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The Hua matrix and inequalities related to contractive matrices

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ABSTRACT

We first deny a conjecture raised in Xu et al. (2011) [14] and then we present some eigenvalue or singular value inequalities related to contractive matrices.

1. Introduction

The Hua matrix is

$$\mathbf{H} = \begin{bmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{bmatrix},$$

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where A, B are $n \times n$ strictly contractive matrices (i.e., matrices whose spectral norm is less than one). This block matrix first appears in Hua's study of the theory of functions of several complex variables; see [9]. The Hua matrix is a source for matrix inequalities. For example, the positivity of the Hua matrix immediately leads to

$$|\det(I - A^*B)|^2 \geq \det(I - A^*A)\det(I - B^*B), \tag{1.1}$$

which is known as Hua's determinantal inequality in the literature (e.g., [16, p. 231]). More examples can be found in [3,4,12]. There is a renewed interest in the Hua matrix and its analogues in recent years; see [2,10,11,13,14]. A remarkable property about the Hua matrix is the positive partial transpose property. That is, the partial transpose of $\mathbf{H}$, viz.,

$$\mathbf{H}^\tau = \begin{bmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{bmatrix}$$

is again positive semidefinite.

In this article, we first address a conjecture raised in [14], then we present some eigenvalue or singular value inequalities involving contractive matrices. The paper is concluded with some comments and a new conjecture along this line of study.

The remaining of this section is devoted to some notation used in this article. Let $\mathbb{M}_n$ be the set of all $n \times n$ complex matrices; the identity matrix of $\mathbb{M}_n$ is denoted by I. For any $X \in \mathbb{M}_n$, X^* stands for the conjugate transpose of X. For two Hermitian matrices X, Y of the same size, we write $X \geq Y$ to mean $X - Y$ is positive semidefinite. Saying that $X \in \mathbb{M}_n$ is contractive is the same as saying $I \geq X^*X$. If the eigenvalues of a square matrix X are all real, then we denote $\lambda_j(X)$ the jth largest eigenvalue of X. The singular values of a complex matrix X are the eigenvalues of $|X| := (X^*X)^{1/2}$, and we denote $\sigma_j(X) := \lambda_j(|X|)$. The geometric mean of two positive definite matrices $X, Y \in \mathbb{M}_n$ is defined as $X\sharp Y := X^{1/2}(X^{-1/2}YX^{-1/2})^{1/2}X^{1/2}$. Its weighted version is defined as $X\sharp_t Y := X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}$, $0 \leq t \leq 1$. It is known that the notion of (weighted) geometric mean could be extended to cover all positive semidefinite matrices; see [5, p. 107].

2. A conjecture in [14]

When considering possible extensions of the Hua matrix to higher number of blocks, it is known (see [2,13]) that in general the following block matrices are no longer positive semidefinite for $m \geq 3$:

$$\mathbf{H}_{(m)} = \begin{bmatrix} (I - A_1^*A_1)^{-1} & (I - A_2^*A_1)^{-1} & \cdots & (I - A_m^*A_1)^{-1} \\ (I - A_1^*A_2)^{-1} & (I - A_2^*A_2)^{-1} & \cdots & (I - A_m^*A_2)^{-1} \\ \vdots & \vdots & & \vdots \\ (I - A_1^*A_m)^{-1} & (I - A_2^*A_m)^{-1} & \cdots & (I - A_m^*A_m)^{-1} \end{bmatrix},$$

$$\mathbf{H}_{(m)}^{\tau} = \begin{bmatrix} (I - A_1^*A_1)^{-1} & (I - A_1^*A_2)^{-1} & \cdots & (I - A_1^*A_m)^{-1} \\ (I - A_2^*A_1)^{-1} & (I - A_2^*A_2)^{-1} & \cdots & (I - A_2^*A_m)^{-1} \\ \vdots & \vdots & & \vdots \\ (I - A_m^*A_1)^{-1} & (I - A_m^*A_2)^{-1} & \cdots & (I - A_m^*A_m)^{-1} \end{bmatrix},$$

where $A_1, \ldots, A_m \in \mathbb{M}_n$, are strictly contractive.

Xu, Xu and Zhang (see [14, Corollary 2]) observed that $\mathbf{H}^{\tau} = \mathbf{H}_{(2)}^{\tau}$ has at least n eigenvalues greater than or equal to 2, and then they made the following conjecture.

Conjecture 2.1. *Let $A_1, \ldots, A_m \in \mathbb{M}_n$ be strictly contractive. Then both $\mathbf{H}_{(m)}$ and $\mathbf{H}_{(m)}^{\tau}$ have at least n eigenvalues greater than or equal to m.*

Ruling out the trivial case $m = 1$, we shall show the conjecture is true if $m = 2$.

Proposition 2.2. *The Hua matrix $\mathbf{H}$ defined in Section 1 has at least n eigenvalues greater than or equal to 2.*

Proof. By [14, Theorem 4],

$$\mathbf{H}^{\tau} \geq \begin{bmatrix} I & I \\ I & I \end{bmatrix}.$$

It follows

$$P\mathbf{H}^{\tau}P^T \geq P \begin{bmatrix} I & I \\ I & I \end{bmatrix} P^T = 2I,$$

where $P := \frac{1}{\sqrt{2}} [I \quad I]$ is a partial isometry. Apparently,

$$P\mathbf{H}^{\tau}P^T = P\mathbf{H}P^T.$$

This yields

$$\lambda_n(\mathbf{H}) \geq \lambda_n(P\mathbf{H}P^T) = \lambda_n(P\mathbf{H}^{\tau}P^T) \geq \lambda_n(2I) = 2,$$

in which the first inequality is by a result of Poincaré (see e.g., [16, p. 271]) and the second inequality is clear (see e.g., [16, Theorem 8.11]). $\square$

When $m \geq 3$, Conjecture 2.1 fails; see the Appendix for a counterexample.

3. New results

For any $X \in \mathbb{M}_n$, it is known that the determinant of X is equal to the product of all its eigenvalues, moreover, $|\det X| = \prod_{j=1}^{n} \sigma_j(X)$. Hua's determinantal inequality (1.1) has the following variants

$$\prod_{j=1}^{n} \sigma_j^2(I - A^*B) \geq \prod_{j=1}^{n} \lambda_j\Big((I - A^*A)(I - B^*B)\Big)$$
$$= \prod_{j=1}^{n} \sigma_j\Big((I - A^*A)(I - B^*B)\Big),$$
$$\prod_{j=1}^{n} \sigma_j(I - A^*B) \geq \prod_{j=1}^{n} \lambda_j\Big((I - A^*A)\sharp(I - B^*B)\Big)$$
$$= \prod_{j=1}^{n} \lambda_j\Big((I - A^*A)^{1/2}(I - B^*B)^{1/2}\Big).$$

In [11, Proposition 3.1], the present author proved the following strengthening of Hua's determinantal inequality.

Proposition 3.1. *Let $A, B \in \mathbb{M}_n$ be contractive. Then for $j = 1, \dots, n$*

$$\sigma_j^2(I - A^*B) \geq \sigma_j\Big((I - A^*A)(I - B^*B)\Big).$$

It is natural to ask whether other variants of Hua's determinantal inequality have similar strengthening. The answer turns out to be yes. We shall prove the following general result.

Theorem 3.2. *Let $\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} \in \mathbb{M}_{2n}$, with each block nonsingular, be positive semidefinite. If $M_{11}^{-1} + M_{22}^{-1} \leq M_{12}^{-1} + (M_{12}^*)^{-1}$, then for each $j = 1, \dots, n$*

$$\sigma_j^2(M_{12}) \leq \lambda_j(M_{11}M_{22}), \tag{3.1}$$
$$\sigma_j^2(M_{12}) \leq \sigma_j(M_{11}M_{22}), \tag{3.2}$$
$$\sigma_j(M_{12}) \leq \lambda_j(M_{11}\sharp M_{22}), \tag{3.3}$$
$$\sigma_j(M_{12}) \leq \lambda_j(M_{11}^{1/2}M_{22}^{1/2}). \tag{3.4}$$

Proof. By a result of Fan and Hoffman [6, p. 73], it is known

$$2\sigma_j(M_{12}^{-1}) \geq \lambda_j(M_{12}^{-1} + (M_{12}^*)^{-1}). \tag{3.5}$$

On the other hand, by a result of Bhatia and Kittaneh [6, p. 262], we have

$$\lambda_j(M_{11}^{-1} + M_{22}^{-1}) \geq 2\sigma_j(M_{11}^{-1/2}M_{22}^{-1/2}) = 2\sqrt{\lambda_j(M_{11}^{-1}M_{22}^{-1})}. \tag{3.6}$$

Combining (3.5), (3.6) with $M_{11}^{-1} + M_{22}^{-1} \leq M_{12}^{-1} + (M_{12}^*)^{-1}$ gives

$$\sigma_j^2(M_{12}^{-1}) \geq \lambda_j(M_{11}^{-1}M_{22}^{-1}),$$

which is equivalent to (3.1) by noting that $\sigma_j(X^{-1}) = \frac{1}{\sigma_{n-j+1}(X)}$, $j = 1, \ldots, n$, for every invertible $X \in \mathbb{M}_n$ and $\lambda_j(X^{-1}) = \frac{1}{\lambda_{n-j+1}(X)}$, $j = 1, \ldots, n$, for every invertible $X \in \mathbb{M}_n$ having all eigenvalues real.

The proof of (3.2) is similar except for that (3.6) should be replaced by

$$\lambda_j(M_{11}^{-1} + M_{22}^{-1}) \geq 2\sqrt{\sigma_j(M_{11}^{-1}M_{22}^{-1})},$$

which is ensured by a result, previously conjectured in [7], recently established by Drury [8].

As $M_{11}^{-1} + M_{22}^{-1} \geq 2M_{11}^{-1}\sharp M_{22}^{-1}$ implies

$$\lambda_j(M_{11}^{-1} + M_{22}^{-1}) \geq 2\lambda_j(M_{11}^{-1}\sharp M_{22}^{-1}),$$

(3.3) can be similarly proved.

The proof of (3.4) is similar except for that (3.6) should be replaced by

$$\lambda_j(M_{11}^{-1} + M_{22}^{-1}) \geq 2\lambda_j(M_{11}^{-1/2}M_{22}^{-1/2}),$$

which is by [7, Eq. (3.12)]. $\square$

This immediately yields

Corollary 3.3. *Let $A, B \in \mathbb{M}_n$ be contractive. Then for $j = 1, \ldots, n$*

$$\sigma_j^2(I - A^*B) \geq \lambda_j\Big((I - A^*A)(I - B^*B)\Big),$$
$$\sigma_j(I - A^*B) \geq \lambda_j\Big((I - A^*A)\sharp(I - B^*B)\Big),$$
$$\sigma_j(I - A^*B) \geq \lambda_j\Big((I - A^*A)^{1/2}(I - B^*B)^{1/2}\Big).$$

We remark that neither (3.1), (3.2), (3.3) nor (3.4) is stronger than the other, although it is known by the theory of majorization (e.g., [16, Chapter 10]) that

$$\begin{aligned}\prod_{j=1}^{k} \lambda_j^2(M_{11}\sharp M_{22}) &\leq \prod_{j=1}^{k} \lambda_j^2(M_{11}^{1/2}M_{22}^{1/2}) \\ &\leq \prod_{j=1}^{k} \sigma_j^2(M_{11}^{1/2}M_{22}^{1/2}) = \prod_{j=1}^{k} \lambda_j(M_{11}M_{22}) \\ &\leq \prod_{j=1}^{k} \sigma_j(M_{11}M_{22})\end{aligned}$$

for $k = 1, \ldots, n$.

The following result complements Proposition 3.1 and Corollary 3.3.

Theorem 3.4. *Let $A, B \in \mathbb{M}_n$ be contractive. Then for $j = 1, \dots, n$*

$$\sigma_j^2(I - AB^*) \geq \lambda_j\Big((I - A^*A)(I - B^*B)\Big),$$
$$\sigma_j^2(I - AB^*) \geq \sigma_j\Big((I - A^*A)(I - B^*B)\Big),$$
$$\sigma_j(I - AB^*) \geq \lambda_j\Big((I - A^*A)\sharp(I - B^*B)\Big),$$
$$\sigma_j(I - AB^*) \geq \lambda_j\Big((I - A^*A)^{1/2}(I - B^*B)^{1/2}\Big).$$

Proof. Similar to the argument in the proof of Theorem 3.2, it suffices to show

$$2\sigma_j(I - AB^*) \geq \lambda_j((I - A^*A) + (I - B^*B)). \tag{3.7}$$

It is known (see [6, p. 262]) that

$$2\sigma_j(AB^*) \leq \lambda_j(A^*A + B^*B),$$

which is equivalent to

$$2U|AB^*|U^* \leq A^*A + B^*B$$

for some unitary matrix $U \in \mathbb{M}_n$. It follows

$$2(I - U|AB^*|U^*) \geq (I - A^*A) + (I - B^*B). \tag{3.8}$$

By a result of Thompson [16, p. 289],

$$\begin{aligned} I &= |I - AB^* + AB^*| \\ &\leq V|I - AB^*|V^* + W|AB^*|W^* \end{aligned}$$

for some unitary matrices $V, W \in \mathbb{M}_n$. Pre-post multiplying both sides with UW^*, WU^*, respectively, gives

$$I - U|AB^*|U^* \leq UW^*V|I - AB^*|V^*WU^*. \tag{3.9}$$

Combining (3.8) and (3.9) and taking eigenvalues on both sides give the required (3.7). □

When taking products for j from 1 to n on both sides in each of the inequality in Theorem 3.4, we would again recover Hua's determinantal inequality, since

$$|\det(I - AB^*)| = |\det(I - BA^*)| = |\det(I - A^*B)|.$$

4. Comments and a conjecture

If $A, B \in \mathbb{M}_n$ are strictly contractive and $U \in \mathbb{M}_n$ is unitary, then by [14, Theorem 1] the block matrix

$$\begin{bmatrix} (I - A^*A)^{-1} & (U - B^*A)^{-1} \\ (U^* - A^*B)^{-1} & (I - B^*B)^{-1} \end{bmatrix}$$

is positive semidefinite. This implies the following generalization of Hua's determinantal inequality

$$|\det(U - A^*B)|^2 \geq \det(I - A^*A)\det(I - B^*B). \tag{4.1}$$

Based on what we discussed in Section 3, it is natural to ask whether (4.1) has the corresponding improvements. However, simulations suggest the following inequalities are not valid in general

$$\begin{aligned} \sigma_j^2(U - A^*B) &\geq \lambda_j\Big((I - A^*A)(I - B^*B)\Big), \\ \sigma_j^2(U - A^*B) &\geq \sigma_j\Big((I - A^*A)(I - B^*B)\Big), \\ \sigma_j(U - A^*B) &\geq \lambda_j\Big((I - A^*A)\sharp(I - B^*B)\Big), \\ \sigma_j(U - A^*B) &\geq \lambda_j\Big((I - A^*A)^{1/2}(I - B^*B)^{1/2}\Big). \end{aligned}$$

Another idea of thought is the weighted extension of the inequalities in Section 3. We are able to prove the following result.

Theorem 4.1. *Let $A, B \in \mathbb{M}_n$ be contractive and let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Then for $j = 1, \ldots, n$*

$$\begin{aligned} \sigma_j^2(I - AB^*) &\geq \lambda_j\Big((I - |A|^p)^{2/p}(I - |B|^q)^{2/q}\Big), \\ \sigma_j(I - AB^*) &\geq \lambda_j\Big((I - |A|^p)\sharp_{\frac{1}{q}}(I - |B|^q)\Big), \\ \sigma_j(I - AB^*) &\geq \lambda_j\Big((I - |A|^p)^{1/p}(I - |B|^q)^{1/q}\Big). \end{aligned}$$

Proof. In view of the proof of Theorem 3.4, it suffices to show

$$\sigma_j(I - |AB^*|) \geq \lambda_j\left(\frac{1}{p}(I - |A|^p) + \frac{1}{q}(I - |B|^q)\right).$$

But this follows immediately from a result of Ando (see [1] or [15, p. 30]) which says that there is a unitary matrix U such that

$$U|AB^*|U^* \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q. \quad \square$$

However, we have not been able to prove other weighted counterparts of the inequalities in Section 3. We make the following conjecture.

Conjecture 4.2. *Let $A, B \in \mathbb{M}_n$ be contractive and let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Then for $j = 1, \ldots, n$*

$$\begin{aligned}
\sigma_j^2(I - A^*B) &\geq \lambda_j\Big((I - |A|^p)^{2/p}(I - |B|^q)^{2/q}\Big), \\
\sigma_j^2(I - A^*B) &\geq \sigma_j\Big((I - |A|^p)^{2/p}(I - |B|^q)^{2/q}\Big), \\
\sigma_j(I - A^*B) &\geq \lambda_j\Big((I - |A|^p)\sharp_{\frac{1}{q}}(I - |B|^q)\Big), \\
\sigma_j(I - A^*B) &\geq \lambda_j\Big((I - |A|^p)^{1/p}(I - |B|^q)^{1/q}\Big); \\
\sigma_j^2(I - AB^*) &\geq \sigma_j\Big((I - |A|^p)^{2/p}(I - |B|^q)^{2/q}\Big).
\end{aligned}$$

Interestingly, numerical simulations we have run so far support the above conjecture. Proving or disproving any of the inequalities in the list would be of great interest.

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Appendix A

A.1. Counterexample to Conjecture 2.1

Take

$$A_1 = \begin{bmatrix} 0.4384 & -0.1847 \\ -0.4632 & -0.2698 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6753 & -0.1516 \\ -0.0252 & 0.3000 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.1932 & 0.2771 \\ 0.5897 & 0.0462 \end{bmatrix}.$$

A calculation gives

$$\mathbf{H}_{(3)} = \begin{bmatrix}
1.6918 & 0.0833 & 0.7633 & 0.0954 & 0.8287 & -0.1518 \\
0.0833 & 1.1238 & -0.1489 & 0.9311 & 0.0780 & 0.9259 \\
0.7633 & -0.1489 & 1.8755 & 0.2005 & 0.8530 & 0.1225 \\
0.0954 & 0.9311 & 0.2005 & 1.1488 & -0.1562 & 0.9502 \\
0.8287 & 0.0780 & 0.8530 & -0.1562 & 1.6452 & 0.1443 \\
-0.1518 & 0.9259 & 0.1225 & 0.9502 & 0.1443 & 1.0983
\end{bmatrix},$$

whose eigenvalues are $0.0677, 0.1198, 0.9271, 1.0994, 2.9759, 3.3934$. So only one eigenvalue of $\mathbf{H}_{(3)}$ is larger than 3.

$$\mathbf{H}_{(3)}^{\tau} = \begin{bmatrix} 1.6918 & 0.0833 & 0.7633 & -0.1489 & 0.8287 & 0.0780 \\ 0.0833 & 1.1238 & 0.0954 & 0.9311 & -0.1518 & 0.9259 \\ 0.7633 & 0.0954 & 1.8755 & 0.2005 & 0.8530 & -0.1562 \\ -0.1489 & 0.9311 & 0.2005 & 1.1488 & 0.1225 & 0.9502 \\ 0.8287 & -0.1518 & 0.8530 & 0.1225 & 1.6452 & 0.1443 \\ 0.0780 & 0.9259 & -0.1562 & 0.9502 & 0.1443 & 1.0983 \end{bmatrix},$$

whose eigenvalues are $0.0747, 0.1157, 0.9085, 1.1153, 2.9758, 3.3934$. Again only one eigenvalue of $\mathbf{H}_{(3)}^{\tau}$ is larger than 3.

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