# Problem Set 4

1. Call $L \subseteq \Sigma^+$ non counting if
$$\exists n_0 \forall n \geq n_0 \forall u, v, w \in \Sigma^* (uv^n w \in L \Leftrightarrow uv^{n+1} w \in L)$$
That is for all $n \geq n_0$, either all $uv^n w$ are in $L$, or none is.
A language $L$ is counting iff it is not non counting.
   - Formulate the condition for a counting language
   - Is $L = (aa)^+$ counting or not?
   - Is $L = (ab)^+$ counting or not?

2. Write second order logic formulae to capture the following:
   (a) There is a path from node $s$ to node $t$ in the graph. The signature is $\tau = \{E\}$.
   (b) Every bounded non empty set has a least upper bound. The signature is $\tau = \{\leq\}$

3. Let $\Sigma$ be a finite alphabet. The atomic formulae in MSO defined over $\Sigma^*$ are $x = y, x < y, S(x, y), X(x)$ and $Q_a(x)$, $a \in \Sigma$. Consider the following logic called $\text{MSO}_0$ having atomic formulae of the following forms:
$$Sing(X), X \subseteq Y, X < Y, S(X, Y), Q_a(X)$$
where
   - $Sing(X)$ means that $X$ is a SO variable of cardinality 1;
   - $X \subseteq Y$ means that every element of the SO variable $X$ is contained in the SO variable $Y$;
   - $X < Y$ means that SO variables $X, Y$ have cardinality 1, and that the element in $Y$ is greater than the element in $X$;
   - $S(X, Y)$ means that SO variables $X, Y$ have cardinality 1, and $Y$ contains the successor of the element in $X$; and,
   - $Q_a(X)$ means that all positions in $X$ are decorated by $a \in \Sigma$.

   If $\varphi$ is an atomic formula in MSO, then $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$, $\forall x\ \varphi$ and $\forall X\ \varphi$ are formulae in MSO. Similarly, if $\varphi$ is an atomic formula in $\text{MSO}_0$, then, $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$ and $\forall X\ \varphi$ are formulae in $\text{MSO}_0$.
   Compare the expressiveness of $MSO$ and $MSO_0$.

4. For the formula $\exists x \forall y (x < y \rightarrow Q_a(y))$ give an equivalent $\text{MSO}_0$ formula.

5. Consider the following NFA $N = (\{0, 1, 2, 3\}, \{a, b\}, \Delta, \{0\}, \{1\})$ with $\Delta(0, b) = \{1\}$, $\Delta(1, a) = \{2\}$, $\Delta(2, a) = \{2\}$, $\Delta(2, b) = \{3\}$ and $\Delta(3, b) = \{0\}$. Write an MSO formula with two SO variables that characterizes $L(N)$.

$b\,(a^+ b^3)^*$