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Research paper

Bäcklund transformation classification, integrability and exact solutions to the generalized Burgers'–KdV equation☆

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ABSTRACT

This paper is concerned with the Bäcklund transformations (BTs) of the nonlinear evolution equations (NLEEs). Based on the homogeneous balance principle (HBP), the existence of the BT of the generalized Burgers'–KdV (B–KdV) equation is classified, then the BTs of the nonlinear equations are given. In general, the method can be used to construct BTs of the nonlinear evolution equations in polynomial form. Furthermore, the integrability and exact explicit solutions to the nonlinear equations are investigated.

1. Introduction

As is well known, the Bäcklund transformation (BT) is an intrinsic property shared by integrable system. Once a Bäcklund transformation of a nonlinear system is obtained, the complete integrability of the system is proved. Furthermore, based on the Bäcklund transformation, many other integrable properties of the system can be considered, such as the Hamiltonian structure, conservation laws (CLs) and soliton solutions. Hence, the Bäcklund transformations of nonlinear systems have long been and will continue to be one of the dominant themes in both nonlinear theory and mathematical physics due to its fundamental importance. In the past few decades, there are noticeable progress in this field, and various methods have been developed, such as the inverse scattering transformation (IST), Darboux transformation (DT) and Lax pair (LP) methods [1–8], the Lie symmetry analysis [9–12], Painlevé test [12–16], and so on.

In this paper, we investigate the Bäcklund transformations of the generalized Burgers'–KdV (B–KdV) equation as follows:

$$u_t + \alpha u^p u_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{1.1}$$

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where $u = u(x, t)$ denotes the unknown function with respect to the space variable x and time t, the parameters $\alpha \neq 0$, β and γ are all arbitrary constants, which denote the nonlinear, dissipative and dispersive coefficients, respectively, p is a positive integer.

We note that Eq. (1.1) is a nonlinear evolution equation (NLEE), which includes a lot of famous nonlinear evolution equations as its special cases. For example, if $p = 1$, then Eq. (1.1) is the classical B–KdV equation as follows

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = 0. \tag{1.2}$$

In particular, if $\beta = 0$, then this equation is the Korteweg–de Vries (KdV) equation

$$u_t + \alpha u u_x + \gamma u_{xxx} = 0. \tag{1.3}$$

If $p = 2$ and $\beta = 0$, then Eq. (1.1) is the modified Korteweg–de Vries (mKdV) equation

$$u_t + \alpha u^2 u_x + \gamma u_{xxx} = 0. \tag{1.4}$$

If $p = 1$ and $\gamma = 0$, then Eq. (1.1) is reduced to the Burgers' equation (BE)

$$u_t + \alpha u u_x + \beta u_{xx} = 0. \tag{1.5}$$

If $p = 2$ and $\gamma = 0$, then Eq. (1.1) is reduced to the modified Burgers' equation (mBE) [17]

$$u_t + \alpha u^2 u_x + \beta u_{xx} = 0, \tag{1.6}$$

and so on.

More generally, we shall consider the evolution equations in polynomial form as follows

$$u_t = P[u], \tag{1.7}$$

where $P[u] = P(x, t, u, u_x, \ldots)$ is a polynomial with respect to their variables.

The above nonlinear evolution equations are of great importance in nonlinear wave theory, integrable system and physical applications [1–7,17,18]. In [11,12], we studied the Bäcklund transformations and exact solutions to some NLEEs by the symmetry analysis method. Recently, the Bäcklund transformations and exact solutions to the other nonlinear equations are considered based on the Painlevé test [15,16]. It is known that the homogeneous balance principle (HBP) is a general and systematic method for dealing with exact explicit solutions to the nonlinear partial differential equations (NLPDEs) in polynomial forms [19,20], and the Bäcklund transformations of such nonlinear equations can be constructed also. However, for some other nonlinear equations such as the B–KdV and modified Burgers' Eqs. (1.2) and (1.6), the homogeneous balance principle is of no help. The main purpose of this paper is to classify the generalized B–KdV Eq. (1.1) for the first time, and give the Bäcklund transformations, exact solutions and integrability. More generally, the method can be used to construct the Bäcklund transformations of the generalized evolution Eq. (1.7).

The rest of this paper is organized as follows: In Section 2, based on the homogeneous balance principle, the existence of the Bäcklund transformations of the nonlinear Eq. (1.1) is classified, then the Bäcklund transformations of the nonlinear evolution equations are constructed. Especially, by construction a Bäcklund transformations of an auxiliary equation, the so called generalized Bäcklund transformation of the the modified Burgers' equation is provided for the first time. In Section 3, the integrability and exact explicit solutions to the nonlinear equations are investigated by using the Bäcklund transformations. In addition, a shock wave solution to the B–KdV equation is obtained by the homogeneous balance principle. Finally, we summarize our new finding and give some remarks in Section 4.

2. BT classification of the generalized B–KdV equation

In general, for a nonlinear system, the Bäcklund transformations can be derived from different procedures, such as the Painlevé test, Lax pairs, Darboux transformations, Hirota's bilinear method and symmetry analysis method, etc. In this section, we shall introduce a direct analytic method for constructing Bäcklund transformations of the above nonlinear equations, the idea of our method is based on the homogenous balance principle [18,19]. Generally, we suppose that Eq. (1.1) has a solution in the following form:

$$u = \frac{\partial^{m+n} f(h)}{\partial x^m \partial t^n} + v, \tag{2.1}$$

where m and n are nonnegative integers, $f = f(h)$ is a composite function with respect to f and $h = h(x, t)$, which to be determined later, $v = v(x, t)$ is a given solution to Eq. (1.1). If we determine the functions $f = f(h)$ and $h = h(x, t)$ (under some condition sometimes), then the Bäcklund transformation of the equation is obtained.

First of all, in view of the arbitrary constant γ in Eq. (1.1), we have the following cases:

(a) $\gamma \neq 0$; or
(b) (b) $\gamma = 0$.

In what follows, we discuss the above cases one by one.

2.1. $\gamma \neq 0$

In this case, by the HBP, we have

$$m = \frac{2}{p}, \quad n = 0. \tag{2.2}$$

So, there are two cases as follows:

(i) If $p = 1$, then we have $m = 2$; and
(ii) If $p = 2$, then we have $m = 1$.

That is to say, only the cases $p = 1$ and $p = 2$ such that Eq. (1.1) admits the possible Bäcklund transformation of the form (2.1).

(i) $p = 1$. In this case, Eq. (1.1) is the general B–KdV equation, i. e.,

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{2.3}$$

where $\alpha\gamma \neq 0$. In view of (2.1), we have a solution to Eq. (2.3) as follows:

$$u = f''(h)h_x^2 + f'(h)h_{xx} + v, \tag{2.4}$$

where $f = f(h)$ and $h = h(x, t)$ to be determined later, $v = v(x, t)$ is a given solution to Eq. (2.3).

Then, substituting (2.4) into Eq. (2.3) and comparing the coefficients with respect to h_x^5, we have the following ordinary differential equation (ODE):

$$\alpha f'' f''' + \gamma f^{(5)} = 0, \tag{2.5}$$

where $f' = df/dh$. Solving this ODE, we get a solution to it as follows

$$f(h) = \frac{12\gamma}{\alpha} \log|h|. \tag{2.6}$$

Substituting (2.6) into (2.4) yields

$$u = \frac{12\gamma}{\alpha}\left(\frac{h_{xx}}{h} - \frac{h_x^2}{h^2}\right) + v. \tag{2.7}$$

Furthermore, comparing the other coefficients with respect to h and its different derivatives, using the assumption that $v = v(x, t)$ be a given solution to Eq. (2.3), then yields

$$\begin{aligned}
&\beta f^{(4)} h_x^4 + f''' h_x[h_x(h_t + \alpha v h_x + \gamma h_{xxx}) + 3(2\beta h_x h_{xx} - \gamma h_{xx}^2 + \gamma h_x h_{xxx})] \\
&\quad + f''\left\{\frac{\partial}{\partial x}[h_x(h_t + \alpha v h_x + \gamma h_{xxx}) + 3(2\beta h_x h_{xx} - \gamma h_{xx}^2 + \gamma h_x h_{xxx})]\right. \\
&\quad \left. + h_x(h_{xt} + \alpha v h_{xx} + \gamma h_{xxxx}) - \beta(3h_{xx}^2 + 2h_x h_{xxx})\right\} \\
&\quad + f'\left[\frac{\partial}{\partial x}(h_{xt} + \alpha v h_{xx} + \gamma h_{xxxx}) + \beta h_{xxxx}\right] = 0.
\end{aligned} \tag{2.8}$$

In view of (2.8), we have the following subcases:

When $\beta \neq 0$, we have $f = 0$ or $h_x = 0$. From (2.4), we get $u \equiv v$, it is trivial. That is, Eq. (2.3), i. e., the general Benny Eq. (1.2) has not any Bäcklund transformation of the form (2.4) in this case.

When $\beta = 0$, Eq. (2.3) becomes the KdV Eq. (1.3). In this case, setting

$$h_x(h_t + \alpha v h_x + \gamma h_{xxx}) - 3\gamma(h_{xx}^2 - h_x h_{xxx}) = 0, \tag{2.9a}$$

$$h_{xt} + \alpha v h_{xx} + \gamma h_{xxxx} = 0. \tag{2.9b}$$

Thus, if $h = h(x, t)$ satisfies (2.9), then (2.7) is a Bäcklund transformation of Eq. (1.3). In other words, Eq. (1.3) admits a Bäcklund transformation (2.7) under the condition (2.9).

(ii) $p = 2$. In this case, Eq. (1.1) is of the form

$$u_t + \alpha u^2 u_x + \beta u_{xx} + \gamma u_{xxx} = 0. \tag{2.10}$$

In view of (2.1), we have a solution to Eq. (2.10) as follows:

$$u = f'(h)h_x + v, \tag{2.11}$$

where $f = f(h)$ and $h = h(x, t)$ to be determined later, $v = v(x, t)$ is a given solution to Eq. (2.10).

Then, substituting (2.11) into Eq. (2.10) and comparing the coefficients with respect to h_x^4, we have the following ODE:

$$\alpha f'^2 f'' + \gamma f^{(4)} = 0. \tag{2.12}$$

Solving this ODE with respect to h, we get a solution to it as follows:

$$f(h) = k\log|h|, \tag{2.13}$$

where $k = \pm\sqrt{-\frac{6\gamma}{\alpha}}$.

Substituting (2.13) into (2.11) yields

$$u = k\frac{h_x}{h} + v. \tag{2.14}$$

Furthermore, similar to the above case $p = 1$, we can get the condition as follows:

$$v = \frac{3\gamma}{k\alpha}\frac{h_{xx}}{h_x} + \frac{\beta}{k\alpha}, \tag{2.15a}$$

$$h_t + \alpha v^2 h_x + \beta h_{xx} + \gamma h_{xxx} = 0, \tag{2.15b}$$

where $k = \pm\sqrt{-\frac{6\gamma}{\alpha}}$. Thus, if $h = h(x,t)$ satisfies (2.15), then (2.14) is a Bäcklund transformation of Eq. (2.10). In other words, Eq. (2.10) admits a Bäcklund transformation (2.14) under the condition (2.15).

In particular, if $\beta = 0$, then Eq. (2.10) becomes the mKdV Eq. (1.4). So, in view of the above argument, we obtain the Bäcklund transformation of the mKdV Eq. (1.4) as follows:

$$u = k\frac{h_x}{h} + v, \tag{2.16}$$

with the condition

$$v = \frac{3\gamma}{k\alpha}\frac{h_{xx}}{h_x}, \tag{2.17a}$$

$$h_t + \alpha v^2 h_x + \gamma h_{xxx} = 0, \tag{2.17b}$$

where $k = \pm\sqrt{-\frac{6\gamma}{\alpha}}$.

2.2. $\gamma = 0$

Under this condition, Eq. (1.1) becomes the following nonlinear equation

$$u_t + \alpha u^p u_x + \beta u_{xx} = 0. \tag{2.18}$$

Similar to the subsection $\gamma \neq 0$, for this equation, by the HBP, we have

$$m = \frac{1}{p}, \quad n = 0. \tag{2.19}$$

So, we get $p = 1$ only, then $m = 1$.

That is to say, only the case $p = 1$ such that Eq. (2.18) admits the Bäcklund transformation of the form (2.1). In this case, Eq. (2.18) becomes the burgers' Eq. (1.5). In view of (2.1), we have a solution to Eq. (1.5) as follows:

$$u = f'(h)h_x + v, \tag{2.20}$$

where $f = f(h)$ and $h = h(x,t)$ to be determined later, $v = v(x,t)$ is a given solution to Eq. (1.5).

Then, substituting (2.20) into Eq. (1.5) and comparing the coefficients with respect to h_x^3, we have the following ODE:

$$\alpha f' f'' + \beta f''' = 0, \tag{2.21}$$

where $f' = df/dh$. Solving this ODE, we get a solution to it as follows

$$f(h) = \frac{2\beta}{\alpha}\log|h|. \tag{2.22}$$

Substituting (2.22) into (2.20) yields

$$u = \frac{2\beta}{\alpha}\frac{h_x}{h} + v. \tag{2.23}$$

Furthermore, comparing the other coefficients with respect to h and its different derivatives, using the assumption that $v = v(x,t)$ be a given solution to Eq. (1.5), then yields

$$\left(f'' h_x + f'\frac{\partial}{\partial x}\right)(h_t + \alpha v h_x + \beta h_{xx}) = 0. \tag{2.24}$$

Referring to (2.24), let $h_t + \alpha v h_x + \beta h_{xx} = 0$, then we obtain the Bäcklund transformation of the Burgers' Eq. (1.5) as follows:

$$u = \frac{2\beta}{\alpha}\frac{h_x}{h} + v, \tag{2.25a}$$

$$h_t + \alpha v h_x + \beta h_{xx} = 0. \tag{2.25b}$$

That is, if $v = v(x, t)$ is a solution to Eq. (1.5), so is (2.25a) under the condition (2.25b). Now, we make further discussion as follows.

First, letting $v = h$ and substituting it into (2.25b), then it becomes Eq. (1.5) with respect to h immediately. Thus, we obtain the Bäcklund transformation of the Burgers' Eq. (1.5) as follows:

$$u = \frac{2\beta}{\alpha}\frac{h_x}{h} + h. \tag{2.26}$$

On the other hand, $v = 0$ is a solution to Eq. (1.5) clearly. Substituting it into (2.25a) and (2.25b), we have

$$u = \frac{2\beta}{\alpha}\frac{h_x}{h}, \tag{2.27}$$

and

$$h_t + \beta h_{xx} = 0. \tag{2.28}$$

That is, the Burgers' Eq. (1.5) is transformed into the linear heat Eq. (2.28) through the transformation (2.27). In fact, these results are the same as the results given by the other methods [7,14].

2.3. Further discussion on $\gamma = 0$

As is well known, (2.27) is the famous Hopf-Cole transformation between Eq. (1.5) and the linear heat Eq. (2.28). Such an equation is called "C-integrable". On the other hand, in view of (2.19), we can see that only $p = 1$ such that $m = 1$ is a positive integer. In other words, if $p > 1$, then m is not an integer, so the homogenous balance principle for constructing Bäcklund transformation of the modified Burgers' Eq. (1.6) failed. For dealing with this case, letting

$$v = u^2, \tag{2.29}$$

and substituting it into Eq. (1.6), we get

$$2vv_t + 2\alpha v^2 v_x - \beta v_x^2 + 2\beta v v_{xx} = 0. \tag{2.30}$$

It is called the auxiliary equation of Eq. (1.6) sometimes.

For this equation, by the HBP, we have

$$m = 1, \quad n = 0. \tag{2.31}$$

So we suppose that Eq. (2.30) has a solution as follows:

$$v = f'(h)h_x + w, \tag{2.32}$$

where $f = f(h)$ and $h = h(x, t)$ to be determined later, $w = w(x, t)$ is a given solution to Eq. (2.30).

Then, substituting (2.32) into Eq. (2.30) and comparing the coefficients with respect to h_x^4, we have the following ODE:

$$2\alpha f'^2 f'' - \beta f''^2 + 2\beta f' f''' = 0, \tag{2.33}$$

where $f' = df/dh$. Solving this ODE, we get a solution to it as follows

$$f(h) = \frac{3\beta}{2\alpha}\log|h|. \tag{2.34}$$

Substituting (2.34) into (2.32) yields

$$v = \frac{3\beta}{2\alpha}\frac{h_x}{h} + w. \tag{2.35}$$

Furthermore, similar to the above case for $p = 1$, we can get the condition as follows:

$$w = -\frac{3h_t}{2\alpha h_x} - \frac{3\beta h_{xx}}{4\alpha h_x}, \tag{2.36a}$$

$$6\beta h_x h_{xt} - 4\alpha w h_x h_t - 4\alpha^2 w^2 h_x^2 + 10\alpha\beta w_x h_x^2 - 3\beta^2 h_{xx}^2 + 6\beta^2 h_x h_{xxx} = 0, \tag{2.36b}$$

$$h_x w_t + w h_{xt} + 2\alpha w w_x h_x + \alpha w^2 h_{xx} - \beta w_x h_{xx} + \beta h_x w_{xx} + \beta w h_{xxx} = 0. \tag{2.36c}$$

Thus, if $h = h(x, t)$ satisfies (2.36), then (2.35) is a Bäcklund transformation of Eq. (2.30). Furthermore, in view of (2.29), we have

$$u^2 = \frac{3\beta}{2\alpha}\frac{h_x}{h} + w, \tag{2.37}$$

where $h = h(x, t)$ satisfies (2.36). In this case, we can say that Eq. (1.6) admits a so called generalized Bäcklund transformation (GBT) under the condition (2.36).

Remark 2.1. In view of (2.7) and (2.9), we can see that the Bäcklund transformation of the KdV Eq. (1.3) can be written as follows

$$u = \frac{12\gamma}{\alpha}(\log h)_{xx} + v, \tag{2.38}$$

where $h = h(x, t)$ satisfies the following system:

$$h_x h_t + \alpha v h_x^2 + 4\gamma h_x h_{xxx} - 3\gamma h_{xx}^2 = 0, \tag{2.39a}$$

$$h_{xt} + \alpha v h_{xx} + \gamma h_{xxxx} = 0. \tag{2.39b}$$

(2.38) is also called the Darboux transformation, which plays a key role in solition theory and integrable system. Similarly, the Bäcklund transformations of mKdV Eq. (1.4) and Burgers' Eq. (1.5) can be written as the form

$$u = k(\log h)_x + v. \tag{2.40}$$

where $k = \pm\sqrt{-\frac{6\gamma}{\alpha}}$ and $k = \frac{2\beta}{\alpha}$, respectively.

Remark 2.2. We note that (2.37) is not the classical BT of the modified Burgers' Eq. (1.6), so it does not imply integrability of the equation. However, we can deal with the exact solutions to the equation based on the formula (see Section 3).

3. Integrability and explicit solutions

In Section 2, we consider the Bäcklund transformations of the nonlinear PDEs. In this section, we investigate the integrability and exact solutions to the nonlinear equations.

3.1. Exact solutions through the BTs

First, we deal with exact solutions to the equations based on the Bäcklund transformations.

Clearly, $v = 0$ is a solution to Eq. (1.3). In view of the Bäcklund transformation (2.7) of this equation with the condition (2.9), let $h_{xx} = 0$, solving system (2.9), we have $h = ax + b$. Substituting it into (2.7), we obtain the exact solution to Eq. (1.3) as follows

$$u(x, t) = -\frac{12a^2\gamma}{\alpha(ax + b)^2}, \tag{3.1}$$

where $a \neq 0$ and b are arbitrary constants.

Similarly, $v = 0$ is also a solution to Eq. (1.4). In view of the Bäcklund transformation (2.16) of this equation with the condition (2.17), so we have $h_{xx} = 0$. Substituting it into (2.17b) we get $h = ax + b$. Thus, we obtain the exact solution to Eq. (1.4) as follows

$$u(x, t) = \pm\sqrt{-\frac{6\gamma}{\alpha}}\frac{a}{ax + b}, \tag{3.2}$$

where $a \neq 0$ and b are arbitrary constants.

The exact solutions to Eq. (1.5) are studied extensively [9–12], so we omit it here. Now, we consider the modified Burgers' Eq. (1.6). Clearly, $w = 0$ is a solution to Eq. (1.6). In view of the Bäcklund transformation (2.35) of Eq. (2.30) with the condition (2.36), let $h_{xx} = 0$, solving system (2.36), we have $h = ax + b$. Substituting it (2.37), we obtain the exact solution to Eq. (1.6) as follows

$$u(x, t) = \pm\sqrt{\frac{3a\beta}{2\alpha(ax + b)}}, \tag{3.3}$$

where $a \neq 0$ and b are arbitrary constants.

Furthermore, we can consider the other types of solutions to the modified Burgers' Eq. (1.6) based on (2.37) and HBP, the details are omitted.

3.2. Exact solutions through the HBP

Now, we consider an important exact solution to the classical B–KdV Eq. (1.2), where $\alpha\gamma \neq 0$. Such a solution plays a significant role in modeling physical applications. Let $u(x, t) = u(\xi)$, where $\xi = x - ct$, c is the wave speed. Then substituting it into Eq. (1.2) and integrating once, we have

$$\frac{1}{2}\alpha u^2 + \beta u' + \gamma u'' - cu + k = 0, \tag{3.4}$$

where k is an integration constant.

First, setting

$$u = v + \frac{c + l}{\alpha}, \tag{3.5}$$

where $l = \sqrt{c^2 - 2\alpha k}$. Substituting it into Eq. (3.4), then we get

$$\frac{1}{2}\alpha v^2 + \beta v' + \gamma v'' + lv = 0, \tag{3.6}$$

where $v' = dv/d\xi$, l is given in (3.5).

Then, by HBP, we suppose that Eq. (3.6) has a solution as follows

$$v = v(\xi) = \frac{Ae^{a\xi}}{(1 + e^{b\xi})^2}, \tag{3.7}$$

where A, a and b are constants to be determined.

Substituting (3.7) into Eq. (3.6), and solving the determined equations, we have $a = 0$, $b = \frac{\beta}{5\gamma}$, $A = -\frac{12\beta^2}{25\alpha\gamma}$ and $l = \frac{6\beta^2}{25\gamma}$. So we get that a solution to Eq. (3.6) is $v = -\frac{12\beta^2}{25\alpha\gamma(1+e^{\frac{\beta}{5\gamma}\xi})^2}$. Thus, we obtain the exact solution to Eq. (1.2) as follows

$$u(x, t) = \frac{6\beta^2}{25\alpha\gamma}\left[1 \pm \sqrt{1 + \frac{625\alpha\gamma^2}{6\beta^4}k} - \frac{2}{\left(1 + e^{\frac{\beta}{5\gamma}(x-ct)}\right)^2}\right], \tag{3.8}$$

where $c = \pm\sqrt{\frac{36\beta^4}{625\gamma^2} + 2\alpha k}$ is the wave speed, and k is an arbitrary constant.

We note that it is a shock wave solution to the classical B–KdV Eq. (1.2) [20], and it plays a key role in physical applications. In addition, the B–KdV equation does not admit the soliton solutions since it is not integrable in the sense of homogeneous balance principle (see Section 3.3 for detail).

Similarly, we can give the exact solution to the modified Burgers' Eq. (1.6) through the HBP procedure, the details are omitted.

3.3. Integrability through the BT classification

From the above discussion, we can see that the procedure based on the homogeneous balance principle is direct and analytic, and it is feasible to construct Bäcklund transformations of the polynomial types of NLEEs (1.7) generally. If we can do so, then the integrability of the nonlinear equations is proved in the sense of HBP. Through the HBP procedure, the BTs of the KdV, mKdV and Burgers' equations are given explicitly. In fact, it is known that these equations are all complete integrable. However, the HBP method failed for constructing BTs of the B–KdV equation and the modified Burgers' equation, respectively. In the case, we prefer to assume that these equations are non-integrable. So we can say that the B–KdV equation and the modified Burgers' equation are not integrable in the sense of HBP (c.f. [16]).

4. Conclusion and remarks

In the current paper, we employ the analytic construction method based on the homogeneous balance principle for dealing with Bäcklund transformations of the nonlinear evolution Eq. (1.1) and its special cases, Eqs. (1.2) – (1.6). The Bäcklund transformations are constructed by the direct method explicitly, and the integrability of the equations are considered. Especially, by constructing Bäcklund transformation of the auxiliary equation, the so called generalized Bäcklund transformation of the modified Burgers' equation is given, and the exact solution is obtained. Furthermore, the shock wave solutions to the general B–KdV equation is provided.

We note that this analytic method is a direct procedure in some extent, and it is a feasible approach to constructing Bäcklund transformations for more generalized nonlinear equations in principle. If we can do so, then the integrability of the equations are proved simultaneously. Thus, the direct construction method implies a new idea for dealing with integrability of the nonlinear evolution equations.

In addition, such Bäcklund transformations play a significant role in further discussions. For example, based on the Bäcklund transformations, the exact solutions, soliton solutions and contact symmetries (see Remark 4.1 for detail) can be considered. These are all interesting and will be the subject of our future studies.

Remark 4.1. Generally, the Bäcklund transformation is a variable transform with respect to the dependent variables, which parallel to the contact transformation from the symmetry analysis point of view [6–12]. For example, if a transformation $u(x,t) \mapsto v(x,t)$ satisfies the system

$$u_x = f(x,t,v,v_x,v_t,\dots), \quad u_t = g(x,t,v,v_x,v_t,\dots), \tag{4.1}$$

then in view of the integrable condition (compatibility condition) $u_{xt} = u_{tx}$, we have

$$\Omega = f_t - g_x + f_v v_t - g_v v_x + (f_{v_x} - g_{v_t})v_{xt} + f_{v_t} v_{tt} - g_{v_x} v_{xx} = 0. \tag{4.2}$$

This integrable condition is satisfied in the following two cases, respectively. First, it is identically equal to zero, that is

$$f_{v_x} - g_{v_t} = f_{v_t} = g_{v_x} = 0, \quad f_t - g_x + f_v v_t - g_v v_x = 0, \tag{4.3}$$

which derives the contact transformation (associated with the contact symmetry). Second, from $\Omega = 0$ derives the so-called Monge–Ampere equation, which derives the Bäcklund transformation in general [6–10].

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