP		SP		EP
x	1		2		$\sqrt{5}$
$f'(x)$		+	0	−	
$f(x)$	loc. min	↗	loc. max	↘	loc. min

Since f is continuous in a closed and bounded D_f, it has a maximum f_{max} and a minimum f_{min}, and takes on all values between the extremes f_{min} and f_{max}.

We get $\begin{cases} f_{min} = \min(f(1), f(\sqrt{5})) = \min(\sqrt{5}, 5) = \sqrt{5} \\ f_{max} = f(2) = 4\sqrt{2} \end{cases}$ Thus $\underline{R_f = [\sqrt{5}, 4\sqrt{2}]}$

8 $$\lim_{x\to-2} \frac{12+4x-x^2}{4+4x+x^2}\cdot\frac{3x+6}{3x+5} = \lim_{x\to-2}\frac{-(x^2-4x-12)}{x^2+4x+4}\cdot\frac{3x+6}{3x+5}$$

the product rule for limits

$$= \lim_{x\to-2}\frac{-(x+2)(x-6)}{(x+2)^2}\left(\frac{3(x+2)}{3x+5}\right) = \lim_{x\to-2}\left(\frac{x+2}{x+2}\right)^2 \cdot \lim_{x\to-2}\left(\frac{-3(x-6)}{3x+5}\right)$$

$$= 1\cdot\left(\frac{-3(-2-6)}{3(-2)+5}\right) = \frac{3\cdot 8}{-1} = \underline{-24}$$

(due to the fact that the function $x \mapsto \frac{-3(x-6)}{3x+5}$ is continuous at -2)

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN1 – 2017-02-17

Maximum points for subparts of the problems in the final examination

1. $9/4$

1p: Correctly worked out the 1st derivative of $g \circ \cos$
1p: Correctly worked out the 2nd derivative of $g \circ \cos$
1p: Correctly evaluated the 2nd derivative at the point $\pi/3$

2. f is bounded below

1p: Correctly concluded that $f(x) \geq 0$
1p: Correctly concluded $f(x) \to \infty$ as $x \to -\infty$, $x \neq n\pi$
1p: Correctly summarized that f is bounded below

3. $f(x) = \ln\left(2\dfrac{x-1}{x+1}\right)$

1p: Correctly found the partial fractions of $2/(x^2-1)$
1p: Correctly found the general antiderivative of f
1p: Correctly adapted the antiderivative to the value at 3

4. $y = 2(e^{4x} + e^{-5x})$

1p: Correctly found one solution of the DE
1p: Correctly found the general solution of the DE
1p: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP

5. $\sum_{n=0}^{\infty} 6(-\tfrac{1}{2})^n = 6 - 3 + \tfrac{3}{2} - \ldots$

1p: Correctly introduced a notation for a geometric series, and correctly in terms of that notation formulated the three conditions which must be valid
2p: Correctly solved the system of conditions (two equations and one inequality), and correctly answered with the series asked for

6. $5/4$

2p: Correctly taken account of the absolute value bars and correctly worked out the topical antiderivatives
1p: Correctly evaluated the antiderivatives at the limits

Note: The student who has not taken account of the absolute value bars obtains **0p**. The student who has found a negative value, and has not commented such a value as being unreasonable obtains **0p**. The student who has found a negative value for the integral, but at least has commented such a value as being unreasonable, may obtain **at most 1p**.

7. $R_f = [\sqrt{5}, 4\sqrt{2}]$

Note: To get full marks, a student should either correctly have concluded from a properly done first-derivative test, or alternatively exhaustively have referred to the two theorems about intermediate values and extreme values respectively.

1p: Correctly differentiated the function f, and correctly concluded about the local extreme points of the function
1p: Correctly found the minimum of f
1p: Correctly found the maximum of f, and correctly stated the range of f

8. The limit exists and is equal to -24

1p: Correctly factorized the expression
1p: Correctly concluded that the limit exists
1p: Correctly found the limit