Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
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MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN2
Datum: 2017-01-12
Hjälpmedel: penna,
sudd och linjal

Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.

1 Visa att $z = i/2$ är en rot till $p(z) = 1 + 4z + 4z^2 + 16z^3$, och faktorisera sedan $p(z)$ i linjära faktorer. (5p)

2 Låt $T : \mathbb{R}^3 \to \mathbb{R}^4$ vara den linjära transformationen $T(\mathbf{x}) = A\mathbf{x}$ där

$$A = \begin{bmatrix} -2 & 1 & -3 \\ 4 & 2 & -2 \\ 2 & 2 & -3 \\ 2 & 0 & 1 \end{bmatrix}, \text{ och } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

a. Bestäm en bas för värderummet till T. (3p)

b. Avgör om $(4, 4, 5, -1)$ tillhör värderummet till T. (2p)

3 Underrummet V till $\mathbb{R}^3$ som ges av $V = \{(x, y, z) \in \mathbb{R}^3 : 4x - 2y + z = 0\}$ har basen $B = \{(1, 4, 4), (-1, 2, 8)\}$.

a. Bestäm koordinatvektorn för $\mathbf{u} = (1, -1, -6)$ i basen B. (3p)

b. Visa att B bildar en bas till V. (3p)

4 Bestäm en matris P som diagonaliserar matrisen A och ange matrisen D som uppfyller $A = PDP^{-1}$, då (5p)

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}.$$

5 För vilka värden på $a \in \mathbb{R}$ blir $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linjärt oberoende då $\mathbf{v}_1 = (1, 2, a, 1)$, $\mathbf{v}_2 = (-1, 1, 0, 1)$, och $\mathbf{v}_3 = (1, a, 1, 1)$. (4p)

Divsion of Mathematics and applied mathematics Mälardalen University Examiner: Mats Bodin	MÄLARDALENS HÖGSKOLA ESKILSTUNA VÄSTERÅS	Examination Vector algebra MAA150 - TEN2 Date: Jan 12, 2017 Exam aids: pencil, eraser and ruler

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

1 Show that $z = i/2$ is a root to $p(z) = 1 + 4z + 4z^2 + 16z^3$, and then factor $p(z)$ into linear factors. (5p)

2 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} -2 & 1 & -3 \\ 4 & 2 & -2 \\ 2 & 2 & -3 \\ 2 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

a. Find a basis for the range of T. (3p)

b. Determine if $(4, 4, 5, -1)$ is in the range of T. (2p)

3 The subspace V of $\mathbb{R}^3$ given by $V = \{(x, y, z) \in \mathbb{R}^3 : 4x-2y+z = 0\}$ has $B = \{(1, 4, 4), (-1, 2, 8)\}$ as basis.

a. Find the coordinate vector of $\mathbf{u} = (1, -1, -6)$ relative to B. (3p)

b. Show that B is a basis for V. (3p)

4 Find a matrix P that diagonalize the matrix A and state the matrix D that satisfies $A = PDP^{-1}$, where (5p)

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}.$$

5 For what values of $a \in \mathbb{R}$ is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent when $\mathbf{v}_1 = (1, 2, a, 1)$, $\mathbf{v}_2 = (-1, 1, 0, 1)$, and $\mathbf{v}_3 = (1, a, 1, 1)$. (4p)

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Assessment criteria for TEN2 2017-01-12

General assessment criteria

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Points may be deducted for erroneous mathematical statements, calculations, or failure to use proper mathematical notation.

Assessment problems

1. **[5 points]**
   Showing that $i/2$ is a root **(1p)**, finding the root $-i/2$ with motivation **(1p)**, concluding that $z^2 + 1/4$ is a factor **(1p)**, finding the remaining root with long division and correct answer **(2p)**

2. **[5 points]**
   a. Correct method; e.g. relevant row-operations **(2p)**, finding a basis **(1p)**
   b. A condition for the vector being in the range of T **(1p)**, checking the condition **(1p)**

3. **[6 points]**
   a. Equation for the coordinates **(1p)**, finding the coordinates **(1p)**, correct coordinate vector **(1p)**
   b. Conditions for B being a basis with motivation **(1p)**, checking the conditions **(2p)**

4. **[5 points]**
   Correct eigenvalues **(1p)**, method of finding the eigenvectors **(2p)**, giving the correct matrices P and D **(2p)**

5. **[4 points]**
   Condition for linear independence **(1p)**, solving the relevant equation **(2p)**, finding values of a with motivation **(1p)**

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① $z_0 = i/2$, $p(z) = 1 + 4z + 4z^2 + 16z^3$

$p(i/2) = 1 + 4\cdot\left(\frac{i}{2}\right) + 4\cdot\left(\frac{i}{2}\right)^2 + 16\left(\frac{i}{2}\right)^3 = 1 + 2i - 2 - 2i = 0$ (1p)

Since $p(z)$ is a real polynomial $\overline{z_0} = \overline{\frac{i}{2}} = -\frac{i}{2}$ is also a root. Then both (1p)

$(z - \frac{i}{2})$ and $(z + \frac{i}{2})$ are factors in $p(z)$

and therefore also

$$(z - \tfrac{i}{2})\cdot(z + \tfrac{i}{2}) = z^2 + \tfrac{1}{4}$$ (1p)

We use long division to find the remaining linear factor. (In total 3 roots since $\deg(P) = 3$.)

$$\begin{array}{l|l} \quad 16z + 4 & \\ \hline 16z^3 + 4z^2 + 4z + 1 & z^2 + \frac{1}{4} \\ -(16z^3 + 4z) & \\ \hline \quad 4z^2 + 1 & \\ \quad -(4z^2 + 1) & \\ \hline \quad\quad 0 & \end{array}$$

So $p(z) = \left(z^2 + \frac{1}{4}\right)\cdot(16z + 4)$.

<u>Answer</u>: $p(z) = (z + \frac{i}{2})\cdot(z - \frac{i}{2})\cdot(16z + 4)$. (2p)

Check: $\left(z^2 + \frac{1}{4}\right)\cdot(16z + 4) = 16z^3 + 4z^2 + 4z + 1 = p(z)$ ok!

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(2) a)

$$\begin{bmatrix} -2 & 1 & -3 \\ 4 & 2 & -2 \\ 2 & 2 & -3 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & -3 \\ 0 & 4 & -8 \\ 0 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} \times -\frac{1}{2} \\ \times \frac{1}{4} \\ \times \frac{1}{3} \\ \ \end{matrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} \ \\ (-1) \\ \ \\ \ \end{matrix} \sim$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2p)

leading ones

Therefore $B = \{(-2,4,2,2), (1,2,2,0)\}$ is a basis for the range of T. (1p)

b) $\bar{v} = (4,4,5,-1)$ is in the range of T iff $\bar{v}$ is a linear combination of $(-2,4,2,2)$ and $(1,2,2,0)$, i.e there exist constants k_1 and k_2 such that

(1p) $k_1 \cdot \begin{bmatrix} -2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 5 \\ -1 \end{bmatrix}$ Solving this gives

$$\left[\begin{array}{cc|c} -2 & 1 & 4 \\ 4 & 2 & 4 \\ 2 & 2 & 5 \\ 2 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 4 \\ 0 & 4 & 12 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(1p)

which has a solution.

Answer a) $B = \{(-2,4,2,2), (1,2,2,0)\}$ is a basis for V.

b) $(4,4,5,-1)$ is in the range of T.

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(3) $B = \{\bar{v}_1, \bar{v}_2\} = \{(1,4,4), (-1,2,8)\}$, $\bar{u} = (1,-1,-6)$

$V = \{(x,y,z) : 4x - 2y + z = 0\}$

a) $(\bar{u})_B = (k_1, k_2) \Leftrightarrow \bar{u} = k_1 \cdot \bar{v}_1 + k_2 \cdot \bar{v}_2$ (1p)

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 4 & 2 & -1 \\ 4 & 8 & -6 \end{array}\right] \overset{-4}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 12 & -10 \end{array}\right] \overset{-2}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 0 \end{array}\right] \times \tfrac{1}{6} \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -5/6 \\ 0 & 0 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 1/6 \\ 0 & 1 & -5/6 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} k_1 = 1/6 \\ k_2 = -5/6 \end{cases}$$ (1p)

Answer: $\underline{\underline{(\bar{u})_B = (1/6, -5/6)_B}}$ (1p)

b) V is a plane in $\mathbb{R}^3$ so $\dim(V) = 2$. Therefore it is enough to show that (i) $B \subset V$ and (ii) B is linearly independent. (1p)

(i) $\bar{v}_1 \in V : 4 \cdot 1 - 2 \cdot 4 + 4 = 0$ ok!

$\bar{v}_2 \in V : 4 \cdot (-1) - 2 \cdot 2 + 8 = 0$ ok! so $B \subset V$ (1p)

(ii) B is linearly independent iff $\begin{bmatrix} 1 & -1 \\ 4 & 2 \\ 4 & 8 \end{bmatrix}$ has rank 2.

$$\begin{bmatrix} 1 & -1 \\ 4 & 2 \\ 4 & 8 \end{bmatrix} \text{(as above)} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ so it has rank 2.

(1p)

Therefore by (i) and (ii) and Thm 4.5.4, B is a basis for V.

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4) $A = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}$. To diagonalize A we need to find eigenvalues and eigenvectors.

(CE) $\det(A - \lambda \cdot I) = \begin{vmatrix} 1-\lambda & 0 \\ 9 & 4-\lambda \end{vmatrix} = (1-\lambda)\cdot(4-\lambda) = 0$

$\iff \lambda = 1$ or $\lambda = 4$.   (1.p)

Eigenvectors: $(A - \lambda I)\bar{v} = \bar{0}$, where $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\boxed{\lambda = 1}$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 9 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\uparrow$ free ; set $\bar{v}_2 = 3t$

$\Rightarrow v_1 = -t$, i.e $\bar{v} = \begin{bmatrix} -t \\ 3t \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

so $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.

$\boxed{\lambda = 4}$ $\left[\begin{array}{cc|c} -3 & 0 & 0 \\ 9 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ (2p)

$\uparrow$ free ; $v_2 = t \Rightarrow v_1 = 0$

Then $\bar{v} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 4$. Then

$P = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ (2p)

Check : $A \cdot P = P \cdot D$?

$A \cdot P = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix}$, $P \cdot D = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix}$ ok!

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(5) $\bar{v}_1 = (1,2,a,1),\ \bar{v}_2 = (-1,1,0,1),\ \bar{v}_3 = (1,a,1,1)$

$\{\bar{v}_1,\bar{v}_2,\bar{v}_3\}$ is linearly independent iff $k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3 = \bar{0}$ (*) only has the trivial solution $k_1 = k_2 = k_3 = 0$. (1p)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & a & 0 \\ a & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & a-2 & 0 \\ 0 & a & 1-a & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & a-2 & 0 \\ 0 & 0 & 1-a & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a-2 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right] \Rightarrow \begin{cases} k_1 + k_3 = 0 \\ k_2 = 0 \\ \boxed{\begin{array}{l} (a-2)k_3 = 0 \\ (a-1)k_3 = 0 \end{array}} \ (*) \end{cases} \quad (2p)$$

Regardless of value of $a \in \mathbb{R}$, (*) implies that $k_3 = 0 \Rightarrow k_1 = k_2 = 0$ so $\{\bar{v}_1,\bar{v}_2,\bar{v}_3\}$ is linearly independent for all $a \in \mathbb{R}$ (1p)