MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2016-02-15 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Determine β such that $\displaystyle\lim_{x\to-2}\left(\frac{1}{x+2}+\frac{\beta}{x^2-4}\right)$ exists. Also, specify the limit!

2. Prove that $y = (x+\sqrt{x})^{-1}$ is a solution of the differential equation

$$2x\,\frac{dy}{dx} + y(1+xy) = 0\,.$$

3. Solve the initial-value problem $y' + 3x^2y = 6x^2,\ \ y(0) = 0.$

4. Prove that the explicitly given terms of the series $\dfrac{2x}{\sqrt{3}}+\sqrt{3}+\dfrac{3\sqrt{3}}{2x}+\ldots$ are the first three in a geometric series. Then assume that the symbol "..." denotes all the other terms of precisely that geometric series. For which x converges the series? Find the sum of the series for these x.

5. Prove that the inverse of the function $x \curvearrowright f(x) = x^2 - 2x + 3,\quad D_f = [0,1]$ exists, and sketch the graphs of f and f^{-1} in the same coordinate system.

6. Evaluate the integral $\displaystyle\int_{-2016}^{2016} x^{2015}\,dx$ and write the result in as simple form as possible.

7. Find the function f such that $f(x) = \displaystyle\int \cot(x)\,dx$ and $f(\pi/6) = 0.$

8. Which point on the curve $y = \sqrt{x+2}$ is closest to the origin?

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN1
Datum: 2016-02-15 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm β så att $\displaystyle\lim_{x\to-2}\left(\frac{1}{x+2}+\frac{\beta}{x^2-4}\right)$ existerar. Specificera även gränsvärdet!

2. Bevisa att $y = (x+\sqrt{x})^{-1}$ är en lösning till differentialekvationen

$$2x\,\frac{dy}{dx} + y(1+xy) = 0\,.$$

3. Lös begynnelsevärdesproblemet $y' + 3x^2y = 6x^2,\ \ y(0) = 0.$

4. Visa att de explicit utskrivna termerna i serien $\dfrac{2x}{\sqrt{3}} + \sqrt{3} + \dfrac{3\sqrt{3}}{2x} + \ldots$ är de tre första i en geometrisk serie. Antag sedan att symbolen "..." betecknar övriga termer i just denna geometriska serie. För vilka x konvergerar serien? Bestäm seriens summa för dessa x.

5. Bevisa att inversen till funktionen $x \curvearrowright f(x) = x^2 - 2x + 3,\quad D_f = [0,1]$ existerar, och skissa graferna till f och f^{-1} i ett och samma koordinatsystem.

6. Beräkna integralen $\displaystyle\int_{-2016}^{2016} x^{2015}\,dx$ och skriv resultatet på en så enkel form som möjligt.

7. Bestäm funktionen f så att $f(x) = \displaystyle\int \cot(x)\,dx$ och $f(\pi/6) = 0$.

8. Vilken punkt på kurvan $y = \sqrt{x+2}$ är närmast origo?

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2016-02-15

① $\lim\limits_{x\to -2}\left(\frac{1}{x+2}+\frac{\beta}{x^2-4}\right)=\lim\limits_{x\to -2}\frac{x-2+\beta}{(x+2)(x-2)}$ exists iff $\beta=4$

Then $\lim\limits_{x\to -2}\frac{x-2+\beta}{(x+2)(x-2)}=\left(\lim\limits_{x\to -2}\frac{x+2}{x+2}\right)\left(\lim\limits_{x\to -2}\frac{1}{x-2}\right)=1\cdot\frac{1}{-2-2}=-\frac{1}{4}$

(since $x\to\frac{1}{x-2}$ is continuous at -2)

② Prove that $y=\frac{1}{x+\sqrt{x}}$ is a solution of $2x\frac{dy}{dx}+y(1+xy)=0$

We get that $\frac{dy}{dx}=-1\frac{1}{(x+\sqrt{x})^2}\left(1+\frac{1}{2}\frac{1}{\sqrt{x}}\right)=-\frac{(2\sqrt{x}+1)}{2\sqrt{x}(x+\sqrt{x})^2}$

Then $LHS_{DE}\left(y=\frac{1}{x+\sqrt{x}}\right)=2x\left[\frac{-(2\sqrt{x}+1)}{2\sqrt{x}(x+\sqrt{x})^2}\right]+\frac{1}{x+\sqrt{x}}\left(1+x\frac{1}{x+\sqrt{x}}\right)$

$=-\sqrt{x}\frac{2\sqrt{x}+1}{(x+\sqrt{x})^2}+\frac{x+\sqrt{x}+x}{(x+\sqrt{x})^2}=\frac{-2x-\sqrt{x}+2x+\sqrt{x}}{(x+\sqrt{x})^2}=\frac{0}{(x+\sqrt{x})^2}$

$=0\underset{\text{comparison}}{=}RHS_{DE}\left(y=\frac{1}{x+\sqrt{x}}\right)$, i.e. $y=\frac{1}{x+\sqrt{x}}$ satisfies the differential equation q.e.d.

③ DE: $y'+3x^2y=6x^2$ IV: $y(0)=0$

Mult. with e^{x^3} in the DE gives $\underbrace{y'\cdot e^{x^3}+y\cdot 3x^2e^{x^3}}_{\frac{d}{dx}\left(ye^{x^3}\right)}=6x^2e^{x^3}$

and $ye^{x^3}=2e^{x^3}+C$

where the initial value gives $0e^0=2e^0+C \Leftrightarrow C=-2$

Thus $y=2\left(1-e^{-x^3}\right)$ solves the IVP.

④ $\frac{2x}{\sqrt{3}}+\sqrt{3}+\frac{3\sqrt{3}}{2x}+\dots$

Since $r_1=\frac{\sqrt{3}}{\frac{2x}{\sqrt{3}}}=\frac{3}{2x}$ and $r_2=\frac{\frac{3\sqrt{3}}{2x}}{\sqrt{3}}=\frac{3}{2x}$, i.e. $r_1=r_2$, the first three terms are the three first of a geometric series. q.e.d.

The series converges iff $\left|\frac{3}{2x}\right|<1 \Leftrightarrow |x|>\frac{3}{2}$

$\Leftrightarrow x<-\frac{3}{2} \vee x>\frac{3}{2}$

The sum of the series for those x is

$$\frac{2x}{\sqrt{3}}\frac{1}{1-\frac{3}{2x}}=\frac{2x}{\sqrt{3}}\frac{2x}{2x-3}=\frac{4x^2}{\sqrt{3}(2x-3)}$$

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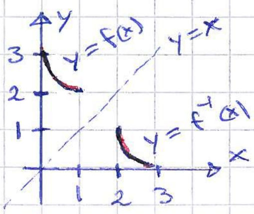
(5) $f(x) = x^2 - 2x + 3 = (x-1)^2 + 2$, $D_f = [0, 1]$

Since $f'(x) = 2(x-1) < 0$ in $[0, 1)$, f is decreasing in D_f which in turn imply that f is invertible. q.e.d.



The curve $y = f^{-1}(x)$ is the mirror picture of $y = f(x)$ with the curve $y = x$ as the mirror.

(Xtra $f^{-1}(x) = 1 - \sqrt{x-2}$, $D_{f^{-1}} = [2, 3]$)

(6) $\displaystyle\int_{-2016}^{2016} x^{2015}\,dx = 0$ since the integrand is an odd function and the interval symmetric about zero.

(7) $f(x) = \int \cot(x)\,dx$ and $f(\pi/6) = 0$

We get $f(x) = \displaystyle\int \frac{\cos(x)}{\sin(x)}\,dx \begin{bmatrix} \sin(x) = u \\ \cos(x)\,dx = du \end{bmatrix} = \int \frac{du}{u} = \ln|u| + C$

$= \ln|\sin(x)| + C$ where $0 = f\left(\frac{\pi}{6}\right) = \ln\left(\frac{1}{2}\right) + C$

i.e. $f(x) = \ln(\sin(x)) + \ln(2) = \ln(2\sin(x))$

where $f\left(\frac{\pi}{6}\right) = 0$ implicitly demands that $D_f = (0, \pi)$.

(8)

The distance between a point on the curve $y = \sqrt{x+2}$ and the origin is given by the function d such that

$d(x) = \sqrt{(x-0)^2 + (\sqrt{x+2} - 0)^2} = \sqrt{x^2 + x + 2}$

where $(x, \sqrt{x+2})$ are the coordinates of a general point of the curve.

We get $d(x) = \sqrt{x^2 + x + 2} = \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}} \geq \underbrace{\sqrt{0^2 + \frac{7}{4}} = \frac{\sqrt{7}}{2}}_{\text{the shortest distance}}$

i.e. the point with the coordinates $\left(-\frac{1}{2}, \sqrt{-\frac{1}{2} + 2}\right) = \left(-\frac{1}{2}, \sqrt{\frac{3}{2}}\right)$ is the point (on the curve) closest to the origin.

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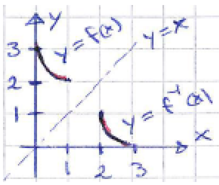
EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

	Examination TEN1 – 2016-02-15	Maximum points for subparts of the problems in the final examination
1.	The limit exists if and only if $\beta = 4$ and is then equal to $-1/4$	**2p**: Correctly brought the terms together with the common denominator $(x+2)(x-2)$, and correctly identified β to be equal to 4 in order for the limit to exist **1p**: Correctly concluded that the limit is equal to $-1/4$
2.	A proof	**1p**: Correctly differentiated the solution function **2p**: Correctly inserted the solution function in the DE, and correctly proven that the function is a solution
3.	$y = 2(1-e^{-x^3})$	**1p**: Correctly found and multiplied by an integrating factor, and correctly rewritten the LHS of the DE as an exact derivative **1p**: Correctly found the general solution of the DE **1p**: Correctly adapted the general solution to the initial value, and correctly summarized the solution of the IVP
4.	The series is a geometric series since it has a quotient. The series converges for $x \in (-\infty, -\frac{3}{2}) \cup (\frac{3}{2}, \infty)$. For those x, the sum of the series is $4x^2(\sqrt{3}(2x-3))^{-1}$	**1p**: Correctly proven that the series is a geometric series **1p**: Correctly found the interval of convergence **1p**: Correctly found the sum of the series
5.	A proof and a sketch 	**1p**: Correctly proven that f is invertible **1p**: Correctly sketched the graph of f **1p**: Correctly sketched the graph of f^{-1}
6.	0	------------------------------------ **One scenario** ------------------------------------ **3p**: Correctly concluded that the integral is for an odd function over an interval symmetric about zero, leading to the conclusion that the integral is equal to zero -------------------------------- **Another scenario** -------------------------------- **1p**: Correctly found an antiderivative of the integrand **2p**: Correctly evaluated the antiderivative at the limits
7.	$f(x) = \ln(2\sin(x))$	**1p**: Correctly applied a substitution which simplifies the expression for the general antiderivative of f **1p**: Correctly found the general antiderivative of f **1p**: Correctly adapted the antiderivative to the value at $\pi/6$
8.	The point with the coordinates $\left(-\frac{1}{2}, \sqrt{\frac{3}{2}}\right)$	**1p**: Correctly for the optimization problem formulated a function of one variable measuring the distance between a point on the curve and the origin **1p**: Correctly, by completing the square in the argument of the square root function and by estimating the square to be ≥ 0 **OR** by a first derivative test, found the first coordinate of the point closest to the origin **1p**: Correctly found the second coordinate of the point closest to the origin