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Linear Preservers of Quadratic Operators

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Abstract. Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space H. In this paper, we get a complete classification of surjective linear maps on $\mathcal{B}(H)$ that preserve quadratic operators in both directions. An analogue result in the setting of finite-dimensional Banach spaces is given.

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1. Introduction

Throughout this paper, X denotes a complex Banach space with dimension greater than two, and $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X.

Let Λ be a subset of $\mathcal{B}(X)$. We shall say that a map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *preserves* Λ *in both directions* if for every $T \in \mathcal{B}(X)$:

$$T \in \Lambda \quad \text{if and only if} \quad \Phi(T) \in \Lambda.$$

Over the last decades, many mathematicians have been attracted by the so-called linear preserver problems that concern the question of characterizing the form of all linear transformations on matrix algebras, or linear bounded operator algebras, that leave invariant a certain subset, and numerous results revealing the algebraic structure of such maps are obtained. For some expositions on linear preserver problems, the reader is referred to [2,5–9,12] and the references therein.

Recall that an operator $T \in \mathcal{B}(X)$ is *quadratic* if there exist two complex numbers $a, b \in \mathbb{C}$, such that $(T - aI)(T - bI) = 0$. Let $\mathcal{A}_2(X)$ denote the set of all quadratic operators in $\mathcal{B}(X)$. It is worth mentioning that several subclasses of quadratic operators are studied in the literature in connection with linear preserver problems:

1. Square-zero operators: $T^2 = 0$, see [2,12];
2. Idempotent operators: $T^2 = T$, see [2,6,7];
3. Skew-idempotent operators: $T^2 = -T$, see [2,6,7];

4. Involution operators: $T^2 = I$, see [2,7].

The quadratic operators find many applications in applied linear algebra, and they was investigated by several mathematicians, see, for instance, [1,3,4,13].

Recently, linear maps preserving square-zero operators in both directions were studied in [2]. More precisely, it was shown that a surjective linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$, where H is an infinite-dimensional complex Hilbert space, preserves the square-zero operators in both directions if and only if it has one of the two following forms:

$$T \mapsto cATA^{-1} \quad \text{or} \quad T \mapsto cAT^{\mathrm{tr}}A^{-1},$$

where $c \in \mathbb{C}$ is non-zero, $A \in \mathcal{B}(H)$ is an invertible operator, and T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis of H. The same paper provides a complete description of all surjective linear maps on $\mathcal{B}(H)$ preserving in both directions, the set of all operators annihilated by a given complex polynomial P with $\deg(P) \geq 2$. Thus, in particular, they determined the structure of all surjective linear maps on $\mathcal{B}(H)$ preserving idempotent operators, or involution operators, in both directions.

It should be noted that in the special case of finite dimension, the linear preservers of square-zero matrices, or matrices annihilated by a given polynomial, are studied in [12] and [7], respectively.

Since square-zero operators, idempotent operators and involution operators are quadratic, the question arises as to what can we say about surjective linear maps preserving quadratic operators in both directions.

In this article, we give a concrete characterization of the linear surjective maps that preserve quadratic operators in both directions. The main results of the present paper can be stated as follows:

Theorem 1.1. *Let H be an infinite-dimensional complex Hilbert space, and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear map. Then, the following assertions are equivalent:*

1. *Φ preserves quadratic operators in both directions.*
2. *There exist a non-zero $c \in \mathbb{C}$, a linear functional ψ on $\mathcal{B}(H)$, and an invertible operator $A \in \mathcal{B}(H)$, such that either*

$$\Phi(T) = cATA^{-1} + \psi(T)I \quad \textit{for all } T \in \mathcal{B}(H),$$

or

$$\Phi(T) = cAT^{\mathrm{tr}}A^{-1} + \psi(T)I \quad \textit{for all } T \in \mathcal{B}(H).$$

In the finite-dimensional case, we have the following result:

Theorem 1.2. *Let $n \geq 3$, and let $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ be a surjective linear map. Then, the following assertions are equivalent:*

1. *Φ preserves quadratic matrices in both directions.*
2. *There exist a non-zero $c \in \mathbb{C}$, a linear form φ on $\mathcal{M}_n(\mathbb{C})$, and an invertible matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that either*

$$\Phi(T) = cUTU^{-1} + \varphi(T)I_n \quad \textit{for all } T \in \mathcal{M}_n(\mathbb{C}),$$

or

$$\Phi(T) = cUT^{\mathrm{tr}}U^{-1} + \varphi(T)I_n \quad \textit{for all } T \in \mathcal{M}_n(\mathbb{C}),$$

where I_n *denotes the* $n \times n$ *identity matrix.*

It should be noted that this theorem does not make sense for $n = 2$. Indeed, every 2×2 complex matrix is quadratic, and so every map on $\mathcal{M}_2(\mathbb{C})$ onto itself preserves quadratic matrices.

The paper is organized as follows. In Sect. 2, we establish some useful results on perturbation of quadratic operators. In particular, we determine the perturbation class of the quadratic operators. These results are needed for proving Theorems 1.1 and 1.2 in the last section.

2. $\mathcal{A}_2(X)$ Under Square-Zero Perturbations

Throughout the sequel, $\mathcal{N}_2(X)$ denotes the set of all square-zero operators in $\mathcal{B}(X)$, and $\mathcal{P}(X)$ denotes the set of all idempotent operators in $\mathcal{B}(X)$.

It is noteworthy that every quadratic operator $T \in \mathcal{B}(X)$ is a linear combination of the identity and either a square-zero operator or an idempotent operator depending on whether its spectrum $\sigma(T)$ consists of one or two points. Indeed, if T is quadratic, then $(T - aI)(T - bI) = 0$ for some $a, b \in \mathbb{C}$, and hence $\sigma(T) \subseteq \{a, b\}$. If $a = b$, then $N = T - aI$ is square-zero and $T = N + aI$; otherwise, $P = (b - a)^{-1}(T - aI)$ is idempotent and $T = (b - a)P + aI$. From this, we get easily that

$$\mathcal{A}_2(X) = (\mathbb{C}I + \mathcal{N}_2(X)) \cup (\mathbb{C}I + \mathbb{C}\mathcal{P}(X)). \tag{2.1}$$

It follows that $\alpha T + \beta I$ is quadratic for every $T \in \mathcal{A}_2(X)$ and $\alpha, \beta \in \mathbb{C}$.

The following example shows that $\mathcal{A}_2(X)$ is not stable under square-zero perturbations.

Example. Let X be a direct sum of two closed subspaces X_1 and X_2 such that $\dim X_1 = 3$. Consider the operators $T, S \in \mathcal{B}(X)$ given by $T_{|X_2} = S_{|X_2} = 0$

$$T_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

with respect to an arbitrary basis of X_1. Clearly, T and S are square-zero operators. However, $(T + S)^3 = 0$ and $(T + S)^2 \neq 0$. Consequently, the operator $T + S$ does not belong to $\mathcal{A}_2(X)$.

In the following proposition, we provide necessary and sufficient conditions for the linear combination of two square-zero operators to be in $\mathcal{A}_2(X)$.

Proposition 2.1. *Let* $N, M \in \mathcal{N}_2(X)$*. Then, the following assertions are equivalent:*

1. $a_1N + a_2M \in \mathcal{A}_2(X)$ *for all* $a_1, a_2 \in \mathbb{C}$.
2. $N + M \in \mathcal{A}_2(X)$.
3. *There exists* $c \in \mathbb{C}$*, such that* $NM + MN = cI$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Suppose that $N + M \in \mathcal{A}_2(X)$. We may assume, without loss of generality that $N + M \neq 0$. Let $a, b \in \mathbb{C}$ be such that $(N + M - aI)(N + M - bI) = 0$. It follows that

$$(a + b)N + (a + b)M - NM - MN - abI = 0. \tag{2.2}$$

Hence, it suffices to show that $a + b = 0$. Suppose to the contrary that $a + b \neq 0$. Then, the left and right multiplications by N in (2.2) give

$$(a + b)NM - NMN - abN = (a + b)MN - NMN - abN = 0. \tag{2.3}$$

This implies that $(a + b)NM = (a + b)MN$, and hence $NM = MN$. Thus, Eq. (2.3) becomes $(a+b)MN - abN = 0$, because $N^2 = 0$. Now, multiplying this equation by M, we get that $abNM = 0$. Clearly, these two last equations imply that $NM = 0$, and so $(N + M)^2 = 0$. Hence, it follows from (2.2) that $N + M = ab(a + b)^{-1}I$. This leads to a contradiction, since $(N + M)^2 = 0$ and $N + M \neq 0$.

(iii) $\Rightarrow$ (i). Assume that $NM+MN = cI$ for some $c \in \mathbb{C}$. Let $a_1, a_2 \in \mathbb{C}$, and let $\lambda \in \mathbb{C}$ be such that $\lambda^2 = a_1a_2c$. One can easily verify that

$$(a_1N + a_2M - \lambda I)(a_1N + a_2M + \lambda I) = 0,$$

and so $a_1N + a_2M$ is a quadratic operator. This completes the proof. $\square$

The following proposition gives necessary and sufficient conditions for the linear combination of a square-zero operator and an idempotent operator to be in $\mathcal{A}_2(X)$.

Proposition 2.2. *Let $N \in \mathcal{N}_2(X)$, $P \in \mathcal{P}(X)$, and $c \in \mathbb{C}$ be non-zero, such that $P \neq I$. Then, the following assertions are equivalent:*

1. *$bN + cP \in \mathcal{A}_2(X)$ for all $b \in \mathbb{C}$.*
2. *$N + cP \in \mathcal{A}_2(X)$.*
3. *There exists $a \in \mathbb{C}$, such that $N - NP - PN = ac^{-1}I$.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Assume that $N + cP \in \mathcal{A}_2(X)$. Clearly, $N + cP \neq 0$. Let $a, b \in \mathbb{C}$ be such that $(N + cP)^2 + b(N + cP) + aI = 0$. It follows that

$$c(c + b)P + cNP + cPN + bN + aI = 0. \tag{2.4}$$

Hence, it suffices to show that $b = -c$. Suppose to the contrary that $c+b \neq 0$. Then, the left and right multiplications by N in (2.4) give

$$c(c + b)NP + cNPN + aN = c(c + b)PN + cNPN + aN = 0. \tag{2.5}$$

This implies that $c(c + b)NP = c(c + b)PN$, and so $NP = PN$. Therefore, we can rewrite (2.4) as $c(c + b)P + 2cNP + bN + aI = 0$, so that

$$c(c + b)P + aI = -2cNP - bN.$$

The operator $-2cNP - bN$ is square-zero because $NP = PN$. Since $P \neq I$, we have $\sigma(P) = \{0, 1\}$, and so

$$\{a, c(c + b) + a\} = \sigma(c(c + b)P + aI) = \sigma(-2cNP - bN) = \{0\}.$$

Thus, $a = c = 0$, the desired contradiction.

(iii) $\Rightarrow$ (i). Assume that $N - NP - PN = ac^{-1}I$ for some $a \in \mathbb{C}$, and let $b \in \mathbb{C}$. One can easily verify that $(bN + cP)^2 - c(bN + cP) + abI = 0$, and so $bN + cP$ is a quadratic operator. This completes the proof. $\square$

For an operator $T \in \mathcal{B}(X)$, write $\ker(T)$ for its kernel and $\mathrm{ran}(T)$ for its range.

As a consequence of the previous propositions, we drive the following corollary:

Corollary 2.3. *Let $A \in \mathcal{A}_2(X)$. The following assertions are equivalent:*

1. *$A \in \mathbb{C}I + \mathcal{N}_2(X)$.*
2. *For every $T \in \mathcal{A}_2(X)$ satisfying $T + A \in \mathcal{A}_2(X)$, we have $T + 2A \in \mathcal{A}_2(X)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $A = N + aI$ where $N \in \mathcal{N}_2(X)$ and $a \in \mathbb{C}$. Let $T \in \mathcal{A}_2(X)$ satisfy $T + A \in \mathcal{A}_2(X)$. Assume first that $T \in \mathbb{C}I + \mathcal{N}_2(X)$. Then, we have $T = M + bI$ where $M \in \mathcal{N}_2(X)$ and $b \in \mathbb{C}$. Since $T + A = N + M + (a + b)I \in \mathcal{A}_2(X)$, we infer that $N + M$ is quadratic. Hence, Proposition 2.1 asserts that $2N + M$ is also quadratic, so that $T + 2A = 2N + M + (b + 2a)I \in \mathcal{A}_2(X)$. Similarly, if $T \in \mathbb{C}I + \mathbb{C}\mathcal{P}(X)$, using Proposition 2.2, we show as above that $T + 2A \in \mathcal{A}_2(X)$.

(ii) $\Rightarrow$ (i). Assume that $A \notin \mathbb{C}I + \mathcal{N}_2(X)$. Then, there exist two different $a_1, a_2 \in \mathbb{C}$, such that $(A - a_1I)(A - a_2I) = 0$. It follows that $X = X_1 \oplus X_2$, where $X_i = \ker(A - a_iI)$. We can assume, without loss of generality that $\dim X_1 \geq 2$. Let Y_1 and Y_2 be two closed non-trivial subspaces, such that $X_1 = Y_1 \oplus Y_2$. With respect to the decomposition $X = Y_1 \oplus Y_2 \oplus X_2$, we can write $A = a_1I \oplus a_1I \oplus a_2I$. Let $T \in \mathcal{B}(X)$ be given by $T = (a_2 - a_1)I \oplus 0 \oplus 0$. Clearly, T and $T + A$ are quadratic operators. However

$$T + 2A = (a_2 + a_1)I \oplus 2a_1I \oplus 2a_2I$$

is not a quadratic operator. This finishes the proof. $\square$

In the following proposition, we determine the perturbation class of the quadratic operators.

Proposition 2.4. *Let $A \in \mathcal{B}(X)$. The following assertions are equivalent:*

1. *$A \in \mathbb{C}I$.*
2. *For every $T \in \mathcal{A}_2(X)$, we have $T + A \in \mathcal{A}_2(X)$.*

Before proving this proposition, we need to establish the following lemma.

Let $z \in X$ and let $f \in X^*$ be non-zero, where X^* denotes the topological dual space. As usual, we denote by $z \otimes f$ the rank one operator given by $(z \otimes f)(x) = f(x)z$ for all $x \in X$. Note that such operator is quadratic and $\sigma(z \otimes f) = \{0, f(z)\}$. Moreover, $z \otimes f$ is square-zero if and only if $f(z) = 0$.

Lemma 2.5. *Let $N \in \mathcal{N}_2(X)$ be non-zero. Then, there exist two closed N-invariant subspaces X_1 and X_2, such that $X = X_1 \oplus X_2$, $\dim X_1$ is either three or four, and*

$$N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{or} \quad N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively.

Proof. We shall discuss two cases.

Case 1. Assume that N is a rank one operator. Put $N = y \otimes f$ with $f(y) = 0$. Let $x, z \in X$ be such that $f(x) = 1$, $f(z) = 0$ and $\{x, y, z\}$ is a linearly independent set. Since $X = \mathrm{Span}\{x, y, z\} + \ker(f)$, there exists a closed subspace $X_2 \subseteq \ker(f)$ such that $X = \mathrm{Span}\{x, y, z\} \oplus X_2$. Clearly, we have $N_{|X_2} = 0$. Setting $X_1 = \mathrm{Span}\{x, y, z\}$, we get that

$$N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case 2. Assume that $\dim \mathrm{ran}(N) \geq 2$. Then, there exist $x_1, x_2 \in X$ such that Nx_1 and Nx_2 are linearly independent. Since $N^2 = 0$, the vectors $\{x_1, Nx_1, x_2, Nx_2\}$ are linearly independent. Let $f_1, f_2 \in X^*$ be such that

$$\begin{cases} f_1(x_1) = f_2(x_2) = 0, & f_1(Nx_1) = f_2(Nx_2) = 1, \\ f_i(N^k x_j) = 0 & \text{for } 1 \leq i \neq j \leq 2 \text{ and } 0 \leq k \leq 1. \end{cases}$$

Consider the operator $P \in \mathcal{B}(X)$ defined by

$$P = x_1 \otimes f_1 N + Nx_1 \otimes f_1 + x_2 \otimes f_2 N + Nx_2 \otimes f_2.$$

One can easily check that $P^2 = P$ and $NP = PN = Nx_1 \otimes f_1 N + Nx_2 \otimes f_2 N$. Thus, $X = \ker(I - P) \oplus \ker(P)$, and $\ker(P)$ is a closed N-invariant subspace. On the other hand, we verify that $\ker(I - P) = \mathrm{Span}\{x_1, Nx_1, x_2, Nx_2\}$. Consequently, taking $X_1 = \mathrm{Span}\{x_1, Nx_1, x_2, Nx_2\}$ and $X_2 = \ker(P)$, we obtain that X_2 is N-invariant subspace and $X = X_1 \oplus X_2$. Since $N^2 = 0$, with respect to the basis $\{x_1, Nx_1, x_2, Nx_2\}$ of X_1, we get that

$$N_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This completes the proof. $\square$

Proof of Proposition 2.4. (i) $\Rightarrow$ (ii) follows immediately from (2.1).

(ii) $\Rightarrow$ (i). Suppose that $T + A \in \mathcal{A}_2(X)$ for every $T \in \mathcal{A}_2(X)$ and that $A \notin \mathbb{C}I$. Then, A is quadratic, and so $(A - aI)(A - bI) = 0$ for some $a, b \in \mathbb{C}$. We shall distinguish two cases.

Case 1. Assume that $a = b$. Then, $(A - aI)^2 = 0$ and $A - aI \neq 0$. Without loss of generality, we may assume that $a = 0$. It follows from the previous lemma that X is a direct sum of two closed A-invariant subspaces X_1 and X_2 such that $\dim X_1$ is either three or four, and

$$A_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad A_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively. Consider the operator $T \in \mathcal{B}(X)$ given by $T_{|X_2} = 0$ and

$$T_{|X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad T_{|X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. Clearly, we have $T^2 = (T + A)^4 = 0$ and $(T + A)^2 \neq 0$. Thus, $T \in \mathcal{A}_2(X)$ and $T + A \notin \mathcal{A}_2(X)$, a contradiction.

Case 2. Assume that $a \neq b$. It follows that $X = \ker(A - aI) \oplus \ker(A - bI)$. Without loss of generality, we can assume that $\dim \ker(A - aI) \geq 2$. Let $\ker(A - aI)$ be a direct sum of two closed non-trivial subspaces L_1 and L_2. With respect to the decomposition $X = L_1 \oplus L_2 \oplus \ker(A - bI)$, we have $A = aI \oplus aI \oplus bI$. Let $S \in \mathcal{B}(X)$ be given by $S = 0 \oplus 2(a - b)I \oplus 2(a - b)I$. Clearly, $S(S - 2(a - b)I) = 0$, and so S is a quadratic operator. However, since

$$S + A = aI \oplus (3a - 2b)I \oplus (2a - b)I$$

we get that $\sigma(S + A) = \{a, 3a - 2b, 2a - b\}$. This spectrum contains more than two points because $a \neq b$, and hence $S + A$ is not quadratic. This contradiction completes the proof. $\square$

3. Proof of Main Theorems

As a consequence of Proposition 2.4 and Corollary 2.3, we have the following result:

Corollary 3.1. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective linear map preserving $\mathcal{A}_2(X)$ in both directions. Then*

1. *$\Phi(I) = \alpha I$ where $\alpha \in \mathbb{C}$ is non-zero.*
2. *Φ is injective.*
3. *Φ preserves $\mathbb{C}I + \mathcal{N}_2(X)$ in both directions.*

Proof. (i) Since Φ is surjective, there exists $A \in \mathcal{B}(X)$ such that $\Phi(A) = I$. Let T be a bounded quadratic operator on X. Since $\Phi(T)$ is also quadratic, then so is $\Phi(T) + I = \Phi(T + A)$. Therefore, the operator $T + A$ is quadratic. Now, Proposition 2.4 infers that $A = cI$ for some non-zero complex number $c \in \mathbb{C}$. Thus, $\Phi(I) = c^{-1}I$.

(ii) Let $F \in \mathcal{B}(X)$ be such that $\Phi(F) = 0$. Then, arguing as above, we obtain that $F \in \mathbb{C}I$. Since $\Phi(I) \neq 0$, then $F = 0$, and so Φ is injective.

(iii) First observe that Φ is bijective and Φ^{-1} satisfies the same properties as Φ. For this reason, it suffices to establish that $\Phi(A) \in \mathbb{C}I + \mathcal{N}_2(X)$

whenever $A \in \mathbb{C}I + \mathcal{N}_2(X)$. Let $A \in \mathbb{C}I + \mathcal{N}_2(X)$, and let $T \in \mathcal{B}(X)$ be a quadratic operator, such that $T + \Phi(A)$ remains quadratic. Then, $\Phi^{-1}(T)$ and $\Phi^{-1}(T) + A$ are quadratic. Thus, we obtain by Corollary 2.3 that $\Phi^{-1}(T) + 2A$ is also quadratic, and hence so is the operator $T + 2\Phi(A)$. Therefore, it follows again by Corollary 2.3 that $\Phi(A) \in \mathbb{C}I + \mathcal{N}_2(X)$. This completes the proof. $\square$

Note that every operator in $\mathbb{C}I + \mathcal{N}_2(X)$ has a unique representation as a linear combination of the identity and a square-zero operator. Indeed, let $N, M \in \mathcal{N}_2(X)$ and $a, b \in \mathbb{C}$ be such that $N + aI = M + bI$. Then, $\{a\} = \sigma(N + aI) = \sigma(M + bI) = \{b\}$, and so $N = M$. However, the representation of elements in $\mathbb{C}I + \mathbb{C}\mathcal{P}(X)$ as linear combination of the identity and an idempotent operator is not unique. Indeed, if $a, b \in \mathbb{C}$ and $P \in \mathcal{P}(X)$, then $I - P$ is idempotent and $aP + bI = -a(I - P) + (a + b)I$.

With these results at hand, we are ready to prove our main theorems.

Proof of Theorem 1.1. (i) $\Rightarrow$ (ii). Suppose that Φ preserves $\mathcal{A}_2(H)$ in both directions. Then, Corollary 3.1 ensures that $\Phi(I) = \alpha I$ where $\alpha \in \mathbb{C}$ is non-zero, Φ is bijective and preserves $\mathbb{C}I + \mathcal{N}_2(H)$ in both directions. In particular, for every $N \in \mathcal{N}_2(H)$ there exist a unique $M \in \mathcal{N}_2(H)$ and a unique $c_N \in \mathbb{C}$ such that $\Phi(N) = M + c_N I$. Since every operator on $\mathcal{B}(H)$ is the sum of five square-zero operator (see [11]), there exists a linear functional ψ on $\mathcal{B}(H)$ such that $\psi(N) = c_N$ for every $N \in \mathcal{N}_2(H)$. Consider the following linear map

$$\Gamma(T) = \Phi(T) - \psi(T)I \quad \text{for all } \; T \in \mathcal{B}(H).$$

Obviously, $\Gamma(\mathcal{N}_2(H)) \subseteq \mathcal{N}_2(H)$ and $\Gamma(I) = aI$ where $a = \alpha - \psi(I)$. Let us show that a is non-zero. Suppose to the contrary that $a = 0$. It follows from [2, Corollary 2.2], which does not require the surjectivity assumption, that

$$2\Gamma(P)^2 = \Gamma(P)\Gamma(I) + \Gamma(I)\Gamma(P) = 0.$$

Hence, $\Phi(P) \in \mathbb{C}I + \mathcal{N}_2(H)$ for every idempotent $P \in \mathcal{P}(H)$. Now, using (2.1), we obtain that $\Phi(\mathcal{A}_2(H)) \subseteq \mathbb{C}I + \mathcal{N}_2(H)$, which contradicts the fact that $\mathcal{A}_2(H) = \Phi(\mathcal{A}_2(H))$.

Next, we shall establish that Γ is surjective and preserves square-zero operators in both directions. Let $T \in \mathcal{B}(H)$, and let $R \in \mathcal{B}(H)$ be such that $T = \Phi(R)$. If we set $S = R + a^{-1}\psi(R)I$, then

$$\Gamma(S) = \Phi(R) + a^{-1}\psi(R)(\alpha - a - \psi(I))I = T.$$

Thus, Γ is surjective. Let $A \in \mathcal{B}(H)$ be such that $\Gamma(A) \in \mathcal{N}_2(H)$. Then, $\Phi(A) = \psi(A)I + \Gamma(A) \in \mathbb{C}I + \mathcal{N}_2(H)$, and so $A \in \mathbb{C}I + \mathcal{N}_2(H)$. Write $A = bI + B$ where $b \in \mathbb{C}$ and $B \in \mathcal{N}_2(H)$. It follows that $\Gamma(B) \in \mathcal{N}_2(H)$ and

$$\Gamma(A) = abI + \Gamma(B) \in \mathcal{N}_2(H).$$

This implies that $b = 0$, and so $A \in \mathcal{N}_2(H)$. Since $\Gamma(\mathcal{N}_2(H)) \subseteq \mathcal{N}_2(H)$, we obtain that Γ preserves $\mathcal{N}_2(H)$ in both directions.

Now, it follows from [2, Theorem 2.3] that there exist a non-zero $c \in \mathbb{C}$ and an invertible operator $A \in \mathcal{B}(H)$ such that either

$$\Gamma(T) = cATA^{-1} \quad \text{for all} \ \ T \in \mathcal{B}(H),$$

or

$$\Gamma(T) = cAT^{\mathrm{tr}}A^{-1} \quad \text{for all} \ \ T \in \mathcal{B}(H).$$

This establishes (ii).

(ii) $\Rightarrow$ (i) is obvious. □

For $T \in \mathcal{M}_n(\mathbb{C})$, we denotes by $\mathrm{tr}(T)$ the trace of T. Let $\mathrm{sl}_n(\mathbb{C}) \subset \mathcal{M}_n(\mathbb{C})$ be the subspace of all matrices with trace zero. Clearly, the linear span of all square-zero matrices in $\mathcal{M}_n(\mathbb{C})$ is $\mathrm{sl}_n(\mathbb{C})$.

Proof of Theorem 1.2. (i) $\Rightarrow$ (ii). Suppose that Φ preserves quadratic matrices in both directions. Then, by Corollary 3.1, we get that $\Phi(I_n) = \alpha I_n$ where $\alpha \in \mathbb{C}$ is non-zero, Φ is bijective, and for every square-zero matrix $N \in \mathcal{M}_n(\mathbb{C})$, there exist a square-zero matrix $M \in \mathcal{M}_n(\mathbb{C})$ and a unique $c_N \in \mathbb{C}$ such that $\Phi(N) = M + c_N I_n$. Since $\mathcal{M}_n(\mathbb{C}) = \mathbb{C}I_n \oplus \mathrm{sl}_n(\mathbb{C})$, there exists a linear form h on $\mathcal{M}_n(\mathbb{C})$ such that $h(I_n) = \alpha - 1$ and $h(N) = c_N$ for every square-zero matrix $N \in \mathcal{M}_n(\mathbb{C})$. Consider the following linear map:

$$\Gamma(T) = \Phi(T) - h(T)I_n \quad \text{for all} \ \ T \in \mathcal{M}_n(\mathbb{C}).$$

Arguing as in the proof of Theorem 1.1, we prove that Γ preserves square-zero matrices in both directions. In particular, Γ maps the subspace $\mathrm{sl}_n(\mathbb{C})$ onto itself.

Hence, it follows from [12, Theorem 1] that there exist a non-zero $c \in \mathbb{C}$ and an invertible matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that either

$$\Gamma(T) = cUTU^{-1} \quad \text{for all} \ \ T \in \mathrm{sl}_n(\mathbb{C}),$$

or

$$\Gamma(T) = cUT^{\mathrm{tr}}U^{-1} \quad \text{for all} \ \ T \in \mathrm{sl}_n(\mathbb{C}).$$

However, since $\Gamma(I_n) = I_n$ and $T - n^{-1}\mathrm{tr}(T)I_n$ belongs to $\mathrm{sl}_n(\mathbb{C})$ for all $T \in \mathcal{M}_n(\mathbb{C})$, we get that either

$$\Gamma(T) = cUTU^{-1} + n^{-1}(1-c)\mathrm{tr}(T)I_n \quad \text{for all} \ \ T \in \mathcal{M}_n(\mathbb{C}),$$

or

$$\Gamma(T) = cUT^{\mathrm{tr}}U^{-1} + n^{-1}(1-c)\mathrm{tr}(T)I_n \quad \text{for all} \ \ T \in \mathcal{M}_n(\mathbb{C}).$$

To conclude, we take $\varphi = h + n^{-1}(1-c)\mathrm{tr}$.

(ii) $\Rightarrow$ (i) is obvious. □

We close this paper by the following remark:

Remark 3.2. Let H and K be infinite-dimensional complex Hilbert spaces. Theorem 1.1 can be formulated without any change for surjective linear maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserving quadratic operators in both directions.

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