INF 556: Topological Data Analysis **Fall 2017**

Lecture 3-4: Introduction to homology

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Disclaimer :
Some typos and errors may remain. Please report them to `theo.lacombe@polytechnique.edu`. Use these notes with caution, especially during the exam (we decline all responsability linked with the use of these notes during the exam session).

Reminder: These notes are a concise summary of the lectures. They do not intend in any case to substitute to your personnal notes and are just an additional support in order to clarify/insist on some points.

Some references: As a complement for these lecture notes, you can check for two books:

[Munkres] *Elements of Algebraic Topology*, by J.Munkres (1984), especially Chapters 1-2.

[EH] *Introduction to Computational Topology*, by H.Edelsbrunner and J.Harer (you can find an excerpt dealing with Homology theory on the course's website).

Keywords: Homotopy, Homology group, Simplicial complexes.

1 Context - Topological spaces

General idea: Classify (regroup) items which are *equivalent* up to an isomorphism (the class of admissible transformations depends on the topic). Basically, you consider that X should be deemed the same as Y if you can transform, in some sense, X into Y.

Example: classification of surfaces.

Definition 1. *$h : X \to Y$ is a* homeomorphism *if there exists a map $h^{-1} : Y \to X$ such that:*

- *both h and h^{-1} are continuous,*
- $h \circ h^{-1} = \mathrm{id}_Y$,
- $h^{-1} \circ h = \mathrm{id}_X$.

Two spaces X and Y are said to be homeomorphic if there is a homeomorphism $h : X \to Y$.

Remark: One can easily check that *being homeomorphic* defines an equivalence relation, that is:

- X is always homeomorphic to itself,
- if X is homeomorphic to Y, then Y is homeomorphic to X,
- if X is homeomorphic to Y and Y to Z, then X is homeomorphic to Z.

Then we have the following theorem (this is out of the scope of this course and is just a cultural example. See on-line for illustrations and details):

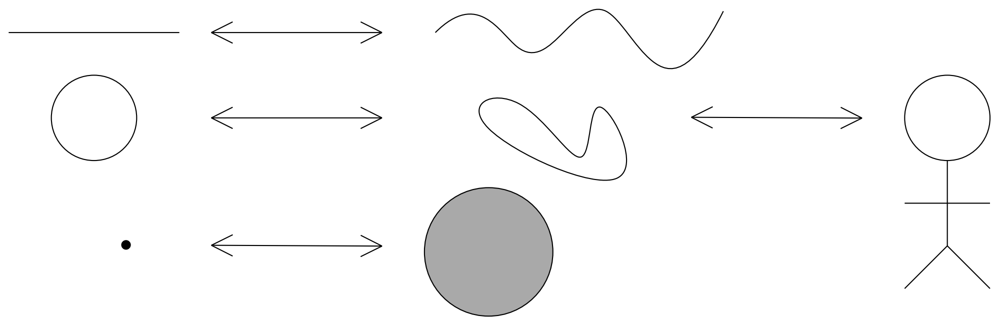

Figure 1: Some example of homotopy equivalent spaces. Left-right-arrows mean "is homotopy equivalent to". Grey means "filled".

Theorem 2 (Classification of surfaces up to homeomorphism - admitted, out of the scope of this course)**.** *Every connected, compact, boundary-free surface is homeomorphic to one of:*

- *the sphere* $\mathbb{S}^2$*,*
- *A connected sum of tori,*
- *A connected sum of projective planes.*

This theorem basically states that, up to homeomorphism, there exists only three different families of compact connected surfaces without boundary. There are weaker notions of equivalence between topological spaces, for instance the following one, which is based on the idea of *continuous deformations* between maps:

Definition 3. $f, g : X \to Y$ *are* homotopic *(denoted* $f \sim g$*) if there is a continuous map* $\varphi : [0,1] \times X \to Y$ *such that* $\varphi(0, \cdot) = f$ *and* $\varphi(1, \cdot) = g$*.*

Definition 4. X *and* Y *are* homotopy equivalent *if there exist continuous maps* $f : X \to Y$ *and* $g : Y \to X$ *such that* $f \circ g \sim id_Y$ *and* $g \circ f \sim id_X$*. Then,* f *(resp.* g*) is said to be a* homotopy equivalence *between* X *and* Y*.*

Proposition 5. X *and* Y *are homeomorphic* $\Rightarrow$ X *and* Y *are homotopy equivalent.*

Proof. Given a homeomorphism $h : X \to Y$ and its inverse $h^{-1} : Y \to X$, take $f = h$ and $g = h^{-1}$. Then, f and g are both continuous, and we have:

$$f \circ g = \mathrm{id}_Y \sim \mathrm{id}_Y$$
$$g \circ f = \mathrm{id}_X \sim \mathrm{id}_X$$

□

2 Intuitive viewpoint on homology

Let X be a topological space. A *path* is a continuous map $[0,1] \to X$. Two points $x, y \in X$ are said to be equivalent in X (denoted $x \sim y$) if there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 6. $\sim$ *is an equivalence relation over* X*.*

Proof. The constant path $\gamma : [0,1] \to \{x\}$ proves the reflexivity of the relation. A path can always be reversed by reparamtrizing the unit segment by $t \mapsto 1-t$, which proves the symmetry of the relation. Finally, paths can be concatenated by letting

$$(\gamma' \cdot \gamma)(t) = \begin{cases} \gamma(2t) & \text{if } t \leq 1/2 \\ \gamma'(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

for any paths $\gamma, \gamma' : [0,1] \to X$ such that $\gamma(1) = \gamma'(0)$, which proves the transitivity of the relation. $\square$

Then, the *path connected components* of X are the equivalence classes of the relation $\sim$.

An alternative formulation is to say that $x, y \in X$ are in the same connected component if there is a path γ such that $\gamma(\partial[0,1]) = \{x, y\}$, where $\partial[0,1]$ denotes the geometric boundary of the segment $[0,1]$, i.e. the 2-point set $\{0,1\}$. Similarly, in higher dimension: a *loop* is a map $\gamma : \mathbb{S}^1 \to X$ that is continuous ($\mathbb{S}^1$ denotes the sphere of dimension 1, i.e. the circle). Two loops γ, γ' are then equivalent if there exists a surface Σ and a map $\nu : \Sigma \to X$ such that $\nu(\partial\Sigma) = \gamma(\mathcal{S}^1) \cup \gamma'(\mathcal{S}^1)$. See Figure 2 for a pictorial view.

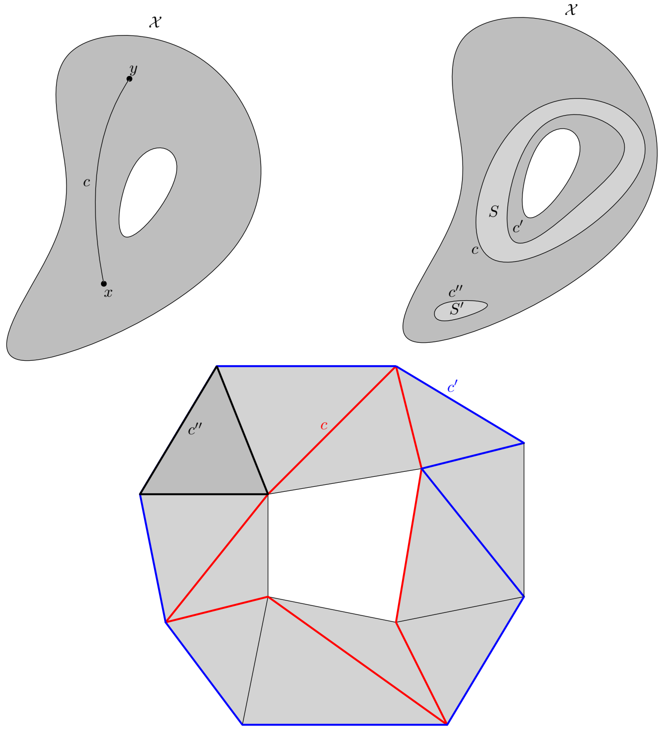

Figure 2: *(left)* Here, points x and y are equivalent in $\mathcal{X}$ in the sense that $\{x\} \cup \{y\}$ is the boundary of a (continuous) path c (included in $\mathcal{X}$). *(right)* Here, path c and c' are equivalent because $c \cup c'$ is the boundary of a continuous surface S included in $\mathcal{X}$. Besides, c'' is *trivial* because $c \cup \emptyset$ is the boundary of a continuous surface in $\mathcal{X}$. *(bottom)* Similar notions exist on a simplicial complex, with a purely combinatorial and algebraic formulation.

More generally:

- a *r-cycle* is a map $\gamma : \Sigma \to X$ where $\dim \Sigma = r$ and $\partial\Sigma = \emptyset$.
- $\gamma \sim \gamma'$ if there exists $\nu : \Omega \to X$ with $\dim \Omega = r+1$ such that $\nu(\partial\Omega) = \gamma(\Sigma) \cup \gamma'(\Sigma')$.

Since such theoretical notions are not handy (and also because in practice, data analysis often starts with a point cloud), we are going to introduce a combinatorial and algebraic formulation that is easier to work with.

3 Simplicial complexes

The general idea is that simplicial complexes extend the standard notion of graph (vertices and edges, that is items with dimension zero and one) by adding some higher dimensional components to it.

Definition 7 (Combinatorial definition of simplicial complexes)**.** *Let V be some finite set (the vertices). A* simplicial complex *on V is a set $K \subseteq \mathcal{P}(V)$ such that: $\forall \tau \subseteq \sigma \subset V, \sigma \in K \Rightarrow \tau \in K$. Here, σ, τ are called* simplices *of K, and τ is a* face *of σ.*

The *dimension* of $\sigma \in K$ is its cardinality minus one: $\dim \sigma := \#\sigma - 1$. Thus, the dimension of a vertex is zero, and an r-dimensional simplex has exactly $r+1$ vertices. The dimension of K is $\dim K = \sup\{\dim \sigma \mid \sigma \in K\}$. We denote by K_r the set of r-dimensional simplices of K. Note that $K_0 = V$, the vertex set.

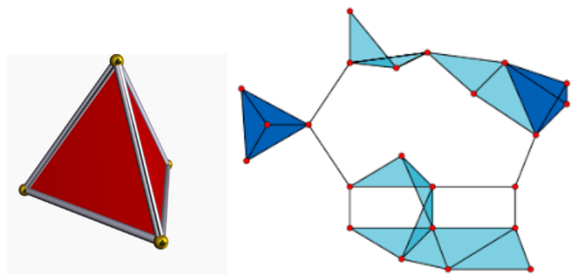

Figure 3: *(left)* A 3-simplex, that is a tetrahedron. *(right)* A 3-simplicial complex. Deep blue represents 3-simplices, teal stands for 2-simplices, and edges are 1-simplices.

To turn a simplicial complex (which is a combinatorial object) into a topological space, we simply 'draw' it (see Figure 3 for an illustration):

Definition 8. *Let K be a simplicial complex. An* immersion *of K into $\mathbb{R}^d$ is simply a map $f : K_0 \to \mathbb{R}^d$, which assigns coordinates to the vertices of K. It induces a map $\bar{f} : K \to \mathcal{P}(\mathbb{R}^d)$, defined by $\sigma = \{v_0, \cdots, v_r\} \mapsto \mathrm{Conv}(f(v_0), \cdots, f(v_r))$, where* Conv *denotes the convex hull.*

Not all 'drawings' are satisfactory: only the ones that do not create extra incidence relations are useful. These are called 'embeddings':

Definition 9. *An immersion $f : K_0 \to \mathbb{R}^d$ is called an* embedding *if for all simplices $\sigma, \tau \in K$ we have $\bar{f}(\sigma) \cap \bar{f}(\tau) = \bar{f}(\sigma \cap \tau)$. This condition implies in particular that the vertices of each simplex have to be affinely independent, and so $d \geq \dim K$.*

Once 'drawn' properly in $\mathbb{R}^d$, a simplicial complex can be equipped with the topology inherited from $\mathbb{R}^d$. The question is then whether this topology depends on how we 'draw' the complex, and the answer is no:

Proposition 10. *For any embeddings $f : K \to \mathbb{R}^d$ and $g : K \to \mathbb{R}^{d'}$, the topological spaces $\bar{f}(K)$ and $\bar{g}(K)$ (equipped with the topologies inherited from their embedding spaces $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively) are homeomorphic.*

Proof. For each simplex $\sigma \in K$, there is a homeomorphism $h_\sigma : \bar{f}(\sigma) \to \bar{g}(\sigma)$ given by barycentric interpolation (which is unique and continuous by the affine independence of the vertices of σ in each 'drawing').

Now, the conditions $\bar{f}(\sigma) \cap \bar{f}(\tau) = \bar{f}(\sigma \cap \tau)$ and $\bar{g}(\sigma) \cap \bar{g}(\tau) = \bar{g}(\sigma \cap \tau)$ imply that the homeomorphisms restrict properly, that is: $h_{\sigma\cap\tau} = h_\sigma|_{\sigma\cap\tau} = h_\tau|_{\sigma\cap\tau}$. As a consequence, we can stitch the local homeomorphisms (defined on the simplices) together to get a global homeomorphism $\bar{f}(K) \to \bar{g}(K)$. $\square$

Thanks to this proposition, we know that the various embeddings of K provide an essentially unique topology on K:

Definition 11. *The* underlying space *(or* topological realization*) of a simplicial complex K, denoted by $|K|$, is the image of K through an arbitrary embedding. It is unique up to homeomorphism.*

Definition 12 (Triangulability)**.** *A topological space X is* triangulable *if there is a simplicial complex K and a homeomorphism $h : X \to |K|$.*

Simplicial maps. Now that we have introduced simplicial complexes as the combinatorial version of topological spaces, we want to do the same with continuous maps. The right concept for this is the one of a simplicial map:

Definition 13. *$f : K \to L$ is* simplicial *if $\exists f_0 : K_0 \to L_0$ such that $\forall \sigma = \{v_0..v_n\} \in K, f(\sigma) = \{f_0(v_0)..f_0(v_n)\}$.*

Note that, in this situation, f is entirely defined by its restriction to the set of vertices, i.e. by f_0. In the following, both functions are identified. The connection between simplicial maps and continuous maps happens at the level of the underlying space:

Lemma 14. *Every simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$, called the* topological realization *of f.*

Proof. Let $|f|(v) = f(v)$ for every vertex $v \in K_0$. Then, extend $|f|$ to the interior of each simplex by barycentric interpolation. This gives a well-defined and continuous map because f maps simplices to simplices. □

Note however that not every continuous map comes from a simplicial map. Nevertheless, it can be 'approximated' in the sense of homotopy:

Theorem 15 (Simplicial Approximation)**.** *Every continuous map $f : |K| \to |L|$ is homotopic to the topological realization $|f'| : |K| \to |L|$ of some simplicial map $f' : K' \to L$ where K' is some simplicial subdivision of K.*

The proof of this result is technical, therefore we refer the reader to [Munkres, §16] for the details. The intuition is that, by subdividing K sufficiently many times, we can guarantee that each simplex of the subdivision K' gets mapped to some small neighborhood of the simplex of K it originates from. Then local homotopies between f and f' can be worked out then stitched together to form a global homotopy. For completeness, we recall the definition of a subdivision, which follows the intuition:

Definition 16. *A subdivision K' of a simplicial complex K is a simplicial complex such that $|K'| = |K|$ and each simplex of K drawn in $|K|$ is the union of the drawings of finitely many simplices of K'.*

4 Simplicial homology

Definition 17 (Orientation)**.** *An* orientation *of a simplex $\sigma = \{v_0, \cdots, v_k\}$ is an ordering of its vertices, i.e. a permutation on its vertex set.*

An oriented simplex is denoted with a pair of square brackets, within which the vertex order matters. For instance, $[v_0, \cdots, v_k]$ and $[v_3, v_2, v_k, \cdots, v_{k-12}]$ are two different orientations of the same simplex. Two orientations $[v_{\pi(0)}, \cdots, v_{\pi(k)}]$ and $[v_{\pi'(0)}, \cdots, v_{\pi'(k)}]$ are equivalent if the permutation $\pi' \circ \pi^{-1}$ is even. For instance, $[v_0, v_1, v_2]$ and $[v_1, v_2, v_0]$ are equivalent, while $[v_0, v_1]$ and $[v_1, v_0]$ are not. This defines two equivalence classes of orientations: the positive one, which contains the identity, and the negative one.

4.1 Chains space

Let K be a (finite) simplicial complex and let $\mathbf{k}$ be a fixed field. Given $r \in \mathbb{N}$, we are interested in the $\mathbf{k}$-linear combinations of r-simplices in K.

Definition 18 (k-chains)**.** *The space of r-chains of K over the field $\mathbf{k}$ is defined by:*

$$C_r(K, \mathbf{k}) := \left\{ \sum_{i=0}^{\#K_r} \alpha_i \sigma_i : \alpha_i \in \mathbf{k}, \sigma_i \in X_k \right\}.$$

$C_r(K, \mathbf{k})$ comes equipped with a $\mathbf{k}$-vector space structure: given any chains $c = \sum_i \alpha_i \sigma_i$ and $c' = \sum_i \beta_i \sigma_i$, and any scalar $\lambda \in \mathbf{k}$:

$$\lambda c + c' := \sum_{i=1}^{\#K_r} \underbrace{(\lambda \alpha_i + \beta_i)}_{\in \mathbf{k}} \sigma_i$$

Remark: Here we assumed K to have finitely many simplices, for simplicity. In the general case where K_r may be infinite, we must restrict the focus to finite $\mathbf{k}$-linear combinations of r-simplices.

4.2 Boundary operator

Definition 19. *Given an (oriented) r-simplex $[v_0..v_k] \in K_r$, we denote by $[v_0.., \widehat{v_j}, ..v_r] \in K_{r-1}$ the $(r-1)$-simplex (which is a face of $[v_0..v_r]$) where the vertex v_j got removed. The* boundary operator ∂_r *is then defined as:*

$$\begin{aligned} \partial_r : C_r(K, \mathbf{k}) &\to C_{r-1}(K, \mathbf{k}) \\ \sigma = [v_0..v_r] &\mapsto \sum_{j=0}^{r} (-1)^j \underbrace{[v_0.., \widehat{v_j}, .., v_r]}_{\in K_{r-1}} \\ (\lambda \sigma + \sigma') &\mapsto \lambda \partial_r \sigma + \partial_r \sigma' \qquad \qquad \textit{(linear extension)} \end{aligned}$$

By convention, we let $\partial_0 = 0$.

The main property of the boundary operator is to be nilpotent, which in plain words means that "the boundary of a boundary is zero":

Proposition 20. $\forall r > 0$, $\partial_r \circ \partial_{r+1} = 0$

Proof. We prove the property on each $(r+1)$-simplex independently. Then, it will extend to all $(r+1)$-chains by linearity. We have:

$$\begin{aligned} \partial_r \circ \partial_{r+1} [v_0, .., v_{r+1}] &= \partial_r \left(\sum_{i=0}^{r+1} (-1)^i [v_0, .., \widehat{v_i}, .., v_{r+1}] \right) = \sum_{i=0}^{r+1} (-1)^i \partial_r [v_0, .., \widehat{v_i}, .., v_{r+1}] \\ &= \sum_{i=0}^{r+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j [v_0, .., \widehat{v_j}, .., \widehat{v_i}, .., v_{r+1}] + \sum_{j=i+1}^{r+1} (-1)^{j-1} [v_0, .., \widehat{v_i}, .., \widehat{v_j}, .., v_{r+1}] \right) \\ &= \sum_{i=0}^{r+1} \sum_{j=0}^{i-1} (-1)^{i+j} [v_0, .., \widehat{v_j}, .., \widehat{v_i}, .., v_{r+1}] + \sum_{i=0}^{r+1} \sum_{j=i+1}^{r+1} (-1)^{i+j-1} [v_0, .., \widehat{v_i}, .., \widehat{v_j}, .., v_{r+1}] \\ &= \sum_{i=0}^{r+1} \sum_{j=0}^{i-1} (-1)^{i+j} [v_0, .., \widehat{v_j}, .., \widehat{v_i}, .., v_{r+1}] + \sum_{i=0}^{r+1} \sum_{j=0}^{i-1} (-1)^{i+j-1} [v_0, .., \widehat{v_j}, .., \widehat{v_i}, .., v_{r+1}] \\ &= \sum_{i=0}^{r+1} \sum_{j=0}^{i-1} (-1)^{i+j} \left([v_0, .., \widehat{v_j}, .., \widehat{v_i}, .., v_{r+1}] - [v_0, .., \widehat{v_j}, .., \widehat{v_i}, .., v_{r+1}] \right) = 0. \end{aligned}$$

□

4.3 Homology groups

As mentioned previously, we want to find *the cycles modulo the boundaries.* Formally, we are interested in the *r-cycles* and *r-boundaries* subgroups of the chain group $C_r(K, \mathbf{k})$:

$$Z_r(K, \mathbf{k}) := \ker \partial_r \quad (r\text{-cycles})$$
$$B_r(K, \mathbf{k}) := \operatorname{im} \partial_{r+1} \quad (r\text{-boundaries}).$$

By Proposition 20, we have $B_r(K, \mathbf{k}) \subseteq Z_r(K, \mathbf{k})$, so we can take the quotient (as a $\mathbf{k}$-vector space):

$$H_k(K, \mathbf{k}) := Z_r(K, \mathbf{k})/B_r(K, \mathbf{k}) = \ker \partial_r / \operatorname{im} \partial_{r+1}. \tag{1}$$

This quotient space is called the *r-th homology group of K over* $\mathbf{k}$. Literally, it represent the space of "cycles modulo boundaries".

4.4 Algorithm to compute homology

Input: A finite simplicial complex K, a field $\mathbf{k}$.

Output: $H_r(K, \mathbf{k})$, $\forall r \geqslant 0$.

Recall that $H_r(K, \mathbf{k}) \simeq \mathbf{k}^{\beta_r}$, so we basically want to find $\beta_r = \dim(H_r(K, \mathbf{k})) = \underbrace{\dim(Z_r)}_{\dim(\ker(\partial_r))} - \underbrace{\dim(B_r)}_{\operatorname{rk}(\partial_{r+1})}$.

If we represent the boundary operator ∂_r for each $r \in \mathbb{N}$ in matrix form (using the simplices as a basis for the chain groups):

$$\begin{array}{c} \\ \nu_1 \\ \vdots \\ \nu_{\#K_{r-1}} \end{array}
\begin{array}{c} \sigma_1 \quad \cdots \quad \sigma_{\#K_r} \\ \left[\begin{array}{ccc} & & \\ & M_r & \\ & & \end{array}\right] \end{array}$$

then we are left with the computation of the rank and nullity of each of these boundary matrices M_r. The rank-nullity theorem gives the following formula:

$$\beta_r = \#K_r - \operatorname{rank} M_r - \operatorname{rank} M_{r+1}.$$

Thus, we are left with only rank computations.

Rank computations. There are different approaches to compute ranks. The one that is easiest to implement is Gaussian elimination, sketched in Figure 4. Despite its poor worst-case complexity ($\mathcal{O}(nm^2)$, where n, m are respectively the numbers of rows and columns in the matrix), it is generally efficient in practice (with a near-linear running time) because the boundary matrix M_r remains sparse throughout the elimination process.

For the sake of the algorithm we introduce the following function that returns the row index of the lowest non-zero entry in a given column j of the matrix M_r:

$$\operatorname{low}(j) = \begin{cases} 0 & \text{if } M_r[i,j] = 0 \ \forall i \\ \max\{i \mid M_r[i,j] \neq 0\} & \text{otherwise} \end{cases}$$

$$\begin{bmatrix}1&1&0\\ \boxed{1}&0&1\\ 0&\boxed{1}&\boxed{1}\end{bmatrix} \xrightarrow{c_3 \leftarrow c_3 - c_2} \begin{bmatrix}1&1&1\\ \boxed{1}&0&\boxed{1}\\ 0&\boxed{1}&0\end{bmatrix} \xrightarrow{c_3 \leftarrow c_3 - c_1} \begin{bmatrix}1&1&0\\ \boxed{1}&0&0\\ 0&\boxed{1}&0\end{bmatrix}$$

Figure 4: Sketch of Gaussian elimination

Then, the details of the elimination process are given in Algorithm 1. Note that we present a lazy version of Gaussian elimination that is sufficient for our purposes. It reduces the matrix M_r until the low function becomes injective on the indices of non-zero columns. Up to a permutation of the columns, this is like reducing the matrix to *column-echelon form*. Upon termination, the rank of M_r is given by the number of non-zero columns.

Algorithm 1 Calculates the rank of matrix M_r using Gaussian elimination

for $j = 1$ to $\#K_r$ **do**
  while $\exists i < j$ s.t. $\mathrm{low}(i) = \mathrm{low}(j) \neq 0$ **do**
    $c_j \leftarrow c_j - \frac{M_r[j,\mathrm{low}(i)]}{M_r[i,\mathrm{low}(i)]} c_i$ // c_l *denotes the l-th column vector of the matrix*
  end while
end for
return $\#\{j \mid \mathrm{low}(j) \neq 0\}$

Proposition 21. *The algorithm terminates.*

Proof. The tricky part is the analysis of the inner **while** loop. Its invariant is that each execution of the loop decreases strictly the row index of the lowest non-zero entry of the currently reduced column. Thus, the number of iterations of the inner **while** loop per column is bounded above by the number of rows and therefore finite. □

4.5 Morphisms

The field $\mathbf{k}$ is fixed. We have an operator on spaces $H_r : K \mapsto H_r(K, \mathbf{k})$, which we want to extend to maps as well: $(f : K \to L) \mapsto H_r(f, \mathbf{k})$.

At the level of chains, a simplicial map $f : K \to L$ induces a *chain map* $f_\# : C_r(K, \mathbf{k}) \to C_r(L, \mathbf{k})$ between the chain spaces of K and L. More precisely, $f_\#$ is defined on each oriented simplex $\sigma = [v_0, .., v_r]$ as follows:

$$f_\#(\sigma) = \begin{cases} [f(v_0), .., f(v_r)] & \text{if } \dim\{f(v_0), .., f(v_r)\} = r \\ 0 & \text{otherwise} \end{cases}$$

Then it is extended to the whole space of chains by linearity:

$$\underbrace{f_\#\left(\sum_i \alpha_i \sigma_i\right)}_{\in C_r(K)} := \sum_i \alpha_i \underbrace{f_\#(\sigma_i)}_{\in C_r(L)}.$$

Proposition 22. *The chain map $f_\#$ commutes with the boundary operator, that is:*

$$\forall r \in \mathbb{N}, \quad f_\# \circ \partial_r = \partial_r \circ f_\#.$$

Proof. We prove the equality on every simplex. Then, it will hold on every chain by linearity. Given

$\sigma = [v_0, .., v_r] \in C_r(K)$, we have:

$$\begin{aligned}
f_\# \circ \partial_r(\sigma) &= f_\# \left(\sum_{i=0}^{r} (-1)^i [v_0, .., \widehat{v_i}, .., v_r] \right) \\
&= \sum_{i=0}^{r} (-1)^i \underbrace{f_\# \left([v_0 .., \widehat{v_i}, .., v_r] \right)}_{= \begin{cases} [f(v_0) .., \widehat{f(v_i)}, .., f(v_r)] & \text{if } \dim\{f(v_0) .., \widehat{f(v_i)}, .., f(v_r)\} = r-1 \\ 0 & \text{otherwise} \end{cases}}
\end{aligned}$$

There are 3 cases:

Case $\dim\{f(v_0), .., f(v_r)\} = r$:

This implies that $\dim\{f(v_0) .., \widehat{f(v_i)}, .., f(v_r)\}) = r-1$ for every i. It follows that

$$\begin{aligned}
f_\# \circ \partial_r(\sigma) &= \sum_{i=0}^{r} (-1)^i [f(v_0) .., \widehat{f(v_i)}, .., f(v_r)] \\
&= \partial_r [f(v_0), .., f(v_r)] = \partial_r \circ f_\#(\sigma).
\end{aligned}$$

Case $\dim\{f(v_0), .., f(v_r)\} = r-1$:

This implies that there is a unique pair of indices $0 \le s < t \le r$ such that $f(v_s) = f(v_t)$. As a result:

$$\dim\{f(v_0), .., \widehat{f(v_i)}, .., f(v_r)\} = \begin{cases} r-1 & \text{if } i \in \{s, t\} \\ r-2 & \text{otherwise} \end{cases}$$

It follows that

$$\begin{aligned}
f_\# \circ \partial_r(\sigma) &= (-1)^s [f(v_0) .., \widehat{f(v_s)}, .., f(v_r)] + (-1)^t [f(v_0) .., \widehat{f(v_t)}, .., f(v_r)] \\
&= \left((-1)^s (-1)^{t+1-s} + (-1)^t \right) [f(v_0) .., \widehat{f(v_t)}, .., f(v_r)] \\
&= 0 = \partial_r \circ f_\#(\sigma).
\end{aligned}$$

Case $\dim\{f(v_0), .., f(v_r)\} \le r-2$:

This implies that $\dim\{f(v_0) .., \widehat{f(v_i)}, .., f(v_r)\}) = r-2$ for every i. Hence,

$$f_\# \circ \partial_r(\sigma) = 0 = \partial_r \circ f_\#(\sigma).$$

$\square$

Proposition 22 implies that we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_r(K) \arrow[r, "\partial_r"] \arrow[d, "f_\#"] & C_{r-1}(K) \arrow[r, "\partial_{r-1}"] \arrow[d, "f_\#"] & \cdots \arrow[r, "\partial_1"] & C_0(K) \arrow[r, "\partial_0"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r] & C_r(L) \arrow[r, "\partial_r"] & C_{r-1}(L) \arrow[r, "\partial_{r-1}"] & \cdots \arrow[r, "\partial_1"] & C_0(L) \arrow[r, "\partial_0"] & 0
\end{tikzcd}$$

In particular, cycles are mapped to cycles, and boundaries to boundaries, through $f_\#$. As a result, $f_\#$ goes to the quotient and induces a (linear) morphism $f_* : H_r(K, \mathbf{k}) \to H_r(L, \mathbf{k}), \forall r$.

Proposition 23 (Functoriality)**.** *We have the following properties:*

(i) *Given* $K \xrightarrow{f} L \xrightarrow{g} M$, $(g \circ f)_* = g_* \circ f_*$.

(ii) *Given* K, $(id_K)_* = id_{H_r(K,\mathbf{k})}$

Sketch of proof. (i) follows from the fact that chain maps compose naturally, i.e. $(g \circ f)_\# = g_\# \circ f_\#$. This equality is easily seen from the definition on a single simplex, then true on all chains by linearity. It implies that the induced maps in homology satisfy $(g \circ f)_* = g_* \circ f_*$.

(ii) follows from the fact that $\mathrm{id}_\# = \mathrm{id}_{C_r(K,\mathbf{k})}$, which, again, is easily seen from the definition. Then the induced map in homology is also the identity. □

5 From simplicial complexes to topological spaces

Note. Proofs in this section are technical and not essential to the course, therefore they are omitted in these notes. The interested reader can refer to [Hatcher] for the details.

How to extend homology from simplicial complexes to triangulable spaces? Given a triangulable space X, can we define its homology in a way that is consistent across all its possible triangulations? The following result gives a positive answer:

Proposition 24. *Given a triangulable space* X*, for any two triangulations* K, L *of* X *the homology groups* $H_r(K,\mathbf{k})$ *and* $H_r(L,\mathbf{k})$ *are isomorphic as vector spaces.*

As a consequence, we can define $H_r(X,\mathbf{k})$ to be $H_r(K,\mathbf{k})$ for any arbitrary triangulation K of X. Similarly, given a continuous map $f : X \to Y$ between triangulable spaces, we can take arbitrary triangulations K, L of X, Y respectively, then compose f with the respective homeomorphisms to get a continuous map $g : |K| \to |L|$. In other words, we define g through the following commutative diagram, where the vertical arrows are homeomorphisms:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\simeq"'] & Y \arrow[d, "\simeq"] \\ {|K|} \arrow[r, "g"'] & {|L|} \end{tikzcd} \tag{2}$$

Note that g itself may not come from a simplicial map $K \to L$. However, by the Simplicial Approximation Theorem 15, we can 'approximate' g by some simplicial map $g' : K' \to L$ where K' is some subdivision of K. Then, we can define $g_* : H_r(|K|,\mathbf{k}) \to H_r(|L|,\mathbf{k})$ through the following commutative diagram, where the vertical arrows are isomorphisms of vector spaces:

$$\begin{tikzcd} H_r(|K'|,\mathbf{k}) \arrow[r, equal] \arrow[d, "\simeq"'] & H_r(|K|,\mathbf{k}) \arrow[r, "g_*"] & H_r(|L|,\mathbf{k}) \arrow[d, "\simeq"] \\ H_r(K',\mathbf{k}) \arrow[rr, "g'_*"] & & H_r(L,\mathbf{k}) \end{tikzcd} \tag{3}$$

Now we can extend the process to $f : X \to Y$ itself by observing that both K and K' are triangulations of X, so they have isomorphic homology groups by Proposition 24. More precisely, we define f_* through the following commutative diagram, where the vertical arrows are isomorphisms of vector spaces:

$$\begin{tikzcd} H_r(X,\mathbf{k}) \arrow[r, "f_*"] \arrow[d, "\simeq"'] & H_r(Y,\mathbf{k}) \arrow[d, "\simeq"] \\ H_r(|K|,\mathbf{k}) \arrow[r, "g_*"] & H_r(|L|,\mathbf{k}) \end{tikzcd} \tag{4}$$

Of course, this definition of f_* is dependent on the choices of triangulations K, L on the one hand, and of subdivision K' on the other hand. However, one can prove that the resulting morphism f_* is actually independent of these choices. The key ingredient is that induced linear maps in homology are invariant under homotopies:

Proposition 25. *For any homotopic maps $f, g : X \to Y$ between triangulable spaces, one has $f_* = g_*$.*

We refer the reader to [Munkres, Chapter 2] for the details of the proofs, which once again are technical.

Conclusion: homology groups of triangulable spaces, and morphisms between them, are uniquely defined. Moreover, morphisms are invariant under homotopy (Prop. 25), and they also satisfy the functoriality axioms of Proposition 23 (immediate from the construction and invariance properties).

Here is a simple application of these results that justifies the use of homology groups as topological invariants (you can see this as a warm-up exercise, therefore we give the details of the proof):

Corollary 26. *If X, Y are homotopy equivalent, then $H_r(X, \mathbf{k})$ and $H_r(Y, \mathbf{k})$ are isomorphic as vector spaces.*

Proof. By definition of homotopy equivalence, there are two functions $f : X \to Y$ and $g : Y \to X$ s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$, where $\sim$ denotes homotopy between maps. By functoriality and homotopy invariance, we have:

$$f_* \circ g_* = (f \circ g)_* = (id_X)_* = id_{H_r(X,\mathbf{k})}$$
$$g_* \circ f_* = (g \circ f)_* = (id_Y)_* = id_{H_r(Y,\mathbf{k})}$$

As a consequence, f_* and g_* are mutual inverses and therefore bijective. It follows that $H_r(X, \mathbf{k})$ and $H_r(Y, \mathbf{k})$ are isomorphic. $\square$

Note that homology does not completely characterize the topology of a space in general (even a triangulable one). More precisely, one can find topological spaces X, Y with $H_r(X, \mathbf{k}) \simeq H_r(Y, \mathbf{k})$ for all $r \in \mathbb{N}$ and any field $\mathbf{k}$, but such that X, Y are not homotopy equivalent. The *Poincaré homology sphere* is such an example: it has the same homology as the sphere $\mathcal{S}^3$, but is not homotopy equivalent to it. Its construction is illustrated in Figure 5.

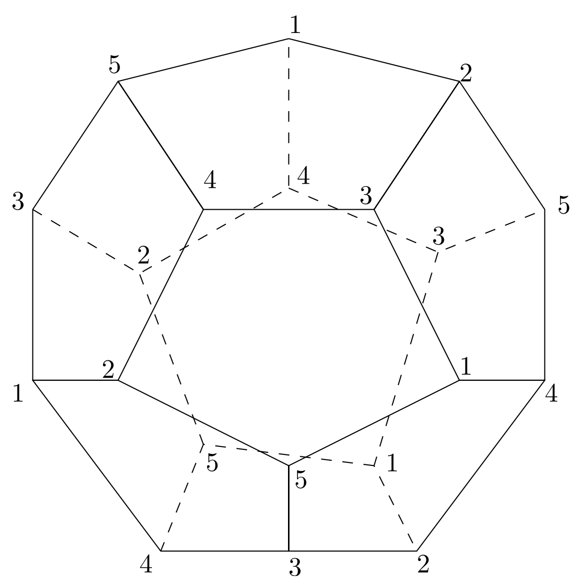

Figure 5: The Poincaré sphere is obtained by gluing the opposite faces of a filled dodecahedron according to the labels shown in the picture.