MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN1
Date: 2015-06-08 *Write time:* 3 hours
Aid: Writing materials

This examination is intended for the examination part TEN1. The examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 3 points. The PASS-marks 3, 4 and 5 require a minimum of 11, 16 and 21 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 11, 13, 16, 20 and 23 respectively. If the obtained sum of points is denoted S_1, and that obtained at examination TEN2 S_2, the mark for a completed course is according to the following

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the GENERAL antiderivative of $x \curvearrowright f(x) = xe^{2x}$.

2. Find the area of the bounded region precisely enclosed by the curves

$$y = 3 - x^2 \quad \text{and} \quad y = x + |x|\,.$$

3. Let f be the function defined by $f(x) = \dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}$. In what intervals is the function convex?

4. Let $f(x) = \arcsin(x)$. State the domain and the range of f and f^{-1} respectively, and sketch in separate figures the graphs of the functions.

5. Solve the initial-value problem $xy' - 2y = \dfrac{3}{x}\,,\ y(1) = 3\,.$

6. Find the coefficients of the power series in x representing $\dfrac{1}{x+2}$. Also, determine the interval of convergence of the power series.

7. Prove that the function f defined by $f(x) = x^5 + x^3 + x$ is invertible, and find the derivative of f^{-1} at the point 3.

8. Determine whether

$$\lim_{x \to 0} \frac{\cos(x) - 1}{e^{2x} - 2x - 1}$$

exists or not. If the answer is NO: Give an explanation of why! If the answer is YES: Give an explanation of why and find the limit!

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen är avsedd för examinationsmomentet TEN1. Provet består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 3 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 11, 16 respektive 21 poäng. Om den erhållna poängen benämns S_1, och den vid tentamen TEN2 erhållna S_2, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm den GENERELLA primitiva funktionen till $x \curvearrowright f(x) = xe^{2x}$.

2. Bestäm arean av det begränsade område som precis innesluts av kurvorna

$$y = 3 - x^2 \quad \text{och} \quad y = x + |x| \, .$$

3. Låt f vara funktionen definierad enligt $f(x) = \dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}$. I vilka intervall är funktionen konvex?

4. Låt $f(x) = \arcsin(x)$. Ange definitionsmängden och värdemängden för f respektive f^{-1}, och skissa i separata figurer graferna till funktionerna.

5. Lös begynnelsevärdesproblemet $xy' - 2y = \dfrac{3}{x}$, $y(1) = 3$.

6. Bestäm koefficienterna i den potensserie i x som representerar $\dfrac{1}{x+2}$. Bestäm även konvergensintervallet för potensserien.

7. Bevisa att funktionen f definierad enligt $f(x) = x^5 + x^3 + x$ är inverterbar, och bestäm derivatan till f^{-1} i punkten 3.

8. Avgör om

$$\lim_{x \to 0} \frac{\cos(x) - 1}{e^{2x} - 2x - 1}$$

existerar eller ej. Om svaret är NEJ: Ge en förklaring till varför! Om svaret är JA: Ge en förklaring till varför och bestäm gränsvärdet!

If you prefer the problems formulated in English, please turn the sheet.

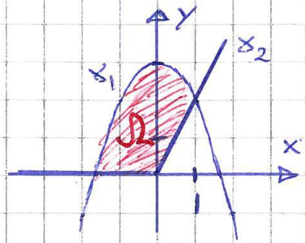
MAA151 / Solutions to the final exam TEN1 2015-06-08

(1) $$\int f(x)\,dx = \int x e^{2x}\,dx = x\left(\tfrac{1}{2}e^{2x}\right) - \int 1\left(\tfrac{1}{2}e^{2x}\right)dx$$

$$= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C = \underline{\tfrac{1}{4}(2x-1)\,e^{2x} + C}$$

<u>where</u> C is a constant.

(2) $$\begin{cases} \gamma_1 : \ y = 3 - x^2 \\ \gamma_2 : \ y = x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases} \end{cases}$$



<u>The intersection</u> $\gamma_1 \cap \gamma_2$: $\begin{cases} x \geq 0 : 3 - x^2 = 2x \\ \text{or} \\ x < 0 : 3 - x^2 = 0 \end{cases}$

$\Leftrightarrow \quad 0 = x^2 + 2x - 3 = (x+3)(x-1),\ x \geq 0 \quad$ OR $\quad 0 = x^2 - 3 = (x+\sqrt{3})(x-\sqrt{3}),\ x < 0$

$\Leftrightarrow \quad (x,y) = (1,2)$ OR $(x,y) = (-\sqrt{3}, 0)$

the area of a triangular region

<u>Thus</u> $$A(\Omega) = \int_{-\sqrt{3}}^{1} (3 - x^2)\,dx - \int_0^1 2x\,dx = \left[3x - \frac{x^3}{3}\right]_{-\sqrt{3}}^{1} - \left[x^2\right]_0^1$$

$$= \left[\left(3 - \tfrac{1}{3}\right) - \left(-3\sqrt{3} + \sqrt{3}\right) - (1 - 0)\right] \text{a.u.} = \underline{\left(\tfrac{5}{3} + 2\sqrt{3}\right) \text{a.u.}}$$

(3) $$f(x) = \frac{1}{x+1} - \frac{1}{(x+1)^2} \quad ; \quad \begin{cases} f'(x) = -\dfrac{1}{(x+1)^2} + \dfrac{2}{(x+1)^3} \\ f''(x) = \dfrac{2}{(x+1)^3} - \dfrac{6}{(x+1)^4} = \dfrac{2(x-2)}{(x+1)^4} \end{cases}$$

<u>Test for convexity/concavity</u>

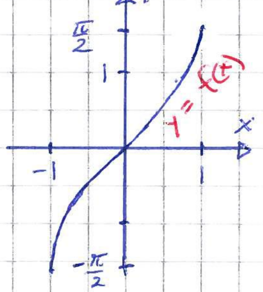
x		-1		2	
$f''(x)$	$-$	$\nexists$	$-$	0	$+$
$f(x)$	$\cap$	$\nexists$	$\cap$	IP	$\cup$

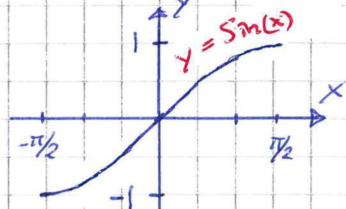
<u>i.e.</u> the function f is convex in the interval $\underline{[2, \infty)}$

(4) $$\begin{cases} f(x) = \arcsin(x) \\ f^{-1}(x) = \operatorname{Sin}(x) \end{cases}$$

$\underline{D_f = [-1, 1], \quad V_f = \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]}$

<u>where</u> $\underline{D_{f^{-1}} = \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \text{ and } V_{f^{-1}} = [-1, 1]}$





<u>Note</u> The function <u>Sin</u> (spelled with a capital 'S') often means the restriction of sin to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

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(5) DE: $xy' - 2y = \frac{3}{x}$, IV: $y(1) = 3$

The DE is a nonhomogeneous linear 1st-order DE which together with the IV give a unique solution on the interval $(0,\infty)$.
The standard form of the DE, i.e. $y' - \frac{2}{x}y = \frac{3}{x^2}$, indicates that an integrating factor is $e^{-\int \frac{2}{x}dx} = e^{-2\ln|x| + c_1} = x^{-2} \cdot c_2$
By choosing e.g. x^{-2}, the DE can be written as
$\frac{d}{dx}(y \cdot x^{-2}) = 3x^{-4}$ which ends up in $y = x^2(-x^{-3} + C) = -\frac{1}{x} + Cx^2$.
The IV gives $3 = y(1) = -\frac{1}{1} + C \cdot 1^2$ i.e. $C = 4$. Thus $y = 4x^2 - \frac{1}{x}$

(6) $$\frac{1}{x+2} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} \underset{\text{iff } |-\frac{x}{2}|<1}{=} \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \underbrace{\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}x^n}_{\sum_{n=0}^{\infty} c_n x^n}$$

i.e. the coefficients c_n of the power series are $c_n = (-1)^n\left(\frac{1}{2}\right)^{n+1}$ for $n \geq 0$
and the interval of convergence is $(-2,2)$
(since $|-\frac{x}{2}| < 1 \Leftrightarrow |x| < 2 \Leftrightarrow -2 < x < 2$).

(7) $f(x) = x^5 + x^3 + x$

Since $f'(x) = 5x^4 + 3x^2 + 1 \geq 1 > 0$, the function f is (strictly) increasing and therefore injective and therefore also invertible (q.e.d.). We notice that $f(1) = 1 + 1 + 1 = 3$, and then get that $1 = f^{-1}(3)$.
The derivative of f^{-1} is given by $(f^{-1})'(y) = \left.\frac{1}{f'(x)}\right|_{x = f^{-1}(y)}$
In our case, we get
$$(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{5 \cdot 1^4 + 3 \cdot 1^2 + 1} = \frac{1}{9}$$

(8) $$\lim_{x\to 0}\frac{\cos(x) - 1}{e^{2x} - 2x - 1} \overset{\text{Maclaurin expansions}}{=} \lim_{x\to 0}\frac{\left(1 - \frac{x^2}{2!} + O(x^4)\right) - 1}{\left(1 + 2x + \frac{(2x)^2}{2!} + O(x^3)\right) - 2x - 1}$$
$$= \lim_{x\to 0}\frac{-\frac{1}{2}x^2 + O(x^4)}{2x^2 + O(x^3)} = \lim_{x\to 0}\frac{x^2\left(-\frac{1}{2} + O(x^2)\right)}{x^2\left(2 + O(x)\right)} = 1\left(\frac{-\frac{1}{2} + 0}{2 + 0}\right) = -\frac{1}{4}$$

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2014/15

Examination TEN1 – 2015-06-08 — Maximum points for subparts of the problems in the final examination

1. $F(x)=\frac{1}{4}(2x-1)e^{2x}+C$ where C is a constant

1p: Correctly worked out the first progressive step in finding the antiderivative by parts
1p: Correctly worked out the second progressive step in finding the antiderivative by parts
1p: Correctly included a constant in an otherwise correctly found antiderivative

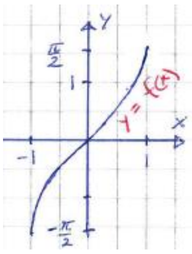
2. $(5/3+2\sqrt{3})$ a.u.

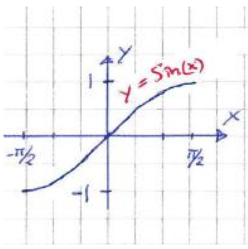
1p: Correctly found the intersection of the two enclosing curves, and correctly formulated an integral for the area
1p: Correctly determined the needed antiderivatives
1p: Correctly found the limits in the integral and the area

3. f is convex in the interval $[2,\infty)$

1p: Correctly found the second derivative of f
1p: Correctly factorized the second derivative of f, and correctly worked out a test for convexity of f
1p: Correctly determined the interval where f is convex

4. $D_f=[-1,1]$ and $V_f=[-\frac{\pi}{2},\frac{\pi}{2}]$
$D_{f^{-1}}=[-\frac{\pi}{2},\frac{\pi}{2}]$ and $V_{f^{-1}}=[-1,1]$





1p: Correctly stated the domains and the ranges of the functions f and f^{-1}
1p: Correctly sketched the graph of the function f
1p: Correctly sketched the graph of the function f^{-1}

5. $y=4x^2-\dfrac{1}{x}$

1p: Correctly written the DE in standard form, correctly determined an integrating factor, and correctly reformulated the left-hand-side of the DE as an exact derivative
1p: Correctly found the general solution of the DE
1p: Correctly adapted the general solution to the initial value, and correctly summarized the solution of the IVP

6. $\dfrac{1}{x+2}=\sum_{k=0}^{\infty}c_k x^k$ where $c_k=(-1)^n(\frac{1}{2})^{n+1}$
The interval of convergence is $(-2,2)$

1p: Correctly expanded $1/(x+2)$ in a power series in x
1p: Correctly identified the coefficients of the power series
1p: Correctly determined the interval of convergence

7. f is invertible since $f'(x)>0$ on D_f
$(f^{-1})'(3)=1/9$

1p: Correctly proved that f is invertible
1p: Correctly found that $f^{-1}(3)=1$
1p: Correctly determined the value of $(f^{-1})'(3)$

8. The limit exists and is equal to $-\frac{1}{4}$

1p: Correctly expanded $\cos(x)$ and e^{2x} in their Maclaurin series
1p: Correctly algebraically prepared for determining the limit
1p: Correctly determined the limit