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Quantum effect of the Kerr-like medium in terms of SU(1, 1) Lie group in interaction with a two-level atom

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HIGHLIGHTS

- The interaction between Casimir operators and su(2) Lie algebra in the presence of Kerr-like medium is considered.
- The solution of the wave function is obtained using the evolution operators.
- It is found that the Kerr-like medium decreases the amplitude and increases the fluctuations.
- The normal squeezing occurs in the first quadrature and it is sensitive to both the Kerr-like medium paraquat as well as the state.

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ABSTRACT

In the present communication, we consider the problem of two quantum systems with the Kerr-like medium nonlinearity. The system is cast form of an interaction between two operators of the form su(1, 1) Lie algebra and su(2) Lie algebra. We obtain the wave function via the evolution operator where we use the Heisenberg equations of motion to derive the constants of motion. We discuss the atomic inversion. It is found that the Kerr-like medium decreases the amplitude and increases the fluctuations. Also we consider different types of squeezing, it is shown that the entropy squeezing is pronounced in the second quadrature, but it shows a small amount in the first quadrature. For the variance squeezing, a small amount occurs in the presence of the Kerr-like medium. However, the normal squeezing occurs in the first quadrature where the squeezing is sensitive to both the Kerr-like medium parameter and the initial state. Furthermore, the degree of entanglement is examined through the linear entropy. It is shown that the function decreases besides rapid fluctuations. The correlation function displays nonclassical behavior in addition to an increase in the amplitude of the fluctuations.

1. Introduction

One of the most subtle and intriguing phenomena in nature is quantum entanglement [1–3]. Moreover, the idea of entanglement has proved to be one of the most fertile fields for investigating properties of quantum mechanics. Recently, it has been used as a physical resource to realize different information tasks, such as quantum cryptography, teleportation, computation, dense coding and metrology [4–8]. In this sense, much work has been devoted to understand how it can be quantified and manipulated, see Ref. [9] and references therein. This, in fact, can be seen in quantum mechanics when the

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superposition principle is applied to the states of a composite system that cannot be separated into the product of states of subsystems. For example, the interaction of an atom with a quantized electromagnetic field leads to the evolution of the field (atomic) quantum entropy and an entanglement of these two subsystems, so that the total state vector cannot be written precisely as the product of time-dependent atomic and field component vectors. This means that entanglement implies the correlation between separated subsystems and indicates the phenomenon of quantum nonlocality. In the meantime, there has been a great interest in the possibility of using optical devices for ultra-light-speed data processing. This is the main object in the quantum information theory, which aims at storing and transferring data [8,10].

One of the promising devices for data transmission is the nonlinear directional coupler that consists of two or more parallel optical waveguides fabricated from some nonlinear material. Both waveguides are placed close enough to permit flux-dependent transfer of energy between them by means of evanescent waves. This flux transfer can be controlled by the device design and the intensity of the input flux as well. In fact, the directional coupler is experimentally implemented, e.g. in planar structures [11], dual optical fibers [12] and certain organic polymers [13]. Among the different types of directional couplers, the directional Kerr nonlinear coupler has received much attention as a result of its application in optics as an intensity-dependent routing switch [14,15]. The directional Kerr nonlinear coupler is useful for low intensity fields and periodical exchange of energy between the guides; but for high intensity fields, energy is trapped by nonlinearity in the guide into which it was initially launched (self-trapping effect).

On the other hand, the su (1, 1) algebra (together with the su (2) algebra) has been much utilized in the theoretical study of the non-classical aspects of light in various quantum optical systems. Note that in a previous communication we have considered the Hamiltonian [16]

$$\frac{\hat{H}}{\hbar} = \omega\hat{K}_z + \lambda_1\left(\hat{K}_+ + \hat{K}_-\right) + \lambda_2\hat{K}_+\hat{K}_-, \tag{1}$$

composed of the su (1, 1) Lie algebra operators, to discuss the squeezing phenomenon, where $\lambda_i, i = 1, 2$ are suitable coupling parameters, ω is a frequency that specifies the model. As one can see, the model does not include any interaction with the atoms. Therefore, for the above reasons, we generalize this model and consider a Hamiltonian model which represents the interaction between a two-level atom and su (1, 1) Lie group in the presence of the Kerr-like medium. This Hamiltonian takes the form

$$\frac{\hat{H}}{\hbar} = \frac{\omega_0}{2}\hat{\sigma}_z + \omega\hat{K}_z + \chi\left(\hat{K}_+\hat{K}_-\right) + g\left(\hat{K}_+\hat{\sigma}_- + \hat{K}_-\hat{\sigma}_+\right) \tag{2}$$

where χ depends in general on the susceptibility, while $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_z$ are the generators of su (1, 1) Lie group and satisfy the commutation relation

$$\left[\hat{K}_z, \hat{K}_\pm\right] = \pm\hat{K}_\pm, \qquad \left[\hat{K}_-, \hat{K}_+\right] = 2\hat{K}_z \tag{3}$$

and the corresponding Casimir operator $\hat{K}$ is given by

$$\hat{K}^2 = \hat{K}_z^2 - \frac{1}{2}\left(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+\right), \tag{4}$$

whereas $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ are the usual Pauli operators (su (2) Lie group generator) which satisfy the relations

$$\left[\hat{\sigma}_z, \hat{\sigma}_\pm\right] = \pm 2\hat{\sigma}_\pm, \qquad \left[\hat{\sigma}_+, \hat{\sigma}_-\right] = \hat{\sigma}_z \tag{5}$$

while g is the coupling parameter between the atom and the quantum system. The main purpose of this article is to discuss some statistical properties as well as to see the effect of the Kerr-like medium. For this reason, we organize the paper as follows: In Section 2, we derive the wave function. The atomic population is discussed in Section 3. Squeezing phenomenon is considered in Section 4. The degree of entanglement is investigated through the purity in Section 5. We deal with the correlation function in Section 6 and finally our conclusion is given in Section 7.

2. Wave function

There are different methods to obtain the wave function. One way is to solve the Schrödinger equation using different methods and techniques [17]. The main task of this section is to derive the solution of the wave function using the evolution operator from which we are able to obtain some statistical properties of the present system. This would enable us to study the behavior of the present system. Thus, we use the model of Eq. (2) as well as the Heisenberg equations of motion for any operator given by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\left[H, Q\right] + \frac{\partial Q}{\partial t} \tag{6}$$

to reach the following set:

$$\mathrm{i}\dot{K}_z = \lambda(\hat{K}_+\hat{\sigma}_- - \hat{K}_-\hat{\sigma}_+), \qquad \mathrm{i}\dot{\sigma}_z = 2\lambda(\hat{K}_-\hat{\sigma}_+ - \hat{K}_+\hat{\sigma}_-). \tag{7}$$

From this set, we deduce that the operator N defined as follows:

$$\hat{N} = \hat{K}_z + \frac{1}{2}\hat{\sigma}_z, \tag{8}$$

is a constant of motion.

Therefore, the Hamiltonian (2) can be cast the form

$$\hat{H}/\hbar = \hat{\Lambda} + \hat{C} \tag{9}$$

where the operators $\hat{N}$ and $\hat{C}$ are given by

$$\begin{aligned}
\hat{\Lambda} &= \omega\hat{N} + \chi\left(\hat{K}_z^2 - \hat{K}_z - \hat{K}^2 + \frac{1}{4}\right),\\
\hat{C} &= \hat{\Gamma}\hat{\sigma}_z + g\left(\hat{K}_+\hat{\sigma}_- + \hat{K}_-\hat{\sigma}_+\right).
\end{aligned} \tag{10}$$

The quantity $\hat{\Gamma}$ is the detuning parameter defined by

$$\hat{\Gamma} = \frac{\omega_0 - \omega}{2} - \chi\left(\hat{N} - \frac{1}{2}\right). \tag{11}$$

It is noted that $\hat{\Gamma}$ depends on the detuning $\omega_0 - \omega$ as well as on χ, $\hat{K}_z$ and $\hat{\sigma}_z$. It is easy to show that the operators $\hat{\Lambda}$ and $\hat{C}$ commute, and hence each of them commute with the Hamiltonian $\hat{H}$. This means that the operators $\hat{\Lambda}$ and $\hat{C}$ are constants of motion. The time evolution operator $\hat{U}(t)$ is given by $\hat{U}(t) = \exp[-\mathrm{i}t\hat{H}/\hbar]$, which can be written in the form

$$U(t) = \exp[-\mathrm{i}\hat{C}t].\exp\left[-\mathrm{i}\hat{\Lambda}t\right]. \tag{12}$$

The exponential operator is calculated to take the form

$$\exp[-\mathrm{i}\hat{C}t] = \begin{bmatrix} \left(\cos\hat{\mu}_1 t - \mathrm{i}\frac{\delta_1}{\hat{\mu}_1}\sin\hat{\mu}_1 t\right) & -\mathrm{i}\lambda\frac{\sin\hat{\mu}_1 t}{\hat{\mu}_1}\hat{K}_- \\ -\mathrm{i}\lambda\hat{K}_+\frac{\sin\hat{\mu}_1 t}{\hat{\mu}_1} & \left(\cos\hat{\mu}_2 t + \mathrm{i}\frac{\delta_2}{\hat{\mu}_2}\sin\hat{\mu}_2 t\right) \end{bmatrix}, \tag{13}$$

where we have used the abbreviations

$$\begin{aligned}
&\mu_j^2 = \delta_j^2 + \nu_j, \quad j = 1, 2 \qquad \hat{\nu}_1 = \lambda^2\hat{K}_-\hat{K}_+, \qquad \hat{\nu}_2 = \lambda^2\hat{K}_+\hat{K}_-,\\
&\hat{\delta}_1 = \left[\frac{\omega_0 - \omega}{2} - \chi\hat{K}_z\right],\\
&\hat{\delta}_2 = \left[\frac{\omega_0 - \omega}{2} - \chi\left(\hat{K}_z - 1\right)\right],
\end{aligned} \tag{14}$$

and dropped the function operator $\exp\left[-\mathrm{i}\hat{\Lambda}t\right]$.

In what follows the atomic coherent state $|\theta, \phi\rangle$, which acquires both excited state $|e\rangle$ and ground state $|g\rangle$ for the two-level atoms, is described by the equation

$$|\theta, \phi\rangle = \cos(\theta/2)|e\rangle + \sin(\theta/2)\exp(-\mathrm{i}\phi)|g\rangle, \tag{15}$$

where θ is the coherence angle and ϕ is the relative phase of the two atomic levels. To reach the excited state, we have to take $\theta \to 0$; whereas, to consider the wave function to describe the particle in the ground state, we have to let $\theta \to \pi$. However, for the su (1, 1) quantum system, we take the Barut–Girardello coherent state of su (1, 1) group which is given by

$$|\beta; k\rangle = \left(\frac{|\beta|^{2k-1}}{I_{2k-1}(2|\beta|)}\right)^{1/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!\Gamma(n+2k)}}|n; k\rangle, \tag{16}$$

where $I_n(x)$ is the nth-order modified Bessel function,

$$I_k(x) = \left(\frac{x}{2}\right)^k\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{n!\Gamma(n+k+1)}. \tag{17}$$

Note that the state $|\beta; k\rangle$ is an eigenstate of $\hat{K}_-$, such that $\hat{K}_-|\beta; k\rangle = \beta|\beta; k\rangle$, where for one-mode bosonic representation k is the Bargmann index.

Assuming that at time $t = 0$, the system is in the state defined by the wave function $|\psi(0)\rangle = |\theta, \phi\rangle \otimes |\beta; k\rangle$. Therefore, if we use Eq. (13), dropping the factor $\exp\left[-i\hat{\Lambda}t\right]$ and after minor calculations, the wave function for $t > 0$ takes the form

$$|\psi(t)\rangle = \left\{\left(\cos\hat{\mu}_1 t - \frac{i\hat{\delta}_1}{\mu_1}\sin\hat{\mu}_1 t\right)\cos\frac{\theta}{2} - ig\frac{\sin\hat{\mu}_1 t}{\mu_1}\hat{K}_-\exp\{-i\phi\}\sin\frac{\theta}{2}\right\}|\beta; k, e\rangle + \left\{\left(\cos\hat{\mu}_2 t + \frac{i\hat{\delta}_2}{\mu_2}\sin\hat{\mu}_2 t\right)\right.$$
$$\left.\times\exp\{-i\phi\}\sin\frac{\theta}{2} - ig\frac{\sin\hat{\mu}_2 t}{\mu_2}\hat{K}_+\cos\frac{\theta}{2}\right\}|\beta; k, g\rangle. \tag{18}$$

Now, we are in a position to write the reduced density matrix for the system which is given by $\hat{\rho}_f(t) = Tr_{atom}|\psi(t)\rangle\langle\psi(t)|$, such that

$$\hat{\rho}_f(t) = |D(t)\rangle\langle D(t)| + |T(t)\rangle\langle T(t)|, \tag{19}$$

where we have defined

$$|D(t)\rangle = \left\{\left(\cos\hat{\mu}_1 t - \frac{i\hat{\delta}_1}{\hat{\mu}_1}\sin\hat{\mu}_1 t\right)\cos\frac{\theta}{2} - i\lambda\frac{\sin\hat{\mu}_1 t}{\mu_1}\hat{K}_-\exp\{-i\phi\}\sin\frac{\theta}{2}\right\}|\beta; k\rangle, \tag{20}$$

and

$$|T(t)\rangle = \left\{\left(\cos\hat{\mu}_2 t + \frac{i\hat{\delta}_2}{\hat{\mu}_2}\sin\hat{\mu}_2 t\right)\exp\{-i\phi\}\sin\frac{\theta}{2} - ig\frac{\sin\hat{\mu}_2 t}{\mu_2}\hat{K}_+\cos\frac{\theta}{2}\right\}|\beta; k\rangle. \tag{21}$$

In what follows we employ the reduced density operator given by Eq. (19) together with Eqs. (20) and (21) to discuss some statistical properties of the present system. This will be seen in the following sections.

3. The atomic population

We concentrate in this section on the atomic inversion which represents the difference between the population of the excited and the ground atomic state. In fact, the use of the atomic inversion enables us to observe the atomic behavior and to measure when the atom reaches its maximal state. Therefore, to discuss the atomic inversion, one needs to calculate the expectation value of the operator $\hat{\sigma}_z$. In this case we have

$$W(t) = \langle\hat{\sigma}_z(t)\rangle = \rho_{11}(t) - \rho_{22}(t), \tag{22}$$

where

$$\rho_{11}(t) = \sum_{n=0}^{\infty}\left(|Q(n)|^2|F_{11}(n,t)|^2\cos^2\frac{\theta}{2} + |Q(n)|^2|F_{12}(n,t)|^2\sin^2\frac{\theta}{2}\right.$$
$$\left.+Re\left[Q(n)Q^*(n+1)F_{11}^*(n+1,t)F_{21}(n,t)\exp\{-i\phi\}\sin\theta\right]\right), \tag{23}$$

and

$$\rho_{22}(t) = \sum_{n=0}^{\infty}\left(|Q(n)|^2|F_{22}(n,t)|^2\sin^2\frac{\theta}{2} + |Q(n)|^2|F_{21}(n,t)|^2\cos^2\frac{\theta}{2}\right.$$
$$\left.+Re\left[Q(n)Q^*(n+1)F_{11}^*(n+1,t)F_{21}(n,t)\exp\{-i\phi\}\sin\theta\right]\right). \tag{24}$$

In the above equations, we have used the abbreviations

$$F_{11}(n,t) = \cos\mu_1 t - \frac{i\,\delta_1}{\mu_1}\sin\mu_1 t, \qquad F_{12}(n,t) = -i\sqrt{\nu_2}\frac{\sin\mu_2 t}{\mu_2}$$
$$F_{22}(n,t) = \cos\mu_2 t + \frac{i\,\delta_2}{\mu 2}\sin\mu_2 t, \qquad F_{21}(n,t) = -i\sqrt{\nu_1}\frac{\sin\mu_1 t}{\mu 1} \tag{25}$$

whereas

$$\mu_1 = \sqrt{\frac{\delta_1^2}{4} + \lambda^2(n+1)(n+2k)}, \qquad \mu_2 = \sqrt{\frac{\delta_2^2}{4} + \lambda^2 n(n+2k-1)}, \tag{26}$$

while

$$\delta_1 = \left(\frac{\omega_0 - \omega}{2}\right) - \chi (n + k) \tag{27}$$

and

$$\delta_2 = \left(\frac{\omega_0 - \omega}{2}\right) - \chi (n + k - 1) \tag{28}$$

in the meantime, we defined the function $Q(n)$ such that

$$Q(n) = \left(\frac{|\beta|^{2k-1}}{I_{2k-1}(2|\beta|)}\right)^{1/2} \frac{\beta^n}{\sqrt{n!\Gamma(n + 2k)}}. \tag{29}$$

In order to discuss the atomic inversion and to see the effect of the Kerr-like medium, we have plotted Fig. 1 for the function $W(t)$ against the scaled time $\lambda t = \tau$. For this reason, we have fixed the value of $\beta = 50$, $k = 1/4$ and $\omega = \omega_0/2 = \lambda\pi/3$ while we consider different values of the other parameters. For example, when we consider the atom in its excited state and $\chi = 0$, the function fluctuates around *zero* and shows a small period of the revival after onset of the interaction. This is followed by a long period of collapse, whence we can see another period of revival, in the meantime the function in this period increases its amplitude and fluctuates between -1 and 1. It is noted that the function is periodic with period π. When the Kerr-like medium is taken into account by taking $\chi/\lambda = 0.5$, the function is shifted upward and displays a short period of collapse followed by successive periods of revivals as in the previous case. It is observed that the amplitudes of the revivals are smaller compared with the case of no Kerr-like medium. The function is also periodic but with short period see Fig. 1(b). The atomic inversion amplitude is decreased and revivals are around zero between -0.5 and 0.5, when the atom starts from the superposition state with $\theta = \pi/2$ and $\phi = \pi/4$, and in the absence of the Kerr-medium, with the periodicity π see Fig. 1(c). Furthermore, it displays a behavior similar to that of the Jaynes–Cummings model (JCM) with coherent trapping; however, here it may be partial trapping. This may be due to the phase ϕ [18]. Finally, when we consider $\chi/\lambda = 0.5$, the atomic inversion shifts downward almost with the same trued as the case of no Kerr-like medium but with slightly smaller period (see Fig. 1(d)). It is worth noting that the atomic inversion for this model behaves almost like the case of the two-photon JCM with coherent input. Thus, it may not be surprising when we recall that the Barut–Girardello state in the realization $\hat{K}_- = \hat{a}^2$ for $k = \frac{1}{4}(\frac{3}{4})$ is the even (odd) coherent state. It is to be noted that the Kerr-like medium as a detuning is shifting the atomic inversion.

4. The phenomenon of squeezing

There are different kinds of squeezing, such as normal squeezing, principle squeezing, variance squeezing, and entropy squeezing. Most of these kinds depend on the nature of the quantum system we are studying. In fact, they are used to examine the variation that would occur in the quadrature variances from which we can measure the classical and nonclassical properties of the system. In the present communication, we shall examine three different kinds of the squeezing phenomenon. For instance, the entropy squeezing which can be used to measure the nonclassical properties in each quadrature variance for system contains Pauli operators.

4.1. Entropy squeezing

This kind of squeezing is used to examine the nonclassical properties of the quantum system which contains atomic operators. It is well known that fluctuations in $\hat{\sigma}_\gamma$ ($\gamma = x$ or y) of the atomic dipole are said to be squeezed in entropy if the Shannon information entropy $H(\hat{\sigma}_\gamma)$ of $\hat{\sigma}_\gamma$, [19–22]

$$\sum_{\gamma=1}^{N+1} H(\hat{\sigma}_\gamma) \geq \frac{N}{2} \ln\left(\frac{N}{2}\right) + \left(1 + \frac{N}{2}\right) \ln\left(1 + \frac{N}{2}\right) \tag{30}$$

where N is an even dimensional Hilbert space, satisfies the condition

$$E(\hat{\sigma}_\gamma) = \left(\exp[H(\hat{\sigma}_\gamma)] - \frac{2}{\sqrt{\exp[H(\hat{\sigma}_z)]}}\right) < 0, \quad \gamma = x, y. \tag{31}$$

To obtain the Shannon information entropies of the atomic operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ for a two-level atom with $N = 2$, we can use the time-dependent atomic density elements by tracing over the quantum system to get

$$\hat{\rho}^{(a)} = Tr_{Field}|\psi(t)\rangle\langle\psi(t)|. \tag{32}$$

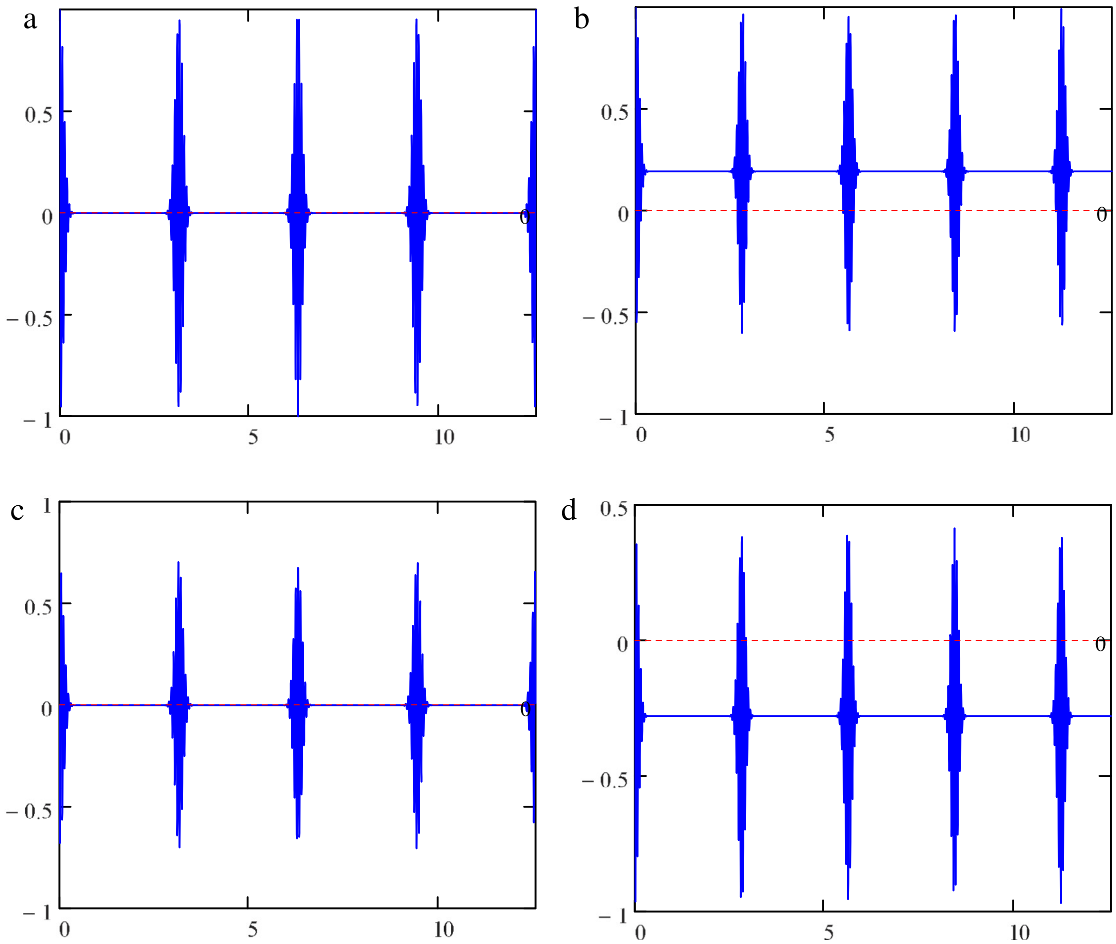

Fig. 1. The atomic inversion for the two-level atom initially in the superposition state in interaction with su (1, 1) quantum system is initially in Barut–Girardella coherent states for fixed value of $\beta = 50$, $k = \frac{1}{4}$ and $\omega = \omega_0 = \lambda\pi/3$. (a) $\chi/\lambda = 0$ and $\theta = 0$, (b) $\chi/\lambda = 0.5$, $\theta = 0$, (c) $\chi/\lambda = 0$, $\theta = \pi/2$, $\phi = \pi/4$ (d) $\chi/\lambda = 0.5$, $\theta = \pi/2$, $\phi = \pi/4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Thus, we have

$$\begin{aligned}\rho_{12}^{(a)} = \rho_{21}^{(a)*} = \sum_{n=0}^{\infty} \Bigg(& Q(n)Q^*(n+1)F_{11}^*(n+1,t)F_{21}(n,t)\exp\{i(x_1(n+1)-x_2(n))t\}\cos^2\frac{\theta}{2} \\ & + |Q(n)|^2 F_{11}^*(n,t)F_{22}(n,t)\exp\{i(x_1(n)-x_2(n))t\}\exp\{-i\phi\}\frac{\sin\theta}{2} \\ & + Q(n)Q^*(n+2)F_{21}^*(n,t)F_{12}(n,t)\exp\{i(x_1(n+1)-x_2(n))t\}\exp\{i\phi\}\frac{\sin\theta}{2} \\ & + Q(n)Q^*(n+1)F_{21}^*(n,t)F_{22}(n,t)\exp\{i(x_1(n)-x_2(n))t\}\sin^2\frac{\theta}{2}\Bigg), \end{aligned} \tag{33}$$

where

$$\begin{aligned} x_1(n) &= \omega\left(n+k+\frac{1}{2}\right) + \chi\left[\left(n+k+\frac{1}{2}\right)^2 - \left(n+k+\frac{1}{2}\right) - k(k-1)\right] + \frac{1}{4}, \\ x_2(n) &= x_1(n-1). \end{aligned} \tag{34}$$

This, in fact, enables us to calculate the expectation values of $\hat{\sigma}_x$ and $\hat{\sigma}_y$ and consequently we can calculate the Shannon information entropy from the following:

$$H(\hat{\sigma}_\gamma) = -\frac{1}{2}\left[\left(1+\langle\hat{\sigma}_\gamma\rangle\right)\ln\frac{1}{2}\left(1+\langle\hat{\sigma}_\gamma\rangle\right) + \left(1-\langle\hat{\sigma}_\gamma\rangle\right)\ln\frac{1}{2}\left(1-\langle\hat{\sigma}_\gamma\rangle\right)\right], \quad \gamma = x, y, z. \tag{35}$$

Therefore, to discuss the entropy squeezing, we have to plot some figures to display the behavior of the quadrature variances. When we considered the atom in its excited state and exclude the effect of the Kerr-like medium, the squeezing occurs in the second quadrature $E_y(\tau)$ after onset of the interaction as it is displayed in Fig. 2(a). The entropy squeezing as the quadrature $E_y(\tau)$ occurs with period π at the position of the atomic inversion revivals (see Fig. 1(a)). However, squeezing occurs in $E_x(\tau)$ with pronounced values at the middle of the collapse periods of the atomic inversion. Fig. 2(b) exhibits the effect of the parameter χ, it is noted that the squeezing occurs at both quadratures. It is more pronounced in $E_y(\tau)$ than $E_x(\tau)$. As time devolves, squeezing disappears in both quadratures as observed in Fig. 2(b). Fig. 2(c) displays the case of $\theta = \pi/2, \phi = \pi/4$ while $\chi = 0$; the squeezing is observed several times in the second quadrature but it is pronounced in the first quadrature $E_x(\tau)$ when it reaches the maximum values. Again note that the squeezing in $E_y(\tau)$ occurs at the revival times while the squeezing in $E_x(\tau)$ occurs at the middle of the collapse periods of the atomic inversion. Finally, the effect of $\chi/\lambda = 0.01$ is included with the superposition state, which is displayed in Fig. 2(d). The squeezing occurs in both quadratures, however with smaller values in $E_y(\tau)$ than in $E_x(\tau)$. It is destroyed in both quadratures after some time. The Kerr-like medium seems to degrade the entropy squeezing.

4.2. Variance squeezing

In this context, we can introduce another type of the squeezing that is the variance squeezing. It is well known that for a quantum mechanical system with two physical observables represented by the Hermitian operators $\hat{A}$ and $\hat{B}$ satisfying the commutation relation $\left[\hat{A}, \hat{B}\right] = \mathrm{i}\hat{C}$, one can write the Heisenberg uncertainty relation in the form

$$\left\langle\left(\Delta\hat{A}\right)^2\right\rangle\left\langle\left(\Delta\hat{B}\right)^2\right\rangle \geq \frac{1}{4}|\langle\hat{C}\rangle|^2 \tag{36}$$

where $\langle\left(\Delta\hat{A}\right)^2\rangle = \left(\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2\right)$ with similar expression for $\left(\Delta\hat{B}\right)^2$.

Consequently, the uncertainty relation for a two-level atom characterized by the Pauli operators, $\hat{\sigma}_x,\ \hat{\sigma}_y$ and $\hat{\sigma}_z$, satisfying the commutation relation $\left[\hat{\sigma}_x, \hat{\sigma}_y\right] = 2\mathrm{i}\hat{\sigma}_z$ can also be written as $\Delta\hat{\sigma}_x\Delta\hat{\sigma}_y \geq |\langle\hat{\sigma}_z\rangle|$. Therefore, the fluctuations in the component $\Delta\hat{\sigma}_\alpha$ of the atomic dipole is squeezed if it satisfies the condition

$$V(\hat{\sigma}_\alpha) = \left(\Delta\hat{\sigma}_\alpha - \sqrt{\left|\langle\hat{\sigma}_z\rangle\right|}\right) < 0, \quad \alpha = x \text{ or } y. \tag{37}$$

In order to see the variance behavior, we have plotted Fig. 3 for different values of the Kerr-like medium parameter. For this reason, we take the atom in its excited state, $\beta = 50$ and $\chi = 0$. In this case, we observe that the squeezing occurs in the second quadrature $V_y(\tau)$ (red color) after onset of the interaction, however it reoccurs with increasing amount at the revival period of the atomic inversion (of Fig. 1(a)) with the same periodicity. In the meantime, there is no squeezing in the first quadrature in $V_x(\tau)$ (blue color), see Fig. 3(a). When we take the effect of the Kerr-like medium into consideration ($\chi \setminus \lambda = 0.01$), we note that the same behaviors, but the squeezing in $V_y(\tau)$ becomes weaker than the previous case and it is destroyed for begin time, see Fig. 3(b). The situation is different when we consider the atom in its superposition state. In the absence of χ the squeezing occurs in the first quadrature $V_x(\tau)$ after onset of the interaction and at revival periods of the atomic inversion with the same periodicity, see Fig. 3(c). On the other hand, for $\chi \setminus \lambda = 0.01$, we observe that the phenomenon of squeezing appeared in the first quadrature $V_x(\tau)$ and absent from $V_y(\tau)$. However, it is not pronounced as in the previous case where it disappeared after few periods, see Fig. 3(d). Thus, we may conclude that a small amount of the Kerr-like medium leads to an exchange between the quadrature variances as well as to decrease the amount of squeezing.

4.3. Normal squeezing

In this subsection, we shall consider the squeezing fluctuations; therefore, we define the Hermitian operators as follows:

$$\hat{K}_x = \frac{1}{2}\left(\hat{K}_+ + \hat{K}_-\right), \qquad \hat{K}_y = \frac{1}{2i}\left(\hat{K}_+ - \hat{K}_-\right) \tag{38}$$

which satisfy

$$\left[\hat{K}_x, \hat{K}_y\right] = -\mathrm{i}\hat{K}_z, \qquad \left[\hat{K}_y, \hat{K}_z\right] = \mathrm{i}\hat{K}_x, \qquad \left[\hat{K}_z, \hat{K}_x\right] = \mathrm{i}\hat{K}_y. \tag{39}$$

From the above relations, we have the uncertainty relation:

$$\left(\Delta\hat{K}_x\right)\left(\Delta\hat{K}_y\right) \geq \frac{1}{2}|\langle\hat{K}_z\rangle|. \tag{40}$$

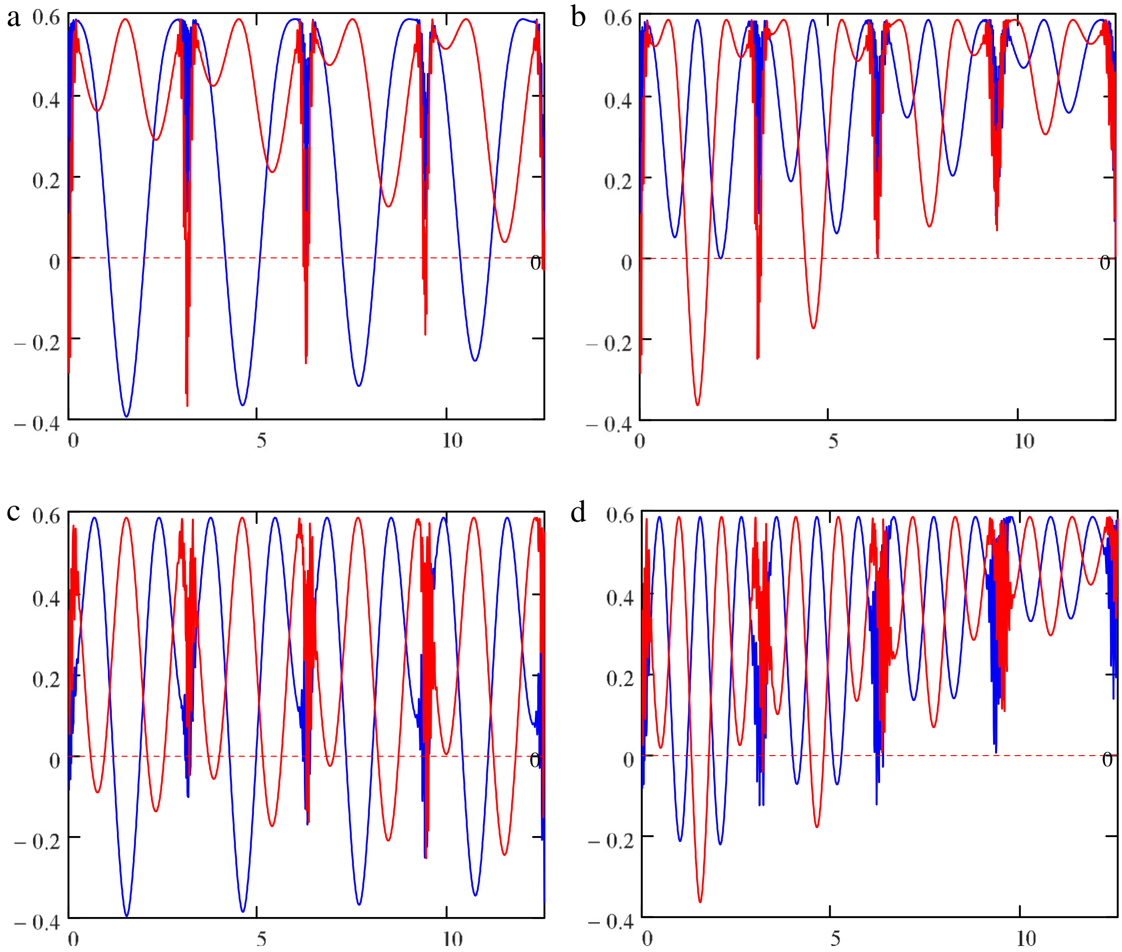

Fig. 2. The entropy squeezing for the two-level atom initially in the superposition state in interaction with su (1, 1) quantum system is initially in Barut–Girardella coherent states for fixed value of $\beta = 50$, $k = \frac{1}{4}$ and $\omega = \omega_0 = \lambda\pi/3$. (a) $\chi/\lambda = 0$ and $\theta = 0$, (b) $\chi/\lambda = 0.01$, $\theta = 0$, (c) $\chi/\lambda = 0$, $\theta = \pi/2$, $\phi = \pi/4$ (d) $\chi/\lambda = 0.01$, $\theta = \pi/2$, $\phi = \pi/4$.

The fluctuation of the $\hat{K}_x$ or $\hat{K}_y$ component is squeezed if

$$W_x(\tau) = \left[\left(\Delta\hat{K}_x\right)^2 - \frac{1}{2}|\langle\hat{K}_z\rangle|\right] < 0, \qquad W_y(\tau) = \left[\left(\Delta\hat{K}_y\right)^2 - \frac{1}{2}|\langle\hat{K}_z\rangle|\right] < 0, \tag{41}$$

respectively.

To consider the normal squeezing, we plot the function $W_x(\tau)$ against the scaled time τ; therefore, we can examine the effect of different atomic state as well as the Kerr-like medium. In Fig. 4(a), the atom is initially in excited state and the Kerr-like effects are absent. It is noted that the normal squeezing occurs only in the component and the squeezing phenomenon occurred only in $\hat{K}_x$, i.e. $W_x(\tau)$ takes negative values at the revival periods of the atomic inversion with the same periodicity. During the collapse periods, there is no squeezing. When the Kerr parameter is considered ($\chi/\lambda = 0.5$), we note that the same trend is displayed but with smaller amounts of squeezing (Fig. 4(b)). When the case of atomic superposition state is considered ($\theta = \pi/2, \phi = \pi/4$), it is observed that squeezing occurs with appreciable values at the revival times and also exists but with small amounts around the middle of the collapse periods. The amounts of squeezing are lesser than in the case of the excited state (see Fig. 4(c)). This picture is changed to stronger squeezing when the Kerr parameter is add ($\chi/\lambda = 0.5$) (see Fig. 4(c)). However, the amount of squeezing start to diminish as time progresses (see Fig. 4(d)). It is interesting to find that the Kerr parameter plays two different roles in normal squeezing phenomenon depending on the atomic state.

5. Linear entropy and entanglement

We devote this section to study the degree of the entanglement of the present system and to see the effect of the Kerr-like medium. The linear entropy is usually used as a tool to discuss the entanglement between quantum systems. It is defined

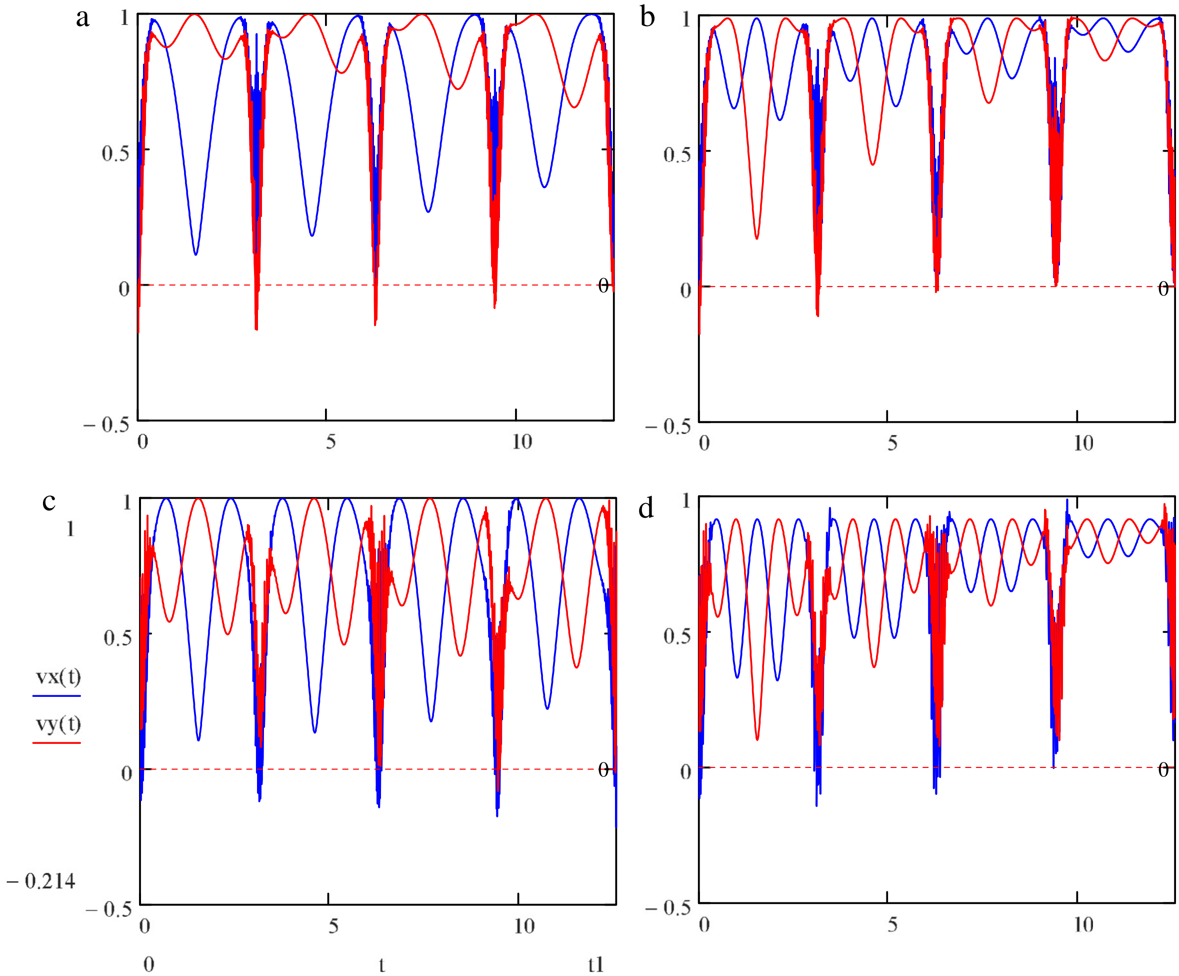

Fig. 3. The variance squeezing for the two-level atom initially in the superposition state in interaction with su (1, 1) quantum system is initially in Barut–Girardella coherent states for fixed value of $\beta = 50$, $k = \frac{1}{4}$ and $\omega = \omega_0 = \pi/3$. (a) $\chi/\lambda = 0$ and $\theta = 0$, (b) $\chi/\lambda = 0.01$, $\theta = 0$, (c) $\chi/\lambda = 0$, $\theta = \pi/2$, $\phi = \pi/4$ (d) $\chi/\lambda = 0.01$, $\theta = \pi/2$, $\phi = \pi/4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

as follows [23–25]

$$\Gamma(\tau) = \left(1 - \Upsilon^2(\tau)\right), \tag{42}$$

where $\Upsilon(\tau)$ is the well known Bloch sphere radius defined as

$$\Upsilon(\tau) = \sqrt{\langle\hat{\sigma}_x(\tau)\rangle^2 + \langle\hat{\sigma}_y(\tau)\rangle^2 + \langle\hat{\sigma}_z(\tau)\rangle^2}. \tag{43}$$

It is remarked that there is a relation between the linear entropy and the uncertainty relation $\mathcal{X} \geq 0$, which can be written in the form

$$\mathcal{X} = \left[1 - \left(\langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2\right) + \langle\hat{\sigma}_x\rangle^2\langle\hat{\sigma}_y\rangle^2\right] \geq 0. \tag{44}$$

One may use Eqs. (42) and (43) together with Eq. (44), from which we have

$$\mathcal{X} = \Gamma(\tau) + \langle\hat{\sigma}_x(\tau)\rangle^2\langle\hat{\sigma}_y(\tau)\rangle^2 \geq 0. \tag{45}$$

As we can see from the above equation, there is a strong relation between the degree of entanglement and the uncertainty relation provided $\langle\hat{\sigma}_x(\tau)\rangle^2\langle\hat{\sigma}_y(\tau)\rangle^2 \neq 0$. On the other hand, the Bloch sphere has been used as a tool in the field of quantum optics; therefore, the simple qubit state is successfully represented, up to an overall phase, by a point on the sphere, whose coordinates are the expectation values of the atomic set operators of the system. From the above equation, it is easy to realize that the function $\Gamma(\tau)$ reaches its maximum when $\Upsilon(\tau) = 0$, where the maximum entanglement occurred. In the meantime, the disentanglement occurs when $\Gamma(\tau) = 0$ corresponding to $\Upsilon(\tau) = 1$ (pure state). Now, we employ

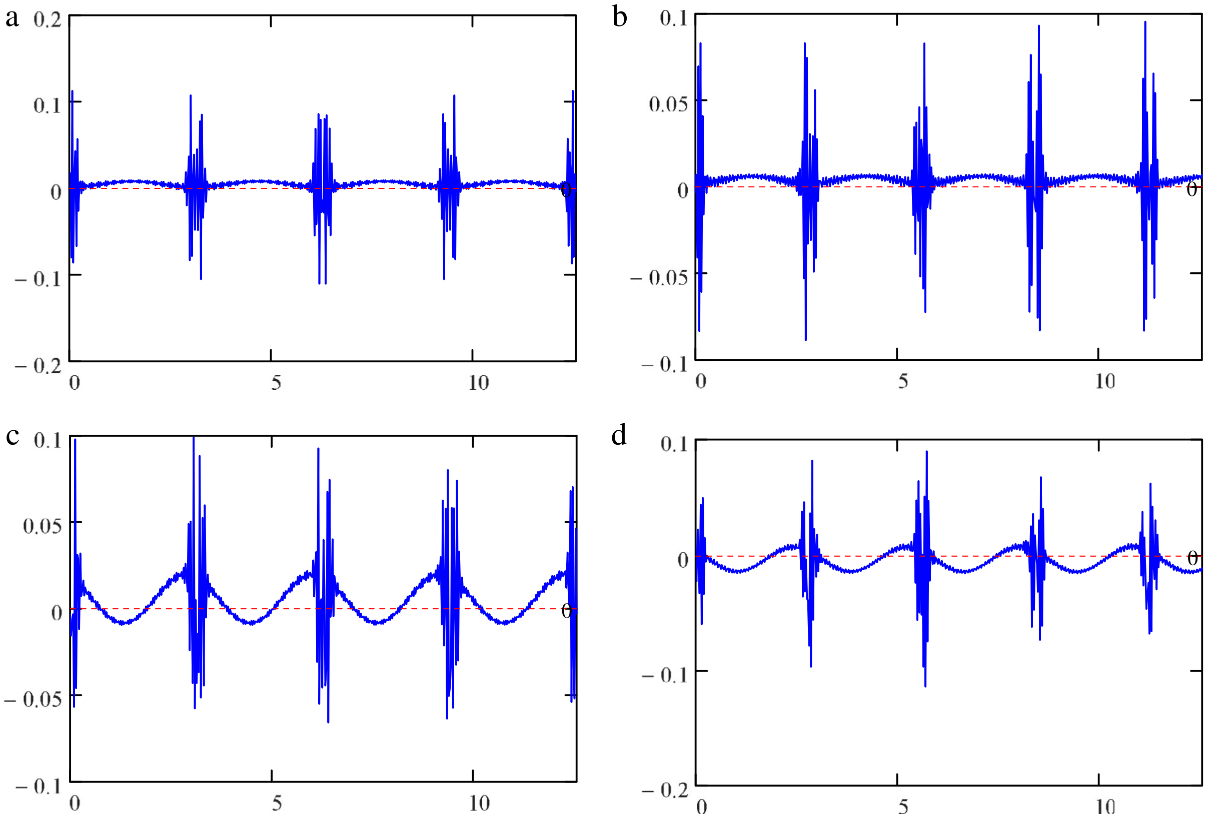

Fig. 4. The normal squeezing for the two-level atom initially in the superposition state in interaction with SU (1, 1) quantum system is initially in Barut–Girardella coherent states for fixed value of $\beta = 50$, $k = \frac{1}{4}$ and $\omega = \omega_0 = \pi/3$. (a) $\chi/\lambda = 0$ and $\theta = 0$, (b) $\chi/\lambda = 0.5$, $\theta = 0$, (c) $\chi/\lambda = 0$, $\theta = \pi/2$, $\phi = \pi/4$ (d) $\chi/\lambda = 0.5$, $\theta = \pi/2$, $\phi = \pi/4$.

the expectation value for the operators $\hat{\sigma}_x(\tau)$, $\hat{\sigma}_y(\tau)$ and $\hat{\sigma}_z(\tau)$ which are obtained in the previous sections to discuss the behavior of the entanglement for the present system.

As we have mentioned previously, the linear entropy has been used as a tool to discuss the degree of entanglement. Since the functions are complicated, we have plotted some figures to display the behavior of the function. We plot the function against the scaled time τ and consider the same parameters as in the previous sections. For the atom in its excited state, we observe that the function displays disentanglement at the periods of revivals and at the middle of the collapse periods of the atomic inversion as well as partial entanglement (see Fig. 5(a)). While almost full entanglement is observed at two points around the revival time. The effect of the Kerr-like parameter is displayed in the Fig. 5(b) where ($\chi/\lambda = 0.5$) is taken into account. Here, we note that these is no disentanglement at any point, whereas strong entanglement is observed at periods during the collapse period and stronger around the revival periods. The picture is changed when we consider the case of superposition of the atomic states by taking ($\theta = \pi/2, \phi = \pi/4$). The case of absent Kerr-medium is recalled in Fig. 5(c) where disentanglement is observed at exactly the same points as in Fig. 5(a) but it near reaches to maximum entanglement. The Kerr-like parameter ($\chi/\lambda = 0.5$) this picture and the system never reaches the complete entanglement, but almost reaches maximum entanglement as in the case of excited state (see Fig. 5(d)).

6. The correlation function

In this section, we consider the behavior of the correlation for the present system using the same data for the atomic inversion. The normalized second order correlation function $g^{(2)}(t)$ is used to measure the bunching and anti-bunching as well as the coherence behavior. It is defined by Ref. [26] in an extension to su (1, 1) operators:

$$g^{(2)}(\tau) = \frac{\left\langle \left(\hat{K}_+\right)^2 \left(\hat{K}_-\right)^2 \right\rangle}{\left\langle \hat{K}_+ \hat{K}_- \right\rangle^2} \tag{46}$$

where the function shows sub-correlated for $g^{(2)}(t) < 1$, and super-correlated for $g^{(2)}(t) > 1$, whereas it shows coherence when $g^{(2)}(t) = 1$. To display the behavior of the correlation function in the presence of Kerr-like medium, we plot some

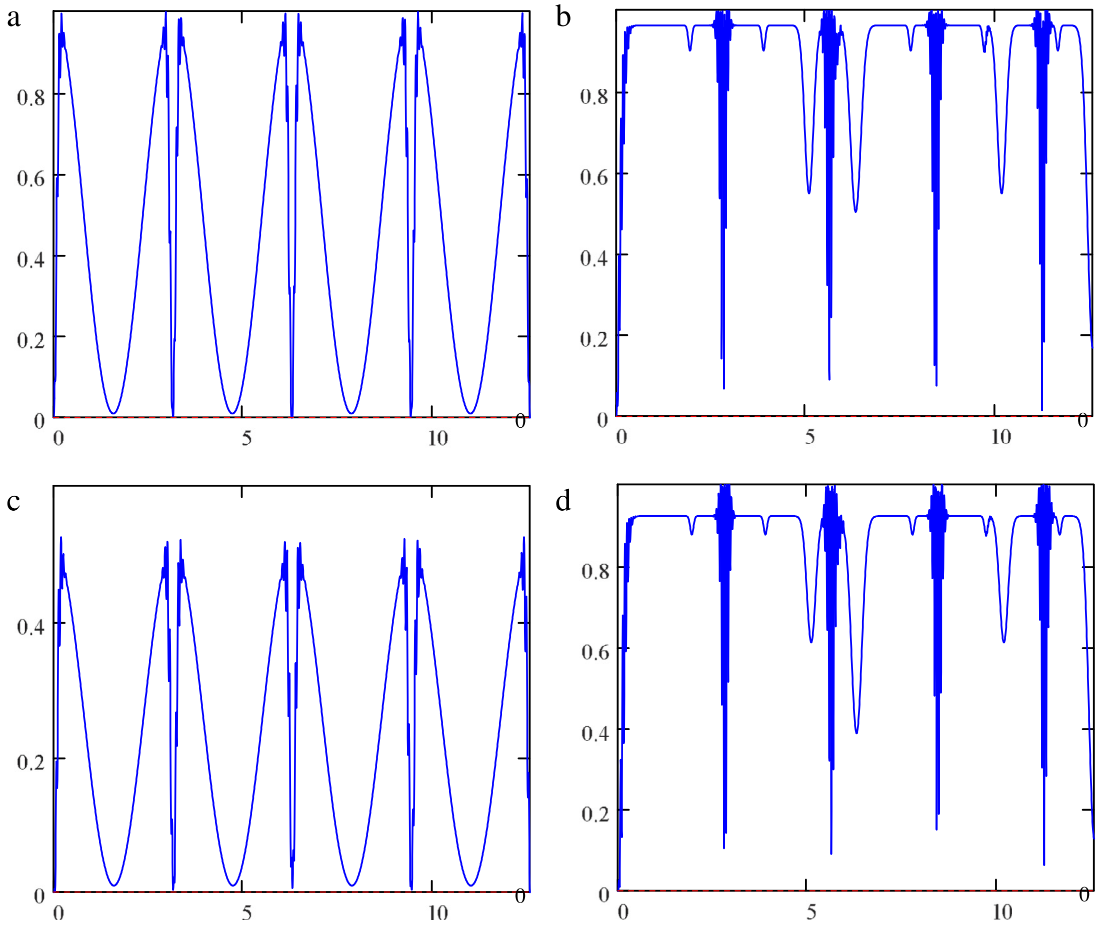

Fig. 5. The linear entropy for the two-level atom initially in the superposition state in interaction with su (1, 1) quantum system is initially in Barut–Girardella coherent states for fixed value of $\beta = 50$, $k = \frac{1}{4}$ and $\omega = \omega_0 = \pi/3$. (a) $\chi/\lambda = 0$ and $\theta = 0$, (b) $\chi/\lambda = 0.5$, $\theta = 0$, (c) $\chi/\lambda = 0$, $\theta = \pi/2$, $\phi = \pi/4$ (d) $\chi/\lambda = 0.5$, $\theta = \pi/2$, $\phi = \pi/4$.

figures against the scaled time $\tau = \lambda t$. For this reason, we plot Fig. 6(a) for the excited state and no Kerr parameter with the same parameters used in the previous section. In this case, we observe that the function fluctuates regularly around *one* and shows several long periods of coherence. These periods of coherence coincide with the collapse periods of the atomic inversion (see Fig. 1(a)). During the revival periods, the function oscillates between anti-correlated and correlated behavior. When the Kerr parameter is added, the same trend is noted with the shorter period of coherence strong side with the scene in the atomic inversion. However, the fluctuations are slighting smaller in the amplitude compared with the previous case (see Fig. 6(b)). In Fig. 6(c), we consider the atom to be initially in the superposition state. Now, we notice a remarkable change. The coherence is only attained at points during the collapse or revival time. During the revival times, the function reaches a highly correlated state. While anticorrelation behavior is occurring at same intervals of the collapse periods, it is noted that at $\tau = 2\pi \& 4\pi$ there is no sharp change to high correlation as at $\tau = \pi \& 3\pi$. The case of addition of the Kerr parameter has only changes of the details especially change during the revival times as observed in Fig. 6(d). It is noted that the correlation function $g^{(2)}$ is greatly affected by the initial state of the atom and by the Kerr parameter. The Kerr parameter lowers the values of the correlations.

7. Conclusion

In the present paper, we considered the interaction between two quantum systems generated by su (1, 1) and su (2) system in the presence of Kerr-like nonlinearity. The time dependent evolution operator is used to find the wave function, and hence we managed to calculate some dynamical operators, more precisely $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$. These operators are employed to examine the atomic inversion where the phenomenon of collapse and revival has been displayed. It has been shown that the Kerr-like medium reduces the amplitude of the function and increases the fluctuations, this in addition shifts above and below the zero value depending on the atomic state. This behavior is realized for the cases in which the atom is in its excited

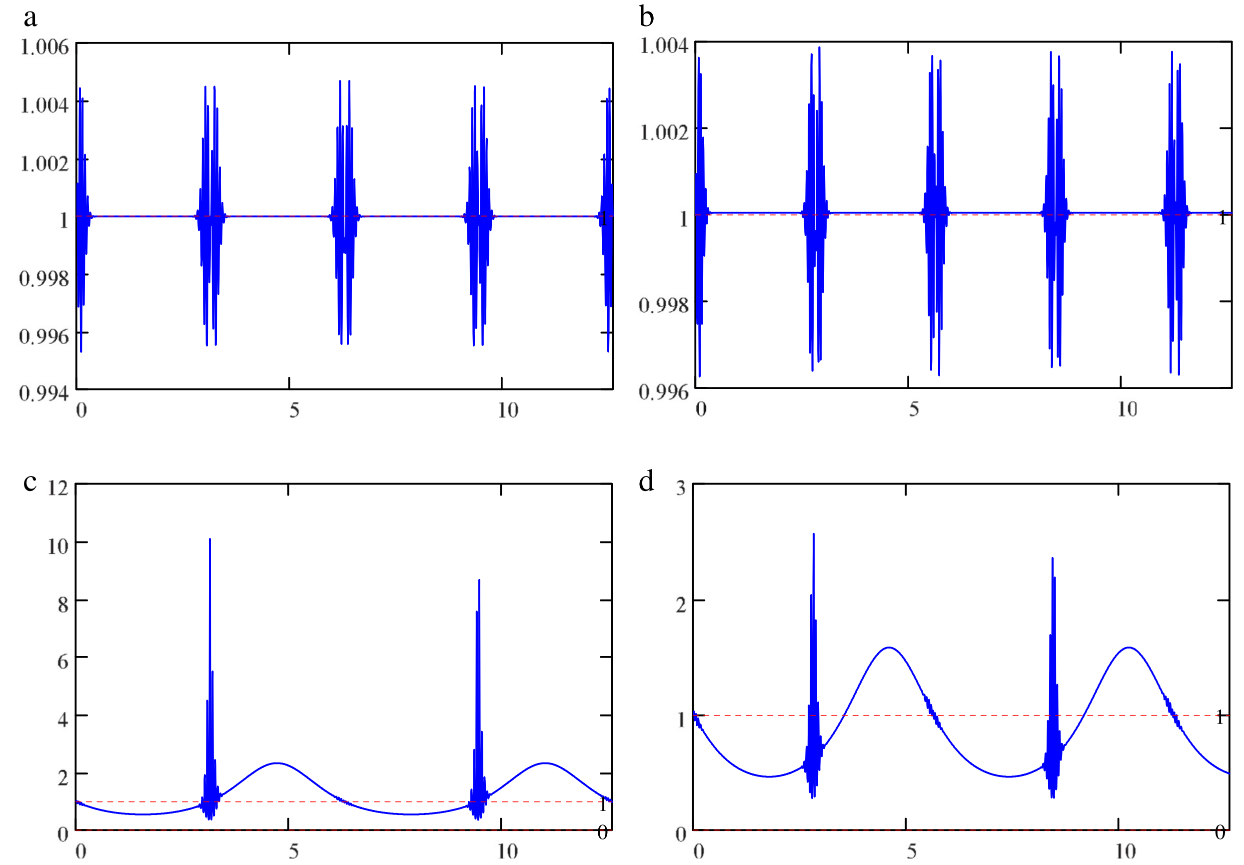

Fig. 6. The correlation function for the two-level atom initially in the superposition state in interaction with su (1, 1) quantum system is initially Barut–Girardella coherent states for fixed value of $\beta = 50$, $k = \frac{1}{4}$ and $\omega = \omega_0/2 = \lambda\pi/3$. (a) $\chi/\lambda = 0$ and $\theta = 0$, (b) $\chi/\lambda = 0.5$, $\theta = 0$, (c) $\chi/\lambda = 0$, $\theta = \pi/2$, $\phi = \pi/4$ (d) $\chi/\lambda = 0.5$, $\theta = \pi/2$, $\phi = \pi/4$.

or superposition states. Also we considered the entropy squeezing, the variance squeezing as well as the normal squeezing. For the entropy squeezing, it is found that it occurs in both quadratures but it is pronounced in the second quadrature provided the atom is in its excited state whereas it is reduced for the Kerr-like medium. On contrary of the entropy squeezing, the variance squeezing appears in the first quadrature when the atom is in its excited state and in the absence of the Kerr-like medium. Similar behavior is realized when the Kerr-like medium takes place. The function exchanged its quadrature when the atom is in the superposition state with an increase in the amount of squeezing in the presence of the Kerr-like medium. For the normal squeezing, the function shows squeezing only in the first quadrature $W_x(\tau)$ which is sensitive to the change in the atomic state as well as a Kerr-like parameter.

Furthermore, we discussed the degree of entanglement using the linear entropy as an indicator where the function displays disentanglement as well as entanglement for all cases we considered. However, the function shows large periods of strong entanglement in the presence of Kerr-like medium. Finally, the second order correlation is examined where correlation and anticorrelation between is exhibited. The effect of Kerr-like medium in this case is to decrease the value of the correlation function and to increase its fluctuations. It is quite noted that this function is highly affected by the state of the atom.

It is worth mentioning to state that quantum real systems will be inevitably influenced by the surrounding environment. There are different ways of dissipation and decoherent. In general, damping will degrade the degree of entanglement [27–31]. For a collection of N 2-level atoms, the trimmed Dicke state decays exponentially [27]. Effect of atomic spontaneous emission and cavity dissipation on the output photon leaks from the cavity [28]. It is to be mentioned that generally entropy increases and entanglement is degraded for Jaynes–Cummings model with cavity damping [29] or without energy relation [30] and when the state of the atom is mixed and the cavity is phase damped [31,32]. Due to the parameters of the Kerr-like medium, such system may find application in communications and information processes.

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References

[1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47 (1935) 777.
[2] E. Schrödinger, Proc. Cambridge Philos. Soc. 31 (1935) 555.
[3] S.L. Braunstein, H.J. Kimble, Phys. Rev. Lett. 80 (1998) 869.
[4] A. Ekert, Phys. Rev. Lett. 67 (1991) 651.
[5] C.H. Bennett, G. Brassard, C. Crepeau, R. Joza, A. Peres, W.K. Wooters, Phys. Rev. Lett. 70 (1993) 1895.
[6] C.H. Bennet, S.J. Weisner, Phys. Rev. Lett. 69 (1992) 2881.
[7] C.F. Roos, M. Chwalla, K. Kim, M. Riebe, R. Blatt, Nature 443 (2006) 316.
[8] A. Ekert, R. Jozsa, Rev. Modern Phys. 68 (1996) 733.
[9] K. Berrada, M. El Baz, Y. Hassouni, Internat. J. Modern Phys. C 21 (2010) 291.
[10] H.-K. Lo, S. Popescu, T. Spiller, Introduction to Quantum Computation and Information, World Scientific, Singapore, 1998.
[11] P. Li Kamwa, J.E. Stitch, N.J. Mason, P.N. Roberts, Electron. Lett. 21 (1985) 26.
[12] R. Jin, C.L. Chuang, H.H. Gibbs, S.W. Koch, J.N. Polky, G.A. Pubanz, Appl. Phys. Lett. 49 (1986) 110.
[13] D.D. Gusovkii, E.M. Dianov, A.A. Mairer, V.B. Neustreuev, E.I. Shklovskii, I.A. Scherbakov, Sov. J. Quantum Electron. 15 (1985) 1523.
[14] P.D. Townsend, G.L. Baker, J.L. Shelburne III, S. Etemad, Proc. SPIE 1147 (1989) 256.
[15] S.M. Jensen, IEE J. Quantum Electron. 18 (1982) 1580.
[16] F. El-Orany, S.S. Hassan, M.S. Abdalla, J. Opt. B: Quantum Semiclass. Opt 5 (2003) 396.
[17] W.P. Schleich, Quantum Optics in Phase Space, Wiely-VCH, Berlin, 2000;
H. Eleach, Y.V. Rostovosky, M.O. Scully, Eur. Phys. Lett. 89 (2010) 50004.
[18] H.-I. Yoo, J.H. Eberly, Phys. Rep. 118 (1985) 239.
[19] J. Sanchez-Ruiz, Phys. Lett. A 201 (1995) 125.
[20] J. Sanchez-Ruiz, Phys. Lett. A 244 (1998) 189.
[21] J. Sanchez-Ruiz, Phys. Lett. A 173 (1993) 233.
[22] V. Majernik, L. Richterek, European J. Phys. 18 (1997) 79.
[23] C.C. Gerry, P.L. Knight, Introductory Quantum Optics, Cambridge University Press, Cambridge, 2005.
[24] F.A.A. El-Orany, J. Modern Opt. 56 (2009) 99.
[25] F.A.A. El-Orany, M.S. Abdalla, J. Phys. A: Math. Theor. 44 (2011) 035302.
[26] M. Ban, Opt. Soc. Amer. B 10 (1993) 1347.
[27] E.A. Sete, A.A. Svidzinsky, H. Eleuch, Z. Yang, R.D. Nevels, M.O. Scully, J. Mod Optics 57 (14–15) (2010) 1311.
[28] H. Eleuch, S. Guérin, H.R. Jauslin, Phys. Rev. A 85 (2012) 013830.
[29] A.-S.F. Obada, H.A. Hessian, A.-B.A. Mohamed, J. Phys. B: At. Mol. Opt. Phys. 41 (2008) 159801.
[30] A.-S.F. Obada, H.A. Hessian, A.-B.A. Mohamed, J. Phys. B: At. Mol. Opt. Phys. 40 (2007) 2241.
[31] A.-S.F. Obada, H.A. Hessian, A.-B.A. Mohamed, Opt. Commun. 280 (2007) 239.
[32] G.S. Agarwal, J. Opt. Soc. Amer. B 5 (1988) 1940.