CS 218 Design and Analysis of Algorithms

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Module 2: Flow networks, Max-flow, Min-cut and applications

Ford-Fulkerson Algorithm

What all do we need to argue about this algorithm?

For each $e \in E$, set $f(e) \leftarrow 0$
Compute G_f
while There is an s to t path π in G_f **do**
  $f' \leftarrow \texttt{Aug}(\pi, f)$
  $f \leftarrow f'$
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end while
Output f

Termination: Why does the algorithm terminate?

Time analysis: What is the running time of the algorithm?

Correctness: Why does it output the maximum flow?

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Other issues worth discussing.

Is it important to choose the right s to t paths? Or would any path be okay?

What is the relationship between Max-Flow and Min-Cut that we have been talking about all along?

Can the capacities be non-integral?

Termination of Ford-Fulkerson Algorithm

Recall that we have assumed that the capacities are integers.

> At every intermediate stage in the algorithm, the flow values and the residual capacities stay integral.
>
> If f is a flow and π is an s to t path in G_f, then $|f'| = |f| + \theta(\pi, f)$.
>
> Let $C_{\max} = \sum_{v \in V, (s,v) \in E} c(s, v)$. Then maximum flow $\leq C_{\max}$.

The above three will suffice to prove the termination of the algorithm.

Termination of Ford Fulkerson Algorithm

Lemma

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Proof.

Before the algorithm starts, the capacities are integral and flows are 0. Say it is true after j-th iteration.

As residual capacities are integral, the value of θ is also integral.

Thus the next flow f' is also integral.

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Proof.

The first edge of π must be an edge out of s in G_f.

It must be a forward edge.

Its flow increases by $\theta(\pi, f)$. Flow on no other edge out of s changes.

Hence $|f'| = |f| + \theta(\pi, f)$.

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Proof.

If all the edges out of s are saturated by flow f then $f = C_{\max}$.

No flow can exceed this value anyway.

Hence $C_{\max}$ is an upper bound on the maximum flow.

Termination of Ford-Fulkerson Algorithm

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Termination assuming the Lemma.

The flow only increases in each iteration of the algorithm.

It increases by at least 1 every time.

It cannot increase beyond $C_{\max}$.

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- ✓ Termination: Why does the algorithm terminate?
- Time analysis: What is the running time of the algorithm?
- Correctness: Why does it output the maximum flow?

Other issues worth discussing.

- Is it important to choose the right s to t paths? Or would any path be okay?
- What is the relationship between Max-Flow and Min-Cut that we have been talking about all along?
- Can the capacities be non-integral?

Time analysis of Ford Fulkerson Algorithm

Lemma

The running time of the algorithm is bounded by $O(C_{max} \cdot |E|)$.

The while loop runs for at most $C_{\max}$ iterations.

In each loop, to maintain residual graph we need $O(m)$ time.

Finding an s to t path π in this graph will need $O(m+n)$ times.

Augmenting takes time $O(n)$. (As $n-1$ vertices along π.)

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Correctness of Ford Fulkerson Algorithm

Notation.

Let $\mathfrak{f}$ be the flow returned by Ford-Fulkerson algorithm.

Let $A \leftarrow \{v \in V \mid \text{ there is a path to } v \text{ from } s \text{ in } G_{\mathfrak{f}}\}$.

Let $B \leftarrow V \setminus A$.

Note that

- A, B partition V.
- $s \in A$. Also $t \in B$. Why?
- When the algorithm terminates, no path from s to t in $G_{\mathfrak{f}}$.
- Therefore (A, B) is a cut.

To show optimality, we need to show two things about $\mathfrak{f}$.

It saturates every edge from A to B.

It avoids every edge from B to A.

Lemma

Let f be any flow in the flow network G. Let (S, T) be any (s, t)-Cut in G. Then $|f| \leq cap(S, T)$. Moreover, $|f| = cap(S, T)$ if and only if f saturates every edge from S to T and avoids every edge from T to S.

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Correctness of Ford Fulkerson Algorithm

Lemma

Let $\mathfrak{f}, A, B$ be as defined on the previous slide. Then $\mathfrak{f}$ saturates every edge from A to B.

Proof.

Suppose there is an edge $(u, v) \in E$ such that $u \in A$ and $v \in B$.

Suppose $\mathfrak{f}(u, v) < c(u, v)$.

Then (u, v) will be a forward edge in $G_{\mathfrak{f}}$.

But then $v \in A$ as per the construction of A. This is a contradiction.

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Let $\mathfrak{f}$ be the flow returned by Ford-Fulkerson algorithm.

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Note that (A, B) is a cut.

To show optimality, we need to show two things about $\mathfrak{f}$.

- ✓ It saturates every edge from A to B.
- It avoids every edge from B to A.

Lemma

Let f be any flow in the flow network G. Let (S, T) be any (s, t)-Cut in G. Then $|f| \leq \mathit{cap}(S, T)$. Moreover, $|f| = \mathit{cap}(S, T)$ if and only if f saturates every edge from S to T and avoids every edge from T to S.

Correctness of Ford Fulkerson Algorithm

Lemma

Let $\mathfrak{f}, A, B$ be as defined on the previous slide. Then $\mathfrak{f}$ avoids every edge from B to A.

Proof.

Suppose there is an edge $(u, v) \in E$ such that $u \in B$ and $v \in A$.

Suppose $\mathfrak{f}(u, v) > 0$.

Then (v, u) will be a backward edge in $G_{\mathfrak{f}}$.

But then $u \in A$ as per the construction of A. This is a contradiction.

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Correctness of Ford Fulkerson Algorithm

Why is $\mathfrak{f}$ the maximum flow?

The algorithm terminates when there is no s to t path left in $G_{\mathfrak{f}}$.

This and all our analysis show that (A, B) cut has the same capacity as $|\mathfrak{f}|$.

As a result, $|\mathfrak{f}|$ is the maximum possible flow in G and $\mathrm{cap}(A, B)$ is the minimum capacity cut in G.

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Which s to t path?

Consider the following example.