Avdelningen för Matematik
och tillämpad matematik
Mälardalens högskola
Examinator: Mats Bodin

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Tentamen Vektoralgebra
MAA150 - TEN1
Datum: 2016-12-01
Hjälpmedel: penna, sudd och linjal

**Lösningarna skall presenteras på ett sådant sätt att räkningar och resonemang blir lätta att följa. Alla svar skall motiveras. Avsluta varje lösning med ett tydligt angivet svar.**

**1** Bestäm det värde på $a \in \mathbb{R}$ så att ekvationssystemet är konsistent och bestäm sedan den allmänna lösningen för detta värde på $a$. (5p)

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 4x_3 & = & 4 \\ -x_1 & + & x_2 & + & 2x_3 & = & 1 \\ x_1 & - & x_2 & - & 2x_3 & = & a \end{array}$$

**2** Låt

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 2 \end{bmatrix}.$$

**a.** Bestäm inversen till $A$. (3p)

**b.** Beräkna $\det(A)$ och $\det(2A^{-4})$ (4p)

**3** Låt $l$ vara linjen som på parameterform ges av $x = 1+3t$, $y = t$, $z = 2-t$. Bestäm ekvationen för planet som innehåller punkten $P(-1, 3, 2)$ och har en normal som är parallell med linjen $l$. (4p)

**4** Givet punkterna $A(2, 2)$, $B(3, -1)$ och vektorn $\mathbf{u} = (1, 3)$.

**a.** Finn vektorformen för linjen, $l$, som går genom punkterna $A$ och $B$. (2p)

**b.** Bestäm $\mathbf{w}_1$ och $\mathbf{w}_2$ så att $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, där $\mathbf{w}_1$ är parallel med linjen $l$ och $\mathbf{w}_2$ är vinkelrät mot $l$. (3p)

**5** Beräkna arean av den triangel vars hörn är $A(2, 1, -1)$, $B(1, 1, 0)$, och $C(-1, 0, 1)$. (4p)

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Mälardalen University
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Examination Vector algebra
MAA150 - TEN1
Date: December 1, 2016
Exam aids: pencil,
eraser and ruler

**All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.**

**1** Determine the value of $a \in \mathbb{R}$ such that the linear system is consistent, and then find the general solution for that value of $a$. (5p)

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 4x_3 & = & 4 \\ -x_1 & + & x_2 & + & 2x_3 & = & 1 \\ x_1 & - & x_2 & - & 2x_3 & = & a \end{array}$$

**2** Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 2 \end{bmatrix}.$$

**a.** Find the inverse of $A$. (3p)

**b.** Evaluate $\det(A)$ and $\det(2A^{-4})$ (4p)

**3** Let $l$ be the line that has parameter form $x = 1 + 3t$, $y = t$, $z = 2 - t$. Find the equation of the plane that passes through the point $P(-1, 3, 2)$ and has a normal that is parallel to the line $l$. (4p)

**4** Given the points $A(2, 2)$, $B(3, -1)$ and the vector $\mathbf{u} = (1, 3)$.

**a.** Find the vector form of the line, $l$, that passes through the points $A$ and $B$. (2p)

**b.** Determine $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to the line $l$ and $\mathbf{w}_2$ is perpendicular to $l$. (3p)

**5** Find the area of the triangle with vertices $A(2, 1, -1)$, $B(1, 1, 0)$, and $C(-1, 0, 1)$. (4p)

# MAA150 Vektoralgebra, HT2016.

### Assessment criterias for TEN1 2016-12-01

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**General assessment criteria**

All solutions should be presented so that calculations and arguments are easy to follow. All answers should be motivated. Each solution should end with a clearly stated answer.

Points may be deducted for erroneous mathematical statements, calculations, or failure to use proper mathematical notation.

**Assessment problems**

1. [**5 points**]
   Relevant row operations on the augmented matrix **(2p)**, finding $a = -1$ **(1p)**, solving including setting a parameter (no point if you find that $t$ is a value) **(1p)**, the correct solution for $a = -1$ **(1p)**.

2. [**7 points**]
   **a.** Relevant method and row operations **(2p)**, correct inverse **(1p)**
   **b.** Evaluating the determinant with a valid method **(2p)**, correct value **(1p)**, evaluating $\det(2A^{-4})$ **(1p)**

3. [**4 points**]
   Finding the normal **(1p)**, relevant method or good figure **(1p)**, finding the equation of the plane **(2p)**

4. [**5 points**]
   **a.** vector in the direction of the line **(1p)**, correct vector form for the line **(1p)**
   **b.** computing relevant projection **(2p)**, correct vectors **(1p)**

5. [**4 points**]
   relevant vectors **(1p)**, computing the cross product **(2p)**, correct area **(1p)**

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(1)
$$\begin{cases} 2x_1 + x_2 - 4x_3 = 4 & (1) \\ -x_1 + x_2 + 2x_3 = 1 & (2) \\ x_1 - x_2 - 2x_3 = a & (3) \end{cases}$$
Solving this linear system

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & 4 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & -2 & a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ -1 & 1 & 2 & 1 \\ 2 & 1 & -4 & 4 \end{array}\right] \overset{(1)\,(-2)}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ 0 & 0 & 0 & a+1 \\ 0 & 3 & 0 & 4-2a \end{array}\right] \sim$$
(2p)

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ 0 & 3 & 0 & 4-2a \\ 0 & 0 & 0 & a+1 \end{array}\right] \quad \text{so} \quad \begin{cases} x_1 - x_2 - 2x_3 = a \\ 3x_2 = 4-2a \\ \boxed{0 = a+1} \end{cases}$$

Therefore the linear system is consistent iff $a=-1$ (1p)

For $a=-1$ we have

$$\begin{cases} x_1 - x_2 - 2x_3 = -1, & (1) \\ 3x_2 = 6 & (2) \end{cases}$$
Set $x_3 = t$ then $x_2 = 2$ (by (2))

and $x_1 = -1 + 2 + 2t$ (by (1)) (1p)

$= 1 + 2t$

$x_3$ free variable

<u>Answer</u>: Consistent iff $a=-1$, then the general solution is
$$\begin{cases} x = 1+2t \\ y = 2 \\ z = t \end{cases}$$
where $t \in \mathbb{R}$. (1p)

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Check

(1) $2\cdot(1+2t) + 2 - 4t = 2+2 = 4$ ok!

(2) $-(1+2t) + 2 + 2t = -1 - 2t + 2 + 2t = 1$ ok!

(3) $(1+2t) - 2 - 2t = 1 + 2t - 2 - 2t = -1$ ok!

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② $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 2 \end{bmatrix}$.

a) $\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -2 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} -1 \quad 2 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right] \sim$

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 0 \end{array}\right] -1 \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right] 1$ (2p)

$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right] -1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & -2 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right]$ (1p)

$\underbrace{\qquad\qquad}_{=A^{-1}}$

b) $\det(A) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 2 \end{vmatrix} \begin{array}{c} -1 \quad 2 \end{array} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \left[\begin{array}{l} \text{Cofactor expansion} \\ \text{column 1} \end{array}\right] =$

$= 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (1 \cdot 0 - 1 \cdot 1) = -1$ (3p)

$\det(2A^{-4}) = 2^3 \cdot \det(A^{-4}) = 8 \cdot \det(A)^{-4} = 8 \cdot (-1)^4 = 8$ (1p)

Answer: a) $A^{-1} = \begin{bmatrix} -4 & 1 & -2 \\ 2 & 0 & 1 \\ -3 & 1 & -1 \end{bmatrix}$

b) $\det(A) = -1$

$\det(2A^{-4}) = 8$

Check: $A \cdot A^{-1} =$

$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 & -2 \\ 2 & 0 & 1 \\ -3 & 1 & -1 \end{bmatrix} =$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ ok!

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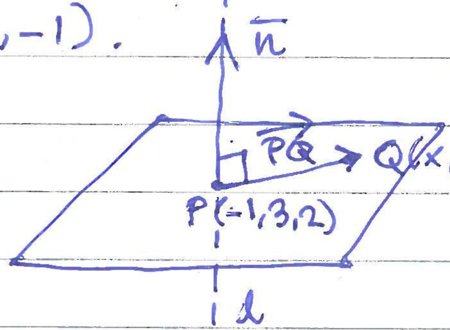
(3) $l: \begin{cases} x = 1+3t \\ y = t \\ z = 2-t \end{cases}$, $t \in \mathbb{R}$. The vector form of $l$ is

$(x,y,z) = (1,0,2) + t\cdot(3,1,-1)$ so the direction of the line is $(3,1,-1)$. Hence we can chose the normal $\bar{n} = (3,1,-1)$.

Illustrative figure.




$\vec{PQ} = (x,y,z) - (-1,3,2) = (x+1, y-3, z-2)$ (2p)

Then $Q(x,y,z)$ belongs to the plane iff.

$$\bar{n} \bullet \vec{PQ} = 0 \iff (3,1,-1)\bullet(x+1, y-3, z-2) = 0 \iff$$

$$\iff \boxed{3(x+1) + (y-3) - (z-2) = 0}$$ (2p)

Answer: $3(x+1) + (y-3) - (z-2) = 0$

or in general form

$3x + y - z + 2 = 0$

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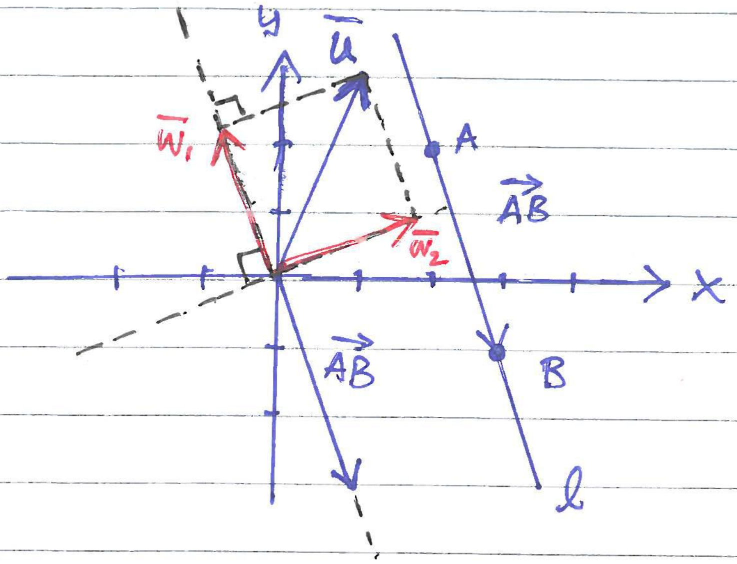
(4)




A (2,2)
B (3,−1)
$\bar{u} = (1,3)$

a) $\overrightarrow{AB} = (3,-1)-(2,2) = (1,-3)$

$l: (x,y) = (2,2) + t\cdot(1,-3)$ where $t \in \mathbb{R}$ (2p)

b) By the projection theorem $\bar{w}_1 = \text{proj}_{\overrightarrow{AB}}\,\bar{u}$ and $\bar{w}_2 = \bar{u} - \bar{w}_1$.

$$\bar{w}_1 = \text{proj}_{\overrightarrow{AB}}\,\bar{u} = \frac{(1,3)\bullet(1,-3)}{\|(1,-3)\|^2}\cdot(1,-3) = \frac{1-9}{1^2+(-3)^2}\cdot(1,-3) =$$

$$= -\frac{8}{10}(1,-3) = \left(-\frac{4}{5}, \frac{12}{5}\right)$$

$$\bar{w}_2 = \bar{u} - \bar{w}_1 = (1,3) - \left(-\frac{4}{5}, \frac{12}{5}\right) = \left(\frac{9}{5}, \frac{3}{5}\right)$$ (2p)

Answer a) $l: (x,y) = (2,2) + t\cdot(1,-3)$, $t \in \mathbb{R}$
b) $\bar{w}_1 = \left(-\frac{4}{5}, \frac{12}{5}\right)$ and $\bar{w}_2 = \left(\frac{9}{5}, \frac{3}{5}\right)$ (1p)

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Check: $\bar{w}_1 + \bar{w}_2 = \left(-\frac{4}{5}, \frac{12}{5}\right) + \left(\frac{9}{5}, \frac{3}{5}\right) = (1,3)$ ok!

$\bar{w}_1 \bullet \bar{w}_2 = \left(-\frac{4}{5}, \frac{12}{5}\right) \bullet \left(\frac{9}{5}, \frac{3}{5}\right) = -\frac{36}{25} + \frac{36}{25} = 0$ ok!

$\bar{w}_1 = -\frac{4}{5}\cdot(1,-3) = -\frac{4}{5}\cdot\overrightarrow{AB}$ ok!

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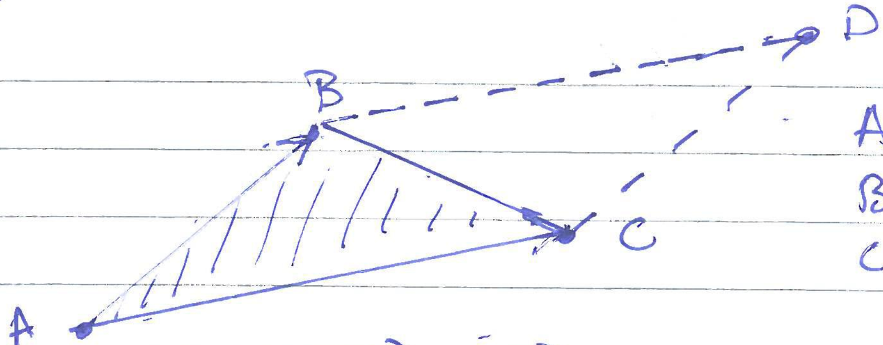
(5)



A(2, 1, -1)
B(1, 1, 0)
C(-1, 0, 1)

Area ABCD $= \|\vec{AB} \times \vec{AC}\|$ so

Area $\Delta ABC = \dfrac{\|\vec{AB} \times \vec{AC}\|}{2}$

$\vec{AB} = (1,1,0) - (2,1,-1) = (-1,0,1)$ (1p)

$\vec{AC} = (-1,0,1) - (2,1,-1) = (-3,-1,2)$

$\vec{AB} \times \vec{AC} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-(-1) \\ -((-1)\cdot 2 - 1\cdot(-3)) \\ (-1)\cdot(-1) - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (2p)

$\|\vec{AB} \times \vec{AC}\| = \|(1,-1,1)\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

Answer: The area of $\Delta ABC$ is $\dfrac{\sqrt{3}}{2}$ area units.

(1p)