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Operational Tau method for singular system of Volterra integro-differential equations

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ABSTRACT

The Legendre spectral Tau matrix formulation is proposed to approximate solution of singular system of Volterra integro-differential equations. The existence and uniqueness solution of this system are investigated by means of the υ-smoothing property of a Volterra integral operator and some projectors. The L^2-convergence of the numerical method is analyzed. It is proved theoretically and demonstrated numerically that the proposed method converges exponentially. Finally, two numerical examples illustrate the theoretical results.

1. Introduction

System of integro-differential equations arise in many mathematical modeling processes such as population growth, one dimensional viscoelasticity and reactor dynamics [1–4]. Singular systems of Volterra integro-differential equations or Volterra integro-differential–algebraic equations (**IDAEs**) are encountered as a differential–algebraic system together with an integral operator which arise in modeling nonlinear electric chains with after-effect [5,6]. In this paper, we consider numerical method for solving the singular systems of Volterra integro-differential equations in the following form:

$$L[X(t)] = A(t)X'(t) + B(t)X(t) + \int_0^t K(t,s)X(s)ds = f(t), \quad X(0) = X_0, \tag{1.1}$$

with $t \in \Omega = [0, 1]$ and $f : \Omega \to R^d (d \geq 1)$. $A(t)$ and $B(t)$ are given $d \times d$ matrices. $K(t, s)$ is the kernel matrix defined in the domain $D = \{(t, s) : 0 \leq s \leq t \leq 1\}$ and $X : \Omega \to R^d$ is the unknown function. We assume that $Rank(A) \geq 1$ and

$$detA(t) = 0, \quad \forall t \in \Omega.$$

The semi-explicit form of the system (1.1) can be described by

$$A(t) = dig(I_{d_1}, O_{d_2}), \quad d_1 + d_2 = d.$$

An initial investigation of these equations indicates that they have properties very similar to differential–algebraic equations (DAEs) [7–13]. If in system (1.1), $K(t, s) \equiv 0$, then we have a linear DAE system. In other words, we can consider linear DAEs as a special form of IDAEs (1.1). Also, it can be shown that IDAE system has properties similar to integral-algebraic equations (IAEs). Author, refers the interested reader to [14–24] for more research works on the IAE systems.

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The study of implementation of implicit Runge–Kutta methods of Pouzet-type for singular systems of Volterra integro-differential equations has been provided by Kauthen [25]. He studied the convergence properties of this numerical method as fully discretized collocation method. Brunner [14] analyzed global and local superconvergence of piecewise polynomial collocation solutions for the semi-explicit index-1 IDAEs. Bulatov [26,27] considered a class of first-order integro-differential equations with a degenerate matrix multiplying the derivative and suggested a numerical solution method based on Euler's implicit method together with a quadrature formula using left rectangles. Also the multistep method to solving a certain class of IDAEs has been provided by Bulatov and Chistyakova [28].

The present paper is devoted to the study of numerical solvability of the integro-differential–algebraic equations (1.1). For this aim, in Section 2, using the υ-smoothing property of a Volterra integral operator and some projectors, we decouple the IDAEs (1.1) into the mixed system of Volterra integro-differential equations (VIDEs) and integral equations (VIEs), and then investigate the existence and uniqueness solution of the obtained system. In Section 3, the operational Tau method as well-known method is applied to approximate the solution of IDAE system (1.1). Convergence analysis of the proposed numerical method is investigated in Section 4 and in Section 5, the results of numerical experiments are compared with analytical solution and with those of other recently published methods to confirm the accuracy and efficiency of the new scheme which is presented in this paper.

2. Existence and uniqueness solution

In this section, we firstly use the υ-smoothing property of a Volterra integral operator and some projectors to decouple the system (1.1) into the inherent system of Volterra integro-differential equations (VIDEs) and a system of Volterra integral equations (VIEs), and then the construction of solution of the system (1.1) is introduced by the existence and uniqueness theorem.

Definition 1 (*[20]*). The Volterra integral operator in (1.1) corresponding to the kernel matrix $K(t,s) = \begin{pmatrix} k_{pq}(t,s) \\ p,q=1,\ldots,d \end{pmatrix}$, with $d \geq 2$, is said to be υ-smoothing if there exist integers $\upsilon_{pq} \geq 1$ with $\upsilon = \max_{1\leq p,q\leq d}\{\upsilon_{pq}\}$ such that the following conditions hold:

(1) $\frac{\partial^i k_{pq}(t,s)}{\partial t^i}\big|_{s=t} = 0,\ t \in \Omega,\ i = 0,\ldots,\upsilon_{pq}-2,$

(2) $\frac{\partial^{\upsilon_{pq}-1} k_{pq}(t,s)}{\partial t^{\upsilon_{pq}-1}}\big|_{s=t} \neq 0,\ t \in \Omega,$

(3) $\frac{\partial^{\upsilon_{pq}} k_{pq}(t,s)}{\partial t^{\upsilon_{pq}}} \in C(D).$

We set $\upsilon_{pq} = 0$ when $k_{pq}(t,s) \equiv 0$.

Now, let the Volterra integral operator in (1.1) be 1-smoothing and $K = K(t,t)$, we rewrite system (1.1) as:

$$A(t)(P(t)X(t))' + B_1(t)(P(t)X(t) + Q(t)X(t)) + V + W = f(t), \tag{2.1}$$

where $Q(t)$ denotes a projector onto $ker\, A(t)$, $P(t) = I - Q(t)$, $Q(t)^2 = Q(t)$, $B_1(t) = B(t) - A(t)P'(t)$ and

$$V = \int_0^t K(t,s)P(s)X(s)ds, \qquad W = \int_0^t K(t,s)Q(s)X(s)ds.$$

Also, system (2.1) can be rewritten as

$$A_1(t)\big(P(t)(P(t)X(t))' + Q(t)X(t)\big) + B_1(t)P(t)X(t) + V + W - KQ(t)X(t) = f(t), \tag{2.2}$$

where $A_1(t) = A(t) + B_1(t)Q(t) + KQ(t)$. Let $u = P(t)X(t),\ v = Q(t)X(t)$ and $det(A_1(t)) \neq 0,\ \forall t \in \Omega$. Multiplying (2.2) by $P(t)A_1^{-1}(t)$ and $Q(t)A_1^{-1}(t)$, respectively, we have the following mixed system of Volterra integro-differential equations and Volterra integral equations

$$\begin{cases} u' + B_2(t)u + B_3(t)v + \int_0^t \bar{K}(t,s)u(s)ds + \int_0^t \bar{K}(t,s)v(s)ds = g_1(t), \\ \quad B_4(t)v + B_5(t)u + \int_0^t \hat{K}(t,s)u(s)ds + \int_0^t \hat{K}(t,s)v(s)ds = g_2(t), \end{cases} \tag{2.3}$$

where $B_2(t) = \Big(P(t)A_1^{-1}(t)B_1(t) - Q(t)P'(t)\Big)$, $B_3(t) = \Big(-P(t)A_1^{-1}(t)K - Q(t)P'(t)\Big)$, $B_4(t) = \Big(I - Q(t)A_1^{-1}(t)K\Big)$, $B_5(t) = Q(t)A_1^{-1}(t)B_1(t)$, $\bar{K}(t,s) = P(t)A_1^{-1}(t)K(t,s)$, $\hat{K}(t,s) = Q(t)A_1^{-1}(t)K(t,s)$, $g_1(t) = P(t)A_1^{-1}(t)f(t)$ and $g_2(t) = Q(t)A_1^{-1}(t)f(t)$.

If $B_4(t)$ be invertible, then differentiating from the second equation of (2.3) and inserting u' from the its first equation and some manipulations, lead to the following second-kind integro-differential equation

$$v' + \tilde{B}_6(t)u + \tilde{B}_7(t)v + \int_0^t \tilde{K}(t,s)u(s)ds + \int_0^t \tilde{K}(t,s)v(s)ds = \tilde{g}_2(t), \tag{2.4}$$

where the meaning of $\tilde{B}_6(t)$, $\tilde{B}_7(t)$, $\tilde{K}(t,s)$ and $\tilde{g}_2(t)$ is clear.

Now, (2.4) together with the first equation of (2.3) is as a regular system of Volterra integro-differential equations. Hence the existence and uniqueness solution of the obtained system can be related to Theorem 3.1.1 from [14]. Finally, the following theorem gives the relevant conditions for the investigation of the unique solution of the singular systems of Volterra integro-differential equations (1.1):

Theorem 1. *Let the Volterra integral operator in (1.1) be 1-smoothing and* $\det(A_1(t)) \neq 0,\ \forall t \in \Omega$. *Assume that for* $\nu \geq 0$

1. $B_2, B_3, g_1 \in C^{\nu}(\Omega)$ *and* $\bar{K}(.,.) \in C^{\nu}(D)$,
2. $B_4, B_5, g_2 \in C^{\nu+1}(\Omega)$ *and* $\hat{K}(.,.) \in C^{\nu+1}(D)$,

   then for any set of consistent initial values u_0, v_0 *the singular system of Volterra integro-differential equations (1.1) possesses a unique solution* $(u, v)^T$, *with* $u \in C^{\nu+1}(\Omega)$ *and* $v \in C^{\nu}(\Omega)$.

Remark 1. In some cases when $B_4(t)$ is not invertible, the existence and uniqueness solution of the singular systems of Volterra integro-differential equations (1.1) may be investigated by considering new strategy. For more details, see the following example

Example 1. Consider IDAE system as:

$$A(t)X'(t) + B(t)X(t) + \int_0^t K(t,s)X(s)ds = f(t), \tag{2.5}$$

where

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}, \qquad K = K(t,t) = \begin{pmatrix} k_{11}(t,t) & k_{12}(t,t) \\ k_{21}(t,t) & k_{22}(t,t) \end{pmatrix}.$$

We can take $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $P = I - Q$ and

$$B_1(t) = B(t) - A(t)P'(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}.$$

The corresponding matrix

$$A_1(t) = A(t) + B_1(t)Q(t) + KQ(t) = \begin{pmatrix} 1 & B_{12} + k_{13} \\ 0 & k_{22} + B_{22} \end{pmatrix},$$

is nonsingular if $k_{22} + B_{22} \neq 0$. Let $k_{22} \neq 0$ and $B_{22} = 0$, then

$$A_1^{-1}(t) = \begin{pmatrix} 1 & \dfrac{B_{12} + k_{13}}{-k_{22}} \\ 0 & \dfrac{1}{k_{22}} \end{pmatrix}.$$

In this case $B_4 = I - QA_1^{-1}K$ is not invertible but

$$B_4 v = v - QA_1^{-1}Kv = 0 \Rightarrow v = QA_1^{-1}Kv,$$

then differentiating from the second equation of (2.3) leads to the following equation

$$v + (B_5(t)u)' + \hat{K}(t,t)u(t) + \int_0^t \frac{\partial \hat{K}(t,s)}{\partial t} u(s)ds + \int_0^t \frac{\partial \hat{K}(t,s)}{\partial t} v(s)ds = g_2'(t). \tag{2.6}$$

Considering (2.6) together with the first equation of (2.3) and using differentiation, we can investigate the existence and uniqueness solution of the obtained system.

3. The Legendre spectral Tau method

We firstly transform system (1.1) into the mixed system (2.3) and then in the Legendre Tau method we seek solution as:

$$u_N(t) = \{u_{iN}\}_{i=1}^d = \left\{ \sum_{k=0}^N (a_i)_k \varphi_k(t) \right\}_{i=1}^d = \{\underline{a}_i\}_{i=1}^d \bigotimes \underline{\varphi} = \{\underline{a}_i\}_{i=1}^d \bigotimes \phi \underline{X}_t, \tag{3.1}$$

$$v_N(t) = \{v_{iN}\}_{i=1}^d = \left\{ \sum_{k=0}^N (b_i)_k \varphi_k(t) \right\}_{i=1}^d = \{\underline{b}_i\}_{i=1}^d \bigotimes \underline{\varphi} = \{\underline{b}_i\}_{i=1}^d \bigotimes \phi \underline{X}_t, \tag{3.2}$$

where $\underline{a}_i = (a_{i0}, a_{i1}, \ldots, a_{iN}, 0, \ldots)$, $\underline{b}_i = (b_{i0}, b_{i1}, \ldots, b_{iN}, 0, \ldots)$ and $\underline{\varphi} = (\varphi_0(t), \varphi_1(t), \ldots)$, $t \in [0, 1]$ is a set of Legendre orthogonal polynomials. Also ϕ is a lower triangular coefficient matrix of Legendre polynomials, $\underline{X}_t = (1, t, t^2, \ldots)^T$, $\underline{\varphi} = \phi \underline{X}_t$ and $\bigotimes$ is a Kronecker product. We approximate $B_l(t)(l = 2, \ldots, 5)$, $g_1(t)$ and $g_2(t)$ by polynomials to any degree of accuracy by interpolation or other suitable methods

$$\begin{aligned} B_l(t) &= \{B_{lij}\}_{i,j=1}^d = \left\{ \sum_{k=0}^{N_{B_l}} (B_{lij})_k t^k \right\}_{i,j=1}^d = \left\{\underline{B}_{lij}\right\}_{i,j=1}^d \bigotimes \underline{X}_t, \quad (l = 2, \ldots, 5) \\ g_l(t) &= \{g_{li}\}_{i=1}^d = \left\{ \sum_{k=0}^{N_{g_l}} (g_{li})_k t^k \right\}_{i=1}^d = \left\{\underline{g}_{li}\right\}_{i=1}^d \bigotimes \underline{X}_t, \quad (l = 1, 2), \end{aligned} \tag{3.3}$$

where $\underline{B}_{lij} = (B_{lij0}, \ldots, B_{lijN_{B_l}}, 0, \ldots)$ and $\underline{g}_{li} = (g_{li0}, \ldots, g_{liN_{g_l}}, 0, \ldots)$. Now, we can consider the approximation of u' by u'_N in the following matrix representation (see [29,30])

$$u'_N(t) = \{\underline{a}_i\}_{i=1}^d \bigotimes \phi \Psi \underline{X}_t = \{\underline{a}_i\}_{i=1}^d \bigotimes \phi \Psi \phi^{-1} \underline{\varphi}, \tag{3.4}$$

where

$$\Psi = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Also, for matrix representations of $B_l(t)u_N(t)$ $(l = 2, 3)$ and $B_l(t)v_N(t)$ $(l = 3, 4)$, we consider the effect of multiplication by the variable t on u_N and v_n as

$$t^k u_N(t) = \{\underline{a}_i\}_{i=1}^d \bigotimes \phi M^k \underline{X}_t, \tag{3.5}$$

$$t^k v_N(t) = \{\underline{b}_i\}_{i=1}^d \bigotimes \phi M^k \underline{X}_t, \tag{3.6}$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

$$\begin{aligned} B_l(t)u_N(t) &= \left(\sum_{j=1}^d B_{lij}(t) u_{jN}(t) \right)_{i=1}^d = \left(\sum_{j=1}^d \left(\underline{B}_{lij}\underline{X}_t\right) \left(\underline{a}_j \phi \underline{X}_t\right) \right)_{i=1}^d \\ &= \left(\sum_{j=1}^d \underline{a}_j \phi B_{lij}(M) \phi^{-1} \underline{\varphi} \right)_{i=1}^d, \quad (l = 3, 4), \end{aligned} \tag{3.7}$$

$$\begin{aligned} B_l(t)v_N(t) &= \left(\sum_{j=1}^d B_{lij}(t) v_{jN}(t) \right)_{i=1}^d = \left(\sum_{j=1}^d \left(\underline{B}_{lij}\underline{X}_t\right) \left(\underline{b}_j \phi \underline{X}_t\right) \right)_{i=1}^d \\ &= \left(\sum_{j=1}^d \underline{b}_j \phi B_{lij}(M) \phi^{-1} \underline{\varphi} \right)_{i=1}^d, \quad (l = 4, 5). \end{aligned} \tag{3.8}$$

For obtaining the matrix vector multiplication representation for the integral term, we consider the following lemma from [31].

Lemma 1. *Assume that functions $x(s)$ and $K(t, s)$ can be expressed as:*

$$x(s) = \sum_{i=0}^{\infty} a_i \varphi_i(s) = \underline{a} \phi \underline{X}_s, \qquad K(t, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} k_{ij} \varphi_i(t) \varphi_j(s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \bar{k}_{ij} t^i s^j,$$

where $\underline{a} = (a_0, a_1, \ldots)$ *and* ϕ, $\underline{X}_s$ *are defined in* (3.1)*, then we have*

$$\int_0^t K(t,s)x(s)ds = \underline{a}\phi\Upsilon\underline{X}_t = \underline{a}\phi\Upsilon\phi^{-1}\underline{\varphi},$$

where

$$\Upsilon = \begin{pmatrix} 0 & \bar{k}_{00} & \bar{k}_{10} + \frac{1}{2}\bar{k}_{01} & \bar{k}_{20} + \frac{1}{2}\bar{k}_{11} + \frac{1}{3}\bar{k}_{02} & \cdots \\ 0 & 0 & \frac{1}{2}\bar{k}_{00} & \frac{1}{2}\bar{k}_{10} + \frac{1}{3}\bar{k}_{01} & \cdots \\ 0 & 0 & 0 & \frac{1}{3}\bar{k}_{00} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & \frac{1}{n}\bar{k}_{00} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Now, we consider the Tau approximation of the integral term of (2.3), as follows:

$$\int_0^t \bar{K}(t,s)u_N(s)ds = \left(\sum_{q=1}^{d}\int_0^t \bar{K}_{pq}(t,s)u_{qN}(s)ds\right)_{p=1}^{d} = \left(\sum_{q=1}^{d}\underline{a}_q\phi\bar{\Upsilon}_{pq}\phi^{-1}\underline{\varphi}\right)_{p=1}^{d}. \tag{3.9}$$

Inserting (3.1)–(3.4) and (3.7)–(3.9) into (2.3), we have

$$\begin{cases} \{\underline{a}_i\}_{i=1}^{d}\bigotimes\phi\Psi\phi^{-1}\underline{\varphi} + \left(\sum_{j=1}^{d}\big(\underline{a}_j\phi B_{2ij}(M) + \underline{b}_j\phi B_{3ij}(M) + (\underline{a}_j + \underline{b}_j)\phi\bar{\Upsilon}_{ij}\big)\phi^{-1}\underline{\varphi}\right)_{i=1}^{d} \\ \quad = \{\underline{g}_{1i}\}_{i=1}^{d}\bigotimes\phi^{-1}\underline{\varphi}, \\ \left(\sum_{j=1}^{d}\big(\underline{b}_j\phi B_{4ij}(M) + \underline{a}_j\phi B_{5ij}(M) + (\underline{a}_j + \underline{b}_j)\phi\hat{\Upsilon}_{ij}\big)\phi^{-1}\underline{\varphi}\right)_{i=1}^{d} = \{\underline{g}_{2i}\}_{i=1}^{d}\bigotimes\phi^{-1}\underline{\varphi}. \end{cases} \tag{3.10}$$

Considering the initial condition from (1.1), we obtain

$$\begin{aligned} &\{\underline{a}_i\}_{i=1}^{d}\bigotimes\phi\underline{X}_0 = P(0)X(0), \\ &\{\underline{b}_i\}_{i=1}^{d}\bigotimes\phi\underline{X}_0 = Q(0)X(0). \end{aligned} \tag{3.11}$$

Because of the orthogonality of $\underline{\varphi} = (\varphi_0(t), \varphi_1(t), \ldots)$, projecting (3.10) on the $\{\varphi_k(t)\}_{k=0}^{N-1}$ together with the equations (3.11) yields a linear system of algebraic equations for the unknown coefficients $\{\underline{a}_i\}_{i=1}^{d}$ and $\{\underline{b}_i\}_{i=1}^{d}$.

4. Convergence analysis

Assume that $H^m(\Lambda)$ denotes the Sobolev space of all functions $\phi(\mathbf{x})(\mathbf{x} = (x_1 \cdots, x_p))$ on $\Lambda = (0,1)^p (p = 1, 2)$ such that $\phi(\mathbf{x})$ and all its weak derivatives up to order m are in $L^2(\Lambda)$, with the norm and the semi-norm as

$$\|\phi\|^2_{H^m(\Lambda)} = \left(\sum_{|\alpha|\le m}\|D^\alpha\phi\|^2_{L^2(\Lambda)}\right)^{\frac{1}{2}},$$

$$|\phi|_{H^{m,N}(\Lambda)} = \left(\sum_{j=\min(m,N+1)}^{m}\sum_{i=1}^{p}\|D_i^j\phi\|^2_{L^2(\Lambda)}\right)^{1/2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a nonnegative multi-index with $D^\alpha\phi = \frac{\partial^{\alpha_1+\cdots+\alpha_p}\phi}{\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}$.

Let $\mathbb{P}_N(\Lambda)$ be the space of all polynomials with degree not exceeding N on Λ. Denote by P_N the orthogonal projective operator from $L^2(\Lambda)$ on to $\mathbb{P}_N(\Lambda)$. Concerning the truncation error of shifted Legendre series, the following estimates hold for all $\phi \in H^m(\Lambda)$, $m \ge 1$, from [32] as:

$$\|\phi - P_N\phi\|_{L^2(\Lambda)} \le CN^{-m}|\phi|_{H^{m,N}(\Lambda)}. \tag{4.1}$$

To prove the error estimate in weighted L^2-norm, we need the generalized Hardy's inequality as

Lemma 2 ([33]). *For all measurable functions $f \geq 0$, the generalized Hardy's inequality*

$$\Big(\int_a^b |(Tf)(x)|^q w_1(x)dx\Big)^{\frac{1}{q}} \leq C\Big(\int_a^b |f(x)|^p w_2(x)dx\Big)^{\frac{1}{p}},$$

holds if and only if

$$\sup_{a<x<b}\Big(\int_x^b w_1(t)dt\Big)^{\frac{1}{q}}\Big(\int_a^x w_2^{1-p'}(t)dt\Big)^{\frac{1}{p'}} < \infty, \quad p' = \frac{p}{p-1},$$

for the case $1 < p \leq q < \infty$. Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x,t)f(t)dt,$$

with $k(x,t)$ a given kernel, w_1 and w_2 weight functions, and $-\infty \leq a < b \leq \infty$.

In this position, we state the following main theorem which reveals the convergence results of the presented scheme in L^2-norm:

Theorem 2. *Assume that the given functions in (1.1) satisfy the conditions of Theorem 1. Let $(u_N, v_N)^T$ be the spectral Legendre Tau approximation of $(u, v)^T$ which is defined by (3.1), (3.2) and (3.10). Then, for all sufficiently large N, we have*

$$\|u_N - u\|_{L^2(0,1)} \to 0,$$
$$\|v_N - v\|_{L^2(0,1)} \to 0.$$

Proof. According to the proposed method in previous section, we have

$$\begin{cases} u_N' + B_{2N}u_N + B_{3N}v_N + \int_0^t \bar{K}_{N,N}(t,s)u_N(s)ds + \int_0^t \bar{K}_{N,N}(t,s)v_N(s)ds = g_{1N}, \\ \quad B_{4N}v_N + B_{5N}u_N + \int_0^t \hat{K}_{N,N}(t,s)u(s)ds + \int_0^t \hat{K}(t,s)v_{N,N}(s)ds = g_{2N}. \end{cases} \tag{4.2}$$

Let $e_1 = u_N - u$ and $e_2 = v_N - v$. Subtracting (4.2) from (2.3) and some manipulations, we get

$$\begin{cases} e_1' + B_2e_1 + (B_{2N} - B_2)u_N + B_3e_2 + (B_{3N} - B_3)v_N + \int_0^t \bar{K}(t,s)e_1(s)ds \\ \quad + \int_0^t (\bar{K}_{N,N}(t,s) - \bar{K}(t,s))u_N(s)ds + \int_0^t (\bar{K}_{N,N}(t,s) - \bar{K}(t,s))v_N(s)ds \\ \quad + \int_0^t \bar{K}(t,s)e_2(s)ds = g_{1N} - g_1, \\ B_4e_2 + (B_{4N} - B_4)v_N + B_5e_1 + (B_{5N} - B_5)u_N + \int_0^t \hat{K}(t,s)e_1(s)ds \\ \quad + \int_0^t (\hat{K}_{N,N}(t,s) - \hat{K}(t,s))u_N(s)ds + \int_0^t (\hat{K}_{N,N}(t,s) - \hat{K}(t,s))v_N(s)ds \\ \quad + \int_0^t \hat{K}(t,s)e_2(s)ds = g_{2N} - g_2. \end{cases} \tag{4.3}$$

Assume that

$$e_1' = \hat{e}_1 \Rightarrow e_1 = e_1(0) + \int_0^t \hat{e}_1(s)ds, \tag{4.4}$$

from (3.11), $e_1(0)=0$, inserting (4.4) into (4.3), we have

$$\begin{cases} \hat{e}_1+(B_{2N}-B_2)u_N+B_3e_2+(B_{3N}-B_3)v_N+\int_0^t (B_2(t)+\bar{K}^*(t,s))\hat{e}_1(s)ds \\ \quad +\int_0^t (\bar{K}_{N,N}(t,s)-\bar{K}(t,s))u_N(s)ds+\int_0^t (\bar{K}_{N,N}(t,s)-\bar{K}(t,s))v_N(s)ds \\ \quad +\int_0^t \bar{K}(t,s)e_2(s)ds=g_{1N}-g_1, \\ B_4e_2+(B_{4N}-B_4)v_N+(B_{5N}-B_5)u_N+\int_0^t (B_5(t)+\hat{K}^*(t,s))\hat{e}_1(s)ds \\ +\int_0^t (\hat{K}_{N,N}(t,s)-\hat{K}(t,s))u_N(s)ds+\int_0^t (\hat{K}_{N,N}(t,s)-\hat{K}(t,s))v_N(s)ds \\ \quad +\int_0^t \hat{K}(t,s)e_2(s)ds=g_{2N}-g_2, \end{cases} \tag{4.5}$$

where $\bar{K}^*(t,s)=\int_s^t \bar{K}(t,\eta)d\eta$ and $\hat{K}^*(t,s)=\int_s^t \hat{K}(t,\eta)d\eta$. The previous two equations can be rewritten in matrix notation

$$\mathbf{D}(t)\mathbf{E}(t)+\int_0^t \mathbf{K}^*(t,s)\mathbf{E}(s)ds=\mathbf{J}(t), \tag{4.6}$$

where

$$\mathbf{D}=\begin{pmatrix} I & B_3 \\ 0 & B_4 \end{pmatrix}, \qquad \mathbf{E}=\begin{pmatrix} \hat{e}_1(t) \\ e_2(t) \end{pmatrix}, \qquad \mathbf{K}^*=\begin{pmatrix} (B_2(t)+\bar{K}^*(t,s)) & \bar{K}(t,s) \\ (B_5(t)+\hat{K}^*(t,s)) & \hat{K}(t,s). \end{pmatrix}.$$

Multiplying (4.6) by $\mathbf{D}^{-1}$ and using the Gronwalls inequality [34], we have

$$\|\mathbf{E}\|_{L^2(0,1)} \le C\|\mathbf{J}\|_{L^2(0,1)}. \tag{4.7}$$

We see that by the inequality (4.1),

$$\begin{aligned} \|(B_{2N}-B_2)u_N\|_{L^2(0,1)} &\le \|(B_{2N}-B_2)\|_{L^2(0,1)}\|u_N\|_{L^2(0,1)} \\ &\le CN^{-m}|B_2|_{H^{m,N}(0,1)}(\|u\|_{L^2(0,1)}+\|e_1\|_{L^2(0,1)}) \\ &\le CN^{-m}|B_2|_{H^{m,N}(0,1)}(\|u\|_{L^2(0,1)}+CN^{-m}|u|_{H^{m,N}(0,1)}), \end{aligned}$$

$$\|(B_{3N}-B_3)v_N\|_{L^2(0,1)} \le CN^{-m}|B_3|_{H^{m,N}(0,1)}(\|v\|_{L^2(0,1)}+CN^{-m}|v|_{H^{m,N}(0,1)}),$$

$$\|(B_{4N}-B_4)v_N\|_{L^2(0,1)} \le CN^{-m}|B_4|_{H^{m,N}(0,1)}(\|v\|_{L^2(0,1)}+CN^{-m}|v|_{H^{m,N}(0,1)}),$$

$$\|(B_{5N}-B_5)u_N\|_{L^2(0,1)} \le CN^{-m}|B_5|_{H^{m,N}(0,1)}(\|u\|_{L^2(0,1)}+CN^{-m}|u|_{H^{m,N}(0,1)}),$$

$$\|(g_{1N}-g_1)\|_{L^2(0,1)} \le CN^{-m}|g_1|_{H^{m,N}(0,1)},$$

$$\|(g_{2N}-g_2)\|_{L^2(0,1)} \le CN^{-m}|g_2|_{H^{m,N}(0,1)}.$$

Applying Lemma 2 and (4.1), yields

$$\begin{aligned} \left\|\int_0^t (\bar{K}_{N,N}(t,s)-\bar{K}(t,s))u_N(s)ds\right\|_{L^2(0,1)} &\le C\|(\bar{K}_{N,N}(t,s)-\bar{K}(t,s))\|_{L^2(0,1)^2}\|u_N\|_{L^2(0,1)} \\ &\le CN^{-m}|\bar{K}(t,s)|_{H^{m,N}(0,1)^2}(\|u\|_{L^2(0,1)}+CN^{-m}|u|_{H^{m,N}(0,1)}), \end{aligned}$$

$$\left\|\int_0^t (\bar{K}_{N,N}(t,s)-\bar{K}(t,s))v_N(s)ds\right\|_{L^2(0,1)} \le CN^{-m}|\bar{K}(t,s)|_{H^{m,N}(0,1)^2}(\|v\|_{L^2(0,1)}+CN^{-m}|v|_{H^{m,N}(0,1)}),$$

$$\left\|\int_0^t (\hat{K}_{N,N}(t,s)-\hat{K}(t,s))u_N(s)ds\right\|_{L^2(0,1)} \le CN^{-m}|\hat{K}(t,s)|_{H^{m,N}(0,1)^2}(\|u\|_{L^2(0,1)}+CN^{-m}|u|_{H^{m,N}(0,1)}),$$

$$\left\|\int_0^t (\hat{K}_{N,N}(t,s)-\hat{K}(t,s))v_N(s)ds\right\|_{L^2(0,1)} \le CN^{-m}|\hat{K}(t,s)|_{H^{m,N}(0,1)^2}(\|v\|_{L^2(0,1)}+CN^{-m}|v|_{H^{m,N}(0,1)}).$$

Finally, the above estimates together with (4.7), reveal the convergence results of the presented numerical scheme. □

5. Numerical experiments and results

Two numerical examples in order to illustrate the validity and the accuracy of the proposed technique are considered. All the computations were performed using software Mathematica®.

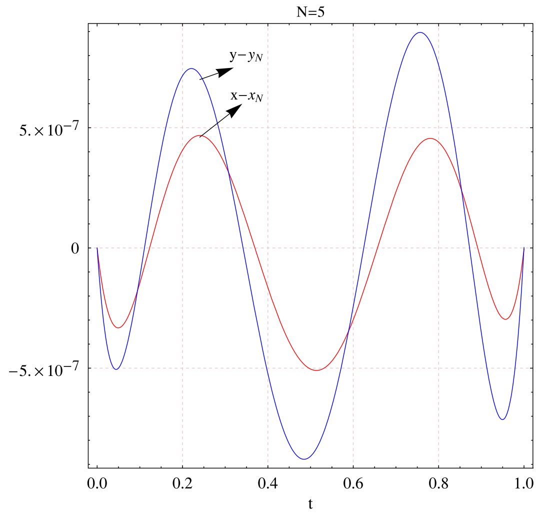

Fig. 1. The Tau approximation errors versus t for $N = 5$ in Example 2.

Table 1
$L^2(0, 1)$ errors for Example 2.

N	2	3	4	5	6
$\|x - x_N\|_{L^2(0,1)}$	4.34×10^{-3}	1.68×10^{-4}	1.37×10^{-5}	3.17×10^{-7}	1.99×10^{-8}
$\|y - y_N\|_{L^2(0,1)}$	2.91×10^{-3}	2.88×10^{-4}	7.78×10^{-6}	5.67×10^{-7}	1.10×10^{-8}
N	7	8	9	10	11
$\|x - x_N\|_{L^2(0,1)}$	3.42×10^{-10}	1.70×10^{-11}	2.33×10^{-13}	9.60×10^{-15}	3.95×10^{-16}
$\|y - y_N\|_{L^2(0,1)}$	6.17×10^{-10}	9.40×10^{-12}	4.23×10^{-13}	5.21×10^{-15}	5.96×10^{-16}

Example 2. Consider IDAE system as:

$$A(t)X'(t) + B(t)X(t) + \int_0^t K(t,s)X(s)ds = f(t), \quad t \in [0, 1], \tag{5.1}$$

where

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \qquad K(t,s) = \begin{pmatrix} (t+s+1) & (t^2+s^2+2) \\ (t+s+4) & (t^2+s^2+1) \end{pmatrix}.$$

$$X(t) = \left(x(t), y(t)\right)^T, \qquad f(t) = \left(f_1(t), f_2(t)\right)^T,$$

and $f_1(t), f_2(t)$ such that the exact solution is:

$$X(t) = \left(\sin t, \cos t\right)^T.$$

The underlying Volterra integral operator in (5.1) is 1-smoothing, $K = K(t,t)$ and $A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus we can take $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $P = I - Q$ and

$$B_1(t) = B(t) - A(t)P'(t) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

The corresponding matrix

$$A_1(t) = A(t) + B_1(t)Q(t) + KQ(t) = \begin{pmatrix} 1 & 2t^2+3 \\ 0 & 2t^2+4 \end{pmatrix},$$

is nonsingular.

Let $(x_N, y_N)^T$ be the Tau approximation of the exact solution $(x, y)^T$ which is given by (3.1) and (3.2). Errors in L^2-norm for several values of N are reported in Table 1 and the plot of the Tau approximation errors for $N = 5$ is shown in Fig. 1.

Table 2
$L^2(0, 1)$ errors for Example 3.

N	2	3	4	5	6
$\lVert x - x_N \rVert_{L^2(0,1)}$	1.49×10^{-2}	7.34×10^{-4}	3.12×10^{-5}	1.21×10^{-6}	4.18×10^{-8}
$\lVert y - y_N \rVert_{L^2(0,1)}$	1.60×10^{-2}	1.01×10^{-3}	5.51×10^{-5}	2.11×10^{-6}	7.48×10^{-8}
$\lVert z - z_N \rVert_{L^2(0,1)}$	6.44×10^{-2}	5.32×10^{-3}	2.72×10^{-4}	1.13×10^{-5}	4.08×10^{-7}
N	7	8	9	10	11
$\lVert x - x_N \rVert_{L^2(0,1)}$	1.27×10^{-9}	3.48×10^{-11}	8.60×10^{-13}	1.94×10^{-14}	7.97×10^{-16}
$\lVert y - y_N \rVert_{L^2(0,1)}$	2.32×10^{-9}	6.41×10^{-11}	1.59×10^{-12}	3.57×10^{-14}	2.30×10^{-15}
$\lVert z - z_N \rVert_{L^2(0,1)}$	1.28×10^{-8}	3.57×10^{-10}	9.07×10^{-12}	4.01×10^{-13}	1.73×10^{-13}

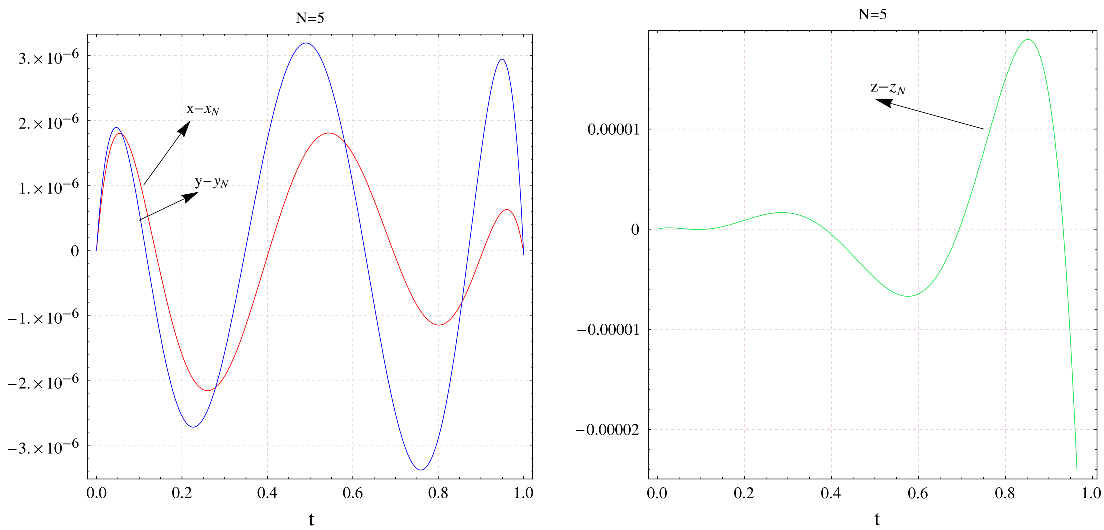

Fig. 2. The Tau approximation errors versus t for $N = 5$ in Example 3.

Example 3. Consider IDAE system from [27] as:

$$A(t)X'(t) + B(t)X(t) + \int_0^t K(t,s)X(s)ds = f(t), \quad t \in [0, 1], \tag{5.2}$$

where

$$A(t) = \begin{pmatrix} (t+1) & 0 & 0 \\ (t+1) & 0 & 0 \\ (t+1) & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & e^t & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad K(t,s) = \begin{pmatrix} e^t & 0 & 1 \\ 1 & 1 & 0 \\ 0 & e^{t+s} & 2 \end{pmatrix},$$

$$X(t) = \left(x(t), y(t), z(t)\right)^T, \quad f(t) = \left(f_1(t), f_2(t), f_3(t)\right)^T,$$

and $f_1(t), f_2(t), f_3(t)$ such that the exact solution is:

$$X(t) = \left(e^t, 2e^t, 3e^t\right)^T.$$

We can take

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = I - Q$$

and

$$B_1(t) = B(t) - A(t)P'(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & e^t & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad K = K(t,t).$$

The corresponding matrix

$$A_1(t) = A(t) + B_1(t)Q(t) + KQ(t) = \begin{pmatrix} (t+1) & 0 & 1 \\ (t+1) & e^t + 1 & 0 \\ (t+1) & e^{2t} & 2 \end{pmatrix},$$

is nonsingular.

Fig. 2 displays the Tau approximation errors for $N = 5$ in Example 3. The L^2-errors for different values of N are reported in Table 2. Indeed, the error obtained with $N = 800$ using the implicit Euler method [27] is the same as that computed using the Tau approximation method with only $N = 4$. However, we note that, for the Tau method, we can calculate the approximation solution in every arbitrary point $t \in [0, 1]$ but for Euler method, we can only compute the approximate solution in the gridpoints t_i. From numerical results in [27] and Table 2, we observe that the results obtained by Tau method are more accurate than the implicit Euler method in this case.

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