Faculty of Science
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Calculus (II)

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For

First Level Students

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Contents

Chapter 1

INTEGRATION

1.1 Antidifferentiation

How can a known rate of inflation be used to determine future prices? What is the velocity of an object moving along a straight line with known acceleration? How can knowing the rate at which a population is changing be used to predict future population levels? In all these situations, the derivative (rate of change) of a quantity is known and the quantity itself is required. Here is the terminology we will use in connection with obtaining a function from its derivative.

> **Antidifferentiation** A function $F(x)$ is said to be an antiderivative of $f(x)$ if
>
> $$F'(x) = f(x)$$
>
> for every x in the domain of $f(x)$. The process of finding antiderivatives is called antidifferentiation or indefinite integration.

Later in this section, you will learn techniques you can use to find antiderivatives. Once you have found what you believe to be an antiderivative of a function, you can always check your answer by differentiating. You should get the original function back. Here is an example

Example 1.1.1. *Verify that* $F(x) = \frac{1}{3}x^3 + 5x + 2$ *is an antiderivative of* $f(x) = x^2 + 5$
Solution: *$F(x)$ is an antiderivative of $f(x)$ if and only if $F'(x) = f(x)$. Differentiate F and you will find that*

$$\begin{aligned} F'(x) &= \frac{1}{3}\left(3x^2\right) + 5 \\ &= x^2 + 5 = f(x) \end{aligned}$$

as required.

A function has more than one antiderivative. For example, one antiderivative of the function $f(x) = 3x^2$ is $F(x) = x^3$, since

$$F'(x) = 3x^2 = f(x)$$

but so are $x^3 + 12$ and $x^3 - 5$ and $x^3 + \pi$, since

$$\frac{d}{dx}\left(x^3 + 12\right) = 3x^2 \quad \frac{d}{dx}\left(x^3 - 5\right) = 3x^2 \quad \frac{d}{dx}\left(x^3 + \pi\right) = 3x^2$$

In general, if F is one antiderivative of f, then so is any function of the form $G(x) = F(x) + C$, for constant C since

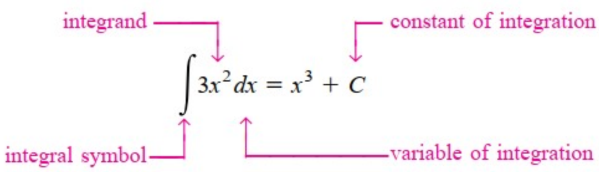
$$\begin{aligned} G'(x) &= [F(x) + C]' \\ &= F'(x) + C' \\ &= F'(x) + 0 \\ &= f(x) \end{aligned}$$

Fundamental Property of Antiderivatives If $F(x)$ is an antiderivative of the continuous function $f(x)$, then any other antiderivative of $f(x)$ has the form $G(x) = F(x) + C$ for some constant C

You have just seen that if $F(x)$ is one antiderivative of the continuous function $f(x)$ then all such antiderivatives may be characterized by $F(x) + C$ for constant C. The family of all antiderivatives of $f(x)$ is written

$$\int f(x)dx = F(x) + C$$

In the context of the indefinite integral $\int f(x)dx = F(x) + C$, the integral symbol is $\int_0^0 f$, the function $f(x)$ is called the integrand, C is the constant of integration, and dx is a differential that specifies x as the variable of integration. These features are displayed in this diagram for the indefinite integral of $f(x) = 3x^2$



For any differentiable function F, we have

$$\int F'(x)dx = F(x) + C$$

since by definition, $F(x)$ is an antiderivative of $F'(x)$. Equivalently,

$$\int \frac{dF}{dx}dx = F(x) + C$$

Rules for Integrating Common Functions

- The constant rule: $\int k dx = kx + C$ for constant k
- The power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for all $n \neq -1$
- The logarithmic rule: $\int \frac{1}{x} dx = \ln|x| + C \quad$ for all $x \neq 0$
- The exponential rule: $\int e^{kx} dx = \frac{1}{k}e^{kx} + C \quad$ for constant $k \neq 0$

Example 1.1.2. *Evaluate*

$$\int \left(2x^5 + 8x^3 - 3x^2 + 5\right) dx$$

Solution:

$$\begin{aligned}
\int \left(2x^5 + 8x^3 - 3x^2 + 5\right) dx &= 2\int x^5 dx + 8\int x^3 dx - 3\int x^2 dx + \int 5dx \\
&= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \\
&= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C
\end{aligned}$$

Example 1.1.3. *Evaluate*

$$\int \left(\frac{x^3 + 2x - 7}{x}\right) dx$$

Solution:

$$\begin{aligned}
\int \left(\frac{x^3 + 2x - 7}{x}\right) dx &= \int \left(x^2 + 2 - \frac{7}{x}\right) dx \\
&= \frac{1}{3}x^3 + 2x - 7\ln|x| + C
\end{aligned}$$

Example 1.1.4. *Evaluate*

$$\int \left(3e^{-5t} + \sqrt{t}\right) dt$$

Solution:

$$\begin{aligned}
\int \left(3e^{-5t} + \sqrt{t}\right) dt &= \int \left(3e^{-5t} + t^{1/2}\right) dt \\
&= 3\left(\frac{1}{-5}e^{-5t}\right) + \frac{1}{3/2}t^{3/2} + C \\
&= -\frac{3}{5}e^{-5t} + \frac{2}{3}t^{3/2} + C
\end{aligned}$$

Example 1.1.5. *Evaluate*

$$\int \frac{dx}{1+e^x}$$

Solution:

$$\begin{aligned}
\int \frac{dx}{1+e^x} &= \int \frac{dx}{1+e^x}\frac{e^{-x}}{e^{-x}} \\
&= \int \frac{e^{-x}dx}{e^{-x}+1} \\
&= -\ln|1+e^{-x}| + C
\end{aligned}$$

Example 1.1.6. *Evaluate*

$$\int \frac{xdx}{1+x^2}$$

Solution:

$$\begin{aligned}
\int \frac{xdx}{1+x^2} &= \frac{1}{2}\int \frac{2xdx}{1+x^2} \\
&= \frac{1}{2}\ln|1+x^2| + C
\end{aligned}$$

1.2 Basic Trigonometric Integrals

Basic Trigonometric Integrals

- $\int \sin kx dx = \frac{-1}{k} \cos x + C$
- $\int \cos kx dx = \frac{1}{k} \sin x + C$
- $\int \sec^2 kx dx = \frac{1}{k} \tan x + C$
- $\int \csc^2 kx dx = \frac{-1}{k} \cot x + C$
- $\int \sec kx \tan kx dx = \frac{1}{k} \sec x + C$
- $\int \csc kx \cot kx dx = \frac{-1}{k} \csc x + C$

Example 1.2.1. *Evaluate*

$$\int \tan x dx$$

Solution:

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln \cos x + C \\ &= \ln |\sec x| + C \end{aligned}$$

Example 1.2.2. *Evaluate*

$$\int \cot x dx$$

Solution:

$$\begin{aligned} \int \cot x dx &= \int \frac{\cos x}{\sin x} dx \\ &= \ln \sin x + C \end{aligned}$$

Example 1.2.3. *Evaluate*

$$\int \sec x dx$$

Solution:

$$\begin{aligned}\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

Example 1.2.4. *Evaluate*

$$\int \csc x dx$$

Solution:

$$\begin{aligned}\int \csc x dx &= \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx \\ &= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx \\ &= -\ln|\csc x + \cot x| + C\end{aligned}$$

Example 1.2.5. *Evaluate*

$$\int (\sin 8t + 20\cos 9t) dt$$

Solution:

$$\begin{aligned}\int (\sin 8t + 20\cos 9t) dt &= \int \sin 8t dt + 20\int \cos 9t dt \\ &= -\frac{1}{8}\cos 8t + \frac{20}{9}\sin 9t + C\end{aligned}$$

1.3 Substitution Method

Integration (antidifferentiation) is generally more difficult than differentiation. There are no sure-fire methods, and many antiderivatives cannot be expressed in terms of elementary functions. However, there are a few important general techniques. One such technique is the Substitution Method, which uses the Chain Rule "in reverse."

Consider the integral $\int 2x\cos\left(x^2\right) dx$. We can evaluate it if we remember the Chain Rule calculation

$$\frac{d}{dx}\sin\left(x^2\right) = 2x\cos\left(x^2\right)$$

This tells us that $\sin\left(x^2\right)$ is an antiderivative of $2x\cos\left(x^2\right)$, and therefore,

$$\int \underbrace{2x}_{\substack{\text{Derivative of}\\ \text{inside function}}} \cos(\underbrace{x^2}_{\substack{\text{Inside}\\ \text{function}}})dx = \sin\left(x^2\right) + C$$

A similar Chain Rule calculation shows that

$$\int \underbrace{\left(1+3x^2\right)}_{\substack{\text{Derivative of}\\ \text{inside function}}} \cos(\underbrace{x+x^3}_{\substack{\text{Inside}\\ \text{function}}})dx = \sin\left(x+x^3\right) + C$$

In both cases, the integrand is the product of a composite function and the derivative of the inside function. The Chain Rule does not help if the derivative of the inside function is missing. For instance, we cannot use the Chain Rule to compute $\int \cos\left(x+x^3\right) dx$ because the factor $\left(1+3x^2\right)$ does not appear.

In general, if $F'(u) = f(u)$ then by the Chain Rule,

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$$

This translates into the following integration formula.

The Substitution Method If $F'(x) = f(x)$, then

$$\int f(u(x))u'(x)dx = F(u(x)) + C$$

Example 1.3.1. *Evaluate*

$$\int 3x^2 \sin\left(x^3\right) dx$$

Solution: *Let $u = x^3$, then $du = 3x^2dx$, hence*

$$\begin{aligned}\int 3x^2 \sin\left(x^3\right) dx &= \int \sin(u)\, du\\ &= -\cos u + C\\ &= -\cos\left(x^3\right) + C\end{aligned}$$

Example 1.3.2. *Evaluate*

$$\int x\left(x^2+9\right)^5 dx$$

Solution: *Let $u = x^2 + 9$, then $du = 2xdx$, hence*

$$\begin{aligned}\int x\left(x^2+9\right)^5 dx &= \frac{1}{2}\int u^5 du\\ &= \frac{1}{12}u^6 + C\\ &= \frac{1}{12}\left(x^2+9\right)^6 + C\end{aligned}$$

Example 1.3.3. *Evaluate*

$$\int \frac{\left(x^2+2x\right)dx}{\left(x^3+3x^2+12\right)^6}$$

Solution: *Let $u = x^3 + 3x^2 + 12$, then $du = (3x^2 + 6x)dx$, hence*

$$\begin{aligned}\int \frac{\left(x^2+2x\right)dx}{\left(x^3+3x^2+12\right)^6} &= \int \left(x^3+3x^2+12\right)^{-6}\left(x^2+2x\right)dx\\ &= \frac{1}{3}\int u^{-6}du\\ &= \left(\frac{1}{3}\right)\left(\frac{u^{-5}}{-5}\right) + C\\ &= -\frac{1}{15}\left(x^3+3x^2+12\right)^{-5} + C\end{aligned}$$

Example 1.3.4. *Evaluate*

$$\int x\sqrt{5x+1}dx$$

Solution: *Let $u = 5x + 1\theta + 5$, then $du = 5dx$, hence*

$$\begin{aligned}\int x\sqrt{5x+1}dx &= \int \left(\frac{1}{5}(u-1)\right)\frac{1}{5}\sqrt{u}du\\ &= \frac{1}{25}\int \left(u^{3/2} - u^{1/2}\right)du\\ &= \frac{1}{25}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{2}{125}(5x+1)^{5/2} - \frac{2}{75}(5x+1)^{3/2} + C\end{aligned}$$

Example 1.3.5. *Evaluate*

$$\int \frac{1}{x}\sin^6(\ln x)\cos(\ln x)dx$$

Solution: *Let $u = \ln x$, then $du = \frac{1}{x}dx$, hence*

$$\int \frac{1}{x}\sin^6(\ln x)\cos(\ln x)dx = \int \sin^6 u \cos u du$$
$$= \frac{1}{7}\sin^7(u) + c \qquad = \frac{1}{7}\sin^7(\ln x) + c$$

Example 1.3.6. *Evaluate*

$$\int \frac{[1+(1/t)]^5}{t^2}dt$$

Solution: *Let $u = \frac{1}{t}$, then $du = \frac{-1}{t^2}dt$, hence*

$$\begin{aligned}\int \frac{[1+(1/t)]^5}{t^2}dt &= -\int u^5 du\\ &= -\frac{u^6}{6} + c\\ &= -\frac{[1+(1/t)]^6}{6} + c\end{aligned}$$

Example 1.3.7. *Evaluate*

$$\int \frac{3x+6}{\sqrt{2x^2+8x+3}}dx$$

Solution: *Let $u = 2x^2 + 8x + 3$, then $du = (4x+8)dx$, hence*

$$\begin{aligned}\int \frac{3x+6}{\sqrt{2x^2+8x+3}}dx &= \int \frac{1}{\sqrt{2x^2+8x+3}}[(3x+6)dx]\\ &= \int \frac{1}{\sqrt{u}\left(\frac{3}{4}du\right)} = \frac{3}{4}\int u^{-1/2}du\\ &= \frac{3}{4}\left(\frac{u^{1/2}}{1/2}\right) + C = \frac{3}{2}\sqrt{u} + c\\ &= \frac{3}{2}\sqrt{2x^2+8x+3} + C\end{aligned}$$

Example 1.3.8. *Evaluate*

$$\int \frac{(\ln x)^2}{x}dx$$

Solution: *Let* $u = \ln x$ *, then* $du = \frac{1}{x}dx$*, hence*

$$\begin{aligned}\int \frac{(\ln x)^2}{x}dx &= \int (\ln x)^2 \left(\frac{1}{x}dx\right) \\ &= \int u^2 du = \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(\ln x)^3 + C\end{aligned}$$

Example 1.3.9. *Evaluate*

$$\int \frac{x}{x-1}dx$$

Solution: *Let* $u = x - 1$ *, then* $du = dx$*, hence*

$$\begin{aligned}\int \frac{x}{x-1}dx &= \int \frac{u+1}{u}du \\ &= \int \left[1 + \frac{1}{u}\right] du \\ &= u + \ln|u| + C \\ &= x - 1 + \ln|x-1| + C\end{aligned}$$

Example 1.3.10. *Evaluate*

$$\int x^3 e^{x^4+2}dx$$

Solution: *Let* $u = x^4 + 2$ *, then* $du = 4x^3dx$*, hence*

$$\begin{aligned}\int x^3 e^{x^4+2}dx &= \int e^{x^4+2}\left(x^3dx\right) \\ &= \int e^{n\left(\frac{1}{4}du\right)} \\ &= \frac{1}{4}e^u + C \\ &= \frac{1}{4}e^{x^4+2} + C\end{aligned}$$

Example 1.3.11. *Evaluate*

$$\int e^{5x+2}dx$$

Solution: *Let $u = 5x + 2$, then $du = 5dx$, hence*

$$\begin{aligned} \int e^{5x+2} dx &= \int e^{5x} e^2 dx \\ &= e^2 \int e^{5x} dx \\ &= e^2 \left[\frac{e^{5x}}{5} \right] + c \\ &= \frac{1}{5} e^{5x+2} + c \end{aligned}$$

1.4 Integrals Resulting in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

The following integration formulas yield inverse trigonometric functions:

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$

Example 1.4.1. *Evaluate*

$$\int \frac{dx}{\sqrt{4 - 9x^2}}$$

Solution: *Let $u = 3x$ and $du = 3dx$, then*

$$\begin{aligned} \int \frac{dx}{\sqrt{4 - 9x^2}} &= \frac{1}{3} \int \frac{du}{\sqrt{4 - u^2}} \\ &= \frac{1}{3} \sin^{-1} \left(\frac{u}{2} \right) + C \\ &= \frac{1}{3} \sin^{-1} \left(\frac{3x}{2} \right) + C \end{aligned}$$

Example 1.4.2. *Evaluate*

$$\int \frac{1}{1 + 4x^2} dx$$

Solution: *Let $u = 2x$ and $du = 2dx$, then*

$$\begin{aligned}
&= \frac{1}{2}\int \frac{1}{1+u^2}du \\
&= \frac{1}{2}\tan^{-1} u + C \\
&= \frac{1}{2}\tan^{-1}(2x) + C
\end{aligned}$$

Example 1.4.3. *Evaluate*

$$\int \frac{1}{1+4x^2}dx$$

Solution: *Let $u = 2x$ and $du = 2dx$, then*

$$\begin{aligned}
\int \frac{dx}{x\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} \\
&= \int \frac{du}{u\sqrt{u^2-1}} \\
&= \sec^{-1}(2x) + C
\end{aligned}$$

Example 1.4.4. *Evaluate*

$$\int \frac{x+\sin^{-1} x}{\sqrt{1-x^2}}dx$$

Solution:

$$\begin{aligned}
\int \frac{x+\sin^{-1} x}{\sqrt{1-x^2}}dx &= \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx - \int \frac{-xdx}{\sqrt{1-x^2}} \\
&= \int \sin^{-1} xd\sin^{-1} x - \frac{1}{2}\int \frac{-2xdx}{\sqrt{1-x^2}} \\
&= \frac{1}{2}\left(\sin^{-1} x\right)^2 - \sqrt{1-x^2} + c
\end{aligned}$$

Example 1.4.5. *Evaluate*

$$\int \frac{1}{x\left(1+\ln^2 x\right)}dx$$

Solution: *Let $u = \ln x$, then $du = \frac{1}{x}dx$, hence*

$$\begin{aligned}
\int \frac{1}{x\left(1+\ln^2 x\right)}dx &= \int \frac{d(\ln x)}{\left(1+\ln^2 x\right)} \\
&= \int \frac{du}{(1+u^2)} \\
&= \tan^{-1} u + c = \tan^{-1}(\ln x) + c
\end{aligned}$$

Example 1.4.6. *Evaluate*

$$\int \frac{x^2 dx}{\sqrt{1-x^6}}$$

Solution: *Let $u = x^3$, then $du = 3x^2 dx$, hence*

$$\begin{aligned}
\int \frac{x^2 dx}{\sqrt{1-x^6}} &= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} \\
&= \frac{1}{3}\sin^{-1} u + c = \frac{1}{3}\sin^{-1} x^3 + c
\end{aligned}$$

Example 1.4.7. *Evaluate*

$$\int \frac{2x^2+3}{x\sqrt{9x^2-4}}dx$$

Solution:

$$\begin{aligned}
\int \frac{2x^2+3}{x\sqrt{9x^2-4}}dx &= \int \left[\frac{2x^2}{x\sqrt{9x^2-4}} + \frac{3}{x\sqrt{9x^2-4}}\right] dx \\
&= \int \frac{2x}{\sqrt{9x^2-4}}dx + \int \frac{3}{x\sqrt{9x^2-4}}dx
\end{aligned}$$

To evaluate $\int \frac{2x}{\sqrt{9x^2-4}}dx$, Let $u = 9x^2 - 4$ and $du = 18x dx$, then

$$\begin{aligned}
\int \frac{2x}{\sqrt{9x^2-4}}dx &= \frac{1}{9}\int \frac{du}{\sqrt{u}} \\
&= \frac{1}{9}\cdot 2\sqrt{u} + c \\
&= \frac{2\sqrt{9x^2-4}}{9} + c
\end{aligned}$$

To evaluate $\int \frac{3}{x\sqrt{9x^2-4}}dx$, Let $u = 3x$ and $du = 3dx$, then

$$\begin{aligned}\int \frac{3dx}{x\sqrt{9x^2-4}} &= 3\int \frac{du}{u\sqrt{u^2-4}} \\ &= 3\sec^{-1}\left|\frac{u}{2}\right| + c \\ &= 3\sec^{-1}\left|\frac{3x}{2}\right| + c\end{aligned}$$

Hence

$$\int \frac{2x^2+3}{x\sqrt{9x^2-4}}dx = \frac{2\sqrt{9x^2-4}}{9} + c + 3\sec^{-1}\left|\frac{3x}{2}\right| + c$$

1.5 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic Integral Formulas

1. $\int \sinh kx dx = \frac{1}{k}\cosh kx + C$
2. $\int \cosh kx dx = \frac{1}{k}\sinh kx + C$
3. $\int \operatorname{sech}^2 kx dx = \frac{1}{k}\tanh kx + C$
4. $\int \operatorname{csch}^2 kx dx = \frac{-1}{k}\coth kx + C$
5. $\int \operatorname{sech} kx \tanh x dx = \frac{-1}{k}\operatorname{sech} kx + C$
6. $\int \operatorname{csch} kx \coth x dx = \frac{-1}{k}\operatorname{csch} kx + C$

Example 1.5.1. *Evaluate*

$$\int x\cosh\left(x^2\right)dx$$

Solution: *Let* $u = x^2$ *and* $du = 2xdx$, *then*

$$\begin{aligned}\int x\cosh\left(x^2\right)dx &= \frac{1}{2}\int \cosh u du \\ &= \frac{1}{2}\sinh u + C \\ &= \frac{1}{2}\sinh\left(x^2\right) + C\end{aligned}$$

Example 1.5.2. *Evaluate*

$$\int \cosh(3x)dx$$

Solution:

$$\begin{aligned}\int \cosh(3x)dx &= \frac{1}{3}\int 3\cdot\cosh(3x)dx \\ &= \frac{1}{3}\sinh(3x) + C\end{aligned}$$

Example 1.5.3. *Evaluate*

$$\int x\sinh\left(x^2+1\right)dx$$

Solution: *Let* $u = x^2 + 1$ *and* $du = 2xdx$, *then*

$$\begin{aligned}\int x\sinh\left(x^2+1\right)dx &= \frac{1}{2}\int \sinh u du \\ &= \frac{1}{2}\cosh u + C \\ &= \frac{1}{2}\cosh\left(x^2+1\right) + C\end{aligned}$$

Example 1.5.4. *Evaluate*

$$\int \tanh x \operatorname{sech}^2 x dx$$

Solution: *Let* $u = \tanh x$, $du = \operatorname{sech}^2 x dx$, *then*

$$\begin{aligned}\int \tanh x \operatorname{sech}^2 x dx &= \int u du \\ &= \frac{u^2}{2} + C \\ &= \frac{1}{2}\tanh^2 x + C\end{aligned}$$

Example 1.5.5. *Evaluate*

$$\int \frac{\cosh x}{3\sinh x + 4}dx$$

***Solution**: Let $u = 3\sinh x + 4$, $du = 3\cosh x dx$, then*

$$\begin{aligned}\int \frac{\cosh x}{3\sinh x + 4}dx &= \frac{1}{3}\int \frac{1}{u}du \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|3\sinh x + 4| + C\end{aligned}$$

Integrals Involving Inverse Hyperbolic Functions

1. $\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C$
2. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C \quad (\text{ for } x > 1)$
3. $\displaystyle\int \frac{dx}{1-x^2} = \tanh^{-1} x + C \quad (\text{ for } |x| < 1)$
4. $\displaystyle\int \frac{dx}{1-x^2} = \coth^{-1} x + C \quad (\text{ for } |x| > 1)$
5. $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x + C \quad (\text{ for } 0 < x < 1)$
6. $\displaystyle\int \frac{dx}{|x|\sqrt{1+x^2}} = -\operatorname{csch}^{-1} x + C \quad (\text{ for } x \neq 0)$

Example 1.5.6. *Evaluate*

$$\int \frac{1}{\sqrt{16+25x^2}}dx$$

***Solution**:*

$$\begin{aligned}\int \frac{1}{\sqrt{16+25x^2}}dx &= \int \frac{1}{\sqrt{16\left(1+\frac{25}{16}x^2\right)}}dx \\ &= \int \frac{1}{4\sqrt{1+\left(\frac{5x}{4}\right)^2}}dx \\ &= \frac{4}{5}\int \frac{\frac{5}{4}}{\sqrt{1+\left(\frac{5x}{4}\right)^2}}dx \\ &= \frac{1}{5}\sinh^{-1}\left(\frac{5x}{4}\right) + C\end{aligned}$$

Example 1.5.7. *Evaluate*

$$\int \frac{\tanh^{-1} x}{x^2 - 1} dx$$

Solution: *Let* $u = \tanh^{-1} x$ *and* $du = \frac{1}{x^2-1} dx$

$$\begin{aligned}
\int \frac{\tanh^{-1} x}{x^2 - 1} dx &= \int -u du \\
&= \frac{-u^2}{2} + C \\
&= \frac{-\left(\tanh^{-1} x\right)^2}{2} + C
\end{aligned}$$

Example 1.5.8. *Evaluate*

$$\int \frac{1}{x\sqrt{x^2 + 16}} dx$$

Solution: *Let* $u =$ *and* $du =$

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2 + 16}} dx &= \int \frac{1}{x\sqrt{16\left(\frac{x^2}{16} + 1\right)}} dx \\
&= \int \frac{1}{4x\sqrt{\left(\left(\frac{x}{4}\right)^2 + 1\right)}} dx \\
&= \int \frac{\frac{1}{4}}{4 \cdot \frac{x}{4}\sqrt{\left(\left(\frac{x}{4}\right)^2 + 1\right)}} dx \\
&= \frac{1}{4}\left[-\operatorname{csch}^{-1}\left(\frac{x}{4}\right)\right]
\end{aligned}$$

Some Useful Relations

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$
3. $\tanh^{-1} x = \dfrac{1}{2} \ln \frac{1+x}{1-x}$
4. $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$
5. $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right)$
6. $\coth^{-1} x = \frac{1}{2} \ln \dfrac{x+1}{x-1}$

1.6 Exercises

Evaluate the following integral

1- $\int e^{2x}dx$ 2- $\int 2^{x}dx$ 3- $\frac{2}{x}dx$

4- $\int \frac{\ln x}{x}dx$ 5- $\int \frac{dx}{x(\ln x)^2}$ 6- $\int \frac{dx}{x\ln x}$

7- $\int \frac{\ln(\sin x)}{\tan x}dx$ 8- $\int \frac{\cos x - x\sin x}{x\cos x}dx$ 9- $\int \ln(\cos x)\tan x dx$

10- $\int e^{\ln x}\frac{dx}{x}$ 11- $\int \frac{e^{\ln(1-t)}}{1-t}dt$ 12- $\int e^{\tan x}\sec^2 x dx$

13- $\int \frac{x\sin\left(x^2\right)}{\cos\left(x^2\right)}dx$ 14- $\int \sec^2(3x+2)dx$ 15- $\int \frac{4x^3}{\left(x^4+1\right)^2}dx$

16- $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}dx$ 17- $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$ 18- $\int \frac{1}{\sqrt{5s+4}}ds$

19- $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}dx$ 20- $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}d\theta$ 21- $\int \frac{t\tan^{-1}\left(t^2\right)}{1+t^4}dt$

22- $\int \frac{e^t\cos^{-1}\left(e^t\right)}{\sqrt{1-e^{2t}}}dt$ 23- $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}dx$ 24- $\int \frac{(\sin x + x\cos x)}{1+x^2\sin^2 x}dx$

25- $\int \frac{1}{x+x\ln^2 x}dx$ 26- $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$ 27- $\int \frac{e^t}{1+e^{2t}}dt$

Chapter 2

TECHNIQUES OF INTEGRATION

In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

2.1 Integration by Parts

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x \sin\left(x^2\right) dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x \sin x dx$ defies us. Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique integration by parts.

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation:

$$\int h'(x)dx = \int \left(g(x)f'(x) + f(x)g'(x)\right) dx$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

Now we solve for $\int f(x)g'(x)dx$:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)dx$ and $dv = g'(x)dx$, we have the more compact form

$$\int udv = uv - \int vdu$$

Theorem 2.1. *Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:*

$$\int udv = uv - \int vdu$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example 2.1.1. *Evaluate*

$$\int x \sin x dx$$

Let

$$\begin{aligned} u &= x & dv &= \sin x dx \\ du &= dx & v &= -\cos x \end{aligned}$$

Then,

$$\begin{aligned} \int x \sin x dx &= (x)(-\cos x) - \int (-\cos x)(1dx) \\ &= -x\cos x + \int \cos x dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

Example 2.1.2. *Evaluate*

$$\int xe^x dx$$

Let

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \end{aligned}$$

Then,

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

Example 2.1.3. *Evaluate*

$$\int x^2 e^x dx$$

Let

$$\begin{aligned}u &= x^2 & dv &= e^x dx \\ du &= 2x dx & v &= e^x\end{aligned}$$

Then,

$$\begin{aligned}\int x^2 e^x dx &= x^2 \cdot e^x - \int 2x \cdot e^x dx \\ &= x^2 e^x - 2 \int xe^x dx\end{aligned}$$

Let

$$\begin{aligned}u &= x & dv &= e^x dx \\ du &= dx & v &= e^x\end{aligned}$$

Then,

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2\left[xe^x - \int 1 \cdot e^x dx\right] \\ &= x^2 e^x - 2\left[xe^x - e^x\right] + C \\ &= x^2 e^x - 2xe^x - 2e^x + C \\ &= e^x\left(x^2 - 2x - 2\right) + C\end{aligned}$$

Example 2.1.4. *Evaluate*

$$\int \ln x dx$$

Let

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x}dx & v &= x \end{aligned}$$

Then,

$$\begin{aligned} \int \ln x dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + C \end{aligned}$$

Example 2.1.5. *Evaluate*

$$\int x \ln x dx$$

Let

$$\begin{aligned} u &= \ln x & dv &= x dx \\ du &= \frac{1}{x}dx & v &= \frac{1}{2}x^2 \end{aligned}$$

Then,

$$\begin{aligned} \int x \ln x dx &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int \frac{x^2}{x} dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \end{aligned}$$

Example 2.1.6. *Evaluate*

$$\int \frac{\ln x}{x^3} dx$$

Let

$$\begin{aligned} u &= \ln x & dv &= x^{-3} dx \\ du &= \frac{1}{x}dx & v &= \frac{-1}{2}x^{-2} \end{aligned}$$

Then,

$$\begin{aligned}
\int \frac{\ln x}{x^3} dx &= \int x^{-3} \ln x dx \\
&= (\ln x)\left(-\frac{1}{2}x^{-2}\right) - \int \left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}dx\right) \\
&= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}dx \\
&= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C \\
&= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C
\end{aligned}$$

Example 2.1.7. *Evaluate*

$$\int \sin 2x e^x dx$$

Let

$$\begin{aligned}
u &= \sin 2x & dv &= e^x dx \\
du &= \cos 2x dx & v &= e^x
\end{aligned}$$

Then,

$$\begin{aligned}
\int \sin 2x e^x dx &= (\sin 2x)(e^x) - \int e^x 2\cos 2x dx \\
&= e^x \sin 2x - 2\int \cos 2x e^x dx
\end{aligned}$$

For the integral part again , let

$$\begin{aligned}
u &= \cos 2x & dv &= e^x dx \\
du &= -\sin 2x dx & v &= e^x
\end{aligned}$$

Then,

$$\begin{aligned}
\int \sin 2x e^x dx &= e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x dx\right) \\
\int \sin 2x e^x dx &= e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x dx \\
5\int \sin 2x e^x dx &= e^x \sin 2x - 2e^x \cos 2x \\
\int \sin 2x e^x dx &= \frac{1}{5}e^x(\sin 2x - 2\cos 2x) + C
\end{aligned}$$

Example 2.1.8. *Evaluate*

$$\int \sin(\ln x)dx$$

Let

$$u = \sin(\ln x) \qquad dv = dx$$
$$du = \frac{1}{x}\cos(\ln x)dx \qquad v = x$$

Then,

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$

For the integral part again , let

$$u = \cos(\ln x) \qquad dv = dx$$
$$du = \frac{-1}{x}\sin(\ln x)dx \qquad v = x$$

Then,

$$\begin{aligned}
\int \sin(\ln x)dx &= x\sin(\ln x) - \left(x\cos(\ln x) - \int -\sin(\ln x)dx\right) \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx \\
2\int \sin(\ln x)dx &= x\sin(\ln x) - x\cos(\ln x) \\
\int \sin(\ln x)dx &= \frac{1}{2}\left[x\sin(\ln x) - x\cos(\ln x)\right]
\end{aligned}$$

Example 2.1.9. *Evaluate*

$$\int \sin^{-1} x dx$$

Let

$$u = \sin^{-1} x \qquad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$$

Then,

$$\begin{aligned}\int \sin^{-1} x dx &= \sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x dx \\ &= x \sin^{-1} x - \int -\frac{-2x}{2\sqrt{1-x^2}} dx \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

Example 2.1.10. *Evaluate*

$$\int \tan^{-1} x dx$$

Let

$$\begin{aligned} u &= \tan^{-1} x & dv &= dx \\ du &= \frac{1}{1+x^2} dx & v &= x \end{aligned}$$

Then,

$$\begin{aligned}\int \tan^{-1} x dx &= x \tan^{-1} x - \int \frac{x}{1+x^2} dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\ &= x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + C\end{aligned}$$

Example 2.1.11. *Evaluate*

$$\int x \tan^{-1} x dx$$

Let

$$\begin{aligned} u &= \tan^{-1} x & dv &= x dx \\ du &= \frac{1}{1+x^2} dx & v &= \frac{1}{2} x^2 \end{aligned}$$

Then,

$$\begin{aligned}\int x\tan^{-1} x dx &= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx \\ &= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx \\ &= \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C\end{aligned}$$

Example 2.1.12. *Evaluate*

$$\int \sec^3 x dx$$

Since

$$\int \sec^3 x dx \int \sec x \sec^2 x dx$$

Let

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x dx \\ du &= \sec x \tan x dx & v &= \tan x \end{aligned}$$

Then,

$$\begin{aligned}\int \sec^3 x dx &= \sec x\tan x - \int \sec x\tan^2 x dx \\ &= \sec x\tan x - \int \sec x(\sec^2 x - 1)dx \\ &= \sec x\tan x - \int \sec^3 x dx + \int \sec x dx \\ 2\int \sec^3 x dx &= \sec x\tan x + \int \sec x dx \\ &= \sec x\tan x + \ln|\sec x + \tan x| \\ \int \sec^3 x dx &= \frac{1}{2}[\sec x\tan x + \ln|\sec x + \tan x|]\end{aligned}$$

Example 2.1.13. *Prove that*

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$$

Let

$$u = x^n \qquad dv = e^x dx$$
$$du = nx^{n-1}dx \qquad v = e^x$$

Then,

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

Example 2.1.14. *Prove that*

$$\int (\ln x)^k dx = x(\ln x)^k - k \int (\ln x)^{k-1} dx$$

and use it to evaluate

$$\int (\ln x)^3 dx$$

Let

$$u = (\ln x)^k \qquad dv = dx$$
$$du = \frac{k}{x}(\ln x)^{k-1}dx \qquad v = x$$

Then,

$$\int (\ln x)^k dx = x(\ln x)^k - k \int (\ln x)^{k-1} dx$$

To evaluate

$$\int (\ln x)^3 dx$$

Here $k = 3$, then

$$\begin{aligned}
\int (\ln x)^3 dx &= x(\ln x)^3 - 3\int (\ln x)^2 dx \\
&= x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\int (\ln x) dx\right) \\
&= x(\ln x)^3 - 3x(\ln x)^2 + 6\int (\ln x) dx \\
&= x(\ln x)^3 - 3x(\ln x)^2 + 6\left(x\ln x - \int (\ln x)^0 dx\right) \\
&= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C
\end{aligned}$$

2.1.1 Exercises

Evaluate the following integral

1-	$\int x^3 e^{2x} dx$	2-	$\int x^3 \ln(x) dx$	3-	$\int t \ln 2t dt$
4-	$\int t \ln t^2 dt$	5-	$\int \frac{\ln x}{x^2} dx$	6-	$\int x^3 e^{x^2} dx$
7-	$\int x^2 \sin x dx$	8-	$\int e^{-x} \sin x dx$	9-	$\int \sec^{-1} x dx$
10-	$\int x 5^x dx$	11-	$\int 3^x \cos x dx$	12-	$\int \sinh^{-1} x dx$
13-	$\int \frac{\ln(\ln x) \ln x dx}{x}$	14-	$\int \sin(\ln x) dx$	15-	$\int \cos x \ln(\sin x) dx$
16-	$\int \frac{\ln(\ln x) dx}{x}$	17-	$\int \frac{(\ln x)^2 dx}{x^2}$	18-	$\int \frac{x \tan^{-1} x dx}{\sqrt{1+x^2}}$
19-	$\int x \sec^2 x dx$	20-	$\int x \sin x \cos x dx$	21-	$\int x \sin^{-1} x dx$
22-	$\int \ln(x^2+1) dx$	23-	$\int \ln(x + \sqrt{x^2+1}) dx$	24-	$\int \left(\sin^{-1} x\right)^2 dx$
25-	$\int \cos x \cosh x dx$	26-	$\int e^{3x} \cos 4x dx$	27-	$\int x^2 \tan^{-1} x dx$

2.2 Trigonometric Integrals

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called . They are an important part of the integration technique called trigonometric substitution, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$

2.2.1 Integrating Products and Powers of $\sin x$ and $\cos x$

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of the form $\displaystyle\int \sin^j x \cos x dx$ or $\displaystyle\int \cos^j x \sin x dx$. After rewriting these integrals, we evaluate them using u -substitution. Before defore describing the general process in detail, let's take a look at the following examples.

Example 2.2.1. *Evaluate*

$$\int \cos^3 x \sin x dx$$

Use u-substitution and let $u = \cos x$ *In this case* $du = \sin x dx$*, Thus,*

$$\begin{aligned}\int \cos^3 x \sin x dx &= -\int u^3 du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C\end{aligned}$$

Evaluate

$$\int \sin^4 x \cos x dx$$

Example 2.2.2. *A Preliminary Example: Integrating* $\displaystyle\int \cos^j x \sin^k x dx$ *Where k is odd*

Evaluate

$$\int \cos^2 x \sin^3 x dx$$

To convert this integral to integrals of the form $\int \cos^j x \sin x dx$, *rewrite* $\sin^3 x = \sin^2 x \sin x$ *and make the substitution* $\sin^2 x = 1 - \cos^2 x$. *Let* $u = \cos x$, *then* $du = -\sin x dx$, *Hence* ,

$$\begin{aligned}
\int \cos^2 x \sin^3 x dx &= \int \cos^2 x \left(1 - \cos^2 x\right) \sin x dx \\
&= -\int u^2 \left(1 - u^2\right) du \\
&= \int \left(u^4 - u^2\right) du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C
\end{aligned}$$

Evaluate

$$\int \sin^2 x \cos^3 x dx$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$ For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1 - \cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1 + \cos(2x)}{2}$$

are invaluable. These identities are sometimes known as power-reducing identities and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$

Example 2.2.3. *Evaluate*

$$\int \sin^2 x dx$$

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, *Then*

$$\begin{aligned}
\int \sin^2 x dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C
\end{aligned}$$

Example 2.2.4. *Evaluate*

$$\int \cos^2 x dx$$

To evaluate this integral, let's use the trigonometric identity $\cos^2 x = \frac{1}{2}(1+\cos(2x))$*, Then*

$$\begin{aligned}\int \cos^2 x dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C\end{aligned}$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ **and** $\cos x$

To integrate $\int \cos^j x \sin^k x dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^k -1x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x dx$

2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)

3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example 2.2.5. *Evaluate*

$$\int \cos^8 x \sin^5 dx$$

Since the power on $\sin x$ *is odd, use strategy 1. Thus,*

$$\begin{aligned}
\int \cos^8 x \sin^5 x dx &= \int \cos^8 x \sin^4 x \sin x dx \\
&= \int \cos^8 x \left(\sin^2 x\right)^2 \sin x dx \\
&= \int \cos^8 x \left(1-\cos^2 x\right)^2 \sin x dx \quad \textit{Let } u = \cos x \textit{ and } du = -\sin x dx \\
&= \int u^8 \left(1-u^2\right)^2 (-du) \\
&= \int \left(-u^8 + 2u^{10} - u^{12}\right) du \\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C^2
\end{aligned}$$

Example 2.2.6. *Evaluate*

$$\int \sin^4 x dx$$

since the power on $\sin x$ *is even* $(k=4)$ *and the power on* $\cos x$ *is even* $(j=0)$, *we must use strategy* 3. *Thus,*

$$\begin{aligned}
\int \sin^4 x dx &= \int \left(\sin^2 x\right)^2 dx \\
&= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx \\
&= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

Evaluate

$$\int \cos^3 x dx, \qquad \int \cos^2(3x) dx$$

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax)$, $\sin(bx)$, $\cos(ax)$ and $\cos(bx)$ These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Integrating Products of Sines and Cosines of Different Angles

$$\sin(ax)\sin(bx) = \frac{1}{2}\left[\cos((a-b)x) - \cos((a+b)x)\right] \tag{2.1}$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin((a-b)x) + \sin((a+b)x)\right] \tag{2.2}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos((a-b)x) + \cos((a+b)x)\right] \tag{2.3}$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example 2.2.7. *Evaluate*

$$\int \sin(5x)\cos(3x)dx$$

Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}[\sin(2x) - \cos(8x)]$ *Thus,*

$$\begin{aligned}\int \sin(5x)\cos(3x)dx &= \frac{1}{2}\int [\sin(2x) - \cos(8x)]dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\sin(8x) + C\end{aligned}$$

Example 2.2.8. *Evaluate*

$$\int \cos(6x)\cos(5x)dx$$

Apply the identity $\cos(6x)\cos(5x) = \frac{1}{2}[\cos(x) + \cos(11x)]$ *Thus,*

$$\begin{aligned}\int \cos(6x)\cos(5x)dx &= \frac{1}{2}\int [\cos(x) + \cos(11x)]dx \\ &= \frac{1}{2}\sin(x) + \frac{1}{22}\sin(11x) + C\end{aligned}$$

2.2.2 Integrating Products and Powers of $\tan x$ and $\sec x$

Before discussing the integration of products and powers of $\tan x$ and $\sec x$ it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\displaystyle\int \sec^2 x dx = \tan x + C$

2. $\displaystyle\int \sec x \tan x dx = \sec x + C$

3. $\displaystyle\int \tan x dx = \ln|\sec x| + C$

4. $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x dx$ or $\int \sec^j x \tan x dx$. As we see in the following example, we can evaluate these new integrals by using u -substitution.

Example 2.2.9. *Evaluate*

$$\int \sec^5 x \tan x dx$$

Start by rewriting $\sec^5 x \tan x$ *as* $\sec^4 x \sec x \tan x$ *,*

$$\begin{aligned}
\int \sec^5 x \tan x dx &= \int \sec^4 x \sec x \tan x dx \quad \textit{Let} \;\; u = \sec x \to du = \sec x \tan x dx \\
&= \int u^4 du \\
&= \frac{1}{5} u^5 + C \\
&= \frac{1}{5} \sec^5 x + C
\end{aligned}$$

We now take a look at the various strategies for integrating products and powers of $\tan x$ and $\sec x$

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x dx$

To integrate $\int \tan^k x \sec^j x dx$ use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$

2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x dx$. (Note: If j is even and k is odd then either strategy 1 or strategy 2 may be used.)

3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x \left(\sec^2 x - 1\right) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.

4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example 2.2.10. *Integrating* $\int \tan^k x \sec^j x dx$ *when j is Even*
Evaluate

$$\int \tan^6 x \sec^4 x dx$$

Since the power on sec x is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ *and use* $\sec^2 x = \tan^2 x + 1$ *to rewrite the first* $\sec^2 x$ *in terms of* $\tan x$*. Thus,*

$$\begin{aligned}
\int \tan^6 x \sec^4 x dx &= \int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x dx \quad \textit{Let } u = \tan x \textit{ and } du = \sec^2 x \\
&= \int u^6 \left(u^2 + 1\right) du \\
&= \int \left(u^8 + u^6\right) du \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C
\end{aligned}$$

Example 2.2.11. *Integrating* $\int \tan^k x \sec^j x dx$ *when k is odd*
Evaluate

$$\int \tan^5 x \sec^3 x dx$$

Rewrite $\tan^5 x \sec^3 x$ *as* $\tan^4 x \sec^2 x \tan x \sec x$*. Thus,*

$$\begin{aligned}
\int \tan^5 x \sec^3 x dx &= \int \left(\tan^2 x\right)^2 \sec^2 x \sec x \tan x dx \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^2 x \sec x \tan x dx \quad \textit{Let } u = \sec x \textit{ and } du = \sec x \tan x dx \\
&= \int \left(u^2 - 1\right)^2 u^2 du \\
&= \int \left(u^6 - 2u^4 + u^2\right) du \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C
\end{aligned}$$

Example 2.2.12. *Integrating* $\int \tan^k x dx$ *when k is odd and* $k \geq 3$
Evaluate

$$\int \tan^3 x dx$$

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x \left(\sec^2 x - 1\right) = \tan x \sec^2 x - \tan x$. *Thus,*

$$\begin{aligned}\int \tan^3 x dx &= \int \left(\tan x \sec^2 x - \tan x\right) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C\end{aligned}$$

2.2.3 Exercises

Evaluate the following integral

1-	$\int \sin^3 x dx$	2-	$\int \cos^3 x dx$	3-	$\int \sin x \cos x dx$
4-	$\int \cos^5 x dx$	5-	$\int \sin^5 x \cos^2 x dx$	6-	$\int \sin^3 x \cos^3 x dx$
7-	$\int \tan^3 x \sec x dx$	8-	$\int \tan^2 x \sec x dx$	9-	$\int \tan^2 x \sec^4 x dx$
10-	$\int \tan^8 x \sec^2 x dx$	11-	$\int \cot^3 x dx$	12-	$\int \cot^5 x \csc^2 x dx$
13-	$\int \frac{\cos^5 x}{\sin^3 x} dx$	14-	$\int \frac{\sin^7 x}{\cos^4 x} dx$	15-	$\int \sin 2x \cos 2x dx$
16-	$\int \cos 4x \cos 6x dx$	17-	$\int \frac{\tan^3(\ln t)}{t} dt$	18-	$\int t \cos^3\left(t^2\right) dt$

19- **Prove the formula** $\int \sec^m x dx = \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x dx$

20- **Prove the formula** $\int \tan^m x dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x dx$

21-	$\int \cot^5 x \csc^5 x dx$	22-	$\int \tan^6 x \sec^4 x dx$	23-	$\int \tan^3 \theta \sec^3 \theta d\theta$

Hint : Use Ex-19 and Ex-20 to solve Ex-21,22,23

2.3 Trigonometric Substitution

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many y still remain inaccessible. The technique of trigonometric substitution comes in very handy when evaluating these integrals. This technique substitution to rewrite these integrals as trigonometric integrals.

2.3.1 Integrals Involving $\sqrt{a^2 - x^2}$

To evaluate integrals involving $\sqrt{a^2 - x^2}$ we make the substitution $x = a \sin\theta$ and $dx = a\cos\theta d\theta$ by making the substitution $x = a\sin\theta$ we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x . Since $\sin\theta = \frac{x}{a}$ we can draw the reference triangle in Figure 2.1 to assist in expressing the values of $\cos\theta$ and $\sin\theta$ the remaining trigonometric functions in terms of x

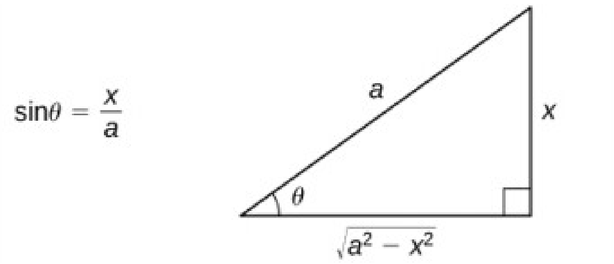


FIGURE 2.1: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}} dx$, $\int \frac{x}{\sqrt{a^2 - x^2}} dx$, and $\int x\sqrt{a^2 - x^2} dx$, they can be integrated directly either by formula or by a simple u -substitution.

2. Make the substitution $x = a \sin\theta$ and $dx = a \cos\theta d\theta$. Note: This substitution yields $\sqrt{a^2 - x^2} = a\cos\theta$

3. Simplify the expression.

4. Evaluate the integral using techniques from the section on trigonometric integrals.

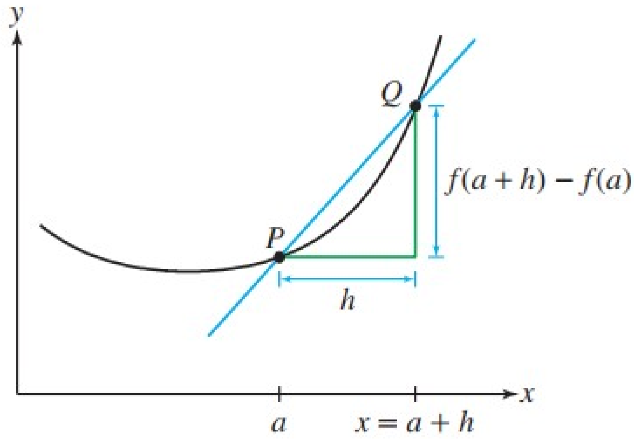
5. Use the reference triangle from Figure 2.1 to rewrite the result in terms of x You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

The following example demonstrates the application of this problem-solving strategy.

Example 2.3.1. *Evaluate*

$$\int \sqrt{9 - x^2} dx$$

Begin by making the substitutions $x = 3 \sin\theta$ *and* $dx = 3\cos\theta d\theta$. *since* $\sin\theta = \frac{x}{3}$, *we can construct the reference triangle shown in the following figure.*

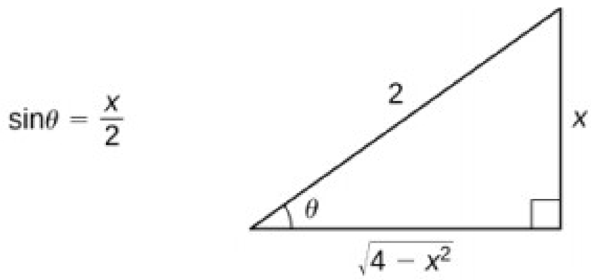


$$
\begin{aligned}
\int \sqrt{9-x^2}dx &= \int \sqrt{9-(3\sin\theta)^2}3\cos\theta d\theta \\
&= \int \sqrt{9\left(1-\sin^2\theta\right)}3\cos\theta d\theta \\
&= \int \sqrt{9\cos^2\theta}3\cos\theta d\theta \\
&= \int 9\cos^2\theta d\theta \\
&= \int 9\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right) d\theta \\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C
\end{aligned}
$$

Example 2.3.2. *Evaluate*

$$\int \frac{\sqrt{4-x^2}}{x}dx$$

Begin by making the substitutions $x = 2\sin\theta$ *and* $dx = 2\cos\theta d\theta$. *since* $\sin\theta = \frac{x}{2}$, *we can construct the reference triangle shown in the following figure.*

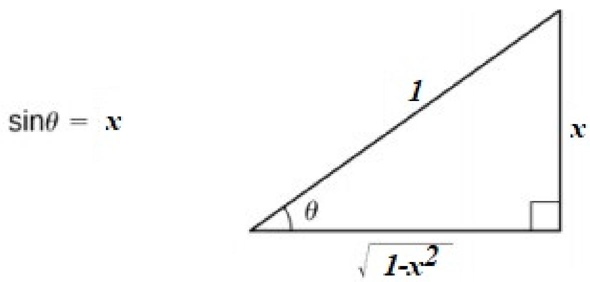


$$
\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x} dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta} 2\cos\theta d\theta \\
&= \int \frac{2\cos^2\theta}{\sin\theta} d\theta \\
&= \int \frac{2\left(1-\sin^2\theta\right)}{\sin\theta} d\theta \\
&= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C \\
&= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x}}{x}\right| + \sqrt{4-x^2} + C
\end{aligned}
$$

Example 2.3.3. *Evaluate*

$$\int x^3\sqrt{1-x^2}dx$$

Begin by making the substitutions $x = \sin\theta$ *and* $dx = \cos\theta d\theta$. *since* $\sin\theta = x$, *we can construct the reference triangle shown in the following figure.*



$$\begin{aligned}
\int x^3\sqrt{1-x^2}dx &= \int \sin^3\theta\cos^2\theta d\theta \\
&= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta d\theta \\
&= \int \left(u^4-u^2\right)du \\
&= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C \\
&= \frac{1}{5}\left(1-x^2\right)^{5/2}-\frac{1}{3}\left(1-x^2\right)^{3/2}+C
\end{aligned}$$

2.3.2 Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$ let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot x$

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.

2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2\left(1+\tan^2\theta\right)} = \sqrt{a^2\sec^2\theta} = |a\sec\theta$. (since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and $\sec\theta>0$ over this interval, $|a\sec\theta| = a\sec\theta$.).

3. Simplify the expression.

4. Evaluate the integral using techniques from the section on trigonometric integrals.

5. Use the reference triangle from Figure 2.2 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x>0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x\leq 0$.)

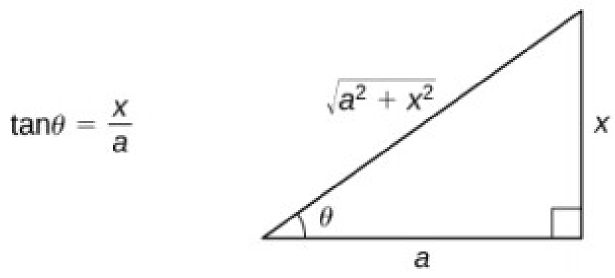
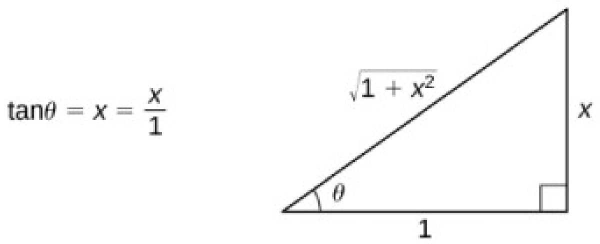


FIGURE 2.2: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x

Example 2.3.4. *Evaluate*

$$\int \frac{1}{\sqrt{1+x^2}}dx$$

Begin with the substitution $x = \tan\theta$ *and* $dx = \sec^2\theta d\theta$. *since tan* $\theta = x$, *draw the reference triangle in the following figure.*

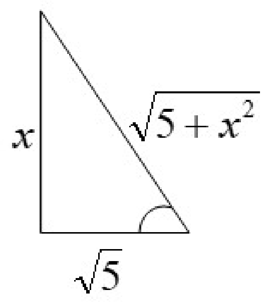


$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\sec\theta}d\theta \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\sqrt{1+x^2} + x| + C\end{aligned}$$

Example 2.3.5. *Evaluate*

$$\int \frac{1}{x^2\sqrt{5+x^2}}dx$$

Begin with the substitution $x = \sqrt{5}\tan\theta$ *and* $dx = \sqrt{5}\sec^2\theta d\theta$. *since tan* $\theta = x$, *draw the reference triangle in the following figure.*



$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{5+x^2}}dx &= \int \frac{\sqrt{5}\sec^2\theta}{5\sqrt{5}\tan^2\theta\sec\theta}d\theta \\
&= \frac{1}{5}\int \sec\theta\cot^2\theta d\theta \\
&= \frac{1}{5}\int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}d\theta \\
&= \frac{1}{5}\int \cos\theta\sin^{-2}\theta d\theta \\
&= \frac{-1}{5}\csc\theta + C \\
&= -\frac{\sqrt{5+x^2}}{5x} + c
\end{aligned}
$$

2.3.3 Integrating Expressions Involving $\sqrt{a^2+x^2}$

The domain of the expression $\sqrt{x^2-a^2}$ is $(-\infty,-a]\cup[a,+\infty)$. Thus, either $x<-a$ or $x>a$. Hence, $\frac{x}{a}\le -1$ or $\frac{x}{a}\ge 1$. since these intervals correspond to the range of $\sec\theta$ on the set $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, it makes sense to use the substitution $\sec\theta=\frac{x}{a}$ or, equivalently, $x=a\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$. The corresponding substitution for dx is $dx=a\sec\theta\tan\theta d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.

2. Substitute $x = a \sec \theta$ and $dx = a \sec \theta \tan \theta d\theta$. This substitution yields

   $$\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2 \left(\sec^2 \theta + 1\right)} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta|$$

   For $x \geq a, \quad |a \tan \theta| = a \tan \theta$ and for $x \leq -a, \quad |a \tan \theta| = -a \tan \theta$

3. Simplify the expression.

4. Evaluate the integral using techniques from the section on trigonometric integrals.

5. Use the reference triangles from Figure 2.3 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

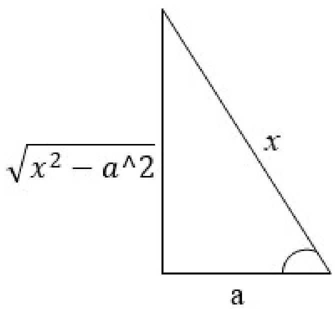
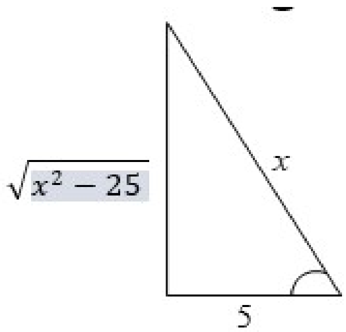


FIGURE 2.3: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x

Example 2.3.6. *Evaluate*

$$\int \frac{1}{x^2\sqrt{x^2-25}}dx$$

Begin with the substitution $x = 5\sec\theta$ *and* $dx = 5\sec\theta\tan\theta d\theta$. *draw the reference triangle in the following figure.*



$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-25}}dx &= \int \frac{5\sec\theta\tan\theta d\theta}{25\sec^2\theta\sqrt{25\sec^2\theta-25}} \\
&= \int \frac{5\sec\theta\tan\theta d\theta}{125\sec^2\theta\tan\theta} \\
&= \int \frac{d\theta}{25\sec\theta} \\
&= \frac{1}{25}\int \cos\theta d\theta \\
&= \frac{1}{25}\sin\theta + c \\
&= \frac{\sqrt{x^2-25}}{25x} + c
\end{aligned}$$

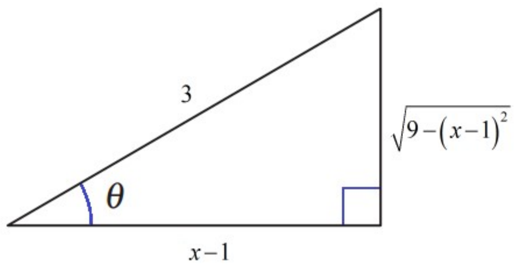
Example 2.3.7. *Evaluate*

$$\int \frac{1}{\sqrt{8-x^2+2x-1+1}}dx$$

Complete the square , we get

$$\begin{aligned}
\int \frac{1}{\sqrt{8-x^2+2x-1+1}}dx &= \int \frac{1}{\sqrt{9-(x^2-2x+1)}}dx \\
&= \int \frac{1}{\sqrt{9-(x-1)^2}}dx
\end{aligned}$$

Begin with the substitution $x - 1 = 3\sin\theta$ *and* $dx = 3\cos\theta d\theta$. *draw the reference triangle in the following figure.*



$$\begin{aligned}\int \frac{1}{\sqrt{9-(x-1)^2}}dx &= \int \frac{3\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}}\\ &= \int \frac{3\cos\theta d\theta}{3\cos\theta}\\ &= \int d\theta\\ &= \theta + c\\ &= \sin^{-1}\left(\frac{x-1}{3}\right) + c\end{aligned}$$

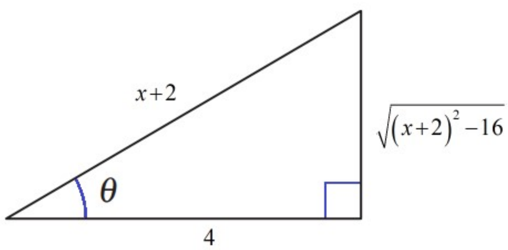
Example 2.3.8. *Evaluate*

$$\int \frac{1}{\sqrt{(x^2+4x+4)-16}}dx$$

Complete the square , we get

$$\int \frac{1}{\sqrt{(x^2+4x+4)-16}}dx = \int \frac{1}{\sqrt{(x+2)^2-16}}dx$$

Begin with the substitution $x+2 = 4\sec\theta$ *and* $dx = 4\sec\theta\tan\theta d\theta$. *draw the reference triangle in the following figure.*



$$\begin{aligned}
\int \frac{1}{\sqrt{(x+2)^2-16}}dx &= \int \frac{4\sec\theta\tan\theta d\theta}{\sqrt{16\sec^2\theta-16}} \\
&= \int \frac{4\sec\theta\tan\theta d\theta}{4\tan\theta} \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta+\tan\theta|+C \\
&= \ln\left|\frac{x+2}{4}+\frac{\sqrt{(x+2)^2-16}}{4}\right|+C
\end{aligned}$$

2.3.4 Exercises

Evaluate the following integrals

1- $\displaystyle\int \frac{x^2dx}{\sqrt{9-x^2}}$ 2- $\displaystyle\int \frac{dt}{(16-t^2)^{3/2}}$ 3- $\displaystyle\int \frac{dx}{x\sqrt{x^2+16}}$

4- $\displaystyle\int \sqrt{12+4t^2}dt$ 5- $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$ 6- $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-25}}$

7- $\displaystyle\int \frac{dy}{y^2\sqrt{5-y^2}}$ 8- $\displaystyle\int x^3\sqrt{9-x^2}dx$ 9- $\displaystyle\int \frac{dx}{\sqrt{25x^2+2}}$

10- $\displaystyle\int \frac{dt}{(9t^2+4)^2}$ 11- $\displaystyle\int \frac{dz}{z^3\sqrt{z^2-4}}$ 12- $\displaystyle\int \frac{dy}{\sqrt{y^2-9}}$

13- $\displaystyle\int \frac{x^2dx}{(6x^2-49)^{1/2}}$ 14- $\displaystyle\int \frac{dx}{(x^2-4)^2}$ 15- $\displaystyle\int \frac{x^2dx}{(x^2-1)^{3/2}}$

16- $\displaystyle\int \frac{x^2dx}{(x^2+1)^{3/2}}$ 17- $\displaystyle\int \frac{dx}{\sqrt{x+6x^2}}$ 18- $\displaystyle\int \frac{dx}{\sqrt{12x-x^2}}$

19- $\displaystyle\int \sqrt{x^2-4x+3}dx$ 20- $\displaystyle\int \frac{dx}{(x^2+6x+6)^2}$

21- $\displaystyle\int \sec^{-1} x dx$ 22- $\displaystyle\int \frac{\sin^{-1} x}{x^2}dx$ 24- $\displaystyle\int \ln\left(x^2+1\right) dx$

2.4 PARTIAL FRACTIONS

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4}+\frac{3}{x+1}=\frac{2(x+1)+3(x-4)}{(x-4)(x+1)}=\frac{5x-10}{x^2-3x-4}$$

However, for purposes of integration, the left side of the previous equation is preferable to the right side since each of the terms is easy to integrate:

$$\int\frac{5x-10}{x^2-3x-4}dx=\int\frac{2}{x-4}dx+\int\frac{3}{x+1}dx=2\ln|x-4|+3\ln|x+1|+C$$

To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side.
In this section, we examine the method of partial fraction decomposition, which allows us to decompose rational functions into sums of simpler, more easily intagrated rational functions. Using this method, we can rewrite an expression such as: $\frac{3x}{x^2-x-2}$ as an expression as $\frac{1}{x+1}+\frac{2}{x-2}$.
The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x)+\frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$ We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int\frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $\dfrac{P(x)}{Q(x)}$ use the following steps:

1. Make sure that degree($P(x)) <$ degree$(Q(x))$. If not, perform long division of polynomials.

2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.

3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.

   - If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct,then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

     $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

   - If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

     $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

   - For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include

     $$\frac{Ax+B}{ax^2+bx+c}$$

   - For each repeated irreducible quadratic factor $\left(ax^2+bx+c\right)^n$, the decomposition must include

     $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2} + \cdots + \frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

   - After the appropriate decomposition is determined, solve for the constants.

   - Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Example 2.4.1. *Evaluate*

$$\int \frac{x^2+3x+5}{x+1}dx$$

Since $\deg\left(x^2+3x+5\right) \geq \deg(x+1)$, *we perform long division to obtain*

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}$$

Then

$$\begin{aligned}\int \frac{x^2+3x+5}{x+1}dx &= \int \left(x+2+\frac{3}{x+1}\right)dx \\ &= \frac{1}{2}x^2+2x+3\ln|x+1|+C\end{aligned}$$

Example 2.4.2. *Evaluate*

$$\int \frac{3x+2}{x^3-x^2-2x}dx$$

Since $\deg(3x+2) < \deg\left(x^3-x^2-2x\right)$, *we begin by factoring the denominator of* $\frac{3x+2}{x^3-x^2-2x}$. *We can see that* $x^3-x^2-2x = x(x-2)(x+1)$. *Thus, there constants* $A, \quad B$, *and* C *satisfying*

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2)$$

Then $A=-1, \quad B=4/3, \quad C=-1/3$, *hence*

$$\begin{aligned}\int \frac{3x+2}{x^3-x^2-2x}dx &= \int \left(-\frac{1}{x}+\frac{4}{3}\cdot\frac{1}{(x-2)}-\frac{1}{3}\frac{1}{(x+1)}\right)dx \\ &= -\ln|x|+\frac{4}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C\end{aligned}$$

Example 2.4.3. *Evaluate*

$$\int \frac{x^2+3x+1}{x^2-4}dx$$

since degree $\left(x^2+3x+1\right) \geq$ *degree* $\left(x^2-4\right)$, *we must perform long division of polynomials. This results in*

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. *We have*

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Then

$$x+5 = A(x+2) + B(x-2)$$

Solving for A *and* B *using either method, we obtain* $A = 11/4$ *and* $B = 1/4$

$$\begin{aligned}\int \frac{x^2+3x+1}{x^2-4}dx &= \int \left(1 + \frac{11}{4}\cdot\frac{1}{x-2} + \frac{1}{4}\cdot\frac{1}{x+2}\right)dx \\ &= x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C\end{aligned}$$

Example 2.4.4. *Evaluate*

$$\int \frac{\cos x}{\sin^2 x - \sin x}dx$$

Let's begin by letting $u = \sin x$. *Consequently,* $du = \cos x dx$. *After making these substitutions, we have*

$$\int \frac{\cos x}{\sin^2 x - \sin x}dx = \int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)}$$

Applying partial fraction decomposition to $1/u(u-1)$ *gives* $\dfrac{1}{u(u-1)} = -\dfrac{1}{u} + \dfrac{1}{u-1}$

$$\begin{aligned}\int \frac{\cos x}{\sin^2 x - \sin x}dx &= -\ln|u| + \ln|u-1| + C \\ &= -\ln|\sin x| + \ln|\sin x - 1| + C\end{aligned}$$

Example 2.4.5. *Evaluate*

$$\int \frac{x-2}{(2x-1)^2(x-1)}dx$$

We have degree($x-2) <$ *degree* $\left((2x-1)^2(x-1)\right)$, *so we can proceed with the decomposition. since* $(2x-1)^2$ *is a repeated linear factor, include* $\dfrac{A}{2x-1} + \dfrac{B}{(2x-1)^2}$ *in the decomposition. Thus,*

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}$$

After getting a common denominator and equating the numerators, we have

$$x-2=A(2x-1)(x-1)+B(x-1)+C(2x-1)^2$$

We then use the method of equating coefficients to find the values of $A,\quad B$, *and* C.

$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C)$$

Equating coefficients yields $2A+4C=0,\quad -3A+B-4C=1$, *and* $A-B+C=-2$. *Solving this system yields* $A=2,\quad B=3$, *and* $C=-1$

$$\begin{aligned}\int\frac{x-2}{(2x-1)^2(x-1)}dx&=\int\left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx\\&=\ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C\end{aligned}$$

Example 2.4.6. *Evaluate*

$$\int\frac{2x-3}{x^3+x}dx$$

since $\deg(2x-3)<\deg\left(x^3+x\right)$, *factor the denominator and proceed with partial fraction decomposition. since* $x^3+x=x\left(x^2+1\right)$ *contains the irreducible quadratic factor* x^2+1, *include* $\frac{Ax+B}{x^2+1}$ *as part of the decomposition, along with* $\frac{C}{x}$ *for the linear term* x. *Thus, the decomposition has the form*

$$\frac{2x-3}{x\left(x^2+1\right)}=\frac{Ax+B}{x^2+1}+\frac{C}{x}$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x-3=(Ax+B)x+C\left(x^2+1\right)$$

Solving for A,B, *and* C, *we get* $A=3,\quad B=2$, *and* $C=-3$

$$\begin{aligned}\int\frac{2x-3}{x^3+x}dx&=\int\left(\frac{3x+2}{x^2+1}-\frac{3}{x}\right)dx\\&=3\int\frac{x}{x^2+1}dx+2\int\frac{1}{x^2+1}dx-3\int\frac{1}{x}dx\\&=\frac{3}{2}\ln\left|x^2+1\right|+2\tan^{-1}x-3\ln|x|+C\end{aligned}$$

Example 2.4.7. *Evaluate*

$$\int \frac{dx}{x^3-8}$$

We can start by factoring $x^3-8=(x-2)\left(x^2+2x+4\right)$. *We see that the quadratic factor* x^2+2x+4 *is irreducible since* $2^2-4(1)(4)=-12<0$. *Using the decomposition described in the problem-solving strategy, we get*

$$\frac{1}{(x-2)\left(x^2+2x+4\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

After obtaining a common denominator and equating the numerators, this becomes

$$1=A\left(x^2+2x+4\right)+(Bx+C)(x-2)$$

Applying either method, we get $A=\frac{1}{12}, B=-\frac{1}{12}$, *and* $C=-\frac{1}{3}$

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\int \frac{1}{x-2}dx-\frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx$$

We can see that $\displaystyle\int \frac{1}{x-2}dx=\ln|x-2|+C$, *but* $\displaystyle\int \frac{x+4}{x^2+2x+4}dx$ *requires a bit more effort. Let's begin by completing the square on* x^2+2x+4 *to obtain*

$$x^2+2x+4=(x+1)^2+3$$

By letting $u=x+1$ *and consequently* $du=dx$, *we see that*

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}dx &= \int \frac{x+4}{(x+1)^2+3}dx \\
&= \int \frac{u+3}{u^2+3}du \\
&= \int \frac{u}{u^2+3}du+\int \frac{3}{u^2+3}du \\
&= \frac{1}{2}\ln\left|u^2+3\right|+\frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}+C \\
&= \frac{1}{2}\ln\left|x^2+2x+4\right|+\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C
\end{aligned}$$

2.4.1 Exercises

Evaluate the following integrals

1- $\displaystyle\int \frac{\left(x^3+2x^2+1\right)dx}{x+2}$

2- $\displaystyle\int \frac{\left(x^2+2\right)dx}{x+3}$

3- $\displaystyle\int \frac{x\,dx}{3x-4}$

4- $\displaystyle\int \frac{\left(x^3+1\right)dx}{x^2+1}$

5- $\displaystyle\int \frac{dx}{(x-2)(x-4)}$

6- $\displaystyle\int \frac{(x+3)dx}{x+4}$

7- $\displaystyle\int \frac{dx}{(x-2)(x-3)(x+2)}$

8- $\displaystyle\int \frac{(2x-1)dx}{x^2-5x+6}$

9- $\displaystyle\int \frac{dx}{x(2x+1)}$

10- $\displaystyle\int \frac{\left(x^2+11x\right)dx}{(x-1)(x+1)^2}$

11- $\displaystyle\int \frac{3dx}{(x+1)\left(x^2+x\right)}$

12- $\displaystyle\int \frac{x^2dx}{x^2+9}$

13- $\displaystyle\int \frac{\left(4x^2-21x\right)dx}{(x-3)^2(2x+3)}$

14- $\displaystyle\int \frac{8dx}{x(x+2)^3}$

15- $\displaystyle\int \frac{4x^2-20}{(2x+5)^3}dx$

16- $\displaystyle\int \frac{\left(x^2-x+1\right)dx}{x^2+x}$

17- $\displaystyle\int \frac{x\,dx}{x^4+1}$

18- $\displaystyle\int \frac{e^x dx}{e^{2x}-e^x}$

19- $\displaystyle\int \frac{\sec^2\theta d\theta}{\tan^2\theta-1}$

20- $\displaystyle\int \frac{\sqrt{x}dx}{x-1}$

21- $\displaystyle\int \frac{dx}{x^{1/2}-x^{1/3}}$

22- $\displaystyle\int \frac{dx}{x^{5/4}-4x^{3/4}}$

23- $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y+\sin y-6}$

24- $\displaystyle\int \frac{1}{x^4+x}dx$

Chapter 3

Definite Integrals and Application

A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method to calculate the areas and volumes of very general shapes. This method, called integration, is a way to calculate much more than areas and volumes. The definite integral is the key tool in calculus for defining and calculating many important quantities, such as areas, volumes, lengths of curved paths, probabilities, averages, energy consumption, the weights of various objects, and the forces against a dam's floodgates, just to mention a few

3.1 Area and estimating with Finite Sums

The basis for formulating definite integrals is the construction of appropriate approximations by finite sums.

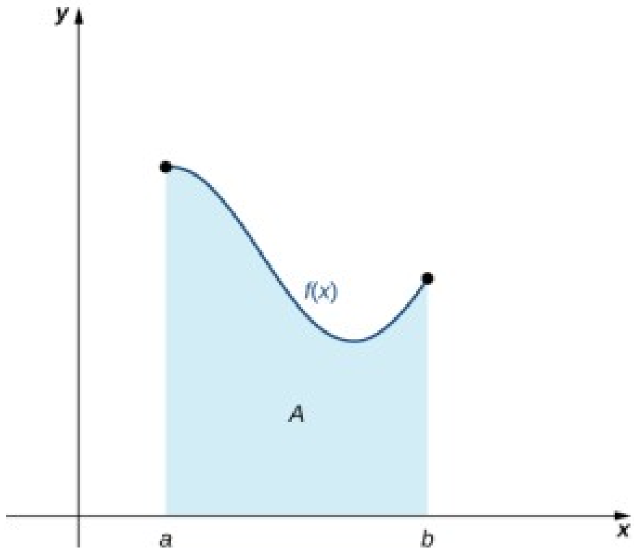
Sums and Powers of Integers

1. $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x -axis below, the line $x = a$ on the left, and the line $x = b$ on the right . How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have lenown area formulas, we can sum these areas and obtain a reasonable



estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}, \quad i = 1, 2, 3, \ldots, n$$

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

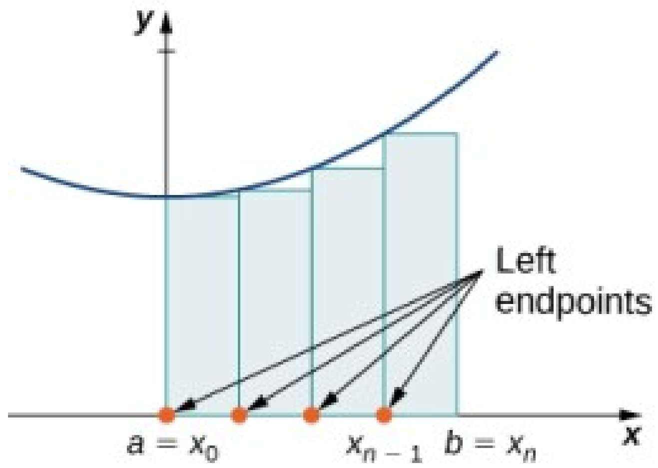
for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a partition of $[a, b]$. If the subintervals all have the same width, the set of points forms a regular partition of the interval $[a, b]$

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation
On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\,\Delta x$. Adding the areas of all these rectangles, we get an approximate for A. We use the notation L_n to denote that this is a left-endpoint approximation of A using n subintervals.

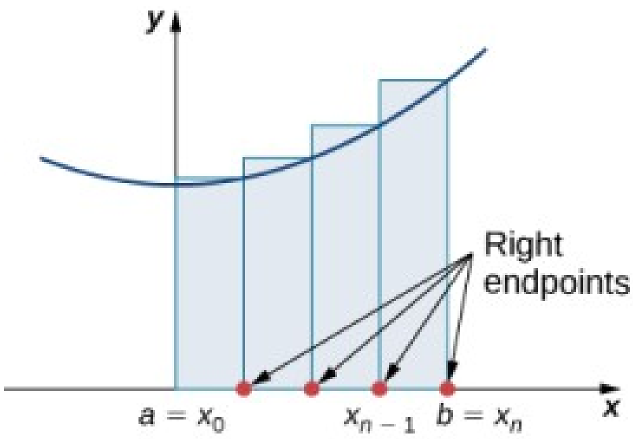
$$\begin{aligned} A \approx L_n &= f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x \\ &= \sum_{i=1}^{n} f(x_{i-1})\,\Delta x \end{aligned}$$



Rule: Right-Endpoint Approximation
Construct a rectangle on each subinterval $[x_i - 1, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$\begin{aligned} A \approx R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\,\Delta x \end{aligned}$$

The notation R_n indicates this is a right-endpoint approximation for A in the following figure

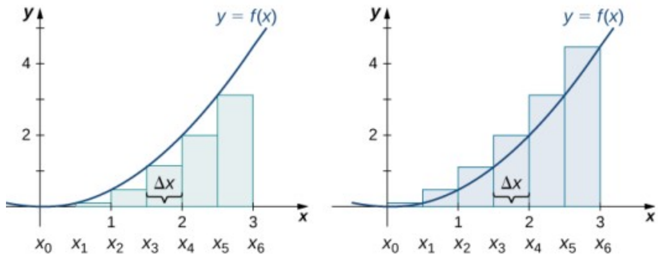


Example 3.1.1. *Find area under* $f(x) = \dfrac{x^2}{2}$ *on* $[0, 3]$ *with* $n = 6$

Solution:*we divide the region represented by the interval* $[0, 3]$ *into six subintervals, each of width* 0.5. *Thus,* $\Delta x = 0.5$. *We then form six rectangles by drawing vertical lines perpendicular to* x_{i-1}, *the left endpoint of each subinterval.*

We determine the height of each rectangle by calculating $f(x_{i-1})$ *for* $i = 1, 2, 3, 4, 5, 6$. *The intervals are* $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. *We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between* $f(x)$ *and the* x *-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation.*

$$
\begin{aligned}
A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \\
&= f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x \\
&= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\
&= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\
&= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\
&= 3.4375
\end{aligned}
$$



This is a right-endpoint approximation of the area under $f(x)$.

$$
\begin{aligned}
A \approx R_6 = \sum_{i=1}^{6} f\left(x_i\right) \Delta x \\
&= f\left(x_1\right) \Delta x + f\left(x_2\right) \Delta x + f\left(x_3\right) \Delta x + f\left(x_4\right) \Delta x + f\left(x_5\right) \Delta x + f\left(x_6\right) \Delta x \\
&= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\
&= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\
&= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\
&= 5.6875
\end{aligned}
$$

Example 3.1.2. *Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.*

The solution is up to the student

Example 3.1.3. *Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods*

The solution is up to the student

3.1.1 Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point c_i in the subinterval $[x_{i-1}, x_i]$, and use $f\left(x_i^*\right)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$ be a Riemann sum for $f(x)$. Then, the area under the curve $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

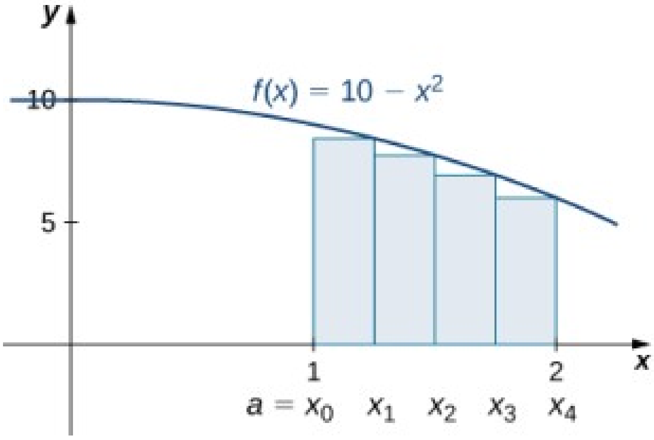
We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for $\{x \neq\}$ Although any choice for $\left\{x_i^*\right\}$ gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for $\left\{x_i^*\right\}$ to guarantee one result or the other.

If we want an overestimate, for example, we can choose $\left\{x_i^*\right\}$ such that for $i = 1, 2, 3, \ldots, n, f\left(x_i^*\right) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$. In other words, we choose $\left\{x_i^*\right\}$ so that for $i = 1, 2, 3, \ldots, n, f\left(x_i^*\right)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select $\left\{x_i^*\right\}$ in this way, then the Riemann sum $\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$ is called an **upper sum**.

Similarly, if we want an underestimate, we can choose $\left\{x_i^*\right\}$ so that for $i = 1, 2, 3, \ldots, n, f\left(x_i^*\right)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left-and right-endpoint approximations.

Example 3.1.4. *Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.*

Solution: *With* $n = 4$ *over the interval* $[1, 2]$, $\Delta x = \frac{1}{4}$. *We can list intervals as* $[1, 1.25], [1.25, 1.5][1.5, 1.7$ *Because the function is decreasing over the interval* $[1, 2]$, *the following figure shows that a lower sum is obtained by using the right endpoints. The Riemann sum is*



$$\begin{aligned}\sum_{k=1}^{4}\left(10-x^2\right)(0.25) &= 0.25\left[10-(1.25)^2+10-(1.5)^2+10-(1.75)^2+10-(2)^2\right]\\ &= 0.25[8.4375+7.75+6.9375+6]\\ &= 7.28\end{aligned}$$

3.2 The Definite Integral

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, realworld problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

Example 3.2.1. *Use the definition of the definite integral to evaluate* $\int_0^2 x^2dx$. *Use a right-endpoint approximation to generate the Riemann sum.*

Solution: *We first want to set up a Riemann sum. Based on the limits of integration, we have* $a = 0$ *and* $b = 2$*. For* $i = 0, 1, 2, \ldots, n$*, let* $P = \{x_i\}$ *be a regular partition of* $[0, 2]$*. Then*

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

since we are using a right-endpoint approximation to generate Riemann sums, for each i*, we need to calculate the function value at the right endpoint of the interval* $[x_{i-1}, x_i]$*. The right endpoint of the interval is* x_i*, and since P is a regular partition,*

$$x_i = x_0 + i\Delta x = 0 + i\left[\frac{2}{n}\right] = \frac{2i}{n}$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}$$

Then the Riemann sum takes the form

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\,\Delta x &= \sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n}\frac{8i^2}{n^3} \\
&= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\
&= \frac{16n^3+24n^2+n}{6n^3} \\
&= \frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}
\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\begin{aligned}
\int_0^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{3}\right) + \lim_{n\to\infty} \left(\frac{4}{n}\right) + \lim_{n\to\infty} \left(\frac{1}{6n^2}\right) \\
&= \frac{8}{3} + 0 + 0 = \frac{8}{3}
\end{aligned}$$

The definite integral is often called, more simply, the integral of f over $[a, b]$. The process of computing integrals is called integration and $f(x)$ is called the integrand. The endpoints a and b of $[a, b]$ are called the limits of integration. Finally, we remark that any variable may be used as a variable of integration (this is a "dummy" variable). Thus, the following three integrals all denote the same quantity:

$$\int_a^b \sin x dx, \quad \int_a^b \sin t dt, \quad \int_a^b \sin u du$$

3.3 Interpretation of the Definite Integral as Signed Area

When $f(x) \geq 0$, the definite integral defines the area under the graph. To interpret the integral when $f(x)$ takes on both positive and negative values, we define the notion of signed area, where regions below the x -axis are considered to have "negative area" that is,

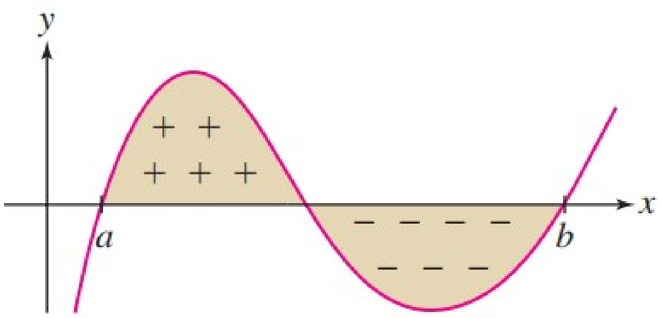


FIGURE 3.1: Signed area is the area above the x−axis minus the area below the x−axis

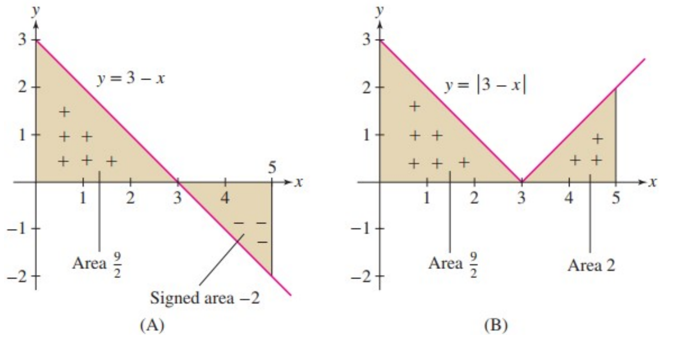
Signed area of a region $=$ (area above x -axis) $-$ (area below x -axis)

Example 3.3.1. *Calculate*

$$\int_0^5 (3-x)dx \quad \textit{and} \quad \int_0^5 |3-x|dx$$

The region between $y=3-x$ and the x -axis consists of two triangles of areas $\frac{9}{2}$ and 2[Figure (A)]. However, the second triangle lies below the x -axis, so it has signed area -2. In the graph of $y = |3-x|$, both triangles lie above the x -axis [Figure (B)] Therefore,

$$\int_0^5 (3-x)dx = \frac{9}{2} - 2 = \frac{5}{2} \quad \int_0^5 |3-x|dx = \frac{9}{2} + 2 = \frac{13}{2}$$



Properties of definite integrals:

1. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b Cf(x)dx = C\int_a^b f(x)dx \quad$ for any constant C
3. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
4. $\int_a^a f(x)dx = 0$
5. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \quad$ for all a, b, c
6. Comparison Theorem: If $f(x) \leq g(x)$ on $[a, b]$, then
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$
7. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ even} \\ 0, & \text{if } f(x) \text{ odd.} \end{cases}$

Example 3.3.2. *Evaluate*

$$\int_1^4 \left(\frac{1}{x} - x^2\right) dx$$

Solution:

$$\begin{aligned} \int_1^4 \left(\frac{1}{x} - x^2\right) dx &= \left(\ln|x| - \frac{1}{3}x^3\right)\Bigg|_1^4 \\ &= \left[\ln 4 - \frac{1}{3}(4)^3\right] - \left[\ln 1 - \frac{1}{3}(1)^3\right] \\ &= \ln 4 - 21 \approx -19.6137 \end{aligned}$$

Example 3.3.3. *Let $f(x)$ and $g(x)$ be functions that are continuous on the interval $-2 \leq x \leq 5$ and that satisfy*

$$\int_{-2}^{5} f(x)dx = 3 \qquad \int_{-2}^{5} g(x)dx = -4 \qquad \int_3^5 f(x)dx = 7$$

Use this information to evaluate each of these definite integrals:

$$a.\ \int_{-2}^{5}[2f(x)-3g(x)]dx \quad b.\ \int_{-2}^{3}f(x)dx$$

Solution:
(a)By combining the difference rule and constant multiple rule and substituting the given information, we find that

$$\begin{aligned}\int_{-2}^{5}[2f(x)-3g(x)]dx &= \int_{-2}^{5}2f(x)dx - \int_{-2}^{5}3g(x)dx\\ &= 2\int_{-2}^{5}f(x)dx - 3\int_{-2}^{5}g(x)dx\\ &= 2(3)-3(-4)=18\end{aligned}$$

(b) According to the subdivision rule

$$\int_{-2}^{5}f(x)dx = \int_{-2}^{3}f(x)dx + \int_{3}^{5}f(x)dx$$

Solving this equation for the required integral $\int_{-2}^{3}f(x)dx$ *and substituting the given information, we obtain*

$$\begin{aligned}\int_{-2}^{3}f(x)dx &= \int_{-2}^{5}f(x)dx - \int_{3}^{5}f(x)dx\\ &= 3-7=-4\end{aligned}$$

Example 3.3.4. *Evaluate*

$$\int_{1/4}^{2}\left(\frac{\ln x}{x}\right)dx$$

Solution:
use the substitution $u=\ln x$ *and* $du=\frac{dx}{x}$ *to transform the limits of integration:*

$$\begin{aligned}&when\ x=\frac{1}{4},\ then\ u=\ln\frac{1}{4}\\ &when\ x=2,\ then\ u=\ln 2\end{aligned}$$

Substituting, we find

$$\begin{aligned}\int_{1/4}^{2} \frac{\ln x}{x} dx &= \int_{\ln 1/4}^{\ln 2} u du = \frac{1}{2} u^2 \Big|_{\ln 1/4}^{\ln 2} \\ &= \frac{1}{2}(\ln 2)^2 - \frac{1}{2}\left(\ln \frac{1}{4}\right)^2 \approx -0.721\end{aligned}$$

Example 3.3.5. *Prove the inequality*

$$\int_1^4 \frac{1}{x^2} dx \leq \int_1^4 \frac{1}{x} dx$$

Solution:

If $x \geq 1$, then $x^2 \geq x$, and $x^{-2} \leq x^{-1}$. Therefore, the inequality follows from the Comparison Theorem, applied with $g(x) = x^{-2}$ and $f(x) = x^{-1}$

3.3.1 Exercises

Use properties of the integral to calculate the integrals

1- $\displaystyle\int_0^4 (6t-3)dt$ 2- $\displaystyle\int_{-3}^2 (4x+7)dx$

3- $\displaystyle\int_0^9 x^2 dx$ 4- $\int_{-3}^3 \left(9x-4x^2\right) dx$

5- $\displaystyle\int_{-a}^1 \left(x^2+x\right) dx$ 6- $\displaystyle\int_{-3}^1 \left(7t^2+t+1\right) dt$

Calculate the integral, assuming that $\displaystyle\int_0^5 f(x)dx = 5, \quad \int_0^5 g(x)dx = 12$

7- $\displaystyle\int_0^5 (f(x)+g(x))dx$ 8- $\displaystyle\int_0^5 \left(2f(x)-\frac{1}{3}g(x)\right) dx$

9- \$\displaystyle\int_{5}^{0}g(x)dx 10- $\displaystyle\int_0^5 (f(x)-x)dx$

Express each integral as a single integral

11- $\displaystyle\int_0^3 f(x)dx + \int_3^7 f(x)dx$ 12- $\displaystyle\int_2^9 f(x)dx - \int_4^9 f(x)dx$

13- $\displaystyle\int_2^9 f(x)dx - \int_2^5 f(x)dx$ 14- $\displaystyle\int_7^3 f(x)dx + \int_3^9 f(x)dx$

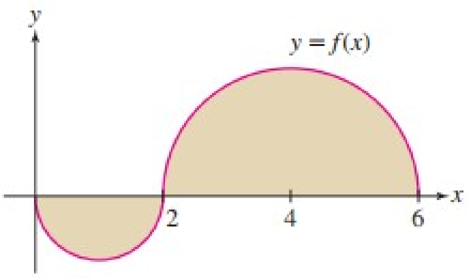
Calculate the integral, assuming that f is integrable and
$\displaystyle\int_1^b f(x)dx = 1-b^{-1}$ for all $b>0$

15- $\displaystyle\int_1^5 f(x)dx$ 16- $\displaystyle\int_3^5 f(x)dx$

17- $\displaystyle\int_1^6 (3f(x)-4)dx$ 18- $\displaystyle\int_{1/2}^1 f(x)dx$

19- Evaluate: (a) $\displaystyle\int_0^2 f(x)dx$ \quad (b) $\displaystyle\int_0^6 f(x)dx$

20- Evaluate: (a) $\displaystyle\int_1^4 f(x)dx$ \quad (b) $\displaystyle\int_1^6 |f(x)|dx$ Evaluate:



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3.4 Area of a Region between Two Curves

Regions Defined with Respect to x

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a,b]$ such that $f(x) \geq g(x)$ on $[a,b]$. We want to find the area between the graphs of the functions, as shown in the following figure. As we did before, we are going to partition the interval on the x -axis and approximate

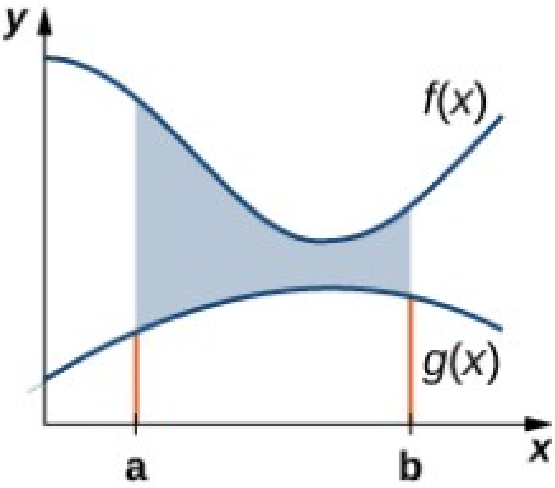


FIGURE 3.2: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g\left(x_i^*\right)$ to $f\left(x_i^*\right)$. The following figure (a) shows the rectangles when x_i^* is selected to be the left endpoint of the . interval and $n = 10$. The following figure (b) shows a representative rectangle in detail. The

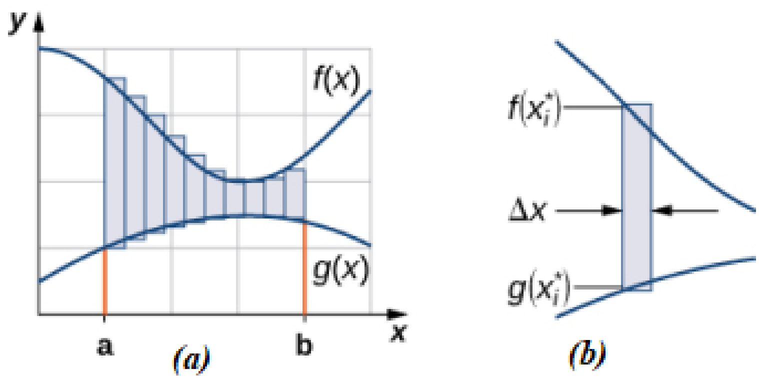


FIGURE 3.3

height of each individual rectangle is $f\left(x_i^*\right) - g\left(x_i^*\right)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} \left[f\left(x_i^*\right) - g\left(x_i^*\right)\right] \Delta x$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f\left(x_i^*\right) - g\left(x_i^*\right)\right] \Delta x = \int_a^b [f(x) - g(x)]dx$$

These findings are summarized in the following theorem.

Theorem 3.1. *Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by*

$$A = \int_a^b [f(x) - g(x)]dx$$

Regions Defined with Respect to y

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure. This time, we are going to partition the interval on the y -axis and use

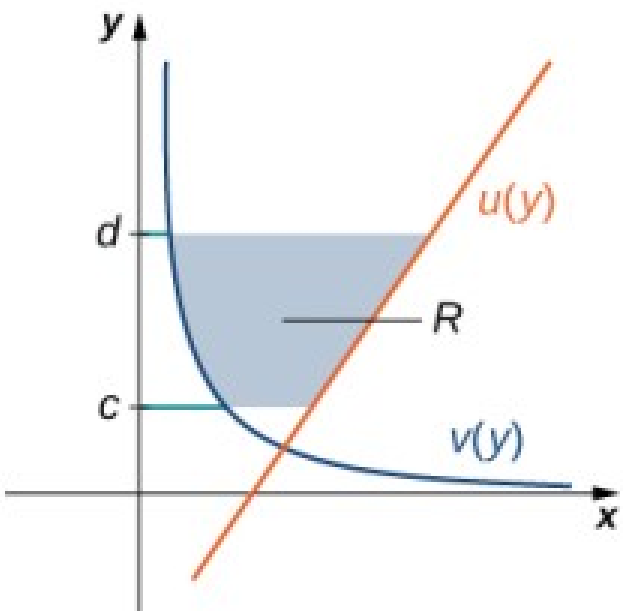


FIGURE 3.4

horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in$

$[y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. The following figure (a) shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$ The following figure (b) shows a representative rectangle in detail.

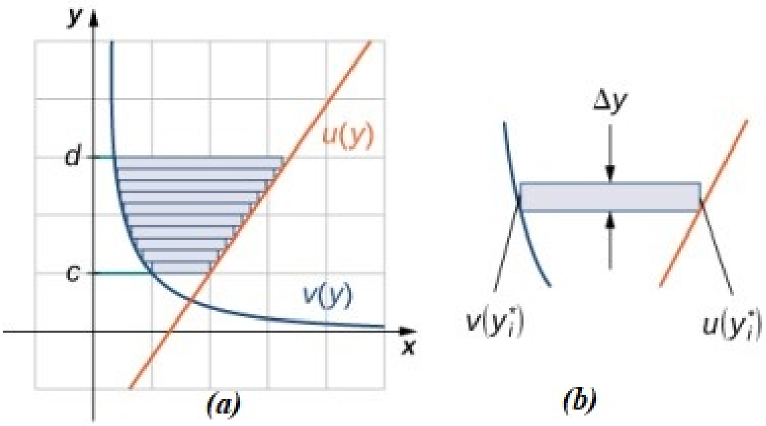


FIGURE 3.5

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y = \int_c^d [u(y) - v(y)]dy$$

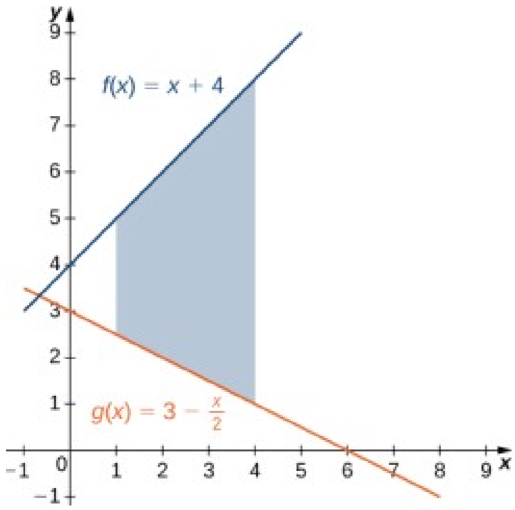
These findings are summarized in the following theorem.

Theorem 3.2. *Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of R is given by*

$$A = \int_c^d [u(y) - v(y)]dy$$

Example 3.4.1. *If R is the region bounded above by the graph of the function $f(x) = x + 4$ and by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of R*

Solution:
The region is depicted in the following figure. The area given by



$$
\begin{aligned}
A &= \int_a^b [f(x) - g(x)]dx \\
&= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right] dx = \int_1^4 \left[\frac{3x}{2} + 1\right] dx \\
&= \left[\frac{3x^2}{4} + x\right]\Bigg|_1^4 = \left(16 - \frac{7}{4}\right) = \frac{57}{4}
\end{aligned}
$$

Example 3.4.2. *If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R*

Solution:*The region is depicted in the following figure. We first need to compute where the graphs of the functions intersect. Setting f (x) = g(x), we get*

$$
\begin{aligned}
f(x) &= g(x) \\
9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\
9 - \frac{x^2}{4} &= 6 - x \\
36 - x^2 &= 24 - 4x \\
x^2 - 4x - 12 &= 0 \\
(x-6)(x+2) &= 0
\end{aligned}
$$

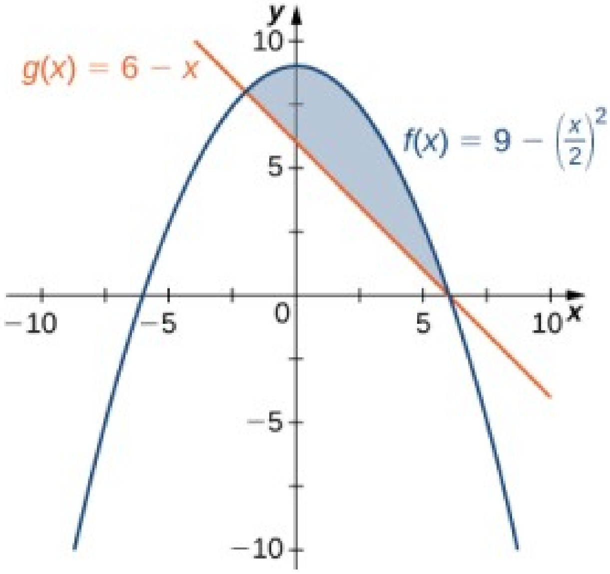


FIGURE 3.6

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain The area given by

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{6} \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right] dx = \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{64}{3} \end{aligned}$$

Example 3.4.3. *If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$ find the area of region R.*

Solution:*The region is depicted in the following figure. The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\quad \cos x \geq \sin x$, so*

$$[f(x) - g(x)] = |\sin x - \cos x| = \cos x - \sin x$$

On the other hand, for $x \in [\pi/4, \pi]$, $\quad \sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x$$

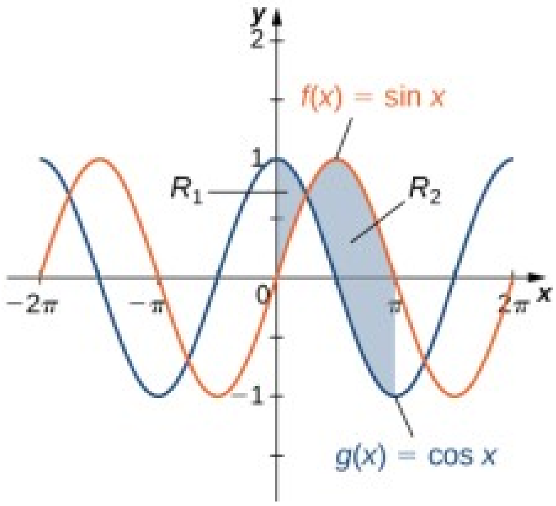


FIGURE 3.7

Then

$$
\begin{aligned}
A &= \int_a^b |f(x) - g(x)| dx \\
&= \int_0^{\pi} |\sin x - \cos x| dx \\
&= \int_0^{x/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\
&= [\sin x + \cos x]|_0^{x/4} + [-\cos x - \sin x]|_{x/4}^{x} \\
&= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}
\end{aligned}
$$

Example 3.4.4. *Find the area of the region R enclosed by the curves* $y = x^3$ *and* $y = x^2$

Solution: *To find the points where the curves intersect, solve the equations simultaneously as follows:*

$$
\begin{aligned}
x^3 &= x^2 \\
x^3 - x^2 &= 0 \\
x^2(x - 1) &= 0 \\
x &= 0, 1
\end{aligned}
$$

The corresponding points $(0, 0)$ *and* $(1, 1)$ *are the only points of intersection. The region R enclosed by the two curves is bounded above by* $y = x^2$ *and below by* $y = x^3$*, over the interval* $0 \leq x \leq 1$*. The*

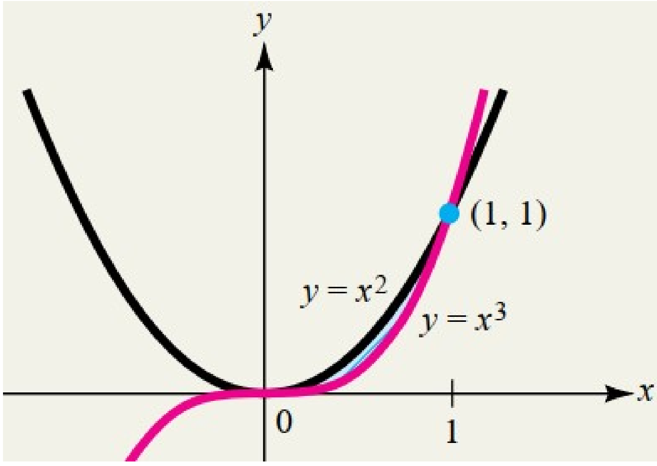


FIGURE 3.8

area of this region is given by the integral

$$\begin{aligned} A &= \int_0^1 \left(x^2 - x^3\right) dx \\ &= \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 \\ &= \left[\frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right] - \left[\frac{1}{3}(0)^3 - \frac{1}{4}(0)^4\right] \\ &= \frac{1}{12} \end{aligned}$$

Example 3.4.5. *Find the area of the region R is the region bounded by the curves* $y = x^2$ $y = -x^2$*, and the line* $x = 1$

Solution: *To find the points where the curves intersect, solve the equations simultaneously as follows:*

$$\begin{aligned} x^2 &= -x^2 \\ 2x^2 &= 0 \\ x &= 0 \end{aligned}$$

the intersection points are $(0, 0)$*, and* $f(x) \geq g(x)$ *on* $[0, 1]$ *The area of this region is given by the*

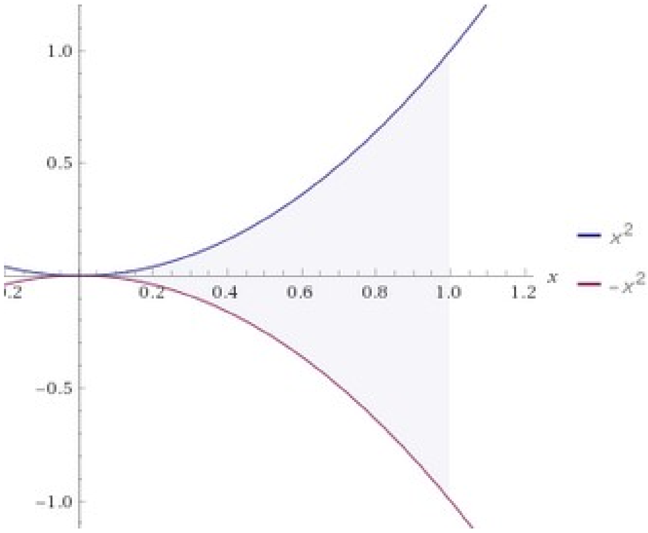


FIGURE 3.9

integral

$$\begin{aligned} Area &= \int_0^1 [f(x) - g(x)]dx \\ &= \int_0^1 \left[x^2 + x^2\right] dx \\ &= \frac{2}{3}x^3\Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

Example 3.4.6. *Find the area of the region R is the region bounded by the curves $f(x) = x^2, \quad g(x) = 2 - x$ and $x-$axis*

Solution: *The intersection points are $(0,0), (1,1)$ and $(2,1)$ Over the interval $[0,1]$, the region is bounded above by $f(x) = x^2$ and below by the x -axis, so we have*

$$\begin{aligned} A_1 &= \int_0^1 x^2 dx \\ &= \frac{x^3}{3}\Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

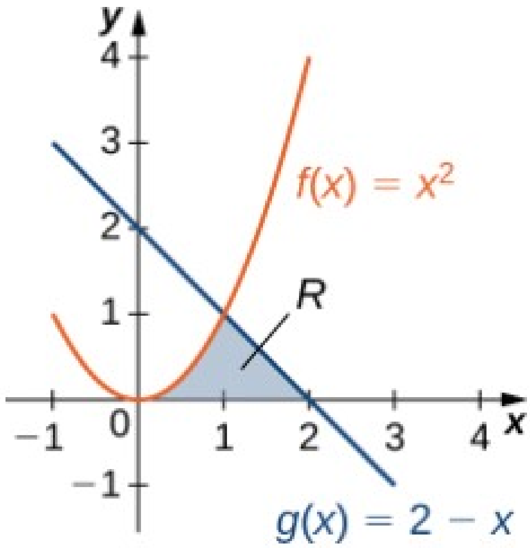


FIGURE 3.10

Over the interval $[1, 2]$*, the region is bounded above by* $g(x) = 2 - x$ *and below by the* x *-axis, so we have*

$$\begin{aligned} A_2 &= \int_1^2 (2 - x)dx \\ &= \left[2x - \frac{x^2}{2}\right]\Big|_1^2 \\ &= \frac{1}{2} \end{aligned}$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Remark 3.3. *It is possible to solve the previous example by integrating with respect to* y

Solution: We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x -axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$. Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y) dy \\ &= \int_0^1 [(2-y) - \sqrt{y}] dy \\ &= \left[2y - \frac{y^2}{2} - \frac{2}{3} y^{3/2} \right] \Bigg|_0^1 \\ &= \frac{5}{6} \end{aligned}$$

Example 3.4.7. *Find the area of the region R is the region bounded by the curves $y = 4 - x^2, \quad y = x^2 - 4$*

Solution: *We will find where the curve intersect and it is calculated when $f(x) = g(x)$.*

$$\begin{aligned} f(x) &= g(x) \\ 4 - x^2 &= x^2 - 4 \\ 2\left(x^2 - 4\right) &= 0 \\ (x+2)(x-2) &= 0 \end{aligned}$$

Hence $x = 2$ and $x = -2$ are point of intersection. First we must determine which curve is on the top. We get that $f(x)$ lies on the top of $g(x)$ implies $f(x) \geq g(x)$ on $[-2, 2]$. The area given by

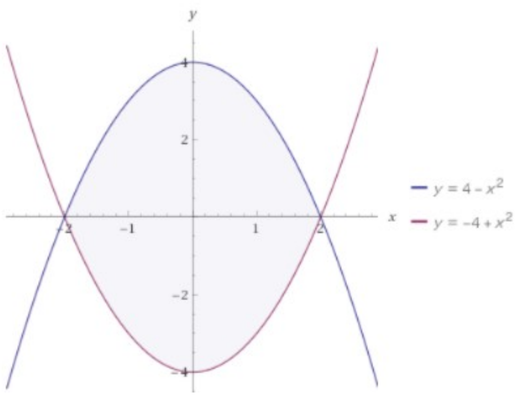


FIGURE 3.11

$$\begin{aligned} A &= \int_{-2}^{2} (f(x) - g(x))dx \\ &= \int_{-2}^{2} \left(\left(4 - x^2\right) - \left(x^2 - 4\right) \right) dx \\ &= \int_{-2}^{2} \left(\left(4 - x^2\right) - \left(x^2 - 4\right) \right) dx \\ &= \int_{-2}^{2} \left(8 - 2x^2\right) dx \\ &= 2\int_{0}^{2} \left(8 - 2x^2\right) dx \\ &= 2\left[8x - \frac{2}{3}x^3\right]_0^2 \\ &= \frac{64}{3} \end{aligned}$$

Example 3.4.8. *Find the area of the region R is the region bounded by* $y = 8 - 3x, \quad y = 6 - x, \quad y = 2$ ***Solution:*** *We solve this example by integration with respect to y,*

$$\begin{aligned} f(y) &= g(y) \\ \frac{1}{3}(8 - y) &= 6 - y \\ 2y &= 10 \end{aligned}$$

then intersection point is $(1, 5)$ *and* $6 - y \geq \frac{1}{3}(8 - y)$ *for* $2 \leq y \leq 5$. *Hence area given by*

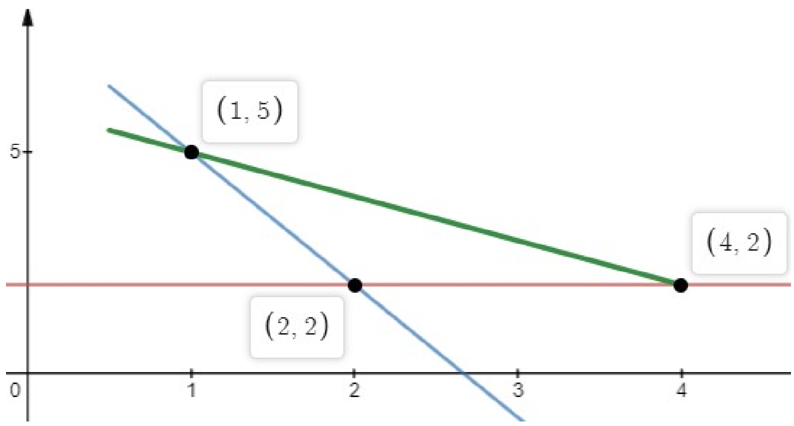


FIGURE 3.12

$$
\begin{aligned}
A &= \int_2^5 \left[6 - y - \frac{8}{3} + \frac{1}{3}y\right] dy \\
&= \int_2^5 \left[\frac{10}{3} - \frac{2}{3}y\right] dx \\
&= \left[\frac{10}{3}y - \frac{1}{3}y^2\right]_2^5 \\
&= \left[\frac{50}{3} - \frac{25}{3} - \frac{20}{3} + \frac{4}{3}\right] \\
&= 3
\end{aligned}
$$

Example 3.4.9. *Find the area of the region R is the region bounded by R is the region between the curve $y = x^3$ and the line $y = 9x$, for $x \geq 0$.*

Solution: *since we have two curves $f(x) = 9x$, and $g(x) = x^3$, then*

$$
\begin{aligned}
x^3 &= 9x \\
x^3 - 9x &= 0 \\
x(x-3)(x+3) &= 0
\end{aligned}
$$

The intersection points are $(0,0), (3,27), (-3,-27)$, we reject $x = -3$ and

$$f(x) \geq g(x) \text{ on } [0,3]$$

The area given by

$$
\begin{aligned}
\text{Area} &= \int_0^3 [f(x) - g(x)]dx \\
&= \int_0^3 \left[9x - x^3\right] dx \\
&= \frac{9}{2}x^2 - \frac{1}{4}x^4\Big|_0^3 \\
&= \frac{81}{2} - \frac{81}{4} \\
&= \frac{81}{4}
\end{aligned}
$$

3.4.1 Exercises

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $x-$axis.

1. $y = x^2$ and $y = -x^2 + 18x$

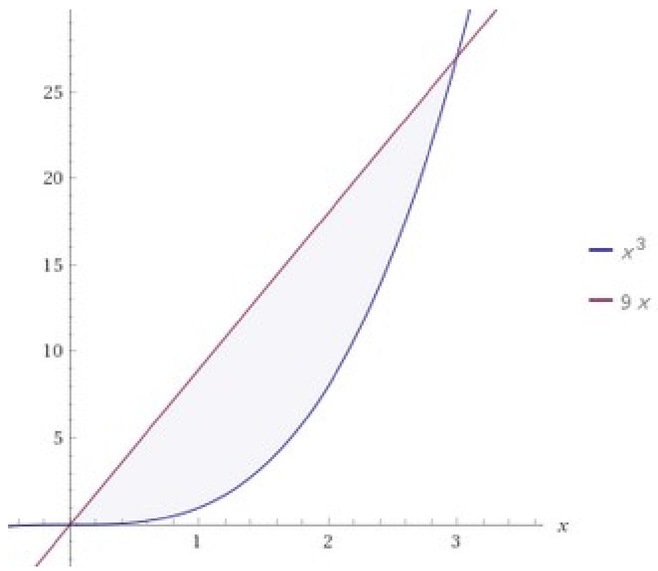


FIGURE 3.13

2. $y = \frac{1}{x}, y = \frac{1}{x^2}$, and $x = 3$

3. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

4. $y = e^x, y = e^{2x-1}$, and $x = 0$

5. $y = e^x, y = e^{-x}, x = -1$ and $x = 1$

For the following exercises, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

1. $y = \sin(\pi x), y = 2x$, and $x > 0$

2. $y = 12 - x, y = \sqrt{x}$, and $y = 1$

3. $y = \sin x$ and $y = \cos x$ over $x = [-\pi, \pi]$

4. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1, 1]$

5. $y = x^2 + 9$ and $y = 10 + 2x$ over $x = [-1, 3]$

6. $y = x^3 + 3x$ and $y = 4x$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $y-$axis

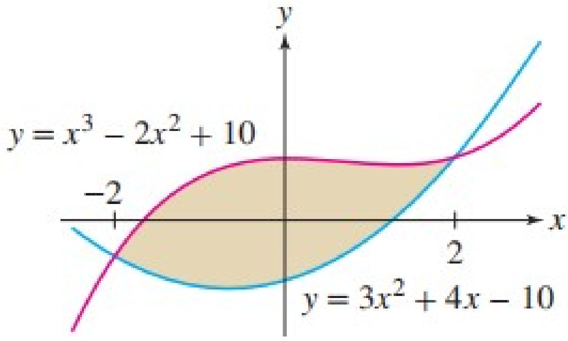
1. $x = y^3$ and $x = 3y - 2$

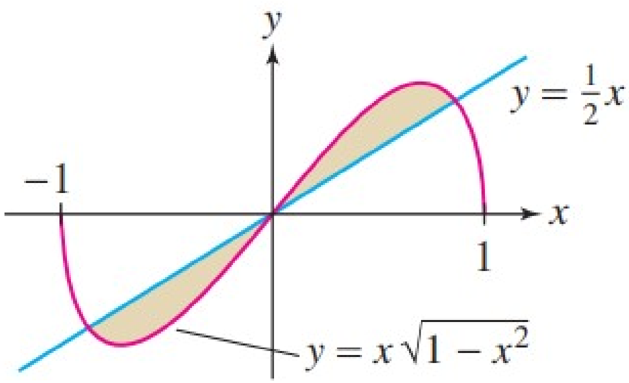
2. $x = 2y$ and $x = y^3 - y$

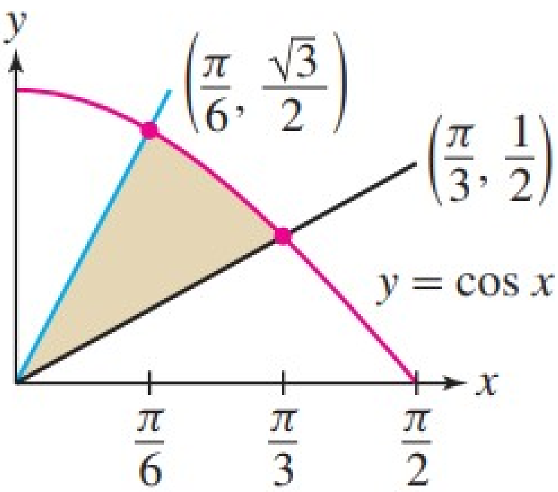
3. $x = -3 + y^2$ and $x = y - y^2$

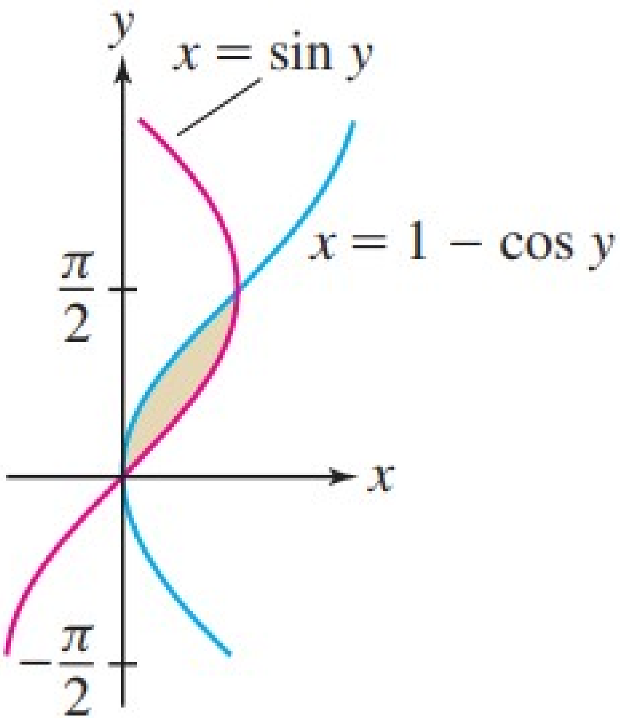
4. $y^2 = x$ and $x = y + 2$

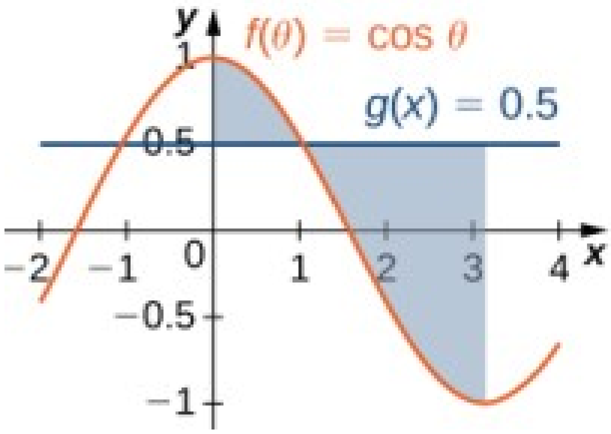
Find the area of the shaded region in Figure

1. 

2. 

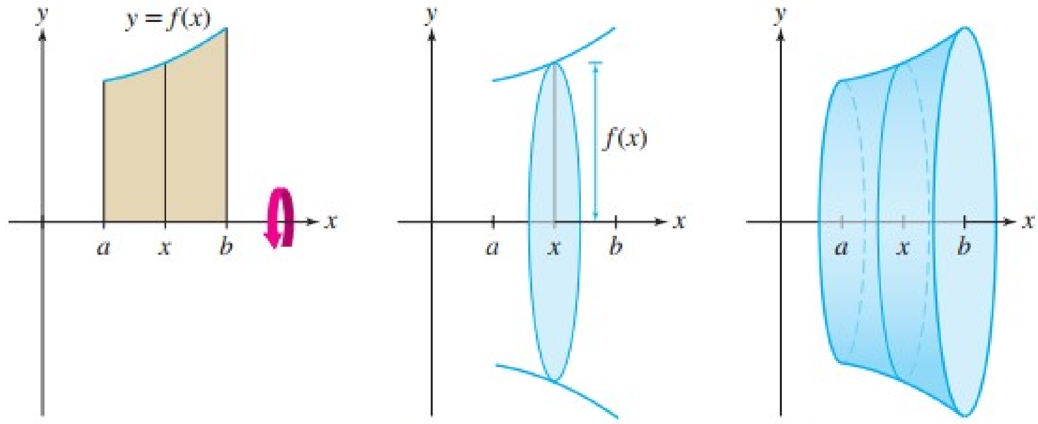
3. 

4. 

5. 

3.5 Volume of a Solid of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a solid of revolution, as shown in the following figure.



(A) (B) Cross section is a circle of radius $f(x)$. (C) Solid of revolution

Suppose that $f(x) \geq 0$ for $a \leq x \leq b$. The solid obtained by rotating the region under the graph about the x -axis has a special feature: All vertical cross sections are circles . In fact, the vertical cross section at location x is a circle of radius $R = f(x)$ and thus

$$\text{Area of the vertical cross section } = \pi R^2 = \pi[f(x)]^2$$

The total volume V is equal to the integral of cross-sectional area. Therefore,

$$V = \int_a^b \pi[f(x)]^2 dx$$

3.5.1 Disk Method

When we use the slicing method with solids of revolution, it is often called the disk method because, for solids of revolution, the slices used to over approximate the volume of the solid are disks.

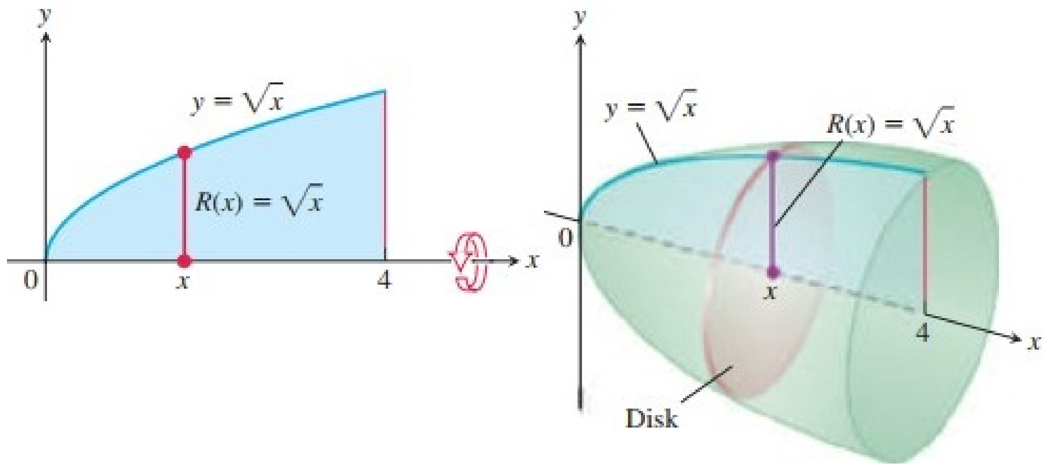
Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2 dx$$

This method for calculating the volume of a solid of revolution is often called the disk method because a cross-section is a circular disk of radius $R(x)$.

Example 3.5.1. *The region between the curve $y = \sqrt{x}, 0 \leq x \leq 4$, and the x -axis is revolved about the x -axis. Find its volume*

Solution:*We draw figures showing the region, a typical radius, and the generated solid.*



The volume is

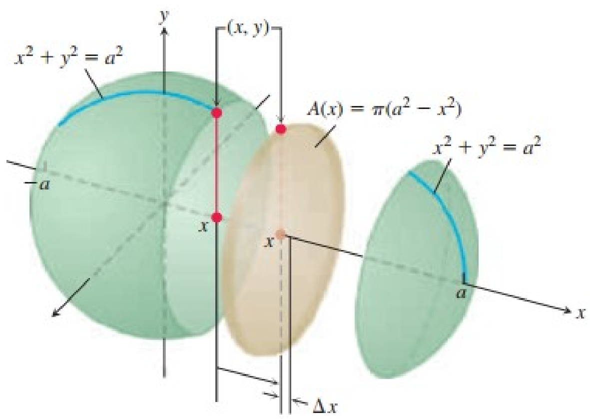
$$
\begin{aligned}
V &= \int_a^b \pi[R(x)]^2 dx \\
&= \int_0^4 \pi[\sqrt{x}]^2 dx \\
&= \pi \int_0^4 x dx \\
&= \pi \frac{x^2}{2}\bigg|_0^4 = \pi \frac{(4)^2}{2} = 8\pi
\end{aligned}
$$

Example 3.5.2. *The circle*

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

Solution: *We imagine the sphere cut into thin slices by planes perpendicular to the $x-$axis*



The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi\left(a^2 - x^2\right)$$

Therefore, the volume is

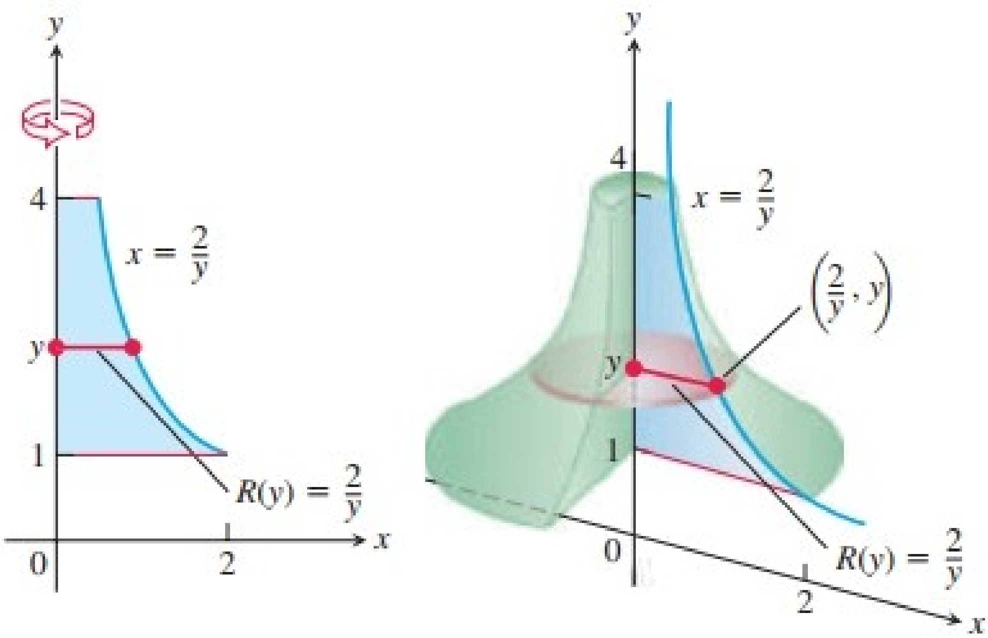
$$\begin{aligned} V &= \int_{-a}^{a} A(x)dx \\ &= \int_{-a}^{a} \pi\left(a^2 - x^2\right)dx \\ &= \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} \\ &= \frac{4}{3}\pi a^3 \end{aligned}$$

Volume by Disks for Rotation About the y-axis

$$V = \int_c^d A(y)dy = \int_c^d \pi[R(y)]^2 dy$$

Example 3.5.3. *Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = 2/y, 1 \leq y \leq 4$, about the y -axis.*

Solution: *We draw figures showing the region, a typical radius, and the generated solid*

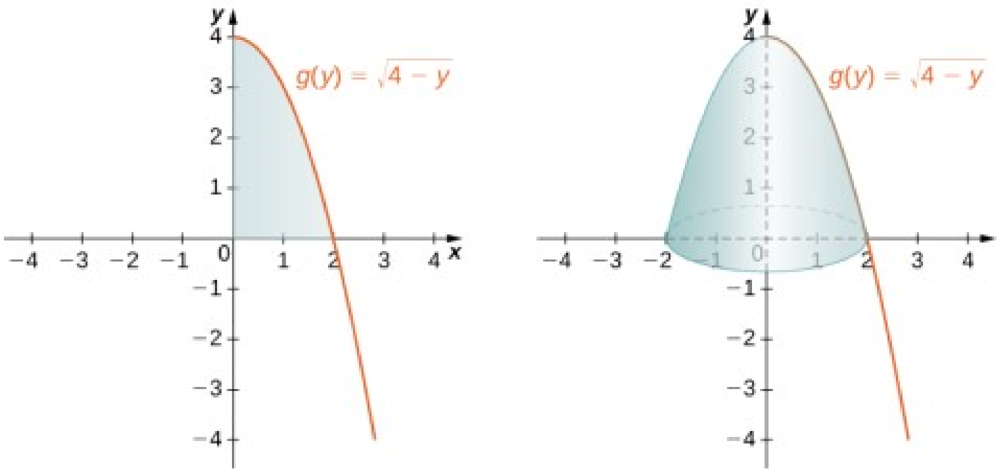


The volume is

$$
\begin{aligned}
V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2 dy \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[2 - y^2\right]^2 dy \\
&= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[4 - 4y^2 + y^4\right] dy \\
&= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \frac{64\pi\sqrt{2}}{15}
\end{aligned}
$$

Example 3.5.4. *Let R be the region bounded by the graph of $g(y) = \sqrt{4-y}$ and the y -axis over the y -axis interval $[0,4]$ Use the disk method to find the volume of the solid of revolution generated by rotating R around the y -axis.*

Solution: *Shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y-axis, the disks are horizontal, rather than vertical.*



To find the volume, we integrate with respect to y. We obtain

$$\begin{aligned} V &= \int_c^d \pi[g(y)]^2 dy \\ &= \int_0^4 \pi[\sqrt{4-y}]^2 dy \\ &= \pi \int_0^4 (4-y) dy \\ &= \pi \left[4y - \frac{y^2}{2}\right]\Bigg|_0^4 = 8\pi \end{aligned}$$

3.5.2 The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the $x-$axis or $y-$axis is selected.

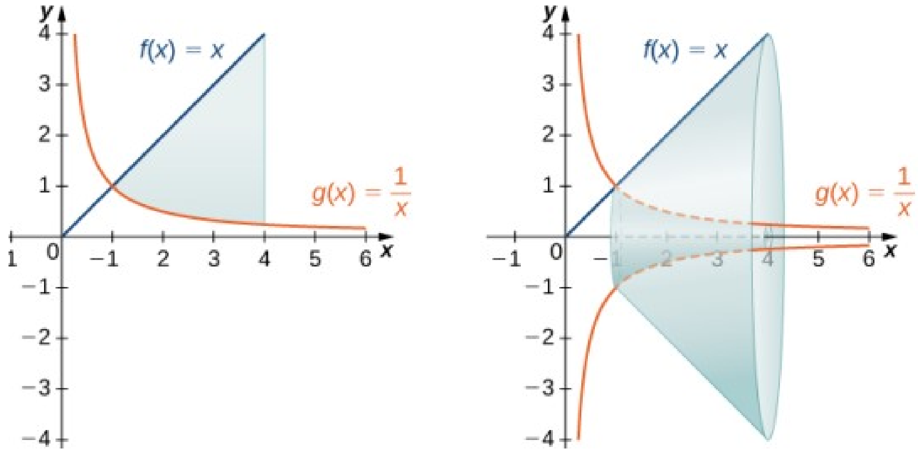
When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center)

The Washer Method Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x -axis given by

$$V = \int_a^b \pi[f(x)]^2 - (g(x))^2 dx$$

Example 3.5.5. *Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x -axis.*

Solution: *The graphs of the functions and the solid of revolution are shown in the following figure*

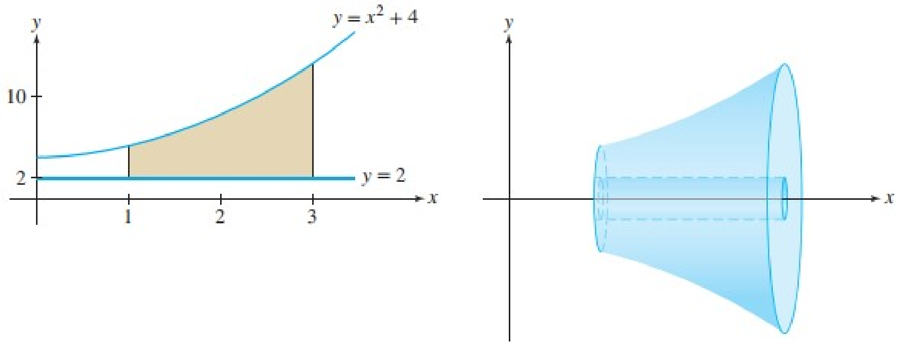


The volume given by

$$\begin{aligned} V &= \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx \\ &= \pi \int_1^4 \left[x^2 - \left(\frac{1}{x} \right)^2 \right] dx \\ &= \pi \left[\frac{x^3}{3} + \frac{1}{x} \right] \Bigg|_1^4 = \frac{81\pi}{4} \text{ units}^3 \end{aligned}$$

Example 3.5.6. *Find the volume V obtained by revolving the region between $y = x^2 + 4$ and $y = 2$ about the x -axis for $1 \leq x \leq 3$*

Solution: *The graph of $y = x^2 + 4$ lies above the graph of $y = 2$. Therefore, $R_{outer} = x^2 + 4$ and $R_{inner} = 2$.*



The volume given by

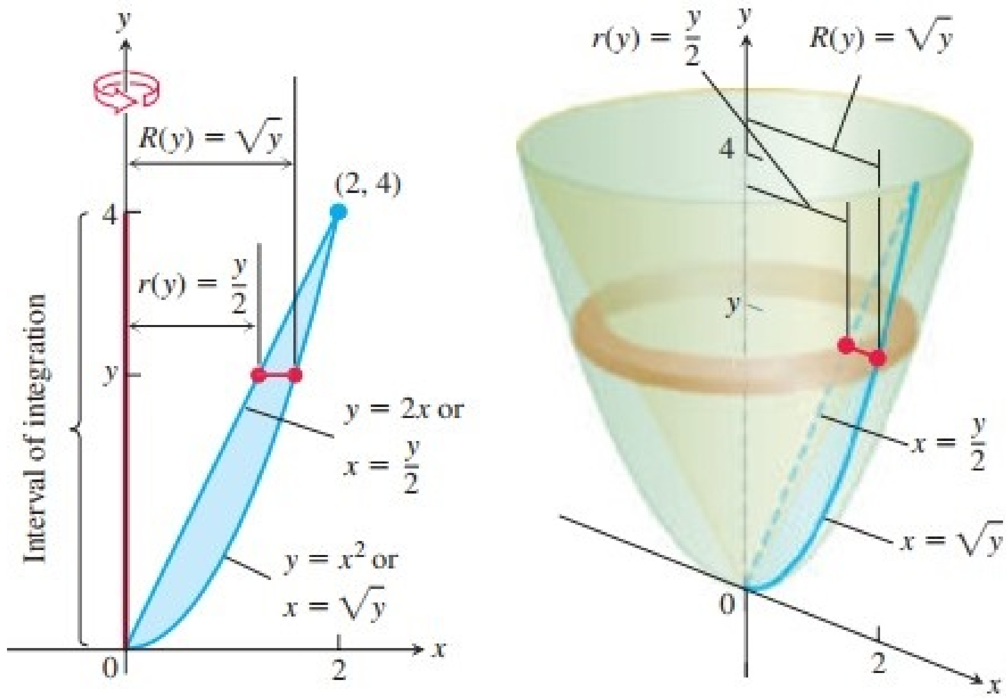
$$
\begin{aligned}
V &= \pi \int_1^3 \left(R_{outer}^2 - R_{inner}^2\right) dx \\
&= \pi \int_1^3 \left(\left(x^2+4\right)^2 - 2^2\right) dx \\
&= \pi \int_1^3 \left(x^4 + 8x^2 + 12\right) dx \\
&= \pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x\right)\Big|_1^3 = \frac{2126}{15}\pi
\end{aligned}
$$

The Washer Method Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$ and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y -axis is given by

$$V = \int_c^d \pi \left[(u(y))^2 - (v(y))^2\right] dy$$

Example 3.5.7. *The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.*

Solution: *First we sketch the region and draw a line segment across it perpendicular to the axis of revolution*

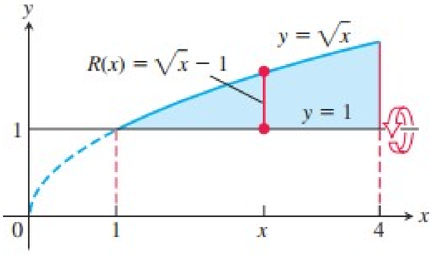


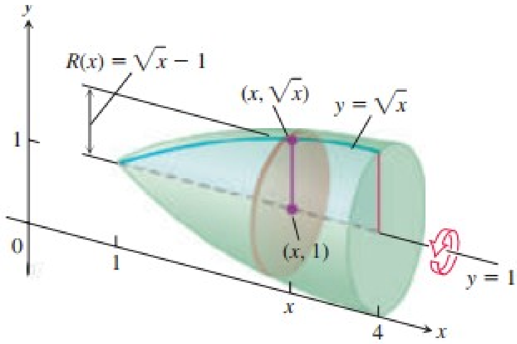
The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$
\begin{aligned}
V &= \int_c^d \pi \left([R(y)]^2 - [r(y)]^2\right) dy \\
&= \int_0^4 \pi \left([\sqrt{y}]^2 - \left[\frac{y}{2}\right]^2\right) dy \\
&= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi
\end{aligned}
$$

Example 3.5.8. *Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1, x = 4$ about the line $y = 1$.*

Solution: *We draw figures showing the region, a typical radius, and the generated solid*



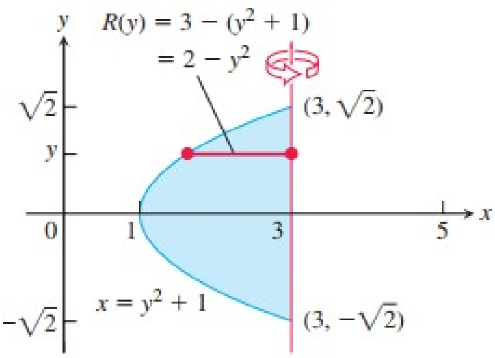


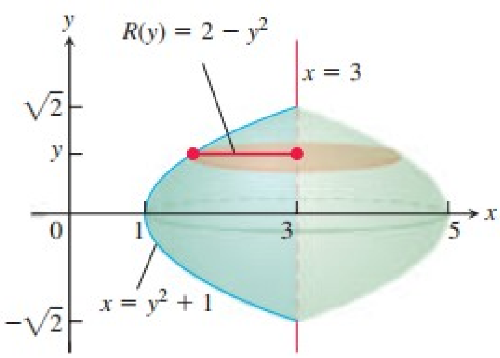
The volume is

$$
\begin{aligned}
V &= \int_1^4 \pi[R(x)]^2 dx \\
&= \int_1^4 \pi[\sqrt{x} - 1]^2 dx \\
&= \pi \int_1^4 [x - 2\sqrt{x} + 1] dx \\
&= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}
\end{aligned}
$$

Example 3.5.9. *Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.*

Solution: *We draw figures showing the region, a typical radius, and the generated solid . Note that the cross-sections are perpendicular to the line $x = 3$ and have y -coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$.*





The volume is

$$\begin{aligned}
V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2 dy \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[2 - y^2\right]^2 dy \\
&= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[4 - 4y^2 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} dy \\
&= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \frac{64\pi\sqrt{2}}{15}
\end{aligned}$$

3.5.3 Exercises

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the $x-$axis.

1- $x + y = 8, x = 0$, and $y = 0$
2- $y = 2x^2, x = 0, x = 4$, and $y = 0$
3- $y = e^x + 1, x = 0, x = 1$, and $y = 0$
4- $y = x^4, x = 0$, and $y = 1$
5- $y = \sqrt{x}, x = 0, x = 4$, and $y = 0$
6- $y = \sqrt{x}, x = 0, x = 4$, and $y = 0$
7- $y = \sin x, y = \cos x$, and $x = 0$
8- $y = \frac{1}{x}, x = 2$, and $y = 3$
9- $x^2 - y^2 = 9$ and $x + y = 9, y = 0$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the $y-$axis.

1- $y = 4 - \frac{1}{2}x, x = 0$, and $y = 0$
2- $y = 2x^3, x = 0, x = 1$, and $y = 0$
3- $y = 3x^2, x = 0$, and $y = 3$
4- $y = \sqrt{4 - x^2}, y = 0$, and $x = 0$
5- $y = \frac{1}{\sqrt{x+1}}, x = 0$, and $x = 3$

3.6 Arc Length and Surface Area

We know what is meant by the length of a straight-line segment, but without calculus, we have no precise definition of the length of a general winding curve. If the curve is the graph of a continuous function defined over an interval, then we can find the length of the curve using a procedure similar to that we used for defining the area between the curve and the $x-$axis. This procedure results in a division of the curve from point A to point B into many pieces and joining successive points of division by straight-line segments. We then sum the lengths of all these line segments and define the length of the curve to be the limiting value of this sum as the number of segments goes to infinity

3.6.1 Arc Length

We define the value of this limiting integral to be the length of the curve

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length } = \int_a^b \sqrt{1+[f'(x)]^2}dx$$

Example 3.6.1. *Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.*

Solution: *We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is*

$$\begin{aligned} \textit{Arc Length } &= \int_a^b \sqrt{1+[f'(x)]^2}dx \\ &= \int_0^1 \sqrt{1+9x}dx \\ &= \frac{1}{9}\int_0^1 9\sqrt{1+9x}dx \\ &= \frac{2}{27}(1+9x)^{3/2}\Big|_0^1 \\ &= \frac{2}{27}[10\sqrt{10}-1] \approx 2.268 \textit{ units.} \end{aligned}$$

Example 3.6.2. *Find the length of the graph of*

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4$$

Solution: *Since*

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

Then

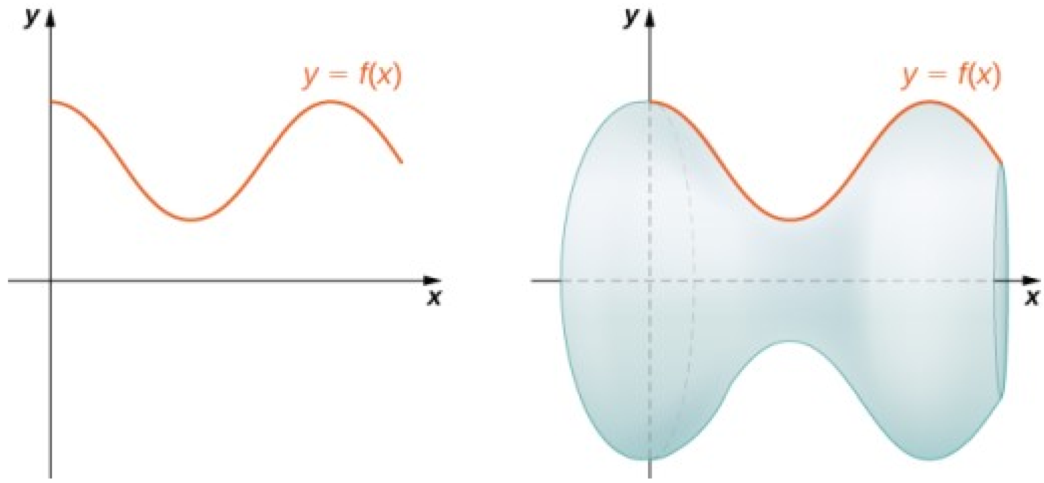
$$
\begin{aligned}
1+\left[f'(x)\right]^2 &= 1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2 \\
&= 1+\left(\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\right) \\
&= \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4} \\
&= \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1+[f'(x)]^2}dx \\
&= \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right) dx \\
&= \left[\frac{x^3}{12}-\frac{1}{x}\right]_1^4 \\
&= \left(\frac{64}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-1\right) \\
&= \frac{72}{12}=6
\end{aligned}
$$

3.6.2 Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. Surface area is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x -axis as shown in the following figure.



Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x -axis is given by

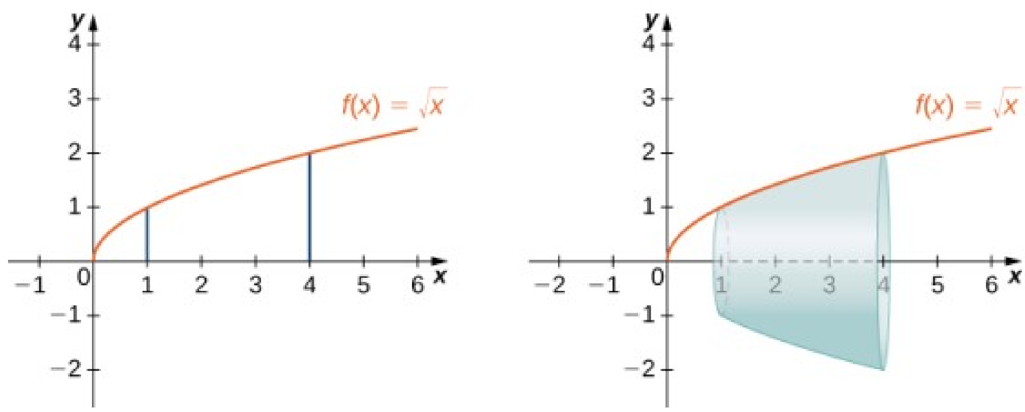
$$\text{Surface Area } = \int_a^b (2\pi f(x)\sqrt{1 + (f'(x))^2})dx$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y -axis is given by

$$\text{Surface Area } = \int_c^d (2\pi g(y)\sqrt{1 + (g'(y))^2})dy$$

Example 3.6.3. *Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x -axis. Round the answer to three decimal places.*

Solution: *The graph of $f(x)$ and the surface of rotation are shown in the following figure*

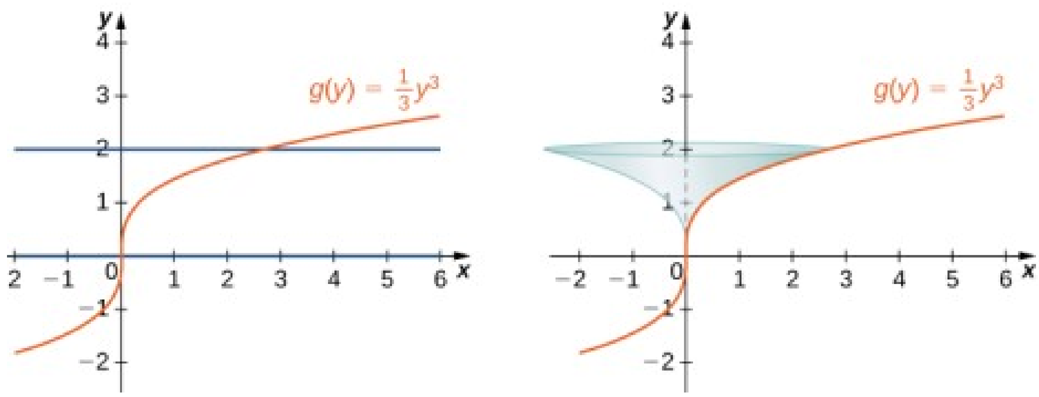


We have $f(x) = \sqrt{x}$. *Then,* $f'(x) = 1/(2\sqrt{x})$ *and* $(f'(x))^2 = 1/(4x)$. *Then,*

$$\begin{aligned}
\textit{Surface Area} &= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})dx \\
&= \int_1^4 (2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}})dx \\
&= \int_1^4 (2\pi\sqrt{x+\frac{1}{4}})dx \\
&= \frac{4}{3}\pi\left(x+\frac{1}{4}\right)^{3/2}\Bigg|_1^4 \\
&= \frac{\pi}{6}[17\sqrt{17}-5\sqrt{5}] \approx 30.846
\end{aligned}$$

Example 3.6.4. *Let* $f(x) = y = \sqrt[3]{3x}$. *Consider the portion of the curve where* $0 \leq y \leq 2$. *Find the surface area of the surface generated by revolving the graph of* $f(x)$ *around the* y *-axis.*

Solution: *Notice that we are revolving the curve around the* y *-axis, and the interval is in terms of* y*, so we want to rewrite the function as a function of* y. *We get* $x = g(y) = (1/3)y^3$. *The graph of* $g(y)$ *and the surface of rotation are shown in the following figure.*

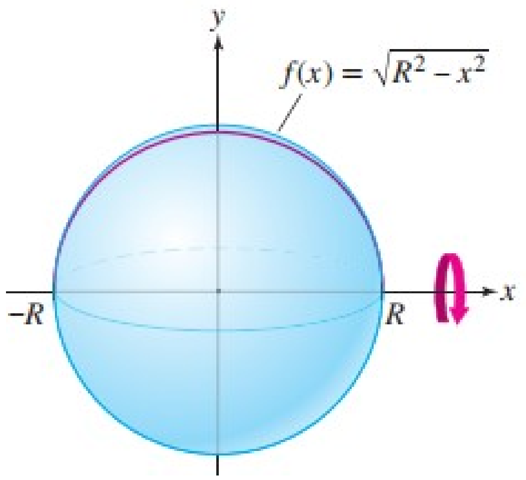


We have $g(y) = (1/3)y^3$, *so* $g'(y) = y^2$ *and* $(g'(y))^2 = y^4$. *Then*

$$\begin{aligned}
\textit{Surface Area} &= \int_c^d (2\pi g(y)\sqrt{1+(g'(y))^2})dy \\
&= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\right)dy \\
&= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right)dy \\
&= \frac{\pi}{9}\left(1+y^4\right)^{3/2}\Big|_0^2 \\
&= \frac{\pi}{9}\left[(17)^{3/2}-1\right] \approx 24.118
\end{aligned}$$

Example 3.6.5. *Calculate the surface area of a sphere of radius R.*

Solution: *The graph of* $f(x) = \sqrt{R^2-x^2}$ *is a semicircle of radius* R . *We obtain a sphere by rotating it about the* x *-axis. We have*



$$\begin{aligned}
f'(x) &= -\frac{x}{\sqrt{R^2-x^2}} \\
1+f'(x)^2 &= 1+\frac{x^2}{R^2-x^2} \\
&= \frac{R^2}{R^2-x^2}
\end{aligned}$$

The surface area integral gives us the usual formula for the surface area of a sphere:

$$
\begin{aligned}
S &= 2\pi \int_{-R}^{R} f(x)\sqrt{1+f'(x)^2}dx \\
&= 2\pi \int_{-R}^{R} \sqrt{R^2-x^2}\frac{R}{\sqrt{R^2-x^2}}dx \\
&= 2\pi R \int_{-R}^{R} dx \\
&= 2\pi R(2R) = 4\pi R^2
\end{aligned}
$$

3.6.3 Exercises

Calculate the arc length over the given interval

1- $y = 3x+1, \quad [0,3]$ **2-** $y = 9-3x, \quad [1,3]$
3- $y = x^{3/2}, \quad [1,2]$ **4-** $y = \frac{1}{3}x^{3/2} - x^{1/2}, \quad [2,8]$
5- $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad [1,2e]$ **6-** $y = \ln(\cos x), \quad \left[0, \frac{\pi}{4}\right]$

Compute the surface area of revolution about the $x-$axis over the interval.

1- $y = x, \quad [0,4]$ **2-** $y = 4x+3, \quad [0,1]$
3- $y = x^3, \quad [0,2]$ **4-** $y = x^2, \quad [0,4]$
5- $y = \left(4 - x^{2/3}\right)^{3/2}, \quad [0,8]$ **6-** $y = e^{-x}, \quad [0,1]$

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