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Positive solutions for a nonlinear algebraic system with nonnegative coefficient matrix

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ABSTRACT

Due to its numerous applications, existence of positive solutions for the algebraic system $x = GF(x)$ has been extensively studied, where G is the coefficient matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear. However, all results require the matrix G to be positive. When G contains a zero-element, positive solutions have not been proved because of the difficulties of cone construction. In the present paper, an existence result is obtained for nonnegative G by introducing a new cone. To show applications of the theorem, two explanatory examples are given. The new result can be naturally extended to some more general systems. In particular, the system can be transformed into an operator equation on a Banach space. Thus, the new method also provides a novel idea for operator equations.

1. Introduction

Consider the nonlinear algebraic system of the form

$$x = GF(x), \tag{1}$$

where $x = col(x_1, x_2, \ldots, x_n)$, $F(x) = col(f(x_1), f(x_2), \ldots, f(x_n))$ and $G = (g_{ij})_{n\times n}$ is an $n\times n$ square matrix with $g_{ij} \geq 0$ for $(i,j) \in [1,n]\times[1,n]$. The notation $[1,n]$ represents the set $\{1, 2, \ldots, n\}$.

Recently, the existence and multiplicity of solutions for system of equations have been investigated under various assumptions, for example, see [1–5]. In applications, positive solutions of a system of equations are important, see [6–10,21] etc. Therefore, existence of positive solutions for system (1) has been extensively studied, see [11–15] and the references therein.

Denote $G \geq 0$ if $g_{ij} \geq 0$ and $G > 0$ if $g_{ij} > 0$ for $(i,j) \in [1,n]\times[1,n]$. We notice that all papers on positive solutions of system (1) required the coefficient matrix $G > 0$, except of [16] and [17]. However, [16] and [17] obtained the existence of nonnegative solutions, that is, some components of the solutions may be zero.

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Is $G > 0$ a necessary condition for a positive solution? In this paper, we show that the answer is negative. Assume that system (1) has a positive solution x^*. Then

$$x_i^* = \sum_{j=1}^{n} g_{ij} f\left(x_j^*\right), \quad i \in [1, n].$$

Clearly, for each $i \in [1, n]$, there exists at least one $j_i \in [1, n]$ such that $g_{ij_i} \neq 0$. Otherwise, if there exists $i_0 \in [1, n]$ such that $g_{i_0 j} = 0$ for all $j \in [1, n]$, we have $x_{i_0}^* = 0$ that contradicts the assumption of x^* is positive. Thus, we assume the following necessary condition holds:

(C_1) For any $i \in [1, n]$, there exists at least one $j_0 \in [1, n]$ such that $g_{ij_0} > 0$.

Can the existence of positive solutions for problem (1) be obtained under (C_1)? Theorem 1 gives an affirmative answer. To prove Theorem 1, a special cone of R^n is constructed and applied to obtain the new existence result for system (1) under the necessary condition. This result improves our recent work, see [12,14,18]. To illustrate the new idea, two explanatory examples are also given. Moreover, the results can be extended to more general cases. For instance, system (1) can be transformed into an operator equation on a Banach space. Thus, the new method provides a novel idea for operator equations, see [18].

Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions: (i) $x \in P$ and $\lambda \geq 0$ implies that $\lambda x \in P$, and (ii) $x \in P$ and $-x \in P$ implies that $x = \theta$, where $\theta \in E$ is the zero element of E . Our main tool is the following Guo–Krasnosel'skii fixed point theorem [19,20].

Lemma 1. *Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to P$ is completely continuous. If either* (H_1) *$\|Ax\| \leq \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$ for $x \in P \cap \partial\Omega_2$, or* (H_2) *$\|Ax\| \geq \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|$ for $x \in P \cap \partial\Omega_2$ holds, then A has at least one fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.*

2. Main result and examples

In the sequel, we will use the following two conditions:

(H_1) For any $i \in [1, n]$, there exist $1 \leq j_1 < j_2 < \cdots < j_{s_i} \leq n$ such that $g_{ij} > 0$ for $j = j_1, j_2, \ldots, j_{s_i}$;

(H_2) There exist $\delta \in (0, 1]$ and $0 < a < b$ such that $f \in C[\delta a, b]$ and $f(u) > 0$ for $u \in [\delta a, b]$, the conditions

$$\frac{m \min_{u\in[\delta a,b]} f(u)}{nM \max_{u\in[\delta a,b]} f(u)} \geq \delta \tag{2}$$

and

$$m \min_{u\in[\delta a,a]} f(u) \geq a \text{ and } nM \max_{u\in[\delta b,b]} f(u) \leq b$$

or

$$Mn \max_{u\in[\delta a,a]} f(u) \leq a \text{ and } m \min_{u\in[\delta b,b]} f(u) \geq b$$

hold, where

$$M = \max\{g_{ij}\} \text{ and } m = \min\{g_{ij} \neq 0\}.$$

Let $P = \{x_i \geq 0, i \in [1, n], x_i \geq \delta |x| \text{ for } i = 1, 2, \ldots, n\}$. For $0 < a < b$, we denote $\Omega_a = \{x : |x_i| < a, x \in R^n\}$ and $\overline{\Omega}_b = \{x : |x_i| \leq b, x \in R^n\}$.

Theorem 1. *Assume that the conditions* (H_1) *and* (H_2) *hold. Then problem* (1) *has at least one positive solution* $x \in P \cap \left(\overline{\Omega}_b \backslash \Omega_a\right)$.

Proof. For

$$0 < \delta \leq \frac{m \min_{u\in[\delta a,b]} f(u)}{Mn \max_{u\in[\delta a,b]} f(u)},$$

clearly, the set

$$P = \{x_i \geq 0, i \in [1,n], x_i \geq \delta |x| \text{ for } i = 1,2,\ldots,n\}$$

is a cone of R^n, where $|x| = \max_{i\in[1,n]} |x_i|$. For $x \in P \cap \left(\overline{\Omega}_b \backslash \Omega_a\right)$, we have

$$y_i = \sum_{j=1}^{n} g_{ij} f(x_j) \leq M \sum_{j=1}^{n} f(x_j) \text{ for } i \in [1,n]$$

and

$$|y| \leq M \sum_{j=1}^{n} f(x_j).$$

On the other hand,

$$\begin{aligned} y_i &= \sum_{j=1}^{n} g_{ij} f(x_j) \geq m \min_{u\in[\delta a,b]} f(u) \\ &\geq \frac{m \min_{u\in[\delta a,b]} f(u)}{nM \max_{u\in[\delta a,b]} f(u)} M \sum_{j=1}^{n} f(x_j) \\ &\geq \frac{m \min_{u\in[\delta a,b]} f(x)}{nM \max_{u\in[\delta a,b]} f(x)} |y| \\ &\geq \delta |y|. \end{aligned}$$

That is, $GF\left(P \cap \left(\overline{\Omega}_b \backslash \Omega_a\right)\right) \subset P$.

Note that the function $f(s)$ is continuous for $s \in [\delta a, b]$, thus, $GF : P \cap \left(\overline{\Omega}_b \backslash \Omega_a\right) \to P$ is completely continuous.

For $x \in P \cap \partial\Omega_a$, we have $a\delta \leq x_i \leq a$ for $i \in [1,n]$ and

$$y_i = \sum_{j=1}^{n} g_{ij} f(x_j) = \sum_{j=j_1,j_2,\ldots,j_{s_i}} g_{ij} f(x_j) \geq m \min_{u\in[\delta a,a]} f(u) \geq a$$

or

$$y_i = \sum_{j=1}^{n} g_{ij} f(x_j) = \sum_{j=j_1,j_2,\ldots,j_{s_i}} g_{ij} f(x_j) \leq Mn \max_{u\in[\delta a,a]} f(u) \leq a.$$

Similarly, for $x \in P \cap \partial\Omega_b$,

$$y_i = \sum_{j=1}^{n} g_{ij} f(x_j) = \sum_{j=j_1,j_2,\ldots,j_{s_i}} g_{ij} f(x_j) \leq nM \max_{u\in[\delta b,b]} f(u) \leq b$$

or

$$y_i = \sum_{j=1}^{n} g_{ij} f(x_j) = \sum_{j=j_1,j_2,\ldots,j_{s_i}} g_{ij} f(x_j) \geq m \min_{u\in[\delta b,b]} f(u) \geq b.$$

In view of Lemma 1, the proof is completed.

The following two examples illustrate the validity of Theorem 1.

Example 1. Consider the system of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3+\sin x_1 \\ 3+\sin x_2 \end{pmatrix}. \tag{3}$$

Let $a = \frac{\pi}{2}$, $b = 10$ and $\delta = \frac{1}{4}$, we have

$$\frac{m \min_{u\in[\delta a,b]}(3+\sin u)}{nM \max_{u\in[\delta a,b]}(3+\sin u)} = \frac{\min_{u\in\left[\frac{\pi}{8},10\right]}(3+\sin u)}{2\max_{u\in\left[\frac{\pi}{8},10\right]}(3+\sin u)} = \frac{1}{4} = \delta.$$

Then

$$m \min_{u\in[\delta a,a]}(3+\sin u) = \min_{u\in\left[\frac{\pi}{8},\frac{\pi}{2}\right]}(3+\sin u) > 3 > a$$

and

$$nM \max_{u\in[\delta b,b]}(3+\sin u) = 2 \max_{u\in\left[\frac{5}{2},10\right]}(3+\sin u) = 8 < b.$$

In view of Theorem 1, system (3) has a positive solution x which satisfies $\frac{\pi}{8} < x_i < 10$ for $i = 1, 2$.

Example 2. Consider the following system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix}. \tag{4}$$

Clearly, $m = 1$, $M = 1$, and $n = 3$. By using Theorem 1, if there exist $\delta \in (0,1]$ and $0 < a < b$ such that $f \in C[\delta a, b]$ and $f(u) > 0$ for $u \in [\delta a, b]$,

$$\frac{\min_{u\in[\delta a,b]} f(u)}{3\max_{u\in[\delta a,b]} f(u)} \geq \delta$$

and

$$\min_{u\in[\delta a,a]} f(u) \geq a \text{ and } 3 \max_{u\in[\delta b,b]} f(u) \leq b$$

or

$$3 \max_{u\in[\delta a,a]} f(u) \leq a \text{ and } \min_{u\in[\delta b,b]} f(u) \geq b$$

hold, then system (4) has a positive solution x which satisfies $\delta a < x_i < b$ for $i = 1, 2$ and 3. We only consider the cases when

$$\frac{\min_{u\in[\delta a,b]} f(u)}{3\max_{u\in[\delta a,b]} f(u)} \geq \delta$$

and

$$\min_{u\in[\delta a,a]} f(u) \geq a \text{ and } 3 \max_{u\in[\delta b,b]} f(u) \leq b$$

hold. Let

$$f(u) = \begin{cases} 10, & u \in (0,1] \\ -u+11, & u \in (1,10) \\ 1, & u \geq 10. \end{cases}$$

We can choose $a = 1$ and $b = 300$. Thus,

$$\frac{\min_{u\in[\delta,b]} f(u)}{3\max_{u\in[\delta,b]} f(u)} = \frac{1}{30} = \delta$$

and

$$\min_{u\in[\delta,1]} f(u) = 10 > 1 \text{ and } 3 \max_{u\in[10,300]} f(u) = 3 \leq 300.$$

In view of Theorem 1, system (4) has a positive solution x which satisfies $1/30 < x_i < 300$ for $i = 1, 2$ and 3. In fact, system (4) has a positive solution

$$x_1 = \frac{22}{3}, x_2 = \frac{11}{3}, x_3 = \frac{22}{3}.$$

Naturally, Theorem 1 can be generalized to the more general cases.

Remark 1. Consider the following 2-dimensional system of the form

$$\begin{cases} x_i = \sum_{j=1}^{n} a_{ij} f(x_j, y_j) \\ y_i = \sum_{j=1}^{n} b_{ij} g(x_j, y_j) \end{cases} \quad \text{for } i \in [1, n]. \tag{5}$$

Define a cone $P = P_1 \times P_2 \subset R^n \times R^n$, where $P_1 = \{x : x_i \geq \delta|x|, x \in R^n\}$ and $P_2 = \{y : y_i \geq \delta|y|, y \in R^n\}$. For $(x, y) \in R^n \times R^n$, define $|(x, y)| = \max\{|x|, |y|\}$. Then $|(\cdot,\cdot)|$ is the norm of $R^n \times R^n$. Similar result can then be obtained. Clearly, our method is also suitable for the k-dimensional systems.

Remark 2. System (1) may be dependent on the variable $\mathbf{x}$. In this case,

$$x_i = \sum_{j=1}^{n} g_{ij} f_j (x_1, x_2, \ldots, x_n) \text{ for } i \in [1, n]. \tag{6}$$

Similarly, we can also prove the existence result of problem (6) when the matrix G has a zero-element.

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