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# Existence of solutions for $2n$th-order nonlinear $p$-Laplacian differential equations

Lorena Saavedra[a,*], Stepan Tersian[b,1]

[a] Instituto de Matemáticas, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela, Galicia, Spain

[b] Department of Mathematics, University of Ruse, Ruse, Bulgaria

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ABSTRACT

The aim of this paper is to study the existence and multiplicity of weak and classic solutions for a $2n$th-order differential equation involving the $p$-Laplacian coupled with periodic boundary conditions. The results are proved by using the minimization argument and an extended Clark's theorem. Some particular cases are shown.



## 1. Introduction

The aim of this paper is to study a $2n$th-order differential equation involving the $p$-Laplacian, with periodic boundary conditions. This kind of problems have been widely studied in the literature. For instance, in [1] it is considered a second order problem, in [2] it is obtained the existence of anti-periodic solutions for a $n$th-order problem. Moreover, in [3] it is studied a fourth order problem involving the $p$-Laplacian with deviating terms.

In this paper, we generalize the results obtained in [4] for a second order problem.

For $p > 1$, let us introduce the function $\varphi_p\colon \mathbb{R} \to \mathbb{R}$, defined by:

$$\varphi_p(t) = \begin{cases} t|t|^{p-2} & t \neq 0, \\ 0 & t = 0, \end{cases}$$

* Corresponding author.
*E-mail addresses:* lorena.saavedra@usc.es (L. Saavedra), sterzian@uni-ruse.bg (S. Tersian).

1 Associate at: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113- Sofia, Bulgaria.

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Let us consider the following problem:

$$\left(\varphi_p\left(u^{(n)}(t)\right)\right)^{(n)}+\sum_{i=1}^{n-1}(-1)^i a_i\left(\varphi_p\left(u^{(n-i)}(t)\right)\right)^{(n-i)}+(-1)^n\left(f(t,u(t))-h(t,u(t))\right)=0, \quad t\in[0,T] \tag{1}$$

coupled with the boundary conditions

$$u(T)-u(0)=\cdots=u^{(2n-1)}(T)-u^{(2n-1)}(0)=0, \tag{2}$$

where $T\geq 0$ and $a_i\geq 0$ for $i=1,\ldots,n-1$.

We introduce the following Banach space:

$$X_p=\left\{u\in W^{n,p}(0,T)\mid u(T)-u(0)=\cdots=u^{(n-1)}(T)-u^{(n-1)}(0)=0\right\}, \tag{3}$$

where $W^{n,p}(0,T)$ is the Sobolev space:

$$W^{n,p}(0,T)=\left\{u\in L^p(0,T):\|u\|_p=\left(\sum_{i=0}^{n}\int_0^T\left|u^{(i)}(t)\right|^p dt\right)^{1/p}<\infty\right\}.$$

The function $u\in C^n([0,T])$ is said to be a classical solution of this problem if $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$ and it satisfies Eq. (1) for $t\in(0,T)$ and periodic conditions (2).

**Remark 1.1.** Realize that, from the derivative chain rule, the assumption $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$ implies that $u^{(n)}\in C^n([0,T])$. Moreover, if $u^{(n)}$ is not of constant sign on $[0,T]$, in order to have $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$, we should ask for $\varphi_p\in C^n(\mathbb{R})$.

So, in particular, $u\in C^{(2n)}([0,T])$. However, we need to study the regularity of the $p$-Laplacian to ensure that a function which verifies $u\in C^{(2n)}([0,T])$ also satisfies $\varphi_p\left(u^{(n)}(\cdot)\right)\in C^n([0,T])$.

For instance, let us consider $n=2$ and $p=3$, for $u\in C^4([0,T])$ we have

$$\left(\varphi_3\left(u''(t)\right)\right)'=\varphi_3'\left(u''(t)\right)u^{(3)}(t)=|u''(t)|\,u^{(3)}(t),$$

hence $\varphi_3\left(u''(\cdot)\right)\in C^1([0,T])$, but $\varphi_3\left(u''(\cdot)\right)\notin C^2([0,T])$ if $u''$ is not of constant sign on $[0,T]$ even if $u^{(3)}\in C^1([0,T])$.

The function $u\in X_p$ is said to be a weak solution of (1)–(2) if for every $v\in X_p$ it is verified the following equality:

$$\begin{aligned}\int_0^T\varphi_p\left(u^{(n)}(t)\right)v^{(n)}(t)\,dt+\sum_{i=1}^{n-1}a_i\int_0^T\varphi\left(u^{(n-i)}(t)\right)v^{(n-i)}(t)\,dt\\+\int_0^T\left(f(t,u(t))-h(t,u(t))\right)v(t)\,dt=0.\end{aligned} \tag{4}$$

The aim of this paper is to ensure the existence of multiple weak solutions of (1)–(2). That is, we look for $u\in X_p$ such that (4) is verified. Then, we ensure that in some cases the weak solution is also a classical solution.

Now, we introduce a condition that $f$ and $h$ must satisfy. Let us consider the following functions:

$$F(t,u)=\int_0^u f(t,s)\,ds, \qquad H(t,u)=\int_0^u h(t,s)\,ds.$$

We say that $f$ and $h$ satisfy condition $(FH)$ if they are continuous functions on $\mathbb{R}^2$ and there exist positive constants $b_1$, $b_2$, $c_1$ and $c_2$, $q > r > 1$ such that for all $u \in \mathbb{R}$ the following inequalities are fulfilled:

$$b_1 |u|^q \le F(t,u) \le b_2 |u|^q, \qquad c_1 |u|^r \le H(t,u) \le c_2 |u|^r. \tag{5}$$

Realize that we can write $b_i = \frac{\tilde{b}_i}{q}$ and $c_i = \frac{\tilde{c}_i}{r}$ for $i = 1, 2$.

Under this condition, we state the main results as follows.

**Theorem 1.2.** *Let $p > 1$, $f$ and $h$ satisfying $(FH)$, $a_i \ge 0$ for $i = 1, \dots, n-1$. Then (1)–(2) has at least a weak solution.*

*If, in addition, we have that $p > r$ and $f$ and $h$ are odd functions, then (1)–(2) has infinitely many pairs of weak solutions $(u_m, -u_m)$, such that $u_m \ne 0$ and $\lim_{m\to+\infty} \max_{t\in[0,T]} |u_m(t)| = 0$.*

We can obtain for some cases the existence of classical solution. Let us denote $i_0 = \min\{i \in \{1, \dots, n-1\} : a_i \ne 0\}$, we have:

**Theorem 1.3.** *Let $p > 1$, $f$ and $h$ satisfying $(FH)$, $a_i = 0$ for $i = 1, \dots, i_0 - 1$, $a_i \ge 0$ for $i = i_0, \dots, n-1$, $i_0 \ge \left\lfloor \frac{n}{2} \right\rfloor$ and $\lceil p \rceil \ge n - i_0 + 2$ or $p = 2k$, where $k \in \mathbb{Z}$. Then (1)–(2) has at least a classical solution.*

*If, in addition, we have that $p > r$ and $f$ and $h$ are odd functions, then (1)–(2) has infinitely many pairs of classical solutions $(u_m, -u_m)$, such that $u_m \ne 0$ and $\lim_{m\to\infty} \max_{t\in[0,T]} |u_m(t)| = 0$.*

Theorem 1.3 for $n = 1$ has been obtained in [4]. In this paper, we use some of the arguments that have been introduced there, to extend the results for the general $n$th-order problem (1)–(2).

This paper is scheduled as follows. At first, we introduce a preliminary section, where we give the variational formulation of problem (1)–(2). We present two results which allow us to ensure the existence of solution. Then, in Section 3, we prove the main results. Next, we show the analogous results to Theorems 1.2 and 1.3 for a problem with impulses. Finally, in Section 4, we show some particular cases where we show the usefulness of the results.

## 2. Preliminaries

First, we introduce an other norm in $X_p$:

$$\|u\| = \left( \int_0^T |u^{(n)}(t)|^p \, dt + \int_0^T |u(t)|^p \, dt \right)^{1/p}.$$

Now, we introduce a result which shows that $\|\cdot\|$ and $\|\cdot\|_p$ are equivalent norms in $X_p$.

**Lemma 2.1.** *If $u \in X_p$, then*

$$\int_0^T \left| u^{(i)}(t) \right|^p dt \le T^{p\,(n-i)} \int_0^T \left| u^{(n)}(t) \right|^p dt,$$

*for $i = 1, \dots, n-1$.*

**Proof.** If $u \in W^{1,p}(0,T)$ and $\int_0^T u(t)\, dt = 0$, then $\int_0^T |u(t)|^p \, dt \le T^p \int_0^T |u'(t)|^p \, dt$, see [5, pp. 8–9].

Now, since $u \in X_p$,

$$\int_0^T u^{(j)}(t)\, dt = u^{(j-1)}(T) - u^{(j-1)}(0) = 0, \quad j = 1, \dots, n-1.$$

We have:

$$\int_0^T \left|u^{(j)}(t)\right|^p dt \le T^p \int_0^T \left|u^{(j+1)}(t)\right|^p dt, \quad j = 1, \ldots, n-1. \tag{6}$$

Applying an induction argument, we prove the result. For $i = n-1$, the result follows from (6).

For $i \in \{2, \ldots, n-1\}$, from (6), we have:

$$\int_0^T \left|u^{(i-1)}(t)\right|^p dt \le T^p \int_0^T \left|u^{(i)}(t)\right|^p dt, \quad i = 1, \ldots, n-1,$$

and the result follows from an induction hypothesis. □

Using this result, the equivalence between the norms $\|\cdot\|$ and $\|\cdot\|_p$ is obvious:

$$\|u\| \le \|u\|_p \le \left(1 + \sum_{i=1}^{n-1} T^{p\,(n-i)}\right)^{1/p} \|u\| := k_1\|u\|.$$

Considering again [5, pp. 8–9], we have the following result.

**Lemma 2.2.** *For $u \in X_p$, the following inequality is fulfilled:*

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)| \le \left(T^{-\frac{1}{p}} + T^{\frac{p-1}{p}}\right) \|u\|_p := k_2\|u\|_p.$$

Now, let us consider the following space:

$$W = \left\{ u \in X_p : \int_0^T u(t)\,dt = 0 \right\},$$

coupled with the norm:

$$\|u\|_W = \left(\int_0^T \left|u^{(n)}(t)\right|^p dt\right)^{1/p}.$$

Taking into account that $\int_0^T u(t)\,dt = 0$ for $u \in W$, we have an analogous result to Lemma 2.1 to show that $\|\cdot\|_p$ and $\|\cdot\|_W$ are equivalent in $W$.

**Lemma 2.3.** *If $u \in W$, then*

$$\int_0^T \left|u^{(i)}(t)\right|^p dt \le T^{p\,(n-i)} \int_0^T \left|u^{(n)}(t)\right|^p dt,$$

*for $i = 0, \ldots, n-1$.*

The equivalence between $\|\cdot\|_p$ and $\|\cdot\|_W$ in $W$ follows directly from Lemma 2.3.

$$\|u\|_W \le \|u\|_p \le \left(1 + \sum_{i=0}^{n-1} T^{p\,(n-i)}\right)^{1/p} \|u\|_W := k_3\|u\|_W.$$

Now, we introduce the variational approach of the problem (1)–(2).

Let us consider the function $\Phi_p(t) = \frac{|t|^p}{p}$. It is clear that $\Phi_p'(t) = \varphi_p(t)$.

Define a functional $I\colon X_p \to \mathbb{R}$, as follows:

$$I(u) := I_1(u) + I_2(u), \tag{7}$$

where

$$I_1(u) = \int_0^T \left[ \Phi_p(u^{(n)}(t)) + \sum_{i=1}^{n-1} a_i \Phi_p(u^{(n-i)}(t)) \right] dt, \tag{8}$$

$$I_2(u) = \int_0^T (F(t, u(t)) - H(t, u(t)))\, dt. \tag{9}$$

The functional $I$ is Gateaux differentiable and for all $u, v \in X_p$, we have:

$$\langle I'(u), v\rangle = \int_0^T \varphi_p\left(u^{(n)}(t)\right) v^{(n)}(t)\, dt + \sum_{i=1}^{n-1} a_i \int_0^T \varphi_p\left(u^{(n-i)}(t)\right) v^{(n-i)}(t)\, dt$$
$$+ \int_0^T (f(t, u(t)) - h(t, u(t)))\, v(t)\, dt.$$

Hence the critical points of $I$ are exactly the weak solutions of (1)–(2).

Now, we introduce two results which we are going to use to prove our result.

First of them, is the classical minimization argument, see for instance [5,6].

**Theorem 2.4.** *Let $E: X \to \mathbb{R}$ be a weakly sequentially lower semicontinuous functional on a reflexive Banach space $X$ and let $E$ have a minimum on $X$, i.e., there exists $u_0 \in X$ such that $E(u_0) = \inf_{u\in X} E(u)$. If $E$ is differentiable, then $u_0$ is a critical point of $E$.*

$I_1$ is a continuous and convex functional, since $\Phi_p$ is a convex functional, then it is weakly sequentially lower semicontinuous functional.

Since the functions $f$ and $h$ satisfy condition $(FH)$ and the embedding $X_p \hookrightarrow\hookrightarrow C([0, T])$ is compact, then $I_2$ is sequentially continuous. Thus, $I$ is a weakly sequentially lower semicontinuous functional.

Moreover, $X_p$ is a uniformly convex Banach space, then it is a reflexive Banach space.

Clark's theorem, see [7], gives the existence of a sequence of critical points for even functionals. This result or some of its modifications have been used to obtain multiplicity results in many cases (see for instance [8,9]). We present a result, which is a generalization of Clark's theorem given in [10].

**Theorem 2.5.** *Let $X$ be a Banach space, $E \in C^1(X, \mathbb{R})$. Assume that $E$ satisfies the $(PS)$ condition, it is even, bounded from below and $E(0) = 0$. If for any $k \in \mathbb{N}$, there exists a $k$-dimensional subspace $X^k$ of $X$ and $\rho_k > 0$ such that $\sup_{X^k\cap S_{\rho_k}} E < 0$, where $S_\rho = \{u \in X,\ \|u\|_X = \rho\}$, then at least one of the following conclusions holds:*

1. *There exists a sequence of critical points $\{u_k\}$ satisfying $E(u_k) < 0$ for all $k$ and $\lim_{k\to\infty} \|u_k\|_X = 0$.*
2. *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point $u$ such that $\|u\|_X = \alpha$ and $E(u) = 0$.*

**Remark 2.6.** Realize that Theorem 2.5 implies the existence of many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$ such that $E(u_m) \leq 0$, $\lim_{m\to+\infty} E(u_m) = 0$ and $\lim_{m\to+\infty} \|u_m\|_X = 0$.

## 3. Main results

This section is devoted to prove Theorems 1.2 and 1.3.

Before proving Theorem 1.2, we are going to enunciate some preliminary results.

**Lemma 3.1** *([11, Lemma 4.2]). If $p \geq 2$, then*

$$|y|^p \geq |x|^p + p\,|x|^{p-2}\,x\,(y-x) + \frac{|y-x|^p}{2^{p-1}-1},$$

*for all points $x, y \in \mathbb{R}$.*

*If $1 < p < 2$, then*

$$|y|^p \geq |x|^p + p\,|x|^{p-2}\,x\,(y-x) + C(p)\,\frac{|y-x|^p}{(|x|+|y|)^{2-p}},$$

*for all points $x, y \in \mathbb{R}$. Where $C(p)$ is a positive constant depending only on $p$.*

**Remark 3.2.** Realize that, from Lemma 3.1, we obtain, directly, the following assertions: If $p \geq 2$, then

$$|x|^{p-2}\,x\,(x-y) \geq \frac{1}{p}\left(|x|^p - |y|^p + \frac{|y-x|^p}{2^{p-1}-1}\right),$$

and, in particular,

$$|x|^{p-2}\,x\,(x-y) + |y|^{p-2}\,y\,(y-x) \geq \frac{2}{p}\left(\frac{|y-x|^p}{2^{p-1}-1}\right),$$

for all points $x, y \in \mathbb{R}$.

If $1 < p < 2$, then

$$|x|^{p-2}\,x\,(x-y) \geq \frac{1}{p}\left(|x|^p - |y|^p + C(p)\,\frac{|y-x|^p}{(|x|+|y|)^{2-p}}\right),$$

and, in particular,

$$|x|^{p-2}\,x\,(x-y) + |y|^{p-2}\,y\,(y-x) \geq \frac{2}{p}\,C(p)\,\frac{|y-x|^p}{(|x|+|y|)^{2-p}},$$

for all points $x, y \in \mathbb{R}$. Where $C(p)$ is a positive constant depending only on $p$.

Using this result, we can prove the following.

**Lemma 3.3.** *If $f$ and $h$ satisfy $(FH)$ condition, then the functional $I\colon X_p \to \mathbb{R}$ defined in (7) satisfies the Palais–Smale condition in $X_p$.*

**Proof.** Let $\{u_m\}$ be a Palais–Smale sequence, i.e., $I(u_m)$ is bounded in $\mathbb{R}$ and $I'(u_m) \to 0$ in $X_p^*$, where $X_p^*$ is the dual space of $X_p$.

First, let us see that $\{u_m\}$ is a bounded sequence in $X_p$.

Consider the function $g(t) = \frac{1}{q}\,\tilde{b}_1 t^q - \frac{1}{r}\,\tilde{c}_2\,t^r$ for $t > 0$. We can prove that in $d_1 := \left(\frac{\tilde{c}_2}{\tilde{b}_1}\right)^{\frac{1}{q-r}}$ is achieved the minimum of $g$ in $(0, +\infty)$. Then

$$g(t) \geq g(d_1) = \frac{r-q}{r\,q}\left(\frac{\tilde{c}_2^q}{\tilde{b}_1^r}\right)^{\frac{1}{q-r}} =: d_2, \quad t > 0. \tag{10}$$

Let us write $u_m = \tilde{u}_m + \bar{u}_m$, where $\tilde{u}_m \in W$ and $\bar{u}_m \in \mathbb{R}$. Using condition $(FH)$, we can estimate $I$ from below on $X_p$ as follows:

$$\begin{aligned} I(u_m) &\geq \int_0^T \Phi_p\left(u_m^{(n)}(t)\right)\,dt + \int_0^T \left(\frac{1}{q}\,\tilde{b}_1\,|u_m(t)|^q - \frac{1}{r}\,\tilde{c}_2\,|u_m(t)|^r\right)dt \\ &\geq \frac{1}{p}\,\|u_m\|_W^p + T\,d_2 = \frac{1}{p}\,\|\tilde{u}_m\|_W^p + T\,d_2. \end{aligned}$$

Then, since $I(u_m)$ is bounded, $\{\tilde{u}_m\}$ is a bounded sequence in $W$. Using the fact that $\|\cdot\|_W$ and $\|\cdot\|$ are equivalent norms in $W$, we conclude that $\{\tilde{u}_m\}$ is also a bounded sequence in $X_p$.

Now, analogously as in [4], it can be proved that $\{\bar{u}_m\}$ is bounded in $\mathbb{R}$ using Lemma 2.2. Thus, it is also bounded in $X_p$. Hence, we can affirm that $\{u_m\}$ is a bounded sequence in $X_p$.

Passing to a subsequence, if it is necessary, we may assume the existence of $u \in X_p$ such that $u_m \rightharpoonup u$ weakly in $X_p$ and $u_m \to u$ in $C([0,T])$.

Since $I'(u_m) \to 0$ in $X_p^*$, taking into account the weak convergence in $X_p$, we have:

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \langle I'(u_m) - I(u), u_m - u\rangle \\
&= \lim_{n\to+\infty} \int_0^T \left(\varphi_p\left(u_m^{(n)}(t)\right) - \varphi_p\left(u^{(n)}(t)\right)\right)\left(u_m^{(n)}(t) - u^{(n)}(t)\right)\, dt \\
&\quad + \lim_{n\to+\infty} \sum_{i=1}^{n-1} a_i \int_0^T \left(\varphi_p\left(u_m^{(n-i)}(t)\right) - \varphi_p\left(u^{(n-i)}(t)\right)\right)\left(u_m^{(n-i)}(t) - u^{(n-i)}(t)\right)\, dt \\
&\quad + \lim_{n\to+\infty} \int_0^T \left(f(t,u_m(t)) - f(t,u(t))\right)\left(u_m(t) - u(t)\right)\, dt \\
&\quad - \lim_{n\to+\infty} \int_0^T \left(h(t,u_m(t)) - h(t,u(t))\right)\left(u_m(t) - u(t)\right)\, dt.
\end{aligned} \tag{11}$$

Since $u_m \to u$ in $C([0,T])$, we have that

$$\begin{aligned}
&\lim_{n\to+\infty} \int_0^T \left(f(t,u_m(t)) - f(t,u(t))\right)\left(u_m(t) - u(t)\right)\, dt = 0, \\
&\lim_{n\to+\infty} \int_0^T \left(h(t,u_m(t)) - h(t,u(t))\right)\left(u_m(t) - u(t)\right)\, dt = 0.
\end{aligned} \tag{12}$$

Now, using Lemma 3.1 and Remark 3.2, we have for each $i = 1, \ldots, n-1$:

(a) for $p \geq 2$,

$$\begin{aligned}
&\left(\varphi_p\left(u_m^{(n-i)}(t)\right) - \varphi_p\left(u^{(n-i)}(t)\right)\right)\left(u_m^{(n-i)}(t) - u^{(n-i)}(t)\right) \\
&\quad = \varphi_p\left(u_m^{(n-i)}(t)\right)\left(u_m^{(n-i)}(t) - u^{(n-i)}(t)\right) + \varphi_p\left(u^{(n-i)}(t)\right)\left(u^{(n-i)}(t) - u_m^{(n-i)}(t)\right) \\
&\quad \geq \frac{2\left|u_m^{(n-i)}(t) - u^{(n-i)}(t)\right|^p}{p\,(2^{p-1}-1)} \geq 0,
\end{aligned}$$

(b) $1 < p < 2$,

$$\begin{aligned}
&\left(\varphi_p\left(u_m^{(n-i)}(t)\right) - \varphi_p\left(u^{(n-i)}(t)\right)\right)\left(u_m^{(n-i)}(t) - u^{(n-i)}(t)\right) \\
&\quad = \varphi_p\left(u_m^{(n-i)}(t)\right)\left(u_m^{(n-i)}(t) - u^{(n-i)}(t)\right) + \varphi_p\left(u^{(n-i)}(t)\right)\left(u^{(n-i)}(t) - u_m^{(n-i)}(t)\right) \\
&\quad \geq \frac{2\,C(p)\left|u_m^{(n-i)}(t) - u^{(n-i)}(t)\right|^p}{p\left(\left|u_m^{(n)}(t)\right| + \left|u^{(n)}(t)\right|\right)^{2-p}} \geq 0.
\end{aligned}$$

This coupled with (11) and (12), implies that

$$\begin{aligned} 0 &= \lim_{n\to+\infty}\int_0^T \left(\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right)\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\,dt \\ &\quad+\lim_{n\to+\infty}\sum_{i=1}^{n-1}a_i\int_0^T \left(\varphi_p\left(u_m^{(n-i)}(t)\right)-\varphi_p\left(u^{(n-i)}(t)\right)\right)\left(u_m^{(n-i)}(t)-u^{(n-i)}(t)\right)\,dt \\ &\geq \lim_{n\to+\infty}\int_0^T \left(\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right)\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\,dt. \end{aligned}$$

If $p\geq 2$, using again Lemma 3.1, we conclude that

$$\begin{aligned} 0 &\geq \lim_{n\to+\infty}\int_0^T \left(\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right)\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\,dt, \\ &\geq \frac{2}{p\,(2^{p-1}-1)}\lim_{n\to+\infty}\int_0^T \left|u_m^{(n)}(t)-u^{(n)}(t)\right|^p\,dt\geq 0. \end{aligned}$$

This, coupled with the fact that $u_m\to u$ in $C([0,T])$, implies that:

$$0=\lim_{n\to+\infty}\int_0^T\left(\left|u_m^{(n)}(t)-u^{(n)}(t)\right|^p+|u_m(t)-u(t)|^p\right)dt=\lim_{n\to+\infty}\|u_m-u\|^p.$$

So, $u_m\to u$ in $X_p$.

If $1<p<2$, we have, using Hölder inequality:

$$\begin{aligned} 0 &\geq \lim_{n\to+\infty}\int_0^T \left(\varphi_p\left(u_m^{(n)}(t)\right)-\varphi_p\left(u^{(n)}(t)\right)\right)\left(u_m^{(n)}(t)-u^{(n)}(t)\right)\,dt, \\ &\geq \lim_{n\to+\infty}\left\{\int_0^T\left|u_m^{(n)}(t)\right|^p dt+\int_0^T\left|u^{(n)}(t)\right|^p dt\right. \\ &\quad\left.-\left(\int_0^T\left|u_m^{(n)}(t)\right|^p dt\right)^{\frac{p-1}{p}}\left(\int_0^T\left|u^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}-\left(\int_0^T\left|u_m^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^T\left|u^{(n)}(t)\right|^p dt\right)^{\frac{p-1}{p}}\right\} \\ &= \lim_{n\to+\infty}\left(\|u_m\|_W-\|u\|_W\right)\left(\|u_m\|_W^{p-1}-\|u\|_W^{p-1}\right)\geq 0. \end{aligned}$$

This coupled with the fact that $u_m\to u$ in $C([0,T])$, implies that $\lim_{n\to+\infty}\|u_m\|=\|u\|$. Now, from the weak convergence in $X_p$, since $X_p$ is uniformly convex, we have that $u_m\to u$ in $X_p$ and the result is proved. □

**Proof of Theorem 1.2.** As we have said, $I$, defined in (7), is a weakly sequentially lower semicontinuous functional on the reflexive Banach Space $X_p$.

Using the arguments of Lemma 3.3, we can prove that

$$I(u)\geq\frac{1}{p}\|u\|_W^p+T\,d_2,$$

so, $\inf_{u\in X_p}I(u)>-\infty$.

Moreover, if we choose a minimizing sequence, i.e. $\{u_m\}\in X_p$, such that $\lim_{n\to+\infty}I(u_m)=\inf_{u\in X_p}I(u)$, we can prove it is bounded in $X_p$ following the same steps as in [4].

Then, from Theorem 2.4, the existence of a critical point of $I$ is proved. This critical point corresponds to a weak solution of (1)–(2).

The multiplicity result follows from Theorem 2.5, using the same arguments as in [4, Theorem 1.1]. For the convenience of the reader, we are going to recall some of these steps.

From Lemma 3.3, $I$ satisfies the $(PS)$ condition.

$I$ is bounded from below in $X_p$ and $I(0) = 0$. Since $f$ and $h$ are odd functions then $I$ is even.

Let $k \in \mathbb{N}$ be arbitrary and $X^k$ be a $k$-dimensional subspace of $X_p$, with basis elements $\{\Theta_1, \dots, \Theta_k\} \subset W \subset X_p$. The separability of $X_p$ allows this construction.

The set $S_\rho^k := \{u = \alpha_1\, \Theta_1 + \cdots + \alpha_k\, \Theta_k : \sum_{j=1}^k |\alpha_j|^p = \rho^k\} \subset X_k$ is homeomorphic to the unit sphere $S^{k-1} \subset \mathbb{R}^k$.

For $u = \sum_{j=1}^k \alpha_j\, \Theta_j$, the expression $\|u\|_{X_k} = \left(\sum_{j=1}^k |\alpha_j|^p\right)^{1/p}$ defines a norm in $X^k$.

From Lemma 2.3, we can use the fact that norms $\|\cdot\|_{X^k}$, $\|\cdot\|_W$, $\|\cdot\|_{L^q}$ and $\|\cdot\|_{L^r}$ are equivalent.

As in [4], using condition $(FH)$ and Lemma 2.1, we obtain that for any $u \in S_\rho^k$

$$\begin{aligned} I(u) &= \int_0^T \left[\Phi_p(u^{(n)}(t)) + \sum_{i=1}^{n-1} a_i\, \Phi_p(u^{(n-i)}(t))\right] dt + \int_0^T (F(t,u(t)) - H(t,u(t)))\, dt \\ &\le \left(1 + \sum_{i=1}^{n-1} T^{p\,i} a_i\right) \frac{1}{p} \|u\|_W^p + b_2 \|u\|_{L^q}^q - c_1\, \|u\|_{L^r}^r. \end{aligned}$$

From the equivalence of the norms, there exists three positive constants, $e_1$, $e_2$ and $e_3$, such that

$$I(u) \le \|u\|_W^r \left(e_1 \|u\|_W^{p-r} + e_2 \|u\|_W^{q-r} - e_3\right).$$

Since $1 < r < p$ and $r < q$ and taking into account the equivalence of the norms $\|\cdot\|_W$ and $\|\cdot\|_{X^k}$, there exists $\rho > 0$ such that

$$\sup_{u \in S_\rho^k} I(u) < 0.$$

Hence, by applying Theorem 2.5, the result is proved. □

Now, we are going to prove Theorem 1.3. In order to do that, we are going to prove that in these cases every weak solution is a classical solution, then, by applying Theorem 1.2, we obtain the result.

Let us denote $i_0 = \min\{i \in \{1, \dots, n-1\} : a_i \neq 0\}$. So, we rewrite (1) as follows:

$$\begin{aligned} &\left(\varphi_p\left(u^{(n)}(t)\right)\right)^{(n)} + \sum_{i=i_0}^{n-1} (-1)^i a_i \left(\varphi_p\left(u^{(n-i)}(t)\right)\right)^{(n-i)} \\ &\quad + (-1)^n \left(f(t,u(t)) - h(t,u(t))\right) = 0, \quad t \in [0,T]. \end{aligned} \tag{13}$$

We have a previous result to obtain the regularity of $\varphi_p$.

**Lemma 3.4.** *Let us consider $\alpha = \lceil p \rceil - 2$, where $\lceil \cdot \rceil$ means the ceiling function then $\varphi_p \in C^\alpha(\mathbb{R})$. Moreover, if $p \in \mathbb{Z}$ is even, then $\varphi_p \in C^\infty(\mathbb{R})$.*

**Proof.** The proof follows from the derivatives of $\varphi_p$, which are given by

$$\begin{cases} \varphi_p^{(2k-1)}(t) = (p-1) \dots (p-2\,k)\, \Phi_{p-2\,k}(t) & t \neq 0,\ k \in \mathbb{Z} \\ \varphi_p^{(2k)}(t) = (p-1) \dots (p-2\,k) \varphi_{p-2\,k}(t) & t \neq 0,\ k \in \mathbb{Z}. \end{cases}$$

So, if $p = 2\,k$, with $k \in \mathbb{Z}$, then $\varphi_p \in C^\infty(\mathbb{R})$.

If $0 < p - 2\,k \le 1$, or which is the same $\lceil p \rceil = 2\,k+1$, then $2\,k-1 = \alpha$ and $\varphi_p \in C^\alpha(\mathbb{R})$.

Finally, if $1 < p - 2\,k < 2$, or equivalently $\lceil p \rceil = 2\,k+2$, then $2\,k = \alpha$ and $\varphi_p \in C^\alpha(\mathbb{R})$. □

**Remark 3.5.** From Lemma 3.4, we obtain, in particular, the following assertions:

- If $p \in (1,2)$, then $\varphi_p \in C(\mathbb{R})$.
- If $p = 2$, then $\varphi_2 \in C^\infty(\mathbb{R})$.
- If $p \in (2,3]$, then $\varphi_p \in C^1(\mathbb{R})$.
- If $p \in (n-1,n]$, then $\varphi_p \in C^{n-2}(\mathbb{R})$. Moreover, if $p = n = 2\,k$, where $k \in \mathbb{Z}$, then $\varphi_{2\,k} \in C^\infty(\mathbb{R})$.

**Lemma 3.6.** *If $i \le \min\left\{\alpha, \frac{n}{2}\right\}$, then the following equality is fulfilled for all $u,\, v \in X_p$:*

$$\int_0^T \varphi_p\left(u^{(i)}(t)\right)\, v^{(i)}(t)\, dt = (-1)^i \int_0^T \left(\varphi_p\left(u^{(i)}(t)\right)\right)^{(i)}\, v(t)\, dt.$$

**Proof.** Since $u \in X_p$, using Lemma 3.4, integrating by parts we have:

$$\begin{aligned}\int_0^T \varphi_p\left(u^{(i)}(t)\right)\, v^{(i)}(t)\, dt &= \sum_{j=1}^{i}(-1)^{j-1}\left(\varphi_p\left(u^{(i)}(t)\right)\right)^{(j-1)}\, v^{(i-j)}(t)\Big]_0^T \\ &\quad + (-1)^i \int_0^T \left(\varphi_p\left(u^{(i)}(t)\right)\right)^{(i)}\, v(t)\, dt.\end{aligned}$$

The result, now follows from the boundary conditions of $u$ and $v$, taking into account that $i \le \frac{n}{2}$. □

**Lemma 3.7.** *If $u \in C^\beta([0,T])$ and $\varphi_p\left(u^{(\beta)}(\cdot)\right) \in C^\beta([0,T])$, then for $k = 1,\ldots,\beta-1$:*

$$\left(\varphi_p\left(u^{(\beta)}(t)\right)\right)^{(k)} = g_k\left(u^{(\beta)}(t),\ldots,u^{(\beta+k-1)}(t)\right) + (p-1)(p-2)\,\Phi_{p-2}\left(u^{(\beta)}(t)\right)\, u^{(\beta+k)}(t), \qquad (14)$$

*where $g_k$ is a $C^{\beta-k}$ function of $k$ variables.*

**Proof.** Let us prove the result by induction. For $k = 1$:

$$\left(\varphi_p\left(u^{(\beta)}(t)\right)\right)' = (p-1)(p-2)\,\Phi_{p-2}(u^{(\beta)})\, u^{(\beta+1)}(t),$$

so, $g_1 \equiv 0$.

Now, assume that (14) is true for $k \in \{1,\ldots,n-2\}$, for $k+1$:

$$\begin{aligned}\left(\varphi_p\left(u^{(\beta)}(t)\right)\right)^{(k+1)} &= \frac{d}{dt} g_k\left(u^{(\beta)}(t),\ldots,u^{(\beta+k-1)}(t)\right) + (p-1)(p-2)\varphi_{p-2}\left(u^{(\beta)}(t)\right)\, u^{(\beta+1)}(t)\, u^{(\beta+k)}(t) \\ &\quad + (p-1)(p-2)\,\Phi_{p-2}\left(u^{(\beta)}(t)\right)\, u^{(\beta+k+1)}(t).\end{aligned}$$

Hence, the result is true considering

$$\begin{aligned}g_{k+1}(u^{(\beta)}(t),\ldots,u^{(\beta+k)}(t)) &= \frac{d}{dt} g_k\left(u^{(\beta)}(t),\ldots,u^{(\beta+k-1)}(t)\right) \\ &\quad + (p-1)(p-2)\varphi_{p-2}\left(u^{(\beta)}(t)\right)\, u^{(\beta+1)}(t)\, u^{(\beta+k)}(t).\end{aligned}$$

Now, we can show a result which ensures that a weak solution is a classical solution.

**Theorem 3.8.** *Let $i_0 \geq \lfloor \frac{n}{2} \rfloor$ and $\lceil p \rceil \geq n - i_0 + 2$ or $p = 2k$, where $k \in \mathbb{Z}$, then every weak solution of (13) coupled with the boundary conditions (2) is also a classical solution.*

**Proof.** Taking into account that $u \in X_p$ is a weak solution then $u \in C^n([0,T])$, from Lemma 3.6, we have for all $v \in X_p$:

$$\int_0^T \varphi_p \left(u^{(n)}(t)\right) v^{(n)}(t)\, dt = \sum_{i=i_0}^{n-1} (-1)^{n-i+1} a_i \int_0^T \left(\varphi_p \left(u^{(n-i)}(t)\right)\right)^{(n-i)} v(t)\, dt - \int_0^T (f(t,u(t)) - h(t,u(t)))\, v(t)\, dt. \tag{15}$$

Hence, following a basic theorem of calculus of variations [5, page 6], taking into account that $f$ and $h$ are continuous functions, we conclude that Eq. (13) is verified for $t \in (0,T)$.

We only have to check the boundary conditions, since $u \in X_p$, then $u(T) - u(0) = \cdots = u^{(n-1)}(T) - u^{(n-1)}(0) = 0$.

Now, integrating in (15) by parts, taking into account that (13) is fulfilled, we obtain that for all $v \in X_p$:

$$\sum_{j=1}^{n} (-1)^{j-1} \left(\varphi_p \left(u^{(n)}(t)\right)\right)^{(j-1)} v^{(n-j)}(t)\Bigg]_0^T = 0.$$

From the arbitrariness of $v \in X_p$, we obtain:

$$\varphi_p \left(u^{(n)}(T)\right) - \varphi_p \left(u^{(n)}(0)\right) = \cdots = \left(\varphi_p \left(u^{(n)}(T)\right)\right)^{(n-1)} - \left(\varphi_p \left(u^{(n)}(0)\right)\right)^{(n-1)} = 0,$$

and, taking into account that $\varphi_p$ is an injective function, $u^{(n)}(T) - u^{(n)}(0) = 0$.

Now, using Lemma 3.7, we obtain the rest of boundary conditions. For $k = 1$, we have

$$\begin{aligned} 0 &= \left(\varphi_p \left(u^{(n)}(T)\right)\right)' - \left(\varphi_p \left(u^{(n)}(0)\right)\right)' \\ &= (p-1)\,(p-2) \left(\Phi_{p-2}(u^{(n)}(T))\, u^{(n+1)}(T) - \Phi_{p-2}(u^{(n)}(0))\, u^{(n+1)}(0)\right). \end{aligned}$$

Since $u^{(n)}(T) - u^{(n)}(0) = 0$, we conclude that $u^{(n+1)}(T) - u^{(n+1)}(0) = 0$.

Now, assume, by induction hypothesis, that for $k = 1, \ldots, n-2$, $u^{(n)}(T) - u^{(n)}(0) = \cdots = u^{(n+k)}(T) - u^{(n+k)}(0) = 0$. From Lemma 3.7, for $k+1$:

$$\begin{aligned} 0 &= \left(\varphi_p \left(u^{(n)}(T)\right)\right)^{(k+1)} - \left(\varphi_p \left(u^{(n)}(0)\right)\right)^{(k+1)} \\ &= (p-1)(p-2) \left(\Phi_{p-2}(u^{(n)}(T))\, u^{(n+k+1)}(T) - \Phi_{p-2}(u^{(n)}(0))\, u^{(n+k+1)}(0)\right), \end{aligned}$$

which implies that $u^{(n+k+1)}(T) - u^{(n+k+1)}(0) = 0$.

Thus, the result is proved. $\square$

The proof of Theorem 1.3 follows directly from Theorems 1.2 and 3.8.

With analogous arguments, we can study the following impulsive problem.

We denote $0 = t_0 < t_1 < \cdots < t_\ell < t_{\ell+1} = T$ and set $J_j = (t_j, t_{j+1})$ for $j = 0, \dots, \ell$.

$$\begin{aligned} &\left(\varphi_p\left(u^{(n)}(t)\right)\right)^{(n)} + \sum_{i=1}^{n-1}(-1)^i a_i \left(\varphi_p\left(u^{(n-i)}(t)\right)\right)^{(n-i)} \\ &\qquad + (-1)^n \left(f(t,u(t)) - h(t,u(t))\right) = 0, \quad t \in \bigcup_{j=0}^{\ell} J_j, \\ &u(T) - u(0) = \cdots = u^{(2n-1)}(T) - u^{(2n-1)}(0) = 0, \\ &\Delta\left(\varphi_p\left(u^{(n)}(t_j)\right)\right) = (-1)^{n+1} g_j\left(u(t_j)\right), \end{aligned} \tag{16}$$

where $T \geq 0$, $a_i \geq 0$ for $i = 1, \dots, n-1$, $\Delta\left(\varphi_p\left(u^{(n)}(t_j)\right)\right) := \varphi_p\left(u^{(n)}\left(t_j^+\right)\right) - \varphi_p\left(u^{(n)}\left(t_j^-\right)\right)$, $u^{(n)}\left(t_j^\pm\right) = \lim_{t\to t_j^\pm} u^{(n)}(t)$, $g_j\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. We obtain the analogous results to Theorems 1.2 and 1.3.

**Theorem 3.9.** *Let $p > 1$, $f$ and $h$ satisfying $(FH)$, $a_i \geq 0$ for $i = 1, \dots, n-1$, $g_j\colon \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying for all $t \in \mathbb{R}$ and $j = 1, \dots, \ell$:*

$$\int_0^t g_j(s)\, ds \geq c,$$

*for a given constant $c \in \mathbb{R}$. Then (16) has at least a weak solution.*

*If, in addition, we have that $p > r$ and $f$, $h$ and $g_j$ are odd functions and*

$$\int_0^t g_j(s)\, ds \leq 0, \quad \forall j = 1, \dots, \ell.$$

*Then (16) has infinitely many pairs of weak solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\lim_{m\to\infty} \max_{t\in[0,T]} |u_m(t)| = 0$.*

**Theorem 3.10.** *Let $p > 1$, $f$ and $h$ satisfying $(FH)$, $a_i \geq 0$ for $i = 1, \dots, n-1$, $i_0 \geq \left\lfloor \frac{n}{2} \right\rfloor$ and $\lceil p \rceil \geq n - i_0 + 2$ or $p = 2k$, where $k \in \mathbb{Z}$, $g_j\colon \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying for all $t \in \mathbb{R}$ and $j = 1, \dots, \ell$:*

$$\int_0^t g_j(s)\, ds \geq c,$$

*for a given constant $c \in \mathbb{R}$. Then (16) has at least a solution.*

*If, in addition, we have that $p > r$ and $f$, $h$ and $g_j$ are odd functions and*

$$\int_0^t g_j(s)\, ds \leq 0, \quad \forall j = 1, \dots, \ell.$$

*Then (16) has infinitely many pairs of solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\lim_{m\to\infty} \max_{t\in[0,T]} |u_m(t)| = 0$.*

## 4. Particular cases

This section is devoted to show the result for different examples.

First, realize that, as a particular case of functions satisfying $(FH)$ condition, we can consider $f(t,u(t)) = b(t)\,\varphi_q(t)$ and $h(t,u(t)) = c(t)\,\varphi_r(t)$, where $b$ and $c$ are positive continuous $T$-periodic functions.

As we have already said, for $n = 1$ the result has been obtained in [4].

Now, let us consider $n = 2$, we have

$$\left(\varphi_p\left(u''(t)\right)\right)'' - a_1\left(\varphi_p\left(u'(t)\right)\right)' + \left(f(t,u(t)) - h(t,u(t))\right) = 0, \quad t \in [0,T] \tag{17}$$

coupled with the boundary conditions

$$u(T) - u(0) = u'(T) - u'(0) = u''(T) - u''(0) = u^{(3)}(T) - u^{(3)}(0) = 0. \tag{18}$$

If $f$ and $h$ satisfy condition $(FH)$, then we can apply Theorem 1.2 to ensure the existence of at least a weak solution. If in addition, $p > r$, we have that there exist many pairs of weak solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\lim_{m\to\infty} \max_{t\in[0,T]} |u_m(t)| = 0$.

Moreover, if $a_1 = 0$ or $p \geq 2$, from Theorem 1.3, we can affirm that the obtained weak solutions are also classical solutions.

Finally, let us consider $n = 3$,

$$\left(\varphi_p\left(u^{(3)}(t)\right)\right)^{(3)} - a_1\left(\varphi_p\left(u''(t)\right)\right)'' + a_2\left(\varphi_p\left(u'(t)\right)\right)' - \left(f(t,u(t)) - h(t,u(t))\right) = 0, \quad t \in [0,T] \tag{19}$$

coupled with the boundary conditions

$$\begin{gathered} u(T) - u(0) = u'(T) - u'(0) = u''(T) - u''(0) = 0, \\ u^{(3)}(T) - u^{(3)}(0) = u^{(4)}(T) - u^{(4)}(0) = u^{(5)}(T) - u^{(5)}(0) = 0. \end{gathered} \tag{20}$$

If $f$ and $h$ satisfy condition $(FH)$, then we can apply Theorem 1.2 to ensure the existence of at least a weak solution. If in addition, $p > r$, we have that there exist many pairs of weak solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\lim_{m\to} \max_{t\in[0,T]} |u_m(t)| = 0$.

Moreover, if $a_1 = a_2 = 0$ or $a_1 = 0$ and $p \geq 2$, from Theorem 1.3, we can affirm that the obtained weak solutions are also classical solutions.

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