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Area-universal drawings of biconnected outerplane graphs

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ABSTRACT

Contact graph representation is a classical graph drawing style where vertices are represented by geometric objects such that edges correspond to contacts between the objects. Contact graph representations using axis-aligned rectilinear polygons are well-investigated. On the other hand, only a scarcity of results and techniques are available for cases using polygons that are not necessarily rectilinear. In this paper, we investigate a type of contact graph representations (named t-TkR) using k-sided convex polygons with their boundaries being t-sided. Given a biconnected outerplane graph, we present a clean necessary and sufficient condition for the graph to admit a t-TkR. We give a linear time algorithm for constructing an area-universal 3-T4R of a given biconnected outerplane graph, which is of interest since most of the previous results on area-universal drawings are with respect to rectilinear settings.

1. Introduction

A *contact graph representation* of a planar graph is a drawing in which vertices are represented by interior-disjoint geometric objects such that edges correspond to contacts between those objects. Following Koebe's circle packing theorem that every planar graph can be drawn as touching circles, a variety of contact graph representations have been proposed and studied in the literature over the years, see, e.g., [3–5,8–10].

Motivated by various applications in floor-planning, cartographic design, and data visualization, *rectilinear duals*, in which all vertices are represented by axis-aligned rectilinear polygons such that the drawing forms a tiling of a rectangle, have received extensive investigation in both VLSI design and graph drawing communities. The *polygonal complexity* of a rectilinear dual is defined as the maximum number of sides of any polygon in the drawing. A rectilinear dual of a graph G is called *area-universal* if it can realize any area-assignment $f : V(G) \to \mathbb{R}_{>0}$ in the sense that for every $v \in V(G)$, the corresponding polygon has area $f(v)$. Designing algorithms for constructing area-universal rectilinear duals of low polygonal complexity has been the focus of a number of recent results (see [3] and its citations).

In practice, it is common to encounter objects displayed as polygons that are not necessarily rectilinear. In contrast to the relatively well-studied rectilinear cases, only a scarcity of results and methods are available for tackling cases for polygons that are not necessarily rectilinear.

To extend the study of rectilinear duals to broader settings, the drawing style *convex polygonal dual* is proposed as a convex polygonal analogue of rectilinear duals [4]. Formally, a convex polygonal dual is a contact representation of a graph in which vertices are represented by convex polygons such that the drawing forms a tiling of a convex polygon. A drawing is called *k-sided* if each vertex is represented by a polygon of at most k sides in the drawing.

Our interests in this paper focus on biconnected outerplane graphs having (t,k)*-touching convex polygon representations*, which are k-sided convex polygonal duals with

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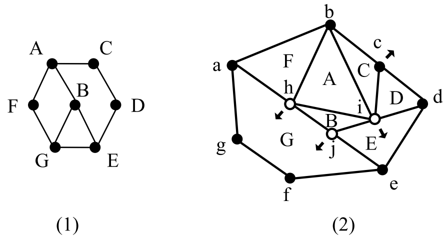


Fig. 1. A graph G and its convex polygonal dual G^d.

their boundary polygons being t-sided. We abbreviate such a representation as t-TkR. For instance, Fig. 1(2) is a 6-T4R.

The purpose of this paper is to study convex polygonal duals for biconnected outerplane graphs. For a biconnected outerplane graph G, we present:

1. A clean necessary and sufficient condition for the existence of a t-TkR, for $k > 3$: G admits a t-TkR iff $3 \le t \le (k-1)|V(G)| - |E(G)| + 1$.
2. A simple linear time algorithm for constructing an area-universal 3-T4R of G.

Related work. The study of representing graphs by touching triangles was initiated in [9]. It is known that triconnected cubic plane graphs [10] and strongly outerplane graphs [8] admit 3-T3R. Plane graphs having straight-line drawings with only triangular faces have been characterized by flat-angle assignments [1] and Schnyder labellings [2]. For contact representations of convex polygons, it was shown in [5] that 6-sided polygons are necessary and sufficient for plane graphs if holes are allowed. In [4], it was shown that convex polygonal duals can be defined in Monadic Second-Order Logic, yielding fixed-parameter tractability results for checking whether various plane graphs admit convex polygonal duals. For area-universal drawings, 12-sided polygons are known to be necessary and sufficient for rectilinear duals [3]. To our best knowledge, the work of [7] on table cartograms is the only result on area-universal drawings in a non-rectilinear setting.

2. Preliminaries

A graph is *planar* iff it can be drawn in the Euclidean plane without edge crossings. A *plane graph* is a planar graph with a fixed combinatorial embedding and a designated outer face. We write $f_O(G)$ to denote the outer face of a plane graph $G = (V, E)$. All the faces other than $f_O(G)$ are called *inner faces*. A vertex (or an edge) is called *boundary* if it is located in $f_O(G)$; otherwise, it is *non-boundary*.

An *outerplanar graph* is a planar graph with a planar embedding in which all vertices belong to the outer face. An outerplanar graph with such an embedding is called an *outerplane graph*. A graph is *biconnected* if removing any single vertex does not render the graph disconnected.

We write $\overline{xy}$ to denote a side of a polygon whose two end points are x and y. See Fig. 1 for an example of a convex polygonal dual. Note that convex polygon G in Fig. 1(2) has four sides, namely, $\overline{ag}$, $\overline{gf}$, $\overline{fe}$ and $\overline{ea}$. Note that the side $\overline{ea}$ consists of three segments (i.e., edges) (e, j), (j, h) and (h, a).

In a convex polygonal dual G^d, *junction points* are points that are endpoints of some segments in the drawing. For convenience, we write $BJ(G^d)$ and $NJ(G^d)$ to denote the sets of boundary and non-boundary junction points of G^d, respectively. In Fig. 1(2), there are 10 junction points with $BJ(G^d) = \{a, b, c, d, e, f, g\}$ and $NJ(G^d) = \{h, i, j\}$. Note that c is interior to one side $\overline{bd}$ of the boundary polygon. The arrows in the drawing indicate 180° angles.

3. Convex polygonal duals of biconnected outerplane graphs

With respect to a t-TkR of a biconnected outerplane graph, we first prove the following lemma which gives an upper bound on the number of sides of the boundary polygon (i.e., t):

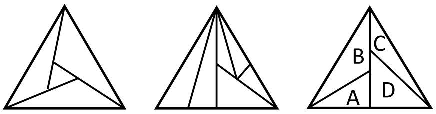
Lemma 1. *Let G be a biconnected outerplane graph. If G admits a t-TkR, then* $3 \le t \le (k-1)|V(G)| - |E(G)| + 1$. *Moreover, the equality* $t = (k-1)|V(G)| - |E(G)| + 1$ *holds iff in the drawing,*

(1) *each polygon is exactly k-sided, and*
(2) *each non-boundary junction point is interior to a side of a polygon.*

Proof. The $t \ge 3$ is obvious since a polygon must have at least 3 sides. Let N be the total number of polygon corners in the t-TkR, say G^d, of G. For convenience, G^d is also referred to as a drawing. As each vertex in $V(G)$ corresponds to a polygon (of at most k sides) in G^d, $N \le k|V(G)|$. Since G is a biconnected outerplane graph, each polygon must intersect the boundary of the drawing in one connected path or a point; otherwise, the vertex corresponding to that polygon will be a cut-vertex in G-violating the assumption of G being biconnected. Since a path of s sides has $s+1$ corners, when a k-sided polygon contains s sides on the boundary of the drawing, it has exactly k-s-1 corners located not along the boundary of the drawing G^d.

Let $N = N_O + N_I$, where N_O denotes the total number of corners located along the boundary of the drawing (i.e., corners associated with boundary junction points), and N_I denotes the total numbers of corners located in the interior of the drawing (i.e., corners associated with non-boundary junction points). First, we show that $N_O \ge |V(G)| + t$. To see this, suppose N_v is the number of sides on the boundary of the drawing that intersect with the polygon corresponding to v. Note that a side can intersect with more than one polygon. For instance, in Fig. 1(2) $N_C = N_D = 1$ and the polygons corresponding to vertices C and D intersect with side $\overline{bcd}$. In view of above, $N_O = \sum_{v \in V(G)} (N_v + 1) = \sum_{v \in V(G)} N_v + |V(G)| \ge t + |V(G)|$.

For N_I, we argue that $N_I \ge \sum_{p \in NJ(G^d)} \deg(p) - |NJ(G^d)|$. Since each junction point can be associated with at most one 180° angle, the number of 180° angles at non-boundary junction points is at most $|NJ(G^d)|$. Hence the above inequality holds. As we note that each of $NJ(G^d)$ corresponds to an inner face of G, according to Euler's formula, $|NJ(G^d)| = |E(G)| - |V(G)| + 1$. For the



(1) (2) (3)

Fig. 2. Sliceability and one-sidedness.

term $\sum_{p \in NJ(G^d)} \deg(p)$ (i.e., the sum of degrees around non-boundary junction points), it is not hard to see that its value equals the number of boundary edges plus two times the number of non-boundary edges in G. So we have $\sum_{p \in NJ(G^d)} \deg(p) = 2|E(G)| - |E(f_O(G))|$. Since G is an outerplane graph, $|E(f_O(G))| = |V(G)|$. To sum up, $N_I \geq (2|E(G)| - |V(G)|) - (|E(G)| - |V(G)| + 1) = |E(G)| - 1$.

Finally, we have $k|V(G)| \geq N = N_O + N_I \geq (|V(G)| + t) + (|E(G)| - 1)$. By re-ordering the terms, we get $t \leq (k-1)|V(G)| - |E(G)| + 1$.

The equality $t = (k-1)|V(G)| - |E(G)| + 1$ is reached iff the two equalities $k|V(G)| = N$ and $N_I = |E(G)| - 2|V(G)| - 1$ are met. The first one holds iff each polygon is exactly k-sided. The second one is met iff each non-boundary junction point has a 180° angle (i.e., is interior to a side of a polygon). □

The above lemma can be seen as a necessary condition for a biconnected outerplane graph to have a t-TkR. Surprisingly, the simple condition is also sufficient when $k \geq 4$, which we will prove later.

A t-TkR is *area-universal* iff for any area-assignment to $V(G)$, there is a combinatorially equivalent[1] drawing realizing that area-assignment. Let $\triangle = \{a, b, c\}$ be a triangle with a, b and c being its three corners. We define the following two operations which subdivide $\triangle$:

1. Adding a new point d inside of $\triangle$, followed by adding three straight lines linking d to a, b, c.
2. Adding a new point d dividing the line $\overline{bc}$, followed by adding a straight line linking a to d.

We call a 3-T3R (i.e., a 3-sided convex polygonal dual with a triangular boundary) *sliceable* iff it can be constructed by applying the above 2 operations to its constituent triangles recursively. A 3-T3R is *one-sided* iff for each straight line in the drawing, one side of the line borders exactly one polygonal region. Following basic geometry, the following lemma is easy to observe:

Lemma 2. *Every one-sided and sliceable 3-T3R is area-universal. Moreover, if the coordinates of the 3 boundary vertices are fixed, the drawing realizing any given area-assignment is unique.*

See Fig. 2 for illustrations of the above concepts. Fig. 2(1) is one-sided but not sliceable; Fig. 2(2) is one-sided and sliceable; Fig. 2(3) is sliceable but not one-sided. Note that Fig. 2(3) is clearly not area-universal since it cannot realize the area-assignment: $f(A) = f(C) = 0.4$, $f(B) = f(D) = 0.1$, for regions A and C would have touched each other.

Given a biconnected outerplane graph G, the plane graph G^* (not the dual graph) is defined as the graph resulting from the following operations:

1. Add a new vertex s in the unbounded face of G, namely, $f_O(G)$, and add an edge between s and each vertex in the boundary face.
2. Take the dual, and the new outer face is designated to the one corresponding to s.

As an illustrating example, Fig. 3(2) shows the plane graph G^* (the outer cycle is depicted in a dotted-line) associated with the graph G depicted in Fig. 3(1). The subgraph of G^* that excludes the edges in the outer cycle is called the *skeleton*. For a biconnected outerplane graph G, the skeleton is always a tree. Notice that each non-leaf vertex in the skeleton has degree at least 3.

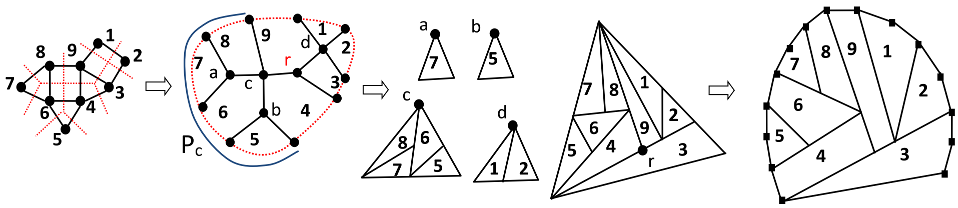
A skeleton can be regarded as a rooted tree by selecting any non-leaf vertex r as its root. For any non-leaf vertex v in the skeleton, we define T_v as the sub-tree rooted at v. We let F_v be the set of faces in G^* such that all their non-boundary edges are contained in $E(T_v)$. For instance, $F_c = \{5, 6, 7, 8\}$ in Fig. 3(2). We write P_v to denote the sub-path of $f_O(G^*)$ formed by including all boundary edges contained in some face $F \in F_v$. See Fig. 3(2) for P_c.

We are now in a position to prove one of the main results in the paper:

Theorem 3. *Every biconnected outerplane graph admits an area-universal 3-T4R, which can be constructed in linear time.*

Proof. The basic idea is that G^* can be regarded as a "sketch" of a contact representation of G. All we have to do is to find a drawing of G^* meeting the requirement of the theorem.

[1] The reader is referred to [6] for more about the notion of combinatorial equivalence in graph contact representations.



(1) (2) (3) (4)

Fig. 3. A graph G, its skeleton with root r and the construction of an area-universal t-T4R.

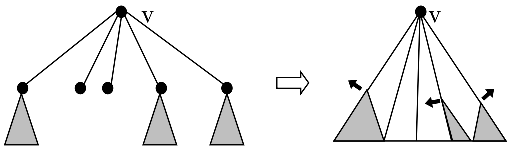


Fig. 4. Illustration of PROCEDURE 1.

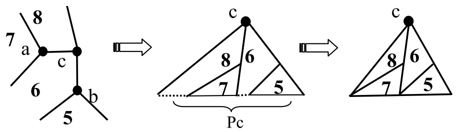


Fig. 5. Applying PROCEDURE 1 to the subtree rooted at c in Fig. 3(2).

The proof is based on a bottom-up approach operating on the skeleton of the input biconnected outerplane graph. When a vertex v in the skeleton is encountered, all vertices in $V(T_v) \setminus \{v\}$ have been processed already. During each iteration, the following invariant is kept:

- For each non-leaf vertex $u \neq r$ that has been processed, the sub-graph G_u of G^* induced by F_u is drawn as an area-universal drawing satisfying:
  1. Each face (in F_u) is either a triangle or a convex quadrangle.
  2. Each non-boundary vertex in $V(T_u) \setminus \{u\}$ is a junction point having an 180° angle in the current sub-drawing.
  3. The outer boundary of the sub-drawing of G_u is a triangle in which u is one of its corner and P_u is one of its sides (hence u is not an 180° corner in any face in F_u).

Let v be the vertex currently being processed. If v is a leaf, we do nothing. If v is non-leaf vertex that is not the root, we do the following:

PROCEDURE 1

1. Let $u_1, \ldots, u_s$ be the children of v.
2. For each u_i that is not a leaf, if one of the two faces incident to the edge $\{v, u_i\}$ is not contained in F_v, we make u_i an 180° corner of the face (indicated by an arrow in the illustration). If both faces are contained in F_v, the choice is arbitrary.
3. For each face $F \in F_v \setminus \bigcup_{1 \leq i \leq s} F_{u_i}$, we contract as many its boundary edges as possible such that F has at least 3 sides in the drawing.
4. Straighten the path P_v.

The idea behind PROCEDURE 1 is depicted in Fig. 4. See Fig. 5 for a showcase of applying PROCEDURE 1 to the subtree rooted at c in Fig. 3(2). Upon encountering c, triangles associated with faces 5 and 7 are available. Step 3 of PROCEDURE 1 (i.e., contracting boundary edges) is shown in the middle of Fig. 5.

It is easy to see from the illustration that the resulting drawing of G_v following the application of PROCEDURE 1 satisfies the invariant. To see that the drawing is area-universal, we first divide each quadrangle into two triangles by adding a straight line linking v to the opposite corner on the boundary of the drawing. Then, if we treat each sub-drawing of G_{u_i} as a single triangle, the drawing of G_v is clearly a one-sided and sliceable 3-T3R (and hence area-universal (Lemma 2)).

What remains to be done is the case when the root r is encountered.

PROCEDURE 2

1. Let $u_1, \ldots, u_s$ be the children of r.
2. Choose a designated face $F \in (F_r \setminus \bigcup_{1 \leq i \leq s} F_{u_i})$; remove F from F_r.
3. Apply PROCEDURE 1 to yield a drawing with a triangular boundary $\Delta = \{r, x, y\}$. Choose a point t interior to the side $\overline{xy}$ such that there is a face in F_r containing both r and t. (See Fig. 6(1–2).)
4. Deform the drawing by changing the boundary triangle from $\{r, x, y\}$ to $\{t, x, y\}$ with r an interior point of the side $\overline{xy}$. (See Fig. 6(2–3).)
5. Subdivide the boundary edge $\{x, y\}$ of F, resulting in two edges $\{x, z\}, \{y, z\}$.
6. Select $\{t, x, z\}$ as designated vertices on the boundary cycle and then straighten everywhere on the boundary cycle except those 3 selected vertices (making the boundary triangular). (See Fig. 6(3–4).)

From the way PROCEDURE 1 operates, the presence of point t in Step 3 of PROCEDURE 2 is easy to observe, so is the preservation of convexity associated with the deformation mentioned in Step 4 of PROCEDURE 2. See Fig. 6 for the idea behind how PROCEDURE 2 works. Similar to PROCEDURE 1, it can be easily seen that the resulting drawing after applying PROCEDURE 2 is an area-universal drawing. The outer boundary of the drawing is a triangle. Each inner face is drawn as a triangle or a convex quadrangle. It is clear that our algorithm takes linear time. Hence the theorem holds. □

See Fig. 3(1–3) for a full example of the algorithm. Theorem 3 is tight in the sense that it fails in general when the underlying graph class is changed to either biconnected 2-outerplane graphs or 1-connected outerplane graphs. Also, for biconnected outerplane graphs, in general, 3-sided polygons are not sufficient to construct convex polygonal duals.

Combining the above algorithm and Lemma 1, we prove the other main theorem of the paper:

Theorem 4. *For a biconnected outerplane graph G, and for $k > 3$, G admits a t-TkR iff $3 \leq t \leq (k-1)|V(G)| - |E(G)| + 1$.*

Proof. We show only the "if" part as the "only-if" part follows from Lemma 1. The case $t = 3$ is a direct result of Theorem 3. We observe that in the resulting drawing of the

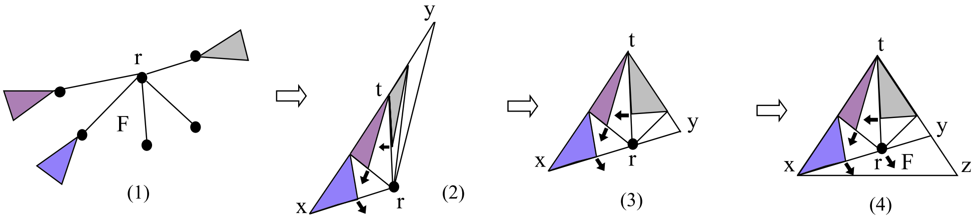


Fig. 6. Illustration of PROCEDURE 2.

algorithm in the proof of Theorem 3, each non-boundary vertex in G^* is assigned to be a 180° corner for some face. Therefore, for the case of $t > 3$, we only need to add sufficient additional corners in the boundary of the drawing while maintaining k-sidedness and convexity for each polygon (in view of Lemma 1). This is achieved by slight perturbations in the boundary. (See Fig. 3(4).) Hence the theorem is concluded. □

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