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Fractal Jacobi Systems and Convergence of Fourier–Jacobi Expansions of Fractal Interpolation Functions

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Abstract. The fractal interpolation function (FIF) is a special type of continuous function on a compact subset of $\mathbb{R}$ interpolating a given data set. They have been proved to be a very important tool in the study of irregular curves arising from financial series, electrocardiograms and bio-electric recording in general as an alternative to the classical methods. It is well known that Jacobi polynomials form an orthonormal system in $\mathcal{L}^2(-1,1)$ with respect to the weight function $\rho^{(r,s)}(x) = (1-x)^r(1+x)^s$, $r > -1$ and $s > -1$. In this paper, a fractal Jacobi system which is fractal analogous of Jacobi polynomials is defined. The Weierstrass type theorem providing an approximation for square integrable function in terms of α-fractal Jacobi sum is derived. A fractal basis for the space of weighted square integrable functions $\mathcal{L}^2_\rho(-1,1)$ is found. The Fourier–Jacobi expansion corresponding to an affine FIF (AFIF) interpolating certain data set is considered and its convergence in uniform norm and weighted-mean square norm is established. The closeness of the original function to the Fourier–Jacobi expansion of the AFIF is proved for certain scale vector. Finally, the Fourier–Jacobi expansion corresponding to a non-affine smooth FIF interpolating certain data set is considered and its convergence in uniform norm and weighted-mean square norm is investigated as well.

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Keywords. Fractal interpolation functions, Fractal Jacobi systems, Schauder basis, Fourier–Jacobi expansions.

1. Introduction

In the real-world applications there are many objects namely financial series, turbulence, sampled signals, etc., which are highly irregular in nature and are not well approximated using smooth functions of classical approximation theory. An important advantage of fractal maps which are continuous but not necessarily differentiable is that they form bases for many functional spaces.

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Barnsley introduced the concept of fractal interpolation function (FIF) using Hutchinson's operator based on an iterated function system (IFS) whose attractor is the graph of a continuous function interpolating certain data set [2,3]. These FIFs are generally self-affine in nature. The result has been extended for non-self-affine FIFs, namely, Hidden variable FIFs in [5] and for partially self-affine (and partially non-self-affine) FIFs, namely Coalescence hidden variable FIFs in [10]. In [4], Barnsley et al. proved existence of a differentiable FIF. For smooth FIFs one can look at [9,11,12] and references therein.

Navascues defined the α-fractal interpolation function f^α as a perturbation of a continuous function f on a compact interval I of $\mathbb{R}$ [20,21]. The function f^α is continuous but nowhere differentiable in nature. Using Theorem 2 of [4], Navascues et al. introduced the smooth α-fractal interpolation function of a continuous function through an IFS with constant scaling factor and a single base function [29]. Recently, Viswanathan and Chand introduced continuously differentiable α-fractal interpolation function as a perturbation of classical continuously differentiable function, using a finite sequence of base functions [35].

The theory and application of α-fractal interpolation function f^α has been extensively studied by Navascues [21,25,26]. Several properties of the operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ defined by $f \mapsto f^\alpha$ have been explored and also been extended to more general spaces like Lebesgue spaces $\mathcal{L}^p(I)(1 \le p < \infty)$. Navascues proposed a general procedure to define non-smooth fractal versions of classical trigonometric approximants and generalized some classical results namely Dini–Lipschitz's theorem on $\mathcal{C}(2\pi)$ [24]. In this case, the fractal maps are defined using IFS theory generalizing the classical 2π-periodic continuous maps on the unit circle and the existence of a Hilbert basis of fractal maps on the circle is proved [23]. The Lipschitz properties of the original function guarantee a good approximation of the represented variable with truncated Chebyshev fractal object for high-frequency sample [25]. In the paper [30], the Legendre expansion of a real sampled signal by means of fractal methods is considered and its pointwise, uniform and mean-square convergence for suitable choice of scale vector are established.

Consider the class of orthonormal polynomials namely Jacobi polynomials [18] $P_k^{(r,s)}(x) = \gamma_k(r,s)x^k + \text{lower degree terms}$, where r and s are real parameters > -1 and $\gamma_k(r,s) > 0$. It is well known that the Jacobi polynomials hold great importance in a hierarchy of orthogonal polynomial classes. Legendre system, Chebyshev system, Gegenbauer system, etc., become special cases of Jacobi polynomial system for certain choices of the real parameters r and s. Using the approximation technique by orthogonal polynomials namely Gegenbauer (ultraspherical), Chebyshev (first and second kind) and Legendre (spherical) many design problems can be solved [17,31]. The Jacobi system forms a Schauder basis/complete orthonormal basis for $\mathcal{L}^2_\rho(-1,1)$ [16]. The domain of uniform convergence of the Fourier–Jacobi expansion of a function depends on the parameters r and s. Results on the uniform convergence of Fourier–Jacobi expansion can be found in [6,18,32] and references therein.

In the present paper, an attempt is made to approximate continuous function with the Fourier–Jacobi expansion of affine FIF. For continuously p-differentiable function, to approximate Fourier–Jacobi expansion of non-affine p-differentiable FIF is used. The content of the paper is organized as follows: the mathematical background of FIF using IFS, the calculus of α-fractal functions, smooth α-fractal functions and introduction to Jacobi system are presented in Sect. 2. In Sect. 3, fractal Jacobi system is defined and using the completeness of Jacobi system a fractal version of classical result namely "Weierstrass type theorem" is proved. For a weighted square integrable function, an expansion in terms of fractal Jacobi polynomials is derived. Also it is proved that fractal Jacobi system forms a Schauder basis for a space of weighted square integrable functions. The convergence analysis of Fourier–Jacobi expansion for an affine FIF as well as non-affine smooth FIF corresponding to certain data set and suitable scale vector is established in Sect. 4.

2. Preliminaries

2.1. Fractal Interpolation Function

Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} \ : \ i = 0, \ 1, \dots, N\}$ be given, where $\Delta : x_0 < x_1 < \cdots < x_N$ is a partition of the closed interval $I = [x_0, x_N]$ and $y_i \in [h_1, h_2] \subset \mathbb{R}, i = 0, 1, \dots, N$. Set $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \dots, N$ and $K = I \times [h_1, h_2]$. Let $L_i : I \to I_i, i = 1, \ 2, \dots, N$, be contraction homeomorphisms such that

$$L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i, \tag{2.1}$$

$$|L_i(c_1) - L_i(c_2)| \le d|c_1 - c_2| \text{ for all } c_1 \text{ and } c_2 \text{ in } I, \tag{2.2}$$

for some $0 \le d < 1$. Furthermore, let $F_i : K \to \mathbb{R}$, $i = 1, 2, \dots, N$, be given continuous functions such that

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \tag{2.3}$$

$$|F_i(x, \xi_1) - F_i(x, \xi_2)| \le |\alpha_i||\xi_1 - \xi_2| \tag{2.4}$$

for all $x \in I$ and for all ξ_1 and ξ_2 in $[h_1, h_2]$, for some $\alpha_i \in (-1, 1), i = 1, 2, \dots, N$. Define mappings $W_i : K \to I_i \times \mathbb{R}, i = 1, 2, \dots, N$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)) \text{ for all } (x, y) \in K.$$

Then

$$\{K; \ W_i(x, y) \ : \ i = 1, 2, \dots, N\} \tag{2.5}$$

constitutes an IFS. Barnsley [2] proved that the IFS $\{K; W_i : i = 1, 2, \dots, N\}$ defined above has a unique attractor G where G is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i = 0, 1, \dots, N$. This function f is called a fractal interpolation function (FIF) or simply fractal function and it is the unique function satisfying the following fixed point equation

$$f(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))) \text{ for all } x \in I_i, \ i = 1, 2, \dots, N. \tag{2.6}$$

The widely studied FIFs so far are defined by the iterated mappings

$$L_i(x) = a_i\, x \;+\; d_i, \quad F_i(x,y) = \alpha_i\, y \;+ q_i(x), \;\; i = 1,\ 2,\ \ldots, N, \tag{2.7}$$

where the real constants a_i and d_i are determined by the condition (2.1) as

$$a_i = \frac{(x_i - x_{i-1})}{(x_N - x_0)} \quad \text{and} \quad d_i = \frac{(x_N x_{i-1} - x_0 x_i)}{(x_N - x_0)}, \tag{2.8}$$

and $q_i(x)$s are suitable continuous functions such that the conditions (2.3) and (2.4) hold. For each i, α_i is a free parameter with $|\alpha_i| < 1$ and is called a vertical scaling factor of the transformation W_i. Then the vector $\alpha = (\alpha_1, \alpha_2,\ \ldots,\ \alpha_N)$ is called the scale vector of the IFS. If $q_i(x)$ is taken as linear then the corresponding FIF is known as affine FIF (AFIF).

2.2. α-Fractal Interpolation Function

Let $\mathcal{C}(I)$ denote the normed space of real-valued continuous functions on I endowed with the uniform norm $\|f\|_\infty = \sup\{|f(x)| \,:\, x \in I\}$. Let $f \in \mathcal{C}(I)$. Consider the case

$$q_i(x) = f(L_i(x)) \;-\; \alpha_i\, b(x) \tag{2.9}$$

where $b(x)$ is a continuous function such that $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$. Let f^α be the continuous function whose graph is the attractor of the IFS (2.7), (2.8) and (2.9). Then, the function f^α is called the α-fractal function associated to f with respect to the function $b(x)$ and the partition Δ [21]. From (2.6) and (2.9), f^α satisfies the following fixed point equation

$$f^\alpha(x) \;=\; f(x) \;+\; \alpha_i\, (f^\alpha - b) \circ L_i^{-1}(x) \text{ for all } x \in I_i, i = 1,\ 2,\ \ldots, N. \tag{2.10}$$

From (2.10), it is easy to deduce the following

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty, \tag{2.11}$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, \ldots, N\}$. For $\alpha = 0$, the fractal function is same as the classical one. The theory of α-fractal function for different choices of $b(x)$ can be found in [20,22,28]. The operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ defined by

$$\mathcal{F}^\alpha(f) = f^\alpha \quad \text{for all} \quad f \in \mathcal{C}(I) \tag{2.12}$$

is linear and bounded for various choices of the continuous function b [21]. In the sequel, b is taken as

$$b = Lf \tag{2.13}$$

where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear and bounded operator with respect to the uniform norm on $\mathcal{C}(I)$ such that $Lf(x_0) = f(y_0)$ and $Lf(x_N) = f(y_N)$. Then for any $f \in \mathcal{C}(I)$ and its fractal function satisfies [28]

$$\|f^\alpha - f\|_\infty \leq |\alpha|_\infty \|f^\alpha - Lf\|_\infty, \tag{2.14}$$

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|_\infty \|f\|_\infty, \tag{2.15}$$

where $\|I - L\|_\infty$ represents the corresponding operator norm as well. If we consider $\mathcal{L}^p$-norm $(1 \le p < \infty)$

$$\|f\|_{\mathcal{L}^p} = \left(\int_I |f|^p \mathrm{d}t\right)^{1/p}$$

such that the operator $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is linear and bounded with respect to $\mathcal{L}^p$-norm on $\mathcal{C}(I)$ then the following results can be found in [28].
For every $f \in \mathcal{C}(I)$ and b defined as (2.13)

$$\|f^\alpha - f\|_{\mathcal{L}^p} \le |\alpha|_\infty \|f^\alpha - Lf\|_{\mathcal{L}^p}, \tag{2.16}$$

$$\|f^\alpha - f\|_{\mathcal{L}^p} \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - Lf\|_{\mathcal{L}^p}. \tag{2.17}$$

The results have been extended in [27] for any $f \in \mathcal{L}^p(I)$, $1 \le p < \infty$ as

$$\|\overline{\mathcal{F}}^\alpha(f) - f\|_{\mathcal{L}^p} \le |\alpha|_\infty \|\overline{\mathcal{F}}^\alpha(f) - \overline{L}f\|_{\mathcal{L}^p}, \tag{2.18}$$

$$\|\overline{\mathcal{F}}^\alpha(f) - f\|_{\mathcal{L}^p} \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - \overline{L}f\|_{\mathcal{L}^p} \tag{2.19}$$

where $\overline{\mathcal{F}}^\alpha : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ and $\overline{L} : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ are the corresponding extensions of $\mathcal{F}^\alpha$ and L with $\|\overline{\mathcal{F}}^\alpha\|_p = \|\mathcal{F}^\alpha\|_p$ and $\|\overline{L}\|_p = \|L\|_p$ where $\|.\|_p$ represents the norm of the operator with respect to the $\mathcal{L}^p$-norm.

2.3. Smooth α-Fractal Interpolation Function

Under certain condition on the base function and the scale vector, the fractal function preserves the smoothness of the corresponding function. For that one needs to define an IFS in such a way that it satisfies the following theorem which proves the existence of differentiable or spline FIF. The differential α-fractal function is used to prove the results in Sect. 4.

Theorem 2.1 (Barnsley et al.) ([4], Theorem 2). *For a given data set $x_0 < x_1 < \cdots < x_N$, let $L_i(x) = a_i x + b_i$ be such that it satisfies (2.1), (2.2) and $F_i(x, y) = \alpha_i y + q_i(x)$ satisfy (2.3), (2.4) for $i = 1, 2, \ldots, N$. Suppose for some $r > 0$, $|\alpha_i| < s a_i^r$, $0 < s < 1$ and $q_i \in \mathcal{C}^r[x_0, x_N]$, $i = 1, 2, \ldots, N$. Let $F_{i,k}(x, y) = \frac{\alpha_i y + q_i^k(x)}{a_i^k}$, $y_{0,k} = \frac{q_1^k(x_0)}{a_1^k - \alpha_1}$, $y_{N,k} = \frac{q_N^k(x_N)}{a_N^k - \alpha_N}$, $k = 1, 2, \ldots, r$. If $F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_0, y_{0,k})$ for $i = 2, 3, \ldots, N$ and $k = 1, 2, \ldots, r$, then $\{(L_i(x)), F_i(x, y)\}_{i=1}^N$ determines a FIF $f \in \mathcal{C}^r[x_0, x_N]$ and f^k is the FIF determined by $\{(L_i(x)), F_{i,k}(x, y)\}_{i=1}^N$ for $k = 1, 2, \ldots, r$.*

To define a smooth α-fractal interpolation function g^α corresponding to a smooth function g, Navascues et al. [29] defined an IFS satisfying Theorem 2.1, in which it is assumed that the constant scale factors $\alpha_i = \alpha_1$ for all $i = 1, 2, \ldots, N$, that is, the scale vector $\alpha = (\alpha_1, \alpha_1, \ldots, \alpha_1)$ and the base function b as Hermite interpolating polynomial such that the kth derivative of b satisfies

$$b^{(k)}(x_0) = g^{(k)}(x_0)$$

$$b^{(k)}(x_N) = g^{(k)}(x_N)$$

for $k = 0, 1, \ldots, N$. Then

$$q_i(x) = g(L_i(x)) - \alpha_1\, b(x)$$

and the IFS,

$$L_i(x) = a_i\, x \;+\; d_i, \;\; F_i(x, y) = \alpha_1\, y \;+ g(L_i(x)) \;-\; \alpha_1\, b(x) \tag{2.20}$$

for $x \in I$ and $i = 1, 2, \ldots, N$. The fractal function g^α corresponding to IFS (2.20) preserves the smoothness of g and satisfies

$$\|g^\alpha - g\|_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|g - b\|_\infty. \tag{2.21}$$

In [35], Viswanathan et al. relaxed the equality condition on the scaling factors α and considered a finite collection of base functions

$$B = \{b_i \in \mathcal{C}(I) | b_i(x_0) = g(x_0), b_i(x_N) = g(x_N), b_i \ne g, i = 1, 2, \ldots, N\}$$

instead of taking a single base function b and defined a smooth α-fractal g^α corresponding to a smooth function g, via the IFS,

$$L_i(x) = a_i\, x \;+\; d_i, \;\; F_i(x, y) = \alpha_i\, y \;+ g(L_i(x)) \;-\; \alpha_i\, b_i(x) \tag{2.22}$$

for $x \in I$ and $i = 1, 2, \ldots, N$. The following theorem can be read in [35].

Theorem 2.2. *Suppose that for some $p \ge 0$, we have $|\alpha_i| \le k a_i^p$ for all $i = 1, 2, \ldots, N$ and $0 < k < 1$. Let $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, \ldots, N\}$, $g \in \mathcal{C}^p$ and the family $B = \{b_i : i = 1, 2, \ldots, N\}$ be such that the derivatives up to the pth order of each of its members agrees with that of g at the end points of the interval. Then the α-fractal function $g^\alpha \in \mathcal{C}^p(I)$ of g with respect to the partition Δ and the family B satisfies*

$$\|g^\alpha - g\|_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \max\{\|g - b_i\|_\infty : i = 1, 2, \ldots, N\}. \tag{2.23}$$

In the following, the Jacobi system is defined.

2.4. Jacobi System

The Jacobi polynomials denoted by $P_n^{r,s}(x)$ and defined by

$$P_n^{(r,s)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+r}{k} \binom{n+s}{n-k}\; (x+1)^k (x-1)^{n-k} \text{ for all } x \in \mathbb{R},$$

for $n = 0, 1, 2, \ldots$; where r and s are real parameters [16]. They include many of the basic special functions, e.g., the Legendre polynomial $P_n(x)$ if $r = s = 0$, the Chebyshev polynomial of first kind $T_n(x)$ if $r = s = -1/2$, etc. With the (Jacobi) weight function $\rho^{(r,s)}(x) = (1-x)^r (1+x)^s$ where $r > -1$ and $s > -1$, the system $\{P_n^{(r,s)}(x)\}_{n=0}^\infty$ is orthonormal in $\mathcal{L}_\rho^2[-1, 1]$, that is,

$$\int_{-1}^{1} P_n^{(r,s)}(x) P_m^{(r,s)}(x) \rho^{(r,s)}(x) \mathrm{d}x \;=\; \delta_{nm}$$

where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \ne m$ [18]. The system $\{P_n^{(r,s)}(x)\}_{n=0}^\infty$ is uniquely defined and is called the Jacobi system of orthonormal polynomials.

If $f\rho^{(r,s)}$ is an integrable function on $[-1,1]$ then f has a Fourier expansion with respect to the Jacobi system of orthonormal polynomials as

$$f(x) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s)}(x) \quad \text{for} \quad x \in [-1,1] \tag{2.24}$$

where

$$a_k^{(r,s)} = \int_{-1}^{1} f(t) P_k^{(r,s)}(t) \rho^{(r,s)}(t) \mathrm{d}t \tag{2.25}$$

is the kth Fourier coefficient of f [18]. Denote the nth partial sum of the expansion in Eq. (2.24) by $S_n^{(r,s)}(f,x)$, that is,

$$S_n^{(r,s)}(f,x) = \sum_{k=0}^{n} a_k^{(r,s)}(f) P_k^{(r,s)}(x) \quad \text{for} \quad x \in [-1,1]. \tag{2.26}$$

Let X be a subspace of $\mathcal{L}_\rho^1$. Define $\|S_n^{(r,s)}\|_X = \sup_{\|f\| \leq 1} \|S_n^{(r,s)}(f,x)\|_X$, called a Lebesgue constant. By virtue of the Lebesgue inequality [19]

$$\|f - S_n^{(r,s)}(f,x)\|_X \leq (1 + \|S_n^{(r,s)}\|_X) E_n(f)_X \tag{2.27}$$

where $E_n(f)_X$ is the distance from f to the best approximation of the function f by algebraic polynomials of degree at most n in the space X. The boundedness of the Lebesgue constant leads to the convergence of the Fourier–Jacobi series for any function in the space X provided that the Weierstrass approximation theorem [13] holds in the space X and also determines the partial sums of the Fourier–Jacobi series $S_n^{(r,s)}(f,x)$ to f in the space X.

3. Fractal Jacobi System

Let us take $I = [-1,1]$ and denote the set of all Jacobi polynomials of degree less than or equal to m where $m \in \mathbb{N} \bigcup \{0\}$ by

$$J_m(I) = \{1 \equiv P_0^{(r,s)}(x), P_1^{(r,s)}(x), \ldots, P_m^{(r,s)}(x)\}$$

and define $J(I) = \bigcup_{m=0}^{\infty} J_m(I)$. Then the fractal analogous of Jacobi polynomials is defined as $P_n^{(r,s,\alpha)}(x) = \mathcal{F}^\alpha(P_n^{(r,s)}(x)), n = 0, 1, \ldots$, where $\mathcal{F}^\alpha$ is a linear bounded operator as given in (2.12).

Then denote $J_m^\alpha(I) = \{P_n^{(r,s,\alpha)}(x) = \mathcal{F}^\alpha(P_n^{(r,s)}(x)), n = 0, 1, \ldots, m\}$ and $J^\alpha(I) = \bigcup_{m=0}^{\infty} J_m^\alpha(I)$. In the next section it is proved that $J^\alpha(I)$ is complete in $\mathcal{L}_\rho^2(I)$.

Example. Let $I = [0,1]$ and $\Delta : 0 < 1/8 < 2/8 < \cdots < 1$ be the partition of I. Also assume that the constant scale factors $\alpha_i = 0.2$ for all $i = 1,\ 2, \ldots, 8$. Then Fig. 1 represents the second-order Legendre polynomial $f(x) = 1.5x^2 - 0.5$ and the corresponding α-fractal f^α for the IFS with the linear base function $b(x)$ in $[0,1]$ such that $b(0) = f(0)$ and $b(1) = f(1)$. Figure 2 represents the second-order Chebyshev polynomial $f(x) = 2x^2 - 1$ of the first kind and the corresponding α-fractal f^α for the IFS with the base function $b(x) = Lf(x) = v(x)f(x)$ where $v(x) = x^2 - x + 1$ for $x \in [0,1]$. Figure 3 represents the second-order Gegenbauer polynomial $f(x) = 3.75x^2 - 0.75$

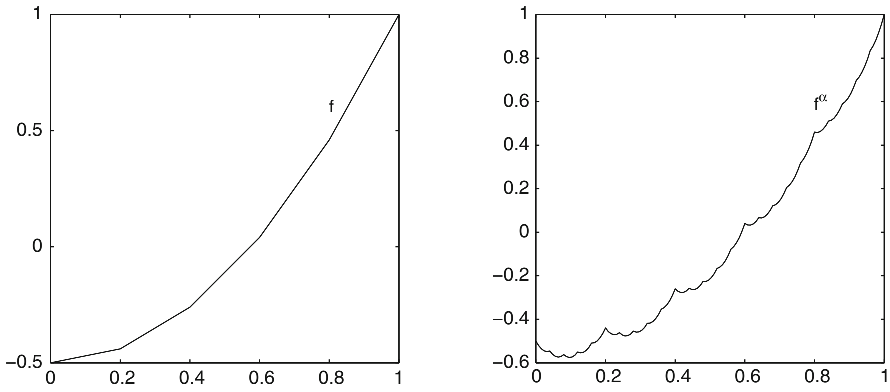

Figure 1. $f(x) = 1.5x^2 - 0.5$ and corresponding f^α

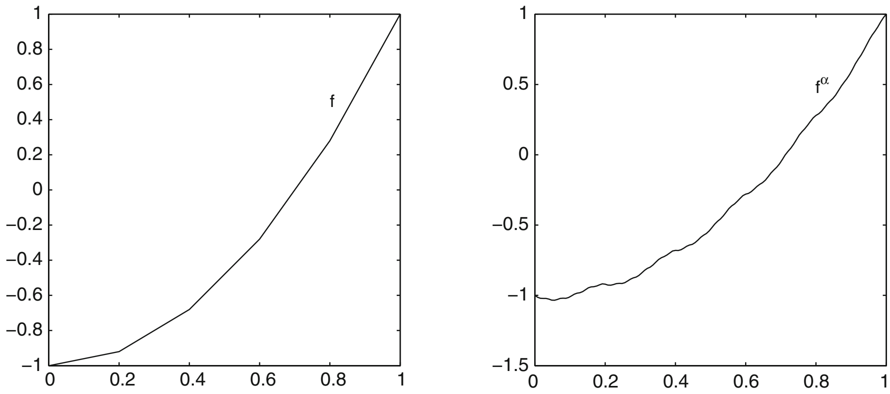

Figure 2. $f(x) = 2x^2 - 1$ and corresponding f^α

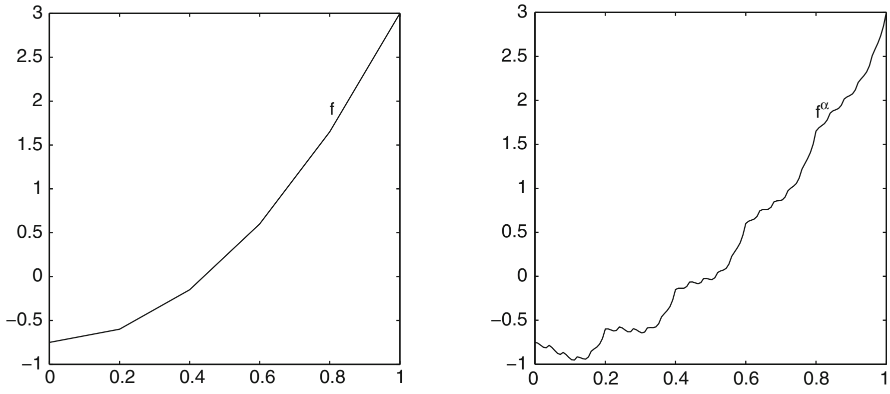

Figure 3. $f(x) = 3.75x^2 - 0.75$ and corresponding f^α

and the corresponding α-fractal f^α for the IFS with the base function $b(x) = Lf(x) = f \circ c(x)$ where $c(x) = \sin(\frac{\pi x}{2})$ for $x \in [0, 1]$.

The following definitions can be read in [16].

Definition 3.1. A sequence $(\phi_n)_{n\in\Lambda}$ in a normed linear space V is called total in V if the class of all finite linear combinations $\sum a_n\phi_n$ is dense in V.

Definition 3.2. A sequence $(\phi_n)_{n\in\Lambda}$ in a Hilbert space H is called complete if the only element of H which is orthogonal to every ϕ_n is the null element, that is

$$\langle f, \phi_n\rangle = 0 \text{ for all } n \in \Lambda \ \Rightarrow f \equiv 0.$$

The following proposition is true for any sequence in a Hilbert space.

Proposition 3.3 [16]. *If (ϕ_n) is any sequence in H, orthogonal or not. Then the following are equivalent:*

(a) *(ϕ_n) is complete.*
(b) *(ϕ_n) is total.*

3.1. Weierstrass Type Theorem for α-Fractal Jacobi Sums

Any continuous function on a compact subset of $\mathbb{R}$ can be uniformly approximated by a sequence of polynomials is a well-known result due to the study by Weierstrass [13]. A similar type of result for a bigger space using sequence of fractal version of Jacobi polynomials is proved below.

Theorem 3.4. *Let $I = [-1, 1]$ and $\epsilon > 0$ be given. Suppose that f is any square integrable function on I with respect to the weight function $\rho^{(r,s)}(x)$. For every $\epsilon > 0$ and any partition $\triangle : -1 = x_0 < x_1 < \cdots < x_N = 1$ of I and linear and bounded operator $L : \mathcal{C}(I) \to \mathcal{C}(I)$ there exists α-fractal Jacobi sum $P^{(r,s,\alpha)}(x) = \sum_{i=1}^{N} a_{k_i} P_{k_i}^{(r,s,\alpha)}(x)$ with $\alpha \neq 0$ such that*

$$\|f - P^{(r,s,\alpha)}\|_{\mathcal{L}^2_\rho} < \epsilon.$$

Proof. Let $\epsilon > 0$ be given and $f \in \mathcal{L}^2_\rho(I)$. Then there exists $g \in \mathcal{C}(I)$ such that

$$\|f - g\|_{\mathcal{L}^2_\rho} < \frac{\epsilon}{3} \tag{3.1}$$

Since $(P_n^{(r,s)})_{n=0}^{\infty}$ is complete in $\mathcal{L}^2_\rho(I)$, it is total there by Proposition 3.3. Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $P^{(r,s)} = \sum_{i=1}^{N} a_{k_i} P_{k_i}^{(r,s)}$ such that

$$\|g - P^{(r,s)}\|_{\mathcal{L}^2_\rho} < \frac{\epsilon}{3}. \tag{3.2}$$

For the Jacobi sum and its fractal, using (2.15)

$$\|P^{(r,s,\alpha)} - P^{(r,s)}\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|I - L\|_\infty\|P^{(r,s)}\|_\infty \tag{3.3}$$

Taking $|\alpha|_\infty < \frac{\frac{\epsilon}{3C}}{\frac{\epsilon}{3C} + \|I-L\|_\infty\|P^{(r,s)}\|_\infty}$ where $C = \left(\int_{-1}^{1} \rho^{r,s}(t)\mathrm{d}t\right)^{1/2}$,

$$\|P^{(r,s,\alpha)} - P^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq \|P^{(r,s,\alpha)} - P^{(r,s)}\|_\infty \left(\int_{-1}^{1} \rho^{r,s}(t)\mathrm{d}t\right)^{1/2} < \frac{\epsilon}{3} \tag{3.4}$$

and combining (3.1), (3.2) and (3.4) we get

$$\|f - P^{(r,s,\alpha)}\|_{\mathcal{L}^2_\rho} < \epsilon.$$

The linearity of the operator $\mathcal{F}^\alpha$ gives the result. $\square$

Remark 3.5. By Theorem 3.4, it follows that

(1) The set of fractal version of Jacobi sums is dense in $\mathcal{L}^2_\rho(I)$.

(2) $J^\alpha(I)$ is complete in $\mathcal{L}^2_\rho(I)$.

In the following, the Schauder basis and basis constant which are used in the sequel to prove the main results are presented.

Definition 3.6 [[8], pp. 24–27]. Let X be a (real) normed space and let (x_n) be a non zero sequence in X. We say that (x_n) is a Schauder basis for X, if for each $x \in X$, there is a unique sequence of scalars (a_n) such that $x = \sum_{n=1}^\infty a_n x_n$, where the series converges in norm to x. We define a sequence of linear maps (P_n) on X by $P_n x = \sum_{i=1}^n a_i x_i$, where $x = \sum_{i=1}^\infty a_i x_i$. The map P_n is a projection onto span$\{x_i : 1 \le i \le n\}$. In addition, since (x_n) is a Schauder basis, it follows that $P_n x \to x$ in norm as $n \to \infty$ for each $x \in X$ and P_n is continuous. Moreover, $K = \sup_n \|P_n\| < \infty$. The number K is called the basis constant of the basis (x_n).

3.2. Expansion in Terms of Fractal Jacobi Polynomials

In this section, an expansion of a weighted square integrable function in terms of fractal Jacobi polynomials is given. To do this a linear and bounded operator is defined on $\mathcal{L}^2_\rho(I)$. Different scale vector α^k for every polynomial is taken in Sect. 3.2 to ensure the boundedness of operator given in (3.5) and also in Sect. 3.3. The perturbation inequality (2.17) for function and its fractal with respect to the weighted p-norm is not true in general. Therefore, the uniform bound on the Jacobi polynomials under certain condition on the real parameters r, s plays an important role, in proving the results.

Let α^k be a sequence of scale vectors such that

$$\sum_{k=0}^{\infty} \frac{|\alpha^k|_\infty}{1 - |\alpha^k|_\infty} < \infty,$$

and assume that L is linear and bounded with respect to the uniform norm.

Proposition 3.7. *For $-1 < r, s \le -1/2$, the operator $T : span(P_k^{(r,s)}(x))_{k=0}^\infty \to span(P_k^{(r,s,\alpha^k)}(x))_{k=0}^\infty$ defined by*

$$T\left(\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right) = \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s,\alpha^{k_i})}(x) \tag{3.5}$$

is linear and bounded.

Proof. The linearity is obvious. Now, the boundedness of T is established below.

$$\left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_k^{(r,s,\alpha^{k_i})}(x)\right\|_{\mathcal{L}^2_\rho} \leq \left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} (P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x))\right\|_{\mathcal{L}^2_\rho} + \left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x))\right\|_{\mathcal{L}^2_\rho} \tag{3.6}$$

Since $(P_k^{(r,s)})_{k=0}^{\infty}$ is an orthonormal basis for $\mathcal{L}^2_\rho$, $a_k^{r,s}$ is a bounded linear functional on $\mathcal{L}^2_\rho$. Therefore,

$$|a_k^{(r,s)}(f)| \leq \|a_k^{(r,s)}\|_2 \|f\|_{\mathcal{L}^2_\rho}. \tag{3.7}$$

Treating $a_{k_i}^{(r,s)}$ as the k_ith Fourier–Jacobi coefficient of $\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x))$ and using (3.7), the first term of inequality (3.6) becomes

$$\left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} (P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x))\right\|_{\mathcal{L}^2_\rho} \leq \sum_{i=1}^{M} \left\|a_{k_i}^{(r,s)}\right\|_2 \left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right\|_{\mathcal{L}^2_\rho} \left\|P_{k_i}^{(r,s,\alpha^{k_i})} - P_{k_i}^{(r,s)}\right\|_{\mathcal{L}^2_\rho}.$$

Therefore,

$$\left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} (P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x))\right\|_{\mathcal{L}^2_\rho} \leq \left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right\|_{\mathcal{L}^2_\rho} \sum_{i=1}^{M} \left\|a_{k_i}^{(r,s)}\right\|_2 \left\|P_{k_i}^{(r,s,\alpha^{k_i}} - P_{k_i}^{(r,s)}\right\|_{\mathcal{L}^2_\rho}. \tag{3.8}$$

But from (2.15)

$$\|P_{k_i}^{(r,s,\alpha^{k_i})} - P_{k_i}^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq \frac{|\alpha^{k_i}|_\infty}{1-|\alpha^{k_i}|_\infty} \|I-L\|_\infty \|P_{k_i}^{(r,s)}\|_\infty \left(\int_{-1}^{1} \rho^{(r,s)}(t)\mathrm{d}t\right)^{1/2}.$$

Using it in (3.8), the left-hand side becomes smaller than

$$\left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right\|_{\mathcal{L}^2_\rho} \sum_{i=1}^{M} \left\|a_{k_i}^{(r,s)}\right\|_2 \left\|P_{k_i}^{(r,s)}\right\|_\infty \frac{|\alpha^{k_i}|_\infty}{1-|\alpha^{k_i}|_\infty} \|I-L\|_\infty \times \left(\int_{-1}^{1} \rho^{(r,s)}(t)\mathrm{d}t\right)^{1/2}. \tag{3.9}$$

Since $(P_k^{(r,s)})$ is a Schauder basis for $\mathcal{L}^2_\rho(I)$, the following inequality hold [15]

$$1 \leq \|a_{k_i}^{(r,s)}\|_2 \|P_{k_i}^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq 2K$$

where K is the basis constant. But $\|P_{k_i}^{(r,s)}\|_{\mathcal{L}^2_\rho} = 1$ and, therefore,

$$1 \leq \|a_k^{(r,s)}\|_2 \leq 2K.$$

The following inequality can be read in [7, page-275]

$$\max_{-1\leq x\leq 1} |P_k^{(r,s)}(x)| \sim 1 \text{ for } -1 < r, s \leq -1/2. \tag{3.10}$$

Then for $\epsilon = 1$ there exists $n_0 \in \mathbb{N}$ such that

$$\|P_k^{(r,s)}\|_\infty \leq 2$$

for $k \geq n_0$. If $M^* = \max\{\|P_0^{(r,s)}\|_\infty, \|P_1^{(r,s)}\|_\infty, \ldots, \|P_{n_0-1}^{(r,s)}\|_\infty, 2\}$, for any k,

$$\|P_k^{(r,s)}\|_\infty \leq M^*.$$

Then

$$\|a_k^{(r,s)}\|_2 \|P_k^{(r,s)}\|_\infty \leq 2KM^*. \tag{3.11}$$

By choosing $(\alpha^k)_{k=0}^\infty$ such that the sum $\sum_{k=0}^\infty \frac{|\alpha^k|_\infty}{1-|\alpha^k|_\infty}$ is finite, say M_1^* and using (3.11) in (3.9), it follows that

$$\left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x) \right\|_{\mathcal{L}^2_\rho} 2KM^* M_1^* l$$

where $l = \|I - L\|_\infty \left(\int_{-1}^{1} \rho^{(r,s)}(t)\mathrm{d}t\right)^{1/2}$. Therefore,

$$\left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_k^{(r,s,\alpha^{k_i})}(x) \right\|_{\mathcal{L}^2_\rho} \leq (1 + 2KM^* M_1^* l) \left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)) \right\|_{\mathcal{L}^2_\rho}.$$

Hence, T is a linear bounded operator. □

Lemma 3.8 (Linear and Bounded Operator Theorem). *Let X be a normed linear space, Y be a Banach space and $A : X \to Y$ be a linear and bounded operator. If X is dense in X' then A can be extended to X' preserving the norm of A.*

Now we are in a position to write an expansion in terms of fractal Jacobi polynomials.

Theorem 3.9. *The map $\overline{T} : \mathcal{L}^2_\rho(I) \to \mathcal{L}^2_\rho(I)$ given by*

$$\overline{T}(f) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s,\alpha^k)}(x) \tag{3.12}$$

where

$$f = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s)}(x)$$

is well defined, linear and continuous for $-1 < r, s \leq -1/2$.

Proof. $(P_k^{(r,s)})_{k=0}^{\infty}$ is complete in $\mathcal{L}^2_\rho(I)$. So it is total there and, therefore, $span(P_k^{(r,s)}(x))_{k=0}^{\infty}$ is dense in $\mathcal{L}^2_\rho(I)$. Then by Lemma 3.8

$$T : span(P_k^{(r,s)}(x))_{k=0}^{\infty} \to span(P_k^{(r,s,\alpha^k)}(x))_{k=0}^{\infty} \subseteq \mathcal{L}^2_\rho(I)$$

$$T\left(\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right) := \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s,\alpha^{k_i})}(x)$$

can be extended to $\overline{T} : \mathcal{L}^2_\rho(I) \to \mathcal{L}^2_\rho(I)$ with $\|\overline{T}\|_2 = \|T\|_2$.

The linearity and boundedness of $\overline{T}$ imply that $\overline{T}(f) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s,\alpha^k)}(x)$ whenever $f = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s)}(x)$. This completes the proof. $\square$

3.3. Schauder Basis for $\mathcal{L}^2_\rho(I)$

In this section, it is proved that the fractal Jacobi system forms a Schauder basis for $\mathcal{L}^2_\rho(I)$ under certain condition on the scale vector. To show this, it is noted that the nth partial sum operator V_n of the continuous operator $V : \mathcal{L}^2_\rho(I) \to \mathcal{L}^2_\rho(I)$ defined by

$$V(f) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f)(P_k^{(r,s)} - P_k^{(r,s,\alpha^k)})$$

is bounded, as

$$\|V_n(f)\| \le \sum_{k=0}^{\infty} |a_k^{(r,s)}(f)| \|(P_k^{(r,s)} - P_k^{(r,s,\alpha^k)})\|_{\mathcal{L}^2_\rho} \le 2KM^*M_1^* l \|f\|_{\mathcal{L}^2_\rho}.$$

Since V_n is of finite rank, V_n is compact. The uniform convergence of V_n to V in operator norm implies that V is compact.

Theorem 3.10. *Suppose that the sequence of scale vectors $\alpha^k \in \mathbb{R}^N$ for $k = 0, 1, 2, \ldots,$ are such that*

$$|\alpha^0|_\infty \ge |\alpha^1|_\infty \ge \cdots$$

and

$$\sum_{k=0}^{\infty} \frac{|\alpha^k|_\infty}{1 - |\alpha^k|_\infty} < \infty.$$

If $|\alpha^0|_\infty < \frac{1}{M^*}\|L\|_\infty^{-1}\left(\int_{-1}^{1} \rho^{(r,s)}(t)\mathrm{d}t\right)^{-1/2}$ and $-1 < r, s \le -1/2$, then $(P_k^{(r,s,\alpha^k)})$ is a Schauder basis for $\mathcal{L}^2_\rho(I)$.

Proof. The proof goes in a similar way to that of Theorem 3.7 in [26]. However, for completeness, we include it here for our settings.

Since $\overline{T} = I - V$ and V is compact, $ker(I - V)$ is finite dimensional. Then

$$\mathcal{L}^2_\rho(I) = ker(I - V) \bigoplus W,$$

where W is a closed subspace of $\mathcal{L}^2_\rho(I)$. To show that $\overline{T}$ is injective, let $\overline{T}(P_k^{(r,s)}) = P_k^{(r,s,\alpha^k)} = 0$. Since $P_k^{(r,s)} \not\equiv 0$, we have

$$\begin{aligned}\|P_k^{(r,s,\alpha^k)} - P_k^{(r,s)}\|_{\mathcal{L}^2_\rho} &\leq \|P_k^{(r,s,\alpha^k)} - P_k^{(r,s)}\|_\infty \left(\int_{-1}^1 \rho^{(r,s)}(t)\mathrm{d}t\right)^{1/2} \\ &\leq |\alpha^k|_\infty \left(\|P_k^{(r,s,\alpha^k)}\|_\infty + \|L\|_\infty \|P_k^{(r,s)}\|_\infty\right) \left(\int_{-1}^1 \rho^{(r,s)}(t)\mathrm{d}t\right)^{1/2}\end{aligned}$$

using (2.14). Therefore,

$$1 \leq |\alpha^0|_\infty \|L\|_\infty \|P_k^{(r,s)}\|_\infty \left(\int_{-1}^1 \rho^{(r,s)}(t)\mathrm{d}t\right)^{1/2},$$

which is a contradiction to the assumption that $|\alpha^0|_\infty < \frac{1}{M^*}\|L\|_\infty^{-1} \left(\int_{-1}^1 \rho^{(r,s)}(t)\mathrm{d}t\right)^{-1/2}$. Therefore, $P_k^{(r,s)} \notin \ ker\,(\overline{T})$ for all k and hence $\overline{T}$ is injective. Since V is compact,

$$dim(ker(I-V)) = dim(\mathcal{L}^2_\rho(I)/Range(I-V)) = 0,$$

and, therefore, $\overline{T}$ is onto. As $\overline{T}$ is a continuous isomorphism, it maps Schauder basis onto Schauder basis. Therefore, $\left(P_k^{(r,s,\alpha^k)}\right)$ is a Schauder basis of $\mathcal{L}^2_\rho(I)$. $\square$

4. Convergence of Fourier–Jacobi Expansion

In this section, we mainly prove the convergence of Fourier–Jacobi series of an affine as well as non-affine fractal interpolation function corresponding to certain data set. Throughout this section, let I denote a closed interval $[-1, 1]$ and Δ denote the partition $-1 = x_0 < x_1 < \cdots < x_N = 1$ of I.

We say that a function f defined on $[a, b]$ belongs to the Dini–Lipschitz class $DL[a, b]$ if

$$w_f(h)\log(1/h) \to 0 \quad \text{as} \quad h \to 0$$

where $w_f(h)$ is the modulus of continuity of f defined for $h \geq 0$ by

$$w_f(h) = \sup_{d(x,y)\leq h} |f(x) - f(y)|.$$

Several propositions on convergence of Fourier–Jacobi series are needed to establish the main results.

Proposition 4.1 ([33], Theorem 4.7, pp. 146, 300). *The condition $f \in DL[-1,1]$ guarantees the uniform convergence of Fourier–Jacobi series of the function f on $[a,b] \subset (-1,1)$.*

Proposition 4.2 ([14], p. 118). *The condition $f \in DL[-1,1]$ is sufficient for the uniform convergence of the Fourier–Jacobi series in the whole interval $[-1,1]$ in the case when $r,s \in (-1,-1/2]$.*

Proposition 4.3 ([1]). *Let* $\max(r,s) > -1/2$ *and* $f^p \in Lip\,\gamma$ *where* $p+\gamma > \max(r,s)+1/2$ *then the Fourier–Jacobi series of the function* f *uniformly converges on* $[-1,1]$.

4.1. Convergence Using Affine FIF

The convergence of Fourier–Jacobi expansion corresponding to an affine FIF (AFIF) interpolating certain data set in weighted mean square norm is proved in this section. Let us consider a set of data $\{(x_n,y_n)\}_{n=0}^{N}$ such that $x_{n-1} < x_n$ for $n = 1,2,\ldots,N$.

Theorem 4.4. *Let* $g \in \mathcal{C}(I)$ *be the original function providing the data with constant step* $h = x_n - x_{n-1}$. *Let* f *be the corresponding affine fractal interpolation function with scale vector* α_h *such that* $|\alpha_h|_\infty < h$. *Then the Fourier–Jacobi expansion of* f *converges in weighted quadratic mean to* g *on* $[a,b] \subset (-1,1)$, *that is,* $S_n^{(r,s)}(f)$ *converges to* g *in* $\mathcal{L}^2_\rho$ *as* $h \to 0$ *and* $n \to \infty$ *on* $[a,b] \subset (-1,1)$.

Proof. Since q_n linear, $q_n \in$ Lip 1, and the corresponding fractal interpolation function f is affine and if $|\alpha_h|_\infty < h$ then $f \in$ Lip 1. ([30], Lemma 6). Then the modulus of continuity

$$w_f(\delta) = \sup_{|x_1-x_2|\le\delta} |f(x_1)-f(x_2)| \;\le\; K\,\delta$$

where K is the Lipschitz constant. Therefore, by definition, f is in $DL([-1,1])$. Using Proposition 4.1, we get

$$\|S_n^{(r,s)}(f) - f\|_\infty \to 0 \text{ as } n\to\infty \text{ on } [a,b]\subset(-1,1).$$

But

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}^2_\rho} \le \|S_n^{(r,s)}(f) - f\|_\infty \left(\int_{-1}^{1} \rho^{r,s}(t)\mathrm{d}t\right)^{1/2}.$$

Hence,

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}^2_\rho} \to 0 \text{ as } n\to\infty \text{ on } [a,b]\subset(-1,1), \tag{4.1}$$

since $\int_{-1}^{1} \rho^{r,s}(t)\mathrm{d}t$ is convergent. Similarly,

$$\|g-f\|_{\mathcal{L}^2_\rho} \le \|g-f\|_\infty \left(\int_{-1}^{1} \rho^{r,s}(t)\mathrm{d}t\right)^{1/2}.$$

The following result can be read in [30]

$$\|g-f\|_\infty \le w_g(h) + \frac{2|\alpha_h|_\infty}{1-|\alpha_h|_\infty}\|g\|_\infty. \tag{4.2}$$

Using it, we get

$$\|g-f\|_{\mathcal{L}^2_\rho} \le \left(w_g(h) + \frac{2|\alpha_h|_\infty}{1-|\alpha_h|_\infty}\|g\|_\infty\right)\left(\int_{-1}^{1} \rho^{r,s}(t)\mathrm{d}t\right)^{1/2}.$$

Due to uniform continuity of g, it happens that $w_g(h) \to 0$ as $h \to 0$. Since $|\alpha_h|_\infty < h$, it follows that

$$\|g-f\|_{\mathcal{L}^2_\rho} \to 0 \qquad \text{as} \qquad h \to 0. \tag{4.3}$$

Using (4.1) and (4.3) in the following inequality

$$\|g - S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho} \leq \|g - f\|_{\mathcal{L}^2_\rho} + \|f - S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho},$$

the result follows. □

In the following theorem, the uniform convergence of Fourier–Jacobi expansion is established.

Theorem 4.5. *Let $g \in \mathcal{C}(I)$ be the original function providing the data with constant step $h = x_n - x_{n-1}$. Then for $|\alpha_h|_\infty < h$, the Fourier–Jacobi expansion defined by the corresponding affine FIF f converges uniformly to g as $h \to 0$ and $n \to \infty$ on $[a, b] \subset (-1, 1)$.*

Proof. If $S_n^{(r,s)}(f)$ is the nth partial sum of the Fourier–Jacobi expansion of f, then

$$\|g - S_n^{(r,s)}(f)\|_\infty \leq \|g - f\|_\infty + \|f - S_n^{(r,s)}(f)\|_\infty. \tag{4.4}$$

Since f is an affine FIF and $|\alpha_h|_\infty < h$, it follows that $f \in DL(I)$. Due to the uniform continuity of g, $w_g(h) \to 0$ as $h \to 0$. Using the fact $w_g(h) \to 0$ as $h \to 0$ in (4.2), the first term in (4.4) tends to zero as $h \to 0$. By Proposition 4.1, the second term tends to zero as $n \to \infty$. Thus, the Fourier–Jacobi expansion of f converges to a given signal g in uniform norm corresponding to these data, with respect to the scale vector α_h. □

Remark 4.6. If the values of r and s are restricted in $(-1, -1/2]$, then the above two theorems are true on the whole interval $[-1, 1]$ in view of Proposition 4.2.

Consider the nth partial sum $S_n^{(r,s)}(f)$ of the Fourier–Jacobi expansion of an affine FIF f corresponding to a data set $\{(x_i, y_i)\}_{i=0}^N$ with the uniform step size $h = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$. As $h \to 0$, the sample size N increases. Let n and N be such that $N = ns$ for some positive constant s. The convergence of the corresponding fractal function $S_n^{(r,s,\alpha)}(f)$ is established in the following theorem.

Theorem 4.7. *Let $g \in \mathcal{C}(I)$ be the original function providing the data and f be the corresponding AFIF with the scale vector α_h, $|\alpha_h|_\infty < h$ where h is the step size. Then the fractal analogue $S_n^{(r,s,\alpha)}(f)$ corresponding to nth partial sum $S_n^{(r,s)}(f)$ converges uniformly to g as $h \to 0$ in intervals $[-1+\delta, 1-\delta] \subset (-1, 1)$ where $\delta > 0$.*

Proof. Denote $P^h = S_n^{(r,s,\alpha)}(f)$. Then,

$$\|g - P^h\|_\infty \leq \|g - f\|_\infty + \|f - S_n^{(r,s)}(f)\|_\infty + \|S_n^{(r,s)}(f) - P^h\|_\infty. \tag{4.5}$$

The following inequality can be obtained from ([30], Prop. 3)

$$\|g - f\|_\infty \leq w_g(h) + \frac{2|\alpha_h|_\infty}{1 - |\alpha_h|_\infty}\|g\|_\infty.$$

The uniform continuity of g implies that $w_g(h) \to 0$ as $h \to 0$. Since α_h tends to zero as well, it together gives that $\|g - f\|_\infty \to 0$ as $h \to 0$.

Since f is an AFIF and $|\alpha_h|_\infty < h$, it follows that $f \in$ Lip 1 as shown in earlier theorem. Since $f \in$ Lip 1,

$$w_f(h) \leq C\,h. \tag{4.6}$$

Using (2.27) for the second term, one can get

$$\|f - S_n^{(r,s)}(f)\|_\infty \leq (1 + \|S_n^{(r,s)}\|)E_n(f).$$

For a function $f \in \mathcal{C}[-1,1]$, $\|S_n^{(r,s)}\|_\infty$ does not exceed $\ln(n+1)$ on $[-1+\delta, 1-\delta]$ (Ref: [34], Section 9.3). Using it along with Jackson's Theorem V in [13], we get

$$\begin{aligned}\|f - S_n^{(r,s)}(f)\|_\infty &\leq (1+\ln(n+1))w_f(\pi/(n+1))\\ &\leq (1+\ln(n+1))C(\pi/(n+1)) \text{ using (4.6)}\end{aligned}$$

Corresponding to these data, the above term tends to zero as h tends to zero ($n \to \infty$ as a consequence of $h \to 0$). For third term using the inequality (2.11), it follows that

$$\|S_n^{(r,s)}(f) - S_n^{(r,s,\alpha)}(f)\|_\infty \leq \frac{|\alpha_h|_\infty}{1-|\alpha_h|_\infty}\|S_n^{(r,s)}(f) - b\|_\infty.$$

Since $|\alpha_h|_\infty < h$, the above term tends to zero as $h \to 0$.

This completes the proof. □

4.2. Convergence Using Smooth FIF

The convergence of Fourier–Jacobi expansion corresponding to a non-affine smooth FIF interpolating certain data set in weighted-mean square norm is proved in this section.

Theorem 4.8. *Suppose $p > \max(r,s) + 1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_n - x_{n-1}$. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with scale vector α_h such that $|\alpha_h|_\infty < a_n^p$ via the IFS (2.20). Then the Fourier–Jacobi expansion of f converges in weighted quadratic mean to g, that is, $S_n^{(r,s)}(f)$ converges to g in $\mathcal{L}^2_\rho$ as $h \to 0$ and $n \to \infty$.*

Proof. Given that $|\alpha_h|_\infty < a_n^p$, $q_n \in \mathcal{C}^p(I)$ and b is a Hermite interpolating polynomial which agrees with the function at the extremes of the interval up to pth derivative. Then by Theorem 2.1, the corresponding IFS (2.20) determines a FIF $f \in \mathcal{C}^p(I)$ (see [29]). The mean value theorem shows that $f^{p-1} \in$ Lip 1. Using Proposition 4.3, we get

$$\|S_n^{(r,s)}(f) - f\|_\infty \to 0 \text{ as } n \to \infty$$

and, therefore,

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}^2_\rho} \to 0 \text{ as } n \to \infty. \tag{4.7}$$

Using the inequality (2.21), for the function and its fractal

$$\|f - g\|_{\mathcal{L}^2_\rho} \leq \frac{|\alpha_h|_\infty}{1-|\alpha_h|_\infty}\|g - b\|_{\mathcal{L}^2_\rho}\left(\int_{-1}^{1}\rho^{r,s}(t)\mathrm{d}t\right)^{1/2}.$$

Therefore,

$$\|f-g\|_{\mathcal{L}^2_\rho} \to 0 \text{ as } h \to 0, \tag{4.8}$$

since a_n goes to zero when h tends to zero, and thus α_h. Using (4.7) and (4.8) in the following inequality

$$\|g-S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho} \le \|g-f\|_{\mathcal{L}^2_\rho} + \|f-S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho},$$

the result follows. □

In the following theorem, the uniform convergence of Fourier–Jacobi expansion is established.

Theorem 4.9. *Suppose $p > \max(r,s)+1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_n - x_{n-1}$. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with constant scale vector α_h such that $|\alpha_h|_\infty < a_n^p$ via the IFS (2.20). Then the Fourier–Jacobi expansion of f converges uniformly to g as $h \to 0$ and $n \to \infty$.*

Proof. If $S_n^{(r,s)}(f)$ is the nth partial sum of the Fourier–Jacobi expansion, then

$$\|g-S_n^{(r,s)}(f)\|_\infty \le \|g-f\|_\infty + \|f-S_n^{(r,s)}(f)\|_\infty. \tag{4.9}$$

Then the first term in the inequality (4.9) tends to zero as $h \to 0$, due to (2.21). Using the mean value theorem, $f \in \mathcal{C}^p(I) \Rightarrow f^{p-1} \in \text{Lip } 1$. Then by Proposition 4.3, the second term in (4.9) tends to zero as $n \to \infty$. Hence, the proof. □

The convergence of Fourier–Jacobi expansion for smooth FIF corresponding to IFS (2.22) is proved in the next two theorems.

Theorem 4.10. *Suppose $p > \max(r,s)+1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_n - x_{n-1}$. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with scale vector α_h such that $|\alpha_h|_\infty < a_n^p$ and a family of base function $B = \{b_i : i = 1, 2 \dots N\}$ are such that the derivatives up to pth order of each of its member agrees with that of g at the end points of the interval. Then the Fourier–Jacobi expansion of f converges in weighted quadratic mean to g as $h \to 0$ and $n \to \infty$.*

Proof. The proof is similar to Theorem 4.8. Here instead of taking (2.21), one need to use (2.23) for the inequality in between the function g and its fractal f. □

Theorem 4.11. *Suppose $p > \max(r,s)+1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_n - x_{n-1}$. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with scale vector α_h such that $|\alpha_h|_\infty < a_n^p$ and a family of base function $B = \{b_i : i = 1, 2 \dots N\}$ are such that the derivatives up to pth order of each of its member agrees with that of g at the end points of the interval. Then the Fourier–Jacobi expansion of f converges uniformly to g as $h \to 0$ and $n \to \infty$.*

Proof. The proof follows on similar lines of the proof of Theorem 4.9. □

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