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Edge disjoint paths in hypercubes and folded hypercubes with conditional faults

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ABSTRACT

It is known that edge disjoint paths is closely related to the edge connectivity and the multicommodity flow problems. In this paper, we study the edge disjoint paths in hypercubes and folded hypercubes with edge faults. We first introduce the F-strongly Menger edge connectivity of a graph, and we show that in all n-dimensional hypercubes (folded hypercubes, respectively) with at most $2n-4(2n-2$, respectively) edges removed, if each vertex has at least two fault-free adjacent vertices, then every pair of vertices u and v are connected by $min\{deg(u), deg(v)\}$ edge disjoint paths, where $deg(u)$ and $deg(v)$ are the remaining degree of vertices u and v, respectively.

1. Introduction

The studies on the edge disjoint paths come up naturally when analyzing connectivity questions or generalizing (integral) network flow problems. Another reason for the grown interest in this area is the variety of applications, e.g. in VLSI-design and interconnection network design. In particular, in the design of a multicomputer (interconnection) system, one important consideration is its fault tolerance, namely its capability of being functional in the presence of failures. The edge connectivity of a connected graph G, denoted by $\lambda(G)$, is the minimum number of edges whose removal from G results in a disconnected graph. The edge connectivity is one of the essential parameters to evaluate the fault tolerance of a network.

To make an overall evaluation on interconnection network with failures, some other measures related to edge connectivity have been studied in recent years. In particular, the extra edge connectivity of hypercubes and folded hypercubes was discussed by several authors in Refs. [4–6,12,13]. In this paper, we consider the classic Menger's Theorem under conditional edge faults.

Theorem 1.1 [7]. *Let x and y be two distinct vertices of a graph G. The minimum size of an x, y edge cut equals the maximum number of edge disjoint x, y-paths.*

Following this theorem, we introduce the F-strong Menger edge connectivity which is similar to the concept on strong Menger connectivity in Oh and Chen [10].

Definition 1.2. A graph G is F-strongly Menger edge connected if for subgraph $G-F$ of G with minimum degree at least 2, $F\subset E(G)$, each pair of vertices u and v in $G-F$ are connected by $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ edge-disjoint fault-free paths in $G-F$, where $deg_{G-F}(u)$ and $deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.

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If no confusion should arise, we call a graph strongly Menger edge connected if it is F-strongly Menger edge connected. In [8–10], Oh and Chen proved that an n-dimensional star graph S_n (an n-dimensional hypercube Q_n, respectively) with at most $n-3$ ($n-2$, respectively) vertices removed is strongly Menger connected. Furthermore, Shih et al. [11] provided a result that if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, the hypercube-like graphs still have the strong Menger property, even if there are up to $2n-5$ vertices fault. Here, we show that all hypercubes (folded hypercubes, respectively) is strong Menger edge connectivity if $|F| \le 2n-4$ ($|F| \le 2n-2$, respectively) edges fault.

2. Preliminary

Due to attractive topological properties, hypercube has been one of the most fundamental interconnection networks. The hypercube Q_n (with $n \ge 2$) is defined as having the vertex set of binary strings of length n. Two vertices are adjacent if and only if their strings differ in exactly 1 bit. So, Q_n is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges.

Basing on the excellent properties of hypercube, a large number of variants have been proposed. One variant has received a great deal of research is folded hypercube, which is obtained by adding an edge to every pair of nodes with complementary address. So, two vertices of folded hypercube, denoted FQ_n, are adjacent if and only if their strings differ in exactly 1 bit or n bits. Moreover, FQ_n is an $(n+1)$-regular graph with 2^n vertices and $(n+1)2^{n-1}$ edges. The interested reader can refer to [1].

For a graph G, let $\lambda(G)$ denote the edge connectivity of G. For a set of edges $F \subset E(G)$, let $G-F$ denote the graph obtained by deleting F from G. For a set of vertices $S \subset V(G)$, let $N'_G(S)$ be the set of edges with exactly one end in S, and $G[S]$ denote the subgraph induced by S. For brevity, for a vertex u of G, we write $N'_G(\{u\})$ as $N'_G(u)$. For a graph G and a vertex $u \in V(G)$, we denote all adjacent vertices of u in G by $N_G(u)$ and the degree of u in G by $deg_G(u)$. Let u, v be two vertices of G, we use $d_G(u, v)$ to represent the distance of u and v. Other fundamental graph-theoretical terminology, the reader is suggested to refer to [3].

Let S_0 (respectively, S_1) denote the set of all the vertices of Q_n which take on value 0 (respectively, 1) on the ith bit position for some i, $1 \le i \le n$. Let $G_0 = G[S_0]$, $G_1 = G[S_1]$, then G_0, G_1 are both isomorphic to Q_{n-1}, and every vertex of G_0 has exactly one neighbor in G_1. Let $M = \{uv \in E(Q_n) | u \in S_0, v \in S_1\}$ be the perfect matching between G_0 and G_1. We use $G_0 \oplus_M G_1$ to denote Q_n. In addition, we sometimes write G_0 and G_1 as Q^{i0}_{n-1} and Q^{i1}_{n-1}, respectively.

By an easy observation, Q_n and FQ_n have the same vertex set. The Hamming distance, denoted by $d_H(u, v)$, between any two vertices u and v of FQ_n is the number of different positions between the binary strings of u and v. It is easy to see that two vertices u and v of folded hypercube FQ_n are adjacent if and only if $d_H(u, v) = 1$ or n. In what follows, we represent $\bar{x_1}\bar{x_2}\ldots\bar{x_n}$ and $x_1 \ldots x_{i-1}\bar{x_i}x_{i+1} \ldots x_n$ as $\bar{u}$ and u_i, respectively, where $\bar{x_i} = 1 - x_i$ for $1 \le i \le n$. In addition, we write $(u_i)_j$ as u_{ij} for $i \ne j$. Let $PM_i = \{(u, u_i) | u \in V(FQ_n)\}$, $PM = \{(u, \bar{u}) | u \in V(FQ_n)\}$. Clearly, G_0 and G_1 mentioned before are also subgraphs of FQ_n, and in FQ_n, they are connected by two specific perfect matchings PM and PM_i. Then, we denote FQ_n as $G_0 \otimes_{PM \cup PM_i} G_1$. We need the following result in the proof of this paper.

Theorem 2.1 [2]. $\lambda(Q_n) = n$.

In the following, we discuss the strong Menger edge connectivity of hypercubes and folded hypercubes.

3. Strong Menger edge connectivity with conditional faults of hypercubes

In this section, we shall show a main result that an n-dimensional hypercube is F-strong Menger edge connected if $|F| \le 2n-4$. We need the following lemmas.

Lemma 3.1. *Let $S \subset E(Q_n)$ be a set of edges with $|S| \le 2n-3$, for $n \ge 2$. There exists a connected component C in $Q_n - S$ with $|V(C)| \ge 2^n - 1$.*

Proof. By induction on n. It is easy to see that the result holds for $n = 2$ and $n = 3$. Assume the lemma holds for $n-1$, $n \ge 4$, we now show that it is true for n.

We may decompose Q_n to $G_0 \oplus_M G_1$. Let S be a set of edges with $|S| \le 2n-3$, for $n \ge 2$, and let $S_0 = S \cap E(G_0)$, $S_1 = S \cap E(G_1)$, $S_2 = S \cap M$. Then $|S_0| + |S_1| + |S_2| = |S| \le 2n-3$. Without loss of generality, we assume that $|S_0| \le |S_1|$. Let C be the largest connected component of $Q_n - S$. It is impossible that both $|S_0|$ and $|S_1|$ are more than $2n-5$. In fact, if $|S_0| > 2n-5$ and $|S_1| > 2n-5$, then $|S| \ge 4n-8$, which contradicts to $|S| \le 2n-3$. We then consider the following two cases.

Case 1. $|S_0| \le 2n-5$ and $|S_1| \le 2n-5$.

It is impossible that both $|S_0|$ and $|S_1|$ are more than $n-2$. In fact, if $|S_0| > n-2$ and $|S_1| > n-2$, then $|S| \ge 2n-2$, which contradicts to $|S| \le 2n-3$.

Subcase 1a. $|S_0| \le n-2$ and $|S_1| \le n-2$.

As $\lambda(G_0) = \lambda(G_1) = n-1$, then $G_0 - S_0$, $G_1 - S_1$ are connected. It follows from $|M| = 2^{n-1}$ and $|S_2| \le 2n-3$ that $|M| > |S_2|$ for $n \ge 4$. Then $G_0 - S_0$ is connected to $G_1 - S_1$, that is , $Q_n - S$ is connected. So, $|V(C)| = |V(G_0 - S_0)| + |V(G_1 - S_1)| = 2^n$.

Subcase 1b. $n-1 \leq |S_1| \leq 2n-5$ and consequently $|S_0|+|S_2|=|S|-|S_1| \leq n-2$.

By induction hypothesis, there exists a connected component C_1 in G_1-S_1, and $|V(C_1)| \geq 2^{n-1}-1$. Since the edge connectivity of G_0 is $n-1$ and $|S_0| \leq n-2$, so G_0-S_0 is connected. It follows from $|M|=2^{n-1}$ and $|S_2| \leq n-2$ that $|M|-|S_2| \geq 2$ for $n \geq 4$. Then C_1 is connected to G_0-S_0. Therefore, $|V(C)| \geq |V(C_1)|+|V(G_0-S_0)| \geq 2^n-1$.

Case 2. $|S_1| > 2n-5$ and consequently $|S_0|+|S_2|=|S|-|S_1| < 2$.

Obviously, G_0-S_0 is connected. Since $|S_2| < 2$, then there are at least $2^{n-1}-1$ vertices of G_1 is connected to G_0-S_0. Hence, $|V(C)| \geq |V(G_0-S_0)|+2^{n-1}-1=2^n-1$.

Combining the above arguments, the proof is complete. □

Remark 3.2. The result of Lemma 3.1 is optimal in that there exists a set of edges S with $|S|=2n-2$ for $n \geq 2$ such that Q_n-S contains a connected component C with $|V(C)| \leq 2^n-2$.

In fact, consider a set of vertices F of two adjacent vertices $\{u, v\}$. Then $|N'_{Q_n}(F)|=2n-2$. Let S be a set of edges of Q_n such that $S=N'_{Q_n}(F)$, then Q_n-S contains a connected component induced by $\{u, v\}$. Therefore, $|V(C)| \leq 2^n-2$.

Lemma 3.3. *Let $S \subset E(Q_4)$ be a set of edges with $|S| \leq 7$. There exists a connected component C in Q_4-S such that $|V(C)| \geq 2^4-2$.*

Proof. We may decompose Q_4 to $G_0 \oplus_M G_1$. Let $S_0=S \cap E(G_0)$, $S_1=S \cap E(G_1)$, $S_2=S \cap M$. Then, $|S_0|+|S_1|+|S_2|=|S| \leq 7$. Without loss of generality, we suppose that $|S_0| \leq |S_1|$. Let C be the largest connected component of Q_4-S. Obviously, there is at most one of $|S_0|$ and $|S_1|$ more than 3 and $|M|=2^3=8$. We then consider six cases.

Case 1. $|S_0| \leq 2$ and $|S_1| \leq 2$.

Since $\lambda(G_0)=\lambda(G_1)=3$, then G_0-S_0, G_1-S_1 are connected. It follows from $|S_2| \leq 7$ and $|M|=8$ that $|M| > |S_2|$. Therefore, G_0-S_0 is connected to G_1-S_1 and $|V(C)|=|V(G_0-S_0)|+|V(G_1-S_1)|=2^4$.

Case 2. $|S_0| \leq 2$ and $|S_1|=3$.

Obviously, G_0-S_0 is connected. By Lemma 3.1, there exists a connected component C_1 in G_1-S_1 such that $|V(C_1)| \geq 2^3-1$. It follows from $|S_2| \leq |S|-|S_1| \leq 4$ and $|M|=8$ that $|M|-|S_2| \geq 4$. So G_0-S_0 is connected to C_1 and $|V(C)| \geq |V(C_1)|+|V(G_0-S_0)| \geq 2^4-1$.

Case 3. $|S_0| \leq 2$ and $|S_1|=4$. Consequently, $|S_2| \leq |S|-|S_1| \leq 3$.

It is easy to see that G_0-S_0 is connected. By an easy observation, either there exists a connected component C_1 in G_1-S_1 that $|V(C_1)| \geq 2^3-2$ or G_1-S_1 is decomposed into two connected components with each of them is isomorphic to Q_2. It follows from $|M|=8$ and $|S_2| \leq 3$ that $|M|-|S_2| \geq 5$. Thus, $|V(C)| \geq 2^4-2$.

Case 4. $|S_0| \leq 2$ and $|S_1| \geq 5$. As a result, $|S_2| \leq |S|-|S_1| \leq 2$.

Obviously, G_0-S_0 is connected. Since $|S_2| \leq 2$, there are at most two vertices of G_1-S_1 disconnected to G_0, hence, $|V(C)| \geq |V(G_0-S_0)|+2^3-2=2^4-2$.

Case 5. $|S_0|=3$ and $|S_1|=3$. Consequently, $|S_2|=|S|-|S_0|-|S_1| \leq 1$.

By Lemma 3.1, G_0-S_0 (G_1-S_1, respectively) contains a connected component C_0 (C_1, respectively) with $|V(C_0)|$ ($|V(C_1)|$, respectively) $\geq 2^3-1$. It follows from $|S_2| \leq 1$ and $|M|=8$ that $|M|-|S_2| \geq 7$. Therefore, C_0 is connected to C_1 and $|V(C)| \geq |V(C_0)|+|V(C_1)| \geq 2^4-2$.

Case 6. $|S_0|=3$ and $|S_1|=4$.

Since $|S|=|S_0|+|S_1|+|S_2| \leq 7$, then $|S_2|=0$. By Lemma 3.1, there exists a connected component C_0 in G_0-S_0 with $|V(C_0)| \geq 2^3-1$. As $|S_2|=0$, then each vertex of C_0 has a neighbor in G_1 by the matching M. Therefore, $|V(C)| \geq 2|V(C_0)| \geq 2^4-2$.

Combining the above cases, the proof is complete. □

Lemma 3.4. *Let $S \subset E(Q_n)$ be a set of edges with $|S| \leq 3n-5$ for $n \geq 4$. There exists a connected component C in Q_n-S such that $|V(C)| \geq 2^n-2$.*

Proof. By induction on n. We first note that the statement is true for $n=4$ by Lemma 3.3. Now assume that $n \geq 5$ and the claim is true for $n-1$, we then show that it holds for n.

As before, we decompose Q_n to $G_0 \oplus_M G_1$. Let $S_0=S \cap E(G_0)$, $S_1=S \cap E(G_1)$, $S_2=S \cap M$. Then, $|S_0|+|S_1|+|S_2|=|S| \leq 3n-5$. Without loss of generality, we suppose that $|S_0| \leq |S_1|$. Let C be the largest connected component of Q_n-S. There is at most one of $|S_0|$ and $|S_1|$ more than $3n-8$. In fact, if $|S_0| > 3n-8$ and $|S_1| > 3n-8$, then $|S| \geq |S_0|+|S_1| \geq 6n-14$, which contradicts to $|S| \leq 3n-5$. We then consider two cases.

Case 1. $|S_0| \leq 3n-8$ and $|S_1| \leq 3n-8$.

It is impossible that both $|S_0|$ and $|S_1|$ are more than $2n-5$. In fact, if $|S_0| > 2n-5$ and $|S_1| > 2n-5$, then $|S| \geq |S_0|+|S_1| \geq 4n-8$, which contradicts to $|S| \leq 3n-5$.

Subcase 1a. $|S_0| \leq 2n-5$ and $|S_1| \leq 2n-5$.

If $|S_0| \le n-2$ and $|S_1| \le n-2$, then by $\lambda(G_0) = \lambda(G_1) = n-1$, we know that $G_0 - S_0$, $G_1 - S_1$ are connected. It follows from $|S_2| \le 3n-5$ and $|M| = 2^{n-1}$ that $|M| > |S_2|$. Therefore, $G_0 - S_0$ is connected to $G_1 - S_1$ and $|V(C)| = |V(G_0 - S_0)| + |V(G_1 - S_1)| = 2^n$.

If $|S_0| \le n-2$ and $n-1 \le |S_1| \le 2n-5$, then $G_0 - S_0$ is connected. By Lemma 3.1, there exists a connected component C_1 in $G_1 - S_1$ such that $|V(C_1)| \ge 2^{n-1} - 1$. It follows from $|S_2| \le |S| - |S_1| \le 2n-4$ and $|M| = 2^{n-1}$ that $|M| - |S_2| \ge 2$. So $G_0 - S_0$ is connected to C_1 and $|V(C)| \ge |V(C_1)| + |V(G_0 - S_0)| \ge 2^n - 1$.

We next assume $n-1 \le |S_0| \le 2n-5$ and $n-1 \le |S_1| \le 2n-5$. Consequently, $|S_2| = |S| - |S_0| - |S_1| \le n-3$. By Lemma 3.1, $G_0 - S_0$ ($G_1 - S_1$, respectively) contains a connected component C_0 (C_1, respectively) with $|V(C_0)|$ ($|V(C_1)|$, respectively) $\ge 2^{n-1} - 1$. It follows from $|S_2| \le n-3$ and $|M| = 2^{n-1}$ that $|M| - |S_2| \ge 3$. Therefore, C_0 is connected to C_1 and $|V(C)| \ge |V(C_0)| + |V(C_1)| \ge 2^n - 2$.

Subcase 1b. $|S_0| \le 2n-5$ and $2n-4 \le |S_1| \le 3n-8$.

If $|S_0| \le n-2$, then $|S_2| \le |S| - |S_1| \le n-1$ and $G_0 - S_0$ is connected. By induction hypothesis, there exists a connected component C_1 in $G_1 - S_1$ that $|V(C_1)| \ge 2^{n-1} - 2$. It follows from $|M| = 2^{n-1}$ and $|S_2| \le n-1$ that $|M| - |S_2| \ge 3$. Then, C_1 is connected to $G_0 - S_0$. So, $|V(C)| \ge |V(C_1)| + |V(G_0 - S_0)| \ge 2^n - 2$.

Assume $n-1 \le |S_0| \le 2n-5$. Since $|S| = |S_0| + |S_1| + |S_2| \le 3n-5$, then only one possibility, that is, $|S_0| = n-1, |S_1| = 2n-4, |S_2| = 0$. By Lemma 3.1, there exists a connected component C_0 in $G_0 - S_0$ with $|V(C_0)| \ge 2^{n-1} - 1$. As $|S_2| = 0$, then each vertex of C_0 has a neighbor in G_1 by the matching M. Therefore, $|V(C)| \ge 2|V(C_0)| \ge 2^n - 2$.

Case 2. $|S_1| > 3n-8$. As a consequence, $|S_0| + |S_2| = |S| - |S_1| \le 2$.

As $\lambda(G_0) = n-1 \ge 4$, so $G_0 - S_0$ is connected. Since $|S_2| \le 2$, there exist at least $2^{n-1} - 2$ vertices of $G_1 - S_1$ are connected to $G_0 - S_0$. Therefore, $|V(C)| \ge |V(G_0 - S_0)| + 2^{n-1} - 2 = 2^n - 2$.

Combining the above cases, the proof is complete. □

Remark 3.5. Lemma 3.4 does not hold for $n = 3$. Consider a 3-dimensional hypercube Q_3, $Q_3 = G_0 \oplus_M G_1$. Let $S = M$. Then $|S| = 4 \le 3*3-5$ but $Q_3 - S$ have two components with each of them is isomorphic to Q_2. So, $|V(C)| = 4 < 2^3 - 2$.

Remark 3.6. The result of Lemma 3.4 is optimal in that there exists a set of edges S with $|S| = 3n-4$ for $n \ge 4$ such that $Q_n - S$ contains a connected component C with $|V(C)| \le 2^n - 3$.

In fact, we consider a set of vertices $F = \{x, y, z\}$, where xy, $yz \in E(Q_n)$. As Q_n contains no odd cycle, $xz \notin E(Q_n)$. Then $|N'_{Q_n}(F)| = 3n-4$. Let S be a set of edges of Q_n such that $S = N'_{Q_n}(F)$, then $Q_n - S$ contains a connected component induced by F. Therefore, $|V(C)| \le 2^n - 3$.

Theorem 3.7. *An n-dimensional hypercube is F-strong Menger edge connected if $|F| \le 2n-4$ and $n \ge 4$.*

Proof. Let F be a conditional faulty edge set such that $\delta(Q_n - F) \ge 2$, and u and v be two vertices in $Q_n - F$. Without loss of generality, we assume $deg_{Q_n-F}(u) \le deg_{Q_n-F}(v)$, so $min\ \{deg_{Q_n-F}(u), deg_{Q_n-F}(v)\} = deg_{Q_n-F}(u)$. We now show that u is connected to v if the number of edges deleted is no more than $deg_{Q_n-F}(u) - 1$ in $Q_n - F$. By Theorem 1.1, this means that u and v are connected by $deg_{Q_n-F}(u)$ edge-disjoint fault-free paths, where $|F| \le 2n-4$.

Suppose on the contrary that u and v are separated by deleting a set of edges E_f, where $|E_f| \le deg_{Q_n-F}(u) - 1$. Due to $deg_{Q_n-F}(u) \le deg_{Q_n}(u) \le n$, we have $|E_f| \le n-1$. We sum up the cardinality of these two sets F and E_f. It follows from $|F| \le 2n-4$ and $|E_f| \le n-1$ that $|F| + |E_f| \le 3n-5$. Let $S = F \cup E_f$, by Lemma 3.4, there exists a connected component C in $Q_n - S$ such that $|V(C)| \ge 2^n - 2$ and $|S| \le 3n-5$. It means that there are at most two vertices in $Q_n - S$ not belonging to C. We then consider two cases.

Case 1. $|V(C)| = 2^n - 1$.

It means that only one vertex is disconnected to C. Since $|E_f| \le deg_{Q_n-F}(u) - 1 \le deg_{Q_n-F}(v) - 1$, neither u nor v can be the only one disconnected vertex, a contradiction.

Case 2. $|V(C)| = 2^n - 2$.

Let a and b be the two vertices in $Q_n - S$ not belonging to C. Assume a is adjacent to b. Since u and v are disconnected in $Q_n - S$, without loss of generality, we can assume $u \in V(C)$, $v \in \{a, b\}$, furthermore, let $v = a$. By simple observation, $N'_{Q_n-F}(\{a, b\}) \subset E_f$. As F is a conditional faulty edge set, then b is incident to at least one edge in E_f. $|E_f| \ge |N'_{Q_n-F}(\{a, b\})| \ge deg_{Q_n-F}(v) - 1 + 1 = deg_{Q_n-F}(v)$, which contradicts to $|E_f| \le deg_{Q_n-F}(v) - 1$.

Then we may assume a is not adjacent to b. Since u and v are disconnected in $Q_n - S$, it is not difficult to find that $\{u, v\} \cap \{a, b\} \ne \emptyset$. Without loss of generality, we let $u = a$, then $N'_{Q_n-F}(u) \subset E_f$ and $|E_f| \ge |N'_{Q_n-F}(u)| = deg_{Q_n-F}(u)$, which contradicts to $|E_f| \le deg_{Q_n-F}(u) - 1$. The proof is complete. □

4. Strong Menger edge connectivity with conditional faults of folded hypercubes

In this section, a main result will be presented that an n-dimensional folded hypercube is strong Menger edge connected with up to $2n-2$ edges fault under the restriction that every vertex has at least two fault-free adjacent vertices. To prove this result, we need some preliminary lemmas. In the following, we provide two properties of FQ_n and discuss the size of largest connected component of folded hypercube with some edges removed.

Lemma 4.1. *Let $FQ_n = G_0 \otimes_{PM \cup PM_1} G_1$, for $n \geq 2$. For any a, $b \in V(G_0)$ with $a \neq b$, if $d_{G_0}(a, b) = n-1$, then $\{a_1, \bar{a}\} = \{b_1, \bar{b}\}$. Otherwise, $\{a_1, \bar{a}\} \cap \{b_1, \bar{b}\} = \emptyset$, where $\{aa_1, bb_1\} \subset PM_1$, $\{a\bar{a}, b\bar{b}\} \subset PM$.*

Proof. Without loss of generality, we assume the binary sequence of a is $0x_2x_3 \ldots x_n$. Then $a_1 = 1x_2x_3 \ldots x_n, \bar{a} = 1\bar{x_2}\bar{x_3} \ldots \bar{x_n} (\bar{x_i} = 1 - x_i)$. If $d_{G_0}(a, b) = n-1$, we have $b = 0\bar{x_2}\bar{x_3} \ldots \bar{x_n}$. Obviously, $b_1 = \bar{a}, \bar{b} = a_1$. That is, $\{a_1, \bar{a}\} = \{b_1, \bar{b}\}$. Furthermore, if $d_{G_0}(a, b) < n-1$, then $\{a_1, \bar{a}\} \cap \{b_1, \bar{b}\} = \emptyset$. In fact, it follows from $a \neq b$ that $a_1 \neq b_1$ and $\bar{a} \neq \bar{b}$. Suppose on the contrary that $\{a_1, \bar{a}\} \cap \{b_1, \bar{b}\} \neq \emptyset$, then we have $a_1 = \bar{b}$ or $\bar{a} = b_1$. Without loss of generality, we assume $a_1 = \bar{b}$, let $a_1 = \bar{b} = 1y_2y_3 \ldots y_n$. Then $a = 0y_2y_3 \ldots y_n$, $b = 0\bar{y_2}\bar{y_3} \ldots \bar{y_n}$, $d_{G_0}(a, b) = n-1$, a contradiction. □

It is easy to see the following, we list it without proof.

Lemma 4.2. *Let FQ_n be an n-dimensional folded hypercube with $n \geq 3$, a, $b \in V(FQ_n)$ and $ab \in E(FQ_n)$. Then a and b have no common adjacent vertices.*

Lemma 4.3. *Let $S \subset E(FQ_n)$ be a set of edges with $|S| \leq 3n-2$, $n \geq 5$. There exists a connected component C in $FQ_n - S$ such that $|V(C)| \geq 2^n - 2$.*

Proof. Consider $FQ_n = G_0 \otimes_{PM \cup PM_1} G_1$ for $n \geq 5$. Let $S \cap E(G_0) = S_0$, $S \cap E(G_1) = S_1$, $S \cap (PM \cup PM_1) = S_2$, then $|S_0| + |S_1| + |S_2| = |S| \leq 3n-2$. Assume C is the largest connected component in $FQ_n - S$. It is impossible that both $|S_0|$ and $|S_1|$ are more than $3n-8$. In fact, if $|S_0| > 3n-8$ and $|S_1| > 3n-8$, then $|S| \geq |S_0| + |S_1| \geq 6n-14$, contradicting to the assumption $|S| \leq 3n-2$ for $n \geq 5$. Without loss of generality, we suppose that $|S_0| \leq |S_1|$. Then we consider two scenarios.

Case 1. $|S_0| \leq 3n-8$ and $|S_1| \leq 3n-8$.

If $|S_0| \leq n-2$ and $|S_1| \leq n-2$, then all of $G_0 - S_0$ and $G_1 - S_1$ are connected (as $\lambda(G_0) = \lambda(G_1) = n-1$). Note that $|PM \cup PM_1| = 2^n > 3n-2 \geq |S_2|$. Thus, $G_0 - S_0$ is connected to $G_1 - S_1$ and $|V(C)| = |V(G_0 - S_0)| + |V(G_1 - S_1)| = 2^n$.

If $|S_0| \leq n-2$ and $n-1 \leq |S_1| \leq 2n-5$, then $G_0 - S_0$ is connected. According to Lemma 3.1, there exists a connected component C_1 in $G_1 - S_1$ with $|V(C_1)| \geq 2^{n-1} - 1$. It follows from $|S_2| \leq |S| - |S_1| \leq 2n-1$ and $|PM \cup PM_1| = 2^n$ that $|PM \cup PM_1| - |S_2| \geq 3$. So $G_0 - S_0$ is connected to C_1 and $|V(C)| \geq |V(C_1)| + |V(G_0 - S_0)| \geq 2^n - 1$.

If $|S_0| \leq n-2$ and $2n-4 \leq |S_1| \leq 3n-8$, then $|S_2| \leq n+2$. Obviously, $G_0 - S_0$ is connected. According to Lemma 3.4, there exists a connected component C_1 in $G_1 - S_1$ with $|V(C_1)| \geq 2^{n-1} - 2$. It follows from $|S_2| \leq n+2$ and $|PM \cup PM_1| = 2^n$ that $|PM \cup PM_1| - |S_2| \geq 5$. So $G_0 - S_0$ is connected to C_1 and $|V(C)| \geq |V(C_1)| + |V(G_0 - S_0)| \geq 2^n - 2$.

If $n-1 \leq |S_0| \leq 2n-5$ and $n-1 \leq |S_1| \leq 2n-5$, then $|S_2| = |S| - |S_0| - |S_1| \leq n$. By Lemma 3.1, $G_0 - S_0$ ($G_1 - S_1$, respectively) contains a connected component C_0 (C_1, respectively) with $|V(C_0)|$ ($|V(C_1)|$, respectively) $\geq 2^{n-1} - 1$. It follows from $|S_2| \leq n$ and $|PM \cup PM_1| = 2^n$ that $|PM \cup PM_1| - |S_2| \geq 5$. Therefore, C_0 is connected to C_1 and $|V(C)| \geq |V(C_0)| + |V(C_1)| \geq 2^n - 2$.

If $n-1 \leq |S_0| \leq 2n-5$ and $2n-4 \leq |S_1| \leq 3n-8$, then $|S_2| = |S| - |S_0| - |S_1| \leq 3$. By Lemmas 3.1 and 3.4, $G_0 - S_0$ ($G_1 - S_1$, respectively) contains a connected component C_0 (C_1, respectively) with $|V(C_0)| \geq 2^{n-1} - 1 (|V(C_1)| \geq 2^{n-1} - 2$, respectively). There are at least $2^n - 6$ edges connect C_0 and C_1 in $G - S_0 - S_1$. As $2^n - 6 - 3 > 0$, then C_0 is connected to C_1 in $G - S$. If $|V(C_0)| = 2^{n-1}$ or $|V(C_1)| \geq 2^{n-1} - 1$, then $|V(C)| \geq 2^n - 2$. We only need to consider the situation when $|V(C_0)| = 2^n - 1, |V(C_1)| = 2^n - 2$. Let $V(G_0) - V(C_0) = \{c\}, V(G_1) - V(C_1) = \{a, b\}$. If $\{ca, cb\} \cap (PM \cup PM_1) = \emptyset$, then $|V(C)| \geq 2^n - 1$. If exactly one of $\{ca, cb\}$ belongs to $(PM \cup PM_1)$, then $|V(C)| \geq 2^n - 2$. Thus, we assume $\{ca, cb\} \subset PM \cup PM_1$. By the definition of FQ_n, $d_{G_1}(a, b) = n-1$. By Lemma 4.1, there exists a vertex in C_0 connected to a and b. It follows that $N'_{G_1}(\{a, b\}) \subset S_1$ and $|S_1| \geq 2(n-1)$. Then $|S_2| \leq 1$ and $|V(C)| = 2^n$.

We next assume $2n-4 \leq |S_0| \leq 3n-8, 2n-4 \leq |S_0| \leq 3n-8$, and $|S_2| \leq 6-n \leq 1$.

By Lemma 3.4, $G_0 - S_0$ ($G_1 - S_1$, respectively) contains a connected component C_0 (C_1, respectively) with $|V(C_0)|$ ($|V(C_1)|$, respectively) $\geq 2^{n-1} - 2$. There are at least $2 * 2^{n-1} - 4 = 2^n - 8$ edges connect C_0 and C_1 in $G - S_0 - S_1$. As $2^n - 8 - 1 > 0$, then C_0 is connected to C_1 in $G - S$. To prove that $|V(C)| \geq 2^n - 2$, we only need to consider two cases. We first assume $|V(C_0)| = 2^{n-1} - 1, |V(C_1)| = 2^{n-1} - 2$. Let $V(G_0) - V(C_0) = \{c\}, V(G_1) - V(C_1) = \{a, b\}$, then $d_{G_1}(a, b) = 1$. In fact, if $d_{G_1}(a, b) \geq 2$, then $N'_{G_1}(\{a, b\}) \subset S_1$ and $|S_1| \geq |N'_{G_1}(\{a, b\})| \geq 2(n-1)$. $|S| \geq |S_0| + |S_1| \geq 4n-6$ contradicts to $|S| \leq 3n-2$. c is connected to at least one vertex of $\{a, b\}$. It is not difficult to find that $|V(C)| = 2^n$. Then we assume $|V(C_0)| = |V(C_1)| = 2^{n-1} - 2$. Let $V(G_0) - V(C_0) = \{a, b\}, V(G_0) - V(C_1) = \{c, d\}$, then $d_{G_0}(a, b) = 1$ and $d_{G_1}(c, d) = 1$. In fact, if $d_{G_0}(a, b) > 1$, $N'_{G_0}(\{a, b\}) \subset S_0$ and $|S_0| \geq |N'_{G_0}(\{a, b\})| \geq 2(n-1)$. $|S| \geq |S_0| + |S_1| \geq 4n-6$, contradicting to $|S| \leq 3n-2$. Then, each vertex of $\{a, b\}$($\{c, d\}$, respectively) is connected to at most one vertex of $\{c, d\}$ ($\{a, b\}$, respectively). Note that Lemma 4.2 and $|S_2| \leq 1$, it is not difficult to find that $|V(C)| = 2^n$.

Case 2. $|S_1| \geq 3n-7$, then $|S_0| + |S_2| \leq 5$.

If $0 \leq |S_0| \leq 3$, then $G_0 - S_0$ is connected. Since each vertex of G_1 has two adjacent vertices in G_0 and $|S_2| \leq 5$, then there are at most two vertices of $G_1 - S_1$ are disconnected to $G_0 - S_0$. Thus, $|V(C)| \geq 2^n - 2$.

Assume $4 \leq |S_0| \leq 5$. Thus $|S_2| \leq 1$ and $|S_0| \leq 5 \leq 2n-5$. By Lemma 3.1, there exists a connected component C_0 in $G_0 - S_0$ with $|V(C_0)| \geq 2^{n-1} - 1$. Since each vertex of G_1 has two adjacent vertices in G_0 and $|S_2| \leq 1$, then there is at most one vertex of G_1 disconnected to $G_0 - S_0$. Thus, $|V(C)| \geq 2^n - 2$.

Combining the above arguments, the proof is complete. □

Remark 4.4. The result of Lemma 4.3 is optimal in that there exists a set of edges S with $|S| = 3n - 1$ for $n \geq 5$ such that $FQ_n - S$ contains a connected component C with $|V(C)| \leq 2^n - 3$.

In fact, consider a set of vertices $F = \{x, y, z\}$, where xy, $yz \in E(FQ_n)$. By Lemma 4.2, there is no triangle in FQ_n, then $xz \notin E(FQ_n)$. By simple calculation, $|N'_{FQ_n}(F)| = 3n - 1$. Let S be a set of edges of FQ_n such that $S = N'_{FQ_n}(F)$, then $FQ_n - S$ contains a connected component induced by F. Therefore, $|V(C)| \leq 2^n - 3$.

By the arguments similar to that of Theorem 3.7, we have the following.

Theorem 4.5. *An n-dimensional folded hypercube is F-strong Menger edge connected if $|F| \leq 2n - 2$ and $n \geq 5$.*

5. Conclusions

In this paper, we introduce the F-strongly Menger edge connectivity and discuss the strongly Menger edge connectivity of hypercubes and folded hypercubes. We show that an n-dimensional hypercube is F-strong Menger edge connected if $|F| \leq 2n - 4$ and $n \geq 4$, and an n-dimensional folded hypercube is F-strong Menger edge connected if $|F| \leq 2n - 2$ and $n \geq 5$. The concept F-strongly Menger edge connectivity is a generalization of Menger's Theorem (edge version). Exploring the F-strong Menger edge connectivity of hypercubes and folded hypercubes for more general $|F|$ is a natural question (we have no counterexample to show that Q_n is not F-strongly Menger edge connected for large n).

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