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On Lee association schemes over $\mathbb{Z}_4$ and their Terwilliger algebra

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A B S T R A C T

Let $F = \{0, 1, 2, 3\}$ and define the set $K = \{K_0, K_1, K_2\}$ of relations on F such that $(x, y) \in K_i$ if and only if $x - y \equiv \pm i \pmod 4$. Let n be a positive integer. We consider the Lee association scheme $L(n)$ over $\mathbb{Z}_4$ which is the extension of length n of the initial scheme (F, K). Let $\mathcal{T}$ denote the Terwilliger algebra of $L(n)$ with respect to the zero codeword of length n. We show that $\mathcal{T}$ is generated by a homomorphic image of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ and the center $Z(\mathcal{T})$. Furthermore, we determine the irreducible modules for $\mathcal{T}$ using the Schur–Weyl duality.

1. Introduction

Codes over $\mathbb{Z}_4$ are an active area of research. Hammons, Kumar, Calderbank, Sloane, and Solé [15] studied $\mathbb{Z}_4$-linear codes to understand via the Gray map the 'duality' of several families of nonlinear binary codes such as the Kerdock codes and the Preparata

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codes. Certain $\mathbb{Z}_4$-codes are also relevant to the study of vertex operator algebras (see [16] for example). The aim of this paper is to explore the algebraic structure of the space $\mathbb{Z}_4^n$ underlying the $\mathbb{Z}_4$-codes of length n.

Codes with the usual Hamming metric are vertex subsets of Hamming association schemes. Associated with a (symmetric) association scheme is a commutative $\mathbb{C}$-algebra, known as the *Bose–Mesner algebra*. The famous *linear programming bound* on the sizes of codes due to Delsarte [7] is based on the representation theory of the Bose–Mesner algebra. In the early 90's, Terwilliger [29–31] introduced a larger, noncommutative $\mathbb{C}$-algebra, called nowadays the *Terwilliger algebra*. Terwilliger used this algebra to the study of the structures of association schemes, but recently there have been several attempts to apply the representation theory of the Terwilliger algebra to coding theory. A prominent example is the *semidefinite programming bound* due to Schrijver [25]. Other examples include [9] and [27].

For $\mathbb{Z}_4$-codes of length n with the Lee metric, we consider the so-called *Lee association scheme* $L(n)$ with vertex set $\mathbb{Z}_4^n$. The structure of the Bose–Mesner algebra of $L(n)$ is known as seen in [12] or [22]. In this paper, we focus on the Terwilliger algebra of $L(n)$ and determine all of its irreducible modules. We plan to discuss applications of the Terwilliger algebra to $\mathbb{Z}_4$-codes in a future paper.

It is helpful to compare our results with some prior work on the Terwilliger algebras of Hamming association schemes. The theory of the Terwilliger algebra has been most successful when the association scheme is both *metric* and *cometric*, and Hamming association schemes possess these two properties. A famous theorem of Leonard [2, p. 263], [20], states that the class of metric and cometric association schemes characterizes the univariate Askey–Wilson orthogonal polynomials and some of its limiting cases. In particular, Hamming association schemes correspond to univariate Krawtchouk polynomials. Go [10] described the irreducible modules of the Terwilliger algebras of the binary Hamming association schemes. She showed (implicitly) that the Terwilliger algebras are in this case homomorphic images of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ of the rank one Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. This relationship between univariate Krawtchouk polynomials and $\mathfrak{sl}_2(\mathbb{C})$ is discussed further by Nomura and Terwilliger [23] at a more abstract linear algebraic level. On the other hand, from the result of Mizukawa and Tanaka [22] it follows that the Lee association scheme $L(n)$ corresponds to bivariate Krawtchouk polynomials, also known as *Rahman polynomials*. Recently, Iliev and Terwilliger [19] studied Rahman polynomials from the point of view of the rank two Lie algebra $\mathfrak{sl}_3(\mathbb{C})$; see also [18]. We show that there is indeed a homomorphism from $\mathcal{U}(\mathfrak{sl}_3(\mathbb{C}))$ to the Terwilliger algebra of $L(n)$, and that the latter is generated by this image together with the center. It follows that in this case every irreducible module of the Terwilliger algebra has the structure of an irreducible $\mathfrak{sl}_3(\mathbb{C})$-module. Our main results are Theorems 5.15, 5.16, and 5.17. The situation here turns out to be much more complicated than in the case of the binary Hamming schemes, and in proving our theorems we invoke several facts from the representation theory of the symmetric groups and the Lie algebras $\mathfrak{sl}_m(\mathbb{C})$.

This paper is organized as follows: In Section 2, we review basic concepts about symmetric association schemes and related algebras. In Section 3, we give a brief background on the representation theory of the symmetric groups and recall some known properties of the Specht modules. In Section 4, we recall some important results in representation theory particularly on the connection between Specht modules and irreducible $\mathfrak{sl}_m(\mathbb{C})$-modules. Also, we describe the irreducible $\mathfrak{sl}_m(\mathbb{C})$-modules from the points of view of highest weight theory and of Weyl modules. In Section 5, we prove our main results.

2. Preliminaries

In this section we briefly review some basic concepts concerning symmetric association schemes and related algebras. We advise the reader to refer to [2,4,6,11,21,29] for a more thorough discussion of the topic. Let (X, R) denote a symmetric association scheme with D classes. We call X the *vertex set* which is nonempty and finite, and $R = \{R_0, \dots, R_D\}$ the set of *associate classes*.

Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ of column vectors with coordinates indexed by X. For every $x \in X$, define $\widehat{x} \in V$ such that y-coordinate of $\widehat{x}$ is δ_{xy} for all $y \in X$. The set $\{\widehat{x} \mid x \in X\}$ forms an orthonormal basis for V with respect to the Hermitian inner product $\langle u, v\rangle = \bar{u}^t v$ for all $u, v \in V$ where $^-$denotes complex conjugate and ^{t} denotes transpose. We call V the *standard module* for (X, R). Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of all matrices over $\mathbb{C}$ with rows and columns indexed by X. Then $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication.

The *Bose–Mesner algebra* M is the commutative subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ with basis consisting of the *associate matrices* $A_0, A_1, \dots, A_D$ such that (x, y)-entries are given by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le i \le D$ and for all $x, y \in X$. Let I and J denote the identity and the all-one matrices in $\mathrm{Mat}_X(\mathbb{C})$, respectively. We note that $A_0 = I$ and $A_0 + A_1 + \cdots + A_D = J$. There exists a second basis for M consisting of the matrices $E_0 = |X|^{-1}J, E_1, \dots, E_D$ such that $E_0 + E_1 + \cdots + E_D = I$ and $E_iE_j = \delta_{ij}E_i$ for all $0 \le i, j \le D$. The matrices $E_0, E_1, \dots, E_D$ are called *primitive idempotents*.

Fix $x \in X$. We recall the *dual Bose–Mesner algebra* $M^*(x)$ *with respect to* x. Define diagonal matrices $E_i^* = E_i^*(x)$ and $A_i^* = A_i^*(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ for each integer $0 \le i \le D$ such that the (y, y)-entries are given by $(E_i^*)_{yy} = (A_i)_{xy}$ and $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for each $y \in X$. Then $M^*(x)$ is the commutative subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ that has two special bases, $\{E_0^*, E_1^*, \dots, E_D^*\}$ and $\{A_0^*, A_1^*, \dots, A_D^*\}$. The matrices $E_0^*, E_1^*, \dots, E_D^*$ are called *dual primitive idempotents with respect to* x and the matrices $A_0^*, A_1^*, \dots, A_D^*$ are called *dual associate matrices with respect to* x.

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ that is generated by M and $M^*(x)$. We call T the *Terwilliger algebra* of (X, R) with respect to x. Note that T is finite-dimensional and is semisimple since T is closed under the conjugate transpose map. On the standard module V, any two non-isomorphic irreducible modules for T are orthogonal. For every irreducible T-module $W \subseteq V$, define the sets

$$W_s = \{0 \leq i \leq D \mid E_i^* W \neq 0\} \text{ and } W_s^* = \{0 \leq i \leq D \mid E_i W \neq 0\}.$$

We call W_s, W_s^* the *support* and *dual support* of W, respectively. We say W is *thin* (resp. *dual thin*) if $\dim(E_i^* W) \leq 1$ for all i (resp. $\dim(E_j W) \leq 1$ for all j). There exists a unique irreducible module for T in V that is both thin and dual thin for which the support and the dual support are both equal to $\{0, 1, \ldots, D\}$. We call this the *primary T-module*. It has a basis consisting of the vectors $v_i = A_i \widehat{x} \in E_i^* V$ for each integer $0 \leq i \leq D$ and another basis consisting of the vectors $v_j^* = A_j^* \mathbf{1} \in E_j V$ for each integer $0 \leq j \leq D$ where $\mathbf{1}$ is the all-one vector in V.

Let (X, R) denote a symmetric association scheme with $R = \{R_0, R_1, \ldots, R_D\}$ and fix a positive integer n. To every pair $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n)$ of elements in $X^{[n]} := X \times \cdots \times X$ (n copies), we associate an ordered tuple $f(u, w) = (f_0(u, w), \ldots, f_D(u, w))$ of nonnegative integers such that

$$f_i(u, w) = |\{1 \leq j \leq n \mid (u_j, w_j) \in R_i\}|$$

for each integer $0 \leq i \leq D$. Define the set $R_\alpha = \{(u, w) \in X^{[n]} \times X^{[n]} \mid f(u, w) = \alpha\}$ for each ordered tuple $\alpha = (\alpha_0, \ldots, \alpha_D)$ of nonnegative integers such that $\alpha_0 + \cdots + \alpha_D = n$ and let $R^{[n]}$ denote the collection of all such R_α. It turns out that the pair $(X^{[n]}, R^{[n]})$ is a symmetric association scheme and is called the *extension of length n of the initial scheme* (X, R). See [7, Section 2.5], [12] and [22] for details and examples. We note that Hamming association schemes are extensions of one-class association schemes.

3. Specht modules

In this section we discuss irreducible modules of the symmetric groups which are called Specht modules. There are a lot of available references for this topic, for instance see [8,13,24,26].

Let k denote a positive integer. A *partition* of k is a tuple $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ of nonnegative integers such that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ and $\sum_{i=0}^{n-1} \lambda_i = k$. In symbols, we write $\lambda \vdash k$. We identify λ with $(\lambda_0, \ldots, \lambda_{n-1}, 0, \ldots, 0)$ so that length is immaterial. We say λ has l *parts* and write $\mathrm{part}(\lambda) = l$ if λ has l nonzero coordinates. We denote by $P(k, l)$ the set of all partitions of k with at most l parts. For convenience, define $P(0, l) := \{(0, \ldots, 0)\}$ for every positive integer l. To each partition $\lambda \vdash k$ we associate a *Ferrers diagram*, an arrangement of k boxes into rows and columns such that the ith row has λ_i boxes for every integer $i \geq 0$.

Definition 3.1. Let λ and ε denote partitions of k. We say that λ *dominates* ε and we write $\lambda \geq \varepsilon$ whenever $\lambda_0 + \cdots + \lambda_i \geq \varepsilon_0 + \cdots + \varepsilon_i$ for every nonnegative integer i. Observe that $\geq$ is a partial order on the set of all partitions of k.

Definition 3.2. Let $\lambda \vdash k$. A *Young tableau of shape* λ (or simply a λ*-tableau*) is an array $\mathtt{t}$ obtained by filling the boxes of the Ferrers diagram associated to λ with integers from 1 to k. A λ-tableau $\mathtt{t}$ is said to be *standard* if its entries are strictly increasing from left to right along each row and from top to bottom along each column. We denote by $\mathsf{STab}(\lambda)$ the set of all standard λ-tableaux.

Two λ-tableaux $\mathtt{t}, \mathtt{t}'$ are said to be *row equivalent* (resp. *column equivalent*) if the entries in the corresponding rows (resp. columns) of $\mathtt{t}$ and $\mathtt{t}'$ are the same up to permutation. Let $\mathfrak{S}_k$ denote the symmetric group on k objects. We observe that $\mathfrak{S}_k$ acts transitively on the set of all λ-tableaux by applying $\sigma \in \mathfrak{S}_k$ to the entries in the boxes. The result of the action of σ on the λ-tableau $\mathtt{t}$ will be denoted by $\sigma\mathtt{t}$. If $\mathtt{t}$ and $\sigma\mathtt{t}$ are row equivalent (resp. column equivalent), we say σ is a *row stabilizer (resp. column stabilizer)* of $\mathtt{t}$. Note that the set of all row stabilizers (resp. column stabilizers) of $\mathtt{t}$ forms a subgroup of $\mathfrak{S}_k$.

Definition 3.3. Let $\lambda \vdash k$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Denote by $R_\mathtt{t}$ and $C_\mathtt{t}$ the subgroups of $\mathfrak{S}_k$ consisting of row and column stabilizers of $\mathtt{t}$, respectively. Define an element $s_\mathtt{t}$ in the group algebra $\mathbb{C}[\mathfrak{S}_k]$ such that

$$s_\mathtt{t} = \sum_{\sigma \in C_\mathtt{t}} sgn(\sigma)\sigma \sum_{\psi \in R_\mathtt{t}} \psi$$

where sgn denotes the sign character of $\mathfrak{S}_k$.

We mention that $s_\mathtt{t}$ is proportional to an idempotent $g_\mathtt{t}$ (see [8, Lemma 5.13.3] or [13, Lemma 9.3.8]). We call $g_\mathtt{t}$ the *normalized Young symmetrizer* associated to $\mathtt{t}$.

Theorem 3.4. *Let $\lambda \vdash k$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Then the subspace $V_\lambda = \mathbb{C}[\mathfrak{S}_k]g_\mathtt{t}$ of $\mathbb{C}[\mathfrak{S}_k]$ is an irreducible module of $\mathbb{C}[\mathfrak{S}_k]$ under left multiplication and is independent of $\mathtt{t}$ up to isomorphism. Moreover, every irreducible module of $\mathbb{C}[\mathfrak{S}_k]$ is isomorphic to V_λ for a unique λ.*

Proof. See Theorem 5.12.2 and Section 5.13 of [8]. □

The spaces V_λ are called the *Specht modules* and the collection $\{V_\lambda \mid \lambda \vdash k\}$ forms a complete set of mutually non-isomorphic irreducible modules for $\mathbb{C}[\mathfrak{S}_k]$.

Remark 3.5. In other references such as [24] and [26], the Specht modules are defined in terms of polytabloids. In this setup, Theorem 3.4 is proven using the so-called submodule theorem (see [24, Theorem 2.4.4] or [26, Theorem 10.2.13]). In addition, the set of all polytabloids associated to a standard λ-tableau forms a basis for the Specht module V_λ (see [24, Theorem 2.5.2]). Thus $\dim(V_\lambda)$ is equal to the number of standard λ-tableaux. This quantity can be computed using known formulas which are as follows:

Consider the Ferrers diagram of a partition $\lambda \vdash k$ such that $\mathrm{part}(\lambda) = l$. To each box we associate an ordered pair (i, j) of nonnegative integers such that $0 \leq i \leq l - 1$ is its row position and $1 \leq j \leq \lambda_i$ is its column position. By the *hook-length* $h_\lambda(i, j)$ we mean the number of boxes (a, b) such that either $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. Then the dimension of V_λ is given by

$$\dim(V_\lambda) = \frac{k!}{\prod h_\lambda(i, j)}$$

where the product ranges to all boxes (i, j) in λ. This formula is called the *hook-length formula* and is proven by Frame, Robinson and Thrall in 1954 (see [24, Section 3.10]). The second one is due to Frobenius and Young, and is called the *determinantal formula*. To state the formula, we set $1/c! = 0$ if $c < 0$. Then the dimension of V_λ is given by

$$\dim(V_\lambda) = k! \det \left[\frac{1}{(\lambda_i - i + j)!} \right]_{i,j=0}^{l-1}.$$

This formula is much older than the hook-length formula (see [24, Section 3.11]).

4. $\mathfrak{sl}(V)$-modules

Throughout this section, let V denote an n-dimensional vector space over $\mathbb{C}$. The Lie algebra $\mathfrak{sl}(V)$ is the vector space over $\mathbb{C}$ of traceless linear operators on V together with the Lie bracket $[x, y] = xy - yx$ for all $x, y \in \mathfrak{sl}(V)$. Fixing an ordered basis for V means we may identify the linear operators on V with $n \times n$ matrices and write $\mathfrak{sl}(V) = \mathfrak{sl}_n(\mathbb{C})$. In this section we discuss the irreducible modules for $\mathfrak{sl}(V)$ from two points of view. The first is described as follows: For every $\sigma \in \mathfrak{S}_k$ and for vectors $v_1, \ldots, v_k \in V$, define $\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ so that $V^{\otimes k}$ becomes a module for $\mathbb{C}[\mathfrak{S}_k]$. Then for every $\lambda \in P(k, n)$ and for every $\mathtt{t} \in \mathsf{STab}(\lambda)$, the space $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}(V)$ and is called a *Weyl module*. The other one is by means of a theorem of the highest weight. This states that every irreducible $\mathfrak{sl}(V)$-module has a unique highest weight and two irreducible modules with the same highest weight are isomorphic. We establish the connection between these two points of view in the latter part of the section. The reader may refer to [8,13,14,17] for more background information.

Let $\mathfrak{gl}(V)$ denote the Lie algebra of linear operators on V with the usual Lie bracket and write $\mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbb{C})$ if an ordered basis for V is fixed. Let I denote the identity operator in $\mathfrak{gl}(V)$. For every $\sigma \in \mathfrak{S}_k$ and for operators $M_1, M_2, \ldots, M_k \in \mathfrak{gl}(V)$, define $\sigma(M_1 \otimes \cdots \otimes M_k) = M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(k)}$ so that $\mathfrak{gl}(V)$ acts on $V^{\otimes k}$ by

$$M(v_1 \otimes \cdots \otimes v_k) = \frac{1}{(k-1)!} \sum_{\sigma \in \mathfrak{S}_k} \sigma(M \otimes I \otimes \cdots \otimes I)(v_1 \otimes \cdots \otimes v_k) \tag{1}$$

for every $M \in \mathfrak{gl}(V)$. We see that the space $V^{\otimes k}$ supports a module structure for both the group algebra $\mathbb{C}[\mathfrak{S}_k]$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(V))$.

Theorem 4.1 *(Schur–Weyl duality). Let $\mathcal{A}$ and $\mathcal{B}$ denote the homomorphic images of $\mathbb{C}[\mathfrak{S}_k]$ and $\mathcal{U}(\mathfrak{gl}(V))$ in End$(V^{\otimes k})$, respectively. Then each of the following statements holds:*

i) *$\mathcal{A}$ and $\mathcal{B}$ are the centralizers of each other.*
ii) *$\mathcal{A}$ and $\mathcal{B}$ are semisimple and in particular $V^{\otimes k}$ is a semisimple $\mathfrak{gl}(V)$-module.*
iii) *$V^{\otimes k} = \bigoplus_{\lambda \vdash k} V_\lambda \otimes L_\lambda$ is a direct sum decomposition into modules for $\mathcal{A} \otimes \mathcal{B}$ where $\{V_\lambda\}$ are Specht modules and $\{L_\lambda\}$ are some non-isomorphic irreducible modules for $\mathfrak{gl}(V)$ or zero.*

Proof. This is found in [8, Theorem 5.18.4] which is in fact a consequence of the double centralizer theorem [8, Theorem 5.18.1]. □

According to Weyl character formula [8, Theorem 5.22.1], $\dim(L_\lambda)$ is zero if and only if part$(\lambda) > n$. Thus $V^{\otimes k}$ decomposes into irreducible $\mathfrak{gl}(V)$-modules

$$V^{\otimes k} = \bigoplus_{\lambda \in P(k,n)} V_\lambda \otimes L_\lambda \cong \bigoplus_{\lambda \in P(k,n)} \dim(V_\lambda)\ L_\lambda. \tag{2}$$

In other words, a complete set of mutually non-isomorphic irreducible $\mathfrak{gl}(V)$-modules on $V^{\otimes k}$ is in bijection with the set $P(k,n)$.

Lemma 4.2. *On the space $V^{\otimes k}$, a complete set of mutually non-isomorphic irreducible modules for $\mathfrak{gl}(V)$ is also a complete set of mutually non-isomorphic irreducible modules for $\mathfrak{sl}(V)$.*

Proof. Let W denote a subspace of $V^{\otimes k}$. Note that every linear operator $M \in \mathfrak{gl}(V)$ can be written as $M = S + cI$ where $S \in \mathfrak{sl}(V)$ and $c \in \mathbb{C}$. From this and (1), we find W is an irreducible $\mathfrak{sl}(V)$-module if and only if it is an irreducible $\mathfrak{gl}(V)$-module. Suppose W is a $\mathfrak{gl}(V)$-module. Let W' denote a $\mathfrak{gl}(V)$-module on $V^{\otimes k}$ and let $f : W \to W'$ be a vector space isomorphism. Then for all $w \in W$ we have

$$\begin{aligned} f(Mw) - Mf(w) &= f((S+cI)w) - (S+cI)f(w) \\ &= f(Sw + ckw) - (Sf(w) + ckf(w)) \\ &= f(Sw) - Sf(w). \end{aligned}$$

Hence f is an isomorphism of $\mathfrak{gl}(V)$-modules if and only if f is an isomorphism of $\mathfrak{sl}(V)$-modules. □

Lemma 4.3. *Let $\lambda \in P(k,n)$. Then for every standard λ-tableau $\mathtt{t}$, the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}(V)$ isomorphic to L_λ.*

Proof. Recall that $g_{\mathtt{t}}$ is an idempotent in $\mathbb{C}[\mathfrak{S}_k]$ and V_λ is isomorphic to $\mathbb{C}[\mathfrak{S}_k]g_{\mathtt{t}}$. Then $\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_k]}(V_\lambda, V^{\otimes k}) \cong g_{\mathtt{t}}(V^{\otimes k})$ by [8, Lemma 5.13.4]. On the other hand, we obtain $L_\lambda \cong \mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_k]}(V_\lambda, V^{\otimes k})$ by the double centralizer theorem. □

Remark 4.4. Fix an ordered basis $\{v_0, v_1, \ldots, v_{n-1}\}$ for V and identify $\mathfrak{sl}(V)$ and $\mathfrak{gl}(V)$ with $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$, respectively. Let $\mathfrak{h}$ denote the set of all complex diagonal matrices of the form $H = \mathrm{diag}(a_0, a_1, \ldots, a_{n-1})$ such that $\sum_{j=0}^{n-1} a_j = 0$. Recall that $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{sl}_n(\mathbb{C})$. For $H = \mathrm{diag}(a_0, \ldots, a_{n-1})$ and $H' = \mathrm{diag}(b_0, \ldots, b_{n-1})$, define an inner product on $\mathfrak{h}$ such that

$$\langle H, H' \rangle = \sum_{j=0}^{n-1} \overline{a_j} b_j.$$

For integers $0 \le r, s \le n-1$, let E_{rs} denote the matrix in $\mathfrak{gl}_n(\mathbb{C})$ that has a 1 on the (r,s)-entry and 0 on all other entries. Define the linear functional $\alpha_{rs} : \mathfrak{h} \to \mathbb{C}$ that sends $H \mapsto (a_r - a_s)$ for all $H \in \mathfrak{h}$. Then for integers $0 \le r, s \le n-1$ such that $r \ne s$ we find $E_{rs} \in \mathfrak{sl}_n(\mathbb{C})$ and

$$[H, E_{rs}] = (a_r - a_s)E_{rs} = \alpha_{rs}(H)E_{rs} \quad \forall H \in \mathfrak{h}.$$

We call α_{rs} a *root* of $\mathfrak{sl}_n(\mathbb{C})$ relative to the Cartan subalgebra $\mathfrak{h}$ with corresponding *root vector* E_{rs}. Clearly, $\alpha_{rs}(H) = \langle E_{rr} - E_{ss}, H \rangle$ for all $H \in \mathfrak{h}$ and thus we can transfer the roots to $\mathfrak{h}$ via the map $\alpha_{rs} \mapsto (E_{rr} - E_{ss})$. Let R denote the set of all roots and let E denote the $\mathbb{R}$-linear span of R. Then R forms a *root system* that is conventionally called A_{n-1}. We abbreviate $H_j = E_{jj} - E_{j+1,j+1}$ for each integer $0 \le j \le n-2$ so that $\{H_0, \ldots, H_{n-2}\}$ is a *base* for E. We comment about finite-dimensional modules for $\mathfrak{sl}_n(\mathbb{C})$. If W is a finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module, then W has a basis consisting of simultaneous eigenvectors for $\mathfrak{h}$. In fact, for each basis vector v there exists $\mu \in \mathfrak{h}$ such that $Hv = \langle \mu, H \rangle v$ for all $H \in \mathfrak{h}$. We call μ a *weight* in W with corresponding *weight vector* v. Recall that $\langle \mu, H_j \rangle$ is an integer for every $0 \le j \le n-2$. There exist elements $\omega_0, \omega_1, \ldots, \omega_{n-2} \in \mathfrak{h}$ called *fundamental weights* where $\langle \omega_i, H_j \rangle = \delta_{ij}$ for all integers $0 \le i, j \le n-2$. Consequently, the weight μ is written as $\mu = \sum_{j=0}^{n-2} \langle \mu, H_j \rangle \omega_j$. We say that a weight μ is *dominant* if $\langle \mu, H_j \rangle$ is nonnegative for all $0 \le j \le n-2$. Suppose μ, μ' are weights in W. We say that μ is *higher than* μ' if there exist nonnegative real numbers $c_0, c_1, \ldots, c_{n-2}$ such that

$$\mu - \mu' = \sum_{j=0}^{n-2} c_j H_j.$$

A weight λ occurring in W is said to be the *highest weight* if λ is higher than any other weight in W. The highest weight theory of irreducible $\mathfrak{sl}_n(\mathbb{C})$-modules states that every irreducible module has a unique highest weight and two irreducible modules are isomorphic if and only if they have the same highest weight. Suppose W is an irreducible $\mathfrak{sl}_n(\mathbb{C})$-module with highest weight λ. The *multiplicity* of the weight μ in W is the dimension of the μ-weight space in W. This quantity is determined using two known formulas: Freudenthal's formula (see [17, Section 22]) and Kostant's formula (see [17, Section 24]).

We resume our discussion on Weyl modules. Recall that $\{v_0, v_1, \dots, v_{n-1}\}$ is a fixed ordered basis for V. By a *simple tensor* we mean a vector $\beta = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k}$ such that $j_1, j_2, \dots, j_k \in \{0, 1, \dots, n-1\}$. For convenience, we view every simple tensor as an ordered k-tuple on $\{0, 1, \dots, n-1\}$. Let $\beta = (j_1, j_2, \dots, j_k)$ denote a simple tensor and define the ordered tuple $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1})$ such that $\varepsilon_r = |\{1 \le s \le k \mid j_s = r\}|$ for every integer $0 \le r \le n-1$. We call ε the *content* of β and we write $\mathsf{cont}(\beta) = \varepsilon$. Denote by $\mathsf{span}(\varepsilon)$ the subspace spanned by simple tensors β with $\mathsf{cont}(\beta) = \varepsilon$.

Definition 4.5. Let $\lambda \in P(k, n)$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Consider a simple tensor $\beta = (j_1, j_2, \dots, j_k)$ and define a substitution $\beta_{\mathtt{t}}$ such that for every integer $1 \le s \le k$ we write j_s in the box labeled s in $\mathtt{t}$. We say β is a $(\lambda, \mathtt{t})$-*semistandard simple tensor* if $\beta_{\mathtt{t}}$ satisfies each of the following conditions:

(i) the numbers are weakly increasing from left to right along each row, and
(ii) the numbers are strictly increasing from top to bottom along each column.

Lemma 4.6. *Let $\lambda \in P(k, n)$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Then the set of all vectors $g_{\mathtt{t}}(\beta)$ where β is a $(\lambda, \mathtt{t})$-semistandard simple tensor forms a basis for $g_{\mathtt{t}}(V^{\otimes k})$. Furthermore, the dimension of $g_{\mathtt{t}}(V^{\otimes k}) \cap \mathsf{span}(\varepsilon)$ is equal to the Kostka number $K_{\lambda,\varepsilon}$ which is the number of distinct $(\lambda, \mathtt{t})$-semistandard simple tensors with content ε.*

Proof. See [5, Theorem 8.11]. □

In view of Lemmas 4.3 and 4.6, the space $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}_n(\mathbb{C})$ with basis consisting of vectors $g_{\mathtt{t}}(\beta)$ such that β is a $(\lambda, \mathtt{t})$-semistandard simple tensor. Pick an arbitrary basis vector $g_{\mathtt{t}}(\beta)$ and suppose $\mathsf{cont}(\beta) = \varepsilon$ where $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{n-1})$. Let $H_0, \dots, H_{n-2}$ and $\omega_0, \dots, \omega_{n-2}$ denote elements of $\mathfrak{h}$ as described in Remark 4.4. Observe that $H_j g_{\mathtt{t}}(\beta) = (\varepsilon_j - \varepsilon_{j+1}) g_{\mathtt{t}}(\beta)$ for every integer $0 \le j \le n-2$ and therefore, ε is viewed as a weight via the map ω that sends $\varepsilon \mapsto \sum_{i=0}^{n-2} (\varepsilon_i - \varepsilon_{i+1}) \omega_i$.

Theorem 4.7. *Let $\lambda \in P(k, n)$ and let $\mathtt{t} \in \mathsf{STab}(\lambda)$. Then the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is an irreducible module for $\mathfrak{sl}_n(\mathbb{C})$ with highest weight $\omega(\lambda)$.*

Proof. The Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ is a highest weight cyclic representation of $\mathfrak{sl}_n(\mathbb{C})$ with weight $\omega(\lambda)$ in view of [14, Definition 7.16]. The theorem then follows immediately from [14, Proposition 7.17]. □

Corollary 4.8. *The set of all dominant weights occurring in the Weyl module* $g_{\mathtt{t}}(V^{\otimes k})$ *is* $\{\omega(\varepsilon) \mid \varepsilon \in P(k,n)$ *and* $\lambda \geq \varepsilon\}$.

Proof. See [5, Lemma 4.5]. □

Remark 4.9. The set of all weights occurring in the Weyl module $g_{\mathtt{t}}(V^{\otimes k})$ together with their corresponding multiplicities can be described as follows: Let $\varepsilon \in P(k,n)$ such that $\lambda \geq \varepsilon$. Thus, ε is a content occurring in $g_{\mathtt{t}}(V^{\otimes k})$ and in particular $\omega(\varepsilon)$ is a dominant weight. Suppose ε' is an n-tuple obtained by permuting the entries of ε. By applying the *Bender–Knuth involution* (see [3]) repeatedly, we see that there exists a bijection between the set of all $(\lambda, \mathtt{t})$-semistandard simple tensors with content ε and that of content ε'. Hence, $\omega(\varepsilon')$ is a weight occurring in $g_{\mathtt{t}}(V^{\otimes k})$ and the multiplicity of $\omega(\varepsilon')$ is equal to $K_{\lambda,\varepsilon}$.

5. Main result

Throughout the section, (F, K) denotes the symmetric association scheme with vertices $F = \{0, 1, 2, 3\}$ and associate classes $K = \{K_0, K_1, K_2\}$ such that $(x, y) \in K_i$ if and only if $x - y \equiv \pm i \pmod 4$. Let A_0, A_1, A_2 and E_0, E_1, E_2 denote the associate matrices and the primitive idempotents, respectively. Write the dual matrices as $A_i^* = A_i^*(0)$ and $E_i^* = E_i^*(0)$ for every integer $0 \leq i \leq 2$. Consider V the standard module for (F, K) and T the Terwilliger algebra of (F, K) with respect to vertex 0. Let W_0 denote the primary T-module with bases $\{v_0, v_1, v_2\}$ and $\{v_0^*, v_1^*, v_2^*\}$, and let W_1 denote the orthogonal complement of W_0 in V. Finally, let T_L denote the Lie algebra over $\mathbb{C}$ obtained by endowing T with the usual Lie bracket.

Definition 5.1. Fix a positive integer n. Let $L(n)$ denote the symmetric association scheme $(F^{[n]}, K^{[n]})$. We refer to $L(n)$ as the *Lee association scheme over* $\mathbb{Z}_4$.

Recall the Bose–Mesner algebra of $L(n)$. Let $(n-i-j, i, j)$ be an ordered triple of nonnegative integers and for each triple, define the matrices $\mathbb{A}_{ij}$ and $\mathbb{E}_{ij}$ in $\mathrm{Mat}_{F^{[n]}}(\mathbb{C})$ by

$$\mathbb{A}_{ij} = \frac{1}{(n-i-j)!i!j!} \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(A_0^{\otimes(n-i-j)} \otimes A_1^{\otimes i} \otimes A_2^{\otimes j}\right),$$

$$\mathbb{E}_{ij} = \frac{1}{(n-i-j)!i!j!} \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(E_0^{\otimes(n-i-j)} \otimes E_1^{\otimes i} \otimes E_2^{\otimes j}\right).$$

Then matrices $\mathbb{A}_{ij}$ are the associate matrices and the matrices $\mathbb{E}_{ij}$ are the primitive idempotents (see [12] or [22]). Now consider the dual Bose–Mesner algebra of $L(n)$ with respect to the zero codeword of length n. For every ordered triple $(n-i-j, i, j)$ of nonnegative integers, we define the diagonal matrices $\mathbb{E}^*_{ij}$ and $\mathbb{A}^*_{ij}$ in $\mathrm{Mat}_{F^{[n]}}(\mathbb{C})$ by

$$\mathbb{E}^*_{ij} = \frac{1}{(n-i-j)!i!j!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(E_0^{*\otimes(n-i-j)} \otimes E_1^{*\otimes i} \otimes E_2^{*\otimes j} \right),$$

$$\mathbb{A}^*_{ij} = \frac{1}{(n-i-j)!i!j!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(A_0^{*\otimes(n-i-j)} \otimes A_1^{*\otimes i} \otimes A_2^{*\otimes j} \right).$$

The matrices $\mathbb{A}^*_{ij}$ are the dual associate matrices and the matrices $\mathbb{E}^*_{ij}$, on the other hand, are the dual primitive idempotents.

Let $\mathcal{T}$ denote the Terwilliger algebra of $L(n)$ with respect to the zero codeword of length n. Recall that $\mathcal{T}$ is the subalgebra of $\mathrm{Mat}_{F^{[n]}}(\mathbb{C})$ generated by the associate and the dual associate matrices. In this section, we determine and describe the irreducible modules for the Terwilliger algebra $\mathcal{T}$. Our technique is inspired by [28].

Lemma 5.2. *With above notation, each of the following relations holds:*

$$\begin{aligned} 4E_1 &= 2A_0 - 2A_2, \\ 4E_2 &= A_0 - A_1 + A_2, \\ A_1^* &= 2E_0^* - 2E_2^*, \\ A_2^* &= E_0^* - E_1^* + E_2^*. \end{aligned}$$

Proof. Label the coordinates of the vectors in V, and the rows and columns of the matrices in $\mathrm{Mat}_F(\mathbb{C})$ with the natural ordering of the vertices in F. Let V_1 (resp. V_2) denote the eigenspace of A_1 corresponding to the eigenvalue $\theta_1 = 0$ (resp. $\theta_2 = -2$). Then V_1 has an orthonormal basis $\{x_1, x_2\}$ and V_2 has an orthonormal basis $\{x_3\}$ where

$$x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \; x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \text{ and } x_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Consequently,

$$\begin{aligned} E_1 &= x_1\bar{x}_1^t + x_2\bar{x}_2^t = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \\ E_2 &= x_3\bar{x}_3^t = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

These prove the first two equations of the lemma. The remaining equations follow immediately from the first two and the definition of dual matrices. □

Lemma 5.3. *With above notation, for $Q \in \{A_1, A_2, E_0^*, E_1^*, E_2^*\}$ the matrix representing $Q|_{W_0}$ with respect to the ordered basis $\{v_0, v_1, v_2\}$ and the matrix representing $Q|_{W_1}$ are given by*

Q	A_1	A_2	E_0^*	E_1^*	E_2^*
$Q\|_{W_0}$	$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
$Q\|_{W_1}$	$[0]$	$[-1]$	$[0]$	$[1]$	$[0]$

Proof. Routine. □

Lemma 5.4. *With above notation, for $Q \in \{A_1^*, A_2^*, E_1, E_2\}$ the matrix representing $Q|_{W_0}$ with respect to the ordered basis $\{v_0, v_1, v_2\}$ and the matrix representing $Q|_{W_1}$ are given by*

Q	A_1^*	A_2^*	E_1	E_2
$Q\|_{W_0}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$.
$Q\|_{W_1}$	$[0]$	$[-1]$	$[1]$	$[0]$

Proof. Follows immediately from Lemma 5.2 and Lemma 5.3. □

Lemma 5.5. *With above notation, the matrices*

$$E_0^*,\ E_1E_1^*,\ E_2^*,\ E_1^*A_2,\ E_0^*A_2,\ A_2E_0^*,\ E_0^*A_1,\ A_1E_0^*,\ E_1^*A_1E_2^*,\ E_2^*A_1E_1^*$$

form a basis for T.

Proof. Use Lemmas 5.3 and 5.4. □

Lemma 5.6. *With above notation, each of the following relations holds:*

$$\begin{aligned}
E_1^*A_2 &= \tfrac{1}{2}A_2 + \tfrac{1}{2}A_2^* - \tfrac{1}{16}[A_1,[A_1,A_2^*]],\\
E_0^*A_2 &= \tfrac{1}{4}A_2 - \tfrac{1}{4}A_2^* + \tfrac{1}{8}[A_1^*,A_2] + \tfrac{1}{32}[A_1,[A_1,A_2^*]],\\
E_0^*A_1 &= \tfrac{1}{4}A_1 - \tfrac{1}{8}[A_1,A_1^*] - \tfrac{1}{8}[A_1,A_2^*] + \tfrac{1}{16}[A_2,[A_1,A_1^*]],\\
E_1^*A_1E_2^* &= \tfrac{1}{4}A_1 - \tfrac{1}{8}[A_1,A_1^*] + \tfrac{1}{8}[A_1,A_2^*] - \tfrac{1}{16}[A_2,[A_1,A_1^*]],\\
E_0^* - E_1E_1^* &= \tfrac{3}{4}A_2^* + \tfrac{1}{4}A_2 + \tfrac{1}{4}A_1^* - \tfrac{1}{32}[A_1,[A_1,A_2^*]],\\
E_2^* - E_1E_1^* &= \tfrac{3}{4}A_2^* + \tfrac{1}{4}A_2 - \tfrac{1}{4}A_1^* - \tfrac{1}{32}[A_1,[A_1,A_2^*]].
\end{aligned}$$

Proof. Use Lemmas 5.3 and 5.4. □

Definition 5.7. Let I denote the identity matrix in T and let $\mathfrak{g}$ denote the Lie subalgebra of T_L consisting of matrices with trace 0. Define the unique element $\Phi \in \mathfrak{g}$ for which $\Phi|_{W_0}$ acts as identity on W_0 and $\Phi|_{W_1}$ acts as the scalar -3 on W_1.

Lemma 5.8. *With above notation, each of the following statements holds:*

i) *The Terwilliger algebra T is a direct sum of $\mathfrak{g}$ and $\mathbb{C}I$.*
ii) *The matrices*

$$E_0^* - E_1E_1^*,\ E_2^* - E_1E_1^*,\ E_1^*A_2,\ E_0^*A_2,\ A_2E_0^*,\ E_0^*A_1,\ A_1E_0^*,\ E_1^*A_1E_2^*,\ E_2^*A_1E_1^*$$

form a basis for $\mathfrak{g}$.
iii) *The Lie algebra $\mathfrak{g}$ is precisely the Lie subalgebra of T_L that is generated by A_1, A_2, A_1^* and A_2^*.*

Proof. If P is a matrix in $\mathfrak{g}$, then $P|_{W_1}$ acts as $-\text{trace}(P|_{W_0})$ on W_1. Thus, $\dim(\mathfrak{g}) = 9$ and (i) holds since $I \notin \mathfrak{g}$. Statement (ii) holds since each matrix has trace 0 and is linearly independent by Lemma 5.5. Statement (iii) follows immediately from (ii) and Lemma 5.6. □

Proposition 5.9. *With above notation, each of the following statements holds:*

i) *The map $\mathfrak{g} \to \mathfrak{gl}(W_0)$ that sends every matrix $P \in \mathfrak{g}$ to the restriction $P|_{W_0}$ is an isomorphism of Lie algebras.*
ii) *The Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is isomorphic to $\mathfrak{sl}(W_0)$.*

Proof. Let τ denote the map in (i). Note that τ is a Lie algebra homomorphism and the spaces $\mathfrak{g}$ and $\mathfrak{gl}(W_0)$ are of equal dimension. Thus, it suffices to show that $\text{Ker}\ \tau = \{0\}$. Suppose $P \in \text{Ker}\ \tau$. Then both P and $P|_{W_0}$ must have trace 0 and so must $P|_{W_1}$. Since $\dim(W_1) = 1$, the matrix P acts as 0 on W_1. This implies that P is the zero matrix. Statement (ii) follows immediately from (i). □

Lemma 5.10.

With above notation, each of the following statements holds:

i) *The Lie algebra $\mathfrak{g}$ is a direct sum of $[\mathfrak{g}, \mathfrak{g}]$ and $\mathbb{C}\Phi$.*
ii) *The Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is precisely the Lie subalgebra of T_L consisting of matrices P such that both $P|_{W_0}$ and $P|_{W_1}$ have trace 0.*

Proof. Let P and Q denote matrices in $\mathfrak{g}$ and observe that both $[P,Q]|_{W_0}$ and $[P,Q]|_{W_1}$ have trace 0. Thus $\Phi \notin [\mathfrak{g},\mathfrak{g}]$ and (i) holds by Proposition 5.9. Statement (ii) follows immediately from (i). □

Lemma 5.11. *With above notation, the Terwilliger algebra T is a direct sum of $[\mathfrak{g},\mathfrak{g}]$ and the center $Z(T)$ of T. In particular, $Z(T)$ is spanned by Φ and I.*

Proof. Observe that $\operatorname{span}\{\Phi, I\} \subseteq Z(T)$. By Lemmas 5.8(i) and 5.10(i), we obtain the direct sum $T = [\mathfrak{g},\mathfrak{g}] \oplus \operatorname{span}\{\Phi, I\}$. Suppose P is a matrix contained in $Z(T) \cap [\mathfrak{g},\mathfrak{g}]$. Then $P|_{W_0}$ has trace 0 and is a scalar multiple of the identity map in $\mathfrak{gl}(W_0)$. Hence, $P|_{W_0}$ is the zero map in $\mathfrak{gl}(W_0)$. Consequently, P is the zero matrix by the isomorphism in Proposition 5.9(i). □

Definition 5.12. Define the unique matrix $\Delta(P)$ in $\mathrm{Mat}_{F^{[n]}}(\mathbb{C})$ for every matrix $P \in T$ such that $\Delta(P)$ is given by

$$\Delta(P) = \frac{1}{(k-1)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(P \otimes I \otimes \cdots \otimes I).$$

Lemma 5.13. *With above notation, the Terwilliger algebra $\mathcal{T}$ has four generators namely $\Delta(A_1), \Delta(A_2), \Delta(A_1^*)$ and $\Delta(A_2^*)$.*

Proof. Let $\mathcal{M}$ denote the subalgebra of the Bose–Mesner algebra of $L(n)$ generated by $\Delta(A_1)$ and $\Delta(A_2)$. For nonnegative integers i,j such that $i+j \leq n$, observe that the matrix $\mathbb{A}_{ij}$ is in the expansion of $[\Delta(A_1)]^i\,[\Delta(A_2)]^j$ and hence, $\mathbb{A}_{ij} \in \mathcal{M}$ by induction on $i+j$. Thus $\mathcal{M}$ contains all the associate matrices and is in fact the Bose–Mesner algebra of $L(n)$. Similarly, we can prove that the subalgebra $\mathcal{M}^*$ generated by $\Delta(A_1^*)$ and $\Delta(A_2^*)$ is in fact the dual Bose–Mesner algebra $\mathcal{M}^*$ of $L(n)$ with respect to the zero codeword of length n. □

Endow $V^{\otimes n}$ with a T_L-module structure such that $P \in T$ acts as $\Delta(P)$ under left multiplication. Consequently, $V^{\otimes n}$ becomes a module for the Lie algebras $\mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}]$.

Proposition 5.14. *With above notation, suppose W is a nonzero subspace of $V^{\otimes n}$ such that Φ acts as a scalar on W. Then the following are equivalent:*

i) *W is an irreducible $\mathcal{T}$-module,*
ii) *W is an irreducible $\mathfrak{g}$-module,*
iii) *W is an irreducible $[\mathfrak{g},\mathfrak{g}]$-module.*

Proof. Immediate from Lemma 5.8(iii), Lemma 5.10(i) and Lemma 5.13. □

Theorem 5.15. *With above notation, there exists a unital algebra homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(W_0))$ to $\mathcal{T}$. Furthermore, $\mathcal{T}$ is generated by the image of $\mathcal{U}(\mathfrak{sl}(W_0))$ and the center $Z(\mathcal{T})$.*

Proof. View $\mathcal{T}$ as a Lie algebra with respect to the usual Lie bracket and consider the map $\mathfrak{g} \to \mathcal{T}$ such that $P \mapsto \Delta(P)$. Note that this map is a well-defined Lie algebra homomorphism (see Lemma 5.8(iii) and Lemma 5.13). Thus there exists a unique unital algebra homomorphism ρ from the universal enveloping algebra of $\mathfrak{g}$ to $\mathcal{T}$ and in particular, ρ is an epimorphism. The theorem follows from Proposition 5.9(ii) and Lemma 5.11. $\square$

Theorem 5.16. *With above notation,*

$$V^{\otimes n} \cong \bigoplus_{k=0}^{n} \bigoplus_{\lambda \in P(k,3)} \binom{n}{k} dim(V_\lambda)\ L_\lambda \otimes (W_1)^{\otimes(n-k)} \tag{3}$$

is a decomposition into irreducible modules for $\mathcal{T}$ such that $\{V_\lambda\}$ are Specht modules, and $\{L_\lambda\}$ are irreducible modules for $\mathfrak{sl}(W_0)$. Moreover, the summands $L_\lambda \otimes (W_1)^{\otimes(n-k)}$ and $L_{\lambda'} \otimes (W_1)^{\otimes(n-k')}$ are isomorphic $\mathcal{T}$-modules if and only if $(k, \lambda) = (k', \lambda')$.

Proof. First we prove the isomorphism (3). Recall that $V = W_0 \oplus W_1$ so we obtain the isomorphism

$$V^{\otimes n} \cong \bigoplus_{k=0}^{n} \binom{n}{k} (W_0)^{\otimes k} \otimes (W_1)^{\otimes(n-k)}.$$

We obtain (3) by applying (2) to $(W_0)^{\otimes k}$. Fix an ordered pair (k, λ) such that $0 \le k \le n$ is an integer and $\lambda \in P(k, 3)$. That the summand $L_\lambda \otimes (W_1)^{\otimes(n-k)}$ is an irreducible $[\mathfrak{g}, \mathfrak{g}]$-module follows from Proposition 5.9(ii) and Lemma 5.10(ii). Moreover, Φ acts on $L_\lambda \otimes (W_1)^{\otimes(n-k)}$ as a scalar multiplication by $4k-3n$. Hence, $L_\lambda \otimes (W_1)^{\otimes k}$ is an irreducible module for $\mathcal{T}$ by Proposition 5.14. The last statement follows from the action of $\Delta(\Phi)$ and Theorem 5.15. $\square$

We view $V^{\otimes n}$ as the standard module for $L(n)$ and observe that a complete set of mutually non-isomorphic irreducible modules for $\mathcal{T}$ on $V^{\otimes n}$ is in bijection with the set of all ordered pairs (k, λ) where k is an integer such that $0 \le k \le n$ and $\lambda \in P(k, 3)$. Recall that the support W_s (resp. dual support W_s^*) of an irreducible $\mathcal{T}$-module W is the set of all triples $(n - i - j, i, j)$ of nonnegative integers such that $\mathbb{E}^*_{ij} W \neq 0$ (resp. $\mathbb{E}_{ij} W \neq 0$).

Theorem 5.17. *With above notation, abbreviate $W = W_{(k,\lambda)} := L_\lambda \otimes (W_1)^{\otimes(n-k)}$ for a fixed integer $0 \le k \le n$ and a fixed partition $\lambda \in P(k, 3)$. Let $P(\lambda)$ denote the set of all partitions in $P(k, 3)$ that are dominated by λ. Then each of W_s and W_s^* is equal to*

$$\{(\mu_0, \mu_1 + n - k, \mu_2) \mid \mu = (\mu_0, \mu_1, \mu_2) \text{ is a permutation of } \varepsilon \text{ for some } \varepsilon \in P(\lambda)\}. \tag{4}$$

Moreover if $\mu = (\mu_0, \mu_1, \mu_2)$ is a permutation of ε for some $\varepsilon \in P(\lambda)$, then each of $\mathbb{E}^*_{ij}W$ and $\mathbb{E}_{ij}W$ has dimension $K_{\lambda,\varepsilon}$ when $i = \mu_1 + n - k$ and $j = \mu_2$.

Proof. We identify W with $g_{\mathtt{t}}(W_0^{\otimes k}) \otimes (W_1)^{\otimes(n-k)}$ for a fixed standard λ-tableau $\mathtt{t}$ in view of Lemma 4.3. Now we give a basis for W consisting of common eigenvectors for the dual primitive idempotents. Fix the ordered basis $\{v_0, v_1, v_2\}$ for W_0 and the basis $\{v\}$ for W_1. Then by Lemma 4.6, the set of all vectors $g_{\mathtt{t}}(\beta) \otimes v^{\otimes(n-k)}$ where β is $(\lambda, \mathtt{t})$-semistandard simple tensor forms a basis for W. Pick a $(\lambda, \mathtt{t})$-semistandard simple tensor β and suppose $\mathsf{cont}(\beta) = (\mu_0, \mu_1, \mu_2)$. Then for every triple $(n - i - j, i, j)$ of nonnegative integers, we obtain

$$\mathbb{E}^*_{ij}\left(g_{\mathtt{t}}(\beta) \otimes v^{\otimes(n-k)}\right) = \begin{cases} g_{\mathtt{t}}(\beta) \otimes v^{\otimes(n-k)} & \text{if } i = \mu_1 + n - k \text{ and } j = \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

We see that W_s is equal to the set (4) by Corollary 4.8 and Remark 4.9. Similarly we get W_s^* by using the ordered basis $\{v_0^*, v_1^*, v_2^*\}$ for W_0 and the basis $\{v\}$ for W_1. The last statement follows immediately from Lemma 4.6 and Remark 4.9. □

Remark 5.18. In [1], there is a method that describes how the multiplicities of the weights in the root system A_2 can be obtained. The set of all weights in an irreducible module for $\mathfrak{sl}_3(\mathbb{C})$ is then partitioned into layers such that weights lying on the same layer have the same multiplicities. In particular, the Kostka number $K_{\lambda,\varepsilon}$ mentioned in Theorem 5.17 only depends on which layer does ε belong. To further explain this, write $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ and let $P(\lambda)$ denote the set of all partitions in $P(k, 3)$ dominated by λ. Let r denote the largest nonnegative integer for which $\lambda^r := (\lambda_0 - r, \lambda_1, \lambda_2 + r) \in P(\lambda)$. Suppose $r > 0$. Then the sequence $\lambda^0, \lambda^1, \ldots, \lambda^r$ determines the layers and the Kostka number $K_{\lambda,\varepsilon}$ for each $\varepsilon \in P(\lambda)$ is given by

$$K_{\lambda,\varepsilon} = \begin{cases} s & \text{if } \lambda^{s-1} \geq \varepsilon \text{ and } \lambda^s \not\geq \varepsilon \text{ for some integer } 1 \leq s \leq r, \\ r + 1 & \text{if } \lambda^r \geq \varepsilon. \end{cases}$$

Suppose $r = 0$. In this case, there is exactly one layer and $K_{\lambda,\varepsilon} = 1$ for each $\varepsilon \in P(\lambda)$. Hence, the corresponding irreducible $\mathcal{T}$-module is both thin and dual thin.

Remark 5.19. Observe that the irreducible $\mathcal{T}$-module $W_{(k,\lambda)} = L_\lambda \otimes (W_1)^{\otimes(n-k)}$ for a positive integer $k \leq n$ and $\lambda \in P(k, 1)$ is isomorphic as an $\mathfrak{sl}_3(\mathbb{C})$-module to the vector space over $\mathbb{C}$ of homogeneous polynomials in mutually commuting indeterminates x, y, z with total degree k. On such an irreducible $\mathcal{T}$-module, we can show the relationship among the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, Rahman polynomials and rank two extension of Leonard pair as discussed in [19].

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