Logic Representation

Virendra Singh
Professor
Computer Architecture and Dependable Systems Lab
Department of Electrical Engineering
Indian Institute of Technology Bombay
http://www.ee.iitb.ac.in/~viren/
E-mail: viren@ee.iitb.ac.in

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CADSL

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
- Canonical Forms in common usage:
  - Truth Table
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)
  - Binary Decision Diagram (BDD)
  - Reed Muller Representation

Representation: Truth Table

Truth Table

X Y Z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

unique.

F =

Logic Expression

$$F = \bar{X}.\bar{Y}.Z + X.\bar{Y}.\bar{Z} + X.\bar{Y}.Z$$
$$+ X.\bar{Y}.Z + X.Y.Z$$

ON set

OFF set

2^n

Representation: Truth Table

Truth Table

X Y Z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

uniqueness

Truth table

→ ON set (min terms)

OFF set

$y = \overline{\bar{a}+\bar{b}+\bar{c}}$

product → $y = a \cdot b \cdot c$

$\bar{y} = (\bar{a}+\bar{b}+\bar{c})$ sum

Minterms

- Minterms are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for *n* variables.
- <u>Example:</u> Two variables (X and Y)produce 2 x 2 = 4 combinations:

  XY (both normal)
  $X\overline{Y}$ (X normal, Y complemented)
  $\overline{X}Y$ (X complemented, Y normal)
  $\overline{X}\,\overline{Y}$ (both complemented)
- Thus there are four minterms of two variables.

Minterms

- Product term containing all variables
  - Possibly complemented
- For n-variable function, are 2^n possible minterms
  - Each truth table row indicates if one minterm included

Minterms for Three Variables

X	Y	Z	Product Term	Symbol	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
0	0	0	$\overline{X}\cdot\overline{Y}\cdot\overline{Z}$	m_0	1	0	0	0	0	0	0	0
0	0	1	$\overline{X}\overline{Y}Z$	m_1	0	1	0	0	0	0	0	0
0	1	0	$\overline{X}Y\overline{Z}$	m_2	0	0	1	0	0	0	0	0
0	1	1	$\overline{X}YZ$	m_3	0	0	0	1	0	0	0	0
1	0	0	$X\overline{Y}\overline{Z}$	m_4	0	0	0	0	1	0	0	0
1	0	1	$X\overline{Y}Z$	m_5	0	0	0	0	0	1	0	0
1	1	0	$XY\overline{Z}$	m_6	0	0	0	0	0	0	1	0
1	1	1	XYZ	m_7	0	0	0	0	0	0	0	1

Functions of Minterms

- Can specify function as sum of minterms
  - Use full minterms, or minterm indices
- Example:
  - $F = X'\,Y'\,Z + X'\,Y\,Z + X\,Y\,Z$
  - $F(X,Y,Z) = m1 + m3 + m7$
  - $F(X,Y,Z) = \Sigma m(1, 3, 7)$
- A function's complement includes all minterms not included in the original

Maxterms

$f(x,y)$

- Maxterms are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for n variables.
- Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:

$f = a \cdot b \cdot c$

$\bar{f} = \bar{a} + \bar{b} + \bar{c}$

$\mathbf{X + Y}$ (both normal)

$\mathbf{X + \overline{Y}}$ (x normal, y complemented)

$\mathbf{\overline{X} + Y}$ (x complemented, y normal)

$\mathbf{\overline{X} + \overline{Y}}$ (both complemented)

Maxterms

- Less common form
- Sum term in which all variables (or their complements) occur
- For **n**-variable function, 2^n possible maxterms

Maxterms for Three Variables

X	Y	Z	Sum Term	Symbol	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	$X+Y+Z$	M_0	0	1	1	1	1	1	1	1
0	0	1	$X+Y+\overline{Z}$	M_1	1	0	1	1	1	1	1	1
0	1	0	$X+\overline{Y}+Z$	M_2	1	1	0	1	1	1	1	1
0	1	1	$X+\overline{Y}+\overline{Z}$	M_3	1	1	1	0	1	1	1	1
1	0	0	$\overline{X}+Y+Z$	M_4	1	1	1	1	0	1	1	1
1	0	1	$\overline{X}+Y+\overline{Z}$	M_5	1	1	1	1	1	0	1	1
1	1	0	$\overline{X}+\overline{Y}+Z$	M_6	1	1	1	1	1	1	0	1
1	1	1	$\overline{X}+\overline{Y}+\overline{Z}$	M_7	1	1	1	1	1	1	1	0

Functions of Maxterms

- Can specify function as product of maxterms
  - Use full maxterms, or maxterm indices
- Example:
  - $F = (X'+Y'+Z)(X'+Y+Z)(X+Y+Z)$
  - $F(X,Y,Z) = M6 \cdot M4 \cdot M0$
  - $F(X,Y,Z) = \Pi\ M(0, 4, 6)$
- A function's complement includes all maxterms not included in the original
- Functions can be converted between minterm and maxterm form using the "other" indices

Maxterms and Minterms

- Examples: Two variable minterms and maxterms.

Index	Minterm	Maxterm
0	$\underbrace{\bar{x}\,\bar{y}}$ m_0	$\underbrace{x + y}$ M_0
1	$\bar{x}\,y$ m_1	$x + \bar{y}$ M_1
2	$x\,\bar{y}$	$\bar{x} + y$
3	$x\,y$	$\bar{x} + \bar{y}$

- The index above is important for describing which variables in the terms are true and which are complemented.

Minterm Function Example

- Example: Find $F_1 = m_1 + m_4 + m_7$
- $F1 = \bar{x}\ \bar{y}\ z + x\ \bar{y}\ \bar{z} + x\ y\ z$

x y z	index	m_1	+	m_4	+	m_7	$= F_1$
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Maxterm Function Example

- **Example: Implement F1 in maxterms:**

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x+y+z)\cdot(x+\bar{y}+z)\cdot(x+\bar{y}+\bar{z})$$
$$\cdot(\bar{x}+y+\bar{z})\cdot(\bar{x}+\bar{y}+z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$ = F1
0 0 0	0	0 · 1 · 1 · 1 · 1 = 0
0 0 1	1	1 · 1 · 1 · 1 · 1 = 1
0 1 0	2	1 · 0 · 1 · 1 · 1 = 0
0 1 1	3	1 · 1 · 0 · 1 · 1 = 0
1 0 0	4	1 · 1 · 1 · 1 · 1 = 1
1 0 1	5	1 · 1 · 1 · 0 · 1 = 0
1 1 0	6	1 · 1 · 1 · 1 · 0 = 0
1 1 1	7	1 · 1 · 1 · 1 · 1 = 1

Canonical Sum of Minterms

- Any Boolean function can be expressed as a <u>Sum of Minterms</u>.
  - For the function table, the <u>minterms</u> used are the terms corresponding to the 1's
  - For expressions, <u>expand</u> all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term ($\mathbf{v} + \bar{\mathbf{v}}$).

$x \cdot 1 = x \cdot (y + \bar{y})$

- Example: Implement $\mathbf{f} = \mathbf{x} + \bar{\mathbf{x}}\,\bar{\mathbf{y}}$ as a sum of minterms.

First expand terms: $\mathbf{f} = \mathbf{x}(\mathbf{y} + \bar{\mathbf{y}}) + \bar{\mathbf{x}}\,\bar{\mathbf{y}}$

Then distribute terms: $\mathbf{f} = \mathbf{x}\mathbf{y} + \mathbf{x}\bar{\mathbf{y}} + \bar{\mathbf{x}}\,\bar{\mathbf{y}}$

Express as sum of minterms: $f = m_3 + m_2 + m_0$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- **Example: Given** $F(x, y, z) = \Sigma_m(1,3,5,7)$

$$\bar{F}(x, y, z) = \Sigma_m(0,2,4,6)$$

$$\bar{F}(x, y, z) = \Pi_M(1,3,5,7)$$

Conversion Between Forms

- **To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:**
  - **Find the function complement by swapping terms in the list with terms not in the list.**
  - **Change from products to sums, or vice versa.**
- **Example:Given F as before:** $F(x, y, z) = \Sigma_m(1,3,5,7)$
- **Form the Complement:** $\bar{F}(x, y, z) = \Sigma_m(0,2,4,6)$
- **Then use the other form with the same indices – this forms the complement again, giving the other form of the original function:** $F(x, y, z) = \Pi_M(0,2,4,6)$

Canonical form.

Truth Table

SOM (Sum of Minterms) ✓

POM (Product of Maxterms)

$2^n \to 2^n$ bit

Textual representation

64 bit

128 bit

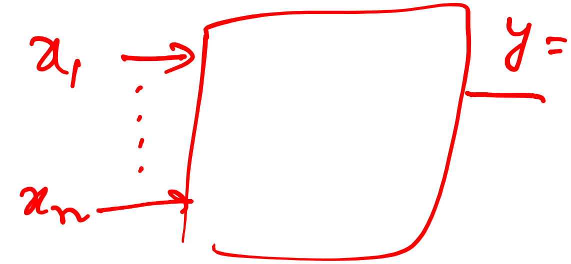
$= 2^{128} \simeq 2^{120} = (2^{10})^{12} = (10^3)^{12} = 10^{36}$ bits

$= 10^{35}$ Bytes

$= 10^{26}$ GB

not scalable ✓

Shannon's Expansion



$f(x_1, x_2 \dots x_n)$

$$= x_i \cdot f(x_1, x_2 \dots x_i = 1 \dots x_n) + \bar{x}_i \cdot f(x_1, x_2 \dots x_i = 0, \dots x_n)$$

$$f(x_1, x_2 . \; x_i = 1 \dots x_n) = f(x_1, x_2 \dots x_n)\big|_{x_i = 1} = f_{x_i}$$

$$f(x_1, x_2 . \; x_i = 0 \dots x_n) = f(x_1, x_2 \dots x_n)\big|_{x_i = 0} = f_{\bar{x}_i}$$

Cofactor

$$f = \begin{cases} f_{x_i} & x_i = 1 \\ f_{\bar{x}_i} & x_i = 0 \end{cases}$$

$$f(x_1, x_2 . x_i . x_n) = x_i f_{x_i} + \bar{x}_i f_{\bar{x}_i} = x_i f_{x_i} \oplus \bar{x}_i \cdot f_{\bar{x}_i}$$

$+ \equiv \oplus$ $\quad a + b = a \oplus b$ $\quad a . b = 0$

Graphical Method

- BDD is canonical form of representation
- Shanon's expansion theorem
- $f(x_1, x_2, \ldots x_i, \ldots\ldots x_n) =$

  $x_i.f(x_1, x_2, \ldots\ ,x_i=1, \ldots\ldots x_n)\ +$

  $x_i'.\ f(x_1, x_2, \ldots\ ,x_i=0, \ldots\ldots x_n)$

$x_1 - - - x_n$

n

f_{x_i}

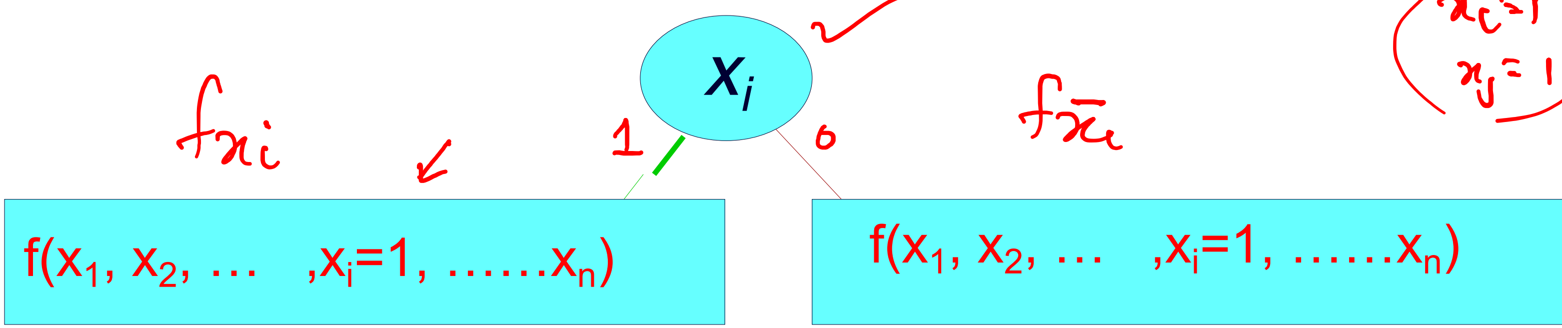
$f_{\bar{x}_i}$

$f_{x_i} = x_j f_{x_i x_j} + \bar{x}_j f_{x_i \bar{x}_j}$

g

$x_i = 1$, $x_j = 1$

$x_i = 1$, $x_j = 0$



Decision Structures

$x_1 < x_2 < x_3$

$f(x_1, x_2, x_3)$

Truth Table

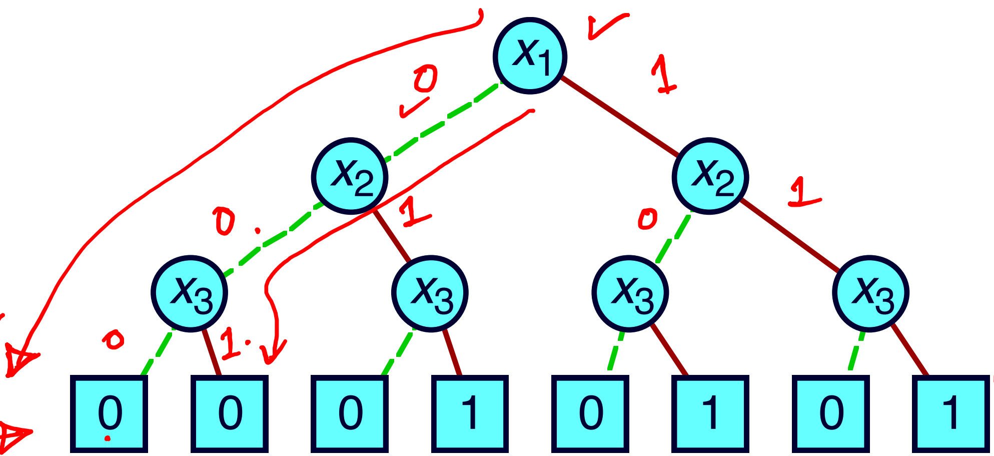
x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

2^n

$2^n - 1$

Decision Tree

2^n paths



2^n leaf

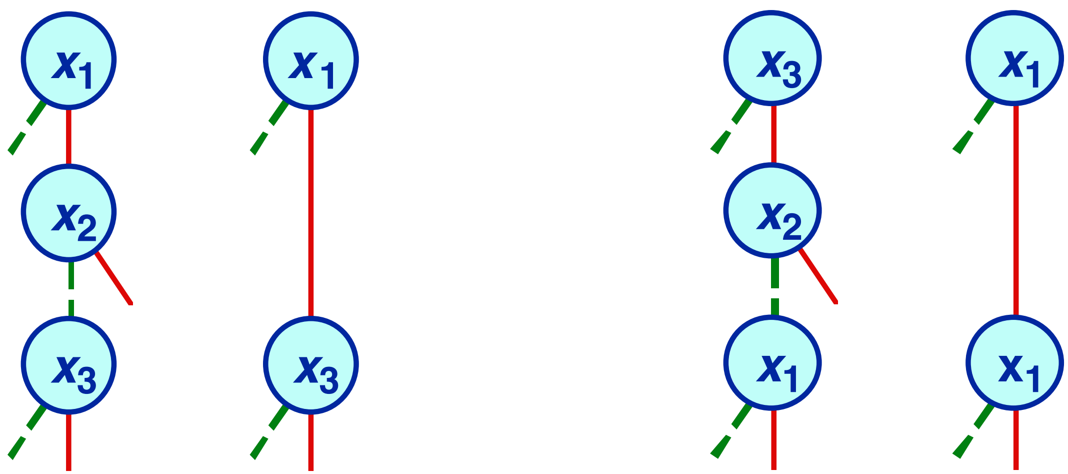
$2^{n+1} - 1$ nodes

- Vertex represents decision
- Follow green (dashed) line for value 0
- Follow red (solid) line for value 1
- Function value determined by leaf value. $\to 2^n$

Variable Ordering

- Assign arbitrary total ordering to variables
  - e.g., $x_1 < x_2 < x_3$
- Variables must appear in ascending order along all paths

OK **Not OK**

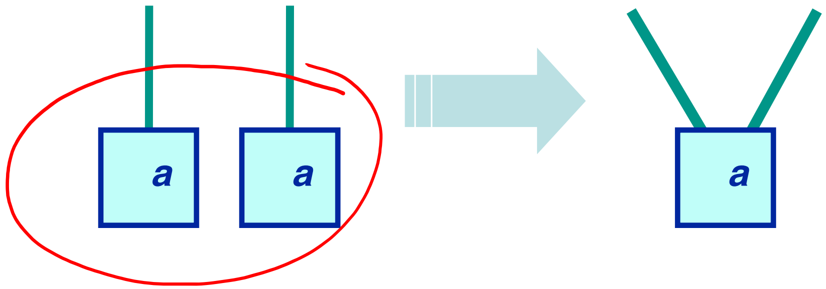


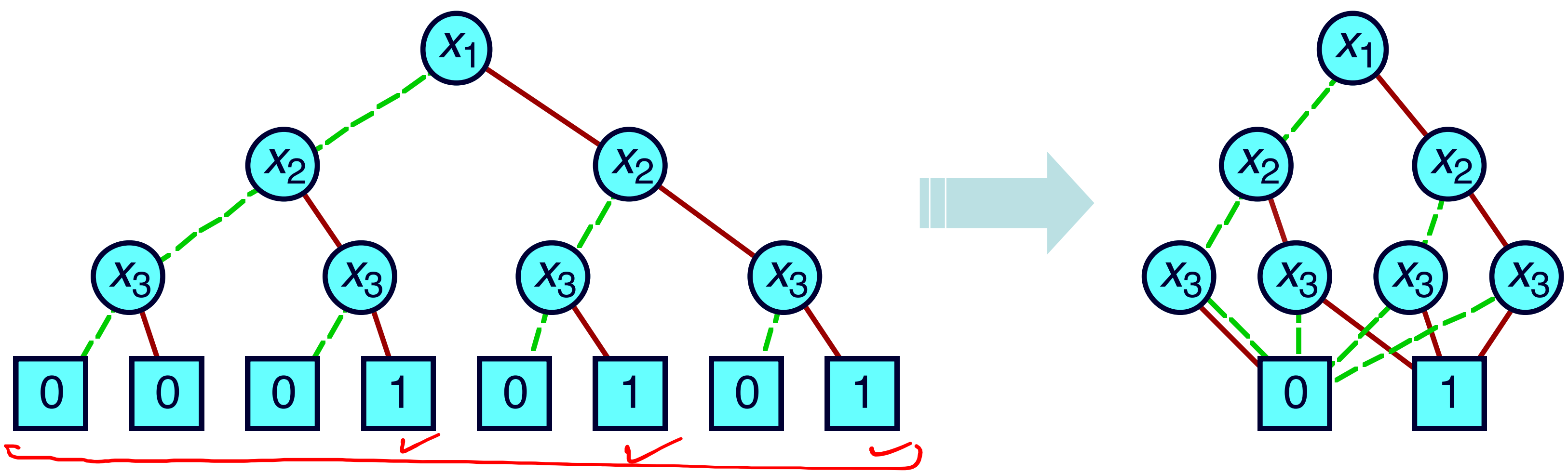
Properties

- No conflicting variable assignments along path
- Simplifies manipulation

Reduction Rule #1

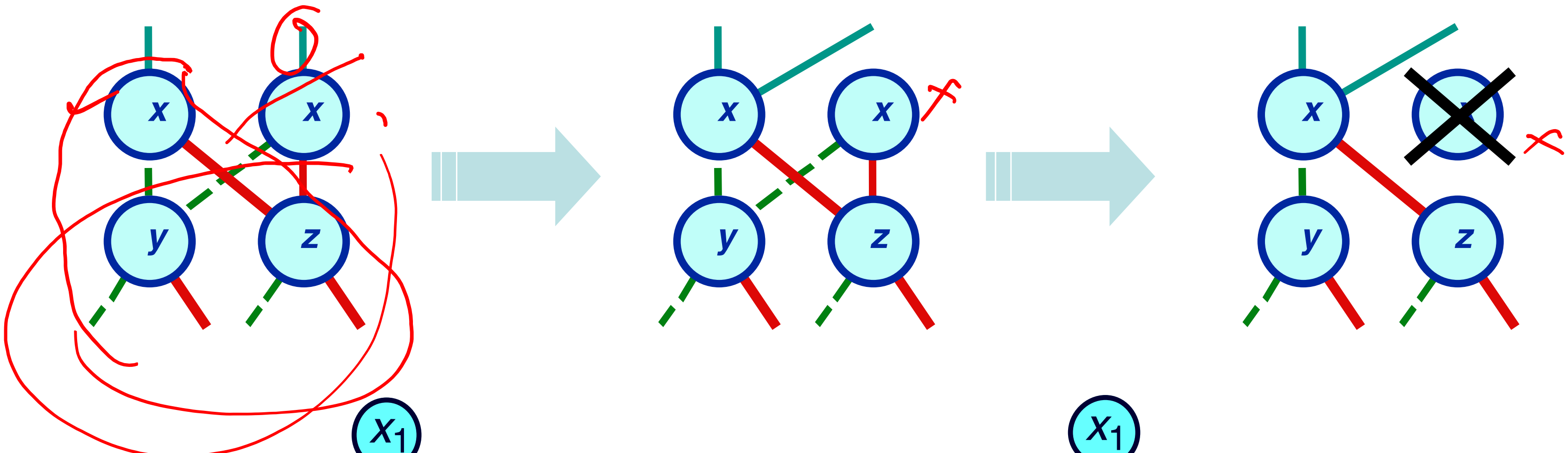
Merge equivalent leaves

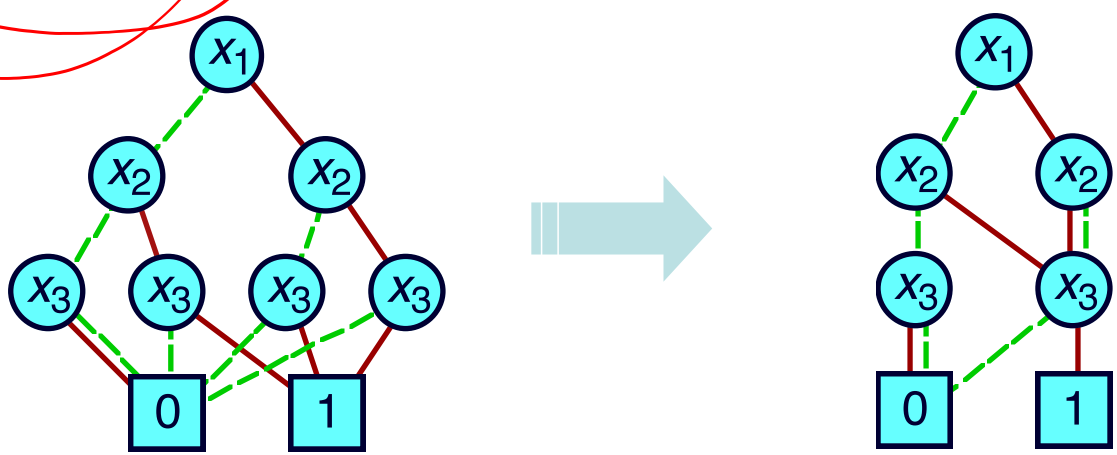




Reduction Rule #2

Merge isomorphic nodes



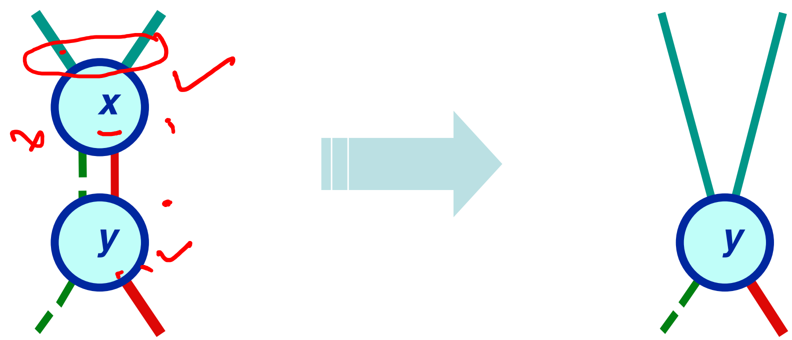


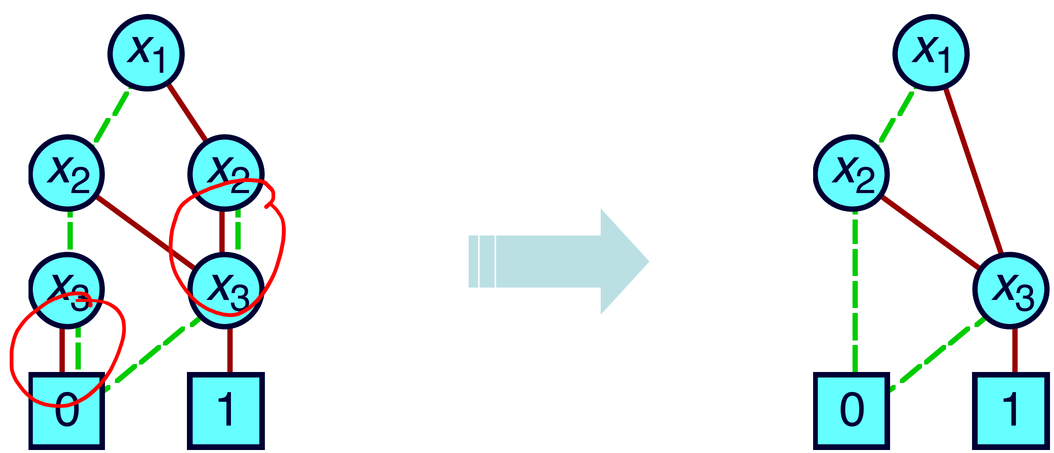
Reduction Rule #3

ROBDD

Canonical

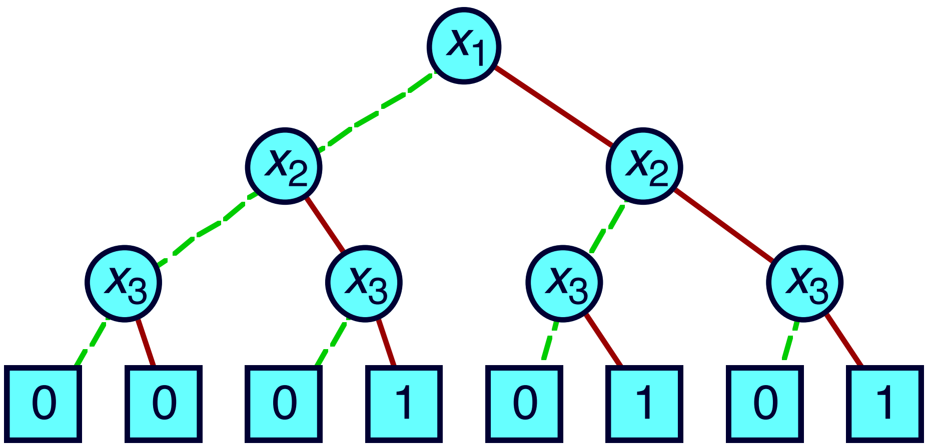
Eliminate Redundant Tests



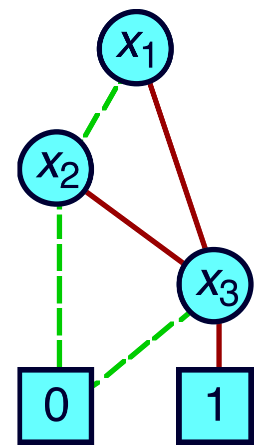


Example OBDD

Initial Graph



Reduced Graph



$(x_1+x_2)đx_3$

- Canonical representation of Boolean function
  - ❖ For given variable ordering
  - ➢ Two functions equivalent if and only if graphs isomorphic
    - o Can be tested in linear time
  - ➢ Desirable property: *simplest form is canonical*.

$$f(x,y) = x \cdot \underbrace{f(x=1,y)}_{f_x} + \bar{x} \cdot \underbrace{f(x=0,y)}_{f_{\bar{x}}}$$

$$= \underbrace{x \cdot f_x}_{\checkmark} + \underbrace{\bar{x}\, f_{\bar{x}}}_{\checkmark}$$

$$x \cdot f_x \cdot \bar{x} \cdot f_{\bar{x}} = 0$$

Orthogonality condition

$$= x \cdot f_x \oplus \bar{x}\, f_{\bar{x}}$$

a	b	a+b	a⊕b
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	0

Thank You