CS 218 Design and Analysis of Algorithms

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Module 1: Basics of algorithms

Shortest path in general directed graphs

Problem Description

Input: Given a directed not necessarily acyclic graph $G = (V, E)$, a weight function $w : E \to \mathbb{Z}$ and designated vertices $s, t \in V$.

Output: the length of the shortest path from s to t.

A related problem Cycle(G, t)

Input: Given a directed graph $G = (V, E)$, a weight function $w : E \to \mathbb{Z}$ and designated vertex $t \in V$.

Output: yes iff there exists a negative cycle with a path reaching t

Another related problem Cycle(G)

Input: Given a directed graph $G = (V, E)$, a weight function $w : E \to \mathbb{Z}$.

Output: yes iff there exists a negative cycle in the graph.

Relationship between Cycle(G, t) and Cycle(G)

Solving Cycle(G, t) is enough to solve Cycle(G).

Given a graph $G = (V, E)$, add a new vertex t_0 to it.

Add directed edges from each vertex $v \in V$ to t_0 of weight 0.

Let us call the new graph G'.

G' has a negative cycle C with a path leading to t_0 if and only if G has a negative cycle.

If G has a negative cycle then G' has a negative cycle with a path to t_0 by construction.

If G' has a negative cycle C, C cannot contain t_0.

G' is the same as G elsewhere. Hence C must exist in G.

Solving Cycle(G, t) suffices to solve Cycle(G)

Lemma

There is no negative cycle in G with a path to t if and only if $\mathbf{Opt}(v, i) = \mathbf{Opt}(v, n-1)$ for each $v \in V$ and $\forall i \leq n$.

If a node v can reach t and is a part of a negative cycle then

$$\lim_{i \to \infty} \mathbf{Opt}(v, i) = -\infty.$$

Lemma

There is no negative cycle in G with a path to t if and only if $\mathbf{Opt}(v, n) = \mathbf{Opt}(v, n-1)$ for each $v \in V$.

For more details see page 302, 303, 304 of the book Kleinberg and Tardos.