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The behavior of fixed point free nonexpansive mappings in geodesic spaces

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ABSTRACT

Dropping the existence of fixed points of a nonexpansive mapping is an interesting and unusual task in metric fixed point theory. Hyperbolic geometry proved to be very relevant in the study of the behavior of fixed point free nonexpansive mappings. In this work we generalize some of the results in that direction in geodesic spaces. More precisely, we show under which additional assumptions the Picard iterative sequence of a mapping defined on a hyperbolic geodesic space tends to a point of the boundary.

1. Introduction

For a wide class of spaces, the boundedness of the closed and convex domain of a nonexpansive self-mapping suffices to expect that T has fixed points. We meet this situation for instance in uniformly convex Banach spaces, complete CAT(κ) spaces or geodesic Ptolemy spaces (see for instance [10,11,16,14]). Otherwise, the results may be completely different. In [12,16,21–23,28] the reader may find a lot of examples of spaces where nonexpansive self-mappings have fixed points even if their domain is unbounded.

The main goal of this paper is to focus on the opposite problem. Namely, how do the fixed point free nonexpansive mappings behave? The fixed point free nonexpansive mappings defined on geodesic δ-hyperbolic spaces are in close relation to the holomorphic functions defined on the interior of bounded closed convex subsets of real and complex Banach spaces; in this case the most important problem is how such functions behave on the boundary of the domain, i.e., the Denjoy–Wolff theorem (see [7,15,16,18]). In very concrete examples of spaces it is known that without additional compactness assumptions of the domain, the Picard iterative sequence for a fixed point free nonexpansive mapping may not converge to the point on the boundary. However, for the special class of firmly nonexpansive mappings, this convergence is guaranteed, for instance, in real and complex Hilbert balls with the hyperbolic metrics as it was shown in [16]. Here we will

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focus on generalizations of such results. More precisely, we will show under which additional assumptions the Picard iterative sequence converges. In our consideration we will use horoballs, i.e., sublevels $f^{-1}(-\infty, a)$ of Busemann (and not only) functions f.

The paper is organized in the following way. Section 2 includes definitions and preliminaries emphasizing the various types of boundaries for geodesic spaces and relations between them. In Section 3 we focus on the behavior of horoballs defined for functions with similar properties as Busemann ones. In Sections 4 and 5 the previous results will be applied to show under which additional assumptions one may expect that the Picard iterative sequence $(T^n x)$ is convergent to a point on the boundary and that this convergence is independent of the choice of $x \in X$, which generalizes results so far known only in very special classes of spaces (see [15,16,18]).

2. Preliminaries

Let us suppose that (X, d) is a geodesic space, i.e., for each pair of points $x, y \in X$ there is an isometric embedding $c\colon [0, d(x, y)] \to X$ such that $c(0) = x$ and $c(d(x, y)) = y$. The image of c is called a metric segment and, if this embedding is unique, it is denoted by $[x, y]$. In this case for any $\alpha \in (0, 1)$ we may define the convex combination $\alpha x + (1-\alpha)y$ as the unique point of the metric segment $[x, y]$ such that $d(x, \alpha x + (1-\alpha)y) = (1-\alpha)d(x, y)$. If the isometric embedding may be extended to $c\colon [0, \infty) \to X$ then its image is called the geodesic ray. Moreover, the space is said to be uniquely geodesic if the uniqueness of embedding holds for any $x, y \in X$. In the sequel we assume that spaces are uniquely geodesic. Next we recall definitions of some subclasses of uniquely geodesic spaces.

Definition 2.1. X is said to be a Busemann space if for each triple of points $x, y, z \in X$ and metric segments $[x, y]$ and $[x, z]$ the inequality

$$d(\alpha x + (1-\alpha)y, \alpha x + (1-\alpha)z) \leq (1-\alpha)d(y, z) \tag{2.1}$$

is satisfied for any $\alpha \in [0, 1]$.

Remark 2.1. Sometimes the space with the property (2.1) is called Busemann convex or hyperbolic (compare with [20,26]).

As a natural example of the Busemann spaces one may consider Hilbert spaces or, more general, the so-called CAT(0) spaces. Let us recall the definition of this class of spaces. Let $\Delta(x_1, x_2, x_3)$ be the triangle consisting of a triple of points $x_1, x_2, x_3 \in X$ (its vertices) and three metric segments joining the vertices (these will be the edges of the triangle). Then one may find a comparison triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane such that $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|$, $i, j \in \{1, 2, 3\}$. We say that $\Delta(x_1, x_2, x_3)$ satisfies the CAT(0) inequality if for all $p, q \in \Delta(x_1, x_2, x_3)$ and their comparison points $\bar{p}, \bar{q} \in \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ the following condition

$$d(p, q) \leq \|\bar{p} - \bar{q}\| \tag{2.2}$$

holds.

Definition 2.2. X is said to be a CAT(0) space if for each triangle $\Delta(x, y, z)$ with $x, y, z \in X$ the CAT(0) inequality (2.2) holds.

CAT(0) spaces satisfy also the following property which will play a crucial role in our considerations.

Let us suppose that (x_n) is a bounded sequence of points of a geodesic space X. Then for each $x \in X$ one may define $r(x,(x_n))$ as

$$r(x,(x_n)) = \limsup_n d(x, x_n).$$

The number $r((x_n)) := \inf_{x \in X} r(x,(x_n))$ is said to be the asymptotic radius of the sequence (x_n) and each point x for which $r(x,(x_n)) = r((x_n))$ is called an asymptotic center of (x_n). If the set of asymptotic centers is a singleton its unique element will be denoted by $A((x_n))$.

Definition 2.3. X is said to satisfy the unique asymptotic center property if each bounded sequence (x_n) has a unique asymptotic center.

Remark 2.2. As mentioned before the unique asymptotic center property holds for CAT(0) spaces but not only. Moreover, let us note that the unique asymptotic center property is independent of the Busemann inequality (2.1). Indeed, let us consider the triangle Δ of the unit sphere S^2 with vertices $(0,0,1)$, $\left(\varepsilon, 0, \sqrt{1-\varepsilon^2}\right)$ and $\left(0, \varepsilon, \sqrt{1-\varepsilon^2}\right)$ where $\varepsilon \in (0,1)$. Then Δ with the spherical distance d_{S^2} defined by

$$d_{S^2}(x,y) = \arccos(x|y),$$

where $(\cdot|\cdot)$ denotes the Euclidean scalar product, is a uniquely geodesic space and satisfies the unique asymptotic center property (see [10, Proposition 4.1]). However, there is no triple of non-collinear points x, y, z for which (2.1) holds.

Now we recall some definitions due to Gromov. We begin with the Gromov product.

Definition 2.4. Let X be a metric space. Then the Gromov product $(y|z)_x$ is a nonnegative number defined by

$$(y|z)_x = \frac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right).$$

Let us note that clearly, we have

$$d(x,y) = (x|z)_y + (y|z)_x.$$

This product can be used to define the concept of hyperbolicity in the Gromov sense.

Definition 2.5. A geodesic space X is called *δ-hyperbolic* for $\delta \geq 0$, if for all triples of points $x, y, z \in X$ the following holds: if y' (belonging to any metric segment joining x and y) and z' (belonging to any metric segment joining x and z) are points with $d(x,y') = d(x,z') \leq (y|z)_x$, then $d(y',z') \leq \delta$.

Remark 2.3. In the main definition we still suppose that X is a geodesic space, however the concept of δ-hyperbolicity may be introduced in a different way for much more general classes of metric spaces (see [17, Section 1.1] and [8, Proposition 2.1.2 and 2.1.3]).

Clearly, there exist CAT(0) spaces which are not δ-hyperbolic, even if Hilbert spaces. But considering CAT(κ) spaces instead of CAT(0) ones, the situation is quite different. We begin with a brief reminder on CAT(κ) spaces.

For $\kappa < 0$, let (M^2_κ, d_κ) denote a complete, simply connected, Riemannian 2-manifold of constant Gaussian curvature κ, so-called model space (see [5, I.2]). As it was done in the case of CAT(0) spaces, for each triangle

$\Delta(x_1, x_2, x_3)$ in the geodesic space X one may consider the comparison triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the model space (M^2_κ, d_κ) with Riemannian distance instead of the Euclidean norm. Then the definition of CAT(κ) inequality does not differ from (2.2), i.e.,

$$d(p, q) \leq d_\kappa(\bar{p}, \bar{q}), \tag{2.3}$$

where $p, q \in \Delta(x_1, x_2, x_3)$; $\bar{p}, \bar{q}$ are their comparison points on $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and d_κ is the Riemannian distance.

Definition 2.6. X is said to be a CAT(κ) space if for each triangle $\Delta(x, y, z)$ with $x, y, z \in X$ the CAT(κ) inequality (2.3) holds.

The reader may find a detailed exposition of this topic in [5]. Here we focus only on the most important property of these spaces from the Gromov hyperbolicity viewpoint.

Proposition 2.1. *(See [5, Proposition III.1.2], [8, Section 1.4.3] and [17, Section 1.5]) Each* CAT(κ) *space, where $\kappa < 0$ is δ-hyperbolic and δ depends only on κ.*

Remark 2.4. The natural examples of Busemann spaces which are not CAT(κ) spaces for any real κ are Banach but not Hilbert spaces (see Proposition I.1.14 in [5]). If we additionally assume that a Banach space X is uniformly convex then X (and each closed and convex subset of X) has the unique asymptotic center property (see [14, p. 92]). Moreover, for any $\bar{x} \in X$ with $\|\bar{x}\| > 0$ one may consider the tube

$$Y = \{x \in X : \exists t \in \mathbb{R}, \|x - t\bar{x}\| \leq R\}.$$

Clearly, Y is also $2R$-hyperbolic space. More interesting examples of δ-hyperbolic spaces can be obtained by gluing sets of this kind.

Next we will introduce three types of boundaries at infinity for a geodesic space X, we will present relations between them and show in which cases they are equal to each other. First, we will focus on the geodesic boundary as the most natural one for uniquely geodesic spaces.

Let us suppose that for two geodesic rays c and c' there is a positive number M such that $d(c(t), c'(t)) \leq M$ for all $t \geq 0$. Then we say that c and c' are asymptotic rays (see [5, Definition II.8.1]).

Definition 2.7. The geodesic boundary $\partial^g X$ of X is a set of equivalence classes of geodesic rays, where two rays are equivalent if and only if they are asymptotic.

Remark 2.5. If we additionally assume that X is a Busemann space then for each $x \in X$ and $\xi \in \partial^g X$ there exists a unique geodesic ray issuing from x and belonging to the equivalence class ξ (see [9, Proposition 5.2]). In the sequel in that case we will use the notion of tending or converging to ξ.

To define the topology on $\partial^g X$ let us assume that X is a Busemann space. Then the extended space $X \cup \partial^g X$ may be equipped with the cone topology. This topology coincides with the natural topology on X. To define the subbasis for points at infinity let us fix a base point $o \in X$. Hence from Remark 2.5 it follows that for any $\xi \in \partial^g X$ there is a unique geodesic ray c issuing from o and tending to ξ. Then the subbasis of the cone topology is the set $\{U(c, R, \varepsilon) : R, \varepsilon > 0\}$, where

$$U(c, R, \varepsilon) = \{x \in X \cup \partial^g X : d(o, x) > R \wedge d(P_R(x), c(R)) < \varepsilon\},$$

where $P_R(x)$ is a unique point of the metric segment $[o, x]$ with $d(o, P_R(x)) = R$ when $x \in X$ or of the geodesic ray issuing from o and tending to x when $x \in \partial^g X$ (compare with [5, II.8.6]). Moreover, the cone topology based on this subbasis is independent of a choice of o, so the topology extended to $\partial^g X$ is well defined (see [9, Lemma 5.3]).

It is worth mentioning that if X is a δ-hyperbolic proper geodesic space, then this boundary coincides with the Gromov boundary – the proof of this fact and necessary definitions may be found in [13, Chapter 7]. Here we only recall that the sequence of points (x_n) of a δ-hyperbolic space X is said to converge to a point (of the Gromov boundary) at infinity if $\lim_{m,n}(x_m|x_n)_x = \infty$ with respect to some (so also to any) base point $x \in X$ and the sequence (y_n) tends to the same point at infinity if $(x_n|y_n)_x \to \infty$ when $n \to \infty$.

Now let us consider the space $C(X)$ of continuous functions on X endowed with the topology of uniform convergence on bounded subsets. For a fixed base point $o \in X$ and for any $x \in X$ we may define the function $b_x : X \to \mathbb{R}$ in the following way

$$b_x(y) = d(x, y) - d(x, o).$$

Then the closure of $\{b_x :\ x \in X\}$ in $C(X)$ is called the Kuratowski compactification and for an unbounded sequence (x_n) the limit $\lim_n b_{x_n}$, if it exists, is called a Busemann function. The reader may find a detailed exposition of this topic, inter alia, independence of the base point o in [4, Chapter II]. Here we focus only on the case when the Kuratowski compactification (equipped in the cone topology) and the geodesic boundary (endowed with the topology of uniform convergence on bounded subsets) coincide.

Proposition 2.2. *(See [4, Proposition 2.5]) Let X be a complete CAT(0) space, $o \in X$ be a fixed point and (x_n) be a sequence in X with $d(o, x_n) \to \infty$. Then (b_{x_n}) converges to a Busemann function f (uniformly on bounded subsets of X) if and only if the metric segments $[o, x_n]$ converge to a geodesic ray σ issuing from o (with respect to the cone topology). Furthermore, we have*

$$f(x) := \lim_{\tau\to\infty}(d(\sigma(\tau), x) - \tau).$$

Remark 2.6. Various examples of spaces where the geodesic boundary and the Kuratowski compactification coincide or do not coincide may be found in [1].

The following lemma which follows directly from Step 2 of [22, Theorem 3.1] plays a key role in our further considerations. For the reader convenience, we sketch its proof.

Lemma 2.3. *Let X be a complete Busemann and δ-hyperbolic space (for some positive δ). Then for each base point o and a sequence (x_n) of X such that $(x_n|x_m)_o \to \infty$ there exists a unique point $\xi \in \partial^g X$ such that (x_n) tends to ξ with respect to the cone topology, i.e., the metric segments $[o, x_n]$ converge to the geodesic ray issuing from o and tending to ξ.*

Proof. Let us consider points $\{u_n\}$ of metric segments $[o, x_n]$ with $d(o, u_n) = 1$ if $d(o, x_n) > 1$. We will show that the sequence (u_n) tends to u^1 with $d(o, u^1) = 1$. Since X is a complete space, it suffices to prove that (u_n) is a Cauchy sequence. Indeed, let us fix $\varepsilon > 0$ and let $(x_n|x_m)_o > M$ with $M > \frac{\delta}{\varepsilon}$ if $m, n > N$. If z_n and z_m are points of the metric segments $[o, x_n]$, $[o, x_m]$ with distance to o equal to $(x_n|x_m)_o$ then $d(z_n, z_m) \leq \delta$ and from the Busemann convexity we get

$$d(u_n, u_m) \leq d(z_n, z_m) \cdot \frac{d(o, u_n)}{d(o, z_n)} \leq \frac{\delta}{M} < \varepsilon. \tag{2.4}$$

In the same way one may check that points of $[o, x_n]$ with distance to o equal to 2 converge to one point u^2 and, moreover, $u^1 \in [o, u^2]$. Repeating these considerations we get that points $\{u^n\}$ form a geodesic ray and the convergence of (x_n) with respect to the cone topology follows directly from inequalities of the same type as (2.4). $\square$

3. Horoballs

Let us suppose that X is a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ and $T\colon X \to X$ is a fixed point free nonexpansive mapping. From [22, Theorem 4.1] it follows that there is a point ξ of the geodesic boundary $\partial^g X$ such that for each $x_0 \in X$ the approximating curve of first kind $\{z_t\colon t \in (0,1)\}$, where z_t is a unique fixed point of the contraction $x \mapsto tTx + (1-t)x_0$, tends to the same point ξ of the geodesic boundary $\partial^g X$ with respect to the cone topology defined on $X \cap \partial^g X$, i.e., the projections of z_t, $t \in (0,1)$ onto each closed ball $B(o,R)$ converge to one point independently of $o \in X$ and $R > 0$. Let us choose $o = x_0$. Since $z_t \in [x_0, Tz_t]$ the projections of Tz_t onto closed balls $B(x_0,R)$ coincide with the projections of z_t. As it was mentioned in Section 2 the cone topology is independent of a choice of a base point o, so this justifies the fact that Tz_t converge to ξ, i.e., for each $o \in X$ the projections onto closed balls $B(o,R)$ form a geodesic ray issuing from o and tending to ξ. Basing on this result we will show when the Picard iterative sequence converges to the same point at infinity and simultaneously we will consider the behavior of horoballs defined for some special functions satisfying similar properties as the Busemann ones. Moreover, under the additional assumption that X is a CAT(0) space, the considered functions will be just the Busemann ones.

Let $x_0 \in X$ and (z_{t_n}), $t_n \to 1$ when $n \to \infty$, be a sequence of points of the curve of first kind such that the sequence $(d(z_{t_n}, Tz_{t_n}))$ tends to a nonnegative number D. The existence of the sequence follows from the fact that $d(z_t, Tz_t) \le d(x_0, Tx_0)$ (see [21, Step 3 in the proof of Th. 4.1]). We define the function $b\colon X \to \mathbb{R}$ in following way:

$$b(x) = \limsup_n \left(d(Tz_{t_n}, x) - d(Tz_{t_n}, x_0)\right). \tag{3.1}$$

Remark 3.1. The function b is well defined since for each $x \in X$ we have that $b(x) \le d(x_0, x)$.

In the sequel we will prove that the horoballs B_M, $M \in \mathbb{R}$, of the form

$$B_M = \{x \in X\colon\ b(x) < -M\},$$

where b is the function defined in (3.1), with fixed $x_0 \in X$ and (z_{t_n}), are T-invariant.

Theorem 3.1. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $T\colon X \to X$ is a fixed point free nonexpansive mapping, then for each function defined in (3.1) all its horoballs B_M are T-invariant.*

Proof. Let us choose a sequence (z_{t_n}), $t_n \to 1$ such that $d(Tz_{t_n}, z_{t_n}) \to D$. The value D will be discussed more precisely in Theorem 3.3. Let us note that we do not assume that D is positive or equal to 0, although we will consider these two cases separately.

- *Case I:* First let us assume that $D = 0$. Then $d(Tz_{t_n}, z_{t_n}) \to 0$, so for each $y \in X$ we get

  $$b(y) := \limsup_n \left(d(Tz_{t_n}, y) - d(Tz_{t_n}, x_0)\right) = \limsup_n \left(d(z_{t_n}, y) - d(z_{t_n}, x_0)\right),$$

  which leads to

  $$b(Ty) = \limsup_n \left(d(Tz_{t_n}, Ty) - d(Tz_{t_n}, x_0)\right) \le \limsup_n \left(d(z_{t_n}, y) - d(Tz_{t_n}, x_0)\right) = b(y).$$

- *Case II:* Now we consider the positive value of D. First we will show that

$$\lim_n \left(d(x, Tz_{t_n}) - d(x, z_{t_n})\right) \to D \tag{3.2}$$

for each $x \in X$. Indeed, let us choose a point u_n of $[x, Tz_{t_n}]$ in such a way that

$$\frac{d(u_n, Tz_{t_n})}{d(x, Tz_{t_n})} = \frac{d(z_{t_n}, Tz_{t_n})}{d(x_0, Tz_{t_n})}.$$

Then from the Busemann convexity it follows that

$$d(u_n, z_{t_n}) \le d(x, x_0) \cdot \frac{d(z_{t_n}, Tz_{t_n})}{d(x_0, Tz_{t_n})}.$$

Since $d(Tz_{t_n}, x_0) \to \infty$ and $d(z_{t_n}, Tz_{t_n}) \to D$, the last inequality gives us

$$d(u_n, z_{t_n}) \to 0,$$

which completes the proof of (3.2). Next let us fix y (and also Ty). Hence on account of (3.2) we have

$$\begin{aligned} b(y) &= \limsup_n \left(d(Tz_{t_n}, y) - d(Tz_{t_n}, x_0)\right) \\ &= \limsup_n \left(d(z_{t_n}, y) - d(Tz_{t_n}, x_0)\right) + D, \end{aligned}$$

and

$$b(Ty) = \limsup_n \left(d(Tz_{t_n}, Ty) - d(Tz_{t_n}, x_0)\right) \le \limsup_n d(z_{t_n}, y) - d(Tz_{t_n}, x_0).$$

Therefore, in both cases we have

$$b(Ty) \le b(y) - D. \qquad \square \tag{3.3}$$

Remark 3.2. From Case *II* of the proof one may deduce that (3.1) may be rewritten independently of the value of D in the following way:

$$b(x) = \limsup_n \left(d(z_{t_n}, x) - d(z_{t_n}, x_0)\right).$$

If we additionally assume that X is a CAT(0) space, then each sequence of points (z_{t_n}), $t_n \to 1$, defines the same Busemann function b (see Proposition 2.2) and we obtain the following corollary.

Corollary 3.2. *Let X be a complete CAT(0) and δ-hyperbolic space for positive δ. If $T\colon X \to X$ is a fixed point free nonexpansive mapping then there is a Busemann function $b\colon X \to \mathbb{R}$ such that all horoballs B_M, $M \in \mathbb{R}$ defined as $B_M := \{x \in X\colon\ b(x) < -M\}$ are T-invariant.*

Proof. It suffices to consider the Busemann function b_c defined by

$$b_c(x) = \lim_\tau \left(d(x, c(\tau)) - \tau\right),$$

where c is the geodesic ray issuing from x_0 in the direction of $\xi \in \partial^g X$ being the limit of the curve of first kind. From Proposition 2.2 it follows that

$$b_c(x) = \lim_{t\to 1} \left(d(x, z_t) - d(x_0, z_t)\right),$$

because points Tz_t (and that z_t) tend to ξ with respect to the cone topology while $t \to 1$. Hence the inequality (3.3) implies T-invariance of horoballs. $\square$

In the sequel we will calculate the precise value of D defined in Theorem 3.1. More precisely, we will prove that the number D is independent of the choice of sequence (t_n) and, moreover, there is a relation between a positive value of D and the approximate fixed point property. First, let us recall that $T\colon X \to X$ is said to have the approximate fixed point property if $\inf_{x\in X} d(x, Tx) = 0$. More about this property can be found in [26] and in the recent paper [25].

Theorem 3.3. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ, $T\colon X \to X$ is a fixed point free nonexpansive mappings and $\{z_t\colon\ z_t = tTz_t + (1-t)x_0,\ t \in (0,1)\}$ – its curve of first kind. Then the following equality holds*

$$\lim_{t\to 1} d(z_t, Tz_t) = \inf_{x\in X} d(x, Tx). \tag{3.4}$$

Proof. Let us fix the sequence (z_{t_n}) for which $\lim_n d(z_{t_n}, Tz_{t_n})$ exists. From (3.3) we know that $b(Tx) \le b(x) - \lim_n d(z_{t_n}, Tz_{t_n})$, so for each $x \in X$:

$$\begin{aligned}
\inf_{z\in X} d(z, Tz) &\le \lim_n d(z_{t_n}, Tz_{t_n}) \ \le\ b(x) - b(Tx)\\
&= \limsup_n \left(d(z_{t_n}, x) - d(z_{t_n}, x_0)\right) - \left(\limsup_n \left(d(z_{t_n}, Tx) - d(z_{t_n}, x_0)\right)\right)\\
&\le \limsup_n \left(d(z_{t_n}, x) - d(z_{t_n}, Tx)\right) \ \le\ d(x, Tx),
\end{aligned}$$

from which it follows that

$$\lim_n d(Tz_{t_n}, z_{t_n}) = \inf_{z\in X} d(Tz, z)$$

and the left-hand limit is independent of a choice of (t_n) which completes the proof of (3.4). $\square$

Let us note that the last result holds independently for both cases of D being positive and equal to 0.

4. The convergence of the Picard iterative sequence

In the proof of the main result of this section we will apply the following result due to Karlsson. Let us note that in case of hyperbolic metric spaces we have to consider the boundary of the Gromov type based on the Gromov product (see Section 2).

Proposition 4.1. *(Proposition 5.1 in [18]) Let (X, d) be a δ-hyperbolic metric space and suppose that T is a nonexpansive mapping such that $d(x, T^n x) \to \infty$. Then the orbit $\{T^n x\colon\ n \in \mathbb{N}\}$ converges to a point γ of the Gromov type boundary based on the Gromov product.*

Theorem 4.2. *Let X be a complete Busemann space with the unique asymptotic center property. Let us also assume that X is δ-hyperbolic for some positive δ and T is a fixed point free nonexpansive mapping. Then there is a point ξ at infinity such that for each $x \in X$ there is a subsequence of the Picard iterative sequence $(T^n x)$ which tends to ξ with respect to the cone topology.*

Moreover, the point ξ is the limit for curves of first kind.

Proof. Let $x \in X$. First let us note that the orbit $\{T^n x\colon\ n \in \mathbb{N}\}$ must be unbounded. Otherwise, the bounded orbit has an exactly one asymptotic center $A((T^n x))$ and from the nonexpansivity of T we get $TA((T^n x)) = A((T^n x))$, a contradiction. Therefore, basing on the proof of Karlsson's result (see [18, Proposition 5.1]) we obtain that there is at least a subsequence $(T^{k_n}x)$ of the Picard iterative sequence which converges in the sense of Gromov product to a point γ of the Gromov boundary, i.e., $(T^{k_n}x|T^{k_m}x)_{x_0} \to \infty$ for each x_0. Since the space X is Busemann convex, from Lemma 2.3 it follows that the sequence $(T^{k_n}x)$ tends to the point γ of the geodesic boundary with respect to the cone topology.

Next we will show that γ is also the limit of the curve of first kind as it was mentioned in Section 3. Suppose on the contrary that denoting by ξ the limit of the curve of first kind $\{z_t\colon z_t = tTz_t + (1-t)x_0, t \in (0,1)\}$, we have $\xi \neq \gamma$. Clearly, it means that there is a positive number M such that

$$(Tz_t|T^{k_n}x)_{x_0} \le M, \qquad n \in \mathbb{N},\ t \in (0,1). \tag{4.1}$$

Indeed, otherwise, there exist subsequences (Tz_{t_m}) and $(T^{k_m}x)$ such that $(Tz_{t_m}|T^{k_m}x)_{x_0} \to \infty$, so from the definition of the Gromov boundary these sequences tend to the same point at infinity and $\gamma = \xi$ which contradicts our assumption.

Since $d(x_0, Tz_t) \to \infty$, $t \to 1$, and $d(x_0, T^{k_n}x) \to \infty$, $n \to \infty$, we may reduce our consideration to t and n such large that the distances to x_0 will be much greater than $M + 10\delta$. Let us fix t. For each $m = k_n$ large enough on account of inequality (4.1) there are the points $u_{t,m}$ of $[x_0, Tz_t]$ and $u'_{t,m}$ of $[T^m x, Tz_t]$ such that $d(Tz_t, u_{t,m}) = d(Tz_t, u'_{t,m}) = (T^m x|x_0)_{Tz_t}$. From the Gromov hyperbolicity of X, $d(u_{t,m}, u'_{t,m}) \le \delta$. Moreover, the equality

$$d(x_0, Tz_t) = (T^m x|Tz_t)_{x_0} + (T^m x|x_0)_{Tz_t}$$

implies that $(T^m x|x_0)_{Tz_t} = d(Tz_t, u_{t,m}) > 10\delta$.

Now on each metric segment $[T^m x, Tz_t]$ there is a point $w_{t,m}$ for which $d(w_{t,m}, Tz_t) = d(z_t, Tz_t)$. From the Busemann convexity of X it follows that

$$\begin{aligned} d(w_{t,m}, z_t) &\le d(u_{t,m}, u'_{t,m}) \cdot \frac{d(Tz_t, z_t)}{d(Tz_t, u_{t,m})} \\ &< \frac{1}{10} d(z_t, Tz_t). \end{aligned} \tag{4.2}$$

Next from the fact that $d(T^{k_n}x, x_0) \to \infty$ (so also $d(T^{k_n}x, Tz_t) \to \infty$) there must be a subsequence of $(T^{k_n}x)$ such that $d(T^{k_n}x, Tz_t) > d(T^{k_n-1}x, Tz_t)$. Since w_{t,k_n-1} was a point from the metric segment $[T^{k_n-1}x, Tz_t]$ and $d(Tz_t, w_{t,k_n-1}) = d(Tz_t, z_t)$, then on account of inequality (4.2) we get

$$\begin{aligned} d(z_t, T^{k_n-1}x) &\le d(z_t, w_{t,k_n-1}) + d(w_{t,k_n-1}, T^{k_n-1}x) \\ &< \frac{1}{10} d(z_t, Tz_t) + d(Tz_t, T^{k_n-1}x) - d(Tz_t, z_t) \\ &< d(Tz_t, T^{k_n}x), \end{aligned}$$

which contradicts the fact that T is nonexpansive.

This proves that the Gromov products $(Tz_t|T^{k_n}x)_{x_0}$ cannot be commonly bounded. So there is a sequence (t_n), $t_n \to 1$, such that $\lim_n \left(Tz_{t_n}|T^{k_n}x\right)_{x_0} = \infty$ and from Lemma 2.3 both sequences (Tz_{t_n}) and $(T^{k_n}x)$ must tend to the same point of the boundary, i.e., $\xi = \gamma$.

Finally, let us note that the convergence of $(T^{k_n}x)$ to ξ is independent of a choice of $x \in X$. $\square$

Clearly, from Proposition 4.1 and Theorem 4.2 it follows that each subsequence $(T^{k_n}x)$ of the orbit with $d(T^{k_n}x, x) \to \infty$ tends to the same point at infinity with respect to the cone topology. Moreover, the result is independent of a choice of $x \in X$. Therefore it is very natural to ask whether for each $x \in X$ the whole Picard iterative sequence $(T^n x)$ is convergent to the point of the boundary. Clearly, this convergence does not hold even in very special CAT(κ) spaces as the following example shows.

Remark 4.1. Let $\mathcal{B}$ be the open unit ball of the complex Hilbert space l^2. Then the space $(\mathcal{B}, \rho)$ with

$$\rho(x, y) = \arg\tanh(1 - \sigma(x, y))^{1/2},$$

where

$$\sigma(x, y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - (x, y)|^2}$$

and (x, y) is the scalar product in l^2, is called the complex Hilbert ball with the hyperbolic metric (see [16, Section II.14 and II.15]). In [27] one may find the example due to Stachura of the fixed point free automorphism $T : \mathcal{B} \to \mathcal{B}$ such that $\limsup_n \|T^n(0)\| = 1$ while $\liminf_n \|T^n(0)\| = 0$.

Since each holomorphic function $f : \mathcal{B} \to \mathcal{B}$ is ρ-nonexpansive (see for instance [15]) as the mapping on $(\mathcal{B}, \rho)$ and, moreover, $(\mathcal{B}, \rho)$ is a CAT(-1) space (see for instance [21]), T can be treated as the fixed point free nonexpansive mapping on the complete CAT(-1) space $(\mathcal{B}, \rho)$ for which the whole orbit $(T^n(0))$ does not converge to the point at infinity.

Next we will show under which additional assumption the whole orbit is convergent.

Theorem 4.3. *Let X be a complete Busemann space with the unique asymptotic center property. Let us also assume that X is δ-hyperbolic for some positive δ and T is a fixed point free nonexpansive mapping. If, additionally, the number $D = \inf_{x \in X} d(x, Tx)$ is positive then for each $x \in X$ the whole orbit $(T^n x)$ tends to the same point at infinity.*

Proof. We have

$$\begin{aligned} \liminf_n (Tz_{t_n}|y)_{x_0} &= \liminf_n \frac{1}{2}\left(d(Tz_{t_n}, x_0) - d(Tz_{t_n}, y)\right) + \frac{1}{2} d(y, x_0) \\ &\geq -\frac{1}{2} b(y) \end{aligned}$$

and on account of (3.2) for each $x \in X$ there must be:

$$b(T^m x) \leq b(x) - m \cdot D,$$

so there is a sequence (Tz_{t_m}), $t_m \to 1$, such that

$$\lim_m (T^m x | Tz_{t_m})_{x_0} \to \infty,$$

and both sequences $(T^m x)$ and (Tz_{t_m}) converge to the point of the Gromov boundary. Finally, from Lemma 2.3 it follows that the orbit $\{T^n x : n \in \mathbb{N}\}$ tends to ξ being the limit of the curve of first kind. □

Remark 4.2. If there is a point $x_0 \in X$ such that $d(x_0, Tx_0) = \inf_{x \in X} d(x, Tx) > 0$ then a result due to Kirk (see [19, Theorem 3]) shows that the convergence of the Picard iterative sequence may be obtained under much weaker assumptions on X.

5. Firmly nonexpansive and averaged mappings

In the last section we focus on the convergence of the Picard iterative sequence in the case of fixed point free firmly nonexpansive and averaged mappings. Let us mention how the firm nonexpansivity is understood in geodesic spaces (compare with [2], [16, Section II.24] and [20]).

Definition 5.1. Let X be a uniquely geodesic space and $T\colon X \to X$. The mapping T is called the λ-firmly nonexpansive mapping, $\lambda \in (0,1)$, if

$$d(Tx, Ty) \leq d\left((1-\lambda)Tx + \lambda x, (1-\lambda)Ty + \lambda y\right)$$

holds for all $x, y \in X$. Moreover, if the following inequality holds for all $\lambda \in (0,1)$, then the mapping is said to be firmly nonexpansive.

Clearly, if we additionally assume that X is a Busemann space, then λ-firmly nonexpansivity implies nonexpansivity independent of λ. So it is natural to ask how orbits of firmly nonexpansive mappings behave if mappings do not have fixed points. We begin with the auxiliary lemma which plays a key role in the main result of this section.

Lemma 5.1. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $T\colon X \to X$ is a fixed point free nonexpansive mapping and there is a point $x_0 \in X$ at which T is asymptotically regular, then for each $x \in X$ the orbit $(T^n x)$ converges to the same point at infinity.*

The mapping T is said to be asymptotically regular at $x \in X$ if $\lim_n d(T^n x, T^{n+1}x) = 0$. If this equality holds for all $x \in X$, then T is said to be asymptotically regular (see [6]).

Proof. Assume not. Then on account of our considerations below Theorem 4.2 it suffices to assume that there is a bounded sequence $(T^{k_n}x)$ for some $x \in X$. But this implies that $(T^{k_n}x)$ must be bounded for any $x \in X$. If T is additionally asymptotically regular at x_0 then the sequence $d(T^{k_n}x_0, T^{k_n+1}x_0)$ tends to 0 and

$$\begin{aligned} d(TA((T^{k_n}x_0)), T^{k_n}x_0) &\leq d(TA((T^{k_n}x_0)), T^{k_n+1}x_0) + d(T^{k_n+1}x_0, T^{k_n}x_0) \\ &\leq d(A((T^{k_n}x_0)), T^{k_n}x_0) + d(T^{k_n}x_0, T^{k_n+1}x_0) \end{aligned}$$

and the uniqueness of the asymptotic center $A((T^{k_n}x_0))$ contradicts the fact that T is fixed point free. $\square$

Theorem 5.2. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $T\colon X \to X$ is a fixed point free λ-firmly nonexpansive mapping (for some $\lambda \in (0,1)$), then for each $x \in X$ the orbit $(T^n x)$ converges to the same point at infinity.*

Proof. On account of Theorem 4.3 it suffices to consider only the case $\inf_{x\in X} d(x, Tx) = 0$. Moreover, since T is λ-firmly nonexpansive, from Theorem 5.1 (and Proposition 2.6) in [2] it follows that

$$\inf_{z\in X} d(z, Tz) = \lim_n d(T^n x, T^{n+1}x), \qquad x \in X. \tag{5.1}$$

Hence, T is asymptotic regular and an application of Lemma 5.1 completes the proof. $\square$

In our last part we will focus on the so-called averaged mappings. Using the same methods as above we will show that in this case the Picard iterative sequence is also convergent to the point at infinity. Let us begin with reminding the definition.

Definition 5.2. A mapping $U\colon X \to X$ defined on a uniquely geodesic space is called averaged if there is a nonexpansive mapping $T\colon X \to X$ and a number $c \in (0, 1)$ such that

$$Ux = cTx + (1 - c)x, \qquad x \in X.$$

First averaged mappings were defined in Banach spaces (see [3,24]) but the definition may be easily reformulated in each space where metric segments (even not necessary unique) are introduced (see [20,24]). However, in the general case one may not expect that the averaged mapping U is also nonexpansive.

In [20, Theorem 4.2] it was shown that for any averaged mapping U defined on a Busemann space X equality (5.1) is satisfied, so one may get the following result.

Theorem 5.3. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $U\colon X \to X$ is a fixed point free averaged mapping, then for each $x \in X$ the orbit $(U^n x)$ converges to the same point at infinity.*

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