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A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations

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ABSTRACT

Based on reproducing kernel theory, a numerical method is proposed for solving variable order fractional boundary value problems for functional differential equations. In the previous works, piecewise polynomial reproducing kernels were employed to solve fractional differential equations. However, the computational cost of fractional order operator acting on such kernel functions is high. In this paper, reproducing kernels with polynomial form will be constructed and applied to solve variable order fractional functional boundary value problems. The method can reduce computation cost and provide highly accurate global approximate solutions.

1. Introduction

Recently, variable fractional differential equations have been used to modeling signal processing, the processing of geographical data and signature verification. The study of such problems has attracted much attention. Razminia, Dizaji and Majd [1] proved the existence of the solution for a generalized fractional differential equation with non-autonomous variable order operators. Sun and Chen [2–5] introduced some important applications of variable fractional derivative.

Due to the existence of variable fractional derivative, it is usually impossible to obtain the analytical solution of such equations. Hence, we must find numerical methods for solving such problems. Fu, Chen and Ling [6] applied the method of approximate particular solutions to both constant- and variable-order time fractional diffusion models. Liu, Shen, Zhang et al. [7–15] proposed various finite difference methods for variable order fractional partial diffusion equations. Sierociuk, Malesza and Macias [16] introduced a numerical scheme for a variable order derivative based on matrix approach. Yu and Ertürk [17] applied a finite difference method to variable order fractional integro-differential equations. Zhao, Sun and Karniadakis [18] derived two second-order approximation formulas for the variable-order fractional time derivatives. Zayernouri and Karniadakis [19] developed fractional spectral collocation methods for linear and nonlinear variable order fractional partial differential equations. Combining Legendre wavelets functions and operational matrices, Chen, Wei et al. [20] presented a numerical method to solve a class of nonlinear variable order fractional differential equations. Atangana [21] gave the Crank–Nicholson scheme for time-fractional variable order telegraph equation. Li and Wu [22] proposed a reproducing kernel method for variable fractional boundary value problems.

Reproducing kernel theory plays an important role in various fields of mathematics, such as probability and statistics, and operator theory. Recently, based on the reproducing kernel theory, a method called the reproducing kernel method

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for solving operator equations was proposed by Cui, Geng et al. [23,24]. Recently, the method has been used and modified by many authors [25–32]. In this paper, we will construct reproducing kernels with polynomial form and apply it to the following variable fractional functional boundary value problems

$$\begin{cases} D^{\alpha(x)}u + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x)) = f(x), \quad -1 < x < 1, \\ u(-1) = \mu_1, \qquad u(1) = \mu_2 \end{cases} \tag{1.1}$$

where $1 < \alpha(x) \leq 2$, $-1 \leq \tau(x) \leq 1$, μ_1, μ_2 are constants, $a(x)$, $b(x)$, $c(x)$, $\tau(x) \in C^2[-1, 1]$, $D^{\alpha(x)}$ denotes the variable order Caputo fractional derivative defined as follows

$$D^{\alpha(x)}u(x) = \frac{1}{\Gamma(2-\alpha(x))}\int_{-1}^{x}(x-t)^{1-\alpha(x)}u''(t)dt. \tag{1.2}$$

Note here that $\tau(x)$ may be larger or smaller than x.

2. Construction of reproducing kernels with polynomial form

Denote by P_n the set of all algebraic polynomials of degree $\leq n$, namely,

$$P_n := span\{1, x, x^2, \ldots, x^n\},$$

equipped with the following inner product and norm

$$(u(y), v(y)) = \int_{-1}^{1} u(y)v(y)\omega(y)dy \tag{2.1}$$

and

$$\|u\| = \sqrt{(u, u)},$$

where ω is a generic weight function.

We say that a finite or infinite sequence of vectors $f_1, f_2, \ldots$ in an inner product space is orthogonal if

$$(f_i, f_j) = 0, \quad (i \neq j).$$

Theorem 2.1. *For any given positive weight function $\omega \in L^1(-1, 1)$, the sequence of polynomials defined as follows is orthogonal:*

$$p_i(x) = (x - a_i)p_{i-1}(x) - b_i p_{i-2}(x), \quad (i \geq 2) \tag{2.2}$$

with $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$a_i = \frac{(xp_{i-1}, p_{i-1})}{(p_{i-1}, p_{i-1})}, \qquad b_i = \frac{(xp_{i-1}, p_{i-2})}{(p_{i-2}, p_{i-2})}.$$

Proof. We proceed with the proof by using an induction argument. We show by induction on n that $(p_n, p_i) = 0$ for $i = 0, 1, \ldots, n-1$.

For $n = 1$,

$$(p_1, p_0) = ((x - a_1)p_0, p_0) = (xp + 0, p_0) - a_1(p_0, p_0).$$

From the definition of a_1, we have $(p_1, p_0) = 0$.

Now we assume the validity of our assertion for $n - 1$, $(n \geq 2)$. Then

$$(p_n, p_{n-1}) = (xp_{n-1}, p_{n-1}) - a_n(p_{n-1}, p_{n-1}) - b_n(p_{n-2}, p_{n-1}) = 0$$

and

$$(p_n, p_{n-2}) = (xp_{n-1}, p_{n-2}) - a_n(p_{n-1}, p_{n-2}) - b_n(p_{n-2}, p_{n-2}) = 0.$$

For any $i \leq n - 3$, one obtains

$$\begin{aligned} (p_n, p_i) &= (xp_{n-1}, p_i) - a_n(p_{n-1}, p_i) - b_n(p_{n-2}, p_i) \\ &= (p_{n-1}, xp_i) \\ &= (p_{n-1}, p_{i+1} + a_{i+1}p_i + b_{i+1}p_{i-1}) = 0 \end{aligned}$$

and the proof is complete. □

Theorem 2.2. *P_n is a reproducing kernel space and its reproducing kernel is*

$$K_n(x, y) = \sum_{i=0}^{n} e_i(x)e_i(y), \tag{2.3}$$

where $e_i(x) = \frac{p_i(x)}{\|p_i\|}$.

Proof. For $\forall\ y \in [-1, 1]$

$$K_n(\cdot, y) = \sum_{i=1}^{n} e_i(\cdot)e_i(y)\ \in P_n.$$

If $u(x)$ is any function in P_n, then

$$u(x) = \sum_{i=0}^{n} \alpha_i e_i(x).$$

For $\forall\ y \in [-1, 1]$

$$\begin{aligned}(u(x), K_n(x, y)) &= \left(\sum_{i=0}^{n} \alpha_i e_i(x), \sum_{j=1}^{n} e_j(x)e_j(y)\right)\\ &= \sum_{i=0}^{n}\sum_{j=1}^{n} \alpha_i e_j(y)(e_i, e_j)\\ &= \sum_{i=0}^{n} \alpha_i e_i(y)\\ &= u(y).\end{aligned}$$

This ends the proof. □

Furthermore, $K_n(x, y)$ can be rewritten as

$$K_n(x, y) = \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{\|p_n\|^2(x - y)}. \tag{2.4}$$

3. Method for (1.1)

Define linear operator L as $Lu(x) = D^{\alpha(x)}u + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x))$. Note that the functions in space P_n do not satisfy the boundary conditions of (1.1). We shall give the approximate solution of (1.1) in space P_n based on reproducing kernel theory.

Let $\{x_i\}_{i=1}^{n-1}$ be nodes in interval $[-1, 1]$. Put $\psi_i(x) = L_s K_n(x, s)|_{s=x_i},\ (i = 1, 2, \ldots, n-1),\ \psi_n(x) = x^{n-1},\ \psi_{n+1}(x) = x^n$. Gram–Schmidt orthogonalization of $\{\psi_i(x)\}_{i=1}^{n+1}$ yields $\{\overline{\psi}_i(x)\}_{i=1}^{n+1}$ which is an orthonormal basis for P_n,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots, n+2). \tag{3.1}$$

Find an approximation solution of (1.1) in space P_n with the following form

$$u_n(x) = \sum_{i=1}^{n-1}(u, \overline{\psi}_i)\overline{\psi}_i + \gamma_1\overline{\psi}_n + \gamma_2\overline{\psi}_{n+1}, \tag{3.2}$$

where γ_1, γ_2 are constants to be determined by boundary conditions of (1.1).

Theorem 3.1. *The approximation solution $u_n(x)$ of (1.1) can be represented in the following form by using the information of right side function $f(x)$*

$$u_n(x) = \sum_{i=1}^{n-1}\sum_{k=1}^{i} \beta_{ik}f(x_k)\overline{\psi}_i + \gamma_1\overline{\psi}_n + \gamma_2\overline{\psi}_{n+1}. \tag{3.3}$$

Proof. According to (3.1), we have

$$(u, \overline{\psi}_i) = \sum_{k=1}^{i} \beta_{ik}(u(x), L_s K_n(x,s)|_{s=x_k}) = \sum_{k=1}^{i} \beta_{ik} L_s(u(x), K_n(x,s))|_{s=x_k}.$$

From the reproducing property of $K_n(x,s)$ and the fact that $Lu(s) = f(s)$, one obtains

$$(u, \overline{\psi}_i) = \sum_{k=1}^{i} \beta_{ik} L_s u(s)|_{s=x_k} = \sum_{k=1}^{i} \beta_{ik} f(x_k).$$

Hence,

$$\begin{aligned} u_n(x) &= \sum_{i=1}^{n+1} (u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{n-1} (u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) + (u(x), \overline{\psi}_n(x))\overline{\psi}_n(x) + (u(x), \overline{\psi}_{n+1}(x))\overline{\psi}_{n+1}(x) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i + \gamma_1 \overline{\psi}_n(x) + \gamma_2 \overline{\psi}_{n+1}(x). \quad \square \end{aligned}$$

Incorporating boundary conditions of (1.1) into approximate solution $u_n(x)$, we get a system of two linear equations

$$u_n(-1) = \mu_1, \qquad u_n(1) = \mu_2 \tag{3.4}$$

which serves to determine $\gamma_1,\ \gamma_2$.

To analyze the error of approximate solutions, we define the residual function as

$$R_n(x) = Lu_n(x) - f(x),$$

where $u_n(x)$ is given by (3.3).

Theorem 3.2. *If $a(x), b(x), c(x), \tau(x), f(x) \in C^4[-1, 1]$, then there exists a positive constant c such that*

$$\|R_n(x)\|_\infty = \max_{x\in[-1,1]} |R_n(x)| \le c\, h^4$$

where $h = \max_{1\le i\le n-1} |x_{i+1} - x_i|$.

Proof. Firstly, we show approximate solution $u_n(x)$ satisfies Eq. (1.1) at x_i, $(1 \le i \le n-1)$, namely, $(Lu_n)(x_j) = f(x_j), j = 1, 2, \ldots, n-1$.

Putting

$$A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k), \quad i = 1, 2, \ldots, n-1, \qquad A_n = \gamma_1, \qquad A_{n+1} = \gamma_2,$$

then $u_n(x)$ is rewritten as

$$u_n(x) = \sum_{i=1}^{n+1} A_i \overline{\psi}_i.$$

From the reproducing property of $K_n(x,y)$, it follows that

$$\begin{aligned} (Lu_n)(x_k) &= \sum_{i=1}^{n+1} A_i L_x \overline{\psi}_i(x)|_{x=x_k} \\ &= \sum_{i=1}^{n+1} A_i L_x (\overline{\psi}_i(s), K_n(x,s))|_{x=x_k} \\ &= \sum_{i=1}^{n+1} A_i (\overline{\psi}_i(s), L_x K_n(x,s))|_{x=x_k} \\ &= \sum_{i=1}^{n+1} A_i (\overline{\psi}_i(s), L_x K_n(x,s)|_{x=x_k}) \\ &= \sum_{i=1}^{n+1} A_i (\overline{\psi}_i(s), \psi_k(s)). \end{aligned}$$

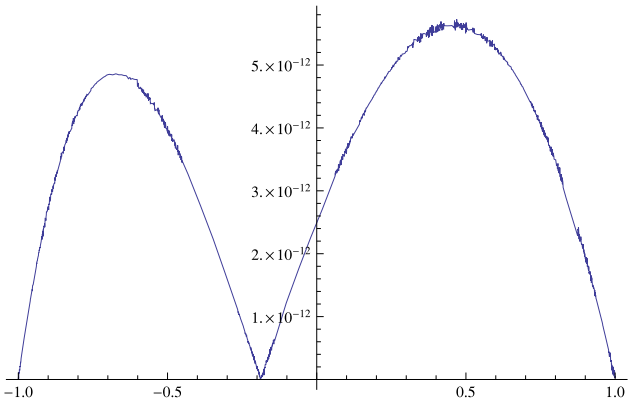

Fig. 1. Absolute errors of approximate solutions for Example 4.1.

Note that

$$\sum_{k=1}^{j}\beta_{jk}(Lu_n)(x_k)=\sum_{i=1}^{n+1}A_i\left(\overline{\psi}_i,\sum_{k=1}^{j}\beta_{jk}\psi_k\right)=\sum_{i=1}^{n+1}A_i(\overline{\psi}_i,\overline{\psi}_j)=A_j. \tag{3.5}$$

Taking $j=1$ in the formula above, we have $(Lu_n)(x_1)=f(x_1)$.

Taking $j=2$ in (3.4), one obtains $(Lu_n)(x_2)=f(x_2)$.

Similarly, we get

$$(Lu_n)(x_j)=f(x_j),\quad j=3,4,\ldots,n-1. \tag{3.6}$$

From the proof of Theorem 3.2 in [24], we have

$$\|R_n(x)\|_\infty=\max_{x\in[-1,1]}|R_n(x)|\le c\,h^4$$

where c is a positive constant, $h=\max_{1\le i\le n-1}|x_{i+1}-x_i|$. □

4. Numerical examples

Example 4.1. Consider the following variable fractional problem [22]

$$\begin{cases}D^{\alpha(x)}u(x)+\cos(x)u'(x)+4u(x)+5u(x^2)=f(x),\quad -1\le x\le 1,\\ u(-1)=1,\qquad u(1)=1\end{cases}$$

where $\alpha(x)=\frac{5+\sin(x)}{4}$, $f(x)=\frac{\Gamma(3)(x+1)^{2-\alpha(x)}}{\Gamma(3-\alpha(x))}+5x^4+4x^2+2x\cos(x)$. Its exact solution is $u(x)=x^2$.

Taking $n=6$, $\omega(x)=\frac{1}{\sqrt{1-x^2}}$, $x_i=-1+\frac{2i}{n+1}$, $i=1,2,\ldots,n-1$, the absolute errors $|u_n(x)-u(x)|$ obtained by the present method are compared with [22] in Table 1. The absolute error between approximate solution $u_n(x)$ and exact solution is shown in Fig. 1. From the errors compared with other methods, it is shown that the present method can provide more accurate numerical results.

Example 4.2. Consider the following variable fractional problem

$$\begin{cases}D^{\alpha(x)}u(x)+e^xu'(x)+2u(x)+8u(e^{x-1})=f(x),\quad -1\le x\le 1,\\ u(-1)=6,\qquad u(1)=6\end{cases}$$

where $\alpha(x)=\frac{6+\cos(x)}{4}$, $f(x)=\frac{\Gamma(5)(x+1)^{4-\alpha(x)}}{\Gamma(5-\alpha(x))}+\frac{\Gamma(3)(x+1)^{2-\alpha(x)}}{\Gamma(3-\alpha(x))}+e^x\left(4x^3+2x\right)+2\left(x^4+x^2+4\right)+8\left(e^{2x-2}+e^{4x-4}+4\right)$. The exact solution is given by $u(x)=x^4+x^2+4$.

Taking $n=7$, $\omega(x)=\frac{1}{\sqrt{1-x^2}}$, $x_i=-1+\frac{2i}{n+1}$, $i=1,2,\ldots,n-1$, the figure of absolute errors $|u_n(x)-u(x)|$ obtained by the present method is shown in Fig. 2.

Table 1
Comparison of absolute errors for Example 4.1.

x	Exact solution	$\lvert u_{10}(x) - u(x)\rvert$ ([22])	$\lvert u_{20}(x) - u(x)\rvert$ ([22])	$\lvert u_6(x) - u(x)\rvert$ (present method)
−0.9	0.81	–	–	2.70×10^{-12}
−0.7	0.49	–	–	4.84×10^{-12}
−0.5	0.25	–	–	3.89×10^{-12}
−0.1	0.01	–	–	1.26×10^{-12}
0.10	0.01	1.43×10^{-6}	1.92×10^{-7}	3.67×10^{-12}
0.30	0.09	2.84×10^{-6}	4.01×10^{-7}	5.22×10^{-12}
0.50	0.25	3.13×10^{-6}	4.53×10^{-7}	5.57×10^{-12}
0.70	0.49	2.53×10^{-6}	3.80×10^{-6}	4.56×10^{-12}
0.90	0.81	9.97×10^{-7}	2.08×10^{-7}	1.98×10^{-12}

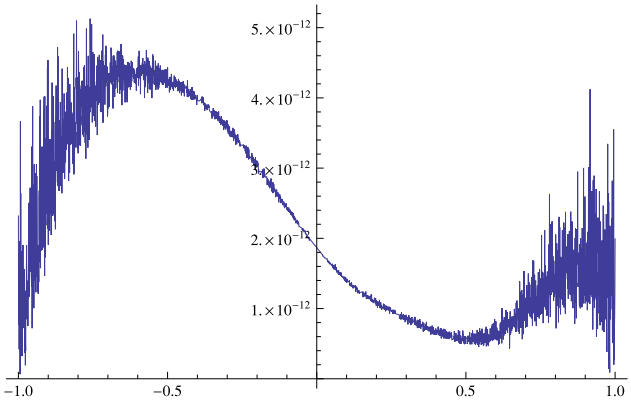

Fig. 2. Absolute errors of approximate solutions for Example 4.2.

5. Conclusion

In this paper, reproducing kernels with polynomial form are constructed. Based on it, an effective numerical method is proposed for variable order fractional functional boundary value problems. The main advantage of the present method lies in the lower computational cost and high accuracy.

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