Midterm practice : Mesh representations, Euler formula, subdivision surfaces.
Date: november 9, 2018

Problem 1. [5 points]

Suppose that V, E, F is a triangulation of a torus with V vertices, E edges and F faces. Argue that the following relations hold:

1. $F = 2V$.
2. $E = 3V$.
3. Average vertex degree is 6.

Solution:
As the mesh is a triangulation, a double counting argument (on the edges) shows that $3F = 2E$. Now it suffices to use Euler formula in the toroidal case ($g = 1$):

$$V - E + F = 2 - 2g = 0$$

and to substitute $E = \frac{3}{2}F$ in order to obtain $F = 2V$ (the second relation is obtained in a similar way).

Counting the half-edges incident to vertices we have (where $deg(v)$ denotes the degree of vertex v) that the average degree is given by:

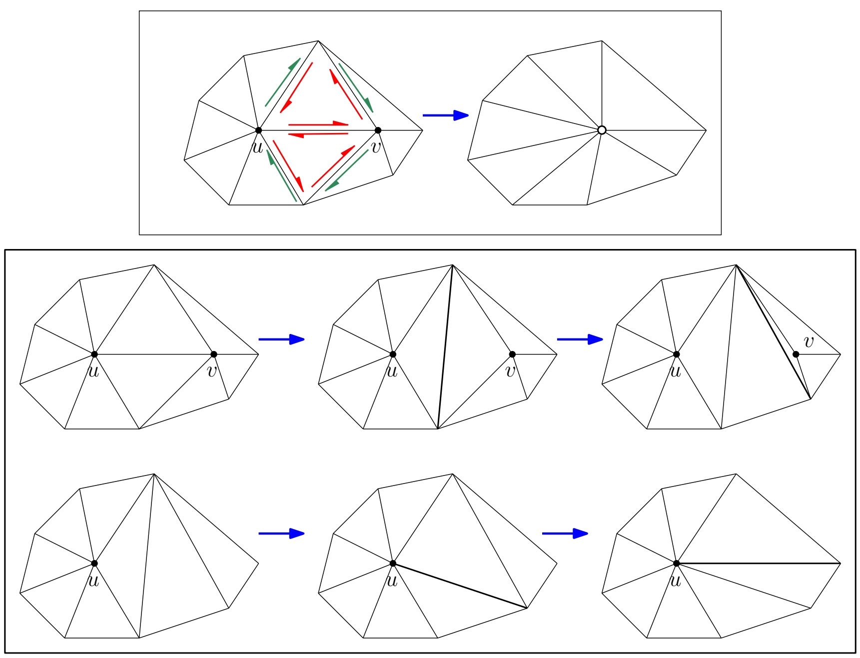
$$\frac{1}{V}\sum_{v\in V} deg(v) = \frac{2E}{V} = 6$$

Problem 2. [5 points]

Given a triangulation, let us consider and edge e whose endpoints are two vertices u and v having degrees d_u and d_v respectively. We are interested in performing the edge contraction operator. We assume to have an half-edge representation of a triangle mesh (the triangulation is simple: no loops, no multiple edges).

- how many references (between half-edges) have to be updated in order to perform the contraction of e?
- give a (combinatorial) configuration where the edge contraction cannot be performed (graphical illustrations are allowed).
- show that the contraction of (u, v) can be achieved by performing a combination of only two operators: *edge flip* and *degree* 3 *vertex removal* (the answer may consist of a sequence of drawings).
- how many edge flips are required in order to perform the edge contraction of (u, v)?

Solution:



Refer to the figure above (first pictures), illustrating which are the half-edges to be removed (red) and the half-edges to be updated. Recall, from TD4, than an edge contraction cannot be performed if the involved edge belongs to a separating triangle.

The pictures above depict how to perform an edge contraction with two series of edge flips combined with a degree 3 vertex removal. Assuming $d_v \leq d_u$, we need $d_v - 3$ edge flips to make v of degree 3. We then remove vertex v, and apply $d_v - 3$ edge flips to obtain the right final configuration.

Observe that we are assuming there are no edges (in the initial triangle mesh) connecting two non consecutive neighbors of v: in that case, some of the edges incident to v could be *not flippable*. But because of planarity, a flippable edge incident to v can always be found.

Problem 3. [5 points]

Let Q be a (closed) quadrangulation of genus g, and T a (closed) triangulation of genus g (g handles and 0 holes), having n vertices, e edges and f faces. Let us apply to Q and T a subdivision scheme C, and denote by Q' and T' respectively the new resulting meshes (after subdivision).

- compute the number of vertices, faces and edges of Q' and T' (with respect to n, e and f) when using Catmull-Clark subdivision;
- using Euler Formula, show that the topology of T' and Q' is preserved (the genus is still g);
- what happen if you change subdivision scheme, applying the Loop subdivision?

Problem 4. [5 points]

Let us consider a planar triangle mesh with n vertices and f faces.

- recall the definition of the *half-edge* and *shared vertex* representations.
- compare the storage performance (memory requirements) of the two data structures above.
- consider the *edge flip* operator: compare the update complexity of this operator for the two representations above.

Solution:

In the half-edge data structures we store $2e = 2(3n - 6)$ half-edges, each containing at least three references: the opposite half-edge, the next half-edge and the incident (destination vertex). Moreover, each vertex stores a reference to an incident (incoming) half-edge. Thus the storage cost (in terms of references) is about $3 \times 6n + n = 19n$ (the cost is higher is we also stores the faces).

In the case of a shared vertex representation we only store, for each triangle face, the indices (or references) of the three incident vertices; so the total cost is $3f = 3(2n - 4)$.

Concerning the flip operation: it can be performed in $O(1)$ time by the half-edge data structure, since only a constant number of references between half-edges require to be updated. While in the shared vertex representation it requires $O(n)$ time just to locate the two triangles which share the given pair of half-edges.