Q1:

A simple example of $y(t) = \sin(\frac{\pi}{2}t)$

$X: \{(x_1, y_1)\}$

Case 1: $\frac{x_1+x_2}{2} = 0$

For left tail 0,

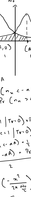
$x_1 = -a$, $x_2 = a$

$\Rightarrow y_1 = -1$, $y_2 = 1$

For right tail 1,

$x_1 = a$, $x_2 = -a$

$\Rightarrow y_1 = 1$, $y_2 = -1$



$P_r(\{\max(|X_1|, |X_2|)\}$ with the x_1, x_2 where $\frac{x_1+x_2}{2} = 0$ & $y_1 = -y_2$ $\}$

$= P_r(\{\max(|y_1|, |y_2|)\}$ where $y_1 = -y_2$ & $\frac{x_1+x_2}{2} = 0$ $\}$

Similarly, $P_r(\{\max(|x_1|, |x_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

Total probability of same: $P_r(\{\max(|x_1|, |x_2|)\}, \{\max(|y_1|, |y_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

$= P_r(\{\max(|x_1|, |x_2|)\}, \{\max(|y_1|, |y_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

$+ P_r(\{\max(|x_1|, |x_2|)\}, \{\max(|y_1|, |y_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

$\Rightarrow P_r = 2 P_r(\{\max(|x_1|, |x_2|)\}, \{\max(|y_1|, |y_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

$\Rightarrow P_r(\{\max(|x_1|, |x_2|)\}, \{\max(|y_1|, |y_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

$= P_r(\{\max(|x_1|, |x_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

Hence sufficient probability, $P_r(\{\max(|x_1|, |x_2|)\}$ where $\frac{x_1+x_2}{2} = 0$ $\}$

$= \int_{\frac{-\sqrt{\pi}}{\sqrt{2}}}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$

$= \int_{\frac{-\sqrt{\pi}}{\sqrt{2}}}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\left(\frac{\pi}{2}\right)}} \exp\left(-\frac{x^2}{2\left(\frac{\pi}{2}\right)}\right) dx$

Substituting $\frac{\pi}{2} = \frac{\sigma^2}{2}$

$= \int_{\frac{-\sqrt{\pi}}{\sqrt{2}}}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{x^2}{\sigma^2}\right) dx$

$= \int_{\frac{-\sqrt{\pi}}{\sqrt{2}}}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{x^2}{\sigma^2}\right) dx$

$= \Phi\left(\sqrt{\pi}, \sqrt{2}\sigma\right)$

Q2

For QPSK

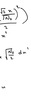
Required probability for first bit being received is given $\{P_r(x(1)=0), P_r(x(1)=1), P_r(x(2)=0), P_r(x(2)=1)\}$

$= \{P_r(x(1)=0), P_r(x(1)=1), P_r(x(2)=0), P_r(x(2)=1), P_r(x(3)=0), P_r(x(3)=1)\}$

$P_r(x(1)=0) + P_r(x(1)=1)$ represents the sign

$P_r(x(1)=0) = P_r(x(1)=0) + P_r(x(1)=-\frac{\sqrt{\pi}}{\sqrt{2}})$ where $x_1 < 0$

$P_r(x(1)=1) = P_r(x(1)=0) + P_r(x(1)=\frac{\sqrt{\pi}}{\sqrt{2}})$ where $x_1 > 0$



Energy of signal for each symbol, $\frac{\pi}{2} + \frac{\pi}{2}$

$\Rightarrow \frac{x_1^2 + x_2^2}{2}$

Hence, $\frac{x_1^2 + x_2^2}{2}$

Right probability $= \frac{1}{2}\left[P_r(x_1 = \frac{\sqrt{\pi}}{\sqrt{2}}) + P_r(x_1 = -\frac{\sqrt{\pi}}{\sqrt{2}}) + P_r(x_2 = \frac{\sqrt{\pi}}{\sqrt{2}}) + P_r(x_2 = -\frac{\sqrt{\pi}}{\sqrt{2}})\right]$

Hence, $= \frac{1}{2}\left[\int_{\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \int_{-\infty}^{-\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \int_{\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \int_{-\infty}^{-\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx\right]$

$= \int_{\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

Substituting $\frac{\pi}{2} = \frac{\sigma^2}{2}$

$= \int_{\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi\left(\frac{\pi}{2}\right)}} \exp\left(-\frac{x^2}{2\left(\frac{\pi}{2}\right)}\right) dx$

$= \int_{\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{x^2}{\sigma^2}\right) dx$

$= \int_{\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{x^2}{\sigma^2}\right) dx$

$= \Phi\left(\sqrt{\pi}, \sqrt{2}\sigma\right)$

Q3

Hence, sufficient probability, $P_r = \frac{1}{2}\left[P_r(x(1)=0) + P_r(x(1)=1) + P_r(x(2)=0) + P_r(x(2)=1) + P_r(x(3)=0) + P_r(x(3)=1)\right]$

Similarly for case 2 bit 1,

$P_r = \frac{1}{2}\left[P_r(x(1)=0) + P_r(x(1)=1) + P_r(x(2)=0) + P_r(x(2)=1) + P_r(x(3)=0) + P_r(x(3)=1)\right]$

Since symmetry, $P_r(x(1)=0) = P_r(x(1)=1)$

$P_r(x(2)=0) = P_r(x(2)=1)$

$P_r(x(3)=0) = P_r(x(3)=1)$

$\Rightarrow P_r = \frac{1}{2}\left[P_r(x(1)=0) + P_r(x(2)=0) + P_r(x(3)=0)\right]$

$= \frac{1}{2}\left[P_r(x(1)=0) + P_r(x(2)=0) + P_r(x(3)=0)\right]$

$= \frac{1}{2}\left[\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)\right]$

$= \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

$P_r = \frac{3}{2}\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) = \frac{3}{2}\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) = \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

Probability for error in 2nd bit,

$P_r = \frac{1}{2}\left[P_r(x(1)=0) + P_r(x(1)=1) + P_r(x(2)=0) + P_r(x(2)=1) + P_r(x(3)=0) + P_r(x(3)=1)\right]$

By symmetry considerations,

$P_r(x(1)=0) = P_r(x(1)=1)$ — ①

$P_r(x(2)=1) = P_r(x(2)=0)$ — ②

$P_r(x(3)=1) = P_r(x(3)=0)$ — ③

$P_r(x(1)=1) = P_r(x(1)=0)$ — ④ .. ① & ④

$P_r(x(2)=0) = P_r(x(2)=1)$ — ⑤ .. ② & ⑤

$P_r(x(3)=0) = P_r(x(3)=1)$ — ⑥ .. ③ & ⑥

$P_r(x(1)=0) = P_r(x_1 > \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(-\frac{\sqrt{\pi}}{\sqrt{2}} < x_1 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(x_1 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\left(\frac{\pi}{2}\right)}} \exp\left(-\frac{x^2}{2\left(\frac{\pi}{2}\right)}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

$P_r(x(1)=1) = P_r(x_1 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(-\frac{\sqrt{\pi}}{\sqrt{2}} < x_1 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(-\frac{\sqrt{\pi}}{\sqrt{2}} < x_1)$

$= \int_{-\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$= \int_{-\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi\left(\frac{\pi}{2}\right)}} \exp\left(-\frac{x^2}{2\left(\frac{\pi}{2}\right)}\right) dx$

$= \int_{-\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \int_{-\frac{\sqrt{\pi}}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

$P_r(x(2)=0) = P_r(\frac{\sqrt{\pi}}{\sqrt{2}} < x_2 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(-\frac{\sqrt{\pi}}{\sqrt{2}} < x_2 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(x_2 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\left(\frac{\pi}{2}\right)}} \exp\left(-\frac{x^2}{2\left(\frac{\pi}{2}\right)}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

$P_r(x(2)=1) = P_r(\frac{\sqrt{\pi}}{\sqrt{2}} < x_2 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(-\frac{\sqrt{\pi}}{\sqrt{2}} < x_2 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= P_r(x_2 < \frac{\sqrt{\pi}}{\sqrt{2}})$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\left(\frac{\pi}{2}\right)}} \exp\left(-\frac{x^2}{2\left(\frac{\pi}{2}\right)}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \int_{-\infty}^{\frac{\sqrt{\pi}}{\sqrt{2}}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{\pi}\right) dx$

$= \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

Total probability for error in 2nd bit, P_r

$= \frac{1}{2}\left[P_r(x(1)=0) + P_r(x(1)=1) + P_r(x(2)=0) + P_r(x(2)=1)\right]$

$= \frac{1}{2}\left[\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)\right]$

$= \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) + \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

$P_r = \frac{3}{2}\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) = \frac{3}{2}\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) = \Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right)$

As we see that P_r & P_r

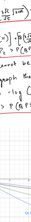
$P_r = P(QPSK) = \left[\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) = \Phi(\cdot)\right], \left[\Phi\left(\sqrt{\frac{\pi}{2}}, \sqrt{\pi}\right) = \Phi(\cdot)\right]$

$P_r = P(QPSK)$

As $\Phi(\cdot)$ is decreasing, $P_r > P(QPSK)$

For P_r, such analogy cannot be supported. Hence, we can conclude from graph that

$-\log(P_r) > -\log(P(QPSK))$

Hence $P_r > P(QPSK)$

Q4