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L. H. SMITH, JR.
Manager,
Compressor Aerodynamic Development Unit,
Advanced Engine and Technology Department,
General Electric Company,
Cincinnati, Ohio.
Mem. ASME

The Radial-Equilibrium Equation of Turbomachinery

An equation is derived that describes the radial variation of circumferential-average flow properties inside of a turbomachine blade row. A physical interpretation is given for each of the several terms in the equation, and the magnitudes of the terms are demonstrated with an example. Annulus taper angle is found to be rather important. The error involved in the conventional axisymmetric (infinite-blade-number) approach is shown to be more related to the loading level, e.g., the lift-coefficient level, than the number of blades. This error is found to be small for the selected example.

Introduction

In modern, high-speed, axial-flow compressors, gas turbines, and steam turbines, the annulus walls are frequently designed with significant taper angles. Also, meridional streamline curvatures of substantial magnitude may be introduced by relatively abrupt changes in annulus taper and also by blade thickness blockage gradients, circumferential vorticity components, and nonradial blade elements. It is the function of a radial-equilibrium equation to describe the radial variation of flow properties, accounting for these phenomena in a satisfactory fashion.

The name radial-equilibrium equation has been given different meanings by different writers. To some, primarily British, it refers to a simplified form of the radial momentum equation in which all acceleration terms except C_u^2/r have been omitted. To others, primarily American, it refers to the complete radial momentum equation arranged in a form suitable for the determination of the flow field in a turbomachine. The term simplified-radial-equilibrium equation is then used to denote the simplified form. The American definition will be used in this paper.

The overall calculation scheme into which the present work is intended to fit may be summarized as follows: The flow is considered at a number of axial stations (z = const planes), and the radial-equilibrium equation, energy equation, and continuity condition are employed at each of them to determine the distribution of flow properties from hub to casing. It is necessary, however, that these distributions obtained separately at each station be consistent from station to station and that the radial acceleration which a fluid particle undergoes as it passes from station to station be accounted for in the radial-equilibrium equation. This is done by assuming shapes for the meridional streamlines consistent with the continuity condition at each station and by expressing the radial acceleration in terms of the streamline slope and curvature. An iterative method of solution is obviously implied.

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This general scheme was first proposed by Traupel [1][1] and independently conceived by Wu [2] and probably also by others. Its use has been widespread. One purpose of the present paper is to give an exact[2] form for the axisymmetric radial-equilibrium equation applicable to stations located exterior to blade rows and involving derivatives with respect to r only, although r_m, which in a sense is also a derivative, is also needed. Such an exact equation was first derived by H. N. Cantrell of the Large Steam Turbine-Generator Department of the General Electric Company in about 1952, and independently by the author in the Aircraft Gas Turbine Division of General Electric in 1954. To the knowledge of the author, the exact form has never before been published. A second purpose is the extension of the equation by the addition of terms that allow its application inside of a blade row. The equation has been used in this form at General Electric since 1959.

The three-dimensional nature of the flow is recognized in the derivations of this paper. The radial-equilibrium equation is first put into a form that contains partial derivatives with respect to r and θ only, although r_m is also used. Next, the circumferential average is taken by integrating over θ from the pressure side of one blade to the suction side of the adjacent blade. The resulting equation has the same form as can be obtained from an axisymmetric analysis that uses body forces to represent the blade action, except for some additional terms that disappear as the blade loading is reduced. This is essentially the same approach as was used by Ruden [3] to eliminate the θ-variable. The axisymmetric method, which has been used extensively by designers, is there-

[1] Numbers in brackets designate References at end of paper.
[2] In this paper, the word exact means within the assumptions listed in the next section.

Nomenclature

a = speed of sound, fps
A = streamtube cross-sectional area, sq ft
c_v = specific heat at constant volume, sq ft/sec² deg R
c_p = specific heat at constant pressure, sq ft/sec² deg R
$\mathbf{C}$ = absolute velocity, $\mathbf{C} = \mathbf{U} + \mathbf{W}$, fps
G, G', G'' = functions of type described by equation (39) ft/sec²
h = static enthalpy, ft²/sec²
$\mathbf{i}$ = unit vector
I = stagnation rothalpy, ft²/sec²; see equation (53)
l = axial chord length, $l = z_{TE} - z_{LE}$, along a streamline, ft
m = distance in meridional direction, ft; equation (6)
$\mathbf{M}$ = relative Mach number, $\mathbf{M} = \mathbf{W}/a$
N = number of blades
p = static pressure, psf
P = relative stagnation pressure, psf
q, u, v, w = mathematical variables defined in Appendix 1
Q' = heat added per unit mass per unit time, ft²/sec³
r = radial coordinate, ft
r_m = radius of curvature of meridional pathline, ft; equation (11)
R = gas constant, ft²/sec² deg R
s = entropy, ft²/sec² deg R
t = time, sec
T = static temperature, deg R
u = distance in circumferential direction, ft
$\mathbf{U}$ = blade speed; $\mathbf{U} = \boldsymbol{\omega} \times \mathbf{r}$, fps
$\mathbf{W}$ = relative velocity, fps
x = distance in instantaneous flow direction, ft

(Continued on next page)

fore at least partly justified by the present paper. Without further knowledge of the flow, the additional terms cannot be evaluated; however, to allow an approximate evaluation, certain quantities are assumed to vary linearly with θ. An example demonstrates the magnitudes of the additional terms for a compressor with moderate blade loading.

Derivation of Three-Dimensional Equation

The following assumptions are made:

1 The fluid is frictionless.

2 The rotor is rigid and rotates with constant angular velocity $\boldsymbol{\omega}$.

3 There is heat added reversibly to the fluid.

4 The fluid is a semiperfect gas; i.e., the equation of state is

$$p = \rho R T \tag{1}$$

with R constant, and the specific heats are dependent upon temperature only. This assumption is not used until equation (24) and is not necessary if there is no heat added.

5 The flow is steady relative to the rotor. This assumption is not used until equation (30).

There may be vorticity, entropy gradients, and stagnation enthalpy gradients in the fluid.

For a frame of reference rotating with constant angular velocity $\boldsymbol{\omega}$ about the z-axis, Newton's second law of motion gives, for a frictionless fluid,

$$-\frac{\nabla p}{\rho} = \frac{D\mathbf{W}}{Dt} - \omega^2\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{W} \tag{2}$$

It is convenient to use the relative cylindrical coordinate system r, θ, and z in which θ is measured relative to the rotating frame and increases in the direction of rotation. Three-component equations can be obtained by taking the scalar product of (2) with the unit vectors in the coordinate directions. Use is also made of the following unit-vector derivatives:

$$\frac{D\mathbf{i}_r}{Dt} = \frac{W_u}{r}\mathbf{i}_u \quad \text{and} \quad \frac{D\mathbf{i}_u}{Dt} = -\frac{W_u}{r}\mathbf{i}_r$$

and of the kinematic relation

$$\frac{D(\;)}{Dt} = W\frac{D(\;)}{Dx} \tag{3}$$

The capital-D operator in these equations signifies that a fluid particle is being followed during the differentiation. There then results

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = W\frac{DW_r}{Dx} - \frac{(W_u + \omega r)^2}{r} \tag{2a}$$

$$-\frac{1}{\rho}\frac{\partial p}{r\partial\theta} = W\frac{DW_u}{Dx} + \frac{W_uW_r}{r} + 2\omega W_r \tag{2b}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = W\frac{DW_z}{Dx} \tag{2c}$$

By use of the substitution

$$C_u = W_u + \omega r \tag{4}$$

equations (2a) and (2b) can also be written

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = W\frac{DW_r}{Dx} - \frac{C_u^2}{r} \tag{2d}$$

and

$$-\frac{1}{\rho}\frac{\partial p}{r\partial\theta} = \frac{W}{r}\frac{D}{Dx}(rC_u) \tag{2e}$$

Now we can also write, for example,

$$W\frac{D(\;)}{Dx} = W_z\frac{D(\;)}{Dz} \tag{5}$$

where it is understood that Dz represents the increase in z-coordinate which the particle undergoes as it moves a distance Dx in the flow direction. An equation like equation (5) may be written for any desired direction. A direction of particular interest here is the meridional direction defined by

$$\mathbf{i}_mDm = \mathbf{i}_zDz + \mathbf{i}_rDr \tag{6}$$

Thus we may write

$$W\frac{D(\;)}{Dx} = W_m\frac{D(\;)}{Dm} \tag{7}$$

Using this relation in equation (2d), we obtain

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = W_m\frac{DW_r}{Dm} - \frac{C_u^2}{r} \tag{8}$$

Furthermore, since

$$W_r = W_m \sin\varphi \tag{9}$$

equation (8) can be written

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{C_u^2}{r} - W_m^2\frac{D\sin\varphi}{Dm} - W_m\sin\varphi\frac{DW_m}{Dm} \tag{10}$$

Now

$$\frac{D\sin\varphi}{Dm} = \cos\varphi\frac{D\varphi}{Dm}$$

Nomenclature

z = axial coordinate, ft

α = absolute swirl angle; $\tan\alpha = C_u/C_z$, rad

β = relative swirl angle; $\tan\beta = W_u/W_z$, rad

β' = angle between intersection of blade surface with r = const cylinder and axial direction, Fig. 2

ϵ = blade-surface lean angle, rad; Fig. 1

θ = circumferential coordinate, rad

κ = D^2r/Dz^2, 1/ft; equation (37)

λ = effective-area coefficient, equation (33)

$\bar{\omega}_z$ = loss-rate coefficient, equation (56)

ρ = mass density, lb-sec²/ft⁴

σ = see equation (41)

φ = meridional angle; $\tan\varphi = W_r/W_z$, rad

ω = rotor angular velocity, rad/sec

Subscripts

h = hub

LE = leading edge

m = meridional component, equation (6)

m = mean blade surface, Fig. 1

0 = relative stagnation condition

p = leading face of blade

r = radial component

s = trailing face of blade

TE = trailing edge

u = circumferential component

x = component in instantaneous flow direction

z = axial component

Operators

d = general differential change

D = differential change following particle

$\bar{D}$ = differential change in direction of average flow, equation (47)

∂ = gradient in indicated direction or at fixed point in coordinate system when used with t

δ = difference from mean value

Δ = blade-to-blade difference; $\Delta(\;) \equiv (\;)_s - (\;)_p$

and since φ is the angle between the axial direction and the direction of a meridional pathline obtained by revolving the particle trajectory into a meridional (θ = const) plane, the quantity $D\varphi/Dm$ is the curvature of a meridional pathline. Thus

$$\frac{D\varphi}{Dm} = -\frac{1}{r_m} \tag{11}$$

The negative sign was chosen arbitrarily in the definition of r_m. With this definition, equation (10) becomes, again using equation (9),

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{C_u^2}{r} + \cos\varphi\frac{W_m^2}{r_m} - W_r\frac{DW_m}{Dm} \tag{12}$$

We pause here for a moment to mention that equation (12) with the last term neglected has been employed frequently as an approximate radial-equilibrium equation for the analysis of axial-flow turbomachines. The approximation is often quite accurate since W_r is usually not large and W_m frequently does not change rapidly in the m-direction. On the other hand, there are cases of interest where this is not true (see section, "Example Demonstrating Magnitude of Terms"), and as we shall see, the proper inclusion of the last term can be accomplished with little additional complication, at least for stations outside of any blade row; see equation (31).

The objective of the balance of this section will be to express DW_m/Dm in terms of the meridional pathline slope and curvature since, in the calculation scheme described in the Introduction, these quantities are presumed known when the radial-equilibrium equation is used. Partial derivatives with respect to θ and t may also be expected to appear because the three-dimensional and unsteady properties of the flow are being retained, at least for a while. In the pursuit of this objective, we will make use of the continuity equation, the equation of state, and the laws of thermodynamics. We will also make further use of the momentum equation.

The most useful form of the continuity equation for present purposes is

$$\frac{W}{\rho}\frac{D\rho}{Dx} + \nabla\cdot\mathbf{W} = 0 \tag{13}$$

In our cylindrical coordinates

$$\nabla\cdot\mathbf{W} = \frac{1}{r}\frac{\partial(rW_r)}{\partial r} + \frac{\partial W_u}{r\partial\theta} + \frac{\partial W_z}{\partial z} \tag{14}$$

Now the derivative of a quantity in the direction of flow (the x-direction) at some instant in time may be expressed by

$$\frac{\partial(\ )}{\partial x} = \frac{W_r}{W}\frac{\partial(\ )}{\partial r} + \frac{W_u}{W}\frac{\partial(\ )}{r\partial\theta} + \frac{W_z}{W}\frac{\partial(\ )}{\partial z} \tag{15}$$

or, rearranging,

$$\frac{\partial(\ )}{\partial z} = \frac{W}{W_z}\frac{\partial(\ )}{\partial x} - \frac{W_r}{W_z}\frac{\partial(\ )}{\partial r} - \frac{W_u}{W_z}\frac{\partial(\ )}{r\partial\theta} \tag{15a}$$

Equation (15a) is now used with equation (14) to eliminate the z-derivative. The result can be expressed as

$$\nabla\cdot\mathbf{W} = W_z\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rW_r}{W_z}\right) + \frac{\partial}{r\partial\theta}\left(\frac{W_u}{W_z}\right)\right] + \frac{W}{W_z}\frac{\partial W_z}{\partial x}$$

Since

$$\frac{W_r}{W_z} = \tan\varphi \quad \text{and} \quad \frac{W_u}{W_z} = \tan\beta$$

this can also be written

$$\nabla\cdot\mathbf{W} = W_z\left[\frac{1}{r}\frac{\partial(r\tan\varphi)}{\partial r} + \frac{\partial\tan\beta}{r\partial\theta}\right] + \frac{W}{W_z}\frac{\partial W_z}{\partial x} \tag{16}$$

Now it is known that

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + W\frac{\partial(\ )}{\partial x} \tag{17}$$

With equation (3), this can be written

$$\frac{\partial(\ )}{\partial x} = \frac{D(\ )}{Dx} - \frac{1}{W}\frac{\partial(\ )}{\partial t} \tag{18}$$

This will be used on the last term in equation (16). Also, since

$$W_z = W_m\cos\varphi \tag{19}$$

we can obtain

$$\frac{\partial W_z}{\partial x} = \frac{D}{Dx}(W_m\cos\varphi) - \frac{1}{W}\frac{\partial W_z}{\partial t}$$

Further analysis using equation (7) gives

$$\frac{\partial W_z}{\partial x} = \frac{W_m^2}{W}\frac{D\cos\varphi}{Dm} + \cos\varphi\frac{W_m}{W}\frac{DW_m}{Dm} - \frac{1}{W}\frac{\partial W_z}{\partial t}$$

When this is substituted into equation (16), we can obtain, with the help of equations (9), (11), and (19),

$$\nabla\cdot\mathbf{W} = W_z\left[\frac{1}{r}\frac{\partial(r\tan\varphi)}{\partial r} + \frac{\partial\tan\beta}{r\partial\theta}\right] + W_r\frac{\sec\varphi}{r_m} + \frac{DW_m}{Dm} - \frac{1}{W_z}\frac{\partial W_z}{\partial t} \tag{20}$$

This is the desired form for $\nabla\cdot\mathbf{W}$.

We next apply the momentum equation in the flow direction. This is obtained by taking the scalar product of $\mathbf{W}$ and equation (2):

$$-\frac{W}{\rho}\frac{\partial p}{\partial x} = \frac{D}{Dt}\left(\frac{W^2}{2}\right) - \omega^2 rW_r \tag{21}$$

Using equations (18), (7), and (3) and

$$W^2 = W_m^2 + W_u^2$$

this can be written

$$-\frac{W}{\rho}\frac{Dp}{Dx} + \frac{1}{\rho}\frac{\partial p}{\partial t} = W_m^2\frac{DW_m}{Dm} + W_uW\frac{DW_u}{Dx} - \omega^2 rW_r \tag{22}$$

Furthermore, the DW_u/Dx-term can be eliminated by use of equation (2b). After collection of terms and use of equation (4), equation (22) becomes

$$-\frac{W}{\rho}\frac{Dp}{Dx} = W_m^2\frac{DW_m}{Dm} - W_r\frac{C_u^2}{r} - \frac{W_u}{\rho}\frac{\partial p}{r\partial\theta} - \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{23}$$

which is the desired form.

If there were no heat addition, the entropy would be constant for a particle as it moves along, and we could employ the relation

$$\frac{Dp/Dx}{D\rho/Dx} = a^2$$

where a is the speed of sound. This relation could then be solved simultaneously with equations (13), (20), and (23) to obtain an expression for DW_m/Dm that does not contain $\nabla\cdot\mathbf{W}$, Dp/Dx, or $D\rho/Dx$. The equation of state would not have been used. However, since we wish to include heat addition, we invoke assumption 4, which allows us to write

$$Tds = \frac{c_v}{R}\frac{dp}{\rho} - c_pT\frac{d\rho}{\rho} \tag{24}$$

and

$$a^2 = \frac{c_p}{c_v}RT \tag{25}$$

Applying equation (24) to a particle as it moves along, and using equation (25), we can write

$$\frac{1}{\rho}\frac{D\rho}{Dx} = \frac{1}{\rho a^2}\frac{Dp}{Dx} - \frac{1}{c_p}\frac{Ds}{Dx} \tag{26}$$

For the reversible addition of heat Q' per unit mass per unit time, the second law of thermodynamics gives

$$\frac{Ds}{Dt} = \frac{Q'}{T} \tag{27}$$

Using equations (27) and (3), equation (26) becomes

$$\frac{1}{\rho}\frac{D\rho}{Dx} = \frac{1}{\rho a^2}\frac{Dp}{Dx} - \frac{Q'}{Wc_pT} \tag{28}$$

We now solve for DW_m/Dm by combining equations (13), (20), (23), and (28) to eliminate $\nabla\cdot\mathbf{W}$, $D\rho/Dx$, and Dp/Dx. This is then substituted into equation (12) and, after combining some of the terms and introducing the Mach number, there results

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial r} = {} & \left(\frac{1 - \mathrm{M}_z^2}{1 - \mathrm{M}_m^2}\right)\left(\frac{C_u^2}{r} + \sec\varphi\,\frac{W_m^2}{r_m}\right) \\ & + \frac{W_r}{1 - \mathrm{M}_m^2}\left\{W_z\left[\frac{1}{r}\frac{\partial(r\tan\varphi)}{\partial r} + \frac{\partial \tan\beta}{r\partial\theta}\right] - \frac{Q'}{c_pT}\right. \\ & \left. + \frac{W_u}{\rho a^2}\frac{\partial p}{r\partial\theta} + \frac{1}{\rho a^2}\frac{\partial p}{\partial t} - \frac{1}{W_z}\frac{\partial W_z}{\partial t}\right\}\end{aligned} \tag{29}$$

This is a general form of the three-dimensional unsteady radial-equilibrium equation.

The next step is to assume that the flow is steady relative to the rotor so that the last two terms in the braces in equation (29) can be dropped. As we do this, it is interesting to make the following observation: Suppose we had decided at the beginning to carry out the derivation using fixed coordinates rather than rotating coordinates. A review of equations (2) through (29) discloses that if this had been done, the result, i.e., the new equation (29), which we will call equation (29*a*), would look the same as equation (29) except the W's would be replaced by C's and β would be replaced by α. Now if the flow is steady relative to the rotor, it would be quite unsteady in the absolute frame, at least inside of a rotor blade row. In fact, we can write

$$\left.\frac{\partial(\ )}{\partial t}\right|_{\substack{\text{absolute}\\ \text{frame}}} = -\omega\,\frac{\partial(\ )}{\partial\theta} \tag{30}$$

when the relative flow is steady. When this is applied to equation (29*a*), the following group of terms appears in the braces:

$$\frac{(C_u - \omega r)}{\rho a^2}\frac{\partial p}{r\partial\theta} + C_z\frac{\partial}{r\partial\theta}\left(\frac{C_u}{C_z}\right) + \frac{\omega r}{C_z}\frac{\partial C_z}{r\partial\theta}$$

Since $C_u = W_u + \omega r$ and $C_z = W_z$, this can be written

$$\frac{W_u}{\rho a^2}\frac{\partial p}{r\partial\theta} + W_z\frac{\partial}{r\partial\theta}\left(\frac{W_u}{W_z}\right)$$

which matches the θ-derivative terms in equation (29). This in essence provides a verification of the unsteady terms in equation (29).

The general calculation scheme described in the Introduction has often been used with calculation stations located only exterior to blade rows. For these locations, it is reasonable to neglect the θ-derivative terms in equation (29), at least on a circumferential-average basis. The heat-addition term might also be neglected here. Under these conditions, equation (29) or (29*a*) can be written

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial r} = {} & \left(\frac{1 - \mathrm{M}_z^2}{1 - \mathrm{M}_m^2}\right)\left(\frac{C_u^2}{r} + \sec^3\varphi\,\frac{C_z^2}{r_m}\right) \\ & + \frac{C_z^2\tan\varphi}{(1 - \mathrm{M}_m^2)r}\frac{\partial(r\tan\varphi)}{\partial r}\end{aligned} \tag{31}$$

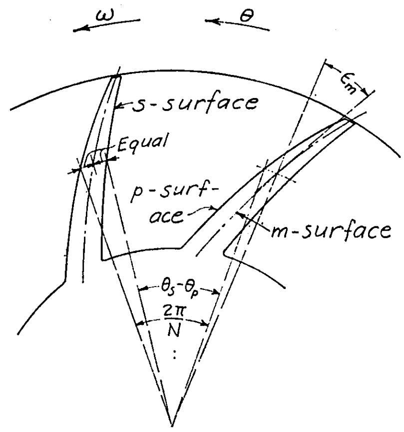

Fig. 1 Section cut by z = const plane

This is the exact form of the axisymmetric radial-equilibrium equation referred to in the Introduction; it is clear from the derivation of equation (29) that if we had assumed steady axisymmetric frictionless adiabatic flow at the outset, equation (31) would have resulted without the necessity of further approximations.

Circumferential-Average Equation

In many engineering applications it is not feasible to treat a problem rigorously because of the great complexity involved. In a turbomachine blade row, the flow is quite three-dimensional in the sense that there are large velocity and thermodynamic property gradients present in all three component directions. Most design systems approach the problem by building up the three-dimensional structure from two or more two-dimensional building blocks. The most important building block to the turbomachinery designer is the blade-to-blade flow model (the cascade flow) which deals directly with the deflecting action of the blades on the fluid. Radial-equilibrium considerations, with which we are concerned here, relate different spanwise locations to one another so that the blade-to-blade flows found at these locations can properly coexist.

Now it would be possible to calculate the radial pressure gradient from equation (29) at any point in the flow field using information from previous approximations; and through a process of iteration these pressure variations could be made consistent with those of the blade-to-blade solutions. To do this correctly would be a formidable task. From a practical point of view, it will often be adequate to satisfy the radial-equilibrium equation only on a circumferential-average basis. This approach will be followed in the balance of this paper. The averages will be formed by integrating equation (29) in the circumferential direction from the pressure side of a blade to the suction side of the adjacent blade. The average of any quantity is defined as (see Fig. 1)

$$\overline{(\ )} \equiv \frac{1}{\theta_s - \theta_p}\int_{\theta_p}^{\theta_s} (\ )d\theta \tag{32}$$

This integration should always be carried out from the leading face of a blade (p-surface) to the trailing face of the blade ahead of it (s-surface). A compressor or pump rotor is shown in Fig. 1; for a turbine rotor the p-surface would actually be the suction surface and the s-surface would be the pressure surface.

Before proceeding with the averaging, it will be helpful to introduce the quantity λ, which is the ratio of open circumference to total circumference. Thus (see Fig. 1)

$$\lambda = \frac{N}{2\pi}(\theta_s - \theta_p) \tag{33}$$

Fig. 1 also defines the mean blade surface, which will be found useful. The pressure and suction surfaces are related to it through the following equations:

$$\theta_p = \theta_m + \frac{\pi}{N}(1 - \lambda) \tag{34}$$

and

$$\theta_s = \theta_m - \frac{\pi}{N}(1 - \lambda) + \frac{2\pi}{N} \tag{35}$$

The lean angle is seen to be given by

$$\tan \epsilon_m = -r\frac{\partial \theta_m}{\partial r} \tag{36}$$

The negative sign is necessary because ϵ is taken as positive for backward-bent blades, as shown in Fig. 1.

We will now proceed with the averaging. Before integrating equation (29), we multiply it by $(1 - \mathrm{M}_m{}^2)$. Also we replace $W_m{}^2$ with $\sec^2 \varphi\, W_z{}^2$ and use the relationship between radius of curvature and the second derivative to obtain

$$\sec \varphi \frac{W_m{}^2}{r_m} = -\frac{D^2 r}{Dz^2} W_z{}^2 \tag{37}$$

Also, the unsteady flow terms are dropped. Then integration of equation (29) yields, using equation (32),

$$F_0 = F_1 + F_2 + F_3 + F_4 + F_5 \tag{38}$$

where

$$F_0 \equiv \overline{\left[(1 - \mathrm{M}_m{}^2)\frac{1}{\rho}\frac{\partial p}{\partial r}\right]} \tag{38a}$$

$$F_1 \equiv \frac{1}{r}\overline{[(1 - \mathrm{M}_z{}^2)C_u{}^2]} \tag{38b}$$

$$F_2 \equiv -\overline{\left[(1 - \mathrm{M}_z{}^2)\frac{D^2 r}{Dz^2}W_z{}^2\right]} \tag{38c}$$

$$F_3 \equiv \frac{1}{r}\overline{\left\{W_r W_z\left[\frac{\partial(r\tan\varphi)}{\partial r} + \frac{\partial \tan\beta}{\partial\theta}\right]\right\}} \tag{38d}$$

$$F_4 \equiv -\overline{\left(W_r\frac{1}{c_p}\frac{Q'}{T}\right)} \tag{38e}$$

$$F_5 \equiv \frac{1}{r}\overline{\left(\mathrm{M}_r\mathrm{M}_u\frac{1}{\rho}\frac{\partial p}{\partial\theta}\right)} \tag{38f}$$

The F-functions are averages of products. These can also be expressed as products of averages as shown in Appendix 1. For example, for the variables q, u, and v, we can write

$$\begin{aligned} F &\equiv \overline{[(1 - q^2)uv]} \\ &= (1 - \bar{q}^2)\bar{u}\bar{v} + G(q, u, v) \end{aligned} \tag{39}$$

The G-functions are somewhat analogous to the Reynolds stresses in turbulence theory. In Appendix 1, it is assumed that the variables like q, u, and v vary linearly with θ, and this permits the writing of specific formulas for the G-functions. These will be presented in the fifth section, and a numerical example will demonstrate the magnitudes involved. It is sufficient here to make the qualitative observation that the G-functions will be small whenever the blade-to-blade variations of q, u, and v are small. We now use equations like equation (39) with equation (38) to obtain

$$\begin{aligned} \overline{\left(\frac{1}{\rho}\right)}\frac{\partial\bar{p}}{\partial r} &= \left(\frac{1 - \overline{\mathrm{M}}_z{}^2}{1 - \overline{\mathrm{M}}_m{}^2}\right)\left(\frac{\bar{C}_u{}^2}{r} - \overline{\frac{D^2 r}{Dz^2}}\bar{W}_z{}^2\right) \\ &+ \frac{\bar{W}_r}{1 - \overline{\mathrm{M}}_m{}^2}\left[\frac{\bar{W}_z\bar{\sigma}}{r} - \overline{\left(\frac{1}{c_p}\right)}\overline{\left(\frac{Q'}{T}\right)}\right] \\ &+ \frac{\overline{\mathrm{M}}_r\overline{\mathrm{M}}_u}{r(1 - \overline{\mathrm{M}}_m{}^2)}\overline{\left(\frac{1}{\rho}\right)}\overline{\frac{\partial p}{\partial\theta}} \\ &+ \frac{-G_0 + G_1 + G_2 + G_3 + G_4 + G_5}{1 - \overline{\mathrm{M}}_m{}^2} \end{aligned} \tag{40}$$

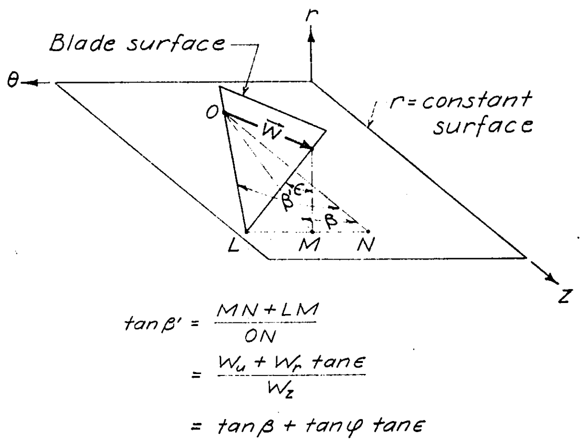

Fig. 2 Relationship between flow angle, β, and angle of intersection of blade with r = const surface, β'

where

$$\sigma \equiv \frac{\partial(r\tan\varphi)}{\partial r} + \frac{\partial\tan\beta}{\partial\theta} \tag{41}$$

The terms in equation (40) containing averages of partial derivatives will now be put into more useful forms. The radial pressure gradient average can be written

$$\begin{aligned} \overline{\frac{\partial p}{\partial r}} &= \frac{1}{\theta_s - \theta_p}\int_{\theta_p}^{\theta_s}\frac{\partial p}{\partial r}d\theta \\ &= \frac{1}{\theta_s - \theta_p}\left(\frac{\partial}{\partial r}\int_{\theta_p}^{\theta_s}p\,d\theta - p_s\frac{\partial\theta_s}{\partial r} + p_p\frac{\partial\theta_p}{\partial r}\right) \end{aligned}$$

Using equations (32) through (36), this becomes

$$\overline{\frac{\partial p}{\partial r}} = \frac{\partial\bar{p}}{\partial r} + \left(\frac{p_s - p_p}{\theta_s - \theta_p}\right)\frac{\tan\epsilon_m}{r} + \left(\bar{p} - \frac{p_p + p_s}{2}\right)\frac{1}{\lambda}\frac{\partial\lambda}{\partial r} \tag{42}$$

The first term on the right-hand side of equation (42) is the desired radial gradient of the average pressure. The second term is related to the radial component of blade force on the fluid. The last term vanishes when the pressure varies linearly with θ and will be neglected.

A similar approach is taken with $\bar{\sigma}$. We can write

$$\begin{aligned} \bar{\sigma} = \frac{1}{\theta_s - \theta_p}\bigg(&\frac{\partial}{\partial r}\int_{\theta_p}^{\theta_s}r\tan\varphi\,d\theta - r\tan\varphi_s\frac{\partial\theta_s}{\partial r} \\ &+ r\tan\varphi_p\frac{\partial\theta_p}{\partial r} + \tan\beta_s - \tan\beta_p\bigg) \end{aligned}$$

Using equation (32) and equations like (36), this becomes

$$\begin{aligned} \bar{\sigma} = \frac{1}{\theta_s - \theta_p}\bigg\{&\frac{\partial}{\partial r}[(\theta_s - \theta_p)r\,\overline{\tan\varphi}] + \tan\beta_s + \tan\varphi_s\tan\epsilon_s \\ &- \tan\beta_p - \tan\varphi_p\tan\epsilon_p\bigg\} \end{aligned} \tag{43}$$

We now use the relationship developed in Fig. 2 to introduce the angle β'. Also, since

$$\tan\beta_s' = r\frac{\partial\theta_s}{\partial z} \tag{44}$$

and

$$\tan\beta_p' = r\frac{\partial\theta_p}{\partial z} \tag{45}$$

we can write, with the help of equation (33),

$$\bar{\sigma} = \frac{\partial(r\,\overline{\tan\varphi})}{\partial r} + \frac{1}{\theta_s - \theta_p}\left(r\,\overline{\tan\varphi}\,\frac{2\pi}{N}\,\frac{\partial\lambda}{\partial r} + \frac{2\pi r}{N}\,\frac{\partial\lambda}{\partial z}\right) \quad (46)$$

If we define a direction in the meridional plane related to $\tan\varphi$, the following expression can be written:

$$\frac{\bar{D}(\ )}{\bar{D}z} = \frac{\partial(\ )}{\partial z} + \overline{\tan\varphi}\,\frac{\partial(\ )}{\partial r} \quad (47)$$

Here $\bar{D}$ represents the change in a quantity in the $\overline{\tan\varphi}$-direction. Using equation (47) on λ in equation (46), and again employing equation (33), we obtain

$$\bar{\sigma} = \frac{\partial(r\,\overline{\tan\varphi})}{\partial r} + \frac{r}{\lambda}\,\frac{\bar{D}\lambda}{\bar{D}z} \quad (48)$$

With a known blade thickness distribution and with the average meridional flow direction known, equation (48) may be evaluated readily.

Finally

$$\overline{\frac{\partial p}{\partial\theta}} = \frac{1}{\theta_s - \theta_p}\int_{\theta_p}^{\theta_s}\frac{\partial p}{\partial\theta}\,d\theta = \frac{p_s - p_p}{\theta_s - \theta_p} \quad (49)$$

Equations (42), (48), and (49) are now substituted into equation (40):

$$\overline{\left(\frac{1}{\rho}\right)}\frac{\partial\bar{p}}{\partial r} = \left(\frac{1-\overline{\mathrm{M}}_z^2}{1-\overline{\mathrm{M}}_m^2}\right)\left(\frac{\bar{C}_u^2}{r} - \frac{\bar{D}^2 r}{\bar{D}z^2}\,\bar{W}_z^2\right)$$
$$+ \frac{\bar{W}_r}{1-\overline{\mathrm{M}}_m^2}\left\{\bar{W}_z\left[\frac{\partial(r\,\overline{\tan\varphi})}{r\partial r} + \frac{1}{\lambda}\,\frac{\bar{D}\lambda}{\bar{D}z}\right] - \overline{\left(\frac{1}{c_p}\right)}\,\overline{\left(\frac{Q'}{T}\right)}\right\}$$
$$+ \overline{\left(\frac{1}{\rho}\right)}\left(\frac{p_s - p_p}{\theta_s - \theta_p}\right)\frac{1}{r}\left(\frac{\overline{\mathrm{M}}_r\overline{\mathrm{M}}_u}{1-\overline{\mathrm{M}}_m^2} - \tan\epsilon_m\right)$$
$$+ \frac{-G_0 + G_1 + G_2 + G_3 + G_4 + G_5}{1-\overline{\mathrm{M}}_m^2} \quad (50)$$

This is an exact form for the radial-equilibrium equation except that the last term in equation (42) was neglected. It is of interest to compare it with the equation that results when the flow is assumed to be axisymmetric at the outset. For that approach, it is necessary to represent the blade action by a distributed body force field and the blade thickness by distributed blockage. The result is given here without derivation:

$$\frac{1}{\rho}\,\frac{\partial p}{\partial r} = \left(\frac{1-\mathrm{M}_z^2}{1-\mathrm{M}_m^2}\right)\left(\frac{C_u^2}{r} - \frac{D^2 r}{Dz^2}\,W_z^2\right)$$
$$+ \frac{W_r}{1-\mathrm{M}_m^2}\left\{W_z\left[\frac{\partial(r\tan\varphi)}{r\partial r} + \frac{1}{\lambda}\,\frac{D\lambda}{Dz}\right] - \frac{Q'}{c_pT}\right\}$$
$$- F_u\,\frac{\mathrm{M}_r\mathrm{M}_u}{1-\mathrm{M}_m^2} + F_r \quad (51)$$

where F_u and F_r are the tangential and radial components of the body force per unit mass. Equations (50) and (51) are seen to agree when it is realized that the pressure difference across the blade is actually a force per unit area of the meridional plane. The distance $r(\theta_s - \theta_p)$ is the third dimension of the volume of fluid on which this force effectively acts, and the presence of the density puts the force on a per unit mass basis. This agreement between equations (50) and (51) means that an axisymmetric treatment of the flow will be correct (based on the assumptions in the second section) if the quantities involved are interpreted as circumferential averages and *if the G-functions in equation (50) may be neglected.* We will investigate the magnitude of these functions in the fifth section.

Effects of Friction

Fluid friction has been kept out of the analysis so far for simplicity. In this section it will be introduced but, for practical reasons, certain approximations will have to be made.

There are essentially two ways in which friction can affect the circumferential-average, radial-equilibrium equation: (*a*) The shear-stress vector at the blade surface generally has a radial component which acts as a force on the fluid and (*b*) the heating effect of internal fluid friction introduces a term somewhat like that due to reversible heat addition. It is recommended here that the shear-stress effect be neglected because with practical blade geometries and loading levels it is very small. This is true even though the drag coefficient might be rather high because we are concerned here only with the part of the drag that is due to skin friction; the pressure drag is already accounted for in the ϵ_m-term in equation (50). The frictional heating effect, however, encompasses all loss sources, including shock waves. An approximate treatment of it will be given here.

Let us consider a case in which there is no external heat addition. The following equation is then usually a good approximation:

$$\frac{DI}{Dt} = \frac{1}{\rho}\,\frac{\partial p}{\partial t} \quad (52)$$

where I is the stagnation rothalpy[3] defined as

$$I \equiv \frac{W^2}{2} + h - \frac{U^2}{2} \quad (53)$$

If equation (52) is used in the analysis of the second section in place of equation (21), and if equation (2*b*) is accepted even though there is friction, then equation (23) becomes

$$-W\,\frac{Dh}{Dx} = W_m^2\,\frac{DW_m}{Dm} - W_r\,\frac{C_u^2}{r} - \frac{W_u}{\rho}\,\frac{\partial p}{r\partial\theta} - \frac{1}{\rho}\,\frac{\partial p}{\partial t} \quad (54)$$

The appropriate equation analogous to equation (26) is

$$\frac{1}{\rho}\,\frac{D\rho}{Dx} = \frac{1}{a^2}\,\frac{Dh}{Dx} - \frac{1}{R}\,\frac{Ds}{Dx} \quad (55)$$

The entropy variation here is due to friction, and it is therefore fitting to express it in terms of a local loss-rate coefficient. The definition selected for this loss-rate coefficient is

$$\bar{\omega}_z \equiv \frac{-l}{P-p}\left[\frac{DP}{Dz} - \frac{DP_{\mathrm{ideal}}}{Dz}\right] \quad (56)$$

where l is the overall change in z-coordinate from leading edge to trailing edge, and DP_{ideal} is the differential change in relative stagnation pressure that occurs as a result of a differential change in radius in the rotating coordinate system for an isentropic process. By use of

$$T_0Ds = c_pDT_0 - \frac{1}{\rho_0}\,DP \quad (57)$$

$$\frac{DP_{\mathrm{ideal}}}{P} = \frac{c_p}{R}\,\frac{DT_0}{T_0} \quad (58)$$

and the equation of state, we can find

$$\frac{1}{R}\,\frac{Ds}{Dx} = \frac{W_z}{W}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l} \quad (59)$$

When this is combined with equation (55), we obtain

$$\frac{1}{\rho}\,\frac{D\rho}{Dx} = \frac{1}{a^2}\,\frac{Dh}{Dx} - \frac{W_z}{W}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l} \quad (60)$$

which is analogous to equation (28). Combination of equations (12), (13), (20), (54), and (60) then leads to an equation like (29), except, in place of the Q'-term, there appears

[3] This appropriate but little-known name was introduced by Wu [4].

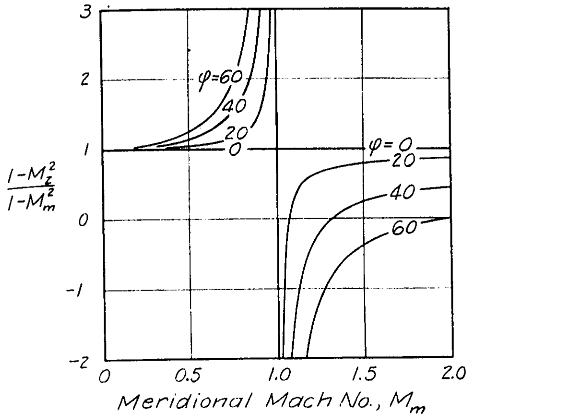

Fig. 3 Coefficient of centripetal acceleration and meridional streamline-curvature terms

$$-\frac{W_r W_z}{1 - \mathrm{M}_m^2}\left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l} \tag{61}$$

The quantity in parentheses is a function of Mach number only.

When circumferential averages are taken, term (61) appears with bars over the quantities that make it up, and a G-function is generated. Since the analysis is approximate anyway, it is not worthwhile to include this G-function; however, it is recommended that term (61) be added to the right-hand sides of equations (50) and (51).

Physical Interpretation of Terms

In this section, we will apply equation (50) to different flow patterns that are easy to comprehend physically. Each pattern will be selected such that all of the terms will be zero except the term that is being studied and one other. The objective is not to provide a separate derivation or analytical check of the term but rather to show that the term is plausible and that such a term is needed. The G-terms will not be considered here.

Before taking up each term separately, we note that all but one contain the quantity $1 - \mathrm{M}_m^2$ in the denominator. This singular behavior as the meridional Mach number approaches unity occurs because we have in effect specified locally the streamwise distribution of meridional stream tube area when we have specified the radial and streamwise distributions of φ and the streamwise distribution of λ. Thus, with the stream-tube area varying in some fixed and generally nonzero manner, we should expect gradients of all quantities in the direction of flow to tend toward infinity as the Mach number approaches unity. If φ is not zero, the infinite streamwise pressure gradient will have a radial component so that $\partial p/\partial r$ will be infinite. When this is realized, it becomes clear that the general scheme described in the Introduction of solving for the flow by iterating on meridional streamline shape is not a good one when the meridional Mach number is near unity. Fortunately, in most practical cases of interest, this is not the case.

Centripetal Acceleration Term

The fact that C_u^2/r appears in a radial-equilibrium equation hardly needs further discussion; however, the coefficient

$$\frac{1 - \mathrm{M}_z^2}{1 - \mathrm{M}_m^2}$$

is unusual and deserves some attention here. A graph of this coefficient is seen in Fig. 3. At low Mach numbers it is nearly unity; it then rises with meridional Mach number slowly at first, but it rapidly becomes infinite as M_m approaches unity. The supersonic portion will not be discussed here; we will endeavor only to show that it is reasonable to expect the coefficient of C_u^2/r to grow above unity as M_m increases subsonically.

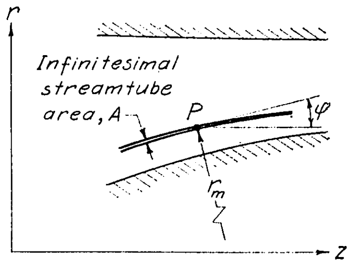

Fig. 4 Axisymmetric flow in an annulus

Consider an adiabatic axisymmetric flow field free of blade effects and with φ positive, such as is shown in Fig. 4. Also, let the meridional streamlines be straight at the point under consideration, let φ vary such that $r \tan \varphi$ is constant radially, and let there be a swirl component in the flow. For this case, all of the terms on the right-hand side of equation (50) are zero except the centripetal acceleration term. Now it is shown in Appendix 2 that when φ varies in this manner and when the curvature is zero, the meridional stream tube area is locally constant. If the flow were incompressible ($\mathrm{M}_m \rightarrow 0$), this would mean that the meridional velocity would not vary in the flow direction and hence, according to equation (12), the C_u^2/r-term would stand unmodified. It is assumed here that the reader accepts the plausibility of equation (12). For compressible flow with subsonic M_m, however, W_m would decrease in the flow direction because the density would be rising due to the reduction of C_u with increasing radius as in any vaneless diffuser. Thus equation (12) tells us that the radial pressure gradient must be some amount larger than is given by C_u^2/r alone.

Meridional Streamline-Curvature Term

The special flow that will be considered here is the same as in the preceding subsection, except there is no swirl and the meridional streamline is locally curved concave toward the axis as shown in Fig. 4. Thus the meridional streamline-curvature term is the only one that appears in the right-hand side of equation (50). Appendix 2 now tells us that the meridional stream tube area is increasing in the flow direction at the rate

$$\frac{1}{A}\frac{\partial A}{\partial m} = \frac{\tan \varphi}{r_m}$$

This causes a streamwise pressure gradient that can be shown by one-dimensional gas-dynamics relations to be given by

$$\frac{1}{\rho}\frac{\partial p}{\partial m} = \frac{W_m^2}{1 - \mathrm{M}_m^2}\frac{\tan \varphi}{r_m}$$

If the radial component of this gradient (obtained by multiplication by $\sin \varphi$) is added to the radial component of the normal pressure gradient ($\cos \varphi\, W_m^2/r_m$), there is obtained

$$\frac{1 - \mathrm{M}_z^2}{1 - \mathrm{M}_m^2}\sec \varphi \frac{W_m^2}{r_m}$$

which is the desired expression; see equation (37). For this term, then, we have not just shown plausibility but have given an alternate derivation.

Slope Gradient Term

For this term and those that follow, we will examine flow fields that have the axis of symmetry infinitely far removed. This will cause the centripetal acceleration term to disappear, and certain other terms will change their forms slightly; e.g., the slope gradient term becomes

$$\frac{W_r W_z}{1 - \mathrm{M}_m^2}\frac{\partial \tan \varphi}{\partial r}$$

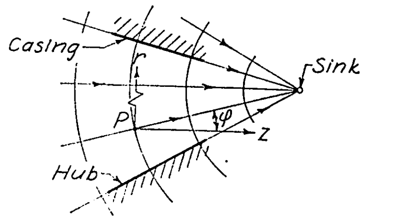

Fig. 5 Two-dimensional plane flow into a sink

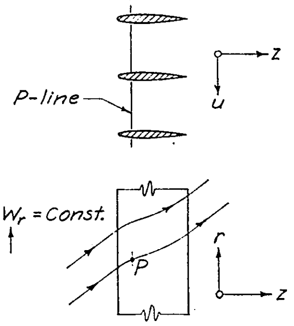

Fig. 6 Streamline curvature caused by thickness blockage

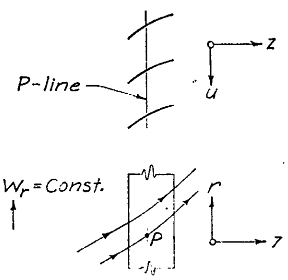

Fig. 7 Streamline curvature caused by density change resulting from flow diffusion

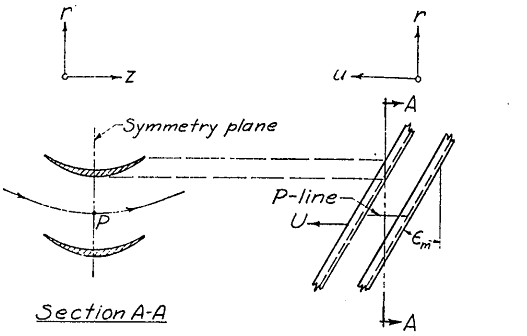

Fig. 8 Streamline curvature caused by radial component of blade force

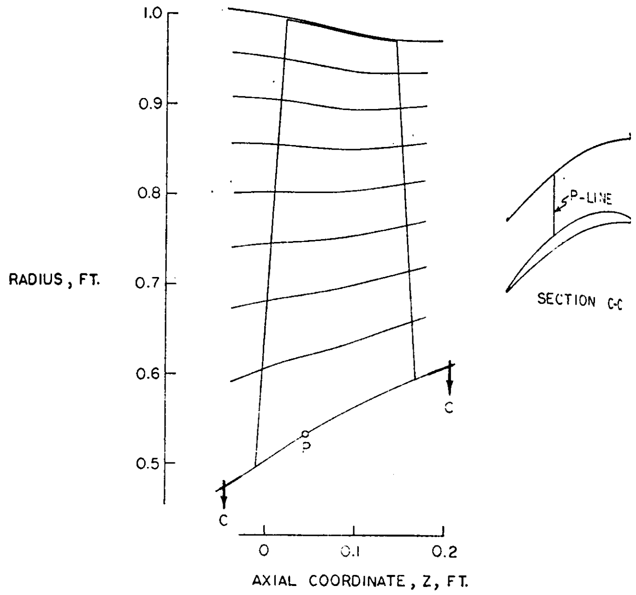

Fig. 9 Compressor rotor of example

Let the flow be that induced by a line sink perpendicular to the meridional plane as in Fig. 5. For this flow pattern, the slope gradient term is the only term appearing on the right-hand side of equation (50). The circles around the sink are constant-pressure contours with the smaller circles having the lower pressures when the meridional Mach number is subsonic. Thus, at point P where φ is positive, we see that $\partial p/\partial r$ must be negative. The slope gradient term is also negative since tan φ decreases with increasing r, so the term looks plausible.

Blockage Gradient Term

For this and the following terms, we will not only have the axis of symmetry infinitely far removed but we will also specify that the hub and casing surfaces are far removed from the portion of the flow field being studied. Furthermore, for this case we let inlet conditions to the blade row be constant along the r-direction, and the blades themselves are specified to be untwisted constant-section, infinite-span profiles. Thus all partial derivatives with respect to r will be zero. The profiles are also uncambered and unstaggered for this case, as shown in Fig. 6. Under these circumstances, all terms on both sides of equation (50) will be zero except the blockage gradient term and the meridional streamline-curvature term.

Now since the normal vectors to all boundary surfaces in the flow field have no radial component, the radial component of velocity W_r will remain unchanged throughout the flow field. The axial component will change, however, because of the thickness blockage, and this will therefore cause the streamlines to curve as shown in Fig. 6 when the axial Mach number is subsonic. At line P, $\overline{D}\lambda/\overline{D}z$ is negative and

$$\overline{\frac{D^2r}{Dz^2}}$$

is negative, so the cancellation of these terms in equation (50) appears plausible.

It is also of interest to note here that it is clearly the *axial* component of Mach number that controls the sign of the streamline curvature; this adds credence of the previous finding that $1 - \mathrm{M}_z^2$ should appear in the coefficient of the streamline-curvature term.

Heat-Addition Term

Let the blading geometry again be as in Fig. 6 except let the blades have zero thickness. If the blades have higher temperature than the fluid, they will heat the fluid, causing the entropy to rise in the direction of flow. This will cause the density to decrease if the axial Mach number is subsonic and, by the continuity principle, the axial velocity will then increase. Since the spanwise velocity W_r remains constant, the streamlines curve toward the z-axis; this is necessary in order for the heat-addition term to balance the streamline curvature term, which is the only other nonzero term in equation (50).

Isentropic Density-Change Term

In order for this term to be nonzero, the fluid must be compressible, the flow must have both radial and swirl components, and blading that deflects the flow must be present. As in the preceding two subsections, we again employ an infinite-span rectilinear blade row so that all partial derivatives in the r-direction vanish. For this kind of configuration, the isentropic density-change term in equation (50) becomes

$$\overline{\left(\frac{1}{\rho}\right)\frac{\Delta p}{\Delta u}\frac{\overline{\mathrm{M}}_r\overline{\mathrm{M}}_u}{1-\overline{\mathrm{M}}_m{}^2}}$$

The axial Mach number is taken to be subsonic.

A moving row of infinitely thin compressor blades is chosen for this example; it translates with the velocity U, as shown in Fig. 7. All terms in equation (50) are seen to be zero except the term being studied and the streamline curvature term. In this example, $\Delta p/\Delta u$ is negative and so is M_u, so the isentropic density-change term is positive. The density rise that occurs because this is a compressor blade row causes a reduction in W_z according to the continuity principle, and this causes the streamlines to curve as shown in Fig. 7, since W_r is constant. The sign of the curvature is seen to be such as to satisfy equation (50).

Radial-Force Term

The geometry selected for the isolation of this term is shown in Fig. 8. Once again, the axis is infinitely far removed. The blading is an infinite-span, two-dimensional, subsonic cascade of symmetrical-impulse compressor blades, but the spanwise direction does not coincide with the r-direction. The cascade is moving in the u-direction as shown. Under these circumstances, the radial-force term in equation (50) has the form

$$-\overline{\left(\frac{1}{\rho}\right)\frac{\Delta p}{\Delta u}}\tan\epsilon_m$$

The P-line is taken in the plane of symmetry where W_r is zero. The only terms that are nonzero in equation (50) are the radial-force term and the streamline-curvature term. (At first glance, we might think that the radial-pressure-gradient term should also be nonzero because $\partial p/\partial r$ is negative at all points along the P-line; this is not the case because $\bar{p}$ is the same no matter where in the symmetry plane the P-line is located. Hence $\partial \bar{p}/\partial r$ is zero.) Since ϵ_m is positive and $\Delta p/\Delta u$ is negative, the radial-force term is positive as it need be to satisfy equation (50) with the curvature shown in Fig. 8.

Loss-Rate Term

The example to demonstrate term (61) is the same as that of the subsection, "Heat-Addition Term," except there is dissipative heating instead of reversible heat addition.

Example Demonstrating Magnitudes of Terms

In this section, we will apply the radial-equilibrium equation at a point in the flow field of a compressor rotor blade row. The object will be to gain knowledge of the magnitudes of the various terms in equation (50), including the G-terms, for a typical example.

The blade row to be studied is shown in Fig. 9. It is assumed that this row operates in air with the standard inlet stagnation conditions of 2116 psf pressure and 519 R temperature. The tip radius at inlet is 1 ft, and the tip speed is 1400 fps. There is no preswirl and no heat addition. Losses are assumed to occur in such a way as to maintain a constant polytropic efficiency along a streamline.

The point to be studied is located on the hub about a third of the way through the blade row. The blade profile is shown in Section C-C; this section is obtained by cutting the blade along the flow surface (hub surface) but viewing it in the radial direction, as recommended in [5]. The blade angles seen in this section are therefore like β in Fig. 2 rather than like β'. The pressure side of the following blade is drawn maintaining the proper circumferential spacing; the angles for this blade are therefore distorted. The trailing edge of the full blade appears unusually thick because a boundary-layer displacement-thickness allowance increasing from the leading edge is included as part of the blade thickness for the calculations.

Distributions of certain quantities of interest are shown in Fig. 10. The λ and rC_u-distributions are input quantities for an axisymmetric calculation pass, and quantities labeled "average" are output from that pass. The suction-surface and pressure-surface relative velocities were calculated to yield the circulation distribution implied by the λ and rC_u-distributions. It was also necessary for this calculation to assume a relationship between the blade-surface velocities and the average velocity; a simple average was assumed here for simplicity. The blade meanline-angle distribution was chosen to represent a reasonable deviation from the calculated average flow direction. Trigonometry then yielded all blade-surface velocity components. The loss-rate coefficient $\bar{\omega}_z$ at P is 0.23; the overall hub blade-element adiabatic efficiency is 91 percent. The lean angle ϵ_m is -4 deg.

The terms on the right-hand side of equation (50), and term (61), were calculated at point P. The results were made nondimensional by dividing by $U_{h\mathrm{TE}}^2/r_{h\mathrm{TE}}$. This quantity was chosen for normalization because it is typical of the maximum value of the radial-pressure-gradient term we would be likely to encounter if we were to use the simplified-radial-equilibrium approximation.

The numerical values obtained are given on the left side of Table 1. It is seen that the familiar centripetal acceleration and meridional-streamline-curvature terms are overshadowed by some of the other less familiar terms. Admittedly, the example was selected to demonstrate this occurrence. Experience has shown that, for points somewhat removed from an annulus wall, the meridional streamline-curvature term tends to compensate for the other terms when they are large, with the net result that the radial pressure gradient is not excessive. The velocity distributions may still differ appreciably from those given by simpler approximations, however.

Evaluation of G-Functions

As shown by equation (39), the G-functions account for the fact that the product of several averaged quantities is different from the average of the product. In Appendix 1, G-function formulas are derived for the kinds of products that exist in equation (38). In the derivation of these formulas, it was assumed that the quantities involved vary linearly with θ. This is a rather bold assumption; accordingly, the formulas should be regarded as representing only first-order approximations.

Using equations (75), (75), (75), (77), and (77), respectively, the following G-functions result:

$$G_0 = \frac{1}{12}\left\{(1-\overline{M}_m^2)\,\Delta\left(\frac{1}{\rho}\right)\Delta\frac{\partial p}{\partial r} - \overline{\left(\frac{1}{\rho}\right)}\,\overline{\frac{\partial p}{\partial r}}\,(\Delta M_m)^2 - 2\overline{M}_m\Delta M_m\left[\overline{\frac{\partial p}{\partial r}}\,\Delta\left(\frac{1}{\rho}\right) + \overline{\left(\frac{1}{\rho}\right)}\,\Delta\frac{\partial p}{\partial r}\right]\right\} - \frac{1}{80}\left[(\Delta M_m)^2\Delta\left(\frac{1}{\rho}\right)\Delta\frac{\partial p}{\partial r}\right] \quad (62)$$

$$G_1 = \frac{1}{12r}\left[(1-\overline{M}_z^2)(\Delta C_u)^2 - (\bar{C}_u\Delta M_z)^2 - 4\overline{M}_z\bar{C}_u\Delta M_z\Delta C_u\right] - \frac{1}{80r}(\Delta M_z\Delta C_u)^2 \quad (63)$$

Table 1 Normalized values of terms in numerical example of Fig. 9

Term	Value	G-function
Centripetal acceleration	0.11	$\frac{G_1}{1-\overline{M}_m^2} = 0.016$
Meridional streamline curvature	0.14	$\frac{G_2}{1-\overline{M}_m^2} = -0.003$
Slope gradient	−1.72	$\frac{G_3}{1-\overline{M}_m^2} = 0.042$
Blockage gradient	−0.42	
Isentropic density change	0.93	$\frac{G_5}{1-\overline{M}_m^2} = 0.009$
Radial force	−0.27	
Loss rate	−0.13	
Radial pressure gradient (Total)	−1.36	$\frac{-G_0}{1-\overline{M}_m^2} = -0.040$

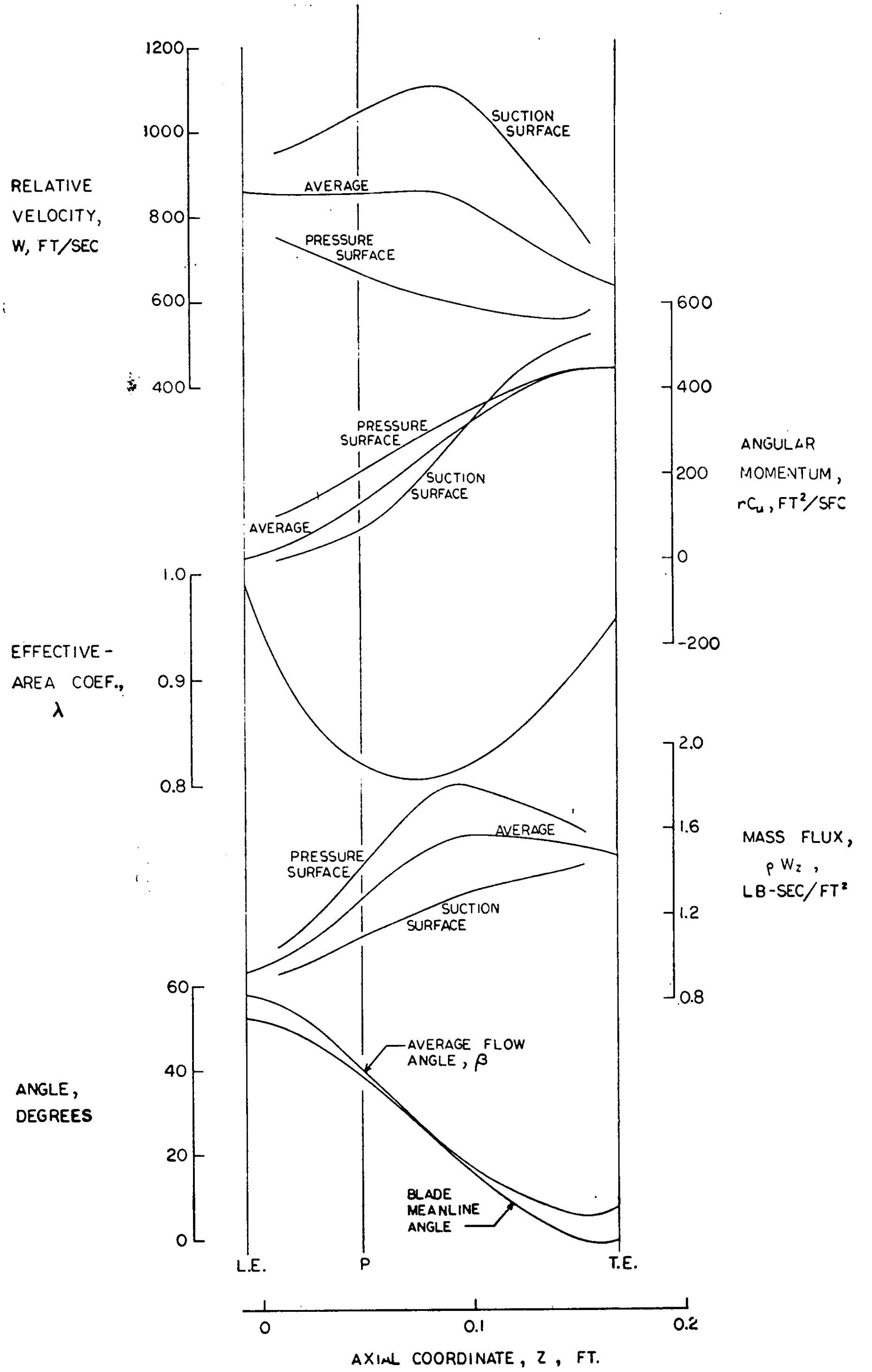

Fig. 10 Variation of properties along hub of example compressor

$$G_2 = -\frac{1}{12}\Big\{(1 - \overline{M}_z^2)[2\overline{W}_z\Delta\kappa\Delta W_z + \bar{\kappa}(\Delta W_z)^2] - \bar{\kappa}\overline{W}_z^2(\Delta M_z)^2 - 2\overline{M}_z\Delta M_z(\overline{W}_z^2\Delta\kappa + 2\bar{\kappa}\overline{W}_z\Delta W_z)\Big\} + \frac{1}{80}\{(\Delta M_z)^2[2\overline{W}_z\Delta\kappa\Delta W_z + \bar{\kappa}(\Delta W_z)^2] + 2\overline{M}_z\Delta M_z\Delta\kappa(\Delta W_z)^2\} \quad (64)$$

where $$\kappa \equiv \frac{D^2r}{Dz^2}$$

$$G_3 = \frac{1}{12r}(\overline{W}_r\Delta W_z\Delta\sigma + \overline{W}_z\Delta W_r\Delta\sigma + \bar{\sigma}\Delta W_r\Delta W_z) \quad (65)$$

$$G_5 = \frac{1}{12}\left[\overline{M}_r\overline{M}_u\Delta\left(\frac{1}{\rho}\right)\Delta\frac{\partial p}{\partial\theta} + \overline{M}_r\overline{\left(\frac{1}{\rho}\right)}\Delta M_u\Delta\frac{\partial p}{\partial\theta} + \overline{M}_r\overline{\frac{\partial p}{\partial\theta}}\Delta M_u\Delta\left(\frac{1}{\rho}\right) + \overline{M}_u\overline{\left(\frac{1}{\rho}\right)}\Delta M_r\Delta\frac{\partial p}{\partial\theta} + \overline{M}_u\overline{\frac{\partial p}{\partial\theta}}\Delta M_r\Delta\left(\frac{1}{\rho}\right) + \overline{\left(\frac{1}{\rho}\right)}\,\overline{\frac{\partial p}{\partial\theta}}\Delta M_r\Delta M_u\right] - \frac{1}{80}\Delta M_r\Delta M_u\Delta\left(\frac{1}{\rho}\right)\Delta\frac{\partial p}{\partial\theta} \quad (66)$$

The G_4-function has been omitted because there seems to be no practical value in calculating it; the variation of heat addition with θ is likely to be highly nonlinear, and heat-addition effects on radial-equilibrium are expected to be small anyway.

It should be noted that each term in the G-functions contains blade-to-blade differences (the Δ-terms) to at least the second degree. Thus the magnitudes of these functions are roughly proportional to the second power of a blade-loading parameter such as lift coefficient. It is of interest to recall that axisymmetric methods are usually justified by resorting to the concept of an infinite number of blades, with the implication that such methods might be poor for blade rows with few blades. The present analysis, however, shows that blade loading is a more meaningful criterion than blade number for the applicability of the axial symmetry approximation.

In the evaluation of G_3, the following alternate formulation for σ will be useful when calculating its value at a blade surface:

$$\sigma = \frac{-r}{\rho W_z}\frac{D(\rho W_z)}{Dz} \tag{67}$$

This was obtained from equation (41) through the use of equations (16), (13), and (5) and the assumption of steady flow. Also, in the evaluation of G_5, equations (2*e*) and (5) can be used to obtain $\partial p/\partial\theta$. In G_0, $\partial p/\partial r$ can be obtained from equation (29).

The G-functions were calculated for point P in Fig. 9, and the results are given on the right side of Table 1. It is seen that they are rather small, and neglect of them seems justified when the loading level is comparable to that of the example. Flows with shock waves are possible exceptions. Also, since each G-function is made up of several terms, there is the possibility that under some circumstances these might all have the same sign and hence accumulate to a significant magnitude. Such a possibility has not been investigated.

Alternative Form for Radial-Equilibrium Equation

So far, we have considered the static pressure as the primary dependent variable to be found from the radial-equilibrium equation. When using the equation to find the variation of flow properties from hub to casing as outlined in the Introduction, the radial distribution of entropy is also known from loss calculations. Knowledge of the pressure and entropy allows calculation of the enthalpy from the equation of state. This then permits calculation of the kinetic energy and hence the velocity from the energy equation, since the stagnation enthalpy is presumed to be known.

In another approach preferred by some workers, the radial-equilibrium equation is used in a form that has the axial or meridional velocity as the primary dependent variable instead of the static pressure. We will now put the equation in that form. For simplicity, the G-functions of the third and fifth sections will be disregarded and the bars above the symbols will be omitted, although it is intended that the symbols still be regarded as representing circumferential averages.

The first and second laws of thermodynamics yield

$$\frac{dp}{\rho} = dh - Tds \tag{68}$$

We eliminate h from equation (68) by means of equation (53) and apply equation (68) along a radial line. This gives

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\partial I}{\partial r} - T\frac{\partial s}{\partial r} - \frac{\partial}{\partial r}\left(\frac{W^2}{2} - \frac{\omega^2 r^2}{2}\right) \tag{69}$$

Using $W^2 = W_u^2 + W_m^2$ and $C_u = W_u + \omega r$, we can manipulate the last term in equation (69) into

$$-\frac{W_u}{r}\frac{\partial(rC_u)}{\partial r} + \frac{C_u^2}{r} - W_m\frac{\partial W_m}{\partial r}$$

Combination of equations (69) with equation (50) and term (61) then yields

$$\frac{\partial I}{\partial r} - T\frac{\partial s}{\partial r} = \frac{M_r^2}{1 - M_m^2}\left(\frac{C_u^2}{r}\right) - \left(\frac{1 - M_z^2}{1 - M_m^2}\right)\frac{D^2 r}{Dz^2}W_z^2 \tag{70}$$

$$+ \frac{W_r W_z}{1 - M_m^2}\left[\frac{\partial(r\tan\varphi)}{r\partial r} + \frac{1}{\lambda}\frac{D\lambda}{Dz} - \frac{Q'}{c_pT} - \left(1 - \frac{p}{P}\right)\frac{\bar{\omega}_z}{l}\right] + \frac{1}{\rho}\left(\frac{p_s - p_p}{\theta_s - \theta_p}\right)\frac{1}{r}\left(\frac{M_rM_u}{1 - M_m^2} - \tan\epsilon_m\right) + \frac{W_u}{r}\frac{\partial(rC_u)}{\partial r} + W_m\frac{\partial W_m}{\partial r} \qquad (70)\ (Cont.)$$

This can be used to determine $\partial W_m/\partial r$.

It is sometimes desirable to use the axisymmetric stream function instead of the radius as the independent variable. This can be accomplished by using

$$\frac{\partial(\ )}{\partial r} = 2\pi r\lambda\rho W_z\left.\frac{\partial(\ )}{\partial\psi}\right|_z \tag{71}$$

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APPENDIX 1

G-Function Derivations

Let q, u, v, and w be functions of θ. We wish to find $G'(q, u, v, w)$, where

$$G'(q, u, v, w) = \overline{(1 - q^2)uvw} - (1 - \bar{q}^2)\bar{u}\bar{v}\bar{w} \tag{72}$$

The bar symbol is defined by equation (32). Let $q(\theta) = \bar{q} + \delta q(\theta)$, $u(\theta) = \bar{u} + \delta u(\theta)$, and so forth, and substitute these into $(1 - q^2)uvw$. The result is then expanded, integrated from θ_p to θ_s, and divided by $(\theta_s - \theta_p)$. Many of the integrals vanish because

$$\int_{\theta_p}^{\theta_s}\delta q d\theta = \int_{\theta_p}^{\theta_s}\delta u d\theta = 0 \tag{73}$$

and so forth. The last term in equation (72) appears and, when it is removed, the remainder is G'. It is necessary to have information about how the variables q, u, v, and w vary with θ in order to evaluate the integrals; it is assumed now that they vary linearly with θ. Thus

$$\delta q = \frac{\Delta q}{\theta_s - \theta_p}(\theta - \bar{\theta}) \tag{74}$$

where

$$\Delta q = q_s - q_p$$

and so forth, and

$$\bar{\theta} = \tfrac{1}{2}(\theta_s + \theta_p)$$

The following integrals are encountered:

$$\frac{1}{(\theta_s - \theta_p)^3} \int_{\theta_p}^{\theta_s} (\theta - \bar{\theta})^2 d\theta = \frac{1}{12}$$

$$\frac{1}{(\theta_s - \theta_p)^5} \int_{\theta_p}^{\theta_s} (\theta - \bar{\theta})^4 d\theta = \frac{1}{80}$$

Integrals vanish that have $\theta - \bar{\theta}$ raised to an odd power for their integrands.

The result is

$$G' = \frac{1}{12} [(1 - \bar{q}^2)(\bar{w}\Delta u\Delta v + \bar{v}\Delta u\Delta w + \bar{u}\Delta v\Delta w) - \bar{u}\bar{v}\bar{w}(\Delta q)^2 - 2\bar{q}\Delta q(\bar{v}\bar{w}\Delta u + \bar{u}\bar{w}\Delta v + \bar{u}\bar{v}\Delta w)] - \frac{1}{80}[(\Delta q)^2(\bar{w}\Delta u\Delta v + \bar{v}\Delta u\Delta w + \bar{u}\Delta v\Delta w) + 2\bar{q}\Delta q\Delta u\Delta v\Delta w] \quad (75)$$

Another type of G-function that is required is

$$G''(q, u, v, w) = \overline{quvw} - \bar{q}\bar{u}\bar{v}\bar{w} \quad (76)$$

An analysis similar to that performed for G' yields

$$G'' = \frac{1}{12} (\bar{q}\bar{u}\Delta v\Delta w + \bar{q}\bar{v}\Delta u\Delta w + \bar{q}\bar{w}\Delta u\Delta v + \bar{u}\bar{v}\Delta q\Delta w + \bar{u}\bar{w}\Delta q\Delta v + \bar{v}\bar{w}\Delta q\Delta u) + \frac{1}{80}\Delta q\Delta u\Delta v\Delta w \quad (77)$$

Other functions that are needed can be obtained from equations (75) and (77). For example, if a triple product is required, we set $\bar{w} = 1$ and $\Delta w = 0$ and use equation (77).

APPENDIX 2

Meridional Stream-Tube Area Variation

It is desired to relate the variation of meridional streamtube area, A, to variations in meridional streamline angle, φ, for the axisymmetric flow field shown in Fig. 4. From the definition of the divergence of a vector field, we have

$$\frac{1}{A}\frac{\partial A}{\partial m} = \nabla \cdot \mathbf{i}_m \quad (78)$$

where $\mathbf{i}_m$ is the unit vector in the meridional flow direction and is therefore given by

$$\mathbf{i}_m = \cos\varphi \mathbf{i}_z + \sin\varphi \mathbf{i}_r \quad (79)$$

Equation (79) is now substituted into equation (78), and use is made of $\nabla \cdot \mathbf{i}_z = 0$ and $\nabla \cdot \mathbf{i}_r = 1/r$. Using also

$$\frac{\partial\varphi}{\partial m} = -\frac{1}{r_m} \quad (80)$$

which is a special case of equation (11) for axisymmetric flow, we can obtain

$$\frac{1}{A}\frac{\partial A}{\partial m} = \cos\varphi \frac{\partial(r\tan\varphi)}{r\partial r} + \frac{\tan\varphi}{r_m} \quad (81)$$

which is the desired result.