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Error analysis of a fractional-step method for magnetohydrodynamics equations

Rong An *, Can Zhou

College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, PR China

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ABSTRACT

This paper focuses on a fractional-step finite element method for the magnetohydrodynamics problems in three-dimensional bounded domains. It is shown that the proposed fractional-step scheme allows for a discrete energy identity. A rigorous error analysis is presented. We derive the temporal and spatial error estimates of $\mathcal{O}(\Delta t + h)$ for the velocity and the magnetic field in the discrete space $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ under the constraint $\Delta t \geq Ch$.

1. Introduction

The incompressible magnetohydrodynamics (MHD) problems are used to describe the flow of a viscous, incompressible and electrically conducting fluid, which are governed by the following time-dependent nonlinear coupled problems:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\mathbf{b}\times \operatorname{curl}\mathbf{b} = \mathbf{f}, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.2}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \frac{1}{Rm}\operatorname{curl}(\operatorname{curl}\mathbf{b}) - \operatorname{curl}(\mathbf{u}\times\mathbf{b}) = 0, \tag{1.3}$$

$$\operatorname{div}\mathbf{b} = 0, \tag{1.4}$$

for $x \in \Omega$ and $t \in (0, T)$ with $T > 0$, where $\Omega \subset \mathbf{R}^3$ is a bounded and simply-connected domain which is either convex or has a $C^{1,1}$ boundary $\partial\Omega$. Re, Rm and S are three positive constants and denote the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The vector-value function $\mathbf{f}$ represents the body forces applied to the fluid. The MHD problems (1.1)–(1.4) couple the incompressible Navier–Stokes equations with Maxwell's equations. Thus, the unknowns in (1.1)–(1.4) are the fluid velocity $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$. We refer to Hughes [1] and Moreau [2] for the understanding of the physical background of the MHD problems. To study (1.1)–(1.4), the appropriate initial and boundary conditions are needed. For the sake of simplicity, in this paper, we consider the following initial and

* Corresponding author.
E-mail addresses: anrong702@aliyun.com, anrong702@gmail.com (R. An).

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boundary conditions:

$$\mathbf{u}(x,0) = \mathbf{u}_0, \qquad \mathbf{b}(x,0) = \mathbf{b}_0 \quad \text{in } \Omega, \tag{1.5}$$

$$\mathbf{u} = 0, \qquad \mathbf{b}\cdot\mathbf{n} = 0, \qquad \operatorname{curl}\mathbf{b}\times\mathbf{n} = 0 \quad \text{on } \partial\Omega\times[0,T], \tag{1.6}$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. It is necessary to require that $\mathbf{u}_0$ and $\mathbf{b}_0$ satisfy the compatibility conditions $\operatorname{div}\mathbf{u}_0 = 0$ and $\operatorname{div}\mathbf{b}_0 = 0$.

The numerical methods for the incompressible MHD problems have received much attention in the last decades. We refer to Gerbeau–Bris–Lelièvre [3] for a review of numerical methods. The mixed finite element approximation was first proposed and studied for the stationary MHD problems in [4], where $\mathbf{H}^1$-conforming elements were used to discretize the magnetic field provided that Ω is either convex or has a $C^{1,1}$ boundary $\partial\Omega$. Inspired by the stabilization method for Stokes problems in [5], a stabilization mixed finite element method for the stationary MHD problems was developed by Gerbeau [6]. For the time-dependent MHD problems (1.1)–(1.6), He proposed a linearized semi-implicit Euler scheme in [7], where $\mathbf{L}^2$-unconditional convergence was proved by using the negative norm technique. For the non-convex domain or Lipschitz polyhedra domain of engineering practice, the magnetic field $\mathbf{b}$ may have regularity below $\mathbf{H}^1(\Omega)$. In this case, the $\mathbf{H}^1$-conforming finite element discretization for $\mathbf{b}$, albeit stable, may not converge to the correct magnetic field. A mixed finite element formulation based on $\mathbf{H}(\text{curl})$-elements (or Nédélec elements) for $\mathbf{b}$ was proposed and studied by Schötzau in [8] for the stationary MHD problems. Other different numerical methods can be found in [9–18] and references cited therein. Roughly speaking, the difficulties in solving the MHD problems numerically are mainly of three kinds: the mixed type of the equations; the incompressible condition and the treatment of the pressure; the nonlinearities of the problems, which are very similar to the incompressible Navier–Stokes problems. In the 1960s, Chorin [19] and Temam [20] proposed a projection method for Navier–Stokes problems, which decoupled the velocity and the pressure in the Navier–Stokes problems. The idea of the projection method is first to compute a velocity field without taking into account incompressibility, and then perform a pressure correction, which is a projection back to the subspace of solenoidal (divergence-free) vector fields. However, the drawback is the appearance of the numerical boundary layer due to the incompatibility of the pressure boundary conditions with those of the original Navier–Stokes problems [21,22]. For the time-dependent MHD problems (1.1)–(1.6), inspired by the projection method in [19,20], Prohl in [14] proposed a projection scheme. However, the projection scheme in [14] does not allow for a discrete energy estimate. To avoid using artificial boundary conditions of pressure type, some fractional-step schemes for the Navier–Stokes problems were introduced and studied in [23,24]. It is a two-step scheme in which the incompressibility and the nonlinearities of the Navier–Stokes problems are split into different steps, and allows the enforcement of the original boundary conditions in all substeps.

In this paper, we propose a two-step fractional-step scheme to approximate the solution of the MHD problems (1.1)–(1.4) with the initial and boundary conditions (1.5)–(1.6). We will prove that the proposed fractional-step scheme allows for a discrete energy identity. To state the main results derived, we introduce the following notations. Let X be a Banach space equipped with norm $\|\cdot\|_X$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0,T]$ with time step $\Delta t = T/N$ and $t_n = n\Delta t$ for $0 \le n \le N$. We denote two discrete norms by

$$\|\mathbf{u}^n\|_{l^2(X)} = \left(\Delta t\sum_{n=1}^{N}\|\mathbf{u}^n\|_X^2\right)^{1/2}, \qquad \|\mathbf{u}^n\|_{l^\infty(X)} = \max_{1\le n\le N}\|\mathbf{u}^n\|_X.$$

It is proved that the time-discrete fractional-step scheme provides the temporal error estimates of $\mathcal{O}(\Delta t)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)\cap l^\infty(\mathbf{L}^2)$ and $\mathcal{O}(\sqrt{\Delta t})$ for the pressure in $l^2(L^2)$. For the fully discrete fractional-step scheme, the finite element error estimates of $\mathcal{O}(\Delta t + h)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)\cap l^\infty(\mathbf{L}^2)$ are obtained under the constraint $\Delta t \ge Ch$.

The remainder of this paper is organized as follows: in the next section, we begin with some notations, lay out some assumptions and recall some known inequalities frequently used. The new linearized projection scheme and the main results are presented in Section 3. Meanwhile, the discrete energy identity is derived in Section 3. The proof containing the main results is given in Sections 4 and 5, which is split into several theorems.

2. Mathematical setting

For the mathematical setting of the MHD problems (1.1)–(1.4) with the initial and boundary conditions (1.5)–(1.6), we introduce some function spaces and their associated norms. For $k \in \mathbb{N}^+$, $1 \le s \le +\infty$, let $W^{k,s}(\Omega)$ denote the standard Sobolev space. The norm in $W^{k,s}(\Omega)$ is defined by

$$\|u\|_{W^{k,s}} = \begin{cases} \left(\displaystyle\sum_{|\beta|\le k}\int_\Omega |\nabla^\beta u|^s dx\right)^{1/s} & \text{for } 1 \le s < +\infty, \\ \displaystyle\sum_{|\beta|\le k}\sup_\Omega |\nabla^\beta u| & \text{for } s = +\infty, \end{cases}$$

where ∇ is the differential operator with respect to $\mathbf{x}$ and

$$\nabla^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}}$$

for the multi-index $\beta = (\beta_1, \beta_2, \beta_3)$, $\beta_1, \beta_2, \beta_3 \geq 0$ and $|\beta| = \beta_1 + \beta_2 + \beta_3$. We define $W_0^{k,s}(\Omega)$ to be the subspace of $W^{k,s}(\Omega)$ of functions with zero trace on $\partial\Omega$. When $s = 2$, we simply use $H^k(\Omega)$ to denote $W^{k,2}(\Omega)$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. The boldface Sobolev spaces $\mathbf{H}^k(\Omega)$, $\mathbf{W}^{k,s}(\Omega)$ and $\mathbf{L}^s(\Omega)$ are used to denote the vector Sobolev spaces $H^k(\Omega)^3$, $W^{k,s}(\Omega)^3$ and $L^s(\Omega)^3$, respectively. For any $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$, $(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx$ denotes $\mathbf{L}^2(\Omega)$ inner product. Let X be a Banach space. Its dual space is denoted by X'. For some $T > 0$ or $T = \infty$, denote Bochner spaces $L^s(0, T; X)$, $1 \leq s < +\infty$, the spaces of measurable functions from the interval $[0, T]$ into X such that

$$\int_0^T \|v(t)\|_X^s dt < +\infty.$$

If $s = +\infty$, the functions in $L^\infty(0, T; X)$ are required to satisfy

$$\sup_{t\in[0,T]} \|v(t)\|_X < +\infty.$$

The symbols $C, C_1, C_2, \ldots$ are used to denote the generic positive constants independent of the time step Δt and the mesh size h.

Let us denote

$$\begin{aligned}
&\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{V} = \mathbf{H}_0^1(\Omega), \quad \mathbf{V}_0 = \{\mathbf{u} \in \mathbf{V},\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\},\\
&\mathbf{X} = \{\mathbf{u} \in \mathbf{H}^1(\Omega),\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \qquad \mathbf{X}_0 = \{\mathbf{u} \in \mathbf{X},\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\},\\
&M = L_0^2(\Omega) = \left\{q \in L^2(\Omega),\ \int_\Omega q dx = 0\right\}.
\end{aligned}$$

Thanks to the Poincaré inequality, the norm in $\mathbf{V}$ can be defined by

$$\|\mathbf{u}\|_V = \left(\int_\Omega |\nabla\mathbf{u}|^2 dx\right)^{1/2}.$$

Introduce two bilinear forms

$$\begin{aligned}
a_1(\mathbf{u}, \mathbf{v}) &= \frac{1}{Re}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}dx \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V},\\
a_2(\mathbf{u}, \mathbf{v}) &= \frac{1}{Rm}\int_\Omega \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v}dx \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{X},\\
d(\mathbf{v}, p) &= \int_\Omega p\operatorname{div}\mathbf{v}dx \quad \forall\, \mathbf{v} \in \mathbf{V}, p \in M
\end{aligned}$$

and a trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}dx \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega).$$

It is easy to verify that if $\operatorname{div}\mathbf{u} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$, the trilinear form $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ satisfies the following important property:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}). \tag{2.1}$$

With the above notations, given $\mathbf{u}_0, \mathbf{b}_0 \in \mathbf{H}$ and $\mathbf{f} \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$, the weak variational formulation of the MHD problems (1.1)–(1.6) is defined to find $(\mathbf{u}, p, \mathbf{b}) \in \mathbf{L}^2(0, T; \mathbf{V}) \times L^2(0, T; M) \times \mathbf{L}^2(0, T; \mathbf{X})$ such that

$$(\mathbf{u}_t, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + S(\mathbf{b}\times\operatorname{curl}\mathbf{b}, \mathbf{v}) - d(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}, \tag{2.2}$$

and

$$(\mathbf{b}_t, \mathbf{w}) + a_2(\mathbf{b}, \mathbf{w}) + \frac{1}{Rm}(\operatorname{div}\mathbf{b}, \operatorname{div}\mathbf{w}) - (\mathbf{u}\times\mathbf{b}, \operatorname{curl}\mathbf{w}) = 0 \quad \forall\, \mathbf{w} \in \mathbf{X}, \tag{2.3}$$

where we use the formula:

$$\int_\Omega \operatorname{curl}\mathbf{u}\cdot\mathbf{v}dx = \int_\Omega \mathbf{u}\cdot\operatorname{curl}\mathbf{v}dx + \int_{\partial\Omega} \mathbf{n}\times\mathbf{u}\cdot\mathbf{v}ds. \tag{2.4}$$

Now, we recall Stokes operator A [25]. Let us denote the orthogonal projection operator by $\mathbb{P}_{\mathbf{H}}$ from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. Then Stokes operator A is defined by

$$A\mathbf{u} = -\mathbb{P}_{\mathbf{H}}\Delta\mathbf{u} \quad \forall\, \mathbf{u} \in \mathbf{V}_0 \cap \mathbf{H}^2(\Omega) := \mathbf{D}(A).$$

The emphasis of this paper is to prove some temporal and spatial error estimates of the time-discrete scheme (3.1)–(3.3) and the fully-discrete scheme (3.8)–(3.11) stated in Section 3. Then we make the following assumptions.

Assumption 1. The initial data $\mathbf{u}_0$, $\mathbf{b}_0$ and $\mathbf{f}$ satisfy

$$\|A\mathbf{u}_0\|_{L^2} + \|\mathbf{b}_0\|_{H^2} + \sup_{0\le t\le T}\left\{\|\mathbf{f}(t)\|_{L^2} + \|\mathbf{f}_t(t)\|_{L^2}\right\} \le C_3.$$

Assumption 2. The MHD problems (1.1)–(1.6) admit a unique local strong solution $(\mathbf{u}, p, \mathbf{b})$ on $[0, T]$ such that

$$\sup_{0\le t\le T}\left\{\|A\mathbf{u}(t)\|_{L^2} + \|\mathbf{b}(t)\|_{H^2} + \|p(t)\|_{H^1} + \|\mathbf{u}_t(t)\|_{L^2} + \|\mathbf{b}_t(t)\|_{L^2}\right\}$$
$$+ \int_0^T \left(\|\nabla\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{H^1}^2 + \|\mathbf{u}_{tt}\|_{H^{-1}}^2 + \|\mathbf{b}_{tt}\|_{H^{-1}}^2 + \tau(t)\|\mathbf{u}_{tt}\|_{L^2}^2 + \tau(t)\|\mathbf{b}_{tt}\|_{L^2}^2\right) dt \le C_4,$$

where $\tau(t) = \min\{t, 1\}$ is a weight function vanishing at $t = 0$.

Assumption 3. For given $\mathbf{g} \in \mathbf{L}^2(\Omega)$, the Stokes problems

$$\begin{aligned} -\Delta\mathbf{v} + \nabla\pi = \mathbf{g}, \qquad \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega,\\ \mathbf{v} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

admit a unique solution $(\mathbf{v}, \pi) \in \mathbf{D}(A) \times H^1(\Omega) \cap M$ satisfying

$$\|A\mathbf{v}\|_{L^2} + \|\nabla\pi\|_{L^2} \le C\|\mathbf{g}\|_{L^2}.$$

Assumption 4. For given $\mathbf{g} \in \mathbf{L}^2(\Omega)$, the magnetic problems

$$\begin{aligned} &\operatorname{curl}\operatorname{curl}\mathbf{v} = \mathbf{g}, && \operatorname{div}\mathbf{v} = 0, && \text{in } \Omega,\\ &\mathbf{v}\cdot\mathbf{n} = 0, && \operatorname{curl}\mathbf{v}\times\mathbf{n} = 0, && \text{on } \partial\Omega \end{aligned}$$

admit a unique solution $\mathbf{v} \in \mathbf{X}_0 \cap \mathbf{H}^2(\Omega)$ satisfying $\|\mathbf{v}\|_{H^2} \le C\|\mathbf{g}\|_{L^2}$.

The following known inequalities are frequently used [26]:

$$\|\mathbf{v}\|_{L^3} \le C\|\mathbf{v}\|_{L^2}^{1/2}\|\mathbf{v}\|_{H^1}^{1/2} \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.5}$$
$$\|\mathbf{v}\|_{L^\infty} \le C\|\mathbf{v}\|_{H^1}^{1/2}\|\mathbf{v}\|_{H^2}^{1/2} \quad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega), \tag{2.6}$$
$$\|\mathbf{v}\|_{H^2} \le C\|A\mathbf{v}\|_{L^2} \quad \forall\, \mathbf{v} \in \mathbf{D}(A), \tag{2.7}$$
$$\|\mathbf{v}\|_{H^2} \le C\|\operatorname{curl}\operatorname{curl}\mathbf{v}\|_{L^2} \quad \forall\, \mathbf{v} \in \mathbf{X}_0 \cap \mathbf{H}^2(\Omega), \tag{2.8}$$
$$\|\mathbf{v}\|_{H^1} \le C(\|\operatorname{curl}\mathbf{v}\|_{L^2} + \|\operatorname{div}\mathbf{v}\|_{L^2}) \quad \forall\, \mathbf{v} \in \mathbf{X}. \tag{2.9}$$

Finally, we recall a discrete version of Gronwall's inequality established in [27].

Lemma 2.1. *Let a_k, b_k, c_k and γ_k, for integers $k \ge 0$, be the nonnegative numbers such that*

$$a_n + \Delta t\sum_{k=0}^{n} b_k \le \Delta t\sum_{k=0}^{n}\gamma_k a_k + \Delta t\sum_{k=0}^{n} c_k + B \quad \text{for } n \ge 0. \tag{2.10}$$

Suppose that $\Delta t\gamma_k < 1$, for all k, and set $\sigma_k = (1 - \Delta t\gamma_k)^{-1}$. Then

$$a_n + \Delta t\sum_{k=0}^{n} b_k \le \exp\left(\Delta t\sum_{k=0}^{n}\gamma_k\sigma_k\right)\left(\Delta t\sum_{k=0}^{n} c_k + B\right) \quad \text{for } n \ge 0. \tag{2.11}$$

Remark 2.1. If the first sum on the right in (2.10) extends only up to $n - 1$, then the estimate (2.11) holds for all $k > 0$ with $\sigma_k = 1$.

3. Fractional-step scheme

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with time step $\Delta t = T/N$ and $t_n = n\Delta t, 0 \le n \le N$, where $[0, T]$ is the maximal time interval such that a local strong solution exists and satisfies Assumption 2. For any sequence of functions $\{g^n\}_{n=0}^N$, we define $D_t g^{n+1} = (g^{n+1} - g^n)/\Delta t$ for $0 \le n \le N-1$. Start with $\mathbf{u}^0 = \mathbf{u}_0$ and $\mathbf{b}^0 = \mathbf{b}_0$. The proposed time discrete fractional-step scheme is the following two-step scheme:

Step I: The first step of the fractional-step scheme consists of finding an intermediate velocity $\mathbf{u}^{n+1/2}$ and $\mathbf{b}^{n+1}$ such that

$$\frac{\mathbf{u}^{n+1/2} - \mathbf{u}^n}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}^{n+1/2} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1/2} + S\mathbf{b}^n \times \text{curl}\, \mathbf{b}^{n+1} = \mathbf{f}(t_{n+1}), \tag{3.1}$$

$$\frac{\mathbf{b}^{n+1} - \mathbf{b}^n}{\Delta t} + \frac{1}{Rm}\text{curl}\,(\text{curl}\, \mathbf{b}^{n+1}) - \text{curl}\,(\mathbf{u}^{n+1/2} \times \mathbf{b}^n) = 0, \qquad \text{div}\, \mathbf{b}^{n+1} = 0 \tag{3.2}$$

with the boundary conditions $\mathbf{u}^{n+1/2} = 0$, $\mathbf{b}^{n+1} \cdot \mathbf{n} = 0$ and $\text{curl}\, \mathbf{b}^{n+1} \times \mathbf{n} = 0$ on $\partial\Omega$.

Step II: Given $\mathbf{u}^{n+1/2}$ from (3.1), the second step of the fractional-step scheme consists of finding $\mathbf{u}^{n+1}$ and p^{n+1} such that

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}}{\Delta t} - \frac{1}{Re}\Delta(\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}) + \nabla p^{n+1} = 0, \quad \text{div}\, \mathbf{u}^{n+1} = 0 \tag{3.3}$$

with the boundary condition $\mathbf{u}^{n+1} = 0$ on $\partial\Omega$.

Since the problems (3.1)–(3.3) consist of three linear problems, their solvabilities should follow easily if we can establish the a priori energy estimates for the solutions to these problems. The following energy identity serves this purpose.

Theorem 3.1. *Let $\mathbf{u}^{n+1/2} \in \mathbf{V}$, $\mathbf{b}^{n+1} \in \mathbf{X}_0$ and $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{V}_0 \times M$ be the weak solutions to the problems (3.1)–(3.3). For $0 \le m \le N-1$, the following energy identity holds:*

$$\begin{aligned}
&\|\mathbf{u}^{m+1}\|_{L^2}^2 + S\|\mathbf{b}^{m+1}\|_{L^2}^2 + \frac{2S\Delta t}{Rm}\sum_{n=0}^{m}\|curl\, \mathbf{b}^{n+1}\|_{L^2}^2 \\
&\quad + \sum_{n=0}^{m}(\|\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}\|_{L^2}^2 + \|\mathbf{u}^n - \mathbf{u}^{n+1/2}\|_{L^2}^2 + S\|\mathbf{b}^{n+1} - \mathbf{b}^n\|_{L^2}^2) \\
&\quad + \frac{\Delta t}{Re}\sum_{n=0}^{m}(\|\nabla \mathbf{u}^{n+1}\|_{L^2}^2 + \|\nabla \mathbf{u}^{n+1/2}\|_{L^2}^2 + \|\nabla(\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2})\|_{L^2}) \\
&= \|\mathbf{u}_0\|_{L^2}^2 + S\|\mathbf{b}_0\|_{L^2}^2 + 2\Delta t\sum_{n=0}^{m}(\mathbf{f}(t_{n+1}), \mathbf{u}^{n+1/2}).
\end{aligned} \tag{3.4}$$

Proof. Multiplying (3.1)–(3.3) by $2\Delta t\mathbf{u}^{n+1/2}$, $2S\Delta t\mathbf{b}^{n+1}$ and $2\Delta t\mathbf{u}^{n+1}$, respectively, and adding three equations and using the following formula

$$2(\mathbf{u} - \mathbf{v}, \mathbf{u}) = \|\mathbf{u}\|_{L^2}^2 + \|\mathbf{u} - \mathbf{v}\|_{L^2}^2 - \|\mathbf{v}\|_{L^2}^2 \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega), \tag{3.5}$$

and the vector relation

$$(\mathbf{u} \times \text{curl}\, \mathbf{v}, \mathbf{w}) = (\mathbf{w} \times \mathbf{u}, \text{curl}\, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega), \tag{3.6}$$

we complete the proof of (3.4). $\square$

As a direct consequence of (2.9), (3.4) and $\text{div}\, \mathbf{b}^{n+1} = 0$, we can prove that $\mathbf{u}^{n+1/2}$, $\mathbf{u}^{n+1}$ and $\mathbf{b}^{n+1}$ are uniformly bounded in $l^2(\mathbf{H}^1)$, $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$, respectively.

Corollary 3.1. *The fractional-step scheme (3.1)–(3.3) is unconditionally stable. Moreover, there exists some $C > 0$ such that*

$$\begin{aligned}
&\|\mathbf{u}^{n+1}\|_{l^\infty(L^2)} + S\|\mathbf{b}^{n+1}\|_{l^\infty(L^2)} + \frac{1}{Re}\|\nabla \mathbf{u}^{n+1}\|_{l^2(L^2)} \\
&\quad + \frac{1}{Re}\|\nabla \mathbf{u}^{n+1/2}\|_{l^2(L^2)} + \frac{S}{Rm}\|\mathbf{b}^{n+1}\|_{l^2(H^1)} \le C,
\end{aligned}$$

where C depends on Re, Rm, S, $\mathbf{u}_0$, $\mathbf{b}_0$ and $\mathbf{f}$.

Next, we give the finite element approximations of (3.1)–(3.3). Let T_h be a quasi-uniform partition of Ω into tetrahedra of diameters by h with $0 < h < 1$. Let $C(\overline{\Omega})$ denote the space of continuous function in $\overline{\Omega} = \Omega \cup \partial\Omega$. For every $K \in T_h$ and a nonnegative integer r, $P_r(K)$ denotes the space of the polynomials on K of degree at most r. We use the boldface spaces

$\mathbf{C}(\overline{\Omega})$ and $\mathbf{P}_r(K)$ to denote the vector spaces $C(\overline{\Omega})^3$ and $P_r(K)^3$. Define the following conforming finite element subspaces of $\mathbf{V}$, M and $\mathbf{X}$ for the velocity, the pressure and the magnetic field, respectively, by

$$\begin{aligned}
\mathbf{V}_h &= \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}),\ \mathbf{v}_h \in \mathbf{P}_2(K),\ \forall K \in T_h\} \cap \mathbf{V},\\
\widetilde{\mathbf{V}}_h &= \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}),\ \mathbf{v}_h \in \mathbf{P}_1(K),\ \forall K \in T_h\} \cap \mathbf{V},\\
\mathbf{X}_h &= \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}),\ \mathbf{v}_h \in \mathbf{P}_1(K),\ \forall K \in T_h\} \cap \mathbf{X},\\
M_h &= \{q_h \in C(\overline{\Omega}),\ q_h \in P_1(K),\ \forall K \in T_h\} \cap M.
\end{aligned}$$

In such a way, the approximation spaces $\mathbf{V}_h$ and M_h satisfy the discrete inf–sup condition, i.e., there exists some positive constant $\beta > 0$ such that [25]

$$\beta \|q_h\|_{L^2} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h/\{\mathbf{0}\}} \frac{d(\mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{L^2}}. \tag{3.7}$$

Under the above notations, the finite element approximations of the fractional-step scheme (3.1)–(3.3) are described as follows. Start with $\mathbf{u}_h^0 = \mathbf{R}_h\mathbf{u}_0$, $\mathbf{b}_h^0 = \mathbf{Q}_h\mathbf{b}_0$, where the projectors $\mathbf{R}_h$ and $\mathbf{Q}_h$ are defined in Section 5. For $0 \leq n \leq N-1$, given $\mathbf{u}_h^n \in \mathbf{V}_h$ and $\mathbf{b}_h^n \in \mathbf{X}_h$, find $\mathbf{u}_h^{n+1/2} \in \widetilde{\mathbf{V}}_h$ and $\mathbf{b}_h^{n+1} \in \mathbf{X}_h$ such that

$$\left(\frac{\mathbf{u}_h^{n+1/2} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + a_1(\mathbf{u}_h^{n+1/2}, \mathbf{v}_h) + b_1(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) + S(\mathbf{b}_h^n \times \operatorname{curl} \mathbf{b}_h^{n+1}, \mathbf{v}_h) = (\mathbf{f}(t_{n+1}), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \widetilde{\mathbf{V}}_h, \tag{3.8}$$

where

$$b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) - \frac{1}{2} b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V},$$

and

$$\left(\frac{\mathbf{b}_h^{n+1} - \mathbf{b}_h^n}{\Delta t}, \mathbf{w}_h\right) + a_2(\mathbf{b}_h^{n+1}, \mathbf{w}_h) + \frac{1}{Rm}(\operatorname{div} \mathbf{b}_h^{n+1}, \operatorname{div} \mathbf{w}_h) - (\mathbf{u}_h^{n+1/2} \times \mathbf{b}_h^n, \operatorname{curl} \mathbf{w}_h) = 0 \quad \forall\, \mathbf{w}_h \in \mathbf{X}_h. \tag{3.9}$$

Then given $\mathbf{u}_h^{n+1/2} \in \widetilde{\mathbf{V}}_h$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times M_h$ such that

$$\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+1/2}}{\Delta t}, \mathbf{v}_h\right) + a_1(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^{n+1}) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \tag{3.10}$$

$$d(\mathbf{u}_h^{n+1}, q_h) = 0 \quad \forall\, q_h \in M_h. \tag{3.11}$$

By using a similar argument for Theorem 3.1, we can prove that the finite element approximation solutions $\mathbf{u}_h^{n+1/2}$, $\mathbf{u}_h^{n+1}$, $\mathbf{b}_h^{n+1}$, p_h^{n+1} satisfy the following discrete energy estimate.

Theorem 3.2. *For $0 \leq n \leq N-1$, let $\mathbf{u}_h^{n+1/2} \in \widetilde{\mathbf{V}}_h$, $\mathbf{b}_h^{n+1} \in \mathbf{X}_h$ and $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times M_h$ be the solutions to the problems (3.8)–(3.10). Then there exists some $C > 0$ such that*

$$\|\mathbf{u}_h^{n+1}\|_{l^\infty(L^2)} + S\|\mathbf{b}_h^{n+1}\|_{l^\infty(L^2)} + \frac{1}{Re}\|\nabla \mathbf{u}_h^{n+1}\|_{l^2(L^2)} + \frac{1}{Re}\|\nabla \mathbf{u}_h^{n+1/2}\|_{l^2(L^2)} + \frac{S}{Rm}\|\mathbf{b}_h^{n+1}\|_{l^2(H^1)} \leq C,$$

where C depends on Re, Rm, S, $\mathbf{u}_0$, $\mathbf{b}_0$ and $\mathbf{f}$.

The emphasis of this paper is to show the following error estimates.

Theorem 3.3. *Suppose that Assumptions 1–4 are satisfied. Then there exists some $\tau_0 > 0$ such that when $\tau < \tau_0$, the following temporal error estimates hold:*

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\|_{l^\infty(L^2)} + \frac{1}{Re}\|\nabla \mathbf{u}(t_{n+1}) - \nabla \mathbf{u}^{n+1}\|_{l^2(L^2)} \leq C\Delta t, \tag{3.12}$$

$$S\|\mathbf{b}(t_{n+1}) - \mathbf{b}^{n+1}\|_{l^\infty(L^2)} + \frac{S}{Rm}\|\mathbf{b}(t_{n+1}) - \mathbf{b}^{n+1}\|_{l^2(H^1)} \leq C\Delta t, \tag{3.13}$$

$$\|p(t_{n+1}) - p^{n+1}\|_{l^2(L^2)} \leq C\sqrt{\Delta t}, \tag{3.14}$$

$$\|\sqrt{\tau(t_{n+1})}(\nabla \mathbf{u}(t_{n+1}) - \nabla \mathbf{u}^{n+1})\|_{l^\infty(L^2)} \leq C\sqrt{\Delta t}, \tag{3.15}$$

$$\|\sqrt{\tau(t_{n+1})}(\mathbf{b}(t_{n+1}) - \mathbf{b}^{n+1})\|_{l^\infty(H^1)} \leq C\sqrt{\Delta t}, \tag{3.16}$$

where C depends on Re, Rm and S, and $\mathbf{u}^{n+1}$, p^{n+1}, $\mathbf{b}^{n+1}$ are the numerical solutions to the problems (3.1)–(3.3) at time-step $n+1$, and $\mathbf{u}(t_{n+1})$, $p(t_{n+1})$, $\mathbf{b}(t_{n+1})$ are the exact solutions evaluated at time t_{n+1}, and $\tau(t_{n+1}) = \min\{1, t_{n+1}\}$ for $0 \leq n \leq N-1$.

Theorem 3.4. *Suppose that Assumptions 1–4 are satisfied. If the time step Δt satisfies $\Delta t \geq Ch$ for some $C > 0$, then there exist some $\tau_0 > 0$ and $h_0 > 0$ such that when $\tau < \tau_0$ and $h < h_0$, the following error estimates hold:*

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|_{l^\infty(L^2)} + S\|\mathbf{b}(t_{n+1}) - \mathbf{b}_h^{n+1}\|_{l^\infty(L^2)} \leq C(\Delta t + h), \tag{3.17}$$

$$\frac{1}{Re}\|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1})\|_{l^2(L^2)} + \frac{S}{Rm}\|\mathbf{b}(t_{n+1}) - \mathbf{b}_h^{n+1}\|_{l^2(H^1)} \leq C(\Delta t + h), \tag{3.18}$$

where C depends on Re, Rm and S, and $\mathbf{u}_h^{n+1}$, p_h^{n+1}, $\mathbf{b}_h^{n+1}$ are the finite element approximation solutions at time-step $n+1$ for $0 \leq n \leq N-1$.

The proof of Theorems 3.3 and 3.4 is split into several lemmas in Sections 4 and 5.

4. Temporal error analysis

In this section, we will prove Theorem 3.3. For $0 \leq n \leq N-1$, let us denote errors by

$$\mathbf{e}_u^{n+1} = \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}, \qquad \mathbf{e}_u^{n+1/2} = \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1/2},$$
$$\mathbf{e}_b^{n+1} = \mathbf{b}(t_{n+1}) - \mathbf{b}^{n+1}, \qquad e_p^{n+1} = p(t_{n+1}) - p^{n+1}.$$

Taking $t = t_{n+1}$ in (1.1)–(1.4) yields

$$\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\Delta t} - \frac{1}{Re}\Delta\mathbf{u}(t_{n+1}) + (\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})$$
$$+ \nabla p(t_{n+1}) + S\mathbf{b}(t_{n+1}) \times \text{curl}\,\mathbf{b}(t_{n+1}) = \mathbf{f}(t_{n+1}) + \mathbf{R}_u^{n+1}, \tag{4.1}$$

$$\text{div}\,\mathbf{u}(t_{n+1}) = 0, \qquad \text{div}\,\mathbf{b}(t_{n+1}) = 0, \tag{4.2}$$

$$\frac{\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)}{\Delta t} + \frac{1}{Rm}\text{curl}\,(\text{curl}\,\mathbf{b}(t_{n+1})) - \text{curl}\,(\mathbf{u}(t_{n+1}) \times \mathbf{b}(t_{n+1})) = \mathbf{R}_b^{n+1}, \tag{4.3}$$

where

$$\mathbf{R}_u^{n+1} = -\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(t - t_n)\mathbf{u}_{tt}(t)dt, \qquad \mathbf{R}_b^{n+1} = -\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(t - t_n)\mathbf{b}_{tt}(t)dt.$$

From Assumption 2, one has

$$\|\mathbf{R}_u^{n+1}\|_{l^2(H^{-1})} + \|\mathbf{R}_b^{n+1}\|_{l^2(H^{-1})} \leq C\Delta t, \tag{4.4}$$

$$\|\sqrt{\tau(t_{n+1})}\mathbf{R}_u^{n+1}\|_{l^2(L^2)} + \|\sqrt{\tau(t_{n+1})}\mathbf{R}_b^{n+1}\|_{l^2(L^2)} \leq C\Delta t. \tag{4.5}$$

Subtracting (3.1) and (3.2) from (4.1) and (4.3), respectively, we obtain

$$\frac{\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n}{\Delta t} - \frac{1}{Re}\Delta\mathbf{e}_u^{n+1/2} + ((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\cdot\nabla)\mathbf{u}(t_{n+1}) + (\mathbf{e}_u^n\cdot\nabla)\mathbf{u}(t_{n+1}) + (\mathbf{u}^n\cdot\nabla)\mathbf{e}_u^{n+1/2}$$
$$+ S(\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) \times \text{curl}\,\mathbf{b}(t_{n+1}) + S\mathbf{e}_b^n \times \text{curl}\,\mathbf{b}(t_{n+1}) + S\mathbf{b}^n \times \text{curl}\,\mathbf{e}_b^{n+1} + \nabla p(t_{n+1}) = \mathbf{R}_u^{n+1} \tag{4.6}$$

and

$$\frac{\mathbf{e}_b^{n+1} - \mathbf{e}_b^n}{\Delta t} + \frac{1}{Rm}\text{curl curl}\,\mathbf{e}_b^{n+1} - \text{curl}\,(\mathbf{e}_u^{n+1/2} \times \mathbf{b}^n)$$
$$- \text{curl}\,(\mathbf{u}(t_{n+1}) \times \mathbf{e}_b^n) - \text{curl}\,(\mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n))) = \mathbf{R}_b^{n+1}. \tag{4.7}$$

First, we show that the fractional-step scheme (3.1)–(3.3) provides the temporal error estimates of $\mathcal{O}(\sqrt{\Delta t})$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$.

Lemma 4.1. *Suppose that Assumptions 1–2 are satisfied. Then there exists some positive constant $C > 0$ such that for every $0 \leq m \leq N-1$,*

$$\|\mathbf{e}_u^{m+1}\|_{L^2}^2 + S\|\mathbf{e}_b^{m+1}\|_{L^2}^2 + \frac{S\Delta t}{Rm}\sum_{n=0}^{m}\|curl\,\mathbf{e}_b^{n+1}\|_{L^2}^2 \quad + \sum_{n=0}^{m}(\|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n\|_{L^2}^2 + \|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^{n+1}\|_{L^2}^2 + S\|\mathbf{e}_b^{n+1} - \mathbf{e}_b^n\|_{L^2}^2)$$
$$+ \frac{\Delta t}{Re}\sum_{n=0}^{m}(\|\nabla\mathbf{e}_u^{n+1}\|_{L^2}^2 + \|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + \|\nabla(\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2})\|_{L^2}^2) \leq C\Delta t, \tag{4.8}$$

where C depends on Re, Rm and S.

Proof. Testing (4.6) and (4.7) by $2\Delta t\mathbf{e}_u^{n+1/2}$ and $2S\Delta t\mathbf{e}_b^{n+1}$, respectively, using (3.6) and adding two resulting equations lead to

$$\begin{aligned}
&\|\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + \|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n\|_{L^2}^2 - \|\mathbf{e}_u^n\|_{L^2}^2 + S\|\mathbf{e}_b^{n+1}\|_{L^2}^2 \\
&\quad + S\|\mathbf{e}_b^{n+1} - \mathbf{e}_b^n\|_{L^2}^2 - S\|\mathbf{e}_b^n\|_{L^2}^2 + \frac{2\Delta t}{Re}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|_{L^2}^2 \\
&\le 2\Delta t|b(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{e}_u^{n+1/2}) + b(\mathbf{e}_u^n, \mathbf{u}(t_{n+1}), \mathbf{e}_u^{n+1/2})| \\
&\quad + 2S\Delta t|((\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) \times \mathrm{curl}\,\mathbf{b}(t_{n+1}) + \mathbf{e}_b^n \times \mathrm{curl}\,\mathbf{b}(t_{n+1}), \mathbf{e}_u^{n+1/2})| \\
&\quad + 2S\Delta t|(\mathbf{u}(t_{n+1}) \times \mathbf{e}_b^n + \mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)), \mathrm{curl}\,\mathbf{e}_b^{n+1})| \\
&\quad + 2\Delta t|(\nabla p(t_{n+1}), \mathbf{e}_u^{n+1/2})| + 2\Delta t|(\mathbf{R}_u^{n+1}, \mathbf{e}_u^{n+1/2}) + S(\mathbf{R}_b^{n+1}, \mathbf{e}_b^{n+1})| \\
&= I_1 + \cdots + I_5.
\end{aligned} \tag{4.9}$$

By using Assumption 2 and Young's inequality, we have

$$\begin{aligned}
I_1 &= 2\Delta t|b(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{e}_u^{n+1/2}) + b(\mathbf{e}_u^n, \mathbf{u}(t_{n+1}), \mathbf{e}_u^{n+1/2})| \\
&\le C\Delta t(\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_{L^2} + \|\mathbf{e}_u^n\|_{L^2})\|A\mathbf{u}(t_{n+1})\|_{L^2}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2} \\
&\le \frac{\Delta t}{8Re}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + CRe\Delta t(\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_{L^2}^2 + \|\mathbf{e}_u^n\|_{L^2}^2).
\end{aligned}$$

By a similar method, we have

$$\begin{aligned}
I_2 &= 2S\Delta t|((\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) \times \mathrm{curl}\,\mathbf{b}(t_{n+1}) + \mathbf{e}_b^n \times \mathrm{curl}\,\mathbf{b}(t_{n+1}), \mathbf{e}_u^{n+1/2})| \\
&\le CS\Delta t(\|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|_{L^2} + \|\mathbf{e}_b^n\|_{L^2})\|\mathbf{b}(t_{n+1})\|_{H^2}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2} \\
&\le \frac{\Delta t}{8Re}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + CReS^2\Delta t(\|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|_{L^2}^2 + \|\mathbf{e}_b^n\|_{L^2}^2)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= 2S\Delta t|(\mathbf{u}(t_{n+1}) \times \mathbf{e}_b^n + \mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)), \mathrm{curl}\,\mathbf{e}_b^{n+1})| \\
&\le CS\Delta t(\|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|_{L^2} + \|\mathbf{e}_b^n\|_{L^2})\|A\mathbf{u}(t_{n+1})\|_{L^2}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|_{L^2} \\
&\le \frac{S\Delta t}{2Rm}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|_{L^2}^2 + CRmS\Delta t(\|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|_{L^2}^2 + \|\mathbf{e}_b^n\|_{L^2}^2).
\end{aligned}$$

Due to the fact that $\mathrm{div}\,\mathbf{e}_u^n = 0$ in Ω and $\mathbf{e}_u^n = 0$ on $\partial\Omega$, we have

$$\begin{aligned}
I_4 &= 2\Delta t|(\nabla p(t_{n+1}), \mathbf{e}_u^{n+1/2})| \\
&= 2\Delta t|(\nabla p(t_{n+1}), \mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n)| \\
&\le C\Delta t\|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n\|_{L^2}\|\nabla p(t_{n+1})\|_{L^2} \\
&\le \frac{1}{2}\|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n\|_{L^2}^2 + C(\Delta t)^2.
\end{aligned}$$

From (2.9) and Young's inequality, one has

$$\begin{aligned}
I_5 &= 2\Delta t|(\mathbf{R}_u^{n+1}, \mathbf{e}_u^{n+1/2}) + S(\mathbf{R}_b^{n+1}, \mathbf{e}_b^{n+1})| \\
&\le C\Delta t(\|\mathbf{R}_u^{n+1}\|_{H^{-1}}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2} + S\|\mathbf{R}_b^{n+1}\|_{H^{-1}}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|_{L^2}) \\
&\le \frac{\Delta t}{8Re}\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + \frac{S\Delta t}{2Rm}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|_{L^2}^2 + C\Delta t(Re\|\mathbf{R}_u^{n+1}\|_{H^{-1}}^2 + RmS\|\mathbf{R}_b^{n+1}\|_{H^{-1}}^2).
\end{aligned}$$

Using the definitions of $\mathbf{e}_u^{n+1}$ and $\mathbf{e}_u^{n+1/2}$, we rewrite (3.3) as

$$\frac{\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2}}{\Delta t} - \frac{1}{Re}\Delta(\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2}) - \nabla p^{n+1} = 0, \qquad \mathrm{div}\,\mathbf{e}_u^{n+1} = 0. \tag{4.10}$$

Testing (4.10) by $2\Delta t\mathbf{e}_u^{n+1}$ leads to

$$\|\mathbf{e}_u^{n+1}\|_{L^2}^2 + \|\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2}\|_{L^2}^2 - \|\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + \frac{\Delta t}{Re}(\|\nabla\mathbf{e}_u^{n+1}\|_{L^2}^2 + \|\nabla(\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2})\|_{L^2}^2 - \|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2) = 0. \tag{4.11}$$

Adding (4.9) and (4.11) and combining these estimates for I_1 to I_5 into the resulting inequality and using discrete Gronwall's inequality (2.11), we conclude that there exists some positive constant $C > 0$ depending on Re, Rm, S such that (4.8) holds. □

Remark 4.1. Assumption 2 and (4.8) imply that $\mathbf{u}^{n+1/2}$, $\mathbf{u}^{n+1}$ and $\mathbf{b}^{n+1}$ are uniform boundedness in $l^\infty(\mathbf{H}^1)$:

$$\|\nabla \mathbf{u}^{n+1}\|_{l^\infty(L^2)} + \|\nabla \mathbf{u}^{n+1/2}\|_{l^\infty(L^2)} + \|\mathbf{b}^{n+1}\|_{l^\infty(H^1)} \le C, \tag{4.12}$$

where C depends on Re, Rm and S.

Using the above uniform boundedness (4.12), we can improve the orders of the temporal error estimates derived in Lemma 4.1 as follows.

Lemma 4.2. *Suppose that Assumptions 1–2 are satisfied. Then there exists some positive constant $C > 0$ such that*

$$\|\mathbf{e}_u^{n+1}\|_{l^\infty(L^2)} + S\|\mathbf{e}_b^{n+1}\|_{l^\infty(L^2)} + \frac{1}{Re}\|\nabla \mathbf{e}_u^{n+1}\|_{l^2(L^2)} + \frac{S}{Rm}\|\mathbf{e}_b^{n+1}\|_{l^2(H^1)} \le C\Delta t, \tag{4.13}$$

$$\|\mathbf{e}_u^{n+1} - \mathbf{e}_u^n\|^2_{l^2(L^2)} \le C(\Delta t)^3, \tag{4.14}$$

where C depends on Re, Rm and S.

Proof. Taking the sum of (4.6) and (4.10) gives

$$\begin{aligned}
&\frac{\mathbf{e}_u^{n+1} - \mathbf{e}_u^n}{\Delta t} - \frac{1}{Re}\Delta \mathbf{e}_u^{n+1} + ((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\cdot\nabla)\mathbf{u}(t_{n+1}) + (\mathbf{e}_u^n\cdot\nabla)\mathbf{u}^{n+1/2} + (\mathbf{u}(t_n)\cdot\nabla)\mathbf{e}_u^{n+1/2} \\
&\quad + S(\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) \times \mathrm{curl}\,\mathbf{b}(t_{n+1}) + S\mathbf{e}_b^n \times \mathrm{curl}\,\mathbf{b}^{n+1} + S\mathbf{b}(t_n)\times \mathrm{curl}\,\mathbf{e}_b^{n+1} + \nabla e_p^{n+1} = \mathbf{R}_u^{n+1}.
\end{aligned} \tag{4.15}$$

Subtracting (3.2) from (4.3) and using

$$\mathbf{u}(t_{n+1}) \times \mathbf{b}(t_{n+1}) - \mathbf{u}^{n+1/2} \times \mathbf{b}^n = \mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) + \mathbf{e}_u^{n+1/2} \times \mathbf{b}(t_n) + \mathbf{u}^{n+1/2} \times \mathbf{e}_b^n$$

yields

$$\begin{aligned}
&\frac{\mathbf{e}_b^{n+1} - \mathbf{e}_b^n}{\Delta t} + \frac{1}{Rm}\mathrm{curl}\,\mathrm{curl}\,\mathbf{e}_b^{n+1} - \mathrm{curl}\,(\mathbf{e}_u^{n+1/2} \times \mathbf{b}(t_n)) - \mathrm{curl}\,(\mathbf{u}^{n+1/2} \times \mathbf{e}_b^n) \\
&\qquad - \mathrm{curl}\,(\mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n))) = \mathbf{R}_b^{n+1}
\end{aligned} \tag{4.16}$$

and $\mathrm{div}\,\mathbf{e}_b^{n+1} = 0$. Testing (4.15) and (4.16) by $2\Delta t\mathbf{e}_u^{n+1}$ and $2S\Delta t\mathbf{e}_b^{n+1}$, respectively, using (3.6) and adding two resulting equations leads to

$$\begin{aligned}
&\|\mathbf{e}_u^{n+1}\|^2_{L^2} + \|\mathbf{e}_u^{n+1} - \mathbf{e}_u^n\|^2_{L^2} - \|\mathbf{e}_u^n\|^2_{L^2} + S\|\mathbf{e}_b^{n+1}\|^2_{L^2} \\
&\quad + S\|\mathbf{e}_b^{n+1} - \mathbf{e}_b^n\|^2_{L^2} - S\|\mathbf{e}_b^n\|^2_{L^2} + \frac{2\Delta t}{Re}\|\nabla \mathbf{e}_u^{n+1}\|^2_{L^2} + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|^2_{L^2} \\
&\le 2\Delta t|b(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{e}_u^{n+1})| + 2S\Delta t|((\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) \times \mathrm{curl}\,\mathbf{b}(t_{n+1}), \mathbf{e}_u^{n+1})| \\
&\quad + 2S\Delta t|(\mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)), \mathrm{curl}\,\mathbf{e}_b^{n+1})| + 2\Delta t|(\mathbf{R}_u^{n+1}, \mathbf{e}_u^{n+1}) + S(\mathbf{R}_b^{n+1}, \mathbf{e}_b^{n+1})| \\
&\quad + 2\Delta t|b(\mathbf{u}(t_n), \mathbf{e}_u^{n+1/2}, \mathbf{e}_u^{n+1}) + b(\mathbf{e}_u^n, \mathbf{u}^{n+1/2}, \mathbf{e}_u^{n+1})| + 2S\Delta t|(\mathbf{u}^{n+1/2} \times \mathbf{e}_b^n, \mathrm{curl}\,\mathbf{e}_b^{n+1}) + (\mathbf{e}_b^n \times \mathrm{curl}\,\mathbf{b}^{n+1}, \mathbf{e}_u^{n+1})| \\
&\quad + 2S\Delta t|(\mathbf{b}(t_n) \times \mathrm{curl}\,\mathbf{e}_b^{n+1}, \mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2})| \\
&= I_6 + \cdots + I_{12}.
\end{aligned} \tag{4.17}$$

By a similar method in the proof of Lemma 4.1, we can bound $I_6 + \cdots + I_9$ as

$$\begin{aligned}
&I_6 + I_7 + I_8 + I_9 \\
&= 2\Delta t|b(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{e}_u^{n+1})| + 2S\Delta t|((\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)) \times \mathrm{curl}\,\mathbf{b}(t_{n+1}), \mathbf{e}_u^{n+1})| \\
&\quad + 2S\Delta t|(\mathbf{u}(t_{n+1}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)), \mathrm{curl}\,\mathbf{e}_b^{n+1})| + 2\Delta t|(\mathbf{R}_u^{n+1}, \mathbf{e}_u^{n+1}) + S(\mathbf{R}_b^{n+1}, \mathbf{e}_b^{n+1})| \\
&\le \frac{\Delta t}{4Re}\|\nabla \mathbf{e}_u^{n+1}\|^2_{L^2} + \frac{S\Delta t}{4Rm}\|\mathrm{curl}\,\mathbf{e}_b^{n+1}\|^2_{L^2} + C\Delta t\left(Re\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|^2_{L^2} + (ReS^2 + RmS)\|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|^2_{L^2}\right) \\
&\quad + C\Delta t\left(Re\|\mathbf{R}_u^{n+1}\|^2_{H^{-1}} + RmS\|\mathbf{R}_b^{n+1}\|^2_{H^{-1}}\right).
\end{aligned}$$

From (2.5)–(2.7), we estimate I_{10} as

$$\begin{aligned}
I_{10} &= 2\Delta t|b(\mathbf{u}(t_n), \mathbf{e}_u^{n+1/2}, \mathbf{e}_u^{n+1}) + b(\mathbf{e}_u^n, \mathbf{u}^{n+1/2}, \mathbf{e}_u^{n+1})| \\
&\le C\Delta t\|A\mathbf{u}(t_n)\|_{L^2}\|\nabla \mathbf{e}_u^{n+1}\|_{L^2}(\|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n\|_{L^2} + \|\mathbf{e}_u^n\|_{L^2}) + C\Delta t\|\nabla \mathbf{u}^{n+1/2}\|_{L^2}\|\nabla \mathbf{e}_u^{n+1}\|_{L^2}\|\mathbf{e}_u^n\|^{1/2}_{L^2}\|\nabla \mathbf{e}_u^n\|^{1/2}_{L^2} \\
&\le \frac{\Delta t}{4Re}\|\nabla \mathbf{e}_u^{n+1}\|^2_{L^2} + \frac{\Delta t}{2Re}\|\nabla \mathbf{e}_u^n\|^2_{L^2} + CRe\Delta t(\|\mathbf{e}_u^{n+1/2} - \mathbf{e}_u^n\|^2_{L^2} + \|\mathbf{e}_u^n\|^2_{L^2}).
\end{aligned}$$

Similarly, from (2.9), I_{11} is bounded by

$$\begin{aligned}
I_{11} &= 2S\Delta t|(\mathbf{u}^{n+1/2}\times \mathbf{e}_b^n, \operatorname{curl}\mathbf{e}_b^{n+1}) + (\mathbf{e}_b^n \times \operatorname{curl}\mathbf{b}^{n+1}, \mathbf{e}_u^{n+1})| \\
&\le CS\Delta t\|\nabla \mathbf{u}^{n+1/2}\|_{L^2}\|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}\|\mathbf{e}_b^n\|_{L^2}^{1/2}\|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}^{1/2} + CS\Delta t\|\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2}\|\nabla\mathbf{e}_u^{n+1}\|_{L^2}\|\mathbf{e}_b^n\|_{L^2}^{1/2}\|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}^{1/2} \\
&\le \frac{\Delta t}{4Re}\|\nabla \mathbf{e}_u^{n+1}\|_{L^2}^2 + \frac{S\Delta t}{4Rm}\|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2 + \frac{S\Delta t}{2Rm}\|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}^2 + C(Re^2RmS^3 + Rm^3S)\Delta t\|\mathbf{e}_b^n\|_{L^2}^2.
\end{aligned}$$

The last term I_{12} satisfies

$$\begin{aligned}
I_{12} &= 2S\Delta t|(\mathbf{b}(t_n)\times \operatorname{curl}\mathbf{e}_b^{n+1}, \mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2})| \\
&\le CS\Delta t\|\mathbf{b}(t_n)\|_{H^2}\|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}\|\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2}\|_{L^2} \\
&\le \frac{S\Delta t}{4Rm}\|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2 + CRmS\Delta t\|\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n+1/2}\|_{L^2}^2.
\end{aligned}$$

Combining these estimates into (4.17) and using the discrete Gronwall's inequality (2.11), (4.4), (4.8), we complete the proof of (4.13) and (4.14). □

The temporal error estimate for the pressure in $l^2(L^2)$ is given in the next lemma by using the inf–sup condition.

Lemma 4.3. *Suppose that Assumptions 1–2 are satisfied. Then there exists some positive constant $C > 0$ such that*

$$\|e_p^{n+1}\|_{l^2(L^2)} \le C\sqrt{\Delta t}, \tag{4.18}$$

where C depends on Re, Rm and S.

Proof. From (4.15) and inf–sup condition, one has

$$\begin{aligned}
\|e_p^{n+1}\|_{L^2} &\le C\sup_{\mathbf{v}\in\mathbf{V}}\frac{d(\mathbf{v}, e_p^{n+1})}{\|\nabla\mathbf{v}\|_{L^2}} \\
&\le \frac{C}{\Delta t}\|\mathbf{e}_u^{n+1} - \mathbf{e}_u^n\|_{L^2} + C\|\nabla\mathbf{e}_u^{n+1}\|_{L^2} + C\|\mathbf{R}_u^{n+1}\|_{H^{-1}} + C(\|\nabla\mathbf{e}_u^n\|_{L^2} + \|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2} + \|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_{L^2}) \\
&\quad + C(\|\operatorname{curl}\mathbf{e}_b^n\|_{L^2} + \|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2} + \|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|_{L^2}).
\end{aligned}$$

Taking the sum of the above inequality from 0 to $N-1$ and using (4.4), (4.8) and (4.14), we complete the proof of Lemma 4.3. □

As a consequence of (4.14) and Assumption 2, we have

$$\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{l^\infty(L^2)} + \|\mathbf{b}^{n+1} - \mathbf{b}^n\|_{l^\infty(L^2)} \le C\Delta t. \tag{4.19}$$

To give the spatial error estimates for the fully discrete fractional-step scheme (3.8)–(3.11), we need to show the regularity results for $\mathbf{u}^{n+1/2}$, $\mathbf{u}^{n+1}$, p^{n+1} and $\mathbf{b}^{n+1}$.

Lemma 4.4. *Suppose that Assumptions 1–4 are satisfied. Then there exists some $\tau_0 > 0$ such that when $\Delta t < \tau_0$, there holds*

$$\frac{1}{Re}\|A\mathbf{u}^{n+1}\|_{l^2(L^2)} + \frac{1}{Re}\|\mathbf{u}^{n+1/2}\|_{l^2(H^2)} + \|\nabla p^{n+1}\|_{l^2(L^2)} + \frac{1}{Rm}\|\mathbf{b}^{n+1}\|_{l^2(H^2)} \le C, \tag{4.20}$$

where C depends on Re, Rm and S.

Proof. Taking the sum of (3.1) and (3.3) leads to

$$-\frac{1}{Re}\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}(t_{n+1}) - D_t\mathbf{u}^{n+1} - (\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1/2} - S\mathbf{b}^n\times\operatorname{curl}\mathbf{b}^{n+1}, \tag{4.21}$$

$$\operatorname{div}\mathbf{u}^{n+1} = 0. \tag{4.22}$$

Using Assumption 3, we get

$$\frac{1}{Re}\|A\mathbf{u}^{n+1}\|_{L^2} + \|\nabla p^{n+1}\|_{L^2} \le C(\|D_t\mathbf{u}^{n+1}\|_{L^2} + \|\mathbf{f}(t_{n+1})\|_{L^2} + S\|\mathbf{b}^n\times\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2} + \|(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1/2}\|_{L^2}). \tag{4.23}$$

The last two terms in the above inequality can be bounded, respectively, by

$$\begin{aligned}
S\|\mathbf{b}^n\times\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2} &\le S\|\mathbf{e}_b^n\times\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2} + S\|\mathbf{b}(t_n)\times\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2} \\
&\le CS(\|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}\|\mathbf{b}^{n+1}\|_{H^2} + \|\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2})
\end{aligned}$$

and

$$\begin{aligned}\|(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1/2}\|_{L^2} &\le C\|\nabla\mathbf{u}^n\|_{L^2}^{1/2}\|A\mathbf{u}^n\|_{L^2}^{1/2}\|\nabla\mathbf{u}^{n+1/2}\|_{L^2}\\ &\le \frac{1}{2Re}\|A\mathbf{u}^n\|_{L^2}+CRe\|\nabla\mathbf{u}^n\|_{L^2},\end{aligned}$$

where we use (4.12). Similarly, from Assumption 4 and (3.2) and the following formula

$$\mathrm{curl}\,(\mathbf{u}\times\mathbf{v})=(\mathrm{div}\,\mathbf{v})\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{u}-(\mathrm{div}\,\mathbf{u})\mathbf{v}, \tag{4.24}$$

we have

$$\begin{aligned}\frac{1}{Rm}\|\mathbf{b}^{n+1}\|_{H^2} &\le C(\|D_t\mathbf{b}^{n+1}\|_{L^2}+\|(\mathbf{u}^{n+1/2}\cdot\nabla)\mathbf{b}^n\|_{L^2})+C(\|(\mathbf{b}^n\cdot\nabla)\mathbf{u}^{n+1/2}\|_{L^2}+\|(\mathrm{div}\,\mathbf{u}^{n+1/2})\mathbf{b}^n\|_{L^2})\\ &\le C(\|D_t\mathbf{b}^{n+1}\|_{L^2}+\|\nabla\mathbf{b}^n\|_{L^3}\|\nabla\mathbf{u}^{n+1/2}\|_{L^2}+\|\mathbf{b}^n\|_{L^\infty}\|\nabla\mathbf{u}^{n+1/2}\|_{L^2})\\ &\le C(\|D_t\mathbf{b}^{n+1}\|_{L^2}+\|\mathrm{curl}\,\mathbf{b}^n\|_{L^2}^{1/2}\|\mathbf{b}^n\|_{H^2}^{1/2})\\ &\le C(\|D_t\mathbf{b}^{n+1}\|_{L^2}+Rm\|\mathrm{curl}\,\mathbf{b}^n\|_{L^2})+\frac{1}{2Rm}\|\mathbf{b}^n\|_{H^2}.\end{aligned} \tag{4.25}$$

Combining (4.23) and (4.25), and taking the sum from 0 to $N-1$ yield

$$\begin{aligned}&\Delta t\sum_{n=0}^{N-1}\left(\frac{1}{Re^2}\|A\mathbf{u}^{n+1}\|_{L^2}^2+\|\nabla p^{n+1}\|_{L^2}^2+\frac{1}{Rm^2}\|\mathbf{b}^{n+1}\|_{H^2}^2\right)\\ &\quad\le C\Delta t\sum_{n=0}^{N-1}\left(\|D_t\mathbf{u}^{n+1}\|_{L^2}^2+\|D_t\mathbf{b}^{n+1}\|_{L^2}^2+\|\mathbf{f}(t_{n+1})\|_{L^2}^2\right)+C\Delta t\left(\|\mathrm{curl}\,\mathbf{b}^{n+1}\|_{L^2}^2+\|\mathrm{curl}\,\mathbf{b}^n\|_{L^2}^2+\|\nabla\mathbf{u}^n\|_{L^2}^2\right)\\ &\qquad+CS^2\Delta t\sum_{n=0}^{N-1}\|\mathrm{curl}\,\mathbf{e}_b^n\|_{L^2}^2\|\mathbf{b}^{n+1}\|_{H^2}^2+\frac{\Delta t}{4}\sum_{n=0}^{N-1}\left(\frac{1}{Re^2}\|A\mathbf{u}^n\|_{L^2}^2+\frac{1}{Rm^2}\|\mathbf{b}^n\|_{H^2}^2\right).\end{aligned}$$

The last term in the above inequality can be absorbed on the left-hand side. From (4.13) and (4.19), we have

$$\Delta t\sum_{n=0}^{N-1}\left(\frac{1}{Re^2}\|A\mathbf{u}^{n+1}\|_{L^2}^2+\|\nabla p^{n+1}\|_{L^2}^2+\frac{1}{Rm^2}\|\mathbf{b}^{n+1}\|_{H^2}^2\right)\le C+C_6S^2(\Delta t)^2\sum_{n=0}^{N-1}\|\mathbf{b}^{n+1}\|_{H^2}^2.$$

Let $\tau_0=\frac{1}{2C_6S^2Rm^2}$. When $\Delta t<\tau_0$, the above inequality implies that

$$\frac{1}{Re}\|A\mathbf{u}^{n+1}\|_{l^2(L^2)}+\|\nabla p^{n+1}\|_{l^2(L^2)}+\frac{1}{Rm}\|\mathbf{b}^{n+1}\|_{l^2(H^2)}\le C. \tag{4.26}$$

By using the regularity result [28] for elliptic problem and (4.26), we can easily get

$$\frac{1}{Re}\|\mathbf{u}^{n+1/2}\|_{l^2(H^2)}\le C,$$

if we notice (4.8) and

$$\|\mathbf{u}^{n+1/2}-\mathbf{u}^n\|_{L^2}\le\|\mathbf{e}_u^{n+1/2}-\mathbf{e}_u^n\|_{L^2}+\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{L^2}.$$

We complete the proof of (4.20). □

Remark 4.2. From (4.20) and Assumption 2, we have

$$\frac{1}{Re}\|A\mathbf{e}_u^{n+1}\|_{l^2(L^2)}+\|\nabla e_p^{n+1}\|_{l^2(L^2)}+\frac{1}{Rm}\|\mathbf{e}_b^{n+1}\|_{l^2(H^2)}\le C. \tag{4.27}$$

Lemma 4.5. *Suppose that Assumptions 1–4 are satisfied. Then when $\tau<\tau_0$, there holds*

$$\|\sqrt{\tau(t_{n+1})}\nabla\mathbf{e}_u^{n+1}\|_{l^\infty(L^2)}+\|\sqrt{\tau(t_{n+1})}curl\,\mathbf{e}_b^{n+1}\|_{l^\infty(L^2)}\le C\sqrt{\Delta t}, \tag{4.28}$$

where τ_0 is defined in Lemma 4.4, and C depends on Re, Rm and S.

Proof. Testing (4.15) by $-2\Delta t A\mathbf{e}_u^{n+1}$ gives

$$\begin{aligned}
&\|\nabla \mathbf{e}_u^{n+1}\|_{L^2}^2 - \|\nabla \mathbf{e}_u^{n}\|_{L^2}^2 + \|\nabla(\mathbf{e}_u^{n+1} - \mathbf{e}_u^{n})\|_{L^2}^2 + \frac{\Delta t}{Re}\|A\mathbf{e}_u^{n+1}\|_{L^2}^2 \\
&\quad\le C\Delta t\|\mathbf{R}_u^{n+1}\|_{L^2}^2 + C\Delta t\|((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\cdot\nabla)\mathbf{u}(t_{n+1})\|_{L^2}^2 + C\Delta t\|(\mathbf{e}_u^n\cdot\nabla)\mathbf{u}^{n+1/2} + (\mathbf{u}(t_n)\cdot\nabla)\mathbf{e}_u^{n+1/2}\|_{L^2}^2 \\
&\qquad + C\Delta t\|(\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n))\times \operatorname{curl}\mathbf{b}(t_{n+1})\|_{L^2}^2 + C\Delta t\|\mathbf{e}_b^n\times\operatorname{curl}\mathbf{b}^{n+1} + \mathbf{b}(t_n)\times\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2 \\
&\quad= C\Delta t\|\mathbf{R}_u^{n+1}\|_{L^2}^2 + I_{13} + \cdots + I_{16}.
\end{aligned} \tag{4.29}$$

From Assumption 2, one has

$$\begin{aligned}
I_{13} &= C\Delta t\|((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\cdot\nabla)\mathbf{u}(t_{n+1})\|_{L^2}^2 \\
&\le C\Delta t\|A\mathbf{u}(t_{n+1})\|_{L^2}^2\|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\|_{L^2}^2 \\
&\le C(\Delta t)^2\int_{t_n}^{t_{n+1}}\|\nabla\mathbf{u}_t\|_{L^2}^2 dt.
\end{aligned}$$

Using (2.6) and (4.12), we get

$$\begin{aligned}
I_{14} &= C\Delta t\|(\mathbf{e}_u^n\cdot\nabla)\mathbf{u}^{n+1/2} + (\mathbf{u}(t_n)\cdot\nabla)\mathbf{e}_u^{n+1/2}\|_{L^2}^2 \\
&\le C\Delta t(\|\mathbf{e}_u^n\|_{L^\infty}^2\|\nabla\mathbf{u}^{n+1/2}\|_{L^2}^2 + \|A\mathbf{u}(t_n)\|_{L^2}^2\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2) \\
&\le C\Delta t(\|\nabla\mathbf{e}_u^n\|_{L^2}\|A\mathbf{e}_u^n\|_{L^2} + \|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2) \\
&\le \frac{\Delta t}{2Re}\|A\mathbf{e}_u^n\|_{L^2}^2 + C\Delta t(\|\nabla\mathbf{e}_u^n\|_{L^2}^2 + \|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2).
\end{aligned}$$

By a similar argument for the above two inequalities, we have

$$\begin{aligned}
I_{15} &= C\Delta t\|(\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n))\times\operatorname{curl}\mathbf{b}(t_{n+1})\|_{L^2}^2 \\
&\le C\Delta t\|\mathbf{b}(t_{n+1})\|_{H^2}^2\|\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)\|_{H^1}^2 \\
&\le C(\Delta t)^2\int_{t_n}^{t_{n+1}}\|\mathbf{b}_t\|_{H^1}^2 dt,
\end{aligned}$$

and

$$\begin{aligned}
I_{16} &= C\Delta t\|\mathbf{e}_b^n\times\operatorname{curl}\mathbf{b}^{n+1} + \mathbf{b}(t_n)\times\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2 \\
&\le C\Delta t(\|\mathbf{e}_b^n\|_{L^\infty}^2\|\operatorname{curl}\mathbf{b}^{n+1}\|_{L^2}^2 + \|\mathbf{b}(t_n)\|_{H^2}^2\|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2) \\
&\le \frac{C_7\Delta t}{2Rm}\|\mathbf{e}_b^n\|_{H^2}^2 + C\Delta t(\|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}^2 + \|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2).
\end{aligned}$$

Combining these estimates into (4.29) yields

$$\begin{aligned}
&\|\nabla\mathbf{e}_u^{n+1}\|_{L^2}^2 - \|\nabla\mathbf{e}_u^n\|_{L^2}^2 + \frac{\Delta t}{Re}\|A\mathbf{e}_u^{n+1}\|_{L^2}^2 \\
&\quad\le C\Delta t\|\mathbf{R}_u^{n+1}\|_{L^2}^2 + \frac{\Delta t}{2Re}\|A\mathbf{e}_u^n\|_{L^2}^2 + \frac{C_7\Delta t}{2Rm}\|\mathbf{e}_b^n\|_{H^2}^2 + C\Delta t(\|\nabla\mathbf{e}_u^n\|_{L^2}^2 + \|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2 + \|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}^2 + \|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2) \\
&\qquad + C(\Delta t)^2\left(\int_{t_n}^{t_{n+1}}\|\nabla\mathbf{u}_t\|_{L^2}^2 dt + \int_{t_n}^{t_{n+1}}\|\mathbf{b}_t\|_{H^1}^2 dt\right).
\end{aligned} \tag{4.30}$$

Testing (4.7) by $2\Delta t\operatorname{curl}\operatorname{curl}\mathbf{e}_b^{n+1}$ leads to

$$\begin{aligned}
&\|\operatorname{curl}\mathbf{e}_b^{n+1}\|_{L^2}^2 - \|\operatorname{curl}\mathbf{e}_b^n\|_{L^2}^2 + \frac{C_7\Delta t}{Rm}\|\mathbf{e}_b^{n+1}\|_{H^2}^2 \\
&\quad\le C\Delta t\|\mathbf{R}_b^{n+1}\|_{L^2}^2 + C\Delta t\|\operatorname{curl}(\mathbf{e}_u^{n+1/2}\times\mathbf{b}^n)\|_{L^2}^2 + C\Delta t\|\operatorname{curl}(\mathbf{u}(t_{n+1})\times\mathbf{e}_b^n)\|_{L^2}^2 \\
&\qquad + C\Delta t\|\operatorname{curl}(\mathbf{u}(t_{n+1})\times(\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)))\|_{L^2}^2 \\
&\quad= C\Delta t\|\mathbf{R}_b^{n+1}\|_{L^2}^2 + I_{17} + I_{18} + I_{19}.
\end{aligned} \tag{4.31}$$

Using (2.5), (2.6) and (4.24), we obtain

$$\begin{aligned}
I_{17} &= C\Delta t\|\text{curl }(\mathbf{e}_u^{n+1/2}\times \mathbf{b}^n)\|_{L^2}^2\\
&\le C\Delta t(\|(\mathbf{e}_u^{n+1/2}\cdot\nabla)\mathbf{b}^n\|_{L^2}^2+\|(\mathbf{b}^n\cdot\nabla)\mathbf{e}_u^{n+1/2}\|_{L^2}^2+\|(\text{div }\mathbf{e}_u^{n+1/2})\mathbf{b}^n\|_{L^2}^2)\\
&\le C\Delta t(\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2\|\mathbf{e}_b^n\|_{H^2}\|\text{curl }\mathbf{e}_b^n\|_{L^2}+\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2\|\mathbf{b}(t_n)\|_{H^2}^2)\\
&\le C(\Delta t)^2\|\mathbf{e}_b^n\|_{H^2}^2+C\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2+C\Delta t\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_{18} &= C\Delta t\|\text{curl }(\mathbf{u}(t_{n+1})\times\mathbf{e}_b^n)\|_{L^2}^2\\
&\le C\Delta t(\|(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{e}_b^n\|_{L^2}^2+\|(\mathbf{e}_b^n\cdot\nabla)\mathbf{u}(t_{n+1})\|_{L^2}^2)\\
&\le C\Delta t\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2\|A\mathbf{u}(t_{n+1})\|_{L^2}^2\le C\Delta t\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
I_{19} &= C\Delta t\|\text{curl }(\mathbf{u}(t_{n+1})\times(\mathbf{b}(t_{n+1})-\mathbf{b}(t_n)))\|_{L^2}^2\\
&\le C\Delta t(\|(\mathbf{u}(t_{n+1})\cdot\nabla)(\mathbf{b}(t_{n+1})-\mathbf{b}(t_n))\|_{L^2}^2+\|((\mathbf{b}(t_{n+1})-\mathbf{b}(t_n))\cdot\nabla)\mathbf{u}(t_{n+1})\|_{L^2}^2)\\
&\le C\Delta t\|\mathbf{b}(t_{n+1})-\mathbf{b}(t_n)\|_{H^1}^2\|A\mathbf{u}(t_{n+1})\|_{L^2}^2\\
&\le C(\Delta t)^2\int_{t_n}^{t_{n+1}}\|\mathbf{b}_t\|_{H^1}^2dt.
\end{aligned}$$

Taking the sum of (4.30) and (4.31) and combining the above estimates into the resulting inequality give

$$\begin{aligned}
&\|\nabla\mathbf{e}_u^{n+1}\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^{n+1}\|_{L^2}^2+\frac{\Delta t}{Re}\|A\mathbf{e}_u^{n+1}\|_{L^2}^2+\frac{C_7\Delta t}{Rm}\|\mathbf{e}_b^{n+1}\|_{H^2}^2\\
&\le \|\nabla\mathbf{e}_u^{n}\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^{n}\|_{L^2}^2+C\Delta t(\|\mathbf{R}_u^{n+1}\|_{L^2}^2+\|\mathbf{R}_b^{n+1}\|_{L^2}^2)+\frac{\Delta t}{2Re}\|A\mathbf{e}_u^{n}\|_{L^2}^2+\frac{C_7\Delta t}{2Rm}\|\mathbf{e}_b^{n}\|_{H^2}^2\\
&\quad+C\Delta t^2\|\mathbf{e}_b^n\|_{H^2}^2+C\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2+C\Delta t(\|\nabla\mathbf{e}_u^n\|_{L^2}^2+\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^{n+1}\|_{L^2}^2)\\
&\quad+C(\Delta t)^2\left(\int_{t_n}^{t_{n+1}}\|\nabla\mathbf{u}_t\|_{L^2}^2dt+\int_{t_n}^{t_{n+1}}\|\mathbf{b}_t\|_{H^1}^2dt\right).
\end{aligned}$$

Multiplying it by $\tau(t_{n+1})$ and using $\tau(t_{n+1})\le\tau(t_n)+\Delta t$ and $|\tau(t_{n+1})|\le 1$, we derive that

$$\begin{aligned}
&\tau(t_{n+1})\|\nabla\mathbf{e}_u^{n+1}\|_{L^2}^2+\tau(t_{n+1})\|\text{curl }\mathbf{e}_b^{n+1}\|_{L^2}^2+\frac{\Delta t}{Re}\tau(t_{n+1})\|A\mathbf{e}_u^{n+1}\|_{L^2}^2+\frac{C_7\Delta t}{Rm}\tau(t_{n+1})\|\mathbf{e}_b^{n+1}\|_{H^2}^2\\
&\le\tau(t_n)\|\nabla\mathbf{e}_u^n\|_{L^2}^2+\tau(t_n)\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2+\Delta t(\|\nabla\mathbf{e}_u^n\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2)+C\Delta t(\tau(t_{n+1})\|\mathbf{R}_u^{n+1}\|_{L^2}^2+\tau(t_{n+1})\|\mathbf{R}_b^{n+1}\|_{L^2}^2)\\
&\quad+\frac{\Delta t}{4Re}\tau(t_n)\|A\mathbf{e}_u^n\|_{L^2}^2+\tau(t_n)\frac{S\Delta t}{4Rm}\|\mathbf{e}_b^n\|_{H^2}^2+C(\Delta t)^2(\|A\mathbf{e}_u^n\|_{L^2}^2+\|\mathbf{e}_b^n\|_{H^2}^2)\\
&\quad+C\Delta t(\|\nabla\mathbf{e}_u^n\|_{L^2}^2+\|\nabla\mathbf{e}_u^{n+1/2}\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2+\|\text{curl }\mathbf{e}_b^{n+1}\|_{L^2}^2)\\
&\quad+C(\Delta t)^2\left(\int_{t_n}^{t_{n+1}}\|\nabla\mathbf{u}_t\|_{L^2}^2dt+\int_{t_n}^{t_{n+1}}\|\mathbf{b}_t\|_{H^1}^2dt\right)+C\|\text{curl }\mathbf{e}_b^n\|_{L^2}^2.
\end{aligned}\tag{4.32}$$

Taking the sum of (4.32) from 0 to $m\le N-1$, and using (4.5), (4.8), (4.13), (4.20), we complete the proof of (4.28). □

Combining the results in this section, we obtain the temporal error estimates (3.12)–(3.16).

5. Spatial error analysis

To show the spatial error estimates, we introduce the projectors $(\mathbf{R}_h,\pi_h):\mathbf{V}\times M\longrightarrow\mathbf{V}_h\times M_h$ defined by

$$\begin{aligned}
a_1(\mathbf{R}_h\mathbf{u}-\mathbf{u},\mathbf{v}_h)-d(\mathbf{v}_h,\pi_hp-p)&=0\quad\forall\,\mathbf{v}_h\in\mathbf{V}_h,\\
d(\mathbf{R}_h\mathbf{u}-\mathbf{u},q_h)&=0\quad\forall\,q_h\in M_h.
\end{aligned}$$

In addition, define $\mathbf{K}_h:\mathbf{V}\longrightarrow\widetilde{\mathbf{V}}_h$ and $\mathbf{Q}_h:\mathbf{X}\longrightarrow\mathbf{X}_h$ by

$$(\nabla(\mathbf{K}_h\mathbf{u}-\mathbf{u}),\nabla\mathbf{v}_h)=0\quad\forall\,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h,$$

$$a_2(\mathbf{Q}_h\mathbf{b}-\mathbf{b},\mathbf{w}_h)+\frac{1}{Rm}(\text{div }(\mathbf{Q}_h\mathbf{b}-\mathbf{b}),\text{div }\mathbf{w}_h)=0\quad\forall\,\mathbf{w}_h\in\mathbf{X}_h.$$

Then for all $(\mathbf{u}, p, \mathbf{b}) \in \mathbf{V} \cap \mathbf{H}^2(\Omega) \times M \cap H^1(\Omega) \times \mathbf{X} \cap \mathbf{H}^2(\Omega)$, the following approximation and stability properties hold:

$$\|\mathbf{u} - \mathbf{R}_h\mathbf{u}\|_{L^2} + h\|\nabla(\mathbf{u} - \mathbf{R}_h\mathbf{u})\|_{L^2} \le Ch^2\|\mathbf{u}\|_{H^2}, \tag{5.1}$$

$$\|\mathbf{u} - \mathbf{K}_h\mathbf{u}\|_{L^2} + h\|\nabla(\mathbf{u} - \mathbf{K}_h\mathbf{u})\|_{L^2} \le Ch^2\|\mathbf{u}\|_{H^2}, \tag{5.2}$$

$$\|\mathbf{b} - \mathbf{Q}_h\mathbf{b}\|_{L^2} + h\|\mathbf{b} - \mathbf{Q}_h\mathbf{b}\|_{H^1} \le Ch^2\|\mathbf{b}\|_{H^2}, \tag{5.3}$$

$$\|p - \pi_h p\|_{L^2} \le Ch\|p\|_{H^1}, \tag{5.4}$$

$$\|\nabla\mathbf{R}_h\mathbf{u}\|_{L^6} + \|\nabla\mathbf{K}_h\mathbf{u}\|_{L^6} + \|\mathbf{Q}_h\mathbf{b}\|_{W^{1,6}} + \|\pi_h p\|_{L^6} \le C. \tag{5.5}$$

For $0 \le n \le N-1$, let us denote errors by

$$\mathbf{E}_u^n = \mathbf{R}_h\mathbf{u}^n - \mathbf{u}^n, \qquad \mathbf{E}_u^{n+1/2} = \mathbf{K}_h\mathbf{u}^{n+1/2} - \mathbf{u}^{n+1/2}, \qquad \mathbf{E}_b^n = \mathbf{Q}_h\mathbf{b}^n - \mathbf{b}^n, \qquad E_p^n = \pi_h p^n - p^n,$$
$$\mathbf{e}_{uh}^n = \mathbf{R}_h\mathbf{u}^n - \mathbf{u}_h^n, \qquad \mathbf{e}_{uh}^{n+1/2} = \mathbf{K}_h\mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2}, \qquad \mathbf{e}_{bh}^n = \mathbf{Q}_h\mathbf{b}^n - \mathbf{b}_h^n, \qquad e_{ph}^n = \pi_h p^n - p_h^n.$$

From (3.1)–(3.3) and (3.8)–(3.11), we obtain that $\mathbf{e}_{uh}^{n+1/2}$, $\mathbf{e}_{bh}^{n+1}$, $(\mathbf{e}_{uh}^{n+1}, e_{ph}^{n+1})$ satisfy the following problems:

$$\begin{aligned}&\left(\frac{\mathbf{e}_{uh}^{n+1/2} - \mathbf{e}_{uh}^n}{\Delta t}, \mathbf{v}_h\right) + a_1(\mathbf{e}_{uh}^{n+1/2}, \mathbf{v}_h) = \left(\frac{\mathbf{E}_u^{n+1/2} - \mathbf{E}_u^n}{\Delta t}, \mathbf{v}_h\right) + b_1(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{v}_h)\\ &\quad - b_1(\mathbf{u}^n, \mathbf{u}^{n+1/2}, \mathbf{v}_h) + S(\mathbf{b}_h^n \times \operatorname{curl}\mathbf{b}_h^{n+1}, \mathbf{v}_h) - S(\mathbf{b}^n \times \operatorname{curl}\mathbf{b}^{n+1}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \widetilde{\mathbf{V}}_h,\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}&\left(\frac{\mathbf{e}_{bh}^{n+1} - \mathbf{e}_{bh}^n}{\Delta t}, \mathbf{w}_h\right) + a_2(\mathbf{e}_{bh}^{n+1}, \mathbf{w}_h) + \frac{1}{Rm}(\operatorname{div}\mathbf{e}_{bh}^{n+1}, \mathbf{w}_h) = \left(\frac{\mathbf{E}_b^{n+1} - \mathbf{E}_b^n}{\Delta t}, \mathbf{w}_h\right)\\ &\quad + (\mathbf{u}^{n+1/2} \times \mathbf{b}^n - \mathbf{u}_h^{n+1/2} \times \mathbf{b}_h^n, \operatorname{curl}\mathbf{w}_h) \quad \forall\, \mathbf{w}_h \in \mathbf{X}_h,\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}&\left(\frac{\mathbf{e}_{uh}^{n+1} - \mathbf{e}_{uh}^{n+1/2}}{\Delta t}, \mathbf{v}_h\right) + a_1(\mathbf{e}_{uh}^{n+1} - \mathbf{e}_{uh}^{n+1/2}, \mathbf{v}_h) - d(\mathbf{v}_h, e_{ph}^{n+1})\\ &\quad = \left(\frac{\mathbf{E}_u^{n+1} - \mathbf{E}_u^{n+1/2}}{\Delta t}, \mathbf{v}_h\right) - a_1(\mathbf{E}_u^{n+1/2}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h,\end{aligned} \tag{5.8}$$

and

$$d(\mathbf{e}_{uh}^{n+1}, q_h) = 0, \quad \forall\, q_h \in M_h. \tag{5.9}$$

The main result in this section is to show the following spatial error estimate based on the regularity estimate (4.20).

Lemma 5.1. *Suppose that Assumptions 1–4 are satisfied. If the time step Δt satisfies $\Delta t \ge Ch$, then there exists some $h_0 > 0$ such that when $h < h_0$, the following spatial error estimate holds:*

$$\|\mathbf{e}_{uh}^{n+1}\|_{l^\infty(L^2)} + S\|\mathbf{e}_{bh}^{m+1}\|_{l^\infty(L^2)} + \frac{1}{Re}\|\nabla\mathbf{e}_{uh}^{n+1}\|_{l^2(L^2)} + \frac{S}{Rm}\|\mathbf{e}_{bh}^{n+1}\|_{l^2(H^1)} \le Ch, \tag{5.10}$$

where C depends on Re, Rm and S.

Proof. Taking $\mathbf{v}_h = 2\mathbf{e}_{uh}^{n+1/2}$ in (5.6) leads to

$$\begin{aligned}&\frac{\|\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2 - \|\mathbf{e}_{uh}^n\|_{L^2}^2 + \|\mathbf{e}_{uh}^{n+1/2} - \mathbf{e}_{uh}^n\|_{L^2}^2}{\Delta t} + \frac{2}{Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2\\ &\quad = 2\left(\frac{\mathbf{E}_u^{n+1/2} - \mathbf{E}_u^n}{\Delta t}, \mathbf{e}_{uh}^{n+1/2}\right) + 2\left(b_1(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{e}_{uh}^{n+1/2}) - b_1(\mathbf{u}^n, \mathbf{u}^{n+1/2}, \mathbf{e}_{uh}^{n+1/2})\right)\\ &\qquad + 2S\left(\mathbf{b}_h^n \times \operatorname{curl}\mathbf{b}_h^{n+1} - \mathbf{b}^n \times \operatorname{curl}\mathbf{b}^{n+1}, \mathbf{e}_{uh}^{n+1/2}\right)\\ &\quad = J_1 + J_2 + J_3.\end{aligned} \tag{5.11}$$

From Hölder inequality and Young's inequality, J_1 satisfies

$$\begin{aligned}J_1 &\le 2(\Delta t)^{-1}(\|\mathbf{E}_u^{n+1/2}\|_{L^2} + \|\mathbf{E}_u^n\|_{L^2})\|\mathbf{e}_{uh}^{n+1/2}\|_{L^2}\\ &\le \frac{1}{4Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2 + Ch^4(\Delta t)^{-2}(\|\mathbf{u}^{n+1/2}\|_{H^2}^2 + \|\mathbf{u}^n\|_{H^2}^2).\end{aligned}$$

The definitions of $\mathbf{e}_{uh}^n$, $\mathbf{e}_{uh}^{n+1/2}$ and the trilinear term b_1 imply that

$$\begin{aligned}
&2\left(b_1(\mathbf{u}_h^n,\mathbf{u}_h^{n+1/2},\mathbf{e}_{uh}^{n+1/2})-b_1(\mathbf{u}^n,\mathbf{u}^{n+1/2},\mathbf{e}_{uh}^{n+1/2})\right)\\
&\quad=2\left(b_1(\mathbf{E}_u^n-\mathbf{e}_{uh}^n,\mathbf{E}_u^{n+1/2},\mathbf{e}_{uh}^{n+1/2})+b_1(\mathbf{E}_u^n-\mathbf{e}_{uh}^n,\mathbf{u}^{n+1/2},\mathbf{e}_{uh}^{n+1/2})+b_1(\mathbf{u}^n,\mathbf{E}_u^{n+1/2},\mathbf{e}_{uh}^{n+1/2})\right).
\end{aligned}$$

By using (5.1), (5.2) and (4.20), we estimate the three terms in the above equation as follows:

$$\begin{aligned}
&b_1(\mathbf{E}_u^n-\mathbf{e}_{uh}^n,\mathbf{E}_u^{n+1/2},\mathbf{e}_{uh}^{n+1/2})\\
&\quad\le C(\|\nabla\mathbf{E}_u^n\|_{L^2}+\|\nabla\mathbf{e}_{uh}^n\|_{L^2})\|\nabla\mathbf{E}_u^{n+1/2}\|_{L^2}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}\\
&\quad\le\frac{1}{6Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+Ch^2\|\mathbf{u}^{n+1/2}\|_{H^2}+CReh^2\|\nabla\mathbf{e}_{uh}^n\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
&b_1(\mathbf{E}_u^n-\mathbf{e}_{uh}^n,\mathbf{u}^{n+1/2},\mathbf{e}_{uh}^{n+1/2})\\
&\quad\le C(\|\mathbf{E}_u^n\|_{L^2}+\|\mathbf{e}_{uh}^n\|_{L^2})\|\mathbf{u}^{n+1/2}\|_{H^2}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}\\
&\quad\le\frac{1}{6Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+Ch^2\|\mathbf{u}^{n+1/2}\|_{H^2}+CRe\|\mathbf{e}_{uh}^n\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
b_1(\mathbf{u}^n,\mathbf{E}_u^{n+1/2},\mathbf{e}_{uh}^{n+1/2})&\le C\|\nabla\mathbf{u}^n\|_{L^2}\|\nabla\mathbf{E}_u^{n+1/2}\|_{L^2}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}\\
&\le\frac{1}{6Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+Ch^2\|\mathbf{u}^{n+1/2}\|_{H^2}.
\end{aligned}$$

Therefor, J_2 satisfies

$$\begin{aligned}
J_2&=2\left(b_1(\mathbf{u}_h^n,\mathbf{u}_h^{n+1/2},\mathbf{e}_{uh}^{n+1/2})-b_1(\mathbf{u}^n,\mathbf{u}^{n+1/2},\mathbf{e}_{uh}^{n+1/2})\right)\\
&\le\frac{1}{2Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+Ch^2\|\mathbf{u}^{n+1/2}\|_{H^2}+CRe(h^2\|\nabla\mathbf{e}_{uh}^n\|_{L^2}^2+\|\mathbf{e}_{uh}^n\|_{L^2}^2)\|\mathbf{u}^{n+1/2}\|_{H^2}^2.
\end{aligned}$$

By a similar method, we have

$$\begin{aligned}
J_3&=2S\left(\mathbf{b}_h^n\times\operatorname{curl}\mathbf{b}_h^{n+1}-\mathbf{b}^n\times\operatorname{curl}\mathbf{b}^{n+1},\mathbf{e}_{uh}^{n+1/2}\right)\\
&=-2S(\mathbf{b}_h^n\times\operatorname{curl}\mathbf{e}_{bh}^{n+1},\mathbf{e}_{uh}^{n+1/2})-2S(\mathbf{e}_{bh}^n\times\operatorname{curl}\mathbf{E}_b^{n+1},\mathbf{e}_{uh}^{n+1/2})\\
&\quad+2S((\mathbf{E}_b^n-\mathbf{e}_{bh}^n)\times\operatorname{curl}\mathbf{b}^{n+1},\mathbf{e}_{uh}^{n+1/2})+2S(\mathbf{E}_b^n\times\operatorname{curl}\mathbf{E}_b^{n+1},\mathbf{e}_{uh}^{n+1/2})+2S(\mathbf{b}^n\times\operatorname{curl}\mathbf{E}_b^{n+1},\mathbf{e}_{uh}^{n+1/2})\\
&\le-2S(\mathbf{b}_h^n\times\operatorname{curl}\mathbf{e}_{bh}^{n+1},\mathbf{e}_{uh}^{n+1/2})+Ch\|\mathbf{e}_{bh}^n\|_{L^2}^2\|\mathbf{b}^{n+1}\|_{H^2}^2\\
&\quad+\frac{1}{4Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+Ch^2\|\mathbf{b}^{n+1}\|_{H^2}+C\|\mathbf{e}_{bh}^n\|_{L^2}^2\|\mathbf{b}^{n+1}\|_{H^2}^2,
\end{aligned}$$

where we use the inverse inequality $\|\mathbf{v}_h\|_{L^3}\le Ch^{-1/6}\|\mathbf{v}_h\|_{L^2}$. Combining these estimates into (5.11) yields

$$\begin{aligned}
&\frac{\|\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2-\|\mathbf{e}_{uh}^n\|_{L^2}^2+\|\mathbf{e}_{uh}^{n+1/2}-\mathbf{e}_{uh}^n\|_{L^2}^2}{\Delta t}+\frac{1}{Re}\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2\\
&\quad\le-2S(\mathbf{b}_h^n\times\operatorname{curl}\mathbf{e}_{bh}^{n+1},\mathbf{e}_{uh}^{n+1/2})+Ch^4(\Delta t)^{-2}(\|\mathbf{u}^{n+1/2}\|_{H^2}^2+\|\mathbf{u}^n\|_{H^2}^2)+Ch^2(\|\mathbf{u}^{n+1/2}\|_{H^2}+\|\mathbf{b}^{n+1}\|_{H^2}^2)\\
&\qquad+CReh^2\|\nabla\mathbf{e}_{uh}^n\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2+CRe\|\mathbf{e}_{uh}^n\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2+C\|\mathbf{e}_{bh}^n\|_{L^2}^2\|\mathbf{b}^{n+1}\|_{H^2}^2.
\end{aligned}\tag{5.12}$$

Taking $\mathbf{v}_h=2\mathbf{e}_{uh}^{n+1}$ in (5.8) and $q_h=e_{ph}^{n+1}$ in (5.9) leads to

$$\begin{aligned}
&\frac{\|\mathbf{e}_{uh}^{n+1}\|_{L^2}^2-\|\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+\|\mathbf{e}_{uh}^{n+1}-\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2}{\Delta t}+\frac{1}{Re}(\|\nabla\mathbf{e}_{uh}^{n+1}\|_{L^2}^2-\|\nabla\mathbf{e}_{uh}^{n+1/2}\|_{L^2}^2+\|\nabla(\mathbf{e}_{uh}^{n+1}-\mathbf{e}_{uh}^{n+1/2})\|_{L^2}^2)\\
&\quad=\left(\frac{\mathbf{E}_u^{n+1}-\mathbf{E}_u^{n+1/2}}{\Delta t},\mathbf{e}_{uh}^{n+1}\right)-a_1(\mathbf{E}_u^{n+1/2},\mathbf{e}_{uh}^{n+1})\\
&\quad\le Ch^4(\Delta t)^{-2}(\|\mathbf{u}^{n+1/2}\|_{H^2}^2+\|\mathbf{u}^{n+1}\|_{H^2}^2)+\frac{1}{2Re}\|\nabla\mathbf{e}_{uh}^{n+1}\|_{L^2}^2+Ch^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2,
\end{aligned}$$

which together with (5.12) yields

$$\begin{aligned}
&\frac{\|\mathbf{e}_{uh}^{n+1}\|_{L^2}^2-\|\mathbf{e}_{uh}^{n}\|_{L^2}^2}{\Delta t}+\frac{1}{Re}\|\nabla\mathbf{e}_{uh}^{n+1}\|_{L^2}^2\\
&\le -2S(\mathbf{b}_h^n\times \operatorname{curl}\mathbf{e}_{bh}^{n+1},\mathbf{e}_{uh}^{n+1/2})+Ch^4(\Delta t)^{-2}(\|\mathbf{u}^{n+1/2}\|_{H^2}^2+\|\mathbf{u}^{n}\|_{H^2}^2+\|\mathbf{u}^{n+1}\|_{H^2}^2)\\
&\quad+Ch^2(\|\mathbf{u}^{n+1/2}\|_{H^2}+\|\mathbf{b}^{n+1}\|_{H^2}^2)+CReh^2\|\nabla\mathbf{e}_{uh}^{n}\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2\\
&\quad+CRe\|\mathbf{e}_{uh}^{n}\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2+C\|\mathbf{e}_{bh}^{n}\|_{L^2}^2\|\mathbf{b}^{n+1}\|_{H^2}^2.
\end{aligned}\tag{5.13}$$

Setting $\mathbf{w}_h=2S\mathbf{e}_{bh}^{n+1}$ in (5.7) leads to

$$\begin{aligned}
&\frac{S\|\mathbf{e}_{bh}^{n+1}\|_{L^2}^2-S\|\mathbf{e}_{bh}^{n}\|_{L^2}^2+S\|\mathbf{e}_{bh}^{n+1}-\mathbf{e}_{bh}^{n}\|_{L^2}^2}{\Delta t}+\frac{2SC}{Rm}\|\mathbf{e}_{bh}^{n+1}\|_{H^1}^2\\
&\le 2S(\mathbf{u}^{n+1/2}\times\mathbf{e}_{bh}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})-2S(\mathbf{u}^{n+1/2}\times\mathbf{E}_{b}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})+2S(\mathbf{e}_{uh}^{n+1/2}\times\mathbf{b}_{h}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})\\
&\quad-2S(\mathbf{E}_{u}^{n+1/2}\times\mathbf{b}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})+2S(\mathbf{E}_{u}^{n+1/2}\times\mathbf{e}_{bh}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})-2S(\mathbf{E}_{u}^{n+1/2}\times\mathbf{E}_{b}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})\\
&\quad+\frac{SC}{2Rm}\|\mathbf{e}_{bh}^{n+1}\|_{H^1}^2+Ch^4(\Delta t)^{-2}(\|\mathbf{b}^{n}\|_{H^2}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2)\\
&\le\frac{SC}{Rm}\|\mathbf{e}_{bh}^{n+1}\|_{H^1}^2+Ch^4(\Delta t)^{-2}(\|\mathbf{b}^{n}\|_{H^2}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2)+2S(\mathbf{e}_{uh}^{n+1/2}\times\mathbf{b}_{h}^{n},\operatorname{curl}\mathbf{e}_{bh}^{n+1})+C\|\mathbf{e}_{bh}^{n}\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2\\
&\quad+C\|\mathbf{E}_{u}^{n+1/2}\|_{L^2}^2\|\mathbf{b}^{n}\|_{H^2}^2+C\|\mathbf{u}^{n+1/2}\|_{H^2}^2\|\mathbf{E}_{b}^{n}\|_{L^2}^2+C\|\mathbf{e}_{bh}^{n}\|_{L^3}^2\|\nabla\mathbf{u}^{n+1/2}\|_{L^2}^2+C\|\nabla\mathbf{E}_{u}^{n+1/2}\|_{L^2}^2\|\mathbf{E}_{b}^{n}\|_{L^2}\|\mathbf{E}_{b}^{n}\|_{H^1}.
\end{aligned}\tag{5.14}$$

Under the condition $\Delta t\ge Ch$, adding (5.13) and (5.14), and taking the sum from $n=0$ to $n=m\le N-1$, we get

$$\begin{aligned}
&\|\mathbf{e}_{uh}^{m+1}\|_{L^2}^2+S\|\mathbf{e}_{bh}^{m+1}\|_{L^2}^2+\Delta t\sum_{n=0}^{m}\left(\frac{1}{Re}\|\nabla\mathbf{e}_{uh}^{n+1}\|_{L^2}^2+\frac{SC}{Rm}\|\mathbf{e}_{bh}^{n+1}\|_{H^1}^2\right)\\
&\le Ch^2\Delta t\sum_{n=0}^{m}\left(\|\mathbf{u}^{n+1/2}\|_{H^2}^2+\|\mathbf{u}^{n+1}\|_{H^2}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2\right)+CReh^2\Delta t\sum_{n=0}^{m}\left(\|\nabla\mathbf{e}_{uh}^{n}\|_{L^2}^2\|\mathbf{u}^{n+1/2}\|_{H^2}^2\right)\\
&\quad+C\Delta t\sum_{n=0}^{m}\left(Re\|\mathbf{u}^{n+1/2}\|_{H^2}^2\|\mathbf{e}_{uh}^{n}\|_{L^2}^2+\|\mathbf{u}^{n+1/2}\|_{H^2}^2\|\mathbf{e}_{bh}^{n}\|_{L^2}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2\|\mathbf{e}_{bh}^{n}\|_{L^2}^2\right)\\
&\le Ch^2+C_8Reh\Delta t\sum_{n=0}^{m}\|\nabla\mathbf{e}_{uh}^{n}\|_{L^2}^2+CRe\Delta t\sum_{n=0}^{m}\left(\|\mathbf{u}^{n+1/2}\|_{H^2}^2\|\mathbf{e}_{uh}^{n}\|_{L^2}^2\right)\\
&\quad+C\Delta t\sum_{n=0}^{m}\left(\|\mathbf{u}^{n+1/2}\|_{H^2}^2\|\mathbf{e}_{bh}^{n}\|_{L^2}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2\|\mathbf{e}_{bh}^{n}\|_{L^2}^2\right).
\end{aligned}$$

Let h be sufficiently small such that $C_8h<\frac{1}{Re^2}$. Applying the discrete Gronwall's inequality to the above inequality, we complete the proof of Lemma 5.1. $\square$

Combining the temporal error estimates (3.12)–(3.16) with the results in Lemma 5.1, we obtain the temporal–spatial error estimates (3.17) and (3.18).

6. Conclusions

In this paper we propose a fractional-step scheme for solving the stationary MHD problems in three-dimensional bounded domains, which is an extension of the previous work for the Navier–Stokes problems studied in [24]. The main feature is that the nonlinearities for velocity and the incompressibility in (1.1)–(1.2) are split into two steps. Compared to the projection scheme [14], the fractional-step scheme studied in this paper allows to enforce the original boundary conditions of the problems in all substeps of the fractional-step scheme, and allows for a discrete energy estimate. We have proved that the time-discrete fractional-step scheme provides the temporal error estimates of $\mathcal{O}(\Delta t)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)\cap l^\infty(\mathbf{L}^2)$ and $\mathcal{O}(\sqrt{\Delta t})$ for the pressure in $l^2(L^2)$. For the fully discrete FEM scheme, the temporal–spatial error estimates of $\mathcal{O}(\Delta t+h)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)\cap l^\infty(\mathbf{L}^2)$ are obtained under the constraint $\Delta t\ge Ch$. However, for the projection scheme, the temporal–spatial error orders derived in [14] are only $\mathcal{O}(\sqrt{\Delta t}+h)$ for the velocity and the magnetic field in $l^2(\mathbf{H}^1)$.

Although some temporal–spatial convergence orders are derived in this paper, there are still two questions worthy of study. One is whether the spatial error estimate for the velocity and the magnetic field in $l^\infty(\mathbf{L}^2)$ can be improved to be optimal. Another is whether there is an optimal error estimate for the pressure in $l^2(L^2)$. We observed that Guillén-González and Redondo-Neble have studied optimal error estimates of the fractional-step scheme for the Navier–Stokes problems in [29,30]. But it is much more complicated to study these optimal error estimates for the MHD problem due to the appearance of the magnetic field equations. In addition, the extension of this fractional-step scheme to higher-order temporal accuracy also an interesting problem, could possibly be achieved using a Crank–Nicolson scheme. However, there needs much higher regularity of the solution to the MHD problems. These objectives are worth studying in our future work.

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