CS 228 : Logic in Computer Science

S. Krishna

First-Order Logic : Syntax

First Order Logic

- A formalism to specify properties of mathematical structures like graphs, partial orders, words, groups, rings, . . . , and the world at large!
- Every dad is older than his child
- Every vertex has atleast two outgoing edges
- There is exactly one vertex with 3 outgoing edges
- Atleast 5 students scored more than 90 marks in a class of 50
- All words starting with the letter a, ending with the letter b, have even length

Signatures

- A vocabulary or signature τ is a set consisting of
  - constants $c_1, c_2, \ldots$
  - Relation symbols $R_1, R_2 \ldots$, each with some arity k, denoted R_i^k
- We look at finite signatures
- $\tau = (E^2, F^3)$ is a finite signature with two relations, E with arity 2 and F with arity 3

Symbols in First Order Logic

Formulae of FO, over signature τ, are sequences of symbols, where each symbol is one of the following:

- The symbol $\perp$ called false
- An element of the infinite set $\mathcal{V} = \{x_1, x_2, \ldots\}$ of variables
- Constants and relations from τ
- The symbol $\rightarrow$ called implication
- The symbol $\forall$ called the universal quantifier
- The symbols (and) called paranthesis

Well formed Formulae

A well-formed formula (wff) over a signature τ is inductively defined as follows:

- $\perp$ is a wff
- If t_1, t_2 are either variables or constants in τ, then $t_1 = t_2$ is a wff
- If t_i is either a variable or a constant, for $1 \leqslant i \leqslant k$ and R is a k-ary relation symbol in τ, then $R(t_1, \ldots, t_k)$ is a wff
- If φ and ψ are wff, then $\varphi \to \psi$ is a wff
- If φ is a wff and x is a variable, then $(\forall x)\varphi$ is a wff

Logical Abbreviations : Boolean Connectives

- $\neg\varphi = \varphi \to \bot$
- $\top = \neg\bot$
- $\varphi \vee \psi = \neg\varphi \to \psi$
- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\exists x.\varphi = \neg(\forall x.\neg\varphi)$
- Precedence of operators : $\neg > \wedge > \vee > \to > \forall$

An Example

Consider the signature $\tau = \{R\}$ where R is a binary relation. The following are FO formulae over this signature.

- $\forall x R(x, x)$ Reflexivity
- $\forall x (R(x, x) \rightarrow \bot)$ Irreflexivity
- $\forall x \forall y (R(x, y) \rightarrow R(y, x))$ Symmetry
- $\forall x \forall y (R(x, y) \rightarrow (R(y, x) \rightarrow (x = y)))$ Anti-symmetry
- $\forall x \forall y \forall z (R(x, y) \rightarrow (R(y, z) \rightarrow R(x, z)))$ Transitivity

First-Order Logic : Semantics

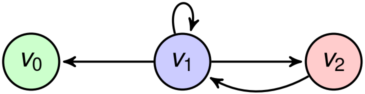
Structures

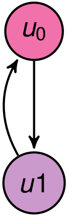
- A structure $\mathcal{A}$ of signature τ consists of
  - A non-empty set A or $u(\mathcal{A})$ called the universe
  - For each constant c in the signature τ, a fixed element $c_{\mathcal{A}}$ is assigned from the universe A
  - For each k-ary relation R^k in the signature τ, a set of k-tuples from A^k is assigned to $R^{\mathcal{A}}$
  - The structure $\mathcal{A}$ is finite if A (or $u(\mathcal{A})$) is finite

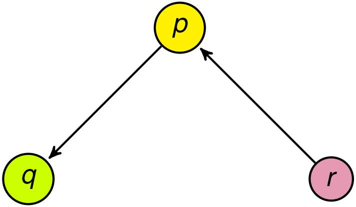
Examples of Structures

A Graph

- A set V of vertices
- A set $E \subseteq V \times V$ of edges







A Graph Structure

- $\tau = \{E\}$, with E binary.
  - A graph structure over τ is $\mathcal{G} = (V, E^{\mathcal{G}})$,
  - The universe $u(\mathcal{G})$ is the set of vertices V
  - The relation E is the edge relation
  - $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.
  - $\forall x \exists y(E(x, y))$

A totally ordered set

- A set S with an order relation
- Relates any two elements of S
- Examples : $(\mathbb{N}, <)$, $(\mathbb{R}, <)$

A Total Order Structure

- $\tau = \{<, S\}$ with $<, S$ binary.
  - A finite order structure over τ is $\mathcal{O} = (O, <^{\mathcal{O}}, S^{\mathcal{O}})$
  - The universe $u(\mathcal{O})$ is the finite ordered set O
  - $<^{\mathcal{O}}$ is the ordering on O and $S^{\mathcal{O}}$ is the successor on O
  - $\mathcal{O} = (O = \{1, 2, 4\}, <^{\mathcal{O}} = \{(1, 2), (1, 4), (2, 4)\}, S^{\mathcal{O}} = \{(1, 2)\})$
  - $\exists x \neg \exists y (S(x, y))$
- Can you write a Partial Order as a structure, where the universe consists of all subsets of a given finite set?

Words

- A word is a sequence of symbols over a (finite) alphabet
- Alphabet $\Sigma = \{a, b, c\}$
- Some words over Σ : $b, aaa, abababa, cacbccc$
- The length of a word is the number of symbols in it

A Word Structure

- $\tau = \{<, S, Q_a, Q_b\}$, where $<, S$ are binary, Q_a, Q_b are unary relations.
  - A word structure $\mathcal{W} = (u(\mathcal{W}), <^{\mathcal{W}}, S^{\mathcal{W}}, {Q_a}^{\mathcal{W}}, {Q_b}^{\mathcal{W}})$
  - The universe $u(\mathcal{W})$ consists of the positions in a word W over symbols a, b
  - $<^{\mathcal{W}}$ is the ordering relation on the positions of W
  - $S^{\mathcal{W}}$ is the successor relation on the positions of W
  - ${Q_a}^{\mathcal{W}}$ is the set of positions labeled a in W
  - ${Q_b}^{\mathcal{W}}$ is the set of positions labeled b in W
  - The structure with $u(\mathcal{W}) = \{0, 1, 2, \dots, 8\}$, ${Q_a}^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, ${Q_b}^{\mathcal{W}} = \{2, 3, 5, 7\}$,
  - $<^{\mathcal{W}} = \{(0, 1), (0, 2), \dots, (7, 8)\}$, $S^{\mathcal{W}} = \{(0, 1), (1, 2), \dots, (7, 8)\}$ uniquely defines the word $W = aabbababa$.
  - $\forall x(Q_b(x) \rightarrow \exists y(x < y \wedge Q_a(y))$