CS 228 : Logic in Computer Science

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Recap

- Given FO formula φ, build an automaton A_φ preserving the language
- Satisfiability of FO reduces to non-emptiness of underlying automaton
- Starting today : non FO-definability

FO Definability

Let φ be a FO formula. Define the quantifier rank of φ denoted $c(\varphi)$

- If φ is atomic ($x = y, x < y, S(x, y), Q_a(x)$) then $c(\varphi) = 0$
- $c(\neg\varphi) = c(\varphi)$
- $c(\varphi \wedge \psi) = max(c(\varphi), c(\psi))$
- $c(\exists\varphi) = c(\varphi) + 1$
- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$
- Formulae of quantifier rank $c + 1$ written as a disjunction of the conjunction of formulae, each formula of the form $\exists x\varphi, \neg\exists x\varphi$ or φ, with $c(\varphi) \leqslant c$. Eliminate repeated disjuncts/conjunts

Number of FO formulae of rank c

Let $\mathcal{V}$ be a finite set of first order variables. Fix a finite signature τ. Let there be m atomic formulae over τ having variables from $\mathcal{V}$.

- If $\mathcal{V}$ has 2 variables x, y, and τ has $Q_a, S, <$.
- Atomic formulae : $\{Q_a(x), Q_a(y), S(x, y), x < y\}$
- $G = \{Q_a(x), \neg Q_a(x), Q_a(y), \neg Q_a(y), S(x, y), \neg S(x, y), x < y, \neg(x < y)\}$
- Each subset of G is a possible conjunct C_i.
- All possible disjuncts using each C_i : formulae in DNF of rank 0

Number of FO formulae of rank c

Let $\mathcal{V}$ be a finite set of first order variables. Fix a finite signature τ. Let there be m atomic formulae over τ having variables from $\mathcal{V}$.

- $2m$ atomic/negated atomic formulae
- Number of conjunctions C_i possible $\leqslant 2^{2m}$
- Number of formulae in DNF $\leqslant 2^{2^{2m}}$ $(c = 0)$

Rank 1

Let there be p formulae φ of rank 0.

- $2p$ formulae of the form $\exists x \varphi, \neg \exists x \varphi$
- 2^{2p} conjunctions of rank 1
- Conjuncting any one of the p formulae of rank 0 gives all conjuncts of rank 1 : $p2^{2p}$ more
- Possible conjuncts of rank 1 is $q = (p+1)2^{2p}$
- Possible disjuncts of these : 2^q

Number of FO formulae of rank c

Let $\mathcal{V}$ be a finite set of first order variables, and let $c \geqslant 0$. There are finitely many FO formulae in DNF with rank c over $\mathcal{V}$.

Some Notation

Given a word $w = a_1 \ldots a_n$, and a finite set of variables $\mathcal{V}$, define a $\mathcal{V}$-enriched-word with respect to w as

- $(a_1, U_1)(a_2, U_2) \ldots (a_n, U_n)$ where
- $\bigcup_i U_i = \mathcal{V}$
- $U_i \cap U_j = \emptyset$

- A $\mathcal{V}$-enriched-word is over the alphabet $\Sigma \times 2^{\mathcal{V}}$
- $(a, \{x\})(b, \{y, z\})(c, \emptyset)(d, \{u, v\})$ is a $\{x, y, z, u, v\}$-enriched word with respect to the word *abcd*.
- We will refer to $\mathcal{V}$-enriched-word structures as $\mathcal{V}$-structures from here on

Notational Semantics

Given a $\mathcal{V}$-structure $w = (a_1, S_1)\ldots(a_n, S_n)$,

- $w \models Q_a(x)$ iff there exists j such that $a_j = a$ and $x \in S_j$
  - $(a, \{y\})(b, \{u, v\})(a, \{x\}) \models Q_a(x)$
- $w \models (x = y)$ iff there exists j such that $x, y \in S_j$
  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \not\models (x = y)$
- $w \models x < y$ iff there exists $i < j$ such that $x \in S_i, y \in S_j$
  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$
- $w \models \exists x Q_a(x)$ iff there exists i such that $(a_1, S_1)\ldots(a_i, S_i \cup \{x\})\ldots(a_n, S_n) \models Q_a(x)$
  - $(b, \{y, z\})(a, \{u\})(c, \emptyset) \models \exists x Q_a(x)$ since $(b, \{y, z\})(a, \{x, u\})(c, \emptyset) \models Q_a(x)$

Notational Semantics

- $(a,\emptyset)(a,\emptyset)(b,\emptyset) \models \forall x \exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$ iff
- $(a,\emptyset)(a,\emptyset)(b,\emptyset) \models \neg\exists x \neg[\exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])]$ iff
- $(a,\emptyset)(a,\emptyset)(b,\emptyset) \not\models \exists x \neg[\exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])]$ iff
  - $(a,\{x\})(a,\emptyset)(b,\emptyset) \not\models \neg[\exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])]$ and
  - $(a,\emptyset)(a,\{x\})(b,\emptyset) \not\models \neg[\exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])]$ and
  - $(a,\emptyset)(a,\emptyset)(b,\{x\}) \not\models \neg[\exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])]$
- $(a,\{x\})(a,\emptyset)(b,\emptyset) \models \exists y (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$ iff
- $(a,\{x\})(a,\emptyset)(b,\{y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$

Similarly, $(a,\emptyset)(a,\{x\})(b,\{y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$ and $(a,\emptyset)(a,\emptyset)(b,\{x,y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$