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Complementarity properties of singular M-matrices

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For a matrix A whose off-diagonal entries are nonpositive, its nonnegative invertibility (namely, that A is an invertible M-matrix) is equivalent to A being a P-matrix, which is necessary and sufficient for the unique solvability of the linear complementarity problem defined by A. This, in turn, is equivalent to the statement that A is strictly semimonotone. In this paper, an analogue of this result is proved for singular symmetric Z-matrices. This is achieved by replacing the inverse of A by the group generalized inverse and by introducing the matrix classes of strictly range semimonotonicity and range column sufficiency. A recently proposed idea of $P_{\#}$-matrices plays a pivotal role. Some interconnections between these matrix classes are also obtained.

1. Introduction

$\mathbb{R}^{n\times n}$ denotes the space of all real square matrices of order n and $\mathbb{R}^n$ denotes the real Euclidean space of real vectors with n coordinates. For $x \in \mathbb{R}^n$, we write $x \geq 0$ to denote

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that all the coordinates of x are nonnegative. This is written as $x \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$. $x > 0$ signifies the fact that all the coordinates of x are positive. A real matrix B is said to be *nonnegative* if all its entries are nonnegative. This is denoted by $B \geq 0$. One of the central objects of interest in this work is the concept of a linear complementarity problem, which we discuss next. For $x, y \in \mathbb{R}^n$, we use $\langle x, y\rangle$ to denote the inner product $x^T y$ and $x \circ y$ to denote the Hadamard entrywise product of x and y. Let $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The *linear complementarity problem* $LCP(A, q)$ is to determine if there exists $x \in \mathbb{R}^n$ such that $x \geq 0$, $y = Ax + q \geq 0$ and $\langle y, x\rangle = 0$. If such a vector x exists, then $LCP(A, q)$ is said to have a *solution*. $SOL(A, q)$ denotes the set of all solutions of $LCP(A, q)$. Various classes of matrices have been introduced to study the existence and uniqueness of solutions of $LCP(A, q)$. Let us recall some of the relevant ones. A real square matrix A is called a *P-matrix* if all its principal minors are positive. It is well known that A is a P-matrix if and only if the implication

$$x \circ Ax \leq 0 \Longrightarrow x = 0$$

holds [3]. A famous result in the theory of linear complementarity problems states that $LCP(A, q)$ has a unique solution for all $q \in \mathbb{R}^n$ if and only if A is a P-matrix [3]. Let us consider the second class of matrices. A real square matrix A is said to be a *strictly semimonotone* matrix if

$$x \geq 0 \text{ and } x \circ Ax \leq 0 \Longrightarrow x = 0.$$

It is well known that A is a strictly semimonotone matrix if and only if $LCP(A, q)$ has a unique solution for all $q \in \mathbb{R}^n_+$ (Theorem 3.9.11) [3]. Any P-matrix is a strictly semimonotone matrix, while the converse could be shown to be false. However, these two classes coincide for a matrix class which we consider next. A is called a *Z-matrix*, if all its off-diagonal entries are nonpositive. Note that if A is a Z-matrix, then $A = sI - B$, for some $s \in \mathbb{R}$ with $s > 0$ and $B \geq 0$. A Z-matrix A is called an *M-matrix* if in the representation as above, one also has $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B. For a Z-matrix A to be a P-matrix, more than fifty characterizations are proved in the literature. We refer to the excellent book [2], for these. In what follows, we list out the conditions that are pertinent to the discussion here.

Theorem 1.1. *[2,12] Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following statements are equivalent:*

(*a*) *A is a P-matrix.*
(*b*) *A^{-1} exists and $A^{-1} \geq 0$.*
(*c*) *A is an invertible M-matrix.*
(*d*) *A is a strictly semimonotone matrix.*

The following reiteration of Theorem 1.1 will be helpful. A strictly semimonotone Z-matrix is also a P-matrix and so the restriction on q to be nonnegative, for uniqueness of solutions of $LCP(A,q)$ stated earlier, is removed. A more important perspective is that an invertible M-matrix has the property that $LCP(A,q)$ has a unique solution for all q. Our primary quest will be to consider an analogue of this assertion for a class of singular M-matrices.

In order to outline the objectives of the article, let us turn our attention to extensions of the four classes of matrices mentioned in Theorem 1.1. For a matrix A, we denote the range space of A and null space of A by $R(A)$ and $N(A)$, respectively. A matrix $A \in \mathbb{R}^{n\times n}$ is called a *$P_{\#}$-matrix* if for each nonzero vector $x \in R(A)$, there is an $i \in \{1,2,\dots,n\}$ such that $x_i(Ax)_i > 0$ [9]. Equivalently, for any $x \in R(A)$, the inequalities $x_i(Ax)_i \leq 0$ for all $i = 1,2,\dots,n$ imply $x = 0$. Clearly, every P-matrix is a $P_{\#}$-matrix and so the $P_{\#}$-matrix notion is a generalization of the P-matrix concept. In order to consider a generalization of statement (b), we must recall the notion of a certain generalized inverse, which we do next. $A \in \mathbb{R}^{n\times n}$ is said to have a *group inverse* if there exists $X \in \mathbb{R}^{n\times n}$ such that the equations $AXA = A$, $XAX = X$ and $AX = XA$ are satisfied. The group inverse is unique, if it exists and coincides with the usual inverse, if the latter exists. The group inverse is denoted by $A^{\#}$. A necessary and sufficient condition for $A^{\#}$ to exist is that $rank(A) = rank(A^2)$ (Theorem 4.2, [1]). We need one more notion. $A \in \mathbb{R}^{n\times n}$ is called *monotone* (see, for instance, [2]) if

$$Ax \geq 0 \Longrightarrow x \geq 0.$$

It is well known that A is monotone if and only if A^{-1} exists and $A^{-1} \geq 0$ [2]. In this connection let us recall that $A \in \mathbb{R}^{n\times n}$ is called *range monotone* [7] if

$$Ax \geq 0 \text{ and } x \in R(A) \Longrightarrow x \geq 0.$$

There, it was shown that A is range monotone if and only if $A^{\#}$ exists and that the following implication holds:

$$x \in \mathbb{R}^n_+ \cap R(A) \Longrightarrow A^{\#}x \geq 0.$$

Thus, a generalization of statement (b) in Theorem 1.1 that we are looking at is the condition that A is range monotone. An extension of the M-matrix concept (used in statement (c) above) that turns out to be the appropriate one for our purpose is the notion of an M-matrix with "property c" [8]. The precise definition will be presented in Section 2.3. Let us propose the fourth matrix class (with an intention of extending statement (d) above) as follows. $A \in \mathbb{R}^{n\times n}$ is said to be *strictly range semimonotone* if the following implication holds:

$$x \in R(A),\ x \geq 0 \text{ and } x \circ Ax \leq 0 \Longrightarrow x = 0.$$

Once again, it is clear that the idea of a strictly range semimonotone matrix could be considered as a singular analogue of the notion of a strictly semimonotone matrix. One of the primary goals of this work is to prove an analogue of Theorem 1.1 for singular M-matrices, where condition (a) is replaced by the statement that A is a $P_{\#}$-matrix, (b) is extended to the condition that A is range monotone, (c) is generalized to the requirement that A is an M-matrix with "property c" and the assertion that A is a strictly range semimonotone matrix, in place of (d). This is shown to be true for symmetric Z-matrices and is proved in Corollary 3.2.

There is another class of matrices that could be included in the discussion. Let us recall this next. Matrix $A \in \mathbb{R}^{n\times n}$ is said to be *column sufficient* [4] if

$$x \circ Ax \leq 0 \Longrightarrow x \circ Ax = 0.$$

In [4], it is shown that column sufficiency of A is equivalent to $LCP(A, q)$ having (possibly empty) convex solution set for all $q \in \mathbb{R}^n$. It is clear that every P-matrix is a column sufficient matrix. For an *invertible Z-matrix* A, it is known that A is strictly semimonotone if and only if A is column sufficient [4,14]. Thus, if A is an invertible Z-matrix, then column sufficiency is equivalent to all the statements of Theorem 1.1. A singular analogue of column sufficient matrices is proposed next. $A \in \mathbb{R}^{n\times n}$ is called a *range column sufficient matrix* if the following implication holds:

$$x \in R(A),\ x \circ Ax \leq 0 \Longrightarrow x \circ Ax = 0.$$

When A is a range column sufficient matrix, we also say that A has the *range column sufficiency* property. In Corollary 3.4, it is shown that range column sufficiency is another equivalent statement that could be included in Corollary 3.2.

Let us briefly recall pertinent recent work related to Theorem 1.1. Some extensions of each of the statements of Theorem 1.1 have been studied in the literature. The concept of a strictly semimonotone matrix in the setting of Euclidean Jordan algebras was considered in [12], where a proof of the equivalence of (b) and (d) is given (Theorem 3.9), [12]. The equivalence of (a) and (d) remains open in this setting. The authors of [9] primarily set out to study a possible extension of the equivalence (a) and (b) for characterizing nonnegativity of the Moore–Penrose inverse (or the group inverse) to what are called as $P_{\dagger}$-matrices. However, this aim was not achieved. (In any case, interesting connections between $P_{\dagger}$-matrices and certain intervals of matrices were obtained there). From the discussion in the earlier paragraph, it is now clear that we have been able to *fill this lacuna*.

The plan of the paper is as follows. In the next section, we present interesting properties of the new matrix classes that are introduced. In Section 3, we prove an important result in Theorem 3.1. This asserts, among other things, that for a Z-matrix A, which is also a $P_{\#}$-matrix it follows that A is range monotone; this in turn, implies that A is a strictly range semimonotone matrix. We prove that the converse holds for matrices of order 2×2 and 3×3, thereby generalizing Theorem 1.1 for (possibly) singular matrices of these orders. Corollary 3.2, as mentioned earlier, presents the sought after extension of Theorem 1.1 for symmetric Z-matrices. For normal Z-matrices, we show that the class of range column sufficient matrices coincides with the class of range monotone matrices which in turn is equivalent to strictly range semimonotone matrices. This is presented in Theorem 3.3. As a consequence of Theorem 3.3, we obtain a result connecting singular M-matrices and certain linear complementarity problems. This is given in Corollary 3.5.

2. Matrix classes

We shall be dealing with four classes of matrices (including the three that were defined in the introduction) that are made use of, in proving the main results. In what follows, we discuss these matrix classes and derive certain interesting properties. Let us begin with the first of these classes. For vectors $x, y \in \mathbb{R}^n$, $x \circ y$ denotes the Hadamard entrywise product. For $v \in \mathbb{R}^n$, let v_i denote its ith coordinate. So, if $x \circ y = z$, then $z_i = x_i y_i$, $1 \leq i \leq n$. It is easy to verify that $P(u \circ v) = P(u) \circ P(v)$ for all $u, v \in \mathbb{R}^n$ for any permutation matrix P. In the rest of the discussion, e^i will denote the vector all of whose entries are zero except the ith coordinate which is one. Let e denote the vector all of whose entries equal 1. For a matrix A, let $A = (a_{ij})$.

2.1. *Strictly range semimonotone matrices*

In this subsection, we consider the notion of strictly range semimonotone matrices and derive some of their properties. Before doing this, however, let us prove an interesting result for the stronger class of strictly semimonotone matrices. Let us recall that a real square matrix A is said to be a *strictly semimonotone* matrix if

$$x \geq 0 \text{ and } x \circ Ax \leq 0 \Longrightarrow x = 0.$$

Thus, if A is strictly semimonotone, then at least one component of Ax is positive for every nonzero $x \geq 0$. We have the following result on the entries of a strictly semimonotone matrix.

Theorem 2.1. *Let $A \in \mathbb{R}^{n \times n}$ be strictly semimonotone. Then*

(a) *The diagonal entries of A are positive.*
(b) *At least one of the row sums of A is positive.*

Proof. (a): Let a_{ii} be a diagonal entry of A. Suppose that $a_{ii} \leq 0$. Since Ae^i is the ith column of A, we have

$$0 \geq a_{ii} e^i = e^i \circ A(e^i).$$

This is a contradiction since $e^i \neq 0$. Hence $a_{ii} > 0$.

(b): Consider the vector e. Then Ae denotes the vector whose entries are the corresponding row sums of A. By the observation made earlier, at least one of the row sums of A is positive. □

Let A be a strictly semimonotone Z-matrix. By Theorem 1.1, it follows that A^{-1} exists. However, in what follows, we give a direct proof of this result. As far as we know, a proof is not available in the literature.

Theorem 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be a strictly semimonotone Z-matrix. Then A is invertible.*

Proof. Let $Ax = 0$ and $x = (x_1, x_2, \ldots, x_n)^T$ so that $0 \leq |x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$. Since A is a Z-matrix, one has $a_{ij} \leq 0$ whenever $i \neq j$. Thus, for all i, j with $i \neq j$ one has

$$a_{ij}|x_i||x_j| \leq a_{ij}x_i x_j.$$

Also, for all i, we have $a_{ii}|x_i||x_i| \leq a_{ii}x_i^2$. Now,

$$\begin{aligned}
(|x| \circ A|x|)_i &= |x_i| \sum_{j=1}^{n} a_{ij}|x_j| \\
&= \sum_{j=1}^{n} a_{ij}|x_i||x_j| \\
&\leq \sum_{j=1}^{n} a_{ij}x_i x_j \\
&= x_i \sum_{j=1}^{n} a_{ij}x_j \\
&= (x \circ Ax)_i.
\end{aligned}$$

Since $Ax = 0$, this yields $|x| \circ A|x| \leq 0$ and by the strict semimonotonicity of A it follows that $x = 0$. This shows that A is invertible. □

Let us now discuss strictly range semimonotone matrices. Recall that $A \in \mathbb{R}^{n\times n}$ is said to be *strictly range semimonotone* if the following implication holds:

$$x \in R(A),\ x \geq 0 \text{ and } x \circ Ax \leq 0 \Longrightarrow x = 0.$$

Clearly, every strictly semimonotone matrix is strictly range semimonotone. If a strictly range semimonotone matrix is invertible, then it is trivially strictly semimonotone. Hence one could think of strictly range semimonotone matrices as singular analogues of strictly semimonotone matrices. First, we obtain a version of Theorem 2.1 for this class.

Theorem 2.3. *Let A be a strictly range semimonotone matrix. Then the following hold:*

(a) *Suppose that e^i is a linear combination of the columns of A for some i, $1 \leq i \leq n$. Then a_{ii} is positive.*
(b) *Suppose that the vector e is a linear combination of the columns of A. Then at least one of the row sums of A is positive.*

Proof. (a): Let e^i be a linear combination of the columns of A. Then $0 \leq e^i \in R(A)$. If $a_{ii} \leq 0$, then $0 \geq a_{ii}e^i = e^i \circ Ae^i$. This is a contradiction.

(b): Since A is strictly range semimonotone, at least one component of Ax is positive for every nonzero $x \in R(A) \cap \mathbb{R}^n_+$. Note that Ae is a vector whose components are the corresponding row sums of A. Hence the result follows. $\square$

Next, we prove a result on the eigenvalues of a strictly range semimonotone Z-matrix. We need a result (Theorem 6) from [10] which states that if A is a Z-matrix and if

$$\lambda := \min\{Re(\mu) : \mu \in \sigma(A)\},$$

then λ is an eigenvalue of A and a nonnegative eigenvector is associated with this eigenvalue. Here $\sigma(A)$ denotes the spectrum of A.

Theorem 2.4. *Let A be a Z-matrix. Suppose that A is strictly range semimonotone. Then all the real eigenvalues of A are nonnegative.*

Proof. Using the just stated result, we have $Ax = \lambda x$ where $0 \neq x \geq 0$ and λ defined as above. If possible, suppose that $\lambda < 0$. Then $A(\frac{1}{\lambda}x) = x$. Thus, $x \in R(A)$. We have $Ax = \lambda x \leq 0$ and so $x \circ Ax \leq 0$. Since A has the strictly range semimonotone property, we have $x = 0$, a contradiction. Thus $\lambda \geq 0$. If μ is any real eigenvalue of A, then $0 \leq \lambda \leq \mu$, completing the proof. $\square$

The next corollary follows from the proof of the previous result. This fact will be used in the proof of Theorem 3.3. Let us recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *semipositive stable* if every eigenvalue of A has nonnegative real part.

Corollary 2.1. *Let A be a Z-matrix. If A is strictly range semimonotone matrix then A is semipositive stable.*

Remark 2.1. The conclusion of Theorem 2.4 does not hold if A is not a Z-matrix. This is shown by the following example. Let

$$A = \begin{pmatrix} a & a & 0 \\ -b & -b & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } a, b > 0.$$

Then $R(A) \cap \mathbb{R}^n_+ = \{(0,0,\gamma)^T : \gamma \geq 0\}$. If $0 \leq x \in R(A)$ and $x \circ Ax \leq 0$, then $x = 0$. Thus, A is a strictly range semimonotone matrix. In particular, if $a = 1$ and $b = 2$, then A is not a Z-matrix. Note that the eigenvalues of A are $0, 1, -1$.

The following example shows that the converse of Theorem 2.4 does not hold. Consider the Z-matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose eigenvalues of A are $0, 1$, each with multiplicity two. Let $x = (2, 0, 1, 0)^T$. Then $0 \leq x = Ay \in R(A)$ where $y = (0, -2, 1, 0)^T$. Further, $x \circ Ax = (0, 0, -1, 0)^T \leq 0$. Hence A is not a strictly range semimonotone matrix.

Let us recall that (Theorem 5.1, [5]) all the real eigenvalues of a Z-matrix A are nonnegative if and only if all the principal minors of A are nonnegative. Therefore, combining this result with Theorem 2.4, we have the following.

Corollary 2.2. *Let A be a strictly range semimonotone Z-matrix. Then all the principal minors of A are nonnegative. In particular, all the diagonal entries of A are nonnegative.*

This corollary will be used in the proof of Theorem 3.2. For a diagonal matrix, the converse is also true and this will be proved in Corollary 3.3.

In the next result, we collect some properties of strictly range semimonotone matrices. First, we show that strictly range semimonotone matrices are permutation invariant. A strictly semimonotone Z-matrix (which is invertible) has the property that its inverse is strictly semimonotone, too. This statement appears to be new and follows from the next item, viz., (b) where we prove a general result for the group inverse. In (c) we obtain a generalization of the result for strictly semimonotone matrices mentioned in the introduction. For a real number λ, we write $\lambda^+ = \max\{\lambda, 0\}$ and $\lambda^- = \lambda^+ - \lambda$. Let $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. We define $x^+ = (x_1^+, x_2^+, \cdots, x_n^+)^T$ and $x^- = x^+ - x$. Then $x^+ \geq 0$ and $x^- \geq 0$. Let us also collect some pertinent details in connection with the group inverse of a matrix. Recall that the existence of the group inverse of a matrix A is characterized by the fact that the ranks of A and A^2 coincide. Another equivalent statement is the condition $N(A) = N(A^2)$. Clearly, it suffices to show that $N(A^2) \subseteq N(A)$, if one wishes to show that $A^{\#}$ exists. Another equivalent statement that will be used here is that $R(A)$ and $N(A)$ are complementary subspaces. Consequently, it follows that if A is *range symmetric* viz., $R(A) = R(A^T)$, then $A^{\#}$ exists. The following property of the group inverse is frequently used in the proofs: If $x \in R(A)$, then $x = AA^{\#}x$. We refer to the book [1] for more details.

Theorem 2.5. *Let $A \in \mathbb{R}^{n\times n}$. Then the following statements hold:*

(a) *If A is a strictly range semimonotone matrix and P is a permutation matrix, then P^TAP is a strictly range semimonotone matrix.*
(b) *Suppose that A is range symmetric. If A is a strictly range semimonotone matrix such that $A^{\#} \geq 0$, then $A^{\#}$ is a strictly range semimonotone matrix.*
(c) *A is a strictly range semimonotone matrix if and only if $SOL(A, q) \cap R(A) = \{0\}$ for all $q \in \mathbb{R}^n_+$.*

Proof. (a) Let $B = P^TAP$. Suppose that $0 \leq x \in R(B)$ and $x \circ Bx \leq 0$. We must show that $x = 0$. Since P is a permutation matrix, we have $P^T = P^{-1}$ and $P(\mathbb{R}^n_+) = \mathbb{R}^n_+$. Since $x \in R(B)$, there exists $y \in \mathbb{R}^n$ such that $x = P^TAPy$. Then $Px = APy$ so that $Px \in R(A)$. Since $x \geq 0$, we have $Px \geq 0$. Let $z \in \mathbb{R}^n_+$. Then there exists $w \in \mathbb{R}^n_+$ such that $Pw = z$. So, we have, $\langle x \circ Bx, w\rangle \leq 0$. Thus,

$$
\begin{aligned}
0 &\geq \langle P^TAPx, x \circ w\rangle \\
&= \langle APx, P(x \circ w)\rangle \\
&= \langle APx, Px \circ Pw\rangle \\
&= \langle APx, Px \circ z\rangle.
\end{aligned}
$$

Thus, $\langle Px \circ APx, z\rangle \leq 0$ for all $z \in \mathbb{R}^n_+$. This implies that $Px \circ APx \leq 0$. Since A is strictly range semimonotone, $Px = 0$ and hence $x = 0$. This completes the proof of (a).

(b): As A is range symmetric, $A^{\#}$ exists. Next, let $u \in R(A^{\#}) = R(A), u \geq 0$ and $u\circ(A^{\#}u) \leq 0$. Set $v = A^{\#}u$. Then $v \in R(A^T) = R(A)$ and $v \geq 0$. Also, $Av = AA^{\#}u = u$. Finally, $0 \geq u \circ (A^{\#}u) = (Av) \circ v = v \circ (Av)$. Since A is strictly range semimonotone, we have $v = 0$ and so $u = 0$. This shows that $A^{\#}$ is strictly range semimonotone.

(c): Let $q \in \mathbb{R}^n_+$. Clearly, $0 \in SOL(A, q) \cap R(A)$. Let $x \in SOL(A, q) \cap R(A)$. Then $0 \leq x \in R(A)$, $Ax + q \geq 0$ and $\langle x, Ax + q\rangle = 0$. This implies that $x \circ (Ax + q) = 0$ and hence $x \circ Ax = -(x \circ q) \leq 0$. Since A is strictly range semimonotone, we have $x = 0$.

Conversely, assume that $SOL(A, q) \cap R(A) = \{0\}$ for all $q \in \mathbb{R}^n_+$. Let $0 \leq x \in R(A)$ such that $x \circ Ax \leq 0$. We claim that $x = 0$. Take $q = (Ax)^+ - Ax = (Ax)^- \geq 0$. Since $x \geq 0$, we have $x = x^+ \in R(A)$. Now $A(x^+) + q = Ax + (Ax)^+ - Ax = (Ax)^+ \geq 0$. From $x\circ Ax \leq 0$, we have $x^+ \circ (Ax)^+ = 0$. This implies that $\langle x^+, (Ax)^+\rangle = \langle x^+, A(x^+) + q\rangle = 0$. Thus, x^+ is a solution of $LCP(A, q)$. By our assumption, $x^+ = 0$ and hence $x = 0$. This proves the result. □

Remark 2.2. We observe that a principal submatrix of a strictly range semimonotone property need not inherit that property. Consider the strictly range semimonotone matrix A of order 3 given in Remark 2.1 and the following (nonsingular) principal submatrix:

$B = \begin{pmatrix} -b & 0 \\ 0 & 1 \end{pmatrix}$ with $b > 0$. Then $x \circ Bx \leq 0$ for a nonzero $x = (1, 0)^T \geq 0$, showing that B is not strictly range semimonotone.

We conclude this subsection with a result that establishes a relationship between strictly range semimonotone matrices and linear complementarity problems.

Theorem 2.6. *Let $A \in \mathbb{R}^{n \times n}$. If A is strictly range semimonotone then $SOL(A, q) \cap R(A)$ is a (possibly empty) bounded set, for all $q \in \mathbb{R}^n$.*

Proof. Suppose for some $q \in \mathbb{R}^n$, $SOL(A, q) \cap R(A)$ is unbounded. Then there exists a sequence $\{x_k\}$ in $SOL(A, q) \cap R(A)$ such that $x_k \neq 0$ and $||x_k|| \to \infty$. Consider the sequence $\{y_k\}$ where $y_k = \frac{x_k}{||x_k||}$. Then $\{y_k\}$ has a convergent subsequence. Without loss of generality, assume that $\{y_k\}$ converges to $y \in R(A)$. Since $x_k \in SOL(A, q)$ and $||x_k|| \to \infty$, we have $0 \neq y \in SOL(A, 0)$ which contradicts (c) of Theorem 2.5. Hence $SOL(A, q) \cap R(A)$ is bounded for all $q \in \mathbb{R}^n$. $\square$

2.2. Range column sufficient matrices

Let us turn our attention to range column sufficient matrices. Let us recall that $A \in \mathbb{R}^{n \times n}$ is called a range column sufficient matrix if $x \in R(A),\ x \circ Ax \leq 0 \Longrightarrow x \circ Ax = 0$. As mentioned in the introduction, $A \in \mathbb{R}^{n \times n}$ is called a column sufficient matrix, if $x \circ Ax \leq 0 \Longrightarrow x \circ Ax = 0$. For range column sufficient matrices, we require that this implication holds only in the subspace $R(A)$. In the next result, we collect certain properties of range column sufficient matrices. First, we obtain results for range column sufficient matrices that are analogous to strictly range semimonotone matrices, as in Theorem 2.3. These are given in (a) and (b). We show in (c) that range column sufficient matrices are closed under the operation of group inversion. This appears to be new even for the subclass of invertible matrices. In (d), we obtain an analogue of the well known result that positive semidefinite matrices are column sufficient [4]. The next item, viz., (e) is motivated by a result for column sufficient matrices (Corollary 6.1, [14]), which states that if $\| S \| \leq 1$, then $I - S$ is column sufficient. Here, $\| \, . \, \|$ denotes the matrix norm induced by the 2-norm for vectors. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *positive semidefinite on* $R(A)$ if $\langle x, Ax \rangle \geq 0$ for all $x \in R(A)$.

Theorem 2.7. *Let $A \in \mathbb{R}^{n \times n}$. Then the following statements hold:*

(a) *Suppose that A is range column sufficient and e^i is a linear combination of the columns of A for some i, $1 \leq i \leq n$. If a_{ii} is non-zero, then it must be positive.*

(b) *Suppose that the vector e is a linear combination of the columns of a range column sufficient matrix A. If a row sum of A is non-zero, then at least one of the row sums of A is positive.*

(*c*) *Suppose that $A^{\#}$ exists. Then A is range column sufficient if and only if $A^{\#}$ is range column sufficient.*
(*d*) *Let A be positive semidefinite on $R(A)$. Then A is range column sufficient.*
(*e*) *Let A be such that $A^{\#}$ exists and $\| A \| \leq 1$. Then the matrix $B = A^{\#}A - A$ is range column sufficient.*

Proof. The proofs for (a) and (b) are similar to the proof of Theorem 2.3 and are skipped.

(c): Since $(A^{\#})^{\#} = A$, it suffices to prove the necessity part. Let $u \in R(A^{\#}) = R(A)$ and $u \circ A^{\#}u \leq 0$. Set $v = A^{\#}u$. Then $v \in R(A^{\#}) = R(A)$ and $u = AA^{\#}u = Av$ so that $v \circ Av = A^{\#}u \circ u \leq 0$. Since A has the range column sufficiency property, it then follows that $u \circ A^{\#}u = v \circ Av = 0$, showing that $A^{\#}$ has the range column sufficiency property.

(d): Let $x \in R(A)$ satisfy $x \circ Ax \leq 0$. Then $\langle x, Ax\rangle \leq 0$. Since A is positive semidefinite on $R(A)$, it follows that $\langle x, Ax\rangle \geq 0$. Hence $\langle x, Ax\rangle = 0$. This implies that $x \circ Ax = 0$ and hence A is range column sufficient.

(e): First, we observe that $R(A^{\#}A) = R(A)$ and so $R(B) \subseteq R(A)$. For $x \in R(B)$ we have

$$\begin{aligned}\langle x, Bx\rangle &= \langle x, A^{\#}Ax\rangle - \langle x, Ax\rangle \\ &= \langle x, x\rangle - \langle x, Ax\rangle,\end{aligned}$$

since $A^{\#}Ax = x$. By the Cauchy–Schwarz inequality, we then have

$$\langle x, Bx\rangle \geq \| x \|^2 - \| x \| \| Ax \| \geq (1 - \| A \|) \| x \|^2 .$$

This shows that B is positive semidefinite on $R(B)$. By (d) above, it follows that B is range column sufficient. □

Remark 2.3. A principal submatrix of a range column sufficient matrix need not be range column sufficient. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $R(A) = \{(\alpha, -\alpha, \beta)^T : \alpha, \beta \in \mathbb{R}\}$ and $\langle x, Ax\rangle \geq 0$ for all $x \in R(A)$. Hence A is positive semidefinite on $R(A)$. By item (d) of Theorem 2.7, A is range column sufficient. Consider the (invertible) principal submatrix $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ of A. Let $x = (1, 0)^T$. Then $0 \neq x \circ Bx = (-1, 0)^T \leq 0$, showing that B is not range column sufficient.

The next result presents a connection to linear complementarity problems.

Theorem 2.8. *Let $A \in \mathbb{R}^{n\times n}$. If A is range column sufficient then $SOL(A,q) \cap R(A)$ is a (possibly empty) convex set, for all $q \in \mathbb{R}^n$.*

Proof. Let $q \in \mathbb{R}^n$, $\lambda \in [0,1]$ and $x^1, x^2 \in SOL(A,q) \cap R(A)$. Then

$$0 \le \lambda x^1 + (1-\lambda)x^2 \in R(A)$$

and

$$A(\lambda x^1 + (1-\lambda)x^2) + q = \lambda(Ax^1 + q) + (1-\lambda)(Ax^2 + q) \ge 0.$$

To prove the result it is enough to show that

$$\langle x^1, Ax^2 + q\rangle = \langle x^2, Ax^1 + q\rangle = 0.$$

Consider

$$\begin{aligned}(x^1 - x^2) \circ A(x^1 - x^2) &= (x^1 - x^2) \circ ((Ax^1 + q) - (Ax^2 + q)) \\ &= -(x^1 \circ (Ax^2 + q) + x^2 \circ (Ax^1 + q)) \\ &\le 0.\end{aligned}$$

Since A is range column sufficient and $x^1 - x^2 \in R(A)$, we have

$$(x^1 - x^2) \circ A(x^1 - x^2) = 0.$$

Because $x^1, x^2, Ax^1 + q$ and $Ax^2 + q$ are nonnegative, we have

$$x^1 \circ (Ax^2 + q) = x^2 \circ (Ax^1 + q) = 0$$

and hence

$$\langle x^1, Ax^2 + q\rangle = \langle x^2, Ax^1 + q\rangle = 0.$$

This completes the proof. □

The proof of the following result is similar to that of item (c) in Theorem 2.5. So, we state it without proof.

Theorem 2.9. *If A is a range column sufficient matrix then $SOL(A,q) \cap R(A) = \{0\}$ for all $q > 0$.*

Remark 2.4. In general, the conclusion of Theorem 2.9 does not hold if $q \ge 0$ (and has a zero component). The following example shows that if A is range column sufficient then

$\mathrm{SOL}(A,q) \cap R(A)$ need not be equal to $\{0\}$ for some $q \geq 0$. Consider $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Then $R(A) = \{(\alpha, 0)^T : \alpha \in \mathbb{R}\}$ and $x^T Ax = 0$ for all $x \in R(A)$. Hence A is positive semidefinite on $R(A)$. By item (d) of Theorem 2.7, A is range column sufficient. Take $q = (0,1)^T$. Then $\{(\beta, 0)^T : \beta \geq 0\} \subseteq \mathrm{SOL}(A,q)$.

In general, a column sufficient matrix need not be strictly semimonotone. For example, the zero matrix is column sufficient but not strictly semimonotone. However, for an invertible Z-matrix A, it has been proved that A is column sufficient if and only if A is strictly semimonotone [4,14]. Further, they are equivalent to $A^{-1} \geq 0$. In the following, we consider a partial generalization of this result.

Theorem 2.10. *Let A be a Z-matrix which is also range monotone. Consider the following statements:*

(a) *A is range column sufficient.*
(b) $x \geq 0, x \in R(A)$ *and* $x \circ Ax \leq 0 \Longrightarrow x \circ Ax = 0$.
(c) *A is strictly range semimonotone.*

Then $(a) \Longrightarrow (b) \Longleftrightarrow (c)$.

Proof. The implications $(a) \Longrightarrow (b)$ and $(c) \Longrightarrow (b)$ are obvious.

$(b) \Longrightarrow (c)$: By item (c) of Theorem 2.5, it is enough to show that $SOL(A,q) \cap R(A) = \{0\}$ for all $q \in \mathbb{R}^n_+$. Let $q \in \mathbb{R}^n_+$ and $x = (x_1, x_2, \cdots, x_n)^T \in SOL(A,q) \cap R(A)$. Then $x \geq 0$ and $x \circ (Ax + q) = 0$. So $x \circ Ax = -(x \circ q) \leq 0$. By our assumption, we have $x \circ Ax = 0$. Let $y = Ax = (y_1, y_2, \cdots, y_n)^T$. If $x_i > 0$, then $y_i = 0$. On the other hand, if $x_i = 0$, then $\langle x, e^i \rangle = 0$. Since A is a Z-matrix, it follows that $y_i = \langle Ax, e^i \rangle \leq 0$. Thus $y \leq 0$. Already, $y \in R(A)$. As mentioned in the introduction, the range monotonicity of A ensures (that $A^{\#}$ exists and) that $x = A^{\#}Ax = A^{\#}y \leq 0$. This implies that $x = 0$, showing that A is strictly range semimonotone. □

Remark 2.5. We do not know whether the implication $(b) \Rightarrow (a)$ of the above theorem holds. Later in Theorem 3.3, we show that all the above conditions are equivalent for a normal Z-matrix.

2.3. Matrices satisfying "property c"

Next, we move on to the third class. A square matrix T is called *semiconvergent* if $lim_{k \to \infty} T^k$ exists. In [8], the author introduced the following subclass of M-matrices. An M-matrix A is said to have *"property c"* if A could be written as $A = sI - B$ for some $B \geq 0$ and $s > 0$ such that the matrix $T = \frac{1}{s}B$ is semiconvergent. Let A be an M-matrix. Then A has "property c" if and only if $A^{\#}$ exists (Theorem 1, [8]). The following result holds, also (Theorem 2) [8]. Let A be a Z-matrix. Then A is an M-matrix with "property c" if and only if $A^{\#}$ exists and that $A^{\#}$ is nonnegative on the range space of A. This last part means that the following implication holds:

$$x \in \mathbb{R}^n_+ \cap R(A) \Longrightarrow A^{\#}x \geq 0.$$

As mentioned in the introduction, this is equivalent to the range monotonicity of A. Let us underscore the importance of these matrices. Let T be the transition matrix for a Markov chain. Then the matrix $A = I - T$ is an M-matrix with "property c" (Theorem 8.4.2, [2]).

Remark 2.6. It is not known if the principal submatrices of an M-matrix with "property c" inherit that property.

2.4. $P_{\#}$-matrices

Finally, we take a relook at the fourth class of matrices that were introduced in [9]. We may recall that $A \in \mathbb{R}^{n\times n}$ is called a $P_{\#}$-matrix if for each nonzero vector $x \in R(A)$, there is an $i \in \{1, 2, \dots, n\}$ such that $x_i(Ax)_i > 0$. Equivalently, for any $x \in R(A)$, the inequalities $x_i(Ax)_i \leq 0$ for all $i = 1, 2, \dots, n$ imply $x = 0$. Using the Hadamard product, we may now paraphrase the above as follows: A is a $P_{\#}$-matrix if and only if

$$x \in R(A),\ x \circ Ax \leq 0 \Longrightarrow x = 0.$$

From this reformulation, it is now apparent that a $P_{\#}$-matrix is both strictly range semimonotone and range column sufficient. In Theorem 3.1, among other things we show that a $P_{\#}$-matrix which is also a Z-matrix, satisfies "property c". In the results to follow, we show that $P_{\#}$-matrices have certain properties that are analogous to P-matrices.

Theorem 2.11. *Let $A \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$ and A be a $P_{\#}$-matrix. We then have the following:*

(a) $A^{\#}$ exists and $A^{\#}$ is a $P_{\#}$-matrix.
(b) $LCP(A, q)$ has at most one solution in $R(A)$.

Proof. (a): Let $x \in R(A) \cap N(A)$. Then $x \circ Ax = 0$ so that $x = 0$. It now follows that $R(A)$ and $N(A)$ are complementary subspaces. Thus, $A^{\#}$ exists. Next, let

$$u \in R(A^{\#}) = R(A) \text{ and } u \circ A^{\#}u \leq 0.$$

Set $v = A^{\#}u$. Then $Av = AA^{\#}u = u$ and so one has

$$0 \geq u \circ A^{\#}u = Av \circ v.$$

We then have $v = 0$ so that $u \in N(A^{\#}) = N(A)$. This means that $u = 0$.

(b): Suppose that $x^1, x^2 \in R(A)$ are two solutions of $LCP(A, q)$. Now

$$\begin{aligned}(x^1 - x^2) \circ A(x^1 - x^2) &= (x^1 - x^2) \circ ((Ax^1 + q) - (Ax^2 + q)) \\ &= -(x^1 \circ (Ax^2 + q) + x^2 \circ (Ax^1 + q)) \leq 0.\end{aligned}$$

Since A is a $P_{\#}$-matrix and $x^1 - x^2 \in R(A)$, it follows that $x^1 = x^2$, completing the proof. □

Remark 2.7. It follows from the definition that a principal submatrix of a P-matrix is also a P-matrix. However, by means of an example, we show that such a property does not hold for $P_{\#}$-matrices. Consider

$$A = \begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $R(A) = \{(2\alpha, -\alpha, \beta)^T : \alpha, \beta \in \mathbb{R}\}$. Let $x \in R(A)$ such that $x \circ Ax \leq 0$. Then $x = 0$. Hence A is a $P_{\#}$-matrix. Consider a principal submatrix $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ of A. Since B is nonsingular, B is a $P_{\#}$-matrix if and only if B is a P-matrix, which it is not.

3. Range monotonicity of A

As mentioned in the introduction, the main objective is to study extensions of Theorem 1.1 for the group generalized inverse. Our results rally around the nonnegativity of the group inverse of A on its range space which, we have seen is the same as saying that A is range monotone. Matrices belonging to this class have been studied in [7,8]. First, we present certain necessary conditions and some other sufficient conditions for a matrix to be range monotone. The precise statements appear in Theorem 3.1. This is one of the main results of this article. In this result, the Perron–Frobenius theorem will be used, which we recall briefly. For more details we refer to [2]. Let $B \in \mathbb{R}^{n\times n}$ be such that $B \geq 0$. Then the spectral radius $\rho(B)$ of B is an eigenvalue of B and there is an eigenvector associated with $\rho(B)$ each of whose coordinates is nonnegative. Observe that the proof of the equivalence of (c) and (d) below is given in [8]. Nevertheless, a proof is provided both for the sake of completeness and for ready reference.

Theorem 3.1. *Let A be a Z-matrix. Consider the following statements:*

(a) *A is a $P_{\#}$-matrix.*
(b) *A is an M-matrix with "property c".*
(c) *A is range monotone.*
(d) *$A^{\#}$ exists and $A^{\#}x \geq 0$ whenever $x \in \mathbb{R}^n_+ \cap R(A)$.*

(e) *$Ax \leq 0$ and $x \in \mathbb{R}^n_+ \cap R(A) \Longrightarrow x = 0$.*
(f) *A is strictly range semimonotone.*

Then $(a) \Longrightarrow (b) \Longleftrightarrow (c) \Longleftrightarrow (d) \Longrightarrow (e) \Longrightarrow (f)$.

Proof. $(a) \Longrightarrow (b)$: Suppose that A is a $P_{\#}$-matrix. By (a) of Theorem 2.11, $A^{\#}$ exists. Since A is a Z-matrix, there exists $s > 0$ such that $A = sI - B$ where $B \geq 0$. We show that $s \geq \rho = \rho(B)$. By the Perron–Frobenius theorem, there exists $0 \neq z \in \mathbb{R}^n_+$ such that $Bz = \rho z$. Thus,

$$Az = (sI - B)z = sz - Bz = (s - \rho)z.$$

Suppose that $s < \rho$. Then $z \in R(A)$, $Az \leq 0$ and $z \circ Az \leq 0$, since $z \geq 0$. Since A is a $P_{\#}$-matrix, we then have $z = 0$, a contradiction. Hence $s \geq \rho$. This means that A is an M-matrix. Since $A^{\#}$ exists, by the result of [8] mentioned in Section 2.3, it follows that A has "property c".

$(b) \Longleftrightarrow (c)$: This follows from the discussion in Section 2.3.

$(c) \Longrightarrow (d)$: Let $A^2x = 0$. Set $y = Ax \in R(A)$ so that $Ay = 0$ and so $y \geq 0$. Replacing y by $-y$, we get $y \leq 0$. Hence $y = Ax = 0$ and so $N(A^2) \subseteq N(A)$. This shows that $A^{\#}$ exists. Let $x \geq 0$ and $x \in R(A)$. Set $z = A^{\#}x$. Then $z \in R(A)$ and $Az = AA^{\#}x = x \geq 0$. Thus $A^{\#}x = z \geq 0$, completing the proof.

$(d) \Longrightarrow (c)$: Let $y = Ax \geq 0$ and $x \in R(A)$. Then $y \in R(A)$ and $A^{\#}Ax = AA^{\#}x = x$. We have $0 \leq A^{\#}y = A^{\#}Ax = x$, proving (d).

$(d) \Longrightarrow (e)$: Let $x \in R(A)$, $x \geq 0$ and $Ax \leq 0$. Set $y = -x$. Then $y \in R(A)$ and $Ay = -Ax \geq 0$. By hypothesis, we have $0 \leq A^{\#}Ay = AA^{\#}y = y$. Thus $0 \leq y = -x$ and so $x \leq 0$, proving that $x = 0$.

$(e) \Longrightarrow (f)$: Let $x \in R(A)$, $x \geq 0$ and $x \circ Ax \leq 0$. If we set $y = Ax$, then $x \circ Ax \leq 0$ transforms into $x_i y_i \leq 0$ for each i. If $x_i > 0$, then $y_i \leq 0$. On the other hand, if $x_i = 0$, then $0 = x_i = \langle x, e^i \rangle$. Since A is a Z-matrix, we then have $y_i = \langle Ax, e^i \rangle \leq 0$. We have shown that $Ax \leq 0$. By (e), we then have $x = 0$, proving (f). □

The following consequence of Theorem 3.1, brings out the first relationship between singular M-matrices and linear complementarity problems.

Corollary 3.1. *Let A be a Z-matrix. If A is an M-matrix with "property c", then $SOL(A, q) \cap R(A) = \{0\}$ for all $q \geq 0$.*

Proof. From the implication $(b) \Longrightarrow (f)$ of Theorem 3.1, one has that A is strictly range semimonotone. By Theorem 2.5, the conclusion now follows. □

Remark 3.1. Consider A as in Remark 2.1 with $a \leq b$. Let $0 \neq y = (a, -b, 0)^T$. Then $y \in R(A)$ and $y \circ Ay = (a^2(a - b), b^2(a - b), 0)^T \leq 0$. Thus A is not a $P_{\#}$-matrix. This shows that a strictly range semimonotone matrix need not be a $P_{\#}$-matrix if it is not a Z-matrix.

Consider the case $a < b$. Then $R(A) \cap N(A) = \{0\}$ and hence $A^{\#}$ exists. On the other hand, if $a = b$, then $(1, -1, 0)^T \in R(A) \cap N(A)$ so that $A^{\#}$ does not exist. This shows that the group inverse of a strictly range semimonotone matrix may not exist, in general.

For matrices of order 2×2 and 3×3, condition (f) above implies (a) and so all the statements are equivalent. This is what is shown next. Hence, for matrices of order 2×2 and 3×3, all the statements of Theorem 3.1 are equivalent (see also Theorem 3.2). Let us also point out that for matrices of higher order, we do not know if this is true. However, for symmetric Z-matrices we show that all the statements of Theorem 3.1 are equivalent and this is presented in Corollary 3.2. While extending the proof of Theorem 3.2 to higher order matrices, the difficult part is to determine the sign of the components of Ax when the corresponding components of x are zero. Alternatively, for $x \in R(A)$, if we prove $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T \in R(A)$, then strictly range semimonotonicity implies $P_{\#}$-property. We do not know whether this condition holds.

Theorem 3.2. *Let A be a Z-matrix which is also a strictly range semimonotone matrix. If A is of order 2×2 or 3×3, then A is a $P_{\#}$-matrix.*

Proof. Let A be a strictly range semimonotone matrix of order 2×2. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

By Corollary 2.2, we have $a_{11} \geq 0$ and $a_{22} \geq 0$. Also, $a_{12} \leq 0$ and $a_{21} \leq 0$. We must show that A is a $P_{\#}$-matrix. Suppose that $x \in R(A)$ and $x \circ Ax \leq 0$. We must show that $x = 0$. Suppose that this is not the case. As A is strictly range semimonotone, we have $x \nleq 0$ and $x \ngeq 0$. Assume without loss of generality (by replacing x by $-x$, if need be) that $x_1 > 0$ and $x_2 < 0$. Let $y = (y_1, y_2)^T = Ax$. As $x \circ y \leq 0$, one has $x_1 y_1 \leq 0$ and since $x_1 > 0$ one has $y_1 \leq 0$. Since $y = Ax$, we have $y_1 = a_{11}x_1 + a_{12}x_2$. By the sign constraints, we have $a_{11}x_1 + a_{12}x_2 \geq 0$ with each term being *nonnegative* and so $y_1 \geq 0$. This means that $y_1 = 0$, which in turn means that $a_{11} = 0 = a_{12}$. Thus the first row of A is zero. Since $x \in R(A)$, it then follows that $x_1 = 0$, a contradiction. This completes the proof that A is a $P_{\#}$-matrix, in this case.

Let us consider the case of a strictly range semimonotone Z-matrix A of order 3×3. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Again, we have $a_{11} \geq 0, a_{22} \geq 0$ and $a_{33} \geq 0$ and all the off-diagonal entries being nonpositive. Suppose that $x \in R(A)$ and $x \circ Ax \leq 0$. We must show that $x = 0$. Suppose that $x \neq 0$. Then $x \nleq 0$ and $x \ngeq 0$ since A is strictly range semimonotone. Then there exists at least one i and at least one j such that $x_i > 0$ and $x_j < 0$.

Case (i): $x_1 = 0$. We may assume without loss of generality that $x_2 > 0$ and $x_3 < 0$. As before set $y = Ax$. Then $x_2 y_2 \leq 0$ and $x_3 y_3 \leq 0$. Then $y_2 \leq 0$ and $y_3 \geq 0$. Since $x_1 = 0$, we have $y_2 = a_{22}x_2 + a_{23}x_3 \geq 0$ with each term being *nonnegative*. This means that $y_2 = 0$ and so we have $a_{22} = 0 = a_{23}$. Also, $y_3 = a_{32}x_2 + a_{33}x_3 \leq 0$ with each term being *nonpositive*. This means that $y_3 = 0$ and so we have $a_{32} = 0 = a_{33}$. Thus the matrix A takes the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}.$$

Since $x \in R(A)$, we then have $x = Az$ for some $z \in \mathbb{R}^3$. Then $0 < x_2 = a_{21}z_1$ and $0 > x_3 = a_{31}z_1$. These inequalities do not hold simultaneously due to the fact that both the numbers a_{21} and a_{31} are nonpositive. Hence we arrive at a contradiction, in the case when $x_1 = 0$.

Case (ii): $x_1 \neq 0$. If either $x_2 = 0$ or $x_3 = 0$, we may proceed as in Case (i) to arrive at a contradiction. So, let us suppose that both x_2 and x_3 are nonzero. Without loss of generality, suppose that $x_1 > 0$. Then, we have $y_1 \leq 0$. Consider the subcase where $x_2 < 0$ and $x_3 < 0$. Now, $y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$, where each term is *nonnegative*, so that $y_1 \geq 0$. Thus $y_1 = 0$ and so we have $a_{11} = a_{12} = a_{13} = 0$. This means that the *first row* of A is zero so that $x_1 = 0$, a contradiction. Since $(-x) \circ A(-x) = x \circ Ax$, the remaining two subcases, viz., $x_2 > 0$, $x_3 < 0$ and $x_2 < 0$, $x_3 > 0$ could be analysed in a similar manner by replacing x with $-x$. This completes the proof when A is of order 3×3 and concludes the proof of the theorem. □

Remark 3.2. Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix. It is useful to observe that the analysis considered in the proof of Theorem 3.2 could be generalized to the case where x has one positive entry and $n - 1$ negative entries or vice versa. For, let $x_i > 0$ for some fixed i and $x_j < 0$ for all $j \neq i$. Since $x \circ Ax \leq 0$, we have $0 \geq (Ax)_i = \sum_{j=1}^{n} a_{ij}x_j$. Note that each term on the right hand sum is nonnegative and so it follows that $(Ax)_i = 0$. The requirement that $x \in R(A)$ forces $x_i = 0$, a contradiction.

We now discuss an important consequence of Theorem 3.1 for symmetric Z-matrices. This is an extension of Theorem 1.1, that we set out to achieve.

Corollary 3.2. *Let A be a symmetric Z-matrix. Then the following statements are equivalent:*

(a) *A is a $P_{\#}$-matrix.*
(b) *A is an M-matrix with "property c".*
(c) *A is range monotone.*

(*d*) $A^{\#}$ *exists and* $A^{\#}x \geq 0$ *whenever* $x \in \mathbb{R}^n_+ \cap R(A)$.
(*e*) $Ax \leq 0$ *and* $x \in \mathbb{R}^n_+ \cap R(A) \Longrightarrow x = 0$.
(*f*) A *is strictly range semimonotone.*

Proof. In view of Theorem 3.1, it is enough to show the implication $(f) \Longrightarrow (a)$. Suppose that A is strictly range semimonotone. Let $x \in R(A)$ such that $x \circ Ax \leq 0$. We claim that $x = 0$. Since A is symmetric, all the eigenvalues of A are real and $A^{\#}$ exists. By Theorem 2.4, all the eigenvalues of A are nonnegative. Therefore, A is a positive semidefinite matrix. That is, $y^T Ay \geq 0$ for all $y \in \mathbb{R}^n$. From $x \circ Ax \leq 0$, we have $x^T Ax \leq 0$. Thus $x^T Ax = 0$. Since A is positive semidefinite and symmetric, it follows that $Ax = 0$. Thus $x \in R(A) \cap N(A)$, so that $x = 0$. This completes the proof. □

In the following, we characterize $P_{\#}$-property for diagonal matrices. This shows that, for a diagonal matrix, the statements in Corollary 3.2 are equivalent to the condition that the diagonal entries are nonnegative.

Corollary 3.3. *Let* $D \in \mathbb{R}^{n\times n}$ *be a diagonal matrix with diagonal entries* $d_1, d_2, \cdots, d_n$. *Then* D *is a* $P_{\#}$ *matrix if and only if* $d_i \geq 0$ *for all* i.

Proof. The necessity part follows from Corollaries 2.2 and 3.2. To prove the sufficiency part, let us assume that $d_i \geq 0$ for all i. Let $x \in R(D)$ be such that $x \circ Dx \leq 0$. Since $x \in R(D)$, we have $x = Dy$ for some $y = (y_1, y_2, \cdots, y_n)^T$. The condition $x \circ Dx \leq 0$ implies that $d_i^3 y_i^2 \leq 0$. Since $d_i \geq 0$, we have $d_i^3 y_i^2 = 0$ and hence $d_i y_i = 0$. Thus $x = 0$. □

Let us recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *normal*, if $AA^T = A^T A$, where A^T denotes the transpose of A. A square complex matrix U is said to be *unitary* if $UU^* = I$, where U^* is the adjoint of U and I is the identity matrix. For two vectors $u = (u_1, u_2, \cdots, u_n)^T$ and $v = (v_1, v_2, \cdots, v_n)^T$ in $\mathbb{C}^n$, we denote the inner product by $\langle u, v\rangle_{\mathbb{C}} = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}$, where $\overline{z}$ is the complex conjugate of z.

We now study an interconnection between range column sufficient and range monotone matrices. For a normal matrix A, we show that range column sufficiency is equivalent to range monotonicity. In order to prove this equivalence, we need the following result. If A is a normal matrix which is also semipositive stable, then A is positive semidefinite (Lemma 5.1, [13]). For the sake of completeness and ready reference, we give a proof which is a modification of the proof of Theorem 2 in [11].

Lemma 3.1. *Let* $A \in \mathbb{R}^{n\times n}$ *be a normal matrix. If* A *is semipositive stable, then* A *is positive semidefinite.*

Proof. Suppose that A is semipositive stable. We show that $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$. Since A is normal, there exist a unitary matrix U and a diagonal matrix D such that $A = U^* DU$ (Theorem 2.5.4, [6]). Let $d_1, d_2, \cdots, d_n$ be the diagonal entries of D. Since

A is semipositive stable, $\mathrm{Re}(d_i) \geq 0$ for all i, where $\mathrm{Re}(d_i)$ is the real part of d_i. Let $x \in \mathbb{R}^n$ and $z = Ux = (z_1, z_2, \cdots, z_n)^T$. Then

$$\langle Ax, x\rangle = \langle Ax, x\rangle_{\mathbb{C}} = \langle DUx, Ux\rangle_{\mathbb{C}} = \langle Dz, z\rangle_{\mathbb{C}} = \sum_{i=1}^{n} d_i |z_i|^2.$$

Since $\langle Ax, x\rangle$ is real, we have

$$\langle Ax, x\rangle = \mathrm{Re}(\sum_{i=1}^{n} d_i |z_i|^2) = \sum_{i=1}^{n} |z_i|^2 \mathrm{Re}(d_i) \geq 0,$$

proving that A is positive semidefinite. □

Theorem 3.3. *Let A be a Z-matrix that is also normal. Then the following statements are equivalent:*

(a) *A is an M-matrix with "property c".*
(b) *A is range monotone.*
(c) *A is strictly range semimonotone.*
(d) *A is range column sufficient.*
(e) *$x \geq 0, x \in R(A)$ and $x \circ Ax \leq 0 \Longrightarrow x \circ Ax = 0$.*

Proof. The equivalence of (a) and (b) is the same as the equivalence of (b) and (c) in Theorem 3.1. The implication $(b) \Longrightarrow (c)$ follows from the implication $(c) \Longrightarrow (f)$ of Theorem 3.1. These implications hold even without the assumption of normality.

$(c) \Longrightarrow (d)$: Suppose that A is strictly range semimonotone. From Corollary 2.1, it follows that A is semipositive stable. Since A is normal, it follows that A is positive semidefinite, by Lemma 3.1. By item (d) of Theorem 2.7, we conclude that A is range column sufficient (A is even column sufficient).

$(d) \Longrightarrow (e)$: Trivial.

$(e) \Longrightarrow (a)$: Since A is normal, A is range symmetric and hence $A^{\#}$ exists (pp. 159, [1]). In view of a result of [8] mentioned in Section 2.3, it is enough to show that A is an M-matrix. However, the argument for this is similar to the implication $(a) \Longrightarrow (b)$ in Theorem 3.1. This completes the proof. □

Combining Corollary 3.2 and Theorem 3.3, we obtain the following consequence, whose proof is immediate.

Corollary 3.4. *Let A be a symmetric Z-matrix. Then any of the statements (a)–(f) of Corollary 3.2 is equivalent to the range column sufficiency of A.*

Remark 3.3. The following example shows that the equivalence between range column sufficient and range monotone matrices does not hold if we drop the assumption of normality. Consider a non-normal M-matrix $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. By Remark 2.4, A is range column sufficient. Let $x = (1,0)^T \geq 0$. Then $x \in R(A)$ and $x \circ Ax \leq 0$. Thus A is not strictly range semimonotone. From Theorem 3.2, it follows that A is not range monotone.

We also have the second connection between singular M-matrices and linear complementarity problems, as described next. This addresses the converse of Corollary 3.1 and is an important consequence of Theorem 3.3.

Corollary 3.5. *Let A be a Z-matrix that is also normal. Then A is an M-matrix with "property c" if and only if $SOL(A,q) \cap R(A) = \{0\}$ for all $q \geq 0$.*

Proof. Follows from (c) of Theorem 2.5 and Theorem 3.3. □

We conclude with an observation.

Remark 3.4. The following example shows that a strictly range semimonotone symmetric Z-matrix which must be a $P_{\#}$-matrix, need not necessarily have a nonnegative group inverse. Consider the symmetric Z-matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then A is strictly range semimonotone, but $A^{\#} = \frac{1}{4}A \ngeq 0$.

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