MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2015-11-30 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Determine to the differential equation $4y'' + y = 0$ the solution satisfying the initial conditions $y(\pi) = 1\,,\ \ y'(\pi) = 2\,.$

2. Classify all local extreme points for the function f defined by $f(x) = \dfrac{x}{x^2+4}\,.$

3. Find the coefficients of the power series in x representing $\dfrac{1}{2x-3}$ in a neighborhood of the point 0. Also, determine the interval of convergence of the power series.

4. It is widely known that the function f, defined by $f(x) = x^9 + x$, has an inverse function which is differentiable. Find an equation for the tangent line τ to the curve $\gamma : y = f^{-1}(x)$ at the point $P : (2, f^{-1}(2))$.

5. Determine whether

$$\lim_{x\to 1} \frac{2-3x+x^2}{1-2x+x^2}$$

exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

6. Find the GENERAL antiderivative of the function $x \curvearrowright f(x) = \dfrac{1}{\sqrt{x}}\,e^{-\sqrt{x}}\,.$

7. Evaluate the integral

$$\int_1^3 \Big(|x-1| + |x-2| + |x-3|\Big)\,dx$$

and write the result in as simple form as possible.

8. Let $f(x) = \ln(x)$. Explain and illustrate how the graphs given by the equations $y = f(2x)$ and $y = f(2+x)$ can be obtained based on the graph of f.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm till differentialekvationen $4y'' + y = 0$ den lösning som satisfierar begynnelsevillkoren $y(\pi) = 1\,,\ \ y'(\pi) = 2\,.$

2. Klassificera alla lokala extrempunkter för funktionen f definierad av

$$f(x) = \frac{x}{x^2+4}\,.$$

3. Bestäm koefficienterna i den potensserie i x som representerar $\dfrac{1}{2x-3}$ i en omgivning till punkten 0. Bestäm även konvergensintervallet för potensserien.

4. Det är allom bekant att funktionen f, definierad av $f(x) = x^9 + x$, har en invers funktion som är deriverbar. Bestäm en ekvation för tangenten τ till funktionskurvan $\gamma : y = f^{-1}(x)$ i punkten $P : (2, f^{-1}(2))$.

5. Avgör om

$$\lim_{x\to 1} \frac{2-3x+x^2}{1-2x+x^2}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

6. Bestäm den GENERELLA primitiva funktionen till funktionen

$$x \curvearrowright f(x) = \frac{1}{\sqrt{x}}\,e^{-\sqrt{x}}\,.$$

7. Beräkna integralen

$$\int_1^3 \Big(|x-1| + |x-2| + |x-3|\Big)\,dx$$

och skriv resultatet på en så enkel form som möjligt.

8. Låt $f(x) = \ln(x)$. Förklara och illustrera hur graferna givna av ekvationerna $y = f(2x)$ och $y = f(2+x)$ kan fås utifrån grafen till f.

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN1 2015-11-30

1 DE: $4y''+y=0$ IV: $y(\pi)=1$, $y'(\pi)=2$

The DE is a linear, homogeneous 2nd-order DE which together with the IV give a unique solution (on $\mathbb{R}$). Since the coefficients are fixed, it is relevant to solve the corresponding auxiliary equation $0=4r^2+1=4(r+\frac{1}{2}i)(r-\frac{1}{2}i)$.

The general solution of the DE is $y=e^{0x}\left(A\cos(\frac{1}{2}x)+B\sin(\frac{1}{2}x)\right)$

(where 0 in e^{0x} is the real part of the complex conjugated pair of roots.)

The initial values give:
$$\begin{cases} 1 = y(\pi) = A\cos\left(\frac{\pi}{2}\right)+B\sin\left(\frac{\pi}{2}\right) = A\cdot 0+B\cdot 1 \\ 2 = y'(\pi) = -\frac{1}{2}A\sin\left(\frac{\pi}{2}\right)+\frac{1}{2}B\cos\left(\frac{\pi}{2}\right) = -\frac{A}{2}\cdot 1+\frac{B}{2}\cdot 0 \end{cases}$$

i.e. $\begin{cases} B=1 \\ A=-4 \end{cases}$ Thus the solution of the IVP is $y=\sin\left(\frac{x}{2}\right)-4\cos\left(\frac{x}{2}\right)$

2 $f(x)=\dfrac{x}{x^2+4}$ Diff. gives $f'(x)=\dfrac{1\cdot(x^2+4)-x\cdot 2x}{(x^2+4)^2}=\dfrac{-x^2+4}{(x^2+4)^2}=-\dfrac{(x+2)(x-2)}{(x^2+4)^2}$

A first derivative test

x		-2		2	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	loc. min	$\nearrow$	loc. max	$\searrow$

i.e. -2 is a local minimum point for f and 2 is a local maximum point.

(-2 is also a minimum point for f and 2 is also a maximum point for f
(but this is not asked for in the problem))

3 $$\frac{1}{2x-3}=-\frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}x}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}x\right)^n=\sum_{n=0}^{\infty}\left[-\frac{1}{3}\left(\frac{2}{3}\right)^n\right]x^n=\sum_{n=0}^{\infty}c_nx^n$$

iff $\left|\frac{2}{3}x\right|<1$

where $c_n=-\frac{1}{3}\left(\frac{2}{3}\right)^n$, $n\geq 0$ are the coefficients of the power series in x. The interval of convergence is $\left(-\frac{3}{2},\frac{3}{2}\right)$

since $\left|\frac{2}{3}x\right|<1 \Leftrightarrow |x|<\frac{3}{2} \Leftrightarrow -\frac{3}{2}<x<\frac{3}{2}$.

4 $f(x)=x^9+x$ We note that $f(1)=2$ and therefore $1=f^{-1}(2)$

Furthermore $(f^{-1})'(2)=\dfrac{1}{f'(1)}=\dfrac{1}{9\cdot 1^8+1}=\dfrac{1}{10}$

Thus an equation for the tangent line τ to the curve γ: $y=f^{-1}(x)$ at the point $P:(2,f^{-1}(2))$ is

τ: $y-f^{-1}(2)=(f^{-1})'(2)\,(x-2)$ i.e. $y-1=\frac{1}{10}(x-2)$

i.e. $y=\frac{1}{10}(x+8)$

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(5) $\lim_{x\to 1} \frac{2-3x+x^2}{1-2x+x^2} = \lim_{x\to 1} \frac{(x-1)(x-2)}{(x-1)^2} = \lim_{x\to 1} \frac{x-1}{x-1}\,\frac{x-2}{x-1}$ does not exist

since the second fraction, i.e. $\frac{x-2}{x-1}$, has no limit when $x\to 1$.

Especially, $\frac{x-2}{x-1} \to +\infty$ when $x\to 1^-$ and $\frac{x-2}{x-1} \to -\infty$ when $x\to 1^+$.

(6) $\int dx\, f(x) = \int dx\, \frac{1}{\sqrt{x}} e^{-\sqrt{x}} \left[\begin{matrix} -\sqrt{x} = u \\ -\frac{1}{2}\frac{1}{\sqrt{x}}dx = du \end{matrix}\right] = \int (-2du)\, e^{u}$

$= -2e^{u} + C = -2e^{-\sqrt{x}} + C$

(7) $\int_1^3 \left(|x-1| + |x-2| + |x-3|\right)dx = \int_1^2 [x-1-(x-2)-(x-3)]dx + \int_2^3 (x-1+x-2-(x-3))dx$

$= \int_1^2 (4-x)dx + \int_2^3 x\,dx = \left[4x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3$

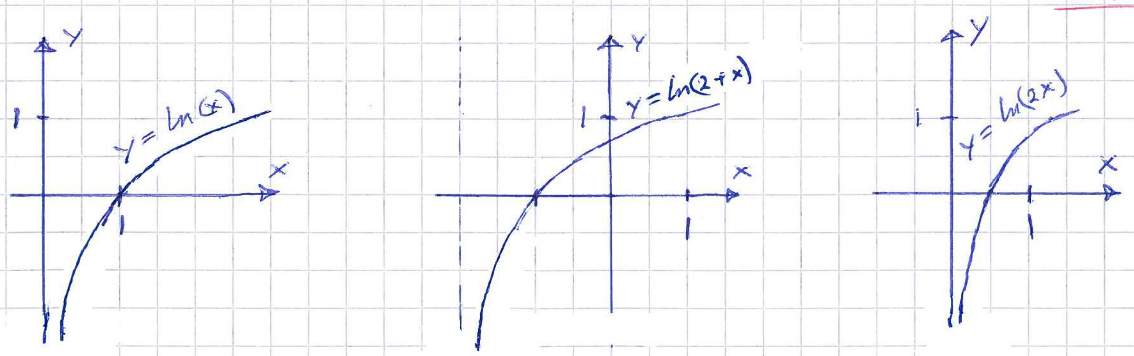
$= \left[(8-2)-\left(4-\frac{1}{2}\right)\right] + \frac{1}{2}(9-4) = 2 + \frac{1}{2} + \frac{5}{2} = 5$

(8) $\begin{cases} \gamma: & y = f(x) \\ \gamma_1: & y = f(2x) \\ \gamma_2: & y = f(2+x) \end{cases}$ where $f = \ln$

parallel with the x-axis

γ_1 is the graph γ compressed "horizontally" by a factor of 2.

γ_2 —— " —— shifted "horizontally" 2 units in the negative x-direction



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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2015/16

Examination TEN1 – 2015-11-30

Maximum points for subparts of the problems in the final examination

1. $y=\sin(x/2)-4\cos(x/2)$

Note: The student who, for the general solution, has found anything else but a linear combination of the trigonometric functions cos and sin obtains **0p**.

1p: Correctly found the general solution of the DE
1p: Correctly differentiated the general solution in preparing for the adaption to the initial values
1p: Correctly adapted the general solution to the initial values, and correctly summarized the solution of the IVP

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2. -2 is a local minimum point for f
2 is a local maximum point for f

1p: Correctly found the stationary points of the function
1p: Correctly classified the local minimum point for f
1p: Correctly classified the local maximum point for f

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3. $$\frac{1}{2x-3}=\sum_{n=0}^{\infty}c_n x^n \quad \text{where } c_n=-\tfrac{1}{3}\left(\tfrac{2}{3}\right)^n$$
The interval of convergence is $\left(-\tfrac{3}{2},\tfrac{3}{2}\right)$

1p: Correctly expanded $1/(2x-3)$ in a power series in x
1p: Correctly identified the coefficients of the power series
1p: Correctly determined the interval of convergence

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4. $\tau: y-1=\tfrac{1}{10}(x-2) \Leftrightarrow y=\tfrac{1}{10}(x+8)$

Note: The student who has failed in finding the value of $(f^{-1})'(2)$, but at least has formulated the equation $\tau: y-1=(f^{-1})'(2)\cdot(x-2)$, will still get the 3rd point.

1p: Correctly found the value of $f^{-1}(2)$
1p: Correctly found the value of $(f^{-1})'(2)$
1p: Correctly formulated an equation for the tangent line τ to the curve γ at the point P

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5. The limit does not exist

3p: Correctly concluded that a limit does not exist

Note: The student who have argued that the limit does not exist based on the fact that the fraction at the limit point is of the type "0/0" obtains **0p**. The student who have claimed that a fraction of the type "1/0" or "0/0" is equal to 0 obtains **0p**, especially if the succeeding conclusion is that *the limit does not exist since the value of it is 0*.

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6. $\int \frac{1}{\sqrt{x}}e^{-\sqrt{x}}\,dx=-2e^{-\sqrt{x}}+C$

where C is a constant

1p: Correctly applied a substitution which simplifies the determination of the general antiderivative of f
1p: Correctly determined an antiderivative of f
1p: Correctly determined the general antiderivative of f

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7. 5

Note: The student who has not taken account of the absolute value bars obtains **0p**. The student who has determined a negative value, and has not commented such a value as being unreasonable, can obtain **at most 1p**. The student who has determined a negative value for the integral, but at least has commented such a value as being unreasonable, can obtain **at most 2p**.

2p: Correctly divided the integral into two integrals, each in which the absolute value bars can be removed, and correctly taken account of the absolute value bars
1p: Provided that the division into two integrals is proper, correctly determined the value of the integral

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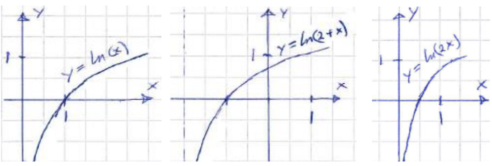
8. The graph given by the equation $y=f(2x)$ is the graph of f compressed horizontally by a factor of 2.

The graph given by the equation $y=f(2+x)$ is the graph of f shifted 2 units in the negative x-direction.



1p: Correctly explained how the graph given by the equation $y=f(2x)$ can be obtained from the graph of f
1p: Correctly explained how the graph given by the equation $y=f(2+x)$ can be obtained from the graph of f
1p: Correctly sketched the graphs given by the equations $y=f(2x)$ and $y=f(2+x)$

Note: A clear and instructive *illustration* of a graph may be accounted for as also being an appropriate *explanation*. However, the student who have sketched the two graphs without any comments supporting the sketches, can obtain **at most 2p**.