

A decorative blue crosshair consisting of a vertical line and a horizontal line intersecting in the upper-left quadrant of the slide.

CS 228 : Logic in Computer Science

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Recap

- ▶ Given FO formula φ , build an automaton A_φ preserving the language
- ▶ Satisfiability of FO reduces to non-emptiness of underlying automaton
- ▶ Starting today : non FO-definability

FO Definability

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- ▶ Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \dots \vee C_n$

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- ▶ Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \dots \vee C_n$
- ▶ Formulae of quantifier rank $c + 1$ written as a disjunction of the conjunction of formulae, each formula of the form $\exists x\varphi, \neg\exists x\varphi$ or φ , with $c(\varphi) \leq c$. Eliminate repeated disjuncts/conjuncts

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- ▶ $G = \{Q_a(x), \neg Q_a(x), Q_a(y), \neg Q_a(y), S(x, y), \neg S(x, y), x < y, \neg(x < y)\}$

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- ▶ All possible disjuncts using each C_i : formulae in DNF of rank 0

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- ▶ $2m$ atomic/negated atomic formulae
- ▶ Number of conjunctions C_i possible $\leq 2^{2m}$
- ▶ Number of formulae in DNF $\leq 2^{2^{2m}}$ ($c = 0$)

Rank 1

Let there be p formulae φ of rank 0.

- ▶ $2p$ formulae of the form $\exists x\varphi, \neg\exists x\varphi$
- ▶ 2^{2p} conjunctions of rank 1
- ▶ Conjoining any one of the p formulae of rank 0 gives all conjuncts of rank 1 : $p2^{2p}$ more
- ▶ Possible conjuncts of rank 1 is $q = (p + 1)2^{2p}$
- ▶ Possible disjuncts of these : 2^q

Number of FO formulae of rank c

Let \mathcal{V} be a finite set of first order variables, and let $c \geq 0$. There are finitely many FO formulae in DNF with rank c over \mathcal{V} .

Some Notation

Given a word $w = a_1 \dots a_n$, and a finite set of variables \mathcal{V} , define a \mathcal{V} -enriched-word with respect to w as

- ▶ $(a_1, U_1)(a_2, U_2) \dots (a_n, U_n)$ where
- ▶ $\bigcup_i U_i = \mathcal{V}$
- ▶ $U_i \cap U_j = \emptyset$

- ▶ A \mathcal{V} -enriched-word is over the alphabet $\Sigma \times 2^{\mathcal{V}}$
- ▶ $(a, \{x\})(b, \{y, z\})(c, \emptyset)(d, \{u, v\})$ is a $\{x, y, z, u, v\}$ -enriched word with respect to the word $abcd$.
- ▶ We will refer to \mathcal{V} -enriched-word structures as \mathcal{V} -structures from here on

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 - ▶ $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$
- ▶ $w \models \exists x Q_a(x)$ iff there exists i such that $(a_1, S_1) \dots (a_i, S_i \cup \{x\}) \dots (a_n, S_n) \models Q_a(x)$
 - ▶ $(b, \{y, z\})(a, \{u\})(c, \emptyset) \models \exists x Q_a(x)$ since
 $(b, \{y, z\})(a, \{x, u\})(c, \emptyset) \models Q_a(x)$

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