CS 218 Design and Analysis of Algorithms

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Module 2: Flow networks, Max-flow, Min-cut and applications

Value of the flow

Lemma

$$|f| = \sum_{v \in V, (s,v) \in E} f(s,v) = f^{\rightarrow}(s) = \sum_{v \in V, (v,t) \in E} f(v,t) = f^{\leftarrow}(t)$$

Proof.

$$\begin{aligned}
|f| &= f^{\rightarrow}(s) + \sum_{v \in V \setminus \{s,t\}} (f^{\rightarrow}(v) - f^{\leftarrow}(v)) \\
&= \sum_{v \in V \setminus \{t\}} (f^{\rightarrow}(v) - f^{\leftarrow}(v)) \\
&= f^{\rightarrow}(V \setminus \{t\}) - f^{\leftarrow}(V \setminus \{t\})
\end{aligned}$$

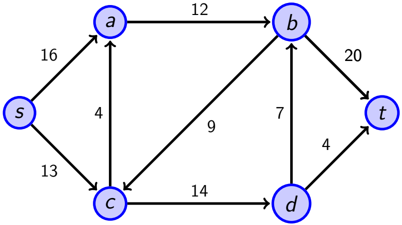
Each edge of the graph appears twice (once +-vely and once -vely). Except the edges entering t which appear once -vely.

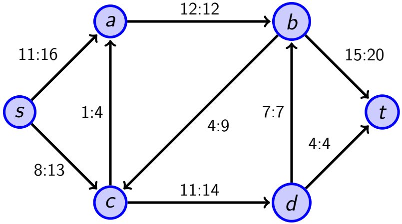
$$= f^{\leftarrow}(t)$$

Example

We will now see an example of a flow network, a flow and the value of a flow.

Flow network.





A flow in the network with value $|f| = 19$.

Maximum Flow Problem

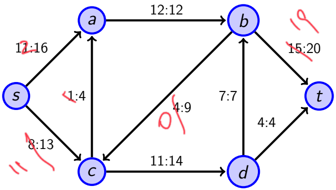
Problem description

Input: a flow network $G = (V, E)$ along with capacity function $c : E \to \mathbb{N}$.

Output: the maximum valued flow that can be transferred in the network.

Recall

The flow must satisfy the capacity constraints and must be conserved at all internal nodes.



An (s, t)-Cut

What is an (s, t)-Cut (or a Cut for brevity).

Given a directed graph $G = (V, E)$ with designated course s, sink t and capacities on the edges given by $c : E \to \mathbb{N}$.

An (s, t)-Cut is given by $S, T \subseteq V$ such that

- $s \in S$, $t \in T$.
- $S \cup T = V$ and $S \cap T = \varnothing$.

Capacity of a cut.

Definition

Given a graph $G = (V, E)$ with capacity function $c : E \to \mathbb{N}$, the capacity of an (s, t)-Cut (S, T) is given by

$$\operatorname{cap}(S, T) = \sum_{u \in S, v \in T \text{ s. t. } (u,v) \in E} c(u, v)$$

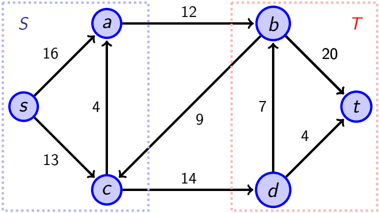
Minimum Cut Problem

Problem description

Input: a network $G = (V, E)$ along with capacity function $c : E \to \mathbb{N}$.

Output: an (s, t)-cut with as small capacity as possible.

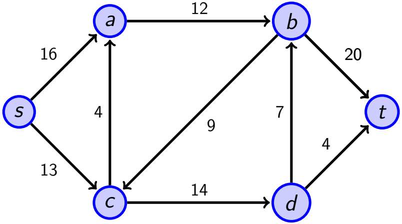
Example.

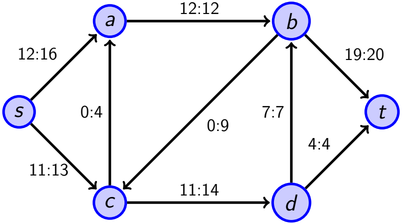


$\text{cap}(S, T) = 12 + 14 = 26.$

Maxflow and Mincut

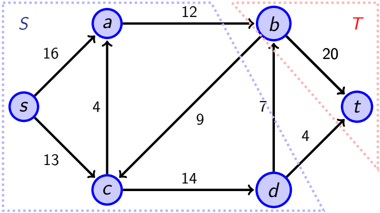
Value of the maximum flow.





Value of the flow = 23.

Value of the minimum cut



Value of the cut = 23

Weak duality

Lemma (Max Flow is at most as much as the Min Cut)

Let f be any flow in the flow network G. Let (S,T) be any (s,t)-Cut in G. Then $|f| \le cap(S,T)$. Moreover, $|f| = cap(S,T)$ if and only if f saturates every edge from S to T and avoids every edge from T to S.

Proof.

$$
\begin{aligned}
|f| &= f^{\rightarrow}(s) \\
&= f^{\rightarrow}(S) - f^{\leftarrow}(S) && \text{due to the conservation constraint} \\
&\le \sum_{u\in S}\sum_{v\in T} f(u,v) && \text{recall, if } (u,v)\notin E \text{ then } f(u,v)=0 \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) && \text{due to the capacity constraint} \\
&= \mathrm{cap}(S,T)
\end{aligned}
$$