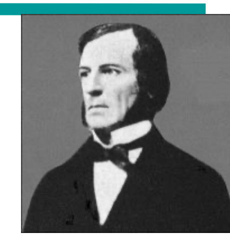
CS-226: Digital Logic Design



Boolean Algebra

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CADSL

Specification: Logic Function

Truth Table

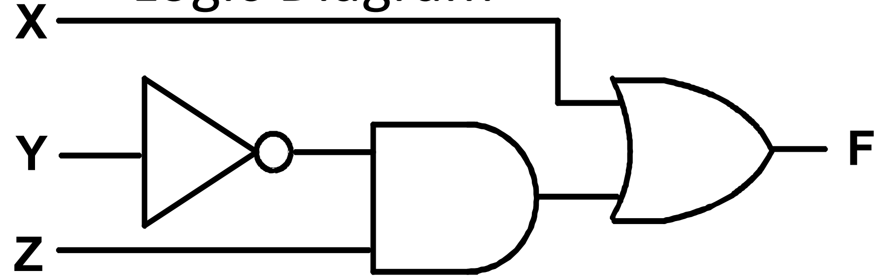
X Y Z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Logic Expression

$$F = \overline{X}.\overline{Y}.Z + X.\overline{Y}.\overline{Z} + X.\overline{Y}.Z + X.\overline{Y}.Z + X.Y.Z$$

$$F = X + \overline{Y}.Z$$

Logic Diagram



ALGEBRA

Algebra

- Algebra is defined as
  1. Set of elements
  2. Set of operators
  3. Number of postulates
- A set of elements is any collection of objects having common properties

  $S = \{a,b,c,d\};\ a \in S, e \notin S$
- A binary operator $*$ defined on a set S of elements is a rule that assigns each pair from S to a unique pair from S. $a * b = c$

An Axiom or Postulate

- A self-evident or universally recognized truth.
- An established rule, principle, or law.
- A self-evident principle or one that is accepted as true without proof as the basis for argument.
- A postulate – Understood as the truth.

Postulates

Postulate 1:

Commutative law: An operator * on S is commutative if

$$a * b = b * a, \qquad \forall a, b \in S$$

Postulate 2:

Associative law: An operator * is associative if

$$a * (b * c) = (a * b) * c, \qquad \forall a, b, c \in S$$

Postulates

Postulate 3

- Identity Element: With respect to an operator * on S if there exists an element e such that

$$e * a = a * e = a, \quad \forall a \in S$$

Postulate 4

- Inverse: For every $a \in S$, if there exists a $b \in S$ such that $a * b = e$

Postulates

Postulate 5

- Distributive law: With respect to two operators * and + if

$$a * (b + c) = (a * b) + (a * c), \qquad \forall a, b, c \in S$$

then * is said to be distributed over +

Example

A set contains four elements:

$x = \{\phi\}$, null set $y = \{1, 2\}$

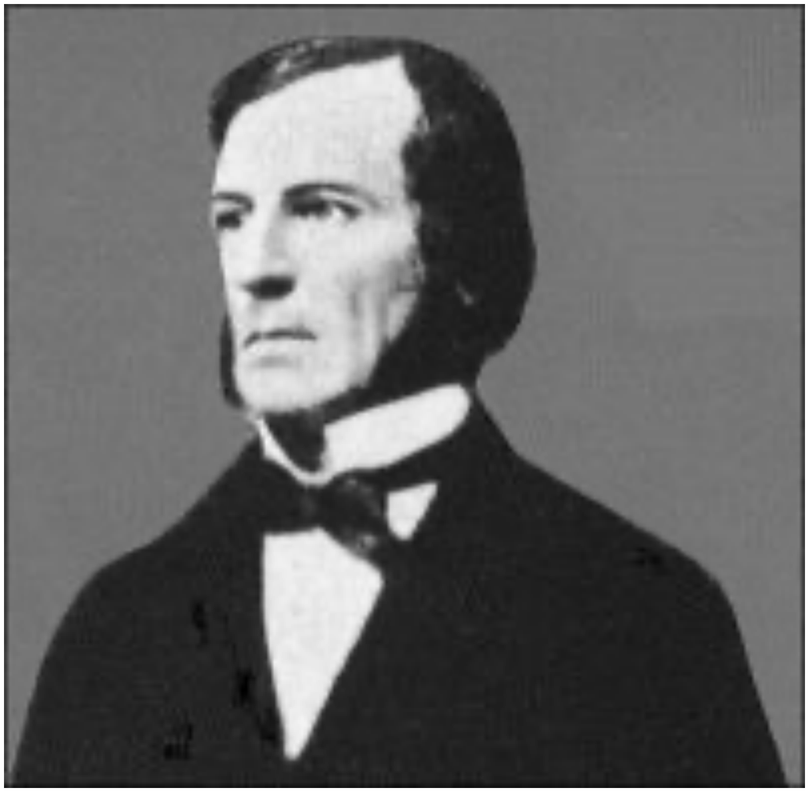
$z = \{3, 4, 5\}$ $w = \{1, 2, 3, 4, 5\}$

Define two operations: union (+) and intersection (·):

+	x	y	z	w
x	x	y	z	w
y	y	y	w	w
z	z	w	z	w
w	w	w	w	w

·	x	y	z	w
x	x	x	x	**x**
y	x	y	x	**y**
z	x	x	z	**z**
w	x	y	z	**w**

George Boole (1815-1864)



- Born, Lincoln, England
- Professor of Math., Queen's College, Cork, Ireland
- Book, *The Laws of Thought*, 1853

Boolean Algebra

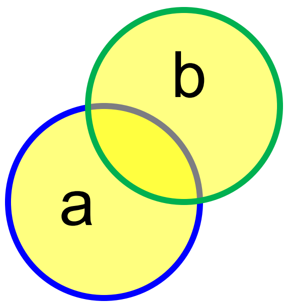
- Boolean Algebra is defined as
  1. Set of elements {0, 1}
  2. Set of operators {+, ., ~}
  3. Number of postulates
- Boolean Algebra: 5-tuple

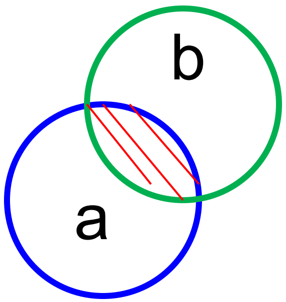
$$\{B, +, ., \sim, 0, 1\}$$

- Closure: If a and b are Boolean then $(a.b)$ and $(a + b)$ are also Boolean

Postulate 1: Commutativity

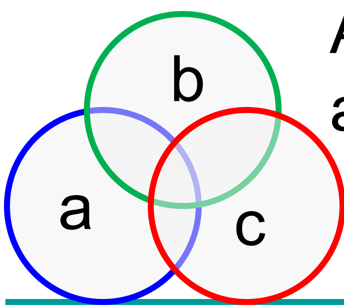
- Binary operators + and · are commutative.
- That is, for any elements *a* and *b* in *B*:
  - $a + b = b + a$
  - $a \cdot b = b \cdot a$





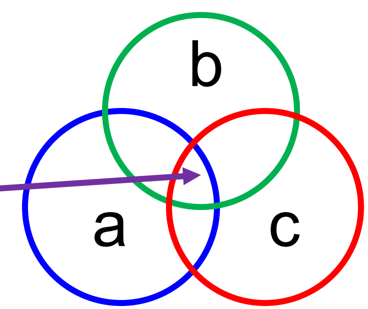
Postulate 2: Associativity

- Binary operators + and · are associative.
- That is, for any elements a, b and c in B:
  - $a + (b + c) = (a + b) + c$
  - $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Example: EE department has three courses with student groups a, b and c



All EE students:
$a + (b + c)$

EE students in all EE courses: $a \cdot (b \cdot c)$



Postulate 3: Identity Elements

- There exist 0 and 1 elements in B, such that for every element a in B
  - $a + 0 = a$
  - $a \cdot 1 = a$
- Definitions:
  - 0 is the identity element for + operation
  - 1 is the identity element for · operation
- Remember, 0 and 1 here should not be misinterpreted as 0 and 1 of ordinary algebra.

Postulate 5: Distributivity

- Binary operator + is distributive over · and · is distributive over +.
- That is, for any elements a, b and c in K:
  - $a + (b \cdot c) = (a + b) \cdot (a + c)$
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- Remember dot (·) operation is performed before + operation:

$$a + b \cdot c = a + (b \cdot c) \neq (a + b) \cdot c$$

Postulate 6: Complement

- A unary operation, *complementation*, exists for every element of B.
- That is, for any element a in B:

$$a + \overline{a} = 1$$

$$a \cdot \overline{a} = 0$$

- Where, 1 is identity element for $\cdot$

  0 is identity element for +

The Duality Principle

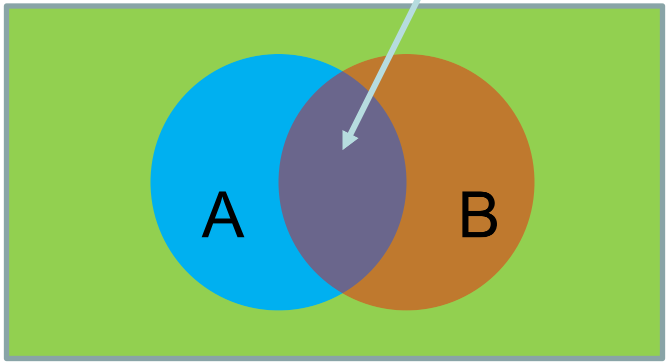
- Each postulate of Boolean algebra contains a pair of expressions or equations such that one is transformed into the other and vice-versa by interchanging the operators, $+ \leftrightarrow \cdot$, and identity elements, $0 \leftrightarrow 1$.
- The two expressions are called the duals of each other.

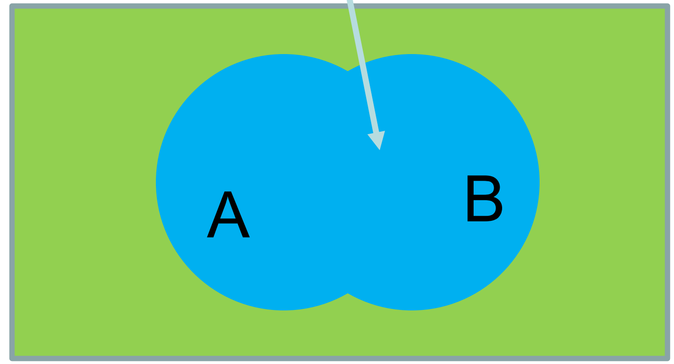
Examples of Duals

Postulate	Duals	
	Expression 1	Expression 2
0	$a, b, a + b \in B$	$a, b, a \cdot b \in B$
3	$a + 0 = a$	$a \cdot 1 = a$
1	$a + b = b + a$	$a \cdot b = b \cdot a$
2	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
4	$a + (b \cdot c) = (a + b) \cdot (a + c)$	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
6	$a + \bar{a} = 1$	$a . \bar{a} = 0$

Examples of Duals

Expressions: $A \cdot B$ $A + B$





Equations:

$$A + (BC) = (A+B)(A+C) \xleftrightarrow{\text{duals}} A\,(B+C) = AB + AC$$

Note: $A \cdot B$ is also written as AB.

Properties of Boolean Algebra

- Properties stated as theorems.

- Provable from the postulates (axioms) of Boolean algebra.

Theorem 1: Idempotency (Invariance)

- For all elements a in B: $a + a = a$; $a.a = a$
- Proof:

$$
\begin{aligned}
a + a &= (a + a).1 && \text{(identity element)} \\
&= (a + a).(a + \bar{a}) && \text{(complement)} \\
&= a + a.\bar{a} && \text{(distributivity)} \\
&= a + 0 && \text{(complement)} \\
&= a && \text{(identity element)}
\end{aligned}
$$

Theorem 1: Idempotency

- For all elements a in B: $a + a = a;\ a\ a = a.$
- Proof:

$$
\begin{aligned}
a.a &= (a.a) + 0 && \text{(identity element)} \\
&= (a.a) + (a.\bar{a}) && \text{(complement)} \\
&= a.(a + \bar{a}) && \text{(distributivity)} \\
&= a.1 && \text{(complement)} \\
&= a && \text{(identity element)}
\end{aligned}
$$

Theorem 2: Null Elements Exist

- $a + 1 = 1$, for + operator.
- $a \cdot 0 = 0$, for $\cdot$ operator.
- Proof: $a + 1 =$ $(a + 1).1$ (identity element)

$= 1.(a + 1)$ (commutativity)

$= (a + \bar{a}).(a + 1)$ (complement)

$= a + \bar{a}\ .1$ (distributivity)

$= a + \bar{a}$ (identity element)

$= 1$ (complement)

Similar proof for $a.0 = 0$.

Theorem 2: Null Elements Exist

- $a + 1 = 1$, for $+$ operator.
- $a \cdot 0 = 0$, for $\cdot$ operator.
- Proof: $a.0$

$=$	$(a.0) + 0$	(identity element)
$=$	$0 + (a.0)$	(commutativity)
$=$	$(a.\bar{a}) + (a.0)$	(complement)
$=$	$a.(\bar{a} + 0)$	(distributivity)
$=$	$a.\bar{a}$	(identity element)
$=$	0	(complement)

Theorem 3: Involution Holds

- $\bar{\bar{a}} = a$
- Proof: $a + \bar{a} = 1$ and $a.\bar{a} = 0$, (complements)

  or $\bar{a} + a = 1$ and $\bar{a}.a = 0$, (commutativity)

  i.e., a is complement of $\bar{a}$

  Therefore, $\bar{\bar{a}} = a$

Theorem 4: Absorption

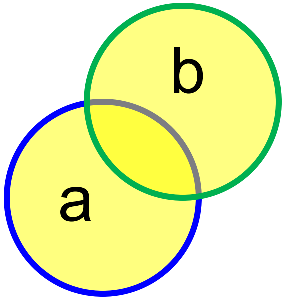
- $a + a.b = a$
- $a.(a + b) = a$
- Proof: $a + a\ b \;=\; a.1 + a.b$ (identity element)

$= a.(1 + b)$ (distributivity)

$= a.1$ (Theorem 2)

$= a$ (identity element)

Similar proof for $a\ (a + b) = a$.

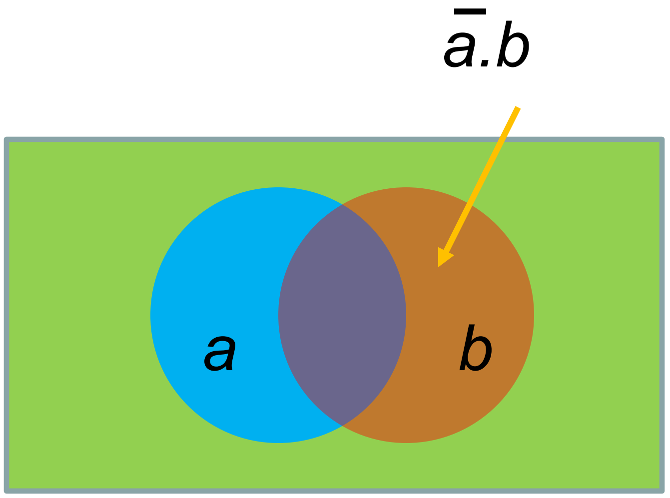


Theorems: Adsorption & Uniting

- Theorem 5: Adsorption

$$a + \bar{a}b = a + b$$

$$a(\bar{a} + b) = ab$$



- Theorem 6: Uniting

$$ab + a\bar{b} = a$$

$$(a + b)(a + \bar{b}) = a$$

Theorem 7: DeMorgan's Theorem

- $\overline{a + b} = \bar{a} \,.\, \bar{b}, \quad \forall a, b \in B$
- $\overline{a \,.\, b} = \bar{a} + \bar{b}, \quad \forall a, b \in B$

1806 - 1871

Generalization of DeMorgan's Theorem:

$$\overline{a + b + \cdots + z} = \bar{a} \cdot \bar{b} \cdots \bar{z}$$

$$\overline{a \cdot b \cdots z} = \bar{a} + \bar{b} + \cdots + \bar{z}$$

DeMorgan's Theorem #1

$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

A	B	$A + B$	$\overline{A + B}$		$\overline{A}$	$\overline{B}$	$\overline{A} . \overline{B}$
0	0	0	1		1	1	1
0	1	1	0		1	0	0
1	0	1	0		0	1	0
1	1	1	0		0	0	0

EQUAL

DeMorgan's Theorem #2

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

A	B	$A \bullet B$	$\overline{A \bullet B}$		$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$
0	0	0	1		1	1	1
0	1	0	1		1	0	1
1	0	0	1		0	1	1
1	1	1	0		0	0	0

EQUAL

Martians and Venusians

- Suppose Martians are blue and Venusians are pink.
- An Earthling identifying itself: "I am not blue or pink."

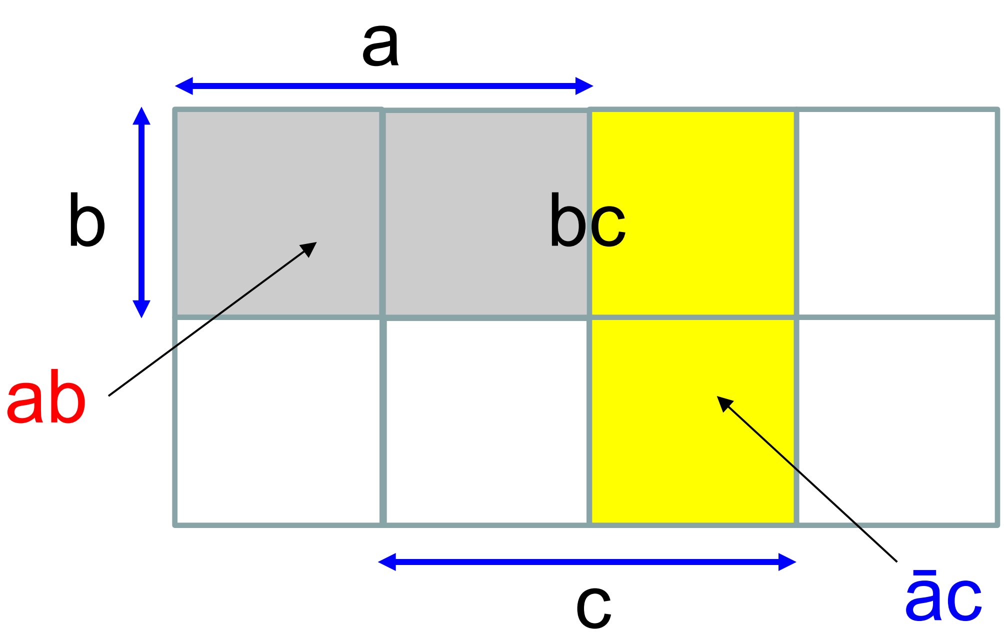
$$\overline{\text{blue} + \text{pink}} = \overline{\text{blue}} \cdot \overline{\text{pink}}$$

- Meaning: "I am not blue and I am not pink."
- Or: "I am not a Martian and I am not a Venusian."

Theorem 8: Consensus

$$ab + \bar{a}c + bc = ab + \bar{a}c$$

$$(a+b)(\bar{a}+c)(b+c) = (a+b)(\bar{a}+c)$$



Theorem 8: Consensus

- $a.b + \bar{a}.c + b.c = a.b + \bar{a}.c$
- *Dual:* $(a+b).(\bar{a}+c).(b+c) = (a+b).(\bar{a}+c)$
- *Proof*

$a.b+\bar{a}.c+b.c = a.b + \bar{a}.c + b.c.\ (a+\bar{a})$ (Complementarity)

$= a.b + \bar{a}.c + a.b.c + \bar{a}.b.c$ *(Commutative)*

$= a.b + a.b.c + \bar{a}.c + \bar{a}.b.c$ *(Absorption)*

$= a.b + \bar{a}.c$

Thank You