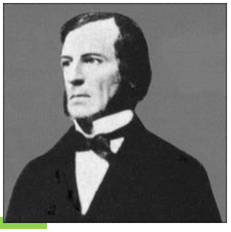
CS-226: Digital Logic Design

Minimization of Logic Expression using Boolean Algebra



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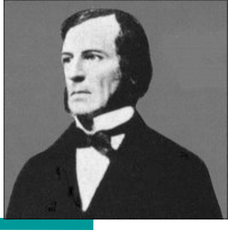
Boolean Algebra



- Boolean Algebra is defined as
  1. Set of elements {0, 1}
  2. Set of operators {+, ., ~}
  3. Number of postulates
- Boolean Algebra: 5-tuple

$$\{B, +, ., \sim, 0,1\}$$

- Closure: If a and b are Boolean then $(a.b)$ and $(a + b)$ are also Boolean

Postulates

Postulate	Duals	
	Expression 1	Expression 2
0	$a, b, a + b \in B$	$a, b, a \cdot b \in B$
3	$a + 0 = a$	$a \cdot 1 = a$
1	$a + b = b + a$	$a \cdot b = b \cdot a$
2	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
4	$a + (b \cdot c)=(a + b) \cdot (a + c)$	$a \cdot (b + c)=(a \cdot b)+(a \cdot c)$
6	$a + \bar{a} = 1$	$a . \bar{a} = 0$

Theorems

Theorem	Duals	
	Expression 1	Expression 2
Idempotency	$a + a = a$	$a \cdot a = a$
Null	$a + 1 = 1$	$a \cdot 0 = 0$
Involution	$\bar{\bar{a}} = a$	$\bar{\bar{a}} = a$
Absorption	$a + a.b = a$	$a \cdot (a + b) = a$
Adsorption	$a + \bar{a}.b = a + b$	$a \cdot (\bar{a} + b) = a.b$
Uniting	$a.b + a.\bar{b} = a$	$(a + b)(a + \bar{b}) = a$

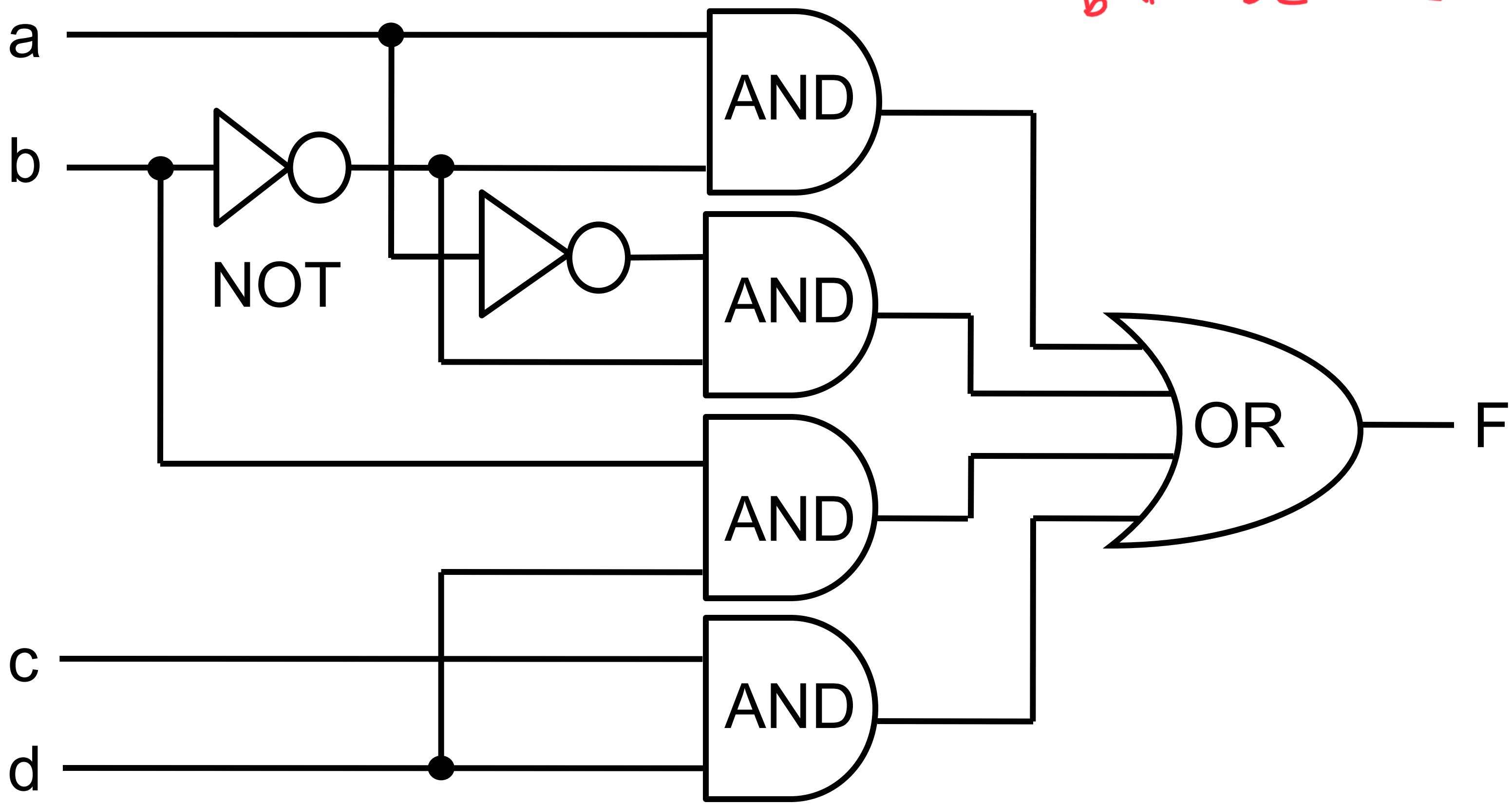
Theorems

Theorem	Duals	
	Expression 1	Expression 2
DeMorgan	$\overline{a + b} = \overline{a} \,.\, \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$
Consensus	$a.b + \overline{a}.c + b.c$ $= a.b + \overline{a}.c$	$(a + b)(\overline{a} + c)\ (b + c)$ $= (a + b).(\overline{a} + c)$

Understanding Minimization

- Logic function: $\mathbf{F = a\bar{b} + \bar{a}\bar{b} + bd + cd}$

$\bar{b} + bd + cd$



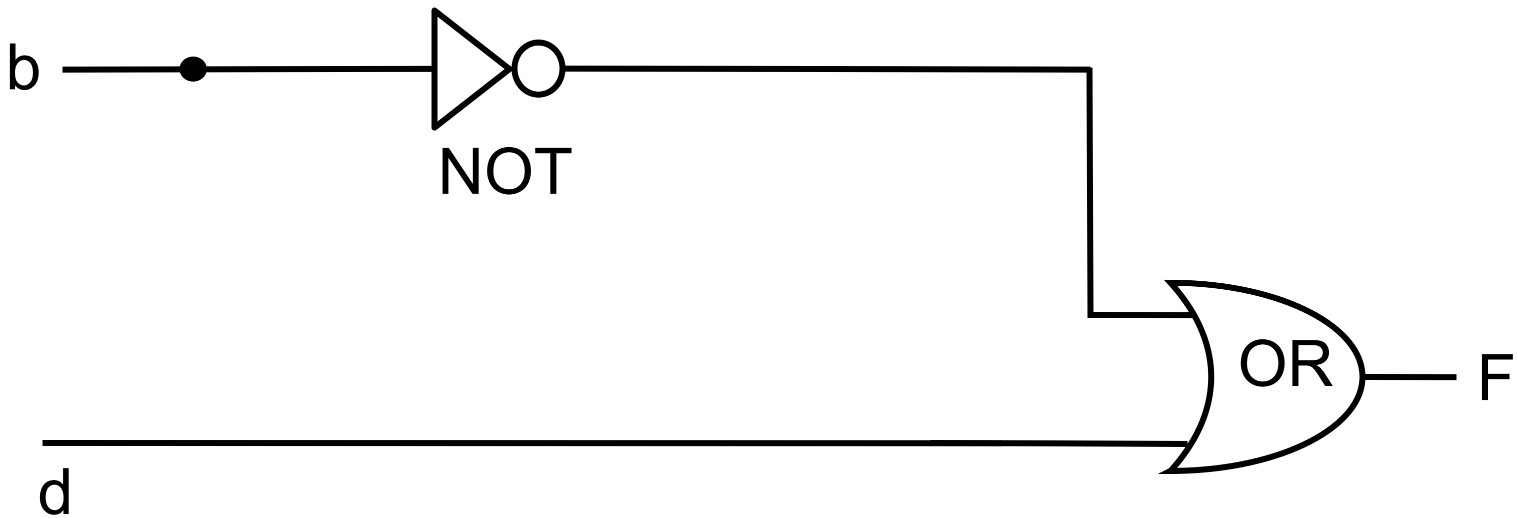
Logic Minimization

- Reducing products:

$F = a\bar{b} + \bar{a}\bar{b} + bd + cd$	Distributivity
$= \bar{b}(a + \bar{a}) + bd + cd$	Complementation
$= \bar{b}1 + bd + cd$	Identity
$= \bar{b}(c + \bar{c}) + bd + cd$	Complementation
$= bd + \bar{b}c + cd + \bar{b}\bar{c}$	Distribitivity
$= bd + \bar{b}c + \bar{b}\bar{c}$	Consensus theorem
$= bd + \bar{b}(c + \bar{c})$	Distributivity
$= bd + \bar{b}$	Complement, identity
$F = \bar{b} + d$	Adsorption

Logic Minimization

- Minimized expression: $F = \overline{b} + d$



Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of literals (complemented and uncomplemented variables):

$$a.b + \bar{a}.c.d + \bar{a}.b.d + \bar{a}.c.\bar{d} + a.b.c.d$$
$$= a.b + a.b.c.d + \bar{a}.c.d + \bar{a}.c.\bar{d} + \bar{a}.b.d$$
$$= a.b + a.b.(c.d) + \bar{a}.c.(d + \bar{d}) + \bar{a}.b.d$$
$$= a.b + \bar{a}.c + \bar{a}.b.d = b\,(a + \bar{a}.d) + \bar{a}.c$$
$$= b.(a + d) + \bar{a}.c$$

5 literals

Specification: Logic Function

Truth Table

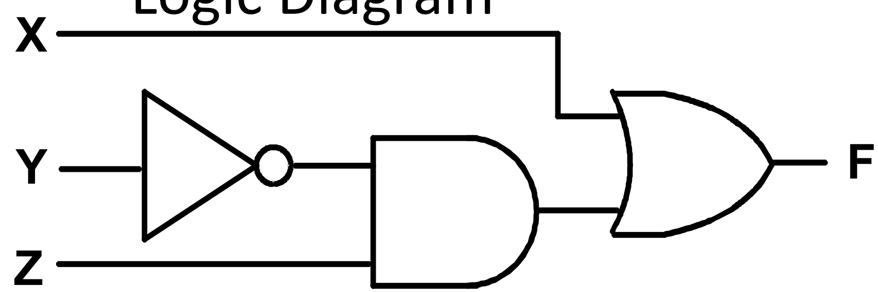
X Y Z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Logic Expression

$$F = \overline{X}.\,\overline{Y}.\,Z + X.\,\overline{Y}.\,\overline{Z} + X.\overline{Y}.Z + X.Y.\overline{Z} + X.Y.Z$$

$$F = X + \overline{Y}.\,Z$$

Logic Diagram



Expression Simplification

- Logic minimization

$$F = \bar{x}.\bar{y}.z + x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.\bar{z} + x.y.z$$

$$F = \bar{x}.\bar{y}.z + x.\bar{y}.(\bar{z} + z) + x.y.(\bar{z} + z)$$

$$F = \bar{x}.\bar{y}.z + x.\bar{y} + x.y = \bar{x}.\bar{y}.z + x.(\bar{y} + y)$$

$$F = \bar{x}.\bar{y}.z + x = \bar{y}.z + x$$

Theorem 7: DeMorgan's Theorem

- $\overline{a + b} = \bar{a} \, . \, \bar{b}, \quad \forall a, b \in B$
- $\overline{a \, . \, b} = \bar{a} + \bar{b}, \quad \forall a, b \in B$

1806 - 1871

Generalization of DeMorgan's Theorem:

$$\overline{a + b + \cdots + z} = \overline{a} \cdot \overline{b} \cdots \overline{z}$$

$$\overline{a \cdot b \cdots z} = \overline{a} + \overline{b} + \cdots + \overline{z}$$

Complementing Functions

- Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal
- Example: Complement $F = \overline{x}.y.\overline{z} + x.\overline{y}.\overline{z}$

$$\overline{F} = (x + \overline{y} + z)(\overline{x} + y + z)$$

DeMorgan’s Theorem (1)

- $\overline{a+b} = \bar{a}\,.\,\bar{b}, \quad \forall a, b \in B$

- $\overline{\overline{a+b}} = a+b = \overline{\bar{a}\,.\,\bar{b}}, \quad \forall a, b \in B$

DeMorgan's Theorem (2)

- $\overline{a\,.\,b} = \bar{a} + \bar{b}, \quad \forall a, b \in B$
- $\overline{\overline{a\,.\,b}} = a.b = \overline{\bar{a} + \bar{b}}, \quad \forall a, b \in B$

Minimum Operator Set

- Minimum number of operators
- { ~, (+ or .)} / $\{\neg, (\wedge\ or\ \vee)\}$

Universal Operator: NAND

- NAND: Composite operator (AND and NOT)
- $\overline{a} = \overline{a\,.\,a}$
- $a\,.\,b = \overline{\overline{a\,.\,b}} = \overline{\overline{(a\,.\,b)}\,.\,\overline{(a\,.\,b)}}$
- $a + b = \overline{\overline{a}\,.\,\overline{b}} = \overline{\overline{(a.a)}\,.\,\overline{(b\,.\,b)}}$

Universal Operator: NOR

- NOR: Composite operator (OR and NOT)
- $\overline{a} = \overline{a + a}$
- $a + b = \overline{\overline{a + b}} = \overline{(a + b)} + \overline{(a + b)}$
- $a \, . \, b = \overline{\overline{a} + \overline{b}} = \overline{\overline{(a + a)} + \overline{(b + b)}}$

Complementing Functions

- Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal
- Example: Complement $F = \overline{x}.y.\overline{z} + x.\overline{y}.\overline{z}$

  $\overline{F} = (x + \overline{y} + z)(\overline{x} + y + z)$

Logic Expression (SOP)

- $F = a.b + c.d$
- $\overline{F} = \left(\overline{a} + \overline{b}\right).\left(\overline{c} + \overline{d}\right)$
- $\overline{\overline{F}} = F = \overline{\left(\overline{a} + \overline{b}\right).\left(\overline{c} + \overline{d}\right)}$

Logic Expression (SOP)

- $F = a.b + c.d$
- $F = \overline{\left(\overline{a} + \overline{b}\right).(\overline{c} + \overline{d})}$

Logic Expression (POS)

- $F = (a + b).(c + d)$
- $\overline{F} = \overline{(a + b)} + \overline{(c + d)}$
- $\overline{F} = (\overline{a}.\ \overline{b}) + (\overline{c}\ .\ \overline{d})$
- $\overline{\overline{F}} = F = \overline{(\overline{a}.\ \overline{b}) + (\overline{c}\ .\ \overline{d})}$

Logic Expression (POS)

- $F = (a + b).(c + d)$
- $F = \overline{(\bar{a}.\ \bar{b}) + (\bar{c}\ .\ \bar{d})}$

Thank You