More on Determining Complexity

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September 9, 2025

This document refers to Q III of Assignment 1.

We've shown that $\sqrt{n\log(n)}$ grows faster than $n^{\frac{1}{3}}\log^2(n^2)$ using L'Hôpital's rule. For large enough n, the sum of the two functions, $T(n) = n^{\frac{1}{3}}\log^2(n^2) + \sqrt{n\log(n)}$, should then approach $\sqrt{n\log(n)}$ because the value of the other term, $n^{\frac{1}{3}}\log^2(n^2)$, becomes insignificant in comparison.

$$T(n) \sim \sqrt{n\log(n)}$$

or we could also write

$$\lim_{n\to\infty} \frac{T(n)}{\sqrt{n\log(n)}} = 1 \tag{1}$$

Since $1 < 2$, there exists some N_1 such that

$$\frac{T(n)}{\sqrt{n\log(n)}} \leq 2, \forall n \geq N_1$$

Multiplying both sides of the inequality by $\sqrt{n\log(n)}$ yields

$$T(n) \leq 2\sqrt{n\log(n)}, \forall n \geq N_1 \tag{2}$$

We note that equation (2) resembles the definition of $O()$. Thus

$$\boxed{T(n) \text{ is } O\left(\sqrt{n\log(n)}\right)}$$

Referring back to equation (1), since $\frac{1}{2} < 1$, then there exists some N_2 such that

$$\frac{T(n)}{\sqrt{n\log(n)}} \geq \frac{1}{2}, \forall n \geq N_2$$

Multiplying both sides of the inequality by $\sqrt{n\log(n)}$ yields

$$T(n) \geq \frac{1}{2}\sqrt{n\log(n)}, \forall n \geq N_2 \tag{3}$$

We note that equation (3) resembles the definition of $\Omega()$. Thus

$$\boxed{T(n) \text{ is } \Omega\left(\sqrt{n\log(n)}\right)}$$

Since $T(n)$ is both $O\left(\sqrt{n\log(n)}\right)$ and $\Omega\left(\sqrt{n\log(n)}\right)$, we finally conclude that

$$\boxed{T(n) \text{ is } \Theta\left(\sqrt{n\log(n)}\right)}$$

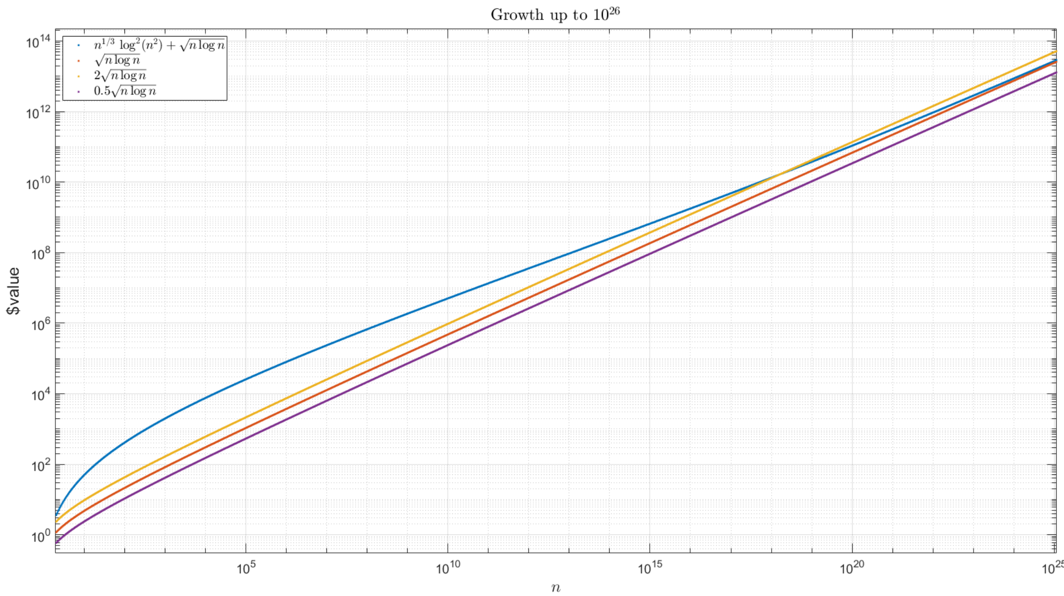

Figure 1: $T(n)$, represented by the blue line, approaches $\sqrt{n \log(n)}$, represented by the orange line. Also, $T(n)$ is bounded between the two multiples of $\sqrt{n \log(n)}$.