CS228 Logic for Computer Science 2021

Lecture 8: k-SAT and XOR SAT

Instructor: Ashutosh Gupta

IITB, India

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Topic 8.1

k-sat

k-sat

Definition 8.1
A k-sat formula is a CNF formula and has at most k literals in each of its clauses

Example 8.1
- $(p \wedge q \wedge \neg r)$ *is 1-SAT*
- $(p \vee \neg p) \wedge (p \vee q)$ *is 2-SAT*
- $(p \vee \neg q \vee \neg s) \wedge (p \vee q) \wedge \neg r$ *is 3-SAT*

3-SAT satisfiablity

Theorem 8.1

For each k-SAT formula F there is a 3-SAT formula F' with linear blow up such that F and F' are equivsatisfiable.

Proof.

Consider F a k-SAT formula with $k \geq 4$.
Consider a clause $G = (\ell_1 \vee \dots \vee \ell_k)$ in F, where ℓ_i are literals.

Let $x_2, \dots, x_{k-2}$ be variables that do not appear in F.
Let G' be the following set of clauses

$$(\ell_1 \vee \ell_2 \vee x_2) \wedge \bigwedge_{i \in 2..k-3} (\neg x_i \vee x_{i+1} \vee \ell_{i+1}) \wedge (\neg x_{k-2} \vee \ell_{k-1} \vee \ell_k).$$

We show G is sat iff G' is sat. ...

Exercise 8.1

Convert $(p \vee \neg q \vee s \vee \neg t) \wedge (\neg q \vee x \vee \neg y \vee z)$ *into a 3-SAT formula*

3-SAT satisfiability(cont. I)

Proof(contd. from last slide).

Recall

$$G' = (\ell_1 \vee \ell_2 \vee x_2) \wedge \bigwedge_{i \in 2..k-3} (\neg x_i \vee x_{i+1} \vee \ell_{i+1}) \wedge (\neg x_{k-2} \vee \ell_{k-1} \vee \ell_k).$$

Assume $m \models G'$:

Assume for each $i \in 1..k$, $m(\ell_i) = 0$.
Due to the first clause $m(x_2) = 1$.
Due to ith clause, if $m(x_i) = 1$ then $m(x_{i+1}) = 1$.
Due to induction, $m(x_{k-2}) = 1$.
Due to the last clause of G', $m(x_{k-2}) = 0$. Contradiction.
Therefore, exists $i \in 1..k$ $m(\ell_i) = 1$. Therefore $m \models G$.

...

3-SAT satisfiability(cont. II)

Proof(contd. from last slide).

Recall

$$G' = (\ell_1 \lor \ell_2 \lor x_2) \land \bigwedge_{i \in 2..k-2} (\neg x_i \lor x_{i+1} \lor \ell_{i+1}) \land (\neg x_{k-2} \lor \ell_{k-1} \lor \ell_k).$$

Assume $m \models G$:

There is a $m(\ell_i) = 1$.

Let $m' \triangleq m[x_2 \mapsto 1, .., x_{i-1} \mapsto 1, x_i \mapsto 0, \dots, x_{k-2} \mapsto 0]$.

Therefore, $m' \models G'_{\text{(why?)}}$.

G' contains $3(k-2)$ literals.

In the worst case, the formula size will increase 3 times. $\square$

Exercise 8.2

a. Complete the above argument.

b. Show a 3-SAT cannot be converted into a 2-SAT via Tseitin's encoding.

c. When is the worst case?

Special classes of formulas

We will discuss the following subclasses whose SAT problems are polynomial

- 2-SAT
- XOR-SAT
- Horn clauses

Commentary: For the curious, Schaefer's dichotomy theorem identified subclasses of SAT problem that are polynomial and all others are NP-complete. The above subclasses play an important role in the theorem. The theorem suggests that no other subclass is polynomial.

Topic 8.2

2-SAT

2-SAT

Definition 8.2

A 2-sat formula is a CNF formula that has only binary clauses

We assume that unit clauses are replaced by clauses with repeated literals.

Example 8.2

- $(\neg p \lor q) \land (\neg q \lor r) \land (\neg r \lor p) \land (r \lor q)$ *is a 2-SAT formula*
- $(p \lor p) \land (\neg p \lor \neg p)$ *is a 2-SAT formula*

Implication graph

Definition 8.3

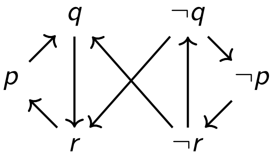
Let F be a 2-SAT formula such that $\mathrm{Vars}(F) = \{p_1, \dots, p_n\}$. *The* ***implication graph*** (V, E) *for* F *is defined as follows.*

$$V = \{p_1, \dots, p_n, \neg p_1, \dots, \neg p_n\} \qquad E = \{(\bar{\ell_1}, \ell_2), (\bar{\ell_2}, \ell_1) | (\ell_1 \lor \ell_2) \in F\},$$

where $\bar{p} = \neg p$ *and* $\overline{\neg p} = p$.

Example 8.3

Consider $(\neg p \lor q) \land (\neg q \lor r) \land (\neg r \lor p) \land (r \lor q)$.



Exercise : implication graph

Exercise 8.3

Draw implication graphs of the following

1. $(p \vee q) \wedge (\neg p \vee \neg q)$
2. $(p \vee \neg q) \wedge (q \vee p) \wedge (\neg p \vee \neg r) \wedge (r \vee \neg p)$
3. $(p \vee p) \wedge (\neg p \vee \neg p)$
4. $(p \vee \neg p) \wedge (p \vee \neg p)$

Properties of implication graph

Consider a formula F and its implication graph (V, E).

Theorem 8.2

If there is a path from ℓ_1 to ℓ_2 in (V, E) then there is a path from $\bar{\ell_2}$ to $\overline{\ell_1}$.

Exercise 8.4

a. Prove the above theorem.

b. Does the above theorem imply
if there is a path from p to $\neg p$ in (V, E) then there is a path from $\neg p$ to p?

Theorem 8.3

For every strongly connected component(scc) $S \subseteq V$ in (V, E), there is another scc S^c, called complementary component, that has exactly the literals that are negation of the literals in S.

Proof.

Due to theorem 8.2. Directly from 8.2 □

Properties of implication graph (contd.)

Theorem 8.4

For each $m \models F$, if there is a path from ℓ_1 to ℓ_2 in (V, E) then if $m(\ell_1) = 1$ then $m(\ell_2) = 1$.

Theorem 8.5

For each $m \models F$ and each scc S in (V, E), either

- $m(\ell) = 1$ *for each* $\ell \in S$ *, or*
- $m(\ell) = 0$ *for each* $\ell \in S$.

Exercise 8.5

Prove the above theorems.

(use theorem in the previous slide)

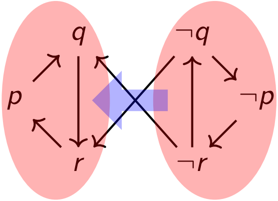
Reduced implication graph

Definition 8.4

For an implication graph (V, E), the reduced implication DAG (V^R, E^R) is defined as follows.

- $V^R = \{S | S \text{ is a scc in } (V,E)\}$
- $E^R = \{(S, S') | \text{there are } \ell \in S \text{ and } \ell' \in S' \text{ s.t. } (\ell, \ell') \in E\}$

Example 8.4



Theorem 8.6

If $(S, S') \in E^R$ then $(S'^c, S^c) \in E^R$.

Exercise 8.6 *Prove the above theorem.*

Commentary: (V^R, E^R) is a graph over scc's of (V, E). Please notice that (V^R, E^R) will always be a directed acyclic graph (DAG).

2-SAT satisfiablity

Theorem 8.7

A 2-SAT formula F is unsat iff there is a scc S in its implication graph (V, E) such that $\{p, \neg p\} \subseteq S$ for some p.

Proof.

Reverse direction

We assume $\{p, \neg p\} \subseteq S$.

$$\begin{tikzcd}
 & \ell_1 \arrow[r, dashed] & \ell_n \arrow[dr] & \\
p \arrow[ur] & & & \neg p \arrow[dl] \\
 & \ell'_k \arrow[ul] & \ell'_1 \arrow[l, dashed] &
\end{tikzcd}$$

There is a path that goes from p to $\neg p$.
Therefore, if p is true then $\neg p$ is true.
Therefore, p must be false.

Due to the path from $\neg p$ to p, if p is false then $\neg p$ is false.
Therefore, p must is true.

Therefore, F is unsat. ...

2-SAT satisfiablity(contd.)

Proof(contd.)

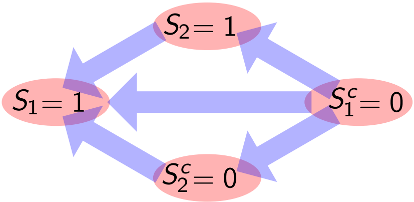
Fwd direction: Let us assume there is no such S. then we prove F is sat

We will construct a model of F as follows.

1. Initially all literals are unassigned.
2. While(some scc in V^R is unassigned)
   2.1 Let $S \in V^R$ be an unassigned scc whose all children are assigned 1.
   2.2 Assign literals of S to 1. Consequently, S^c is assigned 0.

...

Example 8.5



2-SAT satisfiablity(contd.)

Proof(contd.)

We need to show that step 2.1 always finds S with all children assigned 1.

claim: at step 2.1, there is an unassigned node whose all children are assigned

Choose an unassigned node.

Descend down if there is an unassigned child.

Since the DAG is finite, the process will terminate.

claim: an unassigned node can not have a child that is assigned 0.

If S is assigned 1, all its children are already 1.

Therefore, all the parents of S^c are already assigned 0(due to theorem 8.6).

Therefore, no node with 0 model has an unassigned parent. □

Exercise 8.7

Show that the procedure produces a satisfying model.

2-SAT is polynomial

Theorem 8.8
A 2-SAT satisfiability problem can be solved in linear time.

Proof.
Due to the previous theorem. □

Exercise: 2-SAT solving

Exercise 8.8

Find a satisfying model of the following formula

1. $(\neg x \vee \neg y) \wedge (\neg y \vee \neg z) \wedge (\neg z \vee \neg x) \wedge (x \vee \neg w) \wedge (y \vee \neg w) \wedge (z \vee \neg w)$
2. $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge$
   $(p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (p_3 \vee p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6)$

Topic 8.3

XOR SAT

XOR-SAT

Definition 8.5

A formula is XOR-SAT if it is a conjunction of xors of literals.

Example 8.6

$(p \oplus r \oplus s) \wedge (q \oplus \neg r \oplus s) \wedge (p \oplus q \oplus \neg s) \wedge (p \oplus \neg q \oplus \neg r)$ *is a XOR-SAT formula.*

Solving XOR-SAT

Since xors are negation of equality, we may eliminate variables via substitution.

Theorem 8.9

For a variable, p, formula G, and formula F, $(p \oplus G) \wedge F$ *and* $F[\neg G/p]$ *are equivsatisfiable.*

Note : p is not in G

Exercise 8.9

Prove the above theorem.

Commentary: The substitution process is similar to the Gaussian elimination used for solving linear equations.

Example : solving XOR-SAT

Example 8.7

Consider $(p \oplus r \oplus s) \wedge (q \oplus \neg r \oplus s) \wedge (p \oplus q \oplus \neg s) \wedge (p \oplus \neg q \oplus \neg r)$

Eliminate p:

Due to the first xor: $p \Leftrightarrow \neg r \oplus s$

Substitution: $(q \oplus \neg r \oplus s) \wedge (\neg r \oplus s \oplus q \oplus \neg s) \wedge (\neg r \oplus s \oplus \neg q \oplus \neg r)$

Simplification: $(q \oplus \neg r \oplus s) \wedge (\neg r \oplus \neg q) \wedge (s \oplus \neg q)$

Eliminate r:

Due to the second xor: $r \Leftrightarrow \neg q$

Substitution: $(q \oplus \neg\neg q \oplus s) \wedge (s \oplus \neg q)$

Simplification: $s \wedge (s \oplus \neg q)$

Example: solving XOR-SAT (contd.)

Eliminate q:
Due to the second xor: $q \Leftrightarrow s$
After substitution: s

Solution: $m(s) = 1 \qquad m(q) = m(s) = 1 \qquad m(r) = m(\neg q) = 0 \qquad m(p) = m(\neg r \oplus s) = 0$

Exercise 8.10

Find a satisfying model of the following formula

- $(p \oplus r \oplus s) \land (q \oplus r \oplus s) \land (\neg p \oplus q \oplus \neg s) \land (p \oplus \neg q \oplus \neg r)$

Commentary: We construct the model by following the substitution expressions backwards, when we assign a value to q, r, and p.

Topic 8.4

Horn Clauses

Horn clauses

Definition 8.6

A Horn clause is a clause that has the following form

$$\neg p_1 \vee \cdots \vee \neg p_n \vee q,$$

where $p_1, \ldots, p_n \in \mathrm{Vars}$*, and* $q \in \mathrm{Vars} \cup \{\perp\}$*.*
A Horn formula is a set of Horn clauses, which is interpreted as conjunction of the Horn clauses.
The clauses with $\perp$ *literals are called goal clauses and others are called implication clauses.*

Example 8.8

The following set is a Horn formula

$$\{p, \quad \neg q \vee \neg r \vee \neg t \vee p, \quad \neg p \vee q, \quad \neg p \vee \neg r \vee t, \quad \neg p \vee \neg q \vee t, \quad \neg r \vee \perp, \quad \neg p \vee \neg q \vee \neg t \vee \perp\}$$

Implication view of the Horn clauses

We may view a Horn clause

$$\neg p_1 \vee \cdots \vee \neg p_n \vee q$$

as

$$p_1 \wedge \cdots \wedge p_n \Rightarrow q.$$

Example 8.9

The following is an implication view of the Horn formula from previous slide.

$$\{\top \Rightarrow p, \quad q \wedge r \wedge t \Rightarrow p, \quad p \Rightarrow q, \quad p \wedge r \Rightarrow t, \quad p \wedge q \Rightarrow t, \quad r \Rightarrow \bot, \quad p \wedge q \wedge t \Rightarrow \bot\}$$

Note $\top \Rightarrow p$ means p, which is a Horn clause without negative literals

Horn satisfiability

Algorithm 8.1: HORNSAT(Hs,Gs)

Input: Hs: implication clauses, Gs : goal clauses

Output: model/unsat

$m := \lambda x.0$;

while $m \not\models (p_1 \wedge .. \wedge p_n \Rightarrow p) \in Hs$ **do**

  $m := m[p \mapsto 1]$;

  if $m \not\models (q_1 \wedge .. \wedge q_k \Rightarrow \bot) \in Gs$ **then** **return** *unsat* ;

return m

Exercise 8.11

Solve $\{\top \Rightarrow p, \quad q \wedge r \wedge t \Rightarrow p, \quad p \Rightarrow q, \quad p \wedge r \Rightarrow t, \quad p \wedge q \Rightarrow t, \quad r \Rightarrow \bot, \quad p \wedge q \wedge t \Rightarrow \bot\}$

Exercise 8.12

What is the maximum number of times the truth value of a clause in Hs changes during the algorithm? Give a supporting argument for the answer and an example that exhibits the situation.

Topic 8.5

Problems

Unsat XOR-sat

Exercise 8.13
Give an unsat XOR-sat formula that has only xors with more than three arguments.

Unsat 2-CNF**

Exercise 8.14

Let us suppose we have n variables in a 2-CNF problem. What is the maximum number of clauses in the formula such that the formula is satisfiable?

Unsatisfiable core of 2-CNF

Exercise 8.15

An unsatisfiable core of an unsatisfiable CNF formula is a (preferably minimal) subset of the formula that is also unsatisfiable. Give an algorithm to compute a minimal unsatisfiable core of 2-CNF formula.

Horn SAT true to false

0

Exercise 8.16

In the Horn solving algorithm, we started with all false model and incrementally turned the variables true.
a. Is it possible to modify the algorithm such that it starts with all true model and finds satisfying model for the Horn clauses.
b. Can we also start with any initial model?

Topic 8.6

Extra lecture slides : recognizable Horn clauses

Recognizing Horn clauses

Sometimes a set of clauses are not immediately recognizable as Horn clause.

We may convert a CNF into a Horn formula by flipping the negation signs for some variables. Such CNF are called Horn clause renamable.

Definition 8.7

Let F be a CNF formula and m be a model. Let $\mathit{flip}(F, m)$ denote the formula obtained by flipping the variables that are assigned 1 in m.

Example 8.10

$\mathit{flip}((p \lor \neg q \lor \neg s), \{p \mapsto 1, q \mapsto 0, s \mapsto 1, ..\}) = (\neg p \lor \neg q \lor s)$

Exercise 8.17

Calculate $\mathit{flip}((\neg p \lor q \lor \neg s), \{p \mapsto 1, q \mapsto 1, s \mapsto 0, ..\})$

Renaming Horn clauses

Theorem 8.10

A CNF formula $F = \{C_1, \dots, C_n\}$, *where* $C_i = \{\ell_{i1}, \dots, \ell_{i|C_i|}\}$ *is Horn clause renamable iff the following 2-SAT formula is satisfiable.*

$$G = \{\ell_{ij} \vee \ell_{ik} | i \in 1..n \text{ and } 1 \leq j < k \leq |C_i|\}$$

Proof.

Forward direction: there is a model m such that $flip(F, m)$ is a Horn formula

claim: $m \models G$

consider a clause $\ell_{ij} \vee \ell_{ik} \in G$

case $\ell_{ij} = p, \ell_{ik} = q$: one of them must flip,i.e.,$m(p) = 1$ or $m(q) = 1$

case $\ell_{ij} = \neg p, \ell_{ik} = \neg q$: at least one must not flip, i.e., not $m(p) = m(q) = 1$

case $\ell_{ij} = \neg p, \ell_{ik} = q$: if p flips then q must,i.e., if $m(p) = 1$ then $m(q) = 1$

In all the three cases $m \models \ell_{ij} \vee \ell_{ik}$.

Commentary: Proof source : H. Lewis. Renaming a Set of Clauses as a Horn Set. J. of the ACM, 25:134-135, 1978.

Renaming Horn clauses(contd.)

Proof(contd.)

Reverse direction: Let $m \models G$. Let $F' = \mathit{flip}(F, m)$.

claim: F' is a Horn formula

Suppose F' is not a Horn formula.

Then, there are positive literals ℓ'_{ij} and ℓ'_{ik} in clause C_i in F'.

Therefore, $m \not\models \ell_{ij} \vee \ell_{ik}$(why?). Contradiction. □

Exercise 8.18

What is the complexity of checking if a formula is Horn clause renameable?

Exercise 8.19

Can you improve the above complexity?

End of Lecture 8