MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
Department of Applied Mathematics
Examiner: Lars-Göran Larsson

EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus, TEN2
Date: 2017-01-09 *Write time:* 3 hours
Aid: Writing materials, ruler

This examination is intended for the examination part TEN2. The examination consists of five RANDOMLY ORDERED problems each of which is worth at maximum 4 points. The PASS-marks 3, 4 and 5 require a minimum of 9, 13 and 17 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 9, 10, 13, 16 and 20 respectively. If the obtained sum of points is denoted S_2, and that obtained at examination TEN1 S_1, the mark for a completed course is according to the following:

$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5
$S_1 \geq 11,\ S_2 \geq 9$	AND	$S_1 + 2S_2 \leq 32$	$\rightarrow$	E
$S_1 \geq 11,\ S_2 \geq 9$	AND	$33 \leq S_1 + 2S_2 \leq 41$	$\rightarrow$	D
$S_1 \geq 11,\ S_2 \geq 9$	AND	$42 \leq S_1 + 2S_2 \leq 51$	$\rightarrow$	C
		$52 \leq S_1 + 2S_2 \leq 60$	$\rightarrow$	B
		$61 \leq S_1 + 2S_2$	$\rightarrow$	A

Solutions are supposed to include rigorous justifications and clear answers. All sheets of solutions must be sorted in the order the problems are given in.

1. Find the solution of the initial-value problem

$$y'' - 4y' + 4y = 2e^{2x}, \quad y(0) = 1\,, \quad y'(0) = 4\,.$$

2. Find the volume of the solid generated by rotating about the y-axis the bounded region which is precisely enclosed by the curves $y = x^3$ och $y = x^4$.

3. Evaluate the integral

$$\int_{\pi/4}^{\arctan(e)} (1 + \tan^2(x)) \ln(\tan(x))\, dx\,,$$

and write the result in as simple form as possible.

4. Sketch the graph of the function f, defined by

$$f(x) = \frac{x^3}{x^2 - 1}\,,$$

by utilizing the guidance given by asymptotes and stationary points.

5. Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{7/10} + n^{18/25}}{n^{17/10} + n^{8/5}}$ convergent or divergent? Explain!

Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.

MÄLARDALENS HÖGSKOLA
Akademin för utbildning, kultur och kommunikation
Avdelningen för tillämpad matematik
Examinator: Lars-Göran Larsson

TENTAMEN I MATEMATIK
MAA151 Envariabelkalkyl, TEN2
Datum: 2017-01-09 *Skrivtid:* 3 timmar
Hjälpmedel: Skrivdon, linjal

Denna tentamen är avsedd för examinationsmomentet TEN2. Provet består av fem stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 4 poäng. För GODKÄND-betygen 3, 4 och 5 krävs erhållna poängsummor om minst 9, 13 respektive 17 poäng. Om den erhållna poängen benämns S_2, och den vid tentamen TEN1 erhållna S_1, bestäms graden av sammanfattningsbetyg på en slutförd kurs enligt följande:

$S_1 \geq 11,\ S_2 \geq 9$	OCH	$S_1 + 2S_2 \leq 41$	$\rightarrow$	3
$S_1 \geq 11,\ S_2 \geq 9$	OCH	$42 \leq S_1 + 2S_2 \leq 53$	$\rightarrow$	4
		$54 \leq S_1 + 2S_2$	$\rightarrow$	5

Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i.

1. Bestäm lösningen till begynnelsevärdesproblemet

$$y'' - 4y' + 4y = 2e^{2x}, \quad y(0) = 1\,, \quad y'(0) = 4\,.$$

2. Bestäm volymen av den kropp som genereras genom att kring y-axeln rotera det begränsade område som precis innesluts av kurvorna $y = x^3$ och $y = x^4$.

3. Beräkna integralen

$$\int_{\pi/4}^{\arctan(e)} (1 + \tan^2(x))\ln(\tan(x))\,dx\,,$$

och skriv resultatet på en så enkel form som möjligt.

4. Skissa grafen till funktionen f, definierad enligt

$$f(x) = \frac{x^3}{x^2 - 1}\,,$$

genom att använda den vägledning som fås från asymptoter och stationära punkter.

5. Är serien $\displaystyle\sum_{n=1}^{\infty} \frac{n^{7/10} + n^{18/25}}{n^{17/10} + n^{8/5}}$ konvergent eller divergent? Förklara!

If you prefer the problems formulated in English, please turn the page.

MAA151 / Solutions to the final exam TEN2 2017-01-09

1 DE: $y''-4y'+4y=2e^{2x}$ IV: $y(0)=1$, $y'(0)=4$

The auxiliary equation of the corresponding homogeneous DE is $0=r^2-4r+4=(r-2)^2$. Thus $y_H=(A+Bx)e^{2x}$.

For a particular solution, $y_p=x^2a_0e^{2x}$ works.

We get $y_p'=a_0(2x+2x^2)e^{2x}$, $y_p''=a_0(2+8x+4x^2)e^{2x}$

Subst. of y_p into the DE gives

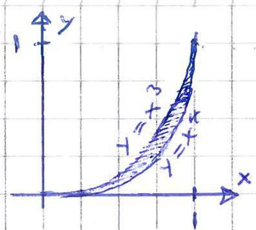
$$a_0\left[2+\cancel{8x}+\cancel{4x^2}-4(\cancel{2x}+\cancel{2x^2})+\cancel{4x^2}\right]e^{2x}=2e^{2x} \iff a_0=1$$

Thus $y=y_H+y_p=(A+Bx+x^2)e^{2x}$ is the general solution of the DE. An adaption to the IV:s gives

$$\begin{cases}1=y(0)=(A+0+0)e^0\\4=y'(0)=(B+0+2A+0+0)e^0\end{cases} \iff \begin{cases}A=1\\B=2\end{cases}$$

The solution of the IVP is thus $\underline{y=(x+1)^2e^{2x}}$

2



The bounded region precisely enclosed by the curves $y=x^3$ and $y=x^4$ is rotated about the y-axis generating a solid. The volume of that solid is denoted V_y.

By the method of slicing, we get

$$V_y=\int_0^1\pi\left(y^{\frac{1}{4}}\right)^2dy-\int_0^1\pi\left(y^{\frac{1}{3}}\right)^2dy=\pi\left[\frac{2}{3}y^{3/2}-\frac{3}{5}y^{5/3}\right]_0^1$$
$$=\pi\left[\left(\frac{2}{3}-\frac{3}{5}\right)-(0-0)\right]=\frac{\pi}{15}$$

By the method of cylindrical shells, we get

$$V_y=\int_0^1 2\pi x\left(x^3-x^4\right)dx=2\pi\left[\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^1=2\pi\left[\left(\frac{1}{5}-\frac{1}{6}\right)-(0-0)\right]=\frac{\pi}{15}$$

Answer: $\underline{\frac{\pi}{15}}$ v.u.

3

$$\int_{\pi/4}^{\arctan(e)}\left(1+\tan^2(x)\right)\ln(\tan(x))\,dx \quad \left[\begin{array}{l}\tan(x)=u\\(1+\tan^2(x))\,dx\overset{*}{=}du\end{array}\right]$$

$$=\int_1^e\ln(u)\,du=\left[u\ln(u)\right]_1^e-\int_1^e u\cdot\frac{1}{u}\,du$$

$$=\left[u(\ln(u)-1)\right]_1^e=e(1-1)-1(0-1)=\underline{1}$$

(* Notice that $\frac{d}{dx}\tan(x)=\frac{1}{\cos^2(x)}=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}=1+\tan^2(x)$)

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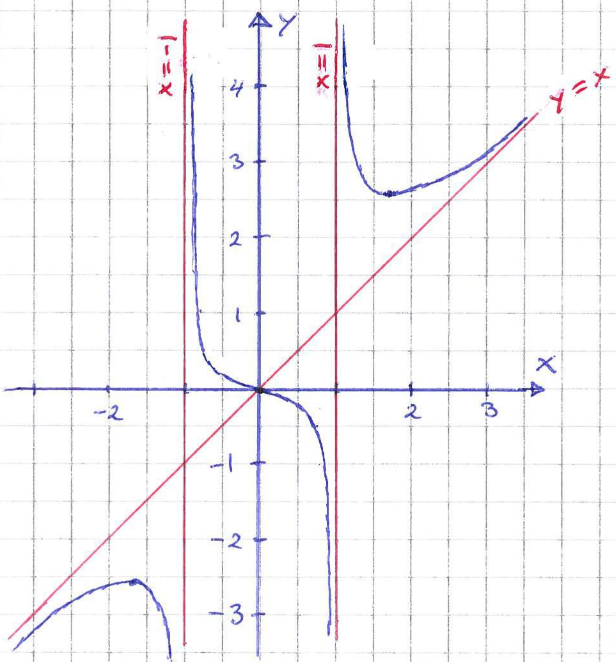
4 $f(x) = \dfrac{x^3}{x^2-1}$ Then $f'(x) = \dfrac{3x^2(x^2-1) - x^3 \cdot 2x}{(x^2-1)^2} = \dfrac{x^2(x+\sqrt{3})(x-\sqrt{3})}{(x+1)^2(x-1)^2}$

A first-derivative test is

x		SP $-\sqrt{3}$		AS -1		SP 0		AS 1		SP $\sqrt{3}$	
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$-$	$\nexists$	$-$	0	$+$
$f(x)$	$\nearrow$	loc max	$\searrow$	$\nexists$	$\searrow$	terr.	$\searrow$	$\nexists$	$\searrow$	loc min	$\nearrow$

$$\begin{cases} x=-1 \text{ is a double-sided vertical asymptote since } f(x)\to\mp\infty \text{ as } x\to(-1)^{\mp} \\ x=1 \text{ —— '' —— —''— —''— —''— } f(x)\to\mp\infty \text{ as } x\to 1^{\mp} \\ y=x \text{ —— '' —— non-vertical —''— —''— } f(x) = x + \frac{x}{x^2-1} \end{cases}$$



$$\begin{cases} f(\pm\sqrt{3}) = \dfrac{\pm 3\sqrt{3}}{3-1} = \pm\dfrac{3}{2}\sqrt{3} \\ f(0) = 0 \end{cases}$$

5 $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{n^{7/10} + n^{18/25}}{n^{17/10} + n^{8/5}} = \dfrac{n^{\frac{18}{25}}\left(n^{\frac{35-36}{50}} + 1\right)}{n^{17/10}\left(1 + n^{\frac{16-17}{10}}\right)}$

$$= \frac{1}{n^{\frac{85-36}{50}}}\left(\frac{1+n^{-\frac{1}{50}}}{1+n^{-\frac{1}{10}}}\right) = \frac{1}{n^{49/50}}\left(\frac{1+n^{-1/50}}{1+n^{-1/10}}\right)$$

Since
$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{a_n}{\left(\frac{1}{n^{49/50}}\right)} = \lim\limits_{n\to\infty}\left(\dfrac{1+n^{-1/50}}{1+n^{-1/10}}\right) = \dfrac{1+0}{1+0} = 1 > 0, \\ \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{49/50}} \text{ is divergent according to the integral test,} \end{cases}$$

the comparison test gives that even the series $\sum_{n=1}^{\infty} a_n$ diverges.

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EXAMINATION IN MATHEMATICS
MAA151 Single Variable Calculus
EVALUATION PRINCIPLES with POINT RANGES
Academic Year: 2016/17

Examination TEN2 – 2017-01-09 | Maximum points for subparts of the problems in the final examination

1. $y=(x+1)^2 e^{2x}$

1p: Correctly identified the differential equation as a non-homogeneous linear DE of second order, and correctly found the general solution y_h of the corr. homog. DE
1p: Correctly proposed a formula for a part. sol. of the DE
1p: Correctly found a particular solution of the DE, and correctly summarized the general solution of the DE
1p: Correctly adapted the general solution of the DE to the IV

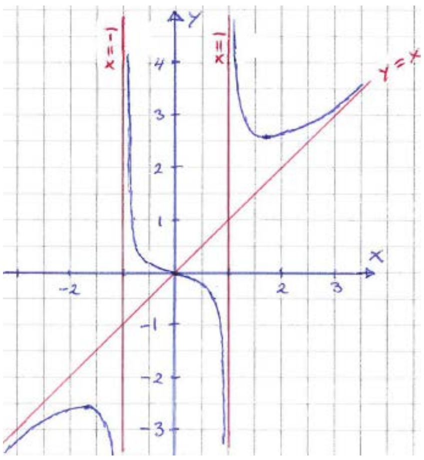
2. $\dfrac{\pi}{15}$ v.u.

2p: Correctly formulated an integral for the volume of the solid obtained by rotating the region about the *y*-axis (irrespective whether the method of slicing or the method of cylindrical shells have been applied)
1p: Correctly found an antiderivative of the integrand
1p: Correctly evaluated the antiderivative at the limits

3. 1

2p: Correctly by the substitution $\tan(x)=u$ translated the integral into $\int_1^e \ln(u)\,du$
1p: Correctly found an antiderivative of the integrand
1p: Correctly evaluated the antiderivative at the limits

4. The graph has a local maximum at $P_1:(-\sqrt{3},-\frac{3\sqrt{3}}{2})$, a terrace at $P_2:(0,0)$, and a local minimum at $P_3:(\sqrt{3},\frac{3\sqrt{3}}{2})$. Asymptotes are $x=-1$, $x=1$, $y=x$



1p: Correctly found the asymptotes of the graph
1p: Correctly classified and illustrated the stationary points of the function
1p: Correctly sketched the graph according to how the graph relates to the asymptotes on their both sides respectively
1p: Correctly completed the sketch of the graph

5. The series diverges

------------ **Another scenario** ------------------
The student who partly wrongly has found that the general term of the series is equal to $n^{-\alpha}B(n)$ where $B(n)\to 1$ as $n\to\infty$ but with $\alpha\neq 49/50$, has the possibility to get the points number 2–4 provided that the following conclusions are properly made on the basis of the resulting expression $n^{-\alpha}B(n)$

1p: Correctly found that the general term a_n of the series is equal to $n^{-49/50}B(n)$ where $B(n)\to 1$ as $n\to\infty$
1p: Correctly found that the comparison test is applicable, and that the series $\Sigma\, n^{-49/50}$ is the one to compare with
1p: Correctly noted that the series $\Sigma\, n^{-49/50}$ is divergent according to the integral test
1p: Correctly concluded by the comparison test that the series $\Sigma\, a_n$ is divergent since the series compared with is divergent