CS228 Logic for Computer Science 2021

Lecture 10: Completeness

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Topic 10.1

Completeness

Completeness

Now let us ask the daunting question!!!!!

Is resolution proof system complete?

In other words,
if Σ is unsatisfiable, are we guaranteed to derive $\Sigma \vdash \perp$ via resolution?

We need a notion of not able to derive something.

Clauses derivable with proofs of depth n

We define the set $Res^n(\Sigma)$ of clauses that are derivable via resolution proofs of depth n from the set of clauses Σ.

Definition 10.1

Let Σ be a set of clauses.

$$Res^0(\Sigma) \triangleq \Sigma$$

$$Res^{n+1}(\Sigma) \triangleq Res^n(\Sigma) \cup \{C \mid C \text{ is a resolvent of clauses } C_1, C_2 \in Res^n(\Sigma)\}$$

Example 10.1

Let $\Sigma = \{(p \vee q), (\neg p \vee q), (\neg q \vee r), \neg r\}$.
$Res^0(\Sigma) = \Sigma$
$Res^1(\Sigma) = \Sigma \cup \{q, p \vee r, \neg p \vee r, \neg q\}$
$Res^2(\Sigma) = Res^1(\Sigma) \cup \{r, q \vee r, p, \neg p, \perp\}$

All derivable clauses

Since there are only finitely many variables in Σ, we can only derive finitely many clauses.
$Res^n(\Sigma)$ must saturate at some time point.

Definition 10.2

Let Σ be a set of clauses. There must be some m such that

$$Res^{m+1}(\Sigma) = Res^m(\Sigma).$$

Let $Res^*(\Sigma) \triangleq Res^m(\Sigma)$.

Completeness

Theorem 10.1
If a finite set of clauses Σ is unsatisfiable, $\bot \in Res^*(\Sigma)$.

Proof.
We prove the theorem using induction over number of variables in Σ.
Wlog, We assume that there are no tautology clauses in Σ. (why?)

base case:
p is the only variable in Σ.
Assume Σ is unsat. Therefore, $\{p, \neg p\} \subseteq \Sigma$.

We have the following derivation of $\bot$.

$$\frac{\Sigma \vdash p \qquad \Sigma \vdash \neg p}{\bot}$$

...

Completeness (contd.)

Proof(contd.)

induction step:

Assume: theorem holds for all the formulas containing variables $p_1, ..., p_n$.

Consider an unsatisfiable set Σ of clauses containing variables $p_1, ...p_n, p$.

Let

- $\Sigma_0 \triangleq$ the set of clauses from Σ that have p.
- $\Sigma_1 \triangleq$ be the set of clauses from Σ that have $\neg p$.
- $\Sigma_* \triangleq$ be the set of clauses from Σ that have neither p nor $\neg p$.

$\Sigma_0 \cap \Sigma_1 = \phi$

[No valid clauses]

$\Sigma = \Sigma_0 \wedge \Sigma_1 \wedge \Sigma_*$

Furthermore, let

- $\Sigma'_0 \triangleq \{C - \{p\} | C \in \Sigma_0\}$
- $\Sigma'_1 \triangleq \{C - \{\neg p\} | C \in \Sigma_1\}$

...

Exercise 10.1

Show $\Sigma'_0 \models \Sigma_0$ *and* $\Sigma'_1 \models \Sigma_1$

Example: projections

Example 10.2

Consider $\Sigma = \{p_1 \vee p, p_2, \neg p_1 \vee \neg p_2 \vee p, \neg p_2 \vee \neg p\}$

$\Sigma_0 = \{p_1 \vee p, \neg p_1 \vee \neg p_2 \vee p\}$
$\Sigma_1 = \{\neg p_2 \vee \neg p\}$
$\Sigma_* = \{p_2\}$

$\Sigma_0' = \{p_1, \neg p_1 \vee \neg p_2\}$
$\Sigma_1' = \{\neg p_2\}$

$(\Sigma_0' \wedge \Sigma_*) \vee (\Sigma_1' \wedge \Sigma_*) = \{p_1, \neg p_1 \vee \neg p_2, p_2\} \vee \{\neg p_2, p_2\}$

Completeness (contd.)

Proof(contd.)

Now consider formula $$\underbrace{(\Sigma_0' \land \Sigma_*) \lor (\Sigma_1' \land \Sigma_*)}_{\text{p is not in the formula}}$$

claim: If $(\Sigma_0' \land \Sigma_*) \lor (\Sigma_1' \land \Sigma_*)$ is sat then Σ is sat.

- Assume for some m, $m \models (\Sigma_0' \land \Sigma_*) \lor (\Sigma_1' \land \Sigma_*)$.
- Therefore, $m \models \Sigma_*$.(why?)
- Case 1: $m \models (\Sigma_1' \land \Sigma_*)$.
  Since all the clauses of Σ_0 have p, $m[p \mapsto 1] \models \Sigma_0$(why?).
  Since Σ_1' and Σ_* have no p, $m[p \mapsto 1] \models \Sigma_1'$ and $m[p \mapsto 1] \models \Sigma_*$.
  Since $\Sigma_1' \models \Sigma_1$, $m[p \mapsto 1] \models \Sigma_1$.
- Case 2: $m \models (\Sigma_0' \land \Sigma_*)$. Symmetrically, $m[p \mapsto 0] \models \Sigma_0 \land \Sigma_1 \land \Sigma_*$.
- Therefore, $\Sigma_0 \land \Sigma_1 \land \Sigma_*$ is sat. ...

Exercise 10.2 *Show Σ and $(\Sigma_0' \land \Sigma_*) \lor (\Sigma_1' \land \Sigma_*)$ are equivsatisfiable but not equivalent.*

Completeness (contd.)

Proof(contd.)

Since Σ is unsat, $(\Sigma'_0 \wedge \Sigma_*) \vee (\Sigma'_1 \wedge \Sigma_*)$ is unsat.

Now we apply the induction hypothesis.
Since $(\Sigma'_0 \wedge \Sigma_*) \vee (\Sigma'_1 \wedge \Sigma_*)$ is unsat and has no p, $\bot \in Res^*(\Sigma'_0 \wedge \Sigma_*)$ and $\bot \in Res^*(\Sigma'_1 \wedge \Sigma_*)$.

Choose a derivation of $\bot$ from both. Now there are two cases.

Case 1: $\bot$ was derived using only clauses from Σ_* in any of the two proofs.
Therefore, $\bot \in Res^*(\Sigma_*)$. Therefore, $\bot \in Res^*(\Sigma_0 \wedge \Sigma_1 \wedge \Sigma_*)$.

Case 2: In both the derivations Σ'_0 are Σ'_1 are involved respectively. ...

Example: choosing derivations

Example 10.3

Recall our example $\Sigma_* = \{p_2\}$, $\Sigma'_0 = \{p_1, \neg p_1 \vee \neg p_2\}$, $\Sigma'_1 = \{\neg p_2\}$.

Proofs for our running example

$$\dfrac{\dfrac{p_1 \quad \neg p_1 \vee \neg p_2}{\neg p_2} \qquad p_2}{\bot} \qquad\qquad \dfrac{\neg p_2 \qquad p_2}{\bot}$$

The above proofs belong to the case 2.

The above proofs do not start from clauses that are from Σ*. So we cannot use them immediately. We need a construction.*

Completeness (contd.)

Proof(contd.)

Case 2: In both the derivations Σ_0' are Σ_1' are involved respectively.(contd.)

Therefore, $p \in Res^*(\Sigma_0 \land \Sigma_*)$ and $\neg p \in Res^*(\Sigma_1 \land \Sigma_*)$.(why?)[needs thinking; look at the example to understand.]

Therefore, $\perp \in Res^*(\Sigma_0 \land \Sigma_1 \land \Sigma_*)$(why?). $\square$

Example 10.4

Recall proofs.

$$\dfrac{\dfrac{\dfrac{p_1 \lor p \qquad \neg p_1 \lor \neg p_2 \lor p}{\neg p_2 \lor p} \qquad p_2}{\perp \lor p} \qquad \dfrac{\neg p_2 \lor \neg p \qquad p_2}{\perp \lor \neg p}}{\perp}$$

Exercise 10.3

Let F be an unsatisfiable CNF formula with n variables. Show that there is a resolution proof of $\perp$ from F that is smaller than $2^{n+1} - 1$.

Commentary: By inserting p in Σ_0' clauses of the left proof we obtain clauses of Σ_0. Therefore, the proof transforms into a proof from $\Sigma_0 \land \Sigma_*$. Since there are no $\neg p$ any where in $\Sigma_0 \land \Sigma_*$, we are guaranteed a leftover p. We need symmetric argument for deriving $\neg p$ from $\Sigma_1 \land \Sigma_*$.

Completeness so far

Theorem 10.2
Let Σ be a finite set of formulas and F be a formula. The following statements are equivalent.

- $\Sigma \vdash F$
- $\emptyset \in Res^*(\Sigma')$, *where Σ' is CNF of $\bigwedge \Sigma \wedge \neg F$*
- $\Sigma \models F$

Proof.

$$
\begin{tikzcd}
 & \Sigma \vdash F \arrow[dr, "\text{Proof rules are sound}"] & \\
\emptyset \in Res^*(\Sigma') \arrow[ur, "\text{Proof can be generated}"] & & \Sigma \models F \arrow[ll, "\text{Last theorem}"]
\end{tikzcd}
$$

$\square$

Exercise 10.4
How is the last theorem applicable here?

Topic 10.2

Finite to Infinite

How do we handle $\Sigma'' \models F$ if Σ'' is an infinite set?

There is an interesting argument.

$$
\begin{tikzcd}
& \Sigma \vdash F \arrow[dr, "\text{Proof rules are sound}"] & \\
\emptyset \in \mathit{Res}^*(\Sigma') \arrow[ur, "\text{Proof can be generated}"] & & \Sigma \models F \arrow[ll, "\text{Last theorem}"'] \arrow[d, leftrightarrow, "\text{For some finite } \Sigma \subseteq \Sigma''"'] \\
& & \Sigma'' \models F
\end{tikzcd}
$$

We prove that if an infinite set implies a formula, then a finite subset also implies the formula.

A theorem on strings

Theorem 10.3

Consider an infinite set S of finite binary strings. There exists an infinite string w such that the following holds.

$$\forall n. \quad |\{w' \in S | w_n \textit{ is prefix of } w'\}| = \infty$$

where w_n is prefix of w of length n.

Proof.

We inductively construct w, and we will keep shrinking S. Initially $w := \epsilon$.

base case:

- Let $S_0 := \{u \in S | u \text{ starts with } 0\}$.
- Let $S_1 := \{u \in S | u \text{ starts with } 1\}$.
- Let $S_\epsilon := S \cap \{w\}$.

Clearly, $S = S_\epsilon \cup S_0 \cup S_1$. Either S_0 or S_1 are infinite.(why?)

If S_0 is infinite, $w := 0$ and $S := S_0$. Otherwise, $w := 1$ and $S := S_1$.

w is prefix of all strings in the shrunk S. ...

A theorem on strings (contd.)

Proof(contd.)

induction step:

Let us suppose we have w of length n and w is prefix of all strings in S.

- Let $S_0 := \{u \in S | u \text{ has } 0 \text{ at } n+1th \text{ position}\}$.
- Let $S_1 := \{u \in S | u \text{ has } 1 \text{ at } n+1th \text{ position}\}$.
- Let $S_\epsilon := S \cap \{w\}$.

Clearly, $S = S_\epsilon \cup S_0 \cup S_1$. Either S_0 or S_1 are infinite.(why?)

If S_0 is infinite, $w := w0$ and $S := S_0$. Otherwise, $w := w1$ and $S := S_1$.
w is prefix of all strings in the shrunk S.

Therefore, we can construct the required w. □

Exercise 10.5

Is the above construction of w **practical***?*

Compactness

Theorem 10.4

A set Σ of formulas is satisfiable iff every finite subset of Σ is satisfiable.

Proof.

Forward direction is trivial.(why?)

Reverse direction:
We order formulas of Σ in some order, *i.e.*, $\Sigma = \{F_1, F_2, ...\}$.
Let $\{p_1, p_2, ...\}$ be ordered list of variables from $\mathrm{Vars}(\Sigma)$ such that

- variables in $\mathrm{Vars}(F_1)$ followed by
- the variables in $\mathrm{Vars}(F_2) - \mathrm{Vars}(F_1)$, and so on.

Due to the rhs, we have models m_n such that $m_n \models \bigwedge_{i=1}^{n} F_i$.

We need to construct a model m such that $m \models \Sigma$. Let us do it! ...

Compactness (contd.) II

Proof(contd.)

We assume $m_n : \mathrm{Vars}(\bigwedge_{i=1}^{n} F_i) \rightarrow \mathcal{B}$.

We may see m_n as finite binary strings, since variables are ordered $p_1, p_2, ...$ and m_n is assigning values to some first k variables.

Let $S = \{m_n \text{ as a string } | n > 0\}$

Due to the previous theorem, there is an infinite binary string m such that each prefix of m is prefix of infinitely many strings in S. ...

Compactness (contd.) III

Proof(contd.)

claim: if we interpret m as a model(how?), then $m \models \Sigma$.

- Consider a formula $F_n \in \Sigma$.
- Let $\bigwedge_{i=1}^{n} F_i$ has k variables.
- Consider m' be the prefix of length k of m.

$$\overbrace{\underbrace{\underbrace{101....}_{m'}\;....}_{m_j}\;.....}^{m}$$

- There must be $m_j \in S$, such that m' is prefix of m_j and $j > n$.(why?)
- Since $m_j \models \bigwedge_{i=1}^{j} F_i$, $m_j \models F_n$.
- Therefore, $m' \models F_n$.
- Therefore, $m \models F_n$. $\square$

Implication is decidable for finite lhs.

Theorem 10.5

If Σ is a finite set of formulas, then $\Sigma \models F$ is decidable.

Proof.

Due to truth tables. $\square$

Two defintions: effectively enumerable and semi-decidable

Definition 10.3

If we can enumerate a set using an algorithm, then it is called effectively enumerable.

Example 10.5

The set of all terminating programs is not effectively enumerable.

Definition 10.4

A yes/no problem is semi-decidable, if we have an algorithm for only one side of the problem.

Commentary: The above definitions will be formally covered in complexity theory and automata theory.

Implication is effectively enumerable.

Theorem 10.6

If Σ is effectively enumerable, then $\Sigma \models F$ is semi-decidable.

Proof.

Due to compactness if $\Sigma \models F$, there is a finite set $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models F$.
Since Σ is effectively enumerable, let $G_1, G_2, ...$ be the enumeration of Σ.
Let $S_n \triangleq \{G_1, \dots, G_n\}$.
There must be a $\Sigma_0 \subseteq S_{k \text{(why?)}}$.
Therefore, $S_k \models F$.
We may enumerate S_n and check $S_n \models F$, which is decidable.
Therefore, eventually we will say yes if $\Sigma \models F$. □

Commentary: If $\Sigma \models F$ does not hold, the above procedure will not terminate. Therefore, implication is only semi-decidable and not decidable.

Topic 10.3

Problems

Slim proofs

For an unsatisfiable CNF formula F, a resolution proof R is a sequence of clauses such that:

- Each clause in R is either from F or derived by resolution from the earlier clauses in R.
- The last clause in R is $\perp$.

Consider the following definitions

- For a clause C and literal ℓ, let $C|_\ell \triangleq \begin{cases} \top & \overset{\ell \in C}{\cancel{C \in \ell}} \\ C - \{\bar{\ell}\} & \text{otherwise.} \end{cases}$
- Let $F|_\ell \triangleq \bigwedge_{C \in F} C|_\ell$.
- Let $width(R)$ and $width(F)$ be the length of the longest clause in R and F, respectively.
- Let $slimest(F) \triangleq min(\{width(R) | R \text{ is resolution proof of unsatisfiability of } F\})$.

Exercise 10.6

Prove the following facts.

1. *if $F|_\ell$ has an unsatisfiability proof, then $F \wedge \ell$ has an unsatisfiability proof.*
2. *if $k \geq width(F)$, $slimest(F|_\ell) \leq k - 1$, and $slimest(F|_{\bar{\ell}}) \leq k$ then $slimest(F) \leq k$.*

End of Lecture 10