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Parametrically defined nonlinear differential equations, differential–algebraic equations, and implicit ODEs: Transformations, general solutions, and integration methods

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ABSTRACT

The study deals with nonlinear ordinary differential equations defined parametrically by two relations; these arise in fluid dynamics and are a special class of coupled differential–algebraic equations. We propose a few techniques for reducing such equations, first or second order, to systems of standard ordinary differential equations as well as techniques for the exact integration of these systems. Several examples show how to construct general solutions to some classes of nonlinear equations involving arbitrary functions. We specify a procedure for the numerical solution of the Cauchy problem for parametrically defined differential equations and related differential–algebraic equations. The proposed techniques are also effective for the numerical integration of problems for implicitly defined equations.

1. Introduction

1.1. Preliminary remarks

Von Mises and Crocco type transformations are often used to reduce the order of boundary layer equations as well as some other nonlinear PDEs. These transformations suggest choosing suitable first- or second-order partial derivatives as new dependent variables [1–7]. In some cases, the resulting equations admit exact solution in implicit or parametric form [8]. In effect, finding exact solutions to the original PDEs reduces to

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integrating ODEs defined parametrically. It is noteworthy that parametrically defined nonlinear differential equations also arise directly in the theory of ODEs [8].

The study [8] described a few classes of nonlinear first- and second-order ordinary differential equations in parametric form that allow the construction of general solutions (parametrically defined equations are not treated in the literature dealing with the standard ODEs; e.g., see [9–14]). These solutions were further used to construct exact solutions of unsteady boundary-layer equations.

The parametrically defined nonlinear differential equations dealt with in the present paper are a special class of coupled differential–algebraic equations (often abbreviated to DAEs). In the literature, the focus is mainly on numerical solution methods for such equations (e.g., see [15–19]).

The present study proposes a few techniques for the exact integration of first- and second-order ODEs defined parametrically, based on reducing them to standard systems of ODEs. By way of example, general solutions are constructed for a few parametrically defined nonlinear differential equations involving arbitrary functions. In addition, the study describes an approach to the numerical solution of the Cauchy problem for parametrically defined nonlinear differential equations and related differential–algebraic equations.

1.2. Ordinary differential equations defined parametrically

In general, an nth-order ordinary differential equation defined parametrically is given by two equations of the form [8]

$$\begin{aligned} &F_1(x, y, y'_x, y''_{xx}, \dots, y_x^{(m)}, t) = 0, \\ &F_2(x, y, y'_x, y''_{xx}, \dots, y_x^{(n)}, t) = 0, \end{aligned} \tag{1}$$

where $y = y(x)$ is the unknown function, $y_x^{(n)} = d^n y/dx^n$, $t = t(x)$ is a functional parameter, and $n \geq m$. In the degenerate case where neither F_1 nor F_2 contains derivatives, relations (1) represent a parametric equation of a plane curve in implicit form.

In what follows, we assume that the functional parameter t cannot generally be explicitly eliminated from Eqs. (1).

2. First-order differential equations defined parametrically

2.1. Some transformations

A first-order ODE defined parametrically is generally given by two equations

$$F_1(x, y, y'_x, t) = 0, \quad F_2(x, y, y'_x, t) = 0. \tag{2}$$

We assume that one of Eqs. (2) can be solved for the derivative y'_x. On eliminating this derivative from the other equation, we rewrite the original equation in the canonical form

$$F(x, y, t) = 0, \quad y'_x = G(x, y, t). \tag{3}$$

Differential–algebraic equations of the form (3) are referred to as a system of semi-explicit DAEs or ODEs with constraints [16,18]. Usually, such equations are reduced to a standard system of ODEs for $y = y(x)$ and $t = t(x)$ by differentiating the first equation in (3) with respect to x [16]. However, an alternative system of ODEs, for $y = y(t)$ and $x = x(t)$, is more convenient for seeking exact solutions to semi-explicit DAEs. The alternative system is derived below.

By taking the full differential of the first equation in (3) and multiplying the second one by dx, we get

$$F_x\,dx + F_y\,dy + F_t\,dt = 0, \quad dy = G(x, y, t)\,dx, \tag{4}$$

where F_x, F_y, and F_t are the respective partial derivatives of $F = F(x, y, t)$. Eliminating dy from (4) yields the first-order equation

$$(F_x + GF_y)x'_t + F_t = 0, \tag{5}$$

while eliminating dx from (4) yields a different first-order equation

$$(F_x + GF_y)y'_t + GF_t = 0. \tag{6}$$

Eqs. (5) and (6) represent a standard system of first-order equations for $x = x(t)$ and $y = y(t)$. If a solution to this system is found, it will also be a solution to the original Eq. (3) in parametric form.

Occasionally, it suffices to use one of the equations of (5) or (6) and the first equation of (3).

Remark 1. If Eqs. (5) and (6) are used, isolated solutions satisfying $F_x + GF_y = 0$ may be lost; this issue calls for further analysis (see also Equation 2).

2.2. General solutions to some ODEs defined parametrically

Equation 1. Consider the nonlinear first-order ODE defined parametrically

$$x = f(t)y + g(t), \quad y'_x = h(t), \tag{7}$$

where $f = f(t)$, $g = g(t)$, and $h = h(t)$ are arbitrary functions. It is a special case of Eq. (3) with $F = x - fy - g$ and $G = h$. Substituting these into (6) gives the linear equation $(1 - fh)y'_t - hf'_t y - hg'_t = 0$. By integrating it while taking into account the first relation in (7), we arrive at the general solution to Eq. (7) in parametric form

$$x = fy + g, \quad y = CE + E\int \frac{hg'_t\,dt}{(1 - fh)E}, \tag{8}$$

where C is an arbitrary constant and $E = \exp\Big(\int \frac{hf'_t\,dt}{1-fh}\Big)$.

Equation 2. Let us look at the equation

$$F(x, y, t) = 0, \quad y'_x = -F_x/F_y, \tag{9}$$

which is a degenerate case of Eq. (3) with $F_x + GF_y \equiv 0$ (see Remark 1). Eliminating dy from (4) while taking into account that $G = -F_x/F_y$ yields

$$F_t\,dt = 0.$$

This equation splits into $dt = 0$ and $F_t = 0$. The former equation solves to give $t = C$, where C is an arbitrary constant, which produces the general solution to the original equation in implicit form

$$F(x, y, C) = 0.$$

The latter equation generates a singular solution which is defined by two algebraic (generally transcendental) equations:

$$F(x, y, t) = 0, \quad F_t(x, y, t) = 0.$$

2.3. A semiinverse method for integrating parametrically defined ODEs

The theorem below provides a semiinverse method for constructing general solutions to parametrically defined ODEs.

Theorem 1. *Suppose the first-order ODE (generating equation)*

$$z'_t = \Phi(z,t) \tag{10}$$

admits a general solution in closed form

$$\phi(z,t,C) = 0, \tag{11}$$

where C is an arbitrary constant. Then the parametrically defined ODE (3) with

$$G(x,y,t) = -\frac{F_t z_x - F_x z_t + \Phi(z,t)F_x}{F_t z_y - F_y z_t + \Phi(z,t)F_y}, \tag{12}$$

where $F = F(x,y,t)$ and $z = z(x,y,t)$ are arbitrary functions, admits the general solution defined implicitly by two relations

$$F(x,y,t) = 0, \quad \phi(z(x,y,t),t,C) = 0. \tag{13}$$

The theorem can be proved by direct verification.

As the generating equation (10), one can use one of the several thousand integrable equations listed in the handbooks [10,13]. The theorem allows one to obtain general solutions to a large number of different parametrically defined ODEs of the form (3). These solutions can be used as test problems to assess the accuracy of relevant numerical methods.

2.4. Cauchy problem. Procedure for numerical solution

Consider the Cauchy problem for the parametric first-order equation (3) with the initial condition

$$y(x_0) = y_0. \tag{14}$$

The initial value of the parameter, $t = t_0$, is found from the algebraic (or, generally, transcendental) equation

$$F(x_0, y_0, t_0) = 0, \tag{15}$$

where x_0 and y_0 are the values appearing in the initial condition (14). This can be done numerically with, for example, the Newton's method.

With the procedure outlined in Section 2.1, we first reduce the parametrically defined differential equation (3) to the standard system of first-order equations (5) and (6) for $x = x(t)$ and $y = y(t)$. Subject to the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{16}$$

this system can then be solved numerically by the Runge–Kutta method or another suitable method [11,12,17,18,20].

Remark 2. The algebraic (or transcendental) equation (15) can generally have several different roots, in which case the original problem (3), (14) will have the same number of respective different solutions.

2.5. First-order ODEs defined implicitly

Let us look at the Cauchy problem for the implicit equation

$$F(x, y, y'_x) = 0 \tag{17}$$

with the initial condition (14).

Substituting $y'_x = t$ reduces Eq. (17) to the parametric equation

$$F(x, y, t) = 0 \quad y'_x = t, \tag{18}$$

which must satisfy condition (14).

Problem (18), (14) is a special case of problem (3), (14) with $G(x, y, t) = t$. It can be solved numerically with the approach mentioned above in Section 2.4.

3. Second-order differential equations defined parametrically

3.1. Some transformations

A parametrically defined second-order ODE is generally given by the two equations

$$F_1(x, y, y'_x, y''_{xx}, t) = 0, \quad F_2(x, y, y'_x, y''_{xx}, t) = 0. \tag{19}$$

We assume that the second derivative y''_{xx} can be eliminated from Eqs. (19) and then y'_x can be isolated to obtain $y'_x = F(x, y, t)$. With this equation, we exclude y'_x from either equation in (19) to get $F_3(x, y, y''_{xx}, t) = 0$. By isolating y''_{xx}, we eventually arrive at the canonical form of Eq. (19):

$$y'_x = F(x, y, t), \quad y''_{xx} = G(x, y, t). \tag{20}$$

Below we outline a technique for integrating such equations; it is assumed that the parameter t can generally not be eliminated from Eqs. (20).

Differentiating the first equation in (20) with respect to t yields $(y'_x)'_t = F_x x'_t + F_y y'_t + F_t$. By virtue of $y'_t = F x'_t$ and $(y'_x)'_t = x'_t y''_{xx}$, we get

$$x'_t y''_{xx} = F_x x'_t + F F_y x'_t + F_t. \tag{21}$$

By eliminating y''_{xx} with the aid of the second equation in (20), we arrive at the first-order equation

$$(G - F_x - F F_y) x'_t = F_t. \tag{22}$$

In view of $y'_t = F x'_t$, we can rewrite this equation as

$$(G - F_x - F F_y) y'_t = F F_t. \tag{23}$$

Equations (22) and (23) represent a standard system of first-order equations for $x = x(t)$ and $y = y(t)$. If this system is solved, the solution is also a solution to the original parametric equation (20). In some cases, one can use either equation (22) or (23) and the first equation in (20).

Remark 3. When Eqs. (22) and (23) are used, isolated solutions, satisfying $G - F_x - F F_y = 0$, may be lost. This issue calls for further analysis (see also Equation 4).

3.2. General solutions to some ODEs defined parametrically

Equation 3. Consider the nonlinear second-order ODE defined parametrically

$$y'_x = \varphi(t), \quad y''_{xx} = \psi(t), \tag{24}$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. This corresponds to Eq. (20) with $F = \varphi(t)$ and $G = \psi(t)$. Then Eq. (22) becomes $\psi(t)x'_t = \varphi'_t(t)$, whose general solution is

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \tag{25}$$

where C_1 is an arbitrary constant. Relation (25) and the first equation of (24) represent a parametrically defined ODE of the form (7) in which

$$f(t) = 0, \quad g(t) = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \quad h(t) = \varphi(t). \tag{26}$$

By inserting (26) in (8), we obtain the general solution of ODE (24) in parametric form

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \quad y = \int \frac{\varphi(t)\varphi'_t(t)}{\psi(t)}\,dt + C_2, \tag{27}$$

where C_1 and C_2 are arbitrary constants (C in (8) has been renamed C_2 here).

Equation 4. Let us look at the special case of Eq. (20) with arbitrary $F = F(x, y, t)$ and G expressed in terms of F as

$$G = F_x + FF_y. \tag{28}$$

Then the expression in parentheses in (22) and (23) becomes zero and Eq. (20) admits the first integral

$$y'_x = F(x, y, C_1),$$

where C_1 is an arbitrary constant. In addition, there is a singular solution satisfying the parametric first-order ODE

$$y'_x = F(x, y, t), \quad F_t(x, y, t) = 0.$$

3.3. First integrals of system (22)–(23)

Equation 5. If

$$G = F_x + FF_y + a(t)b(x)F_t, \tag{29}$$

where $a(t)$, $b(x)$, and $F = F(x, y, t)$ are arbitrary functions, the variables in Eq. (22) separate to give the first integral

$$\int b(x)\,dx = \int \frac{dt}{a(t)} + C_1.$$

Equation 6. If

$$G = F_x + FF_y + a(t)b(y)FF_t, \tag{30}$$

where $a(t)$, $b(y)$, and $F = F(x,y,t)$ are arbitrary functions, the variables in Eq. (23) separate to give the first integral

$$\int b(y)\,dy = \int \frac{dt}{a(t)} + C_1.$$

Remark 4. Using the representation (22), (23), parametrically defined ODEs of the form (20) can be integrated with semiinverse methods similar to that outlined in Section 2.3.

3.4. Cauchy problem. Procedure for numerical solution

Consider the Cauchy problem for the parametrically defined second-order ODE (20) with the initial conditions

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1. \tag{31}$$

The initial value of the parameter, $t = t_0$, is determined by the algebraic (or transcendental) equation

$$y_1 = F(x_0, y_0, t_0), \tag{32}$$

where x_0, y_0, and y_1 are the values appearing in the initial conditions (31); Eq. (32) follows from the first equation in (20). This can be done numerically with, for example, the Newton's method.

Using the technique outlined in Section 3.1, we first reduce the parametric equation (20) to the standard system of first-order equations (22)–(23) for $x = x(t)$ and $y = y(t)$. Then, using the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{33}$$

we can solve this system numerically with the Runge–Kutta method or another suitable method [11,12,17,18,20].

Remark 5. The algebraic (or transcendental) equation (32) can generally have several different roots. In this case, the original problem (20), (31) will have exactly the same number of respective different solutions.

3.5. Exact and numerical integration of equations defined implicitly

Consider the Cauchy problem for the implicit equation

$$y'_x = F(x, y, y''_{xx}) \tag{34}$$

with the initial conditions (31).

By substituting $y''_{xx} = t$, we reduce Eq. (34) to the parametric equation

$$y'_x = F(x,y,t), \quad y''_{xx} = t \tag{35}$$

and initial conditions (33) in which the value of the parameter t_0 is determined by the algebraic (or transcendental) equation (32).

Problem (35), (33) is a special case of problem (20), (33) with $G(x,y,t) = t$. It can be solved numerically using the procedure outlined above.

4. Brief conclusions

The study dealt with nonlinear ordinary differential equations defined parametrically by Eqs. (1). For first- and second-order equations, we have proposed techniques for reducing them to standard systems of ordinary differential equations, in which x and y are unknown functions of the independent variable t, and specified techniques for their exact integration.

We have constructed general solutions for a few classes of first- and second-order ODEs defined parametrically. We have described procedures for the numerical integrations of the Cauchy problem for parametrically defined ordinary differential equations and differential–algebraic equations.

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