TEN1 2018-08-14
MMA130 Mathematical Logic for Computer Science
Duration: 3 hours
Tools: none
Attached: Collection of Formulas (4 pages)
Passing grade requires 15 p or more

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Questions may be answered in English or Swedish. Solutions should be presented in such a way that reasoning and calculations are easy to follow. All answers must be supported by an argument, e.g., if you claim something has a particular property then you must also show that this is the case.

1 Find appropriate predicates and their specifications, then use them to translate the following into predicate logic: (7 p)

a All green things with spirals belong to Pete.

b Pete has exactly one blue thing without spirals.

2 Give a natural deduction proof of $\neg p \to \neg q \vdash (\neg p \to q) \to p$. Provide justifications of all steps. (6 p)

3 A basic board of traffic lights has one red, one yellow (amber, orange), and one green light. Let r, y, and g be propositional atoms denoting that the red, yellow, and green respectively lights are on in a particular group of lights.

a In *old Swedish signalling order*, the lights cycle through the states of red, red+yellow, green, and green+yellow before returning to red. Express as a propositional logic formula the claim that the lights are in one of these four states. (1 p)

b Express in linear time temporal logic the claim that all future states of the lights are among these four states. (1 p)

c Express in linear time temporal logic the claim that the state after red+yellow is green. (1 p)

d Interpret in natural language the claim $\mathrm{G}(r \wedge \neg y \wedge \neg g \to \mathrm{X}r)$. (1 p)

e Express in linear time logic the claim that the red light, whenever it lights up, will remain lit for at least three time steps. (2 p)

f What is the minimal number of states in a transition system (worlds in a Kripke model) that can reproduce the old Swedish signalling order? (1 p)

g Draw a transition system that attains the minimum in (**f**), which additionally satisfies (i) the states with yellow only last for one time step whereas (ii) the states without yellow can last for more than one time step. (2 p)

h Is the linear time temporal logic formula $\mathrm{GF}g$ true for the model given by that transition system? Remember to justify your answer. (1 p)

i Which states of the transition system/model satisfy $\Diamond\Box r$? (1 p)

j Which states of the transition system/model satisfy $\mathrm{E}[y \mathrel{\mathrm{U}} g]$? (1 p)

4 Prove the validity of

$$\forall x\big(P(x,x) \to Q(x)\big) \vdash \exists x \forall y P(x,y) \to \exists x Q(x),$$

where P is a predicate symbol of two arguments and Q is a predicate symbol of one argument. Provide justifications of the steps in your proof. (5 p)

Good luck!

MÄLARDALEN UNIVERSITY
School of Education, Culture
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MMA130, Mathematical Logic for Computer Science

Collection of Formulas

Anna Fedyszak-Koszela

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1 Natural deduction - basic rules

1. $(\wedge\ i)$: $\dfrac{\phi \qquad \psi}{\phi \wedge \psi}$

2. $(\wedge\ e)$: $\dfrac{\phi \wedge \psi}{\phi}$, $\dfrac{\phi \wedge \psi}{\psi}$

3. $(\vee\ i)$: $\dfrac{\phi}{\phi \vee \psi}$, $\dfrac{\psi}{\phi \vee \psi}$

4. $(\vee\ e)$: $\dfrac{\phi \vee \psi \quad \begin{bmatrix} \phi \\ \vdots \\ \chi \end{bmatrix} \quad \begin{bmatrix} \psi \\ \vdots \\ \chi \end{bmatrix}}{\chi}$

5. $(\rightarrow i)$: $\dfrac{\begin{bmatrix} \phi \\ \vdots \\ \psi \end{bmatrix}}{\phi \rightarrow \psi}$

6. $(\rightarrow e)$: $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$

7. $(\neg\ i)$: $\dfrac{\begin{bmatrix} \phi \\ \vdots \\ \bot \end{bmatrix}}{\neg\phi}$

8. $(\neg\ e)$: $\dfrac{\phi \quad \neg\phi}{\bot}$

9. $(\bot\ e)$: $\dfrac{\bot}{\phi}$

10. $(\neg\neg\ e)$: $\dfrac{\neg\neg\phi}{\phi}$

11. (MT): $\dfrac{\phi \to \psi \quad \neg\psi}{\neg\phi}$

12. $(\neg\neg\ i)$: $\dfrac{\phi}{\neg\neg\phi}$

13. (PBC): $\dfrac{\begin{bmatrix} \neg\phi \\ \vdots \\ \bot \end{bmatrix}}{\phi}$

14. (LEM): $\dfrac{}{\phi \vee \neg\phi}$

15. $(copy)$: $\dfrac{\phi}{\phi}$

16. $(=i)$: $\dfrac{}{t = t}$, for any term t

17. *Principle of Substitution* $(= e)$:

$$\frac{t_1 = t_2 \qquad \phi\,[t_1/x]}{\phi\,[t_2/x]},$$

for t_1 free for x in ϕ, and for t_2 free for x in ϕ; all occurrences of t_1 in $\phi\,[t_1/x]$ are replaced by t_2

18. $(\forall x\ i)$:

$$\frac{\begin{bmatrix} x_0 & \\ & \phi\,[x_0/x] \end{bmatrix}}{\forall x \phi},$$

for x_0 - new, doesn't occur anywhere outside its box,
for x_0 - not free in open P before its box

19. ($\forall x\ e$): $\dfrac{\forall x \phi}{\phi\,[t/x]}$,

    for any term t free for x in ϕ; all free occurrences of x in ϕ are replaced by t

20. ($\exists x\ i$): $\dfrac{\phi\,[t/x]}{\exists x \phi}$,

    for some term t free for x in ϕ

21. ($\exists x\ e$):

$$\dfrac{\begin{array}{cllll} & \lceil\ x_0 & \phi\,[x_0/x] & \rceil & 1,\ P \\ & | & \vdots & | & \\ \exists x \phi, & \lfloor & \chi & \rfloor & 2 \end{array}}{\chi}$$

for x_0 - new, doesn't occur anywhere outside its box;

1: x_0 not free in open P before its box,

2: x_0 not free in χ

2 Modal logic

- Let $M = (W, R, L)$ be a Kripke model of basic modal logic, $x \in W$, and ϕ be a formula. We will define when formula ϕ is true in the world x. This is done via a satisfaction relation $x \Vdash \phi$ by structural induction on ϕ:

    1. $x \Vdash T$
    2. *not* $(x \Vdash \bot)$
    3. $x \Vdash p$ iff $p \in L(x)$
    4. $x \Vdash \neg\phi$ iff *not* $(x \Vdash \phi)$
    5. $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
    6. $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
    7. $x \Vdash \phi \to \psi$ iff $x \Vdash \psi$, whenever we have $x \Vdash \phi$
    8. $x \Vdash \phi \leftrightarrow \psi$ iff $(x \Vdash \phi$ iff $x \Vdash \psi)$
    9. $x \Vdash \Box\psi$ iff, for each $y \in W$ with $R(x, y)$, we have $y \Vdash \psi$
    10. $x \Vdash \Diamond\psi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \psi$

    When $x \Vdash \phi$ holds, we say 'x satisfies ϕ' or 'ϕ is true in world x'.

- A model $M = (W, R, L)$ of basic modal logic is said to satisfy a formula ϕ if every state (world) x in the model satisfies it.
  We write: $M \models \phi$ iff, for each $x \in W$, $x \Vdash \phi$.

3 Logic systems with multiple truth values

- Let L be a formal language without non-logical symbols and S_L - the set with all sentences in L. For every positive integer n we define the non-classical logic system $L_n = (L, V_n)$ with $n+1$ truth values and with the valuation $V_n : S_L \longrightarrow \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ by following 5 conditions:

  1. $V_n(\neg\phi) = 1 - V_n(\phi)$ for any sentence $\phi \in S_L$
  2. $V_n(\phi \wedge \psi) = \min\ \{V_n(\phi), V_n(\psi)\}$ for any two sentences $\phi, \psi \in S_L$
  3. $V_n(\phi \vee \psi) = \max\ \{V_n(\phi), V_n(\psi)\}$ for any two sentences $\phi, \psi \in S_L$
  4. $V_n(\phi \to \psi) = \min\ \{1,\ (1 - V_n(\phi) + V_n(\psi))\}$ for any two sentences $\phi, \psi \in S_L$
  5. $V_n(\phi \leftrightarrow \psi) = 1 - |V_n(\phi) - V_n(\psi)|$ for any two sentences $\phi, \psi \in S_L$

- L_2 is a non-classical logic with 3 truth values $\{0, \frac{1}{2}, 1\}$:

ϕ	φ	$\neg\phi$	$\phi \wedge \varphi$	$\phi \vee \varphi$	$\phi \to \varphi$
$x = V(\phi)$	$y = V(\varphi)$	$1 - x$	$min\{x, y\}$	$max\{x, y\}$	$min\{1, 1 - x + y\}$
1	1	0	1	1	1
1	1/2	0	1/2	1	1/2
1	0	0	0	1	0
1/2	1	1/2	1/2	1	1
1/2	1/2	1/2	1/2	1/2	1
1/2	0	1/2	0	1/2	1/2
0	1	1	0	1	1
0	1/2	1	0	1/2	1
0	0	1	0	0	1