MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2015-08-17 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\to$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\to$	4
		$54 \leq S_1 + 2S_2$	$\to$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\to$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\to$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\to$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\to$	B
		$61 \leq S_1 + 2S_2$	$\to$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the length of the curve $y = \ln(x + \sqrt{x^2 - 1})$, $\sqrt{17} \leq x \leq \sqrt{37}$.

2. Sketch the graph of the function f, defined by

$$f(x) = \frac{x^2}{2(x+2)},$$

by utilizing the guidance given by asymptotes and stationary points.

3. Find an equation for the tangent line τ to the curve

$$\gamma : 8xy - x^2y^3 = 12$$

at the point $P : (2, 1)$.

4. Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sin(1/n)}$ convergent or divergent? Explain!

5. Evaluate the integral

$$\int_4^{12} \frac{x+3}{x\sqrt{x-3}}\,dx,$$

and write the result in as simple form as possible.

Om du föredrar frågorna skrivna på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2015-08-17 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm längden av kurvan $y = \ln(x + \sqrt{x^2 - 1}),\ \sqrt{17} \leq x \leq \sqrt{37}$.

2. Skissa grafen till funktionen f, definierad enligt

$$f(x) = \frac{x^2}{2(x+2)}\,,$$

genom att använda den vägledning som fås från asymptoter och stationära punkter.

3. Bestäm en ekvation för tangenten τ till kurvan

$$\gamma : 8xy - x^2y^3 = 12$$

i punkten $P : (2, 1)$.

4. Är serien $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sin(1/n)}$ konvergent eller divergent? Förklara!

5. Beräkna integralen

$$\int_4^{12} \frac{x+3}{x\sqrt{x-3}}\,dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

If you prefer the questions written in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2015-08-17

① Curve: $y = \underbrace{\ln(x+\sqrt{x^2-1})}_{f(x)}$, $\sqrt{17} \le x \le \sqrt{37}$

The length L of the curve is $\displaystyle\int_{\sqrt{17}}^{\sqrt{37}} \sqrt{1+(f'(x))^2}\,dx$

where

$$f'(x) = \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{1}{2}\frac{2x}{\sqrt{x^2-1}}\right) = \frac{\sqrt{x^2-1}+x}{(x+\sqrt{x^2-1})\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

i.e. $$L = \int_{\sqrt{17}}^{\sqrt{37}} \sqrt{1+\left(\frac{1}{\sqrt{x^2-1}}\right)^2}\,dx = \int_{\sqrt{17}}^{\sqrt{37}} \sqrt{\frac{x^2-1+1}{x^2-1}}\,dx = \int_{\sqrt{17}}^{\sqrt{37}} \frac{|x|\,dx}{\sqrt{x^2-1}}$$

$|x| = x$ since $\sqrt{17} \le x \le \sqrt{37}$

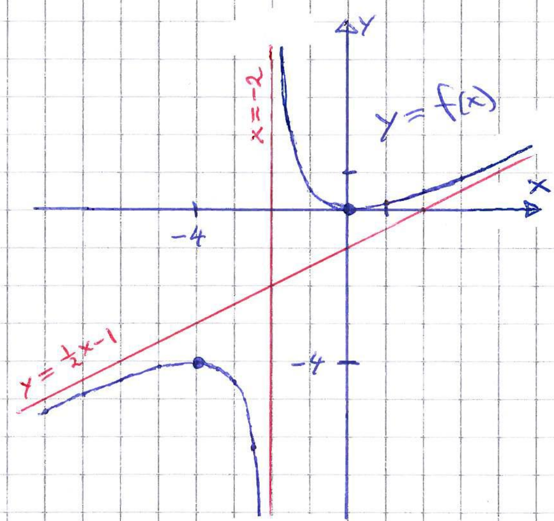
$$= \int_{\sqrt{17}}^{\sqrt{37}} \frac{x\,dx}{\sqrt{x^2-1}} = \left[\sqrt{x^2-1}\right]_{\sqrt{17}}^{\sqrt{37}} = (6-4)\text{ l.u.} = \underline{2\text{ l.u.}}$$

② $f(x) = \dfrac{x^2}{2(x+2)}$, $\quad f'(x) = \dfrac{1}{2}\dfrac{2x(x+2)-x^2\cdot 1}{(x+2)^2} = \dfrac{x^2+4x}{2(x+2)^2} = \dfrac{(x+4)x}{2(x+2)^2}$

First derivative test

$\begin{cases} f(-4) = -4 \\ f(0) = 0 \end{cases}$

x		SP -4		As -2		SP 0	
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$+$
$f(x)$	$\nearrow$	loc. max	$\searrow$	$\nexists$	$\searrow$	loc. min	$\nearrow$



$x=-2$ is double-sided, vertical asymptote of the curve $y=f(x)$ since $f(x) \xrightarrow{x\to -2^{\pm}} \pm\infty$. There is also a non-vertical asymptote $y = \frac{1}{2}x - 1$ since $f(x) = \dfrac{x^2}{2(x+2)}$

$$= \frac{1}{2}\frac{(x+2-2)^2}{x+2} = \frac{1}{2}\left(x+2-4+\frac{4}{x+2}\right)$$

$$= \frac{1}{2}x - 1 + \frac{2}{x+2} \underset{\text{for large } x}{\approx} \frac{1}{2}x - 1$$

③ γ: $8xy - x^2y^3 = 12$ $\qquad$ P: $(2,1)$

To begin with, we note that $8\cdot 2\cdot 1 - 2^2\cdot 1^3 = 16-4 = 12$, i.e. $P \subset \gamma$ (so that the problem is solvable).

Implicit differentiation gives $8\cdot 1y + 8x\cdot y' - 2xy^3 - x^2 3y^2 y' = 0$

Let $\left.\frac{dy}{dx}\right|_P = k$. Then at P, we get

$$0 = 8\cdot 1 + 8\cdot 2\cdot k - 2\cdot 2\cdot 1^3 - 3\cdot 2^2 1^2\cdot k = (8-4) + (16-12)k \Leftrightarrow k = -1$$

Thus an equation for the tangent line τ to γ at P is

$$\tau: \; y - 1 = -1(x-2) \Leftrightarrow y = -x+3 \Leftrightarrow \underline{x+y=3}$$

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(4) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{(-1)^n}{n \sin(1/n)}$

We have that $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \dfrac{1}{n\left(\frac{1}{n} - \frac{1}{6n^3} + \dots\right)}$

$= \lim_{n\to\infty} \dfrac{1}{1 - \frac{1}{6n^2} + \dots} = 1$

i.e. the series does not satisfy a necessary condition for convergence, namely that the terms of a convergent series must have the limit 0 when $n \to \infty$

i.e. the series $\sum_{n=1}^{\infty} a_n$ is divergent

(5) $$\int_4^{12} \frac{x+3}{x\sqrt{x-3}}\,dx \left[\begin{array}{rl} \sqrt{x-3} &= u \\ x &= 3+u^2 \\ dx &= 2u\,du \end{array}\right] = \int_1^3 \frac{u^2+6}{u^2+3}\,\frac{2u\,du}{u}$$

$$= 2\int_1^3 \frac{u^2+3+3}{u^2+3}\,du = 2\int_1^3 \left(1 + \frac{1}{1+\left(\frac{u}{\sqrt{3}}\right)^2}\right) du$$

$$= 2\left[u + \sqrt{3}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3$$

$$= 2\left[\left(3 + \sqrt{3}\underbrace{\arctan(\sqrt{3})}_{\pi/3}\right) - \left(1 + \sqrt{3}\underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}_{\pi/6}\right)\right]$$

$$= 2\left[(3-1) + \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right]$$

$$= 2\left[2 + \sqrt{3}\cdot\frac{\pi}{6}\right] = 4 + \sqrt{3}\,\frac{\pi}{3} = 4 + \frac{\pi}{\sqrt{3}}$$

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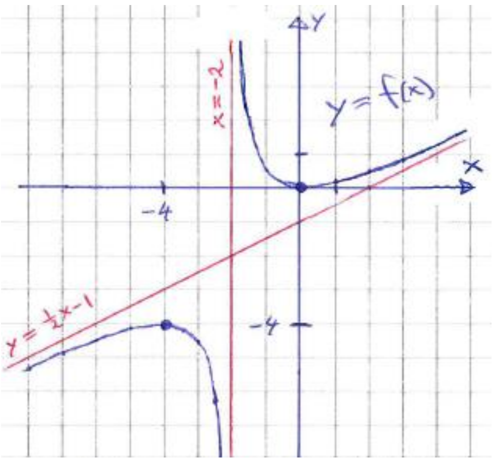
EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2014/15

Examination TEN2 – 2015-08-17 — Maximum points for subparts of the problems in the final examination

1. 2 l.u.

1p: Correctly, in terms of $f'(x)$, formulated an integral whose value is the length of the curve
1p: Correctly differentiated f and correctly rewrited the expression for $f'(x)$ into $(x^2-1)^{-1/2}$
1p: Correctly rewrited the integrand into $x(x^2-1)^{-1/2}$ in preparing for finding the antiderivative
1p: Correctly determined the antiderivative of the integrand and correctly determined the value of the integral

2. The graph has the asymptotes $x=-2$ and $y=\frac{1}{2}x-1$, a local maximum at $(-4,-4)$ and a local minimum at $(0,0)$



1p: Correctly determined the equations for the vertical and the non-vertical asymptotes of the graph
1p: Correctly found the local extreme points of the graph
1p: Correctly sketched the left part of the graph, i.e. the part to the left of the vertical asymptote
1p: Correctly sketched the right part of the graph, i.e. the part to the right of the vertical asymptote

3. $\tau: x+y=3$

2p: Correctly differentiated implicitly the equation with respect to x, all with the purpose of finding the slope at the point P
1p: Correctly determined the slope $dy/dx|_P$ at the point P
1p: Correctly found an equation for the tangent line τ

4. The series is divergent

1p: Correctly, as a starting point, analysed the absolute value of the terms of the series by e.g. correctly expanding $\sin(1/n)$ in a Maclaurin series
1p: Correctly found that the absolute value of the terms of the series has the limit 1 as $n\to\infty$
2p: Correctly concluded that the series diverges since the *necessary* condition $\lim_{n\to\infty} a_n = 0$ for convergence of a series Σa_n is not satisfied

5. $4+\dfrac{\pi}{\sqrt{3}}$

1p: Correctly made a substitution e.g. $\sqrt{x-3}=u$, and correctly worked it out
1p: Correctly worked out the polynomial division and prepared for finding the antiderivative of the integrand
1p: Correctly determined the antiderivative of the integrand
1p: Correctly evaluated the limits of the integral