CS 218 Design and Analysis of Algorithms

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Module 2: Flow networks, Max-flow, Min-cut and applications

Edge Disjoint Paths problem

What are disjoint paths?

Let $G = (V, E)$ be a directed graph. Let $s, t \in V$ be designated vertices, source and sink.

We say that two s to t paths π and π' are edge disjoint, if they do not share any edges in common.

Edge Disjoint Paths problem

Given:	an undirected graph $G = (V, E)$ and $s, t \in V$, designated vertices, source and sink.
Find:	maximum number of edge disjoint paths from s to t

Network flows to solve edge disjoint paths

Using flows to solve Edge Disjoint Path problem

Given a graph $G = (V, E, s, t)$.

Add a capacity of 1 on each edge $e \in E$ to obtain G'.

Lemma

The max flow value of G' is k if and only if G has k edge disjoint paths.

Proof for ($\Leftarrow$) Suppose G has k edge disjoint s to t paths.

In G', set $f(e) = 1$ if the edge belongs to any such path.

Set $f(e) = 0$ otherwise.

The flow value in G' is k.

Network flows to solve edge disjoint paths

Using flows to solve Edge Disjoint Path problem

Given a graph $G = (V, E, s, t)$.

Add a capacity of 1 on each edge $e \in E$ to obtain G'.

Lemma

The max flow value of G' is k if and only if G has k edge disjoint paths.

Proof for ($\Rightarrow$) Suppose G' has a flow of value k.

Then there is an integral flow of value k in G'.

As capacities are 1, the integral flow assigns 0 or 1 to each edge.

We can generate k edge disjoint paths by tracing out 1 edges.

Network connectivity

How connected is the graph?

Let $G = (V, E)$ is a directed graph. Let $s, t \in V$ be designated vertices, source and sink.

We say that $F \subseteq E$ disconnects s from t, if removal of F from the graph disconnects t from s.

Network Connectivity Problem

Given:	a directed graph $G = (V, E)$ and $s, t \in V$, designated vertices, source and sink.
Find:	minimum sized set $F \subseteq E$ such that removal of F disconnects t from s.

Menger's theorem

Theorem (Menger's Theorem)

The maximum number of edge disjoint s to t paths in a graph is equal to the minimum number of edges whose removal disconnects t from s.

Proof.

($\Leftarrow$) Let $F \subseteq E$ be the minimal set of edges such that the removal of F disconnects t from s.
All s to t paths must have at least one edge from F.
Hence, the set of edge disjoint paths must have cardinality at least as much as $|F|$.

($\Rightarrow$) Suppose the graph has k edge disjoint paths.
This means that it has a flow of size k.
By max-flow min-cut theorem, it has a cut (S, T) of capacity k.
Let F be the edges in the cut.
By the definition of the cut, removal of F disconnects t from s.