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Sobolev-type fractional stochastic differential equations with non-Lipschitz coefficients

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ABSTRACT

This paper investigates the existence and uniqueness of mild solutions for a class of nonlinear fractional Sobolev-type stochastic differential equations in Hilbert spaces. In this work, we used the fractional calculus, semigroup theory and stochastic analysis techniques for obtaining the required result. A new set of sufficient condition is established with the coefficients in the equations satisfying some non-Lipschitz conditions, which include classical Lipschitz conditions as special cases. More precisely, the results are obtained by means of standard Picard's iteration. Finally, an example is given to illustrate the obtained theory.

1. Introduction

Fractional differential equations attract many mathematicians and scientists because of their successful applications in various fields of science and engineering. Further, it can efficiently describe dynamical behavior of real life phenomena more accurately than integer order equations. Fractional differential equations have numerous applications in many areas such as in viscoelasticity, electrochemistry, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, flow in porous media, aerodynamics and in different branches of physical and biological sciences [1–3]. Existence and uniqueness of mild solutions for deterministic fractional differential equations has been extensively studied in the literature (see [4,5] and the references therein). Wang et al. [6] studied the existence of mild solutions for the control system governed by fractional delay evolution inclusion in Banach spaces. The existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions is studied in [7] by constructing a new monotone iterative technique.

In recent years, the theory and applications of stochastic differential equations in infinite-dimensional spaces have received much attention [8]. In particular, many mathematical models for dynamic processes in chemical, physical and biological sciences can be described by systems of stochastic differential equations [9]. Recently, the existence of mild solutions of stochastic differential evolution system in Hilbert spaces has been studied by many researchers (see [10,8] and references therein). The existence of mild solutions for a class of impulsive neutral stochastic integro-differential equations with infinite delays is studied in [11], where the Krasnoselskii–Schaefer type fixed point theorem combined with theories

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of resolvent operators is used to obtain the required result. However, there has been little work in the area of fractional stochastic differential equations. Zhang et al. [12] considered a class of fractional stochastic partial differential equations with Poisson jumps, where a set of sufficient conditions for the existence and asymptotic stability in pth moment of mild solutions is obtained by employing a fixed point principle. Recently, Rajivganthi et al. [13] investigated the existence of mild solutions and optimal controls for a class of fractional neutral stochastic differential equations with Poisson jumps in Hilbert spaces, where a new set of sufficient conditions for the existence of mild solutions is obtained by using the successive approximation approach.

Sobolev-type equation appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude and so on [14]. In particular, Sobolev-type equation admits abstract representations in the form of implicit operator-differential equations with an arbitrary operator coefficient multiplying the highest derivative [15]. A detailed study of abstract Sobolev type differential equations has been discussed in the papers [16–18]. The existence result of mild solutions of fractional integrodifferential equations of Sobolev-type with nonlocal condition in a separable Banach space is studied by using the theory of propagation family as well as the theory of the measures of noncompactness and the condensing maps [18]. The existence and uniqueness of mild solution to Sobolev-type fractional nonlocal dynamical equations in Banach spaces is reported in [19], where a new set of conditions is obtained to achieve the required result by using fractional power of operators, a singular version of Gronwall's inequality and Leray–Schauder fixed point theorem.

On the other hand, many mathematical modeling of dynamical systems are based on the description of the properties of Sobolev-type equations. The Sobolev-type fractional models are more adequate than integer order models, so fractional order differential equations of Sobolev type have been investigated by many researchers ([20] and references therein). However, it should be mentioned that to the best of our knowledge, the existence and uniqueness of mild solutions of nonlinear fractional Sobolev-type stochastic differential equations in Hilbert spaces has not been investigated yet and this motivates our study. In order to fill this gap, in this paper, we study the existence and uniqueness of mild solutions for a class of nonlinear fractional Sobolev-type stochastic differential equations under non-Lipschitz conditions by employing Picard type approximate sequences. In particular, the obtained conditions are more general since it includes classical Lipschitz conditions as special cases. Finally, an example is provided to illustrate the obtained theory.

The rest of this paper is organized as follows. In Section 2, we will provide some basic definitions, lemmas of fractional calculus and stochastic analysis theory. Further, the problem formulation and the concept of mild solutions for the considered equations are provided. In Section 3, a set of sufficient conditions is obtained for the existence and uniqueness of mild solutions. In particular, the existence result is established by using stochastic analysis techniques, fractional calculation, semigroup theory and Picard type approximate sequences. In Section 4, an example is given to illustrate the obtained results.

2. Preliminaries

In this section, we provide some basic definitions, notations and lemmas, which will be used throughout the paper. In particular, we present main properties of semigroup theory [16], stochastic analysis theory [21,22] and well known facts in fractional calculus [3].

Let $H,\ E$ be two real separable Hilbert spaces and we denote by $\mathcal{L}(H,E)$ the space of all linear bounded operators from H to E. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, W is a Q-Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with the linear bounded covariance operator $Q \in \mathcal{L}(E,E) = \mathcal{L}(E)$ such that $\mathrm{tr}Q < \infty$, where tr denotes the trace of the operator. Further, we assume that there exist a complete orthonormal system $\{e_n\}_{n\geq 1}$ in E, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda e_n$, $n = 1, 2, \ldots$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \langle e_n, e\rangle\, \beta_n(t), \quad e \in E,\ t \in J \tag{1}$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$ and $\mathcal{F}_T = \mathcal{F}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}E; H)$ be the space of all Hilbert–Schmidt operators from $Q^{\frac{1}{2}}E$ to H with the norm $\|\varphi\|_{\mathcal{L}_2^0} = \mathrm{tr}\,[\varphi Q \varphi^*] < \infty,\ \varphi \in \mathcal{L}(E,H)$. Let $L_2(\mathcal{F}_T, H)$ be the Hilbert space of all $\mathcal{F}_T$-measurable square integrable random variables with values in the Hilbert space H. Let $L_2^{\mathcal{F}}(J,H)$ is the Hilbert space of all square integrable and $\mathcal{F}_t$-adapted processes with values in H. Let $\mathbf{B}_T$ denote the Banach space of all H-valued $\mathcal{F}_t$-adapted processes $X(t,\omega) : J \times \Omega \to H$, which are continuous in t for $a.e.$ fixed $\omega \in \Omega$ and satisfy

$$\|X\|_{\mathbf{B}_T} = \mathbf{E}\left(\sup_{t\in[0,T]} \|X(t,\omega)\|^p\right)^{1/p} < \infty, \quad p \geq 2. \tag{2}$$

In this paper, we study the existence and uniqueness of mild solutions for the following nonlinear fractional stochastic differential equation of Sobolev-type

$$\begin{cases} {}^CD_t^\alpha \left[VX(t)\right] = AX(t)dt + F\left(t, X(t)\right)dt + B\left(t, X(t)\right)\dfrac{dW(t)}{dt}, \quad t \in J = [0, T], \\ X(0) = \zeta, \end{cases} \tag{3}$$

where ${}^CD_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in]1/p, 1[$ with the lower limit zero. The functions F and B are appropriate functions to be specified later. The initial data ζ is an $\mathcal{F}_0$-measurable H-valued random variable independent of W with finite p moments.

To prove our main result, we consider the following hypothesis on the operators $A : D(A) \subset H \to H$ and $V : D(V) \subset H \to H$:

(V1) A and V are linear operators, and A is closed,
(V2) $D(V) \subset D(A)$ and V is bijective,
(V3) $V^{-1} : H \to D(H)$ is compact.

From the assumptions (V1)–(V2) and the closed graph theorem, we get the boundedness of the linear operator $AV^{-1} : H \to H$. Consequently, $-AV^{-1}$ generates a semigroup $\{S(t),\ t \geq 0\}$ in H, $S(t) := e^{-AV^{-1}t}$. Assume that $M := \max_{t\in J} \|S(t)\|$.

Let us recall the following known definitions on fractional calculus, see [2,3].

Definition 1. The fractional integral of order α with the lower limit zero for a function $f : [0, \infty) \to R$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,\ \alpha > 0, \tag{4}$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(y) := \int_0^\infty t^{y-1}e^{-t}dt$.

Definition 2. The Caputo derivative of order α for a function $f \in C^n([0, \infty))$ can be written as

$${}^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0,\ n-1 < \alpha < n. \tag{5}$$

If f is an abstract function with values in H, then the integrals appearing in the above definitions are taken in Bochner's sense.

For $x \in H$, we define two families $\{\mathcal{T}_V(t),\ t \geq 0\}$ and $\{\mathcal{S}_V(t),\ t \geq 0\}$ of operators by

$$\begin{aligned} \mathcal{T}_V(t) &:= \int_0^\infty V^{-1}\Psi_\alpha(\theta)S(t^\alpha\theta)d\theta \\ \mathcal{S}_V(t) &:= \alpha \int_0^\infty V^{-1}\theta\Psi_\alpha(\theta)S(t^\alpha\theta)d\theta, \end{aligned} \tag{6}$$

where

$$\Psi_\alpha(\theta) := \frac{1}{\pi\alpha} \sum_{n=1}^\infty (-\theta)^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in]0, \infty[, \tag{7}$$

is a probability density function defined on $]0, \infty[$, which satisfies $\Psi_\alpha(\theta) \geq 0$ and $\int_0^\infty \Psi_\alpha(\theta)d\theta = 1$.

Lemma 1 ([23]). *The operators $\mathcal{T}_V$ and $\mathcal{S}_V$ defined by (6) have the following properties:*

(i) *For any fixed $t \geq 0$, $\mathcal{T}_V(t)$ and $\mathcal{S}_V(t)$ are linear and bounded operators, and for any $x \in H$*

$$\begin{aligned} \|\mathcal{T}_V(t)x\| &\leq M \left\|V^{-1}\right\| \|x\|, \\ \|\mathcal{S}_V(t)x\| &\leq \frac{M\left\|V^{-1}\right\|}{\Gamma(\alpha)} \|x\|. \end{aligned} \tag{8}$$

(ii) *$\{\mathcal{T}_V(t) : t \geq 0\}$ and $\{\mathcal{S}_V(t) : t \geq 0\}$ are compact.*

In order to define the concept of mild solution for the problem (3), by comparison with the fractional equation given in [23], we associate problem (3) to an integral equation. In this paper, we introduce the following definition of mild solutions of (3).

Definition 3. An H-valued stochastic process $\{X(t) : t \in J\}$ is mild solution of (3) if

(i) $X(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t \in J$,
(ii) For arbitrary $t \in J$ we have, P-a.s.,

$$\int_0^t \left(\left\| (t-s)^{\alpha-1} \mathcal{S}_V(t-s) F(s, X(s)) \right\| + \left\| (t-s)^{\alpha-1} \mathcal{S}_V(t-s) B(s, X(s)) \right\|_{L_2^0}^2 \right) ds < \infty \tag{9}$$

(iii) $X(t)$ satisfies the following equation

$$\begin{aligned} X(t) &= \mathcal{T}_V(t)(V\zeta) + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_V(t-s) F(s, X(s)) ds \\ &\quad + \mathcal{S}_V(t-s) B(s, X(s)) dW(s), \quad P\text{-a.s. for all } t \in J. \end{aligned} \tag{10}$$

The following estimate on the Ito integral is an important tool in obtaining certain estimates in main result.

Lemma 2 ([9]). *For any $p \geq 2$ and let Φ be L_2^0-valued predictable process such that $E\left(\int_0^T \|\Phi(r)\|_{L_2^0}^p dr\right) < +\infty$ we have*

$$\begin{aligned} \mathbf{E}\left(\sup_{\mathbf{s}\in[\mathbf{0},\mathbf{t}]} \left\| \int_0^s \Phi(r) dW(r) \right\|^p \right) &\leq c_p \sup_{\mathbf{s}\in[\mathbf{0},\mathbf{t}]} E\left(\left\| \int_0^s \Phi(r) dW(r) \right\|^p \right) \\ &\leq C_p E\left(\int_0^t \|\Phi(r)\|_{L_2^0} dr \right), \quad t \in [0, T] \end{aligned} \tag{11}$$

where $c_p = \left(\frac{p}{p-1}\right)^p$ and $C_p = \left(\frac{p}{2}(p-1)\right)^{p/2} \left(\frac{p}{p-1}\right)^{p^2/2}$.

3. Existence and uniqueness result

In this section, we study existence and uniqueness of mild solutions to (3). In addition to assumptions (V1)–(V3), we impose the following conditions:

(V4) The functions $F : J \times H \to H, B : J \times H \to L_2^0$ are measurable and continuous in X for each fixed $t \in J$ and there exists a function $N : J \times [0, +\infty) \to [0, +\infty),\ (t, u) \to N(t, u)$ such that

$$E\left(\|F(t, X)\|^p\right) + E\left(\|B(t, X)\|_{L_2^0}^p\right) \leq N(t, E(\|X\|)^p) \tag{12}$$

for all $t \in J$ and all $X \in L^p(\Omega, \mathcal{F}_T, H)$.

(V5) $N(t, u)$ is locally integrable in t for each fixed $u \in [0, +\infty)$ and is continuous non-decreasing in u for each fixed $t \in J$ and for all $\lambda > 0$, $u_0 \geq 0$ the integral equation $u(t) = u_0 + \lambda \int_0^t N(s, u(s)) ds$ has a global solution on J.

(V6) There exists a function $K : J \times [0, +\infty) \to [0, +\infty)$ such that

$$E\left(\|F(t, X) - F(t, Y)\|^p\right) + E\left(\|B(t, X) - B(t, Y)\|_{L_2^0}^p\right) \leq K\left(t, E\left(\|X - Y\|^p\right)\right) \tag{13}$$

for all $t \in J$ and all $X, Y \in L^p(\Omega, \mathcal{F}_T, H)$.

(V7) $K(t, u)$ is locally integrable in t for each fixed $u \in [0, +\infty)$ and continuous non-decreasing in u for each fixed $t \in J$. Moreover, $K(t, 0) = 0$ and if a non-negative continuous function $z(t),\ t \in J$ satisfies

$$\begin{cases} z(t) \leq \sigma \displaystyle\int_0^t K(s, z(s)) ds, & t \in J \\ z(0) = 0 \end{cases} \tag{14}$$

for some $\sigma > 0$, then $z(t) = 0$ for all $t \in J$.

Remark 1. (1) If $K(t, u) = Lu,\ u \geq 0$, where $L > 0$ is a constant, condition (V6) implies global Lipschitz condition.

(2) If K is concave with respect to the second variable for each fixed $t \geq 0$ and

$$\|F(t, x) - F(t, y)\|^p + \|B(t, x) - B(t, y)\|_{L_2^0}^p \leq K\left(t, \|x - y\|^p\right),$$

for all $x, y \in H$ and $t \geq 0$. By Jensen's inequality, (13) is satisfied.

(3) Let $K(t, u) = \eta(t)\vartheta(u),\ t \geq 0, u \geq 0$ where $\eta(t) \geq 0$ is locally integrable and $\vartheta : [0, +\infty) \to [0, +\infty)$ is a continuous, monotone non-decreasing and concave function with $\vartheta(0) = 0,\ \vartheta(u) > 0$ for $u > 0$ and $\int_{0+} 1/\vartheta(u) du = \infty$. It can be easily seen that ϑ satisfies assumption (V6) [21].

Let us give some concrete functions $\vartheta(\cdot)$ and let $\epsilon \in (0, 1)$ be sufficiently small. Further, we define [21]

$$\begin{aligned} \vartheta_1(u) &= \begin{cases} u\log(u^{-1}), & 0 \le u \le \epsilon \\ \epsilon\log(\epsilon^{-1}) + \vartheta_1'(\epsilon_-)(u-\epsilon), & u > \epsilon \end{cases} \\ \vartheta_2(u) &= \begin{cases} u\log(u^{-1})\log\log(u^{-1}), & 0 \le u \le \epsilon \\ \epsilon\log(\epsilon^{-1})\log\log(\epsilon^{-1}) + \vartheta_2'(\epsilon_-)(u-\epsilon), & u > \epsilon, \end{cases} \end{aligned} \tag{15}$$

where ϑ_1' and ϑ_2' are the left derivatives of ϑ_1 and ϑ_2 at the point ϵ. All the functions are concave nondecreasing and satisfy $\int_{0+} \frac{dy}{\vartheta_i(u)} = +\infty$. It should be noted that the Lipschitz condition is a special case of the proposed conditions.

Taking into account the aforementioned definitions and lemmas, we shall derive the existence and uniqueness of mild solution for Eq. (3).

Theorem 1. *Assume that the conditions* (V1)–(V7) *hold, then there exists a solution of* (3) *in* $\mathbf{B}_T$.

First, we aim at proving the existence part of Theorem 1. The proof is based on the Picard type approximate technique. Let us construct the sequence of stochastic process $\{X_n\}_{n\ge 0}$ defined as follows:

$$\begin{cases} X_0(t) = \mathcal{T}_V(t)\,(V\zeta)\,, \\ X_{n+1}(t) = \mathcal{T}_V(t)\,(V\zeta) + G_1(X_n)(t) + G_2(X_n)(t), \quad n \ge 1, \end{cases} \tag{16}$$

where

$$\begin{cases} G_1(X_n)(t) = \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}_V(t-s)F(s, X_n(s))ds \\ G_2(X_n)(t) = \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}_V(t-s)B(s, X_n(s))dW(s). \end{cases} \tag{17}$$

In order to prove the existence result of Theorem 1, the following two lemmas are needed.

Lemma 3. *Under the conditions* (V1)–(V5). $\{X_n\}_{n\ge0}$ *is well defined. Furthermore, it is bounded in* $\mathbf{B}_T$, *i.e.*, $\sup_{n\ge0}\|X_n\|_{\mathbf{B}_T} \le C$, *where* C *is a constant.*

Proof. We have

$$E\|X_{n+1}(t)\|^p \le 3^{p-1}E\|\mathcal{T}_V(t)\,(V\zeta)\|^p + 3^{p-1}E\|G_1(X_n)(t)\|^p + 3^{p-1}E\|G_2(X_n)(t)\|^p\,.$$

Applying Hölder inequality, Lemma 1 and using monotonicity of N the right hand side of the above inequality can be written as

$$\begin{aligned} E\|G_1(X_n)(t)\|^p &\le M^p\frac{\|V^{-1}\|^p}{\Gamma(\alpha)^p}\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1}\int_0^t E\left(\|F(s, X_n(s))\|^p\right)ds \\ &\le k_1\int_0^t N\left(s, E\|X_n(s)\|^p\right)ds \\ &\le k_1\int_0^t N\left(s, \|X\|^p_{\mathbf{B}_s}\right)ds. \end{aligned}$$

Using Lemma 2, Lemma 1, Hölder inequality and monotonicity of N, we find

$$\begin{aligned} E\|G_2(X_n)(t)\|^p &\le C_pE\left(\int_0^t \left\|(t-s)^{\alpha-1}\mathcal{S}_V(t-s)\right\|^2\|B(s, X_n(s))\|^2_{L_2^0}ds\right)^{p/2} \\ &\le M^p\frac{\|V^{-1}\|^p}{\Gamma(\alpha)^p}C_p\left(\frac{p-2}{2\alpha p-p-2}\right)^{\frac{p-2}{2}}T^{\frac{2\alpha p-p-2}{2}}\int_0^t E\|B(s, X_n(s))\|^p_{L_2^0}ds \\ &\le k_2\int_0^t N\left(s, E\|X_n(s)\|^p\right)ds \\ &\le k_2\int_0^t N\left(s, \|X_n\|^p_{\mathbf{B}_s}\right)ds, \end{aligned}$$

where $k_1 = M^p\frac{\|V^{-1}\|^p}{\Gamma(\alpha)^p}\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1}$, $k_2 = M^p\frac{\|V^{-1}\|^p}{\Gamma(\alpha)^p}C_p\left(\frac{p-2}{2\alpha p-p-2}\right)^{\frac{p-2}{2}}T^{\frac{2\alpha p-p-2}{2}}$.

Then, using the above relations in the above estimate, we obtain

$$E\left\|X_{n+1}(t)\right\|^p \le 3^{p-1}M^p\left\|V^{-1}\right\|^p E\left(\|\zeta\|^p\right)+3^{p-1}(k_1+k_2)\int_0^t N\left(s,\|X_n\|^p_{\mathbf{B}_s}\right)ds$$
$$\le C_1+C_2\int_0^t N\left(s,\|X_n\|^p_{\mathbf{B}_s}\right)ds$$

where $C_1=3^{p-1}M^p\left\|V^{-1}\right\|^p E\left(\|\zeta\|^p\right),\ C_2=3^{p-1}(k_1+k_2)$.
Therefore,

$$\|X_{n+1}\|^p_{\mathbf{B}_t}\le C_1+C_2\int_0^t N\left(s,\|X_n\|^p_{\mathbf{B}_s}\right)ds. \tag{18}$$

Now consider the following integral equation:

$$z(t)=C_1+C_2\int_0^t N(s,z(s))ds. \tag{19}$$

By (V5), Eq. (19) has a global solution $z(.)$ on J.
Next we prove $\|X_n\|^p_{\mathbf{B}_t}\le z(t),\ \forall t\in J,\ n\ge 0$, by using the induction argument

$$\|X_0\|^p_{\mathbf{B}_t}=\sup_{0\le s\le t}E\left\|\mathcal{T}_V(t)\left(V\zeta\right)\right\|^p\le M^P\left\|V^{-1}\right\|^p E\left(\|\zeta\|^p\right)\le C_1\le z(t),\quad \forall t\in J.$$

Now, we suppose that $\|X_n(t)\|^p_{\mathbf{B}_t}\le z(t),\ \forall t\in J$. Then by (18), the assumption of the mathematical induction and the non-decreasing property of N in the second variable, we have

$$z(t)-\|X_{n+1}\|^p_{\mathbf{B}_t}\ge C_2\int_0^t\left(N\left(s,z(s)\right)-N\left(s,\|X_n\|^p_{\mathbf{B}_s}\right)\right)ds\ge 0,\quad \forall t\in J.$$

In particular, $\sup_{n\ge0}\|X_n\|_{\mathbf{B}_T}\le z(T)^{1/p},\ i.e.\ \{X_n\}_{n\ge0}$ is well-defined.

Lemma 4. *Under the conditions* (V1)–(V7)*, the sequence* $\{X_n\}_{n\ge0}$ *is a Cauchy sequence in the space* $\mathbf{B}_T$.

Proof. Let $\delta_n(t)=\sup_{m\ge n}\left(\|X_m-X_n\|^p_{\mathbf{B}_t}\right)$. By using the same argument as in Lemma 3, we have

$$\|X_m-X_n\|^p_{\mathbf{B}_t}\le C\int_0^t K\left(s,\|X_{m-1}-X_{n-1}\|^p_{B_s}\right)ds,\quad \forall t\in J$$

where $C=2^{p-1}\left(M^p\frac{\left\|V^{-1}\right\|^p}{\Gamma(\alpha)^p}\left(\frac{p-1}{p\alpha-1}\right)^{p-1}T^{p\alpha-1}+M^p\frac{\left\|V^{-1}\right\|^p}{\Gamma(\alpha)^p}C_p\left(\frac{p-2}{2\alpha p-p-2}\right)^{\frac{p-2}{2}}T^{\frac{2\alpha p-p-2}{2}}\right)$.
This implies that

$$\delta_n(t)\le C\int_0^t K\left(s,\delta_{n-1}(s)\right)ds.$$

It is obvious that the functions $\delta_n,\ n\ge0$, are well defined, uniformly bounded due to Lemma 3 and evidently, monotone non-decreasing. Since $\{\delta_n(t)\}_{n\ge0}$ is a monotone non-increasing sequence for each $t\in J$, there exists a monotone non-decreasing function δ such that $\lim_{n\to\infty}\delta_n(t)=\delta(t)$.
Taking n tends to $+\infty$ in the above inequality, using the Lebesgue convergence theorem, we have

$$\delta(t)\le k\int_0^t K\left(s,\delta(s)\right)ds.$$

It follows from (V7) and (Lemma 2.2 of [24]) that $\delta=0$, for all $t\in J$. But $0\le\|X_m-X_n\|^p_{\mathbf{B}_T}\le\delta_n(T)$ and $\delta_n(T)\to\delta(T)=0$ as $n\to+\infty$. As a result, $\{X_n\}_{n\ge0}$ is a Cauchy sequence in the space $\mathbf{B}_T$.

Proof of Theorem 1. (1) Existence: From Lemma 4, we may denote X by the limit of the sequence $\{X_n\}_{n\ge0}$, repeating the proof of Lemma 4, we know that the right side of the second equality of (16) tends to

$$\mathcal{T}_V(t)(V\zeta)+\int_0^t(t-s)^{\alpha-1}\mathcal{S}_V(t-s)F(s,X(s))ds+\mathcal{S}_V(t-s)B(s,X(s))dW(s), \tag{20}$$

as $n\to\infty$.

(2) Uniqueness: Suppose $X, Y \in \mathbf{B}_T$ are two solutions of Eq. (3), using the same argument as in Lemma 4, we can obtain

$$\|X-Y\|^p_{\mathbf{B}_t} \le C\int_0^t K\left(s, \|X-Y\|^p_{\mathbf{B}_s}\right) ds.$$

Using (V7) again, we have $\|X-Y\|^p_{\mathbf{B}_t} = 0$ for all $t \in J$, then $\|X-Y\|^p_{\mathbf{B}_T} = 0$, which implies that $X = Y$. The proof is completed.

Example 1. Now, we present an example to illustrate the obtained result. Consider the fractional partial differential equations of Sobolev-type

$$\begin{cases} {}^{C}D_t^{3/4}\left[x(t,y)-x_{yy}(t,y)\right] = x_{yy}(t,y)+\widehat{F}(t,x(t,y))+\widehat{B}(t,x(t,y))\dfrac{dW(t)}{dt}, \quad y\in[0,\pi],\ t\in J=[0,1], \\ x(t,0)=x(t,\pi)=0 \\ x(0,y)=x_0(y), \quad y\in[0,\pi], \end{cases} \tag{21}$$

where $x_0(y) \in H$, the function $\widehat{F}, \widehat{B} : J\times R \to R$ is continuous function, $W(t)$ is a Brownian motion. In order to write the above system into the abstract form of (3), let $H := L^2[0,\pi]$ and define the operators $A : D(A) \subset H \to H$ and $V : D(V) \subset H \to H$ by

$$Ax := -x_{yy}, \qquad Vx := x - x_{yy},$$

respectively, where each domain $D(A), D(V)$ is given by

$$D(A) = D(V) = \{x \in H, x, x_y \text{ are absolutely continuous},\ x_{yy} \in H \text{ and } x(0) = x(\pi) = 0\}.$$

Then A and V can be written respectively as [23]

$$Ax := \sum_{n=1}^{\infty} n^2 \langle x, x_n\rangle x_n, \quad x \in D(A)$$

$$Vx := \sum_{n=1}^{\infty}(1+n^2)\langle x, x_n\rangle x_n, \quad x \in D(V),$$

where $x_n(y) = \sqrt{\frac{2}{\pi}}\sin ny,\ n = 1, 2,$ is the orthonormal set of eigenvectors of A. Moreover, for any $x \in H$ we have

$$V^{-1}x = \sum_{n=1}^{\infty}\frac{1}{1+n^2}\langle x, x_n\rangle x_n, \qquad -AV^{-1}x = \sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}\langle x, x_n\rangle x_n$$

$$S(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}\langle x, x_n\rangle x_n$$

$$\mathcal{S}_V(t)(x) = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{1+n^2}\int_0^{\infty}\theta\xi_{3/4}(\theta)e^{\frac{-n^2}{1+n^2}t^{3/4}\theta}d\theta\,\langle x, x_n\rangle x_n.$$

Clearly, V^{-1} is compact, bounded with $\left\|V^{-1}\right\| \le 1$ and $-AV^{-1}$ generates a strongly continuous semigroup $S(t)$ on H with $\|S(t)\| \le e^{-t} \le 1$ [17]. Let $X(t)(y) = x(t,y)$ and $B(t, X(t))(y) = \widehat{B}(t, x(t,y))$. Further, we define $F,\ B : J \times H \to H$ by

$$F(t,x)(y) = \widehat{F}(t, x(y)), \qquad B(t,x)(y) = \widehat{B}(t, x(y))$$

and we assume that the functions F and B satisfy the conditions (V4)–(V7). Therefore, with the above choices, the problem (21) can be written in the abstract form (3); hence, all the hypotheses stated in Theorem 1 are satisfied. Hence, there exists a unique mild solution for the problem (21).

Remark 2. Nonlocal condition is more realistic than the standard initial condition to describe some physical phenomena, so the problem of existence of mild solutions of differential equations with nonlocal condition has been studied extensively for different kinds of problems [25,7,26]. Also, nonlocal conditions can be used to determine the unknown physical parameters in some inverse heat conduction problems. More recently, the existence and uniqueness of mild solutions of Sobolev type fractional nonlocal abstract evolution equations in Banach spaces has been reported in [16] by using the fractional calculus, semigroup theory and Leray–Schauder fixed point technique. The existence of mild solutions for a class of impulsive nonlocal stochastic functional integrodifferential inclusions in a real separable Hilbert space is studied in [27] under the mixed continuous and Caratheodory conditions with the use of fractional operators combined with approximation techniques. However, up to now in the present literature, no work has been reported regarding the existence and uniqueness of mild solutions of stochastic fractional equations of Sobolev-type with nonlocal conditions. In this remark, we consider the existence and uniqueness of mild solutions for the following nonlinear fractional stochastic Sobolev-type differential equation

$$\begin{cases} {}^{C}D_t^{\alpha}\left[VX(t)\right] = AX(t)dt + F\left(t, X(t)\right)dt + B\left(t, X(t)\right)\dfrac{dW(t)}{dt}, \quad t \in J = [0, T], \\ VX(0) = V(\zeta - g(X)), \end{cases} \tag{22}$$

where $g : C([0, T], H) \to D(V) \subset H$ is an appropriate function and other functions are defined as in Eq. (3). In particular, the nonlocal condition can be applied in physics with a better effect than the classical initial condition $X(0) = \zeta$. For example, $g(X)$ can be written as

$$g(X) = \Sigma_{i=1}^{m} c_i X(t_i),$$

where c_i $(i = 1, 2, \ldots, n)$ are given constants and $0 < t_1 < \cdots < t_n \leq T$. The above results can be extended to study the existence and uniqueness of mild solutions of (22) with nonlocal conditions by employing the same techniques as used in Theorem 1.

Remark 3. On the other hand, the impulsive fractional differential equations can be used to model processes which are subject to abrupt changes during the dynamical process, and which cannot be described by the classical differential systems [28,29]. Moreover, impulsive control which is based on the theory of impulsive fractional differential equations has become important one due its promising applications towards controlling systems with chaotic behavior. Therefore, in addition to stochastic effects in fractional systems, impulsive effects likewise exist in real process. Therefore, there has been a significant development in fractional stochastic differential equations with impulses (see [22] and references therein). By adapting the techniques and ideas established in this paper, one can prove the existence and uniqueness of mild solutions to stochastic fractional differential equations of Sobolev-type with impulses.

Remark 4. It should be mentioned that the Rosenblatt process is a selfsimilar process with stationary increments but it is not a Gaussian process. Also, in contrast to the fractional Brownian motion, there are only few works on the investigation of Rosenblatt process due to its complexity of the dependence structures and the property of non-Gaussianity [30]. Therefore, it seems interesting to study fractional stochastic differential equations of Sobolev-type with Rosenblatt process. More recently, Shen and Ren [30] studied the existence and uniqueness result of the mild solution for a class of neutral stochastic partial differential equations with finite delay driven by Rosenblatt process in real separable Hilbert spaces. For more details about Rosenblatt process, one can refer the papers ([31,32] and the references therein). By adapting the techniques and ideas used in this work, it is possible to study the existence and uniqueness of mild solutions to stochastic fractional differential equations of Sobolev-type with Rosenblatt process.

4. Conclusion

In this paper, the problem of existence and uniqueness of mild solutions of fractional stochastic differential equations of Sobolev-type is discussed. In particular, successive approximation approach, fractional calculus and stochastic analysis techniques are used for achieving the required result of aforementioned fractional stochastic equations. Finally, an example is provided to illustrate the obtained theory. Our future work will be focused on investigating the existence and stability results for fractional stochastic functional differential equations of Sobolev-type with Poison jumps and Levy noise.

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