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On the fractal distribution of primes and prime-indexed primes by the binary image analysis

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HIGHLIGHTS

- The fractal nature of prime distribution is studied on binary images of primes.
- The dimension of prime distribution is a non-integer value lower than 2.
- The lacunarity of prime distribution of primes and PIPs have been computed.
- The binary image for prime (and PIP) distribution is similar to a Cantor dust.

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ABSTRACT

In this paper, the distribution of primes and prime-indexed primes (PIPs) is studied by mapping primes into a binary image which visualizes the distribution of primes. These images show that the distribution of primes (and PIPs) is similar to a Cantor dust, moreover the self-similarity with respect to the order of PIPs (already proven in Batchko (2014)) can be seen as an invariance of the binary images. The index of primes plays the same role of the scale for fractals, so that with respect to the index the distribution of prime-indexed primes is characterized by the self-similarity alike any other fractal. In particular, in order to single out the scale dependence, the PIPs fractal distribution will be evaluated by limiting to two parameters, fractal dimension (δ) and lacunarity (λ), that are usually used to measure the fractal nature. Because of the invariance of the corresponding binary plots, the fractal dimension and lacunarity of primes distribution are invariant with respect to the index of PIPs.

1. Introduction

In this paper the distribution of primes is considered by focusing on its similarity with a fractal set (Cantor set). Although the distribution of primes has received much attention in classical literature (see e.g. Refs. [1–3]) only recently there have been many attempts to explain the hidden structure of prime distribution by fractality [4–10]. In almost all these papers the Authors give some proofs that the prime distribution is closely related to fractality. In particular the fractal distribution of prime-indexed primes (PIPs) was the subject of Ref. [5], where the Author "reports the empirical observation of fractal structure in the distribution of prime numbers" by showing that "finite-differenced PIP sequences…exhibit quasi-self similar fractality by prime-index order" [5]. In particular, this Author shows that the order of primes plays the same role

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played by the scale for any fractal. In Ref. [6] a binary map is used to represent the distribution of primes in two dimensions. In the same paper the plot of cumulative frequencies of primes is given [6] and it is shown that this graph is similar to a Cantor function thus suggesting that the distribution of primes within naturals might be a fractal-like distribution. The existence of prime reciprocals in the Cantor set is shown in Ref. [7], for a special set of primes that satisfy the equation $2p+1=3^q$. More in general the existence of a hidden structure is one of the main tasks in signal and time series analysis and it can be checked by giving a proof of the power-law ($1/f$-law or long range) dependence. The power-law is a universal property in almost all physical phenomena, like e.g. phase transitions [11], market fluctuations [12], biology [13], and oceanography [14]. The powerlike behavior was also shown in prime distribution [8]. In Ref. [8] it is proven for the first time that "the distribution of prime numbers displays the $1/f$ noise". Although there is no reason for searching a physical meaning in the set of primes, the power-law or long range dependence, given in Ref. [8], is a fundamental step both for discovering the importance of primes in physics and for deepening the research of power-laws in other mathematical structures.

It can be observed that the source for long-range correlation might be linked with the existence of patchiness in primes. The identification of these patches could be the key point for understanding the large scale structure of primes distribution.

Starting from the point of view, that the distribution of primes is similar to the recursive law of a fractal, in this paper we will investigate this property of primes by analyzing some families of prime-indexed primes (for their definition and properties see also Refs. [5,15–17]).

The main idea is to approach the prime distribution as a discrete time series (a dynamical system) and to define the corresponding correlation matrix (based on a Boolean indicator). Then by evaluating the fractal dimension and lacunarity on a 2D image of this Boolean matrix we derive the fractal properties for primes and prime-indexed primes. This correlation matrix, which gives rise to the so-called recurrence- (or dot-) plots, has been widely used to study the complexity of dynamical systems and time-series (see e.g. Refs. [18–20]). Through the recurrence plots we can identify some typical patterns [18–20] in the prime distribution and their fractal nature will be characterized by computing the fractal parameters of these 2D patterns. We will see that the recurrence plots of prime distribution are similar (but not perfectly equivalent) to the Cantor dust. Moreover, similar to any other fractal, the correlation plot of primes distribution can be characterized by the measures of fractality i.e. fractal dimension δ and lacunarity λ. So that we will characterize the primes and PIPs distribution by computing their fractal dimension and lacunarity of their do-plots.

This paper is organized as follows: preliminary remarks on primes and prime-indexed primes are given in Section 2; Section 3 deals with fractal dimension and lacunarity on the correlation plot for the (Boolean) indicator matrix; in Section 4 the results and discussion on fractal analysis of primes and PIPs are given.

2. Preliminary remarks

2.1. Primes and prime-indexed primes

Let $p_i,\ (i \geq 1,\ p_1 = 2)$ be the ith prime, and

$$\mathbb{P} \overset{\text{def}}{=} \{p_i : i \in \mathbb{N}\},$$

the set of all primes in $\mathbb{N}$; the counting function, $\pi(x) : \mathbb{R} \to \mathbb{N}$, is defined as

$$\pi(x) = |\mathbb{P}_x|, \qquad \mathbb{P}_x \overset{\text{def}}{=} \{p_i \in \mathbb{P} : p_i \leq x\}, \qquad \mathbb{P}_x \subseteq \mathbb{P}. \tag{1}$$

According to Gauss conjecture $\pi(x)$ asymptotically tends to $x/\log x$,

$$\pi(x) \sim x/\log x \tag{2}$$

so that the prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \tag{3}$$

holds.

The sequence of prime indexed primes is the "sequence of primes which have a prime index" [17] or equivalently "the subsequence of $\mathbb{P}$ where the index i is itself prime" [15], so that

$$\mathbb{P}^1 \overset{\text{def}}{=} \{p_{p_i} : (p_i \in \mathbb{P}) \wedge (p_{p_i} \in \mathbb{P})\}.$$

More sets of indexed primes can be iteratively defined [5,16] by taking prime index subsequences, so that there are infinite sets $\mathbb{P}^k$ of PIPs of order k. For instance it is:

$$p_1 = 2,\ p_2 = 3, \ldots, p_{p_1} = p_2 = 3, \ldots, p_{p_{p_1}} = p_{p_2} = p_3 = 5, \ldots$$

so that

$$\begin{aligned} \mathbb{P} &= \{2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 19,\ 23, \ldots\} \\ \mathbb{P}^1 &= \{3,\ 5,\ 11,\ 17,\ 31,\ 41,\ 59,\ 67, \ldots\} \\ \mathbb{P}^2 &= \{5,\ 11,\ 31,\ 59, \ldots\} \\ \vdots\ & \ \vdots\ \ldots. \end{aligned}$$

Table 1
Correlation matrix for primes in the n-length sequence $1 \leq n \leq 13$.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
13	0	1	1	0	1	0	1	0	0	0	1	0	1	$\cdots$
12	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
11	0	1	1	0	1	0	1	0	0	0	1	0	1	$\cdots$
10	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
9	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
8	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
7	0	1	1	0	1	0	1	0	0	0	1	0	1	$\cdots$
6	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
5	0	1	1	0	1	0	1	0	0	0	1	0	1	$\cdots$
4	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
3	0	1	1	0	1	0	1	0	0	0	1	0	1	$\cdots$
2	0	1	1	0	1	0	1	0	0	0	1	0	1	$\cdots$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
u_{hk}	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$

2.2. Correlation matrix

Let $S_1(n) = \{i_1, i_2, \ldots, i_n\}$, $S_2(n) = \{j_1, j_2, \ldots, j_n\}$ be two n-length sequences of ordered naturals, starting at the naturals i_1 and j_1 respectively; in the following, with the slightly different notation $S(i_1, n)$, we will indicate explicitly the starting number i_1 of the n-length sequence. The *indicator function for primes* is defined by the binary map:

$$u : S_1(n) \times S_2(n) \to U(n \times n) \tag{4}$$

so that, for $i_h \in S_1(n)$, $i_k \in S_2(n)$, *the correlation matrix for primes* $U(n \times n)$ is defined as (see e.g. Table 1)

$$\begin{aligned} U(n \times n) &= \{u(i_h, j_k)\} = \{u_{hk}\}, \\ u_{hk} &\overset{\text{def}}{=} \begin{cases} 1 & \text{if } i_h \wedge j_k \text{ are primes} \\ 0 & \text{otherwise} \end{cases} \\ & \quad (i_h \in S_1(n), j_k \in S_2(n)), \ 1 \leq h, k \leq n. \end{aligned} \tag{5}$$

Eqs. (4), (5) can be easily extended to first order PIPs as follows:

Let $S_1^1(n) = \{p_1, p_2, \ldots, p_n\}$, $S_2^1(n) = \{q_1, q_2, \ldots, q_n\}$ be two n-length sequences of ordered primes, the *indicator function for prime indexed primes* is defined by the binary map:

$$u^1 : S_1^1(n) \times S_2^1(n) \to U^1(n \times n) \tag{6}$$

with

$$\begin{aligned} U^1(n \times n) &= \left\{u^1(p_h, p_k)\right\} = \left\{u^1_{hk}\right\} \\ u^1_{hk} &= \begin{cases} 1 & \text{if } (h \wedge k \text{ are primes}) \wedge (p_h \wedge p_k) \text{ are primes} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

The nth order Boolean matrix $\{u_{hk}\}$, $(1 \leq h, k \leq n)$ and of $\{u^1_{hk}\}$, whose elements are 0's and 1's (Fig. 1) are the correlation matrices for primes and PIPs, showing their distribution among naturals and primes respectively.

The distribution of PIPs among naturals is described by the following correlation matrix. Let $S_1(n) = \{i_1, i_2, \ldots, i_n\}$, $S_2(n) = \{j_1, j_2, \ldots, j_n\}$ be two n-length sequences of ordered naturals, the *indicator function for prime indexed primes among naturals* is defined by the binary map:

$$v^1 : S_1(n) \times S_2(n) \to V^1_{\mathbb{N}}(n \times n) \tag{8}$$

with

$$\begin{aligned} V^1_{\mathbb{N}} &= \left\{v^1(i_h, i_k)\right\} = \left\{v^1_{hk}\right\} \\ v^1_{hk} &= \begin{cases} 1 & \text{if } (h \wedge k \text{ are primes}) \wedge (p_h \wedge p_k) \text{ are primes} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

The indicator function shows the existence of correlation between two sequences. In particular, if $S_1(n) = S_2(n)$ or $S_1^1(n) = S_2^1(n)$, this matrix represents the autocorrelation of a single sequence. The non vanishing values of the autocorrelation matrix can be used to mark the primes among naturals (or PIPs among primes or PIPs among naturals), thus being the simplest way to describe their distribution. Moreover, by mapping the correlation matrix into the black and white pixels

$$bw : \{0 \Rightarrow \text{empty dot}, \ 1 \Rightarrow \text{black dot}\}$$

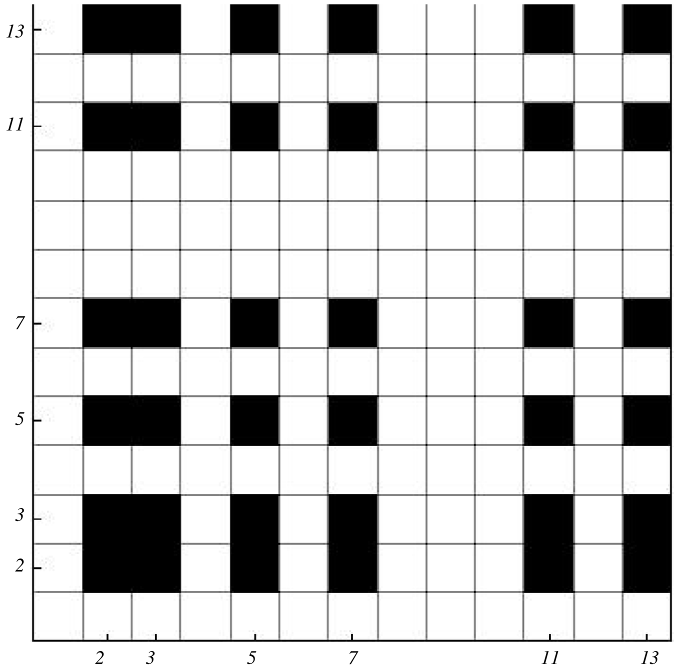

Fig. 1. Binary image (dot-plot) for the correlation matrix of Table 1.

we obtain the two dimensional image $\mathcal{C}_{n\times n}$, also called *dot-plot or binary image*. The indicator matrix can be used to obtain a binary image, i.e. a two dimensional plot. This two dimensional picture can be used to visualize the distribution of primes (PIPs) in the sequence. The dot-plot can be easily drawn by assigning a black dot to 1 and a white dot to 0. As an example, the correlation matrix and the wp map is shown in Table 1 and Fig. 1 respectively, for the first 6 primes. There follows that the maps from the product of sequences to the binary image

$$S(n) \times S(n) \stackrel{u}{\to} U(n \times n) \stackrel{bw}{\to} \mathcal{C}_{n\times n}$$

might be used to visualize the distribution of primes. As an example, let us take the sequence $S(300)$ of the ordered naturals ≤ 300 and the dot-plot associated with the correlation matrix for primes. The binary image (Fig. 2) looks like the Cantor dust [21] however there are some missing symmetries (comparing with the Cantor dust). As already noticed in Ref. [7] only some reciprocals of primes, which fulfill a special equation, belong to a Cantor set. For this reason we can only say that the distribution of primes among naturals is similar to a fractal.

If we consider higher index PIPs we will have some binary images which depend on the index of PIPs. It has been shown [5] that PIPs are self-similar with respect to the index, so that, as suggested in Ref. [5], the index of PIPs can be considered as the equivalent of scale for any other fractal. Fractals are self similar with respect to the scale [21,22] and PIPs are self similar with respect to index as well [5]. So that in order to investigate the fractal nature of primes, as suggested by Ref. [5], we should think about the fundamental concepts of scale as replaced by the index and self-similarity with respect to index. So that the index of PIPs is the equivalent of scale for any other fractal. By the binary image of primes we can visualize the distribution of primes. Moreover, as we can see from the analysis of the dot-plots, the self similarity can be observed when we pass from the dot-plot of PIPs with a given index to the dot-plot of PIPs with the increased index. In doing so, we get another proof of the results, already obtained in Ref. [5], by simply comparing the dot-plots of primes and PIPs.

3. Fractal dimension and lacunarity

In order to characterize the fractal nature of primes distribution, we recall the definition of two fundamental parameters which describes fractals [21,22], i.e. the fractal dimension and the lacunarity [21–24].

Fractal is a geometric object that appears with the same features at different scales of resolution (magnification) (see e.g. Refs. [21,22]). For all magnifications, the fractal object shows occurrences and details that are repeated in accordance with the self-similarity (the smallest part of the object is a scaled image of its integer). In addition of the self-similarity property, a fractal is characterized by its irregularity, fine structure (new details are shown for any resolution) and by a non-integer (fractal) dimension (see e.g. Refs. [21,22]).

The fractal dimension and the lacunarity of a geometrical object indicate how much the embedding space is filled in the object.

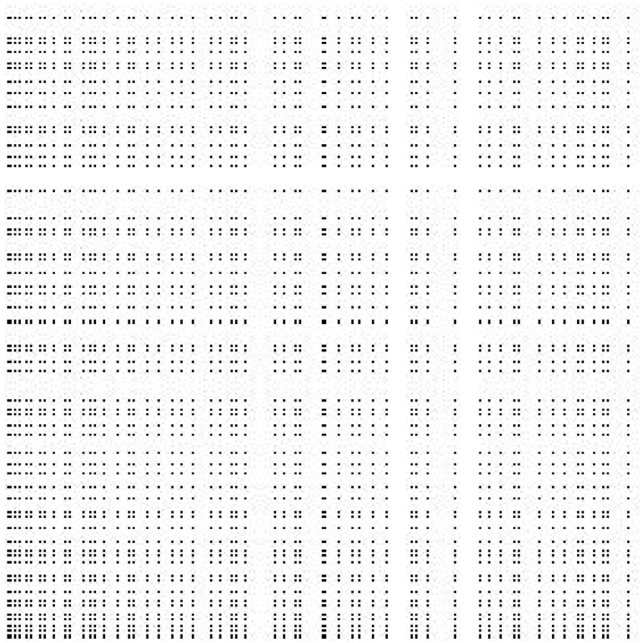

Fig. 2. Dot-plot for the distribution of primes among the first 300 naturals $p_i \in S(300),\ 2 \le p_i < 300$.

The lacunarity describes the heterogeneity of an object by giving a measure of the gaps within it. Homogeneous objects show low values of lacunarity, while heterogeneous objects are characterized by high values. The lacunarity is studied in parallel to the fractal dimension in order to clarify how the space is filled by the patterns and to measure the ratio between fully and empty spaces.

Fractal dimension and lacunarity will be computed on the binary image (dot-plot) of the correlation matrix of primes and PIPs.

3.1. Computational methods for Fractal dimension and Lacunarity

3.1.1. Fractal dimension

Let $S(n)$ be a n-length sequence of ordered naturals, the frequency $\nu(n)$ of primes in $S(n)$ is related to the product $S(n) \times S(n)$ and it is obtained by counting the non zero values of the correlation matrix (5) so that

$$\nu(n) = \nu\,[S(n)] \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} u_{ij}. \tag{10}$$

Fractal dimension is a measure of how the fractal fills the space. There are several definitions of fractal dimension (see e.g. Refs. [21–29]) which however are not always equivalent for fractal objects. In the following the fractal dimension is defined as (see e.g. Ref. [21])

$$\delta(n) = \delta[S(n)] \stackrel{\text{def}}{=} \frac{1}{n-1} \sum_{h=2}^{n} \frac{\log \nu(h)}{\log h}, \quad (n \ge 2,\ \nu(h) \ge 1). \tag{11}$$

In this definition, δ counts the number of black dots with respect to sliding boxes covering the binary image, and describes the information content (i.e. the degree of disorder at each scale) in the correlation matrix and for this it can be also called information dimension [21]. We can easily compute the value of δ in some special cases. For instance when the black dots fill-in the binary plot (i.e. $\nu(n) = n^2$) we have a single black square, with n-length side and

$$u_{ij} = 1, \quad \forall i, j \le h \Longrightarrow \nu(h) = h^2$$

so that the fractal dimension, according to (11), is 2. If instead there is only one black dots, giving $\nu(n) = 1$, the dimension is 0. When the black dots are distributed along the diagonal, it is $\nu(h) = h$ with dimension 1. It can be also seen that if $h < \nu(h) < h^2$ then $1 < \delta(n) < 2$, while it is $\delta(n) < 1$ when $\nu(h) < h$. In the following $\delta^1(n)$ and $\delta^1_{\mathbb{N}}$ will denote the fractal dimension of PIPs among primes and PIPs among naturals respectively, obtained by (11) applied to corresponding frequencies $\nu^1(n)$ and $\nu^1_{\mathbb{N}}(n)$.

Table 2
Fractal dimension for primes (δ), PIPs among primes (δ^1) and PIPs among naturals ($\delta^1_{\mathbb{N}}$), computed on n-length sequences $10 \le n \le 10^9$.

n	δ	δ^1	$\delta^1_{\mathbb{N}}$
10	1.119	1.119	0.471
50	1.321	1.321	0.692
250	1.401	1.401	0.713
450	1.423	1.423	0.718
650	1.436	1.436	0.721
850	1.446	1.446	0.723
1010	1.451	1.451	0.729
10^4	1.462	1.462	0.801
10^5	1.480	1.480	0.812
10^6	1.511	1.511	0.831
10^8	1.543	1.543	0.872
10^9	1.561	1.561	0.893

3.1.2. Lacunarity

Lacunarity is a parameter which gives a measure of gaps in a sequence or within an image and, according to Ref. [28], characterizes the "texture of a fractal". There are several alternative definitions of lacunarity (see e.g. Refs. [21,23,24,29]), so that a universally accepted definition is still missing. In our case, lacunarity will be evaluated by the ratio of the variance and mean of the frequency distribution (10) over some scaled boxes. The algorithm, based on the gliding box algorithm [23,24,26,27], is as follows (see also Refs. [18,19] and references therein).

Let $S(n) = \{i_1,\ i_2,\ \dots,\ i_n\}$ be a n-length sequence of ordered naturals and $S(n) \times S(n)$ be the product which gives rise to the correlation matrix $\{u_{ij}\},\ i = 1, \dots, j = 1, \dots, n$ and corresponding dot-plot $\mathcal{C}_{n\times n}$. In order to compute the lacunarity on the image $\mathcal{C}_{n\times n}$ we need a covering of $\mathcal{C}_{n\times n}$ by some sliding non-overlapping boxes of width r. Let $S(i_h, r),\ S(i_k, r)$ be two r-length $(2 \le r \le n)$ subsequences of $S(n)$, the corresponding $r \times r$ correlation matrix is

$$B_{h,k}(r \times r) = u\left(S(i_h, r) \times S(i_k, r)\right), \quad (1 \le h \le n/r,\ 1 \le k \le n/r).$$

The gliding box is the bw-image of $B_{hk}(r)$ with $(1 \le h \le n/r,\ 1 \le k \le n/r)$.

Let $\nu_{h,k}(r)$ be the frequency in each square box $B_{hk}(r)$, the lacunarity is defined by the ratio of the second and first moment of the frequency distribution [18,19,21–29] which can be simplified into

$$\lambda\left(r\right) = \frac{\sigma^2\left[\nu_{h,k}(r)\right]}{\left[\overline{\nu_{h,k}(r)}\right]^2} + 1, \quad (2 \le h \le n/r,\ 2 \le k \le n/r) \tag{12}$$

where σ^2 is the variance and the bar stands for the mean value.

Lacunarity $\lambda(r)$ will be discussed, in the next section, as a function of the gliding box size r.

4. Results and discussion

In this section we give the results of computations for the fractal dimension and lacunarity on the dot-plots, which are obtained by the correlation matrix of primes, prime-indexed primes (distributed among primes) and prime-indexed primes (distributed among naturals). By the explicit computation of fractal dimension (11) and lacunarity for n-length sequences $S(n),\ n \le 10^9$ of ordered naturals with primes $p_i \le n$ and PIPs $p_{p_i} \le n$ we can see that the fractal dimension computed on the binary images is the same both for primes and for prime-indexed primes distributed among primes (see Table 2). In fact, prime indexed primes are distributed among primes alike the primes are distributed among naturals. If instead we focus on the distribution of PIPs among naturals then the fractal dimension of PIPs ($\delta^1_{\mathbb{N}}$) is nearly half of the corresponding fractal dimension for distribution of PIPs among primes (Table 2). In order to speed up the computation on large correlation matrices ($n \ge 10^3$) with more than 10^9 elements, we have computed the fractal dimension as the average value over sliding squares of 10^3 length, randomly selected. The number of squares was decided according to the stability of the numerical result, at the 10^{-3} approximation.

Concerning the lacunarity, both for primes and PIPs, we have that it rapidly decays to 1, because of the large gaps among primes (and PIPs). If we consider the distribution of PIPs among naturals, lacunarity is nearly 1 starting from very low values of n. Therefore we have

$$\lambda(n) \cong 1, \quad n \ge 25$$

with an approximation of 10^{-2}.

Since, according to (11), (12) the fractal dimension and lacunarity depend on the length n of a sequence, they will have the same values for the prime sequence $S(i_1, n)$ and the PIP sequence $S^1(i_i, n)$.

We can summarize these results as follows:

- The asymptotic value of the fractal (information) dimension for primes and PIPs is a finite value lower than 2, thus confirming the non-integer dimension of the binary image of primes and PIPs correlation matrix
$$\lim_{n\to\infty} \delta(S(n)) = \lim_{n\to\infty} \delta^1(S(n)) = \ell < 2.$$
- The same n-length sequences of primes and PIPs give the same fractal dimension
$$\delta(S(i_1, n)) = \delta^1(S(i_1, n)), \quad \forall i_1,\ n.$$
- The asymptotic value of the fractal (information) dimension for primes among naturals is a finite value $\ell^1_{\mathbb{N}}$ lower than 1 and it is
$$\lim_{n\to\infty} \delta^1_{\mathbb{N}}(S(n)) = \ell^1_{\mathbb{N}} < 1$$
- moreover
$$\ell^1_{\mathbb{N}} \cong \ell.$$
- For a fixed r with $1 \le r \le n$,
$$\lambda(r) = \lambda^1(r), \quad 1 \le r \le n$$
so that the lacunarity for primes coincides with the lacunarity for PIPs on the same n-length sequences and box width.
- The asymptotic value of lacunarity is 1
$$\lim_{r\to\infty} \lambda(r) = \lim_{r\to\infty} \lambda^1(r) < \lim_{r\to\infty} \lambda^1_{\mathbb{N}}(r) \cong 1.$$

There follows that, as already shown in Ref. [5], there is a self-similarity when we pass from primes to PIPs and this can be also seen on the fact that *the correlation matrix is invariant with respect to the index* when we pass from the indicator function for primes to the indicator function for PIPs (among primes). However, since the fractal dimension depends both on the length of sequence and on its starting point, the invariance holds when the n-length sequence of naturals starts from the same natural i_1. As a consequence the dot-plot for primes is the same of PIPs. From the invariance of the correlation matrix there follows the invariance of the fractal measure, (lacunarity and fractal dimension), with respect the order of PIPs. So that the fractal dimension is the same at each index, i.e. the fractal dimension is invariant with respect to the index.

Moreover also the computation of lacunarity and its invariance with respect the order of primes confirm the self-similarity of primes with respect the index.

At last, let us comment on the asymptotic value ℓ of δ for primes (Table 2). It should be noticed that for low values of n, such as $n \le 50$, the value 1.3 of the fractal dimension is quite close to the fractal dimension of a Cantor dust, which is (see e.g. Refs. [21,22])
$$\delta = 2\frac{\log 2}{\log 3} \cong 1.261859.$$
We have computed ℓ for the distribution of primes (and PIPs) among the first 10^9 naturals and we obtained some values of δ lower than 1.66. We expect to have a value ℓ lower than 2 but we cannot properly estimate its value. We cannot even analytically prove that the prime distribution is a Cantor dust, however it was already observed [7] that reciprocals of primes, which satisfy a suitable equation, belong to a Cantor set. We can say that empirically the prime distribution is characterized by a non-integer fractal dimension and a lacunarity with some invariance properties. A suitable subset of primes belong to a Cantor set [7] so that there is the possibility to find some more subsequences of primes on Cantor sets.

5. Conclusion

In recent papers [4–8,10] some empirical proofs of the fractal nature of prime distribution and their power law [9] are explicitly given. In particular in Ref. [5] it has been proven the self-similarity when we pass from primes to prime indexed primes. In the same paper it is shown that, analogously to the invariance of fractals with respect the scale, the prime distribution is a "quasi-self similar fractal" [5] with respect the index of PIPs. Starting from the results of Ref. [5], in our paper we have proposed the fractal analysis on some binary images obtained from the correlation matrices of primes and PIPs respectively. These binary images show some similarities with a Cantor dust [6]. By the explicit computation of the correlation matrix for primes and PIPs we have shown that the correlation matrix depends only on n (and the starting value) and it is invariant with respect to the index. There follows that fractal (information) dimension and lacunarity on the binary images of primes and PIPs have the same values when we compare the prime sequence in $S(i_1, n)$ and the PIPs distribution in the same sequence. The invariance of δ with respect to the index was already proven in Ref. [5], while the invariance of lacunarity has been given here. Moreover from the computation of the asymptotic value of the fractal (information) dimension (both for primes and for PIPs) we can empirically observe that the prime distribution is a fractal (or a "quasi-self similar fractal" [5]) similar to a Cantor dust (at least for some suitable sequences).

References

[1] J.E. Littlewood, Sur la distribution des nombres premieres, C. R. Acad. Sci. Paris 158 (1914) 1869–1872.
[2] W. Narkiewicz, The Development of Prime Number Theory, Springer, 2000.
[3] H.H. Shapiro, Introduction to the Theory of Numbers, John Wiley & Sons, New York, 1983.
[4] S. Ares, M. Castro, Hidden structure in the randomness of the prime number sequence? Physica A 360 (2006) 285–296.
[5] R.G. Batchko, A quasi-self-similar fractal structure in the distribution of prime-indexed primes, 2014. arXiv:1405.2900.
[6] C. Cattani, Fractal patters in prime numbers distribution, Lecture Notes in Comput. Sci. 6017 (2010) 164–176.
[7] C. Salas, On prime reciprocals in the Cantor set, 2011. arXiv:0906-0465v7.
[8] M. Wolf, Multifractality of prime numbers, Physica A (1989) 24–42.
[9] M. Wolf, $1/f$ noise in the distribution of prime numbers, Physica A (1997) 493–499.
[10] M. Wolf, Applications of statistical mechanics in number theory, Physica A (1999) 149–157.
[11] H.E. Stanley, Phase transitions: power laws and universality, Nature 378 (6557) (1995) 554.
[12] X. Gabaix, P. Gopikrishnan, V. Plerou, H.E. Stanley, A theory of power-law distribution in financial market fluctuations, Nature 423 (6937) (2003) 267–270.
[13] C. Gu, C.P. Coomans, K. Hu, F.A.J.L. Scheer, H.E. Stanley, J.H. Meijer, Lack of exercise leads to significant and reversible loss of scale invariance in both aged and Young Mice, Proc. Natl. Acad. Sci. 112 (8) (2015) 2320–2324.
[14] M. Li, Y.C. Li, J.X. Leng, Power-type functions of prediction error of sea level time series, Entropy 17 (7) (2015) 4809–4837.
[15] J. Bayless, D. Klyve, T.O. e Silva, New bounds and computations on prime-indexed primes, Integers 13 (2013) 1–21.
[16] K.A. Broughan, A.R. Barnett, On the subsequence of primes having prime subscripts, J. Integer Seq. 12 (2) (2009) 1–10. article 09.2.3.
[17] R.E. Dressler, S.T. Parker, Prime with a prime subscrit, J. Assoc. Comput. Mach. 22 (2) (1975) 380–381.
[18] C. Cattani, Fractals and hidden symmetries in DNA, Math. Probl. Eng. 2010 (2010) 507056.
[19] C. Cattani, G. Pierro, On the fractal geometry of DNA by the binary image analysis, Bull. Math. Biol. 75 (9) (2013) 1544–1570.
[20] N. Marwan, J. Kurths, Nonlinear analysis of bivariate data with cross recurrence plots, Phys. Lett. A 30 (56) (2002) 299307.
[21] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, third ed., John Wiley & Sons, Ltd., 2014.
[22] B. Mandelbrot, The Fractal Geometry of Nature, W. H. Freeman & Co., 1982.
[23] C. Allain, M. Cloitre, Characterizing the lacunarity of random and deterministic fractal sets, Phys. Rev. A A 44 (1991) 3552–3558.
[24] R.E. Plotnick, R.H. Gardner, W.W. Hargrove, K. Prestegaard, M. Perlmutter, Lacunarity analysis: A general technique for the analysis of spatial patterns, Phys. Rev. E 53 (5) (1996) 5461–5468.
[25] Y. Gefen, Y. Meir, B.B. Mandelbrot, A. Aharony, Geometric implementation of hypercubic lattices with noninteger dimensionality by use of low lacunarity fractal lattices, Phys. Rev. Lett. 50 (145) (1983) 145–148.
[26] P. Grassberg, I. Procaccia, Characterization of strange attractors, Phys. Rev. Lett. 50 (1983) 346. manca.
[27] P. Grassberg, I. Procaccia, Measuring the strangeness of strange attractors, Physica D 9 (1983) 189–208.
[28] B.B. Mandelbrot, Corrélations et texture dans un nouveau modéle d'univers hiérarchisé, basé sur les ensembles trémas, C. R. Acad. Sci. Paris, Ser. A 288 (1979) 81–83.
[29] J. Theiler, Lacunarity in a best estimator of fractal dimension, Phys. Lett. A 133 (4–5) (1988) 195–200.