CS 228 : Logic in Computer Science

Krishna. S

Recap

- Started looking at FO nondefinability
- Defined maximal quantifier depth or quantifier rank of a formula
- For a finite set of variables $\mathcal{V}$, showed that there are finitely many FO formulae of rank r over $\mathcal{V}$
- Introduced some new notations for words, mimicking assignments of values to free variables

Notational Semantics Recap

- $(a_1, \emptyset) \ldots (a_n, \emptyset) \models \exists x \varphi$ iff
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- For a formula $\varphi(x_1, \ldots, x_m)$, $L(\varphi)$ is the set of all $\{x_1, \ldots, x_m\}$ structures satisfying φ
- For a sentence φ, $L(\varphi)$ is the set of all $\emptyset$ structures satisfying φ
- Example : $L(Q_a(x))$ consists of all x-structures of the form $(\Sigma, \emptyset)^*(a, \{x\})(\Sigma, \emptyset)^*$.

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- $(a, \emptyset)(b, \emptyset) \nsim_2 (a, \emptyset)(b, \emptyset)(a, \emptyset)$
- $\sim_r$ is an equivalence relation
- Finitely many equivalence classes : each class consists of words that behave the same way on formulae of rank r

Non-Expressibility in FO : The Game Begins

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- Each player has r pebbles $z_1, \ldots, z_r$

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- A pebble once placed, cannot be removed
- The game ends after r rounds, when both players have used all their pebbles

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  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$
  - Duplicator : $(a, \{z_1, z_2\})(b, \emptyset)$ or $(a, \{z_1\})(b, \{z_2\})$

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- At the end of r-rounds, we have two $\mathcal{V}$-structures (w'_1, w'_2)

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- At the end of r-rounds, we have two $\mathcal{V}$-structures (w_1', w_2')
- Duplicator wins iff for every atomic formula α,
  $w_1' \models \alpha$ iff $w_2' \models \alpha$

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- That is, $w_1' \sim_0 w_2'$
- Spoiler wins otherwise.

Winner

Given two word structures (w_1, w_2), duplicator wins on (w_1, w_2) if for every atomic formula α, $w_1 \models \alpha$ iff $w_2 \models \alpha$

Play continues

- Who won in the earlier play?
- We had
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1, z_2\})(b, \emptyset)$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models (z_1 < z_2)$
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  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models Q_a(z_2)$
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  - $(a, \{z_1\})(b, \{z_2\}) \not\models Q_a(z_2)$
- Spoiler wins in two rounds
- If the game was played only for one round, who will win?

Unique Winner

Given structures w_1, w_2, and a number of rounds r, exactly one of the players win.

Logical Equivalence and Winning

Let w_1, w_2 be $\mathcal{V}$-structures and let $r \geqslant 0$. Then $w_1 \sim_r w_2$ iff Duplicator has a winning strategy in the r-round game on (w_1, w_2).

Logical Equivalence and Winning

Assume $w_1 \sim_r w_2$, and induct on r

- Base : $r = 0$ and $w_1 \sim_0 w_2$. Duplicator wins, since by assumption, w_1, w_2 agree on all atomic formulae.

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- Assume for $r - 1$: $w_1 \sim_{r-1} w_2 \Rightarrow$ Duplicator has a winning strategy in a $r - 1$ round game

Logical Equivalence and Winning

- Now, let $w_1 \sim_r w_2$, and assume spoiler wins the r-round game on (w_1, w_2).
  - Assume spoiler starts on w_1, places a pebble z_1 somewhere on w_1

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  - In response, duplicator places her pebble somewhere on w_2

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  - By assumption, spoiler wins the $r-1$ round game on (w_1', w_2')

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  - By inductive hypothesis, $w_1' \nsim_{r-1} w_2'$

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  - By inductive hypothesis, $w_1' \nsim_{r-1} w_2'$
  - Let ψ be the conjunction of all formulae of rank $r-1$ in normal form that are satisfied by w_1'

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  - Then $w_1' \models \psi$, $w_2' \not\models \psi$

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  - By inductive hypothesis, $w_1' \nsim_{r-1} w_2'$
  - Let ψ be the conjunction of all formulae of rank $r-1$ in normal form that are satisfied by w_1'
  - Then $w_1' \models \psi$, $w_2' \not\models \psi$
  - We thus have
  $$w_1 \models \exists z_1 \psi, w_2 \not\models \exists z_1 \psi$$
  contradicting $w_1 \sim_r w_2$

Logical Equivalence and Winning : Converse

Assume Duplicator wins r-round game on (w_1, w_2) and induct on r

- Base : $r = 0$ and Duplicator wins. Then w_1, w_2 agree on all atomic formulae, and hence $w_1 \sim_0 w_2$

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- Assume for $r - 1$: Duplicator has a winning strategy in a $r - 1$ round game $\Rightarrow w_1 \sim_{r-1} w_2$

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- Now, let duplicator win in the r round game, but $w_1 \not\sim_r w_2$.
  - $w_1 \not\sim_r w_2 \Rightarrow$ there is some formula ψ, $c(\psi) = r$ such that $w_1 \models \psi$, $w_2 \not\models \psi$

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  - Assume $\psi = \exists z_1 \varphi$. Then $c(\varphi) = r - 1$

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  - Since $w_1 \models \exists z_1 \varphi$, spoiler can keep pebble z_1 somewhere in w_1 obtaining w_1' satisfying φ

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  - By assumption, $w_2' \not\models \varphi$

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  - Assume $\psi = \exists z_1 \varphi$. Then $c(\varphi) = r - 1$
  - Since $w_1 \models \exists z_1 \varphi$, spoiler can keep pebble z_1 somewhere in w_1 obtaining w_1' satisfying φ
  - In reply, duplicator keeps pebble z_1 on w_2 obtaining w_2'
  - By assumption, $w_2' \nvDash \varphi$
  - Also, by assumption, duplicator wins the $r - 1$ round game on (w_1', w_2') : this by inductive hypothesis says that $w_1' \sim_{r-1} w_2'$

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  - Assume $\psi = \exists z_1 \varphi$. Then $c(\varphi) = r - 1$
  - Since $w_1 \models \exists z_1 \varphi$, spoiler can keep pebble z_1 somewhere in w_1 obtaining w_1' satisfying φ
  - In reply, duplicator keeps pebble z_1 on w_2 obtaining w_2'
  - By assumption, $w_2' \not\models \varphi$
  - Also, by assumption, duplicator wins the $r - 1$ round game on (w_1', w_2') : this by inductive hypothesis says that $w_1' \sim_{r-1} w_2'$
  - That is, either both w_1', w_2' satisfy φ, or both dont, a contradiction.

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- Let $L = \{v_1, v_2, v_3, \dots\}$ and $\overline{L} = \{w_1, w_2, w_3, \dots\}$
- Play a k round game on $v_i \in L$ and $w_j \notin L$. Let ψ_{v_i,w_j} be the formula of rank k that distinguishes the two words.

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- Consider the formula

$$[\psi_{v_1,w_1} \wedge \psi_{v_1,w_2} \wedge \cdots \wedge \psi_{v_1,w_n} \wedge \dots]$$

$$\vee$$

$$[\psi_{v_2,w_1} \wedge \psi_{v_2,w_2} \wedge \cdots \wedge \psi_{v_2,w_n} \wedge \dots]$$

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$$\vdots$$

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- Hence the disjunction and conjunction are finite
- ψ_L is a proper formula (of finite size)
- ψ_L captures L since each $v \in L$ satisfies $\bigwedge_{w \notin L} \psi_{vw}$ while none of the $w \notin L$ satisfy $\bigwedge_{w \notin L} \psi_{vw}$

FO-definable languages

Given a property $\mathcal{K}$, if for any pair $v \in \mathcal{K}$ and $w \notin \mathcal{K}$, spoiler has a winning strategy in the k-round EF game on v and w, then there is a rank k FO formula $\varphi_{\mathcal{K}}$ that defines the property $\mathcal{K}$.

$$\varphi_{\mathcal{K}} = \bigvee_{v \in \mathcal{K}} \bigwedge_{w \notin \mathcal{K}} \psi_{vw}$$

where ψ_{vw} is as explained in the previous slide.

- Note that k is fixed in the above, and is independent of the choices of the words.

Implications of the Game on FO definability

FO Definability

L is FO definable $\Rightarrow$ there exists an r such that for every (w_1, w_2) pair, such that $w_1 \in L, w_2 \notin L$, spoiler wins in r rounds

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Non FO Definability

For all $r \geqslant 0$, there exists a (w_1, w_2) pair with $w_1 \notin L, w_2 \in L$, duplicator wins in r rounds $\Rightarrow$ L is not FO definable

Non FO[$<$] definability

- $FO[<, S] \subseteq FO[<]$
- Non definability in $FO[<]$ implies non definability in $FO[S, <]$

$(aa)^*$ is not $FO[<]$ Definable

- Assume that there is a sentence φ that defines words of even length, with $c(\varphi) = r$.
- Then, $a^i \models \varphi$ iff i is even
- Show that for all $r > 0$, $a^{2^r} \sim_r a^{2^r - 1}$

$(aa)^*$ is not $FO[<]$ Definable

- Base case : $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$ for $r = 1$
- In one round, duplicator wins on $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$

$(aa)^*$ is not $FO[<]$ Definable

- Show that for all $k \geqslant 2^r - 1$, duplicator has a winning strategy for the r-round game in (a^k, a^{k+1}), for all $r \geqslant 0$
- Induct on r
- If $r = 1$, then on (a, aa) duplicator wins in one round
- Assume now that the claim is true for $\leqslant r - 1$

$(aa)^*$ is not $FO[<]$ Definable

- Let $k \geqslant 2^r - 1$, and consider the structures

$$(a^k, a^{k+1})$$

- Spoiler puts pebble z_1 in one of the words obtaining

$$(a, \emptyset)^s (a, \{z_1\}) (a, \emptyset)^t$$

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- $s \leqslant \frac{k-1}{2}$ or $t \leqslant \frac{k-1}{2}$

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- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

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where $t' = t+1$ or $t' = t-1$.

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- The structures after round 1 are thus

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t}, (a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

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- Hence $min(t, t') \geqslant \frac{k-1}{2} \geqslant 2^{r-1} - 1$
- By inductive hypothesis, duplicator has a winning strategy for the $r - 1$ round game on $(a^t, a^{t'})$.

Duplicator's Win

- Use the duplicator's winning strategy for the $r-1$ round game on $(a^t, a^{t'})$, to obtain a winning strategy in $r-1$ rounds on

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- Whenever spoiler plays on a structure on letter $i \leqslant s+1$, duplicator plays on the same position on the other structure
- When spoiler plays at a position $i > s+1$ in either word, duplicator plays in the part of the other word $> s+1$ using her winning strategy in $(a^t, a^{t'})$

Duplicator's Win

- At the end of r rounds, we have structures w_1', w_2'.
- For $i \leqslant s+1$, pebble z_j appears at position i of w_1' iff pebble z_j appears at position i of w_2'
- Lets erase the first $s+1$ letters in w_1', w_2', obtaining v_1', v_2'
- v_1', v_2' are the words that result after $r' \leqslant r-1$ rounds of play on $(a^t, a^{t'})$. Recall that duplicator won this.
- Show that w_1', w_2' satisfy the same atomic formulae

Duplicator's Win

- Atomic Formulae : $Q_a(z_j)$: Both w'_1, w'_2 satisfy this.
- $w'_1 \models z_i < z_j$.
- If z_i, z_j are in the first $s+1$ letters, then $w'_2 \models z_i < z_j$.
- If z_i, z_j occur in the last $|w'_1| - s - 1$ positions, then $v'_1 \models z_i < z_j$. By duplicator's win in $(a^t, a^{t'})$, $v'_2 \models z_i < z_j$
- If z_i appears among the first $s+1$ letters and z_j after the first $s+1$ letters of w'_1, same is true in w'_2.

Historically Speaking

The games that we saw are due to Ehrenfeucht and Fraïssé

Reference: Finite Automata, Formal Logic and Circuit Complexity, by Howard Straubing.