Tutorial 7

1. For a flow network $G = (V, E)$ and flow f let G_f denote the residual graph. For a parameter $\Delta \in \mathbb{N}$, we define $G_f(\Delta)$ to be the graph obtained from the residual graph G_f by retaining the edges with capacity at least Δ and deleting all the edges of capacity strictly smaller than Δ.

   Now, consider the following algorithm for a graph with integral capacities.

   Set $f(e) = 0$ for all edges $e \in E$.
   Let C be the largest capacity on any edge in G.
   Let Δ be the largest power of 2 such that $\Delta \leq C$.
   while $\Delta \geq 1$ **do**
     Compute $G_f(\Delta)$
     while There exists an s to t path π in $G_f(\Delta)$ **do**
       Compute $\theta(\pi, f)$ in $G_f(\Delta)$
       Set $f' = \text{Aug}(\pi, \theta)$
       Compute $G_{f'}(\Delta)$, set $f \leftarrow f$
     end while
     $\Delta \leftarrow \Delta/2$
   end while
   Output f.

   (a) Prove that the above algorithm terminates.
   (b) Prove that the algorithm correctly computes the maximum flow in the given graph.
   (c) Analyse the running time of the above algorithm.

2. Prove the following statements about flows.

   - f_2 is a flow in G_{f_1} if and only if $f_1 + f_2$ is a flow in G.
   - $|f_1 + f_2| = |f_1| + |f_2|$.
   - f_2 is a max flow in G_{f_1} if and only if $f_1 + f_2$ is a max flow in G.
   - If f^* is a max flow in G and f is any flow then the max flow in G_f has value $|f^*| - |f|$.

3. Consider the following algorithm for a graph with integral capacities.

   Set $f(e) = 0$ for all edges $e \in E$.
   while there exists an s to t path **do**
     Compute G_f
     Let π be a path obtained by using depth first search.
     Compute $\theta(\pi, f)$ in G_f
     Set $f' = \text{Aug}(\pi, \theta)$
     Compute $G_{f'}$, set $f \leftarrow f$
   end while
   Output f.

   (a) Prove that the above algorithm terminates.
   (b) Prove that the algorithm correctly computes the maximum flow in the given graph.
   (c) Analyse the running time of the above algorithm.

4. Reading exercise. Read and understand this image segmentation application of the max-flow problem. http://srmanikandasriram.github.io/files/DSA/Term_ Paper.pdf