جامـــعة بنـــــــــها
كليـــــة العلـــــــــــوم
قسم الرياضيات وعلوم الحاسب

DYNAMICS 1

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for

***First* Level Students**

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Mechanics is the study of force, deformation, and motion, and the relations between them. We care about forces because we want to know how hard to push something to move it or whether it will break when we push on it for other reasons. We care about deformation and motion because we want things to move or not move in certain ways. Towards these ends we are confronted with this general mechanics problem.

In mechanics we try to solve special cases of the general mechanics problem above by idealizing the system, using classical Euclidean geometry to describe deformation and motion, and assuming that the relation between force and motion is described with Newtonian mechanics, or “Newton’s Laws”. Newtonian mechanics has held up, with minor refinement, for over three hundred years. Those who want to know how machines, structures, plants, animals and planets hold together and move about need to know mechanics. In another two or three hundred years people who want to design robots, buildings, airplanes, boats, prosthetic devices, and large or microscopic machines will probably still use the equations and principles we now call Newtonian mechanics.

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Motion along a straight line

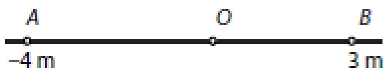
For problems with motion in only one direction, the kinematics is particularly simple. Although we use vectors here because of their help with signs, they are really not needed. Firstly, we must define the three following basics: Position, displacement and distance.

Now, we are only talking about motion in a straight line. For a horizontal line, there are only two directions to consider: right and left. For a vertical line, the two directions are up and down. We choose a point O on the line, which we call the reference point or origin. For convenience, we will measure distance in meters and time in seconds.

Position

The line is coordinated and referenced from a point O, the origin. For a horizontal line, the convention is that positions to the right of O are positive, and positions to the left are negative.

For example:



• The position of the particle at B is 3m.

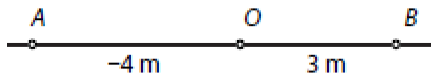
• The position of the particle at A is -4 m.

The position of a particle is often thought of as a function of time, and we write x(t) for the position of the particle at time t.

Displacement

The displacement of a particle moving in a straight line is the change in its position. If the particle moves from the position x(t1) to the position x(t2), then its displacement is x(t2)- x(t1) over the time interval [t1, t2]. In particular, the position of a particle is its displacement from the origin.

For example:



• If a particle moves from O to B, its displacement is 3 m.

• If a particle moves from O to A, its displacement is -4 m.

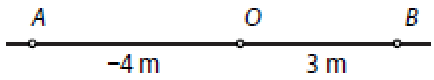
• If a particle moves from A to B, its displacement is 7 m.

• If a particle moves from B to A, its displacement is -7 m.

Position and displacement are vector quantities, that is, they have both magnitude and direction. In this module, we are dealing with vectors in one dimension. The sign of the quantity (positive or negative) indicates its direction. The absolute value of the quantity is its magnitude.

Distance

The distance is the 'actual distance' travelled. Distances are always positive or zero. For example, given the following diagram, if a particle moves from A to B and then to O, the displacement of the particle is 4 m, but the distance travelled is 10 m.



Example 1

A particle moves along a straight line so that its position at time t seconds is x(t) meters, relative to the origin. Assume that x(0)=0, x(3)=2 and x(6)=-5, and that the particle only changes direction when t=3. Find the distance travelled by the particle from time

T=0 to time t=6.

Solution

The distance travelled is 2+7= 9 meters.

Constant velocity

The rate of change of the position of a particle with respect to time is called the velocity of the particle. Velocity is a vector quantity, with magnitude and direction. The speed of a particle is the magnitude of its velocity. If a particle is moving with constant velocity, it does not change direction. If the particle is moving to the right, it has positive velocity, and if the particle is moving to the left, it has negative velocity. In general, if a particle is moving at a constant velocity (rate), then its constant velocity v is determined by the formula

$$v = \frac{x(t_2) - x(t_1)}{t_2 - t_1},$$

where x(ti) is the position at time ti .

Example 2

A particle is moving in a straight line with constant velocity, and its position at time t seconds is x(t) metres. If x(1)=6 and x(5)=-12, find the velocity of the particle.

Solution

The velocity is

$$\frac{x(5)-x(1)}{5-1}=\frac{-12-6}{4}$$

$$=-\frac{9}{2}\ \mathrm{m/s}.$$

Differential equations

A differential equation is an equation that involves derivatives. Thus the equation relating position to velocity is

$$v=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=\frac{dx}{dt}$$

is a differential equation. An ordinary differential equation (ODE) is one that contains ordinary derivatives.

Example 3

Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t)=(3\ \mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\ \mathrm{s}}\right)\right)}{2}$$

we can solve the differential equation $v=\frac{dy}{dt}$ by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[(3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}\right] = \frac{3\pi}{10}\sin\left(\frac{\pi t}{5\,\mathrm{s}}\right)\ \mathrm{m/s}$$

Acceleration

When a particle's velocity changes, then we way that the particle undergoes an acceleration. As with velocity it is usually more important to think about the instantaneous acceleration, given by

$$f = \lim_{\Delta t \to 0}\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Sometimes when the space is a directly function of time, we express the velocity and acceleration as follows:

$$\dot{x} = \frac{dx}{dt} \quad ‘ \quad \ddot{x} = \frac{d^2x}{dt^2}$$

We must observe that, if the velocity is a function in a space x, then the acceleration can be taking the following form:

$$f = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}$$

Velocity and acceleration units

Clearly from the definition of the velocity, that the unit for the velocity is $\frac{\text{distance unit}}{\text{time unit}} = m/s$, whereas the acceleration unit is m/s^2.

Example 4

An electron moving along the x axis has a position given by $x = 16te^{-t}$, where t is in seconds. How far is the electron from the origin when it momentarily stops?

Solution:

To find the velocity of the electron as a function of time, take the first derivative of x(t):

$$v = \frac{dx}{dt} = 16e^{-t}(1-t)$$

again where t is in seconds, so that the units for v are m/s . Now the electron "momentarily stops" when the velocity v is zero. From our expression for v we see that this occurs at t = 1. At this particular time we can find the value of x:

$$x(1) = 16e^{-1} = 5.89\,\mathrm{m}$$

The electron was 5.89m from the origin when the velocity was zero.

Example 5

If the position of a particle is given by $x = 20t - 5t^3$, where x is in meters and t is in seconds, when if ever is the particle's velocity zero?

(b) When its acceleration becomes zero?

(c) When the acceleration becomes negative? Positive?

Solution:

$$v = \frac{dx}{dt} = 20 - 15t^2$$

where, if t is in seconds then v will be in m/s . The velocity v will be zero when

$$20 - 15t^2 = 0$$

which we can solve for t:

$$t = \pm 1.15\,s$$

(b)

$$a(t) = \frac{dv}{dt} = -30t$$

From this, we see that a can be zero only at t = 0.

(c) From the result is part (b) we can also see that a is negative whenever t is positive. a is positive whenever t is negative (assuming that t < 0 has meaning for the motion of this particle).

Example 6

If the position of a particle is given by $x = 3t^2 - t^3$, where x is in meters and t is in seconds, when if ever is the particle's velocity zero? (b) When its position at the acceleration becomes zero?

(c) What the acceleration when the particle's velocity is $3\,m/s$.

Solution:

We know that the particle's velocity can be calculated as

$$v = \frac{dx}{dt} = 6t - 3t^2 \qquad (1)$$

Also, the acceleration becomes,

$$f = \frac{dv}{dt} = 6 - 6t \qquad (2)$$

The velocity may be vanishes at $v = 0$ in Eq. (1), we have:

$$t(2-t) = 0 \Rightarrow \therefore t_1 = 0,\ t_2 = 2$$

Also, the position for the particle at t_1, t_2, we have:

$$x_1 = 0 \qquad ، \qquad x_2 = 4\,m$$

On the other hand, at $f = 0$, we have $t = 1\,s$, so we find that $x = 2\,m$.

Likewise, when the velocity becomes $3\,m/s$, we have:

$$f = 6 - 6 = 0$$

Example 7

If the relation between x and t is given by $x = e^{2t} - 2e^{-2t}$, prove that

$v^2 = 4(x^2 + 8) \qquad ، \qquad f = 4x$

Solution:

By differentiation the position relation we have:

$$v = 2e^{2t} + 4e^{-2t} \qquad (2)$$

Then we observe that

$$v^2 = 4e^{4t} + 16 + 16\,e^{-4t}$$

$$= 4(e^{4t} + 4 + 4\,e^{-4t})$$

But

$$x^2 = e^{4t} - 4 + 4\,e^{-4t}$$

$$\therefore v^2 = 4(x^2 + 8) \qquad (3)$$

To obtain f

$$\therefore f = 4e^{2t} - 8e^{-2t} = 4(e^{2t} - 2e^{-2t}) = 4x$$

$$\frac{dv^2}{dx} = 8x \Rightarrow \therefore f = \frac{1}{2}\frac{dv^2}{dx} = 4x$$

Example 8

A large stone is falling through a layer of mud. At time t seconds, the depth of the stone in metres below the surface is given by

$$x(t) = 20(1 - e^{-t/2})$$

1-Find $\dot{x}(t)$

2-Find $\ddot{x}(t)$

3-Find the position, velocity and acceleration when t=1.

4-What happens to the position, velocity and acceleration as $t \to \infty$?

Solution:

1- $\dot{x}(t) = 10e^{-t/2}$

2- $\ddot{x}(t) = -5e^{-t/2}$

3- $x(1) = 20(1 - e^{-1/2})$, $\dot{x}(1) = 10e^{-1/2}$, $\ddot{x}(1) = -5e^{-1/2}$

The direction of the velocity is down, as it has a positive sign, and the direction of the acceleration is up, as it has a negative sign. The velocity and the acceleration are in opposite directions, so the particle is slowing down.

4- As $t \to \infty$, we have $x(t) \to 20$, $\dot{x}(t) \to 0$, $\ddot{x}(t) \to 0$

Example 9

A particle moves in a straight line. It is initially at rest at the origin. The acceleration of the particle is given by $\ddot{x}(t) = \frac{1}{3}\cos 3t$.

1- Find the velocity at time t.

2- Find the position at time t.

Solution:

1- Starting from $\ddot{x}(t) = \frac{1}{3}\cos 3t$ and integrating once, we obtain

$$\dot{x}(t) = \frac{1}{9}\sin 3t + c_1$$

Since $\dot{x}(0) = 0$, we have $c_1 = 0$. Hence,

$$\dot{x}(t) = \frac{1}{9}\sin 3t$$

2- Integrating again gives

$$x(t) = \frac{-1}{27}\cos 3t + c_2$$

Since $x(0) = 0$, we have $c_2 = \frac{1}{27}$. Hence,

$$x(t) = \frac{-1}{27}\cos 3t + \frac{1}{27}$$

Motion with a constant acceleration

A very useful special case of accelerated motion is the one where the acceleration f is constant. For this case, one can show that the following three important laws are true:

Firstly, suppose that a particle which moves in a positive direction for the x-axis, with a constant acceleration a and an initial velocity $v_\circ$ from a point P which has a position $x_\circ$ from a fixed point o. So, we have:

$$f = \frac{dv}{dt} = a$$

$$\therefore v = \int dv = \int a\,dt = a\,t + c_1$$

Where c_1 is a constant which can be determined from the initial condition which state that:

At $t = 0$ this results in $v = v_\circ$, then $c_1 = v_\circ$, then

$$\therefore v = v_\circ + a\,t \qquad (1)$$

Also, after integration for Eq. (1), we have:

$$\int dx = \int (v_\circ + a\,t)dt \Rightarrow x = v_\circ t + \frac{1}{2}a\,t^2 + c_2$$

but at $t = 0$ this results in $x = x_\circ$, then $c_2 = x_\circ$, then

$$\therefore x = x_\circ + v_\circ t + \frac{1}{2}a\,t^2$$

Likewise, if the particle starts the motion from the origin ($x_\circ = 0$), then

$$x = v_\circ t + \frac{1}{2}a\,t^2$$

On the other hand, to get a relation between v and x, we know that

$$f = v\frac{dv}{dx} = a \Rightarrow \int v\,dv = \int a\,dx + c_3$$

$$\therefore v^2 = 2a\,x + c_4$$

but at $t = 0$ this results in $x = x_\circ$, $v = v_\circ$ then $c_4 = v_\circ^2 - 2ax_\circ$, then

$$\therefore v^2 = v_\circ^2 + 2a(x - x_\circ)$$

Also, if $x_\circ = 0$, we have:

$$\therefore v^2 = v_\circ^2 + 2a\,x$$

Example 10

A body moving with constant acceleration has a velocity of 12.0 cm/s when its x coordinate is 3.00 cm. If it moves a distance −5.00 cm in a time 2 s, what is the magnitude of its acceleration?

Solution:

In this problem we are given the initial coordinate ($x_\circ = 3$ cm), the initial velocity ($v_\circ = 12$ cm/s), the final x coordinate (x = −5 cm) and the elapsed time (2 s). Since we are told that the acceleration is constant, we can solve the equation of the constant acceleration a. We find:

$$\therefore x = x_\circ + v_\circ t + \frac{1}{2} a t^2$$
$$\Rightarrow \frac{1}{2} a t^2 = x - x_\circ - v_\circ t$$
$$\Rightarrow \frac{1}{2} a (2)^2 = -5 - 3 - 12(2)$$
$$\Rightarrow a = -16$$

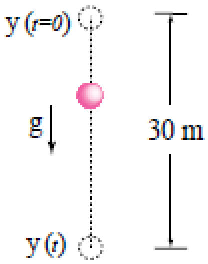
Then the magnitude of the acceleration is 16.0 cm/s^2.

Example 11

A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity $g = 10\, m/s^2$.

Solution:

The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let y denote the distance travelled by the ball. Let us measure y vertically downwards, starting from the height at which the ball starts falling (see the following figure). Under constant acceleration g, we can write the distance travelled as $y = y_\circ + v_\circ t + \frac{1}{2} g t^2$.



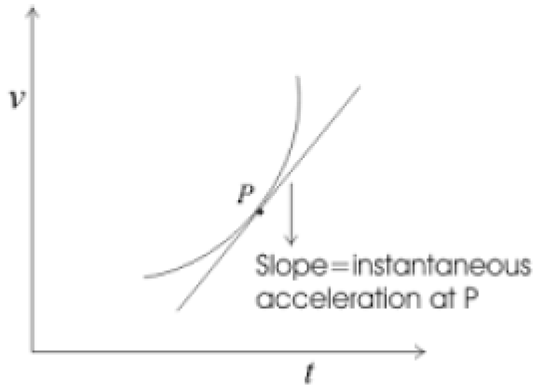
Note that at $t = 0$, $y_\circ = 0$ and $v_\circ = 0$. We are given that at some instant t (that we need to find) $y = 30$m. Thus,

$$y = \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{2(30)}{10}} = 2.45s$$

Variable Acceleration

The acceleration at any instant is obtained from the average acceleration by shrinking the time interval closer zero. As Δt tends to zero average acceleration approaching a limiting value, which is the acceleration at that instant called instantaneous acceleration which is vector quantity.



Hence instantaneous acceleration of a particle at any instant is the rate at which its velocity is changing at that instant. Instantaneous acceleration at any point is the slope of the curve v (t) at that point as shown in figure above.

Example 12

A particle moves from a rest with an incremental acceleration at a constant time rate from $1\, ft/\sec^2$ to $4\, ft/\sec^2$ in a one second. Prove that this point moves a distance $1\, ft$ in this one second.

Solution:

Since the acceleration increases with a constant rate, so the rate of change for the acceleration with respect to time become constant as follows:

$$\frac{df}{dt} = c \Rightarrow \int df = c\int dt + c_1 \Rightarrow f = ct + c_1 \qquad (1)$$

Where c, c_1 are constants which must be determined from the initial conditions at $t = 0$. From the conditions we have:

At $t = 0, f = 1 \Rightarrow c_1 = 1$, and $t = 1, f = 4 \Rightarrow c = 3$, then Equation (1) becomes:

$$f = 3t + 1 \tag{2}$$

Then

$$\frac{dv}{dt} = 3t + 1 \Rightarrow \int dv = \int (3t+1)dt + c_2$$

or

$$v = (3t^2 / 2) + t + c_3 \tag{3}$$

After using the initial conditions $t = 0, v = 0$ which gives $c_3 = 0$, so

$$v = \frac{3}{2}t^2 + t$$

Then

$$\int dx = \int \left(\frac{3}{2}t^2 + t \right) dt + c_4 \quad \text{‘} \quad \therefore x = \frac{1}{2}(t^3 + t^2) + c_4$$

After using the initial conditions $t = 0, x = 0$ which gives $c_4 = 0$, so

$$\therefore x = \frac{1}{2}(t^3 + t^2)$$

The final equation gives the distance which occurred by the particle at any time. Putting $t = 1$, we have:

$$x = (1+1)/2 = 1 \; ft$$

Example 13

If the acceleration for a moving particle is its Also, .$(12t+8)\,m/s^2$ initial velocity is $10\,m/s$ from a point which is away from the origin o with a distance $4m$. Calculate the velocity of the particle and its position with respect to o after a period of time equal to $3s$. Also, when and where the particle velocity becomes $50\,m/s$ and calculate the acceleration at this position.

Solution:

$$\because f=(12t+8) \Rightarrow \therefore \frac{dv}{dt}=12t+8$$

$$\therefore v=6t^2+8t+c$$

But from the boundary conditions which state that at $t=0$, $v=10$, then $c=10$. So, the velocity becomes:

$$\therefore v=6t^2+8t+10 \qquad (1)$$

To get the relation between the position and the time we have:

$$\because \frac{dx}{dt}=6t^2+8t+10 \Rightarrow \therefore x=\int(6t^2+8t+10)dt+c_1$$

$$\therefore x=2t^3+4t^2+10t+c_1$$

But from the boundary conditions which state that at $t=0$, $x=4$, then $c_1=4$.

$$\therefore x=2t^3+4t^2+10t+4 \qquad (2)$$

$$\therefore v(t=3)=6(3)^2+8\times3+10=88\ m/s$$

Also,

$$\therefore x(t=3)=2(3)^3+4(3)^2+10\times3+4=124\ m$$

The time in which the velocity is equal to $50\ m/s$, can be calculated after substituting by $v=50$ in Eq. (1) to get

$$6t^2+8t+10=50 \Rightarrow \therefore (3t+10)(t-2)=0 \Rightarrow t=2\ s$$

The distance at this position becomes

$$\therefore x=2(2)^3+4(2)^2+10\times 2+4=56\ m$$

Also, the acceleration at $t=2$ becomes

$$f=(12t+8)=12\times 2+8=32\ m/s^2$$

<u>Example 14</u>

A particle moves according to the relation $x=a\sqrt{v}-b$, where a,b are constants. If the initial condition is $x=0$ at $t=0$. Calculate the elapsed time until the velocity becomes a double of its initial value. Also, find the acceleration in terms of velocity.

<u>Solution:</u>

Firstly, we get the relation between x and t as follows:

$$x=a\sqrt{v}-b \Rightarrow \therefore v=\left(\frac{x+b}{a}\right)^2 \qquad (1)$$

By separation of variables, then by using the integration we have:

$$\int \frac{dx}{(x+b)^2}=\int \frac{dt}{a^2} \Rightarrow \therefore -\frac{1}{x+b}=\frac{t}{a^2}+c$$

But $x=0$ at $t=0$, then $c=-1/b$

$$\therefore t=\frac{a^2x}{b(x+b)} \qquad (2)$$

The initial velocity can be calculated after putting $x=0$ in Eq. (1) to get:

$$v_{\circ} = \frac{b^2}{a^2}$$

The double values of this velocity can observed at the following position

$$x = a\sqrt{\frac{2b^2}{a^2}} - b = (\sqrt{2} - 1)b$$

By substituting in Eq. (2) we have:

$$t = \frac{(\sqrt{2} - 1)a^2}{\sqrt{2}b}$$

Also, the acceleration can be calculated as:

$$f = v\frac{dv}{dx} = \frac{2v(x + b)}{a^2}$$

But, $x + b = a\sqrt{v}$, then:

$$f = \frac{2v^{3/2}}{a}$$

Problems

1-A particle moves from a rest in the x-coordinate direction with an acceleration $f = 3e^{-t}$. Calculate the velocity and the distance at any time and prove that $x = 3(t - 1) + f$.

2- A particle which moves according to the relation $x = 2t^3 - 3t^2 - 12t + 18$. Calculate the position and its acceleration when its velocity vanishes.

3-A particle is moving in a straight line with constant acceleration of $1.5 m/s^2$. Initially its velocity is 4.5 m/s. Find the velocity of the particle: a) after 1 second b) after 3 seconds c) after t seconds.

4-A car is travelling at 100 km/h =250/9 m/s, and applies its brakes to stop. The acceleration is $-10 m/s^2$. How long does it take for the car to stop?

5-A ball is thrown directly downward with an initial speed of 8.00 m/s from a height of 30m. When does the ball strike the ground?

6-A student throws a set of keys vertically upward to her sorority sister in a window 4 m above. The keys are caught 1.5 s later by the sister’s outstretched hand. (a) With what initial velocity were the keys thrown? (b) What was the velocity of the keys just before they were caught?

7-A stone falls from rest from the top of a high cliff. a second stone is thrown downward from the same height 2.00 s later with an initial speed of 30 m/s. If both stones hit the ground simultaneously, how high is the cliff?

8- A falling object requires 1.5 s to travel the last 30m before hitting the ground. From what height above the ground did it fall?

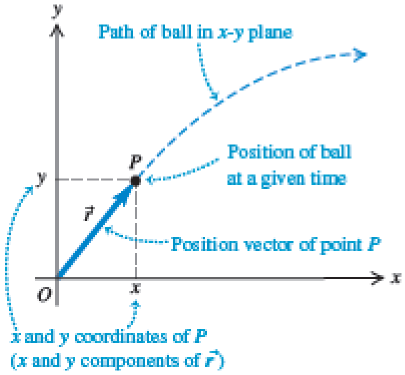
Motion in a Plane

In straight-line problems (such as those in Chapter 1), we developed the formalism we need to describe the position and velocity of a particle as a function of time. We placed particular emphasis on motion with constant acceleration, primarily because an object in free fall has a constant downward acceleration of 9.8 m>s2. In this chapter, we will extend our description of motion to more than one dimension. We’ll focus on motion in two dimensions that is, in a plane but the principles that we develop will also applied to three-dimensional motion. So, instead of simply considering the motion of, let’s say, a baseball that has been thrown straight up into the air, we will expand our analysis to handle the more complex motion of a baseball that has been thrown from home plate to quantities displacement, velocity, and acceleration have two components, one for each axis of our two-dimensional coordinate system. We’ll also generalize the concept of relative velocity to motion in a plane, such as an airplane flying in a crosswind. This chapter represents a merging of the vector language we have learned (in Chapter 1) with kinematic language (which we learned in Chapter 1). As before, we’re concerned with describing motion, not with analyzing its causes. But the language you learn here will be an essential tool in later chapters when you use Newton's laws of motion to study the relationship between force and motion.

Velocity in a Plane

To describe the motion of an object in a plane, we first need to be able to describe the object's position. (In this chapter, as in the preceding one, we assume that the objects we describe can be modeled as particles.) Often, it's useful to use a familiar x-y axis system. For example, when a football player kicks a field goal, the ball (represented by point P) moves in a vertical plane. The ball's horizontal distance from the origin O at any time is x, and its vertical distance above the ground at any time is y. The numbers x and y are called the coordinates of point P. The vector ru from the origin O to point P is called the position vector of point P, and the Cartesian coordinates x and y of point P are the x and y components, respectively, of vector $\underline{r}$. (You may want to review Section 1.8, "Components of Vectors.") The distance of point P from the origin is the magnitude of vector $\underline{r}$: $\underline{r} = x\underline{i} + y\underline{j}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2}.$$



And the velocity vector for the moving point is

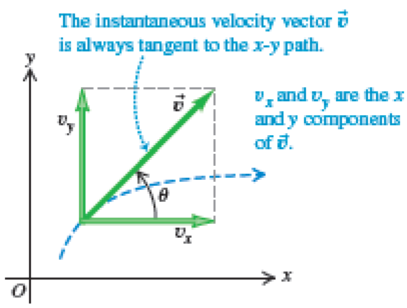
$$\underline{v} = \dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} = \dot{x}\underline{i} + \dot{y}\underline{j}$$

Where, $\dot{x} = \dfrac{dx}{dt}, \dot{y} = \dfrac{dy}{dt}$ are the velocity components in x and y direction; respectively, and the magnitude of the velocity is:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

But the direction of the velocity can be calculated from the following relation:

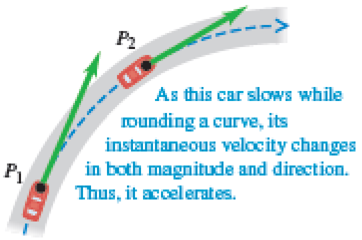
$$\tan^{-1}\left[\frac{\dot{y}}{\dot{x}}\right] = \tan^{-1}\left[\frac{dy/dt}{dx/dt}\right] = \tan^{-1}\left[\frac{dy}{dx}\right]$$



Acceleration in a Plane

Now let's consider the acceleration of an object moving on a curved path in a plane. In Figure 3.4, the vector $\underline{v}_1$ represents the particle's instantaneous velocity at point P1 at time t1, and the vector

$\underline{v}_2$ represents the particle's instantaneous velocity at point P2 at time t2. In general, the two velocities differ in both magnitude and direction.



And the acceleration vector for the moving point is

$$\underline{f} = \dot{\underline{v}} = \frac{d^2 \underline{r}}{dt^2} = \frac{d\dot{x}}{dt}\underline{i} + \frac{d\dot{y}}{dt}\underline{j} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$$

Where, $\ddot{x}, \ddot{y}$ are the acceleration components in x and y direction; respectively, and the magnitude of the acceleration is:

$$f = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

But the direction of the acceleration can be calculated from the following relation:

$$\tan^{-1}\left[\frac{\ddot{y}}{\ddot{x}}\right]$$

Example 1:

A particle moving in a plane according to these equations

$$x = a\cos\omega t \quad ، \quad y = b\sin\omega t$$

Where a, b, ω are constants. Calculate the magnitude and the direction for both the velocity and the acceleration. Also, calculate the trajectory equation for the particle if $a = b$. Likewise, prove that the direction for the velocity is perpendicular on the direction of the acceleration.

Solution:

Firstly, we know that the position vector for the point is:

$$\underline{r} = a\cos\omega t\,\underline{i} + b\sin\omega t\,\underline{j}$$

So, the velocity vector is equal to

$$\underline{v} = \dot{\underline{r}} = \frac{d\underline{r}}{dt} = -a\omega\sin\omega t\,\underline{i} + \omega b\cos\omega t\,\underline{j}$$

And the magnitude for the velocity becomes

$$v = |\underline{v}| = \omega\sqrt{a^2\sin^2\omega t + b^2\cos^2\omega t}$$

Also, the direction for the velocity is:

$$\tan\alpha = \frac{b}{a}\tan\omega t$$

and the acceleration vector is equal to

$$\underline{f} = \frac{d\underline{v}}{dt} = -a\omega^2\cos\omega t\,\underline{i} - b\omega^2\sin^2\omega t\,\underline{j}$$

The magnitude for the acceleration is

$$f = |\underline{f}| = \omega^2\sqrt{a^2\cos^2\omega t + b^2\sin^2\omega t}$$

Also, the direction for the acceleration is:

$$\tan\phi = \frac{b}{a}\tan\omega t$$

If $a = b$, we have:

$$\tan\alpha \times \tan\phi = -1$$

Then, the direction for the velocity is perpendicular to the direction of the acceleration. Also, the trajectory equation for the particle is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Which represents an equation of ellipse.

Example 2:

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The trajectory is such that the components of the rabbit's position with respect to the origin of the coordinate frame are given as function of time by

$$x(t)=-0.31t^2+7.2t+28$$
$$y(t)=0.22t^2-9.1t+30$$

Calculate the velocity and the acceleration for the rabbit at any time.

Solution:

The position vector of the rabbit at time t can be expressed as:

$$\underline{r}=x(t)\underline{i}+y(t)\,\underline{j}$$

From the equations of motion for x(t) and y(t) we can calculate the velocity and acceleration:

$$v_x(t)=-0.62t+7.2, \qquad v_y(t)=0.44t-9.1$$
$$a_x(t)=-0.62m/s^2 \qquad a_y(t)=0.44m/s^2$$

The acceleration of the rabbit is constant (independent of time).

Example 3:

If the parametric equations for the moving particle are

$$x = 5t \quad ، \quad y = 20 - 5t^2$$

Find the trajectory equation for the particle; also find the initial velocity and the velocity when the particle passes through the x-axis. Finally, calculate the acceleration.

Solution:

To get the trajectory equation, we have:

$$t = \frac{x}{5} \quad \Rightarrow \quad \therefore y = 20 - \frac{x^2}{5}$$

Also, the velocity components are

$$\dot{x} = 5 \quad ، \quad \dot{y} = 10t$$

At t=0 we have: $\dot{x} = 5, \dot{y} = 0$, so the initial velocity is equal to 5m/s and in the x-direction.

Likewise, when the particle passes through the x-axis, then $y = 0$, so we have: $t = 2\sec$ and the velocity components becomes $\dot{x} = 5, \dot{y} = -20$ and its value becomes $7\sqrt{5}\, m/s$ and its direction is $\psi = \tan^{-1}(-4)$.

Finally, the acceleration at this case takes the following form

$$\ddot{x} = 0, \ddot{y} = -10\, m/\sec^2$$

Example 4:

If the parametric equations for the moving particle are

$$x = a(2t + \sin 2t) \quad ، \quad y = a(1 - \cos 2t)$$

Prove that this particle moves with a constant acceleration

Solution:

$$\dot{x} = a(2 + 2\cos 2t) \Rightarrow \therefore \ddot{x} = -4a\sin 2t$$

$$\dot{y} = 2a\sin 2t \Rightarrow \therefore \ddot{y} = 4a\cos 2t$$

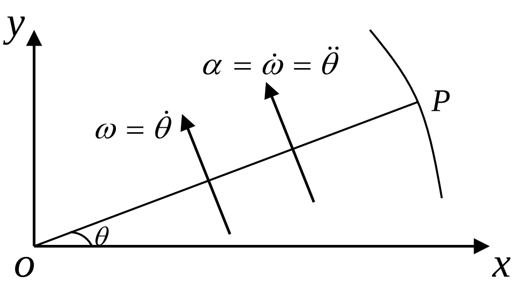
The magnitude for the acceleration is:

$$f = \sqrt{\ddot{x}^2 + \ddot{y}^2} = 4a\sqrt{\sin^2 2t + \cos^2 2t} = 4a$$

Which means that the particle moves with a constant acceleration.

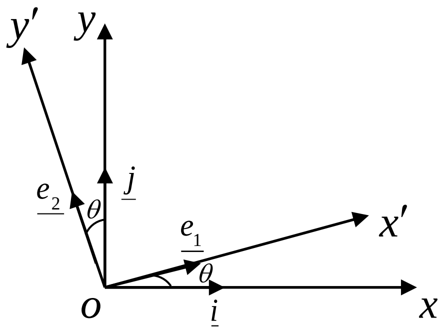
Polar Coordinates

In polar coordinates, the position of a particle A, is determined by the value of the radial distance to the origin, r, and the angle that the radial line makes with an arbitrary fixed line, such as the x axis. Thus, the trajectory of a particle will be determined if we know r and θ as a function of t, i.e. r(t), θ(t). The directions of increasing r and θ are defined by the orthogonal unit vectors e_r and e_θ.



Velocity and acceleration with Polar Coordinate

Assume that o is a fixed point at the plane and their exists two groups of axes. The two fixed axes in the plane (ox, oy) with unit vectors $(\underline{i},\underline{j})$, and the two circular coordinates (ox',oy') with two unit vectors $(\underline{e}_1,\underline{e}_2)$. Assume that the rotating axes start their moving at the case of $\theta=0$. So, we observe that:



$$\underline{e}_1=\cos\theta\underline{i}+\sin\theta\underline{j} \tag{1}$$

$$\underline{e}_2=-\sin\theta\underline{i}+\cos\theta\underline{j} \tag{2}$$

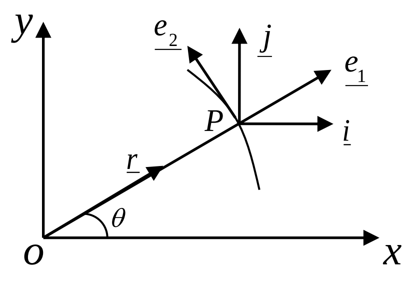
After differentiating the previous equations we have:

$$\frac{d}{dt}(\underline{e}_1)=(-\theta^{\cdot}\sin\theta\underline{i}+\theta^{\cdot}\cos\theta\underline{j})=\theta^{\cdot}\underline{e}_2$$
$$\frac{d}{dt}(\underline{e}_2)=(-\theta^{\cdot}\cos\theta\underline{i}-\theta^{\cdot}\sin\theta\underline{j})=-\theta^{\cdot}\underline{e}_1$$

Where $\theta^{\cdot}$ is the angular velocity about o which can be represented by the symbol ω.

$$\frac{d}{dt}(\underline{e}_1)=\omega\underline{e}_2 \tag{3}$$

$$\frac{d}{dt}(\underline{e}_2)=-\omega\underline{e}_1 \tag{4}$$



Then the position vector of a particle has a magnitude equal to the radial distance, and a direction determined by e_r. Thus,

$$\therefore \underline{r} = \overrightarrow{oP} = r\underline{e_r} = r\underline{e_1}$$

Since the vectors e_r and e_θ are clearly different from point to point, their variation will have to be considered when calculating the velocity and acceleration.

$$\therefore \underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\underline{e_1}) = r^{\cdot}\underline{e_1} + r\frac{d\underline{e_1}}{dt}$$
$$= r^{\cdot}\underline{e_1} + r\theta^{\cdot}\underline{e_2}$$
$$\therefore \underline{v} \equiv (r^{\cdot}, r\theta^{\cdot})$$

We must observe that $r^{\cdot}$ is the velocity component in the r-direction (**the radial velocity component**), but the second $r\theta^{\cdot}$ is the perpendicular velocity component (**the circumferential velocity component**). Similarly, the components of the acceleration can be obtained as follows:

$$\underline{f} = \ddot{r}\underline{e_1} + \dot{r}\dot{\theta}\underline{e_2} + \dot{r}\dot{\theta}\underline{e_2} + r\ddot{\theta}\underline{e_2} - r\dot{\theta}^2\underline{e_1}$$
$$= (\ddot{r} - r\dot{\theta}^2)\underline{e_1} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e_2}$$
$$\underline{f} \equiv \left[(\ddot{r} - r\dot{\theta}^2), \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]$$

Example 5: A particle moves on the curve $r = a + b\sin\theta$ with a constant angular velocity ω, where a and b are constants. Find the acceleration for this particle.

Solution:

$$\dot{r} = b\dot{\theta}\cos\theta = b\omega\cos\theta$$

$$\ddot{r} = -b\dot{\theta}^2\sin\theta + b\ddot{\theta}\cos\theta$$

But $\ddot{\theta} = 0$

$$\therefore \ddot{r} = -b\omega^2\sin\theta$$

Then,

$$f_r = \ddot{r} - r\dot{\theta}^2 = -b\omega^2\sin\theta - a\omega^2 - b\omega^2\sin\theta$$

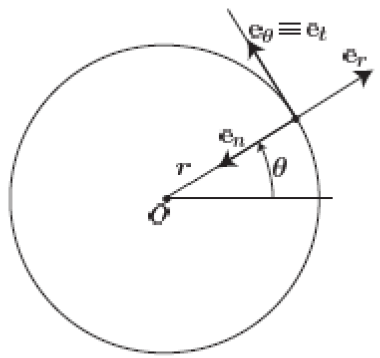
$$= \omega^2(a - 2r)$$

$$f_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2b\omega^2\cos\theta$$

$$\therefore\ f = 2\omega^2\sqrt{b^2 - (r-a)^2}$$

Example 6:

Consider as an illustration, the motion of a particle in a circular trajectory having angular velocity $\omega = \dot{\theta}$, and angular acceleration $\alpha = \dot{\omega}$. Calculate the velocity and the acceleration components.



Solution:

In polar coordinates, the equation of the trajectory is

$$r = R = \text{constant}, \quad \theta = \omega t + \alpha t^2.$$

The velocity components are

$$v_r = \dot{r} = 0, \text{ and } v_\theta = r\dot{\theta} = R(\omega + \alpha t),$$

and the acceleration components are,

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2, \text{ and } a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha.$$

Example 7:

A particle moves with $\dot{\theta} = \omega$ = constant and $r = r_0 e^{\beta t}$, where r_0 and β are constants. Prove that for certain values of β, the particle moves with $a_r = 0$.

Solution:

$$\because a = (\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e_\theta}$$
$$= (\beta^2 - \omega^2) r_0 e^{\beta t}\,\underline{e_r} + 2\beta\omega\, r_0 e^{\beta t}\,\underline{e_\theta}$$

If $\beta = \pm\omega$, the radial part of a vanishes. Then, it seems quite surprising that when $r = r_0 e^{\beta t}$, the particle moves with zero radial acceleration.

Example 8:

If the radial velocity component and the circumferential velocity component are $\mu\theta, \lambda r$, respectively; where μ, λ are constants. Find the trajectory equation and prove that the radial and the circumferential components of acceleration take the following forms:

$$\lambda^2 r - \frac{\mu^2\theta^2}{r} \quad ، \quad \mu\theta(\lambda + \frac{\mu}{r})$$

Solution:

As introduced, we know that the radial and the circumferential components of velocity are:

$$\dot{r} = \lambda r \qquad ، \qquad r\dot{\theta} = \mu\theta$$

After division we have:

$$r\frac{d\theta}{dr} = \frac{\mu}{\lambda}\cdot\frac{\theta}{r} \Rightarrow \therefore \frac{\mu}{\lambda}\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

By integration we get the trajectory equation as following:

$$-\frac{\mu}{\lambda}\ln r = \ln\theta + c$$

Where c is a constant, and the components of accelerations are:

$$f_r = \ddot{r} - r\dot{\theta}^2 = \lambda^2 r - \frac{\lambda^2\theta^2}{r}$$

$$f_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{\mu^2\theta}{r} + \lambda r\frac{\mu\theta}{r} = \mu\theta(\lambda + \frac{\mu}{r})$$

Example 9:

A particle moves with a constant angular velocity ω in a trajectory $r = a\theta$, where a is a constant. Find the velocity and the acceleration of the particle after complete cycle around the pole, if the particle initiated the motion from the pole.

Solution:

$$\because \dot{\theta} = \omega = const. \Rightarrow \therefore \theta = \omega t + c$$

Where c is a constant, which can be determined from the condition $\theta = 0$ at $t = 0$, then $c = 0$

$$\therefore \theta = \omega t$$

After substituting in the trajectory equation we have:

$$r = a\theta = a\omega t$$

This gives:

$$\dot{r} = a\omega \quad ، \quad \ddot{r} = 0$$

and

$$\dot{\theta} = \omega \quad ، \quad \ddot{\theta} = 0$$

Then the radial and circumferential velocity components are:

$$v_r = \dot{r} = a\omega \quad ، \quad v_\theta = r\dot{\theta} = \omega r$$

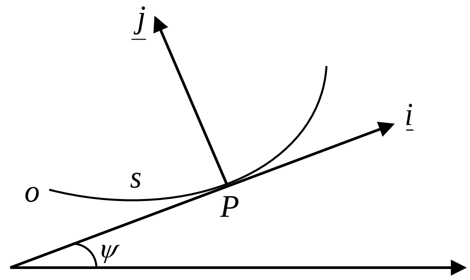
Also, the radial and circumferential acceleration components are:

$$f_r = \ddot{r} - r\dot{\theta}^2 = -\omega^2 r$$

$$f_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2a\omega^2$$

Motion along self-Coordinates (restricted coordinates)

When a particle moves at a restricted movement, such as the movement of a bead on a fixed curved line. The appropriate directions for the analysis that make the study of motion as simple as possible are the direction of the tangent and the vertical direction of the curve, and it is clear that these two directions rotate with the particle in its path in which they keep their orthogonally. The position of the particle on the curve must be determined by the distance s and the angle of the tangent slope of this curve at the point P on a constant straight at the plane ψ .



So, s, ψ are called self-coordinates (restricted coordinates), and $s = s(\psi)$ is called a trajectory self-equation. Then, we take the unit vector $\underline{i}$ in the direction of increasing s, and $\underline{j}$ in the direction of increasing ψ. So, the velocity vector take the tangential direction only as:

$$\underline{v} = \dot{s}\underline{i}$$

And the acceleration vector becomes:

$$\underline{f} = \ddot{s}\underline{i} + \dot{s}\frac{d\underline{i}}{dt}$$

In the same way as in the case of polar coordinates, it can be shown that:

$$\frac{d\underline{i}}{dt} = \dot{\psi}\,\underline{j}$$

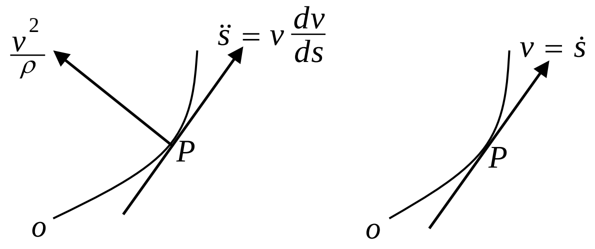
$$\therefore\ \underline{f} = \ddot{s}\underline{i} + \dot{s}\dot{\psi}\,\underline{j}$$

Also, the component $\dot{s}\dot{\psi}$ may be taking the following form:

$$\dot{s}\dot{\psi} = \frac{ds}{dt}\cdot\frac{d\psi}{ds}\cdot\frac{ds}{dt} = \frac{v^2}{\rho}$$

Where $\rho = \dfrac{ds}{d\psi}$ which is called a radius of curvature at the point P.

$$\therefore \underline{f} = \ddot{s}\underline{i} + \frac{v^2}{\rho}\underline{j} = v\frac{dv}{ds}\underline{i} + + \frac{v^2}{\rho}\underline{j}$$



The magnitude $v\frac{dv}{ds}$ is the acceleration component in the tangent direction or in the direction of increasing s, whereas, $\frac{v^2}{\rho}$ is the acceleration component in the direction of the curvature center of curve.

Example 10:

Prove that the angular acceleration for a moving particle can be determined by the following relation:

$$\frac{2v}{\rho}\frac{dv}{ds} - \frac{v^2}{\rho^2}\frac{d\rho}{ds}$$

Solution:

We know that the direction of the movement makes an angle ψ (tangent angle) with a fixed line in the plane, so

$$\because \dot{\psi} = \frac{d\psi}{ds}\cdot\frac{ds}{dt} = \frac{v}{\rho}$$

$$\therefore \ddot{\psi} = \frac{d}{ds}\left(\frac{v}{\rho}\right)\cdot\frac{ds}{dt} = \frac{2v}{\rho}\frac{dv}{ds} - \frac{v^2}{\rho^2}\frac{d\rho}{ds}$$

Example 11:

A particle moves on the curve $s = c\tan\psi$, and the direction of the acceleration always equally split the angle between the tangent of the curve and the vertical. If u it is the magnitude of the velocity at $\psi = 0$, then prove that both the magnitude of the velocity and the acceleration at any position are ue^{ψ} and $\frac{\sqrt{2}}{c}u^2e^{2\psi}\cos^2\psi$.

Solution:

Whereas the direction of the acceleration always equally split the angle between the tangent of the curve and the vertical, then the two components of the acceleration are equally, so

$$\therefore v\frac{dv}{ds} = \frac{v^2}{\rho} = v^2\frac{d\psi}{ds}$$

By separating the variables, we get:

$$\frac{dv}{v} = d\psi \Rightarrow \ln v = \psi + const. \Rightarrow v = Ae^{\psi}$$

But we know that $v = u$ at $\psi = 0$, then $A = u$, then:

$$v = ue^{\psi}$$

To get the acceleration,

$$f_t = v\frac{dv}{ds} = v\frac{dv}{d\psi}\cdot\frac{d\psi}{ds} = v\,ue^{\psi}\cdot\frac{1}{c\sec^2\psi} = \frac{u^2}{c}e^{2\psi}\cos^2\psi$$

So, the magnitude of the acceleration is:

$$f = \frac{\sqrt{2}}{c}u^2e^{2\psi}\cos^2\psi$$

Problems

1- Calculate the velocity, acceleration and the trajectory for a moving particle having a position vector $\underline{r} = \omega t \underline{i} + a \sin \omega t \, \underline{j}$.

2- A velocity vector has a magnitude of 25.0 m/s. If its y component is -13.0 m/s, what are the possible values of its x component?

3- Find the trajectory for a moving particle which have a velocity components $\dot{x} = a + ey$ and $\dot{y} = a + ex$, where a, b, e are constants.

4- A pool ball is rolling along a table with a constant velocity. The components of its velocity vector are $v_x = 0.5$ m/s and $v_y = 0.8$ m/s. Calculate the distance it travels in 0.4 s.

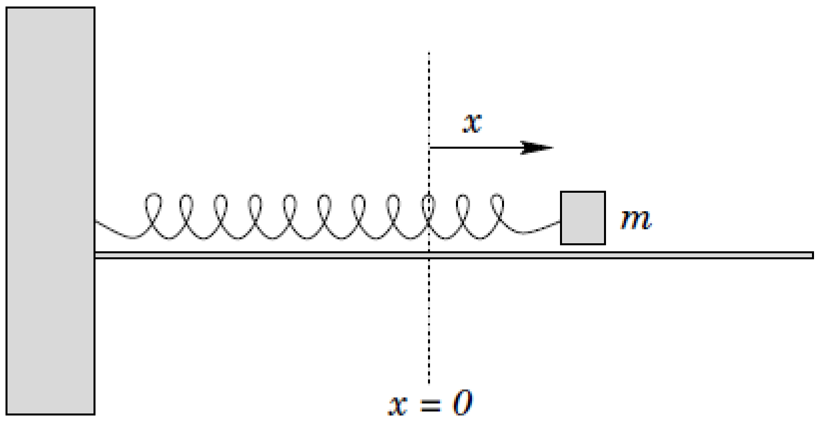
5- Find the acceleration for a moving particle with a constant angular velocity ω if it moves in each of the following polar trajectories: $i) r = a + b\theta$ $ii) r = a + b \sin \theta$

6- A wall clock has a second hand 15.0 cm long. What is the radial acceleration of the tip of this hand?

7- A player kicks a football at an angle of 40.0° from the horizontal, with an initial speed of 12.0 m/s. A second player standing at a distance of 30.0 m from the first (in the direction of the kick) starts running to meet the ball at the instant it is kicked. How fast must he run in order to catch the ball just before it hits the ground?

Simple Harmonic Motion

Consider a mass m which slides over a horizontal frictionless surface. Suppose that the mass is attached to a light horizontal spring whose other end is anchored to an immovable object. See the following Figure.



Let x be the extension of the spring: *i.e.*, the difference between the spring's actual length and its unstretched length. Obviously, x can also be used as a coordinate to determine the horizontal displacement of the mass. The equilibrium state of the system corresponds to the situation where the mass is at rest, and the spring is unextended (*i.e.*, x = 0). In this state, zero net force acts on the mass, so there is no reason for it to start to move. If the system

is perturbed from this equilibrium state (*i.e.*, if the mass is moved, so that the spring becomes extended) then the mass experiences a *restoring force* given by Hooke's law:

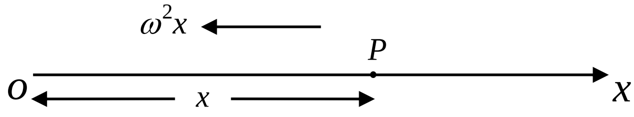
$$f = -k\,x$$

Here, k > 0 is the *force constant* of the spring. The negative sign indicates that f is indeed a restoring force. Note that the magnitude of the restoring force is *directly proportional* to the displacement of the system from equilibrium (*i.e.*,f / x). Of course, Hooke's law only holds for *small* spring extensions. Hence, the displacement from equilibrium cannot be made too large. The motion of this system is representative of the motion of a wide range of systems when they are *slightly* disturbed from a stable equilibrium state. Newton's second law gives following equation of motion for the system:

$$m\ddot{x} = -kx$$

The simple harmonic equation

Assume that o is a fixed point on the straight line and x is the coordinate of the particle position P at the time t with respect to o. Since the amount of the acceleration is proportional to oP, so the amount can be placed on the form $\omega^2(oP)$, where ω is a constant. The proportion constant is taken in this form so that the amount of the acceleration is always positive. If P in the part $x > 0$, then $oP = x$ and the amount of the acceleration becomes $\omega^2 x$ and its direction towards o in the reverse trend of increasing x and thus the equation of movement take the form:



$$\ddot{x} = -\omega^2 x \tag{1}$$

General solution for the particle moving with a simple harmonic motion

To get the solution of Eq. (1), we firstly rewrite it in the following form:

$$v\frac{dv}{dx} = -\omega^2 x \Rightarrow \therefore \int v dv = -\omega^2 \int x dx + c_1$$

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c_1$$

The constant c_1 can be determined from the initial conditions. Suppose the particle starts the motion from the rest at the point A which is placed at the right side of o, where $oA = a$, so at $x = a$, then $v = 0$ which means that

$$\therefore c_1 = \frac{1}{2}\omega^2 a^2$$

$$\therefore v^2 = \omega^2 (a^2 - x^2) \tag{2}$$

By taking the square root of both ends of equation (2) we find that:

$$\therefore v = \frac{dx}{dt} = \pm\omega\sqrt{(a^2 - x^2)}$$

Physically, we take the negative sign, because x decreases with time t

$$\int \frac{-dx}{\sqrt{(a^2 - x^2)}} = \int \omega \, dt + \varepsilon$$

$$\therefore \cos^{-1}\frac{x}{a} = \omega t + \varepsilon$$

$$\therefore x = a\cos(\omega t + \varepsilon) \tag{3}$$

Where ε is a constant which is called the angle of phase and it can be determined after using the initial conditions.

Properties of simple harmonic motion

From the previous relationships which mentioned in the previous item we can deduce the following properties of the simple harmonic motions as follows:

1- As we know $-1 \le \cos(\omega t + \varepsilon) \le 1$ which means that $-a \le x \le a$, that is, the movement of the particle is an oscillating motion between the two points $A(x = a)$ and $A'(x = -a)$ which are equally in the distance from the center of mass o, also the value a is called the amplitude of movement.

2- From the relation $v^2 = \omega^2(a^2 - x^2)$, we observe the vanishing of the velocity at $x = \pm a$ but the velocity becomes maximum at o i.e at $x = 0$. Then $v_{\max} = \omega a$.

3- From the relation $\ddot{x} = -\omega^2 x$ we observe the vanishing of the acceleration at o but it becomes maximum at A, A' with the value $f_{\max} = \omega^2 a$.

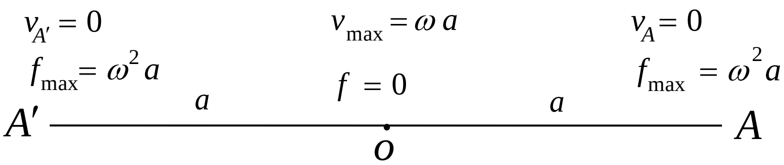
4- Differentiating the equation (3), gives:

$$v = \frac{dx}{dt} = -a\omega \sin(\omega t + \varepsilon)$$

It is clear from this relationship that, at the time t, if the particle passes through P and its velocity has the same value and direction at the following times

$$t + \frac{2\pi}{\omega}, t + \frac{4\pi}{\omega}, t + \frac{6\pi}{\omega}, \ldots$$

The motion of the particle can then be described from the point A as follows:



Periodic time

If the particle moves A to A' and then returns from A' to A, it is said to have done a complete vibration. Then the time taken by the particle in a complete vibration is called a periodic time and it symbolizes with the symbol τ, where

$$\tau = \frac{2\pi}{\omega}$$

Frequency

The total number of oscillations which performed by the particle per second is called the frequency and it is symbolized by the symbol ν, where

$$\nu = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

The general solution for the equation of a simple harmonic motion

As introduced previously, we observe that, the general solution for Eq. (1) $\ddot{x} = -\omega^2 x$ is:

$$x = a\cos(\omega t + \varepsilon)$$

This form contains two constants a, ε which may be determined by the initial conditions. Also, this form can be rewritten in the following form:

$$x = a\cos\omega t\cos\varepsilon - a\sin\omega t\sin\varepsilon$$

Let $A = a\cos\varepsilon$ and $B = -a\sin\varepsilon$

$$\therefore x = A\cos\omega t + B\sin\omega t \qquad (5)$$

The final equation represents the solution for the harmonic motion in a another form, where

$$\therefore A^2 + B^2 = (a\cos\varepsilon)^2 + (-a\sin\varepsilon)^2 = a^2$$

$$\therefore a = \sqrt{A^2 + B^2} \qquad (6)$$

and

$$\tan\varepsilon = -\frac{B}{A} \Rightarrow \varepsilon = \tan^{-1}\left(-\frac{B}{A}\right) \qquad (7)$$

Example 1:

Find the periodic time for the simple harmonic motion which defined by $\ddot{x} = -25x$.

Solution:

After comparison with the equation $\ddot{x} = -\omega^2 x$ we observe that $\omega = 5$, then

$$\therefore \tau = \frac{2\pi}{5} \sec$$

Example 2:

Calculate the maximum velocity and the maximum acceleration for a particle moving a simple harmonic motion with a periodic time $(\pi/4)\sec$ and an amplitude equal to $25cm$.

Solution:

$$\because \tau = \frac{2\pi}{\omega} \Rightarrow \frac{\pi}{4} = \frac{2\pi}{\omega} \Rightarrow \omega = 8$$

$$\because v_{\max} = \omega a \Rightarrow \therefore v_{\max} = 25 \times 8 = 200 cm/\sec$$

$$\because f_{\max} = \omega^2 a \Rightarrow \therefore f_{\max} = 64 \times 25 = 1600 cm/\sec^2$$

Example 3:

If the position x for a moving particle can be determined by

$$x = 0.45 \cos\frac{\pi t}{4} - 0.28 \sin\frac{\pi t}{4} \qquad (1)$$

Prove that the motion represents a simple harmonic motion, and calculate its amplitude, periodic time, maximum acceleration. Also, calculate the initial phase angle.

Solution:

$$v = \dot{x} = -\frac{0.45\pi}{4}\sin\frac{\pi t}{4} - \frac{0.28\pi}{4}\cos\frac{\pi t}{4}$$

$$f = \ddot{x} = -\left(\frac{\pi}{4}\right)^2\left(0.45\cos\frac{\pi t}{4} - 0.28\sin\frac{\pi t}{4}\right)$$

$$\therefore \ddot{x} = -\left(\frac{\pi}{4}\right)^2 x \qquad (1)$$

Which represents a simple harmonic motion.

$$\because A = 0.45 \quad ، \quad B = -0.28$$

$$\therefore a = \sqrt{A^2 + B^2} = \sqrt{(0.45)^2 + (0.28)^2} = 0.53m$$

$$\therefore \omega = \pi / 4 \Rightarrow \therefore \tau = 2\pi / \omega = 8\sec$$

$$\because v_{\max} = \omega a \Rightarrow \therefore v_{\max} = \frac{0.53\pi}{4} m/\sec$$

$$\because f_{\max} = \omega^2 a \Rightarrow \therefore f_{\max} = \frac{0.53\pi^2}{16} m/\sec^2$$

$$\varepsilon = \tan^{-1}\left(-\frac{B}{A}\right) = \tan^{-1}\left(\frac{0.28}{0.45}\right) \Rightarrow \varepsilon = 0.557 Rad$$

Example 4:

A particle moves a simple harmonic motion, if the elapsed distances during a part of the motion in the same direction measured from the center of motion are x_1, x_2, x_3 at the end of three consecutive seconds. Prove that the total periodic time is:

$$2\pi / \cos^{-1}\left(\frac{x_1 + x_3}{x_2}\right)$$

Solution:

$$x_1 = a\sin\omega t \quad ، \quad x_2 = a\sin\omega(t+1)$$

$$x_3 = a\sin\omega(t+2)$$

$$\therefore x_1 + x_3 = a[\sin\omega t + \sin\omega(t+1)]$$

$$= 2a\sin\omega(t+1)\cos\omega = 2x_2\cos\omega$$

$$\therefore \omega = \cos^{-1}[(x_1 + x_3)/x_2]$$

$$\therefore \tau = \frac{2\pi}{\omega} = 2\pi / \cos^{-1}\left(\frac{x_1 + x_3}{x_2}\right)$$

Example 5:

A particle moves in a straight line with an acceleration $\frac{\pi^2}{64}x$ in the direction of the origin. If at $t = 2\sec$, the particle passes through the origin and at $t = 4\sec$ its velocity is $4m/\sec$. Determine the equation of motion, the amplitude of the trajectory, periodic time and the initial phase angle.

Solution:

We know that the equation of motion is:

$$\ddot{x} = -\frac{\pi^2}{64}x$$

This has the following solution

$$x = a\cos(\frac{\pi t}{8} + \varepsilon)$$

$$\frac{dx}{dt} = -\frac{\pi a}{8}\sin(\frac{\pi t}{8} + \varepsilon)$$

But $x = 0$ at $t = 2$, then

$$a\cos(\frac{\pi}{4} + \varepsilon) = 0 \Rightarrow \frac{\pi}{4} + \varepsilon = \frac{\pi}{2} \Rightarrow \therefore \varepsilon = \frac{\pi}{4}$$

Also $\dfrac{dx}{dt} = -4$ at $t = 4$, then

$$\therefore -4 = -\frac{\pi a}{8}\sin(\frac{\pi}{2} + \frac{\pi}{4}) = -\frac{\pi a}{8}\cos\frac{\pi}{4}$$

Likewise, the amplitude is equal to

$$\therefore a = \frac{32\sqrt{2}}{\pi}$$

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/8} 16\sec$$

Finally, the phase angle is $\varepsilon = \pi/4$

Example 6:

If the velocity of a moving particle is obtained from the relation $v^2 = -2x^2 + 4x + 6$. Prove that the motion represents a simple harmonic motion; calculate its center, the maximum acceleration and the frequency.

Solution:

The acceleration of the particle is:

$$\ddot{x} = f = \frac{1}{2}\frac{dv^2}{dx} = -2x + 2 = -(x-1)$$

Putting $y = x - 1$, then $\ddot{y} = \ddot{x}$, and

$$\ddot{y} = -2y = -\omega^2 y$$

Where $\omega = \sqrt{2}$, which represent an equation of a simple harmonic motion, with a center $y = 0$ or $x = 1$. Also, the frequency is $\nu = \omega / 2\pi$. Likewise, the amplitude, we obtain the point in which the velocity vanishes. Then

$$\therefore x^2 - 2x - 3 = (x+1)(x-3) = 0$$

$$\therefore x_1 = -1 \qquad ، \qquad x_2 = 3$$

$$\therefore 2a = x_2 - x_1 = 3 - (-1) = 4m$$

Then the amplitude is equal to $2m$, and the maximum acceleration becomes:

$$\because f_{\max} = \omega^2 a \Rightarrow \therefore f_{\max} = (\sqrt{2})^2 \times 2 = 4m/\sec^2$$

Problems

1-A mass of 0.300 kg is placed on a vertical spring and the spring stretches by 10.0 cm. It is then pulled down an additional 5 cm and then released.

Find (a) the spring constant *k*, (b) the angular frequency ω,

(c) the frequency *f*, (d) the period *T*, (e) the maximum velocity of the vibrating mass.

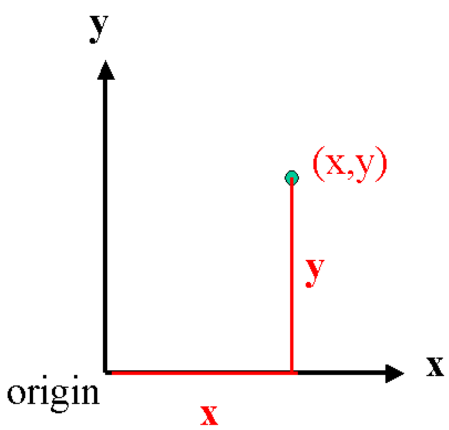
2- A velocity for a particle can be determined by $v^2 = -16x^2 + 32x + 48$. Prove that the motion is a simple harmonic motion and calculate its amplitude, periodic time and the maximum acceleration.

3- A velocity for a particle can be determined by $v^2 = -4x^2 + 16x + 48$. Prove that the motion is a simple harmonic motion and calculate its center and the periodic time.

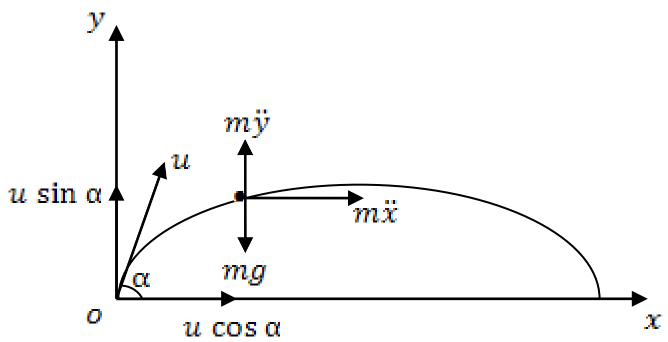
4- A particle moves along the relation $x = 6\cos\frac{5}{2}t + 8\sin\frac{5}{2}t$. Prove that the motion is a simple harmonic motion and calculate its amplitude, initial phase angle, periodic time, the maximum velocity and the maximum acceleration.

Projectile motion

Projectile motion is the motion of an object thrown or projected into the air, subject to only the acceleration of gravity. The object is called a projectile, and its path is called its trajectory. The motion of falling objects, as covered in Problem-Solving Basics for One-Dimensional Kinematics, is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this chapter, we will study projectile motion, which is a special case of two-dimensional motion, such as that of a football or other object for which air resistance is negligible. In a two-dimensional space, an object's position is given by a pair of numbers (coordinates). In Cartesian coordinates there are two orthogonal (at right angles) axes, usually called **x** and **y**. Imagine starting at the origin, you can reach a destination position by first moving along the **x** axis and then along the **y** axis. The destination point is identified as (x, y), where x is the distance moved along the **x** axis and y is the distance moved along the **y** axis. It's just a way of thinking about how a position is represented in Cartesian coordinates. The two parts, x and y, are referred as "components."



Projectile motion results when an object is subject to a single force: the constant force of gravity. Consider a projectile fired with an initial velocity u with a direction of α above the horizontal ox where o is appoint of fired



Acceleration is a vector, and can have any direction. But in the special case of acceleration due solely to gravity, the acceleration is always straight down.

Equations of Motion

The following equations are commonly used equations of motion for an object

$$m\ddot{x} = 0 \qquad (1)$$

$$m\ddot{y} = -mg \qquad (2)$$

From equation (1) we find that

$$\frac{d\dot{x}}{dt} = 0 \Rightarrow \dot{x} = const. = \mathrm{u}\ \cos\alpha \qquad (3)$$

From equation (2) we find that

$$\frac{d\dot{y}}{dt} = -g$$

Separate the variables and make the integral we get

$$\int d\dot{y} = -g\int dt + c$$

$$\therefore \dot{y} = -gt + c \qquad (4)$$

At $t = 0 \Rightarrow \dot{y} = u \sin\alpha$ then $c = u \sin\alpha$

Substitute in the equation (4) we get

$$\therefore \dot{y} = u \sin\alpha - gt \qquad (5)$$

From the equation (3) we find that

$$\int dx = u\cos\alpha \int dt + c_1$$

$x = u\,t\cos\alpha + c_1$

At $t = 0 \Rightarrow x = 0$ then $c_1 = 0$

$$\therefore x = u\,t\cos\alpha \qquad (6)$$

From the equation (5) after the separation of variables and make the integral we find that

$$y = u\,t\sin\alpha - \frac{1}{2}g\text{t}^2 \qquad (7)$$

Characteristics of trajectory :

Flight time

Flight time is the time taken by the projectile to hit the horizontal plane passing through the ejaculation point. To find the flight time, we put $y = 0$ in equation (7) we get

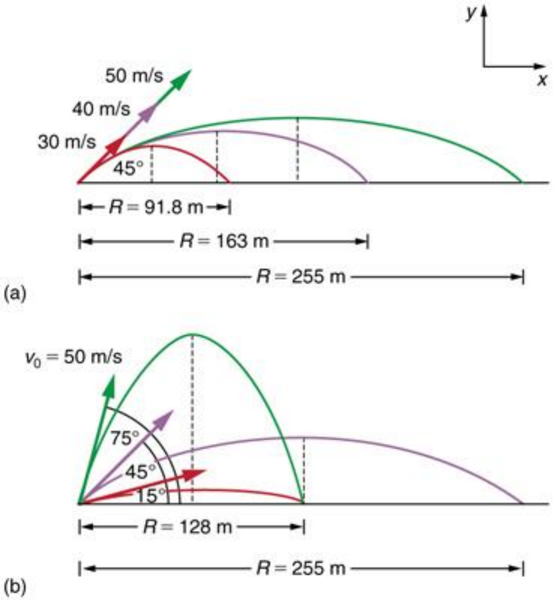
$$\therefore t = \left(\frac{2u}{g}\right)\sin\alpha \qquad (8)$$

This amount represents the time from the moment of ejaculation until it hit the horizontal plane passing through the point of origin O. Because it represents flight time, we exclude time t = 0 because it represents time at the moment of ejaculation.

Horizontal Range

Horizontal Range is the distance between the ejection points and the collision point at the horizontal level in the direction of the ox axis. This distance is the horizontal distance in the flight time. By using equation (8) in equation (6) we get the range, which will be marked with the symbol$R$.

(9) $$R = u\left(\frac{2u}{g}\sin\alpha\right)\cos\alpha = \frac{u^2}{g}\sin 2\alpha$$

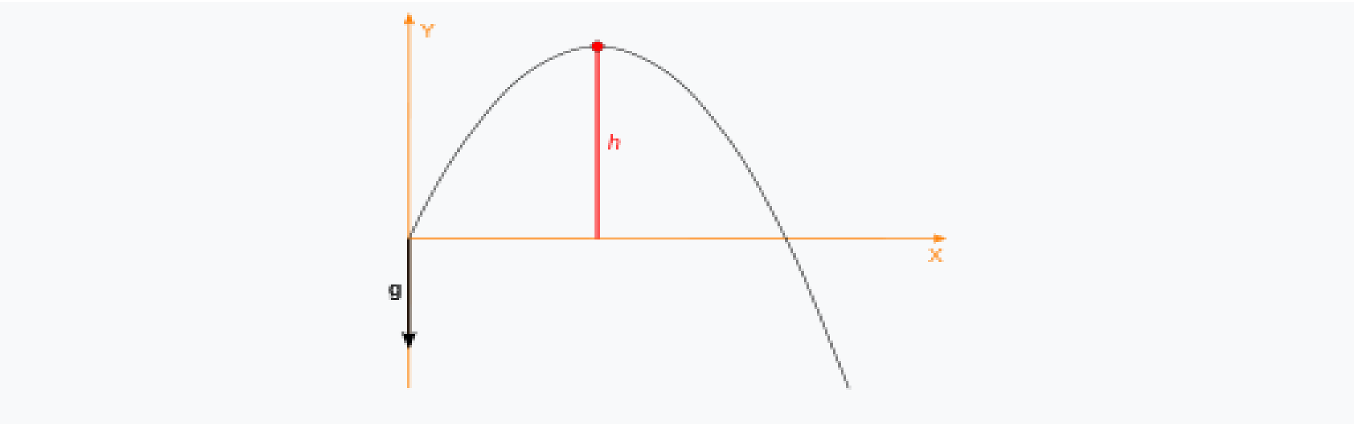


The maximum value of R is when $\sin 2\alpha$ is the largest possible i.e. when $\sin 2\alpha = 1$ and the range is as large as possible and equal

$$R_{\max} = u^2/g$$

1.

aximum Height



The highest height which the object will reach is known as the peak of the object's motion. The increase of the height will last, until vertical component of velocity is zero. Therefore, from the equation (5) after setting $\dot{y} = 0$ we get, Time to reach the maximum height:

$$\therefore t_h = \left(\frac{u}{g}\right)\sin\alpha \qquad (10)$$

From the vertical displacement the maximum height of projectile

$$h = y_{\max} = u\sin\alpha\left(\frac{u}{g}\sin\alpha\right) - \frac{1}{2}g\left(\frac{u}{g}\sin\alpha\right) = \frac{u^2}{2g}\sin^2\alpha \qquad (11)$$

Equation of trajectory

We get the equation of projectile path in Cartesian coordinatesx, y, by deleting time t of the parametric equations (6), (7) as follows:

From the equation (6) we find that

$$t = \frac{x}{u\cos\alpha}$$

Substitute in the equation (7) we get

$$y = x\tan\alpha - \frac{g}{2u^2}x^2\sec^2\alpha$$

$$\therefore y = x\tan\alpha - \frac{g}{2u^2}x^2(1+\tan^2\alpha) \qquad (12)$$

This equation represents a parabola equation and the properties of this parabola can be deduced if the equation is placed on the image

$$\therefore \left(x-\frac{u^2}{g}\cos\alpha\sin\alpha\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2}{2g}\sin^2\alpha\right) \qquad (13)$$

$$\therefore \left(x-\frac{u^2}{g}\sin 2\alpha\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2}{2g}\sin^2\alpha\right)$$

An equation (13) is the equation of a parabola with vertical axis, head upward and the length of the vertical focal is $\frac{2u^2\cos^2\alpha}{g}$

The amount and direction of projectile speed at any moment.

From

$$\dot{x}=u\cos\alpha=v_x$$

$$\dot{y}=u\sin\alpha-gt=v_y$$

∴projectile speed at any moment is

$$v^2=v_x^2+v_y^2=u^2\cos^2\alpha+(u\sin\alpha-gt)^2$$

$$=u^2-2\,g\,u\,t\sin\alpha+g^2t^2$$

$$=u^2-2g\left(u\,t\sin\alpha-\frac{1}{2}gt^2\right)$$

$$=u^2-2gy$$

The direction of the velocity is always in the direction of the tangent of the path, if the angle created by the velocity at any moment with the horizontal is θ , so we have

$$\tan\theta=\frac{v_y}{v_x}=\frac{u\sin\alpha-g\,t}{u\cos\alpha}=\tan\alpha-\frac{g\,t}{u\cos\alpha}$$

But $t = \frac{x}{u\cos\alpha}$

$$\therefore \tan\theta = \tan\alpha - \frac{g\,x}{u^2\cos^2\alpha}$$

Note:

You can get the same previous equation by differential equation of direct path, where the direction of velocity is the slope of the tangent of the curve

$$\therefore \tan\theta = \frac{dy}{dx} = \tan\alpha - \frac{gx}{u^2\cos^2\alpha}$$

An important property:

The velocity has two different directions for throwing to hit a target and this is evident from the equation of the path

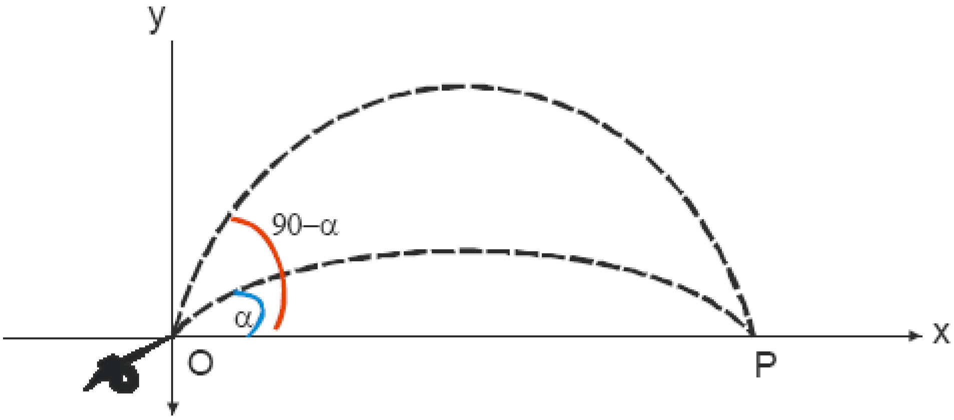
$$\because y = x\tan\alpha - \frac{g}{2u^2}x^2(1+\tan^2\alpha)$$

$$\Rightarrow \frac{2u^2}{gx^2}y = \frac{2u^2}{gx}\tan\alpha - (1+\tan^2\alpha)$$

$$\Rightarrow \tan^2\alpha - \frac{2u^2}{gx}\tan\alpha + \left(1+\frac{2u^2}{gx^2}y\right) = 0$$

It is a second degree equation in ($\tan\alpha$), gives its solution two different values of the α angle

$$\therefore \tan\alpha = \frac{u^2}{gx} \pm \left[\left(\frac{u^2}{gx}\right)^2 - \left(1+\frac{2u^2}{gx^2}y\right)\right]^{\frac{1}{2}} = 0$$



Examples:

Example 1

An object is projected with an initial speed of $640 ft/\sec$ at an angle of 30.0º above the horizontal. Find,

a) Maximum height reached by the object.
b) Flight time and the Horizontal Range.
c) The speed and direction of motion of the particle after $5\sec$ from starting.

The solution:

Initial velocity $u = 640\ \text{ft}/\sec$ with angle $\alpha = 30^\circ$ then the initial velocity

$$(v_\circ)_x = u\cos\alpha = 640\ \cos 30 = 320\sqrt{3}\ ft/\sec$$

$$(v_\circ)_y = u\sin\alpha = 640\sin 30 = 320 ft/\sec$$

$$\dot{x} = u\cos\alpha = 320\sqrt{3}\ ft/\sec$$

$\dot{y} = u \sin\alpha - gt = 320 - 32t \quad ft/\sec$

a) At the maximum height $\dot{y} = 0$, it gives the time to reach to maximum height

$\therefore 0 = 320 - 32t \qquad \Rightarrow \qquad \therefore t = 320/32 = 10\sec$

Substitute in the following equation

$$y = u\, t \sin\alpha - \frac{1}{2} g t^2$$

We get

$$y = 640 \times 10 \times \frac{1}{2} - \frac{1}{2} \times 32 \times 100 = 1600 ft$$

b) Flight time is twice the latency (the time to reach to maximum height) then the flight time is

$T = 2 \times \mathrm{t} = 2 \times 10 = 20\sec$

And Horizontal Range is the horizontal distance traveled in this time

$R = u \cos\alpha \times T = 320\sqrt{3} \times 20 = 6400\sqrt{3}\ ft$

c) After 5 seconds of start

$v_y = 320 - 32 \times 5 = 160\ ft/\sec v_x = 320\sqrt{3}\ ft/\sec$ ،

Then

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(320\sqrt{3})^2 + (160)^2}$$

$= 160\sqrt{13} ft/\sec$

And direction makes the angle with the horizontal

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\frac{160}{320\sqrt{3}} = \tan^{-1}\frac{\sqrt{3}}{6}$$

Example 2

From the top of a tower 208 feet above the surface of the earth, a rocket was launched with initial x− velocity $256\ ft/\sec$ and initial y− velocity$192\ ft/\sec$. Find flight time and how far the point where it collided with the ground at the base of the tower.

The solution:

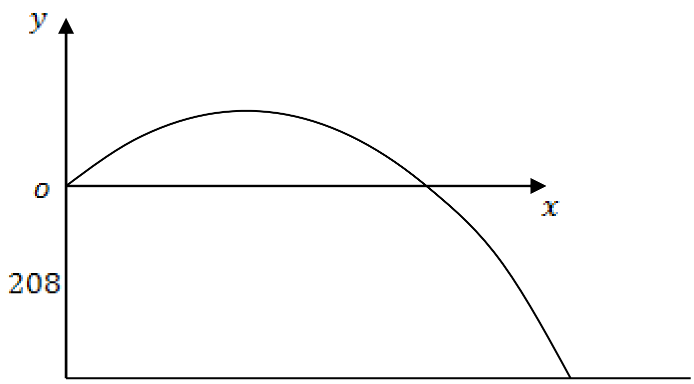
The equations of motion :

Equation of projectile in the horizontal direction is

$$m\ddot{x} = 0 \qquad (1)$$

Equation of motion in vertical direction is

$$m\ddot{y} = -mg \qquad (2)$$



From equations (1), (2) we get

$\dot{x} = const. = 256$ (3)

$\dot{y} = -gt + c$

At t=0 $\Rightarrow$ $\dot{y}$=192 then c=192

$\dot{y} = 192 - gt$ (4)

By integration of the equations (3), (4) we get

$\frac{dx}{dt} = 256 \Rightarrow \int dx = 256 \int dt + c_1 \Rightarrow x = 256t + c_1$

At t=0 $\Rightarrow$ $x = 0$ then $c_1 = 0$

$\therefore x = 256t$ (5)

And

$\frac{dy}{dt} = -gt + 192$

$\int dy = 192 \int dt - g \int tdt + c_2$

$y = 192t - \frac{1}{2}gt^2 + c_2$

At t=0 $\Rightarrow$ $y = 0$ then $c_2 = 0$

$\therefore y = 192t - \frac{1}{2}gt^2$ (6)

To find the flight time we put y = -208 in equation (6)

$16t^2 - 192t - 208 = 0$ $\Rightarrow$ $(t - 13)(t + 1) = 0$

Flight time is t = 13 seconds.

To find the point at which the projectile hits with Earth at the base of the tower, put in equation (5) t = 13 we get

$x = 256 \times 13 = 3328ft$

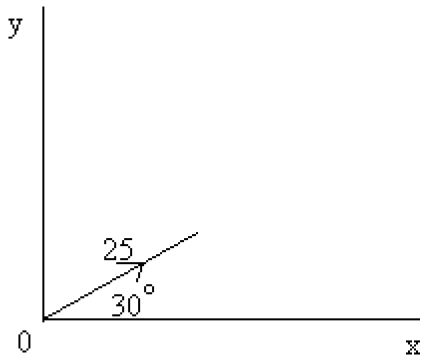
Example 3

A cannon ball is fired at an angle of 30° to the horizontal at a speed of 25 ms^{-1}.

a) How long will it be before the impact?

b) How far will the cannon ball travel before hitting the ground?

The solution:



a) When the particle hits the ground, y = 0.

$$y = u\,t\sin\alpha - \frac{1}{2}gt^2$$

Applying this equation vertically, when the particle hits the ground:

$0 = 25\,T\sin 30 - \frac{1}{2}g\,T^2$ (Where T is the time of flight)

Therefore, $T\left(25\ \sin 30 - \frac{1}{2}g\mathrm{T}\right) = 0$

So T = 0 or T = (50sin30)/g

Therefore the time of flight is 2.55 s

b) The range can be found working out the horizontal distance travelled by the particle after time T found in part (a)

$$x = u\,t\cos\alpha$$

Applying this equation horizontally:

$$\therefore R = 25\, T \cos 30$$

$$= 25 * 2.55 * 0.866$$

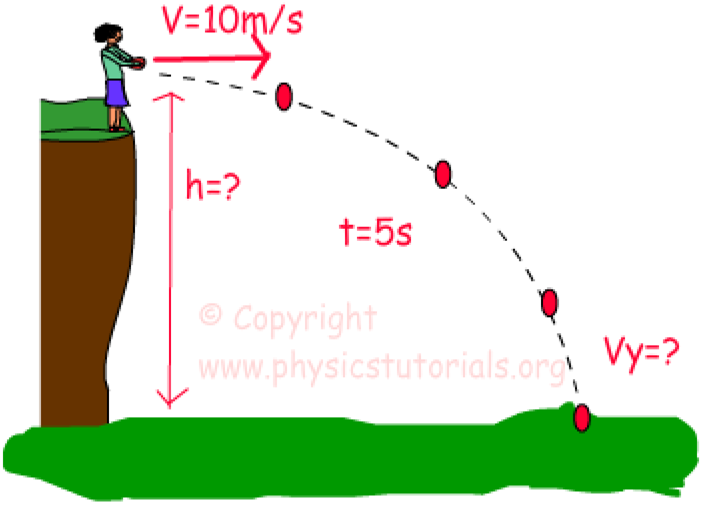
$$= 55.231$$

Therefore the horizontal distance travelled is 55.2 m

Example 4

In the given picture below, Alice throws the ball to the +X direction with an initial velocity 10m/s. Time elapsed during the motion is 5s, calculate the height that object is thrown and v_y component of the velocity after it hits the ground.

The solution:



In vertical direction we have free fall motion

$$y = u\, t \sin \alpha - \frac{1}{2} g t^2$$

$$h = -\left(0 - \frac{1}{2}gt^2\right) = \frac{1}{2} * 9.8 * 5^2 = 122.5\ m$$

In horizontal since our velocity is constant;

$$x = u\,t\cos\alpha$$

$$= 10 * 5 * \ 1 = 50\ \mathrm{m}$$

$$v_y = u\sin\alpha - gt$$

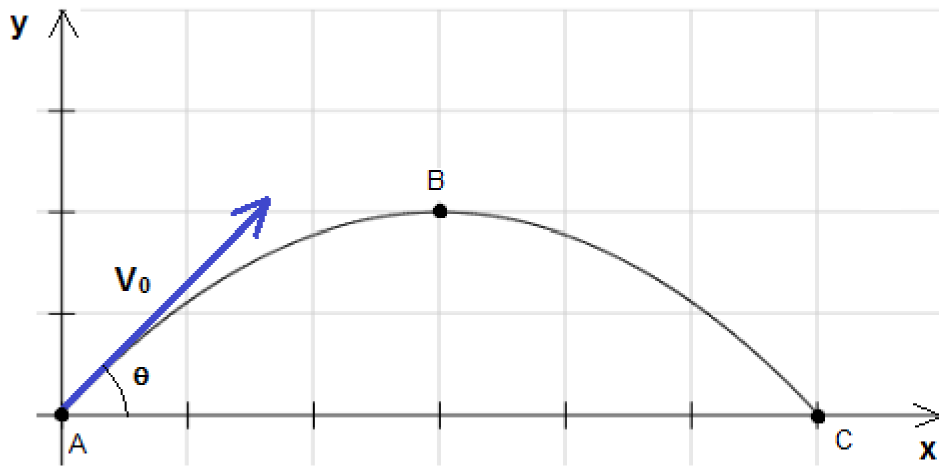
$$= 0 - 9.8 * 5 = \ -49\ m/\mathrm{sec}$$

<u>Example 5</u>

An object is launched at a velocity of 20 m/s in a direction making an angle of 25° upward with the horizontal.

a) What is the maximum height reached by the object?
b) What is the total flight time (between launch and touching the ground) of the object?
c) What is the horizontal range (maximum x above ground) of the object?
d) What is the magnitude of the velocity of the object just before it hits the ground?

<u>The solution:</u>



a) The formulas for the components ν_x and ν_y of the velocity and components x and y of the displacement are given by

$$\nu_x = u \cos \alpha \qquad \nu_y = u \sin \alpha - gt$$

$$x = u\, t \cos \alpha \qquad y = u\, t \sin \alpha - \frac{1}{2} g t^2$$

Or

$V_x = V_0 \cos(\theta)$ $\quad$ $V_y = V_0 \sin(\theta) - g\, t$

$x = V_0 \cos(\theta)\, t$ $\quad$ $y = V_0 \sin(\theta)\, t - (1/2)\, g\, t^2$

in the problem $V_0 = 20$ m/s, $\theta = 25°$ and $g = 9.8$ m/s^2.

The height of the projectile is given by the component y, and it reaches its maximum value when the component V_y is equal to zero. That is when the projectile changes from moving upward to moving downward. (See figure above).

$V_y = V_0 \sin(\theta) - g\, t = 0$

solve for t

$t = V_0 \sin(\theta) / g = 20 \sin(25°) / 9.8 = 0.86$ seconds

Find the maximum height by substituting t by 0.86 seconds in the formula for y then the maximum height is

$y (0.86) = 20 \sin (25°) (0.86) - (1/2) (9.8) (0.86)^2 = 3.64$ meters

b) The time of flight is the interval of time between when projectile is launched: t_1 and when the projectile touches the ground: t_2.

At $t = t_1$ and $t = t_2$, $y = 0$ (ground). Hence

$V_0 \sin(\theta) t - (1/2) g t^2 = 0$

Solve for t

$t(V_0 \sin(\theta) - (1/2) g t) = 0$

two solutions

$t = t_1 = 0$ and $t = t_2 = 2 V_0 \sin(\theta) / g$

Time of flight $= t_2 - t_1 = 2 (20) \sin(25°) / 9.8 = 1.72$ seconds.

c) In part (b) above we found the time of flight $t_2 = 2 V_0 \sin (\theta) / g = 1.72$ sec. The horizontal range is the horizontal distance given by x at $t = t_2$.

$R = x(t_2) = V_0 \cos(\theta) t_2 = 2 V_0 \cos(\theta) V_0 \sin(\theta) / g$

$= V_0^2 \sin (2\theta) / g = 20^2 \sin (2(25°)) / 9.8 = 31.26$ meters

d) The object hits the ground at $t = t_2 = 2\ V0 \sin(\theta) / g = 1.72$ sec (found in part b above)

The components of the velocity at t are given by

$V_x = V_0 \cos(\theta)$ $V_y = V_0 \sin(\theta) - g\ t$

The components of the velocity at $t = 2\ V0 \sin(\theta) / g = 1.72$ sec are given by

$V_x = V_0 \cos(\theta) = 20 \cos(25°)$

$V_y = V_0 \sin(25°) - g\ (2\ V0 \sin(25°) / g) = -\ 20 \sin(25°)$

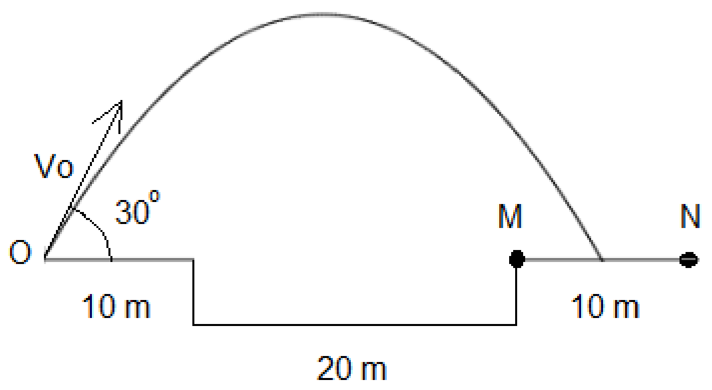
The magnitude V of the velocity is given by

$V = \sqrt{[\ V_x^2 + V_y^2\ ]} = \sqrt{[\ (20 \cos(25°))^2 + (-\ 20 \sin(25°))^2\ ]} = 20$ m/s

Example 6

A projectile is to be launched at an angle of 30° so that it falls beyond the pond of length 20 meters as shown in the figure.

What is the range of values of the initial velocity so that the projectile falls between points M and N?



The solution:

The range is given by $x = V_0^2 \sin(2\theta) / g$

We want to have the range greater than OM and smaller that ON,

with OM = 10 + 20 = 30 m and ON = 10 + 20 + 10 = 40 m

$30 < V_0^2 \sin(2\theta) / g < 40$

$30\ g / \sin(2\theta) < V_0^2 < 40\ g / \sin(2\theta)$

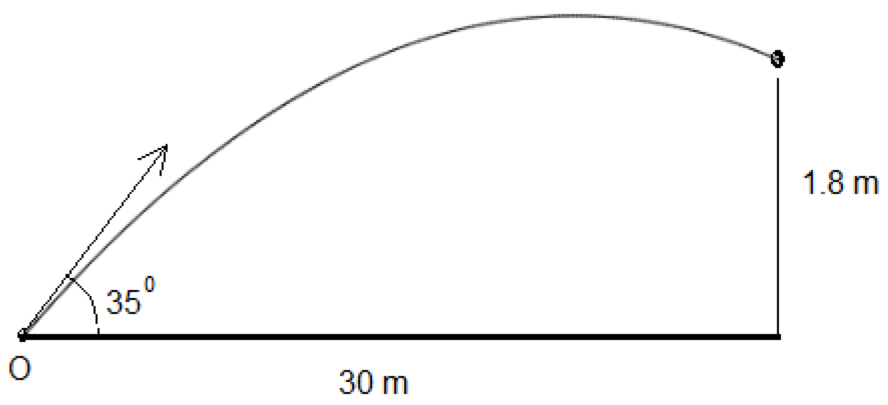
$\sqrt{[\ 30\ g / \sin(2\theta)\ ]} < V_0 < \sqrt{[\ 40\ g / \sin(2\theta)\ ]}$

$18.4\ \text{m/s} < V_0^2 < 21.2\ \text{m/s}$

Example 7

A ball is kicked at an angle of 35° with the ground.

a) What should be the initial velocity of the ball so that it hits a target that is 30 meters away at a height of 1.8 meters?
b) What is the time for the ball to reach the target?



The solution:

a) $x = V_0 \cos(35°)\, t$

$30 = V_0 \cos(35°)\, t$

$t = 30 / V_0 \cos(35°)$

$1.8 = -(1/2)\, 9.8\, (30 / V_0 \cos(35°))^2 + V_0 \sin(35°)(30 / V_0 \cos(35°))$

$V_0 \cos(35°) = 30 \sqrt{[\, 9.8 / 2(30 \tan(35°)-1.8)\,]}$

$V_0 = 18.3$ m/s

b) $t = x / V_0 \cos(35°) = 2.0$ s

Example 8

A ball kicked from ground level at an initial velocity of 60 m/s and an angle θ with ground reaches a horizontal distance of 200 meters.

a) What is the size of angle θ?

b) What is time of flight of the ball?

The solution:

a) Let T be the time of flight. Two ways to find the time of flight

1) $T = 200 / V_0 \cos(\theta)$ (range divided by the horizontal component of the velocity)

2) $T = 2\, V_0 \sin(\theta) / g$

equate the two expressions

$200 / V_0 \cos(\theta) = 2\, V_0 \sin(\theta) / g$

which gives

$2\, V_0^2\, \cos(\theta) \sin(\theta) = 200\, g$

$V_0^2 \sin(2\theta) = 200\, g$

$\sin(2\theta) = 200\ g / V_0^2 = 200\ (9.8) / 60^2$

Solve for θ to obtain

$\theta = 16.5°$

b) Time of flight = $200 / V_0 \cos(16.5°) = 3.48$ s

Example 9

A ball of 600 grams is kicked at an angle of 35° with the ground with an initial velocity V_0.

a) What is the initial velocity V_0 of the ball if its kinetic energy is 22 Joules when its height is maximum?

b) What is the maximum height reached by the ball?

The solution:

a) When the height of the ball is maximum, the vertical component of its velocity is zero; hence the kinetic energy is due to its horizontal component $V_x = V_0 \cos(\theta)$.

$E = (1/2)\ m\ (V_x)^2 = 22$

$22 = (1/2)\ 0.6\ (V_0 \cos(35°))^2$

$V_0 = (1 / \cos(35°)) \sqrt{(44/0.6)} = 10.4$ m/s

b) Initial kinetic energy (just after the ball is kicked)

$E_i = (1/2)\ m\ V_0^2 = (1/2)\ 0.6\ (10.4)^2 = 32.4$ J

The difference between initial kinetic energy and kinetic energy when the ball is at maximum height H is equal to gain in potential energy

$32.4 - 22 = m\, g\, H$

$H = 10.4 / (0.6 * 9.8) = 1.8$ m

Example 10

A projectile starting from ground hits a target on the ground located at a distance of 1000 meters after 40 seconds.

a) What is the size of the angle θ?

b) At what initial velocity was the projectile launched?

The solution:

a) $V_x = V_0 \cos(\theta) = 1000 / 40 = 25$ m/s

Time of flight $= 2\, V_0 \sin(\theta) / g = 40$ s

$V_0 \sin(\theta) = 20 * 9.8 = 196$

Combine the above equation with the equation $V_0 \cos(\theta) = 25$ m/s found above to write

$\tan(\theta) = 196 / 25$

Use calculator to find $\theta = 82.7°$

b) We now use any of the two equations above to find V_0.

$V_0 \cos(\theta) = 25$ m/s

$V_0 = 25 / \cos(82.7°) = 196.8$ m/s

Example 11

The trajectory of a projectile launched from ground is given by the equation

$y = -0.025\, x^2 + 0.5\, x$, where x and y are the coordinate of the projectile on a rectangular system of axes.

Find the initial velocity and the angle at which the projectile is launched.

The solution:

$y = \tan(\theta)\, x - (1/2)\, (g / (V_0 \cos(\theta))^2)\, x^2$

hence

$\tan(\theta) = 0.5 \quad \Rightarrow \quad \theta = \tan^{-1}(0.5) = 26.5\,°$

$-0.025 = -0.5\, (9.8 / (V_0 \cos(26.5\,°))^2)$

Solve for V_0 to obtain $V_0 = 15.6$ m/s

Example 12

Two balls A and B of masses 100 grams and 300 grams respectively are pushed horizontally from a table of height 3 meters. Ball has is pushed so that its initial velocity is 10 m/s and ball B is pushed so that its initial velocity is 15 m/s.

a) Find the time it takes each ball to hit the ground.

b) What is the difference in the distance between the points of impact of the two balls on the ground?

The solution:

a) The two balls are subject to the same gravitational acceleration and therefor will hit the ground at the same time t found by solving the equation

$-3 = -(1/2)\ g\ t^2$

$t = \sqrt{(3(2)/9.8)} = 0.78$ s

b) Horizontal distance X_A of ball A

$X_A = 10 * 0.78 = 7.8$ m

Horizontal distance X_B of ball B

$X_B = 15 * 0.78 = 11.7$ m

Difference in distance X_A and X_B is given by

$|X_B - X_A| = |11.7 - 7.8| = 3.9$ m

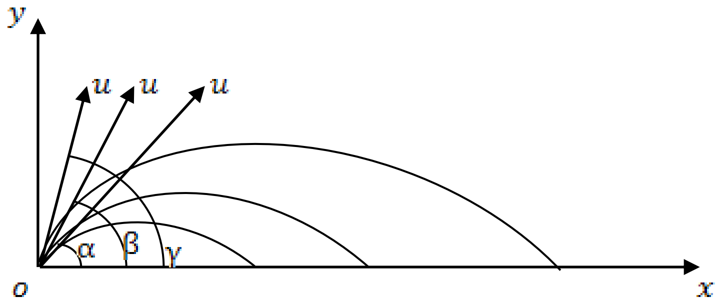
Example 13

A cannon ball is fired at a target at a horizontal plane and it fell before the target at distance (a) when the angle was (α) and when it was shot at the same speed and at an angle (β) it fell after the target at distance (b).

Prove that the angle needed to hit the target is

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right)$$

The solution:



Assume that the correct horizontal distance at the target is hit is x and from the equation

$$R = \frac{u^2}{g} \sin 2\alpha$$

By applying this equation we get

$$x - a = \frac{u^2}{g} \sin 2\alpha \qquad (1)$$

$$x + b = \frac{u^2}{g} \sin 2\beta \qquad (2)$$

from the equations (1), (2) by division we find that

$$\therefore \frac{x-a}{x+b} = \frac{\sin 2\alpha}{\sin 2\beta}$$

$$\Longrightarrow x \sin 2\beta - a \sin 2\beta = x \sin 2\alpha + b \sin 2\beta$$

$$\therefore x(\sin 2\beta - \sin 2\alpha) = a \sin 2\beta + b \sin 2\alpha$$

$$\therefore x = \frac{a \sin 2\beta + b \sin 2\alpha}{\sin 2\beta - \sin 2\alpha} \qquad (3)$$

If θ is the correct angle then

$$x = \frac{u^2}{g} \sin 2\theta \qquad (4)$$

From (3), (4) we get

$\therefore \sin 2\theta = \frac{g}{u^2}\left(\frac{a\sin 2\beta + b\sin 2\alpha}{\sin 2\beta - \sin 2\alpha}\right)$ (5)

But from (1), (2)

$a + b = \frac{u^2}{g}(\sin 2\beta - \sin 2\alpha)$ (6)

By equation (6) in equation (5) we find that

$\therefore \sin 2\theta = (a\sin 2\beta + b\sin 2\alpha)/(a+b)$

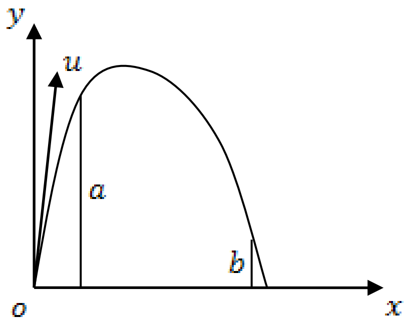
$\therefore \theta = \frac{1}{2}\sin^{-1}\left(\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right)$

Example 13

If a ball is thrown fast enough to make it pass over two wall peaks, the first wall is of height (a) and away distance (b) from the launch point and the second wall is of height (b) and away distance (a) from the launch point.

Prove that the horizontal range is $\frac{a^2+ab+b^2}{a+b}$ and that the launch angle is always greater than $\tan^{-1}(3)$.

The solution:



Equation of trajectory is

$y = x\tan\alpha - \frac{g}{2u^2\cos^2\alpha}x^2$

Since the points (a, b), (b, a) are on the path, then, they achieve the equation of trajectory, and get

(1) $a = b\tan\alpha - \frac{g}{2u^2\cos^2\alpha}b^2$

(2) $$b = a\tan\alpha - \frac{g}{2u^2\cos^2\alpha}a^2$$

By multiplying equation (1) in a^2 and equation (2) in b^2 and subtracting we get

$$a^3 - b^3 = a^2 b\tan\alpha - b^2 a\tan\alpha$$

$$\therefore (a-b)(a^2+ab+b^2) = (a-b)ab\tan\alpha$$

$$\therefore \tan\alpha = \frac{a^2+ab+b^2}{ab}$$

$$\therefore \sin\alpha = \frac{a^2+ab+b^2}{(a+b)\sqrt{a^2+b^2}}$$

$$\therefore \cos\alpha = \frac{ab}{(a+b)\sqrt{a^2+b^2}}$$

The range is given by

(3) $$R = \frac{u^2}{g}\sin 2\alpha = \frac{2u^2}{g}\sin\alpha\ \cos\alpha$$

By multiplying equation (1) in b and equation (2) in a and subtracting we get

$$0 = (b^2-a^2)\tan\alpha - \frac{g}{2u^2\cos^2\alpha}(b^3-a^3)$$

$$(a+b)\tan\alpha = \frac{g}{2u^2\cos^2\alpha}(a^2+ab+b^2)$$

(4) $$\therefore u^2 = \frac{g(a^2+ab+b^2)}{2(a+b)\sin\alpha\cos\alpha}$$

By equation (4) in equation (3) we find that

$$R = \frac{a^2+ab+b^2}{a+b}$$

And

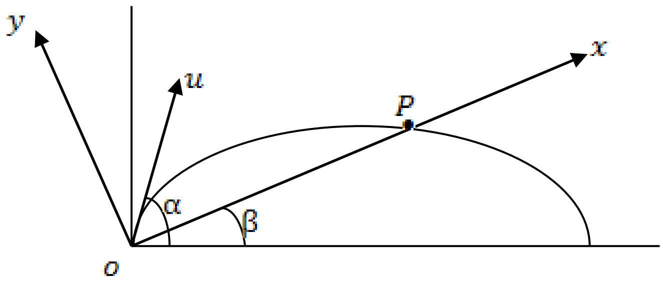
$$\tan\alpha = \frac{a^2+ab+b^2}{ab} = \frac{a^2+3ab-2ab+b^2}{ab}$$

$$= \frac{3ab+(a-b)^2}{ab} = 3 + \frac{(a-b)^2}{ab}$$

i.e. $\tan\alpha > 3 \quad \Rightarrow \quad \alpha > \tan^{-1}3$

The lowest value of angle α is when $a = b$, i.e. when the height of the two walls is equal and in this case; $\alpha = \tan^{-1}3$.

Projectiles on inclined planes



Assume that the sloping surface inclines with the horizontal at an angle β and a particle is projected with velocity u from the bottom of an inclined plane at an angle α with the horizontal, i.e. at angle (α-β) with the surface of incline. The coordinates of point P (the intersection point of the trajectory with the inclined plane) are $(x, x\tan\beta)$. Let R is the range on the inclined plane then

$x = R\cos\beta$

$\therefore R = \frac{x}{\cos\beta} = x\sec\beta$

But from the equation of trajectory and for the coordinates of point P,

$y = x\tan\alpha - \frac{g}{2u^2\cos^2\alpha}x^2$

$x\tan\beta = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

$$x = \frac{2u^2\cos^2\alpha}{g}(\tan\alpha - \tan\beta)$$

And the range on the inclined plane is

$$R = \frac{x}{\cos\beta} = \frac{2u^2}{g} \times \frac{\cos^2\alpha}{\cos\beta}(\tan\alpha - \tan\beta)$$

$$= \frac{2u^2}{g} \times \frac{\cos\alpha}{\cos^2\beta}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$= \frac{2u^2}{g} \times \frac{\cos\alpha}{\cos^2\beta}\sin(\alpha - \beta)$$

In another form

$$R = \frac{u^2}{g\cos^2\beta} \times 2\cos\alpha\sin(\alpha - \beta)$$

$$= \frac{u^2}{g\cos^2\beta} \times [\sin(\alpha + (\alpha - \beta)) - \sin(\alpha - (\alpha - \beta))]$$

Where $\boxed{\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}}$

$$\therefore R = \frac{u^2}{g\cos^2\beta} \times [\sin(2\alpha - \beta) - \sin\beta]$$

To find the maximum range

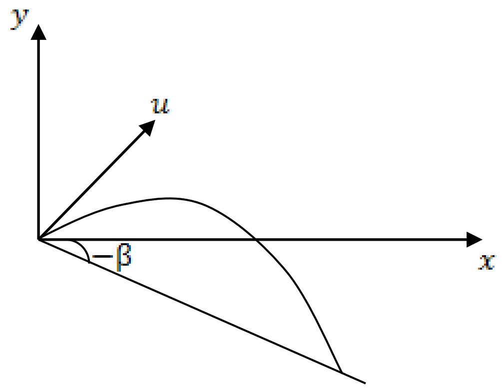
The range will be the maximum when the value of $\sin(2\alpha - \beta)$ is maximum i.e.

$\sin(2\alpha - \beta) = 1 \quad \Rightarrow \quad 2\alpha - \beta = \frac{\pi}{2} \quad \Rightarrow \quad \alpha - \beta = (\pi/2) - \alpha$

Then the maximum range is

(I) $$\therefore R_{\max} = \frac{u^2}{g\cos^2\beta} \times [1 - \sin\beta] = \frac{u^2}{g(1+\sin\beta)}$$

Important note:



If the inclined plane make angle $(-\beta)$ with horizontal then

$$R = \frac{2u^2}{g} \times \frac{\cos\alpha}{\cos^2\beta}\sin(\alpha + \beta)$$

$$= \frac{u^2}{g\cos^2\beta} \times [\sin(2\alpha + \beta) + \sin\beta]$$

And the maximum range in this case is

(II) $$R_{\max} = \frac{u^2}{g(1-\sin\beta)}$$

From (I), (II) shows that the ratio between the maximum range in the inclined plane below the horizontal to the maximum range in the inclined above the horizontal at β angle is

$$\frac{1 + \sin\beta}{1 - \sin\beta}$$

Important note:

To study the motion of projectiles on inclined plane, as in the case of horizontal, the equations of motion are

$m\ddot{x} = -m\,g\sin\beta \quad \Rightarrow \quad \ddot{x} = -g sin\beta \quad \Rightarrow \quad \dot{x} = -g\,t\sin\beta + c_1$

(1)

$m\ddot{y} = -m\,g\cos\beta \quad \Rightarrow \quad \ddot{y} = -\mathrm{g}\cos\beta \quad \Rightarrow \quad \dot{y} = -g\,t\cos\beta + c_2$

(2)

At the starting when t = 0 then

$\dot{x} = u\cos(\alpha - \beta) = c_1$

$\dot{y} = u\sin(\alpha - \beta) = c_2$

And we get

(3) $\dot{x} = u\cos(\alpha - \beta) - g\,t\sin\beta$

(4) $\dot{y} = u\sin(\alpha - \beta) - gt\cos\beta$

And again by integrated we get

$x = ut\cos(\alpha - \beta) - \frac{1}{2}gt^2\sin\beta + c_3$

$y = ut\sin(\alpha - \beta) - \frac{1}{2}gt^2\cos\beta + c_4$

At the starting when t = 0 then

$x = 0 = c_3,$ and $y = 0 = c_4$

Then

(5) $x = ut\cos(\alpha - \beta) - \frac{1}{2}gt^2\sin\beta$

(6) $$y = ut\sin(\alpha-\beta) - \frac{1}{2}gt^2\cos\beta$$

Flight Time:

To find the flight time T we put $y = 0$ in equation (6) we get

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

Put T in equation (5) we get the range at inclined plane as

$$R = \frac{2u^2}{g} \times \frac{\cos\alpha}{\cos^2\beta}\sin(\alpha-\beta)$$

Trajectory equation:

By the same way at horizontal plane we get the trajectory equation at the inclined plane as:

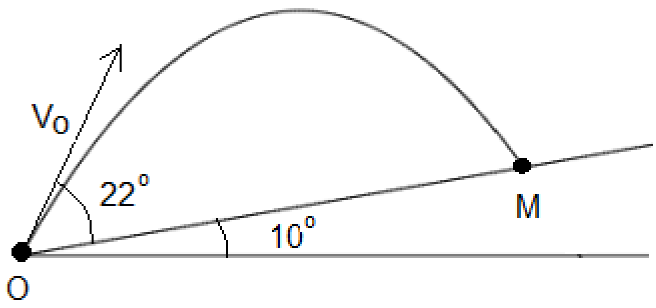
$$y = x\tan(\alpha+\beta) - \frac{g}{2u^2\cos^2(\alpha+\beta)}x^2 \qquad (7)$$

Example 14

A projectile is launched from point O at an angle of 22° with an initial velocity of 15 m/s up an incline plane that makes an angle of 10° with the horizontal. The projectile hits the incline plane at point M.

a) Find the time it takes for the projectile to hit the incline plane.

b) Find the distance OM.



The solution:

a) The x and y components of the displacement are given by

$x = V_0 \cos(\theta)\, t \qquad y = V_0 \sin(\theta)\, t - (1/2)\, g\, t^2$

with $\theta = 22 + 10 = 32°$ and $V_0 = 15$ m/s

The relationship between the coordinate x and y on the incline is given by

$\tan(10°) = y / x$

Substitute x and y by their expressions above to obtain

$\tan(10°) = (V_0 \sin(\theta)\, t - (1/2)\, g\, t^2) / V_0 \cos(\theta)\, t$

Simplify to obtain the equation in t

$(1/2)\, g\, t + V_0 \cos(\theta) \tan(10°) - V_0 \sin(\theta) = 0$

Solve for t

$$t = \frac{V_0 \sin(\theta) - V_0 \cos(\theta) \tan(10°)}{0.5\ \mathrm{g}}$$

$$t = \frac{15\, sin(32°)\; -\; 15\, cos(32°)\, tan(10°)}{0.5\,(9.8)} = 1.16\, s$$

b)

$OM = \sqrt{[(V_0 \cos(\theta)\, t)^2 + (V_0 \sin(\theta)\, t - (1/2)\, g\, t^2)^2]}$

$OM\ (t{=}1.16) = \sqrt{[(15 \cos(32)\, 1.16)^2 + (15 \sin(32)\, 1.16 - (1/2)\, 9.8\, (1.16)^2)^2]}$

$= 15$ meters

Problems

***What information on the (v u a s t)variables (in either the x or the y* direction) can be taken from the following?**

1. If a bullet is fired *horizontally* what does it mean?

Answer:
$u_y = 0$

2. If an object is fired *from a cliff* which is 200 metres above sea-level, how will this affect our approach to answering the question?

Answer:
$S_y = - 200$

3. How would you find *time of flight*?

Answer:
Find t when $s_y = 0$.

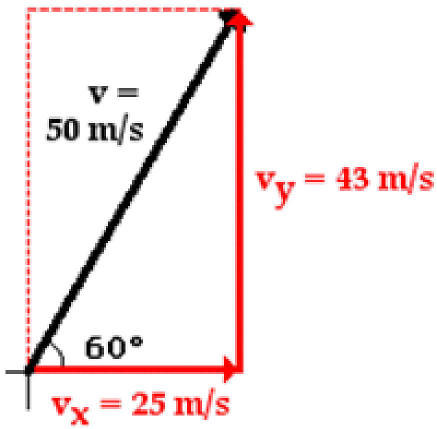
4. How would you find the *range* of a particle?

Answer: Find t when $s_y = 0$ and sub this time in to s_x.

5. How would you find the *Maximum height* reached by a particle?

Answer:At maximum height $v_y = 0$, so find t from this and sub this value in to s_y.
Or

Use the equation $v^2 = u^2 + 2as$ for the y-direction (which becomes $0 = u_y^2 + 2(-9.8)s$)



6. How would you find the *magnitude* of a particle's velocity after 3 seconds?

Answer:
First find v_x and v_y when t = 3. Magnitude is then found from the formula **Magnitude = $\sqrt{(v_x^2 + v_y^2)}$**

7. How would you find the *direction* of a particle after 3 seconds?

Answer:
First find v_x and v_y when t = 3. Direction is then found from the formula
$\boldsymbol{\theta = \text{Tan}^{-1} (v_y / v_x)}$

8. When would we ever take 'g' to be positive?

Answer:
When an object is *given an initial velocity downwards* (this rarely comes up).

9. If a projectile just clears a wall (or hits a dart-board) which is 3 m away and 5 m high then . . .?

Answer:
$S_x = 3$ when $S_y = 5$

10. If two projectiles collide in mid-air then . . .?

Answer:
(i) s_y for first projectile = s_y for second projectile

(ii) s_x for first projectile *plus* s_x for second projectile equals the total distance between them at the beginning.

Motion in a resistance Medium

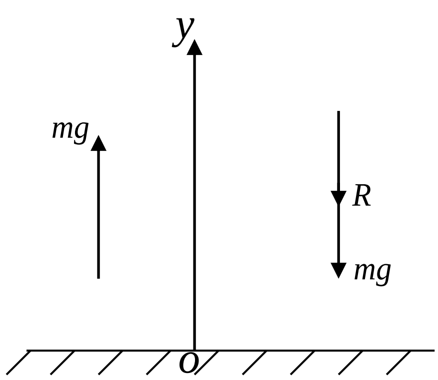
In this chapter, we introduce study for a particle moving vertically with resistance. We will consider the vertical motion of objects through fluids near the Earth's surface where the acceleration due to gravity is assumed to be constant $g = 9.80$ m.s^{-2}.

The motion of falling objects is usually described with constant acceleration. This is only approximately true. For example, in introductory physics textbooks, two objects of different mass when dropped simultaneously from rest will hit the ground at the same time. This is an idealized situation and ignores the effects of the air resisting the motion of the falling objects. Air resistance, a friction which increases with increasing speed, acts against gravity, so the speed of falling objects tends toward a limit called terminal velocity (terminal speed).

For the vertical motion of an object through a fluid, the forces acting on the object are the gravitational force mg (weight) and the resistive force R.

Case1: The particle moving up

In this case, the forces acting on the particle



1- The weight of the particle mg and vertically affects down.

2- Air resistance R and vertically impact down.

Equations of motion

$$mf = -mg - R$$

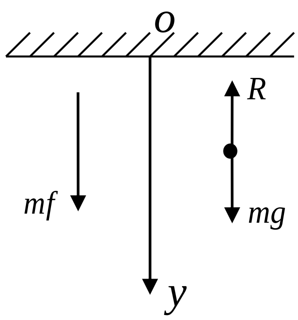
Where f is the acceleration and g acceleration due to gravity and R air resistance

Case2: The particle moving down:

In this case, the forces acting on the particle

1- The weight of the particle mg and vertically affects down.

2- Air resistance R and vertically impact up



Equations of motion

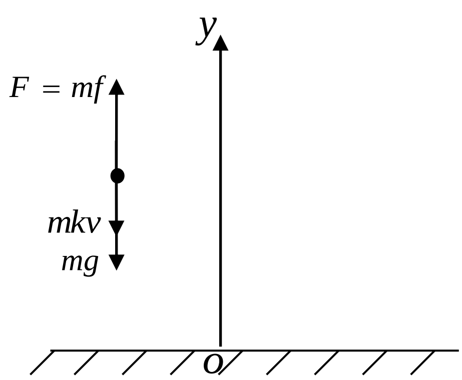
$$mf = mg - R$$

Where f is the acceleration and g acceleration due to gravity and R air resistance.

Example 1:

A particle has been thrown vertically up in the field of its resistance mkv, where v the velocity at any moment, k constant. If the velocity of the particle disappeared after time T from the moment of ejaculation and on height H from ejaculation point. Prove that the initial velocity equal to $gT + kH$.

Solution



Consider oy the vertical axis, since the particle moves upward in the direction of increasing y,then the acceleration takes the form

$\ddot{y} = \dfrac{d^2 y}{dt^2}$

and in the same direction.

Forces acting on the particle

1-The weight of the particle mg and vertically affects down.

2-Air resistance mkv and vertically impact down.

Equations of motion

$$m\ddot{y} = -mg - mkv$$

$$\ddot{y} = -(g + kv) \qquad (1)$$

Substitute $\ddot{y} = \dfrac{dv}{dt}$, separate the variables and integrate, we obtain

$$\frac{dv}{dt} = -k(v + (g/k)) \Rightarrow \int \frac{dv}{v + (g/k)} = -k\int dt + c_1$$

$$\therefore \ln[v + (g/k)] = -kt + c_1 \qquad (2)$$

From the initial conditions $t = 0$, $v = v_\circ$, then we get

$$c_1 = \ln[v_\circ + (g/k)] \qquad (3)$$

From (2) and (3), we obtain

$$t = \frac{1}{k}\ln\left(\frac{v_\circ + (g/k)}{v + (g/k)}\right) \qquad (4)\ 0$$

at $t = T$, then $v = 0$, and (4) takes the form

$$T = \frac{1}{k}\ln\left(\frac{v_\circ k + g}{g}\right) \qquad (5)$$

Eq. (5) give the time when the velocity disappears. To find the Particle height at any velocity, the acceleration must take the form $\ddot{y} = v\dfrac{dv}{dy}$, from (1), we get

$$v\frac{dv}{dy} = -k(v+(g/k)) \Rightarrow \int \frac{vdv}{v+(g/k)} = -k\int dy + c_2$$

$$\therefore \int \frac{v+(g/k)-(g/k)}{v+(g/k)} dv = -ky + c_2$$

$$\therefore \int \left[1 - \frac{(g/k)}{v+(g/k)}\right] dv = -ky + c_2$$

$$\therefore v - \frac{g}{k}\ln(v+(g/k)) = -ky + c_2 \qquad (6)$$

Assume that the particle was extrusion above at initial velocity, $v = v_\circ$, then $y = 0$, and from (6) we get

$$\therefore c_2 = v_\circ - \frac{g}{k}\ln(v_\circ + (g/k))$$

Substituting into (6), we obtain

$$y = \frac{1}{k}\left[v_\circ - \frac{g}{k}\ln(v_\circ + \frac{g}{k}) - v + \frac{g}{k}\ln(v + \frac{g}{k})\right] \qquad (7)$$

Since $v = 0$ when $y = H$, then we get

$$H = \frac{1}{k}\left[v_\circ - \frac{g}{k}\ln\left(\frac{kv_\circ + g}{g}\right)\right] \qquad (8)$$

From (5) and (8), we obtain

$$kH = v_\circ - gT$$

$$\therefore v_\circ = kH + gT \qquad (9)$$

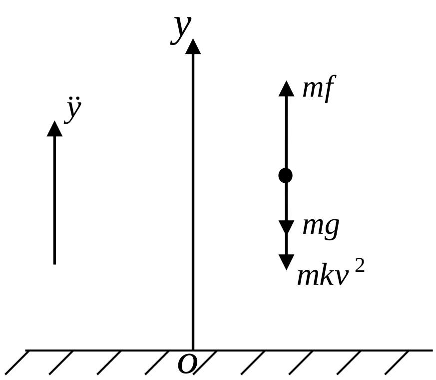
Eq. (9) gives the initial velocity $v_\circ$ at .Tafter the time $y = H$, $v = 0$

Example 2:

A particle has been thrown vertically up with initial velocity U in the field of its resistance proportional to quadratic of particle velocity. If the particle returns to the ejaculation point with velocity U'. Prove that $\frac{1}{U'^2} - \frac{1}{U^2} = \frac{1}{V^2}$ where V the final velocity.

Solution

First: The particle moving up



1-The weight of the particle mg and vertically affects down.

2-Air resistance mkv^2 and vertically impact down

Since the particle moves upward, then the equation of motion takes the form

$$mf = -mg - mkv^2$$

$$\therefore f = -k\left(v^2 + \frac{g}{k}\right) \qquad (1)$$

Using the acceleration form $f = v\frac{dv}{dy}$, separate variables and integrate, we get

$$\therefore v\frac{dv}{dy} = -k\left(v^2 + \frac{g}{k}\right) \Rightarrow \frac{vdv}{v^2 + (g/k)} = -kdy$$

$$\frac{1}{2}\int \frac{dv^2}{v^2 + (g/k)} = -k\int dy + c_1$$

$$\therefore \frac{1}{2}\ln\left[v^2 + (g/k)\right] = -ky + c_1 \qquad (2)$$

Now, we compute the constant c_1, from the initial conditions at $y = 0$, $v = U$.

$$\therefore c_1 = \frac{1}{2}\ln\left[U^2 + (g/k)\right] \tag{3}$$

From (2) and (3), we get

$$y = \frac{1}{2k}\ln\frac{U^2 + (g/k)}{v^2 + (g/k)} \tag{4}$$

Eq. (4) gives the particle height y at any value of velocity v.

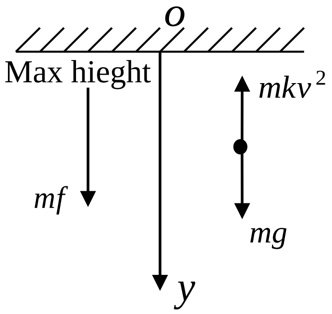
To compute y_{max}, we put $v = 0$ in Eq. (4)

$$y_{max} = \frac{1}{2k} ln \frac{kU^2 + g}{g} \tag{5}$$

Second: The particle moving down

1-The weight of the particle mg and vertically affects down.

2-Air resistance mkv^2 and vertically impact upward



In this case the equations of motion take the form

$$mf = mg - mkv^2$$

$$\therefore f = -k\left(v^2 - \frac{g}{k}\right) \tag{6}$$

Using the acceleration form $f = v\frac{dv}{dy}$, separate variables and integrate, we get

$$v\frac{dv}{dy} = -k\left(v^2 - \frac{g}{k}\right) \Rightarrow \therefore \int \frac{vdv}{\left(v^2 - \frac{g}{k}\right)} = -k\int dy + c_2$$

$$\frac{1}{2}\ln\left(v^2 - \frac{g}{k}\right) = -ky + c_2 \tag{7}$$

To compute c_2 in Eq. (7), at $t = 0$, $v = 0$,then

$$c_2 = \frac{1}{2}\ln(-g/k) \tag{8}$$

from (7) and (8), we get

$$y = \frac{1}{2k}\ln\frac{g}{g-kv^2} \tag{9}$$

Eq. (9) gives the distance between the particle and the point of maximum height. When the particle arrives again to ejaculation point, then $v = U'$ from (9), we obtain

$$y_{\max} = \frac{1}{2k}\ln\frac{g}{g-kU'^2} \tag{10}$$

From (5) and (10), we get

$$\ln\frac{g}{g-kU'^2} = \ln\frac{kU^2+g}{g}$$

$$\therefore g^2 - (g-kU'^2)(g+kU^2) = 0$$

$$U'^2 + \frac{k}{g}U^2U'^2 - U^2 = 0 \tag{11}$$

At the final velocity V (V_{max}), then $f = 0$, and from (6) we get

$$0 = -k(V^2 - \frac{g}{k}) \Rightarrow \therefore V^2 = \frac{g}{k} \tag{12}$$

From (11) and (12), we conclude

$$V^2U'^2 + U^2U'^2 - V^2U^2 = 0$$

by dividing on $V^2U'^2U^2$, we get

$$\frac{1}{U'^2} - \frac{1}{U^2} = \frac{1}{V^2}$$

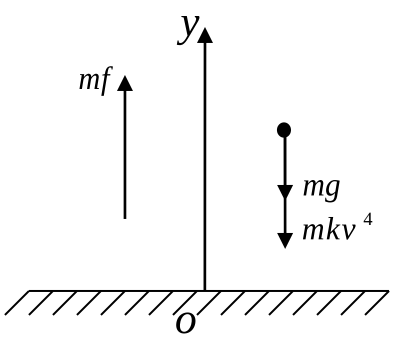
The proof is completed.

Example3:

A particle has been thrown vertically up in the field of its resistance mkv, where v the velocity at any moment, k constant. Prove that the kinetic energy when it is at a distance y from the ejaculation point equal to $T\tan(mgy/T)$ during the climb, and equal to $T\tanh(mgy/T)$ during landing , where $T=(m\sqrt{g/k})/2$

Solution

First: The particle moving up



1-The weight of the particle mg and vertically affects down.

2-Air resistance mkv^4 and vertically impact down

Since the particle moves upward, then the equation of motion takes the form

$$mf=-mg-mkv^4$$

$$\therefore\ f=-k\left(v^4+\frac{g}{k}\right) \qquad (1)$$

Substituting $f=v\frac{dv}{dx}$ in (1) and separate variables, we get

$$\therefore v\frac{dv}{dy}=-k\left(v^4+\frac{g}{k}\right)\Rightarrow\frac{vdv}{v^4+(g/k)}=-kdy$$

$$\frac{1}{2}\int\frac{dv^2}{v^4+(g/k)}=-k\int dy+c_1$$

$$\frac{1}{2}\cdot\frac{1}{\sqrt{g/k}}\tan^{-1}\frac{v^2}{\sqrt{g/k}}=-ky+c_1 \qquad (2)$$

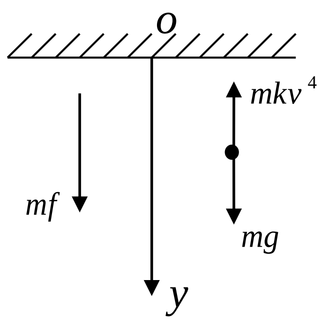
To compute c_1 in (2), at $y = 0. v = 0$. Then $c_1 = 0$, and (2) takes the form

$$v^2=\sqrt{\frac{g}{k}}\tan\frac{-mgy}{(m\sqrt{g/k})/2}$$

The total kinetic energy of the particle $\frac{1}{2}mv^2$

$$E=\frac{1}{2}m\sqrt{\frac{g}{k}}\tan\frac{-mgy}{(m\sqrt{g/k})/2}=-T\tan\frac{-mgy}{T}$$

Second: The particle moving down



1-The weight of the particle mg and vertically affects down.

2-Air resistance mkv^4 and vertically impact upward

In this case the equations of motion take the form

$$mf=mg-mkv^4$$

$$\therefore\ f=k\left(\frac{g}{k}-v^4\right) \qquad (4)$$

Substituting $f = v\frac{dv}{dx}$ in (4) and separate variables, we get

$$v\frac{dv}{dy}=k\left(\frac{g}{k}-v^4\right)\Rightarrow\therefore\int\frac{vdv}{\left(\frac{g}{k}-v^4\right)}=k\int dy+c_2$$

$$\frac{1}{2}\cdot\frac{1}{\sqrt{g/k}}\tanh^{-1}\frac{v^2}{\sqrt{g/k}}=ky+c_2 \qquad (5)$$

To compute c_1 in (5), at $y = 0. v = 0$. Then $c_2 = 0$, and (5) takes the form

$$v^2 = \sqrt{\frac{g}{k}}\tanh\frac{mgy}{(m\sqrt{g/k})/2} = \sqrt{\frac{g}{k}}\tanh\frac{mgy}{T}$$

Where $T = (m\sqrt{g/k})/2$

The kinetic energy during landing

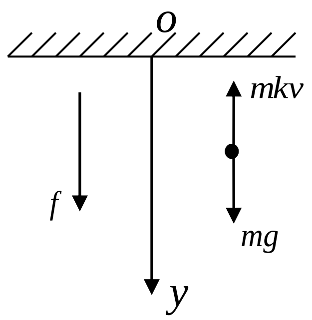
$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\sqrt{\frac{g}{k}}\tanh\frac{mgy}{T} = T\tan\frac{mgy}{T}$$

The proof is completed.

Example4:

Consider the acceleration of a drop filling down vertically equal to $f = g - kv$, where g is the acceleration gravity, k constant. If the time is measured from the moment the drop was falling with velocity u and the distance from the same position was measured y. Find the velocity v and the distance y depending on t. also find the relation between $v.y$

Solution



The equation of motion

$$f = g - k\,v = -k\left(v - \frac{g}{k}\right) \qquad (1)$$

Substituting $f = \frac{dv}{dt}$ into (1), we get

$$\frac{dv}{dt} = -k\left(v - \frac{g}{k}\right)$$

Separate variables and integrate, we obtain

$$\int \frac{dv}{-k\left(v-\frac{g}{k}\right)} = \int dt + c_1$$

$$-\frac{1}{k}\ln\left(v-\frac{g}{k}\right) = t + c_1 \tag{2}$$

At $t = 0$, $v = u$ hence

$$c_1 = -\frac{1}{k}\ \ln\left(u-\frac{g}{k}\right) \tag{3}$$

Substituting (3) into (2), we obtain

$$t = \frac{1}{k}\ln\frac{uk-g}{vk-g} \tag{4}$$

Eq. (4) gives the relation between $v. y$, where

$$v = \left[g+(uk-g)e^{-kt}\right]/\,k \tag{5}$$

Substituting $f = \frac{dv}{dt}$ into (5), we get

$$\frac{dy}{dt} = \frac{1}{k}\left[g+(uk-g)e^{-kt}\right]$$

Separate variables and integrate, we obtain

$$\int dy = \frac{1}{k}\int\left[g+(uk-g)e^{-kt}\right]dt + c_2$$

$$\therefore y = \frac{1}{k}\left[gt-\frac{uk-g}{k}e^{-kt}\right] + c_2 \tag{6}$$

At $t = 0. y = 0$, hence $c_2 = \frac{uk-g}{k^2}$. This with (6) leads to

$$y = \frac{1}{k}\left[gt-\frac{uk-g}{k}(1-e^{-kt})\right] \tag{7}$$

Eq. (7) gives the relation between $t. y$. From (4) and (7), we obtain the relation between $v. y$.

$$y=\frac{1}{k}\left[\frac{g}{k}\ln\frac{uk-g}{vk-g}-\frac{uk-g}{k}\left(1-e^{-\ln\frac{uk-g}{vk-g}}\right)\right]$$

Problems:

(1) A particle has been thrown vertically up with velocity $\frac{g}{k}$ in the field of its resistance mkv, where v the velocity, k constant and m particle mass. Prove that the particle arrive at $g(e-2)/k^2e$ after time $\frac{1}{k}$, and if the particle has been thrown vertically up with velocity $\lambda g/k$, prove that the maximum height equal to $\frac{g}{k^2}[\lambda-\ln(1+\lambda)]$, where λ is a constant.

(2) A particle filling down vertically in the field of its resistance proportional to velocity. If the final velocity equal to V, then prove that the particle moving with velocity $V/2$ after time $(V\ln 2)/g$ and moving a distance of $\frac{V^2}{2g}[2\ln 2-1]$

(3) A particle has been thrown vertically up with velocity u in the field of its resistance $\frac{mgv^2}{3u^2}$, where v the velocity, k constant and m particle mass. Prove that the particle returns to the ejaculation point after time $\frac{\sqrt{3}u}{g}\left[\frac{\pi}{6}+\ln\sqrt{3}\right]$ with velocity $\sqrt{3}u/2$.

Impulse and Newton's second law

How Newton expressed his second law ?

We often think of Newton's second law as F = ma. But Newton originally expressed his second law in a way that relates to momentum P. Newton defined force as the rate of change of momentum—the change in momentum divided by the time interval. Just as velocity is the rate at which displacement changes, force can be expressed as the rate at which momentum changes.

$$F = \frac{\Delta p}{\Delta t}$$

Where:

F	=	force (N)
Δp	=	change in momentum (kg m/s)
Δt	=	change in time (s)

Impulse is the product of force and time interval

Newton's definition of force in terms of momentum leads to a new quantity: *impulse*. Multiplying both sides of $F = \Delta p/\Delta t$ by Δt gives

$F\Delta t = \Delta p$. The product of force and the duration of time the force is applied is called impulse and is usually denoted with the letter J. An impulse applied to an object causes a change in its momentum, Δp

$J = F\Delta t = \Delta p = p_f - p_i$

Where

$J = impulse\ (kg\ m/s)$

$\Delta p = p_f - p_i = momentum\ (kg\ m/s)$

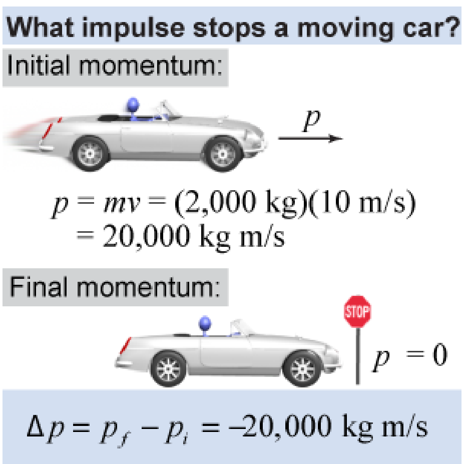
$F = force\ (N)$

$\Delta t = time\ (s)$

The momentum of things around us often change. As a car approaches an intersection and slows down, its momentum decreases. A train departs from the station and its momentum increases as its accelerates, and because of this her momentum frequently changes direction.

Impulse and average force

As we learned on the previous page, impulse J is equivalent to a change in momentum: J = $\Delta p = p_f - p_i$. Imagine a car moving down the street at a velocity of 10 m/s. The driver brakes to a stop; the car's velocity and hence its momentum have changed. The change in momentum, or impulse J, is the final momentum minus the initial momentum. In the illustration at right, the change in momentum is −20,000 kg m/s.



In most problems you will encounter, an object will change its velocity but not it's mass. In such cases, a change in velocity means a change in momentum —and that means an impulse has been applied to the object.

Example 1

A hockey puck that has a mass of 170 g travels with a speed of 30 m/s.

a) What is the momentum of the puck?
b) What impulse must be imparted on the puck by a player who wishes to change the puck's direction by 180°, while keeping the puck moving at the same speed?

<u>Solution:</u>

a) *Asked*: momentum p of the puck

Given: puck mass m = 170 g = 0.17 kg, velocity v = 30 m/s

Relationships: $P = m\,v$

$P = m\,v = (0.17\,\text{kg})\times(30\,\text{m/s}) = 5.1\,\text{kg m/s}$

b) *Asked*: impulse J imparted by the hockey player

Givens: puck initial momentum $P_i = +5.1$ kg m/s,

final momentum $P_f = -5.1$ kg m/s

Relationships: $J = \Delta p = p_f - p_i$

$J = P_f - P_i = -5.1 - 5.1 = -10.2$ kg m/s

Answer: The impulse imparted on the puck is −10.2 kg m/s, which is twice the magnitude of the initial momentum, but opposite in sign.

Example 2

A ball with a mass of 200 g is thrown straight down at the floor. It strikes the floor at a speed of 10.0 m/s and bounces straight up again with a speed of 6.0 m/s. What is the change in the ball's momentum?

Solution:

Letdown be the negative direction.

The initial momentum of the ball will then be negative:

$$p_i = m\,v_i = (0.200\,kg)(-10.0\,m/s) = -2.0\,kg\,m/s$$

The final momentum of the ball will be positive:

$$p_f = m\,v_f = (0.200\,kg)(+6.0\,m/s) = +1.2\,kg\,m/s$$

The change in momentum Δp is therefore

$$\Delta\text{p} = p_f - p_i = (+1.2\text{ kg m/s}) - (-2.0\text{ kg m/s}) = +3.2\text{ kg m/s}$$

Example 3

Sean, whose mass is 60 kg, is riding on a 5.0 kg sled initially traveling at 8.0 m/s. He brakes the sled with a constant force, bringing it to a stop in 4.0 s. What force does he apply?

Solution

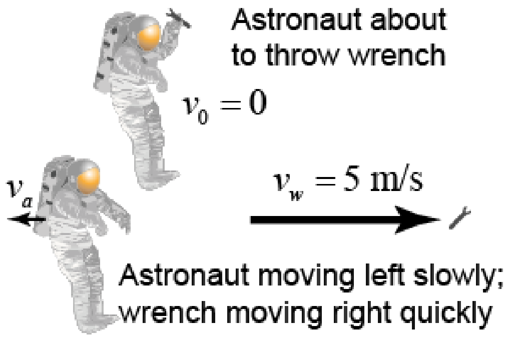
$$J = \Delta p = F\Delta t$$

$$F = \frac{\Delta p}{\Delta t} = \frac{(0) - (65\,)(8.0\,)}{4.0} = \frac{-520}{4.0} = -130\,N$$

Conservation of momentum

If there are no external forces acting on a system, impulses come in pairs that cancel each other out, leaving no net change in the momentum of the system. For a closed system, this implies the law of conservation of momentum:

The total momentum of a closed system remains constant, as long as no net outside force acts upon that system.



Example 3

A motionless 100 kg astronaut is holding a 2 kg wrench while on a spacewalk. To get moving, the astronaut throws the wrench forward at a speed of 5 m/s. How fast does the astronaut move backward?

Solution

Asked: astronaut's final velocity v_a

Given: masses m_w = 2 kg (wrench) and m_a = 100 kg (astronaut); initial velocity v_0 = 0 (both); final velocity v_w = 5 m/s (wrench)

Relationships: momentum p = m v; momentum conservation

Momentum before *equals* momentum after

$$\mathrm{m}_a\, \mathrm{v}_0 + \mathrm{m}_w\, \mathrm{v}_0 = \mathrm{m}_a\, \mathrm{v}_a + \mathrm{m}_w\, \mathrm{v}_w$$

$$\mathbf{0} = \mathrm{m}_a\, \mathrm{v}_a + \mathrm{m}_w\, \mathrm{v}_w$$

Solve for the astronaut's final velocity:

$$v_a = \frac{-m_w\, v_w}{m_a} = -(2\ \mathbf{kg})(5\ \mathbf{m/s})(100\ \mathbf{kg}) = -0.1\ \mathbf{m/s}$$

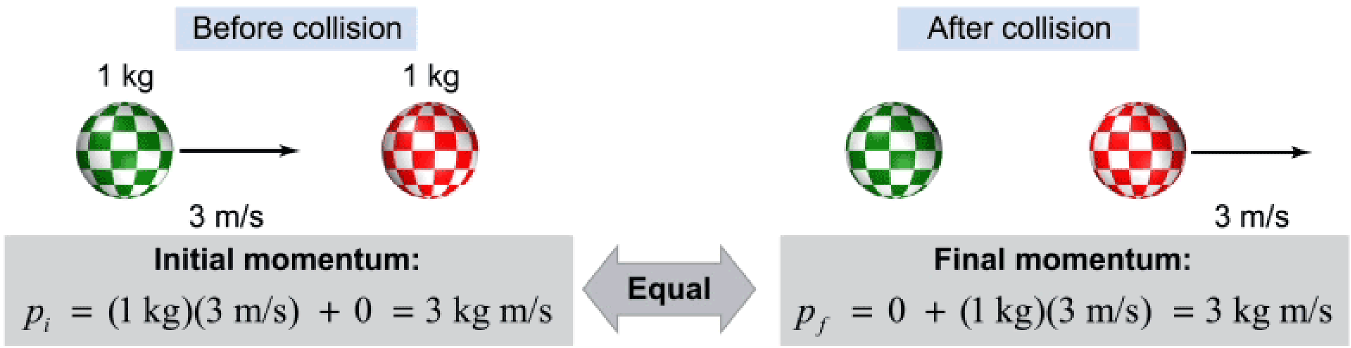
The astronaut moves 0.1 m/s backward (negative direction).

Collisions

Collisions happen in many ways: A cue ball strikes other pool balls, a hiker brushes past branches, a baseball player strikes a fastball, one car rear-ends another car, the space shuttle docks with the space station, or a baker pounds bread dough. People do not usually refer to all of these events as collisions, but in all these cases one object comes in contact with another object. In physics, a collision is any such interaction that causes one or more objects to change its velocity. As you will learn in this section, all collisions can be categorized as ***elastic***, ***inelastic***, or ***perfectly inelastic***.

Conservation of momentum in collisions

Momentum is always conserved for a closed system. The law of conservation of momentum applies to all different kinds of collisions, whether the two objects bounce off each other or stick together.

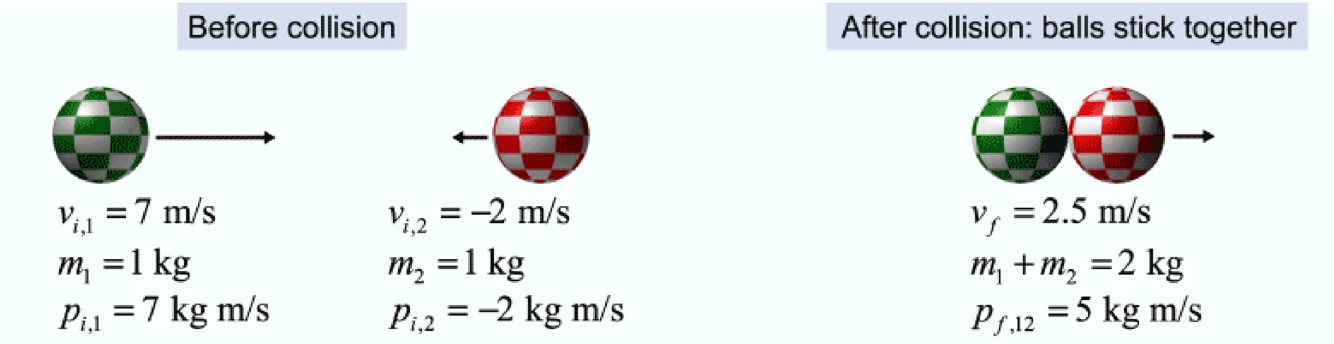


Whenever analyzing a collision, calculate the momentum for all the objects before the collision and equate that to the momentum of all the objects after the collision. If one object has an unknown momentum, then you can determine it by comparing the total momentum before the collision to the total momentum after the collision.

Imagine a collision between two objects with masses m_1 and m_2. The two objects have initial velocities v_{i1} and v_{i2} and final velocities $v_{f1}$and v_{f2}. Momentum conservation for these two colliding objects can be written as

$$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2}$$

There are two basic types of collisions in physics: elastic and inelastic. **In an inelastic collision**, some of the initial kinetic energy of the objects is transformed into heat and/or works to deform the shape of the objects. Auto collisions are nearly always inelastic, because of the damage caused to the cars. In the special case of a perfectly inelastic collision, the two objects stick together after impact.



If the mass of the first particle was m_1 and m_2 is the mass of the second particle and the first particle velocity is u and the second was static. After collision the two bodies stick together and move at one common velocity v and apply in this case the principle Momentum conservation

$$m_1 u + m_2 \times 0 = (m_1 + m_2)v$$

Then the common velocity v is

$$v = \frac{m_1 u}{m_1 + m_2}$$

Note: If the velocity of the second particle is w and in the same direction of the velocity u, then

$$m_1 u + m_2 w = (m_1 + m_2)v$$

Else, If the velocity w of the second particle is in the reverse direction of the velocity u, then

$$m_1 u - m_2 w = (m_1 + m_2)v$$

Now, the loss of **kinetic energy** resulting from the collision is the difference between the kinetic energy before E_1 and after E_2 the collision i.e.

$$\therefore E = E_1 - E_2 = \frac{1}{2}m_1u^2 - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2}m_1u^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1u}{m_1 + m_2}\right)^2$$

$$= \frac{1}{2}m_1u^2 - \frac{1}{2}\times\frac{m_1^2u^2}{m_1 + m_2} = \frac{1}{2}m_1u^2\left[1 - \frac{m_1}{m_1 + m_2}\right]$$

$$= \frac{1}{2}\times\left[\frac{m_1m_2}{m_1 + m_2}\right]u^2 = \frac{1}{2}M\,u^2$$

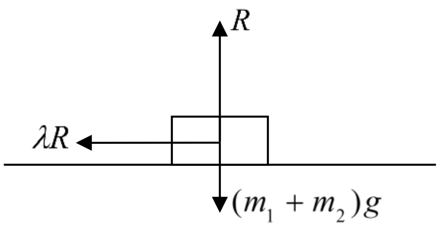
Where, $M = \dfrac{m_1m_2}{m_1 + m_2}$ is the center of mass for the group of two bodies.

Example 4

A block of wood of mass one pound set at the coarse surface. A bullet of mass $1/16$ pound fired with a speed of $510\ ft/\sec$ and settled within the wooden block, found:

1. The common velocity of the wooden block and the bullet after the collision
2. The ratio between the energy lost during collision and the energy lost by friction during movement at the surface

<u>Solution</u>



1. Let the common velocity of the wooden block and the bullet after the collision is v then by the principle Momentum conservation

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v$$

$$\frac{1}{16}\times 510 + 1\times 0 = (\frac{1}{16} + 1)v \Rightarrow \therefore v = 30\ ft/\sec$$

2. Let E is the kinetic energy lost during collision, E_1 is the kinetic energy before collision and E_2 is the kinetic energy after the collision then

$$\therefore E = E_1 - E_2 = \frac{1}{2}\times\frac{1}{16}\times(510)^2 - \frac{1}{2}\times(1+\frac{1}{16})\times(30)^2 = 7650\ lb.ft$$

Andis the energy lost by friction during movement at the E' surface so

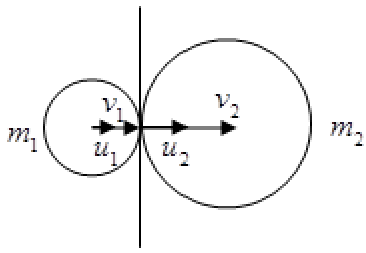
$$E' = \frac{1}{2}(m_1 + m_2)v^2 = \frac{1}{2}\times(1+\frac{1}{16})\times(30)^2 = \frac{3825}{8}\ lb.ft$$

Then we can get the ratio between the energy lost. E/E'

In elastic collision, in this type, the bodies bounce after the collision at different speeds, kinetic energy is conserved as well as momentum. An example of a perfectly elastic collision occurs when an ideal (frictionless) rubber ball bounces off a floor and reaches the same height from which it was initially dropped. A nearly elastic collision occurs in billiards when a fast moving cue ball strikes another ball, causing the cue ball to stop in place and the target ball to move off in the same direction. Real collisions are rarely perfectly elastic however; the amount of kinetic energy lost may be so small that it is often a good approximation to assume perfect elasticity. And there are two types of this collision

1. Direct collision

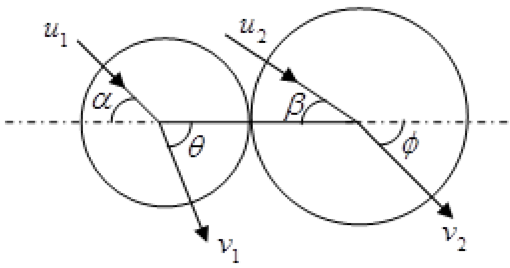
The direction of the movement shall be in the direction of the center-line, i.e. in



the perpendicular direction on the contact line passing through the contact point of two balls.

2. Indirect collision

The direction of movement of one or both of the objects is inclined on the center-line at a certain angle. When two ball are collide, there is no vertical force on the collision line, so the velocities in the vertical direction



$$u_1 \sin\alpha = v_1 \sin\theta \qquad , \qquad u_2 \sin\beta = v_2 \sin\phi$$

Elastic collision problems typically involve two equations: conservation of momentum and conservation of kinetic energy. The momentum equation involves the masses and velocities before and after the collision. The energy equation involves the masses and the velocities squared before and after the collision. The squared velocities make the algebra of solving momentum problems a little more challenging. In problems involving two and three dimensions, momentum must be conserved separately in each direction. Kinetic energy is a scalar however, and there is typically only one kinetic energy equation.

Conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{i1}^2 + \frac{1}{2} m_2 v_{i2}^2 = \frac{1}{2} m_1 v_{f1}^2 + \frac{1}{2} m_2 v_{f2}^2$$

Conservation of momentum:

$$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2}$$

Some algebra with these two equations gives

$$v_{f1} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{i1} + \left(\frac{2\, m_2}{m_1 + m_2}\right) v_{i2}$$

$$v_{f2} = \left(\frac{2\, m_1}{m_1 + m_2}\right) v_{i1} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{i2}$$

Example 5

Say we have mass 1 of $m_1 = 2\,kg$ traveling to the right at $v_{i1} = 6\,m/s$, and a mass 2 of $m_2 = 4\,kg$ traveling left at $v_{i2} = -2\,m/s$. What are their final velocities?

Solution

$$v_{f1} = \left(\frac{2 - 4}{2 + 4}\right) 6 + \left(\frac{2 * 4}{2 + 4}\right)(-2) = \frac{-14}{3}$$

$$v_{f2} = \left(\frac{2 * 2}{2 + 4}\right) 6 + \left(\frac{4 - 2}{2 + 4}\right) (-2) = \frac{10}{3}$$

So what are these numbers telling us? Mass 1 enters the collision headed right (positive velocity) and leaves headed left (negative velocity). Vice versa for mass 2 — enters headed left and leaves headed right. Also, m1 is half the mass of m2, and it enters with 3× the velocity of m2, and leaves only 1.4× as fast.

Example 6

Now, let's do an example of a 2-dimensional elastic collision. That means we will need to work out the x and y coordinates separately, and this will be an opportunity to use the vectors that we talked about previously. Let mass one be $m_1 = 6\,kg$ with an initial velocity of $v_{i1} = 3\,\hat{\imath} + 3\,\hat{\jmath}\ m/s$, and mass two is $m_2 = 3\,kg$ with initial velocity of $v_{i2} = -6\,\hat{\jmath}\ m/s$. Now, we do the same calculation as before, but we need one for x and one for y.

<u>Solution</u>

At X coordinate

$$v_{f1x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{i1x} + \left(\frac{2\,m_2}{m_1 + m_2}\right) v_{i2x}$$

$$v_{f2x} = \left(\frac{2\,m_1}{m_1 + m_2}\right) v_{i1x} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{i2x}$$

$$v_{f1x} = \left(\frac{3}{9}\right) 3 + \left(\frac{6}{9}\right)(0) = 1$$

$$v_{f2x} = \left(\frac{12}{9}\right) 3 + \left(\frac{-3}{9}\right) (0) = 4$$

At Y coordinate

$$v_{f1y} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{i1y} + \left(\frac{2\,m_2}{m_1 + m_2}\right) v_{i2y}$$

$$v_{f2y} = \left(\frac{2\,m_1}{m_1 + m_2}\right) v_{i1y} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{i2y}$$

$$v_{f1y} = \left(\frac{3}{9}\right) 3 + \left(\frac{6}{9}\right)(-6) = -3$$

$$v_{f2y} = \left(\frac{12}{9}\right) 3 + \left(\frac{-3}{9}\right) (-6) = 6$$

So

$$v_{f1} = 1\,\hat{\imath} - 3\,\hat{\jmath} \quad \text{and} \qquad v_{f2} = 4\,\hat{\imath} + 6\,\hat{\jmath}$$

Coefficient of Restitution

When there is a head on Collision between two bodies, the ratio of their relative velocity after collision and their relative velocity before collision is called the Coefficient of restitution. Thus

$$e = \left|\frac{v_{f1} - v_{f2}}{v_{i1} - v_{i2}}\right|$$

For a perfectly elastic collision, the value of e is 1

If $0 < e < 1$, then the collision is inelastic.

For a perfectly inelastic collision, $e = 0$.

If $e > 1$, then the collision is a super-elastic collision.

Elastic and Inelastic Collisions

The comparison between Elastic and inelastic collision is given below:

S. No	ELASTIC COLLISION	INELASTIC COLLISION

1	Momentum Conserved	Momentum Conserved
2	Kinetic energy Conserved	Kinetic energy not conserved
3	Example: Bouncing ball	example: Bullet shot in wood
4	Cannot be Perfect	Can be Perfect.

Exercises

1. Two 1 kg balls, traveling at +1 m/s and −1 m/s, respectively, collide with each other and stick together after impact. What is their velocity after the collision?
2. A 100 kg hockey player, moving at 2 m/s, collides head-on with a 75 kg hockey player moving at 1 m/s. After impact, they become entangled and slide together. What is their velocity after impact?
3. A 3300 kg truck moving at 18 m/s collides with a stationary 1400 kg passenger car, and the two stick together. Assuming that they are free to move afterward, what is the momentum of the truck+car object after the collision?
4. A 4.0 kg ball of clay traveling at 10 m/s collides with a 50 kg ball of clay traveling in the same direction at 2.0 m/s. What is their combined speed if the two balls stick together when they touch?

5. A sphere of mass 10 kg is moving at 3 m/s to the right until it smacks into a second stationary sphere of mass 2 kg. After the collision, both spheres travel to the right: the first sphere at 2.00 m/s, and the second sphere at 5.00 m/s. What kind of collision took place?
6. Ball A has a mass of 5.0 kg. Ball B has twice the mass of ball A. The two balls collide and stick together in a perfectly inelastic collision. After the collision the combined balls are at rest. If the velocity of ball A before the collision was 15 m/s, what was the velocity of ball B before the collision?
7. A ball of mass 0.440 kg moving with a speed of 4.5 m/s collides head-on with a 0.220 kg ball at rest. If the collision is elastic, what will be the speed of each ball after the collision?
8. A 4 kg ball with a velocity of 4 m/s in the positive x-direction has a head-on elastic collision with a stationary 2 Kg ball. What are the velocities of the balls after the collision?
9. A ball with a mass of 100 g is traveling with a velocity of 50 cm/s in the positive x-direction and collides head-on with a 5 Kg ball that was at rest. Assuming that it is elastic, find the velocities (in m/s) of the balls after the collision.

10. A 100 g bullet is fired horizontally into a 14.9 kg block of wood resting on a horizontal surface, and the bullet becomes embedded in the block. If the muzzle velocity of the bullet is 250 m/s, what is the velocity of the block containing the embedded bullet immediately after impact? (Neglect surface friction)