CS228 Logic for Computer Science 2021

Lecture 14: First-order logic

Instructor: Ashutosh Gupta

IITB, India

Compile date: 2021-02-10

Topic 14.1

First-order logic (FOL) syntax

First-order logic(FOL)

First-order logic(FOL)

$=$

propositional logic $+$ quantifiers over individuals $+$ functions/predicates

"First" comes from this property

Example 14.1

Consider argument: Humans are mortal. Socrates is a human. Therefore, Socrates is mortal.

In symbolic form,

$\forall x.(H(x) \Rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$

- $H(x) = x$ *is a human*
- $M(x) = x$ *is mortal*
- $s =$ *Socrates*

A note on FOL syntax

The FOL syntax may appear non-intuitive and cumbersome.

FOL requires getting used to it like many other concepts such as complex numbers.

Connectives and variables

An FOL consists of three disjoint kinds of symbols

- variables
- logical connectives
- non-logical symbols : function and predicate symbols

Variables

We assume that there is a set Vars of variables, which is countably infinite in size.

- Since Vars is countable, we assume that variables are indexed.

$$\mathrm{Vars} = \{x_1, x_2, \ldots, \}$$

- The variables are just names/symbols without any inherent meaning
- We may also sometimes use x, y, z to denote the variables

Now forget all the definitions of the propositional logic. We will redefine everything and the new definitions will subsume the PL definitions.

Logical connectives

The following are a finite set of symbols that are called logical connectives.

formal name	symbol	read as	
true	$\top$	top	0-ary
false	$\bot$	bot	
negation	$\neg$	not	unary
conjunction	$\wedge$	and	binary
disjunction	$\vee$	or	
implication	$\Rightarrow$	implies	
exclusive or	$\oplus$	xor	
equivalence	$\Leftrightarrow$	iff	
equality	$=$	equals	binary predicate
existential quantifier	$\exists$	there is	quantifiers
universal quantifier	$\forall$	for each	
open parenthesis	(		punctuation
close parenthesis	)		
comma	,		

Non-logical symbols

FOL is a parameterized logic

The parameter is a signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$, where

- $\mathbf{F}$ is a set of function symbols and
- $\mathbf{R}$ is a set of predicate symbols (aka relational symbols).

Each symbol is associated with an arity ≥ 0.

We write $f/n \in \mathbf{F}$ and $P/k \in \mathbf{R}$ to explicitly state the arity

Example 14.2

We may have $\mathbf{F} = \{c/0, f/1, g/2\}$ *and* $\mathbf{R} = \{P/0, H/2, M/1\}$.

Example 14.3

We may have $\mathbf{F} = \{+/2, -/2\}$ *and* $\mathbf{R} = \{< /2\}$.

Commentary: We are familiar with predicates, which are the things that are either true or false. However, the functions are the truly novel concept.

Non-logical symbols (contd.)

F and **R** may either be finite or infinite.

Each **S** defines an FOL. We say, consider an FOL with signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ ...

We may not mention **S** if from the context the signature is clear.

Example 14.4

In the propositional logic, $\mathbf{F} = \emptyset$ *and*

$$\mathbf{R} = \{p_1/0, p_2/0, ...\}.$$

Constants and Propositional variable

There are special cases when the arity is zero.

$f/0 \in \mathbf{F}$ is called a constant.

$P/0 \in \mathbf{R}$ is called a propositional variable.

Building FOL formulas

Let us use the ingredients to build the FOL formulas.

It will take a few steps to get there.

- terms
- atoms
- formulas

Syntax : terms

Commentary: Terms are defined using grammar notation. If unfamiliar with the notation Please look https://en.wikipedia.org/wiki/Formal_grammar

Definition 14.1

*For signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$, **S**-terms $T_{\mathbf{S}}$ are given by the following grammar:*

$$t ::= x \mid f(\underbrace{t, \dots, t}_{n}),$$

where $x \in \text{Vars}$ and $f/n \in \mathbf{F}$.

Example 14.5

Consider $\mathbf{F} = \{c/0, f/1, g/2\}$. Let x_is be variables. The following are terms.

- $f(x_1)$
- $g(f(c), g(x_2, x_1))$
- c
- x_1

You may be noticing some similarities between variables and constants

Infix notation

We may write some functions and predicates in infix notation.

Example 14.6

we may write $+(a, b)$ *as* $a + b$ *and similarly* $< (a, b)$ *as* $a < b$.

Syntax: atoms

Definition 14.2

S-*atoms* $A_{\mathbf{S}}$ *are given by the following grammar:*

$$a ::= P(\underbrace{t, \dots, t}_{n}) \mid t = t \mid \bot \mid \top,$$

where $P/n \in \mathbf{R}$.

Exercise 14.1

Consider $\mathbf{F} = \{s/0\}$ *and* $\mathbf{R} = \{H/1, M/1\}$. *Which of the following are atom?*

- $H(x)$
- s
- $M(s)$
- $H(M(s))$

Commentary: Remember: you can nest terms but not atoms.

Equality within logic vs. equality outside logic

We have an equality $=$ within logic and the other when we use to talk about logic.

Both are distinct objects.

Some notations use same symbols for both and the others do not to avoid confusion.

Whatever is the case, we must be very clear about this.

Syntax: formulas

Definition 14.3

S-*formulas* $P_{\mathbf{S}}$ *are given by the following grammar:*

$$F ::= a \mid \neg F \mid (F \wedge F) \mid (F \vee F) \mid (F \Rightarrow F) \mid (F \Leftrightarrow F) \mid (F \oplus F) \mid \forall x.(F) \mid \exists x.(F)$$

where $x \in \text{Vars}$.

Example 14.7

Consider $\mathbf{F} = \{s/0\}$ *and* $\mathbf{R} = \{H/1, M/1\}$

The following is a $(\mathbf{F}, \mathbf{R})$-*formula:*

$$\forall x.(H(x) \Rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$$

Commentary: Notice we have dropped some parenthesis. We will not discuss the minimal parenthesis issue at length.

Unique parsing

For FOL we will ignore the issue of unique parsing,

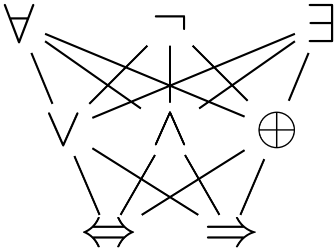
and assume

all the necessary precedence and associativity orders are defined

for ensuring human readability and unique parsing.

Precedence order

We will use the following precedence order in writing the FOL formulas



Example 14.8

The following are the interpretation of the formulas after dropping parenthesis

- $\forall x.H(x) \Rightarrow M(x) = \forall x.(H(x)) \Rightarrow M(x)$
- $\exists z \forall x.\exists y.G(x, y, z) = \exists z.(\forall x.(\exists y.G(x, y, z)))$

Topic 14.2

FOL - semantics

Semantics : models

Definition 14.4

For signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$*, a* $\mathbf{S}$*-model* m *is a*

$$(D_m; \{f_m : D_m^n \rightarrow D_m | f/n \in \mathbf{F}\}, \{P_m \subseteq D_m^n | P/n \in \mathbf{R}\}),$$

where D_m *is a nonempty set. Let* $\mathbf{S}$*-Mods denotes the set of all* $\mathbf{S}$*-models.*

Some terminology

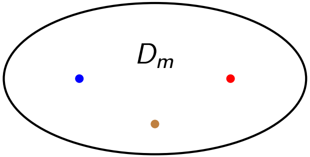
- D_m is called domain of m.
- f_m assigns meaning to f under model m.
- Similarly, P_m assigns meaning to P under model m.

Commentary: Models are also known as interpretations/structures.

Example: model

Example 14.9

Consider $\mathbf{S} = (\{c/0, f/1, g/2\}, \{H/1, M/2\})$.
Let us suppose our model m *has domain* $D_m = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{brown}\bullet}\}$.



We need to assign value to each function.

- $c_m = {\color{blue}\bullet}$
- $f_m = \{{\color{blue}\bullet} \mapsto {\color{brown}\bullet}, {\color{brown}\bullet} \mapsto {\color{red}\bullet}, {\color{red}\bullet} \mapsto {\color{red}\bullet}\}$
- $$\begin{aligned} g_m = \{&({\color{blue}\bullet}, {\color{blue}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{blue}\bullet}, {\color{brown}\bullet}) \mapsto {\color{blue}\bullet}, ({\color{blue}\bullet}, {\color{red}\bullet}) \mapsto {\color{red}\bullet}, \\ &({\color{red}\bullet}, {\color{blue}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{red}\bullet}, {\color{brown}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{red}\bullet}, {\color{red}\bullet}) \mapsto {\color{brown}\bullet}, \\ &({\color{brown}\bullet}, {\color{blue}\bullet}) \mapsto {\color{red}\bullet}, ({\color{brown}\bullet}, {\color{brown}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{brown}\bullet}, {\color{red}\bullet}) \mapsto {\color{blue}\bullet}\} \end{aligned}$$

We also need to assign values to each predicate.

- $H_m = \{{\color{blue}\bullet}, {\color{brown}\bullet}\}$ $\quad M_m = \{({\color{blue}\bullet}, {\color{blue}\bullet}), ({\color{brown}\bullet}, {\color{red}\bullet})\}$

Exercise 14.2

a. How many models are there for the signature with the above domain?
b. Suppose $P/0 \in \mathbf{R}$, *give a value to* P_m.

Semantics: assignments

Recall, We also have variables. Who will assign to the variables?

Definition 14.5

An assignment *is a map* $\nu : \text{Vars} \rightarrow D_m$

Example 14.10

In our running example the domain is $\mathbb{N}$*. We may have the following assignment.*

$$\nu = \{x \mapsto 2, y \mapsto 3\}$$

Semantics: term value

Definition 14.6

For a model m and assignment ν, we define $m^\nu : T_{\mathbf{S}} \rightarrow D_m$ as follows.

$$m^\nu(x) \triangleq \nu(x) \qquad\qquad x \in \text{Vars}$$

$$m^\nu(f(t_1,\dots,t_n)) \triangleq f_m(m^\nu(t_1),\dots,m^\nu(t_n))$$

Example 14.11

Consider $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and term* $s(x) + y$

Consider model $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

$$m^\nu(s(x)+y) = m^\nu(s(x)) +^{\mathbb{N}} m^\nu(y) = succ(m^\nu(x)) +^{\mathbb{N}} 2 = succ(3) +^{\mathbb{N}} 2 = 6$$

Semantics: satisfaction relation

Definition 14.7

We define the satisfaction relation $\models$ among models, assignments, and formulas as follows

- $m, \nu \models \top$
- $m, \nu \models P(t_1, \dots, t_n)$ *if* $(m^\nu(t_1), \dots, m^\nu(t_n)) \in P_m$
- $m, \nu \models t_1 = t_2$ *if* $m^\nu(t_1) = m^\nu(t_n)$
- $m, \nu \models \neg F$ *if* $m, \nu \not\models F$
- $m, \nu \models F_1 \vee F_2$ *if* $m, \nu \models F_1$ *or* $m, \nu \models F_2$

  skipping other propositional connectives
- $m, \nu \models \exists x.(F)$ *if there is* $u \in D_m : m, \nu[x \mapsto u] \models F$
- $m, \nu \models \forall x.(F)$ *if for each* $u \in D_m : m, \nu[x \mapsto u] \models F$

Example: satisfiability

Example 14.12

Consider $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and formula* $\exists z.s(x) + y = s(z)$

Consider model $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

We have seen $m^{\nu}(s(x) + y) = 6$.

$m^{\nu[z \mapsto 5]}(s(x) + y) = m^{\nu}(s(x) + y) = 6.$ *//Since z does not occur in the term*

$m^{\nu[z \mapsto 5]}(s(z)) = 6$

Therefore, $m, \nu[z \mapsto 5] \models s(x) + y = s(z)$.

$m, \nu \models \exists z.s(x) + y = s(z)$.

Satisfiable, true, valid, and unsatisfiable

We say

- F is *satisfiable* if there are m and ν such that $m, \nu \models F$
- Otherwise, F is called unsatisfiable (written $\not\models F$)
- F is *true* in m ($m \models F$) if for all ν we have $m, \nu \models F$
- F is *valid* ($\models F$) if for all ν and m we have $m, \nu \models F$

Exercise: model

Consider $\mathbf{S} = (\{c/0, f/1\}, \{H/1, M/2\})$. Let us suppose model m has $D_m = \{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}\}$ and the values of the symbols in m are

- $c_m = \textcolor{blue}{\bullet}$
- $f_m = \{\textcolor{blue}{\bullet} \mapsto \textcolor{brown}{\bullet}, \textcolor{brown}{\bullet} \mapsto \textcolor{red}{\bullet}, \textcolor{red}{\bullet} \mapsto \textcolor{red}{\bullet}\}$
- $H_m = \{\textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}\}$
- $M_m = \{(\textcolor{blue}{\bullet}, \textcolor{blue}{\bullet}), (\textcolor{brown}{\bullet}, \textcolor{red}{\bullet})\}$

Exercise 14.3

Which of the following hold?

- $m, \{x \mapsto \textcolor{red}{\bullet}\} \models M(f(x), x)$ ✗
- $m, \{\} \models \exists x.H(x)$ ✓
- $m, \{\} \models \exists x.H(f(x))$ ✓
- $m, \{x \mapsto \textcolor{red}{\bullet}\} \models H(x)$ ✗
- $m, \{\} \models \forall x.H(x)$ ✗
- $m, \{\} \models H(c)$ ✓

Extended satisfiability (repeat from PL)

We extend the usage of $\models$. Let Σ be a (possibly infinite) set of formulas.

Definition 14.8

$m, \nu \models \Sigma$ if $m, \nu \models F$ for each $F \in \Sigma$.

Definition 14.9

$\Sigma \models F$ if for each model m and assignment ν if $m, \nu \models \Sigma$ then $m, \nu \models F$.
$\Sigma \models F$ is read Σ implies F. If $\{G\} \models F$ then we may write $G \models F$.

Definition 14.10

Let $F \equiv G$ if $G \models F$ and $F \models G$.

Definition 14.11

Formulas F and G are equisatisfiable if

$$F \text{ is sat} \quad \text{iff} \quad G \text{ is sat.}$$

Commentary: These definitions are identical to the propositional case.

Topic 14.3

Problems

FOL to PL

Exercise 14.4

Give the restrictions on FOL such that it becomes the propositional logic. Give an example of FOL model of a non-trivial propositional formula.

Valid formulas

Exercise 14.5

Prove/Disprove the following formulas are valid.

- $\forall x.P(x) \Rightarrow P(c)$ ✓
- $\forall x.(P(x) \Rightarrow P(c))$ ✗
- $\exists x.(P(x) \Rightarrow \forall x.P(x))$ ✓
- $\exists y \forall x.R(x,y) \Rightarrow \forall x \exists y.R(x,y)$ ✓
- $\forall x \exists y.R(x,y) \Rightarrow \exists y \forall x.R(x,y)$ ✗

Properties of FOL

Exercise 14.6

Show the validity of the following formulas.

1. $\neg\forall x.\ P(x) \Leftrightarrow \exists x.\ \neg P(x)$
2. $\neg\exists x.\ P(x) \Leftrightarrow \forall x.\ \neg P(x)$
3. $(\forall x.\ (P(x) \land Q(x))) \Leftrightarrow \forall x.\ P(x) \land \forall x.\ Q(x)$
4. $(\exists x.\ (P(x) \lor Q(x))) \Leftrightarrow \exists x.\ P(x) \lor \exists x.\ Q(x)$

Exercise 14.7

Show $\forall$ *does not distribute over* $\lor$*.*

Show $\exists$ *does not distribute over* $\land$*.*

Example: non-standard models

Consider $\mathbf{S} = (\{0/0, s/1, +/2\}, \{\})$ and formula $\exists z.s(x) + y = s(z)$

Unexpected model: Let $m = (\{a, b\}^*; \epsilon, append\_a, concat)$.

- The domain of m is the set of all strings over alphabet $\{a, b\}$.
- $append\_a$: appends a in the input and
- $concat$: joins two strings.

Let $\nu = \{x \mapsto ab, y \mapsto ba\}$.
Since $m, \nu[z \mapsto abab] \models s(x) + y = s(z)$, we have $m, \nu \models \exists z.s(x) + y = s(z)$.

Exercise 14.8

- *Show* $m, \nu[y \mapsto bb] \not\models \exists z.s(x) + y = s(z)$
- *Give an assignment* ν *s.t.* $m, \nu \models x \neq 0 \Rightarrow \exists y.\ x = s(y)$.
  Show $m \not\models \forall x.\ x \neq 0 \Rightarrow \exists y.\ x = s(y)$.

Find models

Exercise 14.9

For each *of the following formula give a model that satisfies the formula. If there is no model that satisfies a formula, then report that the formula is unsatisfiable.*

1. $\forall x.\exists y R(x, y) \land \neg\exists x.\forall y R(x, y)$
2. $\neg\forall x.\exists y R(x, y) \land \exists x.\forall y R(x, y)$
3. $\neg\forall x.\exists y R(x, y) \land \neg\exists x.\forall y R(x, y)$
4. $\forall x.\exists y R(x, y) \land \exists x.\forall y R(x, y)$

Similar quantifiers

Exercise 14.10

Show using FOL fol semantics.

- $\exists x.\exists x.\ F \equiv \exists x.\ F$
- $\exists x.\exists y.\ F \equiv \exists y.\exists x.\ F$
- $\forall x.\forall x.\ F \equiv \forall x.\ F$
- $\forall x.\forall y.\ F \equiv \forall y.\forall x.\ F$

Exercise : compact notation for terms

Since we know arity of each symbol, we need not write "," "(", and ")" to write a term unambiguously.

Example 14.13

$f(g(a,b),h(x),c)$ can be written as $fgabhxc$.

Exercise 14.11

Consider $\mathbf{F} = \{f/3, g/2, h/1, c/0\}$ and $x, y \in \mathrm{Vars}$.
Insert parentheses at appropriate places in the following if they are valid term.

- $hc =$
- $gxc =$
- $fhxhyhc =$
- $fx =$

Exercise 14.12

Give an algorithm to insert the parentheses

Commentary: We will not use the compact notation in the course. It makes the formulas very difficult to read.

Exercise: DeBruijn index of quantified variables

DeBruijn index is a method for representing formulas without naming the quantified variables.

Definition 14.12

Each DeBruijn index is a natural number that represents an occurrence of a variable in a formula, and denotes the number of quantifiers that are in scope between that occurrence and its corresponding quantifier.

Example 14.14

We can write $\forall x.H(x)$ *as* $\forall.H(1)$*. 1 is indicating the occurrence of a quantified variable that is bounded by the closest quantifier. More examples.*

- $\exists y \forall x.M(x, y) = \exists \forall.M(1, 2)$
- $\exists y \forall x.M(y, x) = \exists \forall.M(2, 1)$
- $\forall x.(H(x) \Rightarrow \exists y.M(x, y)) = \forall.(H(1) \Rightarrow \exists.M(2, 1))$

Exercise 14.13

Give an algorithm that translates FOL formulas into DeBurjin indexed formulas.

Drinker paradox

Exercise 14.14

Prove

There is someone x such that if x drinks, then everyone drinks.

Let $D(x) \triangleq x$ drinks. Formally

$$\exists x.(D(x) \Rightarrow \forall x.\ D(x))$$

https://en.wikipedia.org/wiki/Drinker_paradox

Exercise: satisfaction relation

Exercise 14.15

Consider $\mathbf{S} = (\{\cup/2\}, \{\in /2\})$ *and formula* $F = \exists x. \forall y.\ \neg y \in x$ *(what does it say to you!)*

Consider $\mathbf{S}$*-model* $m = (\mathbb{N}; \cup_m = max, \in_m = \{(i,j) | i < j\})$ *and* $\nu = \{x \mapsto 2, y \mapsto 3\}$.

$m, \nu \models F$?

Topic 14.4

Extra slides: some properties of models

Homomorphisms of models

Definition 14.13

Consider $\mathbf{S} = (\mathbf{F}, \mathbf{R})$*. Let* m *and* m' *be* $\mathbf{S}$*-models.*
A function $h : D_m \to D_{m'}$ *is a homomorphism of* m *into* m' *if the following holds.*

- *for each* $f/n \in \mathbf{F}$*, for each* $(d_1, .., d_n) \in D_m^n$

$$h(f_m(d_1, .., d_n)) = f_{m'}(h(d_1), .., h(d_n))$$

- *for each* $P/n \in \mathbf{R}$*, for each* $(d_1, .., d_n) \in D_m^n$

$$(d_1, .., d_n) \in P_m \quad \text{iff} \quad (h(d_1), .., h(d_n)) \in P_{m'}$$

Definition 14.14

A homomorphism h *of* m *into* m' *is called isomorphism if* h *is one-to-one.*
m *and* m' *are called isomorphic if an* h *exists that is also onto.*

Example : homomorphism

Example 14.15

Consider $\mathbf{S} = (\{+/2\}, \{\})$.

Consider $m = (\mathbb{N}, +^{\mathbb{N}})$ *and* $m = (\mathcal{B}, \oplus^{\mathcal{B}})$,

$h(n) = n$ *mod* 2 *is a homomorphism of* m *into* m'.

Homomorphism theorem for terms and quantifier-free formulas without $=$

Theorem 14.1

Let h be a homomorphism of m into m'. Let ν be an assignment.

1. *For each term t, $h(m^{\nu}(t)) = m'^{(\nu \circ h)}(t)$*
2. *If formula F is quantifier-free and has no symbol "$=$"*

$$m^{\nu} \models F \quad \textit{iff} \quad m'^{(\nu \circ h)} \models F$$

Proof.

Simple structural induction. $\square$

Exercise 14.16

For a quantifier-free formula F that may have symbol "$=$", show

$$\textit{if} \quad m^{\nu} \models F \quad \textit{then} \quad m'^{(\nu \circ h)} \models F$$

Why the reverse direction does not work?

Homomorphism theorem with $=$

Theorem 14.2

Let h be a homomorphism of m into m'. Let ν be an assignment. If h is isomorphism then the reverse implication also holds for formulas with "$=$".

Proof.

Let us suppose $m'^{(\nu \circ h)} \models s = t$.
Therefore, $m'^{(\nu \circ h)}(s) = m'^{(\nu \circ h)}(t)$.
Therefore, $h(m^{\nu}(s)) = h(m^{\nu}(t))$.
Due to the one-to-one condition of h, $m^{\nu}(s) = m^{\nu}(t)$.
Therefore, $m^{\nu} \models s = t$. $\square$

Exercise 14.17

For a formula F (with quantifiers) without symbol "$=$", show

$$\text{if} \quad m'^{(\nu \circ h)} \models F \quad \text{then} \quad m^{\nu} \models F.$$

Commentary: Note that that implication direction is switched from the previous exercise.

Homomorphism theorem with quantifiers

Theorem 14.3

Let h be a isomorphism of m into m' and ν be an assignment.
If h is also onto, the reverse direction also holds for the quantified formulas.

Proof.

Let us assume, $m^{\nu} \models \forall x.F$.
Choose $d' \in D_{m'}$.
Since h is onto, there is a d such that $d = h(d')$.
Therefore, $m^{\nu[x\mapsto d]} \models F$.
Therefore, $m'^{\nu[x\mapsto d']} \models F$.
Therefore, $m'^{(\nu\circ h)} \models \forall x.\ F$. □

Theorem 14.4

If m and m' are isomorphic then for all sentences F, $m \models F$ iff $m' \models F$.

Commentary: The reverse direction of the above theorem is not true.

End of Lecture 14