Homework 1

Cryptography & Network Security
CS 406 : Spring 2021

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CPA-Secure Symmetric-Key Encryption [Total 100 pts]

1. **Secure Computation with Perfect Secrecy** [15 pts]

   This problem considers a puzzle from Lecture 0, involving three parties, Alice, Bob and Carol. Alice and Bob are given *inputs* $x, y \in D$ (for some finite domain D) and Carol wishes to learn $f(x, y)$ (for some function $f : D \times D \to Z$).

   We shall consider protocols that proceed as follows (specified in terms of a finite set R and three functions g_A, g_B, g_C). After Alice and Bob receive their inputs x and y respectively:

   - Alice picks $r \leftarrow R$ uniformly at random and sends r to Bob.
   - Alice sends a message $\alpha = g_A(x, r)$ to Carol and Bob sends $\beta = g_B(y, r)$ to Carol, where $g_A, g_B : D \times R \to Q$.
   - Carol outputs $z = g_C(\alpha, \beta)$, where $g_C : Q \times Q \to Z$.

   By the nature of the protocol, Alice and Bob learn nothing about each other's inputs (note that r is chosen independently of x).

   (a) State the perfect correctness requirement of the protocol formally, in terms of the sets D, R, and the functions f, g_A, g_B, g_C.
   Answer: For perfect correctness, we need to satisfy

   $$\forall x, y \in D, \forall r \in R, f(x, y) = g_c(g_a(x, r), g_b(y, r))$$

   (b) Formalize a perfect secrecy requirement that Carol learns nothing other than $f(x, y)$ in this protocol, by filling in the blanks below.

   $$\forall \_\_\_\_\_ \quad \Pr_{r \leftarrow R}[\_\_\_\_\_] = \Pr_{r \leftarrow R}[\_\_\_\_\_]$$

   Hint: Carol should not be able to differentiate between (x, y) and (x', y') such that $f(x, y) = f(x', y')$.
   Answer:

   $$\forall x, y, x', y' \in D, \forall \alpha, \beta \in Q \quad \Pr_{r \leftarrow R}\left[(g_a(x, r), g_b(y, r)) = (\alpha, \beta)\right] = \Pr_{r \leftarrow R}\left[(g_a(x', r), g_b(y', r)) = (\alpha, \beta)\right]$$

   (c) Suppose $f(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$ Let R be the set of all permutations over D (so that $|R| = |D|!$), $g_A = g_B = g$ where $g(w, r) = r(w)$ (i.e., apply the permutation r to w). What should g_C be so that the protocol meets the correctness requirement? Also, prove that the perfect secrecy condition above is met.

Answer: $g_c(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$ This is perfectly correct because $\alpha = \beta$ only if $r(x) = r(y)$. Also, r is a permutation and hence must be one to one. Hence $x = y$ as well. We can conclude the same for the case when $\alpha \neq \beta$ as well. For perfect secrecy,

$$\Pr_{r\leftarrow R}[(r(x), r(y)) = (\alpha,\beta)] = \Pr_{r\leftarrow R}[(r(x'), r(y')) = (\alpha,\beta)] \; \forall x, y, x', y' \in D$$

$$\Pr_{r\leftarrow R}[(r(x), r(y)) = (\alpha,\beta)] = \Pr_{r\leftarrow R}[r(x) = \alpha] \Pr_{r\leftarrow R}[r(y) = \beta]$$

$r(x)$ has $|D|$ choices to choose from and hence $\Pr_{r\leftarrow R}[r(x) = \alpha] = 1/|R|$. Similarly,$\Pr_{r\leftarrow R}[r(y) = \alpha] = 1/|R|$. Same holds true when we have (x', y') instead of (x, y). Hence LHS $= 1/|R|^2 =$ RHS

Note: $\Pr_{r\leftarrow R}[r(x) = \alpha]$ *implies for given x the probability* $r(x) = \alpha$

(d) Give a secure protocol for the case when $f : \{0,1\} \times \{0,1\} \to \{0,1\}$ is the AND function (i.e., $f(x, y) = 1$ iff $x = y = 1$). No proof is required.

Hint: You can use the protocol from the previous part that computes $f_{\text{eq}} : D' \times D' \to \{0,1\}$, for an appropriately chosen D'. Alice and Bob would locally map $x, y \in D$ to $x', y' \in D'$, before invoking the protocol for f_{eq}.

Answer: We can define D' to be all 2-bit string and can define two different mapping functions $F_a, F_b : \{0,1\} \to \{0,1\}^2$ from D to D'. For Alice, function $F_a(0) = 01, F_a(1) = 11$ and $F_b(0) = 10, F_b(1) = 11$. Now Alice and Bob can locally map x,y to domain D' and then use the function $f_{eq} : D' \times D' \to \{0,1\}$ for a secure protocol for computing AND function.

2. **IND-CPA and Perfect Correctness $\implies$ SIM-CPA** [15 pts]

Show that a perfectly correct encryption scheme that is IND-CPA secure is SIM-CPA secure.

Hint: You can use a simulator similar to the one we used for showing the analogous result for IND-Onetime and SIM-Onetime. Use a reduction to argue that if the simulation is not good against some PPT adversary and environment, then you can break the IND-CPA security.

3. **Hybrid Argument: Next-Bit Unpredictability implies Pseudorandomness.** [20 pts]

Let Y denote a distribution ensemble over $\{0,1\}^n$, where n is a polynomial function of the security parameter k. Y is said to be *next-bit unpredictable* if, for all PPT algorithms B, $\max_{i\in[n]} |\Pr_{y\leftarrow Y_k}[B(y_1^{i-1}) = y_i] - 1/2|$ is negligible. Y is said to be *pseudorandom* if for all PPT A, $|\Pr_{y\leftarrow Y_k}[A(y) = 0] - \Pr_{y\leftarrow Un}[A(y) = 0]|$ is negligible, where U_n denote the uniform distribution over $\{0,1\}^n$.

Given a PPT distinguisher A, define a PPT predictor B to be as follows:

On input $z \in \{0,1\}^{i-1}$, pick $b \leftarrow \{0,1\}, r \leftarrow \{0,1\}^{n-i}$ and output $A(z||b||r) \oplus b$. (Here $||$ denotes concatenation)

For each $i \in [n]$, define the distribution H_i over n-bit strings as the distribution of the string z produced by taking $y \leftarrow Y_k$, $r \leftarrow U_{n-i}$, and letting $z := y_1^i||r$. Note that $H_0 = U_n$ and $H_n = Y_k$. For parts (a)-(e), fix an $i \in [n]$.

(a) Let $\alpha(z) := \Pr_{y\leftarrow Y_k}[y_1^{i-1} = z]$ and $q(z, b) := \Pr_{r\leftarrow\{0,1\}^{n-i}}[A(z||b||r) = 0]$ for all $z \in \{0,1\}^{i-1}$ and $b \in \{0,1\}$. Compute $\Pr_{y\leftarrow H_{i-1}}[A(y) = 0]$ in terms of these two functions.
Answer: We need to sample for all the values that z can take, and hence apply Bayes rule:

$$\begin{aligned}
\Pr_{y\leftarrow H_{i-1}}[A(y) = 0] &= \sum_{z=\{0,1\}^{i-1}} \Pr_{r\leftarrow\{0,1\}^{n-i+1}}[A(z||r) = 0 \mid y_1^{i-1} = z] * \Pr_{y\leftarrow Y_k}[y_1^{i-1} = z] \\
&= \sum_{z=\{0,1\}^{i-1}} \{\Pr[b = 0] * \Pr_{r\leftarrow\{0,1\}^{n-i}}[A(z||0||r) = 0] + \Pr[b = 1] * \Pr_{r\leftarrow\{0,1\}^{n-i}}[A(z||1||r) = 0]\} * \alpha(z) \\
&= \frac{1}{2} \sum_{z=\{0,1\}^{i-1}} \alpha(z)(q(z, 0) + q(z, 1))
\end{aligned}$$

(b) Also, let $\beta(z) := \Pr_{y \leftarrow Y_k}[y_i = 0 \mid y_1^{i-1} = z]$. Now, compute $\Pr_{y \leftarrow H_i}[A(y) = 0]$.

Answer:

$$\begin{aligned}
\Pr_{y \leftarrow H_i}[A(y) = 0] &= \sum_{z=\{0,1\}^{i-1}} \Pr_{r \leftarrow \{0,1\}^{n-i+1}}[A(z||r) = 0 \mid y_1^{i-1} = z] * \Pr_{y \leftarrow Y_k}[y_1^{i-1} = z] \\
&= \sum_{z=\{0,1\}^{i-1}} (\Pr_{y \leftarrow Y_k}[y_i = 0 \mid y_1^{i-1} = z]q(z,0) + \Pr_{y \leftarrow Y_k}[y_i = 0 \mid y_1^{i-1} = z]q(z,1)) * \alpha(z) \\
&= \sum_{z=\{0,1\}^{i-1}} \alpha(z)(\beta(z)q(z,0) + (1-\beta(z))q(z,1))
\end{aligned}$$

(c) For each $z \in \{0,1\}^{i-1}$, let $\gamma(z) := \Pr[B(z) = 0]$ (where the probability is over the randomness of the algorithm B above). Compute $\gamma(z)$ in terms of the quantities $q(z,0)$ and $q(z,1)$.
Answer:

$$\begin{aligned}
\gamma(z) = \Pr[B(z) = 0] &= \Pr[A(z||b||r) \oplus b = 0] \\
&= \Pr[A(z||b||r) = b] \\
&= \Pr[A(z||0||r) = 0]\Pr[b = 0] + \Pr[A(z||1||r) = 1]\Pr[b = 1] \\
&= \Pr[A(z||0||r) = 0]\Pr[b = 0] + (1 - \Pr[A(z||1||r) = 0])\Pr[b = 1] \\
&= \frac{1}{2}(q(z,0) - q(z,1) + 1)
\end{aligned}$$

(d) Show that $\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i \mid y_1^{i-1} = z] = \frac{1}{2} + 2(\beta(z) - \frac{1}{2})(\gamma(z) - \frac{1}{2})$, for each $z \in \{0,1\}^{i-1}$.

Answer:

$$\begin{aligned}
\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i \mid y_1^{i-1} = z] &= \Pr[B(z) = 0] \Pr_{y \leftarrow Y_k}[y_i = 0 \mid y_1^{i-1} = z] + \Pr[B(z) = 1] \Pr_{y \leftarrow Y_k}[y_i = 1 \mid y_1^{i-1} = z] \\
&= \gamma(z)\beta(z) + (1 - \gamma(z))(1 - \beta(z)) \\
&= 2\gamma(z)\beta(z) - \gamma(z) - \beta(z) + \frac{1}{2} + \frac{1}{2} \\
&= \frac{1}{2} + 2(\gamma(z) - \frac{1}{2})(\beta(z) - \frac{1}{2})
\end{aligned}$$

(e) From the above parts establish that

$$|\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i] - \frac{1}{2}| = |\Pr_{y \leftarrow H_i}[A(y) = 0] - \Pr_{y \leftarrow H_{i+1}}[A(y) = 0]|.$$

Answer: On summing over all z, we get

$$|\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i] - \frac{1}{2}| = |\big[\sum_{z=\{0,1\}^{i-1}} \alpha(z)(\frac{1}{2} + 2(\beta(z) - \frac{1}{2})(\gamma(z) - \frac{1}{2}))\big] - \frac{1}{2}| \tag{1}$$

$$= |\big[\sum_{z=\{0,1\}^{i-1}} \alpha(z)(\frac{1}{2} + (\beta(z) - \frac{1}{2})(q(z,0) - q(z,1)))\big] - \frac{1}{2}| \tag{2}$$

$$= |\sum_{z=\{0,1\}^{i-1}} \alpha(z)(\beta(z) - \frac{1}{2})(q(z,0) - q(z,1)) + \frac{1}{2}(\sum_{z=\{0,1\}^{i-1}} \alpha(z) - 1)| \tag{3}$$

$$= |\sum_{z=\{0,1\}^{i-1}} \alpha(z)(\beta(z)q(z,0) + (1-\beta(z))q(z,1)) - \frac{1}{2}\alpha(z)(q(z,0) + q(z,1))| \tag{4}$$

$$= |\Pr_{y \leftarrow H_i}[A(y) = 0] - \Pr_{y \leftarrow H_{i-1}}[A(y) = 0]| \tag{5}$$

Note: In eqn 3 we have used the fact that $\sum_{z=\{0,1\}^{i-1}} \alpha(z)$ evaulates to 1, this is true because probability of $z = y_1^{i-1}$ over all z shpuld be equal to 1

(f) Using the fact that part (e) holds for all $i \in [n]$, show that

$$|\Pr_{y \leftarrow Y_k}[A(y) = 0] - \Pr_{y \leftarrow U_n}[A(y) = 0]| \leq n \cdot \max_{i \in [n]} |\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i] - \frac{1}{2}|.$$

Answer: Summing up RHS in part (e) for all values of i, we get

$$\begin{aligned}
|\Pr_{y \leftarrow H_0}[A(y) = 0] - \Pr_{y \leftarrow H_n}[A(y) = 0]| &\leq \sum_{i=1}^{n} |\Pr_{y \leftarrow H_i}[A(y) = 0] - \Pr_{y \leftarrow H_{i-1}}[A(y) = 0]| \\
&= \sum_{i=1}^{n} |\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i] - \frac{1}{2}| \\
&\leq n \cdot \max_{i \in [n]} |\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i] - \frac{1}{2}|
\end{aligned}$$

$$|\Pr_{y \leftarrow Y_k}[A(y) = 0] - \Pr_{y \leftarrow U_n}[A(y) = 0]| \leq n \cdot \max_{i \in [n]} |\Pr_{y \leftarrow Y_k}[B(y_1^{i-1}) = y_i] - \frac{1}{2}|$$

If Y is next-bit unpredictable, then the RHS above is negligible (n being polynomial in k), and hence so is the LHS. Since this holds for every PPT adversary A, we conclude that if Y is next-bit unpredictable, then it is pseudorandom.
In the lecture we saw that pseudorandomness implies next-bit unpredictability. The above completes the argument that the two definitions are equivalent.

4. **Impossibility of deterministic CPA-secure encryption.** Suppose a symmetric key encryption scheme has a deterministic encryption algorithm. Give an adversary in the IND-CPA experiment for SKE to show that this scheme cannot be CPA-secure. [5 pts]

A consequence of the above is that the so-called "Electronic Code Book" mode of using a block-cipher is not an IND-CPA secure SKE scheme.

Answer: For this problem, we design an adversary Eve which on first attempt sends $m_0 = a_1, m_1 = a_2, a_1 \neq a_2$, gets back ciphertext $Enc(m_b, K) = c_0$. Here, b can be 0 or 1. We run this algorithm again with messages $m_0 = a_1, m_1 = a_3, a_1 \neq a_3$ We get back $Enc(m_b, K) = c_1$. Now as the SKE scheme has a deterministic encryption algorithm and K has been fixed at the start, each message will have exactly one corresponding ciphertext, hence the adversary can conclude that if c_1 received is equal to c_0 received earlier then $b = 0$, as both ciphertexts correspond to message a_1. If the ciphertexts don't match then adversary has a third attempt with $m_0 = a_2, m_1 = a_3$ and gets back $Enc(m_b, K) = c_2$ Now c_2 must match one of c_0 or c_1. If it matches with c_0 the adversary guesses $b = 0$ else the adversary guesses $b = 1$ accurately. Hence, the adversary guesses b'' accurately ie. $\Pr[b = b'] = 1$ for the third attempt. Hence, the scheme cannot be CPA secure.

5. **One-Timeness of One-Time Pad.** Consider a deterministic "two-message encryption scheme" to be a function $\mathsf{Enc}^2 : \mathcal{K} \times \mathcal{M} \times \mathcal{M} \to \mathcal{C} \times \mathcal{C}$. [5 pts]

(a) Define perfect secrecy for such an encryption scheme.
Answer: For perfect secrecy,

$$\forall m1, m2 \in \mathcal{M}, \forall c1, c2 \in \mathcal{C} \quad \Pr_{k \leftarrow \mathcal{K}}[M = (m1, m2) \mid C = (c1, c2)] = \Pr_{k \leftarrow \mathcal{K}}[M = (m1, m2)]$$

(b) Let $\mathcal{M} = \mathcal{K} = \mathcal{C}$ be the set of n-bit strings. Let $\mathsf{Enc}^2(K, m_1, m_2) = (K \oplus m_1, K \oplus m_2)$, where $\oplus$ is bit-wise xor-ing. Prove that this is **not** perfectly secret, according to your definition.

In particular, using a one-time pad to encrypt two messages will break perfect secrecy.

Answer: Given a ciphertext tuple $C = (c1, c2)$, where $c1, c2 \in \mathcal{C}$, either $c1 = c2$ or $c1 \neq c2$. As the encryption scheme is deterministic and same K is used to get $c1$ and $c2$, if $c1 = c2$ then $m1 = m2$ should hold true. Hence given $C = (c1, c2)$ and consider our prior to be uniformly distributed in the domain $\mathcal{M} \times \mathcal{M}$ then the following must hold:

$$\Pr_{k \leftarrow \mathcal{K}}[M = (m1, m2) \mid C = (c1, c2)] = \Pr_{k \leftarrow \mathcal{K}}[M = (m1', m2') \mid C = (c1', c2')]$$

Suppose for LHS, $c1 = c2$ and in RHS $c1 \neq c2$ then the above equality won't hold as:

$$LHS = \Pr_{k \leftarrow \mathcal{K}}[M = (m1, m2) \mid c1 = c2] = Pr[M1 = (a, a)] = 1/|\mathcal{M}|$$

$$RHS = \Pr_{k \leftarrow \mathcal{K}}[M = (m1, m2) \mid c1' \neq c2'] = Pr[M1 = (a, b), a \neq b] = \frac{1}{|\mathcal{M}|^2 - |\mathcal{M}|}$$

Clearly $LHS \neq RHS$ and hence is **not** perfectly secure.

6. **Statistical Indistinguishability.** [10 pts]

Recall that for two distributions X and Y over n-bit strings, the *statistical difference* (a.k.a. variational distance) between them is denoted by

$$\Delta(X, Y) = \max_{S \subseteq \{0,1\}^n} |\Pr_{x \leftarrow X}[x \in S] - \Pr_{x \leftarrow Y}[x \in S]|.$$

(Alternately, this can be phrased in terms of a statistical test T, which checks if $x \in S$ for some subset S.)

(a) Suppose $G : \{0,1\}^k \to \{0,1\}^n$ is a deterministic function, where $n > k$. Let X be the distribution of the output of $G(s)$ when $s \leftarrow \{0,1\}^k$ is chosen uniformly at random. Let Y be the uniform distribution over $\{0,1\}^n$. Show that $\Delta(X, Y) \geq \frac{1}{2}$. Conclude that the output of a pseudorandom random generator is quite distinguishable from a truly random distribution, if computationally unbounded distinguishers are considered.

(b) Suppose X_k and Y_k are distributions over 2-bit strings (for all integers $k > 0$). Further suppose that for all values of k, $\Delta(X_k, Y_k) \geq 0.1$. Show that X_k and Y_k are *not* computationally indistinguishable.

You may use *non-uniform* PPT distinguishers. i.e., describe a family of distinguishers D_k, each of which runs in time polynomial in k such that $|\Pr_{x \leftarrow X_k}[D_k(x) = 0] - \Pr_{x \leftarrow Y_k}[D_k(x) = 0]| \geq \epsilon(k)$ for some function ϵ that is not negligible.

[Extra Credit] Can you show that X_k and Y_k are in fact distinguishable by a *uniform* PPT distinguisher.

7. **PRG and PRF.** True or False (give reasons): [12 pts]

(a) If $G : \{0,1\}^k \to \{0,1\}^n$ is a PRG, then so is $G' : \{0,1\}^{k+\ell} \to \{0,1\}^{n+\ell}$ defined as $G'(x \circ x') = G(x) \circ x'$ where $x \in \{0,1\}^k$, $x' \in \{0,1\}^\ell$, and $\circ$ denotes concatenation.

Answer: True. We use the Next-bit Unpredictability definition to define psuedo-randomness for $G' : \{0,1\}^{k+\ell} \to \{0,1\}^{n+\ell}$. For i < n, given G is a PRG, we can say

$$|\Pr_{y \leftarrow G}[B(y_1^{i-1}) = y_i] - \frac{1}{2}| = |\Pr_{y \leftarrow G'}[B(y_1^{i-1}) = y_i] - \frac{1}{2}|$$

because $G'(x \circ x') = G(x) \circ x'$ and $i \leq n$. Hence both values are negligible (by definition of PRG). For $n \leq i \leq n + l$, we can also show this to be true. Because x' has been sampled independent of the PRG G or the distribution $\{0,1\}^k$, we can conclude that for $n < i \leq n + l$

$$\Pr_{y \leftarrow G'}[B(y_1^{i-1}) = y_i] = \frac{1}{2}$$

and hence satisfies the definition of Next bit unpredictability. As we have proved for all i and for any PPT adversary B, G' must be a PRG.

(b) If $F : \{0,1\}^k \times \{0,1\}^m \to \{0,1\}^n$ is a PRF, then so is:

i. $F' : \{0,1\}^k \times \{0,1\}^{m+\ell} \to \{0,1\}^{n+\ell}$ defined as $F'(s; x \circ x') = F(s; x) \circ x'$ where $s \in \{0,1\}^k$, $x \in \{0,1\}^m$, $x' \in \{0,1\}^\ell$.
Answer: False.For F' to be PRF we need to show that

$$|\Pr[D^{F'_k(\cdot)}(1^m) = 1] - \Pr[D^{f(\cdot)}(1^m) = 1]| \leq negl(n)$$

where $f \in Func_n$. F' can be proved not to be pseudorandom if values at any two points can be shown correlated to each other. If we consider the distinguisher D, used to distinguish between F' and f, to query the oracle $\mathcal{O}$ on two arbitrary points x_1, x_2 we obtain values $y_1 = \mathcal{O}(x_1), y_2 = \mathcal{O}(x_2)$ The distinguisher will output 1 if and only if last l digits of y_1, y_2 are same. Hence, if $\mathcal{O} = F_k$ then D outputs 1 with probability 1 (as last l digits of each of y_1, y_2 is x'). And if $\mathcal{O} = f$, then D will output 1 with probability 2^{-l}. The equation evaluates to $|1 - 2^{-l}|$ which is not negligible. Hence F' is not PRF.

ii. $F' : \{0,1\}^{k+\ell} \times \{0,1\}^m \to \{0,1\}^{n+\ell}$ defined as $F'(s \circ s'; x) = F(s; x) \circ s'$ where $s \in \{0,1\}^k$, $x \in \{0,1\}^m$, $s' \in \{0,1\}^\ell$.

Answer: False. For this we can again use the definition provided in the previous part. The distinguisher will output 1 if the outputs by the oracle y_1, y_2 have last l digits same. If Oracke is F', then the probability of this occurring is 1 and if it is a random function f then the probability is 2^{-l}. The difference $= |1 - 2^{-l}|$ is not negligible and hence F' is not PRF.

8. **Block Ciphers** [8 pts]

(a) Consider an adversary in the IND-CPA experiment against a symmetric key encryption algorithm implemented using a block-cipher in the CTR mode. Describe a brute-force strategy for the adversary to recover the encryption key.

Answer: Adversary chooses messages m_0, m_1 and gets back $F(m_b, K)$. This adversary can send 2^n such times can hence determine the function used by the CTR mode. This is computationally quite large and as the adversary is not computationally restricted, we can recover such an encryption scheme

(b) A PetaFLOPS computer can execute 10^{15} floating point operations per second. If the adversary uses a 100 PetaFLOPS computer, and the block-cipher used is DES (which uses 56 bit keys), how long would your brute-force strategy take on the average to recover the key? You may suppose that a single evaluation of a block-cipher (DES or AES) takes 10 FLOPs.
What if the block-cipher used is AES with 128-bit keys?

Answer: Average recovery time $= \dfrac{T + 2T + \cdots + NT}{N}$ where T = Time taken to recover 1 key, N = Number of keys $= 2^{56}$.

$$T \times (N+1)/2 = \frac{1 \times 10}{100 \times 10^{15}} \times 2^{55} \approx 3.5\ sec$$

For AES scheme,

$$T \times (N'+1)/2 = \frac{1 \times 10}{100 \times 10^{15}} \times 2^{127} \approx 1.65 \times 10^{22}\ sec \approx 5 \times 10^{14}\ years$$

(c) **[Extra Credit]** The triple-DES (3DES) is a block-cipher that uses the DES block-cipher three times, with three different keys. The output of 3DES with key (K_1, K_2, K_3), on input x is defined as $\text{3DES}_{(K_1,K_2,K_3)}(x) := \text{DES}_{K_1}(\text{DES}^{-1}_{K_2}(\text{DES}_{K_3}(x)))$ where DES_K and DES^{-1}_K stand for the application of the DES block-cipher in the forward and reverse directions. Since DES has 56 bit keys, 3DES has 168 bit keys.
As before, your goal is to design a key-recovery algorithm for an adversary in the IND-CPA experiment for an SKE scheme using 3DES in CTR mode. Your algorithm can also invoke the DES block-cipher locally as a black-box (in either forward or reverse directions) with keys of your own choice.
Can you devise a key-recovery algorithm which invokes the DES block-cipher computation "only" about 2^{112} times. How much memory does your algorithm use?

9. **One-way, but every single bit of the preimage is predictable:** [10 pts]

For any function $f : \{0,1\}^* \to \{0,1\}^*$, define a function g_f as follows $g_f(x, S) = (f(x|_S), S, x|_{\bar{S}})$, where S is a subset of $\{1, 2, \ldots, |x|\}$ of size $\lfloor |x|/2 \rfloor$ (represented as a bit vector of length $|x|$ with $\lfloor |x|/2 \rfloor$ 1's). Here $x|_S$ denotes the string obtained by choosing only those bits from x whose indices are in S and $x|_{\bar{S}}$ is the string containing the remaining bits.

(a) Show that if f is a one-way function, then so is g_f . You may assume that f is length-preserving (i.e., $|f(x)| = |x|$ for all x).

(b) Show that no single bit of the input is a hard-core bit for g_f.