Calculus (2)

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For

First Level

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Contents

Chapter 1

Anti derivatives and Indefinite Integrals

Fundamental Integration Formulas

IF (x) IS A FUNCTION whose derivative $F'(x) = f(x)$ on a certain interval of the x axis, then $F(x)$ is called an *antiderivative* or *indefinite integral* of $f(x)$. The indefinite integral of a given function is not unique; for example, x^2, x^2+5, and x^2-4 are all indefinite integrals of $f(x) = 2x$, since

$\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2+5) = \frac{d}{dx}(x^2-4) = 2x$ **All indefinite integrals of $f(\ ) = 2x$ are**

then included in $F(x) = x2 + C$, where C, called the *constant* of *integration,* is an arbitrary constant.

The symbol $\int (x)dx$ **is used to indicate the indefinite integral of** $f(x)$**. Thus we write**

$\int 2x\, dx = x^2 + c$

1) $\int \frac{d}{dx}[f(x)]dx = f(x) + C$

2) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

3) $\int \alpha f(x)dx = \alpha \int f(x)dx$, α **any constant**

4) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

5) $\int \frac{dx}{x} = \ln |x| + C$

6) (a) $\int e^x dx = e^x + C$ **, (b)** $\int e^{f(x)} . f'(x)dx = e^{f(x)} + C$

7) $\int a^x dx = \frac{a^x}{\ln a} + C, a > 0, a \neq 1$

8) $\int \sin x dx = -\cos x + C$

9) $\int \cos x dx = \sin x + C$

10) $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx = -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

11) $$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$$

12) $$\int \sec x\,dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x\,\tan x}{\sec x + \tan x}dx = \ln|\sec x + \tan x| + C$$

13) $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

14) $\int \sec^2 x\,dx = \tan x + C$

15) $\int \csc^2 x\,dx = -\cot x + C$

16) $\int \sec x \tan x\,dx = \sec x + C$

17) $\int \csc x \cot x\,dx = -\cot x + C$

18) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C = \sin^{-1}\frac{x}{a} + C$

19) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

20) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{x}{a} + C$

21) $\int (a + x)^n\,dx = \frac{(a + x)^{n+1}}{n + 1} + C$

22) $\int \frac{\frac{1}{2} f'(x)}{\sqrt{f(x)}}\,dx = \sqrt{f(x)} + C$

23) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n + 1} + C,\ n \neq -1$

24) **(a)** $\int a^{f(x)} . f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + C,\ a > 0,\ a \neq 1$ **(b)** $\int e^{f(x)} . f'(x)\,dx = e^{f(x)} + C$

25) $\int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin x + C = \sin^{-1} x + C$

26) $\int \frac{dx}{1 + x^2} = \arctan x + C = \tan^{-1} x + C$

27) $\int \frac{dx}{x\sqrt{x^2 - 1}} = arc\sec x + C = \sec^{-1} x + C$

Ex(1) $\int \frac{(\ln x)^3}{} dx = \int (\ln x)^3 . \, ^1 dx = (\ln x)^4 \; +$

$$x \qquad \frac{\;}{x} \qquad \frac{\;\;\;}{4} \quad C$$

Ex(2) $\int x\sqrt{x^2+3}\,dx = \frac{1}{2}\int (x^2+3)^{\frac{1}{2}}.2x\,dx = \frac{1}{2}\,\frac{(x^2+3)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}(x^2+3)^{\frac{3}{2}} + C$

Ex(3)

$$\int \frac{x^2}{x^3-5}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3-5}\,dx = \frac{1}{3}\ln\left|x^3-5\right| + C$$

Ex(4) $\int \frac{dx}{\sqrt[3]{x^2}} = \int (x)^{-2/3} dx = \frac{x^{1/3}}{\frac{1}{3}} + C = 3x^{1/3} + C$

Ex(5) $\int (1-x)\sqrt{x}\, dx = \int \sqrt{x}\, dx - \int x^{3/2} dx = \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} + C = 2\left(\frac{x^{3/2}}{3}\right) - 2\left(\frac{x^{5/2}}{5}\right) + C$

Ex(6) a) $\int (x^3+2)^2 (3x^2) dx = \frac{(x^3+2)^3}{3} + C$

b) $\int (x^3+2)^{1/2}.x^2 dx = \frac{1}{3}\int (x^3+2)^{1/2}.3x^2 dx$

$= \frac{1}{3}\frac{(x^3+2)^{3/2}}{\frac{3}{2}} + C = \frac{2}{9}(x^3+2)^{3/2} + C$

c)

$\int \frac{8x^2}{(x^3+2)^3} dx = \frac{8}{3}\int \frac{3x^2}{(x^3+2)^3} dx$

$= \frac{8}{3}\int (x^3+2)^{-3}.3x^2 dx = \frac{8}{3}\frac{(x^3+2)^{-2}}{-2} + C$

$= \frac{-4}{3}\left(\frac{1}{(x^3+2)^2}\right) + C$

d) $\int \frac{x^2}{\sqrt[4]{x^3+2}} dx = \int (x^3+2)^{-1/4}.x^2 dx$

$= \frac{1}{3}\int (x^3+2)^{-1/4}.3x^2 dx = \frac{1}{3}\frac{(x^3+2)^{3/4}}{\frac{3}{4}} + C$

$= \frac{4}{9}(x^3+2)^{3/4} + C$

Ex(7)

a) $\int e^{-x}dx = -\int e^{-x}(-1)dx = -e^{-x} + C$

b) $\int a^{2x}\,dx = \frac{1}{2}\int a^{2x}.2dx = \frac{1}{2}\frac{a^{2x}}{\ln a} + C$

c) $\int \frac{e^{1/x}}{x^2}dx = -\int e^{1/x}\left(-\frac{1}{x^2}\right)dx = -e^{1/x} + C$

d) $\int (e^x+1)^3 . e^x \, dx = \dfrac{(e^x+1)^4}{4} + C$

e) $\int \dfrac{1}{e^x+1} dx = \int \dfrac{e^{-x}}{1+e^{-x}} dx = -\int \dfrac{-e^{-x}}{1+e^{-x}} dx$

$= -\ln(1+e^{-x}) + C = -\ln(1+\dfrac{1}{e^x}) + C$

$= -\ln(\dfrac{e^x+1}{e^x}) + C = \ln(\dfrac{e^x}{e^x+1}) + C$

$= \ln e^x - \ln(1+e^x) + C = x - \ln(1+e^x) + C$

Ex(8) a) $\int \sin(\dfrac{x}{2}) dx = 2\int \sin(\dfrac{x}{2}) d(\dfrac{x}{2})$

$= -2\cos(\dfrac{x}{2}) + C$

b) $\int \cos(3x) dx = \dfrac{1}{3}\int \cos(3x).3dx$

$= \dfrac{1}{3}\sin(3x) + C$

c) $\int \sin^2 x \cos x dx = \int \sin^2 x d \sin x = \dfrac{\sin^3 x}{3} + C$

d) $\int \tan(2x) dx = \dfrac{1}{2}\int \tan(2x) d(2x)$

$= \dfrac{1}{2}\ln|\sec 2x| + C$

e) $\int x \cot x^2 dx = \dfrac{1}{2}\int \cot x^2 (2x dx)$

$= \dfrac{1}{2}\ln|\sin x^2| + C$

f) $\int \sec \sqrt{x} \dfrac{dx}{\sqrt{x}} = 2\int (\sec x^{1/2})(\dfrac{1}{2} x^{-1/2} dx)$

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$$= 2\ln\left|\sec\sqrt{x} + \tan\sqrt{x}\right| + C$$

g) $\int \sec^2(2ax)\,dx \quad = \frac{1}{2a}\int \sec^2(2ax)(2a\,dx)$

$= \frac{1}{2a}\int \sec^2 u\,du = \frac{1}{2a}\tan u + C$

$= \frac{1}{2a}\tan(2ax) + C$

h) $\int e^x \cos e^x\,dx = \int (\cos e^x)(e^x\,dx)$

$= \int \cos u\,du = \sin u + C = \sin e^x + C$

i) $\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$

$= \int (\csc^2 x - \cot x \csc x)\,dx$

$= -\cot x + \csc x + C$

j) $\int (\sec 4x - 1)^2 dx = \int (\sec^2 4x - 2\sec 4x + 1)\,dx$

$= \frac{1}{4}\int \sec^2 4x\,(4dx) - \frac{2}{4}\int \sec 4x\,(4dx) + \int dx$

$= \frac{1}{4}\tan(4x) - \frac{1}{2}\ln|\sec 4x + \tan 4x| + x + C$

Ex(9) Evalute the following integrals:

(i) $\int \frac{x^2 dx}{\sqrt{1-x^6}} = \frac{1}{3}\int \frac{dx^3}{\sqrt{1-(x^3)^2}} = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{3}\sin^{-1} u + C = \frac{1}{3}\sin^{-1}(x^3) + C$

(ii) $\int \frac{dx}{\quad} = \int \frac{2dx}{\quad} = \int \frac{du}{\quad}$

$x\sqrt{4x^2-9} \qquad 2x\sqrt{(2x)^2-(3)^2} \qquad u\sqrt{u^2-(3)^2}$

$$=\frac{1}{3}\sec^{-1}\left(\frac{u}{3}\right)+C=\frac{1}{3}\sec^{-1}\left(\frac{2x}{3}\right)+C$$

(iii) $\int \frac{dx}{4x^2+9} = \frac{1}{2}\int \frac{d(2x)}{(2x)^2+(3)^2} = \frac{1}{2}\int \frac{du}{u^2+(3)^2}$

$= \frac{1}{2}\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{u}{3}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$

(iv) $\int \frac{dx}{\sqrt{25-16x^2}} = \frac{1}{4}\int \frac{d(4x)}{\sqrt{(5)^2-(4x)^2}} = \frac{1}{4}\int \frac{du}{\sqrt{(5)^2-u^2}}$

$= \frac{1}{4}\sin^{-1}\left(\frac{u}{5}\right)+C = \frac{1}{4}\sin^{-1}\left(\frac{4x}{5}\right)+C$

(v) $\int \frac{dx}{x\sqrt{x^4-1}} = \frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{(x^2)^2-1}} = \frac{1}{2}\int \frac{d(x^2)}{x^2\sqrt{(x^2)^2-1}}$

$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}u + C = \frac{1}{2}\sec^{-1}(x^2)+C$

(vi) $\int \frac{dx}{\sqrt{4-(x+2)^2}} = \int \frac{d(x+2)}{\sqrt{(2)^2-(x+2)^2}} = \int \frac{du}{\sqrt{(2)^2-u^2}}$

$= \sin^{-1}(u)+C = \sin^{-1}(x+2)+C$

(vii) $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \int \frac{de^x}{(e^x)^2+1}$

$= \int \frac{du}{u^2+1} = \tan^{-1}u + C = \tan^{-1}(e^x)+C$

(viii) $\int \frac{3x^3-4x^2+3x}{x^2+1}dx = \int \left(3x-4+\frac{4}{x^2+1}\right)dx$

$= 3\frac{x^2}{2} - 4x + 4\int \frac{1}{1+x^2}dx$

$= \frac{3x^2}{2} - 4x + 4\tan^{-1}x + C$

Ex(10) Evalute the following integrals :

(a) $\int \frac{\sec x \, \tan x}{9+4\sec^2 x}dx = \frac{1}{2}\int \frac{2\sec x \, \tan x}{(3)^2+(2\sec x)^2}dx$

$= \frac{1}{2}\int \frac{du}{(3)^2+u^2} = \frac{1}{2}\frac{1}{3}\tan^{-1}(\frac{u}{3}) + C$

$$= \frac{1}{6}\tan^{-1}(2\sec x) + C$$

(b) $\int \frac{(x+3)dx}{\sqrt{1-x^2}} = -\int \frac{-xdx}{\sqrt{1-x^2}} + 3\int \frac{1}{\sqrt{1-x^2}}dx$

$$= -\sqrt{1-x^2} + 3\sin^{-1}(x) + C$$

(c) $\int \frac{(2x-7)dx}{x^2+9} = \int \frac{2x}{x^2+9}dx - 7\int \frac{1}{(3)^2+x^2}dx$

$$= \ln(x^2+9) - \frac{7}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

(d) $\int \frac{dy}{y^2+10y+30} = \int \frac{dy}{(y^2+10y+25)+5} = \int \frac{dy}{(y+5)^2+(\sqrt{5})^2}$

$$= \int \frac{d(y+5)}{(\sqrt{5})^2+(y+5)^2} = \int \frac{du}{(\sqrt{5})^2+u^2}$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{y+5}{\sqrt{5}}\right) + C$$

(e) $\int \frac{dx}{\sqrt{20+8x-x^2}} = \int \frac{dx}{\sqrt{36-(x^2-8x+16)}} = \int \frac{dx}{\sqrt{(6)^2-(x-4)^2}}$

$$= \int \frac{d(x-4)}{\sqrt{(6)^2-(x-4)^2}} = \int \frac{du}{\sqrt{(6)^2-u^2}} = \sin^{-1}\left(\frac{u}{6}\right) + C$$

$$= \sin^{-1}\left(\frac{x-4}{6}\right) + C$$

(f) $\int \frac{2x+3}{9x^2-12x+8}dx = \frac{1}{9}\int \frac{18x+27}{9x^2-12x+8}dx$

$$= \frac{1}{9}\int \frac{(18x-12)+39}{9x^2-12x+8}dx$$

$$= \frac{1}{9}\int \frac{18x-12}{9x^2-12x+8}dx + \frac{13}{3}\int \frac{dx}{(3x-2)^2+4}$$

$$= \frac{1}{9}\ln(9x^2-12x+8) + \frac{13}{9}\int \frac{d(3x-2)}{(2)^2+(3x-2)^2}dx$$

$$= \frac{1}{9}\ln(9x^2-12x+8) + \frac{13}{9}\cdot\frac{1}{2}\tan^{-1}\left(\frac{3x-2}{2}\right) + C$$

$$= \frac{1}{} \ln(9x^2 - 12x + 8) + \frac{13}{} \tan^{-1}\left(\frac{3x - 2}{}\right) + C$$

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(g) $\int \frac{x+2}{\sqrt{4x-x^2}}dx = \frac{-1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}dx$

$$= \frac{-1}{2}\int \frac{(-2x+4)-8}{\sqrt{4x-x^2}}dx = \frac{-1}{2}\int \frac{-2x+4}{\sqrt{4x-x^2}}dx + 4\int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Exercises : Evalute the following integrals :

(1) $\int \frac{dx}{x^2-1}$

(2) $\int \frac{dx}{1-x^2}$

(3) $\int (x+3)^{20}dx$

(4) $\int \frac{dx}{\sqrt{x+3}}$

(5) $\int (x-1)^2 .xdx$

(6) $\int \frac{xdx}{(x^2+4)^3}$

(7) $\int (x^2-x)^4.(2x-1)dx$

(8) $\int \frac{(x+1)dx}{\sqrt{x^2+2x-4}}$

(9) $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}dx$

(10) $\int (\frac{dx}{2x-1} - \frac{dx}{2x+1})$

(11) $\int (e^x+1)^2dx$

(12) $\int (e^x - x^e)dx$

(13) $\int (e^x+1)^2.e^x dx$

(14) $\int (e^x+e^{-x})^2dx$

(15) $\int \tan^2 xdx$

(16) $\int \sin^3 x \cos xdx$

(17) $\int \tan^5 x .\sec^2 xdx$

(18) $\int \sec^2(\frac{x}{a})\tan(\frac{x}{a})dx$

(19) $\int e^{2\sin 3x}\cos 3xdx$

(20) $\int \cos^4 x \sin xdx$

Chapter 2
Integration by Parts

When u and v are differentiable functions of x ,

$d(uv) = udv + vdu$

$\int d(uv) = \int udv + \int vdu$

$uv = \int udv + \int vdu$

$\int udv = uv - \int vdu$ **(1)**

When *(1)* is to be used in a required integration, the given integral must be separated into
two parts, one part being *U* and the other part, together with *dx,* being *dv.* (For this reason,
integration by use of *(1)* is called *integration* by *parts.)*

Two general rules can be stated:

(a) The part selected as *dv* must be readily integrable.

(b) $\int vdu$ **must not be more complex than** $\int udv$ **.**

<u>Ex</u>(1) Find $\int x^3 e^{x^2} dx$

<u>Solution</u> $\int x^3 e^{x^2} dx = \frac{1}{2}\int x^2 (e^{x^2} .2xdx)$

$= \frac{1}{2}\int x^2 d(e^{x^2}) = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2}\int e^{x^2} .2xdx$

$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C = \frac{e^{x^2}}{2}[x^2 - 1] + C$

$= (\frac{x^2 - 1}{2}) e^{x^2} + C$

Ex(2) Find $\int \ln(x^2 + 2)dx$

Solution $\int \ln(x^2 + 2)dx = x \ln(x^2 + 2) - \int x(\frac{2x}{x^2 + 2})dx$

$= x \ln(x^2 + 2) - \int \frac{2x^2}{x^2 + 2} dx$

$= x \ln(x^2 + 2) - \int [2 - \frac{4}{x^2 + 2}]dx \quad = x \ln(x^2 + 2) - 2x + \frac{4}{\sqrt{2}} \tan^{-1}(\frac{x}{\sqrt{2}}$

$) + C$

$$= x\ \ln(x^2+2) - 2x + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Ex(3) Find $\int x\ \sin x dx$

Solution $\int x\ \sin x dx = -\int x d(\cos x)$

$= -[x\cos x - \int \cos x dx] = -x\cos x + \sin x + C$

Ex(4) Find $\int xe^{x}dx$

Solution $\int xe^{x}dx = \int x d(e^{x}) = xe^{x} - \int e^{x}dx$

$= xe^{x} - e^{x} + C$

Ex(5) Find $\int x^{2}\ln x dx$

Solution $\int x^{2}\ln x dx = \frac{1}{3}\int \ln x d(x^{3}) = \frac{1}{3}[x^{3}\ln x - \int x^{3}\frac{1}{x}dx]$

$= \frac{1}{3}[x^{3}\ln x - \int x^{2}dx] = \frac{1}{3}[x^{3}\ln x - \frac{1}{3}x^{3}] + C$

$= \frac{x^{3}}{3}\ln x - \frac{x^{3}}{9} + C$

Ex(6) Find $\int \sin^{-1}x dx$

Solution $\int \sin^{-1}x dx = x\sin^{-1}x + \int \frac{-x}{\sqrt{1-x^{2}}}dx$

$= x\sin^{-1}x + \sqrt{1-x^{2}} + C$

Ex(7) Find $\int \sin^{2}x dx$

Solution $I = \int \sin^{2}x dx = \int \sin x(\sin x dx)$

$= -\int \sin x d(\cos x) = -[\sin x\cos x - \int \cos^{2}x dx]$

$= -\sin x\cos x + \int (1-\sin^{2}x)dx$

$= -\sin x\cos x + \int dx - \int \sin^{2}x dx$

$2I = x - \sin x\cos x + C_{0}$

$I = \frac{x}{2} - \frac{1}{2}\sin x\cos x + C$

Ex(8) Find $\int \sec^{3}x dx$

Solution $I = \int \sec x.(\sec^{2}x dx)$ $I = \int \sec x d(\tan x) = \sec x.\tan x - \int \tan x.\sec x\tan x dx$

$= \sec x\tan x - \int \sec x(\sec^{2}x - 1)dx$

$= \sec x\tan x - \int \sec^{3}x dx + \int \sec x dx$

$2I = \sec x\tan x + \ln|\sec x + \tan x| + C_{0}$

$$I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Ex(9) Find $\int x^2 \sin x dx$

Solution $I = \int x^2 \sin x dx$

$$= -\int x^2 d(\cos x) = -[x^2 \cos x - 2\int x \cos x dx]$$

$$= -x^2 \cos x + 2\int x d(\sin x)$$

$=-x^2\cos x+2[x\sin x-\int\sin x\,dx]$

$=-x^2\cos x+2x\sin x+2\cos x+C$

Ex(10) Find $\int x^3e^{2x}dx$

Solution $I=\int x^3e^{2x}dx=\frac{1}{2}\int x^3d(e^{2x})$

$=\frac{1}{2}[x^3e^{2x}-\int 3x^2e^{2x}dx]$

$=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\frac{1}{2}\cdot x^2d(e^{2x})$

$=\frac{1}{2}x^3e^{2x}-\frac{3}{4}[x^2e^{2x}-\int 2xe^{2x}dx]$

$=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}\int xd(e^{2x})$

$=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}[xe^{2x}-\int e^{2x}dx]$

$=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}+\frac{3}{4}xe^{2x}-\frac{3}{8}e^{2x}+C$

Exercises: **Find the following integrals**

(1) $\int x \cos x dx$

(2) $\int x \sec^2 3x dx$

(3) $\int \cos^{-1} x dx$

(4) $\int \frac{xe^x}{(1+x)^2} dx$

(5) $\int x^3 \sin x dx$

(6) $\int \sin(\ln x) dx$

Chapter 3
Trigonometric Integrals

THE FOLLOWING IDENTITIES are employed to find some of the trigonometric integrals of this chapter:

(1) $\sin^2 x + \cos^2 x = 1$

(2) $1 + \tan^2 x = \sec^2 x$

(3) $1 + \cot^2 x = \csc^2 x$

(4) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

(5) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

(6) $\sin x \cos x = \frac{1}{2}\sin 2x$

(7) $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$

(8) $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

(9) $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$

(10) $1 - \cos x = 2\sin^2 \frac{1}{2}x$

(11) $1 + \cos x = 2\cos^2 \frac{1}{2}x$

(12) $1 \pm \sin x = 1 \pm \cos(\frac{1}{2}\pi - x)$

TWO SPECIAL SUBSTITUTION RULES are useful in a few simple cases:

1-For $\int \sin^m x \cos^n x dx$ **: if m is odd , substitute u = cos x . if n is odd , substitute u = sin x .**

2- for $\int \tan^m x . \sec^n x dx$ **: if n is even , substitute u = tan x . if m is odd , substitute u = sec x.**

Ex(1) Evaluate each of the following integrals :

(a) $\int \sin^2 x dx = \frac{1}{2}\int (1 - \cos 2x)dx$

$$= \frac{1}{2}[x - \frac{1}{2}\sin 2x] + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(b) $\int \cos^2 3x dx = \frac{1}{2}\int (1 + \cos 6x)dx$

$$= \frac{1}{2}[x + \frac{1}{6}\sin 6x] + C$$

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$$= \frac{1}{2}x + \frac{1}{12}\sin 6x + C$$

(c) $\int \sin^3 x dx = \int \sin^2 x . \sin x dx$

$$= \int (1 - \cos^2 x)\sin x dx = \int \sin x dx + \int \cos^2 x d(\cos x) = -\cos x + \frac{\cos^3 x}{3} + C$$

(d) $\int \cos^5 x dx = \int \cos^4 x . \cos x dx$

$= \int (1-\sin^2 x)^2 . \cos x dx = \int \cos x dx - 2\int \sin^2 x \cos x dx + \int \sin^4 x \cos x dx$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C$

(e) $\int \sin^2 x \cos^3 x dx = \int \sin^2 x (1-\sin^2 x) \cos x dx$

$= \int \sin^2 x . \cos x dx - \int \sin^4 x \cos x dx$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

(f) $\int \sin^4 x dx = \int (\sin^2 x)^2 dx = \int [\frac{1}{2}(1-\cos 2x)]^2 dx$

$= \frac{1}{4}\int (1-\cos 2x)^2 dx = \frac{1}{4}\int [1-2\cos 2x + \cos^2 2x] dx$

$= \frac{1}{4}[x - 2\frac{\sin 2x}{2} + \frac{1}{2}\int (1+\cos 4x) dx]$

$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}[x + \frac{\sin 4x}{4}] + C$

Ex (2) Evaluate the following integrals:

(a) $\int \tan^4 x dx = \int \tan^2 x (\sec^2 x - 1) dx$

$= \int \tan^2 x . \sec^2 x dx - \int \tan^2 x dx$

$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) dx = \frac{1}{3}\tan^3 x - \int \sec^2 x dx + \int dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

(b) $\int \tan^5 x dx = \int \tan^3 x (\sec^2 x - 1) dx$

$= \int \tan^3 x \sec^2 x dx - \int \tan x . \tan^2 x dx$

$= \frac{1}{4}\tan^4 x - \int \tan x (\sec^2 x - 1) dx$

$= \frac{1}{4}\tan^4 x - \int \tan x . \sec^2 x dx + \int \tan x dx$

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$

(c) $\int \sec^4 2x dx = \int \sec^2 2x \,(1+\tan^2 2x\,) dx$

$$= \int \sec^2 2x dx + \int \tan^2 2x . \sec^2 2x dx$$

$$= \frac{1}{2}\tan(2x) + \frac{1}{2}\int \tan^2(2x) d(\tan 2x)$$

$$= \frac{1}{2}\tan(2x) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\tan^3(2x) + C$$

$$= \frac{1}{2}\tan(2x) + \left(\frac{1}{6}\right)\tan^3(2x) + C$$

(d) $$\int \tan^3(3x)\sec^4(3x)dx = \int \tan^3(3x)(1+\tan^2 3x)\sec^2 3x dx$$

$$= \int \tan^3(3x).\sec^2(3x)dx + \int \tan^5(3x).\sec^2(3x)dx$$

$$= \frac{1}{3}\int \tan^3(3x) d\tan(3x) + \frac{1}{3}\int \tan^5(3x) d\tan(3x)$$

$$= \frac{1}{3}\frac{\tan^4(3x)}{4} + \frac{1}{3}\frac{\tan^6(3x)}{6} + C$$

$$= \frac{1}{12}\tan^4(3x) + \frac{1}{18}\tan^6(3x) + C$$

(e) $$\int \tan^2 x \sec^3 x dx = \int (\sec^2 x - 1)\sec^3 x dx$$

$$I = \int \sec^5 x dx - \int \sec^3 x dx \qquad \textbf{(1)}$$

$$I_1 = \int \sec^5 x dx = \int \sec_3 x . \sec_2 x dx$$

$$= \int \sec^3 x d\tan x = \sec^3 x \tan x - \int \tan x . 3\sec^2 x . \sec x \tan x dx$$

$$= \sec^3 x \tan x - 3\int \sec^3 x \tan^2 x dx$$

$$I_1 = \sec^3 x \tan x - 3I \qquad \textbf{(2)}$$

$$I_2 = \int \sec^3 x dx = \int \sec x . \sec^2 x dx$$

$$= \int \sec x d\tan x = \sec x \tan x - \int \tan x . \sec x \tan x dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$2I_2 = \sec x \tan x + \ln|\sec x + \tan x| + C \qquad \textbf{(3)}$$

Substitute from (2) and (3) into (1)

$$I = \sec^3 x \tan x - 3I - \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$4I = \sec^3 x \tan x - \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$I = \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x + \frac{1}{8}\ln|\sec x + \tan x| + C$$

Exercises Evaluate the following integrals:

(1) $\int \cos^2 x dx$

(2) $\int \sin^3 2x dx$

(3) $\int \sin^4 2x dx$

(4) $\int \cos^4 \frac{1}{2} x dx$

(5) $\int \sin^2 x \cos^5 x dx$

(6) $\int \sin^3 x \cos^2 x dx$

(7) $\int \tan^3 x dx$

(8) $\int \tan^4 x \sec^4 x dx$

Chapter 4
Trigonometric Substitutions

SOME INTEGRATIONS may be simplified with the following substitutions:

1. **If an integrand contains** $\sqrt{a^2 - x^2}$ **, substitute** $x = a\sin z$.
2. **If an integrand contains** $\sqrt{a^2 + x^2}$ **, substitute** $x = a\tan z$.
3. **If an integrand contains** $\sqrt{x^2 - a^2}$ **, substitute** $x = a\sec z$.

More generally, an integrand that contains one of the forms $\sqrt{a^2 - b^2x^2}$ **,** $\sqrt{a^2 + b^2x^2}$ **, or** $\sqrt{b^2x^2 - a^2}$ **but no other irrational factor may be transformed into another involving trigonometric functions of a new variable as follows:**

For	Use	To obtain
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b}\sin z$	$a\sqrt{1-\sin^2 z} = a\cos z$
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b}\tan z$	$a\sqrt{1+\tan^2 z} = a\sec z$
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b}\sec z$	$a\sqrt{\sec^2 z - 1} = a\tan z$

In each case, integration yields an expression in the variable z. The corresponding expression in the original variable may be obtained by the use of a right triangle as shown in the solved problems that follow.

Ex(1) (a) Find $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Solution put $x = 2\tan y \Rightarrow dx = 2\sec^2 y \Rightarrow$

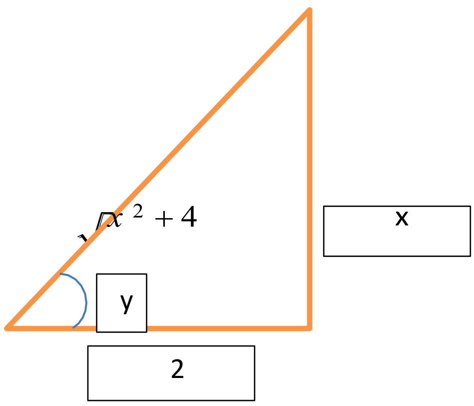
$$I = \int \frac{2\sec^2 y}{4\tan^2 y\sqrt{4+4\tan^2 y}}dy = \int \frac{2\sec^2 y}{(4\tan^2 y)(2\sec y)}dy$$

$$= \frac{1}{4}\int \frac{\sec y}{\tan^2 y}dy = \frac{1}{4}\int \sin^{-2} y \cos y dy$$

$$= \frac{1}{4}\int \sin^{-2} y d\sin y = \frac{1}{4}\left(\frac{1}{\sin y}\right) + C$$

$$= \frac{-1}{4\sin y} + C$$

$$= \frac{-\sqrt{x^2+4}}{4x} + C$$

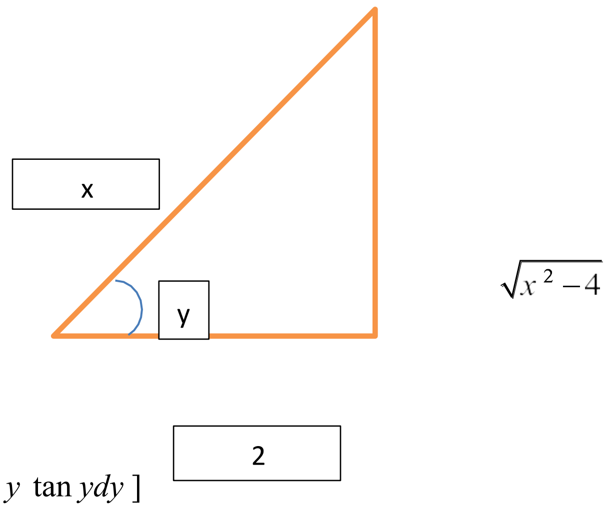


(b) **Find** $\int \frac{x^2}{\sqrt{x^2-4}}dx$

Solution put $x = 2\sec y$

$dx = 2\sec y \tan y dy$

$$I = \int \frac{4\sec^2 y .2\sec y \tan y}{\sqrt{4\sec^2 y - 4}} dy$$



$$= \int \frac{4\sec^2 y .2\sec y \tan y}{2\tan y} dy$$

$$= 4\int \sec^3 y dy = 4\int \sec y .\sec^2 y dy$$

$$= 4\int \sec y d(\tan y) = 4[\sec y \tan y - \int \tan y \sec y \tan y dy]$$

$$I = 4\sec y \tan y - 4\int \sec y (\sec^2 y - 1)dy$$

$$= 4\sec y \tan y - I + 4\int \sec y dy$$

$$\therefore 2I = 4\sec y \tan y + 4\ln|\sec y + \tan y| + C$$

$$I = 2\sec y \tan y + 2\ln|\sec y + \tan y| + C$$

$$= 2\left(\frac{x}{2}.\frac{\sqrt{x^2-4}}{2}\right) + 2\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

$$= \frac{1}{2}x\sqrt{x^2-4} + 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C$$

$$= \frac{x}{2}\sqrt{x^2-4} + 2\ln\left|x + \sqrt{x^2-4}\right| - 2\ln 2 + C$$

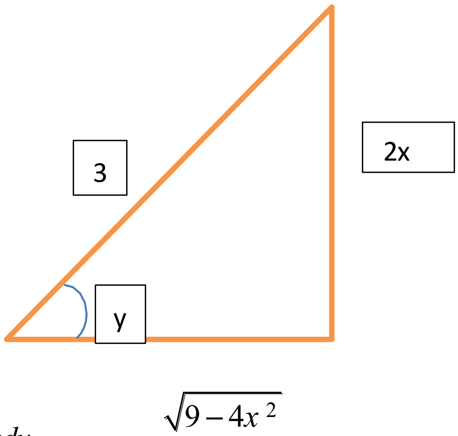
$$\therefore I = \frac{x\sqrt{x^2-4}}{2} + 2\ln\left|x + \sqrt{x^2-4}\right| + C^*,$$

C^* **is a constant**

***(c)* Find** $\int \frac{\sqrt{9-4x^2}}{x}\,dx$

Solution put $x = \frac{3}{2}\sin y \Rightarrow \frac{2x}{3} = \sin y$

$\therefore dx = \frac{3}{2}\cos y\,dy \Rightarrow$



$$I = \int \frac{\sqrt{9-9\sin^2 y}}{\frac{3}{2}\sin y}\,\frac{3}{2}\cos y\,dy = \int \frac{3\cos y\,.\cos y}{\sin y}\,dy$$

$$I = 3\int \frac{\cos^2 y}{\sin y}\,dy = 3\int \frac{(1-\sin^2 y)}{\sin y}\,dy$$

$$= 3\int \csc y\,dy - 3\int \sin y\,dy$$

$$= 3\ln\left|\csc y - \cot y\right| + 3\cos y + C$$

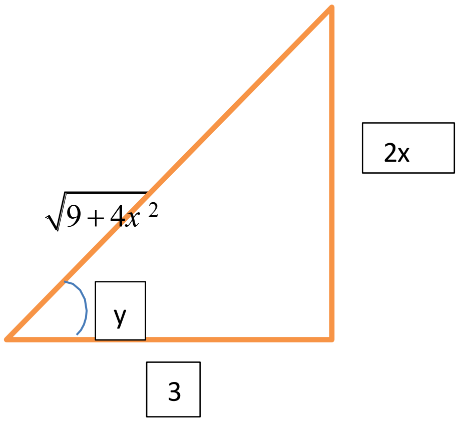
$$= 3\ln\left|\frac{3}{2x} - \frac{\sqrt{9-4x^2}}{2x}\right| + 3\frac{\sqrt{9-4x^2}}{3} + C$$

$$= 3\ln\left|\frac{3-\sqrt{9-4x^2}}{x}\right| + \sqrt{9-4x^2} + C^*,$$

C^* **is a constant**

(d) Find $\int \frac{dx}{x\sqrt{9+4x^2}}$

Solution : put $x = \frac{3}{2}\tan y \Rightarrow \tan y = \frac{2x}{3}$



$dx = \frac{3}{2}\sec^2 y dy \Rightarrow$

$$I = \int \frac{\frac{3}{2}\sec^2 y dy}{(\frac{3}{2}\tan y)(3\sec y)} = \frac{1}{3}\int \csc y dy$$

$$\therefore I = \frac{1}{3}\ln\left|\csc y - \cot y\right| + C^*$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}}{2x} - \frac{3}{2x}\right| + C^*$$

$$\therefore I = \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{x}\right| + C$$

C **is a constant**

Exercises Find each of the following integrals

(1) $\int \frac{\sqrt{25-x^2}}{x}dx$

(2) $\int \frac{dx}{x^2\sqrt{a^2-x^2}}$

(3) $\int \sqrt{x^2+4}dx$

(4) $\int \frac{\sqrt{x^2+a^2}}{x}dx$

(5) $\int \frac{dx}{x^2\sqrt{9-x^2}}$

(6) $\int \frac{dx}{(9+x^2)^2}$

Chapter (5)
Integration by Partial Fractions

A POLYNOMIAL IN *x* is a function of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ **, where the a's are constants ,** $a_0 \neq 0$**, and n, called the *degree* of the polynomial, is a nonnegative integer.**
If two polynomials of the same degree are equal for all values of the variable, then the Coefficients of the like powers of the variable in the two polynomials are equal.
Every polynomial with real coefficients can be expressed (at least, theoretically) as a Product of real linear factors of the form *ax + b* and real irreducible quadratic factors of the form $ax^2 + bx + c$ **(A polynomial of degree 1 or greater is said to be *irreducible* if it cannot be factored into polynomials of lower degree.) By the quadratic formula,** $ax^2 + bx + c$ **is irreducible if and only if and only if** $b^2 - 4ac < 0$ **(In that case, the roots of** $ax^2 + bx + c = 0$ **are not real.)**

Example 1: (a) $x^2 - x + 1$ **is irreducible , since** $(-1)^2 - 4(1)(1) = -3 < 0$

(b) $x^2 - x - 1$ **is not irreducible , since** $(-1)^2 - 4(1)(-1) = 5 > 0$**.**

In fact , $x^2 - x - 1 = (x - \frac{1+\sqrt{5}}{2})\ (x - \frac{1-\sqrt{5}}{2})$

A FUNCTION $F(x) = f(x)/g(x)$ **, where** $f(x)$ **and** $g(x)$ **are polynomials, is called a *rational fraction.***
If the degree of $f(x)$ **is less than the degree of** $g(x)$ **, *F(x)* is called *proper;* otherwise, *F(x)* is called *improper.***
An improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction. Thus, $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$

Every proper rational fraction can be expressed (at least, theoretically) as a sum of simpler fractions *(partial fractions)* whose denominators are of the form $(ax+b)^n$ **and** $(ax^2+bx+c)^n$**, *n* being a positive integer. Four cases, depending upon the nature of the factors of the denominator, arise.**

CASE I: DISTINCT LINEAR FACTORS. To each linear factor *ax + b* occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$**, where *A* is a constant to be determined.**

CASEII: REPEATED LINEAR FACTORS. To each linear factor *ax* + *b* occurring *n* times in the denominator of a proper rational fraction, there corresponds a sum of *n* partial fractions of the form $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\dots+\frac{A_n}{(ax+b)^n}$

where the *A's* are constants to be determined.

CASE III: DISTINCT QUADRATIC FACTORS. To each irreducible quadratic factor ax^2+bx+c occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

CASE IV: REPEATED QUADRATIC FACTORS. To each irreducible quadratic factor ax^2+bx+c occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the A's and B's are constants to be determined.

Ex (a) Find $\displaystyle\int\frac{dx}{x^2-4}$

Solution $\dfrac{1}{x^2-4}=\dfrac{A}{x-2}+\dfrac{B}{x+2}\Rightarrow$

$A(x+2)+B(x-2)=1$ **(*)** $\Rightarrow$

Put x = 2 $\overset{(*)}{\Rightarrow}4A=1\Rightarrow A=1/4$

Put x = -2 $\overset{(*)}{\Rightarrow}-4B=1\Rightarrow B=-1/4$

$$\therefore I=\frac{1}{4}\int\frac{1}{x-2}\,dx-\frac{1}{4}\int\frac{1}{x+2}\,dx$$

$$=\frac{1}{4}[\ln|x-2|-\ln|x+2|]+C$$

$$=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$$

(b) Find $\displaystyle\int\frac{(x+1)}{x^3+x^2-6x}dx$

Solution : $\therefore x+1=A(x-2)(x+3)+Bx(x+3)+Cx(x-2)$ $\quad\dfrac{x+1}{2}\quad 1\quad 1\ 1\ 3\ 1$

$\therefore$

Put x = 0 $\overset{(*)}{\Rightarrow}1=-6A\Rightarrow A=-1/6$

Put x = 2 $\overset{(*)}{\Rightarrow}3=10B\Rightarrow B=3/10$

Put x = -3 $\overset{(*)}{\Rightarrow}-2=15C\Rightarrow C=-2/15$

(*)

$$x(x+3)(x-2) \quad = - \quad 6\ x \quad 10\ \ x-2 \quad 15\ \ x+3$$

$$\Rightarrow I = -\frac{1}{6}\int \frac{1}{x}\,dx + \frac{3}{10}\int \frac{1}{x-2}\,dx - \frac{2}{15}\int \frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C$$

$$= \ln\left(\frac{|x-2|^{3/10}}{|x|^{1/6}\,|x+3|^{2/15}}\right) + C$$

(c) Find $\int \frac{(3x+5)}{x^3 - x^2 - x + 1}dx$

Solution $x^3 - x^2 - x + 1 = (x+1)(x-1)^2$. **Hence,**

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

$$\Rightarrow 3x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1) \qquad \textbf{(*)}$$

Put x = 1 $\overset{(*)}{\Rightarrow} 8 = 2C \Rightarrow C = 4$

Put x = -1 $\overset{(*)}{\Rightarrow} 2 = 4A \Rightarrow A = 1/2$

Put x = 0 $\overset{(*)}{\Rightarrow} 5 = A - B + C = \frac{1}{2} - B + 4$

$$B = 4\frac{1}{2} - 5 = -\frac{1}{2} \Rightarrow B = -1/2$$

$$\therefore \frac{3x+5}{x^3-x^2-x+1} = \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x-1}\right) + 4\left(\frac{1}{(x+1)^2}\right)$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx + 4\int (x+1)^{-2}dx$$

$$= \frac{1}{2}\left(\ln|x+1| - \ln|x-1|\right) - \frac{4}{(x+1)} + C$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| - \frac{4}{(x+1)} + C$$

$$\therefore I = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| - \frac{4}{(x+1)} + C$$

(e) Find $\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}dx$

Solution $x^4 - x^3 - x - 1 = \quad - \quad x + 1$

We write $\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$x^3 - x^2 \quad x \quad x^2(x-1)$

$= \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)} \Rightarrow$

$x+1 = Ax(x-1)+B(x-1)+Cx^2$ **(*)**

Put x = 0 $\overset{(*)}{\Rightarrow} 1 = -B \Rightarrow B = -1$

Put x = 1 $\overset{(*)}{\Rightarrow} 2 = C$

Put x = 2 $\overset{(*)}{\Rightarrow} 3 = 2A + B + 4C$

$= 2A - 1 + 8 \Rightarrow$

$-4 = 2A \Rightarrow A = -2 \Rightarrow$

$$\frac{x+1}{x^2(x-1)} = -2\left(\frac{1}{x}\right) - \left(\frac{1}{x^2}\right) + 2\left(\frac{1}{x-1}\right) \Rightarrow$$

$$I = \frac{x^2}{2} + 2\int \frac{1}{x}\,dx + \int x^{-2}\,dx - 2\int \frac{1}{x-1}\,dx \Rightarrow$$

$$I = \frac{1}{2}x^2 + 2\ln x - \frac{1}{x} - 2\ln|x-1| + C$$

$$I = \frac{1}{2}x^2 - \frac{1}{x} + 2\ln\left|\frac{x}{x-1}\right| + C$$

(f) Find $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx$

Solution

$$x^4+3x^2+2 = (x^2+1)(x^2+2) \Rightarrow$$

$$\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \Rightarrow$$

$$x^3+x^2+x+2 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D) \qquad \textbf{(*)}$$

$$\Rightarrow (A+C) = 1, B+D = 1, 2A+C = 1,$$

$$2B+D = 2 \Rightarrow A = 0, B = 1, C = 1, D = 0 \Rightarrow$$

$$\therefore \int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx = \int \frac{1}{x^2+1}dx + \int \frac{x}{x^2+2}dx$$

$$= \tan^{-1}x + \frac{1}{2}\ln(x^2+2) + C$$

(g) solve the equation $\int \frac{x^2dx}{a^4-x^4} = \int Kdt$

Solution $\frac{x^2}{a^4-x^4} = \frac{x^2}{(a^2-x^2)(a^2+x^2)}$

$$= \frac{x^2}{(a-x)(a+x)(a^2+x^2)}$$

$$x^2 = \frac{A}{a-x} + \frac{B}{a+x} + \frac{(Cx+D)}{a^2+x^2} \Rightarrow$$

$$x^2 = A(a+x)(a^2+x^2) + B(a-x)(a^2+x^2) + (Cx+D)(a-x)(a+x) \qquad \textbf{(*)}$$

Put x = a $\overset{(*)}{\Rightarrow} A(2a)(2a^2) = a^2 \Rightarrow$

$$A=\frac{a^2}{4a^3}\Rightarrow\frac{1}{4a}=A$$

Put x = -a $\overset{(*)}{\Rightarrow} a^2=B(2a)(a^2+a^2)=4a^3B\Rightarrow B=1|4a$

Put x = 0 $\overset{(*)}{\Rightarrow} 0=a^3A+a^3B+a^2D$

$0=a^3(A+B)+a^2D=(a^3)(\frac{1}{\_\_})+a^2D$

$2a$

$$a^2D = -\frac{1}{2}a^2 \Rightarrow D = \frac{-1}{2}$$

Put x = 2a $\overset{(*)}{\Rightarrow} 4a^2 = (3a)(5a^2)A + (-a)(5a^2)B + (2aC + D)(-a)(3a)$

$$4a^2 = 15a^3A - 5a^3B - 3a^2(2aC + D)$$

$$= \frac{15a^3}{4a} - \frac{5a^3}{4a} - 6a^3C - 3a^2\left(-\frac{1}{2}\right)$$

$$4a^2 = \frac{15}{4}a^2 - \frac{5}{4}a^2 - 6a^3C + \frac{3}{2}a^2$$

$$\therefore C = 0$$

$$\therefore \int \frac{x^2}{a^4 - x^4}dx = \frac{1}{4a}\int \frac{dx}{a-x} + \frac{1}{4a}\int \frac{dx}{a+x} - \frac{1}{2}\int \frac{dx}{a^2+x^2}$$

$$= -\frac{1}{4a}\ln|a-x| + \frac{1}{4a}\ln|a+x| - \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) + C = Kt \Rightarrow$$

$$Kt = \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Exercises Evaluate each of the following integrals :

(1) $\int \frac{dx}{x^2-9}$

(2) $\int \frac{xdx}{x^2-3x-4}$

(3) $\int \frac{x^2+3x-4}{x^2-2x-8}dx$

(4) $\int \frac{x^2-3x-1}{x^3+x^2-2x}dx$

(5) $\int \frac{dx}{x^3+x}$

(6) $\int \frac{2x^2+3}{(x^2+1)^2}dx$

Chapter (6)
Integration of Hyperbolic Functions

INTEGRATION FORMULAS

1. $\int \sinh x \, dx = \cosh x + C$

2. $\int \cosh x \, dx = \sinh x + C$

3. $\int \tanh x \, dx = \ln \cosh x + C$

4. $\int \coth x \, dx = \ln |\sinh x| + C$

5. $\int \operatorname{sech}^2 x \, dx = \tanh x + C$

6. $\int \operatorname{csch}^2 x \, dx = -\coth x + C$

7. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$

8. $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$

9. $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1} \frac{x}{a} + C$

10. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} \frac{x}{a} + C$, $x > a > 0$

11. $\int \frac{dx}{a^2-x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C$, $x^2 < a^2$

12. $\int \frac{dx}{x^2-a^2} = -\frac{1}{a} \coth^{-1} \frac{x}{a} +$, $x^2 > a^2$

Evaluate the following integrals:

1. $\int \cosh 2x \, dx = \frac{1}{2} \int \cosh 2x \, d(2x) = \frac{1}{2} \sinh 2x + C$

2. $\int \operatorname{sech}^2 (2x-1)x = \frac{1}{2} \int \operatorname{sech}^2 (2x-1)(2x-1) = \frac{1}{2} \tanh (2x-1) + C$

3. $\int csch\, 3x\, coth\, 3x\, dx = \frac{1}{3} \quad csch\, 3x\, coth\, 3x\, (3x) = -\frac{1}{3}\, csch\, 3x + C$

4. $\int sech\, x\, dx = \int \frac{1}{\quad} dx = \int \quad cosh\, x \qquad cosh\, x$

$\int \qquad \int \quad \int$

$1+sinh^2 x$ $dx =$ $arctan (sinh x) + C$

5. $\int sinh^2 x\, dx = \frac{1}{2}\int (cosh\, 2x - 1)dx = \frac{1}{4}\, sinh\, 2x - \frac{1}{2}\, x + C$

6. $\int tanh^2\, 2x\, dx = \int (1 - sech^2\, 2x)\; dx = x - \frac{1}{2}\, tanh\, 2x + C$

7. $\int e^x\, cosh\, x\, dx = \int e^x \frac{e^x + e^{-x}}{2} dx = \frac{1}{2}\int (e^{2x} + 1)\, dx = \frac{1}{4}\, e^{2x} + \frac{1}{2}\, x + C$

8. $\int x \sinh x\, dx = \int x \frac{e^x - e^{-x}}{2} dx = \frac{1}{2}\int xe^x\, dx - \frac{1}{2}\int xe^{-x}\, dx$

$$= \frac{1}{2}(xe^x - e^x) - \frac{1}{2}(-xe^{-x} - e^{-x}) + C = x\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} + C$$

$$= x \cosh x - \sinh x + C$$

9. $\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\cosh^{-1}\frac{2x}{3} + C$

10. $Find \int \sqrt{x^2 + 4}\, dx$

$let\ x = 2\sinh z.\ then\ dx = 2\cosh z\, dz,\ \sqrt{x^2+4} = 2\cosh z,\ and$

$\int \sqrt{x^2+4} = 4\int \cosh^2 z\, dz = 2\int(\cosh 2z + 1)dz = \sinh 2z + 2z + C$

$= 2\sinh z \cosh z + 2z + C = \frac{1}{2}x\sqrt{x^2+4} + 2\sinh^{-1}\frac{1}{2}x + C$

11. $Find \int \frac{dx}{x\sqrt{1-x^2}}$

$let\ x = sech\, z.\ then\ dx = -sech\, z \tanh z\, dz,\ 1 - x^2 = \tanh z,\ and$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\int \frac{sech\, z \tanh z}{sech\, z \tanh z} dz = -\int dz = -z + C = -sech^{-1} x + C$$

Exercises Evaluate the following integrals:

1. $\int sinh\ 3x\ dx$

2. $\int sech\ 2x\ tanh\ 2x\ dx$

3. $\int csch\ x\ dx$

4. $\int coth^2\ 3x\ dx$

5. $\int sinh^3\ x\ dx$

6. $\int e^x\ sinh\ x\ dx$

7. $\int \frac{dx}{\sqrt{x^2-25}}$

8. $\int \frac{dx}{4-9x^2}$

9. $\int \frac{dx}{16x^2-9}$

10. $\int \sqrt{x^2-9}\ dx$

Chapter (7)
Applications of Indefinite Integrals

WHEN THE EQUATION $y = f(x)$ of a curve is known, the slope m at any point $P(x, y)$ on it is given by $m = f'(x)$. Conversely, when the slope of a curve at a point $P(x, y)$ on it is given by $m = dy/dx = f'(x)$, a family of curves, $y = f(x) + C$, may be found by integration. To single out a particular curve of the family, it is necessary to assign or to determine a particular value of C.
This may be done by prescribing that the curve pass through a given point.

AN EQUATION s = $f(t)$, where s is the distance at time t of a body from a fixed point in its (straight-line) path, completely defines the motion of the body. The velocity and acceleration at time t are given by

$$v = \frac{ds}{dt} = f'(t) \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = f''(t)$$

Conversely, if the velocity (or acceleration) is known at time t, together with the position (or position and velocity) at some given instant, usually at $t = 0$, the equation of motion may be obtained.

Ex (1) Find the equation of the family of curves whose slope at any point is equal to the negative of twice the abscissa of the point. Find the curve of the family which passes through the point (1,1).
We are given that $dy / dx = -2x$. Then $dy = -2x\, dx$, from which $\int dy = \int -2x\,dx$ **and**

$y = -x^2 + C$ **.**
This is the equation of a family of parabolas.
Setting x = 1 and y = 1 in the equation of the family yields 1 = - 1 + C or C = 2. The equation of the curve passing through the point (1 , 1) is then $y = -x^2 + 2$.

Ex (2) Find the equation of the family of curves whose slope at any point $P(x, y)$ is $m = 3x^2 y$ **. Find the equation of the curve of the family which passes through the point (0,8).**
Since $m = \frac{dy}{dx} = 3x^2 y$ **, we have** $\frac{dy}{y} = 3x^2 dx$ **. Then** $\ln y = x^3 + C = x^3 + \ln c$ **and** $y = ce^{x^3}$ **when**

$x = 0$ **and** $y = 8$**, then** $8 = ce^0 = c$ **. The equation of the required curve is** $y = 8e^{x^3}$ **.**

Ex (3) at every point of a certain curve , $y^n = x^2 - 1$**. Find the equation of the curve if it**

passes through the point (1 , 1) and is there tangent to the line x + 12y = 13 .

Here $\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = x^2 - 1$**. Then** $\int \frac{d}{dx}(y')dx = \int (x^2 - 1)dx$ **and** $y' = \frac{x^3}{3} - x + C_1$

At (1, 1), the slope y' of the curve equals the slope $-\frac{1}{12}$ of the line. Then $-\frac{1}{12} = \frac{1}{3} - 1 + C_1$, from which $C_1 = \frac{7}{12}$. Hence $y' = dy/dx = \frac{1}{3}x^3 - x + \frac{7}{12}$, and integration yields

$$\int dy = \int \left(\frac{1}{3}x^3 - x + \frac{7}{12}\right)dx \quad \text{or} \quad y = \frac{1}{12}x^4 - \frac{1}{2}x^2 + \frac{7}{12}x + C_2$$

At (1, 1), $1 = \frac{1}{12} - \frac{1}{2} + \frac{7}{12} + C_2$ and $C_2 = \frac{5}{6}$. The required equation is $y = \frac{1}{12}x^4 - \frac{1}{2}x^2 + \frac{7}{12}x + \frac{5}{6}$.

Ex (4) the family of orthogonal trajectories of a given system of curves is another system of curves, each of which cuts every curve of the given system at right angles. Find the equations of the orthogonal trajectories of the family of hyperbolas $x^2 - y^2 = c$.

At any point p(x , y), the slope of the hyperbola through the point is given by $m_1 = x/y$, and the slope of the orthogonal trajectory through P is given by $m_2 = dy/dx = -y/x$.

Then $\int \frac{dy}{y} = -\int \frac{dx}{x}$ **so that** $\ln|y| = -\ln|x| + \ln C'$ **or** $|xy| = C'$

The required equation is $xy = \pm C'$ or, simply, $xy = C$

Ex (5) A certain quantity q increases at a rate proportional to itself. If q = 25 when t = 0 and q = 75 when t = 2, find q when t = 6.

Since $dq/dt = kq$, we have $dq/q = kdt$. Integration yields $\ln q = kt + \ln c$ or $q = ce^{kt}$.

When t = 0, q = 25 = ce^0; hence, c = 25 and $q = 25e^{kt}$.

When t = 2, $q = 25e^{2k} = 75$; then $e^{2k} = 3 = e^{1.10}$. So $k = 0.55$ and $q = 25e^{0.55t}$

Finally, when t = 6, $q = 25e^{0.55t} = 25e^{3.3} = 25(e^{1.1})^3 = 25(27) = 675$.

Ex (6) A substance is being transformed into another at a rate proportional to the untransformed amount. If the original amount is 50 and is 25 when t = 3, when will $\frac{1}{10}$ of the substance remain untransformed?

Let q represent the amount transformed in time t. then $dq/dt = k(50 - q)$, from which

$\frac{dq}{50-q} = kdt$ **So that** $\ln(50-q) = -kt + \ln c$ **or** $50-q = ce^{k_1}$

When $t = 0, q = 0$ **and** $c = 50$**; thus** $50-q = 50e^{-kt}$ **.**

When $t = 3, 50-q = 25 = 50e^{-3k}$ **; then** $e^{-3k} = 0.5 = e^{-0.69}, k = 0.23$ **and** $50-q = 50-e^{-0.23t}$

When the untransformed amount is 5, $50e^{-0.23t} = 5$**; then** $e^{-0.23t} = 0.1 = e^{-2.30}$ **and** $t = 10$

Exercises

1) **Find the equation of the family of curves having the given slope, and the equation of the curve of the family which passes through the given point, in each of the following:**

(a) $m = 4x ;(1,5)$

(b) $m = \sqrt{x} ;(9,18)$

(c) $m = (x - 1)^3;(3, 0)$

(d) $m = 1/ x^2;(1, 2)$

(e) $m = x / y ;(4, 2)$

(f) $m = x^2 / y^3;(3, 2)$

(g) $m = 2y / x ;(2,8)$

(h) $m = xy / (1+x^2);(3,5)$

(2) (a) For a certain curve, $y^n = 2$**. Find its equation given that it passes through P(2, 6) with slope 10.**

(b) For a certain curve, $y^n = 6x - 8$**. Find its equation given that it passes through P(1, 0) with slope 4.**

(3) A particle moves along a straight line from the origin (at t = 0) with the given velocity v. find the distance the particle moves during the interval between the two given times t.

(a) $v = 4t + 1;0,4$

(b) $v = 6t + 3;1,3$

(c) $v = 3t^2 + 2t;2,4$

(d) $v = \sqrt{t} + 5; 4,9$

(e) $v = 2t - 2;0,5$

(f) $v = t^2 - 3t + 2; 0, 4$

(4) Find the equation of the family of orthogonal trajectories of the system of parabolas $y^2 = 2x + C$ **.**

(5) A particle moves in a straight line from the origin (at t = 0) with given initial velocity v_0 **and acceleration a. find s at time t.**

(a) **a = 32,** v_0 **= 2**

(b) **a = -32;** v_0 **= 96**

(c) **a =** $12t^2 + 6t$ **;** v_0 **= -3**

(d) $a = 1/\sqrt{t}$ **;** v_0 **= 4**

Chapter (8)

The definite integral

The definite integral Let $a \le x \le b$ be an interval on which a given function f(x) is continuous. Divide the interval into n subintervals $h_1, h_2, ..., h_n$ by the insertion of $n-1$ points $\xi_1, \xi_2, ..., \xi_{n-1}$, where $a < \xi_1 < \xi_2 < ... < \xi_{n-1} < b$, and relabel a as ξ_0 and b as ξ_n.

Denote the length of the subinterval h_1 by $\Delta_1 x = \xi_1 - \xi_0$, of h_2 by $\Delta_2 x = \xi_2 - \xi_1$, … , of h_n by $\Delta_n x = \xi_n - \xi_{n-1}$.(This is done in Fig. 8-1. The lengths are directed distances, each being positive in view of the above inequality.) On each subinterval select a point (x_1 on the subinterval h_1, x_2 on h_2,…, x_n on h_n) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k)\Delta_k x = f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + ... + f(x_n)\Delta_n x \qquad \textbf{(8.1)}$$

each term being the product of the length of a subinterval and the value of the function at the selected point on that subinterval. Denote by λ_n the length of the longest subinterval appearing in (8.1). Now let the number of subintervals increase indefinitely in such a manner that $\lambda_n \to 0$. (one way of doing this would be to bisect each of the original subintervals, then bisect each of these, and so on.) then

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\Delta_k x \qquad \textbf{(8.2)}$$

exists and is the same for all methods of subdividing the interval $a \le x \le b$, so long as the condition $\lambda_n \to 0$ is met, and for all choices of the points x_k in the resulting subintervals.

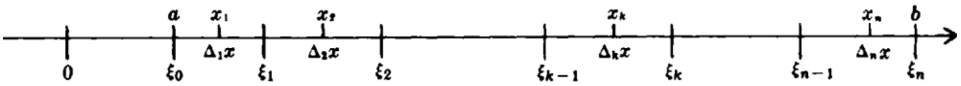


Fig. 8-1

A proof of this theorem is beyond the scope of this book. In Problems 1 to 3 the limit is

evaluated for selected functions $f(x)$. It must be understood, however, that for an arbitrary function this procedure is too difficult to attempt. Moreover, to succeed in the evaluations made here, it is necessary to prescribe some relation among the lengths of the subintervals (we take them all of equal length) and to follow some pattern in choosing a point on each subinterval (for example, choose the left-hand endpoint or the right-hand endpoint or the midpoint of each subinterval).

By agreement, we write $\int_a^b f(x)dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\Delta_k x$

The symbol $\int_a^b f(x)dx$ **is read " the definite integral of f(x), with respect to x, from x = a to x = b." The function f(x) is called the integrand; a and b are called, respectively, the lower and upper limits (boundaries) of integration.**

We have defined $\int_a^b f(x)dx$ **when a < b. the other cases are taken care of by the following definitions:**

$$\int_a^a f(x)dx = 0 \qquad \textbf{(8.3)}$$

If a < b, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$ **(8.4)**

Properties of Definite integrals. If f(x)and g(x) are continuous on the interval of integration $a \le x \le b$, **then**

Property 8.1: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, **for any constant c**

Property 8.2: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)dx$

Property 8.3: $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$, **for a < c < b**

Property 8.4 (first mean-value theorem): $\int_a^b f(x)dx = (b-a)f(x_0)$ **for at least one value** $x = x_0$ **between a and b.**

Property 8.5: if $F(u) = \int_a^u f(x)dx$, **then** $\frac{d}{du}F(u) = f(u)$

Fundamental theorem of integral calculus. If f(x) is continuous on the interval $a \leq x \leq b$, **and if F(x) is any indefinite integral of f(x), then**

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1: (a) Take f(x) = c, a constant, and F(x) = cx; then $\int_a^b c\,dx = cx\Big|_a^b = c(b-a)$**.**

(b) Take f(x) = x and $F(x) = \frac{1}{2}x^2$ **; then** $\int_0^5 x\,dx = \frac{1}{2}x^2\Big|_0^5 = \frac{25}{2} - 0 = \frac{25}{2}$ **.**

(c) Take $f(x)=x^3$ **and** $F(x)=\frac{1}{4}x^4$**; then** $\int_1^3 x^3dx=\frac{1}{4}x^4\Big|_1^3=\frac{81}{4}-\frac{1}{4}=20$.

Use the fundamental theorem of integral calculus to evaluate the following integrals:

1- $\int_{-1}^{1}(2x^2-x^3)dx=\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_{-1}^{1}=\left(\frac{2}{3}-\frac{1}{4}\right)-\left(-\frac{2}{3}-\frac{1}{4}\right)=\frac{4}{3}$

2- $\int_{-3}^{-1}\left(\frac{1}{x^2}-\frac{1}{x^3}\right)dx=\left[-\frac{1}{x}+\frac{1}{2x^2}\right]_{-3}^{-1}=\left(1+\frac{1}{2}\right)-\left(\frac{1}{3}+\frac{1}{18}\right)=\frac{10}{9}$

3- $\int_1^4\frac{dx}{\sqrt{x}}=[2\sqrt{x}]_1^4=2(\sqrt{4}-\sqrt{1})=2$

4- $\int_{-2}^{3}e^{-x/2}dx=[-2e^{-x/2}]_{-2}^{3}=-2(e^{-3/2}-e)=4.9904$

5- $\int_{-6}^{-10}\frac{dx}{x+2}=[\ln|x+2|]_{-6}^{-10}=\ln 8-\ln 4=\ln 2$

6- $\int_{\pi/2}^{3\pi/4}\sin x\,dx=[-\cos x]_{\pi/2}^{3\pi/4}=-(-\frac{1}{2}\sqrt{2}-0)=\frac{1}{2}\sqrt{2}$

7- $\int_{-2}^{2}\frac{dx}{x^2+4}=[\frac{1}{2}\arctan\frac{1}{2}x]_{-2}^{2}=\frac{1}{2}[\frac{1}{4}\pi-(-\frac{1}{4}\pi)]=\frac{1}{4}\pi$

8- $\int_{-5}^{-3}\sqrt{x^2-4}\,dx=\left[\frac{1}{2}x\sqrt{x^2-4}-2\ln\left|x+\sqrt{x^2-4}\right|\right]_{-5}^{-3}=\frac{5}{2}\sqrt{21}-\frac{3}{2}\sqrt{5}-2\ln\frac{3-\sqrt{5}}{5-\sqrt{21}}$

9- $\int_{-1}^{2}\frac{dx}{x^2-9}=\left[\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|\right]_{-1}^{2}=\frac{1}{6}\left(\ln\frac{1}{5}-\ln 2\right)=\frac{1}{6}\ln 0.1$

10- $\int_1^e\ln x\,dx=[x\ln x-x]_1^e=(e\ln e-e)-(\ln 1-1)=1$

11- **Find** $\int_3^6 xy\,dx$ **when** $x=6\cos\theta, y=2\sin\theta$.

We shall express x, y, and dx in the integral in terms of the parameter θ and $d\theta$, change the limits of integration to corresponding values of the parameter, and evaluate the resulting integral. We have, immediately, $dx = -6\sin\theta d\theta$ **. Also, when** $x = 6\cos\theta = 6,$ **then** $\theta = 0;$ **and when** $x = 6\cos\theta = 3,$ **then** $\theta = \pi\ |3.$ **Hence**

0 0

$$\int_{3}^{6} xy\,dx = \int_{\pi/3} (6\cos\theta)(2\sin\theta)(-6\sin\theta)\,d\theta = -72\int_{\pi/3} \sin^2\theta\,\cos\theta\,d\theta$$

$$=[-24\sin^3\theta]_{\pi/3}^{0} = -24\left[0-\left(\frac{\sqrt{3}}{2}\right)^3\right] = 9$$

12- **Find** $\displaystyle\int_{0}^{2\pi} \frac{d\theta}{5+4\cos\theta}$

The substitution $\theta = 2\arctan z$ **(Fig. 8-2) yields** $\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\frac{2dz}{1+z^2}}{5+4\frac{1-z^2}{1+z^2}} = \int \frac{2dz}{9+z^2}$.

To determine the z limits of integration, note that when $\theta = 0, z = 0$; **when** $\theta = 2\pi/3, \arctan z = \pi/3$ **and** $z = \sqrt{3}$. **Then**

$$\int_0^{2\pi/3} \frac{d\theta}{5+4\cos\theta} = 2\int_0^{\sqrt{3}} \frac{dz}{9+z^2} = \frac{2}{3}\left[\arctan\frac{z}{3}\right]_0^{\sqrt{3}} = \frac{\pi}{9}$$

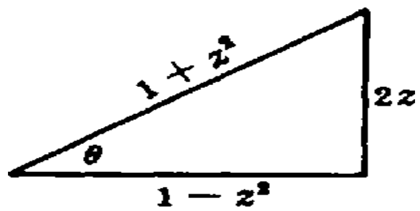


Fig. 8-2

13- Find $\int_0^{\pi/3} \frac{dx}{1-\sin x}$

The substitution $x = 2\arctan z$ **yields** $\int \frac{dx}{1-\sin x} = \int \frac{\frac{2dz}{1+z^2}}{1-\frac{2z}{1+z^2}} = \int \frac{2dz}{(1-z)^2}$. **When** $x = 0, \arctan z = 0$ **and** $z = 0$; **when** $x = \pi/3, \arctan z = \pi/6$ **and** $z = \sqrt{3}/3$. **Then**

$$\int_0^{\pi/3} \frac{dx}{1-\sin x} = 2\int_0^{\sqrt{3}/3} \frac{dz}{(1-z)^2} = \left[\frac{2}{1-z}\right]_0^{\sqrt{3}/3} = \frac{2}{1-\sqrt{3}/3} - 2 = \sqrt{3} + 1$$

Exercises

Use the fundamental theorem to evaluate each integral:

1- $\int_0^2 (2+x)dx$

2- $\int_0^3 (3-2x+x^2)dx$

3- $\int_1^4 (1-u)\sqrt{u}\,du$

4- $\int_0^2 x^2(x^3+1)dx$

5- $\int_0^1 x(1-\sqrt{x})^2 dx$

6- $\int_0^a \sqrt{a^2-x^2}\,dx$

7- $\int_3^4 \frac{dx}{25-x^2}$

8- $\int_2^4 \frac{\sqrt{16-x^2}}{x}dx$

Chapter (9)

Plane Areas by Integration

AREA AS THE LIMIT OF A SUM. If *f (x)* is continuous and nonnegative on the interval $a \le x \le b$ **, the definite integral** $\int_a^b f(x)dx = \lim_{n \to +\infty} \sum_{k=1} f(x_k)\Delta_k x$ **can be given a**

geometric interpretation.
Let the interval $a \le x \le b$ **be subdivided and points** x_k **be selected as in the preceding chapter.**

Through each of the endpoints $\xi_0 = a, \xi_1, \xi_2, \ldots, \xi_n = b$ **erect perpendiculars to the x axis, thus dividing into *n* strips the portion of the plane bounded above by the curve y = *f (x)* , below by the *x* axis, and laterally by the abscissas *x* = a and *x* = b. Approximate each strip as a rectangle whose base is the lower base of the strip and whose altitude is the ordinate erected at the point** x_k **of the subinterval. The area of the kth approximating rectangle, shown in Fig. 9-1,** $f(x_k)\Delta_k x$ **. Hence** $\sum_{k=1}^{n} f(x_k)\Delta_k x$ **is simply the sum of the areas**

of the *n* approximating rectangles.

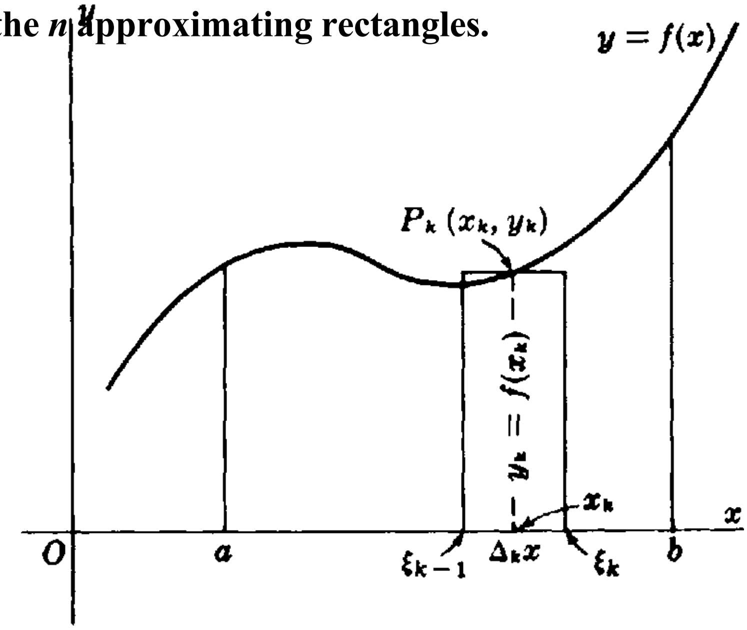


Fig. 9-1

The limit of this sum, as the number of strips is indefinitely increased in the manner prescribed in Chapter 8, is $\int_a^b f(x)dx$ **; it is also, by definition, the area of the portion of the plane described above, or, more briefly, the area under the curve from *x* = a to *x* =**

b.

Similarly, if x = *g(* y) is continuous and nonnegative on the interval $c \le y \le d$ **, the definite integral** $\int_c^d g(y)dy$ **is by definition the area bounded by the curve** ***x* = *g(* y), the *y* axis, and the ordinates y = c and y = *d* .**

If y = f (x) **is continuous and nonpositive on the interval** $a \le x \le b$ **, then** $\int_a^b f(x)dx$ **is**

negative, indicating that the area lies below the x axis. Similarly, if x = g(y) is continuous and nonpositive on the interval $c \le y \le d$ **, then** $\int_c^d g(y)dy$ **is negative, indicating that the area lies to the left of the y axis.**

If y = *f (x)* changes sign on the interval $a \le x \le b$ **, or if x = *g(* y) changes sign on the interval** $c \le y \le d$ **, then the area "under the curve" is given by the sum of two or more definite integrals.**

AREAS BY INTEGRATION. The steps in setting up a definite integral that yields a required area are:

1. Make a sketch showing the area sought, a representative (kth) strip, and the approximating rectangle. We shall generally show the representative subinterval of length Δx **(or** Δy ***)*** **, with the point** x_k **(or** y_k ***)*** **on this subinterval as its midpoint.**

2. Write the area of the approximating rectangle and the sum for the *n* rectangles.

3. Assume the number of rectangles to increase indefinitely, and apply the fundamental theorem of the preceding chapter.

Areas between curves. Assume that f(x) and g(x) are continuous functions such that $0 \le g(x) \le f(x)$ **for** $a \le x \le b$ **. Then the area A of the region R between the graphs of y = f(x) and y = g(x) and between x = a and x = b (see Fig. 9-2) is given by**

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx \qquad \textbf{(9.1)}$$

That is, the area A is the difference between the area $\int_a^b f(x)dx$ **of the region above the x axis and below y = f(x) and the area** $\int_a^b g(x)dx$ **of the region above the x axis and below y = g(x).**

Formula (9.1) holds when one or both of the curves y = f(x) and y = g(x) lie partially or completely below the x axis, that is, when we assume only that $g(x) \le f(x)$ **for** $a \le x \le b$ **, as in Fig. 9-3.**

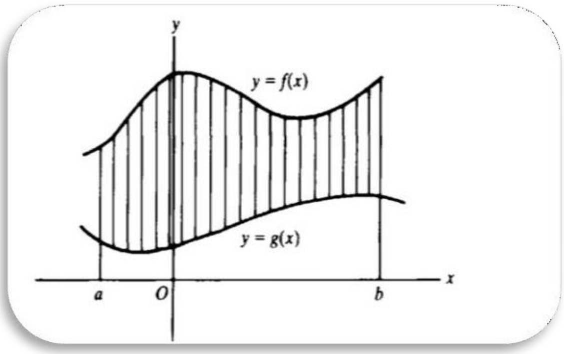


Fig. (9-2)

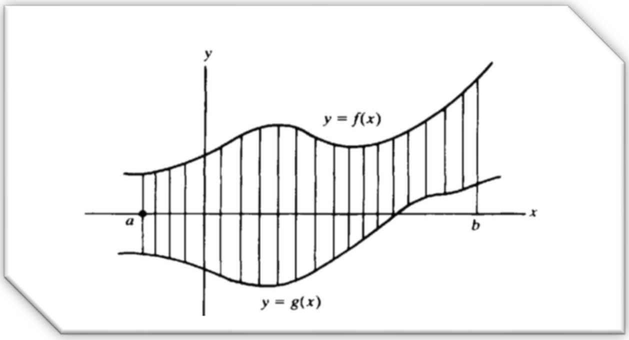


Fig. (9-3)

1. **Find the area bounded by the curve $y = x^2$, the x axis, and the ordinates x = 1 and x = 3.**

Figure 9-4 shows the area KLMN sought, a representative strip RSTU, and its approximating rectangle RVWU. For this rectangle, the base is $\Delta_k x$, the altitude is $y_k = f(x_k) = x_k^2$, and the area is $x_k^2 \Delta_k x$. Then

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} x_k^2 \Delta_k x = \int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = 9 - \frac{1}{3} = \frac{26}{3}$$ **square units**

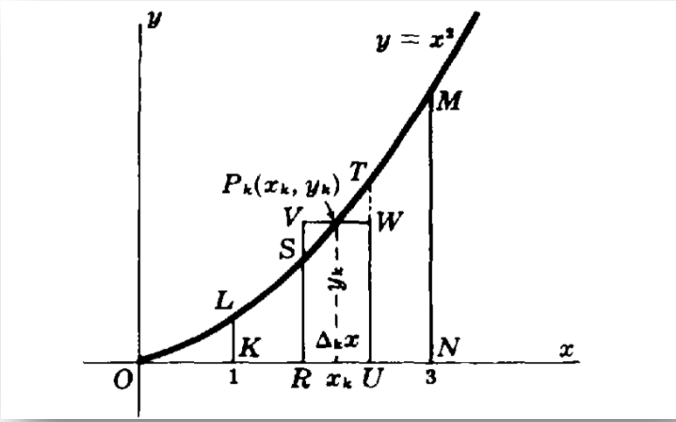


Fig. 9-4

2. **Find the area lying above the x axis and under the parabola $y = 4x - x^2$.**

The given curve crosses the x axis at x = 0 and x = 4. When vertical strips are used, these values become the limits of integration. For the approximating rectangle shown in Fig. 9-5, the width is $\Delta_k x$,

the height is $y_k = 4x_k - x_k^2$, and the area is $(4x_k - x_k^2)\Delta_k x$. Then

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n}(4x_k - x_k^2)\Delta_k x = \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32}{3}$$ **square units**

With the complete procedure, as given above, always in mind, an abbreviation of the work is possible. It will be seen that, aside from the limits of integration, the definite integral can be formulated once the area of the approximating rectangle has been set down.

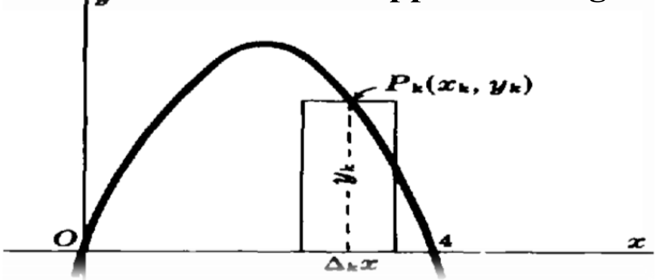


Fig. 9-5

3. **Find the area bounded by the parabola -1 and y = 3.**

$x = 8 + 2y - y^2$, **the y axis, and the lines y =**

Here we slice the area into horizontal strips. For the approximating rectangle shown in Fig. 9-6, the width is Δy, the length is $x = 8+2y-y^2$, and the area is $(8+2y-y^2)\Delta y$. The required area is $A = \int_{-1}^{3}(8+2y-y^2)dy = [8y+y^2-\frac{y^3}{3}]_{-1}^{3} = \frac{92}{3}$ square units

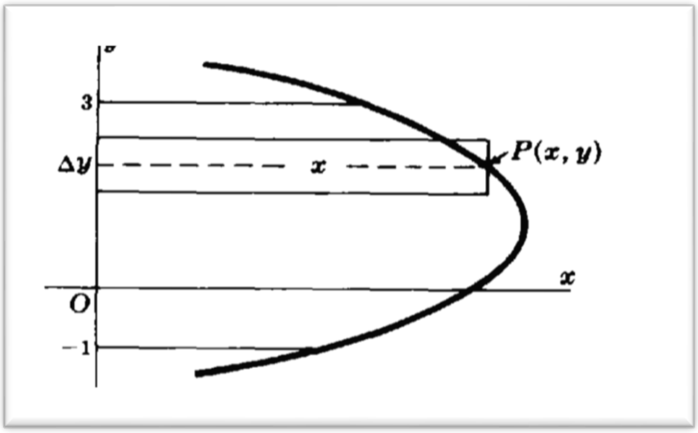


Fig. 9-6

4. Find the area bounded by the parabola $y = x^2 - 7x + 6$, the x axis, and the lines x = 2 and x = 6.

For the approximating rectangle shown in Fig. 9-7, the width is Δx, the height is $-y = -(x^2-7x+6)$, and the area is $-(x^2-7x+6)\Delta x$. The required area id then

$A = \int_{2}^{6} -(x^2-7x+6)dx = -(\frac{x^3}{3} - \frac{7x^2}{2} + 6x)]_{2}^{6} = \frac{56}{3}$ square units

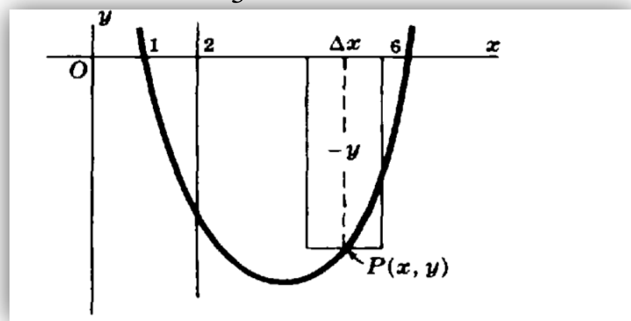


Fig. 9-7

5. Find the area between the curve $y = x^3 - 6x^2 + 8x$ and the x axis.

The curve crosses the x axis at x = 0, x = 2, and x = 4, as shown in Fig. 9-8. For

vertical strips, the area of the approximating rectangle with base on the interval

$0 < x < 2$ **is** $(x^3 - 6x^2 + 8x)\Delta x$ **, and the area of the portion lying above the x axis is given** by $\int_0^2 (x^3-6x^2+8x)dx$. The area of the approximating rectangle with base on the interval $2<x<4$ is $-(x^3-6x^2+8x)\Delta x$. and the area of the portion lying below the x axis is given by $\int_2^4 -(x^3-6x^2+8x)dx$. The required area is, therefore,

$$A = \int_0^2 (x^3-6x^2+8x)dx + \int_2^4 -(x^3-6x^2+8x)dx = \left[\frac{x^4}{4}-2x^3+4x^2\right]_0^2 - \left[\frac{x^4}{4}-2x^3+4x^2\right]_2^4 = 4+4=8$$

square units

The use of two definite integrals is necessary here, since the integrand changes sign on the interval of integration. Failure to note this would have resulted in the incorrect integral $\int_0^4 (x^3-6x^2+8x)dx = 0$.

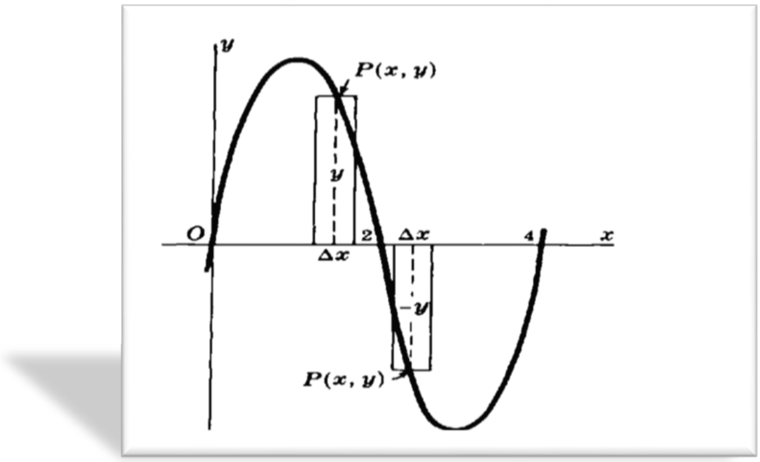


Fig. 9-8

6. Find the area bounded by the parabola $x = 4-y^2$ and the y axis.

The parabola crosses the x axis at the point (4, 0), and the y axis at the point (0, 2) and (0, -2).

We shall give two solutions.

Using horizontal strips: for the approximating rectangle of Fig. 9-9(a), the width is Δy , the length is $4-y^2$, and the area is $(4-y^2)\Delta y$. The limits of integration of the

resulting definite integral are y = -2 and y = 2. However, the area lying below the x axis is equal to that lying above. Hence, we have, for the required area,

$A = \int^2 (4-y^2)dy = 2\int^2 (4-y^2)dy = 2\left[4y - \frac{y^3}{3}\right]^2 = \frac{32}{}$ square units

$\int_{-2}$ $\int_{0}$ $\lfloor$ ^{3} $\rfloor_{0}$ 3

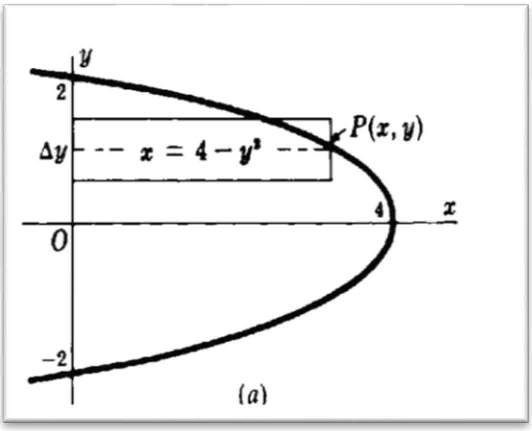


Fig. 9-9(a)

Using vertical strips: for the approximating rectangle of Fig. 9-9(b), the width is Δx **,**

the height is $2y = 2\sqrt{4-x}$ **, and the area is** $2\sqrt{4-x}\,\Delta x$ **. The limits of integration are**

x = 0 and x = 4. Hence the required area is

$$\int_0^4 2\sqrt{4-x}\,dx = \left[-\frac{4}{3}(4-x)^{\frac{3}{2}}\right]_0^4 = \frac{32}{3}$$ **square units**

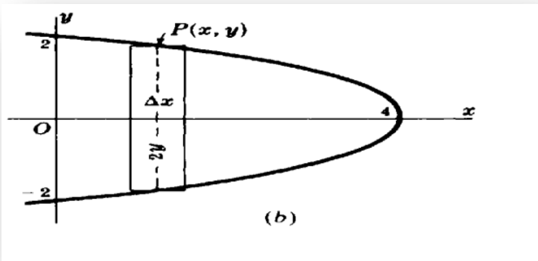
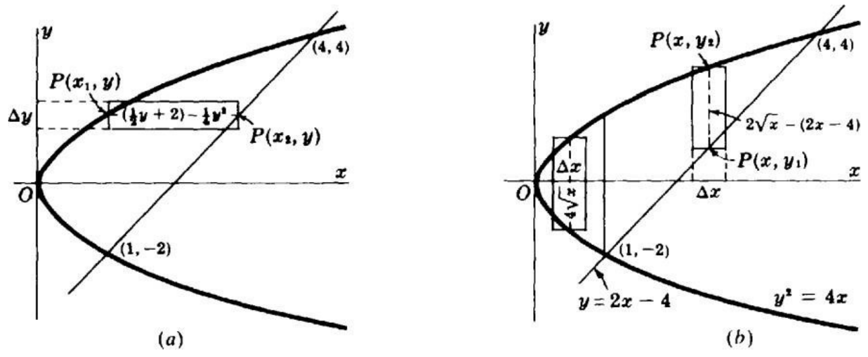


Fig. 9-9 (b)

7. **Find the area bounded by the parabola** $y^2 = 4x$ **and the line** $y = 2x - 4$**.**

The line intersects the parabola at the points (1, -2) and (4, 4). Fig. 9-10 shows clearly that when vertical strips are used, certain strips run from the line to parabola and others from one branch of the parabola to the other branch; however, when horizontal strips are used, each strip runs from the parabola to the line. We give both solutions here to show the superiority of one over the other and to indicate that both methods should be considered before beginning to set up a definite integral.



(a) Fig. 9-10 (b)

Using horizontal strips (Fig. 9-10(a)): For the approximating rectangle of Fig. 9-10(a), the width is Δy, the length is [(value of x of the line) – (value of x of the parabola)] $= (\frac{1}{2}y+2)-\frac{1}{4}y^2 = 2+\frac{1}{2}y-\frac{1}{4}y^2$, and the area is $(2+\frac{1}{2}y-\frac{1}{4}y^2)\Delta y$. The required area is

$$A = \int_{-2}^{4}\left(2+\frac{1}{2}y-\frac{1}{4}y^2\right)dy = \left[2y+\frac{y^2}{4}-\frac{y^3}{12}\right]_{-2}^{4} = 9 \text{ square units}$$

Using vertical strips (Fig. 9-10(b)): Divide the area A into two parts with the line x = 1. For the approximating rectangle to the left of this line, the width is Δx, the height (making use of symmetry) is $2y = 4\sqrt{x}$, and the area is $4\sqrt{x}\,\Delta x$. For the approximating rectangle to the right, the width is Δx, the height is $2\sqrt{x}-(2x-4) = 2\sqrt{x}-2x+4$, and the area is $(2\sqrt{x}-2x+4)\Delta x$. The required area is

$$A = \int_0^1 4\sqrt{x}\,dx + \int_1^4 (2\sqrt{x}-2x+4)\,dx = \left[\frac{8}{3}x^{3/2}\right]_0^1 + \left[\frac{4}{3}x^{3/2}-x^2+4x\right]_1^4$$

$$= \frac{8}{3}+\frac{19}{3} = 9 \text{ square units}$$

8. Find the area bounded by the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$.

The parabolas intersect at the points (0, 0) and (4, 8). It is readily seen in Fig. 9-11 that vertical slicing will yield the simpler solution.
For the approximating rectangle, the width is Δx, the height is [(value of y of the upper boundary) – (value of y of the lower boundary)] $=(6x-x^2)-(x^2-2x) = 8x-2x^2$, and the area is $(8x-2x^2)\Delta x$. The required area is

$$A = \quad (8x - 2x^2)dx = [4x^2 - \frac{2}{\ } x^3]^4 =$$

square units

$\int_0$ $_3$ $_0$ $\overline{3}$

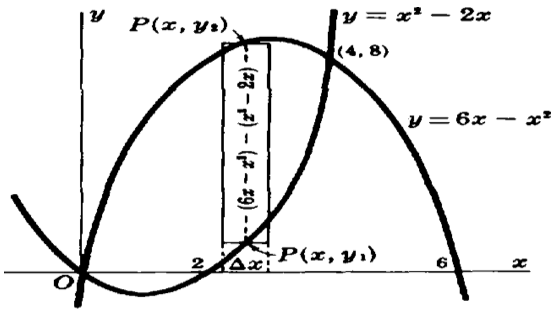


Fig. 9-11

9. Find the area enclosed by the curve $y^2 = x^2 - x^4$.

The curve is symmetric with respect to the coordinate axes. Hence the required area is four times the portion lying in the first quadrant.

For the approximating rectangle shown in Fig. 9-12, the width is Δx, the height is $y = \sqrt{x^2 - x^4} = x\sqrt{1-x^2}$, and the area is $x\sqrt{1-x^2}\,\Delta x$. Hence the required area is

$$A = 4\int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{4}{3}(1-x^2)^{3/2}\right]_0^1 = \frac{4}{3}$$ **square units**

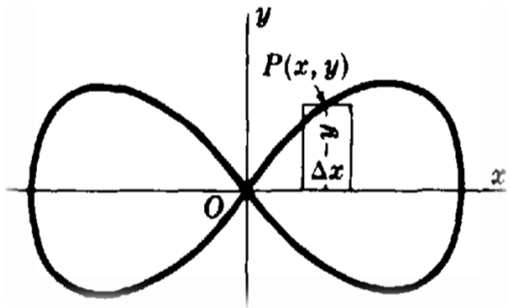


Fig. 9-12

10. Find the smaller area cut from the circle $x^2 + y^2 = 25$ by the line x = 3.

Based on Fig. 9-13,

$$A = \int_3^5 2y\,dx = 2\int_3^5 \sqrt{25-x^2}\,dx = 2\left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\arcsin\frac{x}{5}\right]_3^5$$

$= \left(\frac{25}{2}\pi - 12 - 25\arcsin\frac{3}{5}\right)$ **square units**

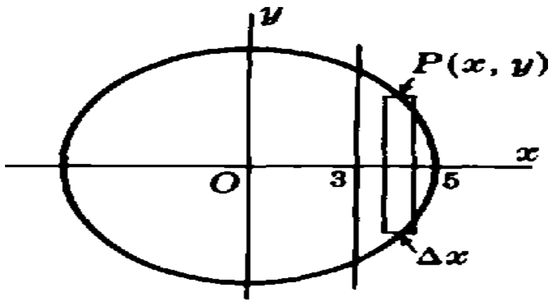


Fig. 9-13

11. Find the area common to the circles $x^2+y^2=4$ **and** $x^2+y^2=4x$ **.**

The circles intersect in the points $(1,\pm\sqrt{3})$**. The approximating rectangle shown in Fig. 9-14 extends from** $x=2-\sqrt{4-y^2}$ **to** $x=\sqrt{4-y^2}$ **. Then**

$$A=2\int_0^{\sqrt{3}}\left[\sqrt{4-y^2}-\left(2-\sqrt{4-y^2}\right)\right]dy=4\int_0^{\sqrt{3}}\left(\sqrt{4-y^2}-1\right)dy$$

$$=4\left[\frac{y}{2}\sqrt{4-y^2}+2\arcsin\frac{1}{2}y-y\right]_0^{\sqrt{3}}=\left(\frac{8\pi}{3}-2\sqrt{3}\right)$$ **square units**

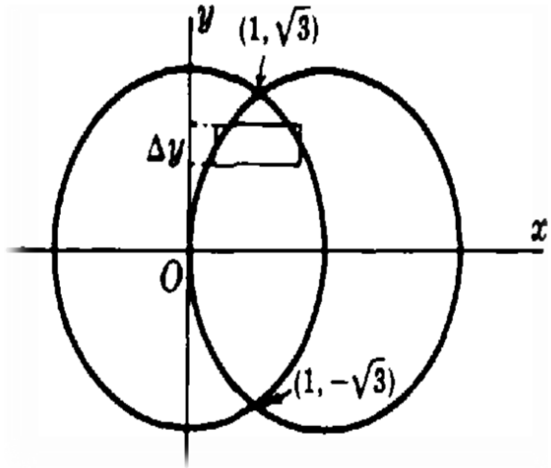


Fig. 9-14

12. Find the area of the loop of the curve $y^2 = x^4(4+x)$. (see Fig. 9-15.)

From the figure, $A = \int_{-4}^{0} 2y\,dx = 2\int_{-4}^{0} x^2\sqrt{4+x}\,dx$ **. Let** $4+x = z^2$**; then**

$$A = 4\int_0^2 (z^2-4)^2 z^2\,dz = 4\left[\frac{z^7}{7} - \frac{8z^5}{5} + \frac{16z^3}{3}\right]_0^2 = \frac{4096}{105}$$ **square units**

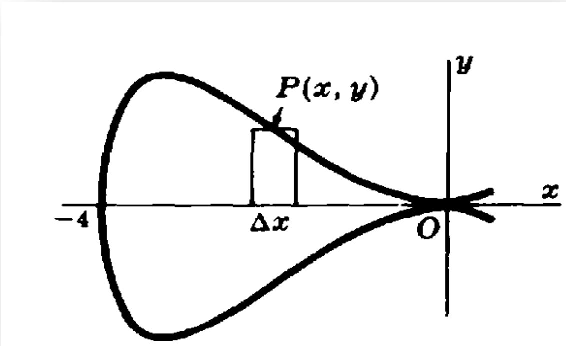


Fig. 9-15

13. Find the area of an arch of the cycloid $x = \theta - \sin\theta,\ y = 1-\cos\theta$ **.**

A single arch is described as θ **varies from 0 to** 2π **(see Fig. 9-16). Then** $dx = (1-\cos\theta)d\theta$ **and** $A = \int_{\theta=0}^{\theta=2\pi} y\,dx = \int_0^{2\pi}(1-\cos\theta)(1-\cos\theta)d\theta = \int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$

$$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi$$ **square units**

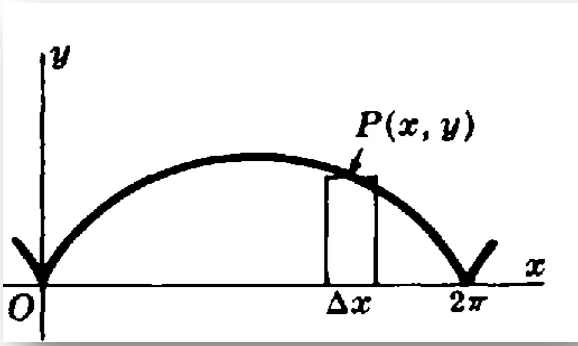


Fig. 9-16

14. Find the area bounded by the curve $x=3+\cos\theta,\ y=4\sin\theta$ **. (see Fig. 9-17)**

The boundary of the shaded area in the figure (one-quarter of the required area) is described from right to left as θ **varies from 0 to** $\frac{1}{2}\pi$ **. Hence,**

$$A=-4\int_{\theta=0}^{\theta=\pi/2} y\,dx=-4\int_0^{\pi/2}(4\sin\theta)(-\sin\theta)\,d\theta=16\int_0^{\pi/2}\sin^2\theta\,d\theta=8\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$=8\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}=4\pi$$ **square units**

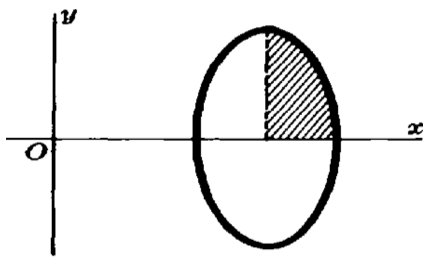


Fig. 9-17

Exercises

Find the area bounded by the given curves, or as described.

(a) $y=x^2, y=0, x=2, x=5$

(b) $y=x^3, y=0, x=1, x=3$

(c) $y=4x-x^2, y=0, x=1, x=3$

(d) $x=1+y^2, x=10$

(e) $x=3y^2-9, x=0, y=0, y=1$

(f) $x=y^2+4y, x=0$

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