Journal of Computational and Applied Mathematics 313 (2017) 129–141

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Existence of solutions for a class of system of functional integral equation via measure of noncompactness

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ARTICLE INFO

Article history:
Received 14 February 2016
Received in revised form 10 July 2016

MSC:
47H09
47H10
34A12

Keywords:
Measure of noncompactness
System of integral equations
Coupled fixed point

ABSTRACT

In this paper, we give some generalization of Darbo's fixed point theorem related to the measure of noncompactness, and present some results on the existence of coupled fixed point theorem for a special class of operators in a Banach space. Our results generalize and extend some comparable results in the literature. Also as an application, we study the existence of solution for a class of the system of nonlinear functional integral equations. Finally an example illustrating the mentioned applicability is also included.

1. Introduction

The study of nonlinear integral equations, is a subject of interest for researchers in nonlinear functional analysis. Integral equations occur in many applications, such as in applied mathematics, and also a lot of problems in physics. On the other hand measure of noncompactness is one of the most useful tools in nonlinear and functional analysis, metric fixed point theory and the theory of operator equations in Banach spaces which was first introduced by Kuratowski in [1]. This concept also used to investigate of functional equation, ordinary and partial differential equations, integral and integro-differential equations. In this context several authors have presented some papers on the existence of solution for nonlinear integral equations which involves the use of measure of noncompactness and many other techniques, for instance see [2–21] and [22–28].

In this paper, we apply, the method related to the technique of measures of noncompactness in order to extend the Darbo's fixed point theorem [18] and to generalize some recent results in the literature. In this regard, we state and prove some existence theorems of coupled fixed point for a class of operators in Banach spaces. Moreover, as an application of this theorems, we study the problem of existence of solutions for a class of system of nonlinear integral equations which satisfy in new certain conditions.

2. Preliminaries

In this section, we recall some definitions, notations and preliminary results which we will use throughout the paper. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|.\|)$ be a real Banach space with zero element 0 and $\bar{B}(x, r)$ denote the closed ball in E centered at x with radius r. The symbol $\bar{B}_r$ stand for the ball $\bar{B}(0, r)$. If X is a nonempty

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http://dx.doi.org/10.1016/j.cam.2016.09.011

subset of E we denote by $\bar{X}$, $ConvX$ the closure and the closed convex hull of X respectively. Finally, let us denote by $\mathcal{M}_E$ the family of nonempty bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

In this paper, we will use axiomatically defined measures of noncompactness as presented in the book [18].

Definition 2.1 (*[18]*). A mapping $\mu : \mathcal{M}_E \longrightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions;

(MNC1) The family $Ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $Ker\mu \subseteq \mathcal{N}_E$.
(MNC2) If $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.
(MNC3) $\mu(\overline{X}) = \mu(X)$.
(MNC4) $\mu(ConvX) = \mu(X)$.
(MNC5) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1)$.
(MNC6) If (X_n) is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subseteq X_n$, $(n \ge 1)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $Ker\mu$ described in (MNC1) said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (MNC6) is a member of the family $Ker\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in Ker\mu$.

Now we present the definition of a coupled fixed point for a bivariate vector function which we need in the proof of main results and a useful theorem in [18] related to the construction of a measure of noncompactness on a finite product space.

Definition 2.2 (*[29]*). An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Theorem 2.3 ([18]). *Suppose $\mu_1, \mu_2, \ldots, \mu_n$ be the measures in Banach spaces $E_1, E_2, \ldots, E_n$ respectively. Moreover assume that the function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ where X_i denotes the natural projection of X into E_i for $i = 1, 2, \ldots, n$.

Now, as a result of Theorem 2.3, we present the following examples.

Example 2.4. Let μ be a measure of noncompactness on a Banach space E, and let the function $F : [0, \infty)^2 \to [0, \infty)$ is convex and $F(x_1, x_2) = 0$ if and only if $x_i = 0$ for $i = 1, 2$. Then

$$\widetilde{\mu}(X) = F(\mu(X_1), \mu(X_2))$$

defines a measure of noncompactness in $E \times E$ where X_i denote the natural projection of X into E.

Example 2.5 (*[10]*). Let μ be a measure of noncompactness on a Banach space E, considering $F(x, y) = x + y$ for any $(x, y) \in [0, \infty)^2$. Then we see that F is convex and $F(x, y) = 0$ if and only if $x = y = 0$, hence all the conditions of Theorem 2.3 are satisfied. Therefore, $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X into E.

Example 2.6 (*[10]*). Let μ be a measure of noncompactness on a Banach space E. If we define $F(x, y) = \max\{x, y\}$ for any $(x, y) \in [0, \infty)^2$, then all the conditions of Theorem 2.3 are satisfied, and $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X into E.

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorems.

Theorem 2.7 (*Schauder[3]*). *Let Ω be a closed, convex subset of a Banach space E. Then every compact, continuous map $T : \Omega \to \Omega$ has at least one fixed point.*

Theorem 2.8 (*Darbo[15]*). *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(T(X)) \le k\mu(X)$$

for any $X \subset \Omega$. Then T has a fixed point.

3. Main results

In this section, we give and prove some theorems for the existence of coupled fixed point to a special class of operators. This basic result will be used in the next section.

First, we introduce the class Γ of all functions $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which have the following properties:

(A_1) $\varphi(t_1 + t_2, s_1 + s_2) \le \varphi(t_1, s_1) + \varphi(t_2, s_2)$ for all $t_1, t_2, s_1, s_2 \in \mathbb{R}_+$,
(A_2) $\varphi(t, s) = 0$ if and only if $s = t = 0$,
(A_3) φ is a lower semicontinuous function on $\mathbb{R}_+ \times \mathbb{R}_+$ *i.e.*, for arbitrary sequences $\{a_n\}$ and $\{b_n\}$ of $\mathbb{R}_+$ we have

$$\varphi(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n) \le \liminf_{n\to\infty} \varphi(a_n, b_n).$$

For example, the functions $\varphi(t, s) = \ln(t + s + 1)$ and $\varphi(t, s) = \max\{t, s\}$ belong to Γ.

Theorem 3.1. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega \times \Omega$ be a continuous function satisfying*

$$\psi\left(\widetilde{\mu}(T(X))\right) \le \psi(\widetilde{\mu}(X)) - \varphi\left(\widetilde{\mu}(X), \widetilde{\mu}(X)\right) \tag{3.1}$$

for any nonempty subset X of $\Omega \times \Omega$, where $\widetilde{\mu}$ is defined by Example 2.4 and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $\varphi \in \Gamma$. Then T has at least one fixed point in $\Omega \times \Omega$.

Proof. By induction we construct the sequence $\{\Omega_n \times \Omega_n\}_{n=1}^{\infty}$ such that $\Omega_0 \times \Omega_0 = \Omega \times \Omega$ and $\Omega_n \times \Omega_n = ConvT(\Omega_{n-1} \times \Omega_{n-1})$, for $n = 1, 2, \ldots$. We have $T(\Omega_0 \times \Omega_0) = T(\Omega \times \Omega) \subseteq \Omega \times \Omega = \Omega_0 \times \Omega_0$, $\Omega_1 \times \Omega_1 = ConvT\,(\Omega_0 \times \Omega_0) \subseteq \Omega_0 \times \Omega_0$, therefore by continuing this process we obtain

$$\cdots \subseteq \Omega_n \times \Omega_n \subseteq \cdots \subseteq \Omega_1 \times \Omega_1 \subseteq \Omega_0 \times \Omega_0.$$

If there exists an integer $N \ge 0$ such that $\widetilde{\mu}(\Omega_N \times \Omega_N) = 0$, then $\Omega_N \times \Omega_N$ is relatively compact and since $T\,(\Omega_N \times \Omega_N) \subseteq ConvT\,(\Omega_N \times \Omega_N) = \Omega_{N+1} \times \Omega_{N+1} \subseteq \Omega_N \times \Omega_N$, therefore, Theorem 2.7 implies that T has a fixed point. So we can assume that $\widetilde{\mu}(\Omega_n \times \Omega_n) > 0$ for all $n \ge 0$. By our assumption, we get

$$\begin{aligned}\psi\left(\widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1})\right) &= \psi\left(\widetilde{\mu}\left(ConvT\,(\Omega_n \times \Omega_n)\right)\right) = \psi\left(\widetilde{\mu}\left(T\,(\Omega_n \times \Omega_n)\right)\right) \\ &\le \psi\left(\widetilde{\mu}\,(\Omega_n \times \Omega_n)\right) - \varphi\left(\widetilde{\mu}(\Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n)\right).\end{aligned} \tag{3.2}$$

Since the sequence $\widetilde{\mu}(\Omega_n \times \Omega_n)$ is nonincreasing and nonnegative real numbers, thus, there is an $r \ge 0$ so that $\widetilde{\mu}(\Omega_n \times \Omega_n) \to r$ as $n \to \infty$. We show that $r = 0$. From (3.2) we obtain

$$\begin{aligned}\psi\,(r) &= \limsup_{n\to\infty} \psi\left(\widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1})\right) \\ &\le \limsup_{n\to\infty} \psi\left(\widetilde{\mu}\,(\Omega_n \times \Omega_n)\right) - \liminf_{n\to\infty} \varphi\left(\widetilde{\mu}(\Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n)\right) \\ &\le \limsup_{n\to\infty} \psi\left(\widetilde{\mu}\,(\Omega_n \times \Omega_n)\right) - \varphi\left(\liminf_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n), \liminf_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n)\right) \\ &= \psi\,(r) - \varphi\,(r, r).\end{aligned}$$

Consequently $\varphi\,(r, r) = 0$ so $r = 0$. Hence we deduce that $\widetilde{\mu}(\Omega_n \times \Omega_n) \to 0$ as $n \to \infty$. Since $\Omega_{n+1} \times \Omega_{n+1} \subseteq \Omega_n \times \Omega_n$, so by axiom (MNC6) of Definition 2.1 we derive that the set $\Omega_\infty \times \Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n \times \Omega_n$ is a nonempty convex closed set, invariant under the operator T and belongs to $Ker\,\mu$. Now by Theorem 2.7 T has at least one fixed point in $\Omega_\infty \times \Omega_\infty$ and hence in $\Omega \times \Omega$. $\square$

Theorem 3.2. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function satisfying*

$$\psi\left(\mu\left(T(X_1 \times X_2)\right)\right) \le \frac{1}{2}\psi\left(\mu(X_1) + \mu(X_2)\right) - \varphi\left(\mu(X_1), \mu(X_2)\right) \tag{3.3}$$

for all $X_1, X_2 \subseteq \Omega \times \Omega$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\psi\,(t + s) \le \psi(t) + \psi(s)$ for all $t, s \in \mathbb{R}_+$ and $\varphi \in \Gamma$. Then T has at least a coupled fixed point.

Proof. First note that, Example 2.5 implies that $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ is a measure of noncompactness in the space $E \times E$, where X_i, $i = 1, 2$ denote the natural projections of X into E. Now define
$\widetilde{T}$ on $\Omega \times \Omega$ by the formula

$$\widetilde{T}(x, y) = (T(x, y), T(y, x))$$

for all $(x, y) \in \Omega \times \Omega$. It is easy to see that $\widetilde{T}$ is continuous on $\Omega \times \Omega$. We claim that $\widetilde{T}$ satisfies all the conditions of Theorem 3.1. To prove this, let $X \subset \Omega \times \Omega$ be a nonempty subset. Then, by (MNC2) and (3.3) we get

$$\begin{aligned}
\psi\left(\widetilde{\mu}(\widetilde{T}(X))\right) &\leq \psi\left(\widetilde{\mu}\left(T(X_1 \times X_2), T(X_2 \times X_1)\right)\right) \\
&= \psi\left(\mu\left(T(X_1 \times X_2)\right) + \mu\left(T(X_2 \times X_1)\right)\right) \\
&\leq \psi\left(\mu\left(T(X_1 \times X_2)\right)\right) + \psi\left(\mu\left(T(X_2 \times X_1)\right)\right) \\
&\leq \frac{1}{2}\psi\left(\mu(X_1) + \mu(X_2)\right) - \varphi\left(\mu(X_1), \mu(X_2)\right) \\
&\quad + \frac{1}{2}\psi\left(\mu(X_2) + \mu(X_1)\right) - \varphi\left(\mu(X_2), \mu(X_1)\right) \\
&= \psi\left(\mu(X_1) + \mu(X_2)\right) - \left[\varphi\left(\mu(X_1), \mu(X_2)\right) + \varphi\left(\mu(X_2), \mu(X_1)\right)\right] \\
&\leq \psi\left(\mu(X_1) + \mu(X_2)\right) - \varphi\left(\mu(X_1) + \mu(X_2), \mu(X_1) + \mu(X_2)\right) \\
&= \psi(\widetilde{\mu}(X)) - \varphi\left(\widetilde{\mu}(X), \widetilde{\mu}(X)\right).
\end{aligned}$$

Hence, by Theorem 3.1 T has at least a coupled fixed point. □

By using the above result, we have the following corollary.

Corollary 3.3. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that there exist nonnegative constants k_1, k_2 with $k_1 + k_2 < 1$*

$$\mu(T(X_1 \times X_2)) \leq \frac{k_1}{2}\mu(X_1) + \frac{k_2}{2}\mu(X_2),$$

for all $X_1, X_2 \subset \Omega$. Then T has at least a coupled fixed point.

Proof. Taking $\psi(t) = t$ and $\varphi(t, s) = \left(\frac{1-k_1}{2}\right)t + \left(\frac{1-k_2}{2}\right)s$ in Theorem 3.2, we obtain the desired conclusion. □

In the next result, we obtain Theorem 2.1 of Aghajani et al. [10]:

Corollary 3.4. *Let $T : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that*

$$\mu\left(T\left(X_1 \times X_2\right)\right) \leq k \max\left\{\mu\left(X_1\right), \mu\left(X_2\right)\right\}$$

for any $X_1, X_2 \subset \Omega$, where μ is an arbitrary measure of noncompactness and k is a constant with $0 \leq k < 1$. Then T has at least a coupled fixed point.

Proof. Taking $\psi(t) = t$ and $\varphi(t, s) = (1 - k)\max\{t, s\}$ in Theorem 3.2, we obtain the desired conclusion. □

Corollary 3.5. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that*

$$\mu(T(X_1 \times X_2)) \leq \frac{\mu(X_1) + \mu(X_2)}{2} - \ln\left(\mu(X_1) + \mu(X_2) + 1\right),$$

for all $X_1, X_2 \subset \Omega$. Then T has at least a coupled fixed point.

Proof. Taking $\psi(t) = t$ and $\varphi(t, s) = \ln(t + s + 1)$ in Theorem 3.2, we obtain the desired conclusion. □

In this part of the paper we will introduce another class of functions and in this direction we present some coupled fixed point theorems.

First we consider the usual order relation "$\preceq$" on $\mathbb{R}_+ \times \mathbb{R}_+$ as follows:

$$(t_1, s_1) \preceq (t_2, s_2) \Leftrightarrow t_1 \leq t_2, s_1 \leq s_2$$

for all $t_1, s_1, t_2, s_2 \in \mathbb{R}_+$.

Now we denote by Φ, the class of all functions $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties

(B_1) ϕ is continuous and nondecreasing function on $\mathbb{R}_+ \times \mathbb{R}_+$,
(B_2) $\phi(t, t) < t$ for all $t > 0$,
(B_3) $\frac{1}{2}\phi(t_1, s_1) + \frac{1}{2}\phi(t_2, s_2) \leq \phi(\frac{t_1+t_2}{2}, \frac{s_1+s_2}{2})$ for all $t_1, t_2, s_1, s_2 \in \mathbb{R}_+$.

For example, the function $\phi(t, s) = k_1 t + k_2 s$, in which $k_1, k_2 \in [0, 1)$ and $k_1 + k_2 < 1$ and the function $\phi(t, s) = \ln\left(1 + \frac{t+s}{2}\right)$, belong to Φ.

Theorem 3.6. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega \times \Omega$ be a continuous function satisfying*

$$\widetilde{\mu}\left(T\left(X\right)\right) \leq \phi\left(\widetilde{\mu}(X), \widetilde{\mu}(X)\right) \tag{3.4}$$

for any nonempty subset X of $\Omega \times \Omega$, where $\widetilde{\mu}$ is defined by Example 2.4 and $\phi \in \Phi$. Then T has at least one fixed point in $\Omega \times \Omega$.

Proof. Similarly as in the proof of Theorem 3.1, we construct the sequence $\{\Omega_n \times \Omega_n\}_{n=1}^{\infty}$ by $\Omega_0 \times \Omega_0 = \Omega \times \Omega$ and $\Omega_n \times \Omega_n = ConvT\left(\Omega_{n-1} \times \Omega_{n-1}\right)$, for $n = 1, 2, \ldots$, such that

$$\cdots \subseteq \Omega_n \times \Omega_n \subseteq \cdots \subseteq \Omega_1 \times \Omega_1 \subseteq \Omega_0 \times \Omega_0.$$

If there exists an integer $N \geq 0$ such that $\widetilde{\mu}(\Omega_N \times \Omega_N) = 0$, then $\Omega_N \times \Omega_N$ is relatively compact and since $T\left(\Omega_N \times \Omega_N\right) \subseteq ConvT\left(\Omega_N \times \Omega_N\right) = \Omega_{N+1} \times \Omega_{N+1} \subseteq \Omega_N \times \Omega_N$, therefore, Theorem 2.7 implies that T has a fixed point. So we can assume that $\widetilde{\mu}(\Omega_n \times \Omega_n) > 0$ for all $n \geq 0$. By our assumption, we get

$$\begin{aligned}\widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1}) &= \widetilde{\mu}\left(ConvT\left(\Omega_n \times \Omega_n\right)\right) = \widetilde{\mu}\left(T\left(\Omega_n \times \Omega_n\right)\right) \\ &\leq \phi\left(\widetilde{\mu}(\Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n)\right).\end{aligned} \tag{3.5}$$

Since the sequence $\widetilde{\mu}(\Omega_n \times \Omega_n)$ is nonincreasing and nonnegative real numbers, thus, there is an $r \geq 0$ so that $\widetilde{\mu}(\Omega_n \times \Omega_n) \to r$ as $n \to \infty$. Now we claim that $r = 0$. On the contrary if $r > 0$ then from (3.5) we obtain

$$\begin{aligned}r &= \lim_{n\to\infty} \widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1}) \\ &\leq \phi\left(\lim_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n), \lim_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n)\right) \\ &= \phi\left(r, r\right) < r.\end{aligned}$$

Which is a contradiction, hence we deduce that $\widetilde{\mu}(\Omega_n \times \Omega_n) \to 0$ as $n \to \infty$. Since $\Omega_{n+1} \times \Omega_{n+1} \subseteq \Omega_n \times \Omega_n$, so by axiom (MNC6) of Definition 2.1 we derive that the set $\Omega_\infty \times \Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n \times \Omega_n$ is a nonempty convex closed set, invariant under the operator T and belongs to $Ker\mu$. Now by Theorem 2.7 T has at least on fixed point in $\Omega_\infty \times \Omega_\infty$ and hence in $\Omega \times \Omega$. □

Theorem 3.7. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function satisfying*

$$\mu\left(T\left(X_1 \times X_2\right)\right) \leq \phi\left(\mu\left(X_1\right), \mu\left(X_2\right)\right) \tag{3.6}$$

for all $X_1, X_2 \subseteq \Omega \times \Omega$, where $\phi \in \Phi$. Then T has at least a coupled fixed point.

Proof. First note that, Example 2.5 implies that $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$ is a measure of noncompactness in the space $E \times E$, where $X_i,\ i = 1, 2$ denote the natural projections of X into E. Now define
$\widetilde{T}$ on $\Omega \times \Omega$ by the formula

$$\widetilde{T}(x, y) = (T(x, y), T(y, x))$$

for all $(x, y) \in \Omega \times \Omega$. It is easy to see that $\widetilde{T}$ is continuous on $\Omega \times \Omega$. We claim that $\widetilde{T}$ satisfies all the conditions of Theorem 3.6. To prove this, let $X \subset \Omega \times \Omega$ be a nonempty subset. Then, by (MNC2) and (3.6) we get

$$\begin{aligned}\widetilde{\mu}(\widetilde{T}(X)) &\leq \widetilde{\mu}\left(T(X_1 \times X_2), T(X_2 \times X_1)\right) \\ &= \mu\left(T(X_1 \times X_2)\right) + \mu\left(T(X_2 \times X_1)\right) \\ &\leq \phi\left(\mu(X_1), \mu(X_2)\right) + \phi\left(\mu(X_2), \mu(X_1)\right) \\ &\leq 2\phi\left(\frac{\mu(X_1) + \mu(X_2)}{2}, \frac{\mu(X_1) + \mu(X_2)}{2}\right).\end{aligned} \tag{3.7}$$

Now from (3.7) and taking $\widetilde{\mu}_1 = \frac{1}{2}\widetilde{\mu}$, we get

$$\widetilde{\mu}_1\left(\widetilde{T}(X)\right) \leq \phi\left(\widetilde{\mu}_1\left(X\right), \widetilde{\mu}_1\left(X\right)\right).$$

Since, $\widetilde{\mu}_1$ is also a measure of noncompactness, So by Theorem 3.6 T has at least a coupled fixed point. □

By using the above result, we have the following corollary

Corollary 3.8. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, μ be an arbitrary measure of noncompactness. Moreover assume that $T : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that there exist nonnegative constants k_1, k_2 with $k_1 + k_2 < 1$*

$$\mu(T(X_1 \times X_2)) \leq k_1\mu(X_1) + k_2\mu(X_2),$$

for all $X_1, X_2 \subset \Omega$. Then T has at least a coupled fixed point.

Proof. Taking $\phi(t, s) = k_1t + k_2s$ in Theorem 3.7, we obtain the desired conclusion. □

4. Applications and examples

In this section, we consider the Banach space $BC(\mathbb{R}_+)$ containing all real functions defined, bounded and continuous on $\mathbb{R}_+$. The norm in $BC(\mathbb{R}_+)$ is defined as the standard supremum norm, i.e

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which is given in [18,19]. In order to define this measure let us fix a nonempty bounded subset of X of $BC(\mathbb{R}_+)$ and a positive number $L > 0$. For $x \in X$ and $\varepsilon \geq 0$ denote by $\omega^L(x, \varepsilon)$, the modulus of continuity of x on the interval $[0, L]$, i.e.,

$$\omega^L(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, L], |t - s| \leq \varepsilon\}.$$

Moreover, let us put

$$\omega^L(X, \varepsilon) = \sup\left\{\omega^L(x, \varepsilon) : x \in X\right\},$$
$$\omega_0^L(X) = \lim_{\varepsilon\to 0} \omega^L(X, \varepsilon),$$
$$\omega_0(X) = \lim_{L\to\infty} \omega_0^L(X).$$

If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Finally, consider the function μ defined on $\mathcal{M}_{BC(\mathbb{R}_+)}$ by formula

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diamX(t), \tag{4.1}$$

where

$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

It is shown (cf. [18,19]) that the function $\mu(X)$ defines sublinear measure of noncompactness in the sense of accepted Definition 2.1.

As an application of Theorem 3.7, we are going to study the existence of solutions for the system of nonlinear integral equations

$$\begin{cases} x(t) = A(t) + h\left(t, x(\xi(t)), y\left(\xi(t)\right)\right) + f\left(\begin{matrix} t, x(\xi(t)), y\left(\xi(t)\right), \\ \varphi\left(\int_0^{\beta(t)} g\left(t, s, x(\eta(s)), y(\eta(s))\right) ds\right)\end{matrix}\right) \\ y(t) = A(t) + h\left(t, y\left(\xi(t)\right), x(\xi(t))\right) + f\left(\begin{matrix} t, y\left(\xi(t)\right), x(\xi(t)), \\ \varphi\left(\int_0^{\beta(t)} g\left(t, s, y\left(\eta(s)\right), x(\eta(s))\right) ds\right)\end{matrix}\right). \end{cases} \tag{4.2}$$

For this purpose, we consider the following situation

(i) the function $A : \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded with $M_1 = \sup\{|A(t)| : t \in \mathbb{R}_+\}$,
(ii) the functions $\xi,\ \eta,\ \beta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$,
(iii) the function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is continuous and there exist positive constants α, δ such that

$$|\varphi(t_1) - \varphi(t_2)| \leq \delta\,|t_1 - t_2|^{\alpha}, \tag{4.3}$$

for any $t_1, t_2 \in \mathbb{R}_+$, and moreover $\varphi(0) = 0$,
(iv) the functions defined by $t \to |f(t, 0, 0, 0)|$ and $t \to |h(t, 0, 0)|$ are bounded on $\mathbb{R}_+$, *i.e.*

$$M_2 = \sup\{|f(t, 0, 0, 0)| : t \in \mathbb{R}_+\} < \infty,$$
$$M_3 = \sup\{|h(t, 0, 0)| : t \in \mathbb{R}_+\} < \infty$$

(v) the functions $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist a function $\phi \in \Phi$, and a nondecreasing continuous function
$\theta : \mathbb{R}_+ \to \mathbb{R}$ with $\theta(0) = 0$ such that

$$\begin{aligned} |h(t,x,y) - h(t,u,v)| &\le \frac{1}{2}\phi(|x-u|, |y-v|), \\ |f(t,x,y,z) - f(t,u,v,w)| &\le \frac{1}{2}\phi(|x-u|, |y-v|) + \theta(|z-w|) \end{aligned} \tag{4.4}$$

for any $t \ge 0$ and for all $x, y, u, v \in \mathbb{R}$,

(vi) the function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{t\to\infty} \int_0^{\beta(t)} |g(t,s,x(\eta(s)),y(\eta(s))) - g(t,s,u(\eta(s)),v(\eta(s)))|\, ds = 0 \tag{4.5}$$

uniformly with respect to $x, y, u, v \in BC(\mathbb{R}_+)$, where

$$M_4 = \sup\left\{ \left| \int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))\, ds \right|^{\alpha} : t \in \mathbb{R}_+, x, y \in BC(\mathbb{R}_+) \right\}. \tag{4.6}$$

(vii) there exists a positive solution ρ of the inequality

$$M_1 + \phi(r,r) + M_2 + M_3 + \theta(\delta M_4) < r. \tag{4.7}$$

Then we can formulate our assertion as follows.

Theorem 4.1. *Let the conditions* (i)–(vii) *hold. Then Eq.* (4.2) *has at least one solution in the space* $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. First we define the operator $F : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ by

$$F(x,y)(t) = A(t) + h(t, x(\xi(t)), y(\xi(t))) + f\left(\begin{matrix} t, x(\xi(t)), y(\xi(t)), \\ \varphi\left(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))\, ds \right) \end{matrix} \right). \tag{4.8}$$

Moreover, the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is equipped with the norm $\|(x,y)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} = \|x\| + \|y\|$ for any $(x,y) \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. Notice that the continuity of $F(x,y)$ for any $(x,y) \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is obvious. Moreover by (4.8), (4.6) (i), (iii), (iv), (v) and triangle inequality, we know that

$$\begin{aligned} |F(x,y)(t)| &\le |A(t)| + |h(t,x,(\xi(t)), y(\xi(t))) - h(t,0,0)| \\ &\quad + \left(\left| f\left(t, x(\xi(t)), y(\xi(t)), \varphi\left(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))\, ds \right) \right) - f(t,0,0,0) \right| \right. \\ &\quad \left. + |f(t,0,0,0)| \right) \\ &\quad + |h(t,0,0)| \\ &\le M_1 + \frac{1}{2}\phi(|x(\xi(t))|, |y(\xi(t))|) + \frac{1}{2}\phi(|x(\xi(t))|, |y(\xi(t))|) \\ &\quad + \theta\left(\left| \varphi\left(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))\, ds \right) \right| \right) + M_2 + M_3 \\ &\le M_1 + \phi(|x(\xi(t))|, |y(\xi(t))|) \\ &\quad + \theta\left(\left| \varphi\left(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))\, ds \right) \right| \right) + M_2 + M_3 \\ &\le M_1 + M_2 + M_3 + \phi(\|x\|, \|y\|) + \theta\left(\delta \left| \int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))\, ds \right|^{\alpha} \right) \\ &\le M_1 + M_2 + M_3 + \phi(\|x\|, \|y\|) + \theta(\delta M_4). \end{aligned} \tag{4.9}$$

Thus F is well defined and the estimate (4.9) and condition (vii) implies that $F\left(\bar{B}_\rho \times \bar{B}_\rho\right) \subseteq \bar{B}_\rho$.

Now, we prove that F is continuous on $\bar{B_\rho} \times \bar{B_\rho}$. For this, take $(x, y) \in \bar{B_\rho} \times \bar{B_\rho}$ and $\varepsilon > 0$ arbitrarily. Moreover consider $(u, v) \in \bar{B_\rho} \times \bar{B_\rho}$ with $\|(x, y) - (u, v)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} < \frac{\varepsilon}{2}$. Now we have

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)| &\le |h(t, x(\xi(t)), y(\xi(t))) - h(t, u(\xi(t)), v(\xi(t)))| \\
&\quad + \left| \begin{array}{l} f\left(t, x(\xi(t)), y(\xi(t)), \varphi\left(\int_0^{\beta(t)} g(t,s,x(\eta(s)), y(\eta(s)))\,ds\right)\right) - \\ f\left(t, u(\xi(t)), v(\xi(t)), \varphi\left(\int_0^{\beta(t)} g(t,s,u(\eta(s)), v(\eta(s)))\,ds\right)\right) \end{array} \right| \\
&\le \frac{1}{2}\phi\left(|x(\xi(t)) - u(\xi(t))|, |y(\xi(t)) - v(\xi(t))|\right) \\
&\quad + \frac{1}{2}\phi\left(|x(\xi(t)) - u(\xi(t))|, |y(\xi(t)) - v(\xi(t))|\right) \\
&\quad + \theta\left(\left| \begin{array}{l} \varphi\left(\int_0^{\beta(t)} g(t,s,x(\eta(s)), y(\eta(s)))\,ds\right) \\ -\varphi\left(\int_0^{\beta(t)} g(t,s,u(\eta(s)), v(\eta(s)))\,ds\right) \end{array} \right|\right) \\
&\le \frac{1}{2}\phi(\|x-u\|, \|y-v\|) + \frac{1}{2}\phi(\|x-u\|, \|y-v\|) \\
&\quad + \theta\left(\delta\left|\int_0^{\beta(t)} (g(t,s,x(\eta(s)), y(\eta(s))) - g(t,s,u(\eta(s)), v(\eta(s))))\,ds\right|^\alpha\right) \\
&\le \phi(\|x-u\|, \|y-v\|) \\
&\quad + \theta\left(\delta\left|\int_0^{\beta(t)} (g(t,s,x(\eta(s)), y(\eta(s))) - g(t,s,u(\eta(s)), v(\eta(s))))\,ds\right|^\alpha\right).
\end{aligned} \tag{4.10}$$

In addition, from (4.5), there exists $L > 0$ such that, if $t > L$, then

$$\theta\left(\delta\left|\int_0^{\beta(t)} g(t,s,x(\eta(s)), y(\eta(s))) - g(t,s,u(\eta(s)), v(\eta(s)))\,ds\right|^\alpha\right) \le \frac{\varepsilon}{2} \tag{4.11}$$

for any $x, y, u, v \in BC(\mathbb{R}_+)$. Now we consider two cases: Case (1^0) if $t > L$, then from (4.10), (4.11), we get

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)| &\le \phi\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{4.12}$$

Case (2^0) if $t \in [0, L]$, thus by the argument similar to those given in (4.10), we have

$$\begin{aligned}
|F(x,y)(t) - F(u,v)(t)| &\le \phi\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) \\
&\quad + \theta\left(\delta\left|\int_0^{\beta(t)} (g(t,s,x(\eta(s)), y(\eta(s))) - g(t,s,u(\eta(s)), v(\eta(s))))\,ds\right|^\alpha\right) \\
&< \frac{\varepsilon}{2} + \theta\left(\delta\left(\beta_L \omega(\varepsilon)\right)^\alpha\right),
\end{aligned} \tag{4.13}$$

where

$$\omega(\varepsilon) = \sup\left\{ \begin{array}{c} |g(t,s,x,y) - g(t,s,u,v)| : t \in [0,L], s \in [0,\beta_L], x, y, u, v \in [-\rho,\rho], \\ \|(x,y) - (u,v)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} < \frac{\varepsilon}{2} \end{array} \right\}$$

and

$$\beta_L = \sup\{\beta(t) : t \in [0, L]\}.$$

By using the continuity of g on $[0, L] \times [0, \beta_L] \times [-\rho, \rho] \times [-\rho, \rho]$, we have $\omega(\varepsilon) \to 0$, as $\varepsilon \to 0$ and by continuity of θ we get

$$\theta\left(\delta\left(\beta_L \omega(\varepsilon)\right)^\alpha\right) \to 0$$

as $\varepsilon \to 0$. Hence from (4.12), (4.13), imply that F is continuous function from $\bar{B_\rho} \times \bar{B_\rho}$ into $\bar{B_\rho}$. Now, we only need to show that F satisfies the conditions of Theorem 3.7. To prove that, let $L,\ \varepsilon \in \mathbb{R}_+$ and X_1, X_2 are arbitrary nonempty subsets of $\bar{B_\rho}$ and take $t_1, t_2 \in\ [0, L]$, such that $|t_1 - t_2| \le \varepsilon$.

Without loss of generality, we may assume that $\beta(t_1) < \beta(t_2)$. We also assume that $(x, y) \in X_1 \times X_2$. Then we get

$$
\begin{aligned}
\left|\begin{matrix} F(x,y)(t_1)- \\ F(x,y)(t_2)\end{matrix}\right| &\le |A(t_1) - A(t_2)| + |h\left(t_2, x(\xi(t_2)), y(\xi(t_2))\right) - h\left(t_2, x(\xi(t_1)), y(\xi(t_1))\right)| \\
&\quad + |h\left(t_2, x(\xi(t_1)), y(\xi(t_1))\right) - h\left(t_1, x(\xi(t_1)), y(\xi(t_1))\right)| \\
&\quad + \left|\begin{matrix} f\left(t_2, x\left(\xi\left(t_2\right)\right), y\left(\xi\left(t_2\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right) \\ -f\left(t_2, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right)\end{matrix}\right| \\
&\quad + \left|\begin{matrix} f\left(t_2, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right) \\ -f\left(t_1, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right)\end{matrix}\right| \\
&\quad + \left|\begin{matrix} f\left(t_1, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right) \\ -f\left(t_1, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_1, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right)\end{matrix}\right| \\
&\quad + \left|\begin{matrix} f\left(t_1, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_2)} g\left(t_1, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right) \\ -f\left(t_1, x\left(\xi\left(t_1\right)\right), y\left(\xi\left(t_1\right)\right), \varphi\left(\int_0^{\beta(t_1)} g\left(t_1, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right)\end{matrix}\right| \\
&\le \omega^L\left(A, \varepsilon\right) + \frac{1}{2}\phi\left(\left|x\left(\xi\left(t_2\right)\right) - x\left(\xi\left(t_1\right)\right)\right|, \left|y\left(\xi\left(t_2\right)\right) - y\left(\xi\left(t_1\right)\right)\right|\right) + \omega_\rho^L\left(h, \varepsilon\right) \\
&\quad + \frac{1}{2}\phi\left(\left|x\left(\xi\left(t_2\right)\right) - x\left(\xi\left(t_1\right)\right)\right|, \left|y\left(\xi\left(t_2\right)\right) - y\left(\xi\left(t_1\right)\right)\right|\right) \\
&\quad + \omega_{\rho,K}^L\left(f, \varepsilon\right) + \theta\left(\left|\begin{matrix} \varphi\left(\int_0^{\beta(t_2)} g\left(t_2, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right) \\ -\varphi\left(\int_0^{\beta(t_2)} g\left(t_1, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\end{matrix}\right|\right) \\
&\quad + \theta\left(\left|\varphi\left(\int_{\beta(t_1)}^{\beta(t_2)} g\left(t_1, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right)\right|\right) \\
&\le \omega^L\left(A, \varepsilon\right) + \omega_\rho^L\left(h, \varepsilon\right) + \phi\left(\omega^L\left(x, \omega^L\left(\xi, \varepsilon\right)\right), \omega^L\left(y, \omega^L\left(\xi, \varepsilon\right)\right)\right) + \omega_{\rho,K}^L\left(f, \varepsilon\right) \\
&\quad + \theta\left(\delta\left|\int_0^{\beta(t_2)} \left(g(t_2, s, x(\eta(s)), y(\eta(s))) - g(t_1, s, u(\eta(s)), v(\eta(s)))\right) ds\right|^\alpha\right) \\
&\quad + \theta\left(\delta\left|\int_{\beta(t_1)}^{\beta(t_2)} g\left(t_1, s, x\left(\eta\left(s\right)\right), y\left(\eta\left(s\right)\right)\right) ds\right|^\alpha\right) \\
&\le \omega^L\left(A, \varepsilon\right) + \omega_\rho^L\left(h, \varepsilon\right) + \phi\left(\omega^L\left(x, \omega^L\left(\xi, \varepsilon\right)\right), \omega^L\left(y, \omega^L\left(\xi, \varepsilon\right)\right)\right) + \omega_{\rho,K}^L\left(f, \varepsilon\right) \\
&\quad + \theta\left(\delta\left(\beta_L\omega_\rho^L\left(g, \varepsilon\right)\right)^\alpha\right) + \theta\left(\delta\left(K\omega^L\left(\beta, \varepsilon\right)\right)^\alpha\right)
\end{aligned} \tag{4.14}
$$

where

$$
\begin{aligned}
\omega^L\left(A, \varepsilon\right) &= \sup\left\{\left|A\left(t_1\right) - A\left(t_2\right)\right| : t_1, t_2 \in [0, L], \left|t_1 - t_2\right| \le \varepsilon\right\}, \\
\omega_\rho^L\left(h, \varepsilon\right) &= \sup\left\{\left|h\left(t_2, x, y\right) - h\left(t_1, x, y\right)\right| : t_1, t_2 \in [0, L], \left|t_1 - t_2\right| \le \varepsilon, x, y \in [-\rho, \rho]\right\} \\
\omega^L\left(\xi, \varepsilon\right) &= \sup\left\{\left|\xi\left(t_1\right) - \xi\left(t_2\right)\right| : t_1, t_2 \in [0, L], \left|t_1 - t_2\right| \le \varepsilon\right\}, \\
\omega^L\left(x, \omega^L(\xi, \varepsilon)\right) &= \sup\left\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \le \omega^L(\xi, \varepsilon)\right\}, \\
K &= \beta_L \sup\left\{\left|g\left(t, s, x, y\right)\right| : t \in [0, L], s \in [0, \beta_L], x, y \in [-\rho, \rho]\right\},
\end{aligned}
$$

$$\omega^L_{\rho,K}(f,\varepsilon)=\sup\left\{\begin{array}{c}|f(t_2,x,y,z)-f(t_1,x,y,z)|:t_1,t_2\in[0,L],|t_1-t_2|\le\varepsilon,\\ x,y\in[-\rho,\rho],\\ z\in[-\delta K^\alpha,\delta K^\alpha]\end{array}\right\},$$

$$\omega^L_{\rho}(g,\varepsilon)=\sup\left\{\begin{array}{c}|g(t_1,s,x,y)-g(t_2,s,x,y)|:t_1,t_2\in[0,L],|t_1-t_2|\le\varepsilon,\\ s\in[0,\beta_L],\\ x,y\in[-\rho,\rho]\end{array}\right\},$$

$$\omega^L(\beta,\varepsilon)=\sup\{|\beta(t_1)-\beta(t_2)|:t_1,t_2\in[0,L],|t_1-t_2|\le\varepsilon\}.$$

Since (x,y) is an arbitrary element of $X_1\times X_2$ in (4.14), we obtain

$$\begin{aligned}\omega^L(F(X_1\times X_2),\varepsilon)\le{}&\omega^L(A,\varepsilon)+\omega^L_\rho(h,\varepsilon)+\phi\left(\omega^L\left(X_1,\omega^L(\xi,\varepsilon)\right),\omega^L\left(X_2,\omega^L(\xi,\varepsilon)\right)\right)\\&+\omega^L_{\rho,K}(f,\varepsilon)+\theta\left(\delta\left(\beta_L\omega^L_\rho(g,\varepsilon)\right)^\alpha\right)+\theta\left(\delta\left(K\omega^L(\beta,\varepsilon)\right)^\alpha\right).\end{aligned}\tag{4.15}$$

Now by the uniform continuity of f,g,h on $[0,L]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\delta K^\alpha,\delta K^\alpha]$, $[0,L]\times[0,\beta_L]\times[-\rho,\rho]\times[-\rho,\rho]$, $[0,L]\times[-\rho,\rho]\times[-\rho,\rho]$ respectively, we have $\omega^L_{\rho,K}(f,\varepsilon)\to0$, $\omega^L_\rho(g,\varepsilon)\to0$, $\omega^L_\rho(h,\varepsilon)\to0$. Also because of the uniform continuity of ξ, β and A on $[0,L]$, we derive that $\omega^L(\xi,\varepsilon)\to0$, $\omega^L(\beta,\varepsilon)\to0$ and $\omega^L(A,\varepsilon)\to0$ as $\varepsilon\to0$. Under the assumptions (v), Since θ is a nondecreasing continuous function with $\theta(0)=0$ and K is finite, hence we have

$$\theta\left(\delta\left(\beta_L\omega^L_\rho(g,\varepsilon)\right)^\alpha\right)+\theta\left(\delta\left(K\omega^L(\beta,\varepsilon)\right)^\alpha\right)\to0,$$

as $\varepsilon\to0$.

Now taking the limit from (4.15) as $\varepsilon\to0$, we get

$$\omega^L_0(F(X_1\times X_2))\le\phi\left(\omega^L_0(X_1),\omega^L_0(X_2)\right).\tag{4.16}$$

By letting $L\to\infty$ in (4.16), we obtain

$$\omega_0(F(X_1\times X_2))\le\phi\left(\omega_0(X_1),\omega_0(X_2)\right).\tag{4.17}$$

In addition, for arbitrary $(x,y),(u,v)\in X_1\times X_2$ and $t\in\mathbb{R}_+$ we have

$$\begin{aligned}\left|\begin{array}{c}F(x,y)(t)-\\F(u,v)(t)\end{array}\right|&\le|h(t,x(\xi(t)),y(\xi(t)))-h(t,u(\xi(t)),v(\xi(t)))|\\&\quad+\left|\begin{array}{l}f\left(t,x(\xi(t)),y(\xi(t)),\varphi\left(\int_0^{\beta(t)}g(t,s,x(\eta(s)),y(\eta(s)))\,ds\right)\right)-\\ f\left(t,u(\xi(t)),v(\xi(t)),\varphi\left(\int_0^{\beta(t)}g(t,s,u(\eta(s)),v(\eta(s)))\,ds\right)\right)\end{array}\right|\\&\le\frac12\phi\left(|x(\xi(t))-u(\xi(t))|,|y(\xi(t))-v(\xi(t))|\right)\\&\quad+\frac12\phi\left(|x(\xi(t))-u(\xi(t))|,|y(\xi(t))-v(\xi(t))|\right)\\&\quad+\theta\left(\left|\begin{array}{l}\varphi\left(\int_0^{\beta(t)}g(t,s,x(\eta(s)),y(\eta(s)))\,ds\right)\\-\varphi\left(\int_0^{\beta(t)}g(t,s,u(\eta(s)),v(\eta(s)))\,ds\right)\end{array}\right|\right)\\&\le\frac12\phi\left(diamX_1(\xi(t)),diamX_2(\xi(t))\right)+\frac12\phi\left(diamX_1(\xi(t)),diamX_2(\xi(t))\right)\\&\quad+\theta\left(\delta\left|\int_0^{\beta(t)}(g(t,s,x(\eta(s)),y(\eta(s)))-g(t,s,u(\eta(s)),v(\eta(s))))\,ds\right|^\alpha\right)\\&\le\phi\left(diamX_1(\xi(t)),diamX_2(\xi(t))\right)\\&\quad+\theta\left(\delta\left|\int_0^{\beta(t)}(g(t,s,x(\eta(s)),y(\eta(s)))-g(t,s,u(\eta(s)),v(\eta(s))))\,ds\right|^\alpha\right).\end{aligned}\tag{4.18}$$

Since (x, y), (u, v) and t are arbitrary in (4.18), we conclude that

$$diamF\left(X_1 \times X_2\right)(t) \le \phi\left(diamX_1\left(\xi(t)\right), diamX_2\left(\xi(t)\right)\right) + \theta\left(\delta\left|\int_0^{\beta(t)}\left(g(t,s,x(\eta(s)),y(\eta(s))) - g(t,s,u(\eta(s)),v(\eta(s)))\right)ds\right|^{\alpha}\right). \tag{4.19}$$

Therefore by (ii) and $\xi(t) \to \infty$ as $t \to \infty$ in the inequality (4.19), then using (4.5) we earn

$$\limsup_{t\to\infty} diamF\left(X_1 \times X_2\right)(t) \le \phi\left(\limsup_{t\to\infty} diamX_1\left(\xi(t)\right), \limsup_{t\to\infty} diamX_2\left(\xi(t)\right)\right). \tag{4.20}$$

Now, combining (4.17), (4.20) we obtain

$$\begin{aligned}
&\omega_0(F(X_1 \times X_2)) + \limsup_{t\to\infty} diamF\left(X_1 \times X_2\right)(t) \\
&\quad\le \phi\left(\omega_0\left(X_1\right), \omega_0\left(X_2\right)\right) \\
&\qquad+ \phi\left(\limsup_{t\to\infty} diamX_1\left(\xi(t)\right), \limsup_{t\to\infty} diamX_2\left(\xi(t)\right)\right) \\
&\quad\le 2\phi\left(\begin{array}{c}\dfrac{\omega_0\left(X_1\right) + \limsup\limits_{t\to\infty} diamX_1\left(\xi(t)\right)}{2}, \\ \dfrac{\omega_0\left(X_2\right) + \limsup\limits_{t\to\infty} diamX_2\left(\xi(t)\right)}{2}\end{array}\right).
\end{aligned} \tag{4.21}$$

Therefore from (4.21)

$$\frac{1}{2}\mu\left(F\left(X_1 \times X_2\right)\right) \le \phi\left(\frac{\mu\left(X_1\right)}{2}, \frac{\mu\left(X_2\right)}{2}\right),$$

and taking $\mu_1 = \frac{1}{2}\mu$, we have

$$\mu_1\left(F\left(X_1 \times X_2\right)\right) \le \phi\left(\mu_1\left(X_1\right), \mu_1\left(X_2\right)\right),$$

where μ is the measure of noncompactness defined in (4.1). Thus by Theorem 3.7, F has at least a coupled fixed point in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ and the proof is complete. □

Example 4.2. Consider the following functional-system of integral equations

$$\begin{cases}
x(t) = \dfrac{12}{35}e^{-t^2} + \dfrac{t^2}{2\left(1+t^2\right)} + \left(\dfrac{e^{-t^2}}{4} + \dfrac{t^4}{12\left(1+t^4\right)}\right)\ln\left(1+\left|x\left(t^2\right)\right|\right) + \left(\dfrac{1}{8} + \dfrac{e^{-t}}{16}\right)\ln\left(1+\left|y\left(t^2\right)\right|\right) + \\
\qquad \ln\left(1 + \dfrac{\int_0^{\sqrt{t}} \frac{\ln\left(1+\sqrt[4]{s}\left|x\left(\sqrt[3]{t}\right)\right|\right)\ln\left(1+\sqrt[4]{s^3}\left|y\left(\sqrt[3]{t}\right)\right|\right)+s\left(1+x^2\left(\sqrt[3]{t}\right)\right)\left(1+y^2\left(\sqrt[3]{t}\right)\right)}{e^{t^2}\left(1+x^2\left(\sqrt[3]{t}\right)\right)\left(1+y^2\left(\sqrt[3]{t}\right)\right)}ds}{3}\right) \\
y(t) = \dfrac{12}{35}e^{-t^2} + \dfrac{t^2}{2\left(1+t^2\right)} + \left(\dfrac{e^{-t^2}}{4} + \dfrac{t^4}{12\left(1+t^4\right)}\right)\ln\left(1+\left|y\left(t^2\right)\right|\right) + \left(\dfrac{1}{8} + \dfrac{e^{-t}}{16}\right)\ln\left(1+\left|x\left(t^2\right)\right|\right) + \\
\qquad \ln\left(1 + \dfrac{\int_0^{\sqrt{t}} \frac{\ln\left(1+\sqrt[4]{s}\left|y\left(\sqrt[3]{t}\right)\right|\right)\ln\left(1+\sqrt[4]{s^3}\left|x\left(\sqrt[3]{t}\right)\right|\right)+s\left(1+y^2\left(\sqrt[3]{t}\right)\right)\left(1+x^2\left(\sqrt[3]{t}\right)\right)}{e^{t^2}\left(1+y^2\left(\sqrt[3]{t}\right)\right)\left(1+x^2\left(\sqrt[3]{t}\right)\right)}ds}{3}\right).
\end{cases} \tag{4.22}$$

With the following choices, it is evident that this system is a special case of the system of integral equations of (4.2) with

$$h(t,x,y) = \frac{t^2}{2\left(1+t^2\right)} + \frac{e^{-t^2}}{4}\ln\left(1+|x|\right) + \frac{1}{8}\ln\left(1+|y|\right)$$

$$f(t,x,y,z) = \frac{1}{7}e^{-t^2} + \frac{t^4}{12\left(1+t^4\right)}\ln\left(1+|x|\right) + \frac{e^{-t}}{16}\ln\left(1+|y|\right) + \ln\left(1+\frac{|z|}{3}\right)$$

$$g(t,s,x,y)=\frac{\ln\left(1+\sqrt[4]{s}\,|x(t)|\right)\ln\left(1+\sqrt[4]{s^3}\,|y(t)|\right)+s\left(1+x^2(t)\right)\left(1+y^2(t)\right)}{e^{t^2}\left(1+x^2(t)\right)\left(1+y^2(t)\right)}$$

$$A(t)=\frac{1}{5}e^{-t^2},\ \xi(t)=t^2,\ \eta(t)=\sqrt[3]{t},\ \beta(t)=\sqrt{t},\ \varphi(x)=\ln\left(1+\frac{|x|}{3}\right),$$

$$\phi(t,s)=\ln\left(1+\frac{t+s}{2}\right),\ \theta(t)=\frac{t}{3}.$$

Now we show that all the conditions of Theorem 4.1 are satisfied for Eq. (4.22).

Conditions (i), (ii) and (iii) are clear, and it is easy to see that $M_1=\frac{1}{5}$, $\delta=\frac{1}{3}$ and $\alpha=1$.

(iv) Clearly, the function $|f(t,0,0,0)|=\frac{1}{7}e^{-t^2}$ is bounded and $M_2=\frac{1}{7}$, also the function $|h(t,0,0)|=\frac{t^2}{2(1+t^2)}$ is bounded and $M_3=\frac{1}{2}$.

(v) Obviously, f and h are continuous.

Now suppose that $t\in\mathbb{R}_+$ and $x,y,z,u,v,w\in\mathbb{R}$ with $|x|\geq|u|,\ \ |y|\geq|v|$. Then by using the Mean Value Theorem for the function $\varphi(x)=\ln\left(1+\frac{|x|}{3}\right)$ and the fact that $\phi(t,s)=\ln\left(1+\frac{t+s}{2}\right)\in\Phi$, we can get the following estimate

$$\begin{aligned}
|f(t,x,y,z)-f(t,u,v,w)| &\leq \frac{t^4}{12\left(1+t^4\right)}\left(\left|\ln(1+|x|)-\ln(1+|u|)\right|\right)\\
&\quad+\frac{e^{-t}}{16}\left(\left|\ln(1+|y|)-\ln(1+|v|)\right|\right)\\
&\quad+\left|\ln\left(1+\frac{|z|}{3}\right)-\ln\left(1+\frac{|w|}{3}\right)\right|\\
&\leq \frac{t^4}{12\left(1+t^4\right)}\left|\ln\left(\frac{1+|x|}{1+|u|}\right)\right|+\frac{e^{-t}}{16}\left|\ln\left(\frac{1+|y|}{1+|v|}\right)\right|+\frac{1}{3}\left||z|-|w|\right|\\
&\leq \frac{1}{4}\left|\ln\left(1+\frac{|x|-|u|}{1+|u|}\right)\right|+\frac{1}{4}\left|\ln\left(1+\frac{|y|-|v|}{1+|v|}\right)\right|+\frac{1}{3}|z-w|\\
&\leq \frac{1}{4}\ln(1+|x-u|)+\frac{1}{4}\ln(1+|y-v|)+\frac{1}{3}|z-w|\\
&\leq \frac{1}{2}\ln\left(1+\frac{|x-u|+|y-v|}{2}\right)+\frac{1}{3}|z-w|\\
&=\frac{1}{2}\phi(|x-u|,|y-v|)+\theta(|z-w|).
\end{aligned}\tag{4.23}$$

Moreover in a similar manner we can get

$$|h(t,x,y)-h(t,u,v)|\leq\phi(|x-u|,|y-v|).\tag{4.24}$$

The case $|u|\geq|x|,\ \ |v|\geq|y|$ can be done in the same manner for (4.23) and (4.24).

(vi) Clearly, g is continuous. Moreover, for each $t,s\in\mathbb{R}_+$ and $x,y,u,v\in\mathbb{R}$ we have

$$|g(t,s,x,y)-g(t,s,u,v)|\leq\frac{2s}{e^{t^2}}.$$

Hence

$$\lim_{t\to\infty}\int_0^{\sqrt{t}}|g(t,s,x(\eta(s)),y(\eta(s)))-g(t,s,u(\eta(s)),v(\eta(s)))|\,ds\leq\lim_{t\to\infty}\int_0^{\sqrt{t}}\frac{2s}{e^{t^2}}ds=\lim_{t\to\infty}\frac{t}{e^{t^2}}=0,$$

uniformly with respect to $x,y,u,v\in BC(\mathbb{R}_+)$. Moreover, we have

$$\begin{aligned}
\left|\int_0^{\beta(t)}g(t,s,x(\eta(s)),y(\eta(s)))\,ds\right| &\leq\int_0^{\sqrt{t}}|g(t,s,x(\eta(s)),y(\eta(s)))|\,ds\\
&\leq\int_0^{\sqrt{t}}\frac{2s}{e^{t^2}}ds=\frac{t}{e^{t^2}}
\end{aligned}$$

for any $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Thus

$$\begin{aligned} M_4 &= \sup\left\{\left|\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)))ds\right| : t \in \mathbb{R}_+, x, y \in BC(\mathbb{R}_+)\right\} \\ &\le \sup\left\{\frac{t}{e^{t^2}} : t \ge 0\right\} = \frac{1}{\sqrt{2}e^{\frac{1}{2}}} = 0.428881\ldots. \end{aligned} \tag{4.25}$$

(vii) By choosing $M_4 = \frac{1}{\sqrt{2}e^{\frac{1}{2}}}$ from (4.25) along with $M_1 = \frac{1}{5}$, $M_2 = \frac{1}{7}$, $M_3 = \frac{1}{2}$ and $\delta = \frac{1}{3}$ in the inequality (4.7), we obtain the following inequality

$$\frac{59}{70} + \ln(1+r) + \frac{1}{9\sqrt{2}e^{\frac{1}{2}}} < r.$$

Now, it is easy to check that for each number $r \ge 2$ we have

$$r - \ln(1+r) - \frac{59}{70} - \frac{1}{9\sqrt{2}e^{\frac{1}{2}}} > 0.$$

Thus for a numeric value of ρ, we can take $\rho = 2$. Consequently, all the conditions of Theorem 4.1 are satisfied and the system of integral equations of (4.2) has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

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