Journal of Computational and Applied Mathematics 313 (2017) 318–342

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Analysis of a Galerkin approach applied to a system of coupled Schrödinger equations

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ARTICLE INFO

Article history:
Received 1 June 2015
Received in revised form 21 July 2016

MSC:
35Q60
65M12
65T50
65M70

Keywords:
Galerkin method
The coupled nonlinear Schrödinger equations
Fourier basis
Optical fiber

ABSTRACT

We study error and convergence rate of a fully discrete Fourier–Galerkin scheme to approximate the solutions of a system consisting of two coupled cubic Schrödinger equations with cross modulation, which is a model for the propagation of a nonlinear pulse in a linearly birefringent Kerr optical fiber. We also establish a result on existence and uniqueness of a solution of the corresponding initial value problem.

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1. Introduction

In this paper, we consider the numerical simulation of the coupled nonlinear Schrödinger equations (CNLS)

$$i\frac{\partial u}{\partial t} + i\delta\frac{\partial u}{\partial x} + K\frac{\partial^2 u}{\partial x^2} + \sigma_1 u + a|u|^2 u + b|v|^2 u + ev^2\overline{u} = 0 \tag{1.1}$$

$$i\frac{\partial v}{\partial t} - i\delta\frac{\partial v}{\partial x} + K\frac{\partial^2 v}{\partial x^2} + \sigma_2 v + c|v|^2 v + d|u|^2 v + eu^2\overline{v} = 0, \tag{1.2}$$

subject to initial conditions

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x),$$

and periodic boundary conditions

$$u(x + L, t) = u(x, t), \qquad v(x + L, t) = v(x, t).$$

System (1.1)–(1.2) is a model for the propagation of a light ray inside a long optical fiber, taking into account birefringence, cross modulation and nonlinear effects (Kerr phenomenon). Here $\delta, K, a, b, c, d, e, \sigma_1, \sigma_2$ are positive real constants. The functions u, v are the complex amplitudes or envelopes of the two components of the electric field inside the optical

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http://dx.doi.org/10.1016/j.cam.2016.09.030
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fiber [1,2]. System (1.1)–(1.2) have been considered for instance by Menyuk in [1,3]. Evangelides [4] employed the full CNLS system, but with random coefficients, to study the polarization of solitons in a long fiber whose linear birefringence changes randomly over its length. Muñoz and Quiceno [2] studied numerically modulational stability/instability under small perturbations on initial data of a family of periodic solutions, using a spectral Galerkin–Fourier approach for the space and a second-order finite difference scheme to discretize the temporal evolution. The effect of the nonlinear terms preceded by the parameter e on the modulation instability mechanism of periodic solutions was investigated in [2]. The CNLS system (with $e = 0$) has been further extensively studied in the field of optics [5–16].

It is worthwhile to point out that the CNLS system (with $\delta = e = \sigma_1 = \sigma_2 = 0$) also arises in other fields of application. For instance, Tan and Liu [17] have derived this system for unstable baroclinic waves in a two-layer atmospheric model. Tan [18] derived the same system as an atmospheric model for Rossby waves. The CNLS system appears also in beam propagation inside photorefractive crystals and the interaction of water waves (see [19–22] and references therein).

Since available analytical solutions for the CNLS system are limited, the numerical methods are important tools to understand the long-time behavior of solutions and instability/stability of standing wave states of system (1.1)–(1.2), which are interesting and important problems in applications. In the present paper, we will develop a rigorous analysis of the convergence of both the semidiscrete and the fully discrete schemes used recently by Muñoz and Quiceno in [2] to study modulational instability of periodic solutions of the CNLS system. The rates of convergence of the resulting schemes are $O(N^{s-r})$ and $O(N^{s-r} + \Delta t^2)$, where r, s with $r > s$ depend only on the smoothness of the exact solution, Δt is the time step and N is the number of spatial Fourier modes. In previous works [23] (and references therein) and [24–27], where finite difference schemes, symplectic and multisymplectic methods, variational iteration method, Fourier pseudospectral methods or Galerkin methods, were employed to discretize the CNLS system (with $e = 0$), the convergence of the numerical schemes was corroborated using some numerical simulations with exact solutions of the system. However rigorous error estimations and convergence results were not developed. Further, we show that the semidiscrete scheme conserves some physical quantities of the CNLS system, such as L^2 and energy norms. Through some numerical examples we illustrate that these quantities are also approximately conserved by the fully discrete scheme. This is an important property for a numerical scheme to solve system (1.1)–(1.2), since the failure of conservation of invariants of evolution can lead to blow up of the computed solution [28]. A theory on existence and uniqueness of the Cauchy problem, corresponding to the CNLS system is presented using semigroup theory, Fourier analysis and Banach's fixed point theory. These are our main theoretical contributions. To our best knowledge, the issues mentioned above have not been considered in earlier works on the CNLS system.

The rest of this paper is organized as follows. In Section 2, we introduce notation and preliminaries required in the paper. In Section 3, existence and uniqueness of a solution to the Cauchy problem (1.1)–(1.2) is discussed using semigroup theory and Banach's fixed point principle. Section 4 develops a study of convergence of the semidiscrete formulation for approximating a solution of system (1.1)–(1.2). In Section 5, we address the numerical properties of the fully discrete scheme for solving system (1.1)–(1.2). In Section 6, we present some numerical simulations corroborating the analytical results obtained in the paper. Finally, Section 7 contains the conclusions of our work.

2. Preliminaries

Through this article we will work with L-periodic functions, where L is a positive real number. For $1 \le p < \infty$, we will denote by $L^p(0, L)$ to the space

$$L^p(0, L) = \left\{ f : [0, L] \to \mathbb{C} : f \text{ is a measurable function such that } \|f\|_{L^p(0,L)} < \infty \right\},$$

where

$$\|f\|_{L^p(0,L)} = \left(\int_0^L |f(x)|^p \, dx \right)^{1/p}, \quad 1 \le p < \infty.$$

When $p = 2$ we have that

$$\langle f, g \rangle = \int_0^L f(x)\overline{g(x)} dx,$$

is an inner product in $L^2(0, L)$ and $\|f\|_{L^2(0,L)} = \langle f, f \rangle^{1/2}$. Henceforth we write L^2 to refer to the space $L^2(0, L)$ and by $\|f\|$, we denote the norm of f in the space L^2. The space of all functions of class C^k that are L-periodic is denoted by $C^k_{per}(0, L)$, $k = 0, 1, 2, \ldots$. Further $C_{per} = C_{per}(0, L) = C^0_{per}(0, L)$ is the space of all continuous functions of period L.

We will denote by $\mathscr{P}$ to the space of all C^∞-functions $\varphi : \mathbb{R} \to \mathbb{C}$ that are L-periodic.

We say that $T : \mathscr{P} \to \mathbb{C}$, defines a periodic distribution, i.e., $T \in \mathscr{P}'$ if T is linear and there exists a sequence $(\Psi_n)_{n\in\mathbb{N}} \subset \mathscr{P}$ such that

$$T(\varphi) = \lim_{n\to\infty} \int_0^L \Psi_n(x)\varphi(x)dx, \quad \forall \varphi \in \mathscr{P}.$$

Let $s \in \mathbb{R}$. The Sobolev space, denoted by $H^s_{per} = H^s_{per}(0, L)$, is defined as

$$H^s_{per}(0, L) = \left\{ f \in \mathscr{P}' : \|f\|_s^2 = L \sum_{n\in\mathbb{Z}} (1 + k^2)^s \left|\widehat{f}(n)\right|^2 < \infty \right\},$$

where $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ represents the Fourier transform of f defined by

$$\widehat{f}(n) = \frac{1}{L}\langle f, e^{-2\pi inx/L}\rangle.$$

In case that $f \in C_{per}$, we can rewrite $\hat{f}(n)$ as

$$\hat{f}(n) = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}dx, \quad n \in \mathbb{Z}.$$

It can be shown that for all $s \in \mathbb{R}$, H^s_{per} is a Hilbert space with respect to the inner product $\langle\cdot, \cdot\rangle_s$ defined as follows:

$$\langle f, g\rangle_s = L \sum_{n\in\mathbb{Z}} (1 + n^2)^s \widehat{f}(n)\overline{\widehat{g}(n)}.$$

In particular, when $s = 0$, we get the Hilbert space denoted by $L^2_{per} = H^0_{per}$. It is important to note that this space is isometrically isomorphic to $L^2(0, L)$. Further we recall that Parseval's identity holds, i.e., for $f \in C_{per}$

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{1}{L}\|f\|^2,$$

or equivalently,

$$\langle f, g\rangle = L \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}}(n) = L\langle \hat{f}, \hat{g}\rangle.$$

For the product space $H^s_{per} \times H^s_{per}$, we have that

$$\|U\|_{H^s\times H^s} = (\|u_1\|_s^2 + \|u_2\|_s^2)^{1/2},$$

with $U = (u_1, u_2)^T$ is a norm. Sometimes we will also use the equivalent norm

$$\|U\|_{H^s\times H^s} = \|u_1\|_s + \|u_2\|_s.$$

Let $N \in 2\mathbb{Z}$ and consider the finite dimensional space S_N defined by

$$S_N = \text{span}\left\{ \frac{1}{\sqrt{L}} e^{\frac{2\pi inx}{L}} : -N/2 \le n \le N/2 \right\}.$$

Remember that the family $\{\frac{1}{\sqrt{L}}e^{\frac{2\pi inx}{L}} : n \in \mathbb{Z}\}$ is orthonormal and complete in $L^2_{per}(0, L)$. Let P_N be the orthogonal projection $P_N : L^2_{per}(0, L) \to S_N$ on the space S_N,

$$P_N g := \sum_{n=-N/2}^{N/2} \hat{g}_n \phi_n,$$

with

$$\phi_n(x) = e^{\frac{2\pi inx}{L}}, \qquad \hat{g}_n = \frac{1}{L}\int_0^L g(x)\overline{\phi}_n(x)dx.$$

This operator has the following properties (see [29–31]): For any $g \in L^2_{per}(0, L)$,

$$\langle P_N g - g, \phi\rangle = 0, \quad \forall \phi \in S_N.$$

Furthermore, given integers $0 \le s \le r$, there exists a constant C independent of N such that, for any $g \in H^r_{per}(0, L)$

$$\|P_N g - g\|_s \le CN^{s-r}\|g\|_r. \tag{2.1}$$

Finally, the following result is very important to establish convergence of the numerical schemes for system (1.1)–(1.2) proposed in the present work.

Lemma 2.1 (*Gronwall's Lemma*). *Let* (ξ_i) *be a sequence satisfying*

$$|\xi_{i+1}| \le (1 + \delta)|\xi_i| + B, \quad \delta > 0, \ B \ge 0, \ i = 0, 1, 2, \ldots$$

then

$$|\xi_n| \le e^{n\delta}|\xi_0| + \frac{e^{n\delta} - 1}{\delta}B.$$

Proof. See [32]. □

3. Existence and uniqueness

In this section, we study existence and uniqueness of a solution to the system of coupled Schrödinger equations

$$i\frac{\partial u}{\partial t}+i\delta\frac{\partial u}{\partial x}+K\frac{\partial^2 u}{\partial x^2}+\sigma_1 u+a|u|^2u+b|v|^2u+ev^2\overline{u}=0 \tag{3.1}$$

$$i\frac{\partial v}{\partial t}-i\delta\frac{\partial v}{\partial x}+K\frac{\partial^2 v}{\partial x^2}+\sigma_2 v+c|v|^2v+d|u|^2v+eu^2\overline{v}=0, \tag{3.2}$$

subject to initial conditions

$$u(x,0)=u_0(x),\qquad v(x,0)=v_0(x),$$

and periodic boundary conditions

$$u(x+L,t)=u(x,t),\qquad v(x+L,t)=v(x,t).$$

We can write the system above for $s\geq 2$ as

$$\partial_t U=AU+F(U),\qquad U(0)=U_0=\begin{pmatrix}u_0\\ v_0\end{pmatrix}, \tag{3.3}$$

where

$$U=\begin{pmatrix}u\\ v\end{pmatrix}\in H^s_{per}\times H^s_{per},\qquad A=\begin{pmatrix}iK\partial^2_{xx}-\delta\partial_x & 0\\ 0 & iK\partial^2_{xx}+\delta\partial_x\end{pmatrix},$$

and

$$F(U)=\begin{pmatrix}F_1(U)\\ F_2(U)\end{pmatrix}=\begin{pmatrix}i\sigma_1 u+ia\,|u|^2\,u+ib\,|v|^2\,u+iev^2\overline{u}\\ i\sigma_2 u+ic\,|v|^2\,v+id\,|u|^2\,v+ieu^2\overline{v}\end{pmatrix}.$$

3.1. Linear semigroup

Now consider the linear problem

$$\partial_t U=AU,\qquad U(0)=U_0. \tag{3.4}$$

For $u, v\in H^s_{per}$, $s\geq 2$, we have the expansions

$$u(x,t)=\sum_n u_n(t)e^{iw_nx},\qquad v(x,t)=\sum_n v_n(t)e^{iw_nx},$$

with $w_n=\frac{2\pi n}{L}$ in Eq. (3.4), we obtain that

$$\sum_n u_n'(t)e^{iw_nx}=-\delta\sum_n iw_nu_n(t)e^{iw_nx}-K\sum_n iw_n^2u_n(t)e^{iw_nx},$$
$$\sum_n v_n'(t)e^{iw_nx}=\delta\sum_n iw_nv_n(t)e^{iw_nx}-K\sum_n iw_n^2v_n(t)e^{iw_nx}.$$

Therefore

$$u_n(t)=e^{-i(\delta w_n+Kw_n^2)t}u_{0,n},$$
$$v_n(t)=e^{i(\delta w_n-Kw_n^2)t}v_{0,n},$$

where $u_{0,n}=u_n(0)$, $v_{0,n}=v_n(0)$. Let us define for $t\geq 0$ and

$$U=\begin{pmatrix}u\\ v\end{pmatrix}=\sum_n\begin{pmatrix}u_n\\ v_n\end{pmatrix}e^{iw_nx},$$

the family of linear operators

$$S(t)(U)=\sum_n\begin{pmatrix}e^{i\alpha_nt} & 0\\ 0 & e^{i\beta_nt}\end{pmatrix}e^{iw_nx}\begin{pmatrix}u_n\\ v_n\end{pmatrix},$$

where $\alpha_n=-\delta w_n-Kw_n^2$ and $\beta_n=\delta w_n-Kw_n^2$. Therefore the solution of problem (3.4) can be expressed as

$$U(t)=S(t)U_0.$$

Theorem 3.1. *$(S(t))_{t\geq 0}$ is a C_0-semigroup in $H^{s-2}_{per}\times H^{s-2}_{per}$, $s\geq 2$. Furthermore the linear operator $A: H^s_{per}\times H^s_{per}\to H^{s-2}_{per}\times H^{s-2}_{per}$, $s\geq 2$ is its infinitesimal generator.*

Proof. First note that for $t=0$ and

$$U=\sum_n \begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x},$$

we have that

$$S(0)U=\sum_n \begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x}=U.$$

For $t,\tau\geq 0$, we have that

$$\begin{aligned} S(\tau+t)U &= \sum_n \begin{pmatrix} e^{i\alpha_n(\tau+t)} & 0\\ 0 & e^{i\beta_n(\tau+t)}\end{pmatrix} e^{iw_n x}\begin{pmatrix} u_n\\ v_n\end{pmatrix}\\ &=\sum_n \begin{pmatrix} e^{i\alpha_n\tau} & 0\\ 0 & e^{i\beta_n\tau}\end{pmatrix}\begin{pmatrix} e^{i\alpha_n t} & 0\\ 0 & e^{i\beta_n t}\end{pmatrix}\begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x}=S(\tau)S(t)U.\end{aligned}$$

Moreover

$$\begin{aligned}\|S(t)U\|^2_{H^{s-2}\times H^{s-2}} &= \left\|\sum_n \begin{pmatrix} e^{\alpha_n ti} & 0\\ 0 & e^{\beta_n ti}\end{pmatrix}\begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x}\right\|^2_{H^{s-2}\times H^{s-2}}\\ &\leq L\sum_n (1+n^2)^{s-2}(|e^{\alpha_n ti}u_n|^2+|e^{\beta_n ti}v_n|^2)\leq \|U\|^2_{H^{s-2}\times H^{s-2}}.\end{aligned}$$

Thus $S(t)$ is a bounded operator in $H^{s-2}_{per}\times H^{s-2}_{per}$ for every $t\geq 0$.

Next

$$\begin{aligned}\|S(t)U-U\|^2_{H^{s-2}\times H^{s-2}} &\leq \left\|\sum_n \begin{pmatrix} e^{\alpha_n ti} & 0\\ 0 & e^{\beta_n ti}\end{pmatrix}\begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x}-\sum_n\begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x}\right\|^2_{H^{s-2}\times H^{s-2}}\\ &=\left\|\sum_n \begin{pmatrix} (e^{\alpha_n ti}-1)u_n\\ (e^{\beta_n ti}-1)v_n\end{pmatrix} e^{iw_n x}\right\|^2_{H^{s-2}\times H^{s-2}}\\ &\leq L\sum_n (1+n^2)^{s-2}(|(e^{\alpha_n ti}-1)u_n|^2+|(e^{\beta_n ti}-1)v_n|^2).\end{aligned}$$

Since $|\exp(\alpha_n ti)-1|\to 0$, $|\exp(\beta_n ti)-1|\to 0$, as $t\to 0$, we conclude that

$$\|S(t)U-U\|_{H^{s-2}\times H^{s-2}}\to 0,\quad t\to 0.$$

On the other hand, since we can express the operator A as

$$AU=\sum_n \begin{pmatrix} -iKw_n^2-\delta iw_n & 0\\ 0 & -iKw_n^2+\delta iw_n\end{pmatrix}\begin{pmatrix} u_n\\ v_n\end{pmatrix} e^{iw_n x},$$

for $U\in\mathcal{D}(A)=H^s_{per}\times H^s_{per}$, we have that

$$\begin{aligned}\|A(U)\|^2_{H^{s-2}\times H^{s-2}} &= L\sum_n (1+n^2)^{s-2}(|(-iKw_n^2+\delta iw_n)u_n|^2+|(-iKw_n^2-\delta iw_n)v_n|^2)\\ &\leq LC\sum_n (1+n^2)^{s}(|u_n|^2+|v_n|^2)=C\|U\|^2_{H^s\times H^s}.\end{aligned}$$

This means that A is a bounded operator from $\mathcal{D}(A)=H^s_{per}\times H^s_{per}$ to $H^{s-2}_{per}\times H^{s-2}_{per}$.

To see that A is the infinitesimal generator of the semigroup $S(t)$, $t\geq 0$, we compute

$$\begin{aligned}\left\|\frac{S(t)U-U}{t}-AU\right\|^2_{H^{s-2}\times H^{s-2}} &= \left\|\frac{1}{t}\sum_n \begin{pmatrix} (e^{i\alpha_n t}-1)u_n\\ (e^{i\beta_n t}-1)v_n\end{pmatrix} e^{iw_n x}-\begin{pmatrix} (iK\partial_x^2-\delta\partial_x)u\\ (iK\partial_x^2+\delta\partial_x)v\end{pmatrix}\right\|^2_{H^{s-2}\times H^{s-2}}\\ &\times L\sum_n (1+n^2)^{s-2}\Big[\Big|\Big(\frac{e^{i\alpha_n t}-1}{t}\Big)u_n-(-iKw_n^2-\delta i)u_n\Big|^2\\ &+\Big|\Big(\frac{e^{i\beta_n t}-1}{t}\Big)v_n-(-iKw_n^2+\delta i)v_n\Big|^2\Big].\end{aligned}$$

In virtue of

$$\lim_{t\to 0}\frac{e^{i\alpha_n t}-1}{t}=\frac{d}{dt}e^{i\alpha_n t}\Big|_{t=0}=i\alpha_n,$$

and

$$\lim_{t\to 0}\frac{e^{i\beta_n t}-1}{t}=\frac{d}{dt}e^{i\beta_n t}\Big|_{t=0}=i\beta_n,$$

we conclude that

$$\left\|\frac{S(t)U-U}{t}-AU\right\|_{H^{s-2}\times H^{s-2}}\to 0,$$

as $t\to 0$, and thus the operator A is the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$. $\square$

3.2. Nonlinear problem

In this section we will consider existence and uniqueness of a solution to the nonlinear system

$$\partial_t U=AU+F(U), \tag{3.5}$$

subject to

$$U(0)=U_0=\begin{pmatrix}u_0\\ v_0\end{pmatrix}.$$

Lemma 3.2. *If $U,V\in C([0,T],H^s_{per}\times H^s_{per})$, $s>1/2$, then for $U=(u_1,v_1)^T$, $V=(u_2,v_2)^T$,*

$$\|F(U)-F(V)\|_{H^s\times H^s}\leq\|U-V\|_{H^s\times H^s}\,L\left(\|u_1\|_s,\|v_1\|_s,\|u_2\|_s,\|v_2\|_s\right),$$

where L is a second degree polynomial depending on $\|u_1\|_s,\|u_2\|_s,\|v_1\|_s,\|v_2\|_s$.

Proof. Let $U=(u_1,v_1)^T$, $V=(u_2,v_2)^T$. Here we will use the norm

$$\|U\|_{H^s\times H^s}=\|u_1\|_s+\|v_1\|_s,$$

for the product space $H^s_{per}\times H^s_{per}$. Then

$$\begin{aligned}
&\|F(U)-F(V)\|_{H^s\times H^s}\\
&\quad=\|i\sigma_1(u_1-u_2)+ia(|u_1|^2u_1-|u_2|^2u_2^2)+ib(|v_1|^2u_1-|v_2|^2u_2)+ie(v_1^2\overline{u_1}-v_2^2\overline{u_2})\|_s\\
&\qquad+\|i\sigma_2(v_1-v_2)+ic(|v_1|^2v_1-|v_2|^2v_2^2)+id(|u_1|^2v_1-|u_2|^2v_2)+ie(u_1^2\overline{v_1}-u_2^2\overline{v_2})\|_s\\
&\quad\leq|\sigma_1|\,\|u_1-u_2\|_s+|a|(\|u_1\|_s^2+\|u_1\|_s\|u_2\|_s+\|u_2\|_s^2)\|u_1-u_2\|_s+|\sigma_2|\,\|v_1-v_2\|_s\\
&\qquad+|c|(\|v_1\|_s^2+\|v_1\|\,\|v_2\|_s+\|v_2\|_s^2)\|v_1-v_2\|_s+|b|\Big[\|v_1\|_s^2\|u_1-u_2\|_s+|\,\|v_1\|_s^2-\|v_2\|_s^2|\,\|u_2\|_s\Big]\\
&\qquad+|d|\Big[\|u_1\|_s^2\|v_1-v_2\|_s+|\,\|u_1\|_s^2-\|u_2\|_s^2|\,\|v_2\|_s\Big]\\
&\qquad+|e|\Big[\|v_1\|_s^2\|\overline{u_1}-\overline{u_2}\|_s+|\|v_1\|_s^2-\|v_2\|_s^2|\,\|\overline{u_2}\|_s+\|u_2\|_s^2\|\overline{v_1}-\overline{v_2}\|_s+|\,\|u_1\|_s^2-\|u_2\|_s^2|\,\|\overline{v_2}\|_s\Big].
\end{aligned}$$

Therefore

$$\begin{aligned}
\|F(U)-F(V)\|_{H^s\times H^s}\leq&\Big[|\sigma_1|+|a|(\|u_1\|_s^2+\|u_1\|_s\|u_2\|_s+\|u_2\|_s^2)+|b|\,\|v_1\|_s^2\\
&+|d|(\|u_1\|_s+\|u_2\|_s)\|v_2\|_s+|e|\,\|v_1\|_s^2+|e|(\|u_1\|_s+\|u_2\|_s)\|v_2\|_s\Big]\|u_1-u_2\|_s\\
&+\Big[|\sigma_2|+|c|(\|v_1\|_s^2+\|v_1\|_s\|v_2\|_s+\|v_2\|_s^2)+|b|(\|v_1\|_s+\|v_2\|_s)\|u_2\|_s+|d|\,\|u_1\|_s^2\\
&+|e|(\|v_1\|_s+\|v_2\|_s)\|u_2\|_s+|e|\,\|u_1\|_s^2\Big]\|v_1-v_2\|_s.
\end{aligned}$$

Let

$$\begin{aligned}
L(\|u_1\|_s,\|v_1\|_s,\|u_2\|_s,\|v_2\|_s):=&\,|\sigma_1|+|a|(\|u_1\|_s^2+\|u_1\|_s\|u_2\|_s+\|u_2\|_s^2)+|b|\,\|v_1\|_s^2\\
&+|d|(\|u_1\|_s+\|u_2\|_s)\|v_2\|_s+|e|\,\|v_1\|_s^2+|e|(\|u_1\|_s+\|u_2\|_s)\|v_2\|_s\\
&+|\sigma_2|+|c|(\|v_1\|_s^2+\|v_1\|_s\|v_2\|_s+\|v_2\|_s^2)+|b|(\|v_1\|_s+\|v_2\|_s)\|u_2\|_s+|d|\,\|u_1\|_s^2\\
&+|e|(\|v_1\|_s+\|v_2\|_s)\|u_2\|_s+|e|\,\|u_1\|_s^2.
\end{aligned}$$

We conclude that

$$\|F(U) - F(V)\|_{H^s \times H^s} \leq L(\|u_1\|_s, \|v_1\|_s, \|u_2\|_s, \|v_2\|_s)\|U - V\|_{H^s \times H^s}. \quad \square$$

Theorem 3.3. *Let $s \geq 2$. If $U \in C([0,T]; H^s_{per} \times H^s_{per})$ is a solution of (3.5), then U satisfies the integral equation*

$$U(t) = S(t)U_0 + \int_0^t S(t-\tau)F(U(\tau))d\tau. \tag{3.6}$$

Reciprocally, if $U \in C([0,T]; H^s_{per} \times H^s_{per})$ is a solution of (3.6), then $U \in C^1([0,T]; H^{s-2}_{per} \times H^{s-2}_{per})$ and satisfies (3.5) in the sense that

$$\lim_{h \to 0^+} \left\| \frac{U(t+h) - U(t)}{h} - AU(t) - F(U(t)) \right\|_{H^{s-2} \times H^{s-2}} = 0.$$

Proof. Suppose $U \in C([0,T]; H^s_{per} \times H^s_{per})$ is a solution of (3.5), then for $0 \leq \tau \leq t$

$$S(t-\tau)U_\tau(\tau) - S(t-\tau)AU(\tau) = S(t-\tau)F(U(\tau)).$$

Since $(S(t))_{t\geq 0}$ is a C_0-semigroup in $H^{s-2}_{per} \times H^{s-2}_{per}$, and $U \in H^s_{per} \times H^s_{per} = \mathcal{D}(A)$ we have that

$$\frac{d}{d\tau}(S(\tau)f) = ASf = SAf,$$

for any $f \in H^s_{per} \times H^s_{per}$. Therefore

$$\frac{d}{d\tau}(S(t-\tau)U(\tau)) = S(t-\tau)F(U(\tau)).$$

Integrating in $[0,t]$ and using that $U(0) = U_0$ yields that

$$S(0)U - S(t)U(0) = \int_0^t S(t-\tau)F(U)(\tau)d\tau.$$

Thus we conclude that

$$U = S(t)U_0 + \int_0^t S(t-\tau)F(U(\tau))d\tau.$$

On the other hand, let $U \in C([0,T]; H^s_{per} \times H^s_{per})$ be a solution of (3.6). We will show that

$$\lim_{h \to 0^+} \left\| \frac{U(t+h) - U(t)}{h} - AU(t) - F(U(t)) \right\|_{H^{s-2} \times H^{s-2}} = 0.$$

In fact,

$$\begin{aligned}
&\left\| \frac{U(t+h) - U(t)}{h} - AU(t) - F(U(t)) \right\|_{H^{s-2} \times H^{s-2}} \\
&\quad = \left\| \frac{1}{h}\Big[S(t+h)U_0 + \int_0^{t+h} S(t+h-\tau)F(U(\tau))d\tau - \int_0^t S(t-\tau)F(U(\tau))d\tau \right. \\
&\qquad \left. - S(t)U_0\Big] - S(t)AU_0 - \int_0^t S(t-\tau)AF(U(\tau))d\tau - F(U(t)) \right\|_{H^{s-2} \times H^{s-2}} \\
&\quad = \left\| S(t)\left[\frac{1}{h}(S(h) - I) - A\right]U_0 \right\|_{H^{s-2} \times H^{s-2}} + \left\| \int_0^t S(t-\tau)\left[\frac{1}{h}(S(h) - I) - A\right]F(U(\tau))d\tau \right\|_{H^{s-2} \times H^{s-2}} \\
&\qquad + \left\| \frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(U(\tau))d\tau - F(U(t)) \right\|_{H^{s-2} \times H^{s-2}}.
\end{aligned}$$

The first term goes to zero if $h \to 0^+$, since $(S(t))_{t\geq 0}$ is a C_0-semigroup in $H^{s-2}_{per} \times H^{s-2}_{per}$ and the operator A is its infinitesimal generator. For the second term, we have

$$\begin{aligned}
&\left\| \int_0^t S(t-\tau)\left[\frac{1}{h}(S(h) - 1) - A\right]F(U(\tau))d\tau \right\|_{H^{s-2} \times H^{s-2}} \\
&\quad = \left\| \left[\frac{1}{h}(S(h) - I) - A\right]\int_0^t S(t-\tau)F(U(\tau))d\tau \right\|_{H^{s-2} \times H^{s-2}} \to 0,
\end{aligned}$$

as $h \to 0^+$.

Next

$$\left\|\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(U(\tau))d\tau - F(U(t))\right\|_{H^{s-2}\times H^{s-2}}$$
$$\le \frac{1}{|h|}\int_t^{t+h} \|S(t+h-\tau)F(U(\tau)) - F(U(t))\|_{H^{s-2}\times H^{s-2}}\, d\tau$$
$$= \|S(t+h-\gamma)F(U(\gamma)) - F(U(t))\|_{H^{s-2}\times H^{s-2}}\,,$$

by the mean value theorem for integrals, for some $t \le \gamma \le t+h$. Therefore

$$\left\|\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(U(\tau))d\tau - F(U(t))\right\|_{H^{s-2}\times H^{s-2}} \le \|S(h)S(t-\gamma)(F(U(\gamma)) - F(U(t)))\|_{H^{s-2}\times H^{s-2}} + \|(S(t+h-\gamma) - I)F(U(t))\|_{H^{s-2}\times H^{s-2}}.$$

Since $S(t)$ is a C_0-semigroup in $H^{s-2}_{per}\times H^{s-2}_{per}$ and $\gamma \to t$ when $h \to 0^+$, we know that

$$\lim_{h\to 0^+} \|(S(t+h-\gamma) - I)F(U(t))\|_{H^{s-2}\times H^{s-2}} = 0.$$

Furthermore by Lemma 3.2, we obtain that

$$\|S(h)S(t-\gamma)(F(U(\gamma)) - F(U(t)))\|_{H^{s-2}\times H^{s-2}} \le \|F(U(\gamma)) - F(U(t))\|_{H^{s-2}\times H^{s-2}}$$
$$\le \|F(U(\gamma)) - F(U(t))\|_{H^{s}\times H^{s}} \le C\|U(\gamma) - U(t)\|_{H^{s}\times H^{s}} \to 0,$$

as $h \to 0^+$ $(\gamma \to t^+)$ due to the continuity of U. Using the results above, we conclude that

$$\lim_{h\to 0^+} \left\|\frac{U(t+h) - U(t)}{h} - AU(t) - F(U(t))\right\|_{H^{s-2}\times H^{s-2}} = 0. \quad \square$$

Theorem 3.4. *$U_0 \in H^s_{per}\times H^s_{per}$, $s > 1/2$. Then there exists $T_s > 0$ and $U \in C\left([0,T], H^s_{per}\times H^s_{per}\right)$ satisfying the integral equation (3.6).*

Proof. We define the complete subset of $C([0,T], H^s_{per}\times H^s_{per})$ defined by

$$E_M = \{U = (u,v) \in C([0,T], H^s_{per}\times H^s_{per}) : \sup_{t\in[0,T]} \|U(t) - S(t)U_0\|_{H^s\times H^s} \le M\},$$

with the norm

$$\|U\|_{C([0,T],H^s\times H^s)} = \|(u,v)\|_{C([0,T],H^s\times H^s)} = \sup_{t\in[0,T]} (\|u(t)\|_s + \|v(t)\|_s).$$

Observe that F is continuous in E_M due to Lemma 3.2.

For $U = (u,v)^T \in E_M$, we define the operator

$$\Psi(U(t)) := S(t)U_0 + \int_0^t S(t-\tau)F(U(\tau))d\tau.$$

Let $U(t) = (u(t), v(t))^T$. Then

$$\|\Psi(U(t+h)) - \Psi(U(t))\|_{H^s\times H^s} = \left\|S(t+h)U_0 + \int_0^{t+h} S(t+h-\tau)F(U(\tau))d\tau - S(t)U_0 - \int_0^t S(t-\tau)F(U(\tau))d\tau\right\|_{H^s\times H^s}$$
$$\le \left\|\int_0^t (S(t+h-\tau) - S(t-\tau))\, F(U(\tau))d\tau\right\|_{H^s\times H^s} + \left\|\int_t^{t+h} S(t+h-\tau)F(U(\tau))d\tau\right\|_{H^s\times H^s}.$$

In first place, observe that

$$\|(S(t+h) - S(t))U_0\|_{H^s\times H^s} \to 0, \quad h \to 0,$$

due to $S(t)$ is a C_0 semigroup in $H^s\times H^s$. Further, let us see that

$$\|(S(t+h-\tau) - S(t-\tau))\, F(U(\tau))\|_{s\times s} \le 2\,\|F(U(\tau))\|_{H^s\times H^s}\,,$$

and $\lim_{h\to 0} \|(S(t+h-\tau)-S(t-\tau))\, F(U(\tau))\|_{H^s\times H^s} = 0$. Then by Lebesgue's dominated convergence theorem

$$\lim_{h\to 0}\int_0^t \|(S(t+h-\tau)-S(t-\tau))\, F(U(\tau))\|_{H^s\times H^s}\, d\tau = 0.$$

For the last term, $\|S(t+h-\tau)F(U(\tau))\|_{H^s\times H^s} \le \|F(U(\tau))\|_{H^s\times H^s}$, being as $U\in E_M$, we have that

$$\begin{aligned}\|U(\tau)\|_{H^s\times H^s} &= \|u\|_s+\|v\|_s \le \|U(\tau)-S(\tau)U_0\|_{H^s\times H^s}+\|S(\tau)U_0\|_{H^s\times H^s}\\ &\le M+\|U_0\|_{H^s\times H^s}.\end{aligned}\tag{3.7}$$

Therefore,

$$\begin{aligned}\|S(t+h-\tau)F(U(\tau))\|_{H^s\times H^s} &\le \|F(U(\tau))\|_{H^s\times H^s}\\ &= |\sigma_1|\,\|u\|_s+|a|\,\|u\|_s^3+|b|\,\|v\|_s^2\,\|u\|_s+|e|\,\|v\|_s^2\,\|u\|_s+|\sigma_2|\,\|v\|_s+|c|\,\|v\|_s^3\\ &\quad + |d|\,\|u\|_s^2\,\|v\|_s+|e|\,\|u\|_s^2\,\|v\|\\ &\le C\left[\|u\|_s+\|v\|_s+(\|u\|_s+\|v\|_s)^3\right]\\ &\le C\left[M+\|U_0\|_{H^s\times H^s}+(M+\|U_0\|_{H^s\times H^s})^3\right].\end{aligned}$$

Then,

$$\lim_{h\to 0}\int_t^{t+h} \|S(t+h-\tau)F(U(\tau))\|_{H^s\times H^s}\, d\tau = 0.$$

Hence it follows that $\Psi(U)\in C\left([0,T];H^s_{per}\times H^s_{per}\right)$ if $U=(u,v)\in E_M$.

Now let us see that there is $T_1>0$ such that if $0<T<T_1$ and $U\in E_M$ then $\Psi(U)\in E_M$.

$$\begin{aligned}\|\Psi\,(U(t))-S(t)U_0\|_{H^s\times H^s} &= \left\|\int_0^t S(t-\tau)F\,(U(\tau))\,d\tau\right\|_{H^s\times H^s} \le \int_0^t \|F(U(\tau))\|_{H^s\times H^s} d\tau\\ &\le \int_0^t L(\|u\|\,,\|v\|\,,0,0)\|U(\tau)\|_{H^s\times H^s} d\tau.\end{aligned}\tag{3.8}$$

If $U\in E_M$ from (3.7) and (3.8), it follows that

$$\|\Psi\,(U(t))-S(t)U_0\|_{H^s\times H^s} \le TL\Big(M+\|U_0\|_{H^s\times H^s}\,,M+\|U_0\|_{H^s\times H^s}\,,0,0\Big)\,(M+\|U_0\|_{H^s\times H^s}).$$

Thus choosing

$$T_1 := \frac{M}{L\Big(M+\|U_0\|_{H^s\times H^s}\,,M+\|U_0\|_{H^s\times H^s}\,,0,0\Big)\,\big(M+\|U_0\|_{H^s\times H^s}\big)},$$

we get the desired result.

Finally, we see that there exists T_2 such that Ψ is a contraction in E_M for $T<T_2$. In fact,

$$\begin{aligned}\|\Psi\,(U(t))-\Psi\,(V(t))\|_{H^s\times H^s} &\le \int_0^t \|F\,(U(\tau))-F\,(V(\tau))\|_{H^s\times H^s}\, d\tau\\ &\le \int_0^t \|U(\tau)-V(\tau)\|_{H^s\times H^s}\, L\Big(M+\|U_0\|_{H^s\times H^s}\,,M+\|U_0\|_{H^s\times H^s}\,,0,0\Big)d\tau\\ &\le L\Big(M+\|U_0\|_{H^s\times H^s}\,,M+\|U_0\|_{H^s\times H^s}\,,0,0\Big)T\sup_{t\in[0,T]}\|U(t)-V(t)\|_{H^s\times H^s}.\end{aligned}$$

Thus choosing

$$T_2 := \frac{1}{L\Big(M+\|U_0\|_{H^s\times H^s}\,,M+\|U_0\|_{H^s\times H^s}\,,0,0\Big)},$$

we get that Ψ is a contraction on E_M. Thus with $T_s<\min\{T_1,T_2\}$ and applying Banach's fixed-point theorem in E_M, we get the desired result. □

Theorem 3.5. *The solution obtained in the previous theorem is unique and depends continuously on the initial condition U_0.*

Proof. Let $U = \binom{u_1}{v_1}, V = \binom{u_2}{v_2} \in C\left([0,T]; H^s_{per} \times H^s_{per}\right)$ solutions of (3.6) with initial data $U_0 = \binom{u_0}{v_0}, V_0 = \binom{\phi_0}{\psi_0}$. Then

$$\begin{aligned}\|U(t)-V(t)\|_{H^s\times H^s} &= \left\| S(t)\begin{pmatrix} u_0-\phi_0 \\ v_0-\psi_0\end{pmatrix} + \int_0^t S(t-\tau)[F(U(\tau))-F(V(\tau))]d\tau \right\|_{H^s\times H^s} \\ &\le \|U_0-V_0\|_{H^s\times H^s} + \int_0^t \|F(U(\tau))-F(V(\tau))\|_{H^s\times H^s}\, d\tau \\ &\le \|U_0-V_0\|_{H^s\times H^s} + \int_0^t L\left(\|u_1\|_s, \|v_1\|_s, \|u_2\|_s, \|v_2\|_s\right)\|U(\tau)-V(\tau)\|_{H^s\times H^s}\, d\tau \\ &\le \|U_0-V_0\|_{H^s\times H^s} + L\left(\|u_1\|_s, \|v_1\|_s, \|u_2\|_s, \|v_2\|_s\right)\int_0^t \|U(\tau)-V(\tau)\|_{H^s\times H^s}\, d\tau.\end{aligned}$$

Let

$$K = \sup_{t\in[0,T]} L(\|u_1(t)\|_s, \|v_1(t)\|_s, \|u_2(t)\|_s, \|v_2(t)\|_s).$$

Thus from Gronwall's inequality, we arrive at

$$\|U(t)-V(t)\|_{H^s\times H^s} \le \|U_0-V_0\|_{H^s\times H^s}\,(1+Kte^{Kt}),$$

for all $0 \le t \le T$. Hence the uniqueness and continuous dependence of the solution on initial conditions follows. □

Theorem 3.6. *Any solution* (u, v) *of the CNLS system* (1.1)–(1.2) *satisfies*

$$\partial_t|u|^2 + \partial_t|v|^2 + 2K\partial_x Im(\overline{u}u_x) + 2K\partial_x Im(\overline{v}v_x) + 2\delta\partial_x|u|^2 - 2\delta\partial_x|v|^2 = 0.$$

Proof. Let us observe that the CNLS system can be written as

$$\partial_t u = -\delta\partial_x u + Ki\partial_x^2 u + \sigma_1 iu + ai|u|^2u + bi|v|^2u + eiv^2\overline{u}, \tag{3.9}$$

$$\partial_t v = \delta\partial_x v + Ki\partial_x^2 v + \sigma_2 iv + ci|v|^2v + di|u|^2v + eiu^2\overline{v}. \tag{3.10}$$

Now from Eq. (3.9)

$$\begin{aligned}\partial_t|u|^2 &= \partial_t(u\overline{u}) = \partial_t(u)\overline{u} + u\partial_t(\overline{u}) \\ &= 2\mathrm{Re}(\partial_t u\overline{u}) = 2\mathrm{Re}\Big(-\delta\partial_x u\overline{u} + Ki\partial_x^2 u\,\overline{u} + \sigma_1 i|u|^2 + ai|u|^4 + bi|v|^2|u|^2 + eiv^2\overline{u}^2\Big) \\ &= -2\delta\partial_x|u|^2 + 2K\mathrm{Re}(i\partial_x^2 u\,\overline{u}) + 2e\mathrm{Re}(iv^2\overline{u}^2) = -2\delta\partial_x|u|^2 - 2K\mathrm{Im}(\partial_x^2 u\,\overline{u}) - 2e\mathrm{Im}(v^2\overline{u}^2).\end{aligned}$$

On the other hand, from

$$\partial_x(\overline{u}u_x) = |\partial_x u|^2 + \overline{u}\partial_x^2 u,$$

we have that

$$\overline{u}\partial_x^2 u = \partial_x(\overline{u}u_x) - |\partial_x u|^2.$$

Therefore,

$$\partial_t|u|^2 = -2\delta\partial_x|u|^2 - 2K\partial_x\mathrm{Im}(\overline{u}u_x) - 2e\mathrm{Im}(v^2\overline{u}^2).$$

Analogously from Eq. (3.10)

$$\partial_t|v|^2 = 2\delta\partial_x|v|^2 - 2K\partial_x\mathrm{Im}(\overline{v}v_x) - 2e\mathrm{Im}(u^2\overline{v}^2).$$

Adding the last two equations, we arrive at

$$\partial_t|u|^2 + \partial_t|v|^2 + 2K\partial_x\mathrm{Im}(\overline{u}u_x) + 2K\partial_x\mathrm{Im}(\overline{v}v_x) + 2\delta\partial_x|u|^2 - 2\delta\partial_x|v|^2 = 0. \quad □$$

4. Analysis of the semidiscrete approximation

The semidiscrete Galerkin-type scheme for approximating a solution to system (3.1)–(3.2) consists in finding functions $u_N,\ v_N : [0,T] \to S_N$ such that

$$\begin{aligned}&\langle i\partial_t u_N, \phi\rangle + \langle i\delta\partial_x u_N, \phi\rangle + \left\langle K\partial_x^2 u_N, \phi\right\rangle + \langle\sigma_1 u_N, \phi\rangle + \left\langle a|u_N|^2u_N, \phi\right\rangle \\ &\quad + \left\langle b|v_N|^2u_N, \phi\right\rangle + \left\langle ev_N^2\overline{u_N}, \phi\right\rangle = 0,\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\langle i\partial_t v_N, \varphi\rangle - \langle i\delta\partial_x v_N, \varphi\rangle + \left\langle K\partial_x^2 v_N, \varphi\right\rangle + \langle \sigma_2 v_N, \varphi\rangle + \left\langle c|v_N|^2 v_N, \varphi\right\rangle \\
&\quad + \left\langle d|u_N|^2 v_N, \varphi\right\rangle + \left\langle e u_N^2 \overline{v_N}, \varphi\right\rangle = 0, \\
&u_N(0) = P_N(u_0), \qquad v_N(0) = P_N(v_0),
\end{aligned} \tag{4.2}$$

for all $\phi,\ \varphi \in S_N$. Since $u_N,\ v_N \in S_N$ we can write

$$u_N(t) = \sum_{n=-N/2}^{N/2} \hat{u}_n(t)\exp(iw_n x), \qquad v_N(t) = \sum_{n=-N/2}^{N/2} \hat{v}_n(t)\exp(iw_n x),$$

where

$$w_n = \frac{2\pi n}{L}, \qquad \hat{u}_n(t) = \frac{1}{L}\int_0^L u_N(x,t)\exp(-iw_n x)dx,$$

and

$$\hat{v}_n(t) = \frac{1}{L}\int_0^L v_N(x,t)\exp(-iw_n x)dx.$$

Substituting the expressions for $u_N(t)$ and $v_N(t)$ in Eqs. (4.1) and (4.2), and taking L^2-inner product with $\phi = \varphi = \exp(iw_m x)$, $m = -N/2, \cdots, N/2$, we can reduce (4.1)–(4.2) to the system of ordinary differential equations

$$\begin{aligned}
&i\partial_t \widehat{u}_m + (-\delta w_m - K w_m^2 + \sigma_1)\widehat{u}_m + \langle a|u_N|^2 u_N + b|v_N|^2 u_N + e v_N^2 \overline{u_N}, \exp(iw_m x)\rangle = 0, \\
&\hat{u}_m(0) = \hat{u}_{0,m},
\end{aligned}$$

and

$$\begin{aligned}
&i\partial_t \widehat{v}_m + (\delta w_m - K w_m^2 + \sigma_2)\widehat{v}_m + \langle c|v_N|^2 v_N + d|u_N|^2 v + e u_N^2 \overline{v_N}, \exp(iw_m x)\rangle = 0 \\
&\hat{v}_m(0) = \hat{v}_{0,m}.
\end{aligned}$$

Here

$$\begin{aligned}
\hat{u}_{0,m} &= \frac{1}{L}\int_0^L u_0(x)\exp(-iw_m x)dx, \\
\hat{v}_{0,m} &= \frac{1}{L}\int_0^L v_0(x)\exp(-iw_m x)dx,
\end{aligned}$$

and we have used the orthogonality of the basis of the space S_N.

Let s be a positive integer. Then substituting ϕ and φ by $\phi^{(m)}$ and $\varphi^{(m)}$ (mth derivative), $m = 2, 4, 6, \ldots, 2s$, in Eqs. (4.1) and (4.2), summing the resulting equations and using integration by parts, we obtain that the solution of the semidiscrete scheme (u_N, v_N) also satisfies that

$$\begin{aligned}
&\langle i\partial_t u_N, \phi\rangle_s + \langle i\delta\partial_x u_N, \phi\rangle_s - \langle K\partial_x u_N, \partial_x\phi\rangle_s + \langle \sigma_1 u_N, \phi\rangle_s \\
&\quad + \left\langle a|u_N|^2 u_N + b|v_N|^2 u_N + e v_N^2 \overline{u_N}, \phi\right\rangle_s = 0, \\
&\langle i\partial_t v_N, \varphi\rangle_s - \langle i\delta\partial_x v_N, \varphi\rangle_s - \langle K\partial_x v_N, \partial_x\varphi\rangle_s + \langle \sigma_2 v_N, \varphi\rangle_s \\
&\quad + \left\langle c|v_N|^2 v_N + d|u_N|^2 v_N + e u_N^2 \overline{v_N}, \varphi\right\rangle_s = 0, \\
&u_N(0) = P_N(u_0), \qquad v_N(0) = P_N(v_0).
\end{aligned} \tag{4.3}$$

Theorem 4.1. *The solution (u_N, v_N) of the semidiscrete scheme (4.1)–(4.2) satisfies,*

$$\frac{\partial}{\partial t}\int_0^L (|u_N|^2 + |v_N|^2)dx = 0. \tag{4.4}$$

Furthermore, if $b = d$ in system (3.1)–(3.2), we have that

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\int_0^L \Big|\frac{i\delta}{2K}u_N + \partial_x u_N\Big|^2 + \Big|-\frac{i\delta}{2K}v_N + \partial_x v_N\Big|^2 dx - \frac{a}{2K}\int_0^L |u_N|^4 dx \\
&\qquad - \frac{c}{2K}\int_0^L |v_N|^4 dx - \frac{b}{K}\int_0^L |u_N|^2|v_N|^2 dx - \frac{e}{K}\int_0^L \mathrm{Re}(u_N^2\overline{v_N}^2)dx\Big) = 0.
\end{aligned} \tag{4.5}$$

Proof. Let $\phi = u_N$ in (4.1),

$$\langle i\partial_t u_N, u_N\rangle + \langle i\delta\partial_x u_N, u_N\rangle + \left\langle K\partial_x^2 u_N, u_N\right\rangle + \langle \sigma_1 u_N, u_N\rangle + \left\langle a|u_N|^2 u_N, u_N\right\rangle \tag{4.6}$$

$$+ \left\langle b|v_N|^2 u_N, u_N\right\rangle + \left\langle e v_N^2 \overline{u_N}, u_N\right\rangle = 0. \tag{4.7}$$

Multiplying by $-i$ and taking real part of the equation, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|u_N\|^2 &= -\delta \mathrm{Re}\langle \partial_x u_N, u_N\rangle + K\mathrm{Re}\left(i\int_0^L \partial_x^2 u_N\overline{u_N}dx\right) + \sigma_1 \mathrm{Re}\left(i\|u_N\|^2\right)\\ &\quad + \mathrm{Re}\left(ai\int_0^L |u_N|^4\,dx\right) + \mathrm{Re}\left(bi\int_0^L |v_N|^2\,|u_N|^2\,dx\right) + \mathrm{Re}\left(ei\int_0^L v_N^2\overline{u_N}^2dx\right)\\ &= -\frac{\delta}{2}\int_0^L \frac{\partial}{\partial x}(u_N\overline{u_N})dx - K\,\mathrm{Re}\left(i\left\|\frac{\partial u_N}{\partial x}\right\|^2\right) + e\,\mathrm{Re}\left(i\int_0^L v_N^2\overline{u_N}^2dx\right)\\ &= e\,\mathrm{Re}\left(i\int_0^L v_N^2\overline{u_N}^2dx\right).\end{aligned}$$

Now letting $\varphi = v_N$ in (4.2)

$$\langle i\partial_t v_N, v_N\rangle - \langle i\delta\partial_x v_N, v_N\rangle + \left\langle K\partial_x^2 v_N, v_N\right\rangle + \langle \sigma_2 v_N, v_N\rangle \tag{4.8}$$

$$+\left\langle c|v_N|^2 v_N, v_N\right\rangle + \left\langle d|u_N|^2 v_N, v_N\right\rangle + \left\langle e u_N^2\overline{v_N}, v_N\right\rangle = 0. \tag{4.9}$$

Multiplying by $-i$ and taking real part, we get that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|v_N\|^2 &= \delta \mathrm{Re}\langle \partial_x v_N, v_N\rangle + K\mathrm{Re}\left(i\int_0^L \partial_x^2 v_N\overline{v_N}dx\right) + \sigma_2 \mathrm{Re}\left(i\|v_N\|^2\right)\\ &\quad + \mathrm{Re}\left(ci\int_0^L |v_N|^4\,dx\right) + \mathrm{Re}\left(di\int_0^L |u_N|^2\,|v_N|^2\,dx\right) + \mathrm{Re}\left(ei\int_0^L u_N^2\overline{v_N}^2dx\right)\\ &= \frac{\delta}{2}\int_0^L \frac{\partial}{\partial x}(v_N\overline{v_N})dx - K\mathrm{Re}\left(i\left\|\frac{\partial v_N}{\partial x}\right\|^2\right) + e\,\mathrm{Re}\left(i\int_0^L u_N^2\overline{v_N}^2dx\right)\\ &= e\,\mathrm{Re}\left(i\int_0^L u_N^2\overline{v_N}^2dx\right).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\|u_N\|^2 + \|v_N\|^2\right) &= e\,\mathrm{Re}\left(i\int_0^L (v_N^2\overline{u_N}^2 + u_N^2\overline{v_N}^2)dx\right)\\ &= 2e\,\mathrm{Re}\left(i\int_0^L \mathrm{Re}(v_N^2\overline{u_N}^2)dx\right)\\ &= 0.\end{aligned}$$

As a consequence,

$$\frac{\partial}{\partial t}\left(\|u_N\|^2 + \|v_N\|^2\right) = 0.$$

On the other hand, letting $\phi = -\partial_t u_N, \varphi = -\partial_t v_N$ in Eqs. (4.1)–(4.2), we have that

$$\begin{aligned}&-i\int_0^L \partial_t u_N\overline{\partial_t u_N}dx - i\delta\int_0^L \partial_x u_N\partial_t\overline{u_N}dx + K\int_0^L \partial_x u_N\partial_x\partial_t\overline{u_N}dx\\ &\quad -\sigma_1\int_0^L u_N\partial_t\overline{u_N}dx - a\int_0^L |u_N|^2 u_N\partial_t\overline{u_N}dx - b\int_0^L |v_N|^2 u_N\partial_t\overline{u_N}dx - e\int_0^L v_N^2\overline{u_N}\partial_t\overline{u_N}dx = 0,\\ &-i\int_0^L \partial_t v_N\overline{\partial_t u_N}dx + i\delta\int_0^L \partial_x v_N\partial_t\overline{v_N}dx + K\int_0^L \partial_x v_N\partial_x\partial_t\overline{v_N}dx\\ &\quad -\sigma_2\int_0^L v_N\partial_t\overline{v_N}dx - c\int_0^L |v_N|^2 v_N\partial_t\overline{v_N}dx - b\int_0^L |u_N|^2 v_N\partial_t\overline{v_N}dx - e\int_0^L u_N^2\overline{v_N}\partial_t\overline{v_N}dx = 0.\end{aligned}$$

Thus adding and taking real part of both sides of the equation above, we obtain

$$\begin{aligned}&\int_0^L (-i\delta\partial_x u_N\partial_t\overline{u_N} + i\delta\partial_x\overline{u_N}\partial_t u_N)dx + K\int_0^L \partial_t(\partial_x u_N\partial_x\overline{u_N})dx\\ &\quad -\sigma_1\partial_t\int_0^L |u_N|^2dx - a\int_0^L |u_N|^2\partial_t(|u_N|^2)dx - b\int_0^L |v_N|^2\partial_t(|u_N|^2)dx\end{aligned}$$

$$
\begin{aligned}
&+\int_0^L (i\delta\partial_x v_N\partial_t\overline{v_N}-i\delta\partial_x\overline{v_N}\partial_t v_N)dx+K\int_0^L \partial_t(\partial_x v_N\partial_x\overline{v_N})dx\\
&-\sigma_2\int_0^L \partial_t(|v_N|^2)dx-c\int_0^L |v_n|^2\partial_t(|v_N|^2)dx-b\int_0^L |u_N|^2\partial_t(|v_N|^2)dx\\
&-e\int_0^L (v_N^2\overline{u_N}\partial_t\overline{u_N}+\overline{v_N}^2u_N\partial_t u_N)dx-e\int_0^L (u_N^2\overline{v_N}\partial_t\overline{v_N}+\overline{u_N}^2v_N\partial_t v_N)dx=0.
\end{aligned}
$$

Since $\partial_t(\int_0^L |u_N|^2+|v_N|^2dx)=0$, we obtain that

$$
\begin{aligned}
&\int_0^L(-i\delta\partial_x u_N\partial_t\overline{u_N}+i\delta\partial_x\overline{u_N}\partial_t u_N)dx+K\partial_t\int_0^L |\partial_x u_N|^2dx-a\int_0^L |u_N|^2\partial_t|u_N|^2dx\\
&-b\int_0^L \partial_t(|u_N|^2|v_N|^2)dx+\int_0^L(i\delta\partial_x v_N\partial_t\overline{v_N}-i\delta\partial_x\overline{v_N}\partial_t v_N)dx+K\partial_t\int_0^L |\partial_x v_N|^2dx\\
&-c\int_0^L |v_N|^2\partial_t|v_N|^2dx-\frac{e}{2}\int_0^L(v_N^2\partial_t\overline{u_N}^2+\overline{v_N}^2\partial_t u_N^2)dx-\frac{e}{2}\int_0^L(u_N^2\partial_t\overline{v_N}^2+\overline{u_N}^2\partial_t v_N^2)dx=0.
\end{aligned}
$$

Then we have that

$$
\begin{aligned}
&\int_0^L\left(-\frac{i\delta}{K}\partial_x u_N\partial_t\overline{u_N}+\frac{i\delta}{K}\partial_x\overline{u_N}\partial_t u_N\right)dx+\partial_t\int_0^L |\partial_x u_N|^2dx-\frac{a}{2K}\partial_t\int_0^L |u_N|^4dx\\
&-\frac{b}{K}\int_0^L \partial_t(|u_N|^2|v_N|^2)dx+\int_0^L\left(\frac{i\delta}{K}\partial_x v_N\partial_t\overline{v_N}-\frac{i\delta}{K}\partial_x\overline{v_N}\partial_t v_N\right)dx+\partial_t\int_0^L |\partial_x v_N|^2dx\\
&-\frac{c}{2K}\partial_t\int_0^L |v_N|^4dx-\frac{e}{2K}\partial_t\int_0^L(v_N^2\overline{u_N}^2+u_N^2\overline{v_N}^2)dx=0.
\end{aligned}
\tag{4.10}
$$

Observe that

$$
\begin{aligned}
\partial_t\int_0^L\left|\frac{i\delta}{K}u_N+\partial_x u_N\right|^2dx=&\frac{\delta^2}{K^2}\partial_t\int_0^L |u_N|^2dx+\partial_t\int_0^L |\partial_x u_N|^2dx\\
&+\int_0^L\Big(\frac{i\delta}{2K}\partial_t u_N\partial_x\overline{u_N}+\frac{i\delta}{2K}u_N\partial_x\partial_t\overline{u_N}\Big)dx+\int_0^L\Big(-\frac{i\delta}{2K}\partial_x\partial_t u_N\overline{u_N}-\frac{i\delta}{2K}\partial_x u_N\partial_t\overline{u_N}\Big)dx.
\end{aligned}
$$

Using integration by parts, we deduce that

$$
\partial_t\int_0^L\left|\frac{i\delta}{2K}u_N+\partial_x u_N\right|^2dx=\partial_t\int_0^L |\partial_x u_N|^2dx+\int_0^L\Big(\frac{i\delta}{K}\partial_t u_N\partial_x\overline{u_N}-\frac{i\delta}{K}\partial_x u_N\partial_t\overline{u_N}\Big)dx.
$$

Analogously we obtain that

$$
\partial_t\int_0^L\left|-\frac{i\delta}{2K}v_N+\partial_x v_N\right|^2dx=\partial_t\int_0^L |\partial_x v_N|^2dx+\int_0^L\Big(\frac{i\delta}{K}\partial_x v_N\partial_t\overline{v_N}-\frac{i\delta}{K}\partial_t v_N\partial_x\overline{v_N}\Big)dx.
$$

Using the equations above, Eq. (4.10) and the fact that $\mathrm{Re}(u_N^2\overline{v_N}^2)=\frac{1}{2}(v_N^2\overline{u_N}^2u_N^2\overline{v_N}^2)$, we obtain

$$
\begin{aligned}
&\partial_t\int_0^L\left|\frac{i\delta}{2K}u_N+\partial_x u_N\right|^2dx+\partial_t\int_0^L\left|-\frac{i\delta}{2K}v_N+\partial_x v_N\right|^2dx-\frac{a}{2K}\partial_t\int_0^L |u_N|^4dx\\
&-\frac{b}{K}\partial_t\int_0^L |u_N|^2|v_N|^2dx-\frac{c}{2K}\partial_t\int_0^L |v_N|^4dx-\frac{e}{K}\partial_t\int_0^L \mathrm{Re}(u_N^2\overline{v_N}^2)dx=0.\quad\square
\end{aligned}
$$

Theorem 4.2. *Let $s\geq 2$ be an integer and $(u,v)\in C^1\left([0,T],H^r_{per}(0,L)\times H^r_{per}(0,L)\right)$ be a classical solution of system (3.1)–(3.2) corresponding to initial data (u^0,v^0) in $H^r_{per}(0,L)\times H^r_{per}(0,L)$, for some integer $r>s$, and assume that $\|u(t)\|_s+\|v(t)\|_s\leq 2B$, for some constant $B>0$ and any $t\in[0,T]$. Then the semidiscrete problem (4.1)–(4.2) has a unique solution $(u_N,v_N)\in C^1([0,T],S_N\times S_N)$, which satisfies for N large enough and some constant $C>0$ independent of N, t,*

$$
\|u(t)-u_N(t)\|_s+\|v(t)-v_N(t)\|_s\leq CN^{s-r},
\tag{4.11}
$$

for any $0\leq t\leq T$.

Proof. From the classical theory of ordinary differential equations, one can show that the initial value problem (4.1)–(4.2) has a unique solution $u_N, v_N \in C^1([0, T_N], S_N)$.

On the other hand, by applying the orthogonal projection P_N to Eqs. (3.1)–(3.2) and taking inner product in H^s_{per} with $\phi, \varphi \in S_N$, we obtain that

$$\begin{aligned} &i\left\langle P_N\left(\frac{\partial u}{\partial t}\right), \phi\right\rangle_s + i\delta\left\langle P_N\left(\frac{\partial u}{\partial x}\right), \phi\right\rangle_s + K\left\langle P_N \frac{\partial^2 u}{\partial x^2}, \phi\right\rangle_s + \sigma_1 \langle P_N u, \phi\rangle_s \\ &\quad + a\left\langle P_N\left(|u|^2 u\right), \phi\right\rangle_s + b\left\langle P_N\left(|v|^2 u\right), \phi\right\rangle_s + e\left\langle P_N\left(v^2\overline{u}\right), \phi\right\rangle_s = 0, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &i\left\langle P_N \frac{\partial v}{\partial t}, \varphi\right\rangle_s - i\delta\left\langle P_N \frac{\partial v}{\partial x}, \varphi\right\rangle_s + K\left\langle P_N \frac{\partial^2 v}{\partial x^2}, \varphi\right\rangle_s + \sigma_2 \langle P_N v, \varphi\rangle_s \\ &\quad + c\left\langle P_N\left(|v|^2 v\right), \varphi\right\rangle_s + d\left\langle P_N\left(|u|^2 v\right), \varphi\right\rangle_s + e\left\langle P_N\left(u^2\overline{v}\right), \varphi\right\rangle_s = 0. \end{aligned} \tag{4.13}$$

Subtracting Eq. (4.1) from (4.12) yields that

$$\begin{aligned} &i\left\langle P_N\left(\frac{\partial u}{\partial t}\right) - \frac{\partial u_N}{\partial t}, \phi\right\rangle_s + i\delta\left\langle P_N\left(\frac{\partial u}{\partial x}\right) - \frac{\partial u_N}{\partial x}, \phi\right\rangle_s + K\left\langle P_N \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_N}{\partial x^2}, \phi\right\rangle_s \\ &\quad + \sigma_1 \langle P_N u - u_N, \phi\rangle_s + a\left\langle P_N\left(|u|^2 u\right) - |u_N|^2 u_N, \phi\right\rangle_s \\ &\quad + b\left\langle P_N\left(|v|^2 u\right) - |v_N|^2 u_N, \phi\right\rangle_s + e\left\langle P_N\left(v^2\overline{u}\right) - v_N^2\overline{u_N}, \phi\right\rangle_s = 0. \end{aligned} \tag{4.14}$$

Let $\eta = P_N u - u_N$. We can rewrite Eq. (4.14) as

$$\begin{aligned} &i\left\langle \frac{\partial \eta}{\partial t}, \phi\right\rangle_s + i\delta\left\langle \frac{\partial \eta}{\partial x}, \phi\right\rangle_s + K\left\langle \frac{\partial^2 \eta}{\partial x^2}, \phi\right\rangle_s + \sigma_1 \langle \eta, \phi\rangle_s + a\left\langle |u|^2 u - |u_N|^2 u_N, \phi\right\rangle_s \\ &\quad + b\left\langle |v|^2 u - |v_N|^2 u_N, \phi\right\rangle_s + e\left\langle v^2\overline{u} - v_N^2\overline{u_N}, \phi\right\rangle_s = 0. \end{aligned} \tag{4.15}$$

Now, letting $\phi = \eta$, we have that

$$\begin{aligned} &i\left\langle \frac{\partial \eta}{\partial t}, \eta\right\rangle_s + i\delta\left\langle \frac{\partial \eta}{\partial x}, \eta\right\rangle_s + K\left\langle \frac{\partial^2 \eta}{\partial x^2}, \eta\right\rangle_s + \sigma_1 \langle \eta, \eta\rangle_s + a\left\langle |u|^2 u - |u_N|^2 u_N, \eta\right\rangle_s \\ &\quad + b\left\langle |v|^2 u - |v_N|^2 u_N, \eta\right\rangle_s + e\left\langle v^2\overline{u} - v_N^2\overline{u_N}, \eta\right\rangle_s = 0. \end{aligned} \tag{4.16}$$

Multiplying the equation above by $-i$, taking real part and using that

$$\begin{aligned} \langle \partial_x \eta, \eta\rangle_s &= \sum_{n\in\mathbb{Z}} (1+n^2)^s \widehat{\partial_x \eta}(n)\overline{\hat{\eta}(n)} = \sum_{n\in\mathbb{Z}} (1+n^2)^s \frac{2\pi i n}{L} \hat{\eta}(n)\overline{\hat{\eta}(n)} \\ &= \sum_{n\in\mathbb{Z}} (1+n^2)^s \frac{2\pi i n}{L} |\hat{\eta}(n)|^2 = 0, \end{aligned}$$

and

$$\mathrm{Re}\,(iK\langle \eta_{xx}, \eta\rangle_s) = \mathrm{Re}\,(-iK\langle \eta_x, \eta_x\rangle_s) = \mathrm{Re}\left(-iK\,\|\eta_x\|_s^2\right) = 0,$$

we find that Eq. (4.16) implies that

$$\frac{1}{2}\frac{\partial}{\partial t}\|\eta\|_s^2 = \mathrm{Re}\left(ai\left\langle |u|^2 u - |u_N|^2 u_N, \eta\right\rangle_s\right) + \mathrm{Re}\left(bi\left\langle |v|^2 u - |v_N|^2 u_N, \eta\right\rangle_s\right) \tag{4.17}$$

$$\quad + \mathrm{Re}\left(ei\left\langle v^2\overline{u} - v_N^2\overline{u_N}, \eta\right\rangle_s\right). \tag{4.18}$$

Thus we have that

$$\frac{\partial}{\partial t}\|\eta\|_s^2 \le C\left(\left|\left\langle |u|^2 u - |u_N|^2 u_N, \eta\right\rangle_s\right| + \left|\left\langle |v|^2 u - |v_N|^2 u_N, \eta\right\rangle_s\right| + \left|\left\langle v^2\overline{u} - v_N^2\overline{u_N}, \eta\right\rangle_s\right|\right). \tag{4.19}$$

Using the identity $\hat{\overline{\beta}}(-n) = \overline{\hat{\beta}}(n)$, we have that

$$\begin{aligned} \|\overline{\beta}\|_s^2 &= L\sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{\overline{\beta}}(n)|^2 = L\sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{\overline{\beta}}(-n)|^2 \\ &= L\sum_{n\in\mathbb{Z}} (1+n^2)^s |\overline{\hat{\beta}}(n)|^2 = \|\beta\|_s^2. \end{aligned}$$

Thus the identity

$$|f|^2\bar{f} - |g|^2\bar{g} = |f|^2(\bar{f} - \bar{g}) + \bar{f}\bar{g}(f - g) + |g|^2(\bar{f} - \bar{g}),$$

and the fact that H^s_{per} is a Banach algebra for $s > \frac{1}{2}$, implies that

$$\begin{aligned}\left\| |u|^2 u - |u_N|^2 u_N \right\|_s &= \left\| |u|^2\bar{u} - |u_N|^2\bar{u}_N \right\|_s \le \left\| |u|^2(\bar{u} - \bar{u}_N) \right\|_s + \|\bar{u}\bar{u}_N(u - u_N)\|_s + \left\| |u_N|^2(\bar{u} - \bar{u}_N) \right\|_s \\ &\le C\left(\|u\|_s^2 \|u - u_N\|_s + \|u\|_s \|u_N\|_s \|u - u_N\|_s + \|u_N\|_s^2 \|u - u_N\|_s \right) \\ &\le C\left(\|u\|_s^2 + \|u\|_s \|u_N\|_s + \|u_N\|_s^2 \right) \|u - u_N\|_s .\end{aligned}$$

Here we also used that

$$\left\| |f|^2 \right\|_s = \left\| f\bar{f} \right\|_s \le C \|f\|_s^2 .$$

For the second term in (4.19), we have that

$$\begin{aligned}\left| \langle |v|^2 u - |v_N|^2 u_N, \eta \rangle_s \right| &\le \left\| |v|^2 u - |v_N|^2 u_N \right\| \|\eta\|_s \\ &\le \left(\left\| |v|^2(\overline{u} - \overline{u_N}) \right\|_s + \|v\overline{u_N}(\overline{v} - \overline{v_N})\|_s + \|\overline{u_N}\,\overline{v_N}(v - v_N)\|_s \right) \|\eta\|_s \\ &\le C\left(\|v\|_s^2 \|u - u_N\|_s + \|v\|_s \|u_N\|_s \|v - v_N\|_s + \|u_N\|_s \|v_N\|_s \|v - v_N\|_s \right) \|\eta\|_s .\end{aligned}$$

For the third term in Eq. (4.19), we have that

$$\begin{aligned}\left| \langle v^2\overline{u} - v_N^2\overline{u_N}, \eta \rangle_s \right| &\le \left\| v^2\overline{u} - v_N^2\overline{u_N} \right\|_s \|\eta\|_s \\ &\le \left(\left\| v^2(\overline{u} - \overline{u_N}) \right\|_s + \|v\overline{u_N}(v - v_N)\|_s + \|v_N\overline{u_N}(v - v_N)\|_s \right) \|\eta\|_s \\ &\le C\left(\|v\|_s^2 \|u - u_N\|_s + \|u_N\|_s \|v\|_s \|(v - v_N)\|_s + \|u_N\|_s \|v_N\|_s \|v - v_N\|_s \right) \|\eta\|_s .\end{aligned}$$

Suppose that for some constant $B > 0$

$$\|u(t)\|_s + \|v(t)\|_s \le B,$$

for any $0 \le t \le T$, and let $0 < T_N < T$ be the maximal time for which

$$\|u_N(t)\|_s + \|v_N(t)\|_s \le 2B, \tag{4.20}$$

for $0 \le t \le T_N$. We get for some constant $C > 0$ independent of N that

$$\frac{1}{2}\frac{\partial}{\partial t} \|\eta\|_s^2 \le C\left(\|u - u_N\|_s + \|v - v_N\|_s \right) \|\eta\|_s , \tag{4.21}$$

for any $0 \le t \le T_N$.

On the other hand, subtracting Eq. (4.2) from (4.13), we get that

$$\begin{aligned}& i\left\langle P_N\left(\frac{\partial v}{\partial t}\right) - \frac{\partial v_N}{\partial t}, \phi \right\rangle_s - i\delta \left\langle P_N\left(\frac{\partial v}{\partial x}\right) - \frac{\partial v_N}{\partial x}, \phi \right\rangle_s + K\left\langle P_N\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v_N}{\partial x^2}, \phi \right\rangle_s \\ &\quad + \sigma_2 \langle P_N v - v_N, \phi \rangle_s + c\left\langle P_N\left(|v|^2 v\right) - |v_N|^2 v_N, \phi \right\rangle_s + d\left\langle P_N\left(|u|^2 v\right) - |u_N|^2 v_N, \phi \right\rangle_s \\ &\quad + e\left\langle P_N\left(u^2\overline{v}\right) - u_N^2\overline{v_N}, \phi \right\rangle_s = 0.\end{aligned} \tag{4.22}$$

Let $\sigma = P_N v - v_N$. We can rewrite Eq. (4.22) as

$$\begin{aligned}& i\left\langle \frac{\partial \sigma}{\partial t}, \phi \right\rangle_s - i\delta \left\langle \frac{\partial \sigma}{\partial x}, \phi \right\rangle_s + K\left\langle \frac{\partial^2 \sigma}{\partial x^2}, \phi \right\rangle_s + \sigma_2 \langle \sigma, \phi \rangle_s + c\left\langle P_N\left(|v|^2 v\right) - |v_N|^2 v_N, \phi \right\rangle_s \\ &\quad + d\left\langle P_N\left(|u|^2 v\right) - |u_N|^2 v_N, \phi \right\rangle_s + e\left\langle P_N\left(u^2\overline{v}\right) - u_N^2\overline{v_N}, \phi \right\rangle_s = 0.\end{aligned} \tag{4.23}$$

Letting $\phi = \sigma$ in the equation above, multiplying by $-i$, taking real part of the resulting equation, we arrive at

$$\frac{1}{2}\frac{\partial}{\partial t} \|\sigma\|_s^2 = \mathrm{Re}\left(ci\left\langle P_N\left(|v|^2 v\right) - |v_N|^2 v_N, \sigma \right\rangle_s \right) + \mathrm{Re}\left(di\left\langle P_N\left(|u|^2 v\right) - |u_N|^2 v_N, \sigma \right\rangle_s \right) \tag{4.24}$$

$$+ \mathrm{Re}\left(ei\left\langle P_N\left(u^2\overline{v}\right) - u_N^2\overline{v_N}, \sigma \right\rangle_s \right) . \tag{4.25}$$

Therefore, we obtain that

$$\frac{\partial}{\partial t} \|\sigma\|_s^2 \le C\left(\left| \left\langle |v|^2 v - |v_N|^2 v_N, \sigma \right\rangle_s \right| + \left| \left\langle |u|^2 v - |u_N|^2 v_N, \sigma \right\rangle_s \right| + \left| \left\langle u^2\overline{v} - u_N^2\overline{v_N}, \sigma \right\rangle_s \right| \right) .$$

Using a similar technique as above, we can show that

$$\left\| |v|^2 v - |v_N|^2 v_N \right\|_s \le C \left(\|v\|_s^2 + \|v\|_s \|v_N\|_s + \|v_N\|^2 \right) \|v - v_N\|_s ,$$

$$\| |u|^2 v - |u_N|^2 v_N \|_s \le C \left(\|u\|_s^2 \|v - v_N\|_s + \|u\|_s \|v_N\|_s \|u - u_N\|_s + \|u_N\|_s \|v_N\|_s \|u - u_N\|_s \right) ,$$

and

$$\|u^2\overline{v} - u_N^2 v_N\|_s \le C \left(\left\|u^2\right\|_s^2 \|v - v_N\|_s + \|u\|_s \|v_N\|_s \|u - u_N\|_s + \|u_N\|_s \|v_N\|_s \|u - u_N\|_s \right) .$$

Using again (4.20), we get for some constant $C > 0$ independent of N that

$$\frac{1}{2}\frac{\partial}{\partial t} \|\sigma\|_s^2 \le C \left(\|u - u_N\|_s + \|v - v_N\|_s \right) \|\eta\|_s ,$$

for any $0 \le t \le T_N$.
From the last inequality and Eq. (4.21), we have that

$$\begin{aligned}
\frac{\partial}{\partial t}(\|\eta\|_s + \|\sigma\|_s) &\le C \left(\|u - u_N\|_s + \|v - v_N\|_s \right) \\
&\le C \left(\|u - P_N u\|_s + \|P_N u - u_N\|_s + \|v - P_N v\|_s + \|P_N v - v_N\|_s \right) \\
&\le C(N^{s-r} + \|\eta\|_s + \|\sigma\|_s).
\end{aligned}$$

By Gronwall's lemma and taking into account that $\eta(0) = \sigma(0) = 0$, we get the estimate

$$\|\eta(t)\|_s + \|\sigma(t)\|_s \le CN^{s-r}, \quad 0 \le t \le T_N, \tag{4.26}$$

where C is a constant independent of N.

Since $u_N = u - (u - P_N u + \eta)$, $v_N = v - (v - P_N v + \sigma)$, and using inequality (2.1), one gets that

$$\begin{aligned}
\|u_N(t)\|_s + \|v_N(t)\|_s &\le \|u(t)\|_s + \|u(t) - P_N u(t)\|_s + \|\eta(t)\|_s \\
&\quad + \|v(t)\|_s + \|v(t) - P_N v(t)\|_s + \|\sigma(t)\|_s \le B + CN^{s-r}, \quad 0 \le t \le T_N.
\end{aligned}$$

Due to $r > s$, we conclude that for N large enough

$$\|u_N(t)\|_s + \|v_N(t)\|_s < 2B, \quad 0 \le t \le T_N. \tag{4.27}$$

Thus, the solution of the semidiscrete formulation u_N, v_N can be extended satisfying (4.27) for $t \in [0, T_N + \epsilon]$, for some $\epsilon > 0$. This contradicts the maximality of T_N. Thus inequalities (4.26), (4.20) are satisfied until $t = T$, and using the triangular inequality and inequality (2.1), we obtain that

$$\begin{aligned}
\|u(t) - u_N(t)\|_s + \|v(t) - v_N(t)\|_s &\le \|u(t) - P_N u(t)\|_s + \|P_N u(t) - u_N(t)\|_s \\
&\quad + \|v(t) - P_N v(t)\|_s + \|P_N v(t) - v_N(t)\|_s \le CN^{s-r},
\end{aligned}$$

for any $0 \le t \le T$. $\square$

5. Analysis of the fully-discrete scheme

The fully discrete solution is to find a sequence $\{(u_N^n, v_N^n)\}_{n\in\mathbb{N}}$ of elements of $S_N \times S_N$ satisfying for all $\phi, \varphi \in S_N$ and for $n = 1, 2, \ldots\ldots, M - 1$ that

$$\begin{aligned}
&i\left\langle \frac{u_N^{n+1} - u_N^{n-1}}{2\Delta t}, \phi \right\rangle + i\delta \left\langle \frac{\partial_x u_N^{n+1} + \partial_x u_N^{n-1}}{2}, \phi \right\rangle - K \left\langle \frac{\partial_x u_N^{n+1} + \partial_x u_N^{n-1}}{2}, \partial_x \phi \right\rangle \\
&+\sigma_1 \left\langle \frac{u_N^{n+1} + u_N^{n-1}}{2}, \phi \right\rangle + a\left\langle \left|u_N^n\right|^2 u_N^n, \phi \right\rangle + b\left\langle \left|v_N^n\right|^2 u_N^n, \phi \right\rangle + e\left\langle (v_N^n)^2 \overline{u_N^n}, \phi \right\rangle = 0,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&i\left\langle \frac{v_N^{n+1} - v_N^{n-1}}{2\Delta t}, \varphi \right\rangle - i\delta \left\langle \frac{\partial_x v_N^{n+1} + \partial_x v_N^{n-1}}{2}, \varphi \right\rangle - K \left\langle \frac{\partial_x v_N^{n+1} + \partial_x v_N^{n-1}}{2}, \partial_x \varphi \right\rangle \\
&+\sigma_2 \left\langle \frac{v_N^{n+1} + v_N^{n-1}}{2}, \varphi \right\rangle + c\left\langle \left|v_N^n\right|^2 v_N^n, \varphi \right\rangle + d\left\langle \left|u_N^n\right|^2 v_N^n, \varphi \right\rangle + e\left\langle (u_N^n)^2 \overline{v_N^n}, \varphi \right\rangle = 0,
\end{aligned} \tag{5.2}$$

subject to

$$u_N^0 = P_N(u_0), \qquad v_N^0 = P_N v(v_0),$$

where Δt is a time step chosen together with a positive integer M, such that $M\Delta t = T$. Furthermore, we define $t_n = n\Delta t$, $n = 0, 1, \ldots, M$. We recall that in analogy to the semidiscrete formulation, we have that for $s \geq 0$ integer, Eqs. (5.1)–(5.2) are also valid changing the inner product of L^2_{per} by the inner product of H^s_{per}.

The following theorem shows that the CNLS system (3.1)–(3.2) can be reduced to the case $\delta = 0$, by means of an appropriate change of variables.

Theorem 5.1. *Let (u, v) a solution of the system*

$$\begin{cases} i\dfrac{\partial u}{\partial t} + i\delta\dfrac{\partial u}{\partial x} + K\dfrac{\partial^2 u}{\partial x^2} + \sigma_1 u + a|u|^2 u + b|v|^2 u + ev^2\overline{u} = 0 \\ i\dfrac{\partial v}{\partial t} - i\delta\dfrac{\partial v}{\partial x} + K\dfrac{\partial^2 v}{\partial x^2} + \sigma_2 v + c|v|^2 v + d|u|^2 v + eu^2\overline{v} = 0, \end{cases}$$

and let α be a real constant. Then the functions $\widetilde{u}$ and $\widetilde{v}$ defined by

$$\begin{aligned} \widetilde{u}(x,t) &= \exp\left(-i\alpha t + \frac{i\delta x}{2K}\right) u(x,t), \\ \widetilde{v}(x,t) &= \exp\left(-i\alpha t - \frac{i\delta x}{2K}\right) v(x,t), \end{aligned} \tag{5.3}$$

satisfy the system

$$\begin{cases} i\dfrac{\partial \widetilde{u}}{\partial t} + K\dfrac{\partial^2 \widetilde{u}}{\partial x^2} + \gamma_1\widetilde{u} + a|\widetilde{u}|^2\widetilde{u} + b|\widetilde{v}|^2\widetilde{u} + e\widetilde{v}^2\overline{\widetilde{u}}\exp\left(\dfrac{2i\delta x}{K}\right) = 0 \\ i\dfrac{\partial \widetilde{v}}{\partial t} + K\dfrac{\partial^2 \widetilde{v}}{\partial x^2} + \gamma_2\widetilde{v} + c|\widetilde{v}|^2\widetilde{v} + d|\widetilde{u}|^2\widetilde{v} + e\widetilde{u}^2\overline{\widetilde{v}}\exp\left(-\dfrac{2i\delta x}{K}\right) = 0, \end{cases} \tag{5.4}$$

where

$$\begin{aligned} \gamma_1 &= \frac{\delta^2}{4K} + \sigma_1 - \alpha, \\ \gamma_2 &= \frac{\delta^2}{4K} + \sigma_2 - \alpha. \end{aligned}$$

Theorem 5.2. *Let $s \geq 2$ be an integer, and $(u, v) \in C^1\left([0,T], H^r_{per}(0,L) \times H^r_{per}(0,L)\right)$ be a classical solution of (3.1)–(3.2), where $r > s$, and assume that $\|u(t)\|_s + \|v(t)\|_s \leq 2B$, for some constant $B > 0$ and any $t \in [0,T]$. Let $(u_N, v_N) \in C^3([0,T], S_N \times S_N)$ be the solution of the semidiscrete problem (4.3) and let $\{(u^n_N, v^n_N)\}$ be the solution of (5.1)–(5.2). If $u^0_N = u_N(0)$, $v^0_N = v_N(0)$ and $u^1_N,\ v^1_N$ satisfy that $\left\|u^1_N - u_N(\Delta t)\right\|_s \leq C\Delta t^2$, $\left\|v^1_N - v_N(\Delta t)\right\|_s \leq C\Delta t^2$, then with the assumption that $u_0,\ v_0 \in H^r_{per}(0,L)$, there exists a constant C independent of N and Δt such that if N large enough and Δt small enough, we have that*

$$\max_{0\leq n\leq M}\left\{\left\|u^n_N - u_N(t_n)\right\|_s + \left\|v^n_N - v_N(t_n)\right\|_s\right\} \leq C\Delta t^2.$$

Proof. By virtue of Theorem 5.1, it is enough to demonstrate the result for the particular case of the system

$$\begin{cases} i\dfrac{\partial u}{\partial t} + K\dfrac{\partial^2 u}{\partial x^2} + \sigma_1 u + a|u|^2 u + b|v|^2 u + e_1(x)v^2\overline{u} = 0 \\ i\dfrac{\partial v}{\partial t} + K\dfrac{\partial^2 v}{\partial x^2} + \sigma_2 v + c|v|^2 v + d|u|^2 v + e_2(x)u^2\overline{v} = 0, \end{cases} \tag{5.5}$$

where

$$e_1(x) = e\exp\left(\frac{2i\delta x}{K}\right), \qquad e_2(x) = e\exp\left(-\frac{2i\delta x}{K}\right).$$

The semidiscrete formulation for system (5.5) is

$$\begin{aligned} &\langle i\partial_t u_N, \phi\rangle + \langle i\delta\partial_x u_N, \phi\rangle + \left\langle K\partial_x^2 u_N, \phi\right\rangle + \langle\sigma_1 u_N, \phi\rangle + \left\langle a|u_N|^2 u_N, \phi\right\rangle \\ &\quad + \left\langle b|v_N|^2 u_N, \phi\right\rangle + \left\langle e_1 v_N^2\overline{u_N}, \phi\right\rangle = 0, && (5.6) \\ &\langle i\partial_t v_N, \varphi\rangle - \langle i\delta\partial_x v_N, \varphi\rangle + \left\langle K\partial_x^2 v_N, \varphi\right\rangle + \langle\sigma_2 v_N, \varphi\rangle + \left\langle c|v_N|^2 v_N, \varphi\right\rangle \\ &\quad + \left\langle d|u_N|^2 v_N, \varphi\right\rangle + \left\langle e_2 u_N^2\overline{v_N}, \varphi\right\rangle = 0, && (5.7) \\ &u_N(0) = P_N(u_0), \qquad v_N(0) = P_N(v_0), \end{aligned}$$

for all $\phi,\ \varphi \in S_N$.

Moreover, the corresponding fully discrete formulation of system above is

$$i\left\langle \frac{u_N^{n+1}-u_N^{n-1}}{2\Delta t}, \phi \right\rangle - K\left\langle \frac{\partial_x u_N^{n+1}+\partial_x u_N^{n-1}}{2}, \partial_x\phi \right\rangle$$

$$+\sigma_1\left\langle \frac{u_N^{n+1}+u_N^{n-1}}{2}, \phi \right\rangle + a\left\langle \left|u_N^n\right|^2 u_N^n, \phi \right\rangle + b\left\langle \left|v_N^n\right|^2 u_N^n, \phi \right\rangle + \left\langle e_1(\cdot)(v_N^n)^2\overline{u_N^n}, \phi \right\rangle = 0, \tag{5.8}$$

$$i\left\langle \frac{v_N^{n+1}-v_N^{n-1}}{2\Delta t}, \varphi \right\rangle - K\left\langle \frac{\partial_x v_N^{n+1}+\partial_x v_N^{n-1}}{2}, \partial_x\varphi \right\rangle$$

$$+\sigma_2\left\langle \frac{v_N^{n+1}+v_N^{n-1}}{2}, \varphi \right\rangle + c\left\langle \left|v_N^n\right|^2 v_N^n, \varphi \right\rangle + d\left\langle \left|u_N^n\right|^2 v_N^n, \varphi \right\rangle + \left\langle e_2(\cdot)(u_N^n)^2\overline{v_N^n}, \varphi \right\rangle = 0, \tag{5.9}$$

$$u_N^0 = P_N(u_0), \qquad v_N^0 = P_N(v_0).$$

We remark that Theorem 4.2 is also valid for the corresponding semidiscrete formulation (5.6)–(5.7) of system (5.5).

Let (u_N, v_N) be the solution of the semidiscrete formulation (5.6)–(5.7). Thus for all $\phi \in S_N$, we have that

$$\begin{aligned}
& i\left\langle u_N(t_{n+1}) - u_N(t_{n-1}), \phi\right\rangle_s + K\Delta t\left\langle \partial_x^2 u_N(t_{n+1}) + \partial_x^2 u_N(t_{n-1}), \phi\right\rangle_s + \sigma_1\Delta t\left\langle u_N(t_{n+1}) + u_N(t_{n-1}), \phi\right\rangle_s \\
&\quad + 2a\Delta t\left\langle |u_N(t_n)|^2 u_N(t_n), \phi\right\rangle_s + 2b\Delta t\left\langle |v_N(t_n)|^2 u_N(t_n), \phi\right\rangle_s + 2\Delta t\left\langle e_1(\cdot)(v_N(t_n))^2\overline{u_N}(t_n), \phi\right\rangle_s \\
&\ = i\left\langle u_N(t_{n+1}) - u_N(t_{n-1}), \phi\right\rangle_s + K\Delta t\left\langle \partial_x^2 u_N(t_{n+1}) + \partial_x^2 u_N(t_{n-1}), \phi\right\rangle_s + \sigma_1\Delta t\left\langle u_N(t_{n+1}) + u_N(t_{n-1}), \phi\right\rangle_s \\
&\quad - 2i\Delta t\left\langle \partial_t u_N(t_n), \phi\right\rangle_s - 2K\Delta t\left\langle \partial_x^2 u_N(t_n), \phi\right\rangle_s - 2\sigma_1\Delta t\left\langle u_N(t_n), \phi\right\rangle_s \\
&\ = i\left\langle u_N(t_{n+1}) - u_N(t_{n-1}) - 2\Delta t\partial_t u_N(t_n), \phi\right\rangle_s + K\Delta t\left\langle \partial_x^2 u_N(t_{n+1}) + \partial_x^2 u_N(t_{n-1}) - 2\partial_x^2 u_N(t_n), \phi\right\rangle_s \\
&\quad + \sigma_1\Delta t\left\langle u_N(t_{n+1}) + u_N(t_{n-1}) - 2u_N(t_n), \phi\right\rangle_s \\
&\ := \left\langle \theta^n, \phi\right\rangle_s,
\end{aligned}$$

namely,

$$\begin{aligned}
& i\left\langle u_N(t_{n+1}) - u_N(t_{n-1}), \phi\right\rangle_s + K\Delta t\left\langle \partial_x^2 u_N(t_{n+1}) + \partial_x^2 u_N(t_{n-1}), \phi\right\rangle_s + \sigma_1\Delta t\left\langle u_N(t_{n+1}) + u_N(t_{n-1}), \phi\right\rangle_s \\
&\quad + 2a\Delta t\left\langle |u_N(t_n)|^2 u_N(t_n), \phi\right\rangle_s + 2b\Delta t\left\langle |v_N(t_n)|^2 u_N(t_n), \phi\right\rangle_s \\
&\quad + 2\Delta t\left\langle e_1(v_N(t_n))^2\overline{u_N}(t_n), \phi\right\rangle_s = \left\langle \theta^n, \phi\right\rangle_s.
\end{aligned} \tag{5.10}$$

We know that

$$\begin{aligned}
& u_N(t_{n+1}) - u_N(t_{n-1}) = 2\Delta t\partial_t u_N(t_n) + O(\Delta t^3), \\
& \partial_x^2 u_N(t_{n+1}) + \partial_x^2 u_N(t_{n-1}) = 2\partial_x^2 u_N(t_N) + O(\Delta t^2),
\end{aligned}$$

and

$$u_N(t_{n+1}) + u_N(t_{n-1}) = 2u_N(t_n) + O(\Delta t^2).$$

Therefore,

$$\left\|\theta^n\right\|_s \le C\Delta t^3.$$

Let us define $e^n \in S_N$ as

$$e^n = u_N^n - u_N(t_n), \quad n = 0, 1, 2, \ldots.$$

Note that $e^n \in S_N$. Subtracting (5.10) from (5.8) (with inner product of H^s_{per}), we get that e^n satisfies, for all $\phi \in S_N$, the following equation:

$$\begin{aligned}
& i\left\langle e^{n+1} - e^{n-1}, \phi\right\rangle_s - K\Delta t\left\langle \partial_x e^{n+1} + \partial_x e^{n-1}, \partial_x\phi\right\rangle_s + \sigma_1\Delta t\left\langle e^{n+1} + e^{n-1}, \phi\right\rangle_s \\
&\quad + 2a\Delta t\left\langle \left|u_N^n\right|^2 u_N^n - |u_N(t_n)|^2 u_N(t_n), \phi\right\rangle_s + 2b\Delta t\left\langle \left|v_N^n\right|^2 u_N^n - |v_N(t_n)|^2 u_N(t_n), \phi\right\rangle_s \\
&\quad + 2\Delta t\left\langle e_1[(v_N^n)^2\overline{u_N^n} - v_N^2(t_n)\overline{u_N}(t_n)], \phi\right\rangle_s = -\langle \theta^n, \phi\rangle_s.
\end{aligned} \tag{5.11}$$

Letting $\phi = e^{n+1}$ in the above equation and multiplying by $-i$:

$$\begin{aligned}
&\left\|e^{n+1}\right\|_s^2 - \langle e^{n-1}, e^{n+1}\rangle_s + iK\Delta t\langle \partial_x e^{n+1}, \partial_x e^{n+1}\rangle_s + iK\Delta t\left\langle \partial_x e^{n-1}, \partial_x e^{n+1}\right\rangle_s - i\sigma_1\Delta t\left\|e^{n+1}\right\|_s^2\\
&\quad - i\sigma_1\Delta t\langle e^{n-1}, e^{n+1}\rangle_s - 2ai\Delta t\left\langle \left|u_N^n\right|^2 u_N^n - |u_N(t_n)|^2 u_N(t_n), e^{n+1}\right\rangle_s\\
&\quad - 2bi\Delta t\left\langle \left|v_N^n\right|^2 u_N^n - |v_N(t_n)|^2 u_N(t_n), e^{n+1}\right\rangle_s - 2i\Delta t\left\langle e_1[(v_N^n)^2\overline{u_N^n} - v_N^2(t_n)\overline{u_N}(t_n)], e^{n+1}\right\rangle_s\\
&= i\left\langle \theta^n, e^{n+1}\right\rangle_s .
\end{aligned} \tag{5.12}$$

Analogously, letting $\phi = -e^{n-1}$ in Eq. (5.11) and multiplying by i:

$$\begin{aligned}
&-\left\|e^{n-1}\right\|_s^2 + \langle e^{n+1}, e^{n-1}\rangle_s + iK\Delta t\langle \partial_x e^{n+1}, \partial_x e^{n-1}\rangle_s + iK\Delta t\left\langle \partial_x e^{n-1}, \partial_x e^{n-1}\right\rangle_s - i\sigma_1\Delta t\left\|e^{n-1}\right\|_s^2\\
&\quad - i\sigma_1\Delta t\langle e^{n+1}, e^{n-1}\rangle_s - 2ai\Delta t\left\langle \left|u_N^n\right|^2 u_N^n - |u_N(t_n)|^2 u_N(t_n), e^{n-1}\right\rangle_s\\
&\quad - 2bi\Delta t\left\langle \left|v_N^n\right|^2 u_N^n - |v_N(t_n)|^2 u_N(t_n), e^{n-1}\right\rangle_s - 2i\Delta t\left\langle e_1[(v_N^n)^2\overline{u_N^n} - v_N^2(t_n)\overline{u_N}(t_n)], e^{n-1}\right\rangle_s\\
&= i\left\langle \theta^n, e^{n-1}\right\rangle_s .
\end{aligned} \tag{5.13}$$

Summing Eqs. (5.12), (5.13) and taking real part, we obtain that

$$\begin{aligned}
&\|e^{n+1}\|_s^2 - \|e^{n-1}\|_s^2 - \mathrm{Re}(2ai\Delta t\langle |u_N^n|^2 u_N^n - |u_N(t_n)|^2 u_N(t_N), e^{n+1} + e^{n-1}\rangle_s) - \mathrm{Re}(2bi\Delta t\langle |v_N|^2 u_N^n\\
&\quad - |v_N(t_n)|^2 u_N(t_n), e^{n+1} + e^{n-1}\rangle_s) - \mathrm{Re}(2i\Delta t\langle e_1[(v_N^n)^2\overline{u_N^n} - v_N^2(t_n)\overline{u_N}(t_n)], e^{n+1} + e^{n-1}\rangle_s)\\
&= \mathrm{Re}(i\langle \theta^n, e^{n+1} + e^{n-1}\rangle_s).
\end{aligned} \tag{5.14}$$

We assume that $\|u(t)\|_s + \|v(t)\|_s \le B$, for all $t \in [0, T]$, where $B > 0$ is a constant independent of N. Let $0 \le n^* < M$ be the greatest integer such that $\left\|u_N^m\right\|_s + \left\|v_N^m\right\|_s \le 2B$, for all $m \le n^*$. The existence of n^* can be ensured inasmuch as

$$\begin{aligned}
\left\|u_N^0\right\|_s + \left\|v_N^0\right\|_s &= \|u_N(0)\|_s + \|v_N(0)\|_s = \|P_N u_0\|_s + \|P_N v_0\|_s\\
&\le \|u_0\|_s + \|v_0\|_s \le 2B.
\end{aligned}$$

Then for all $n \le n^*$ we get that

$$\begin{aligned}
\left\| \left|u_N^n\right|^2 u_N^n - |u_N(t_n)|^2 u_N(t_n)\right\|_s &\le C\left(\left\|u_N^n\right\|_s^2 + \left\|u_N^n\right\|_s \|u_N(t_n)\|_s + \|u_N(t_n)\|_s^2\right)\left\|u_N^n - u_N(t_n)\right\|_s\\
&\le C\left\|u_N^n - u_N(t_n)\right\|_s ,
\end{aligned}$$

$$\left\| \left|v_N^n\right|^2 u_N^n - |v_N(t_n)|^2 u_N(t_n)\right\|_s \le C\left(\left\|u_N^n - u_N(t_n)\right\|_s + \left\|v_N^n - v_N(t_n)\right\|_s\right),$$

and

$$\left\| e_1[(v_N^n)^2\overline{u_N^n} - v_N^2(t_n)\overline{u_N}(t_n)]\right\|_s \le C\left(\left\|u_N^n - u_N(t_n)\right\|_s + \left\|v_N^n - v_N(t_n)\right\|_s\right).$$

Setting $f^n := v_N^n - v_N(t_n)$, from (5.14) follows that

$$\left\|e^{n+1}\right\|_s^2 \le \left\|e^{n-1}\right\|_s^2 (1 + C\Delta t) + \left\|e^{n+1}\right\|_s^2 C\Delta t + C\Delta t\left(\left\|e^n\right\|_s^2 + \left\|f^n\right\|_s^2\right) + C\Delta t^5. \tag{5.15}$$

Analogously, for all $\varphi \in S_N$

$$\begin{aligned}
&i\langle v_N(t_{n+1}) - v_N(t_{n-1}), \varphi\rangle_s + K\Delta t\left\langle \partial_x^2 v_N(t_{n+1}) + \partial_x^2 v_N(t_{n-1}), \varphi\right\rangle_s + \sigma_2\Delta t\langle v_N(t_{n+1}) + v_N(t_{n-1}), \varphi\rangle_s\\
&\quad + 2c\Delta t\left\langle |v_N(t_n)|^2 v_N(t_n), \varphi\right\rangle_s + 2d\Delta t\left\langle |u_N(t_n)|^2 v_N(t_n), \varphi\right\rangle_s + 2\Delta t\left\langle e_2(\cdot)(u_N(t_n))^2\overline{v_N}(t_n), \varphi\right\rangle_s\\
&= i\langle v_N(t_{n+1}) - v_N(t_{n-1}) - 2\Delta t\partial_t v_N(t_n), \varphi\rangle_s + K\Delta t\left\langle \partial_x^2 v_N(t_{n+1}) + \partial_x^2 v_N(t_{n-1}) - 2\partial_x^2 v_N(t_n), \varphi\right\rangle_s\\
&\quad + \sigma_2\Delta t\langle v_N(t_{n+1}) + v_N(t_{n-1}) - 2v_N(t_n), \varphi\rangle_s\\
&= \left\langle \alpha^n, \varphi\right\rangle_s ,
\end{aligned} \tag{5.16}$$

where $\|\alpha_n\|_s \le C\Delta t^3$.
Using $\varphi = f^{n+1}$ and $\varphi = -f^{n-1}$ in Eqs. (5.9), (5.16), we get

$$\left\|f^{n+1}\right\|_s^2 \le \left\|f^{n-1}\right\|_s^2 (1 + C\Delta t) + \left\|f^{n+1}\right\|_s^2 C\Delta t + C\Delta t\left(\left\|f^n\right\|_s^2 + \left\|e^n\right\|_s^2\right) + C\Delta t^5. \tag{5.17}$$

Thus from (5.15) and (5.17), we have the estimate

$$\left\|e^{n+1}\right\|_s^2 + \left\|f^{n+1}\right\|_s^2 \leq (1+C\Delta t)\left(\left\|e^{n-1}\right\|_s^2 + \left\|f^{n-1}\right\|_s^2\right) + C\Delta t\left(\left\|e^{n+1}\right\|_s^2 + \left\|f^{n+1}\right\|_s^2\right) + C\Delta t\left(\left\|e^{n}\right\|_s^2 + \left\|f^{n}\right\|_s^2\right) + C\Delta t^5.$$

Letting $A^{n+1} = \left\|e^{n+1}\right\|_s^2 + \left\|f^{n+1}\right\|_s^2$, it follows that

$$A^{n+1}(1 - C\Delta t) \leq (1 + C\Delta t)\,A^{n-1} + C\Delta t A^n + C\Delta t^5.$$

Note that if Δt is chosen small enough, we have

$$\frac{1+C\Delta t}{1-C\Delta t} \leq 1 + 4C\Delta t, \qquad \frac{C}{1-C\Delta t} \leq 2C.$$

As a consequence,

$$A^{n+1} \leq (1 + C\Delta t)A^{n-1} + C\Delta t A^n + C\Delta t^5.$$

If we define $B^{n+1} = A^{n+1} + A^n$, and adding A^n on both sides of the inequality above, we have for Δt small enough and $0 \leq n \leq n^*$,

$$B^{n+1} \leq B^n(1 + C\Delta t) + C\Delta t^5. \tag{5.18}$$

From hypotheses $e^0 = f^0 = 0$, and $\|e^1\|_s$, $\|f^1\|_s$ are both of order $O(\Delta t^2)$. Therefore $A^0 = 0$ and $B^1 = O(\Delta t^4)$. We can conclude from Gronwall's Lemma 2.1 and estimate (5.18) that $B^{n^*+1} = O(\Delta t^4)$. Therefore for all $0 \leq n \leq n^* + 1$

$$\left\|e^{n}\right\|_s^2 + \left\|f^{n}\right\|_s^2 \leq C\Delta t^4. \tag{5.19}$$

Thus, as a consequence of Theorem 4.2

$$\begin{aligned}
\left\|u_N^{n^*+1}\right\|_s + \left\|v_N^{n^*+1}\right\|_s &\leq \left\|u_N^{n^*+1} - u_N(t_{n^*+1})\right\|_s + \left\|v_N^{n^*+1} - v_N(t_{n^*+1})\right\|_s + \|u_N(t_{n^*+1}) - u(t_{n^*+1})\|_s \\
&\quad + \|v_N(t_{n^*+1}) - v(t_{n^*+1})\|_s + \|u(t_{n^*+1})\|_s + \|v(t_{n^*+1})\|_s \\
&\leq \left\|e^{n^*+1}\right\|_s + \left\|f^{n^*+1}\right\|_s + \|u_N(t_{n^*+1}) - u(t_{n^*+1})\|_s + \|v_N(t_{n^*+1}) - v(t_{n^*+1})\|_s \\
&\quad + \|u(t_{n^*+1})\|_s + \|v(t_{n^*+1})\|_s \\
&\leq C\Delta t^2 + CN^{s-r} + B \\
&< 2B,
\end{aligned}$$

if Δt is small enough and N large enough. This contradicts the maximality of n^* previously assumed. So $\left\|u_N^m\right\|_s + \left\|v_N^m\right\|_s \leq B$, for all $0 \leq m \leq M$ and inequality (5.19) is valid until $n = M$. Therefore,

$$\max_{0\leq n\leq M}\left(\left\|e^n\right\|_s + \left\|f^n\right\|_s\right) \leq C\Delta t^2. \quad \square$$

Remark 1. Using Theorems 4.2, 5.2 and triangle inequality, we can conclude that

$$\max_{0\leq n\leq M}\left\{\left\|u_N^n - u(t_n)\right\|_s + \left\|v_N^n - v(t_n)\right\|_s\right\} \leq C\left(N^{s-r} + \Delta t^2\right).$$

6. Numerical experiments

In this section, we develop some experiments to check the accuracy of the numerical solver introduced in this paper. In first place, we conduct a computer simulation using the exact periodic plane wave solution of system (3.1)–(3.2) for $a = c$, $b = d$, $\sigma_1 = \sigma_2$, $\delta = 0$ (see [2]):

$$\begin{aligned}
u(x,t) &= u_0 e^{i(kt - wx)}, \\
v(x,t) &= v_0 e^{i(kt - wx)},
\end{aligned} \tag{6.1}$$

where $k = -Kw^2 + au_0^2 + (b+e)v_0^2 + \sigma_1$, $a = c = 1$, $b = d = 2/3$, $e = 1/3$, $\sigma_1 = \sigma_2 = 1$, $u_0 = 0.5$, $v_0 = 0.3$, $K = 0.7$ and $w = 0.1$. Therefore this solution has x-period given by $L = 2\pi/w \approx 62.83$. The numerical parameters are $N = 2^6$, $\Delta t = 1e-3$. In Fig. 1 we show the numerical simulation obtained at $t = 100$ using the parameters given above. We observe good agreement with the exact solution given in (6.1). We find that the error in the maximum norm is about $1e-3$.

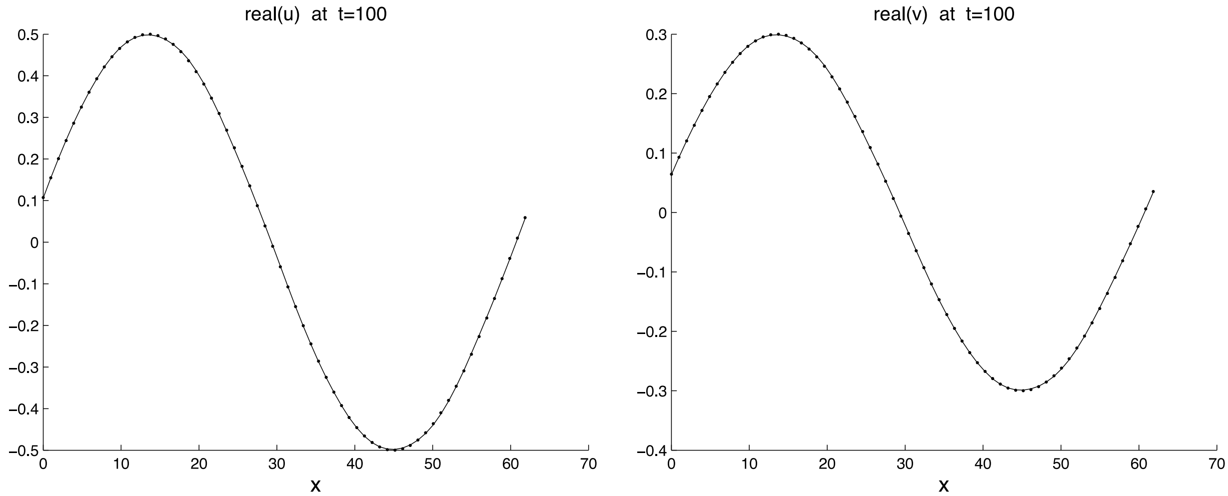

Fig. 1. Solid line: numerical solution. In points: periodic standing wave (6.1).

In second place, when $e = 0$ system (3.1)–(3.2) has standing wave solutions in the form (see [1,2]):

$$\begin{aligned} u(x,t) &= A_1 \exp\left[i\alpha_1 t - \frac{i\delta}{2K}(x - x_0)\right] \operatorname{sech}(x - x_0), \\ v(x,t) &= A_2 \exp\left[i\alpha_2 t + \frac{i\delta}{2K}(x - x_0)\right] \operatorname{sech}(x - x_0), \end{aligned} \tag{6.2}$$

with

$$A_1 = \sqrt{\frac{2K(b-c)}{bd-ac}}, \qquad A_2 = \sqrt{\frac{2K(d-a)}{bd-ac}},$$
$$\alpha_1 = K + \sigma_1 + \frac{\delta^2}{4K}, \qquad \alpha_2 = K + \sigma_2 + \frac{\delta^2}{4K},$$

and $x_0 = 30$, $\sigma_1 = \sigma_2 = 0$, $b = d = 2/3$, $K = 0.5$, $\delta = 0.5$. In this case, to enable the application of the Fourier-spectral scheme (5.1)–(5.2) to this nonperiodic setting, we approximate the initial value problem (3.1)–(3.2), with $x \in \mathbb{R}$, $t > 0$, by the periodic Cauchy problem for $x \in [0, L]$, $t > 0$, with large spatial period L. This type of approximation can be justified by the decay of the solutions of the unrestricted problem as $|x| \to \infty$. In our numerical experiment with the standing wave solution (6.2), we set $L = 70$, and the numerical parameters are $\Delta t = 1e - 3$, $N = 2^{10}$. In Fig. 2 we compare the numerical and the exact solution (6.2) at $t = 100$, and see that the corresponding profiles coincide with good accuracy of $1e - 5$ in the maximum norm.

6.1. *Order in space of the Fourier method*

We want to validate the spectral order of convergence in space of the numerical scheme proposed in the present paper. In Fig. 3, we fix a small time step $\Delta t = 1e - 5$ and increase the number of points in space. We use the standing wave solution given in (6.2) with the same model's parameters as in the experiment above. We start with $N = 2^8$ and the increase by 2 until we get $N = 2^{11}$. For every value of N we compute the numerical solution until time $t = 1$. We can see from Fig. 3, as shown in Theorem 5.2, the fully discrete method (5.1)–(5.2) used in the present paper has spectral accuracy in space and the error decreases very rapidly approximately as $N^{-10.24}$ in contrast with pure finite difference methods, for instance.

6.2. *Order in time of the Fourier method*

We now validate numerically the order in time for the numerical scheme (5.1)–(5.2) proposed in the present paper. We use again the standing wave solution (6.2) with the same model's parameters as in the previous numerical simulation. We choose $N = 2^{20}$ ($\Delta x = L/N \approx 6.7e - 5$) large enough so that the error in space does not dominate the total error. By starting with $\Delta t = 0.1$ and decreasing the time step by $1/10$ until $\Delta t = 1e - 4$, the numerical solution is computed until $t = 1$. We get Fig. 4, from where we can see that the error in time of the numerical scheme is of order 2, in accordance with Theorem 5.2.

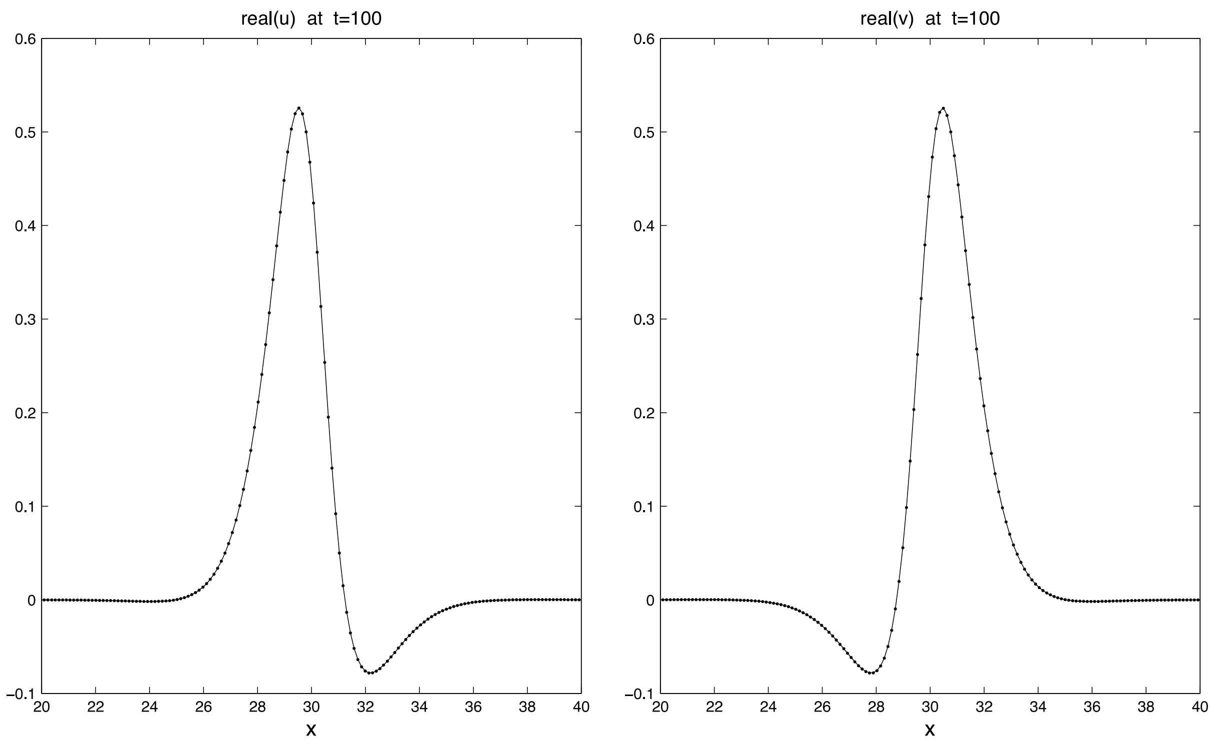

Fig. 2. Solid line: numerical solution. In points: standing wave (6.2).

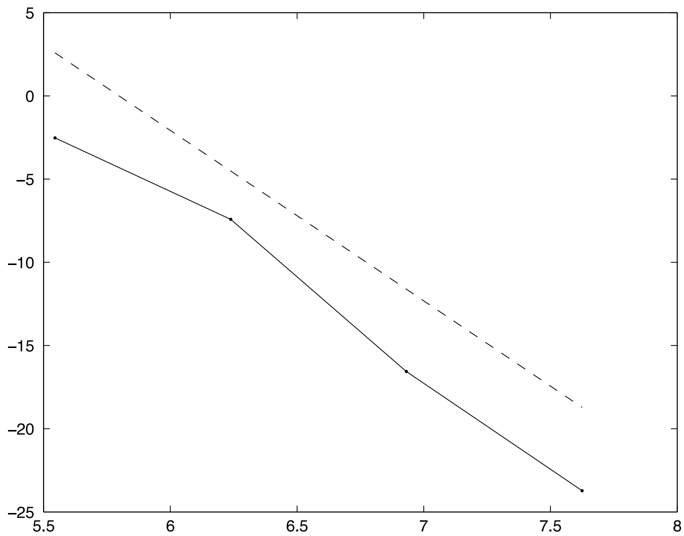

Fig. 3. Plot of the decimal logarithm of the maximum error against $\log_{10} N$. The time step is fixed at $\Delta t = 1e-5$. We see that the plot is approximately a line with slope -10.24.

6.3. Checking conservation laws

In Section 4, we established that the semidiscrete scheme given in (4.1)–(4.2) conserves in time the quantities

$$
\begin{aligned}
E_1(t) &= \int_0^L (|u|^2 + |v|^2)dx, \\
E_2(t) &= \int_0^L \Big(\Big| \frac{i\delta}{2K} u + \partial_x u \Big|^2 + \Big| -\frac{i\delta}{2K} v + \partial_x v \Big|^2 - \frac{a}{2K}|u|^4 - \frac{c}{2K}|v|^4 - \frac{b}{K}|u|^2|v|^2 - \frac{e}{K}\mathrm{Re}(u^2\overline{v}^2) \Big) dx.
\end{aligned}
\tag{6.3}
$$

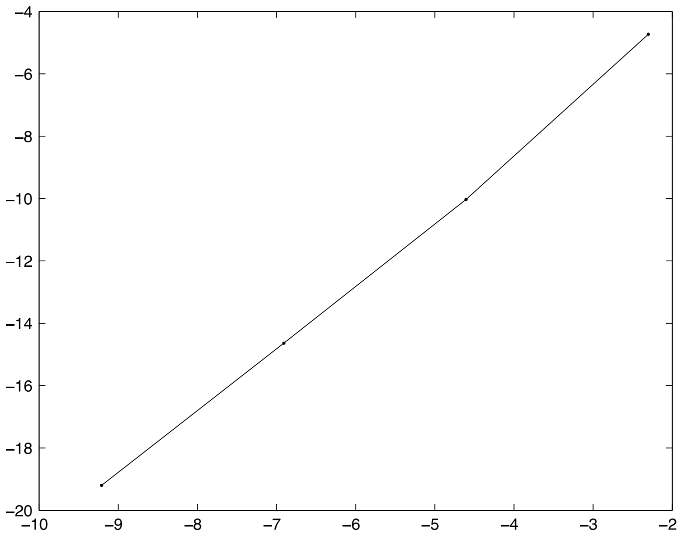

Fig. 4. Plot of the decimal logarithm of the maximum error against $\log_{10}\Delta t$. The number of points in space is fixed at $N = 2^{20}$. We see that the plot is approximately a line with slope 2.09.

We recall that the conservation of $E_2(t)$ requires that $b = d$. In Figs. 5 and 6, we illustrate numerically that the fully discrete scheme (5.1)–(5.2) conserves approximately the discrete versions of $E_1(t)$, $E_2(t)$ given by

$$\begin{aligned} C_1(t) &= \Delta x \sum_{k=0}^{N-1} (|u(k\Delta x,t)|^2 + |v(k\Delta x,t)|^2), \\ C_2(t) &= \Delta x \sum_{k=0}^{N-1} \Big(\Big| \frac{i\delta}{2K} u(k\Delta x,t) + \partial_x^h u(k\Delta x,t) \Big|^2 + \Big| -\frac{i\delta}{2K} v(k\Delta x,t) + \partial_x^h v(k\Delta x,t) \Big|^2 \\ &\quad - \frac{a}{2K}|u(k\Delta x,t)|^4 - \frac{c}{2K}|v(k\Delta x,t)|^4 - \frac{b}{K}|u(k\Delta x,t)|^2|v(k\Delta x,t)|^2 - \frac{e}{K}\mathrm{Re}(u(k\Delta x,t)^2\overline{v}(k\Delta x,t)^2) \Big), \end{aligned} \tag{6.4}$$

where $\partial_x^h u$ is defined by

$$\partial_x^h u(x,t) = \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}.$$

The initial pulses are given by

$$u(x,0) = v(x,0) = \pi\sqrt{2}\Big(1 + \frac{1}{10}\cos(\pi x)\Big),$$

and thus the period is $L = 2$. Other numerical parameters are $\Delta t = 1e-4$, $N = 2^{10}$, $K = \delta = 1/2$, $\sigma_1 = \sigma_2 = 1$, $a = c = 1$, $b = d = 2/3$, $e = 1/3$. From these numerical experiments, we see that the fully discrete numerical solver (5.1)–(5.2) conserves in time the quantities $C_1(t)$, $C_2(t)$ well.

7. Conclusions

In this paper, a numerical scheme was introduced to approximate the solutions of the system of two coupled Schrödinger equations (3.1)–(3.2). The numerical discretization in space was performed using a Fourier-spectral technique, and in time using a second-order two-step finite difference scheme. We showed rigorously the convergence of both the semidiscrete and fully discrete numerical solver. By using some numerical experiments, the spectral accuracy in space and the second order in time of the scheme were verified in perfect accordance with our analytical results. Furthermore, we found that the fully discrete scheme conserves approximately L^2 and energy norms. These features make scheme (5.1)–(5.2) a useful tool to conduct numerical simulations with Schrödinger type systems to investigate different properties of their solutions and wave phenomena, such as orbital stability under small perturbations on initial data, and study of propagation of standing waves and periodic solutions, among others.

Acknowledgments

This work was supported by Departamento de Matemáticas, Universidad del Valle, Calle 13 No. 100-00, Cali-Colombia and Colciencias under the Research Project C.I. 988.

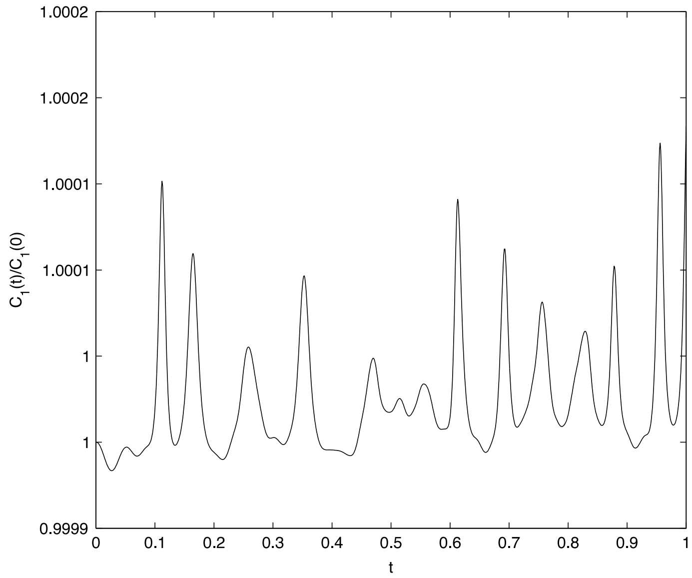

Fig. 5. The time evolution of quantity $C_1(t)$ for the fully discrete method (5.1)–(5.2). Observe that $C_1(t)/C_1(0)$ remains near 1.

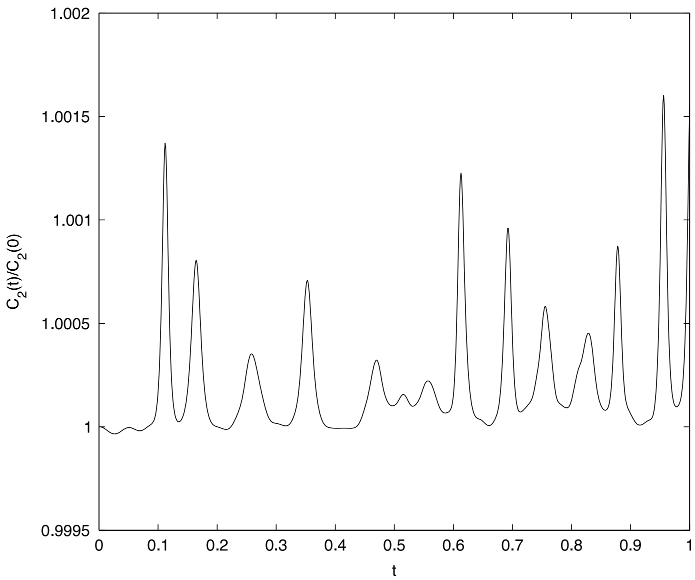

Fig. 6. The time evolution of quantity $C_2(t)$ for the fully discrete method (5.1)–(5.2). Observe that $C_2(t)/C_2(0)$ remains near 1.

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