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Research paper

Probabilistic solutions of nonlinear oscillators excited by combined colored and white noise excitations

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ABSTRACT

In this paper, single-degree-of-freedom (SDOF) systems combined to Gaussian white noise and Gaussian/non-Gaussian colored noise excitations are investigated. By expressing colored noise excitation as a second-order filtered white noise process and introducing colored noise as an additional state variable, the equation of motion for SDOF system under colored noise is then transferred artificially to multi-degree-of-freedom (MDOF) system under white noise excitations with four-coupled first-order differential equations. As a consequence, corresponding Fokker-Planck-Kolmogorov (FPK) equation governing the joint probabilistic density function (PDF) of state variables increases to 4-dimension (4-D). Solution procedure and computer programme become much more sophisticated. The exponential-polynomial closure (EPC) method, widely applied for cases of SDOF systems under white noise excitations, is developed and improved for cases of systems under colored noise excitations and for solving the complex 4-D FPK equation. On the other hand, Monte Carlo simulation (MCS) method is performed to test the approximate EPC solutions. Two examples associated with Gaussian and non-Gaussian colored noise excitations are considered. Corresponding band-limited power spectral densities (PSDs) for colored noise excitations are separately given. Numerical studies show that the developed EPC method provides relatively accurate estimates of the stationary probabilistic solutions, especially the ones in the tail regions of the PDFs. Moreover, statistical parameter of mean-up crossing rate (MCR) is taken into account, which is important for reliability and failure analysis. Hopefully, our present work could provide insights into the investigation of structures under random loadings.

1. Introduction

It is known that some noises in the real world, such as seismic ground motion [1], wind velocity, sea wave, or noise in biological systems [2,3], are colored or non-white noise processes. These physical processes are generally characterized by correlation functions with finite correlation length, and are clearly different from white noise processes with zero correlation length. Alternatively, these colored or non-white noise excitations can be expressed as filtered white noise processes and modeled as responses of the first- or second-order linear or nonlinear oscillators excited by Gaussian white noise excitations [4]. One of the simplest examples of finite time correlation noise is the Ornstein-Uhlenbeck process, $C(t)$, with an exponential

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correlation function of the form [5,6]

$$E[C(t)C(s)] = \frac{D}{\tau}\exp\left[-\frac{|t-s|}{\tau}\right], \tag{1}$$

where $E[\cdot]$ is the expectation operator, τ denotes the correlation time, D is corresponding power spectral density (PSD). Such exponentially correlated Gaussian process, C, can be obtained by passing Gaussian white noise $W(t)$ through a first-order low pass filter of the following form [7–9]

$$\dot{C}(t) = -\frac{1}{\tau_1}C + \frac{1}{\tau_1}W(t). \tag{2}$$

This equation is a linear filter and hence the filtered white noise is also Gaussian. Non-Gaussian process can be obtained by filtering white noise through a nonlinear filter.

Excited by colored noise, state variables of systems are not Markov process. Fokker-Planck-Kolmogorov (FPK) equation cannot be used directly to obtain probabilistic characters of system responses. However, if state variables of the interested system and colored noise are assumed to be initially uncorrelated and colored noise is introduced as an auxiliary state variable, then the joint response process consisting of state variables and colored noise is Markovian. Hence analytical methods for Markov process are appropriate [10,11]. However, this mathematical artefact augments the dimensionality of state-space and gives rise to four-dimensional (4-D) FPK equation. It seems to increase the complexity of problem to a greater degree since it is known that solving multi-dimensional FPK equation is a challenge. Corresponding solution procedure is much more sophisticated.

So far, exact probability density function (PDF) solutions have been available merely to two-dimensional FPK equations on a small scale. Many approximate methods and numerical methods are also just effective for two-dimensional FPK equations. Only a few approximate or numerical methods can be used for multi-dimensional FPK equations. Equivalent linearization (EQL) method, proposed by Booton on the control of electronic systems, was broadly adopted for analyzing weakly nonlinear systems under external excitations [12,13]. This method is based on the assumption that system responses are Gaussian. Hence the first and second moments of system responses can well characterize the interested systems. Furthermore, when the nonlinearity becomes strong or when there is multiplicative excitation, moment closure method or cumulant-neglect closure method, as an generalization of EQL method, is developed to approximate higher-order statistical moments or cumulants [14–16]. The EQL or generalized EQL method has already been employed for multi-dimensional systems, and it can definitely be used to 4-D FPK equations for systems excited by colored noise. But statistical moments obtained with these methods cannot completely characterize probabilistic properties of state variables. With the help of information on statistical moments of system responses, exponential polynomial closure (EPC) method, broadly applied for cases of single-degree-of-freedom (SDOF) systems under white noise excitations, can be facilitated to solve two-dimensional FPK equations. It has been verified effective in many nonlinear systems of polynomial type [17–19]. Furthermore, it is considered as the foundation for analyzing multi-degree-of-freedom (MDOF) systems with split-state-space-EPC (3S-EPC) method [20,21]. In this paper, the EPC method is further improved and developed for the case of system under combined white and colored noises and for solving 4-D FPK equation. Much more complex computer program and more computational effort are required. Numerical method Monte Carlo simulation (MCS) method has often been used to test the approximate solutions [22,23]. The MCS can be employed for analyzing multi-dimensional systems, but the computational effort is huge, particular in the cases of strong nonlinearity or small probability events.

The focus of this paper is on the investigation of systems under combined white and colored noise excitations with the developed EPC method. Firstly, by introducing colored noise as an additional state variable, the second-order equation of motion with colored noise is modified to four coupled first-order differential equations. Based on the modified equation of motion, corresponding 4-D FPK equation results in. Secondly, the EPC method, originally with approximate solution containing two state variables for cases of white noise excitations, is developed to extend variables to four state variables with additional unknown coefficients for cases of colored noise excitations. As a result, much more sophisticated solution procedure and computer programmer are produced. On the other hand, MCS method is performed to verify the efficiency of the developed EPC method. Thirdly, two examples are considered. One example is about nonlinear oscillator under Gaussian colored noise, the other is about nonlinear oscillator under non-Gaussian colored noise. Stationary approximate PDF solutions with the developed EPC method, EQL method and MCS method are obtained and compared with each other. Moveover, statistical parameter mean-up crossing rate (MCR) is taken into account.

2. Problem formulation and developed EPC solution procedure

As white noise process has infinity energy, it cannot exist in practice. To demonstrate the real situations, colored noise excitation is taken into account. Consider the following oscillator

$$\ddot{X} + h_0(X,\dot{X}) + h_i(X)W_i(t) = C(t), \tag{3}$$

where $X, \dot{X}$ and $\ddot{X}$ represent displacement, velocity and acceleration, $h_0(X,\dot{X})$ and $h_i(X)$ are deterministic linear or nonlinear functions. $W_i(t)$ are independent Gaussian white noise excitations with PSDs S_{ii}, $C(t)$ is colored noise excitation with band-limited PSD function $S(\omega)$. For the sake of simplifying the analysis, white noise $W_i(t)$ and colored noise $C(t)$ are assumed to be mutually independent of each other. Further, colored noise can be expressed as a filtered white noise process

$$\ddot{C} + f(C, \dot{C}) = W(t), \tag{4}$$

where $\dot{(\cdot)}$ is time derivative of $(\cdot)$, $f(C, \dot{C})$ is a linear or nonlinear function of $(C, \dot{C})$, $W(t)$ is zero-mean Gaussian white noise with $E[W(t)W(t+\tau)] = S_0\delta(\tau)$. Here $\delta(\tau)$ is Dirac-delta function and S_0 is spectral density for the white noise. If function $f(\cdot)$ is linear like $f(\cdot) = 2\zeta\omega_0\dot{c} + \omega_0^2 c$, colored noise $C(t)$ is Gaussian and its PSD can be expressed as

$$S(\omega) = \frac{2S_0}{[\omega_0^2 - \omega^2]^2 + 4\zeta^2\omega_0^2\omega^2}. \tag{5}$$

Here we pay attention to the positive side of frequency range and multiply factor 2 to its PSD. On the other hand, if function $f(\cdot)$ is nonlinear, corresponding $C(t)$ is non-Gaussian and its PSD can be obtained with EQL method [9]

$$S(\omega) = \frac{4c_e k_e m_{20}}{[k_e - \omega^2]^2 + c_e^2\omega^2}, \tag{6}$$

where c_e and k_e are equivalent damping and stiffness coefficients, which are acquired with least mean square method. m_{20} is the second-order moment of colored noise state variable.

From above Eqs. (5) and (6), it is seen that PSDs for the colored noise $C(t)$ are band limited. Generally, the band-width of colored noise is characterized by $2\varepsilon\omega_0$, and consequently it depends mainly on the stiffness of the filter. For example in engineering, if the foundation soil is very soft, the band-width of earthquake force modeled as white noise filtered by such foundation soil will become very narrow, and most of earthquake energy is converged into some local frequencies. On the contrary, if the foundation soil is hard, the band-width of filtered earthquake force is wide, and earthquake energy is dispersed over a wide range of frequency domain.

By expressing colored noise as an additional state variable, Eqs. (3) and (4) can be further stated as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_0(x_1, x_2) - h_i(x_1)W_i(t) + x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -f(x_3, x_4) + W(t). \end{aligned} \tag{7}$$

As a result, the equation of motion for SDOF system under colored noise is transferred artificially to MDOF system under white noise with four-coupled first-order differential equations. From these equations, some observation may be pointed out. Firstly, in the case of dynamical system enforced by colored noise excitations, colored noise process is treated as an additional state variable. As a result, interested problem can be settled by solving the corresponding FPK equation. Secondly, colored noise can be obtained by a linear/nonlinear filter through white noise process. Generation of colored noise in this way can also be used to model real random loadings with known statistical properties [9]. Thirdly, such mathematical treatment augments state-space dimensionality and transforms original problem of SDOF system artificially to MDOF system problem.

Since combined process consisting of state variable and colored noise is a Markov vector, stationary PDF p of combined process is governed by the following reduced FPK equation

$$\frac{\partial}{\partial x_j}(A_j p) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(B_{ij} p) = 0, \tag{8}$$

where x_j are state variables, repeated indices (i and j) mean Einstein's summation convention unless stated otherwise. The first and second derivative moments A_j and B_{ij} can be derived from Eq. (7) as follows

$$\begin{aligned} &A_1 = x_2 \quad A_2 = -h_0(x_1, x_2) \quad A_3 = x_4 \quad A_4 = -f(x_3, x_4) \\ &B_{22} = h_i^2(x_1)S_{ii} \quad B_{44} = S_0. \end{aligned} \tag{9}$$

As a result, the reduced FPK equation can be rewritten as

$$x_2\frac{\partial p}{\partial x_1} - h_0(x_1, x_2)\frac{\partial p}{\partial x_2} - \frac{\partial h_0(x_1, x_2)}{\partial x_2} + x_4\frac{\partial p}{\partial x_3} - f(x_3, x_4)\frac{\partial p}{\partial x_4} - \frac{\partial f(x_3, x_4)}{\partial x_4}p - \frac{1}{2}S_{ii}h_i^2(x_1)\frac{\partial^2 p}{\partial x_2^2} - \frac{1}{2}S_0\frac{\partial^2 p}{\partial x_4^2} = 0, \tag{10}$$

where p is the joint PDF of state variables (x_1, x_2, x_3, x_4). It is noticed that the FPK equation is in four-dimensional form, and involves state vector with four components since colored noise is treated as an additional state variable. Solving such 4-D FPK equation is much more difficult and even challenging. However, in the case of white noise excitations, state vector contains only two components and the resulted FPK equation is two-dimensional. This FPK equation can be readily solved by many approximate methods or even with exact solutions available in some cases.

In order to solve Eq. (10) approximately, exponential-polynomial closure (EPC) method is employed. It is previously for SDOF systems under white noise excitations with approximate solution

$$\tilde{p}'(\boldsymbol{a};\boldsymbol{x}) = B\exp[Q_n'(a_{ij};x_1,x_2)], \tag{11}$$

where B is normalization constant, n are polynomial orders, Q_n' are polynomial functions expressed as

$$Q_n'(a_{ij};x_1,x_2) = \sum_{i=1}^{n}\sum_{j=0}^{i} a_{ij}x_1^{i-j}x_2^{j}, \tag{12}$$

where a_{ij} are unknown coefficients. It is calculated that the number N_p of total polynomial terms is $N_p = n(n+3)/2$. In the case of colored noise, it is noticed that state variables in Eq. (10) are increased to four. Consequently, the EPC method has to be improved and developed to suit such case. State variables need to be extended and additional polynomial terms are produced. As a result, approximate solution with the developed EPC method is expressed as

$$\tilde{p}(\boldsymbol{a};\boldsymbol{x}) = B\exp[Q_n(a_{ijkm};x_1,x_2,x_3,x_4)], \tag{13}$$

where polynomial functions Q_n are

$$\begin{aligned} Q_n(a_{ijkm};x_1,x_2,x_3,x_4) &= \sum_{i=1}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\sum_{m=0}^{n} a_{ijkm}x_1^{i}x_2^{j}x_3^{k}x_4^{m} \\ &\quad i+j+k+m = 1,2,\ldots,n. \end{aligned} \tag{14}$$

It should be noticed that when the polynomial order $n = 2$, the number of polynomial terms is $N_p = 14$, and the number goes up to $N_p = 52$ when the polynomial order $n = 4$. The numbers are much bigger than the ones $n(n+3)/2$ in the case of white noise excitation. As a result, the procedure for coding the computer programs becomes much complicated, and requires much more computational effort.

By substituting approximate solution Eq. (13) into Eq. (10), residual error inevitably results

$$\begin{aligned} \Delta(x_1,x_2,x_3,x_4;\tilde{p}) &= \Bigg\{x_2\frac{\partial Q_n}{\partial x_1} - h_0(x_1,x_2)\frac{\partial Q_n}{\partial x_2} - \frac{\partial h_0(x_1,x_2)}{\partial x_2} + x_4\frac{\partial Q_n}{\partial x_3} - f(x_3,x_4)\frac{\partial Q_n}{\partial x_4} - \frac{\partial f(x_3,x_4)}{\partial x_4} \\ &\quad -\frac{1}{2}S_{ii}h_i^2(x_1)\left[\frac{\partial^2 Q_n}{\partial x_2^2} + \left(\frac{\partial Q_n}{\partial x_2}\right)^2\right] - \frac{1}{2}S_0\left[\frac{\partial^2 Q_n}{\partial x_4^2} + \left(\frac{\partial Q_n}{\partial x_4}\right)^2\right]\Bigg\}\tilde{p}(\boldsymbol{a};\boldsymbol{x}) \\ &= \delta(a_{ijkm};x_1,x_2,x_3,x_4)\tilde{p}(\boldsymbol{a};\boldsymbol{x}), \end{aligned} \tag{15}$$

where $\Delta(x_1,x_2,x_3,x_4;\tilde{p})$ is residual error, $\delta(a_{ijkm};\ x_1,\ x_2,\ x_3,\ x_4)$ represents

$$\begin{aligned} \delta(a_{ijkm};x_1,x_2,x_3,x_4) &= x_2\frac{\partial Q_n}{\partial x_1} - h_0(x_1,x_2)\frac{\partial Q_n}{\partial x_2} - \frac{\partial h_0(x_1,x_2)}{\partial x_2} + x_4\frac{\partial Q_n}{\partial x_3} - f(x_3,x_4)\frac{\partial Q_n}{\partial x_4} - \frac{\partial f(x_3,x_4)}{\partial x_4} \\ &\quad -\frac{1}{2}S_{ii}h_i^2(x_1)\left[\frac{\partial^2 Q_n}{\partial x_2^2} + \left(\frac{\partial Q_n}{\partial x_2}\right)^2\right] - \frac{1}{2}S_0\left[\frac{\partial^2 Q_n}{\partial x_4^2} + \left(\frac{\partial Q_n}{\partial x_4}\right)^2\right]. \end{aligned} \tag{16}$$

In Eq. (15), $\Delta(x_1,x_2,x_3,x_4;\tilde{p})$ cannot be vanished due to nonzero approximate solution $\tilde{p}$. The only way to eliminate the residual error is to vanish $\delta(a_{ijkm};\ x_1,\ x_2,\ x_3,\ x_4)$. Herein Galerkin method is employed

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(a_{ijkm};x_1,x_2,x_3,x_4)w_h dx_1dx_2dx_3dx_4 \\ &= 0 \quad (h = 1,2,\ldots,N_p), \end{aligned} \tag{17}$$

where w_h are weighting functions. Numerical analysis evidences that effective choice for weighting functions w_h is

$$w_h = x_1^i x_2^j x_3^m x_4^n f(\boldsymbol{x};\boldsymbol{m},\boldsymbol{\sigma}) \quad h = i+j+m+n = 1,2,\ldots,N_p, \tag{18}$$

where $f(\boldsymbol{x};\ \boldsymbol{m},\ \boldsymbol{\sigma})$ is multivariate Gaussian PDF. Statistical mean vector $\boldsymbol{m}(m_i)$ and statistical variance matrix $\boldsymbol{\sigma}(\sigma_{ij})$ are obtained with EQL method.

With Galerkin method, N_p nonlinear algebraic equations are obtained from Eq. (17). Although Gaussian moment properties greatly simplify the multi-fold integration procedure, numerical solution procedure is still tedious since nonlinear algebraic equations are involved with N_p unknown parameters a_{ijkm} in Eq. (17). In the numerical solution procedure, it should be noticed that statistical variance matrix $\boldsymbol{\sigma}(\sigma_{ij})$ has to be adjusted at every iteration step to ensure the convergence.

To investigate the accuracy of the EPC approximate method, MCS with a sample size of 10^9 is performed. The 4th order Rung-Kutta method is used in the numerical simulation.

For the computational time, the EPC solution procedure only takes several seconds and most of the computational time is spent on the EQL procedure, which is about half an hour. Meanwhile, the computer programming code can be further improved to save computational time. With the same computer and under the same operating environment, the time spent with MCS method is about an hour when the sample size is 10^9. Numerical analysis shows that the improved EPC method with polynomial order $n = 4$ does not take a big advantage in the computational time compared to the MCS method. However, we still turn to the EPC method for the following reasons. First, it is well known that small probabilities at the tail regions of the PDF results are very important for the reliability or failure analysis, but there are always fluctuations for the results obtained from the MCS method due to the limitation of sample size. It will cause great numerical errors in the reliability or failure analysis. Higher accuracy of the MCS method with larger sample size often involves with much more expensive computational effort, such as several minutes for a sample size 10^7 increasing up to about one hour for a sample size 10^9. What's more, the EPC method can be extended to 3S-EPC method for the more general MDOF dynamical systems with much higher efficiency than the MCS method.

For the reliability and failure analysis, it is necessary to get MCR, the rate at which a differentiable stationary response process $X(t)$ crosses a level $X = x$ with a positive slope. Based on the PDFs of system response, the MCR of response is given by Rice's formula [24,25]

$$\mu^+(x) = \int_0^\infty \dot{x} p_{x\dot{x}}(x, \dot{x}) d\dot{x}, \tag{19}$$

where $\mu^+(x)$ is the mean up-crossing intensity across level x.

3. Numerical examples and results discussions

Duffing oscillator, the most investigated nonlinear oscillator, is taken as a typical example to illustrate the effectiveness of the improved EPC method. The EPC method can also be used for other oscillators of polynomial nonlinearity, including oscillators subject to excitations in polynomial forms of filtered normal processes. Due to the limitation of paper length, here only Duffing oscillator excited by independent additive colored noise and multiplicative white noise excitations is considered. In the first example, Gaussian colored noise is considered, which is acquired by filtering a linear oscillator to Gaussian white noise excitation. In the second example, non-Gaussian colored noise is obtained through a nonlinear filter to Gaussian white noise.

The transverse motion of a simply-supported uniform column with large deformation subject to axial compressive force and transverse concentrated force at the middle of the column is governed by an ordinary differential equation as follows [26]

$$\ddot{X} + \alpha\dot{X} + \beta X + \varepsilon X^3 = c_1 X W_1(t) + C(t), \tag{20}$$

where $W_1(t)$ is axial force modeled as Gaussian white noise with PSD $S_{11} = 0.2$, $C(t)$ is concentrated force modeled as colored noise. Other system parameters are given as: $\alpha = 0.2, \beta = 1, \varepsilon = 0.5, c_1 = -1$.

3.1. Example 1: Gaussian colored noise

In this example, $C(t)$ is Gaussian colored noise got by a linear filter

$$\ddot{C} + 2\zeta\omega_0\dot{C} + \omega_0^2 C = W(t), \tag{21}$$

where ζ is damping ratio, W is Gaussian white noise and the PSD is $S_0 = 1$. With Gaussian white noise excitation, the response C is still Gaussian and its PSD can be expressed as

$$S(\omega) = \frac{2S_0}{[\omega_0^2 - \omega^2]^2 + 4\zeta^2\omega_0^2\omega^2}, \tag{22}$$

where parameters of the filter are set as $\zeta = 0.25, \omega_0 = 0.6$. The result for the PSD of the colored noise $C(t)$ is shown in Fig. 1. It is seen that Gaussian colored noise $C(t)$ is a narrow-banded process with central frequency close to ω_0. In this sense the linear oscillator in Eq. (21) is equivalent to a narrow-banded filter. It was mentioned earlier that the band width of colored noise depends mainly on the stiffness. In engineering if the foundation soil is very soft, earthquake force modeled as white noise going through such foundation soil will be narrow banded, and almost all the energy is converged into some local frequencies. This easily makes structures with nature frequency close to local frequencies destruct deadly. We should avoid or modify such circumstance.

In this example, Duffing oscillator with strong nonlinearity under Gaussian colored noise is investigated with EQL, MCS and developed EPC ($n = 4$) methods, respectively. Since PDF values in the tail regions play an important role in the reliability analysis, logarithmic PDF results ($Logp(x_1)$ and $Logp(x_2)$) are employed. The results obtained from different methods are compared and displayed in Figs. 2 and 3. From these two figures, it is seen that results from EQL method deviate a lot from the simulated ones, while the results with developed EPC ($n = 4$) method agree well with the simulated ones. It is

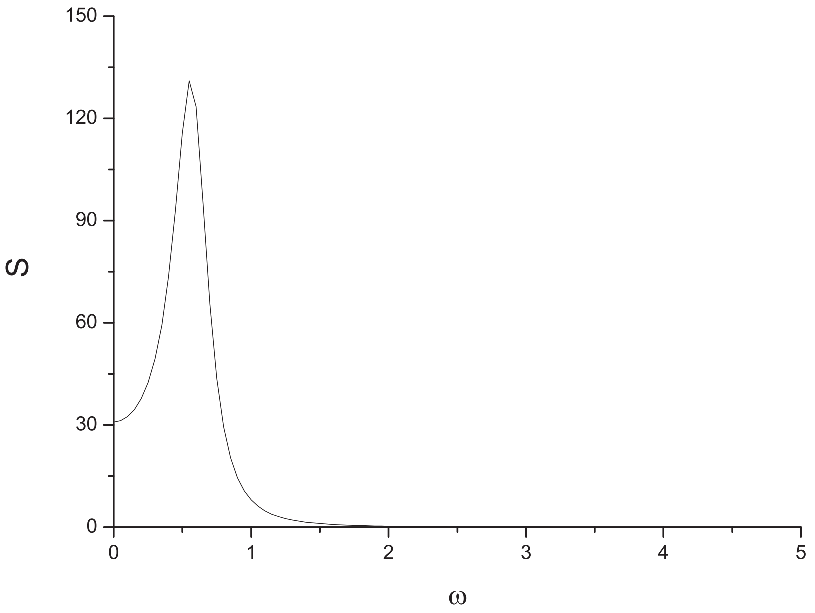

Fig. 1. The PSD for Gaussian colored noise $C(t)$ in Example 1.

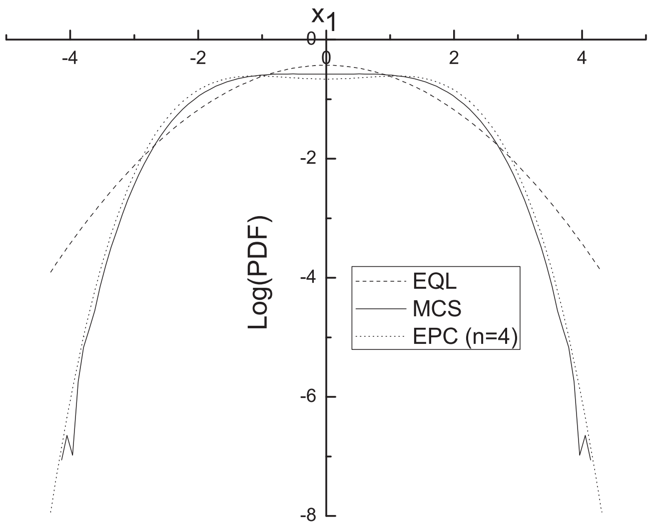

Fig. 2. Log(PDF) of displacement in Example 1.

also noticed apparently that the PDF of velocity is typically non-Gaussian. Further, crossing parameter MCR for the response is computed according to Eq. (19) and the results are shown in Fig. 4. A good agreement is also observed in the crossing intensity estimated.

3.2. Example 2: Non-Gaussian colored noise

In this example, $C(t)$ is non-Gaussian colored noise obtained through a nonlinear filter

$$\ddot{C} + 2\zeta\omega_0\dot{C} + \omega_0^2 C + \varepsilon_2 C^3 = W(t), \tag{23}$$

where parameter ε_2 denotes the degree of non-linearity, $W(t)$ is Gaussian white noise with PSD $S_0 = 1$. Since Eq. (23) is a nonlinear filter, the response C is non-Gaussian with PSD

$$S(\omega) = \frac{4\zeta\omega_0(\omega_0^2 + 3\varepsilon_2 m_{20})m_{20}}{[(\omega_0^2 + 3\varepsilon_2 m_{20} - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2]}, \tag{24}$$

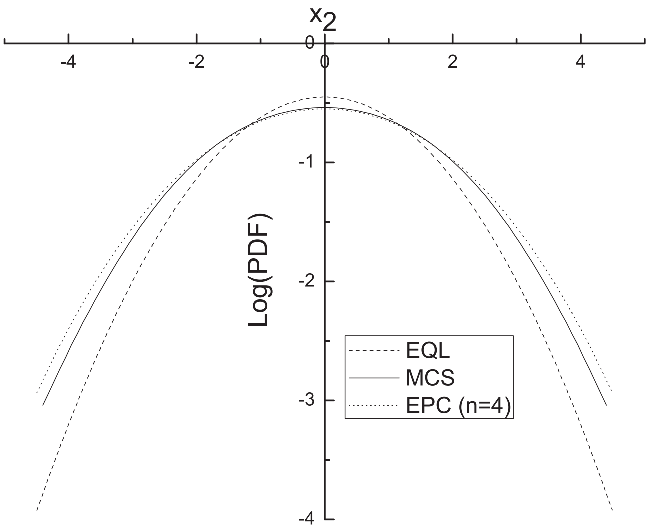

Fig. 3. Log(PDF) of velocity in Example 1.

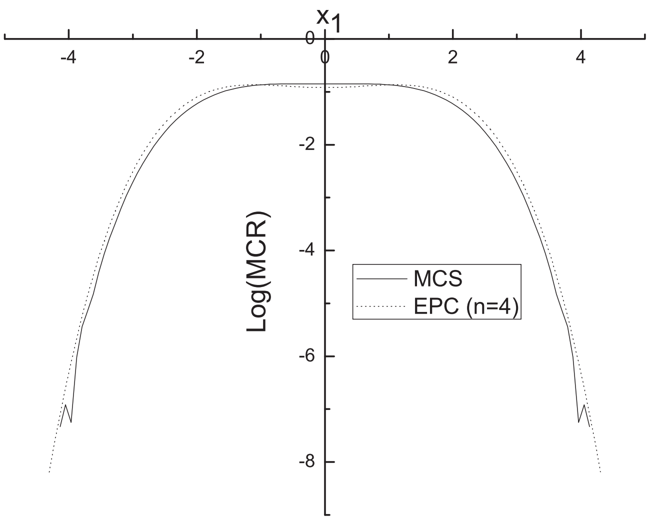

Fig. 4. Log(MCR) of displacement in Example 1.

where m_{20} is the second moment of colored noise state variable. System parameters are setting as $\zeta = 0.25, \omega_0 = 1, \varepsilon_2 = 0.5$. As a result, Duffing oscillator in Eq. (20) can be equivalently excited by a colored noise $C(t)$ with PSD $S(\omega)$, in which $m_{20} = E[c^2(t)]$ can be obtained by

$$m_{20} = E[c^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2 p(c, \dot{c}) dc d\dot{c} \tag{25}$$

with the exact joint PDF $p(c, \dot{c})$

$$p(c, \dot{c}) = B \exp\left\{-\left[\frac{1}{2}\dot{c}^2 + \frac{\omega_0^2}{2}c^2 + \frac{\varepsilon_2}{4}c^4\right]\right\} \tag{26}$$

where B is a normalization constant.

According to Eq. (24), PSD versus frequency for the colored noise $C(t)$ is shown in Fig. 5. It is observed that non-Gaussian colored noise is a moderately narrow banded process with central frequency deviating from ω_0 due to nonlinear term $\varepsilon_2 c^3$. Under such excitation, Duffing oscillator is analyzed by the EQL, developed EPC ($n = 4$) and MCS methods. The results are present in Figs. 6 and 7. These results suggest that the PDF predicted by the EQL method fails to give an acceptable match to the simulated one in the tails of probability distribution, while the PDF obtained by the developed EPC method better estimates the one in the tails.

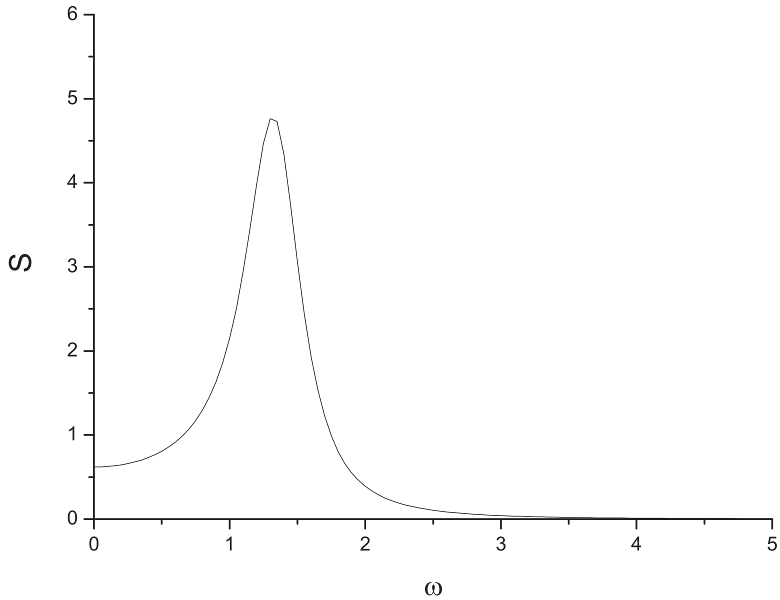

Fig. 5. The PSD for non-Gaussian colored noise $C(t)$ in Example 2.

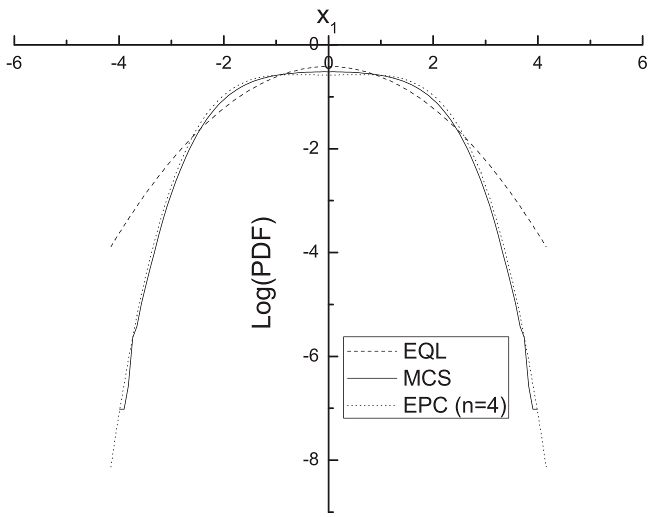

Fig. 6. Log(PDF) of displacement in Example 2.

Based on the PDFs of system responses, MCR can be calculated and it is displayed in Fig. 8. It is implied better fitting between the results from the developed EPC method and ones from the MCS.

4. Conclusions

In this paper, Duffing oscillator combined to Gaussian white noise and Gaussian/non-Gaussian colored noise excitations is investigated. By introducing colored noise as an additional state variable, the second-order equation of motion is transferred to four coupled first-order differential equations. Under the assumption that state variables and colored noise are uncorrelated initially, combined process of state variables and colored noise is a Markovian process. The FPK equation can be adopted to investigate probabilistic properties of the Duffing systems. Due to the additional state variable of colored noise, corresponding FPK equation increases to four-dimension. The EPC method is developed and improved to solve such FPK equation with approximate solution involving four state variables. Two examples are considered and PSDs are separately given for the narrow-banded Gaussian and non-Gaussian colored noise excitations. By comparison with numerical solutions from MCS method and approximate solutions from EQL method, it is indicated that the developed EPC method provides

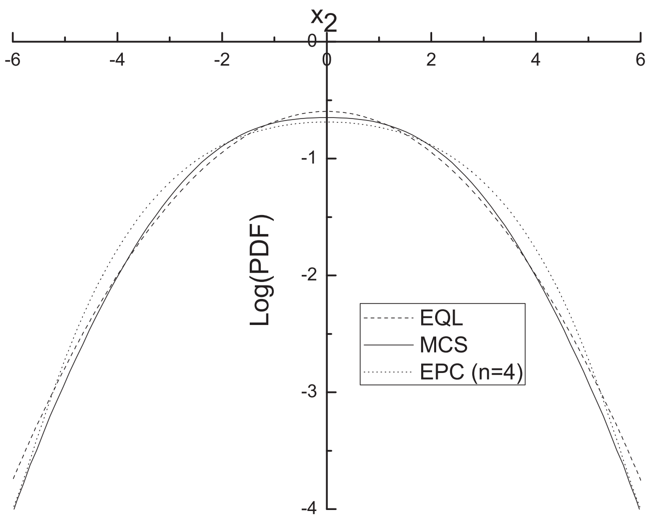

Fig. 7. Log(PDF) of velocity in Example 2.

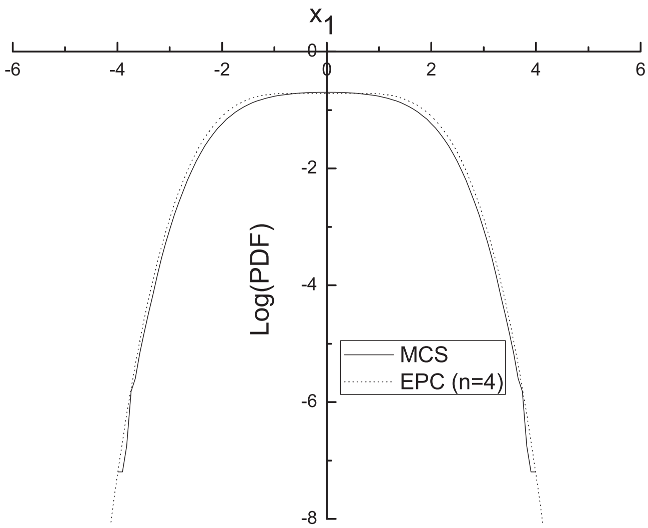

Fig. 8. Log(MCR) of displacement in Example 2.

relatively accurate PDF solutions. Furthermore, it should be noticed that when the band width of colored noise becomes very narrow, non-convergent approximate solutions will be easily produced.

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