CS 218 Design and Analysis of Algorithms

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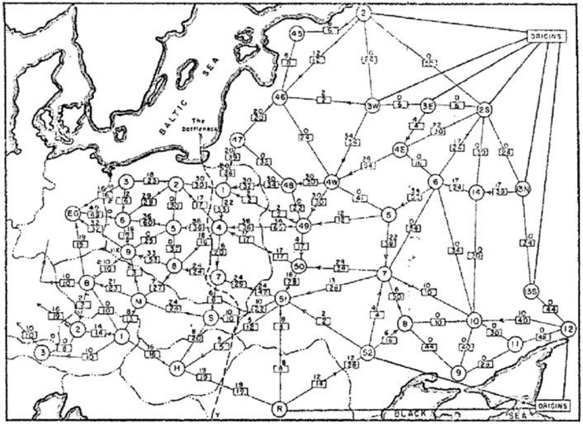
Module 2: Flow networks, Max-flow, Min-cut and applications

Applications of MaxFlow MinCut

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems.* Alexander Schrijver in Math Programming, 91: 3, 2002.

Taken from the textbook of Jeff Erickson

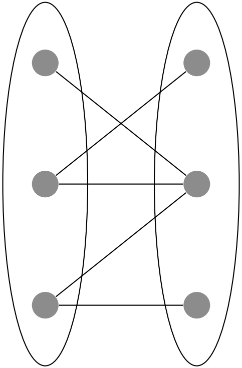
The maximum bipartite matching problem

Problem description

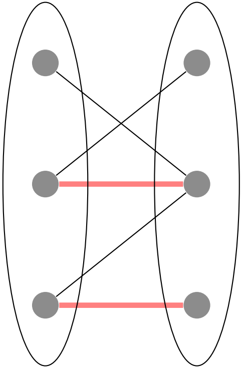
Input:	an undirected graph bipartite $G = (V = (X \cup Y), E)$
Output:	the largest set $M \subseteq E$ s. t. for any $e, e' \in M$ e, e' do not share any vertex in common.

Example

The given graph



Not a maximum matching



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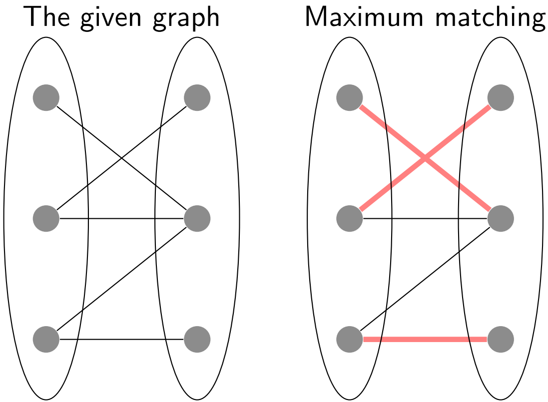
Example



Using Ford-Fulkerson for Maximum Biptatite Matching

Strategy

We start with the input instance of the bipartite maximum matching.

That is, we start with an undirected bipartite graph $G = (V = (X \cup Y), E)$.

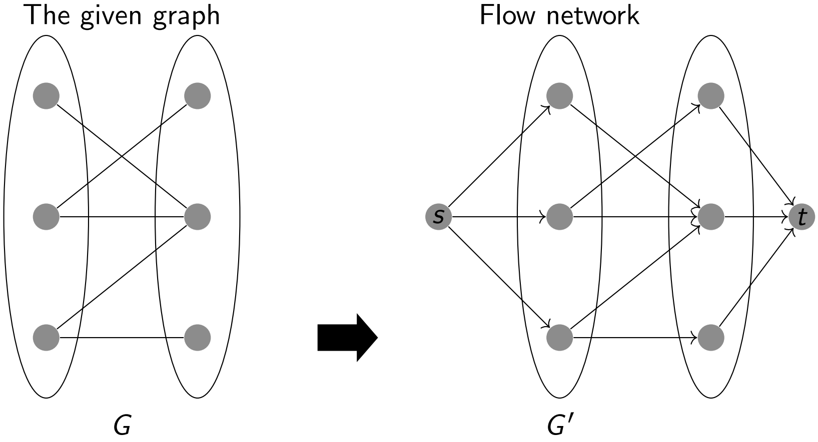
From here, we create an instance of a flow network G'.

That is, we come up with a directed graph G' with designated source, sink and capacities on the edges.

We show that the value of the maximum flow in G' equals the size of the maximum matching in G.

Using Ford-Fulkerson for Maximum Bipratite Matching

Strategy



$c(e) = 1$ for each edge in G'

Using Ford-Fulkerson for Maximum Bipratite Matching

Strategy

Given $G = (V = (X \cup Y), E)$. We construct $G' = (V', E')$ as follows.

$V' = V \cup \{s, t\}$, where s, t are two new vertices.

For any $u \in X$ and $v \in Y$, if $(u, v) \in E$ then add a directed edge from u to v in G'.

For each $u \in X$, add a directed edge from s to u in G'.

For each $v \in Y$, add a directed edge from v to t in G'.

Let the capacity of every edge in G' be equal to 1.

Using Ford-Fulkerson for Maximum Bipratite Matching

Lemma

Let G be a bipartite graph. Let G' be the flow network constructed from G as above. Then,

- *If M is a matching in G, then there is an integer valued flow f in G' such that $|f| = |M|$.*

- *Conversely, if f is an integer valued flow in G', then there is a matching M in G such that $|M| = |f|$.*

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Proof.

For each edge $e = (u, v) \in M$ then let $f(s, u) = f(u, v) = f(v, t) = 1$.
For all other edges $e \notin M$, let $f(e) = 0$.

Observe that f defines a flow: capacity constraints and conservation constraints are satisfied.

Moreover, the value of the flow $|f|$ equals the edges in M, i.e. $|M|$.

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Proof.

Here, given an integral flow f of value $|f|$, we need to give a matching of that size.

This direction is slightly subtle.

We will make 4 observations. These put together will give us the proof.

Using Ford-Fulkerson for Maximum Bipratite Matching

Lemma

Let G be a bipartite graph. Let G' be the flow network constructed from G as above. Then, If f is an integer valued flow in G', then there is a matching M in G such that $|M| = |f|$.

We will show the following 4 things. Put together they will prove the above statement.

- $f(e) \in \{0, 1\}$ for each $e \in E'$.

Let M be a set formed as follows: $e \in M$ if and only if $e \in E' \cap E$ and $f(e) = 1$.

- M has $|f|$ many edges.
- Each vertex in X participates in at most 1 edge in M.
- Each vertex in Y participates in at most 1 edge in M.

All the above put together we get the desired result.

Using Ford-Fulkerson for Maximum Bipratite Matching

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Proof.

As the capacities are all 1.

And flow values are integral, they can be either 0 or 1.

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- Each vertex in Y participates in at most 1 edge in M.
- M has $|f|$ many edges.

Proof.

Suppose there is a vertex in $u \in X$ that has two edges incident on it in M.
Say they are (u, v) and (u, v').

Then as per the construction of M, $f(u, v) = 1$ and $f(u, v') = 1$.
That is, the total flow leaving u is 2.

By flow conservation, the total flow entering u must also be 2.
But in G' there is only 1 edge entering u and has capacity 1. A contradiction.

Using Ford-Fulkerson for Maximum Bipratite Matching

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- M has $|f|$ many edges.

Proof.

Easy to prove using an argument similar/symmetric to the one on the previous slide.

Try to fill in this.

Using Ford-Fulkerson for Maximum Bipratite Matching

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- M has $|f|$ many edges.

Proof.

A small recap is needed.

Value of the flow

Lemma

$$|f| = \sum_{v \in V, (s,v) \in E} f(s, v) = f^{\rightarrow}(s) = \sum_{v \in V, (v,t) \in E} f(v, t) = f^{\leftarrow}(t)$$

Proof.

$$\begin{aligned}
|f| &= f^{\rightarrow}(s) + \sum_{v \in V \setminus \{s,t\}} (f^{\rightarrow}(v) - f^{\leftarrow}(v)) \\
&= \sum_{v \in V \setminus \{t\}} (f^{\rightarrow}(v) - f^{\leftarrow}(v)) \\
&= f^{\rightarrow}(V \setminus \{t\}) - f^{\leftarrow}(V \setminus \{t\})
\end{aligned}$$

Each edge of the graph appears twice (once +-vely and once -vely). Except the edges entering t which appear once -vely.

$$= f^{\leftarrow}(t)$$

Flow across the cut

Lemma

Let f be a flow in the flow network G with source s and sink t. Let (S, T) be any cut in G. Then the net flow across the cut, i.e. the flow leaving S minus the flow entering S, equals $|f|$.

Proof.

$$
\begin{aligned}
|f| &= f^{\rightarrow}(s) + \sum_{v \in S \setminus \{s\}} \left(f^{\rightarrow}(v) - f^{\leftarrow}(v)\right) \\
&= f^{\rightarrow}(s) - f^{\leftarrow}(s) + \sum_{v \in S \setminus \{s\}} \left(f^{\rightarrow}(v) - f^{\leftarrow}(v)\right)
\end{aligned}
$$

By regrouping we get

$$
\begin{aligned}
&= \left[f^{\rightarrow}(s) + f^{\rightarrow}(S \setminus \{s\})\right] - \left[f^{\leftarrow}(s) + f^{\leftarrow}(S \setminus \{s\})\right] \\
&= f^{\rightarrow}(S) - f^{\leftarrow}(S)
\end{aligned}
$$

Using Ford-Fulkerson for Maximum Bipratite Matching

We will show the following 4 things. Put together they will prove the above statement.

- $f(e) \in \{0,1\}$ for each $e \in E'$.

Let M be a set formed as follows: $e \in M$ if and only if $e \in E' \cap E$ and $f(e) = 1$.

- Each vertex in X participates in at most 1 edge in M.
- Each vertex in Y participates in at most 1 edge in M.
- M has $|f|$ many edges.

Proof.

Note that for every matched vertex $u \in X$, $f(u,v) = 1$, where u is matched with $v \in Y$.

If $(u,v) \notin M$ then $f(u,v) = 0$.

Let $S = X \cup \{s\}$ and let $T = Y \cup \{t\}$. Note that (S,T) is a cut. And the flow across (S,T), i.e. $f^{\rightarrow}(S) - f^{\leftarrow}(S)$ equals $|M|$.

But from what we saw, $|f| = f^{\rightarrow}(S) - f^{\leftarrow}(S)$.

Hence $|f| = |M|$.

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Running time

The running time will be $C_{\max} \cdot |E| = O(|X| \cdot |E|) = O(|V| \cdot |E|)$. Proof.

Here, given an integral flow f of value $|f|$, we need to give a matching of that size.