CS 228 : Logic in Computer Science

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So Far

- Satisfiability of FO over words
- Model checking
    - System abstracted as a model DFA/NFA A
    - Specification written in FO as formula φ
    - Does system model $\models \varphi$
    - $L(A) \subseteq \underline{L(\varphi)}$?
    - $L(A) \cap \underline{L(\varphi)} = \emptyset$?
- FO-definable $\subseteq$ *REG*, converse?

Monadic Second Order Logic (MSO)

Symbols in MSO

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Symbols in MSO

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- The symbol $\perp$ called false
- An element of the infinite set $\mathcal{V}_1 = \{x_1, x_2, \dots\}$ of first order variables
- An element of the infinite set $\mathcal{V}_2 = \{X_1, X_2, \dots\}$ of second order variables where each variable has arity 1 (new!)
- Constants and relations from τ
- The symbol $\rightarrow$ called implication
- The symbol $\forall$ called the universal quantifier
- The symbols (and) called paranthesis

Well formed Formulae

A well-formed formula (wff) over a signature τ is inductively defined as follows:

Well formed Formulae

A well-formed formula (wff) over a signature τ is inductively defined as follows:

- $\perp$ is a wff
- If t_1, t_2 are either variables or constants in τ, then $t_1 = t_2$ is a wff
- If t_i is either a first order variable or a constant, for $1 \leqslant i \leqslant k$ and R is a k-ary relation symbol in τ, then $R(t_1, \ldots, t_k)$ is a wff
- If t is either a first order variable or a constant, X is a second order variable, then $X(t)$ is a wff
- If φ and ψ are wff, then $\varphi \rightarrow \psi$ is a wff
- If φ is a wff and x is a first order variable, then $(\forall x)\varphi$ is a wff
- If φ is a wff and X is a second order variable, then $(\forall X)\varphi$ is a wff

Free and Bound Variables

- ▶ Free, Bound Variables and Scope same as in FO
- ▶ In a wff $\varphi = \forall X \psi$, every occurrence of X in ψ is bound
- ▶ A sentence is a formula with no free first order and second order variables

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a pair of functions (α_1, α_2), where

- $\alpha_1 : \mathcal{V}_1 \to u(\mathcal{A})$ assigns every first order variable $x \in \mathcal{V}_1$ a value $\alpha_1(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha_1(t)$ is $c^{\mathcal{A}}$.

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- $\alpha_2 : \mathcal{V}_2 \to 2^{u(\mathcal{A})}$ assigns to every second order variable $X \in \mathcal{V}_2$, $\alpha_2(X) \subseteq u(\mathcal{A})$.

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- $\alpha_2 : \mathcal{V}_2 \to 2^{u(\mathcal{A})}$ assigns to every second order variable $X \in \mathcal{V}_2$, $\alpha_2(X) \subseteq u(\mathcal{A})$.

Binding on a Variable

For an assignment $\alpha = (\alpha_1, \alpha_2)$ over $\mathcal{A}$, and $x \in \mathcal{V}_i$, $i = 1, 2$,

$\alpha_i[x \mapsto a]$ is the assignment $\alpha_i[x \mapsto a](y) = \begin{cases} \alpha_i(y), y \neq x, \\ a, y = x \end{cases}$

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We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \perp$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha_1(t_1) = \alpha_1(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha_1(t_1), \ldots, \alpha_1(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha X(t)$ iff $\alpha_1(t) \in \alpha_2(X)$ (new)
- $\mathcal{A} \models_\alpha (\varphi \rightarrow \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\forall X)\varphi$ iff for every $S \subseteq u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[X \mapsto S]} \varphi$ (new)

Examples

Recall the signature for the graph structure, $\tau = \{E\}$

- ► The graph is 3-colorable

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$$\exists X \exists Y \exists Z(\forall x[X(x) \lor Y(x) \lor Z(x)] \land$$

$$\forall x \forall y[E(x, y) \rightarrow \{\neg(X(x) \land X(y)) \land \neg(Y(x) \land Y(y)) \land \neg(Z(x) \land Z(y))\}])$$

Examples

Recall the signature for the graph structure, $\tau = \{E\}$

- The graph has an independent set of size $\geqslant k$

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Recall the signature for the graph structure, $\tau = \{E\}$

► The graph has an independent set of size $\geq k$

$$\exists I\{\forall x \forall y[(\neg(x = y) \land I(x) \land I(y)) \to \neg E(x, y)] \land$$

$$\exists x_1 \ldots x_k[\bigwedge_{i \neq j} \neg(x_i = x_j) \land \bigwedge_i I(x_i)]\}$$

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Recall the signature τ for the word structure, $\tau = \{Q_a, Q_b, <, S\}$ for $\Sigma = \{a, b\}$

► Words of even length

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$$\exists E \exists O \{\forall x[(first(x) \rightarrow E(x)) \wedge (last(x) \rightarrow O(x))]$$

Examples

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$$\exists E \exists O \{\forall x[(\textit{first}(x) \rightarrow E(x)) \wedge (\textit{last}(x) \rightarrow O(x))]$$

$$\wedge \forall x[(E(x) \vee O(x)) \wedge \neg(E(x) \wedge O(x))]$$

Examples

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- Words of even length

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$$\land \forall x[(E(x) \lor O(x)) \land \neg(E(x) \land O(x))]$$

$$\land \forall x \forall y[S(x, y) \land O(x) \to E(y)]$$

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MSO on Words : Satisfiability

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$$X(x) | Q_\Sigma(x) | x = y | x < y | S(x, y)$$

MSO on Words

- Signature $\tau = (Q_\Sigma, <, S)$, domain or universe = set of positions of a word
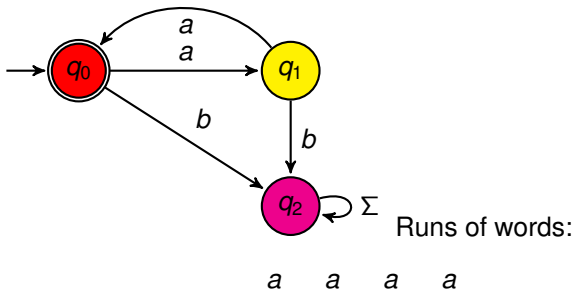- MSO over words: Atomic formulae

$$X(x)|Q_\Sigma(x)|x = y|x < y|S(x, y)$$

- Given a MSO sentence φ, $L(\varphi)$ defined as usual
- A language $L \subseteq \Sigma^*$ is MSO definable iff there is an MSO formula φ such that $L = L(\varphi)$
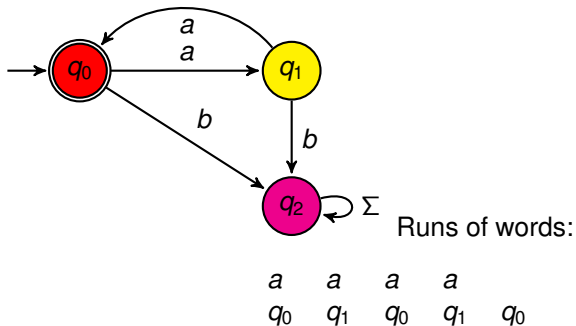- Given an MSO sentence φ, is it satisfiable/valid?

MSO Expressiveness

- Clearly, *FO* $\subseteq$ *MSO*
- *FO* $\subset$ Regular
- *MSO*=Regular

Regular Languages to MSO

Runs of words:

a   a   a   a

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a	a	a	a	
q_0	q_1	q_0	q_1	q_0

Regular Languages to MSO
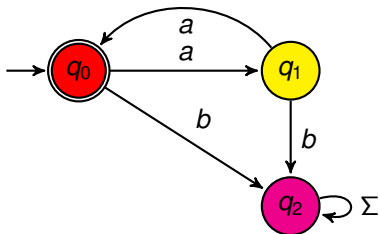
Runs of words:

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Regular Languages to MSO
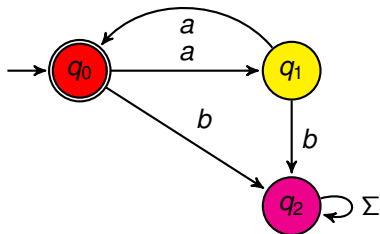
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Regular Languages to MSO

Given a regular language L, and a DFA such that $L = L(A)$,

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\text{Position } x & : & 0 & 1 & 2 & 3 & \\
 & & a & a & b & a & \\
 & & q_0 & q_1 & q_0 & q_2 & q_2
\end{array}
$$

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- For a state $q \in Q$, let X_q=the set of positions of the word where the state is q in the run
- $X_{q_0} = \{0, 2\}, X_{q_1} = \{1\}, X_{q_2} = \{3\}$
- The initial position of any word must belong to $X_{q_0} : 0 \in X_{q_0}$

Regular Languages to MSO

- If a word *wa* is accepted, then
  - The last position x of the word satisfies $Q_a(x)$
  - For some state q, we have $X_q(x)$ and there is a transition $\delta(q, a) = q_f \in F$

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Position x	:	0	1	2	3	
		a	a	b	a	
		q_0	q_1	q_0	q_2	q_2

- $Q_a(3)$ and $3 \in X_{q_2}$. $\delta(q_2, a) = q_2 \notin F$
- If x, y are consecutive positions in the word, and if $X_q(x) \wedge Q_a(x)$, then it must be that $X_t(y)$ such that $\delta(q, a) = t$

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- $X_{q_0}(0)$, $X_{q_1}(1)$ and $Q_a(0)$. $\delta(q_0, a) = q_1$.
- $X_{q_1}(1)$, $X_{q_0}(2)$ and $Q_a(1)$. $\delta(q_1, a) = q_0$.

Regular Languages to MSO

Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$, a word w is accepted iff it satisfies

$$\exists X_0 \exists X_1 \ldots X_n \{ [\forall x (X_0(x) \vee \cdots \vee X_n(x)) \wedge \forall x \bigwedge_{i \neq j} \neg(X_i(x) \wedge X_j(x))] \wedge$$

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Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$, a word w is accepted iff it satisfies

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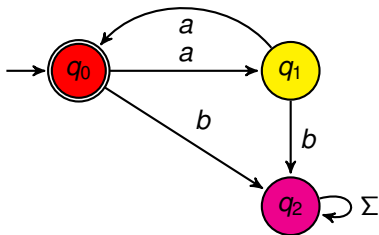
$$[\exists x (\mathit{first}(x) \wedge X_0(x))] \wedge$$

$$\forall x \forall y [S(x, y) \rightarrow \bigvee_{\delta(i,a)=j} [X_i(x) \wedge Q_a(x) \wedge X_j(y)]] \wedge$$

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$$[\exists x (\textit{first}(x) \wedge X_0(x))] \wedge$$

$$\forall x \forall y [S(x, y) \rightarrow \bigvee_{\delta(i,a)=j} [X_i(x) \wedge Q_a(x) \wedge X_j(y)]] \wedge$$

$$\exists x [\textit{last}(x) \wedge \bigvee_{\delta(i,a)=j \in F} [X_i(x) \wedge Q_a(x)]] \}$$

- $w \in L(A)$ iff $w \models \varphi$

$$\exists X_0 \exists X_1 \exists X_2 \{ [\forall x (X_0(x) \lor X_1(x) \lor X_2(x)) \land \forall x [\neg (X_0(x) \land X_1(x)) \land$$
$$\neg (X_0(x) \land X_2(x)) \land \neg (X_1(x) \land X_2(x))] \land$$
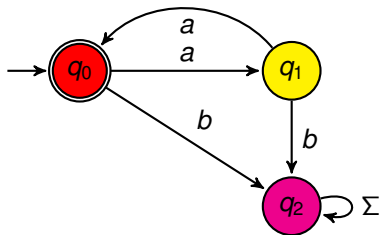
$$\exists X_0 \exists X_1 \exists X_2 \{[\forall x(X_0(x) \lor X_1(x) \lor X_2(x)) \land \forall x[\neg(X_0(x) \land X_1(x)) \land$$

$$\neg(X_0(x) \land X_2(x)) \land \neg(X_1(x) \land X_2(x))] \land [\exists x(\textit{first}(x) \land X_0(x))] \land$$
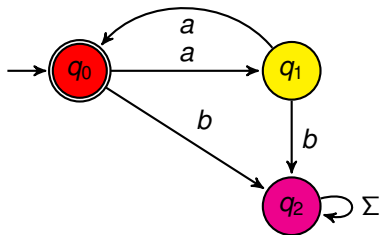
Example : Regular to MSO
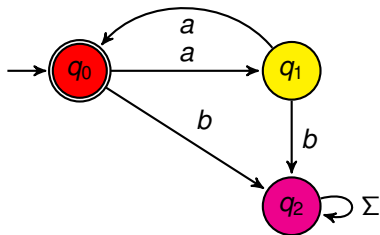
$$\exists X_0 \exists X_1 \exists X_2 \{ [\forall x (X_0(x) \lor X_1(x) \lor X_2(x)) \land \forall x [\neg (X_0(x) \land X_1(x)) \land$$
$$\neg (X_0(x) \land X_2(x)) \land \neg (X_1(x) \land X_2(x))] \land [\exists x (first(x) \land X_0(x))] \land$$

$$\forall x \forall y [S(x, y) \rightarrow [(X_0(x) \land Q_a(x) \land X_1(y)) \lor$$
$$(X_0(x) \land Q_b(x) \land X_2(y)) \lor (X_1(x) \land Q_a(x) \land X_0(y)) \lor$$
$$(X_1(x) \land Q_b(x) \land X_2(y)) \lor (X_2(x) \land Q_a(x) \land X_2(y)) \lor (X_2(x) \land Q_b(x) \land X_2(y))]]$$

Example : Regular to MSO

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$$\neg(X_0(x) \land X_2(x)) \land \neg(X_1(x) \land X_2(x))] \land [\exists x(first(x) \land X_0(x))] \land$$

$$\forall x \forall y[S(x,y) \rightarrow [(X_0(x) \land Q_a(x) \land X_1(y)) \lor$$

$$(X_0(x) \land Q_b(x) \land X_2(y)) \lor (X_1(x) \land Q_a(x) \land X_0(y)) \lor$$

$$(X_1(x) \land Q_b(x) \land X_2(y)) \lor (X_2(x) \land Q_a(x) \land X_2(y)) \lor (X_2(x) \land Q_b(x) \land X_2(y))]]$$

$$\land \exists x[last(x) \land (X_1(x) \land Q_a(x))]\}$$

MSO to Regular Languages

- Every MSO sentence φ over words can be converted into a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.
- Start with atomic formulae, construct DFA for each of them.
- Conjunctions, Disjunctions, Negation easily handled via union, intersection and complementation of respective DFA
- Handling quantifiers?

Atomic Formulae to DFA

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- Think of a word *baab* which satisfies $Q_a(x)$ as $\begin{array}{c} baab \\ 0010 \end{array}$ or $\begin{array}{c} baab \\ 0100 \end{array}$
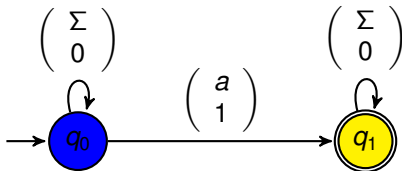
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- The first row is over Σ, and the second row captures a possible assignment to x

Atomic Formulae to DFA

- $Q_a(x)$: All words which have an a. Need to fix a position for x, where a holds.
- Think of a word *baab* which satisfies $Q_a(x)$ as $\begin{matrix} baab \\ 0010 \end{matrix}$ or $\begin{matrix} baab \\ 0100 \end{matrix}$
- The first row is over Σ, and the second row captures a possible assignment to x
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0, 1\}$, and construct an automaton over Σ'.

Atomic Formulae to DFA

- $Q_a(x)$: All words which have an *a*. Need to fix a position for *x*, where *a* holds.
- Think of a word *baab* which satisfies $Q_a(x)$ as $\begin{matrix} baab \\ 0010 \end{matrix}$ or $\begin{matrix} baab \\ 0100 \end{matrix}$
- The first row is over Σ, and the second row captures a possible assignment to *x*
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0, 1\}$, and construct an automaton over Σ'.
- Deterministic, not complete.

Simple Formulae to DFA

- $Q_a(x) \wedge X(x)$ means that the position x is in the set X, and letter a is true when $x = 1$.

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- Think of a word *baab* which satisfies $Q_a(x) \wedge X(x)$ as

 | *baab* | | *baab* |
  |--------|-----|--------|
  | 0010 | or | 0100 |
  | *DD*1*D* | | *D*1*DD* |

  where *D* stands for *dont care*. *X* can have value 0 or 1 at *D*.

Simple Formulae to DFA

- $Q_a(x) \wedge X(x)$ means that the position x is in the set X, and letter a is true when $x = 1$.

- Think of a word *baab* which satisfies $Q_a(x) \wedge X(x)$ as

baab		*baab*
0010	or	0100
*DD*1*D*		*D*1*DD*

  where D stands for *dont care*. X can have value 0 or 1 at D.

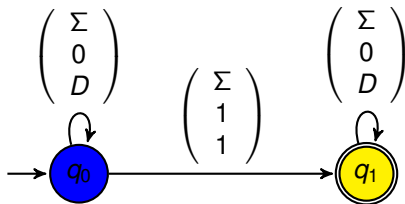- However, the position where $x = 1$ must belong to X.

Simple Formulae to DFA

- The first row is over Σ, and the second row captures a possible assignment to x, and the third row captures a possible assignment to X.
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0, 1\} \times \{0, 1\}$, and construct an automaton over Σ'.
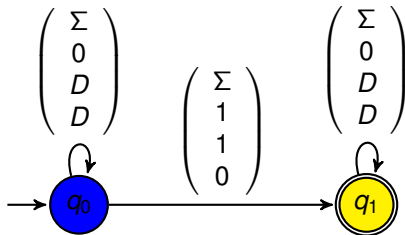- $Q_a(x) \wedge X(x)$: deterministic, not complete
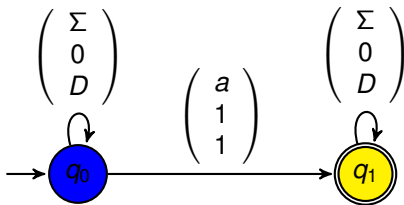
Simple Formulae to DFA

- $X(x)$

Simple Formulae to DFA

- $X(x) \wedge \neg Y(x)$
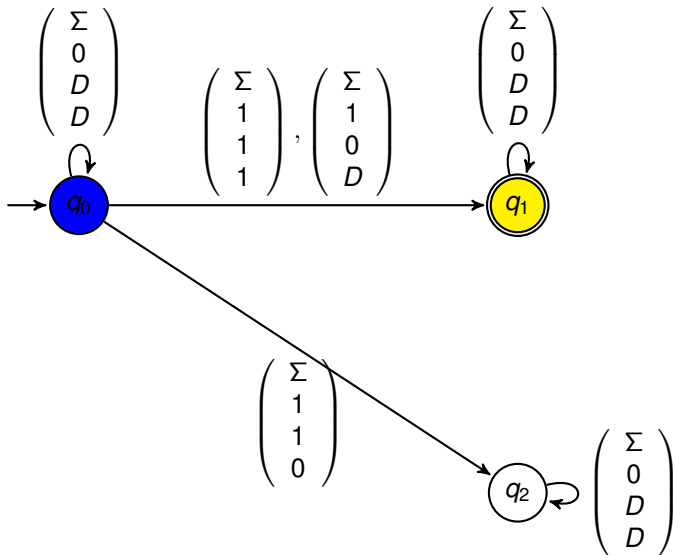- $\Sigma' = \Sigma \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}$

Formulae to DFA

- Given $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$, an MSO formula over Σ, consider the extended alphabet

$$\Sigma' = \Sigma \times \{0, 1\}^{m+n}$$

- Assign values to x_i, X_j at every position as seen in the cases of atomic formulae

- Keep in mind that every x_i can be assigned 1 at a unique position

Handling Quantifiers

- $\exists X \exists Y \forall x [X(x) \rightarrow Y(x)]$

Points to Remember

- Given $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$, construct automaton for atomic MSO formulae over the extended alphabet $\Sigma \times \{0,1\}^{m+n}$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n} Q_{X_1} \ldots Q_{X_m} \varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$, the automaton for $\exists X_i \varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$ is obtained by projecting out the row of X_i

- This may result in an NFA

- Determinize it and complement it to get a DFA for $\neg \exists x_i \varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$

- Intersect with the regular language where each of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are assigned 1 exactly at one position

The Automaton-Logic Connection

Given any MSO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$. If a language L is regular, one can construct an MSO sentence φ such that $L = L(\varphi)$.