Linear Algebra and its Applications 510 (2016) 222–229

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Maximal simultaneously nilpotent sets of matrices over antinegative semirings

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ARTICLE INFO

Article history:
Received 9 August 2015
Accepted 11 August 2016
Available online 24 August 2016
Submitted by R. Brualdi

MSC:
16Y60
05C25

Keywords:
Antinegative semiring
Nilpotent matrix
Simultaneous nilpotence

ABSTRACT

We study the simultaneously nilpotent index of a simultaneously nilpotent set of matrices over an antinegative commutative semiring S. We find an upper bound for this index and give some characterizations of the simultaneously nilpotent sets when this upper bound is met. In the special case of antinegative semirings with all zero divisors nilpotent, we also find a bound on the simultaneously nilpotent index for all nonmaximal simultaneously nilpotent sets of matrices and establish their cardinalities in case of a finite S.

1. Introduction

As semirings allow for more general and joint solutions to various problems, while reducing the time complexity of existing algorithms, they are a very active research area in computer science. This algebraic structure has properties that are quite different from other classical algebraic structures as groups, rings and fields. Thus, over last decades,

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http://dx.doi.org/10.1016/j.laa.2016.08.014

many developments in mathematics have been devoted to the research of semirings. A particular class of canonically ordered semirings (also called dioids) come rather naturally into play in connection with algebraic models for many problems arising from computer science, such as scheduling, network analysis, pathfinding problems in graphs, hierarchical clustering, parsing,...

Nilpotent matrices play a crucial role when studying matrices over semirings. In this work, we continue studying the simultaneous nilpotence of a set of matrices. The definition of the simultaneous nilpotence originates in the study of infinite products of matrices and their convergence to the zero matrix.

A *semiring* is a set S equipped with binary operations $+$ and $\cdot$ such that $(S,+)$ is a commutative monoid with identity element 0 and $(S,\cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S. For a semiring S we denote by $\mathcal{N}(S) \subseteq S$ the set of all nilpotent elements in S and by $Z(S)$ the set of zero-divisors in S.

A semiring is *commutative* if $ab = ba$ for all $a, b \in S$. A semiring S is *antinegative* (sometimes also called a *zerosumfree* semiring or an *antiring*), if the condition $a + b = 0$ implies that $a = b = 0$ for all $a, b \in S$.

For example, $\mathbb{Z}_2$ and Boolean algebra $\mathcal{B} = (\{0,1\}, \vee, \wedge)$ are the smallest nontrivial semirings, with the difference that $\mathcal{B}$ is an antinegative semiring, while $\mathbb{Z}_2$ is actually a ring. The set of nonnegative integers with the usual operations of addition and multiplication is a commutative antinegative semiring. Distributive lattices and fuzzy algebras are commutative antinegative semirings.

The set $M_n(S)$ of $n \times n$ matrices over a semiring S is a semiring as well. Let $E_{i,j}$ denote the zero-one matrix with the only nonzero element in the (i,j)-th position. For a set of matrices $\mathcal{R} \subseteq M_n(S)$ we denote the set of products of k matrices by

$$\mathcal{R}^k = \{A_1 A_2 \ldots A_k;\ A_i \in \mathcal{R}\}.$$

For $1 \le i < j \le k$, the product $A_i A_{i+1} \ldots A_j$ is called a *subproduct* of $A_1 A_2 \ldots A_k$. We say that the set of matrices $\mathcal{R}$ is *simultaneously nilpotent* if $\mathcal{R}^p = \{0\}$ for some positive integer p. By $h(\mathcal{R})$ we denote the smallest integer p such that $\mathcal{R}^p = \{0\}$ and call it the *simultaneously nilpotent index*.

In [1] the first properties of the simultaneously nilpotent set of fuzzy matrices are given. In [2] the authors characterize the simultaneous nilpotence of fuzzy matrices and extend some of the results to the bounded distributive lattices. Further properties of nilpotent fuzzy matrices were investigated in [3]. In [5] the authors give some sufficient and some necessary conditions on elements of $\mathcal{R} \subseteq M_n(S)$ for an antinegative semiring S, that cause $\mathcal{R}$ to be simultaneously nilpotent. Tan [4] proves some characterizations of simultaneously nilpotent matrices over commutative antirings with additional assumption of S being without zero-divisors, $\mathcal{N}(S)$ being simultaneously nilpotent, or when $\mathcal{N}(S) = Z(S)$.

In this paper we continue the investigation of the simultaneously nilpotent index of a simultaneously nilpotent set of matrices over an antinegative commutative semiring S by combining various algebraic and graph theoretical methods. In Section 2 we prove that for a simultaneously nilpotent set $\mathcal{R} \subseteq M_n(S)$ its simultaneously nilpotent index is bounded by $n\, h(\mathcal{N}(S))$. We give some characterizations of the simultaneously nilpotent sets where the upper bound is reached. In Section 3 we examine semirings in which every zero-divisor is nilpotent. We prove that in case that the simultaneously nilpotent index $h(\mathcal{R})$ is not maximal, then it is bounded by $n\, h(\mathcal{N}(S)) - h(\mathcal{N}(S))$. In case S is finite, we also give the exact cardinalities of $\mathcal{R}$ in both extremal cases when $h(\mathcal{R})$ is equal to $n\, h(\mathcal{N}(S))$ or $n\, h(\mathcal{N}(S)) - h(\mathcal{N}(S))$.

2. Simultaneously nilpotent index of matrices

In this section we examine the simultaneously nilpotent index of simultaneously nilpotent set of matrices. We start with few technical lemmas.

Lemma 2.1. *Let S be an antinegative semiring. If $A = [a_{ij}] \in M_n(S)$ is a nilpotent matrix, then $a_{ii} \in \mathcal{N}(S)$.*

Proof. If $a_{ii} \notin \mathcal{N}(S)$, then for every k we have $(A^k)_{ii} = a_{ii}^k + b \neq 0$ since S is antinegative and $a_{ii}^k \neq 0$, a contradiction. □

Lemma 2.2. *Let S be an antinegative commutative semiring and $\mathcal{R} \subseteq M_n(S)$ simultaneously nilpotent. If $A + B \in \mathcal{R}$ then $\mathcal{R} \cup \{A, B\}$ is simultaneously nilpotent as well.*

Proof. Consider the product $\Pi = A_1 \dots A_{h(\mathcal{R})}$, where $A_i \in \mathcal{R} \cup \{A \cup B\}$. For all i, if $A_i = A$ or $A_i = B$, then define $A_i' = (A + B)$ and $A_i' = A_i$ otherwise. Define $\Pi' = A_1' \dots A_{h(\mathcal{R})}'$. Since $A_i' \in \mathcal{R}$ for all i, it follows that $\Pi' = 0$. Since Π is a summand in Π', it follows by antinegativity that $\Pi = 0$ as well. □

The next lemma is straightforward.

Lemma 2.3. *Let S be a semiring and $\mathcal{R} \subseteq M_n(S)$ a simultaneously nilpotent set. If $A, B \in \mathcal{R}$ then $\mathcal{R} \cup \{AB\}$ is simultaneously nilpotent as well.*

In our proofs we will often make a use of the following graph theoretical terminology. Given a simple undirected graph $G = (V(G), E(G))$, the sequence of edges $(x_0, x_1), (x_1, x_2), ..., (x_{k-1}, x_k) \in E(G)$ is called a *path of length k* and is denoted by $(x_0, x_1, \dots, x_k)$. A path $(x_0, x_1, \dots, x_{k-1}, x_0)$ is called a *cycle*. We call a graph G *cyclic* if it contains a cycle and *acyclic* otherwise. We say that graph G is *transitive* if (u, v), $(v, w) \in E(G)$ implies that $(u, w) \in E(G)$ for any $u, v, w \in V(G)$.

The following lemma is well known as the existence of a Hamiltonian path in a tournament graph. For the sake of completeness we also include the proof.

Lemma 2.4. *Let G be an acyclic directed graph such that for every two vertices $u, v \in V(G)$ there exists exactly one of arcs (u, v) or (v, u) in $E(G)$. Then, there exists a path of length $n - 1$ in G.*

Proof. Let $u, v, w \in V(G)$ be distinct vertices in G and suppose $(u, v), (v, w) \in E(G)$. Since G is acyclic, $(w, u) \notin E(G)$. Therefore $(u, w) \in E(G)$ and thus G is transitive. Thus the ordering of the vertices $V(G)$, given by $u \geq v$ if $(u, v) \in E(G)$, is a total order. Therefore, we can denote vertices, so that $v_1 \geq v_2 \geq \ldots \geq v_n$ and so there exists a path $(v_1, v_2, \ldots, v_n)$ in G. □

In the next theorem we provide a bound for the simultaneously nilpotent index of a set of matrices.

Theorem 2.1. *If S is an antinegative commutative semiring and $\mathcal{R} \subseteq M_n(S)$ a simultaneously nilpotent set, then*

$$h(\mathcal{R}) \leq n\, h(\mathcal{N}(S)).$$

Proof. Note that every matrix $A = [a_{ij}] \in \mathcal{R}$ is nilpotent, thus $a_{ii} \in \mathcal{N}(S)$ by Lemma 2.1.

Every matrix $A \in \mathcal{R}$ can be written as a sum of elementary matrices $A = \sum a_{ij} E_{ij}$. By Lemma 2.2, $a_{ij} E_{ij} \in \mathcal{R}$ for all i, j. By the antinegativity of S, there exists a nonzero product Π of $h(\mathcal{R}) - 1$ elementary matrices. If Π contains a subproduct $\pi = bE_{i_1,i_2} E_{i_2,i_3} \ldots E_{i_m,i_1} = bE_{i_1,i_1}$, then it follows by Lemma 2.1 that $b \in \mathcal{N}(S)$. Therefore, the product Π contains at most $h(\mathcal{N}(S)) - 1$ subproducts of the form π, each of length at most n. Note that the only nonzero products of elementary matrices that do not contain any product of the form π, contain at most $n - 1$ factors. It follows that

$$h(\mathcal{R}) - 1 \leq n(h(\mathcal{N}(S)) - 1) + n - 1 = n\, h(\mathcal{N}(S)) - 1,$$

which proves the theorem. □

The next example shows that in general $h(\mathcal{R})$ can be strictly smaller than $n\, h(\mathcal{N}(S))$, even when $\mathcal{R}$ is a maximal simultaneously nilpotent set.

Example 2.1. Consider the semiring $S = \mathcal{B}[x, y, z]$ of polynomials over a Boolean semiring together with the following relations

$$x^2 = x,\ y^2 = y,\ z^2 = 0 \ \text{ and } \ xy = xz = yz = 0.$$

Clearly $\mathcal{N}(S) = zS$, so $h(\mathcal{N}(S)) = 2$. Define $\mathcal{R}$ to be an additive semigroup generated by

$$\{xE_{12}, yE_{21}, zE_{11}, zE_{12}, zE_{21}, zE_{22}\}.$$

Observe that $\mathcal{R}$ is maximal and $h(\mathcal{R}) = 2 < 2 \cdot 2$.

However, there are occasions when this upper bound is met. The next two theorems deal with such cases.

Theorem 2.2. *Let S be an antinegative commutative semiring and $\mathcal{R}$ a maximal simultaneously nilpotent set, such that for every i, j, $1 \leq i, j \leq n$, there exists $a_{ij} \in S \setminus Z(S)$ such that $\mathcal{R}$ contains $a_{ij}E_{ij}$ or $a_{ij}E_{ji}$. Then*

$$h(\mathcal{R}) = n\, h(\mathcal{N}(S)).$$

Proof. Let us denote $h = h(\mathcal{N}(S))$. Define a graph G with vertices $\{1, 2, \ldots, n\}$ and $(i, j) \in E(G)$ if and only if $aE_{ij} \in \mathcal{R}$ for some $a \notin Z(S)$. By Lemma 2.4, there exists a path of length $n - 1$ in G, so there exists a nonzero product matrices $\pi = a_{i_1,i_2}E_{i_1,i_2} \ldots a_{i_{n-1},i_n}E_{i_{n-1},i_n}$, where $a_{i_j,i_{j+1}} \notin Z(S)$ for $j = 1, 2, \ldots, n - 1$.

If $c = c_1 \ldots c_{h-1}$, is a nonzero product of nilpotents in $\mathcal{N}(S)$, then by the maximality of $\mathcal{R}$,

$$(\pi c_1 E_{i_n,i_1})\,(\pi c_2 E_{i_n,i_1}) \ldots (\pi c_{h-1} E_{i_n,i_1})\,\pi$$

is a nonzero product of elements in $\mathcal{R}$. Thus, $h(\mathcal{R}) - 1 \geq n(h - 1) + n - 1 = nh - 1$. It follows by Theorem 2.1 that $h(\mathcal{R}) = n\, h(\mathcal{N}(S))$. □

Theorem 2.3. *Let S be an antinegative commutative semiring such that $\mathrm{Ann}(\mathcal{N}(S)) \subseteq \mathcal{N}(S)$ and let $\mathcal{R} \subseteq M_n(S)$ be a maximal simultaneously nilpotent set. There exists $a \in S \setminus \mathcal{N}(S)$ such that for every $1 \leq i, j \leq n$, $\mathcal{R}$ contains aE_{ij} or aE_{ji} if and only if*

$$h(\mathcal{R}) = n\, h(\mathcal{N}(S)).$$

Proof. Let us denote $h = h(\mathcal{N}(S))$. Choose any nonzero element $c = c_1c_2 \ldots c_{h-1}$ with $c_1, c_2, \ldots, c_{h-1} \in \mathcal{N}(S)$.

First, assume there exists $a \in S \setminus \mathcal{N}(S)$ such that for every $1 \leq i, j \leq n$, $\mathcal{R}$ contains aE_{ij} or aE_{ji}. Note that if $a^k c = 0$, then by the assumption that $\mathrm{Ann}(\mathcal{N}(S)) \subseteq \mathcal{N}(S)$ it follows that $a^k \in \mathcal{N}(S)$ and hence $a \in \mathcal{N}(S)$. Thus, $a^k c \neq 0$ for all k.

Now, let us define a directed graph G with n vertices such that (i, j) is an edge in G if $aE_{ij} \in \mathcal{R}$. Since $a \notin \mathcal{N}(S)$, G is acyclic by Lemma 2.1. By Lemma 2.4 there exists a path $(i = i_1, i_2, \ldots, i_n = j)$ in G. Thus, $a^{n-1}E_{ij} = aE_{i_1,i_2}aE_{i_2,i_3} \ldots aE_{i_{n-1},i_n} \neq 0$ and so $aE_{i_1,i_2}aE_{i_2,i_3} \ldots aE_{i_{n-1},i_n}c_kE_{ji} = a^{n-1}c_kE_{ii} \neq 0$ for $k = 1, \ldots, h - 1$. This gives rise to the product

$$\left(\prod_{k=1}^{h-1} aE_{i_1,i_2}aE_{i_2,i_3} \ldots aE_{i_{n-1},i_n}c_kE_{i_n,i_1}\right) aE_{i_1,i_2}aE_{i_2,i_3} \ldots aE_{i_{n-1},i_n} = a^{nh-1}cE_{ij} \neq 0$$

of $(h - 1)n + (n - 1) = nh - 1$ matrices in $\mathcal{R}$.

To prove the converse statement, assume that $h(\mathcal{R}) = nh$. Thus, there exist matrices $A_1, A_2, \ldots, A_{nh-1} \in \mathcal{R}$, such that $A_1 A_2 \ldots A_{nh-1} \neq 0$. Since every matrix is a sum of scalar multiples of elementary matrices, there exist $a_{i_1}, a_{i_2}, \ldots, a_{i_{nh-1}} \in S$ such that by Lemma 2.2 $a_{i_1}E_{i_1 i_2}, a_{i_2}E_{i_2 i_3}, \ldots, a_{i_{nh-1}}E_{i_{nh-1}, i_{nh}} \in \mathcal{R}$ and

$$\Pi = a_{i_1}E_{i_1 i_2} a_{i_2}E_{i_2 i_3} \ldots a_{i_{nh-1}}E_{i_{nh-1}, i_{nh}} = a_{i_1} a_{i_2} \ldots a_{i_{nh-1}} E_{i_1, i_{nh}} \neq 0.$$

There are at most $h-1$ nilpotents among the elements $a_{i_1}, a_{i_2}, \ldots, a_{i_{nh-1}}$, and furthermore any product of at least h nilpotent elements of S is equal to zero. Consider a cyclic subproduct of Π,

$$\pi = b_{j_1}E_{j_1 j_2} b_{j_2}E_{j_2 j_3} \ldots b_{j_t}E_{j_t, j_1} = bE_{j_1, j_1},$$

where $t \leq n$. By Lemma 2.1, $b \in \mathcal{N}(S)$ and thus there are at most $h-1$ cyclic subproducts of Π of the form π, each of length at most n. In Π there exists a product of subproducts of Π

$$c_{k_1}E_{k_1 k_2} c_{k_2}E_{k_2 k_3} \ldots c_{k_{n-1}}E_{k_{n-1} k_n} = cE_{k_1 k_n} \neq 0,$$

of at least (and thus exactly) $nh-1-n(h-1) = n-1$ elements, where $k_i \neq k_j$ for all i, j and $c, c_{k_i} \notin \mathcal{N}(S)$. Obviously $\{1, 2, \ldots, n\} = \{k_1, \ldots, k_n\}$. Choose (i, j) and denote $i = k_t$ and $j = k_s$. If $t < s$ then choose a subproduct equal to $dE_{k_t k_s} = dE_{ij} \in \mathcal{R}$, and otherwise, if $s < t$, choose $dE_{k_s k_t} = dE_{ji} \in \mathcal{R}$. Note that $c = c'd$. Without loss of generality we can suppose that $dE_{ij} \in \mathcal{R}$. We want to prove that $cE_{ij} \in \mathcal{R}$. Suppose otherwise, $cE_{ij} \notin \mathcal{R}$. By the maximality of $\mathcal{R}$ there exists a nonnilpotent element $xE_{rr} \in \mathcal{R} \cup \{cE_{ij}\}$ for some r, which contains a factor cE_{ij}. By substituting cE_{ij} in the product xE_{rr} with dE_{ij}, we obtain a nilpotent element $yE_{rr} \in \mathcal{R}$, which contains a factor dE_{ij}. Now, $xE_{rr} = c'yE_{rr} \in \mathcal{N}(M_n(S))$, which is a contradiction. □

3. Semirings where all zero-divisors are nilpotent

In the case $Z(S) = \mathcal{N}(S)$ we can find out a bit more about the borderline cases when $h(\mathcal{R})$ is either equal or almost equal to the upper bound $n\, h(\mathcal{N}(S))$.

The next lemma will come in handy when proving the main theorems.

Lemma 3.1. *Let S be an antinegative commutative semiring such that $Z(S) = \mathcal{N}(S)$, and let $\mathcal{R} \subseteq M_n(S)$ be a maximal simultaneously nilpotent set.*

(1) *If $a \in \mathcal{N}(S)$, then $aE_{ij} \in \mathcal{R}$.*
(2) *If $a, b \notin \mathcal{N}(S)$ and $aE_{ij}, bE_{jk} \in \mathcal{R}$, then $ab \notin \mathcal{N}(S)$ and $abE_{ik} \in \mathcal{R}$.*
(3) *If $a \notin \mathcal{N}(S)$ and $aE_{ij} \in \mathcal{R}$, then $bE_{ij} \in \mathcal{R}$ for every $b \in S$.*

Proof. (1) If $aE_{ij} \notin \mathcal{R}$, then the set $\mathcal{R} \cup \{aE_{ij}\}$ is simultaneously nilpotent as well, a contradiction with the maximality of $\mathcal{R}$.

(2) If $a, b \notin \mathcal{N}(S) = Z(S)$, then $ab \notin \mathcal{N}(S) = Z(S)$. Since $aE_{ij}bE_{jk} = abE_{ik}$, the second part of the statement follows by Lemma 2.3.

(3) Suppose $a \notin \mathcal{N}(S)$ and $aE_{ij} \in \mathcal{R}$, but $bE_{ij} \notin \mathcal{R}$. By (1), $b \notin \mathcal{N}(S)$. If $\mathcal{R} \cup \{bE_{ij}\}$ is not a simultaneously nilpotent set, then there exists a non-nilpotent product Π of elements in $\mathcal{R} \cup \{bE_{ij}\}$ that includes a factor bE_{ij}. Since $\Pi = \pi E_{kl}$ is not nilpotent, it follows that $k = l$ and $\pi \notin \mathcal{N}(S)$. By replacing all factors bE_{ij} in product Π by aE_{ij}, we get a product Π' of elements in $\mathcal{R}$. By Lemma 2.3, we have $\Pi' \in \mathcal{R}$. Thus $\Pi' = \pi' E_{kk}$ is nilpotent, hence $\pi' \in \mathcal{N}(S)$. Suppose $\pi = b^k\alpha$, where k is the number of factors of π equal to b. It follows that $\pi' = a^k\alpha$. Since $a \notin \mathcal{N}(S) = Z(S)$, it follows that $\alpha \in Z(S)$, hence $\pi \in Z(S) = \mathcal{N}(S)$, a contradiction. □

We can now exactly determine the cardinality of a maximal simultaneously nilpotent set of maximal index in case S is a finite semiring.

Theorem 3.1. *Let S be a finite antinegative commutative semiring such that $Z(S) = \mathcal{N}(S)$, and let $\mathcal{R} \subseteq M_n(S)$ be a maximal simultaneously nilpotent set. If $h(\mathcal{R}) = n\,h(\mathcal{N}(S))$, then $|\mathcal{R}| = |S|^{\binom{n}{2}}|\mathcal{N}(S)|^{\binom{n+1}{2}}$.*

Proof. Let us denote $h = h(\mathcal{N}(S))$. We define a directed graph G with n vertices such that (i, j) is an edge in G if $aE_{ij} \in \mathcal{R}$ for some $a \notin \mathcal{N}(S)$. Since G is acyclic by Lemma 2.1, there can be at most one directed edge between any two vertices. By maximality, G is transitive, hence $|E(G)| = \binom{n}{2}$. By Lemma 3.1(1,3), the result follows. □

We now investigate the case when $h(\mathcal{R})$ is almost as large as the upper bound $n\,h(\mathcal{N}(S))$.

Theorem 3.2. *Let S be an antinegative commutative semiring such that $Z(S) = \mathcal{N}(S)$, and let $\mathcal{R} \subseteq M_n(S)$ be a maximal simultaneously nilpotent set. If $h(\mathcal{R}) < n\,h(\mathcal{N}(S))$, then*

$$h(\mathcal{R}) \leq n\,h(\mathcal{N}(S)) - h(\mathcal{N}(S)).$$

Moreover, if S is finite and $h(\mathcal{R}) = n\,h(\mathcal{N}(S)) - h(\mathcal{N}(S))$, then

$$|\mathcal{R}| = |S|^{\binom{n}{2}-1}|\mathcal{N}(S)|^{\binom{n+1}{2}+1}.$$

Proof. Let us denote $h = h(\mathcal{N}(S))$. Again, we define a directed graph G with n vertices such that (i, j) is an edge in G if $aE_{ij} \in \mathcal{R}$ for some $a \notin \mathcal{N}(S)$. Let k denote the length of the longest path in G. If $k = n - 1$ then with the similar arguments as in the proof of Theorem 3.1 we can conclude that $h(\mathcal{R}) = nh$, a contradiction. Thus $k \leq n - 2$. Note that each path $(i = i_1, i_2, \ldots, i_{l+1} = j)$ of length l in G corresponds to a product π of

l elements from S, none of which is a zero-divisor. For any set $\{a_1, \ldots, a_m\} \subseteq \mathcal{N}(S)$, by multiplying product π with $a_1 E_{j,j_1}, a_2 E_{j_1,j_2}, \ldots, a_m E_{j_{m-1},j_1} \in \mathcal{R}$, we obtain nonzero product of $l+m$ elements in $\mathcal{R}$. So the length of the longest nonzero product is

$$f(m_1, \ldots, m_s) = k + (k + m_1) + (k + m_2) + \ldots + (k + m_s)$$

where $m_1 + m_2 + \ldots + m_s = h - 1$. Therefore

$$f(m_1, \ldots, m_s) = (s+1)k + h - 1 \leq hk + h - 1. \tag{1}$$

Since $k \leq n-2$, this implies that $f(m_1, \ldots, m_s) \leq h(n-2) + h - 1 = nh - h - 1$, and it follows that $h(\mathcal{R}) \leq nh - h$.

Now, suppose $\mathcal{R}$ is maximal and $h(\mathcal{R}) = nh - h$. By (1) we can deduce that $k = n-2$. Since G is acyclic, there can be at most one directed edge between any two vertices. Lemma 2.4 implies that there exist vertices $u, v \in V(G)$, such that $(u,v), (v,u) \notin E(G)$. Thus, $|E(G)| \leq \binom{n}{2} - 1$.

Since $\mathcal{R}$ is maximal and $Z(S) = \mathcal{N}(S)$, by Lemma 3.1(2) the induced subgraph G' of G on the path $(u_1, u_2, \ldots, u_{n-1})$ of length $n-2$ is transitive. Consider the vertex v not in G' and let t be the largest index such that $u_t \in V(G')$ and $(u_t, v) \in E(G)$, if it exists, and set $t = 0$ otherwise. This implies that $(v, u_i) \notin E(G)$ for $i \leq t$ and the maximality of $\mathcal{R}$ yields $(u_i, v) \in E(G)$ for $i \leq t$. If $(v, u_{t+1}) \in E(G)$, graph G contains a path $(u_1, \ldots, u_t, v, u_{t+1}, \ldots, u_{n-1})$ of length $n-1$, a contradiction. Thus by maximality of t and $\mathcal{R}$ there exists $(v, u_j) \in E(G)$ for all $t+2 \leq j \leq n-1$. Therefore, $|E(G)| = \binom{n-1}{2} + n - 2 = \binom{n}{2} - 1$.

By Lemma 3.1(3) for every $(u_i, u_j) \in E(G)$ we have $aE_{u_i,u_j} \in \mathcal{R}$ for all $a \in S$. This yields $|S|^{\binom{n}{2}-1}$ elementary matrices in $\mathcal{R}$. The number of pairs (u,v), such that $(u,v), (v,u) \notin E(G)$, is equal to $n^2 - \left(\binom{n}{2} - 1\right) = \binom{n+1}{2} + 1$, thus by Lemma 3.1(1) it follows that $|\mathcal{R}| = |S|^{\binom{n}{2}-1} |\mathcal{N}(S)|^{\binom{n+1}{2}+1}$. $\square$

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