Linear Algebra and its Applications 511 (2016) 447–459

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Further results on the minimum rank of regular classes of $(0,1)$-matrices

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ARTICLE INFO

Article history:
Received 1 February 2016
Accepted 16 September 2016
Available online 30 September 2016
Submitted by X. Zhan

MSC:
15A03
05B20
05C35

Keywords:
Regular classes of $(0,1)$-matrices
Minimum rank
Upper bound

ABSTRACT

Let $\mathscr{B}(n,k)$ be the set of all $(0,1)$-matrices of order n with constant line sum k and let $\tilde{\nu}(n,k)$ be the minimum rank over $\mathscr{B}(n,k)$. It is known that $\lceil n/k \rceil \le \tilde{\nu}(n,k) \le \hat{\nu}(n,k) \le \lfloor n/k \rfloor + k$, where $\hat{\nu}(n,k)$ is the rank of a recursively defined matrix $\hat{A} \in \mathscr{B}(n,k)$. Brualdi, Manber and Ross showed that $\tilde{\nu}(n,k) = \lceil n/k \rceil$ if and only if $k|n$. In this paper, we show that $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$ if and only if (n,k) satisfies one of the following three relations: (i) $n \equiv \pm 1 \pmod{k}$, $k = 2$ or 3; (ii) $n = k+1$, $k \ge 2$; (iii) $n = 4q+3$, $k = 4$ and $q \ge 1$. Moreover, we obtain the exact values of $\tilde{\nu}(n,4)$ for all $n \ge 4$ and determine all the possible ranks of regular $(0,1)$-matrices in $\mathscr{B}(n,4)$. We also present some positive integer pairs (n,k) such that $\tilde{\nu}(n,k) < \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$, which gives a positive answer to a question posed by Pullman and Stanford.

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http://dx.doi.org/10.1016/j.laa.2016.09.024

1. Introduction

Let k and n be positive integers with $k \leq n$. Denote by $\mathscr{B}(n,k)$ the set of $(0,1)$-matrices of order n with row sums and column sums equal to k. Such matrices are called *k-regular $(0,1)$-matrices*, or *regular $(0,1)$-matrices* for short. There has been considerable interest in studying extremal values or possible values of some parameters in $\mathscr{B}(n,k)$, such as rank, determinant, boolean rank and nonnegative integer rank, see [2–4,7,10]. The main focus of the present paper is to determine in some cases the minimum rank over $\mathscr{B}(n,k)$ and study some related problems in [2,9] and [10]. This interest is motivated by many interesting connections with graph theory and combinatorics. We denote the rank of a matrix A by $r(A)$. Let

$$\bar{\nu}(n,k) = \max\{r(A) : A \in \mathscr{B}(n,k)\}$$

and

$$\tilde{\nu}(n,k) = \min\{r(A) : A \in \mathscr{B}(n,k)\}.$$

It is of interest to find general formulae for $\bar{\nu}(n,k)$ and $\tilde{\nu}(n,k)$. In [5], Houck and Paul (see also Newman [8]) proved that for all $n > k \geq 1$ and $(n,k) \neq (4,2)$, $\bar{\nu}(n,k) = n$; $\bar{\nu}(4,2) = 3$; $\bar{\nu}(n,n) = 1$.

Compared with the maximum rank problem, the minimum rank problem appears to be more difficult. In [2], Brualdi, Manber and Ross obtained the exact values of $\tilde{\nu}(n,k)$ in some cases. We summarize their results in the following

Theorem 1.1.

(1) *For all $n \geq 2$, $\tilde{\nu}(n,2) = n/2$ or $\lfloor n/2 \rfloor + 2$ according as n is even or odd.*
(2) *For all $n \geq 3$, $\tilde{\nu}(n,3) = n/3$ or $\lfloor n/3 \rfloor + 3$ according as n is divisible by three or not.*
(3) *For all k, $\tilde{\nu}(n,k) \geq \lceil n/k \rceil$, $\tilde{\nu}(n,k) = \lceil n/k \rceil$ if and only if $k|n$.*
(4) *For all even k, if $n \equiv k/2 \pmod{k}$, then $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + 2$.*

We denote by $J_{m,n}$ the $m \times n$ matrix of all 1's. Especially, $J_n = J_{n,n}$ and $j_n = J_{n,1}$. Let A be an $m \times n$ matrix. Then the matrix $A^c = J_{m,n} - A$ is called the *complement* of A. Let I_n be the identity matrix of order n. Denote by $O_{m,n}$ the $m \times n$ zero matrix and $0_n = O_{n,1}$. The following is a consequence of [2, Theorem 2.5] (see also [10, Corollary 1.3]).

Lemma 1.2. *For $A \in \mathscr{B}(n,k)$, if $1 \leq k \leq n-1$, then $r(A) = r(A^c)$. Furthermore, we have $\tilde{\nu}(n,k) = \tilde{\nu}(n,n-k)$.*

In [2], a particular matrix $\hat{A} \in \mathscr{B}(n,k)$ was constructed recursively, whose rank is clearly an upper bound for $\tilde{\nu}(n,k)$. We denote the rank of $\hat{A}$ by $\hat{\nu}(n,k)$. The following lemma (see [9]) gave an algorithm to compute $\hat{\nu}(n,k)$.

Lemma 1.3. *If $n \geq k \geq 1$, then $\hat{\nu}(n,k)$ is the sum of all quotients generated by the Euclidean algorithm as it computes* $\gcd(n,k)$. *Moreover,* $\hat{\nu}(n,k) \leq \lfloor n/k \rfloor + k$.

We summarize the bounds for $\tilde{\nu}(n,k)$ given in [2] and [9] as follows.

Theorem 1.4. *If $n \geq k \geq 1$, then* $\lceil n/k \rceil \leq \tilde{\nu}(n,k) \leq \hat{\nu}(n,k) \leq \lfloor n/k \rfloor + k$.

2. Characterization of the equality $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$

It is natural to ask when the inequalities in Theorem 1.4 are equalities. It was shown in [2] that $\tilde{\nu}(n,k) = \lceil n/k \rceil$ if and only if $k|n$. It was proved in [10] that $\hat{\nu}(n,k) = \lfloor n/k \rfloor + k$ if and only if $n \equiv \pm 1 \pmod{k}$. In this section, we give a sufficient and necessary condition for the equality $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$.

Actually, a necessary condition for $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$ was given in [10].

Lemma 2.1. *Let $n > k > 0$. Then $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$ only if (n,k) satisfies one of the following three relations:*

(i) $n \equiv \pm 1 \pmod{k}$, $k = 2$ *or* 3.

(ii) $n = k + 1$, $k \geq 2$.

(iii) $n = 4q + 3$, $k = 4$, $q \geq 1$.

The following lemma can be found in [10, Theorem 3.9].

Lemma 2.2. $\tilde{\nu}(4q+3, 4) = \hat{\nu}(4q+3, 4) = q + 4$ *for* $q \geq 4$ *and* $q = 1$.

We can see from Theorem 1.1 and Lemma 2.2 that if $\tilde{\nu}(11,4) = 6$ and $\tilde{\nu}(15,4) = 7$, then the conditions in Lemma 2.1 are also sufficient for $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$.

For a given positive integer r, let B_r be the set of all rational numbers q for which there exist k and n with $\frac{k}{n} = q$ and a matrix in $\mathscr{B}(n,k)$ having rank r. Jørgensen (see [6]) gave an algorithm to compute B_r for $1 \leq r \leq 7$.

Lemma 2.3.

$$B_1 = \{1\}.$$

$$B_2 = \{\tfrac{1}{2}\}.$$

$$B_3 = \{\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}\}.$$

$$B_4 = \{\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}\}.$$

$$B_5 = \{\tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{4}{9}, \tfrac{1}{2}, \tfrac{5}{9}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}\}.$$

$$B_6 = \{\tfrac{1}{6}, \tfrac{1}{5}, \tfrac{2}{9}, \tfrac{1}{4}, \tfrac{3}{11}, \tfrac{2}{7}, \tfrac{3}{10}, \tfrac{4}{13}, \tfrac{1}{3}, \tfrac{5}{14}, \tfrac{4}{11}, \tfrac{3}{8}, \tfrac{5}{13}, \tfrac{2}{5}, \tfrac{5}{12}, \tfrac{3}{7}, \tfrac{7}{16}, \tfrac{4}{9}, \tfrac{5}{11}, \tfrac{6}{13}, \tfrac{7}{15}, \tfrac{8}{17}, \tfrac{1}{2}, \tfrac{9}{17}, \tfrac{8}{15},$$
$$\tfrac{7}{13}, \tfrac{6}{11}, \tfrac{5}{9}, \tfrac{9}{16}, \tfrac{4}{7}, \tfrac{7}{12}, \tfrac{3}{5}, \tfrac{8}{13}, \tfrac{5}{8}, \tfrac{7}{11}, \tfrac{9}{14}, \tfrac{2}{3}, \tfrac{9}{13}, \tfrac{7}{10}, \tfrac{5}{7}, \tfrac{8}{11}, \tfrac{3}{4}, \tfrac{7}{9}, \tfrac{4}{5}, \tfrac{5}{6}\}.$$

$$\begin{aligned} B_7 = \{&\tfrac{1}{7}, \tfrac{1}{6}, \tfrac{2}{11}, \tfrac{1}{5}, \tfrac{3}{14}, \tfrac{2}{9}, \tfrac{3}{13}, \tfrac{4}{17}, \tfrac{1}{4}, \tfrac{5}{19}, \tfrac{4}{15}, \tfrac{3}{11}, \tfrac{5}{18}, \tfrac{2}{7}, \tfrac{5}{17}, \tfrac{3}{10}, \tfrac{7}{23}, \tfrac{4}{13}, \tfrac{5}{16}, \tfrac{6}{19}, \tfrac{7}{22}, \tfrac{8}{25}, \tfrac{1}{3}, \tfrac{9}{26}, \\ &\tfrac{8}{23}, \tfrac{7}{20}, \tfrac{6}{17}, \tfrac{5}{14}, \tfrac{9}{25}, \tfrac{4}{11}, \tfrac{7}{19}, \tfrac{10}{27}, \tfrac{3}{8}, \tfrac{11}{29}, \tfrac{8}{21}, \tfrac{5}{13}, \tfrac{7}{18}, \tfrac{9}{23}, \tfrac{11}{28}, \tfrac{2}{5}, \tfrac{11}{27}, \tfrac{9}{22}, \tfrac{7}{17}, \tfrac{12}{29}, \tfrac{5}{12}, \tfrac{13}{31}, \\ &\tfrac{8}{19}, \tfrac{11}{26}, \tfrac{3}{7}, \tfrac{13}{30}, \tfrac{10}{23}, \tfrac{7}{16}, \tfrac{11}{25}, \tfrac{4}{9}, \tfrac{13}{29}, \tfrac{9}{20}, \tfrac{14}{31}, \tfrac{5}{11}, \tfrac{11}{24}, \tfrac{6}{13}, \tfrac{13}{28}, \tfrac{7}{15}, \tfrac{15}{32}, \tfrac{8}{17}, \tfrac{9}{19}, \tfrac{10}{21}, \tfrac{11}{23}, \tfrac{12}{25}, \\ &\tfrac{13}{27}, \tfrac{14}{29}, \tfrac{15}{31}, \tfrac{16}{33}, \tfrac{17}{35}, \tfrac{1}{2}, \tfrac{18}{35}, \tfrac{17}{33}, \tfrac{16}{31}, \tfrac{15}{29}, \tfrac{14}{27}, \tfrac{13}{25}, \tfrac{12}{23}, \tfrac{11}{21}, \tfrac{10}{19}, \tfrac{9}{17}, \tfrac{17}{32}, \tfrac{8}{15}, \tfrac{15}{28}, \tfrac{7}{13}, \tfrac{13}{24}, \tfrac{6}{11}, \\ &\tfrac{17}{31}, \tfrac{11}{20}, \tfrac{16}{29}, \tfrac{5}{9}, \tfrac{14}{25}, \tfrac{9}{16}, \tfrac{13}{23}, \tfrac{17}{30}, \tfrac{4}{7}, \tfrac{15}{26}, \tfrac{11}{19}, \tfrac{18}{31}, \tfrac{7}{12}, \tfrac{17}{29}, \tfrac{10}{17}, \tfrac{13}{22}, \tfrac{16}{27}, \tfrac{3}{5}, \tfrac{17}{28}, \tfrac{14}{23}, \tfrac{11}{18}, \tfrac{8}{13}, \tfrac{13}{21}, \\ &\tfrac{18}{29}, \tfrac{5}{8}, \tfrac{17}{27}, \tfrac{12}{19}, \tfrac{7}{11}, \tfrac{16}{25}, \tfrac{9}{14}, \tfrac{11}{17}, \tfrac{13}{20}, \tfrac{15}{23}, \tfrac{17}{26}, \tfrac{2}{3}, \tfrac{17}{25}, \tfrac{15}{22}, \tfrac{13}{19}, \tfrac{11}{16}, \tfrac{9}{13}, \tfrac{16}{23}, \tfrac{7}{10}, \tfrac{12}{17}, \tfrac{5}{7}, \tfrac{13}{18}, \tfrac{8}{11}, \\ &\tfrac{11}{15}, \tfrac{14}{19}, \tfrac{3}{4}, \tfrac{13}{17}, \tfrac{10}{13}, \tfrac{7}{9}, \tfrac{11}{14}, \tfrac{4}{5}, \tfrac{9}{11}, \tfrac{5}{6}, \tfrac{6}{7}\}. \end{aligned}$$

We use Lemma 2.3 to determine $\tilde{\nu}(11,4)$ and $\tilde{\nu}(15,4)$.

Lemma 2.4. $\tilde{\nu}(11,4) = 6$, $\tilde{\nu}(15,4) = 7$.

Proof. By Theorems 1.1 and 1.4, $\tilde{\nu}(11,4) \geq 4$ and $\tilde{\nu}(15,4) \geq 5$. On the other hand, we can see from Lemma 2.3 that $\frac{4}{11} \notin B_4$, B_5 and $\frac{4}{15} \notin B_5$, B_6. Thus $\tilde{\nu}(11,4) \geq 6$ and $\tilde{\nu}(15,4) \geq 7$. Moreover, since $\frac{4}{11} \in B_6$ and $\frac{4}{15} \in B_7$, it follows that $\tilde{\nu}(11,4) = 6$ and $\tilde{\nu}(15,4) = 7$. $\square$

Now the following theorem follows immediately.

Theorem 2.5. *Let $n > k > 0$. Then $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$ if and only if (n,k) satisfies one of the following three relations:*

(i) $n \equiv \pm 1 \pmod{k}$, $k = 2$ or 3.

(ii) $n = k + 1$, $k \geq 2$.

(iii) $n = 4q + 3$, $k = 4$, $q \geq 1$.

Proof. The necessity follows from Lemma 2.1. For the sufficiency, if (n,k) satisfies (i), then we can see from Theorem 1.1 that $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$. If (n,k) satisfies (ii), then by Lemma 1.2, $\tilde{\nu}(n,k) = \tilde{\nu}(k+1,k) = \tilde{\nu}(k+1,1) = k+1 = \lfloor (k+1)/k \rfloor + k$. If (n,k) satisfies (iii), then Lemmas 2.2 and 2.4 imply that $\tilde{\nu}(n,k) = \lfloor n/k \rfloor + k$. This completes the proof. $\square$

3. Exact values of $\tilde{\nu}(n,4)$

In this section, we determine the exact values of $\tilde{\nu}(n,4)$.

Lemma 3.1. $\tilde{\nu}(4q+1,4) = q + 3$ *for* $2 \leq q \leq 4$.

Proof. By Theorems 1.1 and 1.4, $\tilde{\nu}(4q+1,4) \geq q+2$. On the other hand, we can see from Lemma 2.3 that $\frac{4}{9} \notin B_4$, $\frac{4}{13} \notin B_5$, $\frac{4}{17} \notin B_6$ and $\frac{4}{9} \in B_5$, $\frac{4}{13} \in B_6$, $\frac{4}{17} \in B_7$. Hence, $\tilde{\nu}(4q+1,4) = q+3$ for $2 \leq q \leq 4$. $\square$

We use the techniques in [2] to determine $\tilde{\nu}(4q+1,4)$ for $q \geq 5$. Denote the direct sum of two matrices A and B by $A \oplus B$. The following two lemmas can be found in [2].

Lemma 3.2. *Let $A \in \mathscr{B}(n,k)$ with $n \geq k > 0$. Suppose*

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} B & C \\ & A_2 \end{bmatrix}, \quad r(A) = r(A_1), \tag{1}$$

where C has exactly one 1 in each column. Then

- (P_1) *A_1 has no zero columns.*
- (P_2) *Let $x_i = (x_{i1}, \ldots, x_{in})$, $i = 1, \ldots, t$, be the rows of A_1, and let $y = (y_1, \ldots, y_n)$ be a row of A_2 with $y = \sum\limits_{i=1}^{t} a_i x_i$. Then $\sum\limits_{i=1}^{t} a_i = 1$.*
- (P_3) *If C has no zero rows, then $k|n$.*
- (P_4) *If C has a row with k 1's, then A is permutation equivalent to $J_k \oplus M$ for some $M \in \mathscr{B}(n-k,k)$.*
- (P_5) *In expressing a row A_2 as a linear combination of the rows of A_1, any row in which C has a nonzero entry has coefficient 0 or 1.*

Lemma 3.3. *Let ρ and k be positive integers. Then there exists an integer $N(\rho,k) \leq ((2k-1)/2)\rho$ such that if $n > N(\rho,k)$ and $A \in \mathscr{B}(n,k)$ with $r(A) \leq \rho$, then A is permutation equivalent to $J_k \oplus M$ for some $M \in \mathscr{B}(n-k,k)$.*

We proceed to show that $\tilde{\nu}(4q+1,4) = q+3$ holds for $q \geq 5$, and begin with the following lemma.

Lemma 3.4. *Let $A \in \mathscr{B}(4q+1,4)$ be of the form (1), where the rows of A_1 are linear independent, each column of B has at least two 1's and C has exactly one zero row. For $q \geq 5$, if $r(A) < q+3$, then A has a submatrix equal to J_4.*

Proof. By (P_4), we may assume that no row of C contains four 1's and without loss of generality, we suppose that the last row of C is a zero row. Assume that B contains a 0 in its last row, say in column j. Then B has at least five columns. Let column j of B contain a 1 in row i, and suppose row i of C has a 1 in column k. Thus column k of A_2 contains three 1's, say in rows r_1, r_2 and r_3. By (P_5), each of these rows is a linear combination of the rows of A_1 in which the coefficient of row i is 1 and the coefficients of all but the last row are nonnegative. It follows that column j of A_2 contains 1's in rows r_1, r_2 and r_3. Since each column of B contains at least two 1's, column j of A contains at least five 1's, a contradiction. Thus B has exactly four columns.

Suppose $r(A) \leq q+2$. Then A_1 has at most $q+2$ rows. The number of 1's in C is $n-4$. Since C has exactly one zero row, if $n-4 > 3(q+1)$, C has a row with four 1's. That is, if $q > 6$, C has a row with four 1's and by (P_4), A has a submatrix equal to J_4.

We may now assume $5 \leq q \leq 6$. If $q = 6$, then the possible nonzero row sums of C are $3, 3, 3, 3, 3, 3, 3$. Without loss of generality, we assume A_1 equals

$$\left[\begin{array}{cccc|ccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & & & \\
 & & & & & & & & & & & & & & & & 1 & 1 & 1 & & & & & & \\
 & & & & & & & & & & & & & 1 & 1 & 1 & & & & & & & & & \\
 & & & & & & & & & & 1 & 1 & 1 & & & & & & & & & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

There are three rows of A_2 which have a 1 in the last column. It follows from (P_2) and (P_5) that in expressing these rows as a linear combination of the rows of A_1, the coefficients of rows 1 to 8 are 1, 0, 0, 0, 0, 0, 0, 0 and hence A has a submatrix equal to J_4.

If $q = 5$, then the possible nonzero row sums of C are $3, 3, 3, 3, 3, 2$. In a similar way as that of the case $q = 6$, we conclude that A has a submatrix equal to J_4. $\square$

Lemma 3.5. *$\tilde{\nu}(4q + 1, 4) = q + 3$ for $q \geq 5$.*

Proof. By Theorems 1.4 and 2.5, $\tilde{\nu}(4q + 1, 4) \leq q + 3$. For $q \geq 5$, we assume that $\tilde{\nu}(4q + 1, 4) < q + 3$, then either we obtain a direct contradiction or we show that A has a submatrix equal to J_4. Since $\hat{\nu}(17, 4) = 7$, this will provide a contradiction for all $q \geq 5$. It follows from Lemma 3.3 with $\rho = q + 2$ that when $q > 12$, A has a submatrix equal to J_4. We now consider the cases $q = 5, \ldots, 12$.

As in the proof of Lemma 3.3 with $k = 4$ and $\rho = q + 2$, we may assume (1) holds where C has $q + 2$ rows and $4q - 6 = n - 7$ columns. By (P_4) we may assume C has at most three 1's in each row. Let x_i be the number of rows of C with exactly i 1's $(i = 0, 1, 2, 3)$. If $x_0 = 0$, (P_3) implies $4|n$, a contradiction. We may now suppose $x_0 \geq 1$. Since C has exactly one 1 in each of its $4q - 6$ columns, we obtain

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 &= q + 2, \\ x_1 + 2x_2 + 3x_3 &= 4q - 6. \end{aligned}$$

We claim that for $q \geq 7$, $x_0 \leq 1$. Thus $x_0 = 1$, then by Lemma 3.4, A has a submatrix equal to J_4. Now we need only to consider $q = 5$ and $q = 6$ with $x_0 \geq 2$.

For $q = 6$, we have $x_0 = 2$, $x_1 = x_2 = 0$, $x_3 = 6$. We may assume that

$$A_1 = \begin{bmatrix} M & N \\ K & O_{2,18} \end{bmatrix},$$

where each row of N has exactly three 1's.

If K has a zero column, say column j, then by Lemma 3.6 of [10], column j of B has exactly one 1. We remove this column from B and append it to C, since each row of N has exactly three 1's, then C has a row with four 1's. Thus, by (P_4), A has J_4 as a submatrix. Now we suppose that K has no zero columns and the first row of C has one 1 in column j_1. It follows from (P_2) and (P_5) that in expressing those three rows of A_2 which have a 1 in column j_1 as a linear combination of the rows of A_1, the coefficients are $1, 0, \dots, 0, x, y$. If one of x, y is 1 and the other is -1, then A_2 has a row with a negative entry, a contradiction. Thus, $x = y = 0$ and A has a submatrix equal to J_4.

For $q = 5$, we have $x_0 = 2$, $x_1 = 0$, $x_2 = 1$, $x_3 = 4$. If K has at least two zero columns, then by Lemma 3.6 of [10], B has at least two columns with exactly one 1. We remove these columns from B and append them to C, then C has at least one row with four 1's. Thus, by (P_4), A has J_4 as a submatrix. If K has exactly one zero column, then by Lemma 3.6 of [10], B has a column with exactly one 1. We remove this column from B and append it to C, then either C has a row with four 1's or each row of N has exactly three 1's. For the first possibility, by (P_4), A has a submatrix equal to J_4. For the second, in a similar way as that of the case $q = 6$, we conclude that A has a submatrix equal to J_4. If K has no zero columns, then again, in a similar way as that of the case $q = 6$, we conclude that A has a submatrix equal to J_4. $\square$

Now, by Theorems 1.1 and 2.5, Lemmas 3.1 and 3.5 we have the following

Theorem 3.6. *For $n > 5$,*

$$\tilde{\nu}(n, 4) = \begin{cases} n/4, & \text{if } n \equiv 0 \pmod 4, \\ \lfloor n/4 \rfloor + 3, & \text{if } n \equiv 1 \pmod 4, \\ \lfloor n/4 \rfloor + 2, & \text{if } n \equiv 2 \pmod 4, \\ \lfloor n/4 \rfloor + 4, & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

$\tilde{\nu}(5, 4) = 5$, $\tilde{\nu}(4, 4) = 1$.

Next we solve the problem for all possible ranks of matrices in $\mathscr{B}(n, 4)$. Let $S(n, k)$ denote the set of possible ranks of k-regular $(0, 1)$-matrices of order n. Actually, Brualdi [1, Corollary 3.10.2] showed that $S(n, k)$ is a set of consecutive integers and his proof is not constructive. For each case in Theorem 1.1, Pullman and Stanford (see [9]) constructed a matrix in $\mathscr{B}(n, k)$ of each rank in $S(n, k)$ between the maximum and the minimum. Hence we need only to consider the cases $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$. For these two cases, we construct a matrix in $\mathscr{B}(n, 4)$ having rank r for each r in $S(n, 4)$.

Theorem 3.7. *If $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$, then $S(n, 4)$ is a set of consecutive integers.*

Proof. We construct a matrix $B(n,4;r)$ in $\mathscr{B}(n,4)$ having rank r for each r in $S(n,4)$. Recall that $M(n,k)$ is the nonsingular member of $\mathscr{B}(n,k)$ provided in [9] for all $(n,k) \neq (4,2)$. We use $[a,b]$ to denote the set of integers x such that $a \leq x \leq b$.

If $n \equiv 1 \pmod 4$, we show that $S(n,4) = [\lfloor n/4 \rfloor + 3, n]$ for $n > 5$ and $S(5,4) = \{5\}$. Let

$$A_1 = \begin{bmatrix} 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&0\\ 0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&1&1\\ 0&0&1&1&1&0&0&0&1\\ 1&1&0&0&1&0&0&0&1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 0&1&0&0&1&1&1&0&0\\ 1&0&0&0&0&0&1&1&1\\ 0&0&1&0&1&1&0&1&0\\ 0&0&0&1&1&1&0&0&1\\ 0&0&0&0&1&0&1&1&1\\ 0&0&0&0&0&1&1&1&1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 0&1&0&0&1&1&1&0&0\\ 0&0&1&0&0&1&1&1&0\\ 0&0&0&1&0&0&1&1&1\\ 1&0&0&0&1&1&0&0&1\\ 1&0&0&0&0&1&1&1&0\\ 0&1&0&1&1&0&0&0&1\\ 0&0&1&0&1&0&0&1&1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0\\ 1&0&0&0&1&1&1\\ 1&0&0&0&1&1&1\\ 0&1&1&0&1&1&0\\ 0&0&1&1&1&0&1\\ 0&1&0&1&0&1&1 \end{bmatrix},$$

and

$$A_5 = \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0\\ 0&1&1&0&1&1&0\\ 1&0&0&1&1&0&1\\ 1&1&0&0&0&1&1\\ 0&0&1&0&1&1&1\\ 0&0&0&1&1&1&1 \end{bmatrix}.$$

Let $B(5,4;5) = I_5{}^c$, $B(9,4;5) = A_1$, $B(9,4;6) = J_4 \oplus I_5{}^c$, $B(9,4;7) = A_2$, $B(9,4;8) = A_3$ and $B(9,4;9) = M(9,4)$. This completes the construction for $n \leq 12$.

For all $n \geq 13$ and r in $[\lfloor n/4 \rfloor + 3, n-3]$, let $B(n,4;r) = B(n-4,4;r-1) \oplus J_4$. Moreover, let $B(n,4;n-2) = A_4 \oplus M(n-7,4)$, $B(n,4;n-1) = A_5 \oplus M(n-7,4)$ and $B(n,4;n) = M(n,4)$. This completes the construction for all $n \equiv 1 \pmod 4$.

If $n \equiv 3 \pmod 4$, we show that $S(n,4) = [\lfloor n/4 \rfloor + 4, n]$. Let

$$C_1 = \begin{bmatrix} O & J_{3,4} \\ J_{4,3} & I_4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0&1&1&0&0&1&1\\ 1&0&1&0&0&1&1\\ 1&1&0&0&1&0&1\\ 1&0&0&1&1&1&0\\ 0&1&0&1&1&1&0\\ 0&0&1&1&1&0&1\\ 1&1&1&1&0&0&0 \end{bmatrix},$$

and

$$C_3 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let $B(7,4;5) = C_1$, $B(7,4;6) = C_2$ and $B(7,4;7) = M(7,4)$. For all $n \geq 11$ and r in $[\lfloor n/4 \rfloor + 4, n-3]$, let $B(n,4;r) = B(n-4,4;r-1) \oplus J_4$. Moreover, let $B(n,4;n-2) = C_3 \oplus M(n-6,4)$, $B(n,4;n-1) = C_4 \oplus M(n-6,4)$ and $B(n,4;n) = M(n,4)$. This completes the construction for all $n \equiv 3 \pmod 4$. $\square$

We asked in [10] if $\tilde{\nu}(qk+1,k) = q+k-1$ holds for any $q \geq 2$ and $k \geq 4$. Theorem 3.6 shows that $\tilde{\nu}(qk+1,k) = q+k-1$ holds for $k=4$ and $q \geq 2$. However, we can see from the Appendix that $\tilde{\nu}(13,6) = 6$, $\tilde{\nu}(15,7) = 6$, $\tilde{\nu}(17,8) = 6$, $\tilde{\nu}(19,9) = 7$ and $\tilde{\nu}(21,10) = 7$. Hence, $\tilde{\nu}(qk+1,k) < q+k-1$ for $q \geq 2$ and $6 \leq k \leq 10$, which gives a negative answer to Problem 4.2 in [10]. Similarly, since $\tilde{\nu}(13,7) = 6$, $\tilde{\nu}(15,8) = 6$, $\tilde{\nu}(17,9) = 6$, $\tilde{\nu}(19,10) = 7$, it follows that $\tilde{\nu}(qk-1,k) < q+k-2$ for $q \geq 2$ and $7 \leq k \leq 10$, which gives a negative answer to Problem 4.3 in [10].

4. About the relation $\tilde{\nu}(n,k) < \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$

We can see that each value of (n,k) for which $\tilde{\nu}(n,k)$ is determined in Theorem 1.1 and Lemma 3.1 satisfies

(i) $\tilde{\nu}(n,k) = \hat{\nu}(n,k) = \lfloor n/k \rfloor + k$, or

(ii) $\tilde{\nu}(n,k) < \hat{\nu}(n,k) = \lfloor n/k \rfloor + k$, or

(iii) $\tilde{\nu}(n,k) = \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$.

Each of (i), (ii), (iii) is satisfied for some such value of (n,k). Pullman and Stanford (see [9]) asked if the relation $\tilde{\nu}(n,k) < \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$ is satisfied for some (n,k). We remarked in [10] that if there does not exist any (n,k) such that $\tilde{\nu}(n,k) < \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$, then $\tilde{\nu}(n,k) = \hat{\nu}(n,k)$ holds for any (n,k) with $n \not\equiv \pm 1 \pmod k$. Regretfully, we can see from Table 4 in the Appendix that $\tilde{\nu}(18,8) = 5$. On the other hand, by Lemma 1.3, we have $\hat{\nu}(18,8) = 6$. Hence, $\tilde{\nu}(18,8) < \hat{\nu}(18,8) < \lfloor 18/8 \rfloor + 8$, i.e., the relation $\tilde{\nu}(n,k) < \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$ is satisfied for $(18,8)$. In fact, let $A = J_2 \bigotimes A_1$, where $\bigotimes$ is the Kronecker product. Then $A \in \mathscr{B}(18,8)$ and $r(A) = 5$. There are also some other pairs (n,k) such that $\tilde{\nu}(n,k) < \hat{\nu}(n,k) < \lfloor n/k \rfloor + k$, e.g., $(16,7), (20,9), (22,10)$ and $(24,11)$.

The above examples led us to suspect that $\tilde{\nu}(2k+2,k) < \hat{\nu}(2k+2,k) < \lfloor (2k+2)/k \rfloor + k$ for all $k \geq 7$. We will show that this suspect is true for all even $k \geq 8$.

For $k \geq 2$, let $C_k(x_1, x_2, \ldots, x_n)$ denote the $k \times n$ matrix obtained by circulating the row vector $(x_1, x_2, \ldots, x_n)$ $k-1$ times. That is,

$$C_k(x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_n & x_1 & \cdots & x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-k+2} & x_{n-k+3} & \cdots & x_{n-k+1} \end{bmatrix}.$$

Let $\mathscr{B}^{m\times n}$ be the set of all $m \times n$ $(0,1)$-matrices. Denote by $A[\mu|\nu]$ the submatrix of A with rows indexed by μ and columns indexed by ν.

Theorem 4.1. *Let k and t be positive integers such that $t \geq 2$ and $t|k$. Then*
(i) $\tilde{\nu}(pk+t, k) < \hat{\nu}(pk+t, k) < \lfloor (pk+t)/k \rfloor + k$ holds for all $p \geq 2$ and $k \geq 4t$;
(ii) $\tilde{\nu}(qk-t, k) < \hat{\nu}(qk-t, k) < \lfloor (qk-t)/k \rfloor + k$ holds for all $q \geq 2$ and $k \geq 5t$.

Proof. (i) For $s \geq 4$, let

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix}, \tag{2}$$

where $A_1 = (j_s^T, 0_{s+1}^T) \in \mathscr{B}^{1\times(2s+1)}$, $A_2 = (O_{s-2,s}, C_{s-2}(\underbrace{1,\ldots,1}_{s}, 0)) \in \mathscr{B}^{(s-2)\times(2s+1)}$, $A_3 = \begin{bmatrix} 0_{s-2}^T & j_{s-2}^T & 1 & 0 & 0 & 0 & 1 \\ j_{s-2}^T & 0_{s-2}^T & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathscr{B}^{2\times(2s+1)}$, $A_4 = (J_{s-2,s}, O_{s-2,s+1}) \in \mathscr{B}^{(s-2)\times(2s+1)}$, $A_5 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathscr{B}^{2\times(2s+1)}$, where u_1 and u_2 are equal to the first and the last row of A_2, respectively.

Then we can see from (2) that $A \in \mathscr{B}(2s+1, s)$ and

$$r(A) = r\left(\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}\right) \leq s+1.$$

Next, we will show that $\tilde{A} = A[\{1, 2, \ldots, s+1\} \mid \{1, s, s+1, \ldots, 2s-2, 2s+1\}]$ is nonsingular, which implies $r(A) = s+1$.

Observe that $\tilde{A}$ is of the form

$$\tilde{A} = \begin{bmatrix} j_2^T & 0_{s-1}^T \\ O_{s-2,2} & \tilde{A}_1 \\ \tilde{A}_2 & \tilde{A}_3 \end{bmatrix},$$

where $\tilde{A}_1 = C_{s-2}(\underbrace{1,\ldots,1}_{s-2}, 0)$, $\tilde{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\tilde{A}_3 = \begin{bmatrix} j_{s-4}^T & 1 & 0 & 1 \\ 0_{s-4}^T & 1 & 0 & 1 \end{bmatrix}$.

Now we compute the determinant of $\tilde{A}$. A direct calculation shows that

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ -1 & & & 1 \end{bmatrix} \tilde{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 2 & 0 & 2 \end{bmatrix}. \tag{3}$$

Then it follows from (3) that

$$\det \tilde{A} = (-1)^s \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & 1 \\ 1 & 1 & \cdots & 2 & 0 & 2 \end{vmatrix} := (-1)^s \det \tilde{A}'.$$

Moreover,

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{bmatrix} \tilde{A}' \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{bmatrix} = \begin{bmatrix} s-2 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 \\ 2 & 0 & \cdots & 1 & -1 & 2 \end{bmatrix}. \tag{4}$$

Now we can see from (4) that $\tilde{A}'$ is nonsingular. Hence $\tilde{A}$ is nonsingular and $r(A) = s+1$.

If $t \geq 2$, $t|k$ and $k \geq 4t$, then by the above discussion we know that there exists a matrix $A \in \mathscr{B}(2k/t+1, k/t)$ such that $r(A) = k/t+1$. Let $B = (\bigoplus_{p-2} J_{k/t}) \bigoplus A$ and $C = J_t \bigotimes B$. Then $B \in \mathscr{B}(pk/t+1, k/t)$, $C \in \mathscr{B}(pk+t, k)$ and $r(C) = k/t+p-1$, which implies that $\tilde{\nu}(pk+t, k) \leq k/t+p-1$. Since $\hat{\nu}(pk+t, k) = k/t+p$ and $\lfloor (pk+t)/k \rfloor + k = k+p$, then $\tilde{\nu}(pk+t, k) < \hat{\nu}(pk+t, k) < \lfloor (pk+t)/k \rfloor + k$.

(ii) If $s \geq 5$, then $s-1 \geq 4$. Since $2s-1 = 2(s-1)+1$, by the arguments in (i) we know that there exists some matrix $A' \in \mathscr{B}(2s-1, s-1)$ such that $r(A') = s$. Let $A = A'^{c}$. Then $A \in \mathscr{B}(2s-1, s)$ and it follows from Lemma 1.2 that $r(A) = r(A') = s$. If $k \geq 5t$, then there exists some matrix $M \in \mathscr{B}(2k/t-1, k/t)$ such that $r(M) = k/t$. Let $N = (\bigoplus_{q-2} J_{k/t}) \bigoplus M$ and $P = J_t \bigotimes N$. Then $N \in \mathscr{B}(qk/t-1, k/t)$, $P \in \mathscr{B}(qk-t, k)$ and $r(P) = k/t+q-2$, which implies that $\tilde{\nu}(qk-t, k) \leq k/t+q-2$. Since $\hat{\nu}(qk-t, k) = k/t+q-1$ and $\lfloor (qk-t)/k \rfloor + k = k+q-1$, then $\tilde{\nu}(qk-t, k) < \hat{\nu}(qk-t, k) < \lfloor (qk-t)/k \rfloor + k$. $\square$

Corollary 4.2. *$\tilde{\nu}(2k+2, k) < \hat{\nu}(2k+2, k) < \lfloor (2k+2)/k \rfloor + k$ holds for all even $k \geq 8$.*

Acknowledgements

The authors would like to thank the anonymous referee for helpful suggestions. The research of Zhong was supported by the NSFC grant 11661041, the Jiangxi SF grant 20161BAB211016 and the Science and Technology Project Founded by the Education Department of Jiangxi Province GJJ150645. Ma's research was supported by the NSFC grants 11526139, 11601322, 61573240 and the Shanghai SF grant 15ZR1411500.

Appendix A

By Lemma 2.3, we can determine the exact values of $\tilde{\nu}(n,k)$ for some positive integers n and k, which may be useful to investigate the behavior of $\tilde{\nu}(n,k)$ for some fixed k. (See Tables 1–6.)

Table 1
The values of $\tilde{\nu}(n,5)$ for $5 \le n \le 20$.

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\tilde{\nu}(n,5)$	1	6	5	5	5	2	6	6	6	6	3	7	7	7	7	4

Table 2
The values of $\tilde{\nu}(n,6)$ for $6 \le n \le 22$.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\tilde{\nu}(n,6)$	1	7	4	3	4	6	2	6	5	4	5	7	3	7	6	5	6

Table 3
The values of $\tilde{\nu}(n,7)$ for $7 \le n \le 23$.

n	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$\tilde{\nu}(n,7)$	1	8	6	6	6	6	6	2	6	6	7	7	7	7	3	7	7

Table 4
The values of $\tilde{\nu}(n,8)$ for $8 \le n \le 26$.

n	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$\tilde{\nu}(n,8)$	1	9	5	6	3	6	5	6	2	6	5	7	4	7	6	7	3	7	6

Table 5
The values of $\tilde{\nu}(n,9)$ for $9 \le n \le 27$.

n	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$\tilde{\nu}(n,9)$	1	10	7	4	6	6	4	6	6	2	7	7	5	7	7	5	7	7	3

Table 6
The values of $\tilde{\nu}(n,10)$ for $10 \le n \le 28$.

n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\tilde{\nu}(n,10)$	1	11	6	7	5	3	5	7	5	7	2	7	6	7	6	4	6	7	6

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