Divsion of Mathematics
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Examination Vector algebra
MAA150 - TEN1
Date: January 7, 2016
Exam aids: not any

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Determine the constant b such that the linear system is consistent and then find the general solution. (6p)

$$\begin{aligned} -3x + y - z &= 2 \\ 15x - 5y + 15z &= b \\ 6x - 2y + 7z &= -1 \end{aligned}$$

2 For what real values k is the matrix X invertible? (4p)

$$X = \begin{bmatrix} 3 & k & 0 \\ -1 & -1 & k \\ 1 & -k & 2 \end{bmatrix}$$

3 Find the matrix $(AB^T + 3A)^{-1}$ when (5p)

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}.$$

4 Given that the lines $l_1 : (x, y) = (0, -3) + t(a, 2)$ and $l_2 : 3x + 4y = 2$ in $\mathbb{R}^2$ are parallel

a. find the shortest distance between l_1 and l_2. (3p)

b. determine the constant a. (2p)

5 Verify that $z = 2i$ is a root of the polynomial $p(z) = 9z^4 + 37z^2 + 4$ and factor $p(z)$ into linear factors. (5p)

Avdelningen för Matematik
och tillämpad matematik
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Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2016-01-07
Hjälpmedel: inga

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Bestäm konstanten b så att ekvationsystemet blir konsistent och bestäm sedan den allmänna lösningen. (6p)

$$\begin{aligned} -3\,x + y - z &= 2 \\ 15x - 5y + 15z &= b \\ 6x - 2\,y + 7z &= -1 \end{aligned}$$

2 För vilka rella tal k är matrisen X inverterbar? (4p)

$$X = \begin{bmatrix} 3 & k & 0 \\ -1 & -1 & k \\ 1 & -k & 2 \end{bmatrix}$$

3 Bestäm matrisen $(AB^T + 3A)^{-1}$ då (5p)

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \text{ och } B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}.$$

4 Givet att linjerna $l_1 : (x, y) = (0, -3) + t(a, 2)$ och $l_2 : 3x + 4y = 2$ i $\mathbb{R}^2$ är parallella

a. bestäm det minsta avståndet mellan l_1 och l_2. (3p)

b. bestäm konstanten a. (2p)

5 Verifiera att $z = 2i$ är en rot till polynomet $p(z) = 9z^4 + 37z^2 + 4$ och faktorisera $p(z)$ i linjära faktorer. (5p)

MAA150 Vektoralgebra, ht-15.

Assessment criterias for TEN1 2016-01-07

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Assessment problems

1. Complete solution gives 6 points. Finding b gives 3 points; maximum 2 points for a correct method, where relevant row operations give 1 point, identifying an equation for b and solving for b gives 1 point. Solving the resulting linear system gives 3 points; relevant row operations 1 point, setting free variable 1 point, and correct solution 1 point.

2. Complete solution gives 4 points. A condition that X is invertible gives 1 point. Checking that the condition is satisfied gives 2 points, finding the correct values of k gives 1 point.

3. Complete solution gives 5 points. Finding B^T gives 1 point, $AB^T + 3A$ gives 1 point, and its inverse 3 points; relevant row operations 2 points, and the correct answer 1 point.

4.
- a. Complete solution gives 3 points. Correct method maximum 2 points, computing the distance 1point.
- b. Complete solution gives 2 points. Setting up a correct equation for finding a 1 point and the correct answer 1 point.

5. Complete solution gives 5 points. Verifying the given root gives 1 point, and finding a second root gives 1 point. Polynomial division to find the remaining factors gives 2 points, and the correct answer gives 1 point.

(1)

$$\left[\begin{array}{ccc|c} -3 & 1 & -1 & 2 \\ 15 & -5 & 15 & b \\ 6 & -2 & 7 & -1 \end{array}\right] \begin{matrix} (5) & (2) \end{matrix} \sim \left[\begin{array}{ccc|c} -3 & 1 & -1 & 2 \\ 0 & 0 & 10 & b+10 \\ 0 & 0 & 5 & 3 \end{array}\right] (-1/2)$$

$$\sim \left[\begin{array}{ccc|c} -3 & 1 & -1 & 2 \\ 0 & 0 & 10 & b+10 \\ 0 & 0 & 0 & 3-\frac{b}{2}-5 \end{array}\right] \quad \begin{cases} -3x+y-z=2 \\ 10z = b+10 \\ 0 = -2-\frac{b}{2} \end{cases}$$

The system is consistent iff $0=-2-\frac{b}{2} \Leftrightarrow b=-4$

This gives the system

$$\begin{cases} -3x+y-z=2 \\ 10z=6 \end{cases} \qquad y \text{ is a free variable}$$

$$y=t \Rightarrow z=\frac{6}{10}=\frac{3}{5}, \quad x=\frac{2+z-y}{-3}=\frac{2+\frac{3}{5}-t}{-3}$$

$$=-\frac{13}{15}+\frac{t}{3}$$

Answer: $x=-\frac{13}{15}+\frac{t}{3}$, $y=t$, $z=\frac{3}{5}$ for all $t\in\mathbb{R}$.

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(2) X is invertible $\Leftrightarrow \det(X)\neq 0$

cofactor expand row 1

$$\det(X)=\begin{vmatrix} 3 & k & 0 \\ -1 & -1 & k \\ 1 & -k & 2 \end{vmatrix} (1) = \begin{vmatrix} 4 & 0 & 2 \\ -1 & -1 & k \\ 1 & -k & 2 \end{vmatrix} = 4\cdot\begin{vmatrix} -1 & k \\ -k & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & -1 \\ 1 & -k \end{vmatrix} =$$

$$=4(-2+k^2)+2(k+1)=-8+4k^2+2k+2$$

Then $\det(X)=0 \Leftrightarrow 4k^2+2k-6=0 \Leftrightarrow k^2+\frac{k}{2}-\frac{3}{2}=0$

$$\Leftrightarrow k=-\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{24}{16}}=-\frac{1}{4}\pm\frac{5}{4} \Leftrightarrow k=-\frac{3}{2} \text{ or } k=1$$

Answer: X is invertible if $k\neq-\frac{3}{2}$ and $k\neq 1$.

(3) $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ so

$$AB^T + 3A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 8 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 17 \\ 0 & 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 4 & 17 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{array}\right] (\times \tfrac{1}{5}) \sim \left[\begin{array}{cc|cc} 4 & 17 & 1 & 0 \\ 0 & 1 & 0 & 1/5 \end{array}\right] (-17)$$

$$\sim \left[\begin{array}{cc|cc} 4 & 0 & 1 & -17/5 \\ 0 & 1 & 0 & 1/5 \end{array}\right] (\times \tfrac{1}{4}) \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/4 & -17/20 \\ 0 & 1 & 0 & 1/5 \end{array}\right]$$

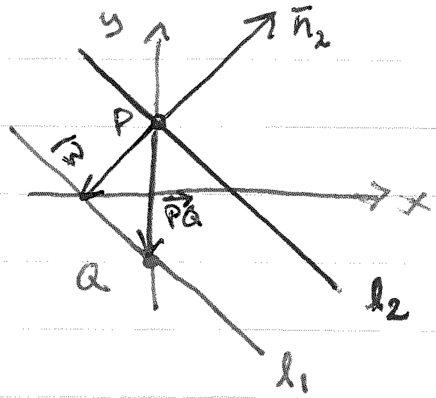
$(AB^T + 3A)^{-1}$

<u>Answer:</u> $(AB^T + 3A)^{-1} = \begin{bmatrix} 1/4 & -17/20 \\ 0 & 1/5 \end{bmatrix}$

4

(a) To find the distance between l_1 and l_2, take $Q=(0,-3)\in l_1$ and any $P\in l_2$, e.g. $x=0 \Rightarrow y=1/2$, i.e. $P=(0,1/2)$

Then the distance d is given by $d=\|\text{proj}_{\bar{n}_2}\vec{PQ}\|$, where $\bar{n}_2=(3,4)$ is a normal to l_2



$$\vec{PQ}=(0,-3)-(0,1/2)=(0,-7/2)$$

$$\bar{w}=\text{proj}_{\bar{n}_2}\vec{PQ}=\frac{(0,-7/2)\cdot(3,4)}{\|(3,4)\|^2}\cdot(3,4)$$

$$\text{so } d=\|\bar{w}\|=\frac{|(0,-7/2)\cdot(3,4)|}{\|(3,4)\|}=\left|-\frac{14}{5}\right|=\frac{14}{5}$$

Answer (a): $\frac{14}{5}$

(b) The lines are parallel iff they have the same normal, then every vector in the direction of l_1 must be orthogonal to $\bar{n}_2=(3,4)$ especially $\bar{v}=(a,2)$, which gives $(3,4)\cdot(a,2)=0$, or

$$3a+8=0 \Rightarrow a=-\frac{8}{3}$$

answer (b): $a=-\frac{8}{3}$

5. $p(z) = 9z^4 + 37z + 4$

(a) $p(2i) = 9\cdot(2i)^4 + 37(2i)^2 + 4 =$
$= 9\cdot 16 - 37\cdot 4 + 4 = 144 - 148 + 4 = 0$ ok!

Therefore $z = 2i$ is a root of $p(z)$.

(b) Since $p(z)$ is a real polynomial, $\bar{z} = \overline{2i} = -2i$ must be a root as well. Therefore

$$(z-2i)(z+2i) = \underline{z^2+4}$$

is a factor of $p(z)$. Polynomial division gives

$$\begin{array}{l|l} \multicolumn{1}{l}{\quad 9z^2+1} & \\ \hline 9z^4+37z^2+4 & \underline{z^2+4} \\ \underline{-(9z^4+36z^2)} & \\ \qquad z^2+4 & \\ \quad \underline{-(z^2+4)} & \\ \qquad\quad 0 & \end{array}$$

so $p(z) = (9z^2+1)(z^2+4)$

$9z^2 + 1 = 0 \Leftrightarrow z^2 = -\frac{1}{9} \Leftrightarrow z = \pm\frac{1}{3}i$
which gives the two remaining roots.

<u>Answer</u>: $p(z) = 9(z-2i)(z+2i)(z-\frac{i}{3})(z+\frac{i}{3})$

$= (z-2i)(z+2i)(3z-i)(3z+i)$