CS 218 Design and Analysis of Algorithms

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Module 2: Flow networks, Max-flow, Min-cut and applications

Flow networks, Max-Flow, Min-Cut and Applications

Where there are graphs, there are flow networks!

Flow Networks, Max-Flow, Min-Cut

The problem has interesting history!

> In the 1950s US Air Force researcher Theodore E. Harris and retired U. S. Army general Frank S. Ross introduced this problem.

Flow Networks, Max-Flow, Min-Cut

They wrote a classified report studying the rail network that linked the Soviet Union to its satellite countries in Eastern Europe.

A map can be found by a simple search.

Flow Networks, Max-Flow, Min-Cut

The map was modelled by a graph as follows.

- It had 44 vertices depicting geographical regions.
- 105 edges represented the (transportation) links between these regions.
- Each edge had a number on it, representing the rate at which material could be sent from one region to the other.
- They wanted to compute the cheapest way to disrupt the network (by removing edges).
- They did it in an ad-hoc way.
- Incidentally, they also managed to compute the maximum amount of good that can be carried from Soviet Russia to Europe!

Flow network

Flow Network

It is a directed graph $G = (V, E)$ with designated source s and sink t.

With each edge $(u, v) \in E$, there a non-negative integer associated. We call it the capacity of (u, v). Denoted as $c(u, v)$.

As s is a source, there are no edges entering s.

Similarly, as t is a sink, there are no edges leaving t.

Network Flow

What does it try to model?

The source can be a source of water.

The sink can be a place where the water collects, like a reservoir.

The other nodes in the network can be the junctions that transmit water coming into them without accumulating anything.

The capacities will indicate the volume of water the pipes can carry. For example, 50 litres per minute.

Maximum rate at which water can be sent from the source to the reservoir without violating the capacity constraints.

Network Flow

What does it try to model?

The source can be a source of current.

The sink can be a place where the current can be used.

The other nodes are the junctions that transmit the current coming into them while respecting the Kirchoff's law.

The amount of current entering a node equals the current leaving it (except at the source or the sink).

The capacities will indicate the the upper bound on the amount of current that can be carried by the wires.

Maximum rate at which the current can be sent from the source to the sink respecting the capacity constraints.

Network Flow

What does it try to model?

The source can be a source of the network packets getting created.

The sink can be where the packets end up at the end, say for example a server.

The other nodes in the network can be the routers that transmit packets coming into them without accumulating anything.

The capacities will indicate the upper bound on the number of packets the network links can carry.

Maximum rate at which network packets can be transferred from the source to the server without violating the capacity constraints.

What is a flow?

Flow in a network

Definition (Flow in a network)

The flow in a network $G = (V, E)$ is a function $f : E \to \mathbb{R}$, that satisfies

Capacity constraints: For each $e \in E$, $0 \le f(e) \le c(e)$.

Flow conservation: For all $u \in V \setminus \{s, t\}$,

$$\sum_{v \in V, (v,u) \in E} f(v, u) = \sum_{v \in V, (u,v) \in E} f(u, v).$$

Definition (Value of the flow)

The value of the flow, denoted as $|f|$ if the total flow going out of the source s, i.e.

$$|f| = \sum_{v \in V, (s,v) \in E} f(s, v).$$

Value of the flow

Some notations.

For the sake of brevity, we use $f^{\leftarrow}(u)$ to denote $\sum_{v \in V, (v,u) \in E} f(v,u)$. And we use $f^{\rightarrow}(u)$ to denote $\sum_{v \in V, (u,v) \in E} f(u,v)$.

Then we have $|f| = f^{\rightarrow}(s)$.

Later on we will see a notion of residual graphs, in which we will have edges entering the source node and we may have non-zero value of $f^{\leftarrow}(s)$.

If $(u,v) \notin E$ then $f(u,v) = f(v,u) = 0$.

For any $U \subseteq V$, we use $f^{\leftarrow}(U)$ to denote the total flow into the vertices in U from vertices in $V \setminus U$.

Similarly, we use $f^{\rightarrow}(U)$ to denote the total flow out of the vertices in U into the vertices in $V \setminus U$.

Value of the flow

Lemma

$$|f| = \sum_{v \in V, (s,v) \in E} f(s,v) = f^{\rightarrow}(s) = \sum_{v \in V, (v,t) \in E} f(v,t) = f^{\leftarrow}(t)$$