# Topological Persistence (in a nutshell)

$X$ topological space

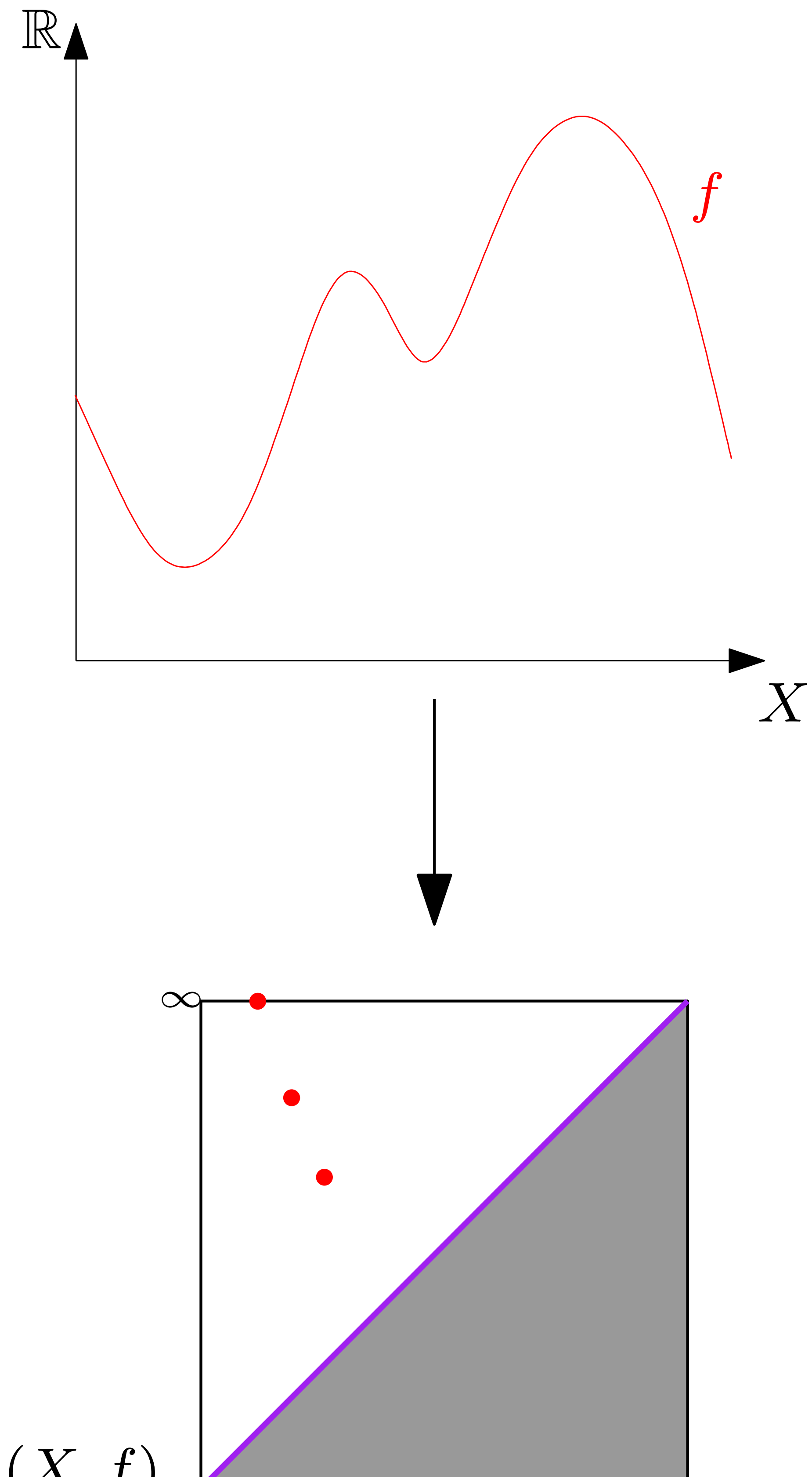
$f : X \to \mathbb{R}$

↓

persistence

↓

$\mathrm{Dg}\, f$




signature: *persistence diagram*

encodes the topological structure of the pair $(X, f)$

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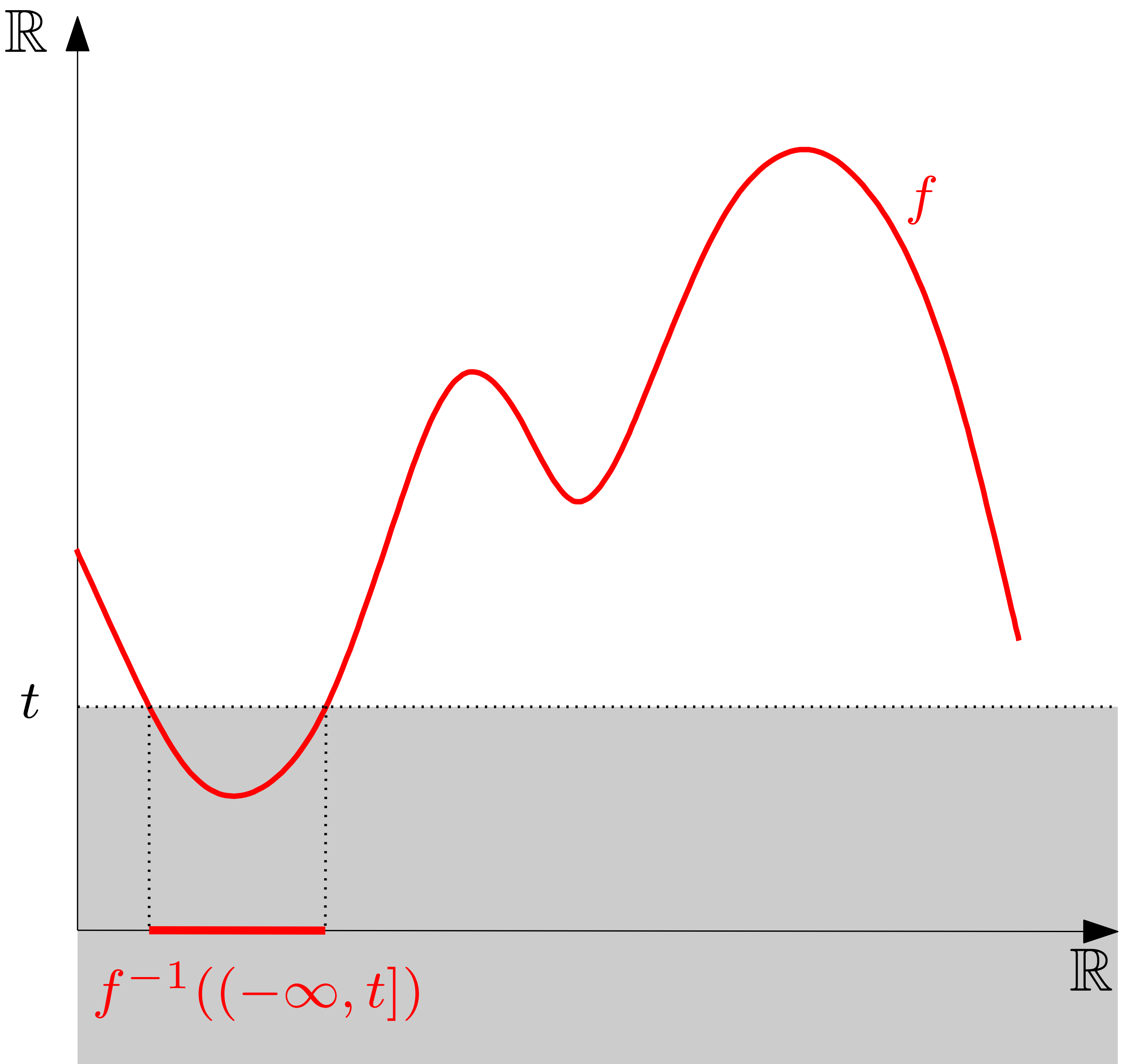
Inside the black box:

- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty,t])$ for $t$ ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



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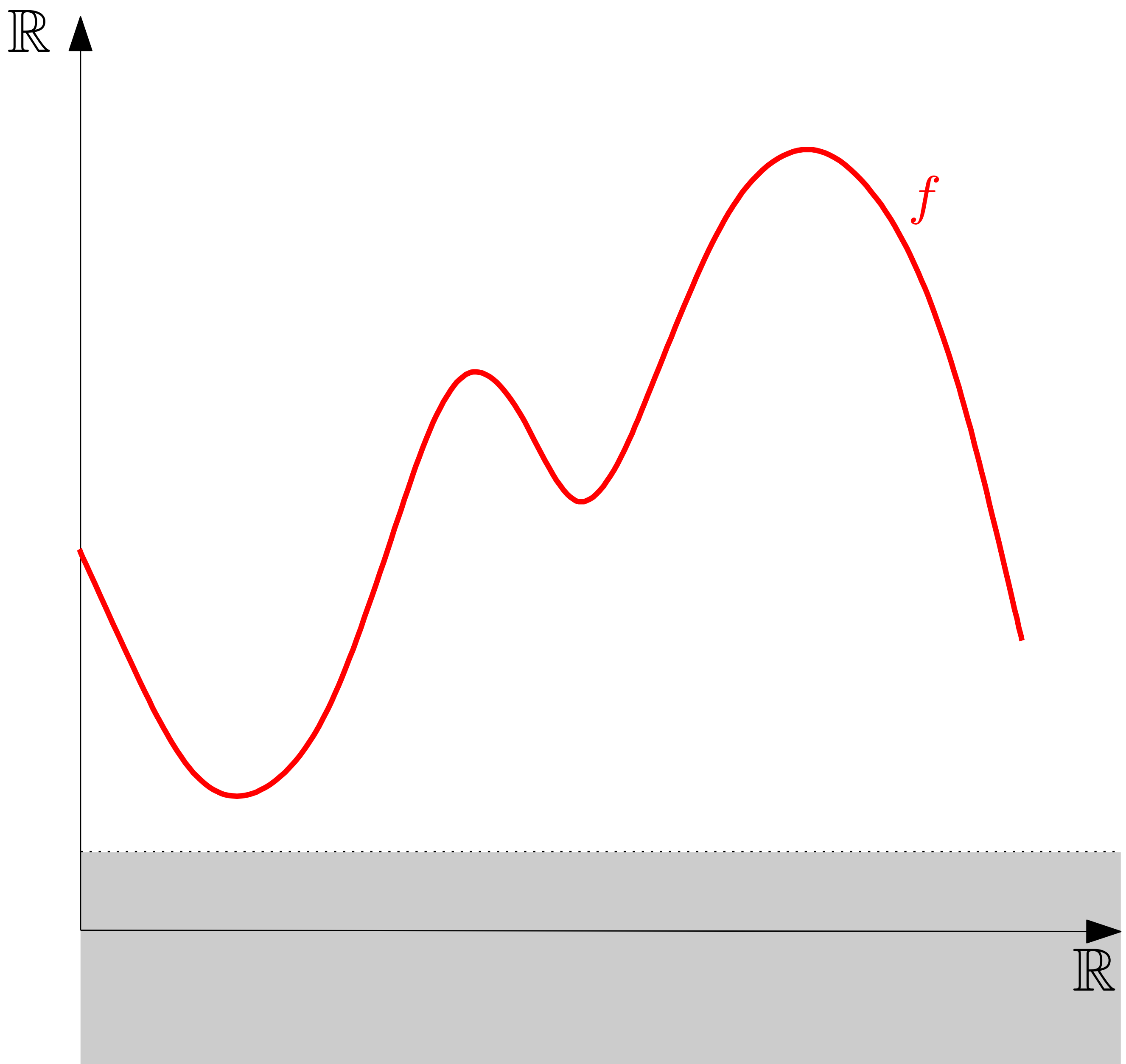

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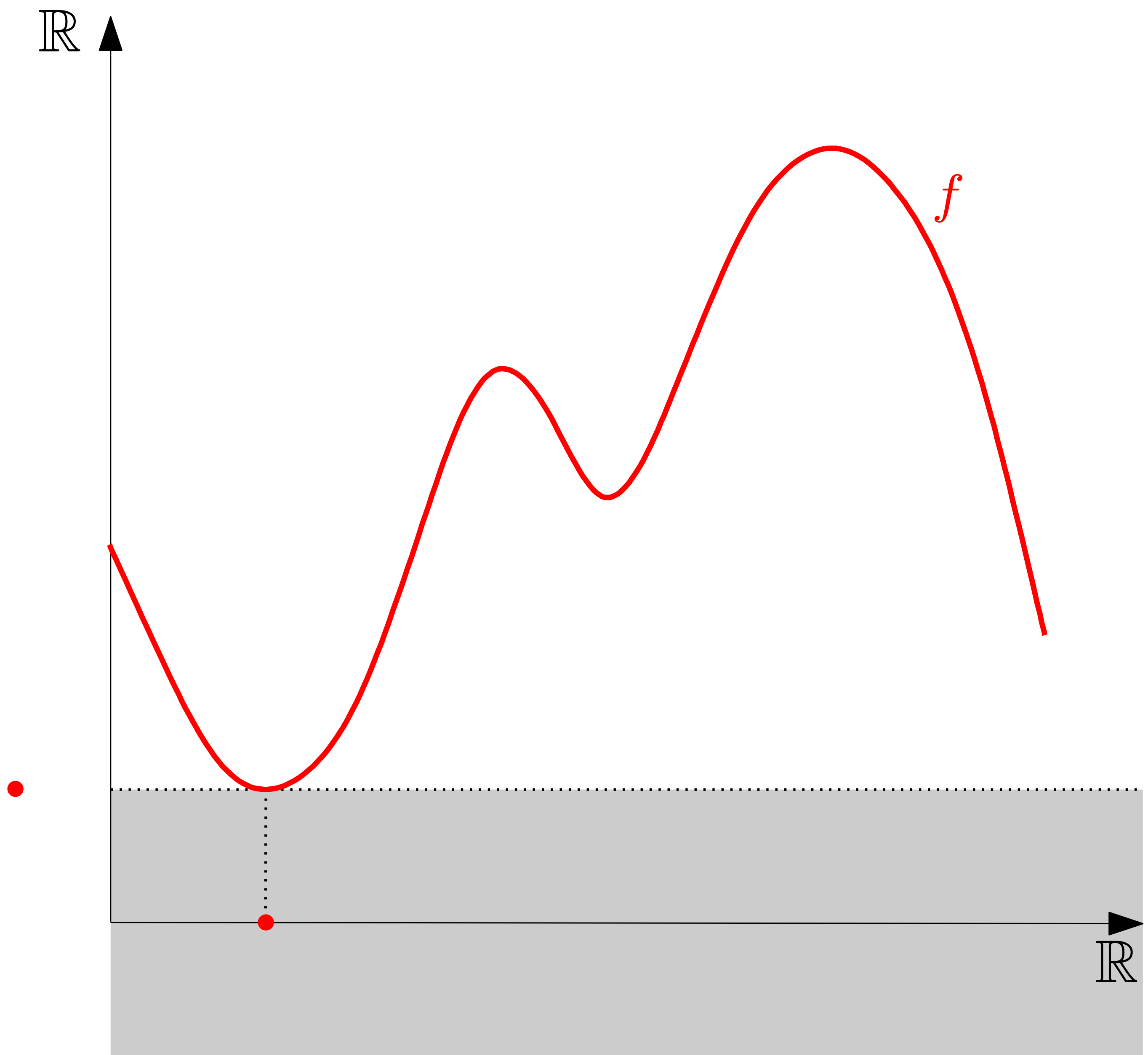

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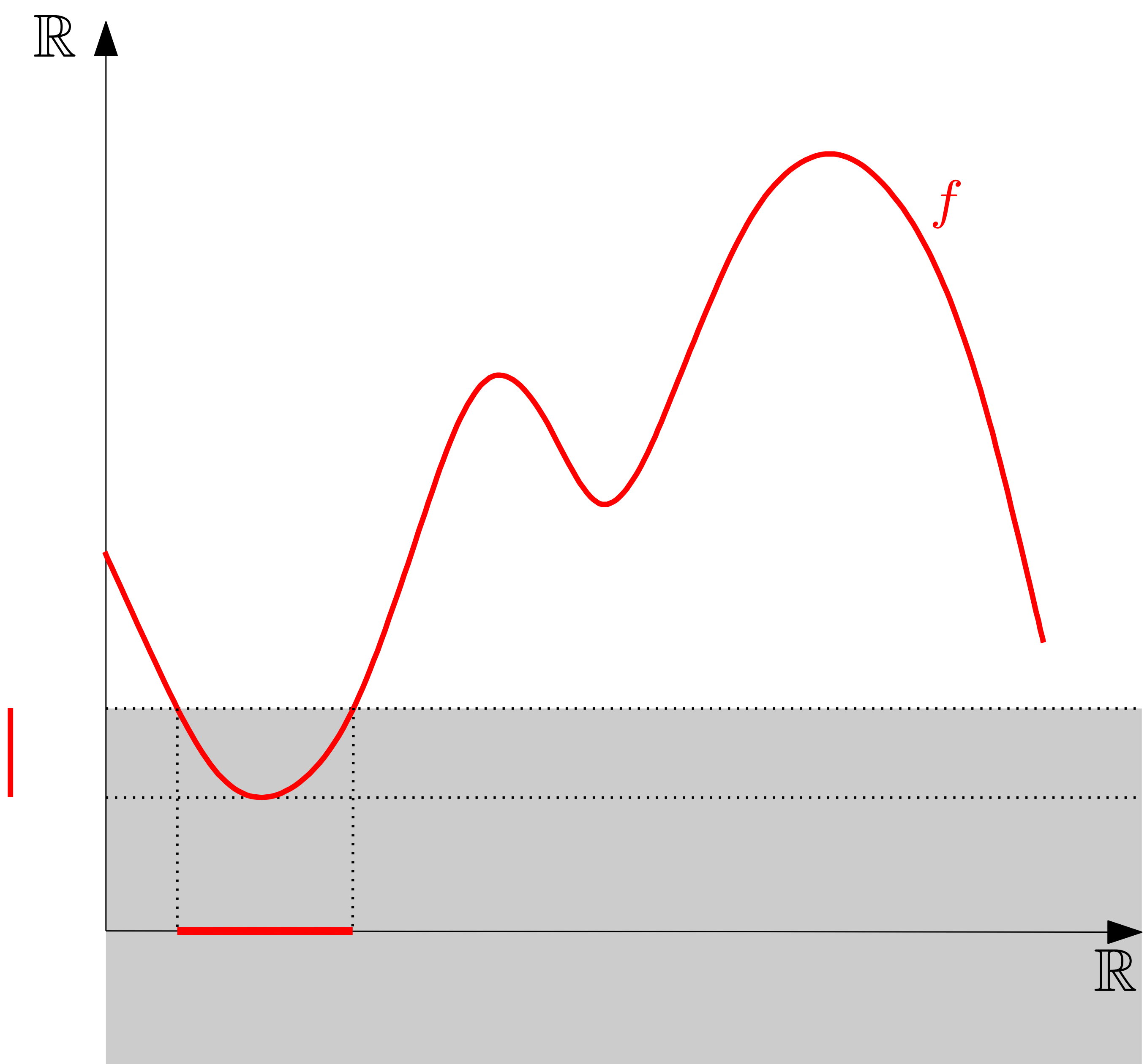

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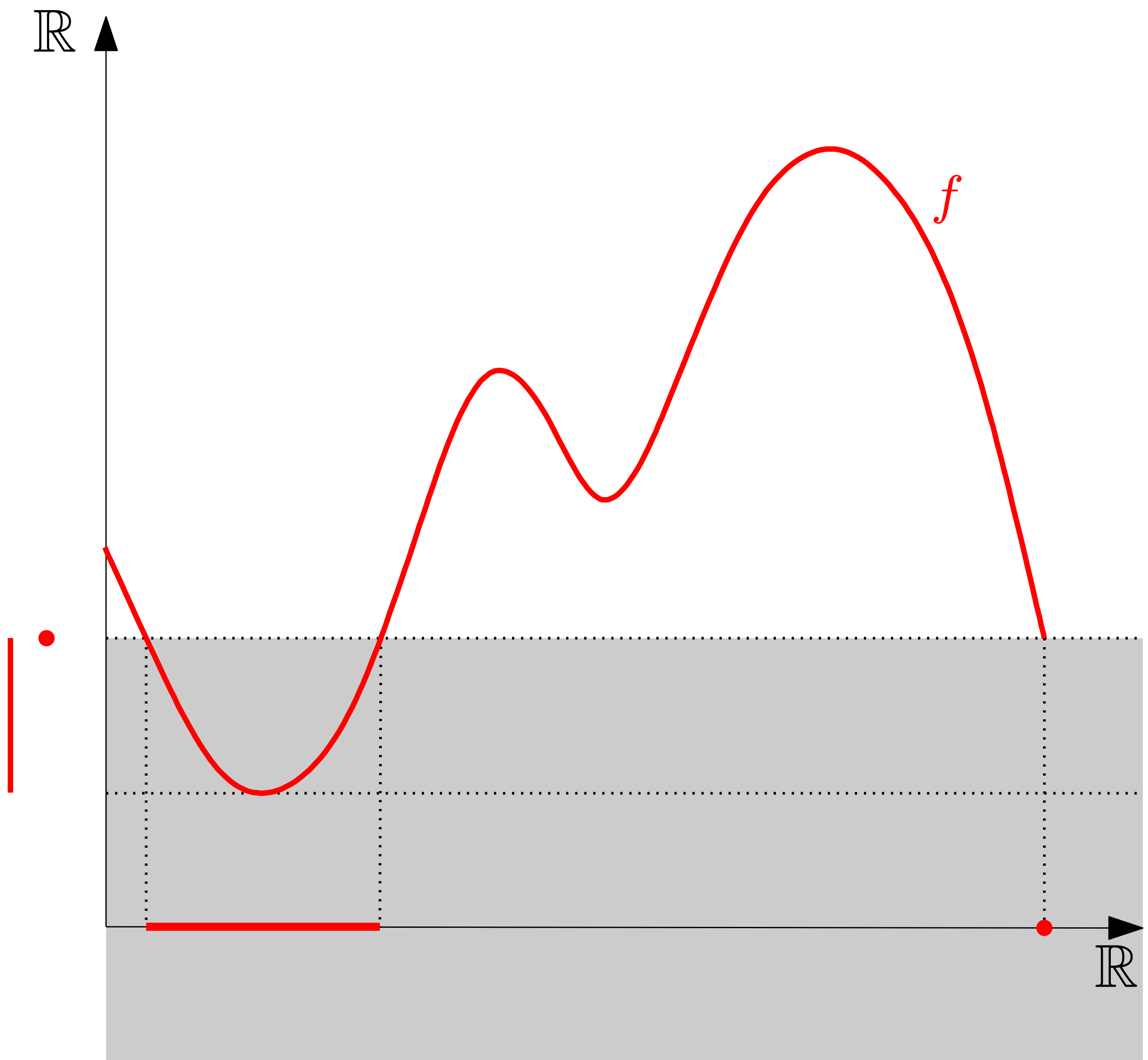

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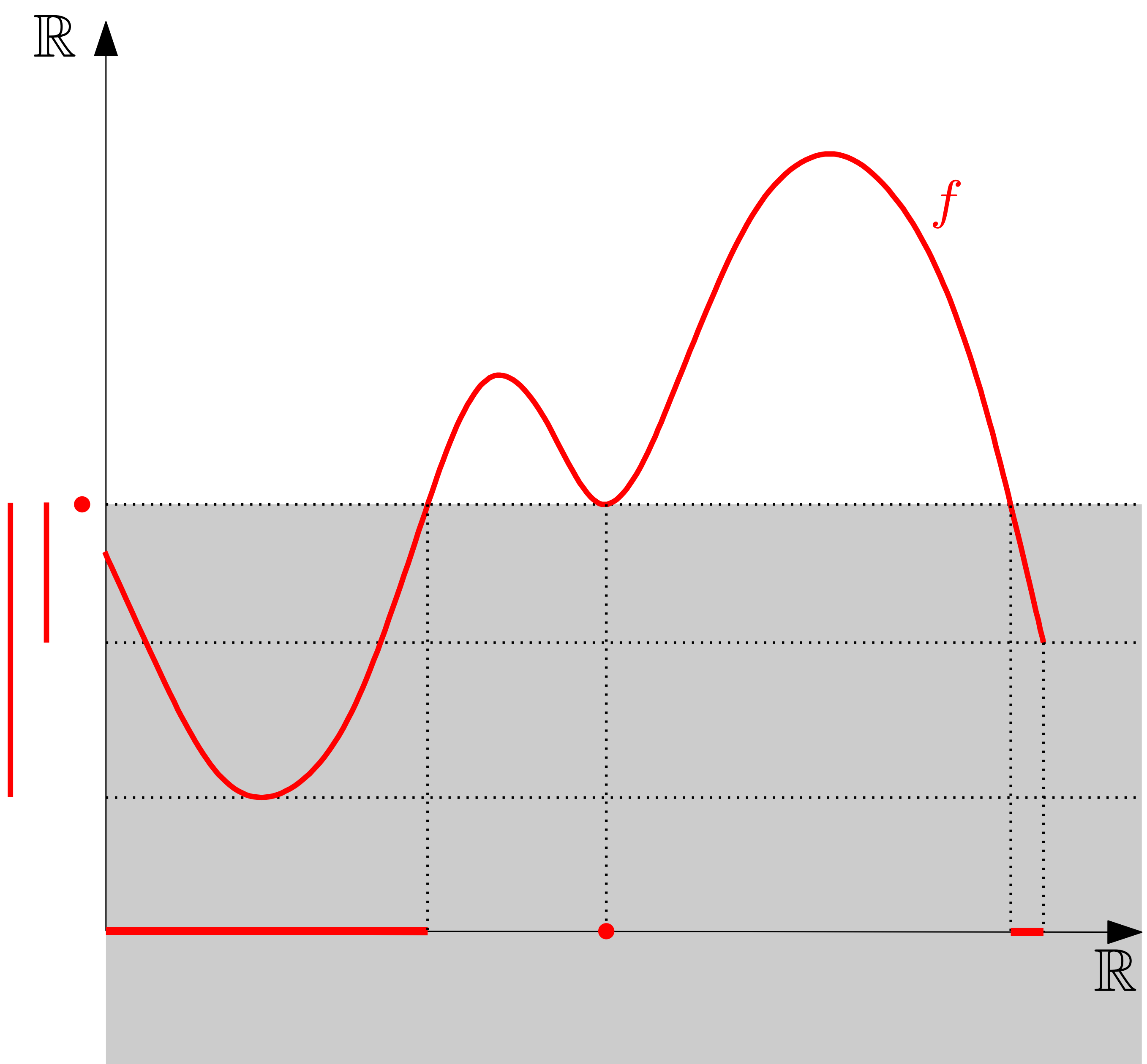

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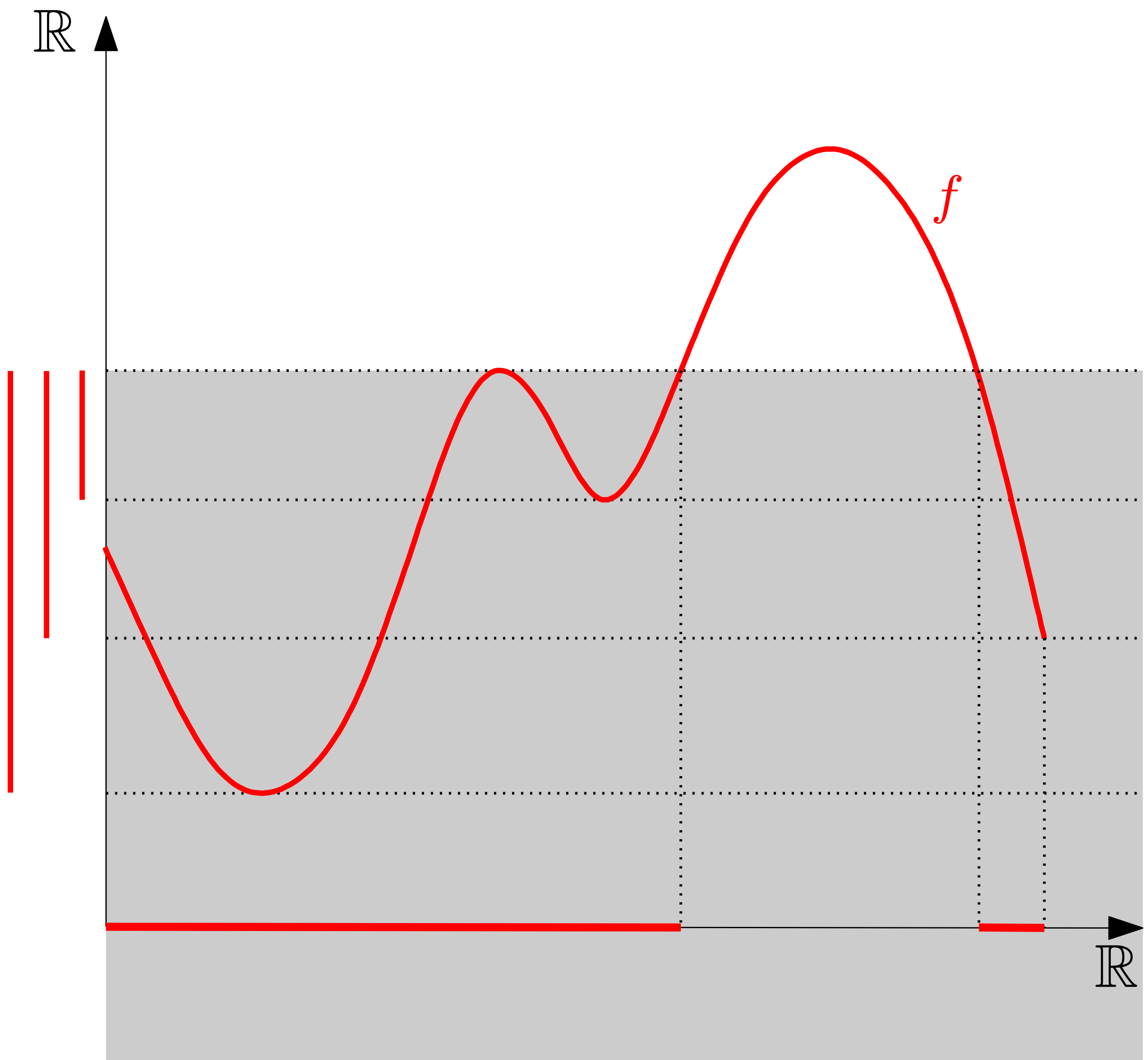

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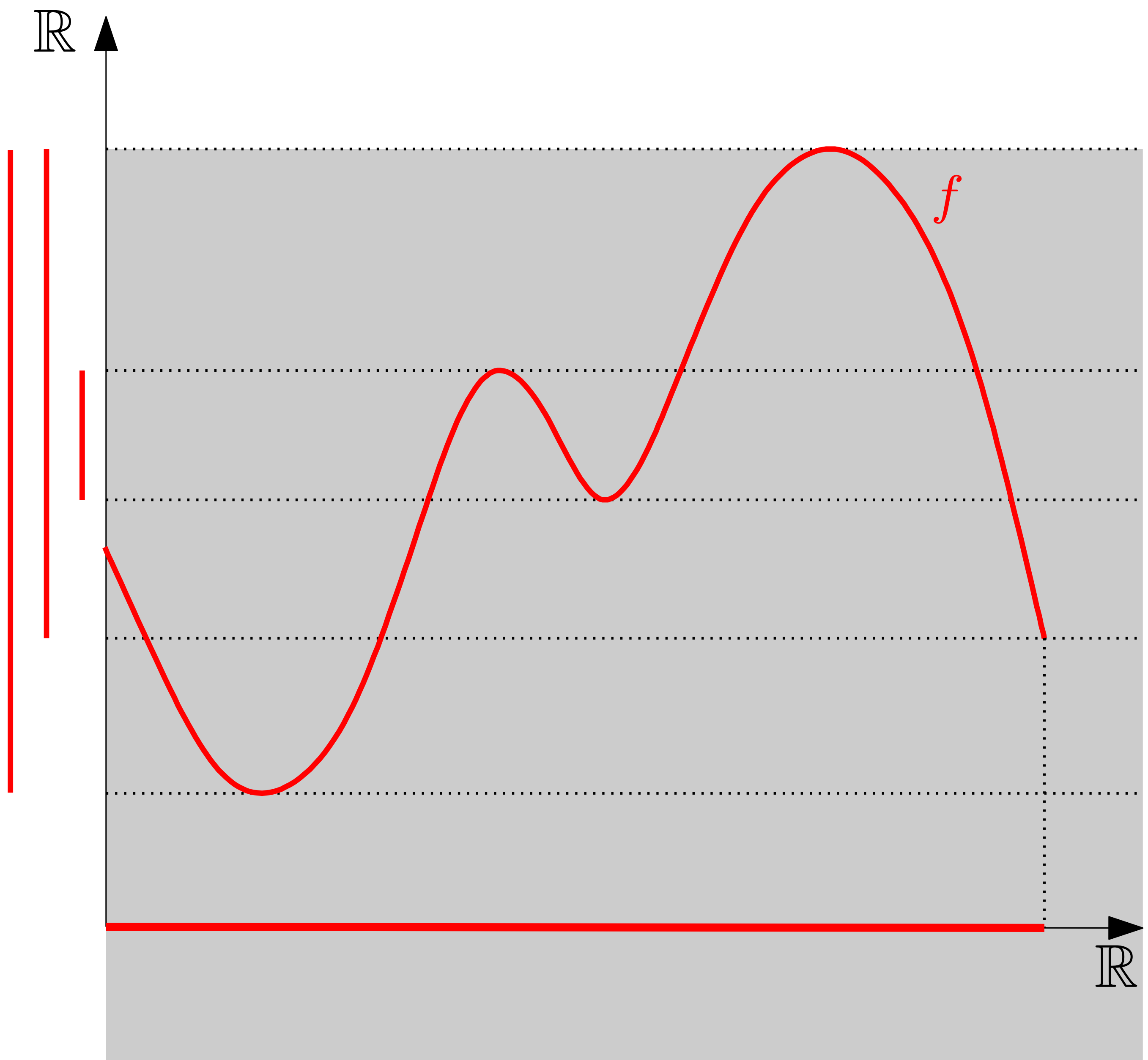

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- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for $t$ ranging over $\mathbb{R}$
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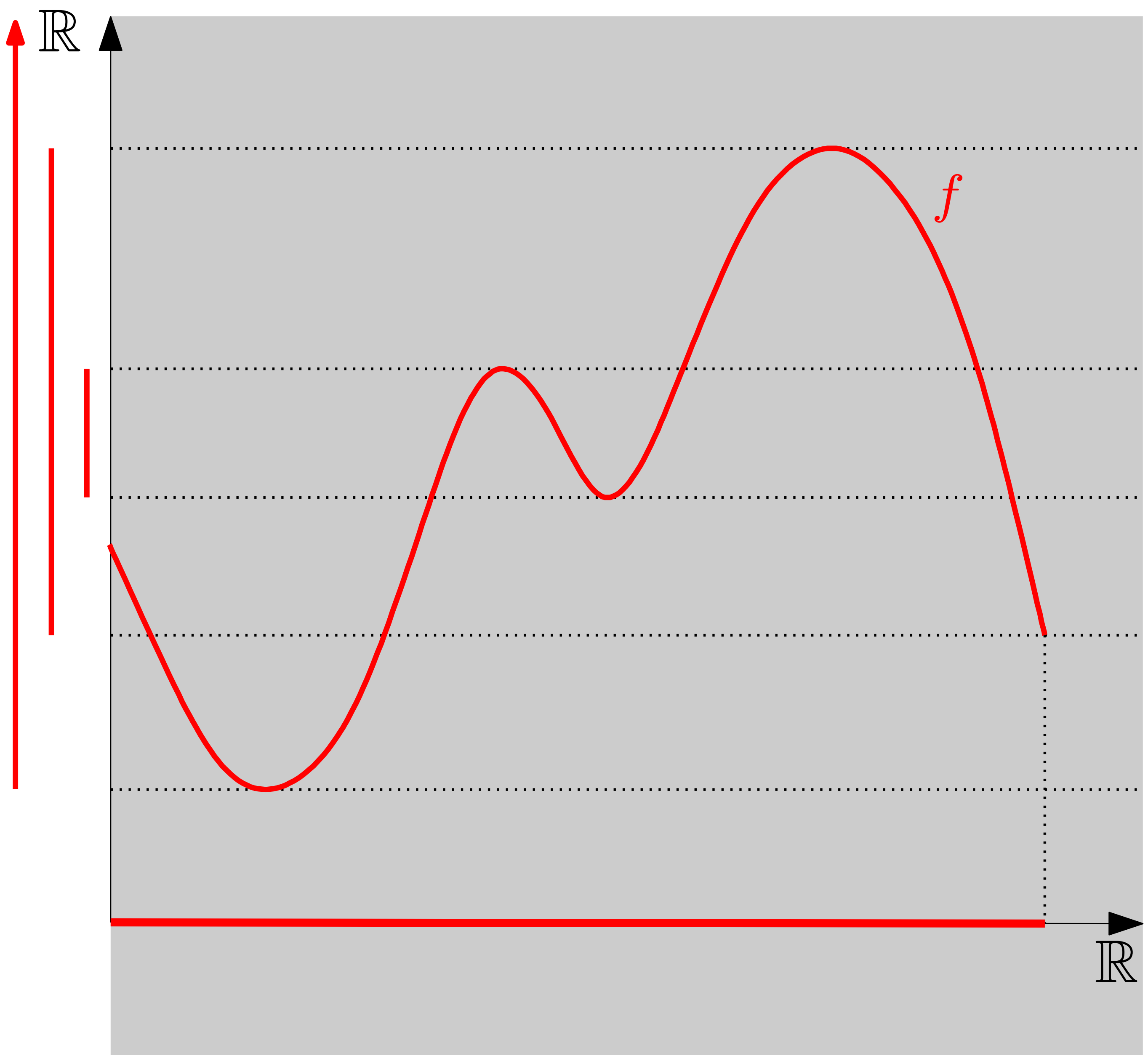

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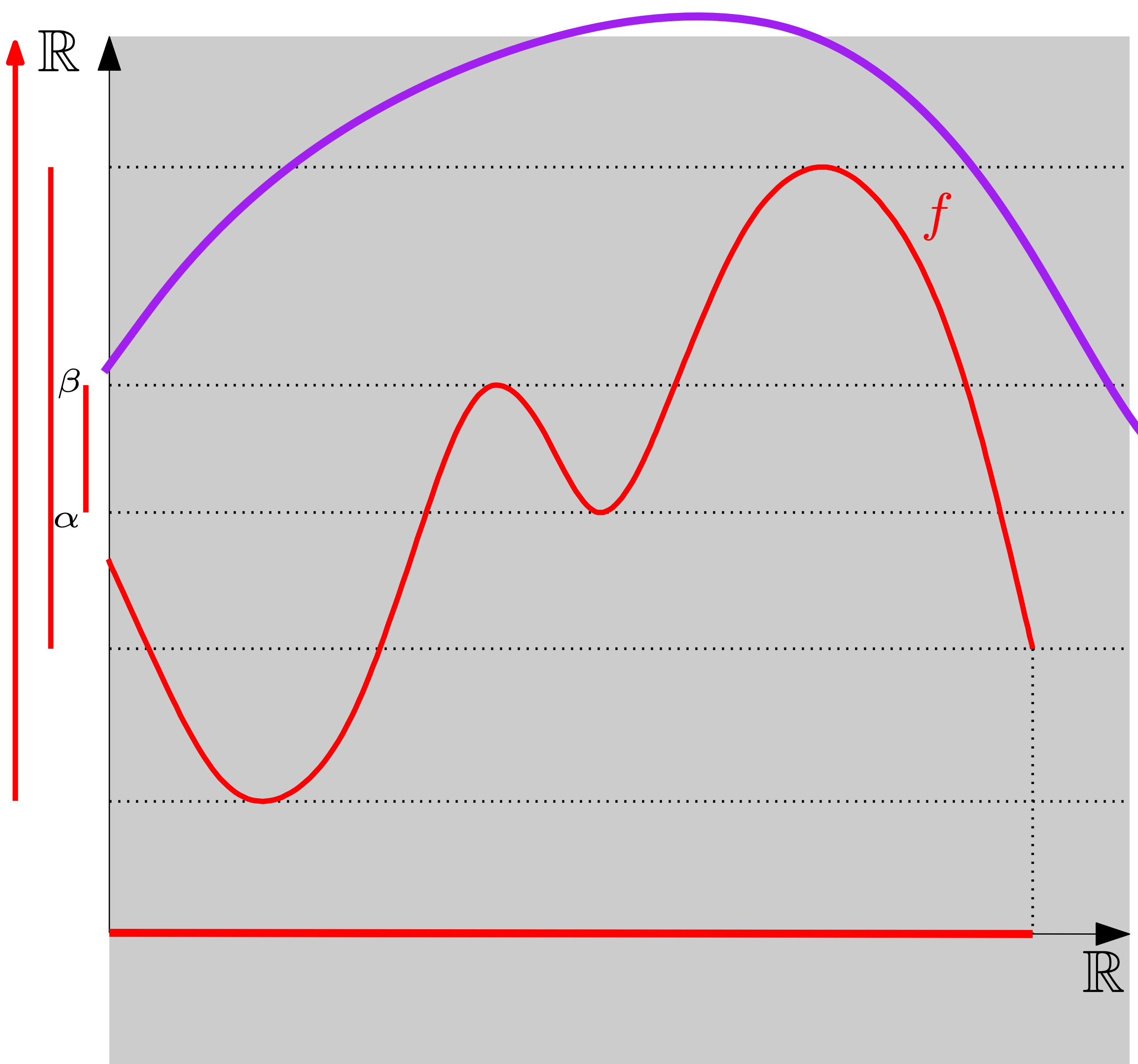
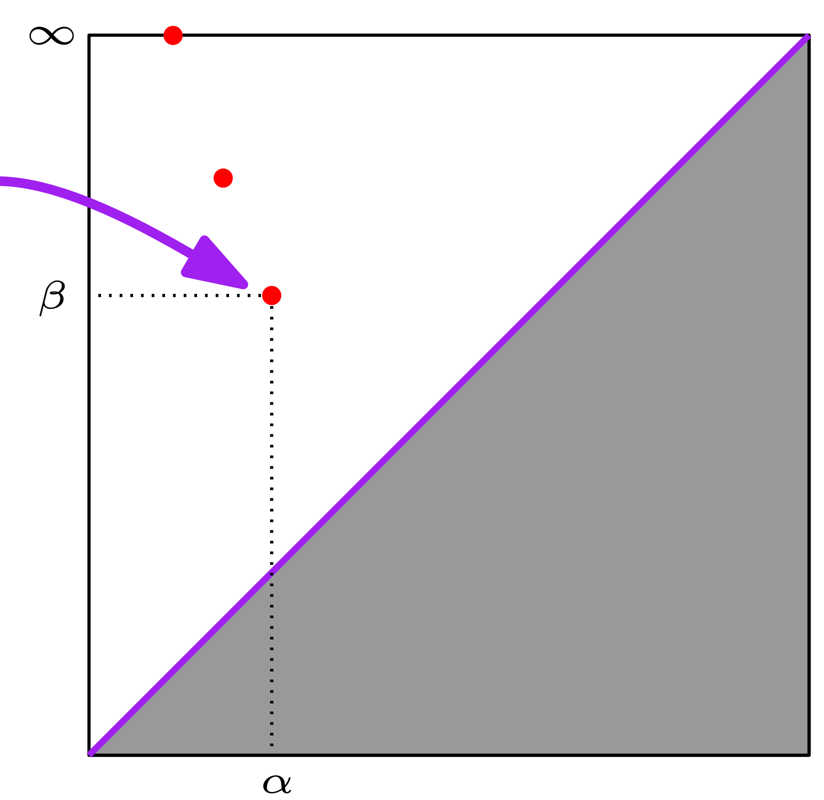
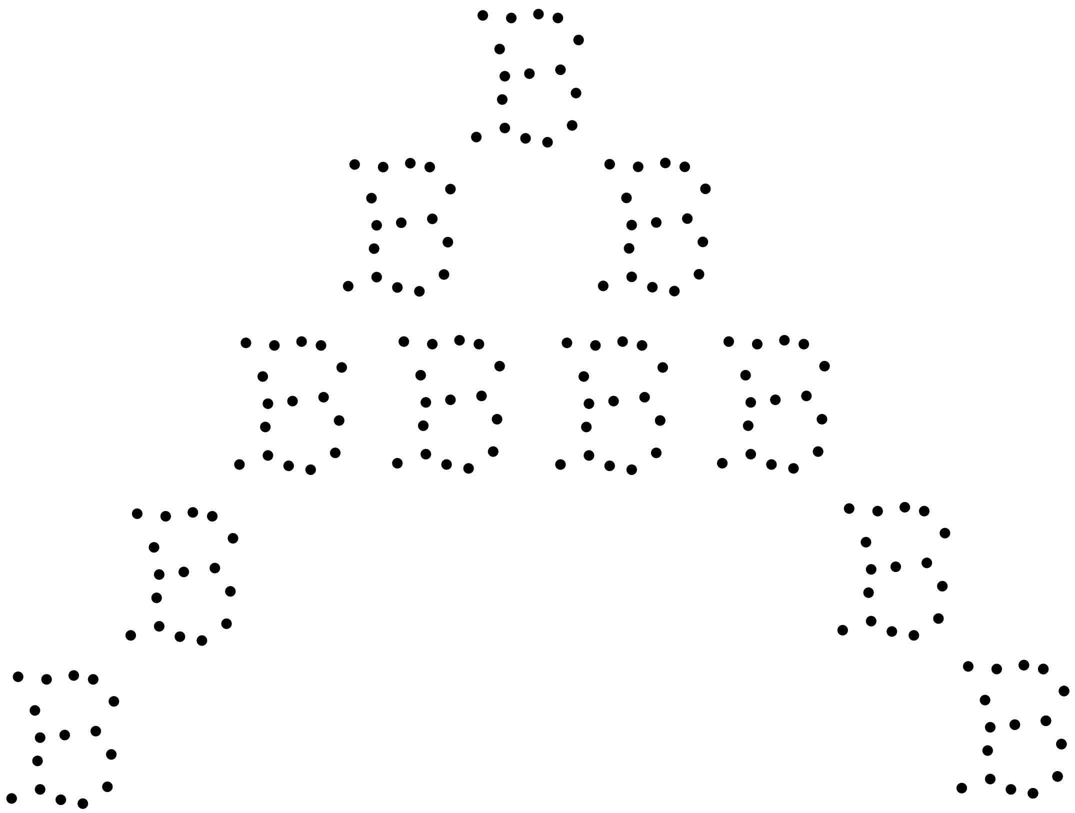


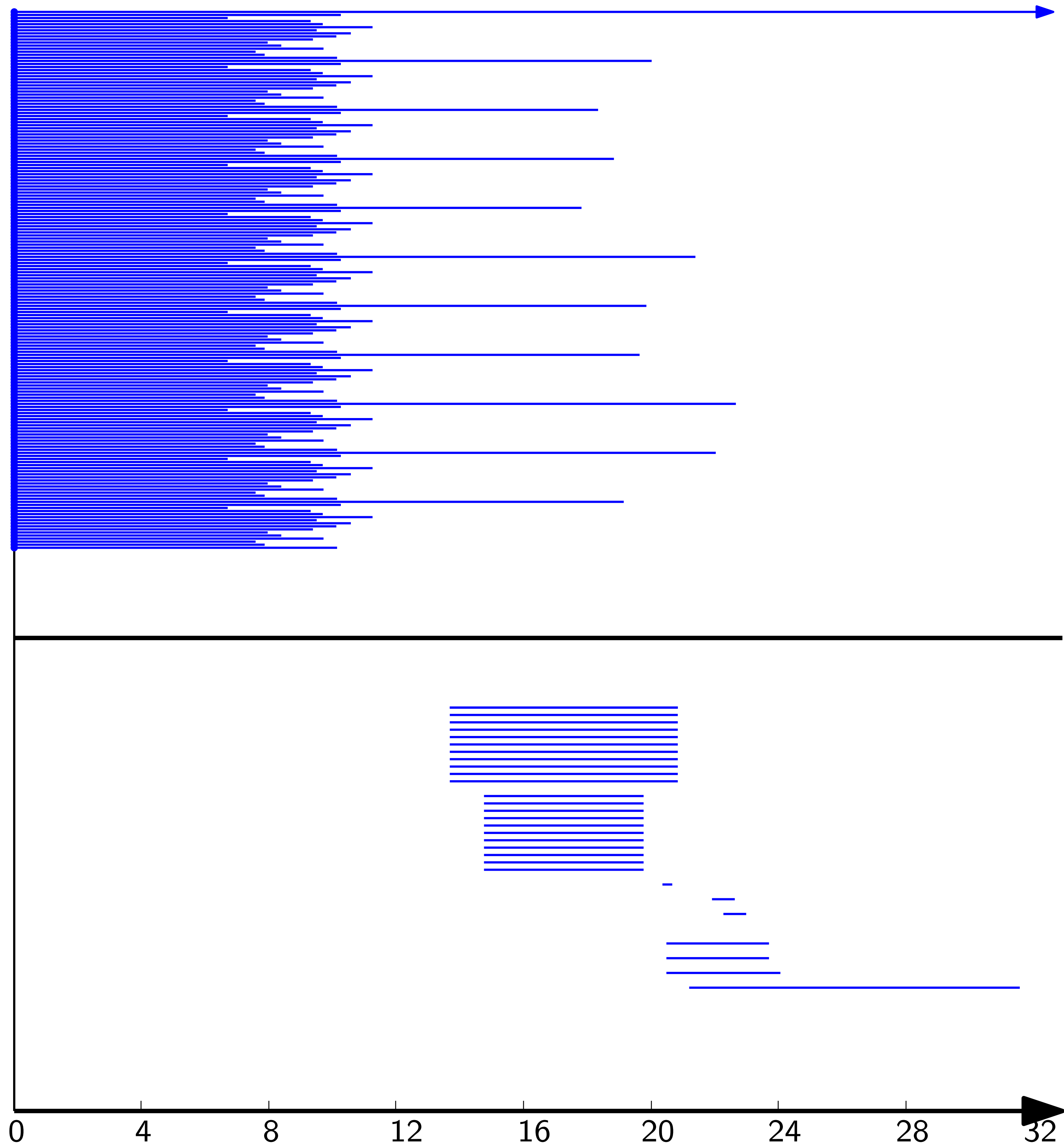

- Alternate representation as a (multi-) set of points in the plane (*diagram*).



# Example: Distance Function

$$f_P : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$





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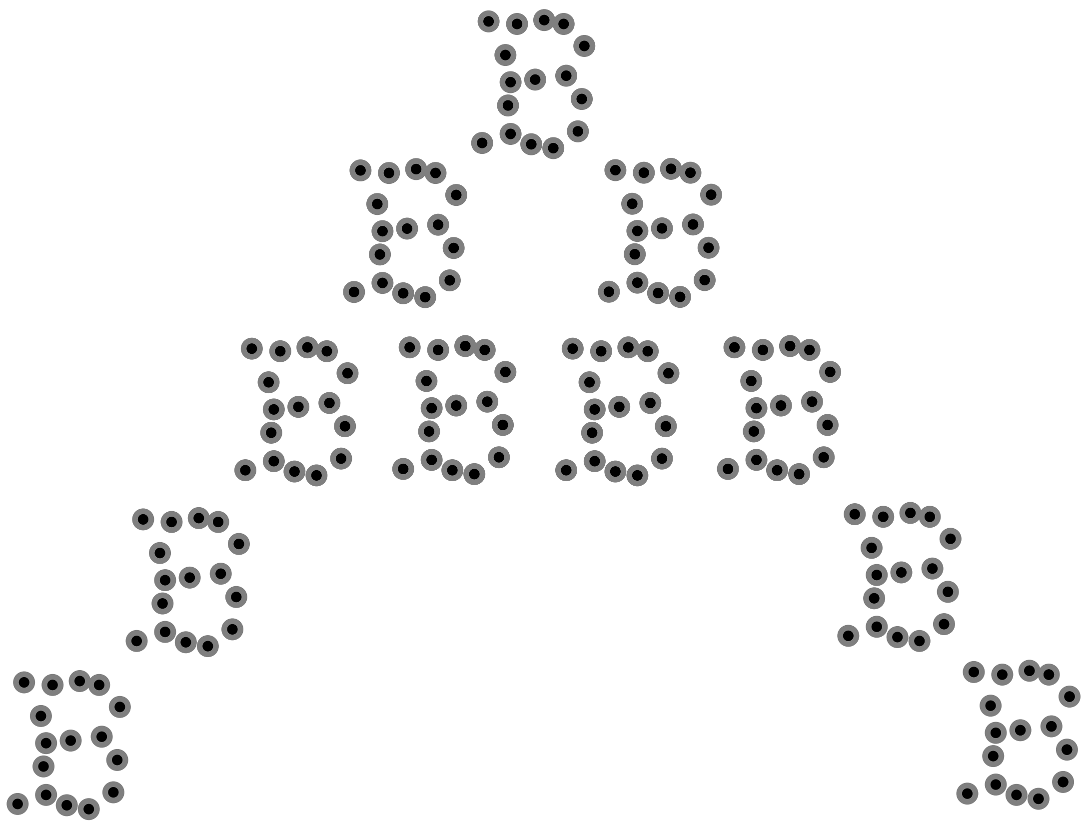

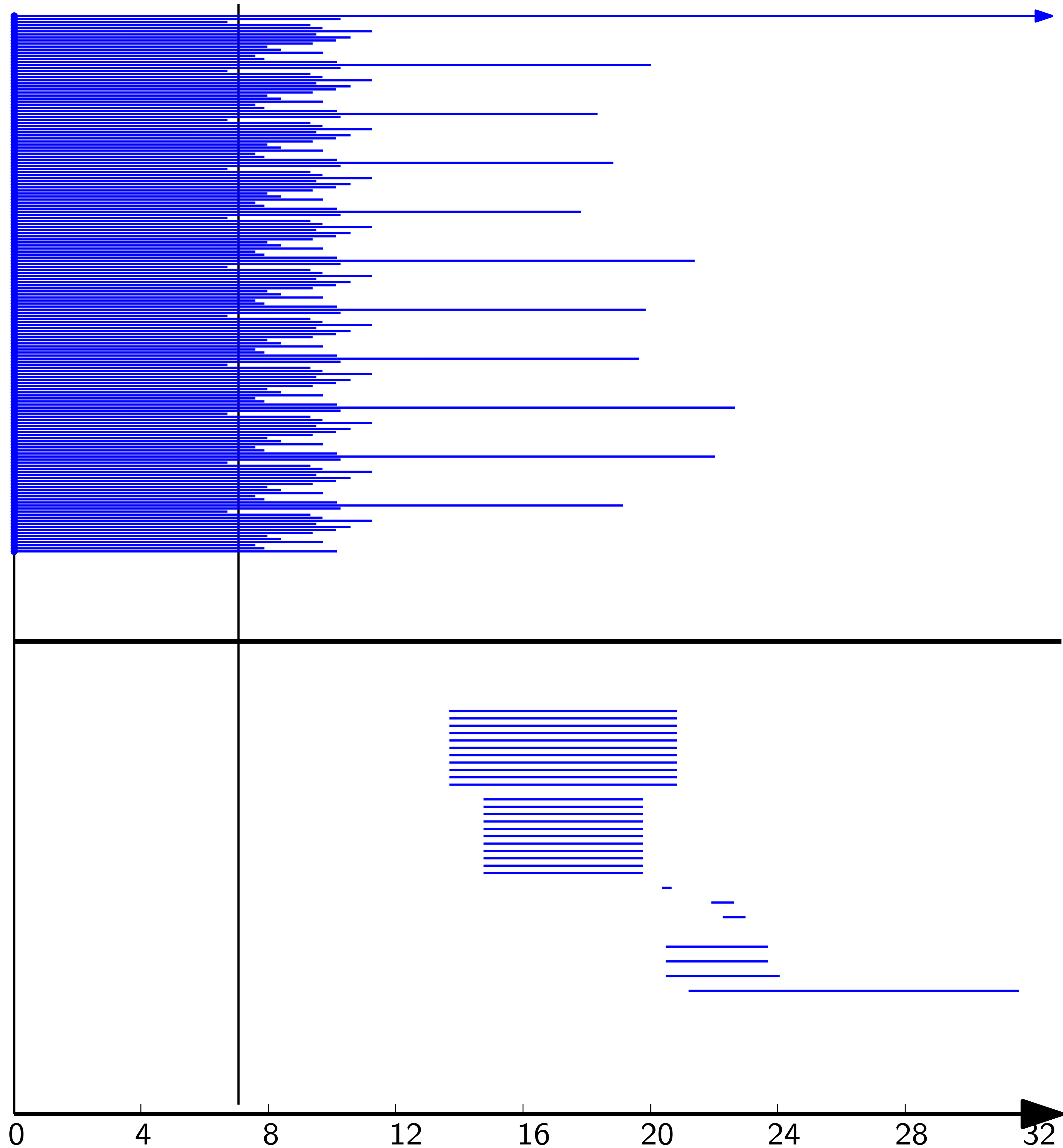

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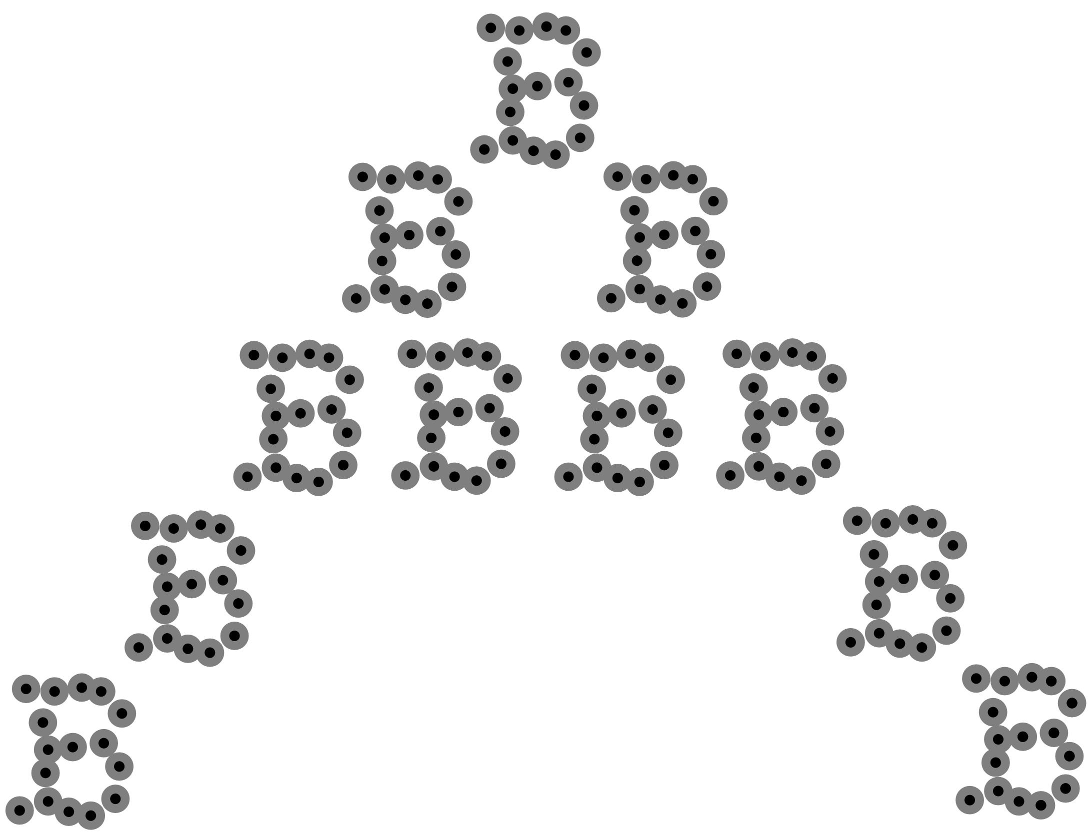

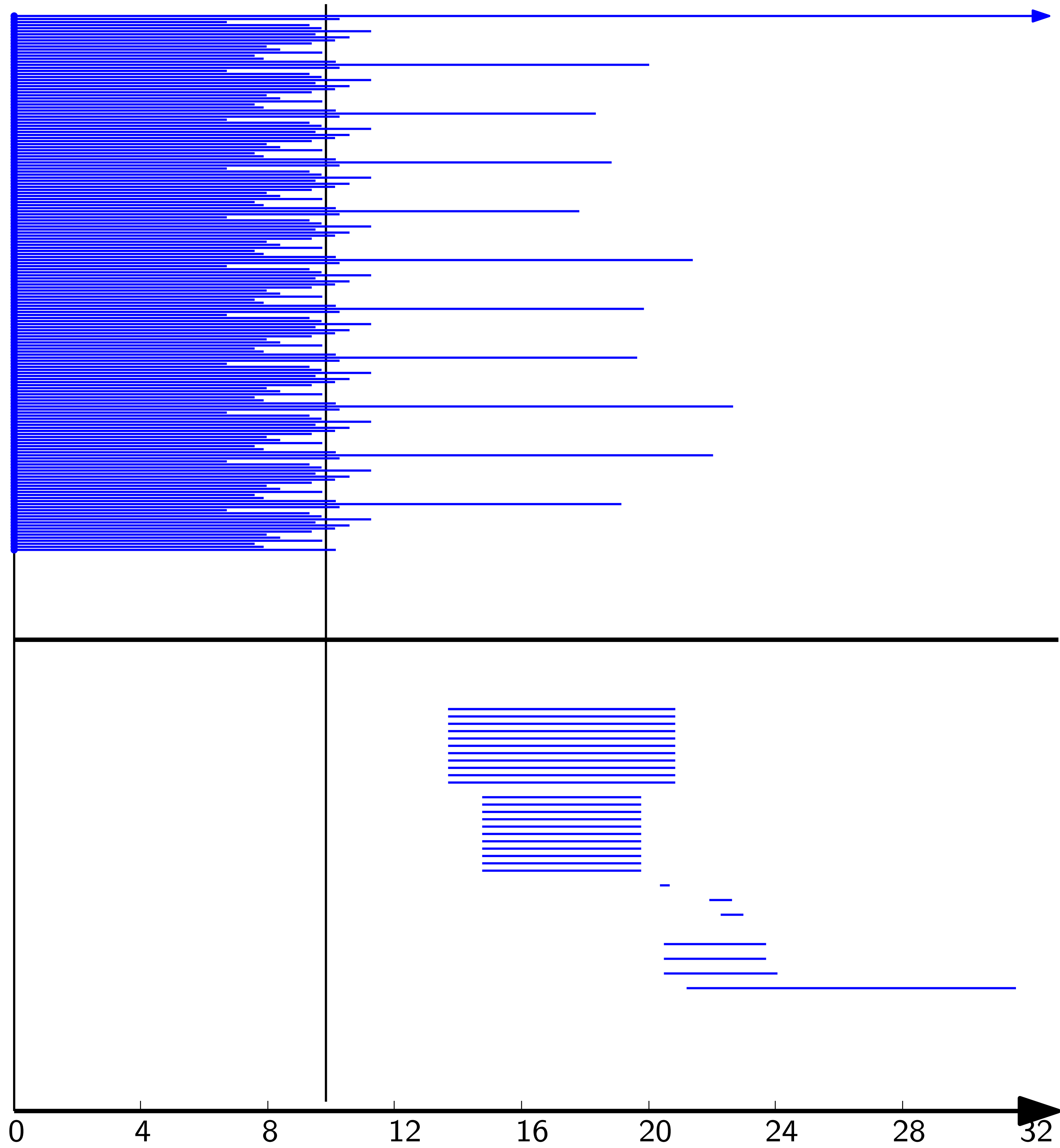

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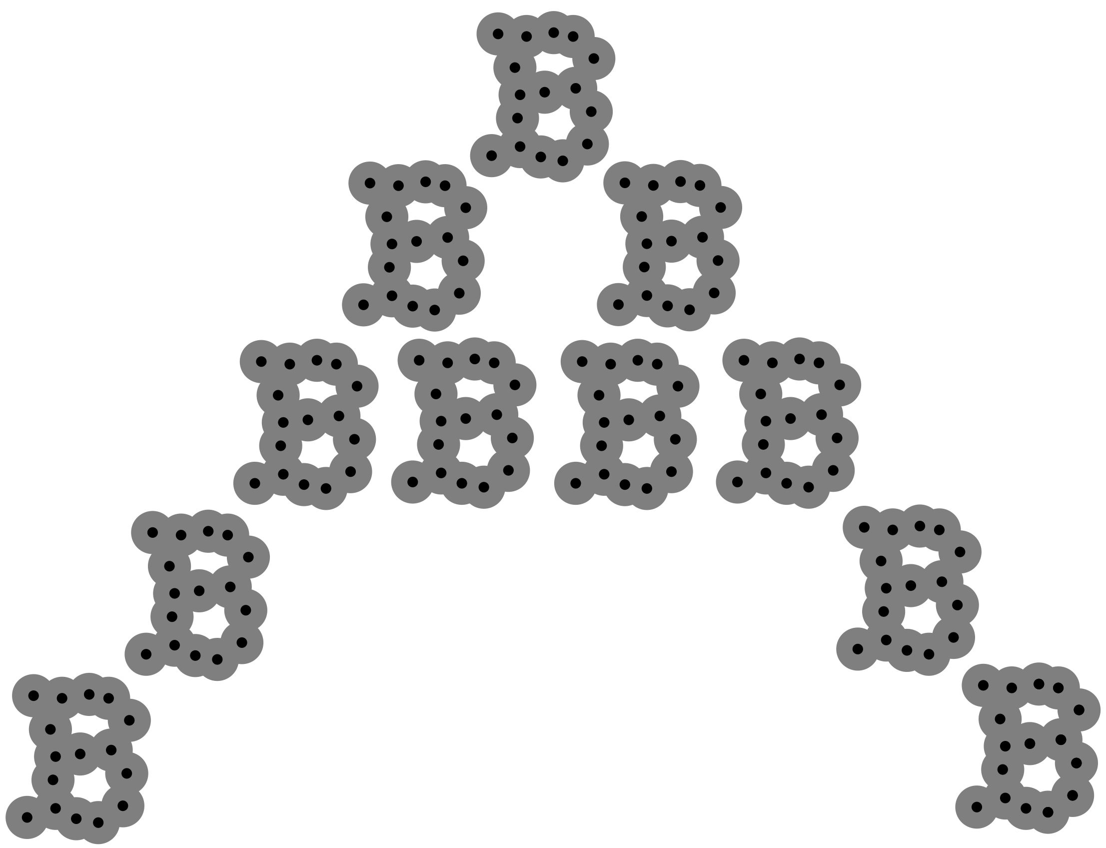

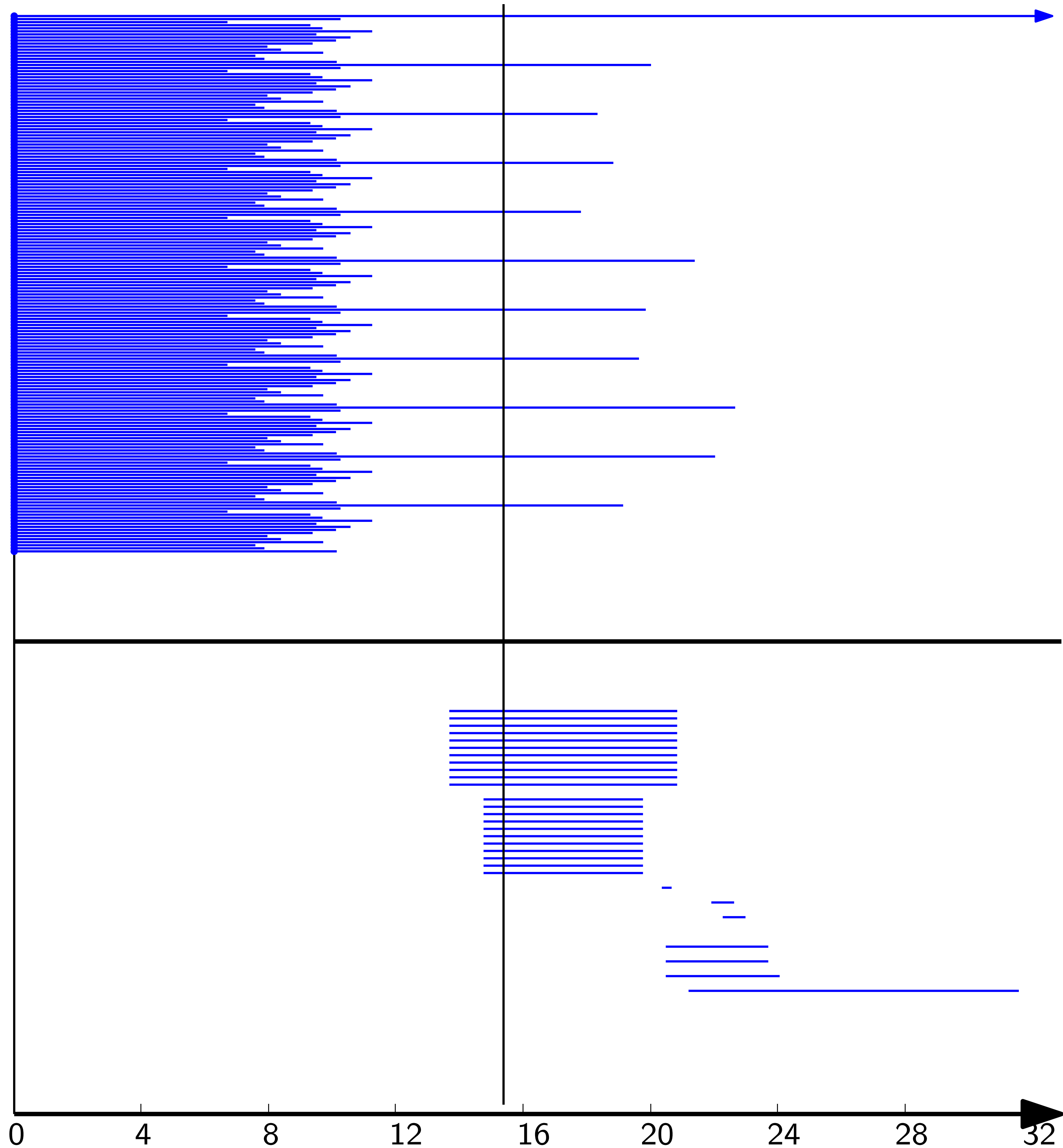

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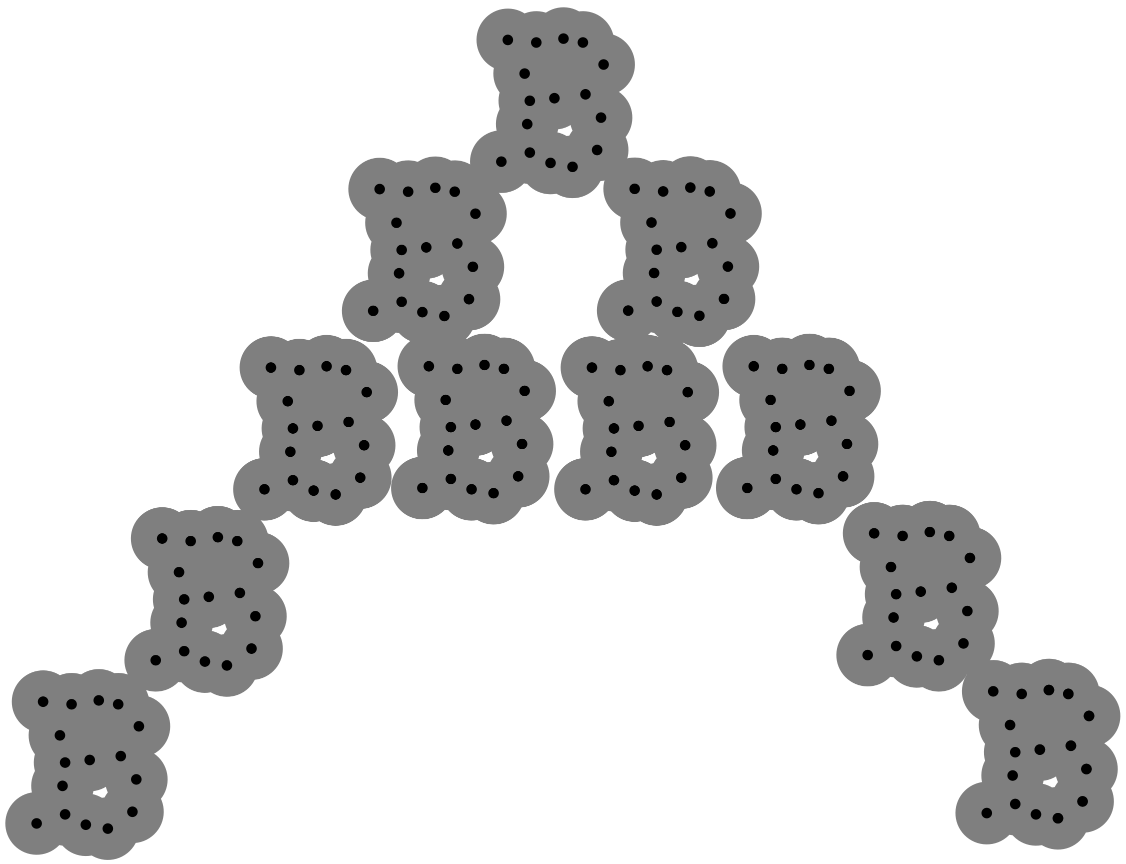

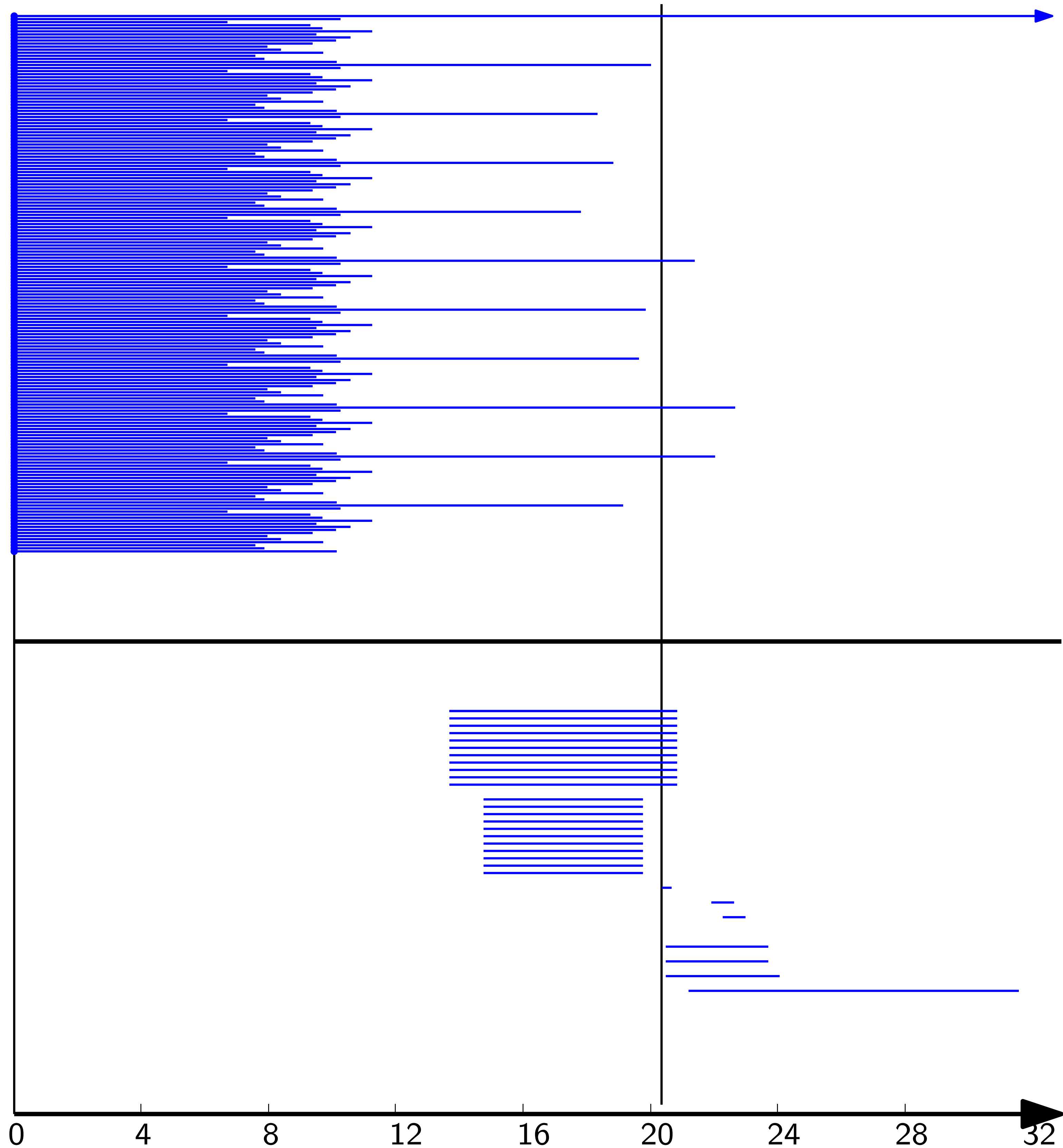

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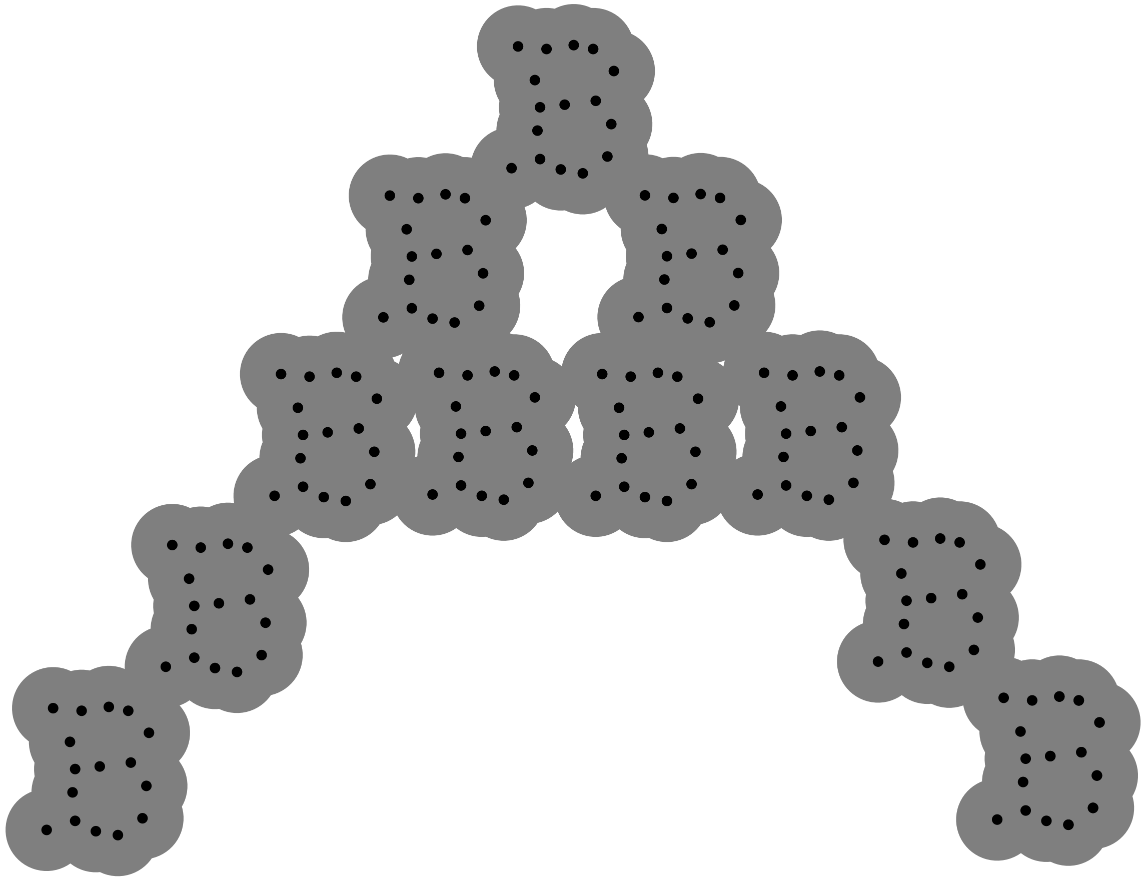

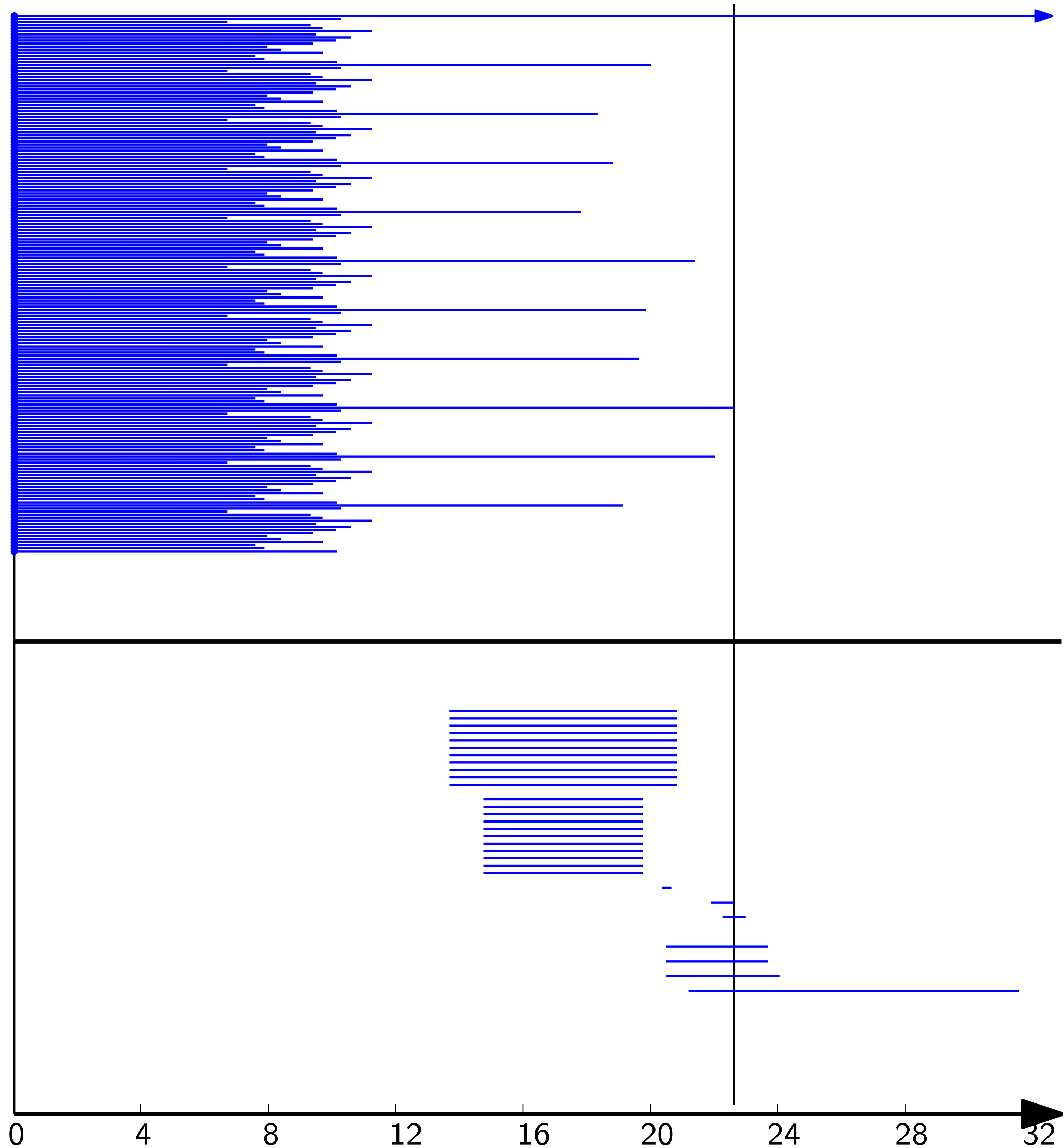

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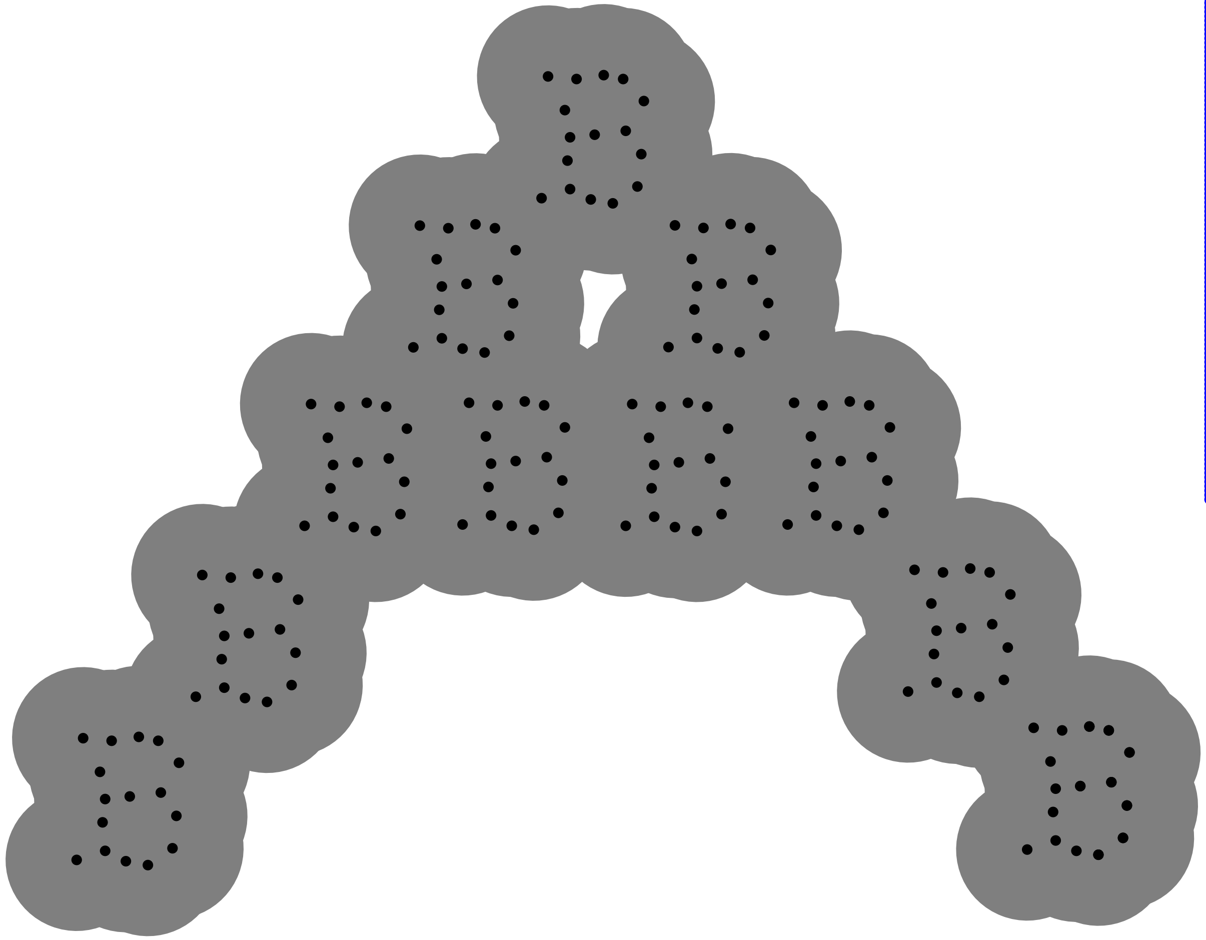

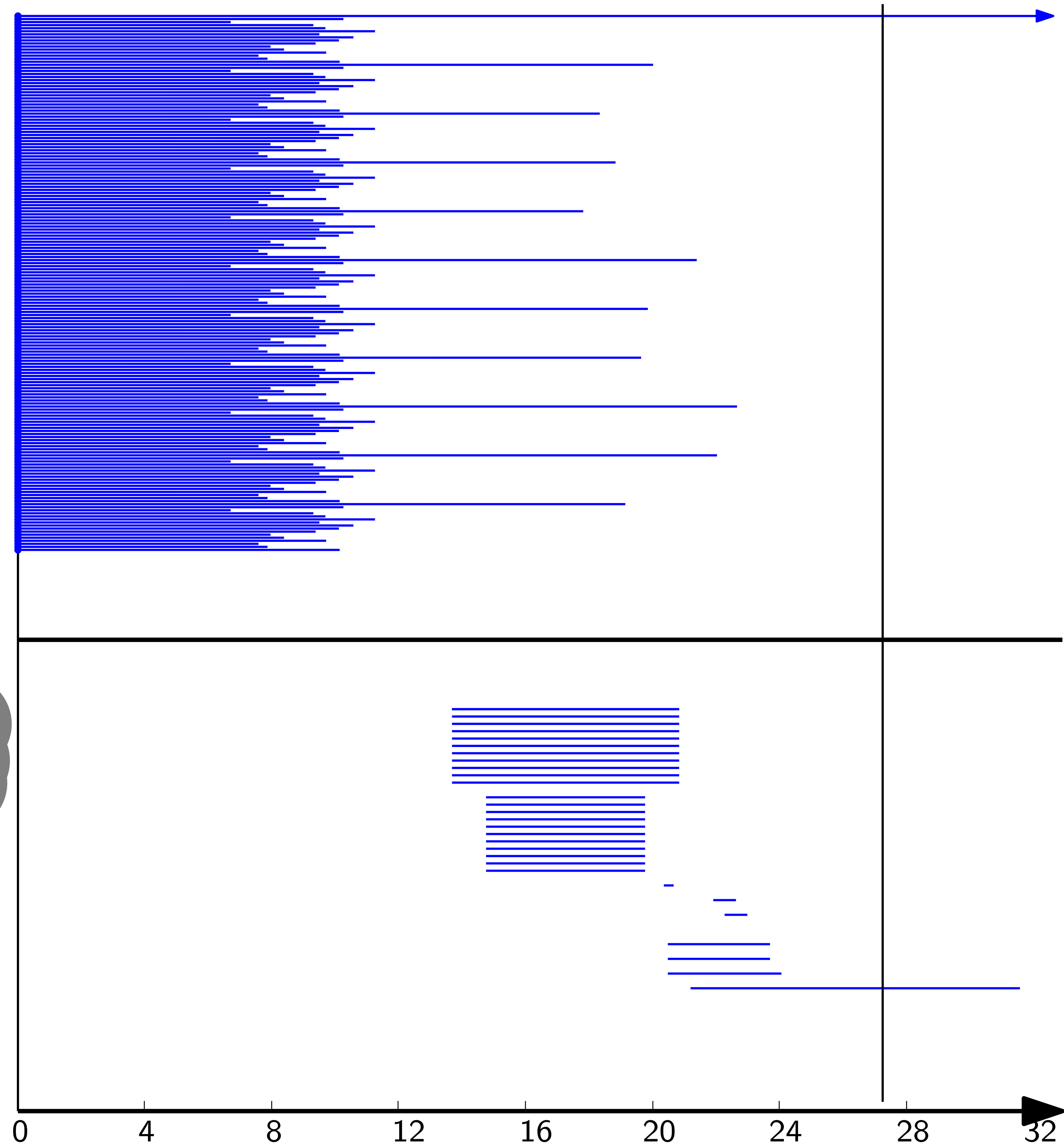

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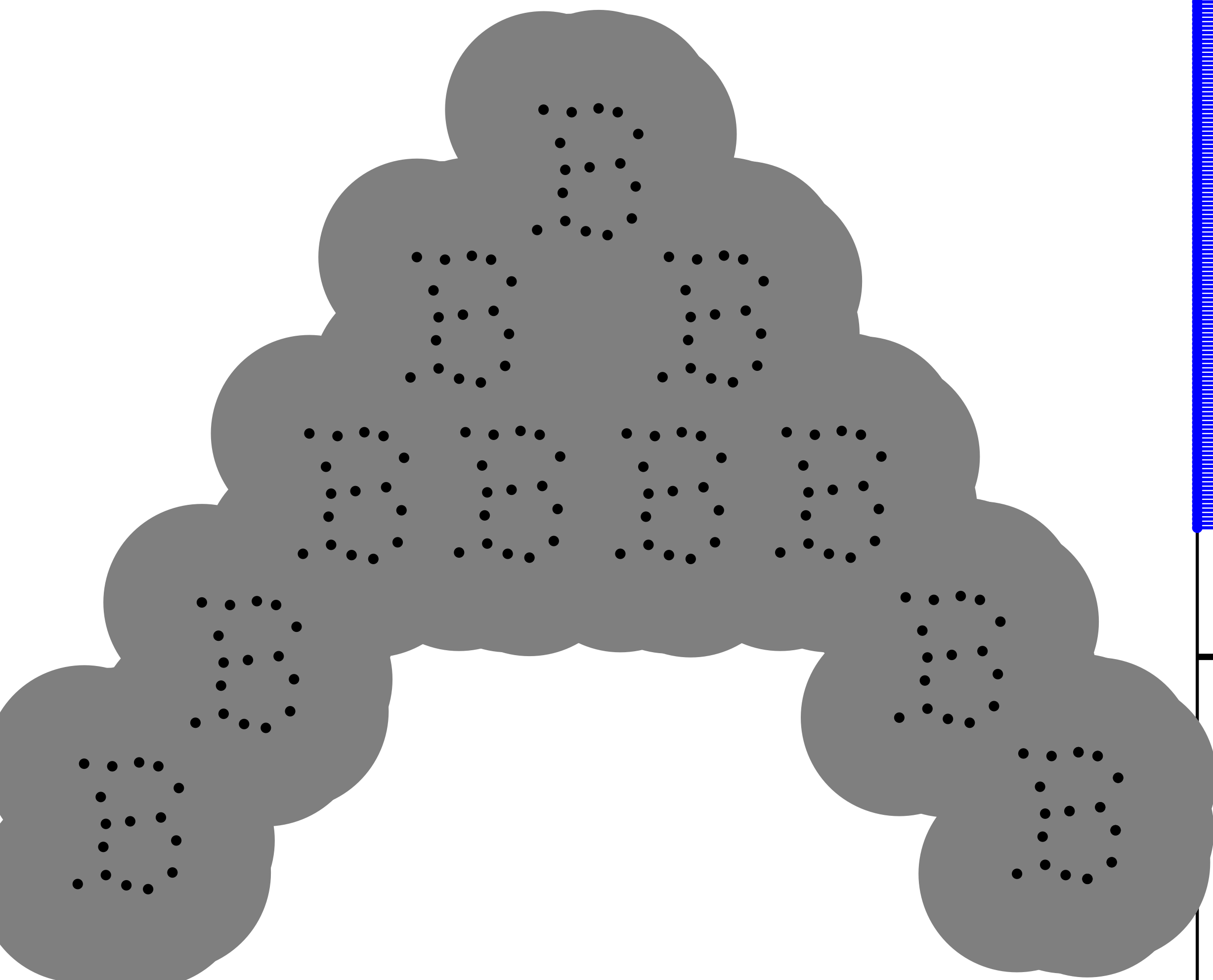

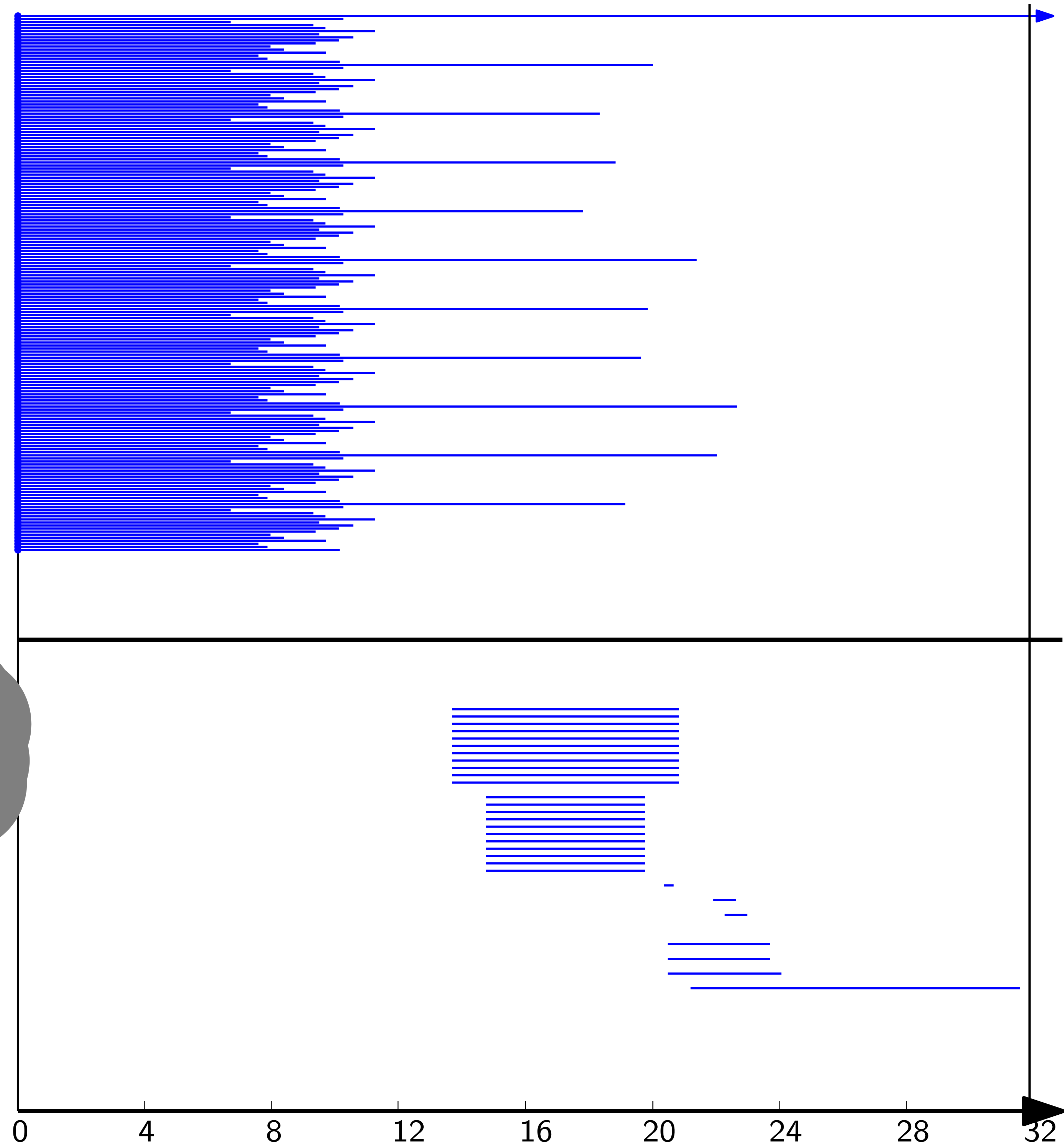

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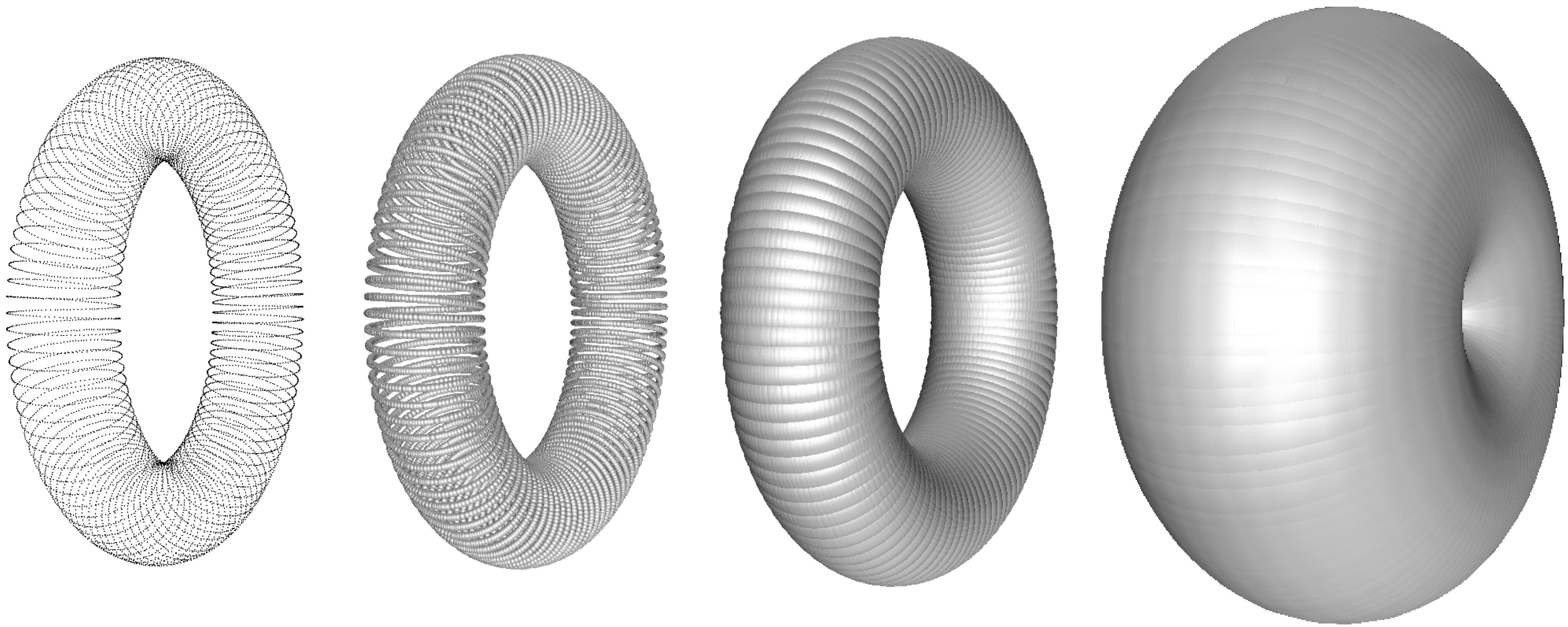

# Topological Persistence (in a nutshell)

**3 pillars:**

1. Decomposition theorems (existence of barcodes / diagrams)

2. algorithm (computation of barcodes / diagrams)

3. stability theorem (use of barcodes as signatures in applications)

# Mathematical viewpoint: homology + quivers

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

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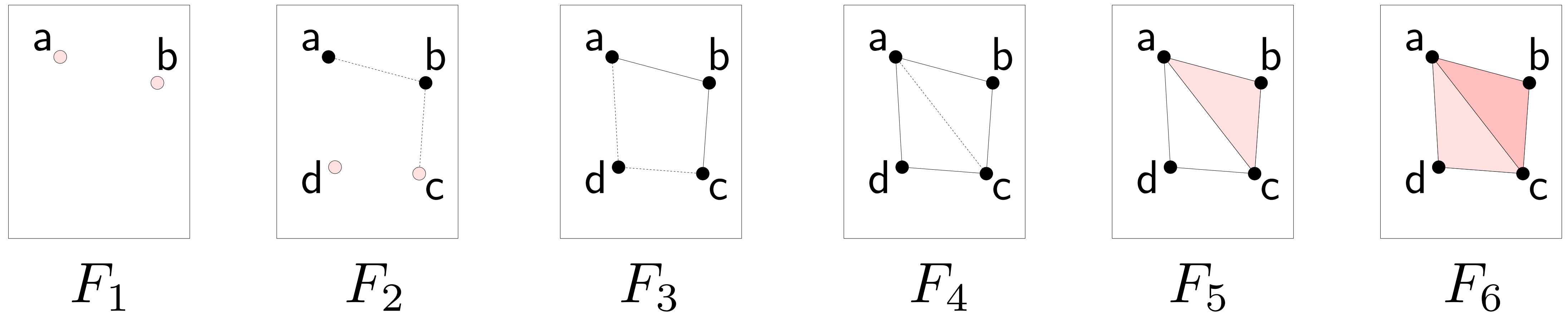
Example 2: *simplicial filtration* (nested family of simplicial complexes)



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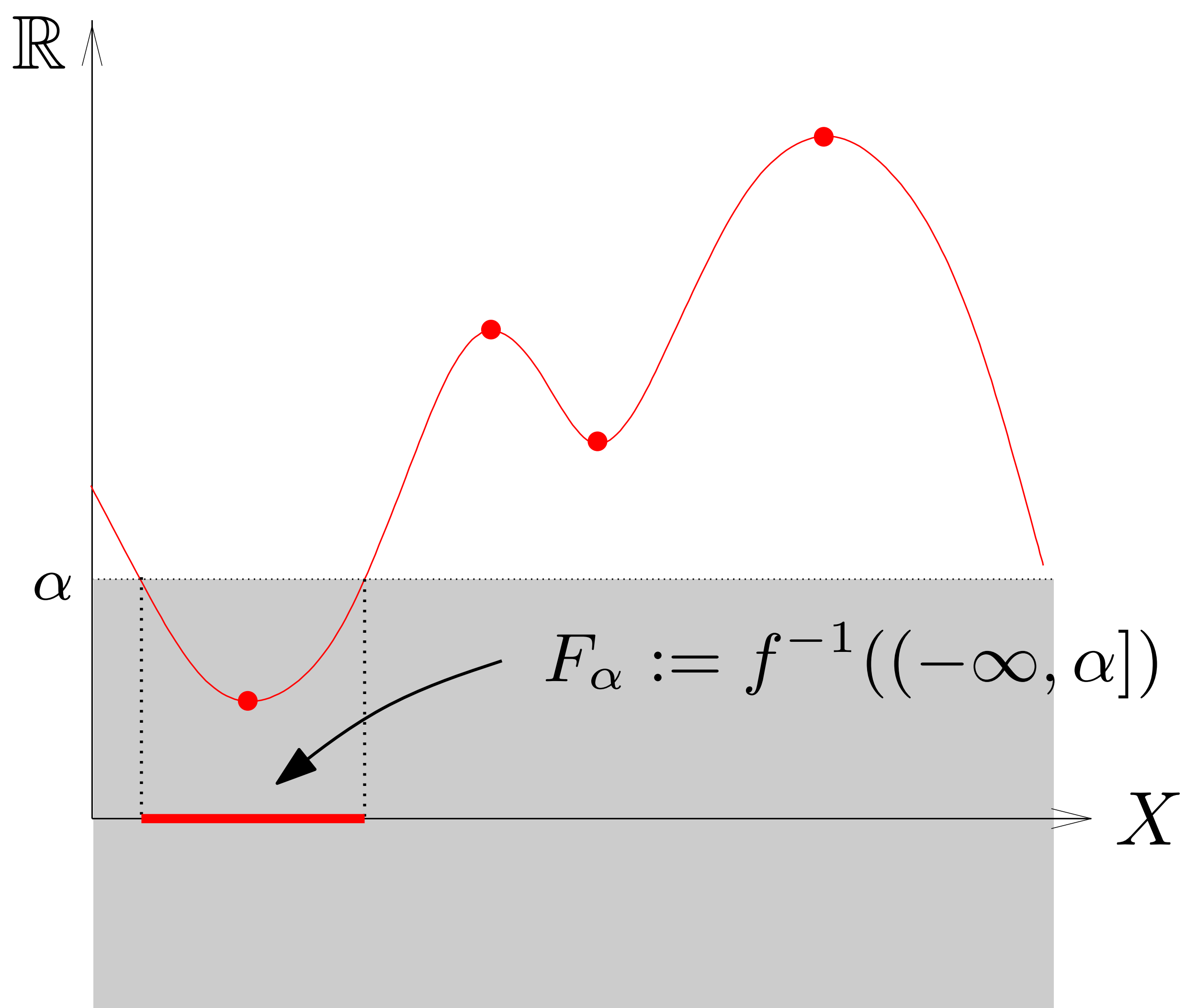
Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *simplicial filtration* (nested family of simplicial complexes)

Example 3: *sublevel-sets filtration* (family of sublevel sets of a function $f : X \to \mathbb{R}$)



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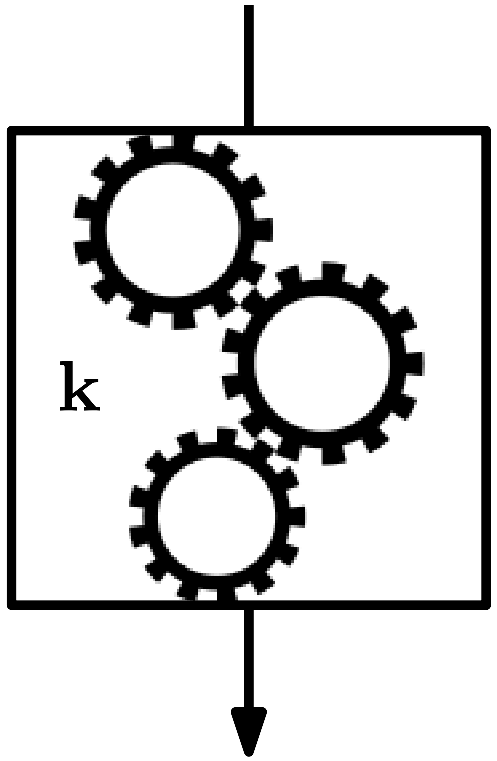



(homology functor)

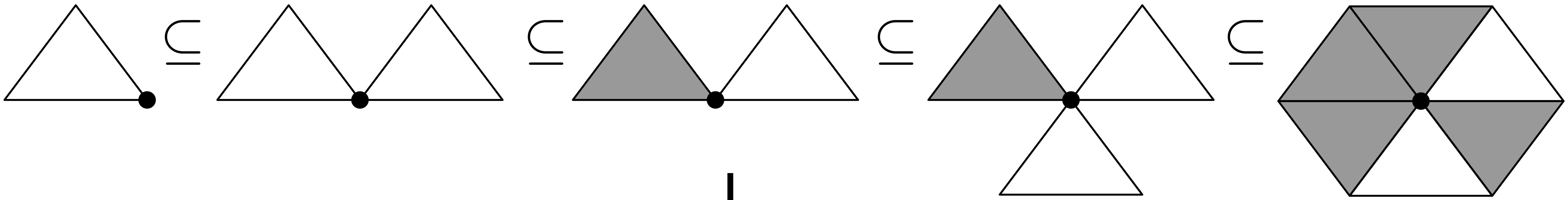
topological level

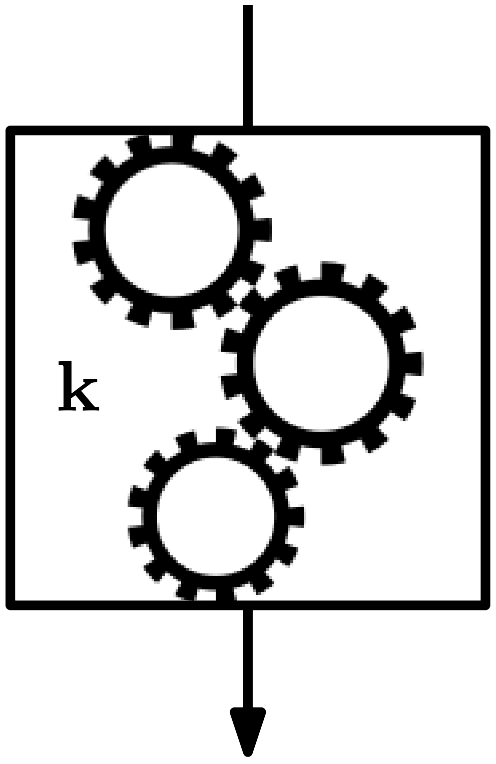
algebraic level

*Persistence module*: $H_*(F_1) \to H_*(F_2) \to H_*(F_3) \to H_*(F_4) \to H_*(F_5) \cdots$

# Mathematical viewpoint: homology + quivers

Example:



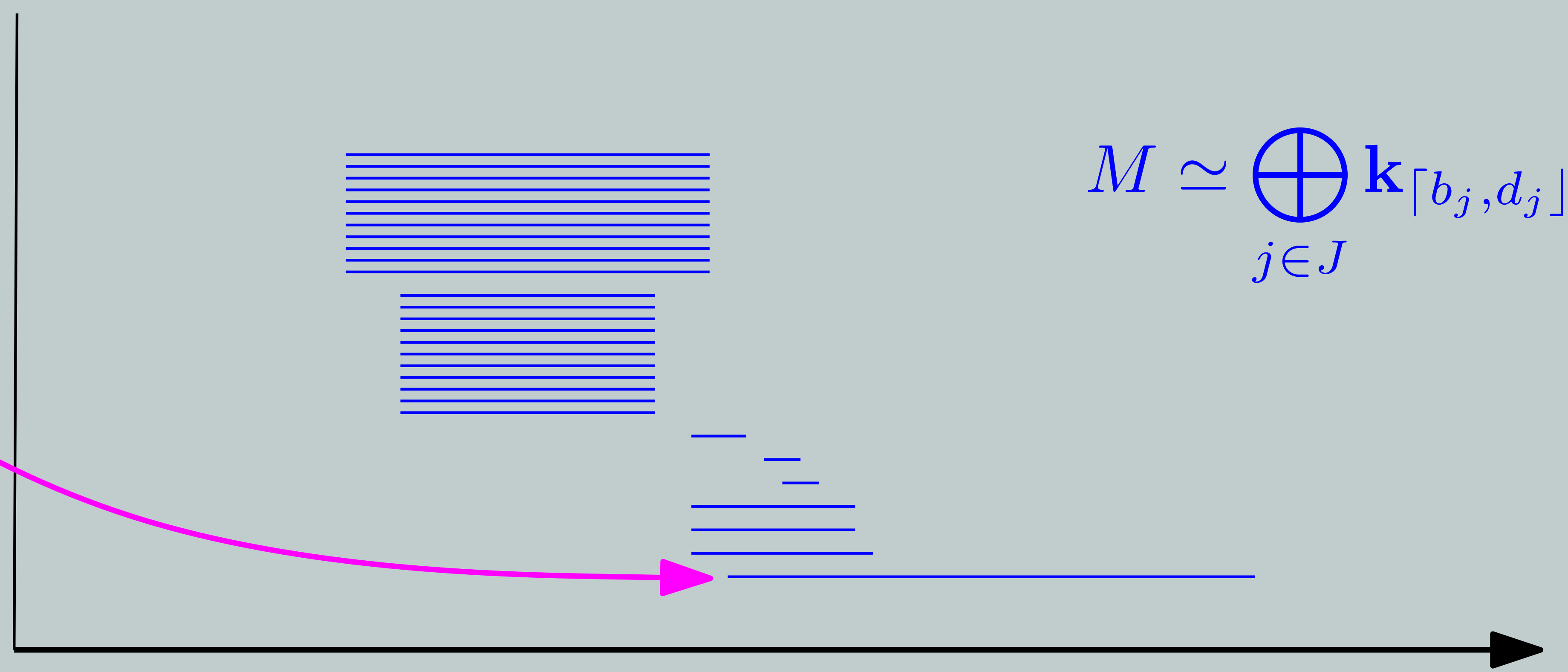

(1-homology functor)

$$\mathbf{k} \xrightarrow{\binom{1}{0}} \mathbf{k}^2 \xrightarrow{(0\ 1)} \mathbf{k} \xrightarrow{\binom{0}{1}} \mathbf{k}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} \mathbf{k}^2 \ \cdots$$

# Mathematical viewpoint: homology + quivers

**Theorem.** Let $M$ be a persistence module over an index set $T \subseteq \mathbb{R}$. Then, $M$ decomposes as a direct sum of *interval modules* $\mathbf{k}_{\lceil b,d \rfloor}$:

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{t<\lceil b,d\rfloor} \xrightarrow{0} \underbrace{\mathbf{k} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbf{k}}_{\lceil b,\, d\rfloor} \xrightarrow{0} \underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{t>\lceil b,d\rfloor}$$



$$M \simeq \bigoplus_{j \in J} \mathbf{k}_{\lceil b_j, d_j \rfloor}$$

(the barcode is a complete descriptor of the algebraic structure of $M$)

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in the following cases:

- $T$ is finite [Gabriel 1972] [Auslander 1974],
- $M$ is *pointwise finite-dimensional* (every space $M_t$ has finite dimension) [Webb 1985] [Crawley-Boevey 2012].

Moreover, when it exists, the decomposition is **unique** up to isomorphism and permutation of the terms [Azumaya 1950].

(Note: this is independent of the choice of field $\mathbf{k}$.)

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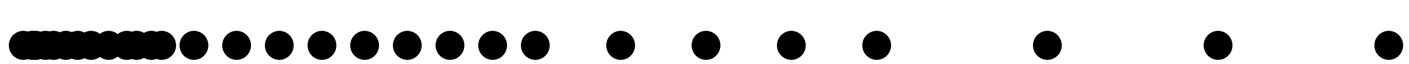
$\exists$ compact sets whose offsets do not induce *pfd* modules.



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- $M$ is *q-tame* ($\operatorname{rank} m_s^t < \infty$ for all $s < t \in T$).

$\rightarrow$ barcode is well-defined, even though $M$ may not be interval-decomposable

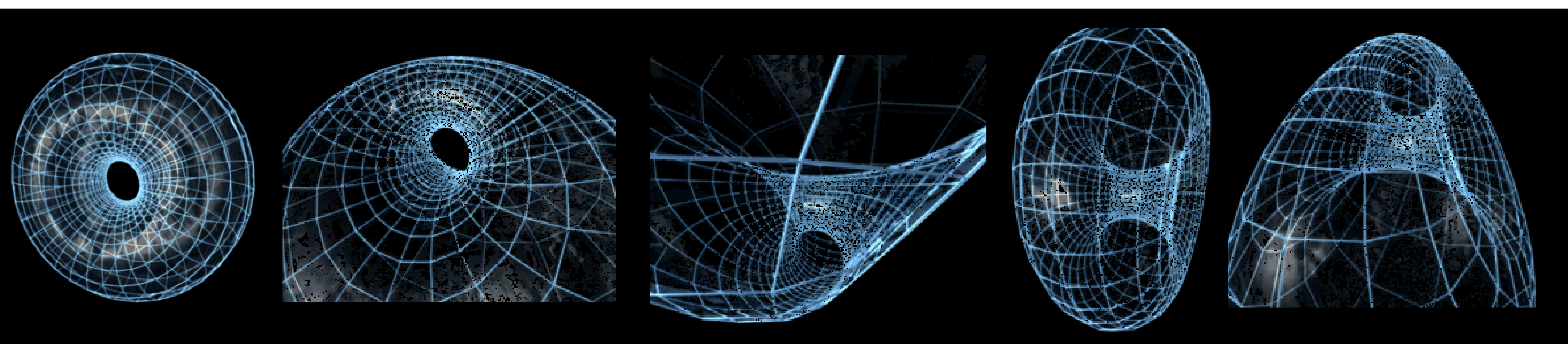
[Chazal, Cohen-Steiner, Glisse, Guibas, O. 2009] [Chazal, de Silva, Glisse, O. 2016]

# Example: Distance Function

$(\mathbb{R} \mod \mathbb{Z})^2$

$$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$$

$\subset \mathbb{S}3 \subset \mathbb{R}^4$



source: `http://http://en.wikipedia.org/wiki/Clifford_torus`

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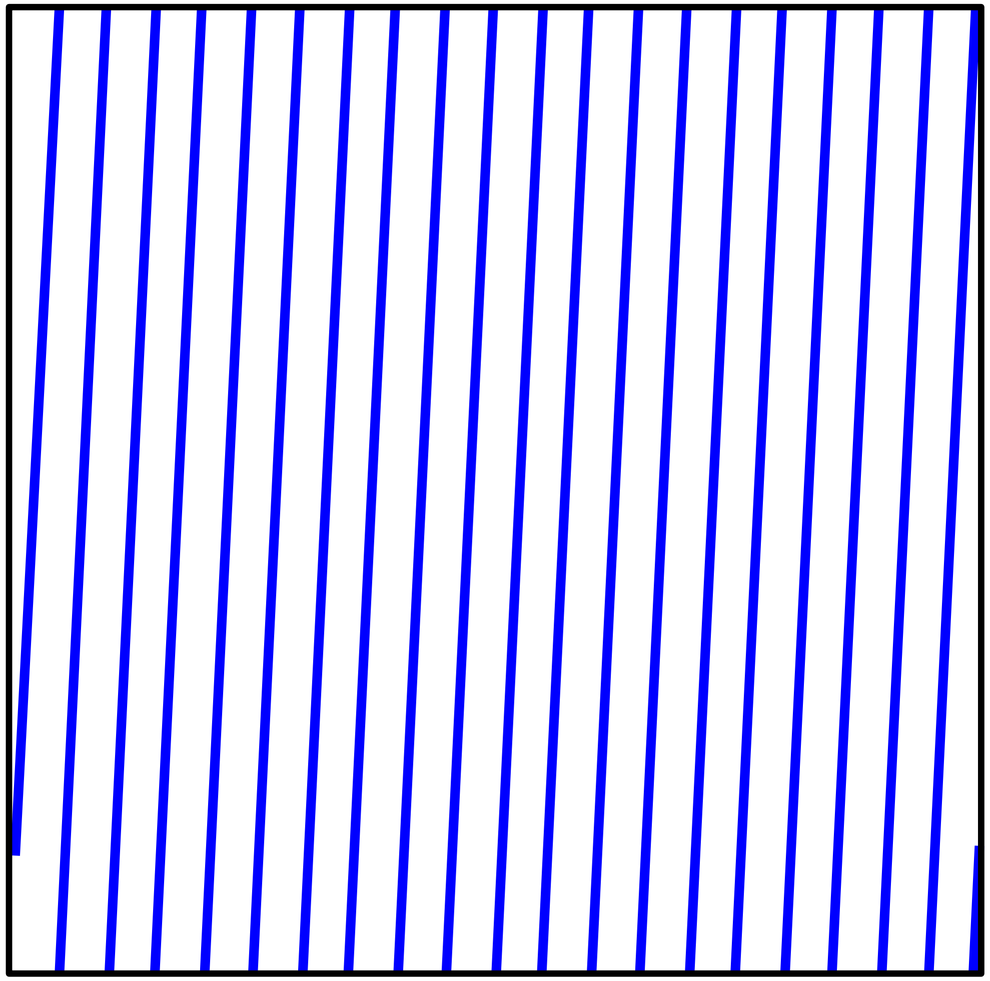

(spiral winding around the flat torus)

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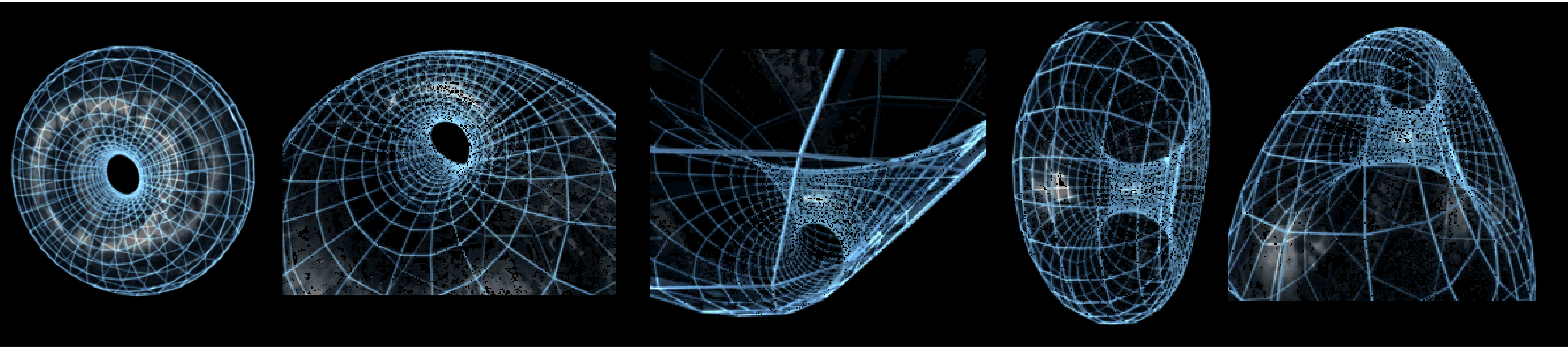

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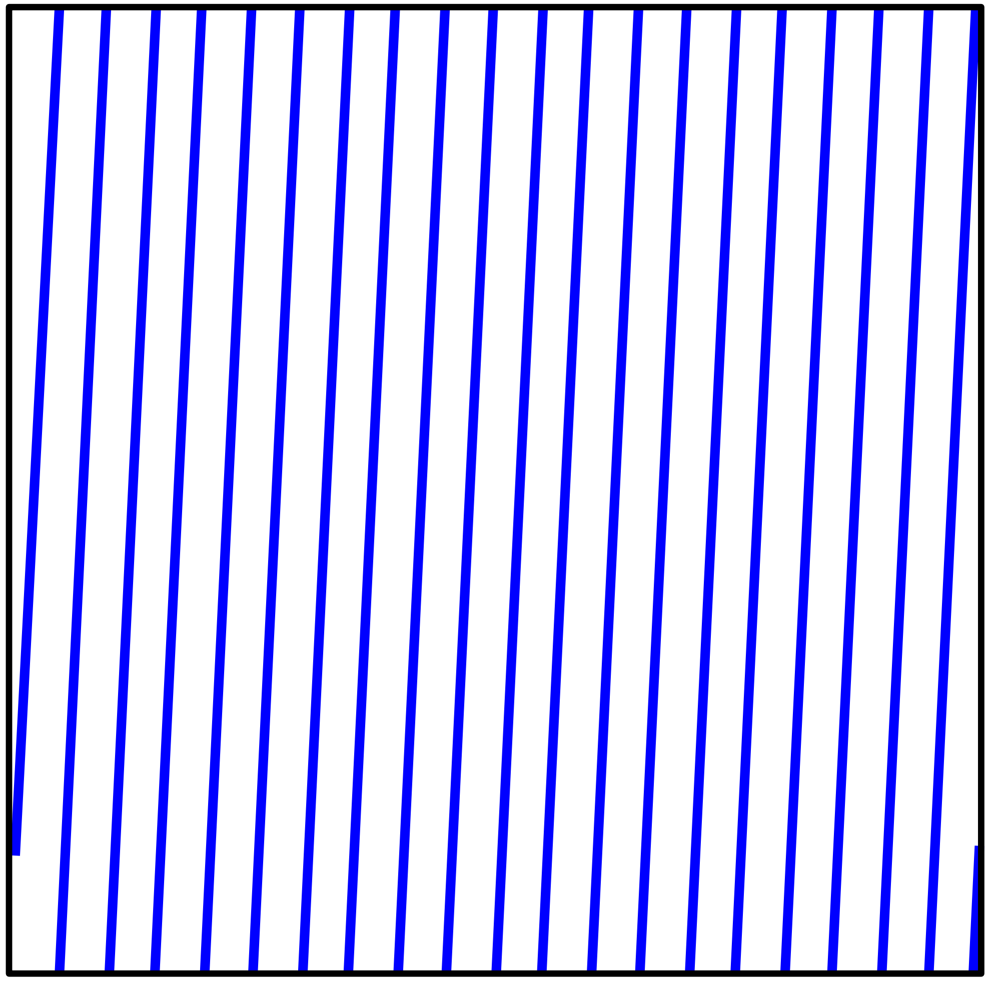

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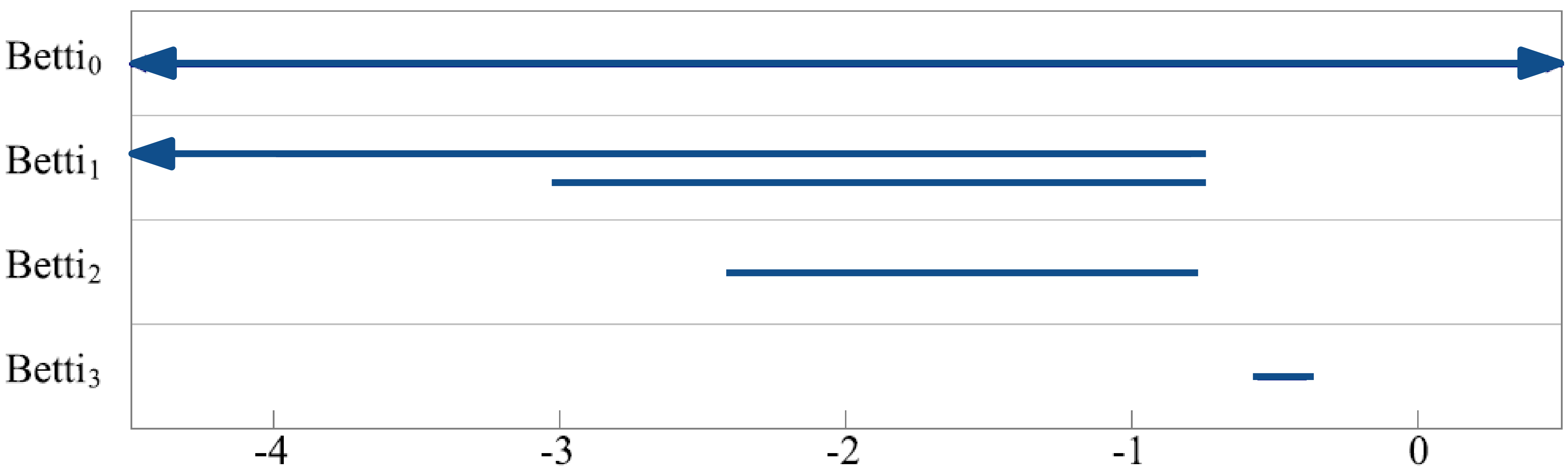

# Stability Properties