# CS228 Logic for Computer Science 2021

## Lecture 18: Terms and unification

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## Topic 18.1

# Game of terms

# CNF formulas and proofs

### Example 18.1

*Recall we had a proof for* $\emptyset \vdash (\forall x.\ (P(x) \lor Q(x)) \Rightarrow \exists x.P(x) \lor \forall x.Q(x))$.

*Let us try to prove it via FOL CNF.*

*We fist take negation of the formula and transform it into FOL CNF. We obtain*

$$\Sigma \triangleq \{\forall x.\ (P(x) \lor Q(x)), \forall x.\neg P(x), \neg Q(c)\}$$

*We have written each clause as a separate formula without dropping quantifiers.*

*We show that we can derive contradiction from* $\Sigma$.

# CNF formulas and proofs

Recall

$$\Sigma \triangleq \{\forall x.\ (P(x) \lor Q(x)), \forall x. \neg P(x), \neg Q(c)\}$$

We can drive contradiction Here is a proof that derives contradiction.

1. $\Sigma \vdash \neg Q(c)$ — Assumption
2. $\Sigma \vdash \forall x.\ (P(x) \lor Q(x))$ — Assumption
3. $\Sigma \vdash P(x) \lor Q(x)$ — $\forall$-Elim applied to 2
4. $\Sigma \vdash \forall x.\ \neg P(x)$ — Assumption
5. $\Sigma \vdash \neg P(x)$ — $\forall$-Elim applied to 4
6. $\Sigma \vdash Q(x)$ — Resolution applied to 3 and 5
7. $\Sigma \vdash \forall x.\ Q(x)$ — $\forall$-Intro applied to 6
8. $\Sigma \vdash Q(c)$ — $\forall$-Elim applied to 7
9. $\Sigma \vdash Q(c) \land \neg Q(c)$ — $\land$-Intro applied to 1 and 8

Step 7 introduced $c$, which is a non-mechanical step, i.e., we need to plan to choose the term.

# Example : an extreme example for finding a magic term.

## Example 18.2

*Let us derive contradiction from the following.*
*Let* $\Sigma = \{\forall x_4, x_3, x_2, x_1.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))\}$

*Let us construct a proof for the above.*

1. $\Sigma \vdash \forall x_4, x_3, x_2, x_1.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))$
2. $\Sigma \vdash \forall x_3, x_2, x_1.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))$ — $\forall$-*Elim applied to 1*
3. $\Sigma \vdash \forall x_2, x_1.\ f(x_1, j(x_4), x_2) \neq f(g(x_2), j(x_4), h(j(x_4), a))$ — $\forall$-*Elim applied to 2*
4. $\Sigma \vdash \forall x_1.\ f(x_1, j(x_4), h(j(x_4), a)) \neq f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$ — $\forall$-*Elim applied to 3*
5. $\Sigma \vdash f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a)) \neq f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$ — $\forall$-*Elim applied to 4*

We need a mechanism to auto detect substitutions such that terms with variables become equal

## Exercise 18.1

*Finish the proof using Reflex and derive contradiction.*

# How to find the magic terms?

In the previous, example we were asked to equate terms

$$f(x_1, x_3, x_2) \text{ and } f(g(x_2), j(x_4), h(x_3, a))$$

by mapping variables $x_1$, $x_2$, $x_3$, and $x_4$ to terms.

The process of equating terms is called unification.

Sometimes, the unification may not even be possible.

## Topic 18.2

# Unification

# Making terms equal by substitution

# Unifier

## Definition 18.1

*For terms $t$ and $u$, a substitution $\sigma$ is a unifier of $t$ and $u$ if $t\sigma = u\sigma$.*
*We say $t$ and $u$ are unifiable if there is a unifier $\sigma$ of $t$ and $u$.*

## Example 18.3

*Find a unifier $\sigma$ of the following terms*

- $x_4\sigma = f(x_1)\sigma$ — $\sigma = \{x_1 \mapsto c, x_4 \mapsto f(c)\}$
- $x_4\sigma = f(x_1)\sigma$ — $\sigma = \{x_1 \mapsto x_2, x_4 \mapsto f(x_2)\}$
- $g(x_1)\sigma = f(x_1)\sigma$ — *not unifiable*
- $x_1\sigma = f(x_1)\sigma$ — *not unifiable*

# More general substitution

**Commentary:** The following definition depends on composition of substitution, which was discussed in earlier lectures. If not clear please look it up.

## Definition 18.2

*Let $\sigma_1$ and $\sigma_2$ be substitutions. $\sigma_1$ is more general than $\sigma_2$ if there is a substitution $\tau$ such that $\sigma_2 = \sigma_1\tau$. We write $\sigma_1 \geq \sigma_2$.*

## Example 18.4

- $\sigma_1 = \{x \mapsto f(y,z)\} \geq \sigma_2 = \{x \mapsto f(c, g(z))\}$ *because* $\sigma_2 = \sigma_1\{y \mapsto c, z \mapsto g(z)\}$.
- $\sigma_1 = \{x \mapsto f(y,z)\} \geq \sigma_2 = \{x \mapsto f(z,z)\}$ *because* $\sigma_2 = \sigma_1\{y \mapsto z\}$.

## Exercise 18.2

*If $\sigma_1 \geq \sigma_2$ and $\sigma_2 \geq \sigma_3$. Then, $\sigma_1 \geq \sigma_3$.*

$\sigma_2 = \sigma_1 \tau_1$

$\sigma_3 = \sigma_2 \tau_2 = (\sigma_1 \tau_1)\tau_2 = \sigma_1(\tau_1 \tau_2) = \sigma_1 \tau_3$

# Most general unifier (mgu)

Is mgu unique? Does mgu always exist?

## Definition 18.3

*Let $t$ and $u$ be terms with variables, and $\sigma$ be a unifier of $t$ and $u$.*
*$\sigma$ is most general unifier(mgu) of $u$ and $t$ if it is more general than any other unifier.*

## Example 18.5

*Consider terms $f(x, g(y))$ and $f(g(z), u)$. The following are unifiers of the terms.*

1. $\sigma_1 = \{x \mapsto g(z), u \mapsto g(y), z \mapsto z, y \mapsto y\}$
2. $\sigma_2 = \{x \mapsto g(c), u \mapsto g(d), z \mapsto c, y \mapsto d\}$
3. $\sigma_3 = \{x \mapsto g(z), u \mapsto g(z), z \mapsto z, y \mapsto z\}$

*where $c$ and $d$ are constants.*

*Please note $\sigma_1 \geq \sigma_2$ and $\sigma_1 \geq \sigma_3$. $\sigma_2 \not\geq \sigma_3$ and $\sigma_3 \not\geq \sigma_2$.(why?)*

# Uniqueness of mgu

### Definition 18.4

*A substitution $\sigma$ is a renaming if $\sigma : \text{Vars} \rightarrow \text{Vars}$ and $\sigma$ is one-to-one*

### Theorem 18.1

$\sigma_1 \geq \sigma_2$

*If $\sigma_1$ and $\sigma_2$ are mgus of $u$ and $t$. Then there is a renaming $\tau$ such that $\sigma_1\tau = \sigma_2$.*

### Proof.

Since $\sigma_1$ is mgu, therefore there is a substitution $\hat{\sigma_1}$ such that $\sigma_2 = \sigma_1\hat{\sigma_1}$.
Since $\sigma_2$ is mgu, therefore there is a substitution $\hat{\sigma_2}$ such that $\sigma_1 = \sigma_2\hat{\sigma_2}$.
Therefore $\sigma_1 = \sigma_1\hat{\sigma_1}\hat{\sigma_2}$. (And also, $\sigma_2 = \sigma_2\hat{\sigma_2}\hat{\sigma_1}$. )

Without loss of generality, for each $y \in \text{Vars}$, if $y \notin FV(x\sigma_1)$ for each $x \in \text{Vars}$, then we assume $y\hat{\sigma_1} = y$.

# Uniqueness of mgu (contd.)

## Proof(contd.)

**claim:** for each $y \in \text{Vars}$, $y\hat{\sigma_1} \in \text{Vars}$

Consider a variable $x$ such that $y \in FV(x\sigma_1)$. Three possibilities for $y\hat{\sigma_1}$.

1. If $y\hat{\sigma_1} = f(..)$, $x\sigma_1\hat{\sigma_1}$ is longer than $x\sigma_1$. Therefore, $x\sigma_1\hat{\sigma_1}\hat{\sigma_2}$ is longer than $x\sigma_1$.
   Contradiction.
2. If $y\hat{\sigma_1} = c$, $\hat{\sigma_2}$ will not be able to rename $c$ back to $y$ in $x\sigma_1$.
3. Therefore, we must have the third possibility, i.e., $y\hat{\sigma_1} \in \text{Vars}$ is a variable.

**claim:** for each $y_1 \neq y_2 \in \text{Vars}$, $y_1\hat{\sigma_1} \neq y_2\hat{\sigma_1}$

Assume $y_1\hat{\sigma_1} = y_2\hat{\sigma_1}$. $\hat{\sigma_2}$ will not be able to rename the variables back to distinct variables.(why?)
Contradiction.

$\hat{\sigma_1}$ is a renaming. □

Topic 18.3

# Unification algorithm

# How to find unifiers?

We need to identify where terms are not in agreement.

Apply substitutions to fix the disagreement.

# Disagreement pairs

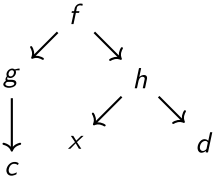
## Definition 18.5

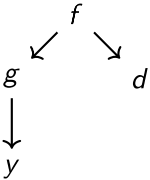
*For terms $t$ and $u$, $d_1$ and $d_2$ are disagreement pair if*

1. *$d_1$ and $d_2$ are subterms of $t$ and $u$ respectively,*
2. *the path to $d_1$ in $t$ is same as ~~and~~ the path to $d_2$ in $u$, and*
3. *roots of $d_1$ and $d_2$ are different.*

## Example 18.6

*Consider terms $t = f(g(c), h(x, d))$ and $u = f(g(y), d)$*




*Disagreement pairs: $h(x, d)$ and $d$*

*Disagreement pairs: $c$ and $y$*

# Robinson algorithm for computing mgu

**Algorithm 18.1:** $\text{MGU}(t, u \in T_S)$

$\sigma := \{\}$;
**while** $t\sigma \neq u\sigma$ **do**
  choose disagreement pair $d_1, d_2$ in $t\sigma$ and $u\sigma$;
  **if** *both $d_1$ and $d_2$ are non-variables* **then** **return** *FAIL* ;
  **if** $d_1 \in \text{Vars}$ **then**
    $x := d_1$; $s := d_2$;
  **else**
    $x := d_2$; $s := d_1$;
  **if** $x \in FV(s)$ **then** **return** *FAIL* ;
  $\sigma := \sigma\{x \mapsto s\}$ // update the substitution
**return** $\sigma$

If MGU is sound and always terminates then mgus for unifiable terms always exist.

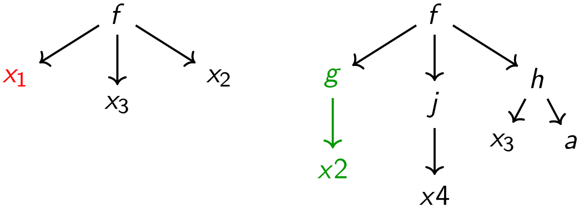
### Exercise 18.3

*Let $\sigma_0, \sigma_1, \ldots$ be the sequence of observed substitutions during the run of* MGU. *Show $\sigma_i \geq \sigma_{i+1}$.*

# Example: run of Robinson's algorithm

## Example 18.7

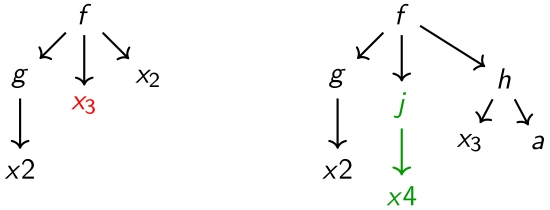
*Consider call* $\text{MGU}(f(x_1, x_3, x_2), f(g(x_2), j(x_4), h(x_3, a)))$
*Initial* $\sigma = \{\}$




*Disagreement pairs* $:= \{\ (x_1, g(x_2)),\ (x_3, j(x_4)),\ (x_2, h(x_3, a))\ \}$
*Choose a disagreement pair:* $(x_1, g(x_2))$
*After update* $\sigma = \{x_1 \mapsto g(x_2)\}$
*Input terms after applying* $\sigma$*:* $f(g(x_2), x_3, x_2)$ *and* $f(g(x_2), j(x_4), h(x_3, a))$

# Example: run of Robinson's algorithm II (contd.)

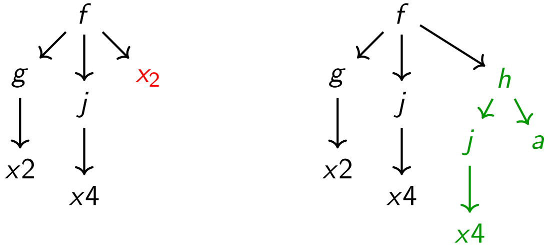
Input terms now:




Disagreement pairs in the new terms:= $\{\ (x_3, j(x_4)),\ (x_2, h(x_3, a))\ \}$
Choose a disagreement pair: $(x_3, j(x_4))$
After update $\sigma = \{x_1 \mapsto g(x_2), x_3 \mapsto j(x_4)\}$
Input terms after applying $\sigma$: $f(g(x_2), j(x_4), x_2)$ and $f(g(x_2), j(x_4), h(j(x_4), a))$

# Example: run of Robinson's algorithm III(contd.)

Input terms now:




Choose the last disagreement pair: $(x_2, h(j(x_4), a))$.

Since the mapping of $x_1$ refers to $x_2$ in old $\sigma$, it is also updated.

After applying new mapping $\sigma := \sigma\{x_2 \mapsto h(j(x_4), a)\}$

$$= \{x_1 \mapsto g(x_2), x_3 \mapsto j(x_4)\}\{x_2 \mapsto h(j(x_4), a)\}$$

$$= \{x_1 \mapsto g(h(j(x_4), a)), x_3 \mapsto j(x_4), x_2 \mapsto h(j(x_4), a)\}$$

Terms after applying $\sigma$: $f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$ and $f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$

Since no disagreement pairs, we are done.

# Unification in proving

## Example 18.8

*Consider again* $\forall x_1, x_2, x_3, x_4.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))$

*Given the above, one may ask*

$$\textit{Are } f(x_1, x_3, x_2) \textit{ and } f(g(x_2), j(x_4), h(x_3, a)) \textit{ unifiable?}$$

If we run the unification algorithm on the above terms, we obtain

- $x_1 \mapsto g(h(j(x_4), a))$
- $x_2 \mapsto h(j(x_4), a)$
- $x_3 \mapsto j(x_4)$

We will integrate unification with a simpler resolution proof system.

The above instantiations are not magic anymore!

Topic 18.4

## Correctness of Robinson algorithm

# Termination of MGU

### Theorem 18.2

MGU *always terminates.*

### Proof.

Total number of variables in $t\sigma$ and $u\sigma$ decreases in every iteration.(why?)

Since initially there were finite variables in $t$ and $u$, MGU terminates. □

# Soundness of MGU

### Theorem 18.3

*MGU$(t, u)$ returns unifier $\sigma$ iff $t$ and $u$ are unifiable. Furthermore, $\sigma$ is a mgu.*

### Proof.

Since MGU must terminate, if $t$ and $u$ are not unifiable then MGU must return FAIL.

Let us suppose $t$ and $u$ are unifiable and $\tau$ is a unifier of $t$ and $u$.
**claim:** $\tau = \sigma\tau$ is the loop invariant of MGU.

**base case:**
Initially, $\sigma$ is identity. Therefore, the invariant holds initially.

**induction step:**
Induction hypothesis: $\tau = \sigma\tau$ holds at the loop head. ...

# Soundness of MGU(contd.)

## Proof(contd.)

**claim:** $t\sigma$ and $u\sigma$ are unifiable.

$$\underbrace{t\sigma\tau =}_{\text{Ind. Hyp.}} \underbrace{t\tau = u\tau}_{\text{Assumption}} \underbrace{= u\sigma\tau}_{\text{Ind. Hyp.}}.$$

**claim:** $x\tau = s\tau$.
Since $t\sigma\tau = u\sigma\tau$, and $x$ and $s$ are disagreement pairs in $t\sigma$ and $u\sigma$, $x\tau = s\tau$.

**claim:** $\{x \mapsto s\}\tau = \tau$.
Choose $y \in Vars$.

- If $y = x$, $y\{x \mapsto s\}\tau = s\tau = x\tau = y\tau$.
- If $y \neq x$, $y\{x \mapsto s\}\tau = y\tau$.

Therefore, $\{x \mapsto s\}\tau = \tau$.

# Soundness of MGU(contd.)

## Proof(contd.)

We now show that if we assume the invariant at the loop head, then FAIL is not returned.

**claim:** no FAIL at the first if condition
One of $d_1$ and $d_2$ is a variable. Otherwise $t\sigma$ and $u\sigma$ are not unifiable.

**claim:** no FAIL at the last if condition
Since $x\tau = s\tau$, $x$ cannot occurs in $s$. Otherwise, no unifier can make them equal(why?).

...

# Soundness of MGU(contd.)

## Proof(contd.)

Since there is no fail, we show that invariant will continue to hold after the iteration.

**claim:** $\sigma\{x \mapsto s\}\tau = \tau$
Since $\{x \mapsto s\}\tau = \tau$, $\sigma\{x \mapsto s\}\tau = \sigma\tau$. By induction hypothesis, $\sigma\{x \mapsto s\}\tau = \tau$.

Due to the invariant $\tau = \sigma\tau$, $\sigma$ is mgu at the termination. $\square$

# Topic 18.5

## Problems

# MGU

## Exercise 18.4

*Find mgu of the following terms*

1. $f(g(x_1), h(x_2), x_4)$ *and* $f(g(k(x_2, x_3)), x_3, h(x_1))$
2. $f(x, y, z)$ *and* $f(y, z, x)$
3. $\textsc{MGU}(f(g(x), x), f(y, g(y)))$

## Exercise 18.5

*Let* $\sigma_1$ *and* $\sigma_2$ *be the MGUs in the above exercise. Give unifiers* $\sigma_1'$ *and* $\sigma_2'$ *for the problems respectively such that they are not MGUs. Also give* $\tau_1$ *and* $\tau_2$ *such that*

1. $\sigma_1' = \sigma_1\tau_1$
2. $\sigma_2' = \sigma_2\tau_2$

# Maximum and minimal substitutions

Exercise 18.6

a. *Give two maximum general* substitutions *and two minimal general* substitutions*.*

b. *Show that maximum general* substitutions *are equivalent under renaming.*

# Multiple unification

### Definition 18.6

*Let $t_1, .., t_n$ be terms. A substitution $\sigma$ is a unifier of $t_1, .., t_n$ if $t_1\sigma = .. = t_n\sigma$.*
*We say $t_1, .., t_n$ are unifiable if there is a unifier $\sigma$ of them.*

### Exercise 18.7

*Write an algorithm for computing multiple unifiers using the binary* MGU.

# Concurrent unification

## Definition 18.7

*Let $t_1, .., t_n$ and $u_1, .., u_n$ be terms. A substitution $\sigma$ is a concurrent unifier of $t_1, .., t_n$ and $u_1, .., u_n$ if*

$$t_1\sigma = u_1\sigma, \quad .., \quad t_n\sigma = u_n\sigma.$$

*We say $t_1, .., t_n$ and $u_1, .., u_n$ are concurrently unifiable if there is a unifier $\sigma$ for them.*

## Exercise 18.8

*Write an algorithm for concurrent unifiers using the binary* MGU.

Topic 18.6

# Extra slides: algorithms for unification

# Robinson is exponential

Robinson algorithm has worst case exponential run time.

### Example 18.9

*Consider unification of the following terms*
$f(x_1, g(x_1, x_1), x_2, ....)$
$f(g(y_1, y_1), y_2, g(y_2, y_2), ....)$

*The mgu:*

- $x_1 \mapsto g(y_1, y_1)$
- $y_2 \mapsto g(g(y_1, y_1), g(y_1, y_1))$
- .... *(size of term keeps doubling)*

*After discovery of a substitution $x \mapsto s$, Robinson checks if $x \in FV(s)$.*
*Therefore, Robinson has worst case exponential time.*

# Martelli-Montanari algorithm

This algorithm is lazy in terms of applying occurs check

**Algorithm 18.2:** MM-MGU($t, u \in T_{\mathbf{S}}$)

$\sigma := \lambda x.x$; $M = \{t = u\}$;
**while** *change in M or* $\sigma$ **do**
  **if** $f(t_1, ...t_n) = f(u_1, ...u_n) \in M$ **then**
    $M := M \cup \{t_1 = u_1, ...t_n = u_n\} - \{f(t_1, ...t_n) = f(u_1, ...u_n)\}$;
  **if** $f(t_1, ...t_n) = g(u_1, ...u_n) \in M$ **then** **return** *FAIL* ;
  **if** $x = x \in M$ **then** $M := M - \{x = x\}$ ;
  **if** $x = t' \in M$ *or* $t' = x \in M$ **then**
    **if** $x \in FV(t')$ **then** **return** *FAIL* ;
    $\sigma := \sigma[x \mapsto t']$; $M := M\sigma$

**return** $\sigma$

**Commentary:** Please find more details on https://pdfs.semanticscholar.org/3cc3/338b59855659ca77fb5392e2864239c0aa75.pdf

# Escalada-Ghallab Algorithm

There is also Escalada-Ghallab Algorithm for unification.

# End of Lecture 18