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Types of signature analysis in reliability based on Hilbert series

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ABSTRACT

The present paper studies multiple failure and signature analysis of coherent systems using the theory of monomial ideals. While system reliability has been studied using Hilbert series of monomial ideals, this is not enough to understand in a deeper sense the ideal structure features that reflect the behavior of the system under multiple simultaneous failures. Therefore, we introduce the lcm-filtration of a monomial ideal, and we study the Hilbert series and resolutions of the corresponding ideals. Given a monomial ideal, we explicitly compute the resolutions for all ideals in the associated lcm-filtration, and we apply this to study coherent systems. Some computational results are shown in examples to demonstrate the usefulness of this approach and the computational issues that arise. We also study the failure distribution from a statistical point of view by means of the algebraic tools described.

1. Algebraic reliability summary

Let n be a positive integer and consider a coherent system S with n components, each of which can be in a finite number of states. The set of possible states of the whole system S can be coded as elements of $\mathbb{N}^n$. The set of possible states contains a distinguished subset $\mathcal{F}$ of failure states which we

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assume to be coherent, meaning closed above under the standard entrywise ordering. The assumption of coherence is equivalent to saying that the failure states are precisely the exponents appearing in the monomials of the monomial ideal $M_{\mathcal{F}}(S) \subseteq \mathbf{k}[x_1, \ldots, x_n]$. This ideal is known as the *failure ideal* of the system S.

Fix now a probability distribution on the set of states of the system. Our goal is to compute the *reliability* of S, i.e. the probability that S is in a working (non-failure) state. Alternatively, we want to compute the *unreliability* of S, i.e. the probability that S is in failure state. The unreliability, or failure probability of S is defined as $\mathrm{E}[\mathbf{1}_{\mathcal{F}}]$ where E denotes expectation, and $\mathbf{1}_{\mathcal{F}}$ is the indicator function of the failure set $\mathcal{F}$, which is equal to 1 for all failure states and 0 otherwise. Note that the expectation of the indicator function of $\mathcal{F}$ expresses the probability that the system is in one of the states of the set $\mathcal{F}$.

The indicator function and the multigraded Hilbert series of the ideal $M_{\mathcal{F}}(S)$ are closely related. Specifically, if $\mathbb{F}$ is any free resolution of $M_{\mathcal{F}}(S)$ with multigraded ranks $\gamma_{i,\mu}$ we have

$$\mathcal{HS}_{M_{\mathcal{F}}(S)}(t, x) = \frac{1 + \sum_{i=1}^{n} (-1)^i t^i (\sum_{\mu \in \mathbb{N}^n} \gamma_{i,\mu} x^{\mu})}{\prod_{j=1}^{n} (1 - x_j)}. \tag{1.1}$$

To simplify our notation, we set

$$\mathcal{H}_{M_{\mathcal{F}}(S)}(t, x) = -\sum_{i=1}^{n} (-1)^i t^i (\sum_{\mu \in \mathbb{N}^n} \gamma_{i,\mu} x^{\mu}), \tag{1.2}$$

and we refer to this as the numerator of the Hilbert series of $M_{\mathcal{F}}(S)$. This numerator is a way of counting the set of all monomials in the ideal $M_{\mathcal{F}}(S)$ by considering this set as the union of the sets of monomials in the ideals generated by each of the minimal generators of $M_{\mathcal{F}}(S)$. This is therefore a special kind of inclusion–exclusion formula.

We also set

$$\mathcal{H}_{M_{\mathcal{F}}(S)}(1, \alpha) = \mathbf{1}_{\mathcal{F}}(\alpha) = -\sum_{i=1}^{n} (-1)^i \sum_{\mu \in \mathbb{N}^n} \gamma_{i,\mu} \mathbf{1}_{x^{\mu}}(\alpha), \tag{1.3}$$

where $\mathbf{1}_{x^{\mu}}(\alpha)$ is the indicator function of the states $\mu \geq \alpha$. If $\mathbb{F}$ is a minimal free resolution of $M_{\mathcal{F}}(S)$ then the ranks $\gamma_{i,\mu}$ are the smallest possible and depend only on $M_{\mathcal{F}}(S)$, in this case we call them the *multigraded Betti numbers* of $M_{\mathcal{F}}(S)$ and denote them by $\beta_{i,\mu}$. The formula (1.3) is potentially useful for determining failure probabilities because it reduces the problem to the computation of the simpler probabilities $\mathrm{E}[\mathbf{1}_{x^{\mu}}(\alpha)]$. By truncating the sum over i we obtain obvious lower and upper bounds for $\mathbf{1}_{\mathcal{F}}(\alpha)$ and thus on failure probabilities. These bounds improve the traditional Bonferroni bounds based on inclusion–exclusion. Among the bounds coming from free resolutions, the tightest ones are obtained when $\mathbb{F}$ is taken to be the minimal free resolution of $M_{\mathcal{F}}(S)$.

From these basic principles, the authors have studied different aspects of the relation between monomial ideals and coherent systems in a series of works. In Sáenz-de Cabezón and Wynn (2010; 2011) several relevant systems, including k-out-of-n systems and variants, or series-parallel systems were studied, obtaining explicit and recursive formulas for the Betti numbers of the failure ideals of those systems. In Sáenz-de Cabezón and Wynn (2015) the Hilbert function of the system ideal is applied to optimal design in reliability. Other works extending the scope of application of this approach are Sáenz-de Cabezón and Wynn (2014) and Mohammadi et al. (2016) devoted to robustness measure of networks and percolation on trees respectively.

In this paper we propose two further steps for the application of multigraded Hilbert series in probability, namely the study of multiple simultaneous failures of the system, and signature analysis. These two problems, not totally unrelated, imply a deeper knowledge of the structure of the system under consideration, beyond the information given by reliability analysis. The algebraic approach has already proven being useful for the analysis of the structure of coherent systems, not only in what respects the reliability of the system but also on system design and measures of components importance (Sáenz-de Cabezón and Wynn, 2010, 2015). The main contributions of the algebraic approach to system reliability have been so far the improvements of the extensively used inclusion–exclusion method, the new algorithms that have been proven to be efficient for several important systems (k-out-of-n and variants or series-parallel systems) and finally a good tool to obtain insight of the system's structure, beyond the mere numerical computation of reliability figures. This approach is a good complement to the wide variety of techniques used by reliability experts (Kuo and Zuo, 2003).

The tools used so far, namely the Hilbert series of the failure ideal of the system are not enough when one needs to study simultaneous failures or signature analysis. We therefore introduce a new algebraic object that provides the necessary insight on the structure of the ideals: the lcm-filtration. With this motivation at hand, the main contribution of the paper is the definition of the lcm filtration of a monomial ideal, the study of its resolutions and Hilbert series and also the performance of actual computations on these objects. The results obtained are then successfully applied to the study of simultaneous failures and signatures in two different paradigmatic examples.

The plan of the paper is the following. In §2 we present the two problems that we are dealing with: multiple failure and signature analysis. In this section, we see that the main algebraic object we need to understand to study these problems is the lcm-filtration of a monomial ideal. The lcm-filtration is studied in §3 where we give an explicit free resolution for the ideals involved in the lcm-filtration of any monomial ideal I. This section uses previous work by Aramova et al. (1997; 1998) on squarefree stable ideals, and by Peeva and Velasco (2011) on frames of monomial resolutions. We study also the behavior of this and other resolutions in examples. Finally, §4 is devoted to estimation of the probability distribution of the multiple failures of a system. During the paper we use two different, paradigmatic examples, consecutive k-out-of-n systems and cut ideals of complete graphs. The structural differences of the behavior of these two systems with respect to multiple failures is made evident by the use of the concepts and techniques developed in the paper.

2. Two steps further

2.1. Multiple failures

Let S be a coherent system in which several minimal failures can occur at the same time. Let Y be the number of such simultaneous failures. The event $\{Y \geq 1\}$ is the event that at least one elementary failure event occurs, which is the same as the event that the system fails. If x^α and x^β are the monomials corresponding to two elementary failure events, then $\operatorname{lcm}(x^\alpha, x^\beta) = x^{\alpha \vee \beta}$ corresponds to the intersection of the two events and we have $Y \geq 2$. The corresponding ideal is $\langle x^\alpha \rangle \cap \langle x^\beta \rangle$. The full event $Y \geq 2$ corresponds to the ideal generated by all such pairs. The argument extends to $Y \geq k$ and to study the tail probabilities $\operatorname{prob}\{Y \geq k\}$. We now discuss these ideals in more detail.

2.1.1. The lcm-*filtration and the survivor*

Let $I \subseteq \mathbf{k}[x_1, \ldots, x_n]$ be a monomial ideal and $\{m_1, \ldots, m_r\}$ be a minimal monomial generating system of I. Let I_k be the ideal generated by the least common multiples of all sets of k distinct monomial generators of I,

$$I_k = \langle \operatorname{lcm}(\{m_i\}_{i \in \sigma}) : \sigma \subseteq \{1, \ldots, r\}, |\sigma| = k \rangle.$$

We call I_k the *k-fold* lcm-*ideal of I*. The ideals I_k form a descending filtration

$$I = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r,$$

which we call the lcm-*filtration* of I.

The survivor functions

$$F(k) = \operatorname{prob}\{Y \geq k\}$$

for a coherent system, are obtained from the multigraded Hilbert function of the k-fold lcm-ideal I_k. In fact, to emphasize the counting we relabel $M_{\mathcal{F}}(S)$ as I_1.

Example 2.1. Consider a sequential (also named consecutive) k-out-of-n system with $n = 5, k = 2$. Then

$$I_1 = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5 \rangle.$$

The numerator of the Hilbert series obtained from the Taylor resolution of I_1 is formed by successively taking the lcm's of pairs, triples and so on, with sign changes and with cancellations across neighboring rows. The indicator function of the failure set of this system, $\mathbf{1}_F(\alpha)$ is the evaluation of the polynomial

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 - (x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_1x_2x_4x_5) + x_1x_2x_3x_4x_5$$

which is the numerator of the Hilbert series of I_1, i.e., $\mathcal{H}_{I_1}(1, x)$. The Taylor resolution (which is equivalent to the full inclusion–exclusion formula) uses all the terms from the full lcm-lattice. A similar analysis gives the numerator of the Hilbert series of I_2, $\mathcal{H}_{I_2}(1, x)$ as:

$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_1x_2x_4x_5 - (x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_1x_2x_3x_4x_5).$$

The above considerations can be summarized in the following lemma.

Lemma 2.2. *Let Y be the number of failure events of a system S. If $G(M_{\mathcal{F}}(S))$ is the set of monomial minimal generators of the failure ideal $M_{\mathcal{F}}(S) = I_1$, then*

$$\mathbb{P}\{Y \geq k\} = E[\mathbf{1}_{M_k}(\alpha)] = \mathcal{H}_{I_k}(1, \alpha),$$

where $\mathbf{1}_{M_k}$ is the indicator function of the exponents of monomials in the k-fold lcm-*ideal*

$$I_k := \langle \mathrm{lcm}(\{m_i\}_{i\in\sigma}) : \sigma \subseteq G(M_{\mathcal{F}}(S)), |\sigma| = k\rangle.$$

As a result, we also obtain identities, lower and upper bounds for multiple failure probabilities from free resolutions in §3.

2.2. Signature analysis

2.2.1. Classical signature

The theory of signature analysis considers a system with n components which fail independently with a common failure time distribution with cumulative distribution function, cdf, $\mathcal{F}(t)$ and density $f(t)$. As time proceeds, components start to fail and one can write the failure times as

$$T_{(1)}, T_{(2)}, \ldots, T_{(n)}.$$

In statistical terminology, these are the order statistics of the full set of failure times. Because of the distributional assumption there are no ties. Now at some (first) integer i the system will fail: $T = T_{(i)}$ where T is the failure time of the system. The signature s_i codes the probabilities

$$\begin{aligned} s_i &= \mathrm{prob}\{T = T_{(i)}\} \\ &= \mathrm{prob}\{T \geq T_{(i)}\} - \mathrm{prob}\{T \geq T_{(i+1)}\}, \quad i = 1, \ldots, m. \end{aligned}$$

Moreover, because the system must eventually fail, $\sum_{i=1}^{n} s_i = 1$. For material on signature analysis see Boland (2001), Samaniego (2007). A main idea of this paper is that we can derive $s_1, \ldots, s_n$ from the failure ideal. The value s_i is the conditional probability that exactly i components have failed, *conditional* on the event that the system has failed. If we use a squarefree (binary) representation, then for the monomials describing individual failures the degree gives the number of component failures. Thus, if $P_i = \sum_{\alpha\in\mathcal{F}, |\alpha|=i} p_\alpha$, where p_α is the probability that the system is in the failure state α, and $P(\mathcal{F}) = \sum_{\alpha\in\mathcal{F}} p_\alpha$ (i.e., the full probability of failure), then

$$s_i = \frac{P_i}{P(\mathcal{F})}.$$

Identities, upper and lower bounds for the P_i are inherited from those for the $P(\mathcal{F})$ simply by intersecting with the event $A_i = \{\alpha : |\alpha| = i\}$, which can be found by extracting all the terms of the same degree i.

We have that $s_i = \frac{\mathrm{E}[\mathbf{1}_{E_i}(\alpha)]}{\mathrm{E}[\mathbf{1}_{\mathcal{F}}(\alpha)]}$, where E_i is the set consisting of α's for which exactly i elements have failed and α is a failure state.

Notation. Let $I = M_{\mathcal{F}}(S) \subset \mathbf{k}[x_1, \ldots, x_n]$ be the failure ideal of the system S. We use $I^{[i]}$ to denote the set of squarefree monomials of degree i in I, and $\langle I^{[i]} \rangle$ for the ideal generated by all such monomials. The indicator function $\mathbf{1}_{E_i}$ is the indicator function of the set of exponents of the monomials in $I^{[i]}$.

Observe that now the exact formulas for the probabilities s_i as well as upper and lower bounds can be obtained from the free resolutions and Hilbert series of $\langle I^{[i]} \rangle$. In particular, s_i is the difference of the evaluation of the numerator of the Hilbert series of $\langle I^{[i]} \rangle$ and that of $\langle I^{[i+1]} \rangle$.

2.2.2. The k-fold signature

Now we focus on multiple simultaneous minimal failures on the system. As time proceeds, components start to fail simultaneously and one can write the failure times as

$$T^k_{(1)}, T^k_{(2)}, \ldots, T^k_{(n)},$$

where $T^k_{(i)}$, $1 \le k \le n$, is the time in which we have k simultaneous minimal failures with i failed components. As before, at some (first) integer i the system will have k simultaneous minimal failures: $T^k = T^k_{(i)}$ where T^k is the time of k minimal failures in the system. The k-signature s^k_i codes the probabilities

$$\begin{aligned} s^k_i &= \mathrm{prob}\{T^k = T^k_{(i)}\} \\ &= \mathrm{prob}\{T^k \ge T^k_{(i)}\} - \mathrm{prob}\{T^k \ge T^k_{(i+1)}\}, \quad i = 1, \ldots, n, \text{ and } k = 1, \ldots, r. \end{aligned}$$

Now the ideal encoding the states that at least k failures occur simultaneously is the k-fold lcm-ideal I_k. Doing the same analysis as we did for the classical signature, we obtain that s^k_i is the difference of the evaluation of the numerator of the Hilbert series of $\langle I^{[i]}_k \rangle$ and $\langle I^{[i+1]}_k \rangle$. This is why we call s^k_i the k-fold signature of the system.

3. The lcm-filtration and its resolutions

3.1. Aramova–Herzog–Hibi resolution and frames

Let $I \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal with r generators, and let I_k be the k-fold lcm-ideal of I. In this section we want to study various free resolutions of I_k. We will use an explicit minimal resolution for the ideal generated by all k-fold products of r variables, which is called the k-out-of-r ideal. Using frame theory, as developed by Peeva and Velasco (2011), we construct resolutions of I_k, and relate them to the Taylor resolution (Taylor, 1966).

3.1.1. The minimal free resolution of the k-out-of-r ideal

Let $I_{k,r} \subseteq R = \mathbf{k}[x_1, \ldots, x_r]$ be the ideal generated by all products of k different variables. This ideal corresponds in algebraic reliability theory, to the ideal of a k-out-of-r system (Sáenz-de Cabezón and Wynn, 2011). It is clear that $I_{k,r}$ is a squarefree stable ideal in the sense of Aramova et al. (1998), hence its minimal free resolution is of the same form of the Eliahou–Kervaire resolution (Eliahou and Kervaire, 1990) as described explicitly by Aramova et al. (1998). Let us recall here some definitions and notation from Aramova et al. (1998). For the squarefree monomial $m = x_{i_1} x_{i_2} \cdots x_{i_d}$ with $i_1 < i_2 < \cdots < i_d$ we denote $\min(m) = i_1$ and $\max(m) = i_d$. For every squarefree monomial ideal I minimally generated by $G(I)$ and for any squarefree monomial in I, there exists a unique pair (a, b) of squarefree monomials in R such that $a \in G(I)$, $m = a \cdot b$ and $\max(a) < \min(b)$. Thus, we have a map g from the set of squarefree monomials in I to the set $G(I)$. This map is given by $g(m) = a$. Now, given $j \in \{1, 2, \ldots, r\}$ with $j \in \mathrm{supp}(m)$ we set $m_j = g(x_j \cdot m)$ and $y(m)_j = (x_j \cdot m)/m_j$. Aramova, Herzog and Hibi gave an explicit resolution of any squarefree monomial ideal using the function g. The cellular realizations of these resolutions are described in Dochtermann and Mohammadi (2014).

We give here an explicit description of the Aramova–Herzog–Hibi resolution of $I_{k,r}$ independent of the map g and based only on the subsets of $\{1, \dots, r\}$.

Proposition 3.1. *Let $I_{k,r} \subseteq \mathbf{k}[x_1, \dots, x_r]$ be the k-out-of-r ideal. The minimal free resolution of $I_{k,r}$ is given by*

$$\mathbf{I}_{k,r} : 0 \longrightarrow F_{r-k} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} I_{k,r} \longrightarrow 0,$$

where:

(1) *Each generator of F_i is labeled by a pair $[\sigma, \tau]$ such that $\sigma, \tau \subseteq \{1, \dots, r\}$, $\sigma \cap \tau = \emptyset$, $|\sigma| = k$, $|\tau| = i$ and $\max(\sigma) > \max(\tau)$.*
(2) *The differential ∂ is given by*

$$\partial([\sigma, \tau]) = \sum_{j\in\tau} (-1)^{\mathrm{sgn}(j,\tau)}(-x_j[\sigma, \tau - j] + x_{\max(\sigma)}[\sigma - \max(\sigma) + j, \tau - j])$$

if $\max(\tau) < \max(\sigma - \max(\sigma))$, or

$$\partial([\sigma, \tau]) = \sum_{j\in\tau} (-1)^{\mathrm{sgn}(j,\tau)}(-x_j[\sigma, \tau - j]) + x_{\max(\sigma)}[\sigma - \max(\sigma) + \max[\tau], \tau - \max(\tau)]$$

if $\max(\tau) > \max(\sigma - \max(\sigma))$. Observe that in the first case we have $2|\tau|$ summands, and in the second case only $|\tau| + 1$ summands.

Proof. Since $I_{k,r}$ is squarefree stable, we can use the description of its minimal free resolution given in Aramova et al. (1998) which has an equivalent labeling of the generators of each free module F_i. Using our notation, this differential is

$$\delta([\sigma, \tau]) = \sum_{j\in\tau} (-1)^{\mathrm{sgn}(j,\tau)}(-x_j[\sigma, \tau - j] + y(\sigma)_j[\sigma_j, \tau - j]),$$

where $\sigma_j = g(\sigma + j)$, $y(\sigma)_j = (\sigma + j) - \sigma_j$ and $[\mu, \rho] = 0$ if $\max(\rho) > max(\mu)$.

To complete the proof we have to show that:

(1) For all $j \in \tau$, $y(\sigma)_j = \max(\sigma)$ and $\sigma_j = \sigma - \max(\sigma) + j$;
(2) If $\max(\tau) > \max(\sigma - \max(\sigma))$ and $j < \max(\tau)$, then $\max(\tau - j) > \max(\sigma - \max(\sigma) + j)$ and hence $[\sigma - \max(\sigma) + j] = 0$.

To prove (1) observe that for $\sigma + j$ the unique δ, γ such that $\sigma + j = \delta \cup \gamma$ with $\delta \in G(I_{s,r})$ and $\max(\delta) < \min(\gamma)$ are $\delta = \sigma - \max(\sigma) + j$ and $\gamma = \max(\sigma)$: First, it is clear that $\delta \cup \gamma = \sigma + j$, and since $j \le \max(\tau) < \max(\sigma)$ then $\max(\delta) < \min(\gamma)$. Then, by definition $\sigma_j = \delta = \sigma - \max(\sigma) + j$ and $y(\sigma)_j = \sigma + j - \sigma_j = \sigma + j - \sigma + \max(\sigma) - j = \max(\sigma)$.

To prove (2) first we note that if $j = \max(\tau)$, then $\max(\sigma - \max(\sigma) + j) \ge j$ and $\max(\tau - j) < j$. Thus $\max(\sigma - \max(\sigma) + j) > \max(\tau - j)$. Now if $j < \max(\tau)$, then on one hand we have that $\max(\sigma - \max(\sigma)) < \max(\tau)$ implies that $\max(\sigma - \max(\sigma) + j)$ is equal to $\max(\max(\sigma - \max(\sigma)), j)$, which is strictly less than $\max(\tau) = \max(\tau - j)$. Hence we have that $[\sigma - \max(\sigma) + j, \tau - j] = 0$. On the other hand, if $\max(\sigma - \max(\sigma)) > \max(\tau)$ then $\max(\sigma - \max(\sigma) + j) = \max(\sigma - \max(\sigma))$ is strictly greater than $\max(\tau) = \max(\tau - j)$. This completes the proof. □

Using $\mathbf{I}_{k,r}$ we construct now a resolution of I_k. We use the techniques and terminology of Peeva and Velasco (2011). First recall that the lcm-lattice of a monomial ideal M denoted by L_M is the lattice whose elements are labeled by the least common multiples of the monomial minimal generators of M. A monomial ideal M in a polynomial ring S is called a *reduction* of another monomial ideal M' in a polynomial ring S' over the same ground field of S, if there exists a map $f : L_{M'} \to L_M$ which is a bijection on the atoms and preserves lcm's. Such a map is called a *degeneration*.

Lemma 3.2. *I_k is a reduction of $I_{k,r}$.*

Proof. Consider the map $f : L_{I_{k,r}} \to L_{I_k}$ that takes σ to m_σ, where $m_\sigma = \mathrm{lcm}(\{m_i\}_{i\in\sigma})$. The map f is clearly a degeneration and hence I_k is a reduction of $I_{k,r}$. Observe that f is not an isomorphism in general. $\square$

We can now construct the f-degeneration $f(\mathbf{I}_{k,r})$ following Peeva and Velasco (2011, Construction 4.3), which by Peeva and Velasco (2011, Theorem 4.6) is a free resolution of I_k. We can also construct the f-homogenization of $\mathbf{I}_{k,r}$ by Peeva and Velasco (2011, Construction 4.10) and since $\mathbf{I}_{k,r}$ is a minimal free resolution of $I_{k,r}$, both f-degeneration and f-homogenization coincide. We denote the so obtained (not necessarily minimal) resolution of I_k by $\mathbf{I}_k$. Let us describe it:

The elements of the basis of $\mathbf{I}_k$ are also labeled by pairs $[\sigma, \tau]$ such that $\sigma, \tau \subseteq \{1, \ldots, r\}$, $\sigma \cap \tau = \emptyset$, $|\sigma| = s$, $|\tau| = i$ and $\max(\sigma) > \max(\tau)$. Observe that the multidegree of element $[\sigma, \tau]$ in $\mathbf{I}_k$ is $f(\sigma \cup \tau)$. The differential ∂ is given by

$$\partial([\sigma,\tau]) = \sum_{j\in\tau}(-1)^{\mathrm{sgn}(j,\tau)}(-\frac{m_{\sigma\cup\tau}}{m_{\sigma\cup\tau-j}}[\sigma,\tau-j] + \frac{m_{\sigma\cup\tau}}{m_{\sigma\cup\tau-\max(\sigma)}}[\sigma-\max(\sigma)+j,\tau-j])$$

if $\max(\tau) < \max(\sigma - \max(\sigma))$, or

$$\partial([\sigma,\tau]) = \sum_{j\in\tau}(-1)^{\mathrm{sgn}(j,\tau)} - \frac{m_{\sigma\cup\tau}}{m_{\sigma\cup\tau-j}}[\sigma,\tau-j] \\ + \frac{m_{\sigma\cup\tau}}{m_{\sigma\cup\tau-\max(\sigma)}}[\sigma-\max(\sigma)+\max[\tau],\tau-\max(\tau)]$$

if $\max(\tau) > \max(\sigma - \max(\sigma))$.

Theorem 3.3. *Let $I = \langle m_1, \ldots, m_r\rangle$ be a monomial ideal. $\mathbf{I}_k$ is a free resolution of I_k for all k. The Betti numbers of $I_{k,r}$ are an upper bound for the Betti numbers of I_k.*

Remark 3.4. $\mathbf{I}_1$ is the Taylor resolution of I.

3.1.2. Resolutions for I_k

For any monomial ideal we can construct different free resolutions. A distinguished one is the unique (up to isomorphism) minimal free resolution, which is not always easy to obtain computationally. For this and other reasons, one usually uses other non-minimal resolutions, that are easier to obtain. A prominent example is Taylor resolution (Taylor, 1966). Taylor resolution begins with any set of monomial generators of the ideal (not necessarily the minimal generating set) and the resolution is described combinatorially. In the case of the ideals I_k involved in the lcm-filtration of I, there are two natural choices for Taylor resolutions, the one that uses the minimal generating set of I_k and the one that uses the (redundant) generating set given by all k-fold lcm's of generators of I. We denote the first one by $\mathbb{T}_k$ and the second one by $\mathbb{T}^k$. $\mathbb{T}_k$ is the usual Taylor resolution of I_k. We also have the minimal free resolution $\mathbb{M}_k$ and the above described resolution $\mathbf{I}_k$.

All three resolutions $\mathbb{T}_k$, $\mathbb{M}_k$ and $\mathbf{I}_k$ are subcomplexes of $\mathbb{T}^k$. $\mathbb{M}_k$ is a subcomplex of $\mathbb{T}^k$, $\mathbb{T}_k$ and $\mathbf{I}_k$. But $\mathbb{T}_k$, the usual Taylor resolution of I_k, and $\mathbf{I}_k$ are not subcomplexes of each other. Therefore $\mathbf{I}_k$ is one of the rare examples of interesting non-sub-Taylor resolutions of a monomial ideal, which also has non-minimal first syzygies. Mermin in (2012, Question 8.1) asks whether there are any (interesting) resolutions of a monomial ideal which are not subcomplexes of the Taylor resolution. The one described above is an example of such resolution.

3.2. Examples

Let us describe now two examples that demonstrate the usefulness of the lcm-filtration to detect structural differences in the ideals (i.e. the systems) under study. The first example shows ideals of

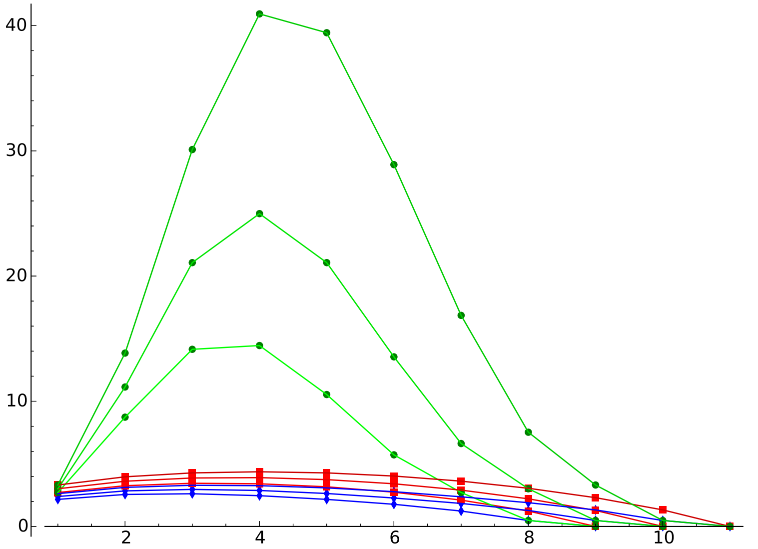

Fig. 1. Sizes of resolutions of $I_{2,n}$ for $n = 10, 11, 12$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

consecutive linear k-out-of-n systems, i.e. systems that fail whenever k *consecutive* components out of n components fail. The second example is the cut ideal of the n-complete graph. This ideal models the behavior of all-terminal reliability of a system of n components, i.e. the system fails whenever there are two disconnected nodes, that is when the graph is disconnected. The behavior of these two systems with respect to multiple simultaneous failures is well illustrated by the study of their lcm-filtrations. These will be our running examples in this paper. We now introduce them and show the behavior of the resolutions of the ideals in their lcm-filtrations.

Example 3.5. Consecutive 2-out-of-n ideals:

The ideal of the consecutive 2-out-of-n system is minimally generated by products of consecutive pairs of the variables, x_1x_2 up to $x_{n-1}x_n$. This corresponds to the edge ideal of the line graph, which has been intensively studied (also from the algebraic reliability point of view, see Sáenz-de Cabezón and Wynn, 2011) and for which the minimal free resolution and Betti numbers are known (He and Van Tuyl, 2010). Fig. 1 shows the behavior of $\mathbb{T}_k$ (green, circles), $\mathbf{I}_k$ (red, squares) and $\mathbb{M}_k$ (blue, diamonds) for the lcm-filtrations of the ideals of the consecutive 2-out-of-n systems for $n = 10, 11, 12$. Each line corresponds to the full filtration of one ideal, the abscissa corresponds to the level k of the filtration, and the ordinate gives the logarithm of the *total size* of the resolution, understood as the sum of all the ranks of the modules in the resolution. In this example we can see that while the generating set of each $\mathbb{T}_k$ is smaller than the corresponding one of $\mathbf{I}_k$, the latter resolution is much closer to the minimal free resolution, except for the latest steps of the filtration.

Example 3.6. Cut ideals of graphs:

Fig. 2 shows the picture of the logarithm of the size of $\mathbb{T}_k$ (green, circles), $\mathbf{I}_k$ (red, squares) and $\mathbb{M}_k$ (blue, diamonds) for the ideals in the lcm-filtration of the ideal of the complete graphs on 4 and 5 vertices. Observe that the picture is completely different from Fig. 1. There are two main differences: In the first place, Taylor resolution is closer to the minimal one than $\mathbf{I}_k$ and that is because the cancellations of non-minimal generators are much more numerous here than in the case of the consecutive k-out-of-n ideals, and therefore the difference of the resolutions is much more evident. Another difference is that the green and blue lines have some horizontal trends. This is because for some values of k, the k-fold lcm ideals are exactly the same, i.e. in this case the filtration has a

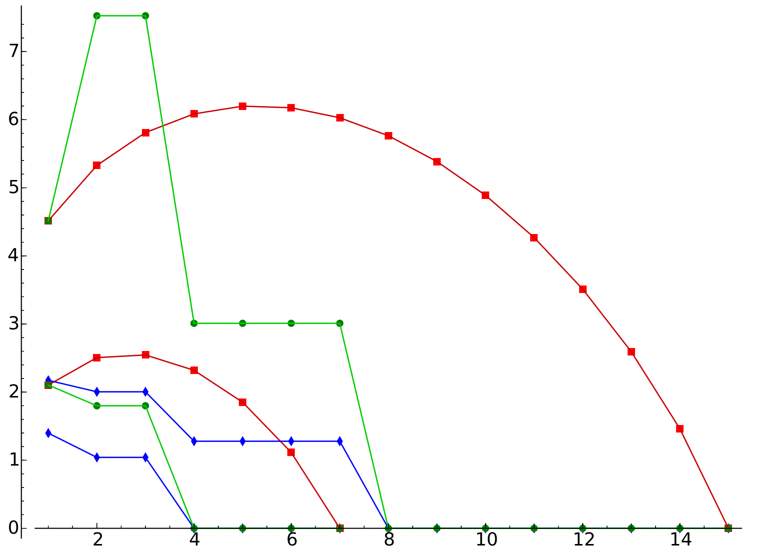

Fig. 2. Sizes of resolutions of the ideals in the lcm filtration of I_4 and I_5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

staircase behavior, it remains constant for some steps and then shrinks and so on. This was not evident just from the generating set of the ideal and has to do with the nature of the cut ideals of the complete graph, which have been extensively studied in Mohammadi (2016a; 2016b), Mohammadi et al. (2016; in preparation), Mohammadi and Shokrieh (2014; 2016). Here we just mention the main results that illustrate the behavior we have just observed, and we refer the interested readers to Mohammadi et al. (in preparation) for more details.

Let K_n be the complete graph on n vertices. Let $I_n \subset \mathbf{k}[x_{ij} : 1 \leq i < j \leq n]$ be the cut ideal of K_n. For integer k, $1 \leq k \leq n$, we let $I_{n,k}$ be the k-fold lcm-ideal of I_n. We denote by $\mathcal{P}_{n,k}$ the set of k-partitions of $[n]$. For any k-partition of $[n]$, we associate a monomial whose support is the set of edges between distinct blocks of the partition. For example for the partition $\sigma = 12|3|4$ of K_4 we associate the monomial $m_\sigma = x_{13}x_{14}x_{23}x_{24}x_{34}$.

We denote by $P_{n,k}$ the ideal minimally generated by the monomials associated to the partitions in $\mathcal{P}_{n,k}$. We have the following relations between the ideals $I_{n,k}$ and the ideals $P_{n,k}$.

Theorem 3.7. *(Mohammadi et al., in preparation, Theorem 3.1) For all integers k and n,* $1 \leq k \leq n$ *we have*

$$I_{n,2^{k-1}} = I_{n,2^{k-1}+1} = \cdots = I_{n,2^k-1} = P_{n,k+1}.$$

3.3. Computational cost

When dealing with the kind of computations we are presenting in this paper, one has to consider computational costs, in particular for applications. The problem of computing the list of multiple failure events of a system (equivalently, the lcms of the generators of a monomial ideal) is intrinsically expensive in term of memory (and time). This is mainly due to the fact that the cardinality of the set of k-fold lcms of the elements of a set grows exponentially with respect to the size of the set. This said, we can analyze the computations needed to compute the Betti numbers of the ideals in the lcm-filtration in the ways proposed in this section. There are some interesting considerations in this respect.

For our analysis we will use our running examples, namely consecutive 2-out-of-n systems and the cut ideal of complete graphs. We will use for our analysis a simple program written in the computer algebra system `Macaulay2` (Grayson and Stillman) running on a MacOsX 10.9 machine with an Intel

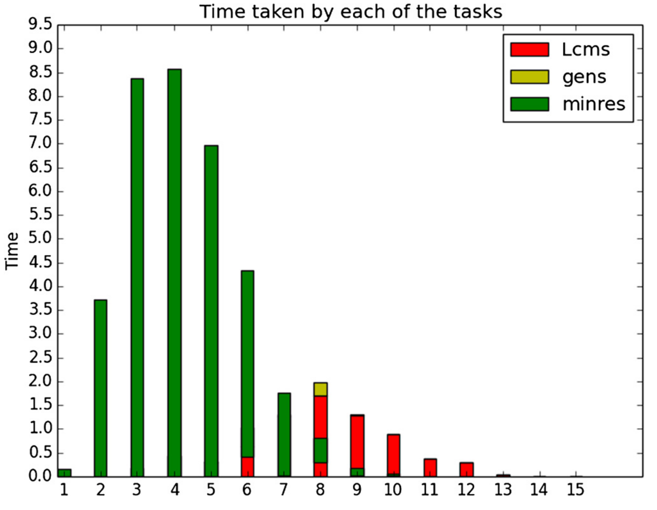

Fig. 3. Costs of each of the tasks for the consecutive 2-out-of-16 i deal. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Core i5 1.4 GHz processor with 4 GB of DDR RAM. Our program computes the sizes of the Taylor, Aramova–Herzog–Hibi and minimal resolutions of the ideals in the lcm-filtration of a given monomial ideal. The Taylor resolutions considered will use the minimal generating sets of the corresponding ideals. The main tasks needed are the following:

(1) Computing the k-fold lcms of the generators of the ideal: The computation of the k-fold lcms of a set of monomials is a theoretically easy task, but since the number of such lcms is large, its computational cost is not irrelevant. For this, we use the inbuilt command `lcm` in `Macaulay2`.
(2) Obtaining the minimal set of generators of the ideal I_k: This step implies autoreduction of the set of generators obtained in the previous step, so that we obtain the minimal generating set. We use the command `monomialIdeal` in `Macaulay2` which, for a given set of monomials, builds the minimal generating set.
(3) Computing the minimal free resolution of I_k: This is in principle the computationally most expensive task. For this we use the `Macaulay2` command `res` and assume we already have the minimal generating set of the corresponding ideal.

Figs. 3 and 4 show the distribution of time of each of these three tasks in the 2-out-of-16 and the cut ideal of complete graph K_5. We have chosen these examples because the number of generators of the original ideal is in both cases 15. The 2-out-of-16 ideal is generated in degree 2, and the number of minimal generators of each of the corresponding ideals is relatively big. On the other hand, the generators of the cut ideal of the complete graph have bigger degrees. In this case, there are many more cancellations between the generators of the lcm-ideals, which are minimally generated by much fewer monomials. These differences have their reflection in the distribution of times of the three tasks mentioned above. The cut ideal of the complete graph uses most of its time in computing the k-fold lcm-ideals. A good optimization of this part of the algorithm is therefore crucial for the application of this method to this kind of ideals. In the case of the 2-out-of-16 ideal, on the contrary, the best part of the computing time is devoted to the computation of the minimal free resolutions. Observe that the total times in both examples are quite similar, the differences are due to the different distribution of computing time among tasks, which reflects the differences in the lcm-filtration of both examples. For each of the examples, the maximum amount of CPU time used by a single ideal is less than 10 seconds. For our purposes, however, the distribution of time among tasks is more important than

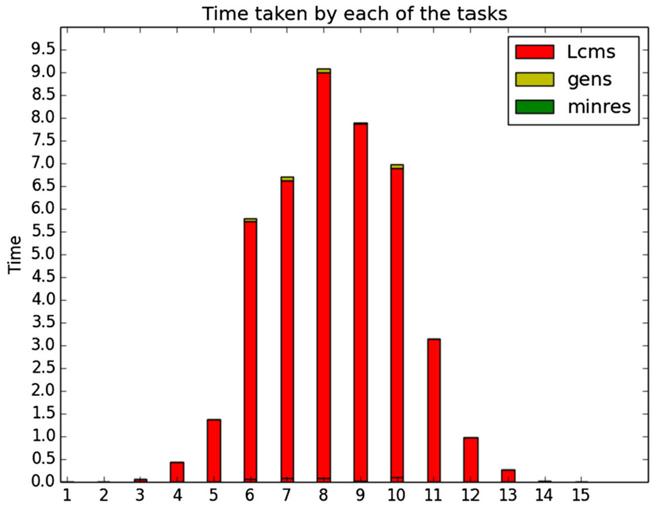

Fig. 4. Costs of each of the tasks for the cut ideal of K_5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the total time for each ideal, since this distribution reflects the structural differences between the aforementioned examples.

The computation of the k-fold lcm-ideals is unavoidable for our purposes, hence the data in our experiments show that for examples like the 2-out-of-16 ideals, computing the Aramova–Herzog–Hibi resolutions makes the computation possible, for the time needed for the minimal free resolution is too big too soon. On the other hand, for examples like the cut ideal of the complete graph K_5, the computation time of the minimal free resolution is (at this size of examples) not so relevant, and one should focus on an efficient implementation of the construction of the k-fold lcm-ideals. In this respect, Theorem 3.7 is a strong result that on one hand reduces the number of ideals to be computed, and on the other hand allows us to substitute the computations of the subsequents ideals by computations of partitions of the set of vertices, for which we can adapt already existing efficient algorithms. The efficient computation of cut ideals for the complete graph is an important point towards the algebraic study of the Erdös–Rényi model of random networks. We focus in more detail on this on our work in progress (Mohammadi et al., in preparation).

4. Failure distributions and signatures

Squarefree monomial ideals have a special role in the study of classical system reliability as explained in §2, and this extends to lcm ideals. The first cut ideal I_1 represents the full system failure set which is the event that at least one minimal cut has occurred. Then, the first lcm-ideal of I_1, namely I_2, is the event that at least two minimal cuts have occurred; and so on to at least k minimal cuts have occurred. For a general probability distribution over the states, and to obtain the probability of one of these events, one would have to sum the raw probabilities of the state making the event. But for simple independence models one can avoid this by the use of elementary probability methods, as explained in the next subsection.

We now have, as promised, two ways of giving a description of failure which are finer than the basic failure event represented by I_1. Each is represented by a special integer count. The first, just discussed, is the number of elementary cuts, under system failure. The second, which is discussed in §2.2.1, is the number of edges which are cut, under system failure. In both cases we can find, under our probability model, the (marginal) distribution of each count as a random variable and also their joint distribution. We do this for a simple example in the final subsection.

4.1. Computing means and higher moments

In the case of binary (squarefree) ideals, which includes the case of the k-out-of-r ideal and the cut ideal of a network in this paper, the computation of the moment of the distribution of Y, the number of elementary cuts, is straightforward.

Lemma 4.1. *Let $\mathcal{C}$ be the set of elementary cuts for a network reliability problem, and let Y be the number of elementary cuts. Then under the Erdös–Rényi independence model with probability p, the expectation of Y considered as a random variable, is given by*

$$E(Y) = \sum_{\alpha \in \mathcal{C}} p^{|\alpha|},$$

where $|\alpha|$ is the degree of α.

Proof. Let X_α be the indicator function for the cut α. Then $Y = \sum_{\alpha \in \mathcal{C}} X_\alpha$ and $\mathrm{E}(Y) = \mathrm{E}(\sum_{\alpha \in \mathcal{C}} X_\alpha) = \sum_{\alpha \in \mathcal{C}} \mathrm{E}(X_\alpha)$. But for any α, $X_\alpha = \prod_{i=1}^n X_i^{\alpha_i}$, where X_i is the indicator function of the failure of the i-th component for $i = 1, \ldots, n$. However, for any indicator function $\mathrm{prob}\{X_\alpha = 1\} = \mathrm{E}(X_\alpha)$ by independence in the Erdös–Rényi model. Thus $\mathrm{E}(X_\alpha) = \mathrm{E}(\prod_{i=1}^n X_i^{\alpha_i}) = \prod_{i=1}^n \mathrm{E}(X_i^{\alpha_i}) = p^{|\alpha|}$ which completes the proof. $\square$

Theorem 4.2. *For the complete graph K_n the mean value is:*

$$\mu_n = \begin{cases} \sum_{k=1}^{[\frac{n}{2}]} \binom{n}{k} p^{k(n-k)} & \text{if } n \text{ is odd;} \\[2ex] \sum_{k=1}^{[\frac{n}{2}]-1} \binom{n}{k} p^{k(n-k)} + \frac{1}{2}\binom{n}{\frac{n}{2}} p^{(\frac{n}{2})^2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Lemma 4.1 we just need to count the number of elementary cuts, i.e. 2-partitions of the graph. By Postnikov and Shapiro (2004, Corollary 6.8) this number is equal to $S(n, 2)$ where $S(n, k)$ denotes the Stirling number of the second kind (i.e. the number of ways to partition a set of n elements into k nonempty subsets). Note that for a partition with k vertices in one part, and $n - k$ vertices in the other part, we have $k(n - k)$ edges between these two parts, and the number of such partitions is $\binom{n}{k}$. In the case that n is even and $k = \frac{n}{2}$, we have to divide this number by two because of double counting. $\square$

Theorem 4.3. *For the sequential k-out-of-n ideal $I_{k,n}$, the mean value is $\mu_{k,n} = (n - k + 1)p^k$.*

Proof. In this case we have $n - k + 1$ cut monomials of degree k. Thus the result follows by Lemma 4.1. $\square$

Remark 4.4. The same method can be used to obtain higher order moments. For the second non-central moment, $\mu_2 = \mathrm{E}(Y^2)$ we write

$$\begin{aligned} \mu_2 &= \mathrm{E}\left(\left(\sum_{\alpha \in \mathcal{C}} X_\alpha\right)^2\right) \\ &= \mathrm{E}\left(\sum_{\alpha \in \mathcal{C}} X_\alpha^2 + \sum_{\substack{\alpha,\beta \in \mathcal{C} \\ \alpha \neq \beta}} X_\alpha X_\beta\right). \end{aligned}$$

Noting that $X_\alpha^2 = X_\alpha$ and $X_\alpha X_\beta = X_{\alpha \wedge \beta}$, and using the argument above we have

$$\mu_2 = \sum_{\alpha \in \mathcal{C}} p^{|\alpha|} + 2 \sum_{\gamma \in \mathcal{C}_2} p^{|\gamma|},$$

where $\mathcal{C}_2$ is the set of monomial generators in the first lcm list, preserving repetitions.

In general, $\mathcal{C}_k$ is the list of $k-$lcm's of the monomials corresponding to the elements of $\mathcal{C}$. We see that the k-th non-central moment can be written in terms of the lcm-ideals up to the k-th level.

4.2. Examples

We develop the case of sequential 2-out-of-6 ideal with all the accompanying lcm-ideals in more details. We compute the Hilbert series $\mathcal{H}_k(x,t)$ based on Aramova–Herzog–Hibi version of the Hilbert series for the lcm-ideals I_k for $k = 1, \ldots, 5$. The ideal is $I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6 \rangle$. The numerators of the Hilbert series are

$$\begin{aligned}
\mathcal{H}_1(t,x) = 1 &- (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6)t \\
&+ (x_1x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_4x_5 + x_1x_2x_5x_6 + x_2x_3x_4 + x_2x_3x_4x_5 \\
&+ x_2x_3x_5x_6 + x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6 + x_1x_2x_3x_4)t^2 \\
&+ (x_1x_2x_3x_4x_5 + x_1x_2x_3x_5x_6 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5x_6 \\
&+ x_1x_2x_4x_5x_6 + x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6 + x_3x_4x_5x_6)t^3 \\
&+ (x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 \\
&+ x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6)t^4 - x_1x_2x_3x_4x_5x_6t^5
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_2(t,x) = 1 &- (x_1x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_4x_5 + x_1x_2x_5x_6 + x_2x_3x_4 + x_2x_3x_4x_5 \\
&+ x_2x_3x_5x_6 + x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6)t + (2x_1x_2x_3x_4 + 2x_1x_2x_3x_4x_5 \\
&+ 2x_1x_2x_3x_5x_6 + 2x_1x_2x_3x_4x_5 + 2x_1x_2x_3x_4x_5x_6 + 2x_1x_2x_4x_5x_6 \\
&+ 2x_2x_3x_4x_5 + 2x_2x_3x_4x_5x_6 + 2x_2x_3x_4x_5x_6 + 2x_3x_4x_5x_6)t^2 \\
&- (3x_1x_2x_3x_4x_5 + 3x_1x_2x_3x_4x_5x_6 + 3x_1x_2x_3x_4x_5x_6 + 3x_1x_2x_3x_4x_5x_6 \\
&+ 3x_2x_3x_4x_5x_6)t^3 + 4x_1x_2x_3x_4x_5x_6t^4
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_3(t,x) = 1 &- (x_1x_2x_3x_4 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_5x_6 + x_1x_2x_3x_4x_5 \\
&+ x_1x_2x_3x_4x_5x_6 + x_1x_2x_4x_5x_6 + x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6 \\
&+ x_3x_4x_5x_6)t + (3x_1x_2x_3x_4x_5 + 3x_1x_2x_3x_4x_5x_6 + 3x_1x_2x_3x_4x_5x_6 \\
&+ 3x_1x_2x_3x_4x_5x_6 + 3x_2x_3x_4x_5x_6)t^2 - 6x_1x_2x_3x_4x_5x_6t^3
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_4(t,x) = 1 &- (x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 + x_1x_2x_3x_4x_5x_6 \\
&+ x_2x_3x_4x_5x_6)t + 4x_1x_2x_3x_4x_5x_6t^2
\end{aligned}$$

$$\mathcal{H}_5(t,x) = 1 - x_1x_2x_3x_4x_5x_6.$$

Under the usual failure model, of independent component (edge) failure with probability p, if Y is the number of failures the survivor functions are given by:

$$\begin{aligned}
P_k &= F(k) \\
&= \mathbb{P}\{Y \geq k\} \\
&= 1 - \mathcal{H}_k(1,p).
\end{aligned}$$

These are given by (see Fig. 5):

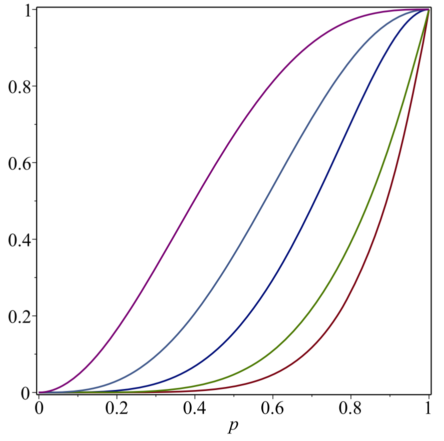

Fig. 5. Plots of $P_k(p)$, $k = 1, \dots, 5$, from left to right, respectively.

$$
\begin{aligned}
P_1 &= 5p^2 - 4p^3 - 3p^4 + 4p^5 - p^6 \\
P_2 &= 4p^3 - 6p^5 + 3p^6 \\
P_3 &= 3p^4 - 2p^6 \\
P_4 &= 2p^5 - p^6 \\
P_5 &= p^6
\end{aligned}
$$

Setting $p_0 = 1 - P_1$, $p_k = P_k - P_{k+1}$, $k = 1, \dots, 4$ and $p_5 = P_5$ we find the (discrete) distribution as

$$
\begin{aligned}
p_0 &= 1 - 5p^2 + 4p^3 + 3p^4 - 5p^5 + p^6 \\
p_1 &= 5p^2 - 8p^3 - 3p^4 + 10p^5 - 4p^6 \\
p_2 &= 4p^3 - 3p^4 - 6p^5 + 5p^6 \\
p_3 &= 3p^4 - 2p^5 - p^6 \\
p_4 &= 2p^5 - 2p^6 \\
p_5 &= p^6
\end{aligned}
$$

The mean of this distribution is $\mu_{6,2} = \sum_{i=0}^{5} i p_i = 5p^2$. The form of the distribution for general sequential k-out-of-n was found relatively recently (see Ling, 1988) together with the general form of the mean, confirmed in our case:

$$\mu_{n,k} = (n - k + 1)p^k.$$

In probability theory and related areas this might be called "the expected number of runs of size k in a Bernoulli sequence of length n and probability p". Care has to be taken, in accessing the literature, concerning the definition of a run. For example whether overlaps are counted, as here, this distinguishes from isolated runs of length k.

For the signature we need to find the intersection $\tilde{I}_k$ of the k-out-of-n ideals for $k = 1, \dots, 6$ with the sequential k-out-of-n ideal I_1, noting that the latter is the basic failure ideal. A simple way to carry out this calculation is to identify for each $k = 1, \dots, 6$ which monomial of degree k occurs in I_1.

For $k = 1$, none of $x_1, \dots, x_6$ lie in I_1 and for $k = 2$ we obtain $\tilde{I}_2 = I_1$. Similarly:

Table 1
Elementary cuts via component failure, for the sequential 2-out-of-6 system.

6					1
5			2	2	
4		3	3		
3		4			
2	5				
$z \diagup y$	1	2	3	4	5

$$\tilde{I}_3 = \langle x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6\rangle$$
$$\tilde{I}_4 = \langle x_1x_2x_3x_4, x_1x_2x_4x_5, x_1x_2x_5x_6, x_2x_3x_4x_5, x_2x_3x_5x_6, x_3x_4x_5x_6\rangle$$
$$\tilde{I}_5 = \langle x_1x_2x_3x_4x_5, x_1x_2x_3x_5x_6, x_1x_2x_4x_5x_6, x_2x_3x_4x_5x_6\rangle$$
$$\tilde{I}_6 = \langle x_1x_2x_3x_4x_5x_6\rangle.$$

The associated cumulative probabilities including $\tilde{I}_2$ are:

$$Q_2 = 5p^2 - 4p^2 - 3p^4 + 4p^5 - p^6$$
$$Q_3 = 4p^3 - 3p^4$$
$$Q_4 = 6p^4 - 6p^5 + p^6$$
$$Q_5 = 4p^5 - 3p^6$$
$$Q_6 = p^6.$$

Again by taking differences we obtain the raw signature probabilities as

$$q_2 = 5p^2 - 8p^3 + 4p^3 - p^6$$
$$q_3 = 4p^3 - 9p^4 + 6p^5 - p^6$$
$$q_4 = 6p^4 - 10p^5 + 4p^6$$
$$q_5 = 4p^5 - 4p^6$$
$$q_6 = p^6.$$

From these we compute the signatures as $s_j = \frac{q_j}{P_1}$ for $j = 2, \dots, 6$.

One purpose of this paper is to present the $\{p_k\}_{k=1}^n$, namely the distribution of the number of elementary cuts as an alternative "signature" to the classical signature distribution, of the number of failed components, in the event of failure. But we can also study systems by looking at several different types of signature, what might be called multivariate signature analysis. To make this point clear we compute, for the current example, the *joint* distribution, that is to say the distribution of the bivariate random variables (Y, Z), where Y is the number of elementary cuts and Z is the number of failed components, conditional on failure.

The case $p = \frac{1}{2}$ corresponds to simple counting since every binary state vector has probability $\frac{1}{2^6}$. Table 1 gives the counts of the number of elementary cuts versus the number of component failures, for the 20 failure states (a blank cell indicates the count is zero). This shows a close association between the two types of signature.

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