CS 228 : Logic in Computer Science

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- To make sense out of a formula, we need structures

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- A structure in PL will just consist of the universe $\{0, 1\}$, since there is no signature. All variables assume values from this boolean universe.

Satisfiability in PL and FO

- The satisfiability of a PL formula depends on the existence of an assignment satisfying it; likewise, the satisfiability of a FO formula φ over signature τ depends on the existence of a structure $\mathcal{A}$ of τ such that φ is true on $\mathcal{A}$.

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  - y is free in $Q(x, y)$ and bound in $R(x, y)$,
  - x is free in $P(x, y)$, and bound in $Q(x, y)$, $R(x, y)$

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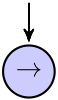
- Given φ, denote by $\varphi(x_1, \ldots, x_n)$, that $x_1, \ldots, x_n$ are the free variables of φ, also *free*(φ)
- A sentence is a formula φ none of whose variables are free

$P(x, y) \rightarrow \forall x((\forall y R(x, y)) \rightarrow Q(x, y))$



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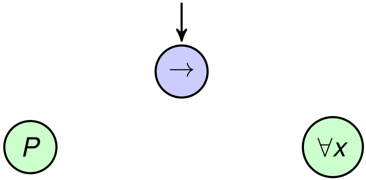

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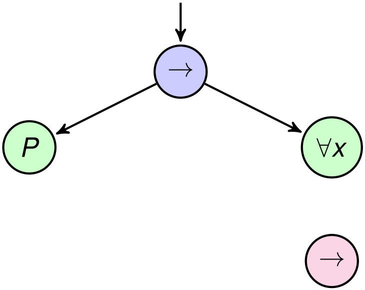

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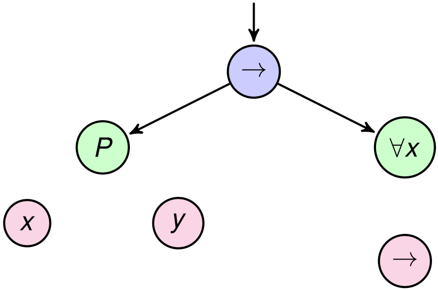

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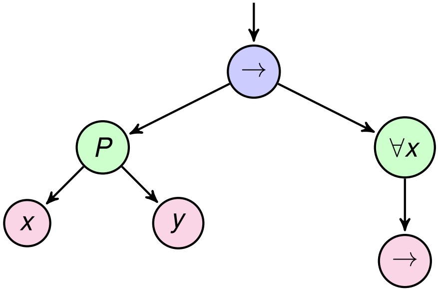

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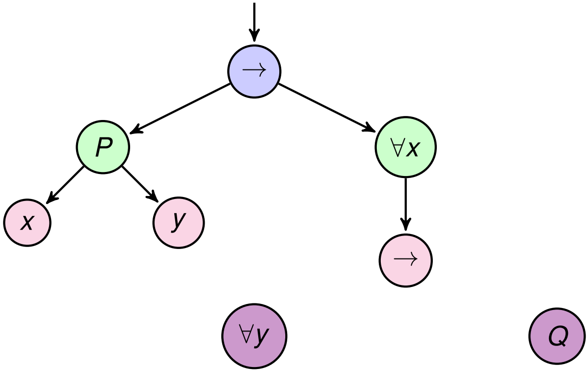

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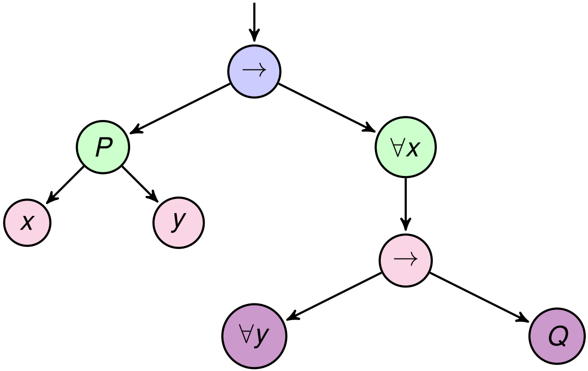

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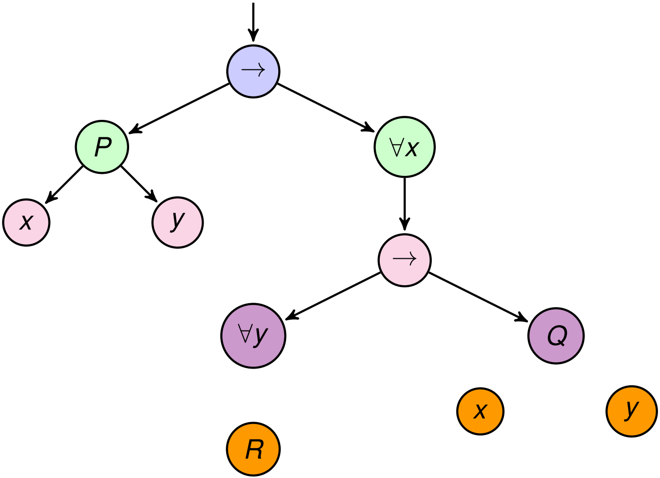

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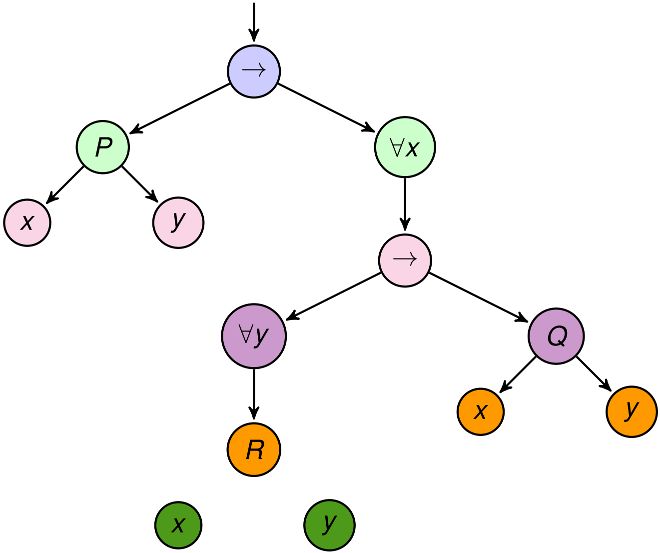

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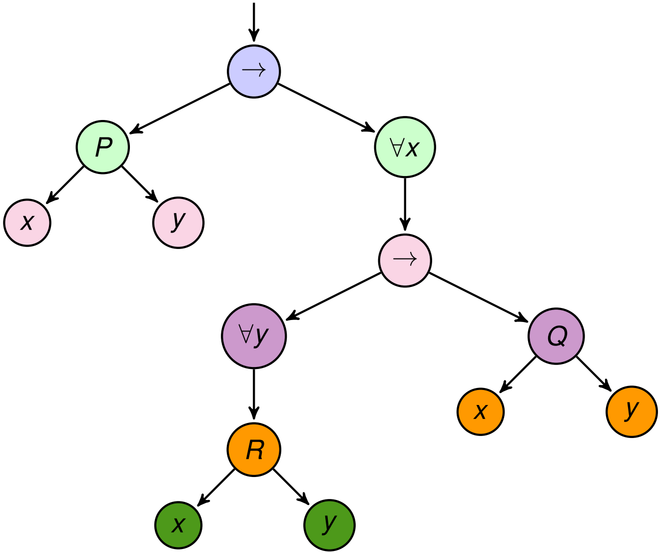

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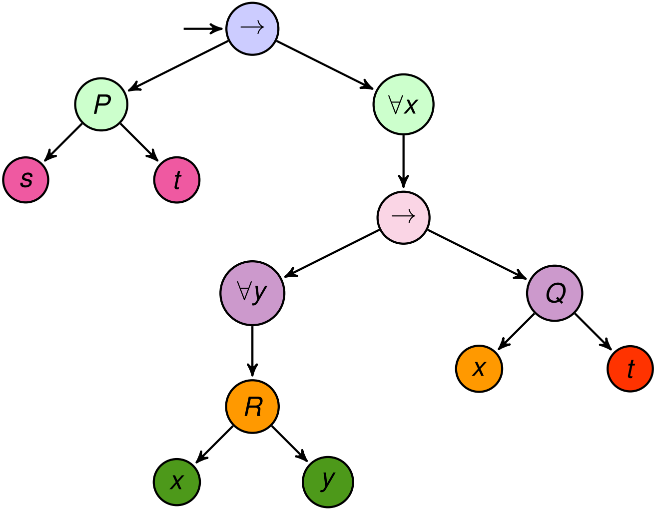
$\varphi(s, t) = P(s, t) \to \forall x((\forall y R(x, y)) \to Q(x, t))$



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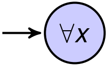

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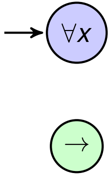

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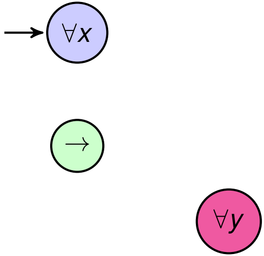

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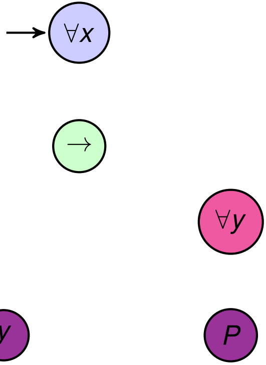

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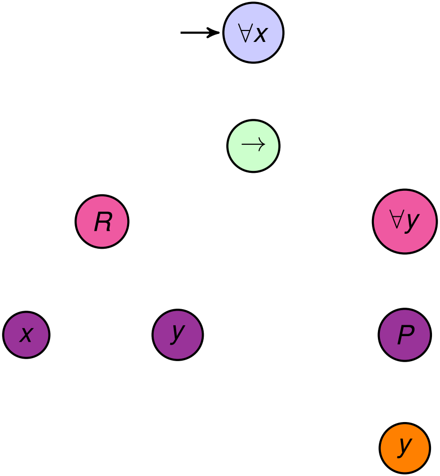

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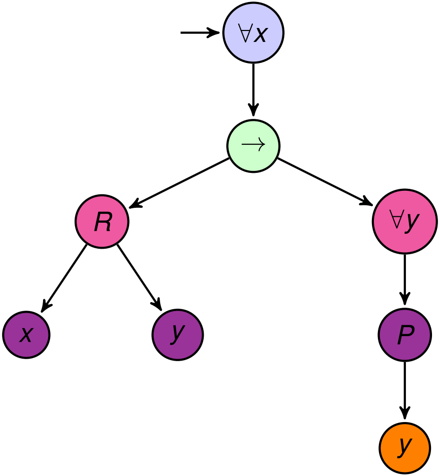

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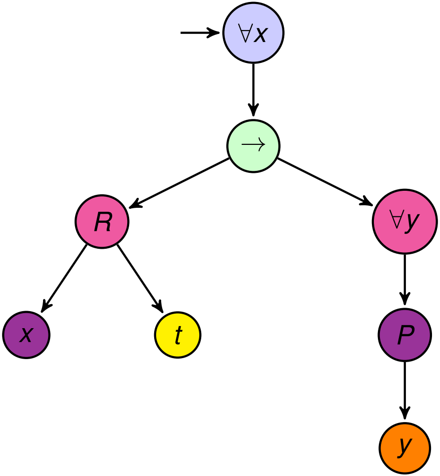

$\varphi(t) = \forall x(R(x,t) \to \forall y P(y))$

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a function $\alpha : \mathcal{V} \to u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha(t)$ is $c^{\mathcal{A}}$

Assignments

Binding on a Variable

For an assignment α over $\mathcal{A}$, $\alpha[x \mapsto a]$ is the assignment

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y), y \neq x, \\ a, y = x \end{cases}$$

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Let $u(\mathcal{A}) = \{a, b, c, d\}$, and consider assignment $\alpha : \{x, y, z\} \to u(\mathcal{A})$ defined by $\alpha(x) = d, \alpha(y) = b, \alpha(z) = c$. Then,

- $\alpha[x \mapsto a]$ is the assignment α' where $\alpha'(x) = a, \alpha'(y) = \alpha(y), \alpha'(z) = \alpha(z)$.
- $\alpha[x \mapsto c]$ is the assignment α'' where $\alpha''(x) = c, \alpha''(y) = \alpha(y), \alpha''(z) = \alpha(z)$.

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- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$

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- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x. Thus, assignments matter only to free variables.

Example of Satisfaction

$\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$

- For any assignment α, $\mathcal{G} \models_\alpha \forall x \forall y (E(x, y) \to E(y, x))$ iff

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$\mathcal{G} = (\{1,2,3\}, E^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\})$

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  - for $\alpha_1 : \alpha_1(x) = 1, \alpha_1(y) = 1, \mathcal{G} \models_{\alpha_1} (E(x,y) \to E(y,x))$,
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  - for $\alpha_9 : \alpha_9(x) = 3, \alpha_9(y) = 3, \mathcal{G} \models_{\alpha_9} (E(x, y) \to E(y, x))$
- There is an assignment α which satisfies
  $\mathcal{G} \models_\alpha \exists x (E(x, y) \wedge E(x, z) \wedge y \neq z)$

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  $\mathcal{G} \models_\alpha \exists x (E(x,y) \wedge E(x,z) \wedge y \neq z)$
  $\alpha(y) = 1, \alpha(z) = 3$, and consider $\alpha[x \mapsto 2]$.

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- Check this: $\mathcal{G} \not\models \exists x \forall y E(x,y)$,

Example of Satisfaction

$\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$

- For any assignment α, $\mathcal{G} \models_\alpha \forall x \forall y(E(x, y) \rightarrow E(y, x))$ iff for all $a \in \{1, 2, 3\}, \mathcal{G} \models_{\alpha[x \mapsto a]} \forall y(E(x, y) \rightarrow E(y, x))$ iff for every $a, b \in \{1, 2, 3\}$, $\mathcal{G} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x, y) \rightarrow E(y, x))$
  - for $\alpha_1 : \alpha_1(x) = 1, \alpha_1(y) = 1, \mathcal{G} \models_{\alpha_1} (E(x, y) \rightarrow E(y, x))$,
  - for $\alpha_2 : \alpha_2(x) = 1, \alpha_2(y) = 2, \mathcal{G} \models_{\alpha_2} (E(x, y) \rightarrow E(y, x))$,
  - for $\alpha_3 : \alpha_3(x) = 1, \alpha_3(y) = 3, \mathcal{G} \models_{\alpha_3} (E(x, y) \rightarrow E(y, x))$,
  - for $\alpha_4 : \alpha_4(x) = 2, \alpha_4(y) = 1, \mathcal{G} \models_{\alpha_4} (E(x, y) \rightarrow E(y, x))$,
  - $\vdots$
  - for $\alpha_9 : \alpha_9(x) = 3, \alpha_9(y) = 3, \mathcal{G} \models_{\alpha_9} (E(x, y) \rightarrow E(y, x))$
- There is an assignment α which satisfies $\mathcal{G} \models_\alpha \exists x(E(x, y) \wedge E(x, z) \wedge y \neq z)$
  $\alpha(y) = 1, \alpha(z) = 3$, and consider $\alpha[x \mapsto 2]$.
- Check this: $\mathcal{G} \not\models \exists x \forall y E(x, y)$, $\mathcal{G} \models \forall x \exists y E(x, y)$

Example of Satisfaction

- $\mathcal{W} = abaaa$ or,
  $\mathcal{W} = (\{0, \ldots, 4\}, <^{\mathcal{W}}, S^{\mathcal{W}}, Q_a^{\mathcal{W}} = \{0, 2, 3, 4\}, Q_b^{\mathcal{W}} = \{1\})$

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    One possibility : $\alpha(x) = 2, \alpha(y) = 3$
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  - There is no assignment α which satisfies
    $\exists x \exists y (Q_b(x) \wedge Q_b(y) \wedge x \neq y)$
  - Prove or disprove : $\mathcal{W} \models \exists x \forall y [Q_b(x) \wedge x < y \wedge Q_a(y)]$
  - Prove or disprove : $\mathcal{W} \models \exists x \forall y [Q_b(x) \wedge x < y \Rightarrow Q_a(y)]$

Satisfiability, Validity, Equivalence and Equisatisfiability

- A formula φ over a signature τ is said to be satisfiable iff for some τ-structure $\mathcal{A}$ and assignment α, $\mathcal{A} \models_\alpha \varphi$

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- Consider $\varphi_1(x) = \forall y R(x, y)$ and $\varphi_2 = \exists x \forall y R(x, y)$.
- It is clear that whenever φ_2 is satisfiable on $\mathcal{A}$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \forall y R(x, y)$, for some $a \in u(\mathcal{A})$. Then one can find the assignment α such that $\mathcal{A} \models_\alpha \varphi_1(x)$, $\alpha(x) = a$.
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- Likewise, if $\mathcal{A} \models_\alpha \varphi_1(x)$, then $\mathcal{A} \models_{\alpha'[x \mapsto \alpha(x)]} \varphi_2$, and $\alpha'(y)$ can be defined as $\alpha(y)$.
- Thus, $\varphi_1(x), \varphi_2$ agree on satisfiability : equisatisfiable.

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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- $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

True or False?

For a sentence φ, and any two assignments α_1 and α_2, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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No free variables!

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- $\varphi = \exists x[(\forall y E(x,y)) \wedge \forall z[(\forall y E(z,y)) \rightarrow z = x]]$. Does φ evaluate to true under some graph structure?
- $\psi = \exists x[Q_a(x) \wedge \forall y[(y < x \wedge Q_b(y)) \rightarrow (z < x \wedge y < z \wedge Q_a(z))]]$. Does ψ evaluate to true under some word structure?