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Limiting behavior of immanants of certain correlation matrix

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ABSTRACT

A correlation matrix is a positive semi-definite Hermitian matrix with all diagonals equal to 1. The minimum of the permanents on singular correlation matrices is conjectured to be given by the matrix Y_n, all of whose non-diagonal entries are $-1/(n-1)$. Also, Frenzen–Fischer proved that $\operatorname{per} Y_n$ approaches to $e/2$ as $n \to \infty$. In this paper, we analyze some immanants of Y_n, which are the generalizations of the determinant and the permanent, and we generalize these results to some other immanants and conjecture most of those converge to 1.

1. Introduction

Let χ be a character of a subgroup G of the symmetric group $\mathfrak{S}_n$, and $A = (a_{ij})$ an $n \times n$ complex matrix. The generalized matrix function associated with G and χ is defined to be

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

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When $G = \mathfrak{S}_n$ and χ is its irreducible character, d_χ^G is called an immanant. When $\chi = \mathrm{sgn}$, the immanant $d_{\mathrm{sgn}}^{\mathfrak{S}_n}$ is the determinant, and when χ is trivial, $d_{\mathrm{triv}}^{\mathfrak{S}_n}$ is the permanent. Since there is a one-to-one correspondence between Young diagrams of n boxes and irreducible representations of $\mathfrak{S}_n$, we simply denote the immanant associated with the character corresponding to the Young diagram λ by d_λ. We also define the normalized generalized matrix function $\overline{d}_\chi^G$ to be $\overline{d}_\chi^G = d_\chi^G/\chi(\mathrm{id})$.

A correlation matrix is a positive semi-definite Hermitian matrix with all diagonals equal to 1. The minimum of the permanents on $n \times n$ singular correlation matrices is conjectured by Pierce [7] to be given by the matrix Y_n, all of whose non-diagonal entries are $-1/(n-1)$. Motivated by this conjecture, Frenzen–Fischer [2] showed that the sequence $\{\operatorname{per} Y_n\}$ is monotonically decreasing for $n \geq 2$ and

$$\lim_{n\to\infty} \operatorname{per} Y_n = \frac{e}{2}. \tag{1}$$

It is easy to see that $\det Y_n = 0$ for all $n \geq 2$. One can notice that the permanent and the determinant are the (normalized) immanants corresponding to (n) and (1^n), respectively. Thus, we shall be interested in the limiting behavior of other normalized immanants of Y_n.

In Section 2, we find the limits of the determinantal and permanental minors of Y_n, which are the key lemmas in this paper. In Section 3 and 4, the immanants associated with hook Young diagrams $(k, 1^{n-k})$ and other Young diagrams are discussed using Littlewood–Richardson's correspondence between Schur functions and immanants. Notice that the number of boxes of Young diagrams increases as $n \to \infty$, so that we describe the behavior of immanants in terms of limit shapes of diagrams. In light of the results, we conjecture that the limits of immanants for many cases converge to 1:

Conjecture 1. *Let $\{\lambda^{(n)}\}$ be a sequence of Young diagrams such that $|\lambda^{(n)}| = n$ and*

$$\lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \cdots,$$

and let $\mu^{(n)}$ be the conjugate of $\lambda^{(n)}$. If $\lim_{n\to\infty} \lambda_1^{(n)}/n = 0$ and $\lim_{n\to\infty} \mu_1^{(n)}/n = 0$, then

$$\lim_{n\to\infty} \overline{d}_{\lambda^{(n)}}(Y_n) = 1.$$

In Section 5, we point out some remarks on the permanental dominance conjecture, which can be the motivation to find the immanants of Y_n. In particular, we conjecture that Y_n maximizes $(\overline{d}_\lambda(A) - \det A)/(\operatorname{per} A - \det A)$ except for $\lambda = (n-1, 1)$.

2. Lemmas on principal minors of Y_n

Let I_n be the $n \times n$ identity matrix and J_n the $n \times n$ matrix with all entries equal to 1. We introduce two formulae for the determinant and the permanent for later use.

Lemma 2.

$$\det(tI_n + aJ_n) = t^{n-1}(t + na).$$

Proof. *It immediately follows from the fact that* $\mathrm{rank} J_n = 1$ *and* $\mathrm{tr} J_n = n$. □

Lemma 3 *(Frenzen–Fischer [2]).*

$$\begin{aligned}\mathrm{per}(tI_n + aJ_n) &= \int_0^\infty (t + ax)^n e^{-x} dx \\ &= n! a^n \sum_{k=0}^{n} \frac{(t/a)^k}{k!}.\end{aligned}$$

Using the lemmas above and Frenzen–Fischer's idea of the proof of (1), we will find the limits of the determinantal and permanental minors of Y_n.

Definition 4. Let $A(i|j)$ be the matrix obtained by deleting the i-th row and the j-th column of A. Let $Y_{n,k}$ denote the $k \times k$ principal submatrix of Y_n, i.e. $Y_{n,k} = n/(n-1)I_k - 1/(n-1)J_k$.

From here, we assume that $k = k(n)$ is a function of n.

Lemma 5. *Suppose that* $k = k(n)$ *and* $\lim_{n\to\infty} k/n = c$ $(0 \le c \le 1)$. *Then,*

$$\lim_{n\to\infty} \det Y_{n,k} = (1 - c)e^c.$$

Proof. It immediately follows from Lemma 2. □

Lemma 6. *Suppose that* $k = k(n)$ *and* $\lim_{n\to\infty} k/n = c$ $(0 \le c \le 1)$. *Then,*

$$\lim_{n\to\infty} \mathrm{per}\, Y_{n,k} = \frac{1}{c+1} e^c.$$

Proof. Using Lemma 3 with $t = n/(n-1)$ and $a = -1/(n-1)$ and substituting x into ny, we can write

$$\begin{aligned}\mathrm{per}\, Y_{n,k} &= \int_0^\infty \left(\frac{n - x}{n - 1}\right)^k e^{-x} dx \\ &= \int_0^\infty \left(\frac{n - ny}{n - 1}\right)^k e^{-ny} \cdot n dy \\ &= P_1 + P_2,\end{aligned}$$

where

$$P_1 = \left(\frac{n}{n-1}\right)^k \int_0^1 (1-y)^k e^{-ny} \cdot n\, dy,$$

$$P_2 = \left(\frac{-n}{n-1}\right)^k \int_1^\infty (y-1)^k e^{-ny} \cdot n\, dy.$$

It holds that $1 - t \leq e^{-t}$ for $0 \leq t < 1$, and $-k/n + 2/n - 1 \leq -1/2$ under the condition $n \geq 4$. Let $t = x/n$, and we have

$$\left(1 - \frac{x}{n}\right)^{k-2} e^{-x} \leq e^{(-\frac{k}{n}+\frac{2}{n}-1)x} \leq e^{-\frac{1}{2}x}.$$

It is obvious that $e^{-\frac{1}{2}x}$ is integrable over $[0, \infty]$. We can apply the dominated convergence theorem to find

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 (1-y)^{k-2} e^{-ny} \cdot n dy &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^{k-2} e^{-x} dx \\
&= \int_0^\infty e^{-(c+1)x} dx \\
&= \frac{1}{c+1},
\end{aligned}$$

so that

$$\lim_{n\to\infty} P_1 = \frac{e^c}{c+1}.$$

While $(-n/(n-1))^{k-2}$ in P_2 is bounded as $n \to \infty$, we have

$$\int_1^\infty (y-1)^{k-2} e^{-ny} dy = \frac{(k-2)!}{n^{k-2}} \cdot \frac{1}{n} e^{-n},$$

which tends to 0 as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} P_2 = \lim_{n\to\infty} \left(\frac{-n}{n-1}\right)^{k-2} \frac{(k-2)!}{n^{k-2}} \cdot e^{-n} = 0. \qquad \square$$

3. Hook immanants

The purpose of this section is to give a formula for the normalized hook immanant $\overline{d}_{(k,1^{n-k})}(Y_n)$.

Lemma 7. *For $2 \leq k \leq n$,*

$$\operatorname{per} Y_{n,k} = \frac{n-k}{n-1} \operatorname{per} Y_{n,k-1} + \frac{n(k-1)}{(n-1)^2} \operatorname{per} Y_{n,k-2}.$$

Proof. Expanding $\operatorname{per} Y_{n,k}$ with the k-th row and k-th column, we have

$$\operatorname{per} Y_{n,k} = \operatorname{per} Y_{n,k-1} + \left(-\frac{1}{n-1}\right)^2 \sum_{1\leq i,j\leq k-1} \operatorname{per} Y_{n,k-1}(i|j). \tag{2}$$

In order to take care of the last term of (2), using the Laplace expansion of the permanent of $Y_{n,k-1}$ along the i-th row, we write

$$\begin{aligned}\operatorname{per} Y_{n,k-1} &= \operatorname{per} Y_{n,k-1}(i|i) + \left(-\frac{1}{n-1}\right) \sum_{j\neq i} \operatorname{per} Y_{n,k-1}(i|j) \\ &= \frac{n}{n-1} \operatorname{per} Y_{n,k-1}(i|i) + \left(-\frac{1}{n-1}\right) \sum_{1\leq j\leq k-1} \operatorname{per} Y_{n,k-1}(i|j),\end{aligned}$$

hence we obtain, for each i,

$$\begin{aligned}\left(-\frac{1}{n-1}\right)^2 \sum_{1\leq j\leq k-1} \operatorname{per} Y_{n,k-1}(i|j) &= -\frac{1}{n-1} \operatorname{per} Y_{n,k-1} + \frac{n}{(n-1)^2} \operatorname{per} Y_{n,k-1}(i|i) \\ &= -\frac{1}{n-1} \operatorname{per} Y_{n,k-1} + \frac{n}{(n-1)^2} \operatorname{per} Y_{n,k-2}.\end{aligned}$$

Adding up for all $i = 1, 2, \cdots, k-1$, and substituting into (2), we have

$$\operatorname{per} Y_{n,k} = \frac{n-k}{n-1} \operatorname{per} Y_{n,k-1} + \frac{n(k-1)}{(n-1)^2} \operatorname{per} Y_{n,k-2}. \qquad \square$$

Next we give a formula for $d_{(k,1^{n-k})}(Y_n)$. To expand hook immanants, we use Littlewood–Richardson's correspondence. Littlewood–Richardson stated that "in a relation between S-functions we may replace the S-functions by the corresponding immanants if at the same time we replace the multiplication sign by a suitable sign of summation" (pp. 135 in [5]), where S-functions are also called Schur functions nowadays. Let us see an example:

Example 8 *(Littlewood–Richardson's correspondence [5]).* Let S_λ be the Schur function corresponding to the partition λ. The identity

$$S_{(k)}S_{(1^{n-k})} = S_{(k,1^{n-k})} + S_{(k+1,n-k-1)}$$

corresponds to

$$\sum_{(N_1,N_2)} \operatorname{per} A[N_1] \det A[N_2] = d_{(k,1^{n-k})}(A) + d_{(k+1,1^{n-k-1})}(A),$$

where the sum is taken over all ordered partitions (N_1, N_2) such that $|N_1| = k$, $|N_2| = n-k$, $N_1 \cup N_2 = \{1, \cdots, n\}$, and $A[N_i]$ is the $|N_i| \times |N_i|$ principal submatrix of A lying in the rows and columns of N_i.

Now we prove the following formula for the normalized hook immanants $\overline{d}_{(k,1^{n-k})}(Y_n)$ in terms of the determinantal and the permanental minors.

Theorem 9. *For* $3 \le k \le n$*,*

$$\overline{d}_{(k,1^{n-k})}(Y_n) = \frac{k-1}{k-2} \operatorname{per} Y_{n,k-2} \det Y_{n,n-k+2}.$$

Proof. We prove by induction on k. First, consider the case $k = 3$. By Example 8 and Lemma 2,

$$d_{(3,1^{n-3})}(Y_n) = (n-2)\left(\frac{n}{n-1}\right)^{n-2}.$$

Note that the dimension of the irreducible representation of $\mathfrak{S}_n$ corresponding to $(k, 1^{n-k})$ is $\binom{n-1}{k-1}$ to see

$$\overline{d}_{(3,1^{n-3})}(Y_n) = \frac{2}{n-1}\left(\frac{n}{n-1}\right)^{n-2}.$$

Using Lemma 2, the right-hand side is equal to the above. Hence, Theorem 9 holds for $k = 3$.

Next, suppose that Theorem 9 holds for k. Using Example 8, Lemma 2 and Lemma 7,

$$\begin{aligned}
&\overline{d}_{(k+1,1^{n-k-1})}(Y_n) \\
&= \frac{1}{\binom{n-1}{k}}\left\{\sum_{(N_1,N_2)} \operatorname{per} Y_n[N_1] \det Y_n[N_2] - \binom{n-1}{k-1}\overline{d}_{(k,1^{n-k})}(Y_n)\right\}
\end{aligned}$$

$$= \frac{\binom{n-1}{k-1}}{\binom{n-1}{k}} \left(\frac{n}{n-1}\right)^{n-k} \left\{ \operatorname{per} Y_{n,k} - \frac{n(k-1)}{(n-1)^2} \operatorname{per} Y_{n,k-2} \right\}$$

$$= \frac{k}{n-1} \left(\frac{n}{n-1}\right)^{n-k} \operatorname{per} Y_{n,k-1}.$$

By Lemma 2, we have

$$\det Y_{n,n-k+1} = \frac{k-1}{n-1} \left(\frac{n}{n-1}\right)^{n-k},$$

hence we prove

$$\overline{d}_{(k+1,1^{n-k-1})}(Y_n) = \frac{k}{k-1} \operatorname{per} Y_{n,k-1} \det Y_{n,n-k+1}. \qquad \square$$

We conclude this section with the following theorem.

Theorem 10. *Suppose that $k = k(n)$ and $\lim_{n\to\infty} k/n = c$ $(0 \leq c \leq 1)$. Then,*

$$\lim_{n\to\infty} \overline{d}_{(k,1^{n-k})}(Y_n) = \frac{c}{c+1} e.$$

Proof. It follows from Lemma 5, Lemma 6 and Lemma 9. $\square$

4. Other immanants and main conjecture

In Section 3, it is observed that hook immanants of Y_n as $n \to \infty$ interpolates $\det Y_n = 0$ and $\lim_{n\to\infty} \operatorname{per} Y_n = e/2$, depending on the ratio of the length of the arm and leg. In particular, one of the interesting points is the appearance of "e". We obtain a similar type of results for some other immanants in this section. Nevertheless, we will also observe that many of those converge to 1 (by canceling e) under some condition.

Let H_k be the completely symmetric functions of degree k, and let E_k be the elementary symmetric functions of degree k. Recall that Schur functions have the expressions

$$S_\lambda = \det\left(H_{\lambda_i+j-i}\right) \text{ and } S_\mu = \det\left(E_{\lambda_i+j-i}\right),$$

where μ is the conjugate Young diagram of λ.

Again, Littlewood–Richardson's correspondence can be applied to this work. We display another brief example below.

Example 11. Noting that $H_k = S_{(k)}$,

$$S_{(4,2,1)} = \begin{vmatrix} H_4 & H_5 & H_6 \\ H_1 & H_2 & H_3 \\ 0 & H_0 & H_1 \end{vmatrix}$$
$$= H_4H_2H_1 + H_6H_1 - H_4H_3 - H_5H_1H_1$$

corresponds to

$$d_{(4,2,1)}(A) = \sum_{(N_1,N_2,N_3)} \operatorname{per} A[N_1] \operatorname{per} A[N_2] \operatorname{per} A[N_3] + \sum_{(N_1,N_2)} \operatorname{per} A[N_1] \operatorname{per} A[N_2]$$
$$- \sum_{(N_1,N_2)} \operatorname{per} A[N_1] \operatorname{per} A[N_2] - \sum_{(N_1,N_2,N_3)} \operatorname{per} A[N_1] \operatorname{per} A[N_2] \operatorname{per} A[N_3].$$

If $A = Y_7$, then

$$d_{(4,2,1)}(Y_7) = \frac{7!}{4!2!1!} \operatorname{per} Y_{7,4} \operatorname{per} Y_{7,2} \operatorname{per} Y_{7,1} + \frac{7!}{6!1!} \operatorname{per} Y_{7,6} \operatorname{per} Y_{7,1}$$
$$- \frac{7!}{4!3!} \operatorname{per} Y_{7,4} \operatorname{per} Y_{7,3} - \frac{7!}{5!1!1!} \operatorname{per} Y_{7,5} \operatorname{per} Y_{7,1} \operatorname{per} Y_{7,1}.$$

Replacing H_k by the elementary symmetric functions E_k, which are equal to $S_{(1^k)}$, all the immanants turn to the ones corresponding to the conjugate Young diagrams:

$$d_{(3,2,1,1)}(Y_7) = \frac{7!}{4!2!1!} \det Y_{7,4} \det Y_{7,2} \det Y_{7,1} + \frac{7!}{6!1!} \det Y_{7,6} \det Y_{7,1}$$
$$- \frac{7!}{4!3!} \det Y_{7,4} \det Y_{7,3} - \frac{7!}{5!1!1!} \det Y_{7,5} \det Y_{7,1} \det Y_{7,1}.$$

Let us consider a sequence of Young diagrams

$$\lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \cdots$$

with $|\lambda^{(n)}| = n$, to describe the limit shape in a similar way to the last section. Suppose that $\ell \in \mathbb{N}$ is fixed, and for $1 \le i \le \ell$,

$$\lim_{n\to\infty} \frac{\lambda_i^{(n)}}{n} = b_i,$$

where $\lambda_i^{(n)}$ is the number of the boxes in i-th row of the diagram $\lambda^{(n)}$. Note that $b_1 \ge b_2 \ge \cdots \ge b_\ell \ge 0$ and $\sum_{i=1}^{\ell} b_i = 1$.

Theorem 12. *Let $\{\lambda^{(n)}\}$ be a sequence of Young diagrams such that $|\lambda^{(n)}| = n$ and*

$$\lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \cdots,$$

and let $\mu^{(n)}$ be the conjugate of $\lambda^{(n)}$. If $\lim_{n\to\infty} \lambda_i^{(n)}/n = b_i$ and $b_1 > b_2 > \cdots > b_\ell \ge 0$, then

$$\lim_{n\to\infty} \overline{d}_{\lambda^{(n)}}(Y_n) = \prod_{i=1}^{\ell} \frac{1}{1+b_i} e,$$

and

$$\lim_{n\to\infty} \overline{d}_{\mu^{(n)}}(Y_n) = \prod_{i=1}^{\ell}(1-b_i)e.$$

Proof. For the simplicity, we write $\lambda^{(n)}$ as λ. By Littlewood–Richardson's correspondence as well as Example 11, $d_\lambda(Y_n)$ is expanded in terms of permanental minors:

$$d_\lambda(Y_n) = n! \begin{vmatrix} \frac{\operatorname{per} Y_{n,\lambda_1}}{\lambda_1!} & \frac{\operatorname{per} Y_{n,\lambda_1+1}}{(\lambda_1+1)!} & \cdots & \frac{\operatorname{per} Y_{n,\lambda_1+\ell-1}}{(\lambda_1+\ell-1)!} \\ \frac{\operatorname{per} Y_{n,\lambda_2-1}}{(\lambda_2-1)!} & \frac{\operatorname{per} Y_{n,\lambda_2}}{\lambda_2!} & \cdots & \frac{\operatorname{per} Y_{n,\lambda_2+\ell-2}}{(\lambda_1+\ell-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\operatorname{per} Y_{n,\lambda_\ell-\ell+1}}{(\lambda_\ell-\ell+1)!} & \frac{\operatorname{per} Y_{n,\lambda_\ell-\ell+2}}{(\lambda_\ell-\ell+2)!} & \cdots & \frac{\operatorname{per} Y_{n,\lambda_\ell}}{\lambda_\ell!} \end{vmatrix}.$$

To normalize the immanant, recall that $\dim\lambda = n!/\prod(\text{hook length of } \lambda)$ and

$$\prod_{(i,j)\in\lambda}(\text{hook length of } \lambda)^{-1} = \begin{vmatrix} \frac{1}{\lambda_1!} & \frac{1}{(\lambda_1+1)!} & \cdots & \frac{1}{(\lambda_1+\ell-1)!} \\ \frac{1}{(\lambda_2-1)!} & \frac{1}{\lambda_2!} & \cdots & \frac{1}{(\lambda_1+\ell-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(\lambda_\ell-\ell+1)!} & \frac{1}{(\lambda_\ell-\ell+2)!} & \cdots & \frac{1}{\lambda_\ell!} \end{vmatrix}$$

(see [3], pp. 50). Therefore,

$$\overline{d}_\lambda(Y_n) = \frac{\begin{vmatrix} \frac{\operatorname{per} Y_{n,\lambda_1}}{\lambda_1!} & \frac{\operatorname{per} Y_{n,\lambda_1+1}}{(\lambda_1+1)!} & \cdots & \frac{\operatorname{per} Y_{n,\lambda_1+\ell-1}}{(\lambda_1+\ell-1)!} \\ \frac{\operatorname{per} Y_{n,\lambda_2-1}}{(\lambda_2-1)!} & \frac{\operatorname{per} Y_{n,\lambda_2}}{\lambda_2!} & \cdots & \frac{\operatorname{per} Y_{n,\lambda_2+\ell-2}}{(\lambda_1+\ell-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\operatorname{per} Y_{n,\lambda_\ell-\ell+1}}{(\lambda_\ell-\ell+1)!} & \frac{\operatorname{per} Y_{n,\lambda_\ell-\ell+2}}{(\lambda_\ell-\ell+2)!} & \cdots & \frac{\operatorname{per} Y_{n,\lambda_\ell}}{\lambda_\ell!} \end{vmatrix}}{\begin{vmatrix} \frac{1}{\lambda_1!} & \frac{1}{(\lambda_1+1)!} & \cdots & \frac{1}{(\lambda_1+\ell-1)!} \\ \frac{1}{(\lambda_2-1)!} & \frac{1}{\lambda_2!} & \cdots & \frac{1}{(\lambda_1+\ell-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(\lambda_\ell-\ell+1)!} & \frac{1}{(\lambda_\ell-\ell+2)!} & \cdots & \frac{1}{\lambda_\ell!} \end{vmatrix}}.$$

Multiplying $(\lambda_1+\ell)!(\lambda_2+\ell-1)!\cdots\lambda_\ell!/n^{1+2+\cdots+\ell-1}$ to both the denominator and the numerator, and noting that as $n\to\infty$ with $\lambda_i/n \to b_i$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \overline{d}_\lambda(Y_n) &= \frac{\det(b_i^{\ell-j})\frac{1}{1+b_1}\cdots\frac{1}{1+b_\ell}e^{b_1}\cdots e^{b_\ell}}{\det(b_i^{\ell-j})} \\ &= \prod_{i=1}^{\ell}\left(\frac{1}{1+b_i}\right)e.\end{aligned}$$

In the similar way, we can prove

$$\lim_{n\to\infty} \overline{d}_\mu(Y_n) = \frac{\det(b_i^{\ell-j})(1-b_1)\cdots(1-b_\ell)e^{b_1}\cdots e^{b_\ell}}{\det(b_i^{\ell-j})}$$
$$= \prod_{i=1}^{\ell}(1-b_i)\,e. \qquad \square$$

It is expected to allow the equality between b_i and b_{i+1} for $i = 1, 2, \cdots, \ell - 1$ in Theorem 12. We can prove one of the simplest cases, λ is a rectangle (ℓ^n), which implies $b_1 = \cdots = b_\ell$.

Theorem 13.

$$\lim_{n\to\infty} \overline{d}_{(\ell^n)}(Y_{\ell n}) = \left(\frac{\ell-1}{\ell}\right)^{\ell} e.$$

Proof. By Lemma 2, for $i = 0, \pm1, \pm2, \cdots, \pm(\ell-1)$,

$$\det Y_{\ell n, n+i} = \left(\frac{\ell n}{\ell n - 1}\right)^{n+i-1}\left(\frac{(\ell-1)n - i}{\ell n - 1}\right),$$

we can write

$$d_{(\ell^n)}(Y_{\ell n})$$
$$= (\ell n)! \begin{vmatrix} \frac{\det Y_{\ell n,n}}{n!} & \frac{\det Y_{\ell n,\ell+1}}{(n+1)!} & \cdots & \frac{\det Y_{\ell n,2\ell-1}}{(n+\ell-1)!} \\ \frac{\det Y_{\ell n,\ell-1}}{(n-1)!} & \frac{\det Y_{\ell n,\ell}}{n!} & \cdots & \frac{\det Y_{\ell n,2+\ell-2}}{(n+\ell-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\det Y_{\ell n,1}}{(n-\ell+1)!} & \frac{\det Y_{\ell n,2}}{(n-\ell+2)!} & \cdots & \frac{\det Y_{\ell n,n}}{n!} \end{vmatrix}$$
$$= (\ell n)!\left(\frac{\ell n}{\ell n-1}\right)^{\ell n-\ell}\left(\frac{1}{\ell n-1}\right)^{\ell} \begin{vmatrix} \frac{(\ell-1)n}{n!} & \frac{(\ell-1)n-1}{(n+1)!} & \cdots & \frac{(\ell-1)n-(\ell-1)}{(n+\ell-1)!} \\ \frac{(\ell-1)n+1}{(n-1)!} & \frac{(\ell-1)n}{n!} & \cdots & \frac{(\ell-1)n-(\ell-2)}{(n+\ell-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(\ell-1)n+\ell-1}{(n-\ell+1)!} & \frac{(\ell-1)n+\ell-2}{(n-\ell+2)!} & \cdots & \frac{(\ell-1)n}{n!} \end{vmatrix}.$$

To normalize the immanant above, we find $\dim(\ell^n)$ using the hook length formula. Writing the hook length in each box, we have

$$\dim(\ell^n) = \frac{(\ell n)!\ 1!\ 2!\ \cdots\ (\ell-1)!}{n!(n+1)!\cdots(n+\ell-1)!}.$$

Thus,

$$\overline{d}_{(\ell^n)}(Y_{\ell n}) = \frac{1}{1!2!\cdots(\ell-1)!}\left(\frac{\ell n}{\ell n-1}\right)^{\ell n-\ell}\left(\frac{1}{\ell n-1}\right)^{\ell} \times \left|\left((n+j-i+1)^{\overline{i-1}}\{(\ell-1)n+i-j\}\right)_{1\le i,j\le \ell}\right|,$$

where $x^{\overline{s}}$ is the rising factorial of x with length s.

Since $(\ell n/(\ell n-1))^{\ell n-\ell}$ approaches e as $n\to\infty$, all we have to prove is that the determinant $\left|\left((n+j-i+1)^{\overline{i-1}}\{(\ell-1)n+i-j\}\right)_{1\le i,j\le \ell}\right|$ is a polynomial with degree ℓ and the coefficient of n^ℓ (that is, the greatest term) is

$$1!2!\cdots(\ell-1)!(\ell-1)^\ell.$$

First, multiply the determinant of

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & \cdots & & (-1)^{\ell-1} \\ 0 & 1 & -2 & 3 & -4 & 5 & \cdots & & (-1)^{\ell-2}(\ell-1) \\ 0 & 0 & 1 & -3 & 6 & -10 & \cdots & & \\ 0 & 0 & 0 & 1 & -4 & 10 & \cdots & & \\ 0 & 0 & 0 & 0 & 1 & -5 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\ & & & & & & & 1 & -(\ell-1) \\ 0 & 0 & 0 & 0 & 0 & & \cdots & 0 & 1 \end{pmatrix} \tag{3}$$

to the determinant from the right, so that it does not change the value:

$$\left|\begin{pmatrix} (n+j-i+1)^{\overline{i-j}} \\ \quad\times\left\{(i-j+2)^{\overline{j-2}}\left(-i+(i-j+1)\,\ell\right)n+(i-j+2)^{\overline{j-1}}(i-j)\right\}\end{pmatrix}_{1\le i,j\le\ell}\right|$$

$$= \begin{vmatrix} (\ell-1)n & -1 & 0 & 0 & \cdots & & 0 \\ & 1!(\ell-2)n & -2! & 0 & \cdots & & 0 \\ & & 2!(\ell-3)n & -3! & \cdots & & 0 \\ & & & 3!(\ell-4)n & \cdots & & \vdots \\ & * & & & \ddots & -(\ell-2)! & 0 \\ & & & & & (\ell-2)!n & -(\ell-1)! \\ & & & & & & 0 \end{vmatrix}.$$

Each of the nonzero term in the expansion of the determinant above is yielded by a permutation with the cycle decomposition

$$\sigma = (12\cdots i_1)(i_1+1\cdots i_2)\cdots(i_{m-1}+1\cdots i_m)(i_m+1\cdots \ell),$$

in which the cycles make the factors with the degrees i_1, i_2-i_1, $\cdots$, $i_{m-1}-i_m$, $\ell-i_m$. Therefore, the degree of the polynomial is exactly ℓ (if the coefficient does not vanish). Our interest is the coefficient of n^ℓ, so it is enough to consider the determinant obtained by picking up the coefficient of the greatest term in each entries, that is,

$$\begin{aligned}
&\left(\text{The coefficient of } n^\ell\right)\\
&= \left|\left\{(i-j+2)^{\overline{j-2}}\left(-i+(i-j+1)\,\ell\right)\right\}\right|_{1\le i,j\le \ell}\\
&= \begin{vmatrix}
\ell-1 & -1 & 0 & 0 & \cdots & & 0\\
\ell-1 & 1!(\ell-2) & -2! & 0 & \cdots & & 0\\
\ell-1 & & 2!(\ell-3) & -3! & \cdots & & 0\\
\ell-1 & & & 3!(\ell-4) & \ddots & & \vdots\\
\ell-1 & & & & \ddots & -(\ell-2)! & 0\\
\ell-1 & & * & & & (\ell-2)! & -(\ell-1)!\\
\ell-1 & & & & \cdots & & 0
\end{vmatrix}.
\end{aligned}$$

Multiplying the determinant of the transpose of the matrix (3) to the determinant above from the left,

$$\begin{aligned}
&\left(\text{The coefficient of } n^\ell\right)\\
&= \left|\left\{(i-j+2)^{\overline{j-2}}\left(-i+(i-j+1)\,\ell\right)\right\}\right|\\
&= \begin{vmatrix}
\ell-1 & -1 & 0 & 0 & \cdots & & 0\\
0 & 1!(\ell-1) & -2! & 0 & \cdots & & 0\\
0 & 0 & 2!(\ell-1) & -3! & \cdots & & 0\\
0 & 0 & 0 & 3!(\ell-1) & \ddots & & \vdots\\
\vdots & & & \ddots & \ddots & -(\ell-2)! & 0\\
0 & 0 & & & 0 & (\ell-2)!(\ell-1) & -(\ell-1)!\\
0 & 0 & & \cdots & 0 & 0 & (\ell-1)!(\ell-1)
\end{vmatrix}\\
&= 1!2!\cdots(\ell-1)!(\ell-1)^\ell. \qquad \square
\end{aligned}$$

One can notice that $((\ell-1)/\ell)^\ell$ converges to $1/e$ as $\ell\to\infty$, so that we observe that

$$\lim_{\ell\to\infty}\lim_{n\to\infty}\overline{d}_{(\ell^n)}(Y_{\ell n}) = \frac{1}{e}\cdot e = 1.$$

This is explained that the shape of the rectangle is infinitely extended vertically first, and horizontally next. The similar result for the immanants corresponding to the conjugate of the rectangles (namely, extended horizontally first and vertically) to approach to 1 as $\ell\to\infty$, is expected to hold:

Conjecture 14.

$$\lim_{n\to\infty} \overline{d}_{(n^\ell)}(Y_{\ell n}) = \left(\frac{\ell}{\ell+1}\right)^\ell e.$$

Consider the coefficients in Theorem 12 again, and notice that the coefficients $\prod_{i=1}^{\ell}(1 \pm b_i)$ are expressed as the approximations of the broken line of the differential equations

$$\frac{dy}{dx} = \pm y \text{ with } y(0) = 1,$$

whose solutions are $y = f_\pm(x) = e^{\pm x}$. This indicates that as $\ell \to \infty$, which implies that the interval of the broken line is getting narrow, $\prod_{i=1}^{\ell}(1 + b_i)$ approaches to $f_+(1) = e$ and $\prod_{i=1}^{\ell}(1 - b_i)$ approaches to $f_-(1) = 1/e$. Therefore, we can conclude that

$$\lim_{\ell\to\infty} \lim_{n\to\infty} \overline{d}_{\lambda^{(n)}}(Y_n) = 1$$

and

$$\lim_{\ell\to\infty} \lim_{n\to\infty} \overline{d}_{\mu^{(n)}}(Y_n) = 1.$$

As we will mention in the next section, the expectation of the immanants of Y_n with the random partitions under the Plancherel measure is also 1. Its limit shape of Young diagrams, given by Logan–Shepp [6] and Kerov–Vershik [11], has the scaling $\lambda_1 \sim 2\sqrt{n}$ as $n \to \infty$. Our conjecture is the following:

Conjecture 15. *Let $\{\lambda^{(n)}\}$ be a sequence of Young diagrams such that $|\lambda^{(n)}| = n$ and*

$$\lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \cdots,$$

and let $\mu^{(n)}$ be the conjugate of $\lambda^{(n)}$. If $\lim_{n\to\infty} \lambda_1^{(n)}/n = 0$ and $\lim_{n\to\infty} \mu_1^{(n)}/n = 0$, then

$$\lim_{n\to\infty} \overline{d}_{\lambda^{(n)}}(Y_n) = 1.$$

Remark 16. The converse of Conjecture 15 does not hold from Theorem 10.

5. Remarks and conjecture on immanantal inequalities

Let us write $A > 0$ (resp. $A \geq 0$) if A is a positive definite Hermitian matrix (resp. a positive semi-definite Hermitian matrix). Schur [8] proved that $\det A \leq \overline{d}_\chi^G(A)$ for $A \geq 0$, and Lieb's permanental dominance conjecture [4] asserts that $\overline{d}_\chi^G(A) \leq \operatorname{per} A$. To improve these inequalities, the author [9] defines

$$F_\chi^G(A) = \frac{\overline{d}_\chi^G(A) - \det A}{\operatorname{per} A - \det A}$$

for $A \geq 0$ such that $\operatorname{per} A \neq \det A$, and

$$R(G,\chi) = \left\{ F_\chi^G(A) \in \mathbb{R} \,|\, A \geq 0 \text{ such that } \operatorname{per} A \neq \det A \right\},$$

and determines the possible values of $R(G,\chi)$ for all subgroups G of $\mathfrak{S}_3$ and their irreducible characters χ.

On such inequality problems, it is sufficient to restrict ourselves to correlation matrices, namely positive semi-definite Hermitian matrices with all diagonals equal to 1. Every positive semi-definite Hermitian matrix A with no zero row can be turned to a correlation matrix DAD with $D = \operatorname{diag}(1/\sqrt{a_{11}}, \cdots, 1/\sqrt{a_{nn}})$ (see [1], for example).

One can easily see that when $A \geq 0$ satisfying $\operatorname{per} A \neq \det A$, and $D = \operatorname{diag}(1/\sqrt{a_{11}}, \cdots, 1/\sqrt{a_{nn}})$,

$$F_\chi^G(A) = F_\chi^G(DAD).$$

Remark 17. Let G be a subgroup of $\mathfrak{S}_n$ and χ a character of G. The maximum and minimum values of $R(G,\chi)$ can be realized by some singular $A \geq 0$.

Conjecture 18. *Suppose that $n \geq 3$.*

(1) For $\lambda \neq (n), (n-1,1), (1^n)$, the maximum value of $R(\mathfrak{S}_n, \lambda)$ is realized at Y_n.
(2) For $\lambda = (n-1,1)$, the maximum value of $R(\mathfrak{S}_n, (n-1,1))$ is realized at

$$Y_3 \oplus I_{n-3} = \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & O & \\ -\frac{1}{2} & -\frac{1}{2} & 1 & & & \\ \hline & & & 1 & & O \\ & O & & & \ddots & \\ & & & O & & 1 \end{array}\right).$$

Conjecture 18 is true for $n \leq 4$ (see [1]). Our computer experiments have given no counterexamples so far.

Moreover, Conjecture 18 has a remarkable connection with Pierce's conjecture [7], which asserts that the minimum value of the permanents on correlation matrices is given by Y_n. Combining all the properties above, this conjecture turns to the following:

Conjecture 19. *The maximum value of $R(\{\mathrm{id}\}, \mathrm{triv})$ is realized at Y_n.*

Remark 20. The minimum value of $R(\{\mathrm{id}\}, \mathrm{triv})$ is $1/n!$, realized at J_n.

Combining with Frenzen–Fischer's result, Conjecture 19 would imply the interesting upper bound:

$$\bigcup_{n=2}^{\infty} R(\{\mathrm{id}\}, \mathrm{triv}) = \left[0, \frac{2}{e}\right].$$

Denote the regular representation of $\mathfrak{S}_n$ by $\mathcal{R}$, and notice that $\overline{d}_{\mathcal{R}}^{\mathfrak{S}_n}(A) = a_{11} \cdots a_{nn}$ because the character $\chi_{\mathcal{R}}$ is given by

$$\chi_{\mathcal{R}}(\sigma) = \begin{cases} n! & (\text{if } \sigma = \mathrm{id}) \\ 0 & (\text{otherwise}). \end{cases}$$

On the other hand, the irreducible decomposition $\mathcal{R} = \oplus_{\lambda \vdash n} V_\lambda^{\oplus \dim V_\lambda}$ gives the Plancherel measure, whose properties are described in [10], over the set of all partitions of n:

$$P(\lambda \vdash n) = \frac{(\dim V_\lambda)^2}{n!}.$$

Therefore, we easily observe that denoting the expectations of $\overline{d}_\lambda(A)$ and $F_\chi^G(A)$ by $E\left[\overline{d}_\lambda(A)\right]$ and $E\left[F_\chi^G(A)\right]$,

$$\overline{d}_{\mathcal{R}}^{\mathfrak{S}_n}(A) = \sum_{\lambda \vdash n} \frac{(\dim V_\lambda)^2}{n!} \overline{d}_\lambda^{\mathfrak{S}_n}(A) = E\left[\overline{d}_\lambda(A)\right],$$

and as $n \to \infty$,

$$E\left[F_\lambda^{\mathfrak{S}_n}(Y_n)\right] = \frac{2}{e}.$$

It asserts that as $|\lambda| = n \to \infty$ under the Plancherel measure,

$$R(\mathfrak{S}_n, \lambda) \to \left[0, \frac{2}{e}\right]$$

with probability 1. However, as we showed and conjectured in Section 4, that many immanants of Y_n converge to 1, and this suggests that the behavior of the permanental dominance conjecture for many immanants as $n \to \infty$ would be

$$R(\mathfrak{S}_n, \lambda^{(n)}) \to \left[0, \frac{2}{e}\right].$$

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