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EXAMINATION IN MATHEMATICS
MAA153 Linear Algebra
Date: 2017-08-16 *Write time:* 5 hours
Aid: Writing materials, ruler

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

1. Let $\mathbb{E}$ be the vector space $\mathbb{R}^3$ equipped with the inner product

$$\langle u|v\rangle = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathsf{A} \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T ,$$

where x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of u and v respectively relative to the ordered basis $(1,0,0), (0,1,0), (0,0,1)$, and where we in the notation make no difference between a 1×1 matrix and its entry. Find the matrix A such that $(1,1,0), (1,0,1), (0,1,1)$ constitutes an ordered ON-basis.

2. The linear operator $F : E^3 \to E^3$ has the matrix

$$\mathsf{A} = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 10 & 0 \\ 3 & 0 & 7 \end{pmatrix} .$$

relative to the standard basis. Prove that F is diagonalizable, i.e. that there exists a basis of eigenvectors of F. Find also a change-of-basis matrix S such that the basis of eigenvectors is orthonormal (ON).

3. The subspaces $\mathbb{U}$ and $\mathbb{V}$ of $\mathbb{R}^4$ are definied according to

$$\begin{cases} \mathbb{U} = \operatorname{span}\{(6,3,-8,4), (11,7,-14,10)\} \\ \mathbb{V} = \operatorname{span}\{(3,-2,5,-4), (4\alpha-5, 4-3\alpha, 3\alpha-1, 8-6\alpha)\} . \end{cases}$$

For which values of α is the dimension of the subspace $\mathbb{U} \cap \mathbb{V}$ not equal to zero? Find, for these values of α, a basis for $\mathbb{U} \cap \mathbb{V}$.

4. Let $M = \{(a,b,c,d) \in \mathbb{E}^4 : 5a+3b = c-2d,\ a+2b = 3c+d,\ 3a+4b = 5c+d\}$. Find an ON-basis (orthonormal basis) for M.

5. Classify the two surfaces $\begin{cases} S_1 : (x-y)^2 - (3x-y+2z)^2 + (y+z)^2 = 1, \\ S_2 : (x-y)^2 - (2x-y+3z)^2 + (y+z)^2 = 1, \end{cases}$
i.e. find the geometric meaning of each equation. Motivate!

6. Find, with respect to the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, the matrix of a linear transformation $F : \mathbb{R}^4 \to \mathbb{R}^3$ for which $\ker(F) = \operatorname{span}\{(2,-1,1,2), (-3,2,1,3)\}$ and $\operatorname{im}(F) = \operatorname{span}\{(1,3,2), (2,4,1)\}$.

7. Prove that the set of functions p_1, p_2, p_3, where $p_n(x) = x^n$, is linearly independent and therefore is a basis for a 3-dimensional linear space $\mathcal{P}$. Then prove that also the three functions p, q, r defined by

$$p(x) = 2x - x^3, \quad q(x) = x + 2x^2 + x^3, \quad r(x) = x - 3x^2 - x^3,$$

is a basis for $\mathcal{P}$, and find relative to this basis the coordinates of $-p_1 + 5p_2 - 4p_3$.

8. Let $\mathcal{M}$ denote the vector space of all 2×2-matrices with real-valued entries. Find, relative to the ordered basis $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ for $\mathcal{M}$ and the ordered standard basis for $\mathbb{R}^3$, the matrix of the linear transformation $F : \mathcal{M} \to \mathbb{R}^3$ defined as

$$F(\left(\begin{smallmatrix}x_1&x_2\\x_3&x_4\end{smallmatrix}\right)) = (2x_1 - x_2 + x_4, x_1 + 3x_2 - x_3 - x_4, -3x_1 + 4x_3 + 2x_4).$$

Also, find out whether F is surjective or not.

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

1. Låt $\mathbb{E}$ vara det det linjära rummet $\mathbb{R}^3$ utrustat med skalärprodukten

$$\langle u|v\rangle = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathsf{A} \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T ,$$

där x_1, x_2, x_3 och y_1, y_2, y_3 är koordinaterna för u respektive v relativt den ordnade basen $(1,0,0), (0,1,0), (0,0,1)$, och där vi beteckningsmässigt inte gör någon skillnad mellan en matris av typ 1×1 och dess element. Bestäm matrisen A så att $(1,1,0), (1,0,1), (0,1,1)$ utgör en ordnad ON-bas.

2. Den linjära operatorn $F : E^3 \to E^3$ ges i standardbasen av matrisen

$$\mathsf{A} = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 10 & 0 \\ 3 & 0 & 7 \end{pmatrix} .$$

Visa att F är diagonaliserbar, dvs att det finns en bas av egenvektorer till F. Bestäm även en basbytesmatris S så att basen av egenvektorer är ortonormerad (ON).

3. Underrummen $\mathbb{U}$ och $\mathbb{V}$ till $\mathbb{R}^4$ är definierade enligt

$$\begin{cases} \mathbb{U} = \operatorname{span}\{(6,3,-8,4), (11,7,-14,10)\} \\ \mathbb{V} = \operatorname{span}\{(3,-2,5,-4), (4\alpha-5, 4-3\alpha, 3\alpha-1, 8-6\alpha)\} . \end{cases}$$

För vilka värden på α är dimensionen av underrummet $\mathbb{U} \cap \mathbb{V}$ skild från noll? Bestäm, för dessa värden på α, en bas för $\mathbb{U} \cap \mathbb{V}$.

4. Låt $M = \{(a,b,c,d) \in \mathbb{E}^4 : 5a + 3b = c - 2d,\ a + 2b = 3c + d,\ 3a + 4b = 5c + d\}$. Bestäm en ON-bas (ortonormerad bas) för M.

5. Klassificera de två ytorna $\begin{cases} S_1 : (x-y)^2 - (3x-y+2z)^2 + (y+z)^2 = 1, \\ S_2 : (x-y)^2 - (2x-y+3z)^2 + (y+z)^2 = 1, \end{cases}$

dvs bestäm den geometriska innebörden av varje ekvation. Motivera!

6. Bestäm, med avseende på standardbaserna för $\mathbb{R}^4$ och $\mathbb{R}^3$, avbildningsmatrisen för en linjär avbildning $F : \mathbb{R}^4 \to \mathbb{R}^3$ för vilken $N(F) = \operatorname{span}\{(2,-1,1,2), (-3,2,1,3)\}$ and $V(F) = \operatorname{span}\{(1,3,2), (2,4,1)\}$.

7. Bevisa att funktionerna p_1, p_2, p_3, där $p_n(x) = x^n$, är linjärt oberoende och därmed utgör en bas för ett 3-dimensionellt vektorrum $\mathcal{P}$. Bevisa sedan att även de tre funktionerna p, q, r definierade genom

$$p(x) = 2x - x^3, \quad q(x) = x + 2x^2 + x^3, \quad r(x) = x - 3x^2 - x^3,$$

är en bas för $\mathcal{P}$, och bestäm relativt denna bas koordinaterna för $-p_1 + 5p_2 - 4p_3$.

8. Låt $\mathcal{M}$ beteckna det linjära rummet av alla 2×2-matriser med reellvärda element. Bestäm, med avseende på den ordnade basen $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ för $\mathcal{M}$ och den ordnade standardbasen för $\mathbb{R}^3$, matrisen för den linjära avbildningen $F : \mathcal{M} \to \mathbb{R}^3$ definierad enligt

$$F\left(\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)\right) = (2x_1 - x_2 + x_4, x_1 + 3x_2 - x_3 - x_4, -3x_1 + 4x_3 + 2x_4).$$

Utred även om F är surjektiv eller inte?

If you prefer the problems formulated in English, please turn the page.

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(1) $\langle u|v\rangle = X^T A Y = (S\tilde{X})^T A (S\tilde{Y}) = \tilde{X}^T S^T A S \tilde{Y}$

where S is the change-of-basis matrix from the (ordered) standard basis $(1,0,0), (0,1,0), (0,0,1)$ to the (-"-) ON-basis $(1,1,0), (1,0,1), (0,1,1)$, i.e. $S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

If $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ and $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ are the coordinates of u and v respectively relative to the ON-basis, the matrix $S^T A S$ is equal to the identity matrix I. Thus, we have that

$$A = (S^T)^{-1} S^{-1} = (SS^T)^{-1} = \left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1}$$

$$= \frac{1}{(8+1+1)-(2+2+2)} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix}$$

i.e. $\langle u|v\rangle = \frac{3}{4}(x_1y_1 + x_2y_2 + x_3y_3) - \frac{1}{4}(x_1y_2 + x_2y_1) - \frac{1}{4}(x_2y_3 + x_3y_2) - \frac{1}{4}(x_3y_1 + x_1y_3)$

(2) $F: E^3 \to E^3$ has the matrix $A = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 10 & 0 \\ 3 & 0 & 7 \end{pmatrix}$ relative to the standard basis.

Since A is symmetric relative to an ON-basis, F is a symmetric operator. The spectral theorem the gives that F is not only diagonalizable (q.e.d.) but relative an ON-basis of eigenvectors.

Eigenvalues $0 = \det(A - \lambda I) = \det\begin{pmatrix} 7-\lambda & 0 & 3 \\ 0 & 10-\lambda & 0 \\ 3 & 0 & 7-\lambda \end{pmatrix} = (10-\lambda)[(\lambda-7)^2 - 3^2]$

$$= -(\lambda-10)(\lambda-7+3)(\lambda-7-3) = -(\lambda-4)(\lambda-10)^2$$

Eigenvectors:

$\lambda_1 = 4$: $A - \lambda_1 I = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 7 & 0 \\ 3 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $X_1 = t_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $t_1 \neq 0$

$\lambda_{2,3} = 10$: $A - \lambda_{2,3} I = \begin{pmatrix} -3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $X_{2,3} = t_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ where t_2, t_3 are not both zero

An ON-basis of eigenvectors is then e.g. $\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(1,0,1), (0,1,0)$ and the change-of-basis matrix S from the standard basis (which here is ON) to the (chosen) ON-basis of eigenvectors is the (orthogonal) matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$.

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(3) $\begin{cases} U = \operatorname{span}\{\overset{u_1}{(6,3,-8,4)}, \overset{u_2}{(11,7,-14,10)}\} \subset R^4 \\ V = \operatorname{span}\{\underset{v_1}{(3,-2,5,-4)}, \underset{v_2}{(4\alpha-5, 4-3\alpha, 3\alpha-1, 8-6\alpha)}\} \subset R^4 \end{cases}$

Every vector w belonging to the subspace $U \cap V$ is a linear combination of u_1, u_2 and of v_1, v_2 i.e. $x_1 u_1 + x_2 u_2 = w = y_1 v_1 + y_2 v_2$, a relation which corresponds to the augmented coefficient matrix

$$\left(\begin{array}{cc|cc} 6 & 11 & 3 & 4\alpha-5 \\ 3 & 7 & -2 & 4-3\alpha \\ -8 & -14 & 5 & 3\alpha-1 \\ 4 & 10 & -4 & 8-6\alpha \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & -3 & 7 & 10\alpha-13 \\ 3 & 7 & -2 & 4-3\alpha \\ -2 & 0 & 1 & 7-3\alpha \\ 1 & 3 & -2 & 4-3\alpha \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & -3 & 7 & 10\alpha-13 \\ 3 & 1 & 12 & 17\alpha-22 \\ -2 & 0 & 1 & -3\alpha+7 \\ 1 & 0 & 5 & 7\alpha-9 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 0 & -3 & 7 & 10\alpha-13 \\ 0 & 1 & -3 & -4\alpha+5 \\ 0 & 0 & 11 & 11\alpha-11 \\ 1 & 0 & 5 & 7\alpha-9 \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & 0 & -2 & -2\alpha+2 \\ 0 & 1 & -3 & -4\alpha+5 \\ 0 & 0 & 1 & \alpha-1 \\ 1 & 0 & 5 & 7\alpha-9 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 2(\alpha-2) \\ 0 & 1 & 0 & -(\alpha-2) \\ 0 & 0 & 1 & \alpha-1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $\begin{cases} x_1 = 2(\alpha-2)y_2 \\ x_2 = -(\alpha-2)y_2 \\ 0 = y_1 + (\alpha-1)y_2 \end{cases}$ i.e. $\begin{cases} x_1 u_1 + x_2 u_2 = (\alpha-2)y_2(2u_1 - u_2) = (\alpha-2)y_2(1,-1,-2,-2) \\ y_1 v_1 + y_2 v_2 = y_2[(1-\alpha)v_1 + v_2] = \ldots = \text{—}\,\,\text{''}\,\,\text{—} \end{cases}$

If $\alpha = 2$, then the only common vector of the subspaces U and V is the zero vector, i.e. $\dim(U \cap V) = 0$

If $\alpha \neq 2$, then a basis for $U \cap V$ is e.g. $(1,-1,-2,-2)$

(4) $M = \{(a,b,c,d) \in E^4 : 5a+3b = c-2d,\ a+2b = 3c+d,\ 3a+4b = 5c+d\}$

$$= \left\{(a,b,c,d) \in E^4 : \underbrace{\begin{pmatrix} 5 & 3 & -1 & 2 \\ 1 & 2 & -3 & -1 \\ 3 & 4 & -5 & -1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

where $A \sim \begin{pmatrix} 0 & -7 & 14 & 7 \\ 1 & 2 & -3 & -1 \\ 0 & -2 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

i.e. the general vector u_M in M is given by

$$u_M = (-c-d, 2c+d, c, d) = c\underbrace{(-1,2,1,0)}_{u_1} + d\underbrace{(-1,1,0,1)}_{u_2}$$

i.e. $M = \operatorname{span}\{u_1, u_2\}$

The Gram-Schmidt orthog. proc. gives an ON-basis e_1, e_2 for M as $e_1 = \frac{1}{\|u_1\|} u_1 = \frac{1}{\sqrt{6}}(-1,2,1,0)$

$$f_2 = u_2 - \langle u_2 | e_1 \rangle e_1 = (-1,1,0,1) - \tfrac{1}{6}(1+2+0+0)(-1,2,1,0)$$

$$= \tfrac{1}{2}[(-2,2,0,2) - (-1,2,1,0)] = \tfrac{1}{2}(-1,0,-1,2)$$

$$e_2 = \frac{1}{\|f_2\|} f_2 = \frac{1}{\|2f_2\|} 2f_2 = \frac{1}{\sqrt{6}}(-1,0,-1,2)$$

Thus $\frac{1}{\sqrt{6}}(-1,2,1,0),\ \frac{1}{\sqrt{6}}(-1,0,-1,2)$ is an ON-basis for M

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5 $\begin{cases} S_1: (x-y)^2-(3x-y+2z)^2+(y+z)^2=1 \\ S_2: (x-y)^2-(2x-y+3z)^2+(y+z)^2=1 \end{cases}$

$\boxed{S_1}$ Let $\begin{cases} x-y = \tilde{x} \\ 3x-y+2z=\tilde{y} \\ y+z=\tilde{z} \end{cases} \Leftrightarrow \begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix} = \begin{pmatrix}1&-1&0\\3&-1&2\\0&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} \Leftrightarrow \tilde{X}=A_1X$

where $A_1 \sim \begin{pmatrix}1&-1&0\\0&2&2\\0&1&1\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}$ i.e. $\tilde{x},\tilde{y},\tilde{z}$ are not the coord:s of a vector rel. a basis and therefore $\tilde{x}^2-\tilde{y}^2+\tilde{z}^2$ doesn't reveal the properties of the quadratic form.

By expanding the quadratic form, we get

$$1 = x^2+y^2-2xy-(9x^2+y^2+4z^2-6xy+12xz-4yz)$$
$$+y^2+z^2+2yz = -8x^2+y^2-3z^2+4xy+6yz-12zx$$
$$=(y+3z+x)^2-12z^2-12x^2-24zx=(y+3z+x)^2-12(z+x)^2$$

i.e. S_1 is a hyperbolic cylinder

$\boxed{S_2}$ $A_2=\begin{pmatrix}1&-1&0\\2&-1&3\\0&1&1\end{pmatrix}\sim\begin{pmatrix}1&-1&0\\0&1&3\\0&1&1\end{pmatrix}\sim\begin{pmatrix}1&-1&0\\0&1&3\\0&0&-2\end{pmatrix}\sim\dots\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

i.e. A_2 has an inverse and is therefore the change-of-basis matrix (from the basis $\tilde{e}_1,\tilde{e}_2,\tilde{e}_3$ to e_1,e_2,e_3)

i.e. $S_2: \tilde{x}^2-\tilde{y}^2+\tilde{z}^2=1$ where $\tilde{x},\tilde{y},\tilde{z}$ are the coord:s relative a basis $\tilde{e}_1,\tilde{e}_2,\tilde{e}_3$

i.e. S_2 is a one-sheeted hyperboloid.

6 $F: R^4 \to R^3$ $\begin{cases} \ker(F) = \text{span}\{(2,-1,1,2),(-3,2,1,3)\} \\ \text{im}(F)=\text{span}\{(1,3,2),(2,4,1)\}\end{cases}$

Let A denote the matrix of F relative the standard bases for R^4 and R^3. The conditions for F are not enough for a unique F, but we make the "simplest" choice $A=\begin{pmatrix}1&2&a&\alpha\\3&4&b&\beta\\2&1&c&\gamma\end{pmatrix}$ i.e. taking the vectors which span $\text{im}(F)$ as the two first columns of A.

Applying the meaning of $\ker(F)$ gives

$$\begin{pmatrix}1&2&a&\alpha\\3&4&b&\beta\\2&1&c&\gamma\end{pmatrix}\begin{pmatrix}2&-3\\-1&2\\1&1\\2&3\end{pmatrix}=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \Leftrightarrow \begin{pmatrix}0+a+2\alpha & 1+a+3\alpha\\ 2+b+2\beta & -1+b+3\beta\\ 3+c+2\gamma & -4+c+3\gamma\end{pmatrix}=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}1&2\\1&3\end{pmatrix}\begin{pmatrix}a&b&c\\ \alpha&\beta&\gamma\end{pmatrix}=\begin{pmatrix}0&-2&-3\\-1&1&4\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}a&b&c\\ \alpha&\beta&\gamma\end{pmatrix}=\begin{pmatrix}1&2\\1&3\end{pmatrix}^{-1}\begin{pmatrix}0&-2&-3\\-1&1&4\end{pmatrix}=\begin{pmatrix}3&-2\\-1&1\end{pmatrix}\begin{pmatrix}0&-2&-3\\-1&1&4\end{pmatrix}=\begin{pmatrix}2&-8&-17\\-1&3&7\end{pmatrix}$$

Thus $A=\begin{pmatrix}1&2&2&-1\\3&4&-8&3\\2&1&-17&7\end{pmatrix}$ represents a linear transf. F with the given specifications of $\ker(F)$ and $\text{im}(F)$.

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7 $P = \text{span}\{p_1, p_2, p_3\}$ where $p_k(x) = x^k$

a) The equation $c_1p_1 + c_2p_2 + c_3p_3 = 0$ evaluated at the points 1, 2, 3 gives $\begin{cases} c_1 + c_2 + c_3 = 0 \\ 2c_1 + 4c_2 + 8c_3 = 0 \\ 3c_1 + 9c_2 + 27c_3 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ 3c_2 + 7c_3 = 0 \\ 6c_2 + 24c_3 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ 3c_2 + 7c_3 = 0 \\ 10c_3 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$ i.e. $c_1p_1 + c_2p_2 + c_3p_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

i.e. the set of p_1, p_2, p_3 is linearly independent q.e.d.

b) $\begin{cases} p(x) = 2x - x^3 = 2p_1(x) - p_3(x) \\ q(x) = x + 2x^2 + x^3 = p_1(x) + 2p_2(x) + p_3(x) \\ r(x) = x - 3x^2 - x^3 = p_1(x) - 3p_2(x) - p_3(x) \end{cases}$

i.e. $(p\ q\ r) = (p_1\ p_2\ p_3)\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -3 \\ -1 & 1 & -1 \end{pmatrix}$

where $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -3 \\ -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 0 & 3 & -1 \end{pmatrix} \sim \ldots \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

i.e. the matrix has full rank

i.e. —"— is a change-of-basis matrix from p_1, p_2, p_3 to p, q, r q.e.d.

c) $-p_1 + 5p_2 - 4p_3 = (p_1\ p_2\ p_3)\begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix}$

$= (p\ q\ r)\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -3 \\ -1 & 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix} = (p\ q\ r)\frac{1}{7}\begin{pmatrix} 1 & 2 & -5 \\ 3 & -1 & 6 \\ 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix} = (p\ q\ r)\frac{1}{7}\begin{pmatrix} 29 \\ -32 \\ -33 \end{pmatrix}$

i.e. $\frac{29}{7}, -\frac{32}{7}, -\frac{33}{7}$ are the coords of $-p_1 + 5p_2 - 4p_3$ relative to the basis p, q, r.

8 $F: M \to R^3$ where $F\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) = (2x_1 - x_2 + x_4,\ x_1 + 3x_2 - x_3 - x_4,\ -3x_1 + 4x_3 + 2x_4)$

We get $\begin{cases} F\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = (2, 1, -3) \\ F\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = (-1, 3, 0) \end{cases}$ $\begin{cases} F\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = (0, -1, 4) \\ F\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = (1, -1, 2) \end{cases}$

Let $\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = m_1,\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = m_2,\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = m_3,\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = m_4 \\ (1,0,0) = e_1,\ (0,1,0) = e_2,\ (0,0,1) = e_3 \end{cases}$

Then $(F(m_1)\ F(m_2)\ F(m_3)\ F(m_4)) = (e_1\ e_2\ e_3)\begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 3 & -1 & -1 \\ -3 & 0 & 4 & 2 \end{pmatrix}$

i.e. the matrix of $F: M \to R^3$ relative to the bases m_1, m_2, m_3, m_4 and e_1, e_2, e_3 is $\begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 3 & -1 & -1 \\ -3 & 0 & 4 & 2 \end{pmatrix} = A$

Since $A \sim \begin{pmatrix} 0 & -7 & 2 & 3 \\ 1 & 3 & -1 & -1 \\ 0 & 9 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 7 & 2 & 3 \\ 7 & 0 & -1 & 2 \\ 0 & 0 & 25 & 20 \end{pmatrix} \sim \begin{pmatrix} 7 & 0 & -1 & 2 \\ 0 & 7 & 2 & 3 \\ 0 & 0 & 5 & 4 \end{pmatrix}$, the matrix A has rank 3 and therefore $\text{Im}(F) = R^3$,

i.e. F is surjective

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School of Education, Culture and Communication
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EXAMINATION IN MATHEMATICS
MAA153 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
Academic year: 2016/17

Examination 2017-08-16 — Maximum points for subparts of the problems in the final examination

1. $A = \begin{pmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix}$

------------------------------- **One scenario** ---

1p: Correctly noted that the matrix of the scalar product relative to the ON-basis is equal to the identity matrix

2p: Correctly noted that the matrix relative to the standard basis is equal to the (symmetric) matrix $(SS^T)^{-1}$, where S is the change-of-basis matrix from the standard basis to the ON-basis

2p: Correctly found the matrix $(SS^T)^{-1}$

---------------------------- **Another scenario** ---------------------------------------

2p: Correctly stated the three orthogonality and the three norm conditions for the ON-basis, i.e.

$$1 = \langle (1,1,0) | (1,1,0) \rangle = a_{11} + 2a_{12} + a_{22} ,$$
$$0 = \langle (1,1,0) | (1,0,1) \rangle = a_{11} + a_{12} + a_{13} + a_{23}$$

etc, where a_{mn} are the elements of A and where $a_{mn} = a_{nm}$ since an inner product for vector spaces over real numbers is symmetric in its arguments

3p: Correctly solved the system of equations, where the number of unknowns are $9 - 3 = 6$

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2. A basis of eigenvectors is e.g. $(1,0,-1)$, $(1,0,1)$, $(0,1,0)$, i.e. F is diagonalizable

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

3p: Correctly proved that F is diagonalizable, by e.g. finding a set of three linearly independent eigenvectors

2p: Correctly found an orthogonal change-of-basis matrix S

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3. $\dim(U \cap V) = \begin{cases} 0 & \text{if } \alpha = 2 \\ 1 & \text{if } \alpha \neq 2 \end{cases}$

If $\alpha \neq 2$, then a basis for $U \cap V$ is e.g. $(1,-1,-2,-2)$

1p: Correctly formulated an equation which answers the question of which vectors belong to the subspace $U \cap V$

2p: Correctly, from e.g. the reduced row-echelon form of the coordinate matrix of the vectors spanning U and V, concluded that $\alpha = 2$ means that $U \cap V$ equals the trivial subspace $\{0\}$ and therefore that $\dim(U \cap V) = 0$

2p: Correctly, from e.g. the reduced row-echelon form of the coordinate matrix of the vectors spanning U and V, concluded that $\alpha \neq 2$ means that $U \cap V$ equals $\operatorname{span}\{(1,-1,-2,-2)\}$ and therefore that $\dim(U \cap V) = 1$

4.	An orthonormal basis for M is e.g. $\frac{1}{\sqrt{6}}(-1,2,1,0)$, $\frac{1}{\sqrt{6}}(-1,0,-1,2)$	**2p**: Correctly found a basis u_1, u_2 for the vector space M **1p**: Correctly initiated a Gram-Schmidt process to find an orthonormal basis e_1, e_2, and correctly as a first step in the process normed u_1 to e_1 **2p**: Correctly continued with a second step in the G-S process by defining, evaluating and normalizing a nonzero vector $u_2 - e_1\langle e_1 \vert u_2\rangle$ orthogonal to e_1, ending up with e_2
5.	S_1 is an hyperbolic cylinder S_2 is a one-sheeted hyperboloid	**3p**: Correctly classified the first surface **2p**: Correctly classified the second surface
6.	The linear transformation F may have the matrix $\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 4 & -8 & 3 \\ 2 & 1 & 17 & 7 \end{pmatrix}$ relative to the standard basis	**2p**: Correctly concluded that the elements of two of the matrix A's four columns must be the coordinates of two (linearly independent) linear combinations of the vectors that span the image of F **1p**: Correctly concluded that the elements of the two other columns are given by the kernel of F, and correctly for the matrix A formulated the equations $\mathsf{A}(2\ -1\ 1\ 2)^T = (0\ 0\ 0\ 0)^T = \mathsf{A}(-3\ 2\ 1\ 3)^T$ **2p**: Correctly solved the equations for the kernel, and correctly summarized the expression for the matrix A
7.	Since the matrix S in the change-of-basis-relation $(p\ q\ r) = (p_1\ p_2\ p_3)\mathsf{S}$ is invertible, the functions p, q, r constitute a basis for the vector space P. The function $-p_1 + 5p_2 - 4p_3$ has the coordinates $(\frac{29}{7}, -\frac{32}{7}, -\frac{33}{7})$ relative to the basis p, q, r, i.e. $-p_1 + 5p_2 - 4p_3 = \frac{29}{7}p - \frac{32}{7}q - \frac{33}{7}r$	**1p**: Correctly proved thay p_1, p_2, p_3 is a basis for the vector space P **2p**: Correctly proved thay p, q, r is a basis for the vector space P **1p**: Correctly found the inverse of the matrix S **1p**: Correctly relative the basis p_1, p_2, p_3 found the coordinates of the function $-p_1 + 5p_2 - 4p_3$
8.	The matrix of F relative to the ordered bases for the domain and the codomain is equal to $\begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 3 & -1 & -1 \\ -3 & 0 & 4 & 2 \end{pmatrix}$. F is surjective	**3p**: Correctly identified the matrix of F relative to the ordered bases for the domain $\mathcal{M}$ and the codomain R^3 **2p**: Correctly concluded that F is surjective due to the fact that the rank of its matrix is equal to $\dim(R^3)$