CS 218 Design and Analysis of Algorithms

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Module 2: Flow networks, Max-flow, Min-cut and applications

IPL winner

There are 4 IPL teams. CSK, DC, KNR and MI.

They have played some games with each other and they are yet to play some more games.

Current win count is as follows.

CSK: 9, DC: 8, KNR: 8, MI: 7.

Each team is yet to play 1 game with each other except CSK and MI.

That is, there are 5 more games to go.

Can MI win or tie for the first position?

IPL winner

Win count. CSK: 9, DC: 8, KNR: 8, MI: 7.

Each team is yet to play 1 game with each other except CSK and MI.

Can MI win or tie for the first position?

MI must win both its games.
CSK must lose both.

At the end of this the win count will be.
CSK: 9, DC: 9, KNR: 9, MI: 9.

With DC and KNR yet to play against each other.
Hence at least one of them will have win count of 10.

MI cannot win.

IPL winner

A different scenario.

	wins	CSK	DC	KNR	MI
CSK	15	–	1	6	4
DC	13	1	–	1	4
KNR	12	6	1	–	4
MI	4	4	4	4	–

Can MI win or tie for the first position?

In this case too, MI cannot win, no matter what.

How do we conclude that?

Requires a slightly delicate argument.

IPL winner

	wins	CSK	DC	KNR	MI
CSK	15	–	1	6	4
DC	13	1	–	1	4
KNR	12	6	1	–	4
MI	4	4	4	4	–

MI can win at most 16 matches.

Let $P = \{\text{CSK, KNR}\}$.

CSK and KNR have together won 27 matches. And they will play 6 more matches against each other.

$27 + 6 = 33$ and $33/2 > 16$.

Hence MI cannot win.

Note that the argument does not go through if we take $P = \{\text{CSK, KNR, DC}\}$.

IPL elimination problem

IPL elimination problem.

Input: A set of teams X. For each $x \in X$, the number of wins so far w_x. For each pair of teams $x, y \in X$, the number of games yet to be played between x and y, g_{xy}, and a team $z \in X$.

Check: Will z be eliminated?

We will rephrase the problem as a max-flow problem.

Then use the max-flow algorithm to solve it.

Also known as "Baseball elimination problem".

IPL winner as a network flow

The approach

From the given w_xs and $g_{x,y}$s values and a team name z, we will create a flow network G_z.

Let $g_z^* = \sum_{x,y \in X \setminus \{z\}} g_{x,y}$.

We will show that G_z has a flow of value strictly less than g^* in it if and only if z is eliminated.

We will argue that to obtain G_z from the input takes polynomial time.

This will overall prove that IPL elimination problem can be solved in polynomial time.

Construction of G_z

We describe G_z by giving its vertices V, edges E, and capacities $c : E \to \mathbb{N}$.

Let $X' = X \setminus \{z\}$.
$V = \{s\} \cup \{t\} \cup \{u_{x,y} \mid x, y \in X'\} \cup \{u_x \mid x \in X'\}$.

$E = E_{\text{source}} \cup E_{\text{mid}} \cup E_{\text{sink}}$, where

$$\begin{aligned}
E_{\text{source}} &= \{(s, u_{x,y}) \mid x, y \in X'\} \\
E_{\text{mid}} &= \cup_{x,y \in X'} \{(u_{x,y}, u_x), (u_{x,y}, u_y)\} \\
E_{\text{sink}} &= \{(u_x, t) \mid x \in X'\}
\end{aligned}$$

The map c is defined as follows.

$$c(e) = \begin{cases} g_{x,y} & \text{if } e = (s, u_{x,y}) \\ \infty & \text{if } e \in E_{\text{mid}} \\ m - w_x & \text{if } e = (u_x, t) \end{cases}$$

Lemma

G_z has a flow of value strictly less than g^* in it if and only if z is eliminated.

Proof.

($\Rightarrow$) Suppose G_z has a flow of value g^*.

This means, there is a way to obtain outcomes where all other teams win at most m more games.

This means that z is not eliminated.

($\Leftarrow$) Conversely, suppose z is not eliminated.

Then using the outcomes of the games, it is possible to set the flow values of the edges in E_{right} such that no flow exceeds $m - w_u$ for any u at the end of all the games.

Thereby achieving a flow of value g^*.

IPL elimination problem

Lemma (Characterising the team that is eliminated)

Suppose team z has been eliminated. Then the following two things hold.

- *z can finish with at most $m := w_z + \sum_{x \in X \setminus \{z\}} g_{z,x}$ wins.*

- *There exists a set $Q \subseteq X$ such that*

$$\sum_{x \in Q} w_x + \sum_{x,y \in Q} g_{x,y} \geq m \cdot |Q|$$