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On maps preserving operators of local spectral radius zero

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ABSTRACT

Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space X. We describe surjective linear maps ϕ on $\mathcal{L}(X)$ that satisfy

$$\mathrm{r}_{\phi(T)}(x) = 0 \Longrightarrow \mathrm{r}_T(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$. We also describe surjective linear maps ϕ on $\mathcal{L}(X)$ that satisfy

$$\mathrm{r}_T(x) = 0 \Longrightarrow \mathrm{r}_{\phi(T)}(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$. Furthermore, we characterize maps ϕ (not necessarily linear nor surjective) on $\mathcal{L}(X)$ which satisfy

$$\mathrm{r}_{\phi(T)-\phi(S)}(x) = 0 \text{ if and only if } \mathrm{r}_{T-S}(x) = 0$$

for every $x \in X$ and $T, S \in \mathcal{L}(X)$.

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1. Introduction

Let $\mathcal{L}(X)$ be the algebra of all bounded operators on a complex Banach space X. The local spectral radius of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$ is defined by

$$\mathrm{r}_T(x) = \limsup_{n\to+\infty} \| T^n x \|^{\frac{1}{n}} .$$

Recall that the quasi-nilpotent part of an operator $T \in \mathcal{L}(X)$ is given by

$$\mathrm{H}_0(T) := \{x \in X : \limsup_{n\to+\infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

The problem of describing linear or additive maps on $\mathcal{L}(X)$ preserving the local spectra has been initiated by A. Bourhim and T. Ransford in [5], and continued by several authors; see for instance [2–4,6–8] and the references therein.

In [8], C. Costara described surjective linear maps on $\mathcal{L}(X)$ which preserve operators of local spectral radius zero at points of X. He showed that if $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ is a linear and surjective map such that for every $x \in X$ and $T \in \mathcal{L}(X)$, we have

$$\mathrm{r}_{\phi(T)}(x) = 0 \text{ if and only if } \mathrm{r}_T(x) = 0,$$

then there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.

This result has been extended by Bourhim and Mashreghi in [4] where it is shown that if ϕ is a surjective (not necessarily linear) map on $\mathcal{L}(X)$ that satisfies

$$\mathrm{r}_{\phi(T)-\phi(S)}(x) = 0 \text{ if and only if } \mathrm{r}_{T-S}(x) = 0,$$

for every $x \in X$ and $T, S \in \mathcal{L}(X)$, then there are a nonzero scalar $\mu \in \mathbb{C}$ and an operator $A \in \mathcal{L}(X)$ such that $\phi(T) = \mu T + A$ for all $T \in \mathcal{L}(X)$.

In this paper, we start by studying surjective linear maps ϕ on $\mathcal{L}(X)$ such that either

$$\mathrm{H}_0(\phi(T)) \subset \mathrm{H}_0(T)$$

for all $T \in \mathcal{L}(X)$, or

$$\mathrm{H}_0(T) \subset \mathrm{H}_0(\phi(T))$$

for all $T \in \mathcal{L}(X)$. This will give characterizations of surjective linear maps ϕ on $\mathcal{L}(X)$, that preserve operators of local spectral radius zero in one direction; i.e.

$$\mathrm{r}_{\phi(T)}(x) = 0 \Longrightarrow \mathrm{r}_T(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$, or

$$\mathrm{r}_T(x) = 0 \Longrightarrow \mathrm{r}_{\phi(T)}(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$.

We shall also give a similar result to the one in [4], without assuming that ϕ is surjective. That is, we shall characterize maps $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ that satisfy

$$\mathrm{H}_0(\phi(T) - \phi(S)) = \mathrm{H}_0(T - S) \quad \text{for all} \;\; T, S \in \mathcal{L}(X).$$

2. Preliminaries

For $T \in \mathcal{L}(X)$, we will denote by $\mathrm{N}(T), \mathrm{R}(T), \sigma(T)$ and $\sigma_p(T)$ the kernel, the range, the spectrum and the point spectrum of T respectively.

Let x be a nonzero vector in X and f be a nonzero functional in the topological dual X^* of X. We denote, as usual, by $x \otimes f$ the rank one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. Note that $x \otimes f$ is a projection if and only if $f(x) = 1$, and it is nilpotent if and only if $f(x) = 0$. We denote by $\mathrm{span}\,\{x\}$ and $\mathrm{N}(f)$, respectively, the subspace spanned by x and the kernel of f. We write $\mathcal{F}_1(X)$ for the set of all rank one operators on X.

We give some properties of quasi-nilpotent part of an operator in $\mathcal{L}(X)$; see [1,9].

Lemma 2.1. *Let $T \in \mathcal{L}(X)$. The following statements hold.*

(i) $\mathrm{N}(T) \subset \mathrm{H}_0(T)$.
(ii) $x \in \mathrm{H}_0(T)$ *if and only if* $Tx \in \mathrm{H}_0(T)$.
(iii) $\mathrm{N}(T - \lambda) \cap \mathrm{H}_0(T) = \{0\}$ *for every complex scalar* $\lambda \neq 0$.
(iv) T *is quasi-nilpotent if and only if* $\mathrm{H}_0(T) = X$.
(v) *If* T *is bounded below then* $\mathrm{H}_0(T) = \{0\}$.
(vi) *If* $\mathrm{H}_0(T) = \{0\}$ *then* T *is injective.*

The following lemma is a useful elementary result about perturbations by rank one operators.

Lemma 2.2. *([10]) Let $T \in \mathcal{L}(X)$ be an invertible operator, let x be a nonzero vector in X and f be a nonzero functional in X^*. Then $T - x \otimes f$ is not invertible if and only if $f(T^{-1}x) = 1$.*

Note that the quasinilpotent part of a rank one operator has an extensive use in the sequel. Obviously, we have

$$f(x) = 0 \Longleftrightarrow \mathrm{H}_0(x \otimes f) = X, \text{ and}$$
$$f(x) \neq 0 \Longleftrightarrow \mathrm{H}_0(x \otimes f) = \mathrm{N}(f).$$

Also, it is easy to see that

$$f(x) = 1 \Longleftrightarrow \mathrm{H}_0(I - x \otimes f) = \operatorname{span}\{x\}, \text{ and}$$
$$f(x) \neq 1 \Longleftrightarrow \mathrm{H}_0(I - x \otimes f) = \{0\}.$$

3. Linear maps preserving, in one direction, operators of local spectral radius zero at non-fixed vectors

The following lemma characterizes the rank one operators in terms of the spectrum and the point spectrum, which is due to A.R. Sourour [10].

Lemma 3.1. *For $F \in \mathcal{L}(X)$, the following assertions are equivalent.*

(i) *F is of rank at most 1.*
(ii) *For every $T \in \mathcal{L}(X)$, there exists a compact subset K_T of the complex plane, such that*

$$\sigma(T + \alpha F) \cap \sigma(T + \beta F) \subset K_T$$

for all scalars $\alpha \neq \beta$.
(iii) *For every $T \in \mathcal{L}(X)$, there exists a compact subset K_T of the complex plane, such that*

$$\sigma_p(T + \alpha F) \cap \sigma_p(T + \beta F) \subset K_T$$

for all scalars $\alpha \neq \beta$.

Lemma 3.2. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map. If ϕ satisfies one of the following assertions:*

(i) $\mathrm{H}_0(\phi(T)) \subset \mathrm{H}_0(T)$ *for all $T \in \mathcal{L}(X)$, or*
(ii) $\mathrm{H}_0(T) \subset \mathrm{H}_0(\phi(T))$ *for all $T \in \mathcal{L}(X)$,*

then there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(I) = \mu I$, where I stands for the identity operator on X.

Proof. (i) Suppose that there is a nonzero vector $x \in X$ such that x and $\phi(I)x$ are linearly independent. Let $f \in X^*$ such that $f(x) = 1$ and $f(\phi(I)x) = 0$. So $\phi(I)x \otimes f$ is nilpotent and by Lemma 2.1 (iv), $\mathrm{H}_0(\phi(I)x \otimes f) = X$. Since ϕ is surjective, there is $T \in \mathcal{L}(X)$ such that $\phi(I)x \otimes f = \phi(T)$. We have so

$$X = \mathrm{H}_0(\phi(I)x \otimes f) = \mathrm{H}_0(\phi(T)) \subset \mathrm{H}_0(T).$$

This implies, by Lemma 2.1 (iv), that T is quasi-nilpotent, hence $I - T$ is invertible and it follows by Lemma 2.1 (v), that $\mathrm{H}_0(I - T) = \{0\}$. On the other hand, we have

$$\begin{aligned} x \in \mathrm{N}(\phi(I) - \phi(I)x \otimes f) \subset \mathrm{H}_0(\phi(I) - \phi(I)x \otimes f) &= \mathrm{H}_0(\phi(I) - \phi(T)) \\ &= \mathrm{H}_0(\phi(I - T)) \\ &\subset \mathrm{H}_0(I - T) \\ &= \{0\}, \end{aligned}$$

a contradiction. Thus $\phi(I) = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$. The fact that μ is nonzero comes from (i).

(ii) Since ϕ is surjective, there is $S \in \mathcal{L}(X)$ such that $I = \phi(S)$. Suppose that there is a nonzero vector $x \in X$ such that x and Sx are linearly independent. Let $f \in X^*$ such that $f(x) = 1$ and $f(Sx) = 0$. We have

$$X = \mathrm{H}_0(Sx \otimes f) \subset \mathrm{H}_0(\phi(Sx \otimes f)).$$

Then $\phi(Sx \otimes f)$ is quasi-nilpotent, hence $I - \phi(Sx \otimes f)$ is invertible and so $\mathrm{H}_0(I - \phi(Sx \otimes f)) = \{0\}$. On the other hand, we have

$$\begin{aligned} x \in \mathrm{N}(S - Sx \otimes f) \subset \mathrm{H}_0(S - Sx \otimes f) \subset \mathrm{H}_0(\phi(S) - \phi(Sx \otimes f)) \\ = \mathrm{H}_0(I - \phi(Sx \otimes f)) \\ = \{0\}, \end{aligned}$$

a contradiction. Therefore x and Sx are linearly dependent for every $x \in X$. Hence there exists a scalar $\mu' \in \mathbb{C}$ such that $S = \mu' I$. Since $S \neq 0$, then $\mu' \neq 0$. Thus $\phi(I) = \phi(\frac{1}{\mu'}S) = \frac{1}{\mu'}\phi(S) = \frac{1}{\mu'}I$. As desired. $\square$

Lemma 3.3. *Let $A, B \in \mathcal{L}(X)$ such that A is injective and B is invertible.*
If $\mathrm{H}_0(A + F) \subset \mathrm{H}_0(B + F)$ for all $F \in \mathcal{F}_1(X)$ then $A = B$.

Proof. Let $A, B \in \mathcal{L}(X)$ such that A is injective and B is invertible. Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$.

Suppose that $\mathrm{H}_0(A + F) \subset \mathrm{H}_0(B + F)$ for all $F \in \mathcal{F}_1(X)$. For $F = -Ax \otimes f$, we have

$$\begin{aligned} \operatorname{span}\{x\} = \mathrm{N}(I - x \otimes f) = \mathrm{N}(A(I - x \otimes f)) &= \mathrm{N}(A - Ax \otimes f) \\ &\subset \mathrm{H}_0(A - Ax \otimes f) \\ &\subset \mathrm{H}_0(B - Ax \otimes f). \end{aligned}$$

Then $\mathrm{H}_0(B - Ax \otimes f) \neq \{0\}$ and so by Lemma 2.1 (v), $B - Ax \otimes f$ is not invertible. Lemma 2.2 gives that

$$f(B^{-1}Ax) = 1 = f(x).$$

Consequently, $B^{-1}Ax = x$. Since this holds for each x, then $A = B$. $\square$

Theorem 3.4. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map. Then the following assertions are equivalent.*

(i) $\mathrm{H}_0(\phi(T)) \subset \mathrm{H}_0(T)$ *for all* $T \in \mathcal{L}(X)$.
(ii) $\mathrm{H}_0(T) \subset \mathrm{H}_0(\phi(T))$ *for all* $T \in \mathcal{L}(X)$.
(iii) *There exists a nonzero scalar* $\mu \in \mathbb{C}$ *such that* $\phi(T) = \mu T$ *for all* $T \in \mathcal{L}(X)$.

Proof. (i) $\Rightarrow$ (iii) By Lemma 3.2, we have $\phi(I) = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$. Let us consider the map ψ defined by

$$\psi(T) = \frac{1}{\mu}\phi(T) \text{ for all } T \in \mathcal{L}(X).$$

The map ψ satisfies (i). By Lemma 2.1 (v), (vi), we have

$$\begin{aligned} T \text{ is invertible} &\Longrightarrow \mathrm{H}_0(T) = \{0\} \\ &\Longrightarrow \mathrm{H}_0(\psi(T)) = \{0\} \\ &\Longrightarrow \psi(T) \text{ is injective,} \end{aligned}$$

for every $T \in \mathcal{L}(X)$. It follows, since $\psi(I) = I$, that

$$\sigma_p(\psi(T)) \subset \sigma(T)$$

for all $T \in \mathcal{L}(X)$. Let $F \in \mathcal{L}(X)$ be an operator of rank one. We have so

$$\sigma_p(\psi(T) + \alpha\psi(F)) \cap \sigma_p(\psi(T) + \beta\psi(F)) \subset \sigma(T + \alpha F) \cap \sigma(T + \beta F)$$

for every $T \in \mathcal{L}(X)$ and all scalars $\alpha \neq \beta$. Using Lemma 3.1, we get that for every $T \in \mathcal{L}(X)$, there exists a compact subset $K_T \subset \mathbb{C}$ such that

$$\sigma_p(\psi(T) + \alpha\psi(F)) \cap \sigma_p(\psi(T) + \beta\psi(F)) \subset K_T$$

for all scalars $\alpha \neq \beta$. Since ψ is surjective, we conclude by Lemma 3.1, that $\psi(F)$ is of rank at most one.

Let $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$. There exist $y \in X$ and $g \in X^*$ such that

$$\psi(x \otimes f) = y \otimes g.$$

Since

$$\mathrm{H}_0(y \otimes g) = \mathrm{H}_0(\psi(x \otimes f)) \subset \mathrm{H}_0(x \otimes f) = \mathrm{N}(f),$$

it follows that $\mathrm{H}_0(y \otimes g) \neq X$, so by Lemma 2.1 (iv), $y \otimes g$ is not quasi-nilpotent, and then $g(y) \neq 0$. This gives that $\mathrm{N}(g) \subset \mathrm{N}(f)$, and so f and g are linearly dependent; i.e.

$$g = af$$

for some nonzero scalar $a \in \mathbb{C}$. We have

$$\begin{aligned}\{0\} \neq \operatorname{span}\{y\} = \mathrm{H}_0(I - \frac{1}{g(y)} y \otimes g) &= \mathrm{H}_0(\psi(I) - \frac{1}{g(y)}\psi(x \otimes f)) \\ &= \mathrm{H}_0(\psi(I - \frac{1}{g(y)} x \otimes f)) \\ &\subset \mathrm{H}_0(I - \frac{1}{g(y)} x \otimes f).\end{aligned}$$

By Lemma 2.1 (v) and Lemma 2.2, we obtain that $f(\frac{1}{g(y)}x) = 1$, and therefore $\operatorname{span}\{y\} \subset \operatorname{span}\{x\}$. Then x and y are linearly dependent; i.e.

$$y = bx$$

for some nonzero scalar $b \in \mathbb{C}$. Since $f(x) = g(y) = af(bx) = abf(x)$, then $ab = 1$. Hence

$$\psi(x \otimes f) = ax \otimes (bf) = abx \otimes f = x \otimes f.$$

Thus

$$\psi(F) = F$$

for all non-nilpotent rank one operator $F \in \mathcal{L}(X)$.

As ϕ is linear, and every nilpotent rank one operator is a sum of two non-nilpotent rank one operator, we deduce that $\psi(F) = F$ for all rank one operator $F \in \mathcal{L}(X)$.

Let $T \in \mathcal{L}(X)$ and $\lambda \notin \sigma(T) \cup \sigma(\psi(T))$. We have

$$\mathrm{H}_0(\psi(T) - \lambda I + F) = \mathrm{H}_0(\psi(T - \lambda I + F)) \subset \mathrm{H}_0(T - \lambda I + F)$$

for all $F \in \mathcal{F}_1(X)$.

Lemma 3.3 gives that $\psi(T) = T$, as desired.

(ii) $\Rightarrow$ (iii) By Lemma 3.2, we have $\phi(I) = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$. Let the map ψ given by

$$\psi(T) = \frac{1}{\mu}\phi(T) \text{ for all } T \in \mathcal{L}(X).$$

The map ψ satisfies (ii). By Lemma 2.1 (v), (vi), we have

$$\begin{aligned}\psi(T) \text{ is invertible} &\Longrightarrow \mathrm{H}_0(\psi(T)) = \{0\} \\ &\Longrightarrow \mathrm{H}_0(T) = \{0\} \\ &\Longrightarrow T \text{ is injective}\end{aligned}$$

for every $T \in \mathcal{L}(X)$. It follows, since $\psi(I) = I$, that

$$\sigma_p(T) \subset \sigma(\psi(T))$$

for all $T \in \mathcal{L}(X)$. Let $F \in \mathcal{L}(X)$ such that $\psi(F)$ is of rank one. We have

$$\sigma_p(T + \alpha F)) \cap \sigma_p(T + \beta F) \subset \sigma(\psi(T) + \alpha\psi(F)) \cap \sigma(\psi(T) + \beta\psi(F))$$

for every $T \in \mathcal{L}(X)$ and all scalars $\alpha \neq \beta$. Using Lemma 3.1, and the fact that ψ is surjective, we get that for every $T \in \mathcal{L}(X)$, there exists a compact subset $K_T \subset \mathbb{C}$ such that

$$\sigma_p(T + \alpha F) \cap \sigma_p(T + \beta F) \subset K_T$$

for all scalars $\alpha \neq \beta$. Thus we conclude by Lemma 3.1, that F is of rank one.

Let $y \in X$ and $g \in X^*$ such that $g(y) \neq 0$. Then there exist $x \in X$ and $f \in X^*$ such that $\psi(x \otimes f) = y \otimes g$. Since

$$\mathrm{H}_0(x \otimes f) \subset \mathrm{H}_0(\psi(x \otimes f)) = \mathrm{H}_0(y \otimes g) = \mathrm{N}(g) \neq X,$$

the operator $x \otimes f$ is non-nilpotent, so $f(x) \neq 0$. Therefore $\mathrm{N}(f) \subset \mathrm{N}(g)$ and then f and g are linearly dependent. We have

$$\begin{aligned}
\{0\} \neq \operatorname{span}\{x\} = \mathrm{H}_0(I - \frac{1}{f(x)} x \otimes f) &\subset \mathrm{H}_0(\psi(I - \frac{1}{f(x)} x \otimes f)) \\
&= \mathrm{H}_0(\psi(I) - \frac{1}{f(x)} \psi(x \otimes f)) \\
&= \mathrm{H}_0(I - \frac{1}{f(x)} y \otimes g).
\end{aligned}$$

This gives that $g(\frac{1}{f(x)} y) = 1$ and so $\operatorname{span}\{x\} \subset \operatorname{span}\{y\}$. Hence x and y are linearly dependent. As above, we get that

$$\psi(F) = F$$

for all rank one operator $F \in \mathcal{L}(X)$.

Let $T \in \mathcal{L}(X)$ and $\lambda \notin \sigma(T) \cup \sigma(\psi(T))$. We have

$$\mathrm{H}_0(T - \lambda I + F) \subset \mathrm{H}_0(\psi(T - \lambda I + F)) = \mathrm{H}_0(\psi(T) - \lambda I + F)$$

for all $F \in \mathcal{F}_1(X)$. Lemma 3.3 gives that $\psi(T) = T$, as desired. $\square$

One may restate the result of Theorem 3.4 in the following form:

Theorem 3.5. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective linear map. Then the following assertions are equivalent.*

(i) $\mathrm{r}_{\phi(T)}(x) = 0 \Longrightarrow \mathrm{r}_T(x) = 0$, *for every $x \in X$ and $T \in \mathcal{L}(X)$.*
(ii) $\mathrm{r}_T(x) = 0 \Longrightarrow \mathrm{r}_{\phi(T)}(x) = 0$, *for every $x \in X$ and $T \in \mathcal{L}(X)$.*
(iii) *There exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.*

Proof. (i) $\Longrightarrow$ (iii) Let $x \in X$ and $T \in \mathcal{L}(X)$. We have

$$x \in \mathrm{H}_0(\phi(T)) \Longleftrightarrow \mathrm{r}_{\phi(T)}(x) = 0 \Longrightarrow \mathrm{r}_T(x) = 0 \Longleftrightarrow x \in \mathrm{H}_0(T)$$

and so

$$\mathrm{H}_0(\phi(T)) \subset \mathrm{H}_0(T).$$

Thus, Theorem 3.4 completes the proof.

(ii) $\Longrightarrow$ (iii) It is similar to (i) $\Longrightarrow$ (iii).

Clearly, (iii) implies both (i) and (ii). □

4. Maps preserving operators of local spectral radius zero at non-fixed vectors

In this section, we establish a similar result to the one given by [4, Theorem 4.1]. The only difference is that the map ϕ is not assumed surjective. Note that the proof of [4, Theorem 4.1] is broke into seven steps, and the surjectivity condition on ϕ is used only in step 2. The following Lemma gives the same result as the one obtained in step 2 of the proof of [4, Theorem 4.1], without supposing ϕ to be surjective.

Lemma 4.1. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map satisfying:*

$$\mathrm{r}_{\phi(T)-\phi(S)}(x) = 0 \text{ if and only if } \mathrm{r}_{T-S}(x) = 0, \tag{1}$$

for all $x \in X$ and $T, S \in \mathcal{L}(X)$. Then for every nonzero scalar $\lambda \in \mathbb{C}$, there exists a nonzero scalar $\mu_\lambda \in \mathbb{C}$ such that

$$\phi(\lambda I) = \mu_\lambda I + \phi(0).$$

Proof. It is clear that (1) is equivalent to

$$\mathrm{H}_0(\phi(T) - \phi(S)) = \mathrm{H}_0(T - S)$$

for all $T, S \in \mathcal{L}(X)$. Let us consider $\varphi(T) = \phi(T) - \phi(0)$ for all $T \in \mathcal{L}(X)$. The map φ satisfies (1) and in particular,

$$\mathrm{H}_0(\varphi(T)) = \mathrm{H}_0(T) \quad \text{for all} \quad T \in \mathcal{L}(X). \tag{2}$$

Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$. Let $0 \neq \lambda \in \mathbb{C}$ and set $F = x \otimes f$. We have

$$\mathrm{H}_0(\varphi(\lambda I) - \varphi(\lambda F)) = \mathrm{H}_0(\lambda I - \lambda F) = \mathrm{H}_0(I - F) = \mathrm{span}\,\{x\}.$$

Lemma 2.1 (ii) gives that $\varphi(\lambda I)x - \varphi(\lambda F)x \in \mathrm{span}\,\{x\}$. Hence there exists a scalar $\alpha_\lambda \in \mathbb{C}$ such that $\varphi(\lambda I)x - \varphi(\lambda F)x = \alpha_\lambda x$ and so $x \in \mathrm{N}(\varphi(\lambda I) - \varphi(\lambda F) - \alpha_\lambda)$. Assuming that $\alpha_\lambda \neq 0$, it follows by Lemma 2.1 (iii), that $\mathrm{N}(\varphi(\lambda I) - \varphi(\lambda F) - \alpha_\lambda) \cap \mathrm{span}\,\{x\} = \{0\}$, a contradiction. Then $\alpha_\lambda = 0$ and so

$$\varphi(\lambda I)x = \varphi(\lambda F)x.$$

On the other hand, using (2), we have

$$\mathrm{H}_0(\varphi(\lambda F)) = \mathrm{H}_0(\lambda F) = \mathrm{H}_0(F) = \mathrm{N}(f).$$

Since $x \notin \mathrm{N}(f)$, then by Lemma 2.1 (ii), we get that $\varphi(\lambda F)x \notin \mathrm{N}(f)$ and so $\varphi(\lambda I)x \notin \mathrm{N}(f)$ i.e., $f(\varphi(\lambda I)x) \neq 0$. Therefore $\varphi(\lambda I)x$ and x are linearly dependent and so there exists a nonzero scalar $\mu_\lambda \in \mathbb{C}$ such that

$$\varphi(\lambda I) = \mu_\lambda I. \qquad \square$$

According to Lemma 4.1 and steps 3 to 7 of the proof of [4, Theorem 4.1], we get the following theorem.

Theorem 4.2. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a map. Then the following assertions are equivalent.*

(i) $\mathrm{r}_{\phi(T)-\phi(S)}(x) = 0$ *if and only if* $\mathrm{r}_{T-S}(x) = 0$, *for all* $x \in X$ *and* $T, S \in \mathcal{L}(X)$.
(ii) *There exists a nonzero scalar* $\mu \in \mathbb{C}$ *such that* $\phi(T) = \mu T + \phi(0)$ *for all* $T \in \mathcal{L}(X)$.

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References

[1] P. Aiena, Fredholm and Local Spectral Theory, with Application to Multipliers, Kluwer Acad. Publishers, 2004.
[2] A. Bourhim, Surjective linear maps preserving local spectra, Linear Algebra Appl. 432 (2010) 383–393.
[3] A. Bourhim, J. Mashreghi, A survey on preservers of spectra and local spectra, Contemp. Math. 638 (2015) 45–98.

[4] A. Bourhim, J. Mashreghi, Local spectral radius preservers, Integral Equations Operator Theory 76 (1) (2013) 95–104.

[5] A. Bourhim, T. Ransford, Additive maps preserving local spectrum, Integral Equations Operator Theory 55 (2006) 377–385.

[6] J. Bračič, V. Müller, Local spectrum and local spectral radius at a fixed vector, Studia Math. 194 (2009) 155–162.

[7] C. Costara, Automatic continuity for linear surjective mappings decreasing the local spectral radius at some fixed vector, Arch. Math. 95 (6) (2010) 567–573.

[8] C. Costara, Linear maps preserving operators of local spectral radius zero, Integral Equations Operator Theory 73 (1) (2012) 7–16.

[9] K.B. Laursen, M.M. Neumann, Introduction to Local Spectral Theory, Clarendon, Oxford, 2000.

[10] A.R. Sourour, Invertibility preserving linear maps on $L(X)$, Trans. Amer. Math. Soc. 348 (1996) 13–30.