CS 228 : Logic in Computer Science

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Summary

- Started looking at FO nondefinability
- Defined maximal quantifier depth or quantifier rank of a formula
- Showed that there are finitely many FO formulae of rank r
- Introduced some new notations for words, mimicking assignments of values to free variables

Notational Semantics Recap

- $(a_1, \emptyset) \ldots (a_n, \emptyset) \models \exists x \varphi$ iff
- There is some position i such that $(a_1, \emptyset) \ldots (a_i, \{x\}) \ldots (a_n, \emptyset) \models \varphi$
- For a formula $\varphi(x_1, \ldots, x_m)$, $L(\varphi)$ is the set of all $\{x_1, \ldots, x_m\}$ structures satisfying φ
- For a sentence φ, $L(\varphi)$ is the set of all $\emptyset$ structures satisfying φ
- Example : $L(Q_a(x))$ consists of all x-structures $(\Sigma, \emptyset)^*(a, \{x\})(\Sigma, \emptyset)^*$.

Logical Equivalence

- Let w_1, w_2 be two $\mathcal{V}$-structures and let $r \geqslant 0$.
- Write $w_1 \sim_r w_2$ iff w_1, w_2 satisfy the same set of FO formulae of rank $\leqslant r$.
- $(a, \emptyset)(b, \emptyset) \sim_0 (a, \emptyset)(b, \emptyset)(a, \emptyset)$
- $(a, \emptyset)(b, \emptyset) \nsim_2 (a, \emptyset)(b, \emptyset)(a, \emptyset)$
- $\sim_r$ is an equivalence relation
- Finitely many equivalence classes : each class consists of words that behave the same way on formulae of rank r

Non-Expressibility in FO : The Game Begins

Come, Lets Play

- Given two $\mathcal{V}$-structures w_1, w_2, lets play a game on the pair of words w_1, w_2
- There are 2 players : Spoiler and Duplicator
- Play for r-rounds, $r \geqslant 0$
- Spoiler wants to show that w_1, w_2 are different ($w_1 \nsim_r w_2$)
- Duplicator wants to show that they are same ($w_1 \sim_r w_2$)
- Each player has r pebbles $z_1, \ldots, z_r$

Moves of the Game

- At the start of each round, spoiler chooses a structure.
- Duplicator gets the other structure
- Spoiler places his pebble say z_i on one of the positions of his chosen word
- Duplicator must keep the pebble z_i on one of the positions of her word
- A pebble once placed, cannot be removed
- The game ends after r rounds, when both players have used all their pebbles

A Play

- $w_1 = (a, \emptyset)(b, \emptyset)$ and $w_2 = (a, \emptyset)(b, \emptyset)(a, \emptyset)$
- 2 rounds, so 2 pebbles : z_1, z_2
- Spoiler picks w_2, duplicator picks w_1
- Round 1:
  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \emptyset)$
  - Duplicator : $(a, \{z_1\})(b, \emptyset)$
  - After round 1, we have two $\{z_1\}$ structures (w_1', w_2')
- Round 2:
  - Spoiler continues on the structure w_2'
  - Duplicator gets w_1' to play
  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$
  - Duplicator : $(a, \{z_1, z_2\})(b, \emptyset)$ or $(a, \{z_1\})(b, \{z_2\})$

Winner

- Start with two $\emptyset$ structures (w_1, w_2)
- r-round game, pebble set $\mathcal{V} = \{z_1, \ldots, z_r\}$
- Each round changes the structures
- At the end of r-rounds, we have two $\mathcal{V}$-structures (w_1', w_2')
- Duplicator wins iff for every atomic formula α,
  $w_1' \models \alpha$ iff $w_2' \models \alpha$
- That is, $w_1' \sim_0 w_2'$
- Spoiler wins otherwise.

Winner

Given two word structures (w_1, w_2), duplicator wins on (w_1, w_2) if for every atomic formula α, $w_1 \models \alpha$ iff $w_2 \models \alpha$

Play continues

- Who won in the earlier play?
- We had
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1, z_2\})(b, \emptyset)$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models (z_1 < z_2)$
  - $(a, \{z_1, z_2\})(b, \emptyset) \not\models (z_1 < z_2)$ or
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1\})(b, \{z_2\})$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models Q_a(z_2)$
  - $(a, \{z_1\})(b, \{z_2\}) \not\models Q_a(z_2)$
- Spoiler wins in two rounds
- If the game was played only for one round, who will win?

Unique Winner

Given structures w_1, w_2, and a number of rounds r, exactly one of the players win.

Logical Equivalence and Winning

Let w_1, w_2 be $\mathcal{V}$-structures and let $r \geqslant 0$. Then $w_1 \sim_r w_2$ iff Duplicator has a winning strategy in the r-round game on (w_1, w_2).

Logical Equivalence and Winning

Assume $w_1 \sim_r w_2$, and induct on r

- Base : $r = 0$ and $w_1 \sim_0 w_2$. Duplicator wins, since by assumption, w_1, w_2 agree on all atomic formulae.
- Assume for $r - 1$: $w_1 \sim_{r-1} w_2 \Rightarrow$ Duplicator has a winning strategy in a $r - 1$ round game

Logical Equivalence and Winning

- Now, let $w_1 \sim_r w_2$, and assume spoiler wins the r-round game on (w_1, w_2).
  - Assume spoiler starts on w_1, places a pebble z_1 somewhere on w_1
  - The resultant structure is w_1'
  - In response, duplicator places her pebble somewhere on w_2
  - The resultant structure is w_2'
  - By assumption, spoiler wins the $r-1$ round game on (w_1', w_2')
  - By inductive hypothesis, $w_1' \nsim_{r-1} w_2'$
  - Let ψ be the conjunction of all formulae of rank $r-1$ in normal form that are satisfied by w_1'
  - Then $w_1' \models \psi$, $w_2' \not\models \psi$
  - We thus have

$$w_1 \models \exists z_1 \psi, w_2 \not\models \exists z_1 \psi$$

contradicting $w_1 \sim_r w_2$

Logical Equivalence and Winning : Converse

Assume Duplicator wins r-round game on (w_1, w_2) and induct on r

- Base : $r = 0$ and Duplicator wins. Then w_1, w_2 agree on all atomic formulae, and hence $w_1 \sim_0 w_2$
- Assume for $r - 1$: Duplicator has a winning strategy in a $r - 1$ round game $\Rightarrow w_1 \sim_{r-1} w_2$

Logical Equivalence and Winning : Converse

- Now, let duplicator win in the r round game, but $w_1 \nsim_r w_2$.
  - $w_1 \nsim_r w_2 \Rightarrow$ there is some formula ψ, $c(\psi) = r$ such that $w_1 \models \psi, w_2 \not\models \psi$
  - Assume $\psi = \exists z_1 \varphi$. Then $c(\varphi) = r - 1$
  - Since $w_1 \models \exists z_1 \varphi$, spoiler can keep pebble z_1 somewhere in w_1 obtaining w_1' satisfying φ
  - In reply, duplicator keeps pebble z_1 on w_2 obtaining w_2'
  - By assumption, $w_2' \not\models \varphi$
  - Also, by assumption, duplicator wins the $r - 1$ round game on (w_1', w_2') : this by inductive hypothesis says that $w_1' \sim_{r-1} w_2'$
  - That is, either both w_1', w_2' satisfy φ, or both dont, a contradiction.

FO-definable languages

Assume L is FO-definable, and $L = L(\varphi)$ with rank of φ being k.

- Let $L = \{v_1, v_2, v_3, \dots\}$ and $\overline{L} = \{w_1, w_2, w_3, \dots\}$
- Play a k round game on $v_i \in L$ and $w_j \notin L$. Let ψ_{v_i, w_j} be the formula of rank k that distinguishes the two words.
- Consider the formula

$$[\psi_{v_1,w_1} \wedge \psi_{v_1,w_2} \wedge \dots \wedge \psi_{v_1,w_n} \wedge \dots]$$

$$\vee$$

$$[\psi_{v_2,w_1} \wedge \psi_{v_2,w_2} \wedge \dots \wedge \psi_{v_2,w_n} \wedge \dots]$$

$$\vee$$

$$\vdots$$

FO-definable languages

$$\psi_L = \bigvee_{v \in L} \bigwedge_{w \notin L} \psi_{vw}$$

- Each ψ_{vw} has rank atmost k
- Upto equivalence, there are finitely many formulae of rank k
- Hence the disjunction and conjunction are finite
- ψ_L is a proper formula (of finite size)
- ψ_L captures L since each $v \in L$ satisfies $\bigwedge_{w \notin L} \psi_{vw}$ while none of the $w \notin L$ satisfy $\bigwedge_{w \notin L} \psi_{vw}$

FO-definable languages

Given a property $\mathcal{K}$, if for any pair $v \in \mathcal{K}$ and $w \notin \mathcal{K}$, spoiler has a winning strategy in the k-round EF game on v and w, then there is a rank k FO formula $\varphi_{\mathcal{K}}$ that defines the property $\mathcal{K}$.

$$\varphi_{\mathcal{K}} = \bigvee_{v \in \mathcal{K}} \bigwedge_{w \notin \mathcal{K}} \psi_{vw}$$

where ψ_{vw} is as explained in the previous slide.

- Note that k is fixed in the above, and is independent of the choices of the words.

Implications of the Game on FO definability

FO Definability

L is FO definable $\Rightarrow$ there exists an r such that for every (w_1, w_2) pair, such that $w_1 \in L$, $w_2 \notin L$, spoiler wins in r rounds

Non FO Definability

For all $r \geqslant 0$, there exists a (w_1, w_2) pair with $w_1 \notin L$, $w_2 \in L$, duplicator wins in r rounds $\Rightarrow$ L is not FO definable