CS 228 : Logic in Computer Science

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First-Order Logic : Syntax

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- All words starting with the letter a, ending with the letter b, have even length

Signatures

- A vocabulary or signature τ is a set consisting of
  - constants $c_1, c_2, \ldots$
  - Relation symbols $R_1, R_2 \ldots$, each with some arity k, denoted R_i^k
- We look at finite signatures
- $\tau = (E^2, F^3)$ is a finite signature with two relations, E with arity 2 and F with arity 3

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- The symbols (and) called paranthesis

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- If φ is a wff and x is a variable, then $(\forall x)\varphi$ is a wff

Logical Abbreviations : Boolean Connectives

- $\neg\varphi = \varphi \to \bot$
- $\top = \neg\bot$
- $\varphi \vee \psi = \neg\varphi \to \psi$
- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\exists x.\varphi = \neg(\forall x.\neg\varphi)$
- Precedence of operators : $\neg > \wedge > \vee > \to > \forall$

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- $\forall x \forall y \forall z (R(x, y) \rightarrow (R(y, z) \rightarrow R(x, z)))$

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- $\forall x \forall y \forall z (R(x, y) \rightarrow (R(y, z) \rightarrow R(x, z)))$ Transitivity

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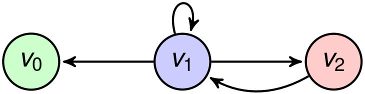
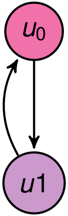
  - The structure $\mathcal{A}$ is finite if A (or $u(\mathcal{A})$) is finite

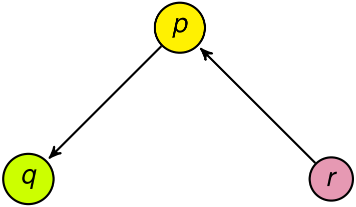
Examples of Structures

A Graph

- A set V of vertices
- A set $E \subseteq V \times V$ of edges







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  - $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.
  - $\forall x \exists y (E(x, y))$

A totally ordered set

- A set S with an order relation
- Relates any two elements of S
- Examples : $(\mathbb{N}, <)$, $(\mathbb{R}, <)$

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  - $\mathcal{O} = (O = \{1, 2, 4\}, <^{\mathcal{O}} = \{(1, 2), (1, 4), (2, 4)\}, S^{\mathcal{O}} = \{(1, 2)\})$
  - $\exists x \neg \exists y (S(x, y))$
- Can you write a Partial Order as a structure, where the universe consists of all subsets of a given finite set?

Words

- A word is a sequence of symbols over a (finite) alphabet
- Alphabet $\Sigma = \{a, b, c\}$
- Some words over Σ : $b, aaa, abababa, cacbccc$
- The length of a word is the number of symbols in it

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  - The structure with $u(\mathcal{W}) = \{0, 1, 2, \dots, 8\}$, ${Q_a}^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, ${Q_b}^{\mathcal{W}} = \{2, 3, 5, 7\}$,
  - $<^{\mathcal{W}} = \{(0,1), (0,2), \dots, (7,8)\}$, $S^{\mathcal{W}} = \{(0,1), (1,2), \dots, (7,8)\}$ uniquely defines the word $W = aabbababa$.
  - $\forall x(Q_b(x) \rightarrow \exists y(x < y \wedge Q_a(y))$