Notation: $\mathbb{N} = \{0,\ 1,\ 2,\ \ldots\}$ denotes the set of nonnegative integers.

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1. Given: $\varphi = \forall x \exists y R(x,y) \wedge \exists y \forall x \neg R(x,y)$.

φ is satisfiable over the following structure: $\mathcal{A} = (\mathbb{N}, R^{\mathcal{A}})$, where $R^{\mathcal{A}} = \{(x,y) \in \mathbb{N}^2 : y = x+1\}$. Note that $\mathbb{N}$ is infinite and countable.
It is clear that given any x, if we choose $y = x+1$, then $y \in \mathbb{N}$ and we have that $R(x,y)$.
Also, given $y = 0 \in \mathbb{N}$, and given any $x \in \mathbb{N}$, it not true that $R(x,0)$. (There is no natural number n such that $n+1 = 0$.)

2.
(i) $\varphi_B(x,y) = \big(\forall z[P(z,x) \leftrightarrow P(z,y)]\big) \wedge \neg F(x) \wedge \neg(x = y)$.
(All parents of x are parents of y and vice-versa and x is not a female. Moreover, x is not a brother of x.)
(ii) $\varphi_A(x,y) = F(x) \wedge \Bigg(\forall z\big(P(z,y) \rightarrow \exists w[P(z,w) \wedge P(w,y)]\big)\Bigg)$
(x is a female and given any parent of x, that parent is the grandparent of y. I have assumed aunt to either be father's sister or mother's sister.)
(iii) $\varphi_C(x,y) = \forall g \forall p\big([P(g,p) \wedge P(p,x)] \rightarrow [\exists p'(P(g,p') \wedge P(p',y) \wedge \neg p = p')]\big)$.
(Note that I've assumed first cousin.)
(iv) $\varphi_O(x) = \forall p \forall x'(P(p,x) \rightarrow x = x')$.
(If p is a parent of x and x' is a human such that p is a parent of x', then x and x' must be the same.)
(v) Guess - $\varphi_H(x,y)$ which says that "x is a husband of y" cannot be defined.

3.
(i) $\text{Zero}(a) = \forall z[a + z = z] \equiv \exists z[a + z = z]$.
(ii) $\text{One}(a) = \forall z[a \times z = z]$.
Note that in this case, the above is not equivalent to the formula obtained for replacing '$\forall$' with '$\exists$.' (Why?)
(iii) $\text{Even}(a) = \exists z[z + z = a]$.
(iv) $\text{Odd}(a) = \neg\,\text{Even}(a)$.
(v) $\text{Prime}(a) = \neg\,\text{One}(a) \wedge \forall x \forall y[(x \times y = a) \rightarrow (\text{One}(x) \wedge \text{One}(y))]$
(vi) Before tackling Goldbach's conjecture, let us first define the following formula:
$\text{Two(a)} = \exists z[\text{One}(z) \wedge z + z = a]$.

$$\forall n[(\text{Even}(n) \wedge \neg\,\text{Zero}(n) \wedge \neg\,\text{Two}(n)) \rightarrow (\exists p \exists q(\text{Prime}(p) \wedge \text{Prime}(q) \wedge n = p + q))].$$

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4. Let $\mathcal{A}$ be a structure of τ. Let $c_{\mathcal{A}}$ denote the fixed element that is assigned to c.
(A1) $\forall x \forall y \forall z[op(x, op(y,z)) = op(op(x,y),z)]$.
(A2) $\forall x(op(x, c_{\mathcal{A}}) = x)$.
(A3) $\forall x \exists y(op(x,y) = c_{\mathcal{A}})$.
(A4) $\forall x \forall y \forall z[op(x,z) = op(y,z) \rightarrow x = y]$.

5.
(i) Take the structure $\mathcal{A}$ whose universe just consists of 0. It trivially satisfies ψ.
(ii) Let $\mathcal{A}$ be the structure with universe as $\{0,1\}$ and $+$ defined as:

$$0+0=0,\ 0+1=0,\ 1+0=0,\ 1+1=0.$$

This clearly does not satisfy φ_2.
(iii) No, that is not the case. For example, take $\mathcal{A}$ whose universe consists of all 2×2 invertible matrices with

real entries with $+$ defined to multiplication and 0 to be the identity element. Then, it is clear that this $\mathcal{A}$ satisfies ψ but not α as there exist invertible matrices which do not commute.
A simpler (but more abstract) is the S_3 group. You may look it up.
(iv) (a) Let $\mathcal{A} = \{0,\ 1\}$ and let $+$ be defined as

$$0+0=0,\ 0+1=1,\ 1+0=1,\ 1+1=1.$$

This satisfies $\varphi_1 \wedge \varphi_2$ but not ψ.
(b) Let $\mathcal{A} = \{0,\ 1,\ 2\}$ and let $+$ be defined as

$+$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Then, $\mathcal{A}$ satisfies $\varphi_2 \wedge \varphi_3$ but not ψ as $1+(1+2) = 1+1 = 0 \neq 2 = 0+2 = (1+1)+2$.
(c) Let $\mathcal{A}$ be the structure with universe as $\{0, 1\}$ and $+$ defined as:

$$0+0=0,\ 0+1=0,\ 1+0=0,\ 1+1=0.$$

This satisfies $\varphi_1 \wedge \varphi_3$ but not ψ.

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7. Let the structure be $\mathcal{G}$ with $u(\mathcal{G}) = \{1, 2\}$. And $E^{\mathcal{G}} = \{(1,1), (1,2), (2,1)\}$.
This satisfies the latter but not the former.
It does not satisfy the former as one can take $x = 2$, then no matter what y is, z takes all possible values. In particular, z takes the value 2. As $(2,2) \notin E^{\mathcal{G}}$, we have it that $E(x,z)$ is not true. Thus, the former sentence is not true.
For the latter sentence, choose $x = 1$. For $y = 1$, choose $z = 1$ and for $y = 2$, choose $z = 1$. Thus, we are done.

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6. The above example illustrates the difference.

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8. (a) $\varphi_{45} = (\exists^{\geq 45} x(x = x)) \wedge \neg(\exists^{\geq 46} x(x \neq x))$.

(b) $\forall x_1 \forall x_2 \ldots \forall x_{n-1} \exists x_n \big[(x_n = x_n) \wedge (x_n \neq x_1) \wedge (x_n \neq x_2) \wedge \cdots \wedge (x_n \neq x_{n-1})\big]$.

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9. By 8. (b), we have shown that $\exists^{\geq n}$ can actually be written as a FO formula. Thus, we can use it freely. So, the answer is simply $(\exists^{\geq n} x(x = x)) \wedge \neg(\exists^{\geq (m+1)} x(x \neq x))$.