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Finite dimensional invariant subspaces for algebras of linear operators and amenable Banach algebras

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ABSTRACT

We study a finite dimensional invariant subspace property similar to Fan's Theorem on semigroups for arbitrary Banach algebras $\mathcal{A}$ in terms of amenability of $\mathcal{X}(\mathcal{A},\phi)$, the closed subalgebra of $\mathcal{A}$ generated by the set of all maximal elements in $\mathcal{A}$ with respect to a character ϕ. As a consequence, we offer some applications to the measure algebra $M(G)$ and the generalized Fourier algebra $A_p(G)$ of a locally compact group G.

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1. Introduction

The Lorentz transformations, as the result of attempts by Lorentz and others, are invertible linear mappings on $\mathbb{R}^4$ that explain how the speed of light was observed to be independent of the reference frame. In fact, they preserve the quadratic form

$$J(\vec{x}) = x^2 + y^2 + z^2 - c^2t^2,$$

for all $\vec{x} = (x, y, z, t) \in \mathbb{R}^4$, where the constant c is the speed of light.

For any Lorentz transformation T, there is a three dimensional subspace V of $\mathbb{R}^4$ such that

$$T(\vec{x}) = \vec{x} \qquad \text{and} \qquad J(\vec{x}) \geq 0$$

for all $\vec{x} \in V$. Pontrjagin [29], Iovihdov [7], Krein [8,9] and Naimark [23,24] investigated infinite-dimension versions of the above result.

Let E be a separated locally convex space and let X be a subset of E containing an n-dimensional subspace. In [4], Fan obtained the following finite-dimensional invariant subspace property $P(n)$ for n-dimensional subspaces contained in X: If $\mathfrak{J} = \{T_s : s \in S\}$ is a representation of a left amenable (discrete) semigroup S as continuous linear operators from E into E such that $T_s(L)$ is an n-dimensional subspace contained in X whenever L is an n-dimensional subspace contained in X and $s \in S$, and if there exists a closed $\mathfrak{J}$-invariant subspace H in E of codimension n with the property that $(x+H) \cap X$ is compact and convex for each $x \in E$, then there exists an n-dimensional subspace L_0 contained in X such that $T_s(L_0) = L_0$ for all $s \in S$.

The origin of Fan's Theorem goes back to the investigations of Pontrjagin, Iovihdov, Krein and Naimark, concerning invariant subspaces for Lorentz transformations on a Hilbert space; see [7–9,23,24,29].

Lau [13], in 1983, extended Fan's result to topological semigroups by showing that a topological semigroup S is left amenable (i.e., the left uniformly continuous functions on S has a left invariant mean) if and only if S has the property $P(n)$ for all natural numbers; see also [16].

More recently, Lau and Zhang [17] established a finite dimensional invariant subspace property similar to Fan's Theorem for the class of Lau algebras in terms of left amenability; see also [18]. Recall that a Lau algebras is a Banach algebra $\mathcal{L}$ which is the unique predual of a W^*-algebra M and the identity element of M is a character on $\mathcal{L}$; the large class of Lau algebras was introduced and studied by Lau [12] who called them F-algebras. Later on, in his useful monograph, Pier [28] introduced the name "Lau algebra". The class of Lau algebras includes the Fourier algebra $A(G)$ and the group algebra $L^1(G)$ of a locally compact group G, as well as the measure algebra $M(S)$ of a locally compact topological semigroup S; see Lau [12]. It also includes the Fourier–Stieltjes algebra $B(G)$ of any topological group G; see Lau and Ludwing [15].

In this paper, we continue with the more general setting of Banach algebras $\mathcal{A}$ to obtain some fixed point characterizations for amenability of $\mathcal{A}$ with respect to a character ϕ on $\mathcal{A}$ in terms of a finite-dimensional invariant subspace property $F(n, \phi)$. As a consequence of our main result, we prove that a locally compact group G is amenable if and only if the measure algebra $M(G)$ satisfies $F(n, \psi_\rho)$ for all natural numbers n and $\rho \in \widehat{G}$, where $\widehat{G}$ is the dual group of G and ψ_ρ is the character on $M(G)$ induced by ρ. Moreover, we prove that G_d, the group G equipped with the discrete topology, is amenable if and only if the measure algebra $M(G)$ satisfies $F(n, \psi_{1,d})$ for all natural numbers n, where $\psi_{1,d}$ is the discrete augmentation character on $M(G)$. Finally, we prove that the generalized Fourier algebras $A_p(G)$ always satisfy $F(n, \phi)$ for all natural numbers n and characters ϕ on $A_p(G)$.

2. Fixed point property

Let $\mathcal{A}$ be a Banach algebra and denote by $\Delta(\mathcal{A})$ the set of all characters on $\mathcal{A}$; that is, non-zero bounded multiplicative linear functionals on $\mathcal{A}$. For $\phi \in \Delta(\mathcal{A})$, Kaniuth, Lau and Pym [10,11] introduced and investigated a notion of amenability for Banach algebras called ϕ-amenability; see also [21]. In fact, $\mathcal{A}$ is said to be ϕ-amenable if there exists a functional m on $\mathcal{A}^*$ satisfying

$$m(\phi) = 1, \qquad m(f \cdot a) = \phi(a)m(f),$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where $f \cdot a \in \mathcal{A}^*$ is defined by $(f \cdot a)(b) = f(ab)$ for all $b \in \mathcal{A}$. Such functional m is called a ϕ-mean. For some related works see [2,3,25–27]. The notion of ϕ-amenability is a considerable generalization of left amenability of Lau algebras introduced and investigated in Lau [12].

Following [11], we denote by $P_1(\mathcal{A}, \phi)$ the set of all ϕ-maximal elements of $\mathcal{A}$; that is, elements $a \in \mathcal{A}$ with

$$\|a\| = \phi(a) = 1.$$

Then, as readily checked, $P_1(\mathcal{A}, \phi)$ with the multiplication of $\mathcal{A}$ is a semigroup. A representation of $P_1(\mathcal{A}, \phi)$ on a set C is a map $T : P_1(\mathcal{A}, \phi) \times C \longrightarrow C$ denoted by $(a, x) \longmapsto T_a(x)$ such that

$$T_{ab}(x) = T_a(T_b(x)),$$

for all $a, b \in P_1(\mathcal{A}, \phi)$ and $x \in C$. Let us remark that $P_1(\mathcal{A}, \phi)$ with the metric topology inherited from $\mathcal{A}$ is a topological semigroup; that is, a semigroup with a Hausdorff topology under which the binary operation is continuous.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. If $\mathcal{A}$ has a ϕ-mean in $\overline{P_1(\mathcal{A}, \phi)}^{w^*}$, then every jointly continuous representation of the metric topological semigroup $P_1(\mathcal{A}, \phi)$ on a compact Hausdorff space C has a common fixed point in C.*

Proof. Let $\ell^\infty(P_1(\mathcal{A},\phi))$ denote the set of all bounded complex-valued functions on $P_1(\mathcal{A},\phi)$ with the supremum norm and pointwise multiplication and let $LUC(P_1(\mathcal{A},\phi))$ denote the space of bounded complex-valued left uniformly continuous functions on $P_1(\mathcal{A},\phi)$; that is, all functions $f \in \ell^\infty(P_1(\mathcal{A},\phi))$ for which the map

$$a \longmapsto \ell_a f$$

from $P_1(\mathcal{A},\phi)$ into $\ell^\infty(P_1(\mathcal{A},\phi))$ is norm continuous, where $(\ell_a f)(b) = f(ab)$ for all a, b in $P_1(\mathcal{A},\phi)$.

By Mitchell's Theorem [20], it is sufficient to prove that there exists an element ψ in $\Delta(LUC(P_1(\mathcal{A},\phi)))$ which is left invariant; that is,

$$\psi \circ \ell_a = \psi$$

for all $a \in P_1(\mathcal{A},\phi)$. For this end, let us remark that, by the Banach–Alaoglu Theorem, $\Delta(LUC(P_1(\mathcal{A},\phi)))$ is a weak*-compact subset of $LUC(P_1(\mathcal{A},\phi))^*$. Define

$$\begin{gathered} T : P_1(\mathcal{A},\phi) \times \Delta(LUC(P_1(\mathcal{A},\phi))) \longrightarrow \Delta(LUC(P_1(\mathcal{A},\phi))) \\ (a,\psi) \longmapsto T_a(\psi), \end{gathered}$$

where

$$T_a(\psi) := \psi \circ \ell_a$$

for all $a \in P_1(\mathcal{A},\phi)$ and $\psi \in \Delta(LUC(P_1(\mathcal{A},\phi)))$. It is obvious that T is a separately continuous representation. Now, let us show that the mapping $a \longmapsto T_a(\psi)$ is uniformly continuous for each $\psi \in \Delta(LUC(P_1(\mathcal{A},\phi)))$. Since $f \in LUC(P_1(\mathcal{A},\phi))$, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$\|\ell_a f - \ell_b f\| < \epsilon \tag{1}$$

whenever

$$\|a - b\| < \delta.$$

Furthermore, for $a, b \in P_1(\mathcal{A},\phi)$, $f \in LUC(P_1(\mathcal{A},\phi))$ and $\psi \in \Delta(LUC(P_1(\mathcal{A},\phi)))$, we have

$$\langle T_a(\psi) - T_b(\psi), f\rangle = \langle \psi, \ell_a f - \ell_b f\rangle. \tag{2}$$

It follows from (1) and (2) that the mapping $a \longmapsto T_a(\psi)$ is uniformly continuous for all ψ in $\Delta(LUC(P_1(\mathcal{A},\phi)))$. Now, suppose that $m \in \overline{P_1(\mathcal{A},\phi)}^{w^*}$ is a ϕ-mean on $\mathcal{A}^*$. By Theorem 1.4 of [10] and its proof, there is a net (a_α) in $P_1(\mathcal{A},\phi)$ such that $\|aa_\alpha - a_\alpha\| \longrightarrow 0$ for all $a \in P_1(\mathcal{A},\phi)$. Fix $\psi \in \Delta(LUC(P_1(\mathcal{A},\phi)))$ and consider the net

$$(T_{a_\alpha}\psi) \subseteq \Delta(LUC(P_1(\mathcal{A},\phi))).$$

Since $\Delta(LUC(P_1(\mathcal{A},\phi)))$ is compact, there is a subnet of $(T_{a_\alpha}\psi)$ which converges to an element ψ_0 of $\Delta(LUC(P_1(\mathcal{A},\phi)))$. So, without loss of generality, we may assume that

$$T_{a_\alpha}\psi \to \psi_0.$$

Since the representation T is uniformly continuous, it follows that

$$T_{aa_\alpha}\psi - T_{a_\alpha}\psi \to 0$$

for all $a \in P_1(\mathcal{A},\phi)$. Hence, for each $a \in P_1(\mathcal{A},\phi)$ we have

$$\begin{aligned}\psi_0 \circ \ell_a &= T_a\psi_0\\ &= T_a \lim_\alpha T_{a_\alpha}\psi\\ &= \lim_\alpha T_{aa_\alpha}\psi\\ &= \lim_\alpha T_{a_\alpha}\psi\\ &= \psi_0.\end{aligned}$$

This means that $P_1(\mathcal{A},\phi)$ has the required fixed point property. $\square$

3. Finite dimensional invariant subspaces

Let E be a separated locally convex vector space and let X be a subset of E. As in Lau [13], $L_n(X)$ denotes the set of all n-dimensional subspaces of E contained in X.

Let $T : P_1(\mathcal{A},\phi) \times E \longrightarrow E$ be a linear representation; that is, a representation of the semigroup $P_1(\mathcal{A},\phi)$ for which the maps $x \longmapsto T_a(x)$ are linear on E for all $a \in P_1(\mathcal{A},\phi)$. Then $L_n(X)$ is called $P_1(\mathcal{A},\phi)$-invariant under T if

$$T_a(L_n(X)) \subseteq L_n(X)$$

for all $a \in P_1(\mathcal{A},\phi)$; that is, $T_a(L) \in L_n(X)$ for all $L \in L_n(X)$.

In general, $P_1(\mathcal{A},\phi)$ can be quite small, even empty; see Examples 5.4 of [11]; we denote by $\Delta_1(\mathcal{A})$ the set of all $\phi \in \Delta(\mathcal{A})$ with $P_1(\mathcal{A},\phi) \neq \emptyset$.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \Delta_1(\mathcal{A})$. For each natural numbers n, we say that the semigroup $P_1(\mathcal{A},\phi)$ has the property $F(n,\phi)$ if the following holds:

Let E be a separated locally convex vector space and let $T : P_1(\mathcal{A},\phi) \times E \longrightarrow E$ be a linear representation such that

(1) The mapping $a \mapsto T_a(x)$ is continuous for each fixed $x \in E$.

(2) T is jointly continuous on compact subsets of E.

Let X be a subset of E and let H be a closed $P_1(\mathcal{A}, \phi)$-invariant subspace of E with codimension n having the following properties:

(3) $L_n(X)$ is nonempty and $P_1(\mathcal{A}, \phi)$-invariant.

(4) $(x + H) \cap X$ is compact for all $x \in E$.

Then there is $L_0 \in L_n(X)$ such that $T_a(L_0) = L_0$ for all $a \in P_1(\mathcal{A}, \phi)$.

We are now ready to prove our main result. First, let us denote by $\mathcal{X}(\mathcal{A}, \phi)$ the closed linear span of $P_1(\mathcal{A}, \phi)$ which is a closed subalgebra of $\mathcal{A}$.

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta_1(\mathcal{A})$. Then the following statements are equivalent.*

(a) *The semigroup $P_1(\mathcal{A}, \phi)$ has the property $F(n, \phi)$ for all natural numbers n.*

(b) *The semigroup $P_1(\mathcal{A}, \phi)$ has the property $F(1, \phi)$.*

(c) *$\mathcal{X}(\mathcal{A}, \phi)$ has a $\phi|_{\mathcal{X}(\mathcal{A},\phi)}$-mean in $\overline{P_1(\mathcal{A}, \phi)}^{w^*}$.*

Proof. Suppose that (b) holds. Take $E := \mathcal{X}(\mathcal{A}, \phi)^{**}$ endowed with the weak* topology and define

$$T : P_1(\mathcal{A}, \phi) \times \mathcal{X}(\mathcal{A}, \phi)^{**} \longrightarrow \mathcal{X}(\mathcal{A}, \phi)^{**}$$
$$(a, m) \longmapsto T_a(m)$$

where

$$T_a(m) := a \cdot m$$

for all $a \in P_1(\mathcal{A}, \phi)$ and $m \in \mathcal{X}(\mathcal{A}, \phi)^{**}$ and "$\cdot$" is the left $\mathcal{X}(\mathcal{A}, \phi)$-module natural action on $\mathcal{X}(\mathcal{A}, \phi)^{**}$. It is easy to check that the linear representation T is jointly continuous on weak* compact sets of $\mathcal{X}(\mathcal{A}, \phi)^{**}$. Put

$$Y := \overline{P_1(\mathcal{A}, \phi)}^{w^*}.$$

Let X be the union of all one-dimensional subspaces of $\mathcal{X}(\mathcal{A}, \phi)^{**}$ generated by elements m of Y and define

$$H = \{n \in \mathcal{X}(\mathcal{A}, \phi)^{**} : n(\phi) = 0\}.$$

Then the nonempty set

$$L_1(X) = \{\mathbb{C}m : m \in Y\}$$

is $P_1(\mathcal{A}, \phi)$-invariant. Moreover, for each $n' \in \mathcal{X}(\mathcal{A}, \phi)^{**}$ the set

$$(n' + H) \cap X = \{n'(\phi)m : m \in Y\}$$

is weak* compact and H is a closed $P_1(\mathcal{A}, \phi)$-invariant subspace of $\mathcal{X}(\mathcal{A}, \phi)^{**}$ with codimension one. By $F_1(1, \phi)$, there is $m_0 \in Y$ such that

$$T_a(\mathbb{C}m_0) = \mathbb{C}m_0$$

for all $a \in P_1(\mathcal{A}, \phi)$. That is, $a \cdot m_0 = \lambda_a m_0$ for some $\lambda_a \in \mathbb{C}$. Since m_0 and $a \cdot m_0$ belong to Y, we must have $\lambda_a = 1$. Since $\mathcal{X}(\mathcal{A}, \phi)$ is the closed linear span of $P_1(\mathcal{A}, \phi)$, we conclude that

$$b \cdot m_0 = \phi(b)m_0$$

for all $b \in \mathcal{X}(\mathcal{A}, \phi)$, which gives (c).

Now, suppose that (c) holds and let H be a closed $P_1(\mathcal{A}, \phi)$-invariant subspace of E with codimension n and $\pi : E \longrightarrow E/H$ be the quotient map. For each $a \in P_1(\mathcal{A}, \phi)$, the map $T_a : E \longrightarrow E$ induces a map

$$\tilde{T}_a : E/H \longrightarrow E/H$$

such that

$$\pi \circ T_a = \tilde{T}_a \circ \pi.$$

Consider a subspace L of X with dimension n. By Definition 3.1 (4) and the fact that the only compact vector space is zero, $\pi|_L$ is injective. Hence $\pi(L) = E/H$ and $\pi|_L$ is invertible. Also, it is not hard to see that $\tilde{T}_a$ is an invertible operator on E/H. Let $\{y_1, \cdots, y_n\}$ be a basis for E/H. Then

$$\{(T_a \circ (\pi|_L)^{-1} \circ \tilde{T}_a^{-1})(y_1), \cdots, (T_a \circ (\pi|_L)^{-1} \circ \tilde{T}_a^{-1})(y_n)\}$$

is a basis for $T_a(L)$ and

$$(\pi \circ T_a \circ (\pi|_L)^{-1} \circ \tilde{T}_a^{-1})(y_i) = (\tilde{T}_a \circ \pi \circ (\pi|_L)^{-1} \circ \tilde{T}_a^{-1})(y_i) = y_i \tag{3}$$

for $i = 1, 2, \cdots, n$. Choose a fix basis $\{y_1, \cdots, y_n\}$ for E/H. Then there exists a unique basis

$$\{x_1^L, \cdots, x_n^L\}$$

for L such that for each $i = 1, 2, \cdots, n$,

$$\pi(x_i^L) = y_i.$$

Since $L_n(X)$ is $P_1(\mathcal{A},\phi)$-invariant,

$$T_a(L) \in L_n(X)$$

and

$$\pi(T_a(L)) = E/H$$

for all $a \in P_1(\mathcal{A},\phi)$ and $L \in L_n(X)$. Therefore, there exists a unique basis

$$\{x_1^{T_a(L)}, \cdots, x_n^{T_a(L)}\}$$

for $T_a(L)$ such that for each $i = 1, 2, \cdots, n$,

$$\pi(x_i^{T_a(L)}) = y_i.$$

By assumption, the sets $(\pi^{-1}(y_i) \cap X)$ for $i = 1, 2, \cdots, n$ are compact subsets of E. Let

$$\mathcal{K} := \{(x_1^L, \cdots, x_n^L) : L \in L_n(X)\ \} \subseteq \prod_{i=1}^{n}(\pi^{-1}(y_i) \cap X).$$

Since $\mathcal{K}$ is closed, by the Tychynoff Theorem, $\mathcal{K}$ is a compact Hausdorff space. Now, define

$$\Omega := \{\Psi_a : \mathcal{K} \longrightarrow \mathcal{K} : a \in P_1(\mathcal{A},\phi)\}$$

where Ψ_a is defined by

$$\Psi_a((x_1^L, \cdots, x_n^L)) := (x_1^{T_a(L)}, \cdots, x_n^{T_a(L)})$$

for all $(x_1^L, \cdots, x_n^L) \in \mathcal{K}$. It is clear that Ω is a representation of $P_1(\mathcal{A},\phi)$ on $\mathcal{K}$. By the relation (3) and the fact that

$$\{x_1^{T_a(L)}, \cdots, x_n^{T_a(L)}\}$$

is the only basis for $T_a(L)$ satisfying

$$\pi(x_i^{T_a(L)}) = y_i$$

for $i = 1, 2, \cdots, n$, we have

$$\Psi_a((x_1^L, \cdots, x_n^L)) := (\ (T_a \circ (\pi|_L)^{-1} \circ \tilde{T}_a^{-1})(y_1), \cdots, (T_a \circ (\pi|_L)^{-1} \circ \tilde{T}_a^{-1})(y_n)\ ).$$

Now, we show that Ω is jointly continuous. To prove this, first note that if $\{y_1, \cdots, y_n\}$ is a basis for E/H, then

$$\{\tilde{T}_a^{-1}(y_1), \cdots, \tilde{T}_a^{-1}(y_n)\}$$

is also a basis in E/H for any $a \in P_1(\mathcal{A}, \phi)$. So, there exist the scalars $\alpha_i^j(a)$ with $1 \le i, j \le n$, depend only on $a \in P_1(\mathcal{A}, \phi)$ such that

$$\begin{pmatrix} \alpha_1^1(a) & \cdots & \alpha_1^n(a) \\ \vdots & & \vdots \\ \alpha_n^1(a) & \cdots & \alpha_n^n(a) \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \tilde{T}_a^{-1}(y_1) \\ \vdots \\ \tilde{T}_a^{-1}(y_n) \end{pmatrix}.$$

We therefore have

$$\begin{aligned} \Psi_a((x_1, \cdots, x_n)) &= \left((T_a \circ (\pi|_L)^{-1}) \left(\sum_{i=1}^n \alpha_1^i(a) y_i \right), \cdots, (T_a \circ (\pi|_L)^{-1}) \left(\sum_{i=1}^n \alpha_n^i(a) y_i \right) \right) \\ &= \left(T_a \left(\sum_{i=1}^n \alpha_1^i(a) x_i \right), \cdots, T_a \left(\sum_{i=1}^n \alpha_n^i(a) x_i \right) \right) \end{aligned}$$

for all $a \in P_1(\mathcal{A}, \phi)$ and $(x_1, \cdots, x_n) \in \mathcal{K}$, where L is the unique element of $L_n(X)$ containing $\{x_1, \cdots, x_n\}$ as its basis. It follows that

$$\Psi_a((x_1, \cdots, x_n)) = T_a(\begin{pmatrix} \alpha_1^1(a) & \cdots & \alpha_1^n(a) \\ \vdots & & \vdots \\ \alpha_n^1(a) & \cdots & \alpha_n^n(a) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}).$$

We now show that the mapping $a \longmapsto \tilde{T}_a$ from $P_1(\mathcal{A}, \phi)$ to $B(E/H)$, the algebra of bounded operators on E/H, is continuous. Indeed, as E/H is finite dimensional, all the topologies on $B(E/H)$ coincide.

Therefore, it suffices to show that $a \longmapsto \langle \tilde{T}_a(y), y^* \rangle$ is continuous for $y \in E/H$ and $y^* \in (E/H)^*$; this also follows from the weak continuity of the mapping $a \mapsto T_a(x)$ for each fixed $x \in E$ and the fact that

$$\langle \tilde{T}_a(y), y^* \rangle = \langle \pi(T_a(x)), y^* \rangle = \langle T_a(x), y^* \circ \pi \rangle.$$

It follows that the mapping $a \longmapsto \tilde{T}_a^{-1}$ from $P_1(\mathcal{A}, \phi)$ to $B(E/H)$ is continuous. Therefore, the functions

$$a \longrightarrow \alpha_j^i(a) = \langle \tilde{T}_a^{-1}(y), y_i^* \rangle$$

on $P_1(\mathcal{A}, \phi)$ for $i, j = 1, 2, \cdots, n$ are continuous, where $\{y_1^*, \cdots, y_n^*\}$ is the basis dual to $\{y_1, \cdots, y_n\}$.

Let $(a_\alpha) \subseteq P_1(\mathcal{A}, \phi)$ and $((x_1^\alpha, \cdots, x_n^\alpha)) \subseteq K$ be such that

$$a_\alpha \longrightarrow a \in P_1(\mathcal{A}, \phi)$$

and

$$(x_1^\alpha, \cdots, x_n^\alpha) \longrightarrow (x_1, \cdots, x_n) \in K.$$

Then we have

$$\alpha_i^j(a_\alpha) \longrightarrow \alpha_i^j(a)$$

for $i, j = 1, \cdots, n$ and consequently

$$\Psi_{a_\alpha}((x_1^\alpha, \cdots, x_n^\alpha)) \to \Psi_a((x_1, \cdots, x_n)),$$

since the original representation T is jointly continuous on compact sets. This shows that Ω is a jointly continuous representation. By Proposition 2.1, we conclude that there is a common fixed point, say $(x_1^0, \cdots, x_n^0)$, for Ω. Let $\{x_1^0, \cdots, x_n^0\}$ be a basis for $L_0 \in L_n(X)$. From the definition of Ψ_a, it follows that the basis $\{x_1^0, \cdots, x_n^0\}$ for L_0 is also a basis for $T_a(L_0)$ for all $a \in P_1(\mathcal{A}, \phi)$. This leads us to $L_0 = T_a(L_0)$ for all $a \in P_1(\mathcal{A}, \phi)$, and so (a) holds. $\square$

Proposition 3.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta_1(\mathcal{A})$ and let $P_1(\mathcal{A}, \phi)$ be weak* dense in $P_1(\mathcal{A}^{**}, \phi^{**})$. If $\mathcal{A}$ is ϕ-amenable with a ϕ-mean of norm one, then the semigroup $P_1(\mathcal{A}, \phi)$ has the property $F(n, \phi)$ for all natural numbers n.*

Proof. The proof is essentially the same as the proof of Theorem 3.2 $(c) \Longrightarrow (a)$. We therefore omit it. $\square$

The following example shows that for a Banach algebra $\mathcal{A}$ and a character $\phi \in \Delta(\mathcal{A})$, weak* density of $P_1(\mathcal{A}, \phi)$ in $P_1(\mathcal{A}^{**}, \phi^{**})$ does not imply that $\mathcal{X}(\mathcal{A}, \phi) = \mathcal{A}$.

Example 1. Let $\mathcal{A}$ be the set of all upper-triangular 3×3 matrices

$$[\lambda_{ij}] := \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

over $\mathbb{C}$, where $i, j \in \{1, 2, 3\}$. Then $\mathcal{A}$ with the usual operations and the norm

$$\|[\lambda_{ij}]\| = \sum_{j=1}^{3} \sum_{i=1}^{3} |\lambda_{ij}|$$

is a Banach algebra. An easy computation shows that

$$\Delta(\mathcal{A}) = \{\phi_1, \phi_2, \phi_3\},$$

where for $k = 1, 2, 3$

$$\phi_k([\lambda_{ij}]) = \lambda_{kk}$$

for all $[\lambda_{ij}] \in \mathcal{A}$. Consider $i, j, k \in \{1, 2, 3\}$ and let

$$E_{kk} = [\lambda_{ij}]$$

be the matrices over $\mathbb{C}$ such that $\lambda_{ij} = 1$ if $i = j = k$ and otherwise $\lambda_{ij} = 0$. For $k \in \{1, 2, 3\}$, we have

$$P_1(\mathcal{A}, \phi_k) = \{E_{kk}\};$$

indeed, if $[\lambda_{ij}] \in P_1(\mathcal{A}, \phi_k)$, then

$$\phi_k([\lambda_{ij}]) = \lambda_{kk} = 1 \tag{4}$$

and

$$\|[\lambda_{ij}]\| = \sum_{j=1}^{3} \sum_{i=1}^{3} |\lambda_{ij}| = 1; \tag{5}$$

it follows from (4) and (5) that $[\lambda_{ij}] = E_{kk}$. Since $\mathcal{A}$ is of finite dimension, then

$$\overline{P_1(\mathcal{A}, \phi_k)}^{w^*} = P_1(\mathcal{A}^{**}, \phi_k^{**}).$$

But, we have

$$\mathcal{X}(\mathcal{A}, \phi_k) = \left\{ \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \lambda_{11} \in \mathbb{C} \right\}$$

which is not equal to $\mathcal{A}$.

4. Application to group algebras

For a locally compact group G with a left Haar measure λ_G and $1 \leq p \leq \infty$, let $L^p(G)$ denote the set of all equivalence classes of measurable functions $f : G \longrightarrow \mathbb{C}$ such that

$$\|f\|_p = \int_G |f(x)|^p d\lambda_G(x) < \infty.$$

Let $\widehat{G}$ denote the set of all continuous homomorphisms ρ from G into the circle group $\mathbb{T}$, and define $\phi_\rho \in \Delta(L^1(G))$ to be the character induced by ρ on $L^1(G)$; that is,

$$\phi_\rho(f) = \int_G \overline{\rho(x)} f(x)\ d\lambda_G(x)$$

for all $f \in L^1(G)$. It is known that there are no other characters on $L^1(G)$; that is,

$$\Delta(L^1(G)) = \left\{\phi_\rho : \rho \in \widehat{G}\ \right\};$$

see Corollary 23.7 of [5]. Let us recall that, G is called amenable if $L^1(G)$ is ϕ_1-amenable; or equivalently, there is a ϕ_1-mean with norm one in $L^1(G)^{**}$; see [14] for details. It is clear that

$$P_1(L^1(G), \phi_\rho) = \left\{ f \in L^1(G) : \quad \overline{\rho} f \geq 0, \int_G \overline{\rho} f d\lambda = 1 \right\}.$$

Moreover, let $M(G)$ denote the bounded regular Borel measures on G. For each $\rho \in \widehat{G}$, the functional ψ_ρ defined by

$$\psi_\rho(\mu) = \int_G \overline{\rho(x)} d\mu(x)$$

for all $\mu \in M(G)$, is a character on $M(G)$.

In the following example, we state a characterization of amenability of G in terms of a finite dimensional invariant subspace property $F(n, \phi)$ for all natural numbers n.

Proposition 4.1. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *G is amenable.*

(b) *The metric topological semigroup $P_1(M(G), \psi_\rho)$ has the property $F(n, \psi_\rho)$ for all natural numbers n and $\rho \in \widehat{G}$.*

(c) *The metric topological semigroup $P_1(L^1(G), \phi_\rho)$ has the property $F(n, \phi_\rho)$ for all natural numbers n and $\rho \in \widehat{G}$.*

Proof. Consider the map $\Psi : M(G) \longrightarrow M(G)$ defined by

$$\Psi(\mu) = \rho\mu$$

for $\rho \in \widehat{G}$ and $\mu \in M(G)$, where

$$(\rho\mu)(f) = \int_G \rho(x) f(x) d\mu(x)$$

for all f in $C_0(G)$; the set of all continuous functions vanishing at infinity. Then Ψ is an isometric isomorphism. For every $\mu \in M(G)$ and $\rho \in \widehat{G}$, the function $\overline{\rho}\mu$ can be written as a linear combination

$$\overline{\rho}\mu = \sum_{j=1}^{4} c_j \mu_j,$$

where $c_j \in \mathbb{C}$, $\mu_j \geq 0$ and $\|\mu_j\| = 1$ for $1 \leq j \leq 4$. Hence

$$\mu = \Psi(\overline{\rho}\mu) = \sum_{j=1}^{4} c_j \Psi(\mu_j) = \sum_{j=1}^{4} c_j \rho\mu_j$$

and $\rho\mu_j \in P_1(M(G), \psi_\rho)$. So $P_1(M(G), \psi_\rho)$ spans $M(G)$; that is

$$\mathcal{X}(M(G), \psi_\rho) = M(G).$$

It is known that $P_1(M(G), \psi_1)$ is dense in $P_1(M(G)^{**}, \psi_1^{**})$ in the weak* topology $M(G)^{**}$; see [14], Lemma 2.1. It follows easily that $P_1(M(G), \psi_\rho)$ is dense in $P_1(M(G)^{**}, \psi_\rho^{**})$ in the weak* topology of $M(G)^{**}$. Now, we only need to show that G is amenable if and only if for each $\rho \in \widehat{G}$, there is a ψ_ρ-mean on $M(G)^*$. For this end, we note that Ψ is an isometric isomorphism. Therefore, there is a ψ_1-mean with norm one on $M(G)^*$ if and only if there is a ψ_ρ-mean with norm one on $M(G)^*$. So, by Theorem 3.2, we have (a) is equivalent to (b).

Similarly, we can show that

$$\mathcal{X}(L^1(G), \phi_\rho) = L^1(G)$$

and $P_1(L^1(G), \phi_1)$ is weak* dense in $P_1(L^1(G)^{**}, \phi_1^{**})$. So, a similar argument as the above shows that (a) is equivalent to (c). $\square$

Let G be a locally compact group and let $M(G) = M_d(G) \oplus M_c(G)$ be the direct sum decomposition of the measure algebra $M(G)$ into its closed subalgebra $M_d(G)$ of discrete measures and its closed ideal $M_c(G)$ of continuous measures; see [5], Theorem 19.20, p. 273. That is, every measure $\mu \in M(G)$ has a unique decomposition $\mu = \mu_d + \mu_c$, where $\mu_d \in M_d(G)$ and $\mu_c \in M_c(G)$.

The discrete augmentation character on $M(G)$ is defined by

$$\psi_{1,d}(\mu) = \sum_{x \in G} \mu(\{x\})$$

for all $\mu \in M(G)$. Note that $\psi_{1,d}$ is in fact the augmentation character ψ_1 on $M(G_d)$, where G_d is the group G equipped with the discrete topology; see [1], p. 215.

Proposition 4.2. *Let G be a locally compact group. Then G_d is amenable if and only if $P_1(M(G), \psi_{1,d})$ has the property $F(n, \psi_{1,d})$ for all natural numbers n.*

Proof. Note that

$$\{\delta_x : x \in G\} \subseteq P_1(M(G), \psi_{1,d}) \subseteq M_d(G);$$

in fact, for each $\mu \in P_1(M(G), \psi_{1,d})$ we have

$$\begin{aligned}\|\mu_d\| + \|\mu_c\| &= \|\mu\| \\ &= \psi_{1,d}(\mu) \\ &= \psi_{1,d}(\mu_d + \mu_c) \\ &= \psi_{1,d}(\mu_d),\end{aligned}$$

and

$$\psi_{1,d}(\mu_d) \leq \|\mu_d\|.$$

Therefore $\mu_c = 0$. It follows that

$$\mathcal{X}(M(G), \psi_{1,d}) = M_d(G).$$

Now, suppose that $P_1(M(G), \psi_{1,d})$ has the property $F(n, \psi_{1,d})$ for all natural numbers n. Then Theorem 3.2 shows that $M_d(G)$ has a $\psi_{1,d}$-mean in $\overline{P_1(M(G), \psi_{1,d})}^{w^*}$. Since

$$M_d(G) = \ell^1(G),$$

$\ell^1(G)$ has a ϕ_1-mean with norm one in $\ell^1(G)^{**}$. Therefore G_d is amenable.

Conversely, suppose that G_d is amenable. Then $\ell^1(G)$ has a ϕ_1-mean of norm one. It follows that $M_d(G)$ has a $\psi_{1,d}$-mean of norm one. So, by Theorem 3.2, $P_1(M(G), \psi_{1,d})$ has the property $F(n, \psi_{1,d})$ for all natural numbers n. $\square$

Let G be a locally compact group and let $q \in (1, \infty)$ be such that $1/p + 1/q = 1$. Then the Banach algebra $A_p(G)$ consists of all functions of the form

$$\sum_{i=1}^{\infty} g_i * \widehat{f_i}$$

with $f_i \in L^p(G)$, $g_i \in L^q(G)$ and

$$\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty,$$

where for each $i \geq 1$,

$$\widehat{f_i}(x) = f_i(x^{-1})$$

for all $x \in G$. The norm on $A_p(G)$ is defined as

$$\|u\|_{A_p} = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q : u = \sum_{i=1}^{\infty} g_i * \widehat{f_i}\right\}$$

for all $u \in A_p(G)$. It is known that $\Delta(A_p(G))$ can be canonically identified with G. More precisely, the map $x \longmapsto \phi_x$, where $\phi_x(u) = u(x)$ for $u \in A_p(G)$, is a homeomorphism from G onto $\Delta(A_p(G))$; see [22] for more details.

Proposition 4.3. *Let G be a locally compact group. The metric topological semigroup $P_1(A_p(G), \phi_x)$ has the property $F(n, \phi_x)$ for all $x \in G$ and natural numbers n.*

Proof. Let $e \in G$ be the identity element of G, and recall from [19], Lemma 1.1 and Remark 1.2, that

$$\overline{P_1(A_p(G), \phi_e)}^{w^*} = \{m \in A_p(G)^{**} : \|m\| = m(\phi_e) = 1\},$$

and

$$\mathcal{X}(A_p(G), \phi_e) = A_p(G).$$

Now, suppose that $x \in G$ and L_x is the left translation by x on $A_p(G)$ defined by

$$L_x u(y) = u(x^{-1}y)$$

for all $u \in A_p(G)$ and $y \in G$. Then as shown in [6], p. 216,

$$P_1(A_p(G), \phi_x) = L_x(P_1(A_p(G), \phi_e))$$

and

$$\overline{P_1(A_p(G), \phi_x)}^{w^*} = \{m \in A_p(G)^{**} : \|m\| = m(\phi_x) = 1\}.$$

In [22], Lemma 3.1, it is proved that for each $x \in G$, $A_p(G)$ has a ϕ_x-mean in $\overline{P_1(A_p(G), \phi_x)}^{w^*}$. It follows from Theorem 3.2 that $P_1(A_p(G), \phi_x)$ has the property $F(n, \phi_x)$ for all $x \in G$ and natural numbers n. $\square$

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