→ $e_1 = \cos(\theta)\, \underline{i} + \sin(\theta)\, \underline{j}$

→ $e_2 = -\sin(\theta)\, \underline{i} + \cos(\theta)\, \underline{j}$

$$\boxed{\frac{d\theta}{dt} = \dot{\theta}}$$

$\therefore \dfrac{de_1}{dt} = -\dot{\theta}\sin(\theta)\, \underline{i} + \dot{\theta}\cos(\theta)\, \underline{j}$

$$\boxed{\therefore \frac{de_1}{dt} = \dot{\theta} e_2}$$

$\therefore \dfrac{de_2}{dt} = -\dot{\theta}\cos(\theta)\, \underline{i} - \dot{\theta}\sin(\theta)\, \underline{j}$

$$\boxed{\therefore \frac{de_2}{dt} = -\dot{\theta} e_1}$$

$\because \vec{r} = r e_1$ ← position vector

$\therefore \vec{v} = \dfrac{dr}{dt} = \dot{r} e_1 + r\dot{\theta} e_2$

$$\boxed{\therefore \vec{v} = \dot{r} e_1 + r\dot{\theta} e_2}$$

$\therefore \vec{f} = \dfrac{dv}{dt} = \ddot{r} e_1 + \dot{r}\dot{\theta} e_2 + (\dot{r}\dot{\theta} + r\ddot{\theta}) e_2 - r\dot{\theta}^2 e_1$

$$\boxed{\therefore \vec{F} = (\ddot{r} - r\dot{\theta}^2) e_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta}) e_2}$$

ex(1):

If the radial velocity component $(\lambda r, \mu\theta)$ find trajectory and prove that component of acceleration is $\left(\lambda^2 r - \dfrac{\mu^2\theta^2}{r}, \mu\theta\left(\lambda + \dfrac{\mu}{r}\right)\right)$

Solution:

$\therefore \dot{r} = \lambda r$ → $\therefore r\dot{\theta} = \mu\theta$ → $\dfrac{r\dot{\theta}}{\dot{r}} = \dfrac{\mu\theta}{\lambda r}$

$\therefore \frac{\dot{\theta}}{\theta} = \frac{m\,\dot{r}}{\lambda r^2}$

$-\frac{m}{\lambda} \ln|r| = \ln|\theta| + c$ (integration)

$\dot{\theta} = \frac{m\theta}{r}$

- $\ddot{r} = \lambda \dot{r} = \lambda^2 r \rightarrow (1)$
- $r\dot{\theta}^2 = r \cdot \left(\frac{m\theta}{r}\right)^2 = \frac{m^2\theta^2}{r} \rightarrow (2)$

- From 1, 2 :-

$\therefore F_r = \ddot{r} - r\dot{\theta}^2 = \lambda^2 r - \frac{m^2\theta^2}{r}$

- $r\ddot{\theta} = r \cdot \left(\frac{m^2\theta}{r^2}\right) \rightarrow (3)$
- $2\dot{r}\dot{\theta} = 2\lambda r \cdot \left(\frac{m\theta}{r}\right) = 2\lambda\,(m\theta) \rightarrow (4)$

- From 4, 3 :-

$\therefore F_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{m^2\theta}{r} + 2\lambda(m\theta)$

$= m\theta\left(2\lambda + \frac{m}{r}\right)$