Game Theory And Its Applications

Sambit Behera

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1 Preface

Welcome to my Summer Of Science report on Game Theory And Its Applications. This is an attempt to summarise all the sub-topics I cover in this journey. I would mostly follow the book **An Introduction To Game Theory** by **Levent Koçkesen** and **Efe A. Ok**. Also, I would be referring to Game Theory course on Coursera for additional knowledge. I hope this report would be helpful to gain a decent understanding of Game Theory.

2 Introduction

2.1 Game Theory

In a game as simple as Rock, Paper, Scissors, when two people play their respective moves at a time, they hope to play the move that is in favour of them and eventually win the game. The strategy applied by both the users, like looking for a pattern in their opponents' previous moves to predict their next move, is what makes up a part of this topic. Largely popularised by the famous movie *A Beautiful Mind*, based on the life of Nobel winning laureate **John Nash**, Game Theory is essentially the science of strategy, or at least the optimal decision-making of independent and competing actors in a strategic setting.

2.2 Basic Terminologies

- **Players:**
  The strategic decison-makers in the context of the game. These can be as small as individuals and as large as governments or multi-national companies.
- **Actions:**
  These are the choices available to the player from which she has to choose.
- **Payoff:**
  Sounds like a reward, it acts as a motivating factor behind the actions of the players and the reason for their participation.
- **Rational:**
  An individual is considered *rational* if she has well defined objectives (or preferences) over the set of possible outcomes and she implements the best available *strategy* to pursue them. In reality, assumption of rationality might be an unrealistic one. These limitations is what gives birth to the concept of *bounded rationality* which is an active area of research currently.
- **Strategy:**
  A proper set of action plans chosen by a player in a certain setting, whose outcome depends not only on her action, but on others' too.
- **Rules:**
  A set of statements that clarifies, demarcates and/or interprets the proceedings of a game.
- **Utility Function:**
  It is a mathematical measure that tells how much a player likes or does not like a given situation. It describes not only their attitude towards a definitive event, but also describe the preferences towards a distribution of such outcomes.
- **Common Knowledge:**
  As we consider all players in the game to be *rational*, everyone of the players knows about the model, everybody knows that everybody knows about the model, everybody knows that everybody knows that everybody knows it, and so on.

- **Best Response:**
  It is the best strategic response that a player makes according to others' strategies in order to achieve maximum payoff possible.

2.3 Defining Games

There's basically two standard representations of a game:

1. Normal Form (or Matrix Form, Strategic form)
2. Extensive Form (*we will talk about this topic later*)

Normal Form

It lists what players receive as payoffs as a function of their actions. Actions in these games are considered to be simultaneous.

Key Ingredients:

- Players: $N = \{1, \ldots, n\}$ is a finite set of n, indexed by i
- Action Set for Player $i = A_i$
  $a = (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ is an action profile
- Utility function or Payoff Function for player $i : u_i : A \mapsto \mathbb{R}$
  $u = \{u_1, \ldots, u_n\}$ is a profile of utility functions.

2.4 Few Examples

Two Player Game as a *matrix*:

The "row" player is considered Player 1 and "column" player as Player 2. Rows correspond to actions $a_1 \in A_1$ and columns to $a_2 \in A_2$. We list the payoff values for both the players as a *tuple* in the cells, the row player being first and then the column player.

So, as a basic example, we consider a simple Rock, Paper, Scissors game. In a matrix form, it would look like this:

1/2	rock	paper	scissors
rock	0,0	0,1	1,0
paper	1,0	0,0	0,1
scissors	0,1	1,0	0,0

- The above matrix, along with the actions and payoffs is called a *game*.
- In this game, available actions are rock, paper and scissors
- The payoff is the point you will score after each move is played, $eg : u_1(paper, scissors) = 0$
- Here, the common knowledge is that everyone is aware of the moves and their respective outcomes.

A Large Collective Action Game

If there are large number of players, we cannot draw up a table and find out the payoffs for each of them. Let's consider the following scenario-
We consider a city of population of 10 *million* people. Some of them are unhappy about the government and organize a revolt on a sunny afternoon on a certain Friday.

- Players: $N = \{1, \dots, 10,000,000\}$
- Action Set for player i $A_i = \{Revolt, Abstain\}$
- Utility function for player i
  - $u_i(a) = 1$ if $\#\{j : a_j = Revolt\} \geq 2,000,000$
  - $u_i(a) = -1$ if $\#\{j : a_j = Revolt\} < 2,000,000$ and $a_i = Revolt$
  - $u_i(a) = 0$ if $\#\{j : a_j = Revolt\} < 2,000,000$ and $a_i = Abstain$
- Any player i has an option to *Revolt* or *Abstain* on the day of the revolution.
- A revolution would be considered successful if there are atleast 2*million* people revolting that day
- If the revolt is successful then each person gets a payoff value of 1.
- If the revolt is unsuccessful, and the player had participated in it, then there would be a capital punishment by the dictator and the payoff value for that individual would be -1
- If the player did not participate in the unsuccessful revolt, then the payoff value would be 0.

Games of Cooperation

Players should not always have opposing interests to each other and might even cooperate for their best response. We say that the players have *exactly the same* interests if:

$$\forall a \in A, \forall i, j, u_i(a) = u_j(a)$$

Coordination Game

Let's consider a situation where we want to know which side of the road we should drive on. Consider the following matrix:

1/2	Left	Right
Left	1,1	-1,-1
Right	-1,-1	1,1

Hence, we observe that if both the drivers going in the opposite direction choose the same side, then they can avoid a collision and hence have a payoff of 1.

Battle Of The Sexes

A game does not always have to be completely competitive or coordinative. The most interesting games are the ones that combine the elements of both.

Consider a couple who want to go to watch a movie. They have two options: Movie A or Movie B. The Husband wants to watch A and the Wife, B. But most importantly, they want to go together or else no one's happy. Hence the following matrix:

Husband/Wife	Movie A	Movie B
Movie A	2,1	0,0
Movie B	0,0	1,2

Here, if they watch movie A, then the Husband gets to watch his favourite movie and also be together, hence a payoff of 2, and wife 1. Similar reasoning holds if they watch movie B. But, if they watch different movies, then both are unhappy and hence have a payoff of 0.

3 Dominant Strategies and Equilibrium

In certain situations, a player may have strategic options to choose for his best move, and each move would have its corresponding payoff. As the player is considered *rational*, he would prefer the move with the best possible payoff. Hence, this move is said to *dominate* over the others. There are various forms of dominance.

3.1 Strongly Dominant Strategy

Let s_i and s_i' be two strategies for player i, and let S_{-i} be the set of all possible strategy profile for the other players, then s_i is said to *strongly dominate* over s_i' if,

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \forall s_{-i} \in S_{-i}$$

Hence, it is obvious that player i will always prefer strategy s_i over s_i'.

3.2 Very Weakly Dominant Strategy

We say that strategy s_i is *weakly dominant* over s_i' if,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \forall s_{-i} \in S_{-i}$$

Here, the dominance is not strict.
There also are other kinds of dominant strategies that lie between the above two extremes., which we would talk about later in the report.

3.3 Dominance and Equilibrium

3.3.1 Strongly Dominant Strategy Equilibrium

A strategy profile $(s_1^*, \ldots, s_n^*)$ is called a *strongly dominant strategy equilibrium* of a game, if $\forall i = 1, 2, \ldots, n$ the strategy s_i^* is a strongly dominant strategy for player i.

3.3.2 Very Weakly Dominant Strategy Equilibrium

A strategy profile $(s_1^*, \ldots, s_n^*)$ is called a *very weakly dominant strategy equilibrium* of a game, if $\forall i = 1, 2, \ldots, n$ the strategy s_i^* is a very weakly dominant strategy for player i.

4 Nash Equilibrium

4.1 Definition

It is a stable state of a game involving the interaction of different participants, in which no participant can gain by a unilateral change of strategy if the strategies of the others remain unchanged.

Nash Equilibrium of a game G in strategic form is defined as any outcome $(a_1^*, \dots, a_n^*)$ such that

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i$$

holds for each player i. The set of all Nash equilibria of G is denoted by $N(G)$.

4.2 Best Response

We define the *Best Response correspondence*[1] of a player i in a strategic form game as the correspondence$B_i : A_{-i} \rightrightarrows A_i$ given by

$$B_i(a_{-i}) = \{a_i \in a_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \ \ \forall b_i \in A_i\}$$

4.3 Pure Strategy Nash Equilibrium

Given a normal form game $\mathcal{T} = \langle \mathcal{N}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, the action profile $A^* = (a_1^*, \dots, a_n^*)$ is called a *pure strategy Nash Equilibrium of* $\mathcal{T}$ *if* $\forall i, a_i^*$ belongs to the set of *best response* of a_{-i}^*.

In words, we can say that this Nash Equilibrium strategy is the best response to the Nash Equilibrium strategies of the other players.

Application on Some Example Games

Prisoners' Dilemma

Consider the following story:
Two suspects are arrested and put into different cells before the trial. The district attorney, who is pretty sure that both of the suspects are guilty but lacks enough evidence, offers them the following deal: If both of them confess and implicate the other (labeled C), then each will be sentenced to, say, 3 years of prison time. If one confesses and the other does not (labeled N), then the "rat" goes free for his cooperation with the authorities and the non-confessor is sentenced to 4 years of prison time. Finally, if neither of them confesses, then both suspects get to serve 1 year.

Now, Let's analyse this case. The corresponding matrix will be-

1/2	C	N
C	-3,-3	0,-4
N	-4,0	-1,-1

- First consider that the suspect 2 chooses to confess and implicate the other *ie* chooses C. Then, the best response for suspect 1 would be to confess rather than not, as it has lesser prison time.

[1]By definition, a *correspondence* f from A to B assigns to each $x \in A$ a *subset* of B, and hence we write $f : A \rightrightarrows B$

- If we consider suspect 2 to not confess, then suspect 1 has the option to "rat" out 1 and walk free. We can consider similar cases for suspect 2.

We see that no matter what the other suspect does, it is in the best interest of each suspect to rat out the other as this is the best response in any situation. Hence, we can say that choosing C is the *strongly dominant strategy.* Hence, we can conclude that CC is the *pure strategy Nash Equilibrium* here.

Let's consider the **Coordination Game** described in **section 2.4**. Here's the matrix for it-

1/2	Left	Right
Left	1,1	-1,-1
Right	-1,-1	1,1

We can see that we have *two Nash equilibria* here. If one of the drivers goes to the left, it's the best response to go to the left. And conversely, if the the other driver goes to the right, then the first driver is best off going to the right as well. And the others are not Nash equilibria.

4.4 Pareto Dominance and Optimality

Till now, we have been considering the whole game scenario as a participant in it. But there should be a proper approach from the point of view of an outside observer who has no obligation to any of the players. The observer might prefer a certain outcome as kind of a social good of the participants.

Sometimes one outcome o is atleast as good for every agent as another outcome o', and there is some agent who *strictly* prefers o to o'. In this case, it is reasonable to say that o is better than o'.

We say that o *Pareto-dominates* o'

Pareto optimality

An outcome o^* is **Pareto-optimal** if there is no other outcome that *Pareto-dominates* it.

Proposition 1: It is possible for a game to have more than one Pareto-optimal outcome.

Proof: A game might have exactly same payoff values that are also Pareto-dominant over others for different action combinations, hence none of them is Pareto-dominant over each other and both are Pareto-optimal.

Proposition 2: Every game has atleast one Pareto-optimal outcome.

Proof: For some outcome to not be Pareto-optimal it has to be dominated by some other outcome. And if it is dominated by some outcome, then the dominant outcome would be considered for Pareto-optimality and so goes on the cycle. Hence, a game has to have a Pareto-optimal outcome.

Matching Pennies

Let's consider a new example game called *Matching Pennies.* . It is played between two players, Even and Odd. Each player has a penny and must secretly turn the penny to heads or tails. The

players then reveal their choices simultaneously. If the pennies match (both heads or both tails), then Even keeps both pennies and hence his payoff is $+1$ and it is -1 for Odd. But if the pennies do not match, then the payoff is $+1$ for Odd and -1 for Even.
Here's the following matrix for it-

Even/Odd	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

Here, we can see that in search of best response, we are lead into a cycle. Hence, there is no pure strategy Nash Equilibrium. Now, if we consider Pareto Dominance, no move dominates another move, hence *by definition*, every strategy is a *Pareto Optimal* strategy.

5 Mixed Strategies and Nash Equilibrium

5.1 Mixed Strategy

Consider a scenario, where there is a conflict between the Mafia and the Army. The Mafia wants to attack one of several trade routes randomly for loot, but would lose if the Army is guarding it. If the Army plays a fixed strategy to guard only certain routes (as we do in pure strategy equilibrium) then the Mafia will be able to understand the plan and attack the ones not being guarded at a particular time. Hence, the Army has to play mixed strategy to make it difficult for the Mafia to figure out a way to loot.

Let's reconsider the *Matching Pennies* game in the previous section. It would be a pretty bad idea to play any deterministic strategy in this game. So, a basic approach would be to confuse the opponent by playing randomly. In this way, there is no sure approach and hence we have a *positive probability* for each action being played.

So, we re-define a **strategy** s_i for agent i as any probability distribution over the actions A_i-

- **pure strategy:** only one action is played with positive probability.
- **mixed strategy:** more than one action is being played with positive probability
  - These actions are called the **support** of the mixed strategy

If the set of all *strategies* for i be S_i, then the set of *all strategy profiles* will be $S = S_1 \times \ldots \times S_n$

Payoff/Utility under Mixed strategies

If all the players follow mixed strategy profile $s \in S$, the payoff won't be as simple as reading it from the game matrix. Instead, we would use the idea of *expected utility* as follows:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

where,

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Here, $u_i(s)$ is the utility of a player i who played the strategy profile s and Pr is the probability that we get to an action a given we choose strategy profile s.

5.2 Best Response and Nash Equilibrium

Previously in **section 4**, we used actions to determine the best response and hence Nash Equilibrium. Now, we can generalise the definition to use strategies to determine the best response.

Best Response:

$$s_i^* \in BR(s_{-i}) \ iff \ \forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

Hence, there we can be multiple best responses.

Nash Equilibrium

$$s = \langle s_1, \ldots, s_n \rangle \textit{ is a Nash Equilibrium iff } \forall i, s \in BRs_{-i}$$

If we reconsider the *Matching Pennies* game, we had no pure strategy Nash Equilibrium. But, if consider a probabilistic approach to the choice of Head or Tail, *ie* a 0.5 probability for both, then we have a mixed strategy Nash Equilibrium.

5.3 Computing Mixed Nash Equilibrium

A mixed strategy profile is a Nash Equilibrium if and only if the utility of the profile is same for both players and is greater than all other possible utilities for different actions.

Let's consider the *Battle Of The Sexes* game in section 2.4.

Husband/Wife	Movie A	Movie B
Movie A	2,1	0,0
Movie B	0,0	1,2

Nash equilibrium is achieved if one player sets his probability such that the options for other player become indifferent, *ie* any of the actions will give equal payoff to the second player. We can ensure that by solving a linear equation to find the corresponding probability.

Assume the Wife opts for Movie A with probability p and B with probability $1-p$. Then,

$$\begin{aligned} u_1(A, s_2^*) &= 2(p) + 0(1-p) = 2p && (1)\\ u_1(B, s_2^*) &= 0(p) + 1(1-p) = 1-p && (2)\\ u_1(B, s_2^*) &= u_1(A, s_2^*) && (3)\\ 2p &= 1-p && (4)\\ p &= \frac{1}{3} && (5) \end{aligned}$$

Similarly, to make the options for wife indifferent, we have to find the probability q with which Husband chooses the movie A

$$\begin{aligned} u_2(A, s_1^*) &= 1(q) + 0(1-q) = 1q && (6)\\ u_2(B, s_1^*) &= 0(q) + 2(1-q) = 2-2q && (7)\\ u_2(B, s_1^*) &= u_2(A, s_1^*) && (8)\\ q &= 2-2q && (9)\\ q &= \frac{2}{3} && (10) \end{aligned}$$

Hence, the strategy profile$\left(\left(\frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}\right)\right)$ results in Nash Equilibrium.

5.4 Theorem - Nash, 1950

Every finite game has a Nash Equilibrium