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Automorphisms of the endomorphism algebra of a free module

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Let R be a commutative ring with identity $1 \in R$ and V a free R-module of arbitrary rank. Let $End_R(V)$ denote the R-algebra of all R-linear endomorphisms of V. We show that all R-algebra automorphisms of $End_R(V)$ are inner if R is a Bezout domain. We also consider 2-local automorphisms of $End_R(V)$.

1. Introduction

Let A and B be central simple algebras over the field F. The celebrated Skolem–Noether Theorem goes back to [11] and [10] and states that any two algebra homo-

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morphisms $\varphi, \psi : A \to B$ are conjugate, i.e. there exists a unit $b \in B$ such that $\varphi(x) = b\psi(x)b^{-1}$ for all $x \in A$. It is an immediate consequence of this theorem that all automorphisms of the algebra $Mat_{n\times n}(F)$, the ring of $n \times n$ matrices over F, are inner automorphisms. This result has been extended to many more classes of algebras. For example, it was shown in [6] that all automorphisms of $M = Mat_{n\times n}(R)$ are inner, if R is a unique factorization domain (UFD). Moreover, for any (!) commutative ring, the group of outer automorphisms $Out(M) = Aut(M)/Inn(M)$ is abelian and bounded by n. For Dedekind domains R, [6] also contains a nice description of $Out(M)$ in terms of the class group of R. Since there exist non-Noetherian Bezout domains R, which are not UFD, we are able to show a little bit more:

- If R is a Bezout domain, then all automorphisms of $Mat_{n\times n}(R)$ are inner.

The paper [6, Example 6] also contains an example of some $\tau \in Aut(Mat_{2\times 2}(\mathbb{Z}[\sqrt{-5}\,]))$ that is not inner. We will construct a similar example pointing out how maximal, non-principal ideals of the Dedekind domain come into play.

Let R be a commutative ring with identity $1 \in R$ and V some free R-module. Then Isaac's result can be stated as: If R is a UFD and V has **finite** rank, then all automorphisms of $End_R(V)$ are inner. Let $Fin_R(V) = \{\varphi \in End_R(V) : \varphi(V)$ is contained in a finitely generated submodule of $V\}$. In his classic monograph [7, Isomorphism Theorem, page 79] Jacobson shows that if A is a subalgebra of $End_D(V)$ containing $Fin_D(V)$, where D is a division ring and V a D-vector space of arbitrary dimension, then any automorphism of A is induced (via conjugation) by some semi-linear map on V. We will extend this result by replacing D by suitable commutative rings. We will show:

- Let R be a Bezout domain and V any free R-module. If A is a subalgebra of $End_R(V)$ with $Fin_R(V) \subseteq A$, then all automorphisms of A are restrictions of some inner automorphism of $End_R(V)$.

There is a large amount of literature on automorphisms of certain finite dimensional algebras, see, for example, [2], [4] or [9]. Most similar results on infinite dimensional algebras all seem to be connected in some way to analysis, like operator algebras on Banach spaces or C^*-algebras.

Let A be an algebra over the commutative ring R and $\varphi : A \to A$ an R-linear map. Moreover, let $n \in \mathbb{N}$. We call φ a n-local automorphism if for any n elements $x_i \in A$ there is some $\alpha \in Aut(A)$ such that $\varphi(x_i) = \alpha(x_i)$ for all $1 \le i \le n$. If $n = 1$, then φ is called a local automorphism. While any n-local automorphism is clearly injective, surjectivity does not follow in general. In a forthcoming paper [1] we will present examples of fields F and F-algebras A for which there exists an F-linear map $\varphi : A \to A$ which is an n-local automorphism for each $n \ge 1$, but φ is not surjective and thus not an automorphism.

In the last section we will show:

- Let R be a Bezout domain of characteristic $char(R) \neq 2$ and V any free R-module. Then any surjective, 2-local automorphism of $End_R(V)$ is an inner automorphism.

We will also obtain some results on local derivations on $End_V(R)$.

2. The algebra $End_R(V)$

In this section we will adhere to the following:

Notation 1. Let R be a commutative ring with identity $1 \in R$. Let V be a free R-module with basis B and $End_R(V)$ be the R-algebra of all R-endomorphisms of V. We define $Fin_R(V) = \{\varphi \in End_R(V) : \varphi(V)$ is contained in a finitely generated submodule of $V\}$. Then $Fin_R(V)$ is a two-sided ideal of $End_R(V)$.

For any element $v \in V$, there exists a finite subset $supp(v) \subseteq B$ such that $v = \sum_{b \in supp(v)} v_b b$ for some $v_b \in R$. We define special elements $\varepsilon_b \in A$ for all $b \in B$, by $\varepsilon_b(b) = b$ and $\varepsilon_b(c) = 0$ for all $b \neq c \in B$. Of course, ε_b is just the natural projection of V onto Rb. For any $a, b \in B$, define $\varepsilon_{ab} \in A$ by $\varepsilon_{ab}(v) = v_b a$ for all $v \in V$. Note that $\varepsilon_{ab}(b) = a$ and $\varepsilon_{bb} = \varepsilon_b$. Moreover, **all our maps operate from the left**. If follows from the definitions, that $\varepsilon_a, \varepsilon_{ab} \in Fin_R(V)$ for all $a, b \in B$.

The letter A will always denote a subalgebra of the R-algebra $End_R(V)$ such that $id_V \in A$ and $Fin_R(V) \subseteq A$.

Proposition 1. *The element $\varepsilon_b \in End_R(V)$ is idempotent for all $b \in B$, and is a primitive idempotent element if and only if R has only the trivial idempotents* 0 *and* 1.

Proof. Obviously, ε_b is idempotent. Suppose that ε_b is primitive and $e \in R$ is idempotent. Then $e\varepsilon_b$ is idempotent and $\varepsilon_b(e\varepsilon_b) = e\varepsilon_b = (e\varepsilon_b)\varepsilon_b$. Since ε_b is primitive, we infer $e\varepsilon_b = 0$ or $\varepsilon_b = e\varepsilon_b$. In the first case, we get $eb = 0$ and $b = eb$ in the second. Thus $e = 0$ or $e = 1$.

Now suppose that R has only the trivial idempotents and assume that $\pi \in End_R(V)$ is an idempotent with $\varepsilon_b\pi = \pi\varepsilon_b = \pi$. This implies $\varepsilon_b\pi\varepsilon_b = \pi$. Let $\pi(b) = \sum_{c \in B} \lambda_c c$. For any $v \in V$ we have $\pi(v) = \varepsilon_b(\pi(v_b b)) = \varepsilon_b(v_b \sum_{c \in B} \lambda_c c) = v_b \lambda_b b = (\lambda_b \varepsilon_b)(v)$ and thus $\pi = \lambda_b \varepsilon_b$ and it follows that $\lambda_b b = \pi(b) = \pi^2(b) = \lambda_b^2 b$. This shows that $\lambda_b \in R$ is an idempotent element and thus $\lambda_b = 0$ or $\lambda_b = 1$. We infer $\pi = 0$ or $\pi = \varepsilon_b$ and ε_b is primitive. $\square$

Remark 2.1. If α is a surjective local automorphism, then α^{-1} is a local automorphism:

Since α is bijective, the map α^{-1} exists. Let $\sigma \in A$ and $\theta \in Aut(A)$ with $\sigma = \alpha(\alpha^{-1}(\sigma)) = \theta(\alpha^{-1}(\sigma))$. We infer $\alpha^{-1}(\sigma) = \theta^{-1}(\sigma)$ and α^{-1} is a local automorphism.

Here is our key:

Lemma 2.2. *Let R be a commutative ring with identity element such that R has only the trivial idempotents and all indecomposable, projective R-modules are isomorphic to R_R. Let A be a subalgebra of $End_R(V)$ such that $Fin_R(V) \subseteq A$ and $id_V \in A$. Let $\alpha : A \to A$ be a map such that:*

(1) α is a surjective local automorphism of A.

(2) If $p, q \in A$ are idempotents and $a \in A$, then $\alpha(paq) = \alpha(p)\alpha(a)\alpha(q)$.

Then there exists some invertible element $\tau \in End_R(V)$ such that $\alpha(\sigma) = \tau^{-1}\sigma\tau$ for all $\sigma \in A$.

Proof. Note that $\{\varepsilon_b : b \in B\}$ is a complete set of orthogonal, primitive idempotents in A. Let $\varphi_b = \alpha(\varepsilon_b)$ for all $b \in B$. By (1), the set $\{\varphi_b : b \in B\}$ is a set of primitive idempotent elements of A. Let $\varepsilon, \delta \in A$ be orthogonal primitive idempotents. Then $0 = \alpha(0) = \alpha(\varepsilon\varepsilon\delta) = \alpha(\varepsilon)\alpha(\varepsilon)\alpha(\delta) = \alpha(\varepsilon)\alpha(\delta)$ by (2). This shows that $\{\varphi_b : b \in B\}$ is a set of primitive, orthogonal idempotent elements of A and thus $\mathrm{Im}(\varphi_a) \subseteq \ker(\varphi_b)$ for any $a \neq b \in B$. Note that $\mathrm{Im}(\varphi_b)$ is an indecomposable direct summand of V and thus $\mathrm{Im}(\varphi_b) = Rg_b$ for some $g_b \in V$.

Claim 1: Let $v \in V$. Then $\varphi_b(v) = 0$ for all but finitely many elements $b \in B$.

Fix $c \in B$. For $v \in V$, define $\eta_v \in End_R(V)$ by $\eta_v(b) = \begin{cases} v \text{ if } b = c \\ 0 \text{ if } b \neq c \end{cases}$. Note that $\eta_v = \eta_v\varepsilon_c$. Assume that $\varphi_b(v) \neq 0$. Then $\varphi_b\eta_v\varepsilon_c \neq 0$ and thus $\alpha(\varepsilon_b)\alpha(\alpha^{-1}(\eta_v))\alpha(\alpha^{-1}(\varepsilon_c)) \neq 0$. It follows from (2), that $\varepsilon_b\alpha^{-1}(\eta_v)\alpha^{-1}(\varepsilon_c) \neq 0$. Now $\alpha^{-1}(\varepsilon_c)$ is a primitive idempotent of $End_R(V)$ whose image is an indecomposable direct summand of V. By (2) we infer that $im(\alpha^{-1}(\varepsilon_c)) = Rw$ for some $w \in V$ and thus $\varepsilon_b(\alpha^{-1}(\eta_v)(w)) \neq 0$, which can happen only for the finitely many elements $b \in supp(\alpha^{-1}(\eta_v)(w))$, and the claim follows.

Claim 2: $V = \left(\bigoplus_{b\in B} \mathrm{Im}(\varphi_b)\right) \oplus \left(\bigcap_{b\in B} \ker(\varphi_b)\right)$.

Let $v \in V$. By Claim 1, the set $S = \{b \in B : \varphi_b(v) \neq 0\}$ is finite. Let $w = v - \sum_{b\in S} \varphi_b(v)$. Note that $\varphi_a(w) = 0$ for all $a \in B - S$. If $a \in S$, then $\varphi_a(w) = \varphi_a(v) - \varphi_a(\varphi_a(v)) = 0$ and it follows that $w \in \bigcap_{b\in B} \ker(\varphi_b)$. This shows that $V = \left(\bigoplus_{b\in B} \mathrm{Im}(\varphi_b)\right) + \left(\bigcap_{b\in B} \ker(\varphi_b)\right)$. Obviously, $\left(\bigoplus_{b\in B} \mathrm{Im}(\varphi_b)\right) \cap \left(\bigcap_{b\in B} \ker(\varphi_b)\right) = \{0\}$.

Claim 3: $\bigcap_{b\in B} \ker(\varphi_b) = \{0\}$.

Let $w \in \bigcap_{b\in B} \ker(\varphi_b)$. Define $\pi \in End_R(V)$ by $\pi(b) = w$ for all $b \in B$. Then $\pi \in Fin_R(V) \subset A$ and $\varphi_b\pi = 0$ for all $b \in B$. Since α is surjective, there exists $\sigma \in A$ such that $\pi = \alpha(\sigma)$ and it follows by (2) that $0 = \varphi_b\pi = \varphi_b\pi id_V = \alpha(\varepsilon_b)\alpha(\sigma)\alpha(id_V) = \alpha(\varepsilon_b\sigma)$ and thus that $\varepsilon_b\sigma = 0$ for all $b \in B$ This implies that $\sigma = 0$ and $\pi = \alpha(\sigma) = 0$ follows. This shows that $w = 0$, and we infer $\bigcap_{b\in B} \ker(\varphi_b) = \{0\}$.

We infer that $V = \bigoplus_{b\in B} \mathrm{Im}(\varphi_b) = \bigoplus_{b\in B} Rg_b$. Now define $\gamma \in End_R(A)$ by $\gamma(b) = g_b$ for all $b \in B$. Then γ is invertible and $(\varphi_b\gamma)(u) = \varphi_b\left(\sum_{a\in supp(u)} u_a g_a\right) = \varphi_b(u_b g_b) = u_b g_b$ for $u = \sum_{a\in supp(u)} u_a a$. On the other hand, $(\gamma\varepsilon_b)(u) = \gamma(u_b b) = u_b g_b$. We infer $\varepsilon_b = \gamma^{-1}\varphi_b\gamma$. Let $\widehat{\gamma} : End_R(V) \to End_R(V)$ denote the conjugation by γ. Then $\widehat{\gamma}\alpha : A \to End_R(V)$ is a map that still satisfies clause (2) of the hypothesis. We call that map α again and have that

$\alpha(\varepsilon_a) = \varepsilon_a$ for all $a \in B$.

Assume that the bijective map $\widehat{\gamma}\alpha : A \to \widehat{\gamma}(A)$ has the property that $\widehat{\gamma}\alpha(\varepsilon_{ab}) = s_{ab}\varepsilon_{ab} \in \widehat{\gamma}(A)$ for all $a, b \in B$. Then $(\widehat{\gamma}\alpha)^{-1} = \alpha^{-1}\widehat{\gamma^{-1}} : \widehat{\gamma}(A) \to A$ and none of the s_{ab} are zero divisors in R. It follows that $\varepsilon_{ab} = s_{ab}[(\widehat{\gamma}\alpha)^{-1}(\varepsilon_{ab})]$ and we infer that $(\widehat{\gamma}\alpha)^{-1}(\varepsilon_{ab}) \in R\varepsilon_{ab}$ and thus $(\widehat{\gamma}\alpha)^{-1}(\varepsilon_{ab}) = r_{ab}\varepsilon_{ab}$ for some $r_{ab} \in R$. Then $r_{ab}s_{ab} = 1$, which shows that s_{ab} is a unit in R.

Let $\alpha(\varepsilon_{ab})(b) = \sum_c w_{bc}c$ for some $w_{bc} \in R$. Note that $\varepsilon_a\varepsilon_{ab}\varepsilon_b = \varepsilon_{ab}$ and thus $\varepsilon_a\alpha(\varepsilon_{ab})\varepsilon_b = \alpha(\varepsilon_{ab})$ by (2).

We compute: $(\varepsilon_a\alpha(\varepsilon_{ab})\varepsilon_b)(u) = \varepsilon_a(\alpha(\varepsilon_{ab})(u_b b)) = \varepsilon_a(u_b \sum_c w_{bc}c) = u_b w_{ba} a$. Moreover $(w_{ba}\varepsilon_{ab})(u) = w_{ba}(u_b a)$ and we infer:

$\alpha(\varepsilon_{ab}) = s_{ab}\varepsilon_{ab}$ for all $a, b \in B$ and some $s_{ab} = w_{ba} \in R$. We have shown in the above paragraph that the elements s_{ab} are units in R.

By hypothesis, $\alpha : A \to \widehat{\gamma}(A)$ is bijective and the inverse map is also a local automorphism and $\varepsilon_{ab} = \alpha^{-1}(\alpha(\varepsilon_{ab})) = s_{ab}\alpha^{-1}(\varepsilon_{ab})$. Thus $a = \varepsilon_{ab}(b) \in s_{ab}V$ and s_{ab} is a unit element of R and $s_{bb} = 1$ for all $b \in B$.

Note that for $a \neq b \neq c \in B$ the elements $\varepsilon_a + \varepsilon_{ab}$ and $\varepsilon_b + \varepsilon_{bc}$ are idempotents and α acts multiplicatively on idempotents by (2). Note that $(\varepsilon_a + \varepsilon_{ab})(\varepsilon_b + \varepsilon_{bc}) = \varepsilon_{ab} + \varepsilon_{ac}$ and it follows that $s_{ab}\varepsilon_{ab} + s_{ac}\varepsilon_{ac} = \alpha(\varepsilon_{ab} + \varepsilon_{ac}) = \alpha(\varepsilon_a + \varepsilon_{ab})\alpha(\varepsilon_b + \varepsilon_{bc}) =$

$= (\varepsilon_a + s_{ab}\varepsilon_{ab})(\varepsilon_b + s_{bc}\varepsilon_{bc}) = s_{ab}\varepsilon_{ab} + s_{ab}s_{bc}\varepsilon_{ac}$ and $s_{ac}\varepsilon_{ac} = s_{ab}s_{bc}\varepsilon_{ac}$. We infer $s_{ab}s_{bc} = s_{ac}$ for all $a \neq b \neq c$ in B.

Let $\psi = \sum_{x\in B} s_{tx}\varepsilon_x$ and $u \in V$. Then $\psi^{-1} = \sum_{x\in B} s_{tx}^{-1}\varepsilon_x$. We compute: $(\psi\alpha(\varepsilon_{ab})\psi^{-1})(u) = (\psi\alpha(\varepsilon_{ab}))(\sum_{x\in s(u)} u_x s_{tx}^{-1} x) =$

$= \psi(s_{ab}\varepsilon_{ab}(\sum_{x\in s(u)} u_x s_{tx}^{-1} x)) = \psi(s_{ab}\varepsilon_{ab}(u_b s_{tb}^{-1} b)) =$

$= \psi(s_{ab}u_b s_{tb}^{-1} a) = s_{ta}s_{ab}s_{tb}^{-1}u_b a = s_{tb}s_{tb}^{-1}u_b a = u_b a = \varepsilon_{ab}(u)$.

This shows that we may assume $\alpha(\varepsilon_{ab}) = \varepsilon_{ab}$ for all $a, b \in B$.

Claim 4: Let $\varphi, \psi \in End_R(V)$. Then $\varphi = \psi$ if and only if $\varepsilon_a\varphi\varepsilon_b = \varepsilon_a\psi\varepsilon_b$ for all $a, b \in B$.

Let $\varepsilon_b(b) = b \in B$. Let $\varphi(b) = \sum_c \lambda_{bc}c$ and $\psi(b) = \sum_c \mu_{bc}c$.

We have $(\varepsilon_a\varphi\varepsilon_b)(u) = u_b(\varepsilon_a(\varphi(b))) = u_b(\varepsilon_a(\sum_c \lambda_{bc}c)) = u_b\lambda_{ba}a = (\lambda_{ba}\varepsilon_{ab})(u)$ and thus $\varepsilon_a\varphi\varepsilon_b = \lambda_{ba}\varepsilon_{ab}$. In the same fashion we have $\varepsilon_a\psi\varepsilon_b = \mu_{ba}\varepsilon_{ab}$ and we infer $\lambda_{ba} = \mu_{ba}$ for all $a, b \in B$. This shows that $\varphi(b) = \psi(b)$ for all $b \in B$ and thus $\varphi = \psi$.

Now let $\varphi \in End_R(V)$. Then $\varepsilon_a\varphi\varepsilon_b = \lambda_{ba}\varepsilon_{ab}$ as seen above. Now apply α to get $\varepsilon_a\alpha(\varphi)\varepsilon_b = \lambda_{ba}\alpha(\varepsilon_{ab}) = \lambda_{ba}\varepsilon_{ab} = \varepsilon_a\varphi\varepsilon_b$ for all $a, b \in B$. By the above, we get $\alpha(\varphi) = \varphi$ for all $\varphi \in End_R(V)$ and thus $\alpha = id_{End_R(V)}$.

This shows that the initial local automorphism α is the composition of two conjugations and thus α is an inner automorphism.

3. Automorphisms of $End_R(V)$

Let $1 \in R$ be a commutative ring, V a free R-module and η an R-algebra automorphism of $A = End_R(V)$. Then η satisfies conditions (1) and (2) of our Lemma 2.2 and we instantly obtain:

Theorem 3.1. *Let $1 \in R$ be a commutative ring, V a free R-module and $id_V \in A$ a subalgebra of $End_R(V)$ such that $Fin_R(V) \subseteq A$. Let η be an R-algebra automorphism of A. If each indecomposable projective R-module is isomorphic to R_R, and R has only the trivial idempotent elements, then η is induced by an inner automorphism of $End_R(V)$, i.e. there is a unit γ in $End_R(V)$ such that $\eta(\sigma) = \gamma^{-1}\sigma\gamma$ for all $\sigma \in A$.*

Corollary 1. *Let $1 \in R$ be a commutative ring, V a free R-module and η an R-algebra automorphism of $A = End_R(V)$. If each indecomposable projective R-module is isomorphic to R_R, and R has only the trivial idempotent elements, then η is an inner automorphism of A.*

Suppose that our ring has finite Goldie dimension and each indecomposable projective R-module is isomorphic to R_R. Assume that e is a non-trivial idempotent of R. Then $R = eR \oplus (1-e)R$. If e is not primitive, then there exists an idempotent $e_1 \notin \{0, e\}$ such that $ee_1 = e_1e = e_1$. It follows that $R = e_1R \oplus (e-e_1)R \oplus (1-e)R$. If e_1 is not primitive, there is some idempotent $e_2 \notin \{0, e_1\}$ such that $e_1e_2 = e_2e_1 = e_2$ and so on. Since R has finite Goldie dimension, this process has to terminate at a primitive idempotent e_n. But then e_nR is an indecomposable projective module, with e_nR isomorphic to R_R, which is absurd since $(e_nR)(1-e_n) = \{0\}$ but $R(1-e_n) \neq 0$. This proves:

Corollary 2. *Let $1 \in R$ be a commutative ring with finite Goldie dimension, V a free R-module and η an R-algebra automorphism of $End_R(V)$. If each indecomposable projective R-module is isomorphic to R_R, then η is an inner automorphism of A.*

Corollary 3. *Let $1 \in R$ be an integral domain such that all projective modules are free. Then all automorphisms of $End_R(V)$ are inner.*

By [3, p. 199, Corollary 1.13], all projective modules over a Bezout domain are free. This gives us our extension of Jacobson's Isomorphism Theorem for free modules over rings:

Corollary 4. *Let R be a Bezout domain. Let V be a free R-module and $id_V \in A$ a subalgebra of $End_R(V)$ such that $Fin_R(V) \subseteq A$. Then all automorphisms A are induced by inner automorphisms of $End_R(V)$.*

Corollary 5. *Let F be a field and V any F-vector space. Then all F-algebra automorphisms of $End_F(V)$ are inner automorphisms.*

Remark 3.2. Let R be a Bezout domain and $A_0 = R \cdot id_V + Fin_R(V)$ a subalgebra of $End_R(V)$. As we just showed, any automorphism of A_0 is induced by some inner automorphism $\widehat{\gamma}$ of $End_R(V)$, where γ is a unit of $End_R(V)$. Since $Fin_R(V)$ is an ideal of $End_R(V)$, any unit γ of $End_R(V)$ has the property that $\widehat{\gamma}(A_0) \subseteq A_0$ and thus induces an automorphism of A_0. It follows that $End_R(V)$ and A_0 have (naturally) isomorphic groups of automorphisms, even though A_0 is a very small subalgebra of $End_R(V)$ if V has infinite rank.

We will now present a counterexample to show that "Bezout" can not be replaced by "Dedekind".

Let $R = \mathbb{Z} \oplus \mathbb{Z}\sqrt{-5}$ be the ring of algebraic integers in the quadratic field extension $\mathbb{Q}[\sqrt{-5}\,]$.

Isaacs [6, Example 6] introduced the matrix $m = \begin{bmatrix} 1+i\sqrt{5} & -2 \\ -2 & 1-i\sqrt{5} \end{bmatrix} \in Mat_{2\times 2}(R)$. Then $m^{-1} = \frac{1}{2}\begin{bmatrix} 1-i\sqrt{5} & 2 \\ 2 & 1+i\sqrt{5} \end{bmatrix} \notin Mat_{2\times 2}(R)$ but m still induces an automorphism of $Mat_{2\times 2}(R)$, which is not inner.

We will present an example of a non-inner automorphisms of $Mat_{2\times 2}(R)$ that arises from the fact that $R \oplus R = P \oplus Q$ where P and Q are non-cyclic, projective R-modules.

Let $\lambda = 1 + i\sqrt{5}$. Then $\lambda^2 = 2\lambda - 6$ and $\lambda^{-1} = \frac{1}{6}(2 - \lambda)$.

Then $P = 2R + \lambda R$ is a maximal ideal of R. Let $V = R \oplus R$, a free R-module of rank 2. Consider the R-linear homomorphism $\varphi : V \to P$ by $\varphi(r, s) = 2r + \lambda s$. We compute the kernel $\widetilde{Q}$ of this map:

$\widetilde{Q} = \{(r, s) \in V : r \in R, s = -2\lambda^{-1}r \in R\}$. Let $Q = \{a + bi\sqrt{5} : a, b \in \mathbb{Z}, a \equiv b \bmod 3\} = 3R + \lambda\mathbb{Z} = 3R + \lambda R$ be a maximal ideal of R. An easy computation shows that $\widetilde{Q}^T = \left\{\begin{bmatrix} r \\ -2r\lambda^{-1} \end{bmatrix} : r \in Q\right\} = \begin{bmatrix} 1 \\ -2\lambda^{-1} \end{bmatrix} Q = \begin{bmatrix} 1 \\ \frac{1}{3}\lambda - \frac{2}{3} \end{bmatrix} Q$ and thus $\widetilde{Q} \cong Q$ as R-modules. Note that $\widetilde{Q}$ is a direct summand of V, since P is a projective R-module.

Now let $\pi_{22} = \begin{bmatrix} 4-\lambda & 3+\lambda \\ \lambda & \lambda-3 \end{bmatrix} \in Mat_{2\times 2}(R)$. A direct computation shows that π_{22} is idempotent and $\pi_{22}\widetilde{Q} = \{0\}$.

Now put $\pi_{11} = 1 - \pi_{22} = \begin{bmatrix} \lambda-3 & -\lambda-3 \\ -\lambda & 4-\lambda \end{bmatrix}$. Note that $\pi_{11}\widetilde{Q} = \widetilde{Q}$ and $\widetilde{Q}$ is **not** a cyclic R-module. Moreover $\pi_{11}V = \begin{bmatrix} -\frac{1}{2}i\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix} P = \begin{bmatrix} -\frac{1}{2}\lambda \\ 1 \end{bmatrix} P = \widetilde{P}$ and $V = \widetilde{P} \oplus \widetilde{Q}$.

Now define $\pi_{21} = \begin{bmatrix} -\lambda-3 & 5-3\lambda \\ 3-\lambda & \lambda+3 \end{bmatrix}$ and $\pi_{12} = \begin{bmatrix} -\lambda & 3-\lambda \\ 2 & \lambda \end{bmatrix}$.

We will show that the π_{ij} satisfy the 16 matrix unit identities $\pi_{ij}\pi_{ab} = \begin{cases} \pi_{ib} \text{ if } j = a \\ 0 \text{ otherwise} \end{cases}$ for all $1 \le i, j, a, b \le 2$. Since we already know that $\{\pi_{11}, \pi_{22}\}$ is a pair of orthogonal idempotents, we only need to verify that

(1) $\pi_{11}\pi_{12} = \pi_{12}$,

(2) $\pi_{11}\pi_{21} = 0$,

(3) $\pi_{12}\pi_{21} = \pi_{11}$,

(4) $\pi_{21}\pi_{12} = \pi_{22}$ and:

(5) $\pi_{21}\pi_{11} = \pi_{21}$.

It is easy to show that the other identities will follow.

We compute;

(1) $\pi_{11}\pi_{12} = \begin{bmatrix} \lambda-3 & -\lambda-3 \\ -\lambda & 4-\lambda \end{bmatrix} \begin{bmatrix} -\lambda & 3-\lambda \\ 2 & \lambda \end{bmatrix} =$

$= \begin{bmatrix} -(2\lambda-6)+\lambda-6 & -2(2\lambda-6)+3\lambda-9 \\ (2\lambda-6)-2\lambda+8 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 3-\lambda \\ 2 & \lambda \end{bmatrix} = \pi_{12}$

(2) $\pi_{11}\pi_{21} = \begin{bmatrix} \lambda-3 & -\lambda-3 \\ -\lambda & 4-\lambda \end{bmatrix} \begin{bmatrix} -\lambda-3 & 5-3\lambda \\ 3-\lambda & \lambda+3 \end{bmatrix} =$

$= \begin{bmatrix} 0 & -4(2\lambda-6)+8\lambda-24 \\ 2(2\lambda-6)-4\lambda+12 & 2(2\lambda-6)-4\lambda+12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(3) $\pi_{12}\pi_{21} = \begin{bmatrix} -\lambda & 3-\lambda \\ 2 & \lambda \end{bmatrix} \begin{bmatrix} -\lambda-3 & 5-3\lambda \\ 3-\lambda & \lambda+3 \end{bmatrix} =$

$= \begin{bmatrix} 2(2\lambda-6)-3\lambda+9 & 2(2\lambda-6)-5\lambda+9 \\ -(2\lambda-6)+\lambda-6 & (2\lambda-6)-3\lambda+10 \end{bmatrix} = \begin{bmatrix} \lambda-3 & -\lambda-3 \\ -\lambda & 4-\lambda \end{bmatrix} = \pi_{11}.$

(4) $\pi_{21}\pi_{12} = \begin{bmatrix} -\lambda-3 & 5-3\lambda \\ 3-\lambda & \lambda+3 \end{bmatrix} \begin{bmatrix} -\lambda & 3-\lambda \\ 2 & \lambda \end{bmatrix} =$

$= \begin{bmatrix} (2\lambda-6)-3\lambda+10 & -2(2\lambda-6)+5\lambda-9 \\ (2\lambda-6)-\lambda+6 & 2(2\lambda-6)-3\lambda+9 \end{bmatrix} = \begin{bmatrix} 4-\lambda & \lambda+3 \\ \lambda & \lambda-3 \end{bmatrix} = \pi_{22}.$

(5) $\pi_{21}\pi_{11} = \begin{bmatrix} -\lambda-3 & 5-3\lambda \\ 3-\lambda & \lambda+3 \end{bmatrix} \begin{bmatrix} \lambda-3 & -\lambda-3 \\ -\lambda & 4-\lambda \end{bmatrix} =$

$= \begin{bmatrix} 2(2\lambda-6)-5\lambda+9 & 4(2\lambda-6)-11\lambda+29 \\ -2(2\lambda-6)+3\lambda-9 & \lambda+3 \end{bmatrix} = \begin{bmatrix} -\lambda-3 & 5-3\lambda \\ 3-\lambda & \lambda+3 \end{bmatrix} = \pi_{21}.$

We infer that the π_{ij}, $1 \leq i, j \leq 2$, satisfy all the matrix units identities.

Let $M = \begin{bmatrix} \lambda-3 & -\lambda & -3-\lambda & 4-\lambda \\ -\lambda-3 & 3-\lambda & 5-3\lambda & 3+\lambda \\ -\lambda & 2 & 3-\lambda & \lambda \\ 4-\lambda & \lambda & \lambda+3 & \lambda-3 \end{bmatrix} =$

$= \begin{bmatrix} i\sqrt{5}-2 & -i\sqrt{5}-1 & -i\sqrt{5}-4 & 3-i\sqrt{5} \\ -i\sqrt{5}-4 & 2-i\sqrt{5} & 2-3i\sqrt{5} & i\sqrt{5}+4 \\ -i\sqrt{5}-1 & 2 & 2-i\sqrt{5} & i\sqrt{5}+1 \\ 3-i\sqrt{5} & i\sqrt{5}+1 & i\sqrt{5}+4 & i\sqrt{5}-2 \end{bmatrix}$. It turns out that $\det(M) = 1$.

Since each column of M represents a π_{ij}, we infer that $Mat_{2\times 2}(R) = span_R\{\pi_{ij} : 1 \leq i \leq j\} = \bigoplus_{1\leq i,j\leq 2} \pi_{ij}R$. This shows that the R-linear map $\tau : Mat_{2\times 2}(R) \to Mat_{2\times 2}(R)$ with $\tau(\varepsilon_{ij}) = \pi_{ij}$ for all $1 \leq i, j \leq 2$ is an R-algebra isomorphism. If τ is inner, then $\pi_{22} = \gamma^{-1}\varepsilon_{22}\gamma$ for some invertible $\gamma \in End_R(V)$. Note that this implies that $\pi_{22}(V)$ is a cyclic R-module, but $\pi_{22}(V) = \widetilde{Q} \cong Q$ is not cyclic. This contradiction shows that τ is an automorphism of $End_R(V)$ that is **not** an inner automorphism. (The corresponding author would like to mention that Maple was used to verify the computations in this section.)

Remark 3.3. Let R be an integral domain with F the field of fractions of R. Let $\mathcal{C}(R)$ denote the ideal class group of R, i.e. the set of all invertible, fractional ideals of R, modulo the principal ideals. Isaacs [6, Theorem 13] showed that the group of outer automorphisms of $Mat_{n\times n}(R)$ is isomorphic to a subgroup of $\mathcal{C}(R)$ for all n. If R is a Dedekind domain, then that group is isomorphic to the n-torsion part of $\mathcal{C}(R)$ [6, Corollary 18]. Fundamental to this is the fact that for any $m \in Mat_{2\times 2}(F)$, there is some $0 \neq s \in R$ such that $sm \in Mat_{2\times 2}(R)$. The analog to this is no longer true if the free module V has infinite rank, which might make it difficult to extend these results to the infinite rank case. We will not pursue this in the present paper.

4. Local automorphism of $End_R(V)$

Recall that a map φ from a ring S to a ring T is called a Jordan homomorphism if φ is additive and $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$ for all $a, b \in S$.

Herstein has shown in [5, Lemma 2] that for $char(T) \neq 2$ and any $a, b \in S$, one has $\varphi(bab) = \varphi(b)\varphi(a)\varphi(b)$ for any Jordan homomorphism φ.

Suppose $\varphi : S \to T$ is an additive map such that $\varphi(x^2) = (\varphi(x))^2$ for all $x \in S$. Then $\varphi(a)^2 + \varphi(a)\varphi(b) + \varphi(b)\varphi(a) + \varphi(b)^2 =$

$= (\varphi(a+b))^2 = \varphi((a+b)^2) = \varphi(a^2) + \varphi(ab) + \varphi(ba) + \varphi(b^2)$ and it follows that φ is a Jordan homomorphism.

Remark 4.1. Let φ be a 2-local automorphism of the algebra A. Then $\varphi(a^n) = (\varphi(a))^n$ for all $a \in A$ and $n \in \mathbb{N}$. Moreover, φ is a Jordan homomorphism.

Proof. Let $a \in A$. Then there exists some automorphism θ of A such that $\varphi(a) = \theta(a)$ and $\varphi(a^2) = \theta(a^2)$. Then $\varphi(a)^2 = \theta(a)\theta(a) = \theta(a^2) = \varphi(a^2)$. The rest follows by an easy induction over n. $\square$

Let S, T be rings and $\eta : S \to T$ an additive map. We call η zero product preserving, if whenever $a, b \in S$ with $ab = 0$, then $\eta(a)\eta(b) = 0$.

Hadwin and Li stated the following lemma for Banach algebras, which actually holds for all for any rings with identity and characteristic different from 2:

Lemma 4.2. *[4, Lemma 3.4] Let S, T be rings, $char(T) \neq 2$, with identity element 1 and let $\alpha : S \to T$ be a zero product preserving Jordan homomorphism with $\eta(1) = 1$. Then $\alpha(pm) = \alpha(p)\alpha(m)$ for all $m \in S$ and all idempotent elements $p \in S$.*

We need this property on both sides, and, for the convenience of the reader, give a proof of:

Corollary 6. *Let S, T be rings, $char(T) \neq 2$, with identity element 1 and let $\alpha : S \to T$ be a zero product preserving Jordan homomorphism with $\eta(1) = 1$. Then $\alpha(pmq) = \alpha(p)\alpha(m)\alpha(q)$ for all $m \in S$ and all idempotent elements $p, q \in S$.*

Proof. Let $a, b \in S$ such that $ab = 0$. Then

(1) $\alpha(ba) = \alpha(b)\alpha(a)$:

Note that $\alpha(ba) = \alpha(ab + ba) = \alpha(a)\alpha(b) + \alpha(b)\alpha(a) = \alpha(b)\alpha(a)$ since α is a zero product preserving Jordan homomorphism.

(2) Let $m \in S$ and let q be an idempotent element of S. Then $\alpha((1-q)mq) = \alpha((1-q)m)\alpha(q)$:

Note that $(mq)(1-q) = 0$ and apply (1).

(3) $\alpha(q)\alpha(qm) = \alpha(qm)$:

We have $\alpha(1) = 1$ and $(1-q)(qm) = 0$ and thus $\alpha(1-q)\alpha(qm) = 0$. We infer $\alpha(qm) - \alpha(q)\alpha(qm) = 0$ and (3) follows.

Now we have: $\alpha(mq) - \alpha(qmq) = \alpha((1-q)(mq)) \overset{(2)}{=} \alpha((1-q)m)\alpha(q) =$
$= (\alpha(m) - \alpha(qm))\alpha(q) \overset{(3)}{=} (\alpha(m) - \alpha(q)\alpha(qm))\alpha(q) =$
$= \alpha(m)\alpha(q) - \alpha(q)\alpha(qm)\alpha(q) = \alpha(m)\alpha(q) - \alpha(q(qm)q) =$
$= \alpha(m)\alpha(q) - \alpha(qmq)$ by Herstein's result. We infer $\alpha(mq) = \alpha(m)\alpha(q)$. Now we have $\alpha(pmq) = \alpha(pm)\alpha(q) = \alpha(p)\alpha(m)\alpha(q)$ by the previous result. □

We now return to $End_R(V)$ and apply Lemma 2.2 to get:

Theorem 4.3. *Let $1 \in R$ be a commutative ring, V a free R-module and $id_V \in A$ a subalgebra of $End_R(V)$ such that $Fin_R(V) \subseteq A$. Let $\alpha : A \to A$ be a map such that:*
(1) α is a surjective 2-local automorphism of A.
(2) R has characteristic different from 2 and only the trivial idempotents.
(3) Whenever P is an indecomposable projective R-module, then $P \cong R_R$.
Then α is induced by an inner automorphism of $End_R(V)$.

Recall that (2) holds if R is a Bezout domain.
Similar to the previous section, we get:

Corollary 7. *Let R be a Bezout domain of characteristic different from 2, V any free R-module and $\alpha : End_R(V) \to End_R(V)$ a surjective 2-local algebra automorphism. Then α is an inner automorphism.*

5. Derivations of $End_R(V)$

Let $1 \in R$ be a commutative ring and V some R-module. Let $A = End_R(V)$ and $V^* = Hom_R(V, R)$. For $u \in V$ and $f \in V^*$ define a map $\tau_{u,f} : V \to V$ by $\tau_{u,f}(x) = f(x)u$ for all $x \in V$. It is easy to check that $\tau_{u,f} \in A$ for all $u \in V, f \in V^*$. Moreover, for any $\varphi \in A$ we have $\varphi \circ \tau_{u,f} = \tau_{\varphi(u),f}$ and $\tau_{u,f} \circ \varphi = \tau_{u,f\circ\varphi}$.

We adopt the proof of [9, Theorem 1.1] to show:

Theorem 5.1. *Let $1 \in R$ be a commutative ring and V some R-module such that V contains a free direct summand. Then any derivation $\delta : End_R(V) \to End_R(V)$ is inner, i.e. there exists some $\mu \in End_R(V)$ such that $\delta(\varphi) = \mu \circ \varphi - \varphi \circ \mu$ for all $\varphi \in End_R(V)$.*

Proof. Let $y_0 \in V$ such that $y_0 R$ is a direct summand of V. Then there exists $f_0 \in V^*$ such that $f_0(y_0) = 1$. Define $\mu : V \to V$ by $\mu(\nu) = (\delta(\tau_{v,f_0}))(y_0)$. Direct verification shows that $\mu \in End_R(V)$. Let $\varphi \in End_R(V)$. We compute:

$\delta(\tau_{\varphi(v),f_0}) = \delta(\varphi \circ \tau_{v,f_0}) = \delta(\varphi) \circ \tau_{v,f_0} + \varphi \circ \delta(\tau_{v,f_0})$. Apply these maps to y_0 and we get:

$(\delta(\tau_{\varphi(v),f_0}))(y_0) = (\delta(\varphi) \circ \tau_{v,f_0})(y_0) + (\varphi \circ \delta(\tau_{v,f_0}))(y_0)$ and we infer $\mu(\varphi(v)) = \delta(\varphi)(f_0(y_0)v) + \varphi(\mu(v))$ for all $v \in V$. Since $f_0(y_0) = 1$, we have $(\mu \circ \varphi)(v) = (\delta(\varphi))(v) + (\varphi \circ \mu)(v)$ for all $v \in V$ and thus $\delta(\varphi) = \mu \circ \varphi - \varphi \circ \mu$ for all $\varphi \in End_R(V)$, i.e. δ is an inner derivation. □

The previous result shows that $End_R(V)$ has only inner derivations for "most" R-modules V. Now we adopt the clever argument in the proof of Theorem 3 in [8].

Let $1 \in R$ be a commutative ring and V a free R-module with basis $B = \{b_i : i \in \mathbb{N}\}$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be some map. Define $\widetilde{\pi} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $\widetilde{\pi}(i;j) = (\pi(i), \pi(j))$ for all $i, j \in \mathbb{N}$. We define $\widehat{\pi} \in End_R(V)$ by $\widehat{\pi}(b_j) = b_{\pi(j)}$ for all $j \in \mathbb{N}$. Let $\alpha \in End_R(V)$. Then $\alpha(b_j) = \sum\limits_i b_i \alpha(i,j)$ for some $\alpha(i,j) \in R$. An easy computation shows that if $\alpha \in C_{End_R(V)}(\widehat{\pi})$, then $\alpha(i,j) = \alpha(\pi(i), \pi(j))$ for all $i, j \in \mathbb{N}$ and $\alpha(i,j) = 0$ for all $i \notin im(\pi), j \in im(\pi)$. Now choose a subset $\Delta = \{d_i : i \in \mathbb{N}\} \subseteq R$ such that $d_i - d_j$ is not a zero divisor in R for all $i \neq j \in \mathbb{N}$. Define $\widehat{\Delta} \in End_R(V)$ by $\widehat{\Delta}(b_j) = d_j b_j$. Note that $C_{End_R(V)}(\widehat{\Delta}) = \{\alpha \in End_R(V) : \alpha(i,j) = 0 \text{ for all } i \neq j \in \mathbb{N}\}$.

Now let $\sigma : \mathbb{N} \to \mathbb{N}$ be the shift map, i.e. $\sigma(i) = i+1$ for all $i \in \mathbb{N}$. Then $C_{End_R(V)}(\widehat{\sigma}) = \{\alpha \in End_R(V) : \alpha(1,j) = 0 \text{ for all } j \geq 2 \text{ and } \alpha(i,1) = \alpha(i+k, k+1) \text{ for all } k \in \mathbb{N}\}$, i.e. if $\alpha \in C_{End_R(V)}(\widehat{\Delta}) \cap C_{End_R(V)}(\widehat{\sigma})$, then there exists some $\lambda \in R$ with $\alpha = \lambda \cdot id_V$. Note that for any $\alpha \in C_{End_R(V)}(\widehat{\sigma})$ we have $\alpha(1,1) = \alpha(i,i)$ for all $i \in \mathbb{N}$.

Now let $\delta : End_R(V) \to End_R(V)$ be a 2-local derivation. This means that for all $\beta, \gamma \in End_R(V)$ there exists some derivation $\delta_{\beta,\gamma}$ such that $\delta(\beta) = \delta_{\beta,\gamma}(\beta)$ and $\delta(\gamma) = \delta_{\beta,\gamma}(\gamma)$.

Note that $\delta - \delta_{\widehat{\sigma},\widehat{\Delta}}$ is a 2-local derivation and we may assume that $\delta(\widehat{\Delta}) = 0 = \delta(\widehat{\sigma})$. By Theorem 5.1, we have that each derivation of $End_R(V)$ is inner. Thus there is some $\beta_\tau \in End_R(V)$ such that

$\delta(\tau) = \delta_{\widehat{\Delta},\tau}(\tau) = \tau \circ \beta_\tau - \beta_\tau \circ \tau$ and $0 = \delta(\widehat{\Delta}) = \widehat{\Delta} \circ \beta_\tau - \beta_\tau \circ \widehat{\Delta}$ and thus $\beta_\tau \in C_{End_R(V)}(\widehat{\Delta})$.

Moreover, there is some $\gamma_\tau \in End_R(V)$ such that

$\delta(\tau) = \delta_{\widehat{\sigma},\tau}(\tau) = \tau \circ \gamma_\tau - \gamma_\tau \circ \tau$ and $0 = \delta(\widehat{\sigma}) = \widehat{\sigma} \circ \beta_\tau - \beta_\tau \circ \widehat{\sigma}$ and thus $\gamma_\tau \in C_{End_R(V)}(\widehat{\sigma})$.

We infer $\delta(\tau) = \tau \circ \beta_\tau - \beta_\tau \circ \tau = \tau \circ \gamma_\tau - \gamma_\tau \circ \tau$.

Recall that $\varepsilon_{ij} \in End_R(V)$ with $\varepsilon_{ij}(b_k) = \begin{cases} b_i \text{ for } j = k \\ 0 \text{ otherwise} \end{cases}$. Note that $\delta(\varepsilon_{ij}) = \varepsilon_{ij} \circ \beta_{\varepsilon_{ij}} - \beta_{\varepsilon_{ij}} \circ \varepsilon_{ij} = \varepsilon_{ij} \circ \gamma_{\varepsilon_{ij}} - \gamma_{\varepsilon_{ij}} \circ \varepsilon_{ij}$ where $\beta_{\varepsilon_{ij}}(b_k) = d_k b_k$ for some $d_k \in R$.

Thus $\delta(\varepsilon_{ij})(b_k) = (\varepsilon_{ij} \circ \beta_{\varepsilon_{ij}} - \beta_{\varepsilon_{ij}} \circ \varepsilon_{ij})(b_k) = (\varepsilon_{ij} \circ \beta_{\varepsilon_{ij}})(b_k) - (\beta_{\varepsilon_{ij}} \circ \varepsilon_{ij})(b_k) =$
$= d_k(\varepsilon_{ij}(b_k)) - \beta_{\varepsilon_{ij}}(\varepsilon_{ij}(b_k)) = \begin{cases} d_j b_i - d_i b_i \text{ if } k = j \\ 0 \text{ otherwise} \end{cases} = (d_j - d_i)(\varepsilon_{ij}(b_k))$ and we have:

$\delta(\varepsilon_{ij}) = (d_j - d_i)\varepsilon_{ij}$.

This implies $\delta(\varepsilon_{ij}) = \varepsilon_i \circ \delta(\varepsilon_{ij}) \circ \varepsilon_j = \varepsilon_i \circ (\varepsilon_{ij} \circ \gamma_{\varepsilon_{ij}} - \gamma_{\varepsilon_{ij}} \circ \varepsilon_{ij}) \circ \varepsilon_j = \varepsilon_{ij} \circ \gamma_{\varepsilon_{ij}} \circ \varepsilon_j - \varepsilon_i \circ \gamma_{\varepsilon_{ij}} \circ \varepsilon_{ij}$. This shows that $\delta(\varepsilon_{ij})(b_k) = 0$ for all $k \neq j$ and $\delta(\varepsilon_{ij})(b_j) = (\varepsilon_{ij} \circ \gamma_{\varepsilon_{ij}})(b_j) - (\varepsilon_i \circ \gamma_{\varepsilon_{ij}})(b_i)$.

Now let $\gamma_{\varepsilon_{ij}}(b_j) = \sum\limits_k b_k g_{kj}$ for some $g_{kj} \in R$. Then $\delta(\varepsilon_{ij})(b_j) = \varepsilon_{ij}\left(\sum\limits_k b_k g_{kj}\right) - \varepsilon_i\left(\sum\limits_k b_k g_{ki}\right) = b_i g_{jj} - b_i g_{ii} = 0$ since $\gamma_{\varepsilon_{ij}} \in C_{End_R(V)}(\widehat{\sigma})$.

This shows that $\delta(\varepsilon_{ij}) = 0$ for all $i, j \in \mathbb{N}$.

Now consider the derivation $\delta_{\tau,\varepsilon_{ij}}$. We compute $\delta_{\tau,\varepsilon_{ij}}(\varepsilon_{ij} \circ \tau \circ \varepsilon_{ij}) = \delta_{\tau,\varepsilon_{ij}}(\varepsilon_{ij}) \circ \tau \circ \varepsilon_{ij} + \varepsilon_{ij} \circ \delta_{\tau,\varepsilon_{ij}}(\tau \circ \varepsilon_{ij}) =$
$= \delta_{\tau,\varepsilon_{ij}}(\varepsilon_{ij}) \circ \tau \circ \varepsilon_{ij} + \varepsilon_{ij} \circ (\delta_{\tau,\varepsilon_{ij}}(\tau) \circ \varepsilon_{ij} + \tau \circ \delta_{\tau,\varepsilon_{ij}}(\varepsilon_{ij})) = \varepsilon_{ij} \circ \delta_{\tau,\varepsilon_{ij}}(\tau) \circ \varepsilon_{ij} = \varepsilon_{ij} \circ \delta(\tau) \circ \varepsilon_{ij}$.

Note that $\varepsilon_{ij} \circ \tau \circ \varepsilon_{ij} = t\varepsilon_{ij}$ for some $t = \tau(j, i) \in R$.

Thus $\varepsilon_{ij} \circ \delta(\tau) \circ \varepsilon_{ij} = \delta_{\tau,\varepsilon_{ij}}(\varepsilon_{ij} \circ \tau \circ \varepsilon_{ij}) = \delta_{\tau,\varepsilon_{ij}}(t\varepsilon_{ij}) = t\, \delta_{\tau,\varepsilon_{ij}}(\varepsilon_{ij}) = t\delta(\varepsilon_{ij}) = 0$ and we have that $\varepsilon_{ij} \circ \delta(\tau) \circ \varepsilon_{ij} = 0$ for all $i, j \in \mathbb{N}$ and thus $\delta(\tau) = 0$ for all $\tau \in End_R(V)$.

We have shown:

Theorem 5.2. *Let $1 \in R$ be a commutative ring and V a free R-module of finite or countably infinite rank. Assume that R contains an infinite subset S such that $s - t$ is not a zero divisor for all $s \neq t \in S$. Then any 2-local derivation of $End_R(V)$ is an inner derivation.*

Corollary 8. *Let R be an infinite integral domain and V a free R-module of finite or countably infinite rank. Then any 2-local derivation of $End_R(V)$ is an inner derivation.*

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