CS228 Logic for Computer Science 2021

Lecture 17: FOL - conjunctive normal form

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CNF normalization steps

We can convert any FOL sentence into a first-order logic conjunctive normal form(FOL CNF).

We will define FOL CNF by following the process of transformation.

The following transformations results in the CNF.

1. Rename apart : rename variables for each quantifier
2. Negation normal form : pushing negation inside
3. Prenex form : pulling quantifiers to front
4. Skolemization: remove existential quantifiers (only satisfiability preserving)
5. CNF transformation: turn the quantifier-free part of the sentence into CNF
6. Syntactical removal of universal quantifiers: a CNF with free variables.

Topic 17.1

Step 1: Rename apart

What is in a name?

Theorem 17.1

If $x, y \notin FV(F(z))$, *then* $\forall x.F(x)$ *and* $\forall y.F(y)$ *are provably equivalent .*

Proof.

1. $\{\forall x.F(x)\} \vdash \forall x.F(x)$ — Assumption
2. $\{\forall x.F(x)\} \vdash F(y)$ — $\forall$-Instantiation applied to 1
3. $\{\forall x.F(x)\} \vdash \forall y.F(y)$ — $\forall$-Intro applied to 2, since $y \notin FV(\forall x.F(x))$

We can run the proof both directions. $\square$

Exercise 17.1

a. *Prove: if* $x, y \notin FV(F(z))$, *then* $\exists x.F(x)$ *and* $\exists y.F(y)$ *are provably equivalent.*

b. *Give proof for renaming a quantified variable to a fresh name that is not on the top.*

Step 1: rename apart

Definition 17.1

A formula F is renamed apart if no quantifier in F use a variable that is used by another quantifier or occurs as free variable in F.

Due to the previous theorem, we can assume that every quantifier has different variable. If not, we can rename quantified variables apart.

Example 17.1

Consider formula $\neg(\exists x.\forall y R(x,y) \Rightarrow \forall y.\exists x(R(x,y) \wedge P(x)))$. After renaming apart we obtain the following

$$\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \wedge P(w)))$$

Topic 17.2

Step 2: Negation normal form

Relating $\forall$ and $\exists$

Theorem 17.2 *If we have $\Sigma \vdash \neg\exists x.\neg F(x)$, we can prove $\Sigma \vdash \forall x.F(x)$.*

Proof.

1. $\Sigma \vdash \neg\exists x.\neg F(x)$ — Premise
2. $\Sigma \cup \{\neg F(y)\} \vdash \neg F(y)$ — Assumption (choose fresh $y_{\text{(why?)}}$)
3. $\Sigma \cup \{\neg F(y)\} \vdash \exists x.\neg F(x)$ — $\exists$-Intro
4. $\Sigma \vdash F(y)$ — propositional rules applied to 1 and 3
5. $\Sigma \vdash \forall x.F(x)$ — $\forall$-Intro on 4

$\square$

Exercise 17.2

a. *Prove: if we have $\Sigma \vdash \neg\forall x.F(x)$, we can prove $\Sigma \vdash \exists x.\neg F(x)$.*

b. *Prove: If we have $\Sigma \vdash \neg\exists x.F(x)$, we can prove $\Sigma \vdash \forall x.\neg F(x)$.*

(Hint: Replace $\neg F(.)$ by $F(.)$ in the above proof)

Step 2: negation normal form(NNF)

Definition 17.2

A formula F is in negation normal form if all the negation symbols in the formula occur in form of atomic formulas.

Due to the previous theorems and the properties of propositional connectives, we can translate any formula in negation normal form.

Example: negation normal form

Example

~~Exercise~~ 17.3

We convert $\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \land P(w)))$ *into NNF form as follows*

$$
\begin{aligned}
\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \land P(w))) &\equiv (\exists x.\forall y.R(x,y) \land \neg\forall z.\exists w.(R(w,z) \land P(w))) \\
&\equiv (\exists x.\forall y.R(x,y) \land \exists z.\neg\exists w.(R(w,z) \land P(w))) \\
&\equiv (\exists x.\forall y.R(x,y) \land \exists z.\forall w.\neg(R(w,z) \land P(w))) \\
&\equiv (\exists x.\forall y.R(x,y) \land \exists z.\forall w.(\neg R(w,z) \lor \neg P(w)))
\end{aligned}
$$

Commentary: The reverse direction of the above equivalences are proven in the extra slides of this lecture.

Topic 17.3

Step 3: Prenex form

No occurrence; no issues

Theorem 17.3

Let x be a variable such that $x \notin FV(F)$. Then F, $\exists x.F$, and $\forall x.F$ are provably equivalent.

Proof.

We have already seen $\forall x.F$ to $\exists x.F$.

Proving from F to $\forall x.F$

1. $\Sigma \vdash F$ Premise
2. $\Sigma \vdash \forall x.\ F$ $\forall$-Intro applied to 1

Since x is not in F, we choose $y, z \notin FV(\Sigma \cup \{F\})$ and say $F(z)\{z \mapsto y\} = F$. $\forall$-Intro conditions are met.(why?)

Proving from $\exists x.F$ to F

1. $\Sigma \vdash \exists x.F$ Premise
2. $\Sigma \cup \{F\} \vdash F$ Assumption
3. $\Sigma \vdash F \Rightarrow F$ propositional rules applied to 2
4. $\Sigma \vdash \exists x.F \Rightarrow F$ $\exists$-Elim applied to 3
5. $\Sigma \vdash F$ propositional rules applied to 4 and 1

$\square$

Commentary: Please check if the side conditions of $\exists$-Elim are met in step 4 of the right proof. Why absence of x is important in the proof?

No occurrence; we can pull quantifiers to top

Theorem 17.4

If $x \notin FV(G)$, then $\exists x.F(x) \wedge \exists x.G$ and $\exists x.(F(x) \wedge G)$ are provably equivalent.

Proof.

Reverse direction is trivial. Consider the forward direction.

Commentary: If x occurs in G, which step of the following proof does not work?

1. $\Sigma \vdash \exists x.F(x) \wedge \exists x.G$ — Premise
2. $\Sigma \vdash \exists x.G$ — propositional rules applied to 1
3. $\Sigma \vdash G$ — previous theorem applied to 2
4. $\Sigma \cup \{F(x)\} \vdash F(x) \wedge G$ — propositional rules applied to 3
5. $\Sigma \cup \{F(x)\} \vdash \exists x.(F(x) \wedge G)$ — $\exists$-Intro applied to 4
6. $\Sigma \vdash F(x) \Rightarrow \exists x.(F(x) \wedge G)$ — $\Rightarrow$-Intro applied to 5
7. $\Sigma \vdash \exists x.F(x) \Rightarrow \exists x.(F(x) \wedge G)$ — $\exists$-Elim applied to 6
8. $\Sigma \vdash \exists x.(F(x) \wedge G)$ — propositional rules applied to 7 and 1 □

Exercise 17.4

If $x \notin FV(G)$, then $\forall x.F(x) \vee \forall x.G$ and $\forall x.(F(x) \vee G)$ are provably equivalent.

Step 3: prenex form

Definition 17.3

A formula F is in prenex form if all the quantifiers of the formula occur as prefix of F. The quantifier-free suffix of F is called matrix of F.

Due to the previous theorems, we move quantifiers to the front.

Exercise 17.5

Show that the following equivalences hold.

- $\forall x.F \Rightarrow G \equiv \exists x.(F \Rightarrow G)$
- $\exists x.F \Rightarrow G \equiv \forall x.(F \Rightarrow G)$
- $F \Rightarrow \forall x.G \equiv \forall x.(F \Rightarrow G)$
- $F \Rightarrow \exists x.G \equiv \exists x.(F \Rightarrow G)$

Example: prenex form

Example

~~Exercise~~ 17.6

We convert $(\exists x.\forall y.R(x,y) \land \exists z.\forall w.(\neg R(w,z) \lor \neg P(w)))$ *into prenex form as follows*

- $(\exists x.\forall y.R(x,y) \land \exists z.\forall w.(\neg R(w,z) \lor \neg P(w)))$
- $\exists z.(\exists x.\forall y.R(x,y) \land \forall w.(\neg R(w,z) \lor \neg P(w)))$
- $\exists z.\forall w.(\exists x.\forall y.R(x,y) \land (\neg R(w,z) \lor \neg P(w)))$
- $\exists z.\forall w.\exists x.(\forall y.R(x,y) \land (\neg R(w,z) \lor \neg P(w)))$
- $\underbrace{\exists z.\forall w.\exists x.\forall y.}_{\textit{Quantifiers}}\underbrace{(R(x,y) \land (\neg R(w,z) \lor \neg P(w)))}_{\textit{body/matrix of the formula}}$

We move quantifier forward step by step.

In the standard definition of prenex, the body need not be in NNF. Our body is in NNF due to the order of steps we have followed.

Topic 17.4

Step 4: Skolemization

Step 4: Skolemization

Skolemization removes existential quantifiers from the prenex sentence and only the universal quantifiers are left.

Example 17.2

Let us suppose. We know "for every man there is a woman".

$$\forall m.\ \exists w.\ Relationship(m, w)$$

To satisfy the sentence, we need to find a woman for each man.

In other words, there is a function f: $Men \rightarrow Women$.

In terms of FOL, we may write

$$\forall m. Relationship(m, f(m))$$

The replacement of $\exists$ by a function is called skolemization. And f are called skolem functions.

Introduction of skolem function with free variables

Theorem 17.5

Let F be a $\mathbf{S}$-formula, $FV(F) = \{x, y_1, \ldots, y_n\}$ and $f/n \in \mathbf{F}$ does not occur in F. For each model m', there is a model m such that

Commentary: m is not with any assignment, which means for any assignment.

$$m \models \exists x.F \Rightarrow F(f(y_1, \ldots, y_n)).$$

and m and m' only differ on interpretation of f.

Proof.

Consider a model m'. We will construct m. Before, let us construct an interpretation $f' : D^n_{m'} \to D_{m'}$ of f as follows.

$$f'(d_1, ..., d_n) \triangleq \begin{cases} d & \text{if } m', \{y_1 \mapsto d_1, ..., y_n \mapsto d_n\} \models \exists x.F, \\ & \text{Choose } d \in D_{m'} \text{ such that } m', \{y_1 \mapsto d_1, ..., y_n \mapsto d_n, x \mapsto d\} \models F \\ \\ d & \text{otherwise choose any } d \in D_{m'} \end{cases}$$

Why d exists?

Introduction of skolem function with free variables(contd.)

Proof(contd.)

Let us define $m \triangleq m'[f \mapsto f']$.

Since f does not occur in F, if $m, \nu \models \exists x.F$ then $m', \nu \models \exists x.F$.

Due to the construction of m,

$$m, \nu \models F\{x \mapsto f(y_1, \ldots, y_n)\}_{\text{(why?)}}.$$

$\square$

Exercise 17.7

Show there is m such that $m \models F\{x \mapsto f(y_1, \ldots, y_n)\} \Rightarrow \forall x.F$

Introduction of skolem functions under quantifiers

Theorem 17.6

Let $F(x)$ be a $(\mathbf{F}, \mathbf{R})$-formula with $FV(F) = \{x, y_1, \dots, y_n\}$ and $f/n \in \mathbf{F}$ such that f does not occur in $F(x)$.

$$\forall y_1, \dots, y_n . \exists x . F(x) \text{ is sat} \qquad \text{iff} \qquad \forall y_1, \dots, y_n .\ F(f(y_1, \dots, y_n)) \text{ is sat}$$

Proof.

Forward direction:

Assume $m' \models \forall y_1, ...y_n . \exists x . F(x)$. Therefore, $m' \models \exists x . F(x)_{\text{(why?)}}$.

Due to the last theorem, there is m such that $m \models \exists x . F(x) \Rightarrow F(f(y_1, \dots, y_n))$.

Since m and m' only differ on f, $m \models \exists x . F(x)$.

Therefore, $m \models F(f(y_1, .., y_n))$. Therefore, $m \models \forall y_1, ..., y_n .\ F(f(y_1, .., y_n))$. ...

Introduction of skolem functions under quantifiers(contd.)

Proof.

Reverse direction

1. $\{\forall y_1, \ldots, y_n.\ F(f(y_1, \ldots, y_n))\} \vdash \forall y_1, \ldots, y_n.\ F(f(y_1, \ldots, y_n))$ Assumption
2. $\{\forall y_1, \ldots, y_n.\ F(f(y_1, \ldots, y_n))\} \vdash F(f(y_1, \ldots, y_n))$ $\forall$-Elim
3. $\{\forall y_1, \ldots, y_n.\ F(f(y_1, \ldots, y_n))\} \vdash \exists x.F(x)$ $\exists$-Intro
4. $\{\forall y_1, \ldots, y_n.\ F(f(y_1, \ldots, y_n))\} \vdash \forall y_1, \ldots, y_n.\ \exists x.F(x)$ $\forall$-Intro

$\square$

Skolemization of prenex sentence

Since the quantifiers are in prenex form, all $\exists$s can be removed using skolem functions.

Skolemization should be applied from out to inside, i.e.,

remove outermost $\exists$ first.

Example 17.3

Let us skolemize the following sentence

- $\exists z.\forall w.\exists x.\forall y.(R(x,y) \land (\neg R(w,z) \lor \neg P(w)))$
- *Since there are no universals before $\exists z$, we introduce a function $c/0$.*
  $\forall w.\exists x.\forall y.(R(x,y) \land (\neg R(w,c) \lor \neg P(w)))$
- *Since there is a universal $\forall w$ before $\exists x$, we introduce a function $f/1$.*
  $\forall w.\forall y.(R(f(w),y) \land (\neg R(w,c) \lor \neg P(w)))$

Topic 17.5

Step 5-6: FOL CNF

Step 5: convert body of the sentence to CNF

Consider the following skolemized prenex sentence,

$$\forall x_1, \ldots, x_n.\ F.$$

Since F is quantifier-free, we can use propositional logic methods to convert F into CNF

$$\forall x_1, \ldots, x_n.\ C_1 \wedge \cdots \wedge C_k.$$

Example 17.4

In our running example, the body of the sentence was already in CNF

$$\forall w. \forall y. (R(f(w), y) \wedge (\neg R(w, c) \vee \neg P(w))).$$

Exercise 17.8

We may use Tseitin encoding to obtain CNF, which introduces fresh propositional predicates. What is the quantifier over the fresh propositional variables?

Step 6: drop of explicit mention of quantifiers

Consider the following skolemized prenex clauses,

$$\forall x_1, \dots, x_n.\ C_1 \wedge \dots \wedge C_k.$$

Since $\forall$ distributes over $\wedge$, we translate to

$$(\forall x_1, \dots, x_n.\ C_1) \wedge \dots \wedge (\forall x_1, \dots, x_n.\ C_k).$$

We may view the above sentence as conjunction of clauses

$$C_1 \wedge \dots \wedge C_k,$$

without any explicit mention of quantifiers.
We will assume that all the free variables are universally quantified.

Example 17.5

We write the sentence as $R(f(w), y) \wedge (\neg R(w, c) \vee \neg P(w))$

Topic 17.6

Problems

Skolemization

Exercise 17.9

Demonstrate that skolemization does not produce equivalent formula.

Define $f(x)$ in $\exists$ form

Minimize skolem functions

Exercise 17.10

The order of quantifiers determines the number of parameters in the skolem functions. Give a greedy and efficient(linear) strategy for producing prenex formula such that the total number of parameters in skolem functions is minimal?

FOL CNF

Exercise 17.11

Convert the following formulas in FOL CNF

- $\exists z.(\exists x.Q(x,z) \vee \exists x.P(x)) \Rightarrow \neg(\neg\exists x.P(x) \wedge \forall x.\ \exists z.Q(z,x)\ )$
- $\neg\exists n.\ \forall w.\ (Gtn(w) \Rightarrow \exists x,y,z.\ (f(x,y,z) = w \wedge NonEmpty(y) \wedge \neg Gtn(f(x,y)) \wedge (\forall k.\ L(x,y,z,k))))$

$\neg Gtn(w) \vee$

Convert into CNF

Exercise 17.12

Consider the following formulas

$$\begin{aligned}\Sigma = \{\ & \forall x, y, z.\ (z \in x \Leftrightarrow z \in y) \Rightarrow x \approx y, \\ & \forall x, y.\ (x \subseteq y \Leftrightarrow \forall z.\ (z \in x \Rightarrow z \in y)), \\ & \forall x, y, z.\ (z \in x - y \Leftrightarrow (z \in x \wedge z \notin y))\}.\end{aligned}$$

Convert the following formula into FOL CNF.

$$\bigwedge \Sigma \wedge \neg \forall x, y.\ x \subseteq y \Rightarrow \exists z.(y - z \approx x)$$

Theorem prover

Exercise 17.13

Download EPROVER *a first order theorem prover from the following web page.*

`http://wwwlehre.dhbw-stuttgart.de/~sschulz/E/Usage.html`

Run the prover to prove the validity of the following sentence.

$$\forall x.\exists y.\forall z.\exists w.(R(x,y) \vee \neg R(w,z))$$

Report the proof generated by the prover. Explain the proof steps.

Topic 17.7

Extra slides: proofs for pulling negations out

Relating $\forall$ and $\exists$ (reverse direction)

Commentary: $c \neq c$ is proxy for false. Reflex rule allows us to have a representation of true without having true a symbol. It may feel like cheating.

Theorem 17.7 *If we have $\Sigma \vdash \forall x.F(x)$, we can prove $\Sigma \vdash \neg\exists x.\neg F(x)$.*

Proof.

1. $\Sigma \vdash \forall x.F(x)$ — Premise
2. $\Sigma \cup \{\neg F(x)\} \vdash \forall x.F(x)$ — Monotonic applied to 1
3. $\Sigma \cup \{\neg F(x)\} \vdash F(x)$ — $\forall$-Elim applied to 2
4. $\Sigma \cup \{\neg F(x)\} \vdash \neg F(x) \wedge F(x)$ — propositional rules applied to 3
5. $\Sigma \vdash \neg F(x) \Rightarrow c \neq c$ — Contra applied to 4
6. $\Sigma \vdash \exists x.\neg F(x) \Rightarrow c \neq c$ — $\exists$-Elim applied to 5
7. $\Sigma \vdash c = c$ — Reflex
8. $\Sigma \vdash \neg\exists x.\neg F(x)$ — propositional rules applied to 6 and 7 $\square$

Exercise 17.14

Prove: If we have $\Sigma \vdash \forall x.\neg F(x)$, we can derive $\Sigma \vdash \neg\exists x.F(x)$. *Hint : Replace $F(.)$ by $\neg F(.)$ in the above.*

End of Lecture 17