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The numerical solution of nonlinear two-dimensional Volterra–Fredholm integral equations of the second kind based on the radial basis functions approximation with error analysis

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ABSTRACT

In this paper, we present a numerical method for solving two-dimensional nonlinear Volterra–Fredholm integral equations of the second kind. The method approximates the solution by the discrete collocation method based on radial basis functions (RBFs) constructed on a set of disordered data. The proposed method is meshless, since it does not require any background mesh or domain elements. Error analysis of this method is also investigated. Numerical examples which compare the proposed method with 2D-TFs method [4] approve its supremacy in terms of accuracy and computational cost. Using various RBFs we have concluded that MQ-RBF is the best choice for the proposed method.

1. Introduction

Integral equations are used as mathematical models for many varied physical situations, and they also occur as reformulations of other mathematical problems. Nonlinear integral equations have been studied in connection with many diverse topics such as vehicular traffic, biology, the theory of optimal control, economics, etc [12,13].

We consider the two-dimensional Volterra–Fredholm integral equation

$$u(x,t) - \int_0^t \int_0^x k_1(x,t,y,z)\psi_1(y,z,u(y,z))dydz + \int_0^1 \int_0^1 k_2(x,t,y,z)\psi_2(y,z,u(y,z))dydz = f(x,t). \quad (1)$$

The functions $f(x, t)$, $k_1(x, t, y, z)$ and $k_2(x, t, y, z)$ are assumed to be given smooth real valued functions on $\Omega = [0, 1] \times [0, 1]$ and $D = \{(x, t, y, z) : 0 \le y \le x \le 1, 0 \le z \le t \le 1\}$, respectively and $u(x, t)$ is the solution to be determined. Classical theorems on the existence and uniqueness of the solution of two-dimensional nonlinear integral equations can be found in Abdou et al. [1,2]. Theoretical results and numerical approaches on one dimensional integral equations have been investigated by several authors [14,15] but a few studies have been devoted to two-dimensional integral equations. Han and Wang [16] studied the iterated collocation method to solve two-dimensional nonlinear Volterra integral equations. The Nystrom method was used to solve two-dimensional nonlinear Fredholm integral equations in Han and Wang [17,18]. Recently a numerical method based on the Tau method is presented for solving two-dimensional nonlinear Volterra–Fredholm integral equations [21].

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Table 1
Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GA)	$\phi(r) = \exp(-cr^2), \quad c > 0$
Multiquadrics (MQ)	$\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$
Inverse multiquadrics (IMQ)	$\phi(r) = (r^2 + c^2)^{\frac{-1}{2}}$
Thin plate splines	$\phi(r) = (-1)^{k+1} r^{2k} \log(r), \quad k \in \mathbb{N}$

In recent years, meshless methods have been used in many different areas ranging from artificial intelligence, image processing, neural networks, and sampling theory, etc. Radial basis functions (RBFs) are probably best known for their applications to problems with scattered data. The multiquadric (MQ) RBF interpolation method was developed by Roland Hardy who described and named the method in a paper appeared in 1971 [6]. Among the tested RBFs, the MQ-RBFs gave the most accurate results. In 1990 Kansa first used the MQ-RBFs method to solve differential equations [7,8]. Kansa's method was recently extended to solve various ordinary and partial differential equations [9–11]. We refer interested reader to Refs. [22–24] for applications of meshless methods for finding numerical solution of integral equations.

The main purpose of this paper is to present a numerical method based on radial basis functions approximation for numerical solution of nonlinear two-dimensional Volterra–Fredholm integral equations. The outline of this paper is as follows: In Section 2 we review some basic formulations and properties of the radial basis functions approximations. In Section 3, we present a computational method for nonlinear two-dimensional Volterra–Fredholm integral equations utilizing the RBF approximation. We provide the error analysis for the method in Section 4. Finally, numerical examples are given in Section 5.

2. Radial basis functions approximation

A meshfree method does not require a mesh to discretize the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. Radial basis functions is one of the most developed meshless methods that has attracted attention in recent years and form a primary tool for multivariate interpolation [11]. It is also receiving increasing attention for solving PDE's on irregular domains. We quote the following definitions from Ref. [3].

Definition 2.1. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|_2)$, for all $\mathbf{x} \in \mathbb{R}^d$.

Some of the most popular RBFs are given in Table 1. In order to explain the multivariate scattered data interpolation by radial basis functions, consider the following definition:

Definition 2.2. A real-valued continuous even function Φ is conditionally positive definite of order m, if for all sets $X = \{x_1, ...x_N\} \subset \mathbb{R}^d$ of distinct points, and all vectors $\mathbf{\Lambda} = [\lambda_1, \ldots, \lambda_N]^T \in \mathbb{R}^N$ satisfying $\sum_{i=1}^N \lambda_i p(x_i) = 0$ for all polynomials of degree less then m the quadratic form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \Phi(x_i - x_j), \tag{2}$$

is positive.

In interpolation of the scattered data using RBFs, the approximation of a function f on a set $X = \{x_1, \ldots, x_N\}$ usually has the form:

$$\mathcal{P}f = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|) + \sum_{l=1}^{M} d_l p_l(x), \quad x \in \Omega, \tag{3}$$

where p_1, ..., p_M form a basis for the $M = \binom{d+m-1}{m-1}$ dimensional linear space Π^d_{m-1} of polynomials of total degree less than or equal to $m-1$ in d variables. The interpolation problem is to find $\lambda_i, \quad i = 1, \ldots, N$ and $d_l, \quad l = 1, \ldots, M$ such that the interpolant $\mathcal{P}f$ through all data satisfies

$$\mathcal{P}f(x_i) = f(x_i), \quad i = 1, \ldots, N, \qquad \text{and} \qquad \sum_{i=1}^{N} \lambda_i p_j(x_i) = 0, \; j = 1, \ldots, M. \tag{4}$$

One can write this system in matrix form as

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}, \tag{5}$$

where A and P are the $N \times N$ and $N \times M$ matrices with the elements $A_{ij} = \phi(\|x_i - x_j\|), i, j = 1, \ldots, N$ and $P_{ij} = p_j(x_i), i = 1, \ldots, N, j = 1, \ldots, M$, respectively. Furthermore $\Lambda = [\lambda_1, \ldots, \lambda_N]^T, \mathbf{d} = [d_1, \ldots, d_M]^T$ and $\mathbf{f} = [f(x_1), \ldots, f(x_N)]^T$.

Definition 2.3 ([3]). The fill distance of a given set $X = \{x_1, \ldots, x_N\}$ consisting of pairwise distinct points in Ω can be defined as

$$h_{X,\Omega} = \sup_{x\in\Omega} \min_{x_j\in X} \|x - x_j\|,$$

which indicates how well the data in the set X fill out the domain Ω.

Definition 2.4 ([3]). A set Ω is said to satisfy an interior cone condition if there exists an angle $\theta \in (0, \frac{\pi}{2})$ and a radius $r > 0$ such that for every given $x \in \Omega$ a unit vector $\eta(x)$ exists such that the cone

$$C(x, \eta(x), \theta, r) = \{x + \lambda y : y \in \mathbb{R}^d, ||y||_2 = 1, y^T\eta(x) \geq \cos(\theta), \lambda \in [0, r]\},$$

is contained in Ω.

Definition 2.5 ([3]). Let H be a real Hilbert space of $u : \Omega \to \mathbb{R}$. A function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is called reproducing kernel for H if

1. $\Phi(., y) \in H$ for all $y \in \Omega$
2. $u(y) =< u, \Phi(., y) >_H$ for all $u \in H$ and all $y \in \Omega$. We define the linear space

$$H_\Phi(\Omega) = span\{\Phi(., y) : y \in \Omega\},$$

and equip this space with the bilinear form

$$\left\langle \sum_{j=1}^{N} c_j\Phi(., x_j), \sum_{k=1}^{N} d_k\Phi(., y_k) \right\rangle_\Phi = \sum_{j=1}^{N}\sum_{k=1}^{N} c_j d_k \Phi(x_j, y_k). \tag{6}$$

Theorem 2.6 ([3]). *Let Φ be a radial basis and strictly positive definite function. Then the bilinear form $<, >_\Phi$ defines an inner product on $H_\Phi(\Omega)$. Moreover $H_\Phi(\Omega)$ is a pre-Hilbert space with reproducing Φ.*

Definition 2.7 ([3]). The native space $\mathcal{N}_\Phi(\Omega)$ of Φ is defined to be the completion of $H_\Phi(\Omega)$ with respect to the Φ-norm $\|.\|_\Phi$, so that $\|u\|_\Phi = \|u\|_{\mathcal{N}_\Phi}(\Omega)$ for all $u \in \mathcal{N}_\Phi(\Omega)$.

Native spaces for conditionally positive definite RBFs can also be generalized but we will not go into the details here.

Theorem 2.8 ([3]). *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, satisfying an interior cone condition. Suppose that Φ is conditionally positive definite with respect to Π^d_{m-1} and the reproducing kernel $\Phi(., .)$ is in $C^{2k}(\Omega \times \Omega)$. Suppose the interpolant to $u \in H_\Phi(\Omega)$ is based on the m-unisolvent set $X = \{x_1, ..., x_N\}$ and denoted by $\mathcal{P}_N u$. Moreover, if derivatives of Φ of order $2k$ are continuous on $\bar{\Omega} \times \bar{\Omega}$, then there exist positive constants h_0, C and C_1 such that*

$$\|u - \mathcal{P}_N u\| \leq CC_\Phi^{\frac{1}{2}} h^k_{X,\Omega}\|u\|_{H_\Phi(\Omega)}, \tag{7}$$

provided $h_{X,\Omega} \leq h_0$. The number C_Φ is defined by

$$C_\Phi = \max_{\beta=2k} \|D^\beta\Phi\|_{L^\infty(B(0,2C_1h_{X,\Omega}))}. \tag{8}$$

It has been shown that the associated interpolation matrix for multiquadric RBFs is non-singular [3]. In this paper, we use multiquadric RBFs for solving (1). For the theoretical developments of the RBFs in scattered data interpolation, Madych and Nelson [20] showed that the MQ-RBF interpolant employs exponential convergence with minimal semi-norm errors.

3. Description of the method

In this section, the RBF collocation method is used to solve (1). To apply the method, we estimate the unknown function $u(x, t)$ by the RBF interpolation as

$$u(x, t) \simeq \sum_{i=1}^{N} \lambda_i\Phi_i(x, t) + \sum_{l=1}^{M} d_l p_l(x, t), \tag{9}$$

where $\Phi_i(x, t) = \phi(\|(x, t) - (x_i, t_i)\|)$ are the shape functions of the RBF method corresponding to set X.

By substituting (9) into (1) we have

$$\sum_{i=1}^{N} \lambda_i\Phi_i(x, t) + \sum_{l=1}^{M} d_l p_l(x, t) - \int_0^t \int_0^x k_1(x, t, y, z)\psi_1\left(y, z, \sum_{i=1}^{N} \lambda_i\Phi_i(y, z) + \sum_{l=1}^{M} d_l p_l(y, z)\right)dydz$$
$$+ \int_0^1 \int_0^1 k_2(x, t, y, z)\psi_2(y, z, \sum_{i=1}^{N} \lambda_i\Phi_i(y, z) + \sum_{l=1}^{M} d_l p_l(y, z))dydz = f(x, t). \tag{10}$$

In order to use the Legendre–Gauss integration we transform the intervals $[0, t]$, $[0, x]$ into $[0, 1]$ by means of the transformation $y = x\theta, z = t\eta$. So Eq. (10) may be restated as

$$\sum_{i=1}^{N}\lambda_i\Phi_i(x,t)+\sum_{l=1}^{M}d_l p_l(x,t)-\int_0^1\int_0^1 xtk_1(x,t,x\theta,t\eta)\psi_1\left(x\theta,t\eta,\sum_{i=1}^{N}\lambda_i\Phi_i(x\theta,t\eta)+\sum_{l=1}^{M}d_l p_l(x\theta,t\eta)\right)d\theta d\eta$$
$$+\int_0^1\int_0^1 k_2(x,t,y,z)\psi_2\left(y,z,\sum_{i=1}^{N}\lambda_i\Phi_i(y,z)+\sum_{l=1}^{M}d_l p_l(y,z)\right)dydz,=f(x,t). \tag{11}$$

If (11) holds at collocation points (x_j, t_j) we have

$$\sum_{i=1}^{N}\lambda_i\Phi_i(x_j,t_j)+\sum_{l=1}^{M}d_l p_l(x_j,t_j)-\int_0^1\int_0^1 x_jt_jk_1(x_j,t_j,x_j\theta,t_j\eta)\psi_1\left(x_j\theta,t_j\eta,\sum_{i=1}^{N}\lambda_i\Phi_i(x_j\theta,t_j\eta)+\sum_{l=1}^{M}d_l p_l(x_j\theta,t_j\eta)\right)d\theta d\eta$$
$$+\int_0^1\int_0^1 k_2(x_j,t_j,y,z)\psi_2\left(y,z,\sum_{i=1}^{N}\lambda_i\Phi_i(y,z)+\sum_{l=1}^{M}d_l p_l(y,z)\right)dydz=f(x_j,t_j). \tag{12}$$

To approximate the integrals in O'Regan (12) we use a n-points numerical integration formula relative to $\{(\theta_q, \eta_\gamma)\}$, $\{(y_q, z_\gamma)\}$ and weights $\{(w_q, w_\gamma)\}$. So we have

$$\sum_{i=1}^{N}\tilde{\lambda}_i\Phi_i(x_j,t_j)+\sum_{l=1}^{M}\tilde{d}_l p_l(x_j,t_j)-\sum_{q=1}^{n}\sum_{\gamma=1}^{n}w_qw_\gamma x_jt_jk_1(x_j,t_j,x_j\theta_q,t_j\eta_\gamma)$$
$$\psi_1\left(x_j\theta_q,t_j\eta_\gamma,\sum_{i=1}^{N}\tilde{\lambda}_i\Phi_i(x_j\theta_q,t_j\eta_\gamma)+\sum_{l=1}^{M}\tilde{d}_l p_l(x_j\theta_q,t_j\eta_\gamma)\right)$$
$$+\sum_{q=1}^{n}\sum_{\gamma=1}^{n}w_qw_\gamma k_2(x_j,t_j,y_q,z_\gamma)\psi_2(y_q,z_\gamma,\sum_{i=1}^{N}\tilde{\lambda}_i\Phi_i(y_q,z_\gamma)+\sum_{l=1}^{M}\tilde{d}_l p_l(y_q,z_\gamma))=f(x_j,t_j). \tag{13}$$

To determine the unknown coefficients $\tilde{\lambda}_i$ and $\tilde{d}_l$ from N equations resulting from (13), a set of M extra equations is needed, that is

$$\sum_{i=1}^{N}\tilde{\lambda}_i p_l(x_i,t_i)=0,\quad 1\le l\le M. \tag{14}$$

In this way, a nonlinear system of algebraic equations is obtained which can be solved by any nonlinear solver; we have used the *fsolve* command of Maple. In fact any efficient algorithm for solving systems of nonlinear equations such as Newton method, etc... can be utilized. Thus the values of $u(x, t)$ at any point $(x, t) \in [0, 1] \times [0, 1]$ can be approximated by

$$\hat{u}(x,t)=\sum_{i=1}^{N}\tilde{\lambda}_i\Phi_i(x,t)+\sum_{l=1}^{M}\tilde{d}_l p_l(x,t). \tag{15}$$

The above discussion gives us the following algorithm for obtaining the approximate solution of Eq. (1):

Algorithm of the proposed method:

Step 1. Input the given functions $f(x, t)$, $k_1(x, t, y, z)$, $k_2(x, t, y, z)$, ψ_1 and ψ_2.
Step 2. Choose RBF-generating function $\phi(r)$ from Table 1 together with N and M; we suggest MQ-RBFs.
Step 3. Choose the set of source points (x_i, t_i) and the shape parameter c.
Step 4. Solve the $(N + M) \times (N + M)$ system of nonlinear Eqs. (13) and (14) to obtain $\tilde{\lambda}_i$ and $\tilde{d}_l$.
Step 5. Substitute the obtained $\tilde{\lambda}_i$ and $\tilde{d}_l$ in (15) to obtain the approximate solution $\hat{u}(x, t)$.

4. Error analysis

In this section, we investigate the error analysis of the proposed method. Let the operator $\mathcal{K}$ be defined as

$$\mathcal{K}u=\int_0^t\int_0^x k_1(x,t,y,z)\psi_1(y,z,u(y,z))dydz+\int_0^1\int_0^1 k_2(x,t,y,z)\psi_2(y,z,u(y,z))dydz. \tag{16}$$

So, we can write (1) in abstract form as

$$(I-\mathcal{K})u=f. \tag{17}$$

Let $\mathcal{P}_N : C(\Omega) \to V_N$ be the collocation operator defined by

$$\mathcal{P}_N u = \sum_{i=1}^{N} \lambda_i \Phi_i(x,t) + \sum_{l=1}^{M} d_l p_l(x,t), \tag{18}$$

where $V_N = \text{span}\{\Phi_1, \Phi_2, \ldots, \Phi_N, p_1, p_2, \ldots, p_M\}$. So (12) is equivalent to solving the abstract form

$$(I - \mathcal{P}_N \mathcal{K}) u_N = \mathcal{P}_N f. \tag{19}$$

By introducing a sequence of numerical integral operator $\mathcal{K}_N$ by

$$\begin{aligned}\mathcal{K}_N u = &\sum_{q=1}^{M_N}\sum_{\gamma=1}^{M_N} w_q w_\gamma x t k_1(x,t,x\theta_q,t\eta_\gamma)\psi_1(x\theta_q,t\eta_\gamma,u(x\theta_q,t\eta_\gamma)) \\ &+ \sum_{q=1}^{M_N}\sum_{\gamma=1}^{M_N} w_q w_\gamma k_2(x,t,y_q,z_\gamma)\psi_2(y_q,z_\gamma,u(y_q,z_\gamma)),\end{aligned} \tag{20}$$

we can write (13) in the operator form

$$(I - \mathcal{P}_N \mathcal{K}_N)\hat{u}_N = \mathcal{P}_N f. \tag{21}$$

Let $\mathcal{T}_N$ be the operator defined by

$$\mathcal{T}_N u = \mathcal{P}_N \mathcal{K} u + \mathcal{P}_N f. \tag{22}$$

Then (19) can be written as

$$u_N = \mathcal{T}_N u_N. \tag{23}$$

Also suppose that $\hat{\mathcal{T}}_N$ be the operator defined by

$$\hat{\mathcal{T}}_N u = \mathcal{P}_N \mathcal{K}_N u + \mathcal{P}_N f. \tag{24}$$

Then (21) can be written as

$$\hat{u}_N = \hat{\mathcal{T}}_N \hat{u}_N. \tag{25}$$

Theorem 4.1 ([3]). *Let $\Phi(x)$ be a multiquadric RBF. Suppose that $\Omega \subset \mathbf{R}^2$ is open and bounded satisfying an interior cone condition. Denote the interpolant of a function $u \in \mathcal{N}_\Phi(\Omega)$ based on $X = \{x_1, \ldots x_N\}$ by $\mathcal{P}_N u$. Then for some constant c we have*

$$\|u - \mathcal{P}_N u\|_{L^\infty(\Omega)} \le e^{\frac{-c}{h_{X,\Omega}}} \|u\|_{\mathcal{N}_\Phi(\Omega)}. \tag{26}$$

Now, we give the following theorem from Vainikko [5] that is used to obtain the error analysis of the presented method.

Theorem 4.2 ([5]). *Let T and $\tilde{T}$ be continuous over an open set Ω in Banach space X. Let the equation*

$$u = \tilde{T}u \tag{27}$$

has an isolated solution $\tilde{u}_0$ in Ω and let the following conditions be satisfied:

(a) *The operator T is Frechet differentiable in some neighborhood of the point $\tilde{u}_0$ while the linear operator $I - T'(\tilde{u}_0)$ is continuously invertible,*

(b) *Suppose that for some $\delta > 0$ and $0 < q < 1$ the following inequalities are valid (the number $\delta > 0$ assumed to be so small that the sphere $\|u - \tilde{u}_0\| \le \delta$ is contained within Ω)*

$$\sup_{\|u-\tilde{u}_0\|\le\delta} \|(I - T'(\tilde{u}_0))^{-1}(T'(u) - T'(\tilde{u}_0))\| \le q, \tag{28}$$

$$\alpha = \|(I - T'(\tilde{u}_0))^{-1}(T(\tilde{u}_0) - \tilde{T}(\tilde{u}_0))\| \le \delta(1-q), \tag{29}$$

then the equation $u = Tu$ has a unique solution u_0 in the sphere $\|u - \tilde{u}_0\| \le \delta$. Moreover, the inequality

$$\frac{\alpha}{1+q} \le \|u_0 - \tilde{u}_0\| \le \frac{\alpha}{1-q}, \tag{30}$$

is valid.

Theorem 4.3 ([19]). *Let $u_0 \in C[0, 1]$ be an isolated solution of*

$$u = \mathcal{K}u + f. \tag{31}$$

Assume that 1 is not an eigenvalue of the linear operator $\mathcal{T}'(u_0)$. Then for sufficiently large N, the operator $(I - \hat{\mathcal{T}}'_N(u_0))^{-1}$ is invertible and there exists constant $L > 0$ independent of N such that $\|(I - \hat{\mathcal{T}}'_N(u_0))^{-1}\| \le L$.

In the following theorem we give the conditions for the existence of an approximate solutions $\hat{u}_N$ in the neighborhood of u_0.

Theorem 4.4. *Let $u_0 \in C([0,1]\times[0,1])$ be an isolated solution of*

$$u = \mathcal{K}u + f. \tag{32}$$

Also let $\mathcal{P}_N$ be interoplatory projection. Assume that 1 is not an eigenvalue of the linear operator $\mathcal{T}'(u_0)$, then for sufficiently large N the approximate solution $\hat{u}_N$ of (21) is unique in $B(u_0,\delta)=\{u : \|u-u_0\| \le \delta\}$ for some $\delta > 0$. Moreover, there exists a constant $0 < q < 1$ independent of N such that

$$\frac{\beta_N}{1+q} \le \|u_0 - \hat{u}_N\| \le \frac{\beta_N}{1-q}, \tag{33}$$

where $\beta_N = \|(I-\hat{\mathcal{T}}'_N(u_0))^{-1}(\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0))\|$.

Proof. From Theorem (4.3) we know that $(I-\hat{\mathcal{T}}'_N(u_0))^{-1}$ exists and it is uniformly bounded i.e., there exists a constant $L > 0$ such that

$$\|(I-\hat{\mathcal{T}}'_N(u_0))^{-1}\| \le L. \tag{34}$$

Now considering the facts $\|\mathcal{P}_N\| \le p$ and $\|\mathcal{K}'_N\| \le \gamma$ for any $u \in B(u_0,\delta)$ it holds

$$\|\hat{\mathcal{T}}'_N(u) - \hat{\mathcal{T}}'_N(u_0)\| = \|\mathcal{P}_N\mathcal{K}'_N u - \mathcal{P}_N\mathcal{K}'_N u_0\| \le \|\mathcal{P}_N\|\|\mathcal{K}'_N u - \mathcal{K}'_N u_0\| \le p\gamma\delta. \tag{35}$$

Thus we have

$$\sup_{\|u-\tilde{u}_0\|\le\delta,} \|(I-\hat{\mathcal{T}}'_N(u_0))^{-1}(\hat{\mathcal{T}}'_N(u)-\hat{\mathcal{T}}'_N(u_0))\| \le Lp\gamma\delta \le q, \ (say) \tag{36}$$

where $0 < q < 1$, which proves Eq. (28) of Theorem (4.2) for δ sufficiently small. Furthermore we have

$$\begin{aligned}\beta_N &= \|(I-\hat{\mathcal{T}}'_N(u_0))^{-1}(\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0))\| \\ &\le \|(I-\hat{\mathcal{T}}'_N(u_0))^{-1}\|\|\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0)\| \le L\|\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0)\|.\end{aligned} \tag{37}$$

Now we want to prove that $\|\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0)\| \to 0$ as $N \to \infty$.

We have

$$\begin{aligned}\|\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0)\| &= \|\mathcal{P}_N\mathcal{K}_N u_0 + \mathcal{P}_N f - \mathcal{K}u_0 - f\| \\ &= \|\mathcal{P}_N\mathcal{K}_N u_0 + \mathcal{P}_N f - \mathcal{K}u_0 - f + \mathcal{P}_N(\mathcal{K}u_0+f) - \mathcal{P}_N(\mathcal{K}u_0+f)\| \\ &\le \|\mathcal{P}_N\{\mathcal{K}_N u_0 + f - \mathcal{K}u_0 - f\}\| + \|(\mathcal{P}_N - I)(\mathcal{K}u_0+f)\| \\ &\le p\|\mathcal{K}_N - \mathcal{K}\|\|u_0\| + \|\mathcal{P}_N u_0 - u_0\| \to 0 \text{as} N \to \infty.\end{aligned} \tag{38}$$

For sufficient large N we have $\beta_N \le \delta(1-q)$. Hence Eq. (29) of Theorem (4.2) is satisfied. Then applying Theorem (4.2) we get

$$\frac{\beta_N}{1+q} \le \|\hat{u}_N - u_0\| \le \frac{\beta_N}{1-q}. \tag{39}$$

The proof is complete. □

Theorem 4.5. *Let $u_0 \in \mathcal{N}_\Phi(\Omega)$ be an isolated solution of (1) and $\hat{u}_N$ be the RBFs collocation approximation of u_0. Then we have*

$$\|\hat{u}_N - u_0\|_{L^\infty(\Omega)} \le \frac{Lp}{1-q}\|\mathcal{K}_N u_0 - \mathcal{K}u_0\|_{L^\infty(\Omega)} + \frac{L}{1-q}e^{-\frac{c}{h_{X,\Omega}}}\|u_0\|_{\mathcal{N}_\Phi(\Omega)}. \tag{40}$$

Proof. From Theorem (4.4) we have

$$\|\hat{u}_N - u_0\| \le \frac{\beta_N}{1-q}. \tag{41}$$

Hence we have from Theorems (4.1), (4.3) and using the estimate (38)

$$\begin{aligned}\|\hat{u}_N - u_0\|_{L^\infty(\Omega)} &\le \frac{\beta_N}{1-q} = \frac{\|(I-\hat{\mathcal{T}}'_N(u_0))^{-1}(\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0)\|_{L^\infty(\Omega)}}{1-q} \\ &\le \frac{L}{1-q}\|\hat{\mathcal{T}}_N(u_0)-\mathcal{T}(u_0)\|_{L^\infty(\Omega)} \\ &\le \frac{L}{1-q}\|\mathcal{P}_N\|_{L^\infty(\Omega)}\|\mathcal{K}_N - \mathcal{K}\|_{L^\infty(\Omega)}\|u_0\|_{L^\infty(\Omega)} + \frac{L}{1-q}\|\mathcal{P}_N u_0 - u_0\|_{L^\infty(\Omega)} \\ &\le \frac{Lp}{1-q}\|\mathcal{K}_N - \mathcal{K}\|_{L^\infty(\Omega)}\|u_0\|_{L^\infty(\Omega)} + \frac{L}{1-q}e^{-\frac{c}{h_{X,\Omega}}}\|u_0\|_{\mathcal{N}_\Phi(\Omega)}.\end{aligned} \tag{42}$$

□

Table 2
Error comparison of the proposed method using MQ-RBFs with 2D-TFs method [4] for Example 1 for $c = 4$ and $N = 16$.

Test point (x, t)	$e(x, t)$ of proposed method	$e(x, t)$ of 2D-TFs method [4]
(0, 0)	1×10^{-4}	2×10^{-4}
(.1, .1)	3×10^{-4}	7×10^{-3}
(.2, .2)	1×10^{-4}	5×10^{-3}
(.3, .3)	2×10^{-4}	6×10^{-3}
(.4, .4)	6×10^{-4}	7×10^{-3}
(.5, .5)	9×10^{-4}	7×10^{-3}
(.6, .6)	1×10^{-3}	8×10^{-3}
(.7, .7)	5×10^{-4}	3×10^{-2}
(.8, .8)	8×10^{-4}	5×10^{-2}
(.9, .9)	1×10^{-3}	6×10^{-2}

Table 3
$\|e\|_\infty$ for various RBFs and different values of N and c for Example 1.

N	$c = 2.0$			$c = 4.0$		
	MQ	GA	IMQ	MQ	GA	IMQ
9	3.5×10^{-2}	4×10^{-1}	8×10^{-2}	6.0×10^{-3}	5×10^{-1}	2×10^{-2}
16	3.0×10^{-3}	9×10^{-2}	9×10^{-3}	6.0×10^{-4}	9×10^{-1}	1.8×10^{-3}
25	5.0×10^{-4}	2.5×10^{-2}	2×10^{-3}	1.8×10^{-5}	8×10^{-2}	1.4×10^{-4}
36	1.0×10^{-4}	6×10^{-3}	2.5×10^{-4}	7.0×10^{-6}	3×10^{-2}	1.8×10^{-5}

5. Numerical experiments

In this section, some comparative examples are provided to show the strength of the proposed method in approximating the solution of two-dimensional Volterra–Fredholm integral equations. In order to find the best type of RBFs for the proposed method, we have used various RBFs. Moreover we use $p_1(x, t) = 1$, $p_2(x, t) = x$ and $p_3(x, t) = t$ as a basis for Π_1^2. Also, for the numerical quadrature rule we have used the 5-point Gauss–Legendre quadrature formula. Accuracy of the estimated solutions can be worked out by measuring $\|e\|_\infty$ error norm which is defined by

$$\|e\|_\infty = max|u(x, t) - \hat{u}(x, t)|, \quad (x, t) \in [0, 1] \times [0, 1],$$

where $\hat{u}(x, t)$ is the approximation of $u(x, t)$. Computations have been done by Maple 13. Before we start, it is noteworthy that we have to solve a nonlinear $(N + 3) \times (M + 3)$ system of equations while the statement after Eq. (66) in Babolian et al. [4] shows that 2D-TFs method requires solving a nonlinear $4m_1m_2 \times 4m_1m_2$ system of equations which is more expensive.

Example 1. ([4]) As the first example consider

$$u(x, t) + \int_0^t \int_0^x (x + t + y + z)u^2(y, z)dydz + \int_0^1 \int_0^1 (xt + yz^2)u(y, z)dydz = f(x, t), \tag{43}$$

where

$$f(x, t) = x^2 + \frac{17}{6}xt + \frac{7}{9}x^3t^4 + \frac{29}{18}t^3x^4 + \frac{6}{5}t^2x^5 + \frac{11}{30}x^6t + \frac{1}{4}.$$

The exact solution of this equation is $u(x, t) = x^2 + 2xt$. Table 2 compares the proposed method with 2D-TFs method [4]. Table 3 provides $\|e\|_\infty$ for various RBFs and different values of N and c. Figs. 1 and 2 represent the behavior of errors of the computed solution using the proposed collocation method.

Example 2. ([4]) Consider the following nonlinear integral equation:

$$u(x, t) - \int_0^t \int_0^x u(y, z)dydz - \int_0^1 \int_0^1 16\exp(x + t + y + z)u^3(y, z)dydz = f(x, t), \tag{44}$$

where

$$f(x, t) = -\exp(x + t) - 1 + \exp(x) + \exp(t) + 2\exp(x + t + 4) - \exp(x + t + 8)$$

with the exact solution $u(x, t) = \exp(x + t)$. Table 4 compares the proposed method with 2D-TFs method [4]. Table 5 provides $\|e\|_\infty$ for various RBFs and different values of N and c.

Example 3. Consider the following nonlinear integral equation:

$$u(x, t) + \int_0^t \int_0^x \cos(z)u^2(y, z)dydz = x\sin(t) + \frac{1}{9}\sin^3(t)x^3. \tag{45}$$

The exact solution of this equation is $u(x, t) = x\sin(t)$. The numerical results obtained using the presented method are shown in Figs. 3 and 4. Table 6 provides $\|e\|_\infty$ for various RBFs and different values of N and c.

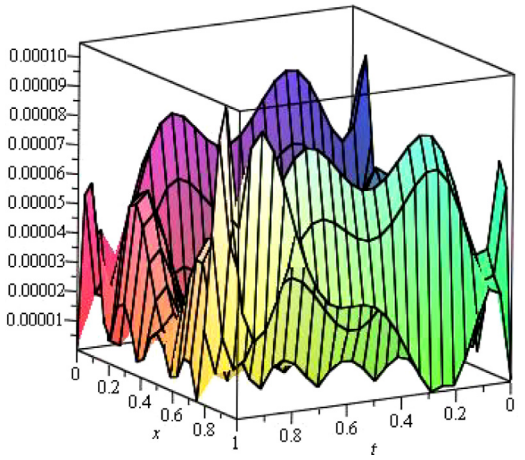

Fig. 1. The error function graph for Example 1 with $c = 2$ and $N = 36$.

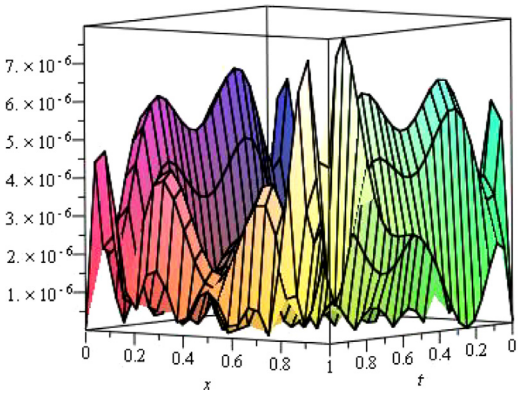

Fig. 2. The error function graph for Example 1 with $c = 4$ and $N = 36$.

Table 4
Error comparison of the proposed method using MQ-RBFs and 2D-TFs method [4] for Example 2 for $c = 4$ and $N = 16$.

Test point (x, t)	$e(x, t)$ of proposed method	$e(x, t)$ of 2D-TFs method [4]
(0, 0)	9×10^{-6}	2×10^{-2}
(.1, .1)	5×10^{-4}	1×10^{-2}
(.2, .2)	4×10^{-4}	2×10^{-2}
(.3, .3)	8×10^{-5}	3×10^{-2}
(.4, .4)	1×10^{-4}	3×10^{-2}
(.5, .5)	2×10^{-4}	8×10^{-2}
(.6, .6)	1×10^{-4}	5×10^{-2}
(.7, .7)	1×10^{-4}	9×10^{-2}
(.8, .8)	4×10^{-4}	1×10^{-1}
(.9, .9)	2×10^{-4}	1×10^{-2}

Table 5
$\|e\|_\infty$ for various RBFs and different values of N and c for Example 2.

N	$c = 2.0$			$c = 3.0$		
	MQ	GA	IMQ	MQ	GA	IMQ
4	8×10^{-1}	6×10^{-1}	8×10^{-1}	8×10^{-1}	9×10^{-1}	9×10^{-1}
9	4×10^{-2}	6×10^{-1}	1.4×10^{-1}	3×10^{-2}	8×10^{-1}	8×10^{-2}
16	7×10^{-3}	1.8×10^{-1}	1.8×10^{-2}	3.5×10^{-3}	3×10^{-1}	8×10^{-3}
25	1.2×10^{-3}	4×10^{-2}	4×10^{-3}	7×10^{-4}	9×10^{-2}	1.2×10^{-3}

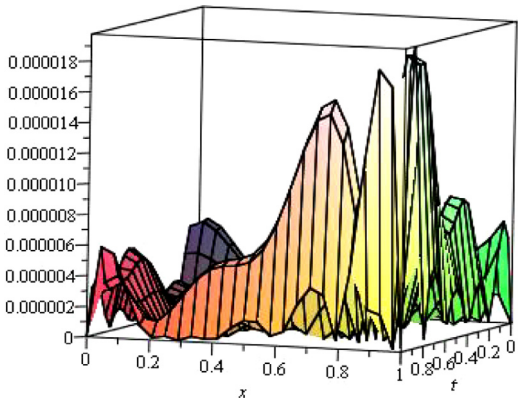

Fig. 3. The error function graph for Example 3 with $c = 2$ and $N = 36$.

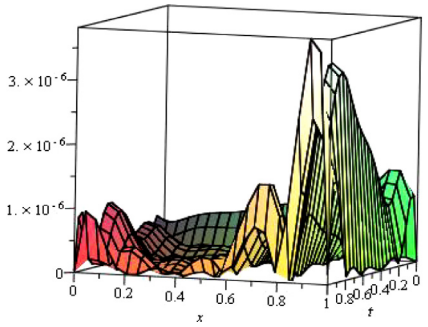

Fig. 4. The error function graph for Example 3 with $c = 3$ and $N = 36$.

Table 6
$||e||_\infty$ for various RBFs and different values of N and c for Example 3.

N	$c = 2.0$			$c = 3.0$		
	MQ	GA	IMQ	MQ	GA	IMQ
9	1.0×10^{-2}	1×10^{-1}	1.8×10^{-2}	7.0×10^{-3}	1.4×10^{-1}	2×10^{-2}
16	7.0×10^{-4}	1.8×10^{-2}	1×10^{-3}	2.0×10^{-4}	3×10^{-2}	4×10^{-4}
25	2.5×10^{-4}	8×10^{-3}	5×10^{-4}	9.0×10^{-5}	1.6×10^{-2}	1.5×10^{-4}
36	1.8×10^{-5}	1×10^{-3}	4×10^{-5}	3.6×10^{-6}	3×10^{-3}	7×10^{-6}

6. Conclusion and further research

The presented method transforms a nonlinear two-dimensional Volterra–Fredholm integral equation into a system of nonlinear algebraic equations. The error analysis is provided for the method. This method does not need any background interpolation or approximation cells and so it is a meshless method. These advantages make this method very simple and affordable from computational point of view. The convergence and accuracy of the method are examined in some examples. Examples 1 and 2 approve the supremacy of the proposed method in comparison to 2D-TFs method [4]. Experiments show that MQ-RBFs give the best results for the proposed method. Generalizing the proposed method for local fractional integral equations introduced in [25,26] is the subject of our future work.

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