TEN1 2019-06-14
MMA130 Mathematical Logic for Computer Science
Duration: 3 hours
Tools: none
Attached: Collection of Formulas (4 pages)
Passing grade requires 15 p or more

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Questions may be answered in English or Swedish. Solutions should be presented in such a way that reasoning and calculations are easy to follow. All answers must be supported by an argument, e.g., if you claim something has a particular property then you must also show that this is the case.

1 In your own words,

a explain the purpose of Gödel numbers, (1 p)

b describe in detail some system of Gödel numbers for formulae (there are several such systems described in the literature, which differ in details but generally serve the same purpose; you do not need to reproduce any particular system, but your system must support encoding the formulae of Peano arithmetic), (3 p)

c explain the role played by recursive functions in the proof of Gödel's incompleteness theorem. (1 p)

d Encode the formula $\forall x \neg (Sx = 0)$ (first axiom of Peano arithmetic: 0 is not the successor of any natural number) using the Gödel number system you described above. (1 p)

2 Give a natural deduction proof that $p \wedge q, \neg r \to \neg p \vdash r$. Provide justifications of all steps. (6 p)

3 Russell's paradox in set theory starts with considering the set R of all sets that do not contain themselves as elements.

a Using set membership $\in$ as only predicate, write down a predicate logic formula expressing the claim that R is the set of all sets that do not contain themselves. (For simplicity, assume that everything is a set.) (2 p)

b Write down a proof, with justification of all steps, that the Russell set R does not exist. Hint: Consider the matter of whether $R \in R$. (4 p)

4 Consider the Kripke model $M = (W, R, L)$ where $W = \{a, b, c, d\}$, $L(a) = \{p\}$, $L(b) = \varnothing$, $L(c) = \{q\}$, $L(d) = \{p, q\}$, and $R = \bigl\{(a, d), (b, a), (b, b), (c, a), (d, b), (d, c)\bigr\}$.

a Draw a graph for M. (2 p)

b Determine the set of worlds where $p \to q$ is satisfied. (1 p)

c Determine the set of worlds where $\Diamond(p \to q)$ is satisfied. (1 p)

d Determine the set of worlds where $\Box p \to q$ is satisfied. (1 p)

e Determine the set of worlds where $\mathrm{F}p$ is satisfied. (1 p)

5 Check the validity in the non-classical logic L_2 (see attached pages) of the following statements.

a $\models p \to \neg\neg p$ (1 p)

b $\models \neg\neg p \to p$ (1 p)

c $\models p \vee \neg p$ (1 p)

d $p \models \neg p \to \bot$. (1 p)

e $\models (p \wedge \neg p) \to \bot$. (1 p)

f $p \to r, q \to r \models (p \vee q) \to r$. (1 p)

Good luck!

MÄLARDALEN UNIVERSITY
School of Education, Culture
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MMA130, Mathematical Logic for Computer Science

Collection of Formulas

Anna Fedyszak-Koszela

November 10, 2015

1 Natural deduction - basic rules

1. $(\wedge\ i)$: $\dfrac{\phi \qquad\qquad \psi}{\phi \wedge \psi}$

2. $(\wedge\ e)$: $\dfrac{\phi \wedge \psi}{\phi}\ ,\ \dfrac{\phi \wedge \psi}{\psi}$

3. $(\vee\ i)$: $\dfrac{\phi}{\phi \vee \psi}\ ,\ \dfrac{\psi}{\phi \vee \psi}$

4. $(\vee\ e)$: $$\frac{\phi \vee \psi \quad \begin{array}{c} \lceil\ \phi\ \rceil \\ |\ \vdots\ | \\ \lfloor\ \chi\ \rfloor \end{array} \quad \begin{array}{c} \lceil\ \psi\ \rceil \\ |\ \vdots\ | \\ \lfloor\ \chi\ \rfloor \end{array}}{\chi}$$

5. $(\rightarrow i)$: $$\frac{\begin{array}{c} \lceil\ \phi\ \rceil \\ |\ \vdots\ | \\ \lfloor\ \psi\ \rfloor \end{array}}{\phi \rightarrow \psi}$$

6. $(\rightarrow e)$: $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$

7. $(\neg\ i)$: $$\frac{\begin{array}{c} \lceil\ \phi\ \rceil \\ |\ \vdots\ | \\ \lfloor\ \bot\ \rfloor \end{array}}{\neg\phi}$$

8. $(\neg\ e)$: $\dfrac{\phi \quad \neg\phi}{\bot}$

9. $(\bot\ e)$: $\dfrac{\bot}{\phi}$

10. $(\neg\neg\ e)$: $\dfrac{\neg\neg\phi}{\phi}$

11. (MT): $\dfrac{\phi \to \psi \quad \neg\psi}{\neg\phi}$

12. $(\neg\neg\ i)$: $\dfrac{\phi}{\neg\neg\phi}$

13. (PBC): $\dfrac{\begin{bmatrix} \neg\phi \\ \vdots \\ \bot \end{bmatrix}}{\phi}$

14. (LEM): $\dfrac{}{\phi \vee \neg\phi}$

15. $(copy)$: $\dfrac{\phi}{\phi}$

16. $(=\ i)$: $\dfrac{}{t = t}$, for any term t

17. *Principle of Substitution* $(=\ e)$:

$$\frac{t_1 = t_2 \qquad \phi\,[t_1/x]}{\phi\,[t_2/x]}\ ,$$

for t_1 free for x in ϕ, and for t_2 free for x in ϕ; all occurrences of t_1 in $\phi\,[t_1/x]$ are replaced by t_2

18. $(\forall x\ i)$:

$$\frac{\begin{bmatrix} x_0 & \\ & \phi\,[x_0/x] \end{bmatrix}}{\forall x \phi}\ ,$$

for x_0 - new, doesn't occur anywhere outside its box,
for x_0 - not free in open P before its box

19. $(\forall x\ e)$: $\dfrac{\forall x \phi}{\phi\,[t/x]}$,

    for any term t free for x in ϕ; all free occurrences of x in ϕ are replaced by t

20. $(\exists x\ i)$: $\dfrac{\phi\,[t/x]}{\exists x \phi}$,

    for some term t free for x in ϕ

21. $(\exists x\ e)$:

$$\dfrac{\exists x\phi, \quad \begin{array}{llll} \lceil\, x_0 & \phi\,[x_0/x] & \rceil & 1,\ P \\ | & \vdots & | & \\ \lfloor & \chi & \rfloor & 2 \end{array}}{\chi}$$

    for x_0 - new, doesn't occur anywhere outside its box;

    1: x_0 not free in open P before its box,

    2: x_0 not free in χ

2 Modal logic

- Let $M = (W, R, L)$ be a Kripke model of basic modal logic, $x \in W$, and ϕ be a formula. We will define when formula ϕ is true in the world x. This is done via a satisfaction relation $x \Vdash \phi$ by structural induction on ϕ:

    1. $x \Vdash T$
    2. *not* $(x \Vdash \bot)$
    3. $x \Vdash p$ iff $p \in L(x)$
    4. $x \Vdash \neg\phi$ iff *not* $(x \Vdash \phi)$
    5. $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
    6. $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
    7. $x \Vdash \phi \rightarrow \psi$ iff $x \Vdash \psi$, whenever we have $x \Vdash \phi$
    8. $x \Vdash \phi \leftrightarrow \psi$ iff $(x \Vdash \phi$ iff $x \Vdash \psi)$
    9. $x \Vdash \Box\psi$ iff, for each $y \in W$ with $R(x, y)$, we have $y \Vdash \psi$
    10. $x \Vdash \Diamond\psi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \psi$

    When $x \Vdash \phi$ holds, we say 'x satisfies ϕ' or 'ϕ is true in world x'.

- A model $M = (W, R, L)$ of basic modal logic is said to satisfy a formula ϕ if every state (world) x in the model satisfies it.
  We write: $M \models \phi$ iff, for each $x \in W$, $x \Vdash \phi$.

3 Logic systems with multiple truth values

- Let L be a formal language without non-logical symbols and S_L - the set with all sentences in L. For every positive integer n we define the non-classical logic system $L_n = (L, V_n)$ with $n+1$ truth values and with the valuation $V_n : S_L \longrightarrow \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ by following 5 conditions:

  1. $V_n(\neg\phi) = 1 - V_n(\phi)$ for any sentence $\phi \in S_L$
  2. $V_n(\phi \wedge \psi) = \min \{V_n(\phi), V_n(\psi)\}$ for any two sentences $\phi, \psi \in S_L$
  3. $V_n(\phi \vee \psi) = \max \{V_n(\phi), V_n(\psi)\}$ for any two sentences $\phi, \psi \in S_L$
  4. $V_n(\phi \to \psi) = \min \{1, (1 - V_n(\phi) + V_n(\psi))\}$ for any two sentences $\phi, \psi \in S_L$
  5. $V_n(\phi \leftrightarrow \psi) = 1 - |V_n(\phi) - V_n(\psi)|$ for any two sentences $\phi, \psi \in S_L$

- L_2 is a non-classical logic with 3 truth values $\{0, \frac{1}{2}, 1\}$:

ϕ	φ	$\neg\phi$	$\phi \wedge \varphi$	$\phi \vee \varphi$	$\phi \to \varphi$
$x = V(\phi)$	$y = V(\varphi)$	$1 - x$	$min\{x, y\}$	$max\{x, y\}$	$min\{1, 1 - x + y\}$
1	1	0	1	1	1
1	1/2	0	1/2	1	1/2
1	0	0	0	1	0
1/2	1	1/2	1/2	1	1
1/2	1/2	1/2	1/2	1/2	1
1/2	0	1/2	0	1/2	1/2
0	1	1	0	1	1
0	1/2	1	0	1/2	1
0	0	1	0	0	1