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Analysis of splitting methods for solving a partial integro-differential Fokker–Planck equation[☆]

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ABSTRACT

A splitting implicit-explicit (SIMEX) scheme for solving a partial integro-differential Fokker–Planck equation related to a jump-diffusion process is investigated. This scheme combines the Chang–Cooper method for spatial discretization with the Strang–Marchuk splitting and first- and second-order time discretization methods. It is proved that the SIMEX scheme is second-order accurate, positive preserving, and conservative. Results of numerical experiments that validate the theoretical results are presented.

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1. Introduction

The Fokker–Planck (FP) equation governs the time evolution of the probability density function (PDF) of stochastic processes and plays a fundamental role in many problems involving random quantities [14,23,24,30]. The FP equation has been first applied to problems with randomness given by Brownian motion; in this case, the derivation of the FP equation, some methods of solution and its application to diffusion models can be found in [10,14,28,30].

In the last decade, one can see a growing interest in stochastic processes with jumps. This class of problems includes Lévy processes, whose increasing popularity stems, e.g., from the need of modeling the market behavior beyond the Black–Scholes framework [8].

In this work, we consider a jump-diffusion Markov process $X_t \in \mathbb{R}^d$, for $t \in I = [t_0, t_f]$, that solves the following stochastic initial value problem

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t + dP_t, \\ X_{t_0} = X_0, \end{cases} \tag{1}$$

where $X_0 \in \mathbb{R}^d$ is a given initial random data. This stochastic differential equation (SDE) relates the infinitesimal increments of the stochastic process X_t to both deterministic and random increments, given by the multidimensional Wiener process

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$W_t \in \mathbb{R}^m$ and the compound Poisson process $P_t \in \mathbb{R}^d$. We denote with $\lambda \in \mathbb{R}^+$ the rate of the time events of the compound Poisson process and with $g(y)$ the PDF of the size of its jumps. The density $g(y)$ is nonnegative and normalized, $\int_{\mathbb{R}^d} g(y)dy = 1$. The deterministic functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ represent the drift and the diffusion coefficients, respectively. We assume that the matrix σ is full-rank. The solvability of (1) follows under growth and regularity conditions on b and σ; see, e.g., [3,21].

A powerful tool to examine the stochastic problem (1) is the investigation of the time dependent PDF $f(x, t)$ of X_t, because it characterizes the statistics of the process over the time interval I. The PDF of X_t is defined in $\mathbb{R}^d \times I$ and it is governed by the following partial integro-differential equation (PIDE) of FP type

$$\partial_t f(x,t) = \mathcal{L}f(x,t) + \mathcal{I}f(x,t), \tag{2}$$

where the two linear operators $\mathcal{L}$ and $\mathcal{I}$ are defined as follows

$$\begin{aligned}\mathcal{L}f(x,t) &:= -\sum_{i=1}^{d} \partial_{x_i}(b_i(x)f(x,t)) + \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\left(a_{ij}(x)f(x,t)\right)\\ \mathcal{I}f(x,t) &:= \lambda\int_{\mathbb{R}^d} f(y,t)g(x-y)dy - \lambda f(x,t),\end{aligned} \tag{3}$$

where $a_{ij}(x)$ are the elements of a matrix defined as $a_{ij}(x) := \frac{1}{2}\sum_{k=1}^{d}\sigma_{ik}(x)\sigma_{jk}(x)$. Since the diffusion coefficient σ is full rank, the matrix a is positive definite. For more details about the link between PIDEs of FP type and Markov processes, see [10,14,32]. From a probabilistic point of view, integro-differential operators are closely linked to the concept of group generator of a stochastic process; for a detailed discussion about the generator of a Lévy process, see [3].

Classical solutions of initial-boundary problems containing PIDEs have been examined in the context of Hölder spaces and uniformly parabolic operators [15–17]. More in general, existence of solutions is considered in the framework of viscosity solutions [11]. Numerical schemes for viscosity solutions of PIDEs are discussed, e.g., in [4] and references therein.

In financial applications, PIDEs naturally arise when option prices in jump diffusion models have to be computed [1,5,8,9,27]. An overview on several numerical schemes for PIDEs arising in the pricing problem of financial derivatives can be found in [13]. However, a rigorous numerical analysis of these schemes is often not available.

In this work, we consider an initial-boundary value problem governed by (2)–(3) with the aim of computing the PDF of X_t. Since $f(x, t)$ is the PDF of a stochastic process, it must be positive and its integral over the evolution domain of the process must be equal to 1. These two properties hold for the continuous problem endowed with suitable initial and boundary conditions [14,32]; we require that these two structural properties have to be valid also for the discretized FP equation.

With the aim of solving the FP equation related to a ionized plasma, Chang and Cooper [7] proposed a conservative finite difference scheme that turns out to be one of the most appropriate schemes for discretizing the FP spatial operator. A complete numerical analysis of this scheme has been carried out in [26]. Conservative numerical schemes for our FP problem in case of a jump-diffusion process have been less investigated.

The purpose of our work is to provide a numerical analysis of a conservative scheme that solves the FP problem for a jump-diffusion process, with appropriate initial and boundary conditions. We consider the framework of the method of lines (MOL) [6,31]. The differential operator $\mathcal{L}$ in (3) is discretized with the Chang–Cooper (CC) scheme, and the integral operator $\mathcal{I}$ in (3) is approximated by a quadrature formula. This leads to a large system of ordinary differential equations, that we approximate with the the Strang–Marchuk (SM) splitting method [18,19,25,33]. The SM method decomposes the original problem in a sequence of different sub-problems with simpler structure, which are separately solved and linked to each other through initial conditions and final solutions. A splitting method allows to solve each sub-problem implicitly or explicitly, depending on the nature of the sub-problem. After performing the SM splitting, we carry out the time integration with a first- and a second-order time-differencing method. Our discretization procedure with the two different time-discretization schemes leads to the SIMEX1 and SIMEX2 schemes, respectively. For clarity, our discretization workflow is summarized in Fig. 1. We remark that splitting methods and finite differences are frequently used by practitioners [13]. However, less attention has been put on positivity and conservation properties, and the numerical analysis has focused mainly on time-approximation properties. In this paper, we prove that the combination of the CC scheme with the SM method results in accurate discretization schemes that guarantee conservativeness of the total probability and non-negativity of the PDF solutions.

This paper is organized as follows. The next section defines our FP problem, including a source term for analysis purposes. Correspondingly, we investigate existence and uniqueness of solutions to this problem in the case of unbounded and bounded domains. In Section 3, we illustrate the CC scheme and the SM splitting method, and their combination for constructing our SIMEX schemes. Section 4 is devoted to the convergence analysis of our SIMEX1 and SIMEX2 schemes. For these schemes, we prove stability in time and second-order accuracy in space. Further, we prove first- and second-order accuracy in time for the SIMEX1 and SIMEX2 schemes, respectively. In Section 5, we prove that our numerical schemes guarantee non-negativity and conservation of total probability of the PDF solution. Section 6 presents results of numerical experiments that validate our theoretical results. The Appendix illustrates a possible choice for the truncation of the domain of definition of the Fokker–Planck problem. A section of conclusion completes this work.

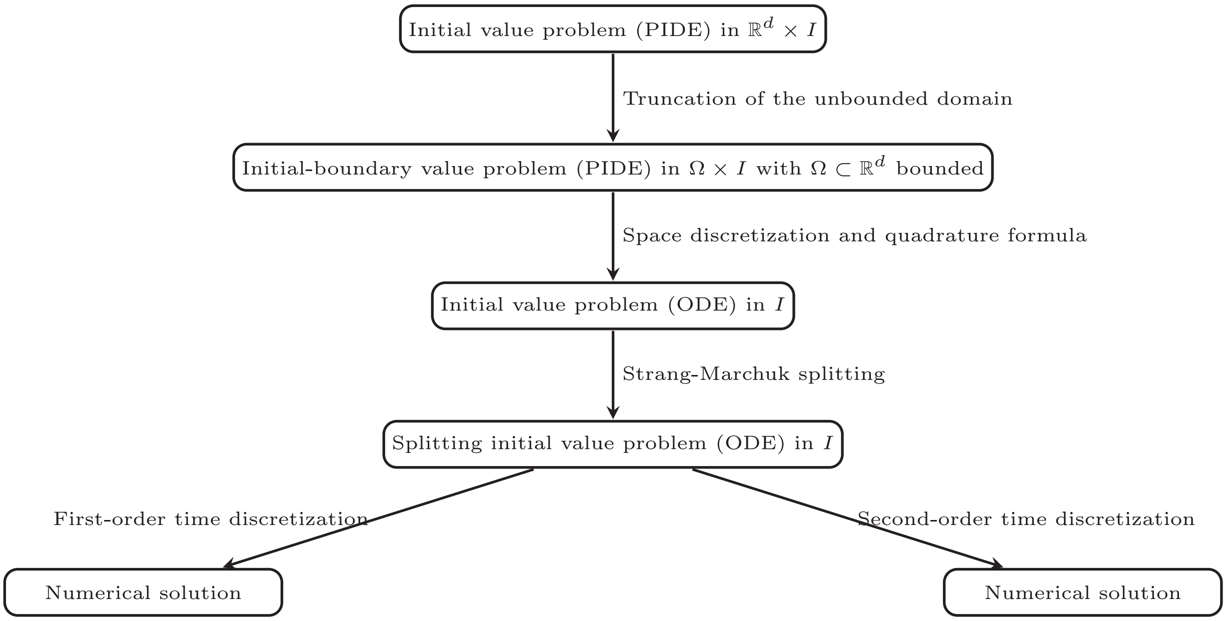

Fig. 1. Discretization workflow.

2. The PIDE Fokker–Planck problem

We consider the following initial value problem

$$\begin{cases} \partial_t f(x,t) = \mathcal{L} f(x,t) + \mathcal{I} f(x,t) + \psi(x,t) \\ f(x,t_0) = f_0(x), \end{cases} \tag{4}$$

where $x \in \mathbb{R}^d$ and $t \in I$. The source term ψ is included for the purpose of numerical error analysis. Notice that the space variable x has the same range of the considered stochastic process X_t in (1).

The existence and uniqueness of classical solutions to (4) is ensured by the next theorem, whose proof can be found in [15]. Let $\alpha \in (0, 1)$. The space $\mathcal{C}^{\alpha;\frac{\alpha}{2}}(\mathbb{R}^d \times I)$ refers to the functions that are Hölder continuous on $\mathbb{R}^d \times I$, with Hölder exponents α and $\alpha/2$ with respect to the space and time variables, respectively. The space $\mathcal{C}^{2,\alpha}(\mathbb{R}^d)$ refers to the $\mathcal{C}^2$ functions that are Hölder continuous on $\mathbb{R}^d$, with Hölder exponent α.

Theorem 1. *Let the coefficients a and b of the differential operator $\mathcal{L}$ in (3) belong to the space of Hölder continuous function $\mathcal{C}^{\alpha}(\mathbb{R}^d)$. Let the PDF g in the integral operator $\mathcal{I}$ in (3) satisfy the following condition*

$$\int_{\mathbb{R}^d} \frac{\|y\|^2}{(1+\|y\|)} g(y) dy = K_0 < \infty.$$

Then for any $f_0 \in \mathcal{C}^{2,\alpha}(\mathbb{R}^d)$ and for any source term $\psi \in \mathcal{C}^{\alpha;\frac{\alpha}{2}}(\mathbb{R}^d \times I)$, the problem (4) admits a unique solution $f \in \mathcal{C}^{2,\alpha;1,\frac{\alpha}{2}}(\mathbb{R}^d \times I)$ satisfying

$$\|f\|_{\mathcal{C}^{2,\alpha;1,\frac{\alpha}{2}}(\mathbb{R}^d \times I)} \le K\Big(\|f_0\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^d)} + \|\psi\|_{\mathcal{C}^{\alpha;\frac{\alpha}{2}}(\mathbb{R}^d \times I)} \Big),$$

where the constant K does not depend on ψ and f_0.

Our main focus is (4) with $\psi = 0$, whose solution is the PDF of the process X_t governed by (1). Therefore, in the case of zero source term ψ, f satisfies the following conditions

$$\begin{aligned} &(1)\ f(x,t) \ge 0 \text{ for each } (x,t) \in \mathbb{R}^d \times I, \\ &(2) \int_{\mathbb{R}^d} f(x,t) dx = 1 \text{ for each time } t \in I, \end{aligned} \tag{5}$$

provided that the initial data f_0 in (4) is the PDF of the initial data X_0. The positivity is ensured by standard maximum principle arguments [15]. We discuss a setting such that the total probability remains constant along time evolution. The differential operator $\mathcal{L}$ in (3) can be written in divergence form as follows

$$\mathcal{L} f(x,t) = \nabla \cdot \mathcal{F}(x,t), \tag{6}$$

where $\mathcal{F}(x,t)$ is defined as follows

$$\begin{aligned}\mathcal{F}(x,t) &:= B(x)f(x,t) + C(x)\nabla f(x,t) \\ B_i(x) &:= \sum_{i=1}^{d} \partial_{x_j} a_{ij}(x) - b_i(x) \\ C_{ij}(x) &:= a_{ij}(x).\end{aligned} \tag{7}$$

Notice that $\mathcal{F}(x,t)$ represents the d-dimensional flux in case of a stochastic process without jumps [32].

Remark 1. Let us consider the FP problem as in (4) with $\psi = 0$. Let us assume the following conditions.

1. $$\lim_{\|x\|\to\infty} \mathcal{F}(x,t) = 0 \qquad \forall t \in I.$$
2. The density g satisfies
$$\int_{\mathbb{R}^d} g(x)dx = 1.$$

Then

$$\int_{\mathbb{R}^d} f(x,t) = \int_{\mathbb{R}^d} f_0(x)dx \qquad \forall t \in I.$$

We first note that

$$\int_{\mathbb{R}^d} \mathcal{I}f(x,t)dx = 0.$$

Applying Fubini's theorem, we have that

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{I}f(x,t)dx &= \lambda \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(y,t)g(x-y)dy - f(x,t)\right]dx \\ &= \lambda\left[\int_{\mathbb{R}^d} f(y,t)\left(\int_{\mathbb{R}^d} g(x-y)dx\right)dy - \int_{\mathbb{R}^d} f(x,t)dx\right] = 0.\end{aligned}$$

Hence, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(x,t)dx = \int_{\mathbb{R}^d} \partial_t f(x,t)dx = \int_{\mathbb{R}^d} [\nabla \cdot \mathcal{F}(x,t) + \mathcal{I}f(x,t)]dx = 0,$$

that proves conservation of the total probability.

In the absence of barriers, the domain of the stochastic process X_t in (1) is unbounded. However, in the presence of barriers and also for computational purposes, a bounded domain Ω is considered. In the latter case, one assumes that within the time horizon under consideration, the process is essentially localized in a bounded region that could be estimated by Monte Carlo simulation. Alternatively, one could use the methodology proposed in the Appendix of this paper.

On the other hand, since we address the issue of conservation properties of our numerical schemes, we consider a bounded domain with reflecting boundary conditions (zero flux) such that conservation of total probability is required already at the continuous level.

We focus on the following FP problem

$$\begin{cases} \partial_t f(x,t) = \mathcal{L}f(x,t) + \mathcal{I}f(x,t) + \psi(x,t) & \text{for } (x,t) \in \Omega \times I \\ f(x,0) = f_0(x) \text{ for } x \in \Omega \\ \mathcal{F}(x,t) = 0 \text{ for } (x,t) \in \partial\Omega \times I, \end{cases} \tag{8}$$

where $\Omega := (r, s)$, f_0 is the PDF of X_0 in (1) and $\mathcal{F}$ is defined in (7). Notice that in the numerical validation, we use a sufficiently large domain such that (8) represents a valid approximation to (4).

Theorem 1 can be extended to the initial-boundary FP problem (8) by requiring further assumptions on the boundary $\partial\Omega$. A complete proof of the following theorem can be found in [15].

Theorem 2. *Let $\alpha \in (0, 1)$ and let the coefficients of the differential operator $\mathcal{L}$ in (3) belong to the space of Hölder continuous function $\mathcal{C}^{\alpha}(\Omega \times I)$. Let the PDF g in the integral operator $\mathcal{I}$ in (3) satisfy the following condition*

$$\int_{\mathbb{R}^d} \frac{\|y\|^2}{(1+\|y\|)} g(y)dy = K_0 < \infty.$$

Assume that the boundary $\partial\Omega$ belongs to the class $\mathcal{C}^{2,\alpha}$.

Then for any $f_0 \in \mathcal{C}^{2,\alpha}(\Omega)$, *compatible with the boundary conditions, and for any source term* $\psi \in \mathcal{C}^{\alpha;\frac{\alpha}{2}}(\Omega \times I)$, *the problem* (8) *has a unique solution* $f \in \mathcal{C}^{2,\alpha;1,\frac{\alpha}{2}}(\Omega \times I)$ *satisfying*

$$\|f\|_{\mathcal{C}^{2,\alpha;1,\frac{\alpha}{2}}(\Omega\times I)} \le K\Big(\|f_0\|_{\mathcal{C}^{2,\alpha}(\Omega)} + \|\psi\|_{\mathcal{C}^{\alpha;\frac{\alpha}{2}}(\Omega\times I)}\Big),$$

where the constant K does not depend on ψ *and* f_0.

3. Discretization scheme of the PIDE

In this section, we discuss the implementation of our SIMEX schemes.

Consider the domain $\Omega = (r, s)$ and the time interval $I = [t_0, t_f]$. We set the mesh sizes h and δt as follows

$$h := \frac{s-r}{N-1} \text{ and } \delta t := \frac{t_f - t_0}{M}, \text{ with } N, M \in \mathbb{N}.$$

We consider uniform meshes in space and time. We have

$$\begin{aligned} \Omega_h &:= \{x_j = r + (j-1)h,\ j = 1, \dots, N\} \subset \bar{\Omega} \\ I_{\delta t} &:= \{t_n = t_0 + n\delta t,\ n = 0, \dots, M\} \subset I. \end{aligned} \tag{9}$$

First, we carry out the spatial discretization of the integro-differential terms of $\mathcal{L}$ and $\mathcal{I}$ defined in (3), that together lead to a large system of ODEs. Exploiting the divergence form (6) of the differential operator $\mathcal{L}$, we discretize the spatial derivative of $\mathcal{F}$ defined by (7) using the CC scheme. This is a cell-centered finite-volume scheme with a special technique to determine the fluxes at the cell boundaries, $x_{j\pm\frac{1}{2}}$, $j = 1, \dots, N$, and the unknown variables are considered on the grid points x_j, $j = 1, \dots, N$. In the following, $\Phi_j(t)$ and $\Phi_{j\pm\frac{1}{2}}(t)$ denote the time-continuous restrictions of a generic function $\Phi(x, t)$ to the grid points x_j and $x_{j\pm\frac{1}{2}} := \frac{x_j + x_{j\pm1}}{2}$, respectively.

We have the following discretization formula

$$\partial_x \mathcal{F}(x_j, t) \approx \frac{\mathcal{F}_{j+\frac{1}{2}}(t) - \mathcal{F}_{j-\frac{1}{2}}(t)}{h},$$

where

$$\mathcal{F}_{j+\frac{1}{2}}(t) = \Big((1-\delta_j)B_{j+\frac{1}{2}} + \frac{1}{h}C_{j+\frac{1}{2}}\Big)f_{j+1}(t) - \Big(\frac{1}{h}C_{j+\frac{1}{2}} - \delta_j B_{j+\frac{1}{2}}\Big)f_j(t).$$

The parameter $\delta_j \in [0, 1]$ is defined as follows

$$\delta_j := \frac{1}{hB_{j+\frac{1}{2}}/C_{j+\frac{1}{2}}} - \frac{1}{\exp\{hB_{j+\frac{1}{2}}/C_{j+\frac{1}{2}}\} - 1}. \tag{10}$$

The integral over Ω in the PIDE is approximated by the midpoint rule [12]. Hence the discretization of the integral operator takes the following form

$$\mathcal{I}f(x_j, t) \approx \lambda\Bigg(h\sum_{i=1}^{N} f(x_i, t)g(x_j - x_i) - f(x_j, t)\Bigg).$$

The space discretization above gives the following MOL approximation

$$\begin{cases} f_{\mathrm{SD}}'(t) = (\mathcal{A} + \mathcal{G})f_{\mathrm{SD}}(t) + \Psi(t) \\ f_{\mathrm{SD}}(t_0) = f_{\mathrm{SD}}(0), \end{cases} \tag{11}$$

where $\mathcal{A}$ and $\mathcal{G}$ are defined below. Notice that (11) is a system of ordinary differential equations parametrized by the space mesh size h in $\mathcal{A}$ and $\mathcal{G}$. In other words, $f_{\mathrm{SD}}(t) = \{f_{SD,1}(t), \dots, f_{SD,N}(t)\} \in \mathbb{R}^N$ can be viewed as a grid function, where each component describes the time evolution of f_{SD} on the correspondent grid point of Ω_h. The initial value $f_{\mathrm{SD}}(0)$ and the source term Ψ represent the restriction on the grid Ω_h of the sufficiently smooth initial data f_0 and of the source term ψ in (8), respectively.

The matrix $\mathcal{A}$ in (11) follows from the CC spatial discretization. By setting $w_j = \exp\{hB_{j+\frac{1}{2}}/C_{j+\frac{1}{2}}\}$, $\beta_j = C_{j+\frac{1}{2}}/h - \delta_j B_{j+\frac{1}{2}} = B_{j+\frac{1}{2}}/(w_j - 1)$, the tridiagonal matrix $\mathcal{A}$ is defined as follows

$$\mathcal{A}_{ij} = \begin{cases} \beta_{i-1}/h & j = i-1, \quad 2 \le i \le N, \\ -(\beta_{i-1}w_{i-1} + \beta_i)/h & j = i, \quad 1 \le i \le N, \qquad \beta_0 = \beta_N = 0, \\ \beta_i w_i/h & j = i+1, \quad 1 \le i \le N-1, \\ 0 & \text{otherwise}. \end{cases} \tag{12}$$

Remark 2. The conditions $\beta_0 = \beta_N = 0$ correspond to the zero-flux boundary conditions. Imposing $\mathcal{F}_{\frac{1}{2}}(t) = 0$ and $\mathcal{F}_{N+\frac{1}{2}}(t) = 0$, we obtain the following

$$\sum_{j=1}^{N} (\mathcal{A} f_{\mathrm{SD}}(t))_j = \frac{\mathcal{F}_{N+\frac{1}{2}}(t) - \mathcal{F}_{\frac{1}{2}}(t)}{h} = 0. \tag{13}$$

Remark 3. Note that $\mathcal{A}_{ii} \le 0$, while $\mathcal{A}_{ij} \ge 0$ for $i \ne j$.

The matrix $\mathcal{G}$ in (11) is defined as follows

$$\mathcal{G} := \lambda(G - I),$$

where I denotes the N-dimensional identity matrix and G is the matrix with normalized columns as follows

$$G_{ij} := \frac{h\,g(x_i - x_j)}{\sum_{k=1}^{N} h\,g(x_k - x_j)}. \tag{14}$$

The choice of the normalization in (14) is discussed in Section 5; see also [2].

The next step of our discretization procedure is to consider the semi-discrete system (11), for which we use an operator splitting method, exploiting the fact that the semidiscretized system (11) is naturally decoupled into two linear operators. The idea behind a splitting method is to divide the evolution problem into simpler sequential sub-problems that are separately solved with different methods. Setting δt also as the splitting time step, we apply the Strang–Marchuk (SM) splitting scheme [18,19,25,33]. In the following, we refer to the time-continuous solution of the splitting scheme as $f_{\mathrm{SP}}(t)$. The initial data is set as follows, $f_{\mathrm{SP}}(t_0) := f_{\mathrm{SD}}(t_0)$, where f_{SP} is the splitting solution and f_{SD} is the solution of the semi-discretized system (11) without splitting.

In each time interval $[t_n, t_{n+1}]$, given the splitting solution $f_{\mathrm{SP}}(t_n)$, the following subproblems, connected via the initial conditions, are solved

$$\begin{aligned}
&1. \begin{cases} \phi_1'(t) = \mathcal{A}\phi_1(t) \\ \phi_1(t_n) = f_{\mathrm{SP}}(t_n) \end{cases} && t \in [t_n, t_{n+\frac{1}{2}}] \\
&2. \begin{cases} \phi_2'(t) = \mathcal{G}\phi_2(t) + \Psi(t) \\ \phi_2(t_n) = \phi_1(t_{n+\frac{1}{2}}) \end{cases} && t \in [t_n, t_{n+1}] \\
&3. \begin{cases} \phi_3'(t) = \mathcal{A}\phi_3(t) \\ \phi_3(t_{n+\frac{1}{2}}) = \phi_2(t_{n+1}) \end{cases} && t \in [t_{n+\frac{1}{2}}, t_{n+1}]. \\
&4. \left\{ f_{\mathrm{SP}}(t_{n+1}) := \phi_3(t_{n+1}) \right.
\end{aligned} \tag{15}$$

This system of continuous-time equations is approximated by time discretization. The fully discrete numerical solution will be referred to as $\hat{f} = (f_j^n)$, $j = 1, \dots, N$, $n = 0, \dots, M$. We propose to use two different time discretization methods that together with the space discretization give the schemes named SIMEX1 and SIMEX2.

In SIMEX1, the solution of the first and third step of (15) is carried out with the implicit Euler method, while the second step is explicit, in order to avoid the drawback of inverting dense matrices. Given f^n at time t_n, the three initial value problems in (15) read as follows

$$\begin{aligned}
&1. \frac{f^{n+\frac{1}{2}} - f^n}{\delta t/2} = \mathcal{A} f^{n+\frac{1}{2}} \\
&2. \frac{f^{(n+\frac{1}{2})^*} - f^{n+\frac{1}{2}}}{\delta t} = \mathcal{G} f^{n+\frac{1}{2}} + \Psi(t_n) \\
&3. \frac{f^{n+1} - f^{(n+\frac{1}{2})^*}}{\delta t/2} = \mathcal{A} f^{n+1},
\end{aligned} \tag{16}$$

where the unknown are sequentially solved: $f^n \to f^{n+\frac{1}{2}} \to f^{(n+\frac{1}{2})^*} \to f^{n+1}$.

The time discretization of (15) in SIMEX2 is carried out with the predictor corrector method. Given f^n at time t_n, the discretization of the three initial value problems in (15) take the following form

$$\begin{aligned}
&1. \begin{cases} \bar{f}^{n+\frac{1}{2}} = f^n + \dfrac{\delta t}{2}\mathcal{A} f^n \\ f^{n+\frac{1}{2}} = f^n + \dfrac{\delta t}{4}\left[\mathcal{A} f^n + \mathcal{A}\bar{f}^{n+\frac{1}{2}}\right] \end{cases} \\
&2. \begin{cases} \bar{f}^{n+\frac{1}{2}^*} = f^{n+\frac{1}{2}} + \delta t[\mathcal{G} f^{n+\frac{1}{2}} + \Psi(t_n)] \\ f^{n+\frac{1}{2}^*} = f^{n+\frac{1}{2}} + \dfrac{\delta t}{2}\left[\mathcal{G} f^{n+\frac{1}{2}} + \Psi(t_n) + \mathcal{G}\bar{f}^{n+\frac{1}{2}^*} + \Psi(t_{n+1})\right] \end{cases}
\end{aligned} \tag{17}$$

3. $$\begin{cases} \bar{f}^{n+1} = f^{n+\frac{1}{2}^*} + \dfrac{\delta t}{2} \mathcal{A} f^{n+\frac{1}{2}^*} \\ f^{n+1} = f^{n+\frac{1}{2}^*} + \dfrac{\delta t}{4} \left[\mathcal{A} f^{n+\frac{1}{2}^*} + \mathcal{A} \bar{f}^{n+1} \right]. \end{cases}$$

4. Convergence analysis

In this section, we investigate stability and accuracy properties of the SIMEX1 and SIMEX2 schemes. After determining the order of convergence of the spatial discretization, we focus on the rate of convergence in time.

The discrete L^2-scalar product of two grid functions u and v on $\Omega_h \times I_{\delta t}$ is defined as follows [20,34]

$$(u, v)_{L^2_{h,\delta t}} := h\delta t \sum_{j=1}^{N} \sum_{n=0}^{M} u_j^n v_j^n,$$

with associated the norm $\|u\|_{L^2_{h,\delta t}} := \sqrt{(u, u)_{L^2_{h,\delta t}}}$. In a similar fashion, the discrete L^2_h inner product and norm are defined for functions w, z on the spatial grid Ω_h as follows

$$(w, z)_{L^2_h} := h \sum_{j=1}^{N} w_j z_j \text{ and } \|w\|_{L^2_h} := \sqrt{(w, w)_{L^2_h}}.$$

We aim at comparing the continuous PIDE solution f of (8) and the numerical solution $\hat{f}$, which is defined on the grid points of $\Omega_h \times I_{\delta t}$. We have the following inequality

$$\|f - \hat{f}\|_{L^2_{h,\delta t}} \le \|f_h - f_{SD}\|_{L^2_{h,\delta t}} + \|f_{SD} - f_{SP}\|_{L^2_{h,\delta t}} + \|f_{SP} - \hat{f}\|_{L^2_{h,\delta t}}, \tag{18}$$

where $f_h(t) = f(\bar{x}, t) \in \mathbb{R}^N$ is the PIDE solution restricted to $\bar{x} \in \Omega_h$, f_{SD} solves (11) and f_{SP} is obtained as in (15). In (18), the $L^2_{h,\delta t}$ norms are computed after evaluating the continuous functions at the points of the meshes Ω_h and $I_{\delta t}$ defined in (9).

In the following, we provide bounds for each of the three norms of (18). Specifically, we prove in Proposition 2 that $\|f_h - f_{SD}\|_{L^2_{h,\delta t}} = \mathcal{O}(h^2)$. In Proposition 3, we obtain $\|f_{SD} - f_{SP}\|_{L^2_{h,\delta t}} = \mathcal{O}(\delta t^2)$. Further, for the SIMEX1 scheme, we prove in Proposition 5 that $\|f_{SP} - \hat{f}\|_{L^2_{h,\delta t}} = \mathcal{O}(\delta t)$; and in Proposition 7, we obtain $\|f_{SP} - \hat{f}\|_{L^2_{h,\delta t}} = \mathcal{O}(\delta t^2)$ for the SIMEX2 scheme.

For our analysis, we assume that the PIDE solution f is 4 times continuously derivable with respect to the space variable and that the function B that defines the $\mathcal{F}$ in (7) is Lipschitz continuous, with Lipschitz constant equal to L. Moreover, we assume that the PDF g of the integral operator $\mathcal{I}$ in (3) is two times differentiable.

Next, we aim at a bound for the first addend in (18). For each $j = 1, \dots, N$, we define the following time-continuous quantities

$$\begin{aligned} \varepsilon_j(t) &:= f_h(t)_j - f_{SD\,j}(t), \\ \alpha_j(t) &:= \partial_t f(x_j, t) - ([\mathcal{A} + \mathcal{G}] f_h(t))_j - \Psi_j(t), \end{aligned} \tag{19}$$

called spatial discretization error and spatial truncation error, respectively. The spatial truncation error α is the residual obtained by inserting the exact solution f in the semidiscretized equation (11).

Notice that f satisfies both the PIDE in (8) restricted to the line (x_j, t) and the following equation

$$\partial_t f_h(t)_j = ((\mathcal{A} + \mathcal{G}) f_h(t))_j + \Psi_j(t) + \alpha_j(t),$$

where the spatial truncation error α is

$$\alpha_j(t) = [\mathcal{L} f(x_j, t) - (\mathcal{A} f_h(t))_j] + [\mathcal{I} f(x_j, t) - (\mathcal{G} f_h(t))_j]. \tag{20}$$

Proposition 1. *If the solution f of (8) has continuous space derivatives up to the fourth order and g in (3) is twice continuously differentiable, the spatial truncation error α defined in (19) is consistent of order 2 as follows*

$$\|\alpha(t)\|_{L^2_h} = \mathcal{O}(h^2).$$

Proof. In [26], under the assumption that f has continuous space derivatives up to the fourth order, it is proved that

$$\mathcal{L} f(x_j, t) - (\mathcal{A} f_h(t))_j = \mathcal{O}(h^2).$$

Further, we consider the second addend in (20). We have

$$\begin{aligned} \mathcal{I} f(x_j, t) - (\mathcal{G} f_h(t))_j &= \lambda \left(\int_\Omega f(y, t) g(x_j - y) dy - (G f_h(t))_j \right) \\ &= \lambda \left(\int_\Omega f(y, t) g(x_j - y) dy - \sum_{i=1}^{N} \frac{h g(x_j - x_i)}{\sum_{k=1}^{N} h g(x_k - x_i)} f(x_i, t) \right) \end{aligned}$$

$$
\begin{aligned}
&= \lambda\left(\int_\Omega f(y,t)g(x_j-y)dy - \sum_{i=1}^N h\,g(x_j-x_i)f(x_i,t)\right)\\
&\quad + \lambda\left(\sum_{i=1}^N h\,g(x_j-x_i)f(x_i,t) - \sum_{i=1}^N \frac{h\,g(x_j-x_i)}{\sum_{k=1}^N h\,g(x_k-x_i)} f(x_i,t)\right)\\
&= \mathcal{O}(h^2) + \lambda\sum_{i=1}^N \frac{h\,g(x_j-x_i)f(x_i,t)}{\sum_{k=1}^N h\,g(x_k-x_i)}\left(\sum_{k=1}^N h\,g(x_k-x_i)-1\right) = \mathcal{O}(h^2)
\end{aligned}
$$

We notice that the last term consists of the midpoint rule to calculate the integrals over Ω of the following two functions

$$\varphi_1(y) := f(y,t)g(x_j-y) \text{ and } \varphi_2(y) := g(y-x_i), \text{ for each } j=1,\dots,N.$$

The error associated with the midpoint rule is given by

$$\int_\Omega \varphi_l(y)dy - \sum_{k=1}^N h\,\varphi_l(x_k) = \mathcal{O}(h^2) \text{ for } l=1,2.$$

In particular, we have $|\sum_{k=1}^N h\,g(x_k-x_i)-1| = \mathcal{O}(h^2)$, hence

$$\alpha_j(t) = \mathcal{O}(h^2) \text{ for each } j=1,\dots,N.$$

Since $\|\alpha(t)\|^2_{L^2_h} = \sum_{j=1}^N h|\alpha_j(t)|^2$, we can conclude that $\|\alpha(t)\|_{L^2_h} = \mathcal{O}(h^2)$. □

Remark 4. Notice that $\sum_{i=1}^N G_{ij} = 1$ by construction. Furthermore, the midpoint rule approximation implies

$$\sum_{j=1}^N G_{ij} = 1 + \mathcal{O}(h^2).$$

Proposition 2. *Let f_h be the solution to (8) and f_{SD} be the solution to (11). Then the following holds $\|f_h - f_{SD}\|_{L^2_{h,\delta t}} = \mathcal{O}(h^2)$.*

Proof. By definition of spatial truncation error in (19), we have that $\varepsilon(t)$ satisfies the following initial value problem

$$\begin{cases} \varepsilon'(t) = [\mathcal{A}+\mathcal{G}]\varepsilon(t) + \alpha(t) \\ \varepsilon(t_0) = 0. \end{cases} \tag{21}$$

We estimate the logarithmic norm $\mu_2(\mathcal{A}+\mathcal{G})$. For this purpose, we show that $\langle \mathcal{A}\varepsilon(t), \varepsilon(t)\rangle_{L^2_h} \le 0$, for all $\varepsilon(t) \in L^2_h$. We note that there exist two matrices L and U defined as follows

$$L_{ij} = \begin{cases} -1 & i=2,\dots,N,\ j=i-1 \\ 1 & i=1,\dots N,\ j=i \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{ij} = \begin{cases} -\beta_i w_i & i=1,\dots,N-1,\ j=i+1 \\ \beta_i & i=1,\dots,N-1,\ j=i \\ 0 & \text{otherwise,} \end{cases}$$

such that $-h\mathcal{A} = LU$. Notice that $U_{NN} = 0$. Therefore the determinant of U, and thus of $\mathcal{A}$, is zero, which means that $\mathcal{A}$ has a zero eigenvalue. Furthermore, by inspection, we see that all other leading principal minors of U are positive. Then, we conclude that the matrix $\mathcal{A}$ has $N-1$ negative eigenvalues and a zero eigenvalue, and $\mu_2(\mathcal{A}) = 0$.

For the matrix $\mathcal{G}$ we have the following

$$
\begin{aligned}
\langle \mathcal{G}\varepsilon(t), \varepsilon(t)\rangle_{L^2_h} &= \lambda\langle G\varepsilon(t), \varepsilon(t)\rangle_{L^2_h} - \lambda\langle \varepsilon(t), \varepsilon(t)\rangle_{L^2_h}\\
&= \lambda h\sum_{i=1}^N\sum_{j=1}^N G_{ij}\varepsilon_i(t)\varepsilon_j(t) - \lambda h\sum_{j=1}^N \varepsilon_j^2(t)\\
&\le \lambda\frac{h}{2}\sum_{i=1}^N\left(\sum_{j=1}^N G_{ij}\right)\varepsilon_i^2(t) + \lambda\frac{h}{2}\sum_{j=1}^N\left(\sum_{i=1}^N G_{ij}\right)\varepsilon_j^2(t) - \lambda\|\varepsilon(t)\|^2_{L^2_h}\\
&\le c\lambda h^2\|\varepsilon(t)\|^2_{L^2_h},
\end{aligned} \tag{22}
$$

hence $\mu_2(\mathcal{G}) \le c\lambda h^2$. With this preparation, we obtain

$$\|e^{\mathcal{A}\delta t}\|_{L_h^2} = 1 \tag{23}$$

$$\|e^{\mathcal{G}\delta t}\|_{L_h^2} \leq e^{c\lambda h^2 \delta t} \tag{24}$$

$$\mu_2(\mathcal{A}+\mathcal{G}) \leq c\lambda h^2, \qquad \|e^{(\mathcal{A}+\mathcal{G})\delta t}\|_{L_h^2} \leq e^{c\lambda h^2 \delta t} \tag{25}$$

Notice that, in particular, we have

$$\|(I-\delta t\mathcal{A})^{-1}\| = 1. \tag{26}$$

By using the standard error analysis [19] to Eq. (21) we conclude that

$$\|\varepsilon(t)\|_{L_h^2} \leq e^{c\lambda h^2 t}\|\varepsilon(0)\|_{L_h^2} + \frac{e^{c\lambda h^2 t}-1}{c\lambda h^2}\max_{0\leq s\leq t}\|\alpha(s)\|$$

and therefore $\|f_h - f_{\mathrm{SD}}\|_{L^2_{h,\delta t}} = \mathcal{O}(h^2)$. $\square$

We now aim at proving a bound for the second addend in (18) related to the splitting method. Consider the matrices $\mathcal{A}$ and $\mathcal{G}$ as in (12) and (14), respectively, and define the operator $\mathcal{S}$ as follows

$$\mathcal{S} := e^{\frac{\delta t}{2}\mathcal{A}}e^{\delta t\mathcal{G}}e^{\frac{\delta t}{2}\mathcal{A}}. \tag{27}$$

Consider the time interval $[t_n, t_{n+1}]$. We apply four times the variation of constants formula for ODEs [19] to integrate the ODE systems (15). Therefore, the splitting solution can be formally written as follows

$$f_{\mathrm{SP}}(t_{n+1}) = \mathcal{S}f_{\mathrm{SP}}(t_n) + e^{\frac{\delta t}{2}\mathcal{A}}e^{\frac{\delta t}{2}\mathcal{G}}\int_0^{\frac{\delta t}{2}} e^{(\frac{\delta t}{2}-s)\mathcal{G}}\Psi(t_n+s)ds + e^{\frac{\delta t}{2}\mathcal{A}}\int_0^{\frac{\delta t}{2}} e^{(\frac{\delta t}{2}-s)\mathcal{G}}\Psi(t_{n+\frac{1}{2}}+s)ds. \tag{28}$$

We define with d_n the local truncation splitting error for each $n = 0, \dots, M-1$, which is the residual obtained at time t_n by inserting the exact solution of the semidiscretized system (11) in the formal expression of the splitting solution (28) as follows

$$d_n := f_{\mathrm{SD}}(t_{n+1}) - \mathcal{S}f_{\mathrm{SD}}(t_n) - e^{\frac{\delta t}{2}\mathcal{A}}e^{\frac{\delta t}{2}\mathcal{G}}\int_0^{\frac{\delta t}{2}} e^{(\frac{\delta t}{2}-s)\mathcal{G}}\Psi(t_n+s)ds - e^{\frac{\delta t}{2}\mathcal{A}}\int_0^{\frac{\delta t}{2}} e^{(\frac{\delta t}{2}-s)\mathcal{G}}\Psi(t_{n+\frac{1}{2}}+s)ds. \tag{29}$$

Define the global splitting error at time t_n as $E_n := f_{\mathrm{SD}}(t_n) - f_{\mathrm{SP}}(t_n)$. Subtracting (28) from (29), we obtain the following relation

$$E_{n+1} = \mathcal{S}E_n + d_n. \tag{30}$$

Exploiting the linearity of the solution operator $\mathcal{S}$ and the fact that $E_0 = 0$, we recursively apply (30), and obtain

$$E_n = \sum_{k=0}^{n}\mathcal{S}^{n-k}d_k. \tag{31}$$

In the remainder of this section, we make use of the following two facts. Given $z \in \mathbb{R}$ and a matrix $M \in \mathbb{R}^{N\times N}$, we have the following two properties. The exponential of the matrix zM is defined by the convergent power series

$$e^{zM} = \sum_{k=0}^{+\infty}\frac{(zM)^k}{k!}. \tag{32}$$

For $z \to 0$,

$$(I - zM)^{-1} = I + zM + z^2M^2 + \mathcal{O}(z^3). \tag{33}$$

Proposition 3. *Let Ψ in (11) be of class $\mathcal{C}^1([t_0, t_f])$. Then $\|f_{\mathrm{SD}} - f_{\mathrm{SP}}\|_{L^2_{h,\delta t}} = \mathcal{O}(\delta t^2)$.*

Proof. We first determine the order of the splitting error d_n. By applying twice to (11) the variation of constant formula [19], we have that the exact solution f_{SD} of (11) can be written as follows

$$f_{\mathrm{SD}}(t_{n+1}) = e^{\delta t(\mathcal{A}+\mathcal{G})}f_{\mathrm{SD}}(t_n) + e^{\frac{\delta t}{2}(\mathcal{A}+\mathcal{G})}\int_0^{\frac{\delta t}{2}} e^{(\frac{\delta t}{2}-s)(\mathcal{A}+\mathcal{G})}\Psi(t_n+s)ds + \int_0^{\frac{\delta t}{2}} e^{(\frac{\delta t}{2}-s)(\mathcal{A}+\mathcal{G})}\Psi(t_{n+\frac{1}{2}}+s)ds. \tag{34}$$

By using of the matrix series (32) and exploiting that $\Psi \in \mathcal{C}^1([t_0, t_f])$, we rewrite (28) and (34) as follows,

$$\begin{aligned} f_{\mathrm{SP}}(t_{n+1}) &= \mathcal{S}f_{\mathrm{SP}}(t_n) \\ &\quad + \delta t\left(I + \frac{\delta t}{2}(\mathcal{A}+\mathcal{G}) + \frac{\delta t^2}{8}\left(\mathcal{A}^2 + 2\mathcal{A}\mathcal{G} + \mathcal{G}^2\right)\right)(\Psi(t_n) + \mathcal{O}(\delta t)) \\ &\quad + \mathcal{O}(\delta t^4) \end{aligned} \tag{35}$$

and

$$f_{\mathrm{SD}}(t_{n+1}) = e^{\delta t(\mathcal{A}+\mathcal{G})} f_{\mathrm{SD}}(t_n) + \delta t\left(I + \frac{\delta t}{2}(\mathcal{A}+\mathcal{G}) + \frac{\delta t^2}{8}(\mathcal{A}+\mathcal{G})^2\right)(\Psi(t_n) + \mathcal{O}(\delta t)) + \mathcal{O}(\delta t^4). \tag{36}$$

Therefore, by using (29) the local splitting error can be rewritten as follows

$$d_n = (e^{\delta t(\mathcal{A}+\mathcal{G})} - \mathcal{S}) f_{\mathrm{SD}}(t_n) + \frac{\delta t^3}{8}(\mathcal{G}\mathcal{A} - \mathcal{A}\mathcal{G})\Psi(t_n) + \mathcal{O}(\delta t^4). \tag{37}$$

By applying (32) to $e^{\frac{\delta t}{2}\mathcal{A}}$ and $e^{\delta t\mathcal{G}}$, we have that

$$\mathcal{S} = I + \delta t(\mathcal{A}+\mathcal{G}) + \frac{\delta t^2}{2}(\mathcal{A}+\mathcal{G})^2 + \frac{\delta t^3}{6}\left(\mathcal{A}^3 + \mathcal{G}^3 + \frac{3}{4}\mathcal{A}^2\mathcal{G} + \frac{3}{2}\mathcal{A}\mathcal{G}\mathcal{A} + \frac{3}{2}\mathcal{A}\mathcal{G}^2 + \frac{3}{4}\mathcal{G}\mathcal{A}^2 + \frac{3}{2}\mathcal{G}^2\mathcal{A}\right) + \mathcal{O}(\delta t^4). \tag{38}$$

Furthermore, we have

$$e^{\delta t(\mathcal{A}+\mathcal{G})} = I + \delta t(\mathcal{A}+\mathcal{G}) + \frac{\delta t^2}{2}(\mathcal{A}+\mathcal{G})^2 + \frac{\delta t^3}{6}(\mathcal{A}+\mathcal{G})^3 + \mathcal{O}(\delta t^4),$$

it is clear that $e^{\delta t(\mathcal{A}+\mathcal{G})} - \mathcal{S} = \mathcal{O}(\delta t^3)$. Hence, $d_n = \mathcal{O}(\delta t^3)$ for each $n = 0, \ldots, M-1$.

Next, we consider the L_h^2 norm of (31), hence

$$\|E_n\|_{L_h^2} \le \sum_{k=0}^{n} e^{c\lambda h^2 \delta t(n-k)} \|d_k\|_{L_h^2} \le c_1\, \delta t^2,$$

where we used (24). Thus $\|E_n\|_{L_h^2} = \mathcal{O}(\delta t^2)$, and

$$\|f_{\mathrm{SD}} - f_{\mathrm{SP}}\|_{L^2_{h,\delta t}} = \sqrt{\sum_{n=0}^{M} \delta t \|E_n\|^2_{L_h^2}} \le c_2\, \delta t^2,$$

that completes the proof. □

In the remainder of this section, we aim at proving a bound for the third term in (18), where the full numerical solution $\hat{f} = f_j^n,\ j = 1, \ldots, N,\ n = 0, \ldots, M,$ is given by either the Euler discretization or by the predictor corrector scheme. By the definition of the operator $\mathcal{S}$ in (27), we write the solution of (15) in a convenient form as follows

$$f_{\mathrm{SP}}(t_{n+1}) = \mathcal{S} f_{\mathrm{SP}}(t_n) + e^{\frac{\delta t}{2}\mathcal{A}} \int_0^{\delta t} e^{(\delta t - s)\mathcal{G}} \Psi(t_n + s) ds, \tag{39}$$

where we applied three times the variation of constants formula [19] to (15).

Next, we write (16) in a compact form. Given f^n, the computation of f^{n+1} is carried out as follows

$$f^{n+1} = \left(I - \frac{\delta t}{2}\mathcal{A}\right)^{-1}\left((I + \delta t\mathcal{G})(I - \frac{\delta t}{2}\mathcal{A})^{-1} f^n + \delta t\Psi(t_n)\right) = R_1(\mathcal{A}, \mathcal{G}, \delta t) f^n + \delta t\left(I - \frac{\delta t}{2}\mathcal{A}\right)^{-1}\Psi(t_n), \tag{40}$$

where

$$R_1(\mathcal{A}, \mathcal{G}, \delta t) := \left(I - \frac{\delta t}{2}\mathcal{A}\right)^{-1}(I + \delta t\mathcal{G})\left(I - \frac{\delta t}{2}\mathcal{A}\right)^{-1}$$

is the amplification factor.

Remark 5. The non singularity and positivity of the matrix $I - \frac{\delta t}{2}\mathcal{A}$ is guaranteed also under the condition $\delta t \le 2/L$, where L is the Lipschitz constant of B. See [26] for details.

For each time window $[t_n, t_{n+1}]$ we define the respective time truncation error T_n, obtained by inserting the formal splitting solution f_{SP} defined in (39) in the numerical approximation given by (40). We have

$$T_n := f_{\mathrm{SP}}(t_{n+1}) - R_1(\mathcal{A}, \mathcal{G}, \delta t) f_{\mathrm{SP}}(t_n) - \delta t\left(I - \frac{\delta t}{2}\mathcal{A}\right)^{-1}\Psi(t_n). \tag{41}$$

Proposition 4. *(Consistency of SIMEX1) The truncation error (41) is of order $\mathcal{O}(\delta t^2)$.*

Proof. Recall the definitions of the splitting solution and truncation error, in (39) and in (41), respectively. Exploiting the fact that $\Psi \in \mathcal{C}^1(I)$ and making use of (32), we have that

$$T_n = (\mathcal{S} - R_1(\mathcal{A}, \mathcal{G}, \delta t)) f_{\mathrm{SP}}(t_n) + \delta t^2 \left(\frac{\mathcal{G}}{2} + \mathcal{O}(\delta t) \right) \Psi(t_n) + \mathcal{O}(\delta t^3). \quad (42)$$

We consider the Taylor expansion of $(I - \frac{\delta t}{2} A)^{-1}$ as in (33) and note that the amplification factor R_1 can be rewritten as follows

$$R_1(\mathcal{A}, \mathcal{G}, \delta t) = I + \delta t(\mathcal{A} + \mathcal{G}) + \frac{\delta t^2}{2} \left(\frac{3}{2}\mathcal{A}^2 + \mathcal{A}\mathcal{G} + \mathcal{G}\mathcal{A} \right) + \mathcal{O}(\delta t^3).$$

Recalling the expansion of $\mathcal{S}$ in (38), we can state that $\mathcal{S} - R_1(\mathcal{A}, \mathcal{G}, \delta t) = \mathcal{O}(\delta t^2)$.

These observations lead to the conclusion that $T_n = \mathcal{O}(\delta t^2)$, hence the proof is completed. □

We define, for each $n = 1, \ldots, M$, the time discretization error e_n as follows

$$e_n := f_{\mathrm{SP}}(t_n) - f^n,$$

such that by subtracting (41) from (40), we obtain the following relation

$$e_{n+1} = R_1(\mathcal{A}, \mathcal{G}, \delta t) e_n + T_n. \quad (43)$$

Proposition 5. *(Accuracy of SIMEX1) If δt is chosen such that $\|R_1(\mathcal{A}, \mathcal{G}, \delta t)\|_{L^2_h} \leq 1 + c\lambda h^2 \delta t$, then $\|f_{\mathrm{SP}} - \hat{f}\|_{L^2_{h,\delta t}} = \mathcal{O}(\delta t)$.*

Proof. First, we notice that the bound for $\|R_1\|$ in the hypothesis results from the definition of R_1 and (26). We consider the L^2_h norm of (43), obtaining

$$\|e_{n+1}\|_{L^2_h} \leq \|R_1(\mathcal{A}, \mathcal{G}, \delta t)\|_{L^2_h} \|e_n\|_{L^2_h} + \|T_n\|_{L^2_h} \leq (1 + c\lambda h^2 \delta t)\|e_n\|_{L^2_h} + \|T_n\|_{L^2_h},$$

This recursive relation gives the following

$$\|e_n\|_{L^2_h} \leq e^{c\lambda h^2 T} \left(\|e_0\|_{L^2_h} + \sum_{k=0}^{n-1} \|T_k\|_{L^2_h} \right),$$

that is the stability of the discrete operator R_1. Thus, $\|e_n\|_{L^2_h} = \mathcal{O}(\delta t)$. By noting that $\|f_{\mathrm{SP}} - \hat{f}\|^2_{L^2_{h,\delta t}} = \sum_{n=1}^{M} \delta t \|e_n\|^2_{L^2_h}$, the proof is completed. □

Next, we write (17) in a compact form. Given f^n, the computation of f^{n+1} is carried out as follows

$$f^{n+1} = R\left(\frac{\delta t}{2}\mathcal{A}\right) R(\delta t \mathcal{G}) R\left(\frac{\delta t}{2}\mathcal{A}\right) f^n + R\left(\frac{\delta t}{2}\mathcal{A}\right) \bar{\Psi}_n, \quad (44)$$

where

$$\bar{\Psi}_n := \frac{\delta t}{2}[(I + \delta t \mathcal{G})\Psi(t_n) + \Psi(t_{n+1})]$$

and the function R is the amplification factor; given a matrix M and $z \in \mathbb{R}$, R is defined as

$$R(zM) := I + zM + \frac{z^2}{2} M^2.$$

For each time window $[t_n, t_{n+1}]$ we define the respective time truncation error T_n, obtained by inserting the formal splitting solution f_{SP} defined in (39) in the numerical approximation given by (44)

$$T_n := f_{\mathrm{SP}}(t_{n+1}) - R\left(\frac{\delta t}{2}\mathcal{A}\right) R(\delta t \mathcal{G}) R\left(\frac{\delta t}{2}\mathcal{A}\right) f_{\mathrm{SP}}(t_n) - R\left(\frac{\delta t}{2}\mathcal{A}\right) \bar{\Psi}_n. \quad (45)$$

Proposition 6. *(Consistency of SIMEX2) The truncation error (45) is of order $\mathcal{O}(\delta t^3)$.*

Proof. Recall the definitions of the splitting solution and of truncation error, in (39) and in (45), respectively. Exploiting the fact that $\Psi \in \mathcal{C}^1(I)$ and making use of (32), we have

$$T_n = (\mathcal{S} - R_2(\mathcal{A}, \mathcal{G}, \delta t)) f_{\mathrm{SP}}(t_n) + \delta t^3 \frac{\mathcal{G}^3}{6} \Psi(t_n) + \mathcal{O}(\delta t^4), \quad (46)$$

where

$$R_2(\mathcal{A}, \mathcal{G}, \delta t) := R\left(\frac{\delta t}{2}\mathcal{A}\right) R(\delta t \mathcal{G}) R\left(\frac{\delta t}{2}\mathcal{A}\right).$$

By considering the Taylor expansion of $e^{\frac{\delta t}{2}\mathcal{A}}$ and $e^{\delta t\mathcal{G}}$ as in (32) up to the fourth order, we note that the solution operator $\mathcal{S}$ can be written as follows

$$\mathcal{S} = R_2(\mathcal{A}, \mathcal{G}, \delta t) + \frac{\delta t^3}{6}\left(\frac{\mathcal{A}^3}{2} + \mathcal{G}^3\right) + \mathcal{O}(\delta t^4).$$

These observations lead to the conclusion that $T_n = \mathcal{O}(\delta t^3)$, hence the proof is completed. □

We define for each $n = 1, \ldots, M$ the time discretization error e_n as follows

$$e_n := f_{\mathrm{SP}}(t_n) - f^n,$$

such that by subtracting (45) from (44), we obtain the following relation

$$e_{n+1} = R_2(\mathcal{A}, \mathcal{G}, \delta t)e_n + T_n. \tag{47}$$

Proposition 7. *(Accuracy of SIMEX2) If δt is chosen such that $\|R_2(\mathcal{A}, \mathcal{G}, \delta t)\|_{L_h^2} \le e^{c\lambda h^2 \delta t}$, then $\|f_{\mathrm{SP}} - \hat{f}\|_{L^2_{h,\delta t}} = \mathcal{O}(\delta t^2)$.*

Proof. First, we notice that the bound for $\|R_2\|$ in the hypothesis results from the definition of R_2, (23) and (24). We apply the L_h^2 norm to (47), obtaining

$$\|e_{n+1}\|_{L_h^2} \le \|R_2(\mathcal{A}, \mathcal{G}, \delta t)\|_{L_h^2}\|e_n\|_{L_h^2} + \|T_n\|_{L_h^2} \le e^{c\lambda h^2\delta t}\|e_n\|_{L_h^2} + \|T_n\|_{L_h^2},$$

This recursive relation gives the following

$$\|e_n\|_{L_h^2} \le e^{c\lambda h^2 T}\left(\|e_0\|_{L_h^2} + \sum_{k=0}^{n-1}\|T_k\|_{L_h^2}\right),$$

that is the stability of the discrete operator R_2. Thus, $\|e_n\|_{L_h^2} = \mathcal{O}(\delta t^2)$. By noting that $\|f_{\mathrm{SP}} - \hat{f}\|^2_{L^2_{h,\delta t}} = \sum_{n=1}^{M}\delta t\|e_n\|^2_{L_h^2}$, the proof is completed. □

5. Positivity and conservativeness of the SIMEX schemes

In this section, we prove that the SIMEX1 and SIMEX2 schemes are conservative and positive preserving. First, we focus on the SIMEX1 scheme, where the time discretization is given by the Euler scheme. Given the numerical solution f^n at time t_n, we compute f^{n+1} as follows

$$\begin{aligned} &1.\ \frac{f^{n+\frac{1}{2}} - f^n}{\frac{\delta t}{2}} = \mathcal{A}f^{n+\frac{1}{2}}\\ &2.\ \frac{f^{n+\frac{1}{2}^*} - f^{n+\frac{1}{2}}}{\delta t} = \mathcal{G}f^{n+\frac{1}{2}}\\ &3.\ \frac{f^{n+1} - f^{n+\frac{1}{2}^*}}{\frac{\delta t}{2}} = \mathcal{A}f^{n+1}. \end{aligned} \tag{48}$$

Proposition 8. *Let us consider (48). Assume that $\delta t \le \min\{\frac{1}{\lambda}, \frac{2}{L}\}$, with λ rate of jumps of the compound Poisson process P in (1) and let L be the Lipschitz constant of the function B that defines $\mathcal{F}$ in (7). If $f_j^n \ge 0$ for each $j = 1, \ldots N$, then $f_j^{n+1} \ge 0$ for each $j = 1, \ldots, N$.*

Proof. Let use consider each step of (48).

1. Given $f_j^n \ge 0$ for each $j = 1, \ldots, N$, then $f_j^{n+1/2} \ge 0$ for each $j = 1, \ldots, N$. In fact, the evolution matrix of Step 1. is given by $I - \frac{\delta t}{2}\mathcal{A}$. According to the definition of $\mathcal{A}$ given in (12), the evolution matrix is a non singular M-matrix provided that it is diagonal dominant. This property is satisfied when $\delta t/(2h) < |\beta_i(w_i - 1) - \beta_{i-1}(w_{i-1} - 1)|^{-1} = |B_{i+1/2} - B_{i-1/2}|^{-1}, \forall i$, that is true for $\delta t \le 2/L$. Since, for non-singular M-matrix it is $M^{-1} \ge 0$, the assertion is proved. An alternative proof can be found in [26].
2. Step 2 in (48) can be recast, for each $j = 1, \ldots, N$, as follows
$$f_j^{(n+\frac{1}{2})^*} = \lambda\delta t(Gf^{n+\frac{1}{2}})_j + (1 - \lambda\delta t)f_j^{n+\frac{1}{2}},$$
where $f_j^{n+\frac{1}{2}} \ge 0$ by hypothesis. Since G has nonnegative components and δt is such that both $\lambda\delta t$ and $(1 - \lambda\delta t)$ are nonnegative, then $f_j^{(n+\frac{1}{2})^*} \ge 0$ for each $j = 1, \ldots, N$.
3. In Step 3, given $f_j^{(n+\frac{1}{2})^*} \ge 0$ for each $j = 1, \ldots, N$, then $f_j^{n+1} \ge 0$, according to the argument used for analyzing Step 1. □

Proposition 9. *Let us consider (48). The total probability is preserved, in the sense that*

$$\sum_{j=1}^{N} f_j^n = \sum_{j=1}^{N} f_j^{n+1}.$$

Proof. Let use separately consider each step of (48).

1. As shown in Remark 2,

$$\sum_{j=1}^{N} (\mathcal{A} f^{n+\frac{1}{2}})_j = 0 \text{ for each } n = 0, \dots, M,$$

due to the zero-flux boundary conditions. By summing over $j = 1, \dots, N$ both sides of the first equation in (48), we have that

$$\sum_{j=1}^{N} f_j^{n+\frac{1}{2}} = \sum_{j=1}^{N} f_j^{n+1}.$$

2. Let us consider

$$\frac{f_j^{(n+\frac{1}{2})^*} - f_j^{n+\frac{1}{2}}}{\delta t} = (\mathcal{G} f^{n+\frac{1}{2}})_j.$$

Summing over j and reshaping, we have that

$$\sum_{j=1}^{N} f_j^{(n+\frac{1}{2})^*} = \sum_{j=1}^{N} f_j^{n+\frac{1}{2}} + \delta t \sum_{j=1}^{N} (\mathcal{G} f^{n+\frac{1}{2}})_j,$$

thus it is sufficient to show that $\sum_{j=1}^{N} (\mathcal{G} f^{n+\frac{1}{2}})_j = 0$. This follows straightforwardly from the construction of the matrix G, in fact

$$\begin{aligned} \sum_{j=1}^{N} (\mathcal{G} f^{n+\frac{1}{2}})_j &= \sum_{j=1}^{N} \lambda \left[\sum_{k=1}^{N} \left(G_{jk} f_k^{n+\frac{1}{2}} \right) \right] - \lambda \sum_{j=1}^{N} f_j^{n+\frac{1}{2}} \\ &= \lambda \left[\sum_{k=1}^{N} f_k^{n+\frac{1}{2}} \left(\sum_{j=1}^{N} G_{jk} \right) - \sum_{j=1}^{N} f_j^{n+\frac{1}{2}} \right] = 0, \end{aligned}$$

since $\sum_{j=1}^{N} G_{jk} = 1$ independently by k, as defined in (14).

3. In the same fashion as in the first step, we claim that

$$\sum_{j=1}^{N} f_j^{n+1} = \sum_{j=1}^{N} f_j^{(n+\frac{1}{2})^*}.$$

□

Next, we focus on the SIMEX2 scheme, where the time discretization is given by the predictor-corrector scheme. Given the numerical solution f^n at time t_n, the three steps required to compute f^{n+1} are as follows

$$\begin{aligned} &1. \begin{cases} \bar{f}^{n+\frac{1}{2}} = f^n + \dfrac{\delta t}{2} \mathcal{A} f^n \\ f^{n+\frac{1}{2}} = f^n + \dfrac{\delta t}{4} \left[\mathcal{A} f^n + \mathcal{A} \bar{f}^{n+\frac{1}{2}} \right] \end{cases} \\ &2. \begin{cases} \bar{f}^{n+\frac{1}{2}^*} = f^{n+\frac{1}{2}} + \delta t \mathcal{G} f^{n+\frac{1}{2}} \\ f^{n+\frac{1}{2}^*} = f^{n+\frac{1}{2}} + \dfrac{\delta t}{2} \left[\mathcal{G} f^{n+\frac{1}{2}} + \mathcal{G} \bar{f}^{n+\frac{1}{2}} \right] \end{cases} \\ &3. \begin{cases} \bar{f}^{n+1} = f^{n+\frac{1}{2}^*} + \dfrac{\delta t}{2} \mathcal{A} f^{n+\frac{1}{2}^*} \\ f^{n+1} = f^{n+\frac{1}{2}^*} + \dfrac{\delta t}{4} \left[\mathcal{A} f^{n+\frac{1}{2}^*} + \mathcal{A} \bar{f}^{n+1} \right]. \end{cases} \end{aligned} \tag{49}$$

Proposition 10. *Let us consider (49). Let us suppose that $\delta t \le \min\{\frac{1}{\lambda}, \frac{2}{\max_j |\mathcal{A}_{jj}|}\}$, where λ is the rate of the compound Poisson process P in (1), and $\mathcal{A}_{jj}$ are the diagonal elements of $\mathcal{A}$ as defined in (12). If $f_j^n \ge 0$ for each $j = 1, \dots, N$, then $f_j^{n+1} \ge 0$ for each $j = 1, \dots, N$.*

Proof. Let use separately consider each step of (49).

1. This step can be rewritten as

$$f^{n+\frac{1}{2}} = \bar{\mathcal{A}} f^n,$$

where

$$\bar{\mathcal{A}} := I + \frac{\delta t}{2}\mathcal{A} + \frac{\delta t^2}{8}\mathcal{A}^2 = \frac{1}{8}\left[4I + (2I + \delta t \mathcal{A})^2\right]$$

and I is the N-dimensional identity matrix.
Given $f_j^n \geq 0$ for each $j = 1, \ldots, N$, then $f_j^{n+1/2} \geq 0$ for each $j = 1, \ldots, N$, provided that $\bar{\mathcal{A}}$ has positive entries. This condition holds if $2I + \delta t \mathcal{A}$ has positive entries; thanks to Remark 3, it is sufficient to choose $(2 + \delta t \mathcal{A}_{jj}) \geq 0$ for each j, which is satisfied since $\delta t \leq \frac{2}{\max_j |\mathcal{A}_{jj}|}$ by hypothesis.

2. The intermediate step in (49) can be rewritten as follows

$$f^{(n+\frac{1}{2})^*} = \frac{1}{2}\left(I + (I + \delta t \mathcal{G})^2\right) f^{n+\frac{1}{2}},$$

where I is the N-dimensional identity matrix. The time step $\delta t \geq 0$ is chosen such that $(1 - \lambda \delta t) \geq 0$ and thus $I + \delta t \mathcal{G} = (1 - \lambda \delta t) I + \lambda \delta t \mathcal{G}$ has positive entries. Hence we have that $f_j^{(n+\frac{1}{2})^*} \geq 0$ for each $j = 1, \ldots, N$.

3. Given $f_j^{(n+\frac{1}{2})^*} \geq 0$ for each $j = 1, \ldots, N$, then $f_j^{n+1} \geq 0$ for each $j = 1, \ldots, N$, by the same reasoning of the first step. □

Remark 6. We obtain an estimate for $\max_j |\mathcal{A}_{jj}|$ as follows.

$$|\mathcal{A}_{jj}| = (\beta_i + \beta_{i-1} w_{i-1})/h = \frac{C_{j+1/2}}{h^2} \frac{r_{j+1/2}}{e^{r_{j+1/2}} - 1} + \frac{C_{j-1/2}}{h^2} \frac{-r_{j-1/2}}{e^{-r_{j-1/2}} - 1}$$

where $r(x) = hB(x)/C(x)$ is the Peclét number, and $r_{j+1/2} = r(x_{j+1/2})$. Therefore,

$$\max_j |\mathcal{A}_{jj}| \leq \frac{\max_x(C(x))}{h^2}(\max(1, 1 - r(x)) + \max(1, 1 + r(x))) \leq$$

$$\frac{\max_x(C(x))}{h^2}(2 + \max_x(|r(x)|)).$$

Hence, the positivity bound to δt in the previous theorem becomes

$$\delta t \leq \min\left\{\frac{1}{\lambda}, \frac{2h^2}{\max_x\{C(x)\}(2 + h \max_x\{|B(x)|/C(x)\}}\right\}.$$

We observe that for vanishing space step size h, the time step size must vanishes with order 2. In small diffusion regime, i.e. $C(x) \simeq 0$, it scales linearly as $\delta t < 2h/\max_x\{B(x)\}$.

Proposition 11. *Let us consider (49). The total probability is conserved, in the sense that*

$$\sum_{j=1}^{N} f_j^n = \sum_{j=1}^{N} f_j^{n+1}.$$

Proof. Let use separately consider each step of (49).

1. Let us consider the second equation. Summing over j, we obtain

$$\sum_{j=1}^{n} f_j^{n+\frac{1}{2}} = \sum_{j=1}^{n} f_j^n + \frac{\delta t}{4}\left[\sum_{j=1}^{n}(\mathcal{A} f^n)_j + \sum_{j=1}^{n}(\mathcal{A} \bar{f}^{n+\frac{1}{2}})_j\right].$$

Thanks to the construction of the matrix $\mathcal{A}$ in (12) and Remark 2, we have that

$$\sum_{j=1}^{N} f_j^n = \sum_{j=1}^{N} f_j^{n+\frac{1}{2}}.$$

2. Let us consider the second equation. Summing over j and reshaping, we have that

$$\sum_{j=1}^{N} f_j^{(n+\frac{1}{2})^*} = \sum_{j=1}^{N} f_j^{n+\frac{1}{2}} + \frac{\delta t}{2}\left[\sum_{j=1}^{N}(\mathcal{G} f^{n+\frac{1}{2}})_j + \sum_{j=1}^{N}(\mathcal{G} \bar{f}^{n+\frac{1}{2}^*})_j\right],$$

With same arguments as in Proposition 8, we claim that $\sum_{j=1}^{N}(\mathcal{G} f^{n+\frac{1}{2}})_j = 0$ and that $\sum_{j=1}^{N}(\mathcal{G} \bar{f}^{n+\frac{1}{2}^*})_j = 0$ and hence

$$\sum_{j=1}^{N} f_j^{(n+\frac{1}{2})^*} = \sum_{j=1}^{N} f_j^{n+\frac{1}{2}}.$$

Table 1
$L^2_{h,\delta t}$-error of the scheme SIMEX1.

N	M	$\|f-\hat{f}\|_{h,\delta t}$
100	100	5.39×10^{-3}
200	400	1.09×10^{-3}
400	800	2.90×10^{-4}
800	1600	7.37×10^{-5}

Table 2
$L^2_{h,\delta t}$-error of the scheme SIMEX2.

N	M	$\|f-\hat{f}\|_{h,\delta t}$
200	200	1.10×10^{-3}
400	400	2.93×10^{-4}
800	800	7.45×10^{-5}
1600	1600	1.87×10^{-5}

3. By the same reasoning for the first step, we have that

$$\sum_{j=1}^{N} f_j^{n+1} = \sum_{j=1}^{N} f_j^{(n+\frac{1}{2})^*}.$$

□

6. Numerical experiments

In this section, we present results of numerical experiments for the FP problem (8). We set $b(x)=-x$ and $\sigma=\sqrt{2}$, i.e. $a(x)=1$. The jumps of the compound Poisson process are chosen to have rate $\lambda=5$ and the distribution g is chosen to be $g\sim\mathcal{N}(3,0.2^2)$. We consider the domain $\Omega=(-15,30)$ and $I=[0,1]$. In order to test the performance of the SIMEX schemes, we set the solution to (8) as the following a moving Gaussian

$$f(x,t)=\frac{1}{\sqrt{2\pi}\tilde{\sigma}}\exp\left\{-\frac{(x-\mu t)^2}{2\tilde{\sigma}^2}\right\},$$

with $\mu=10$ and $\tilde{\sigma}=3$, hence compute analytically the corresponding source term $\psi(x,t)$, as discussed in Section 4.

In Table 1, we report the norm of the SIMEX1 solution error as a function of the mesh size. We see that the scheme is first-order accurate in time and second-order accurate in space, as proved in Section 4.

In Table 2, we present results for the same test case, obtained with the SIMEX2 method. We have second-order convergence in time and space.

Next, we use the setting of the previous experiment and $\psi=0$, to compute the PDF of the process. Note that this setting corresponds to the Ornstein–Uhlenbeck process with jumps [29]. The analytical form of the PDF of this process is not known in closed form. We consider the process X in the time interval $[t_0,t_f]=[0,1]$ with initial $X_0\sim\mathcal{N}(15,3)$. We set the domain $\Omega=(-20,50)$, resulting from the estimate given in the Appendix.

Fig. 2 depicts the PDF of the process X_t at time $T=1$ computed using the SIMEX2 scheme and the empirical PDF [22]. To determine the empirical PDF, the initial-value SDE problem is solved by applying the Euler–Maruyama method [21] in the time interval [0, 1], obtaining $M=10^5$ sample paths with terminal values $X^m_{t_f}$, $m=1,\dots,M$. The state domain is divided in $K=100$ intervals. For each interval I_k, $k=1,\dots,K$ the value y_k of the empirical PDF in the midpoint of I_k is defined as

$$y_k := \{\#m \text{ such that } X^m_{t_f}\in I_k\}/(M\Delta x), \qquad K\,\Delta x=|\Omega|.$$

7. Conclusion

In this work, two numerical approximation schemes for solving a Fokker–Planck equation related to a jump-diffusion process, with jumps given by a compound Poisson process, were investigated. These schemes discretize this partial integro-differential equation combining a finite-difference method proposed by Chang and Cooper and the Strang–Marchuk operator splitting method, where the time discretization was done by applying either the Euler method or the predictor-corrector method. It was proved that these numerical schemes are respectively first and second accurate, positive-preserving and conservative. Results of numerical experiments were presented that validated the theoretical estimates.

Appendix. Identification of the computational domain

In this appendix, we present a methodology to estimate the size of a computational domain for the case in which f describes the time evolution of the PDF of a one dimensional Ornstein–Uhlenbeck process with jumps given by a compound

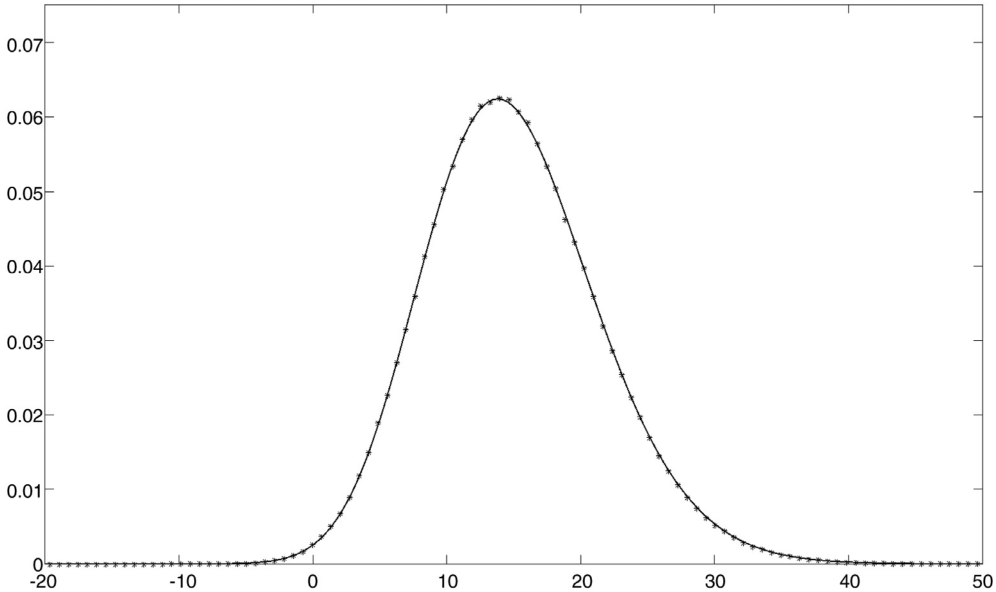

Fig. 2. The PDF of an Ornstein–Uhlenbeck process with jumps. Comparison between the solution of the SIMEX2 scheme (solid line; $N = 400$ and $M = 400$) and the empirical PDF estimated with the Monte Carlo method with $M = 10^5$ sample paths (stars).

Poisson process. Consider the following initial-value problem

$$\begin{cases} \partial_t f(x,t) = \dfrac{\sigma^2}{2}\partial_x^2 f(x,t) + \gamma\partial_x(xf(x,t)) - \lambda f(x,t) + \lambda(f*g)(x,t), & (x,t) \in \mathbb{R}\times I \\ f(x,0) = f_0(x), & x \in \mathbb{R} \end{cases}$$

where $\sigma, \gamma \in \mathbb{R}$, $\gamma \geq 0$, λ rate of jumps and g density of the jump amplitude distribution. The convolution $(f{*}g)(x, t)$ is defined as follows

$$(f*g)(x,t) := \int_{\mathbb{R}} f(y,t)g(x-y)dy,$$

for each $t \in I$.

Using the Fourier transforms of $f(x, t)$ and $g(x)$, given by $h(\omega,t) = \int_{-\infty}^{\infty} e^{i\omega x} f(x,t)dx$ and $W(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} g(x)dx$, respectively, we have that $h(\omega, t)$ satisfies the following initial-value problem

$$\begin{cases} \partial_t h(\omega,t) = -\gamma\omega\partial_\omega h(\omega,t) - P(\omega)h(\omega,t), & (\omega,t) \in \mathbb{R}\times I \\ h(\omega,0) = h_0(\omega), & \omega \in \mathbb{R}, \end{cases} \tag{50}$$

where $P(\omega) := \frac{\sigma^2}{2}\omega^2 + \lambda(1 - W(\omega))$ and $h_0(\omega)$ is the Fourier transform of the initial data $f_0(x)$.

After performing a change of variable for $\omega > 0$, and noting that the following calculations can be carried out also for the case $\omega < 0$ by defining $\bar{h}(\omega,t) := h(-\omega,t)$, we have that

$$h(\omega,t) = h_0(e^{-\gamma t}\omega)\exp\left\{-\frac{1}{\gamma}\int_{\omega e^{-\gamma t}}^{\omega}\frac{P(\omega')}{\omega'}d\omega'\right\}$$

solves (50).

Defining with $M^k[g] = \int_{-\infty}^{\infty} x^k g(x)dx$ the kth moment of g, the Taylor expansion of P on $\omega = 0$ reads as follows

$$P(\omega) = \frac{\sigma^2}{2}\omega^2 - \lambda\sum_{k=1}^{\infty}\frac{i^k M^k g}{k!}\omega^k,$$

since

$$W(\omega) = \sum_{k=0}^{\infty}\frac{W^{(k)}(0)}{k!}\omega^k = \sum_{k=0}^{\infty}\frac{i^k M^k[g]}{k!}\omega^k.$$

Hence, the solution of (50) can be written as follows

$$h(\omega,t) = h_0(e^{-\gamma t}\omega)\exp\{z(\omega,t)\},$$

where

$$z(\omega,t) := i\frac{\lambda}{\gamma}(M^1[g])\omega(1-e^{-\gamma t}) - \frac{\sigma^2+\lambda M^2[g]}{4\gamma}\omega^2(1-e^{-2\gamma t}) + \frac{\lambda}{\gamma}\sum_{k=3}^{\infty}\frac{i^k M^k[g]}{k(k!)}\omega^k(1-e^{-k\gamma t}).$$

This form for $h(\omega, t)$ allows us to calculate $M_t^1[f]$ and $M_t^2[f]$, that is, the first- and second-time dependent moments of $f(x, t)$.

We are interested in these two quantities at a time near to equilibrium, i.e. $T \gg 1/\gamma$. Therefore we suppose that the support of the PDF is vanishing outside the interval of size $[M_T^1[f] - 5\sqrt{M_T^2[f]}, M_T^1[f] + 5\sqrt{M_T^2[f]}]$. We exploit the fact that $h(0,t) = 1$ for each t to state the following

$$\lim_{t\to\infty} M_t^1 f = \lim_{t\to\infty} -ih'(0,t) = \frac{\lambda}{\gamma}(M^1[g])$$

and that

$$\lim_{t\to\infty} M_t^2 f = -\lim_{t\to\infty} h''(0,t) = \frac{\lambda^2}{\gamma^2}\left(M^1[g]\right)^2 + \frac{\sigma^2+\lambda M^2[g]}{2\gamma}.$$

Notice that the moments of the density g of the jump amplitude play a key role in the width of the relevant range of the dynamics of the considered process.

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