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# Oscillation of functional trinomial differential equations with positive and negative term☆

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ABSTRACT

In the paper, we establish new technique for investigation of properties of trinomial differential equations with positive and negative term

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0.$$

We offer new criteria for asymptotic properties of nonoscillatory solutions for the studied equations. We support our results with illustrative examples.



## 1. Introduction

We shall study asymptotic properties of the third order trinomial differential equation with positive and negative term

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0. \tag{E}$$

Throughout the paper, it is assumed that the following hypothesis hold true

($H_1$) $a(t)$, $b(t)$, $p(t)$, $q(t)$, $\tau(t)$, $\sigma(t) \in C([t_0, \infty))$ are positive;
($H_2$) $\int_{t_0}^{\infty} \frac{1}{b(s)}\, ds = \int_{t_0}^{\infty} \frac{1}{a(s)}\, ds = \infty$;
($H_3$) $\sigma(t) \geq t$, $\sigma(t)$ is nondecreasing, $\lim_{t\to\infty} \tau(t) = \infty$;
($H_4$) $f(u), h(u) \in C(\mathbb{R})$, $uf(u) > 0$, $uh(u) > 0$ for $u \neq 0$, $f$ is bounded, $h$ is nondecreasing.

By a solution of (E) we mean a function $x(t)$ with $b(t)(a(t)x'(t))' \in C^1([T_x, \infty))$, $T_x \geq t_0$, which satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Eq. (E) is said to be oscillatory if all its solutions are oscillatory.

Setting either $p(t) \equiv 0$ or $q(t) \equiv 0$ Eq. (E) reduces to simpler binomial differential equations

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) = 0, \tag{$E_f$}$$

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and

$$\Big(b(t)\big(a(t)x'(t)\big)'\Big)' - q(t)h(x(\sigma(t))) = 0. \tag{$E_h$}$$

Properties of both equations have been studied by many authors. See papers of Baculikova and Džurina [5,6], Candan and Dahiya [10], Grace et al. [14], Thandapany and Li [20], Tiryaki and Atkas [21].

The well known lemma of Kiguradze [15] implies that the solutions' spaces of ($E_f$) and ($E_h$) are absolutely different. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of considered equations, then for ($E_f$) the set $\mathcal{N}$ has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2,$$

where positive solution

$$x(t) \in \mathcal{N}_0 \Longleftrightarrow a(t)x'(t) < 0, \quad b(t)\big(a(t)x'(t)\big)' > 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' < 0$$

$$x(t) \in \mathcal{N}_2 \Longleftrightarrow a(t)x'(t) > 0, \quad b(t)\big(a(t)x'(t)\big)' > 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' < 0.$$

While, for ($E_h$) the set $\mathcal{N}$ has the following reduction

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3,$$

with positive solution

$$x(t) \in \mathcal{N}_1 \Longleftrightarrow a(t)x'(t) > 0, \quad b(t)\big(a(t)x'(t)\big)' < 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' > 0$$

$$x(t) \in \mathcal{N}_3 \Longleftrightarrow a(t)x'(t) > 0, \quad b(t)\big(a(t)x'(t)\big)' > 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' > 0.$$

It is understandable that in generally the nonoscillatory solutions' space of (E) with positive and negative part is unclear.

In this paper, we offer new method that overcome those difficulties caused by presence of negative and positive terms of (E). Throughout the paper, we assume that

$$(H_5)\ \int_{t_0}^{\infty} \frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(s)} \int_s^{\infty} p(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty,$$

and this assumption implies that the negative term is dominating and the structure of the nonoscillatory solutions of (E) is similar that of ($E_h$).

## 2. Main results

In this paper, we reduce investigation of trinomial equation onto the oscillation of the suitable first order differential equation. We establish new comparison method for investigating properties of trinomial differential equation with positive and negative term.

**Theorem 1.** *Assume that for every $k > 0$*

$$\int_{t_0}^{\infty} q(s)\left| h\left( \pm k \int_{t_0}^{\sigma(s)} \frac{1}{a(u)} \int_{t_0}^{u} \frac{1}{b(v)}\,\mathrm{d}v\,\mathrm{d}u \right) \right| \mathrm{d}s = \infty. \tag{2.1}$$

*If the first order advanced differential equation*

$$y'(t) - \left[ \frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(u)} \int_u^{\infty} q(s)\,\mathrm{d}s\,\mathrm{d}u \right] h(y(\sigma(t))) = 0 \tag{$E_0$}$$

*is oscillatory, then every nonoscillatory solution $x(t)$ of (E) satisfies either*

$$\lim_{t\to\infty} \frac{|x(t)|}{\int_{t_1}^{t} \frac{1}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\,\mathrm{d}u\,\mathrm{d}s} = \infty. \tag{2.2}$$

*or*

$$\begin{aligned} &\lim_{t\to\infty} x(t) = 0 \quad \text{with} \\ &|x(t)| \le k \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} \frac{1}{b(u)} \int_u^{\infty} p(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s, \qquad k > 0. \end{aligned} \tag{2.3}$$

**Proof.** Assume that (E) possesses a nonoscillatory solution $x(t)$. Without loss of generality we may assume that $x(t)$ is eventually positive. We introduce the auxiliary function

$$w(t) = x(t) - \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} \frac{1}{b(u)} \int_u^{\infty} p(v) f(x(\tau(v)))\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s. \tag{2.4}$$

Note that condition $(H_5)$ and the fact that $f(u)$ is bounded implies that $w(t)$ exists for all $t$ and so definition of $w(t)$ is correct. Moreover, $x(t) > w(t)$ and

$$\Big(b(t)\big(a(t)w'(t)\big)'\Big)' = q(t)h(x(\sigma(t))) > 0. \tag{$E_p$}$$

Therefore, condition $(H_2)$ together with a modification of Kiguradze's lemma imply that either

$$w(t) \in \mathcal{N}_1 \iff a(t)w'(t) > 0, \quad b(t)\big(a(t)w'(t)\big)' < 0,$$

or

$$w(t) \in \mathcal{N}_3 \iff a(t)w'(t) > 0, \quad b(t)\big(a(t)w'(t)\big)' > 0,$$

or

$$w(t) < 0,$$

eventually, let us say for $t \geq t_1$. First assume that $w(t) \in \mathcal{N}_1$. An integration of ($E_p$) yields

$$\begin{aligned} -b(t)\big(a(t)w'(t)\big)' &\geq \int_t^\infty q(s)h(x(\sigma(s)))\,\mathrm{d}s \geq \int_t^\infty q(s)h(w(\sigma(s)))\,\mathrm{d}s \\ &\geq h(w(\sigma(t)))\int_t^\infty q(s)\,\mathrm{d}s \end{aligned} \tag{2.5}$$

Integrating again, we are led to

$$a(t)w'(t) \geq h(w(\sigma(t)))\int_t^\infty \frac{1}{b(u)}\int_u^\infty q(s)\,\mathrm{d}s\,\mathrm{d}u.$$

Thus, $w(t)$ is a positive solution of the differential inequality

$$w'(t) - \left[\frac{1}{a(t)}\int_t^\infty \frac{1}{b(u)}\int_u^\infty q(s)\,\mathrm{d}s\,\mathrm{d}u\right]h(w(\sigma(t))) \geq 0.$$

It follows from Lemma 2.3 in [7] that the corresponding differential equation ($E_0$) also has a positive solution. A contradiction and the case $w(t) \in \mathcal{N}_1$ is impossible.

Now, we assume that $w(t) \in \mathcal{N}_3$. We claim that $\lim_{t\to\infty} b(t)\big(a(t)w'(t)\big)' = \infty$. If we admit that $\lim_{t\to\infty} b(t)\big(a(t)w'(t)\big)' = \ell < \infty$, then integrating ($E_p$) from $t_1$ to $\infty$, we have

$$\ell > \int_t^\infty q(s)h(x(\sigma(s)))\,\mathrm{d}s > \int_t^\infty q(s)h(w(\sigma(s)))\,\mathrm{d}s. \tag{2.6}$$

On the other hand, there exists some $k > 0$ such that $b(t)\big(a(t)w'(t)\big)' > k$, eventually, which implies

$$\big(a(t)w'(t)\big)' \geq \frac{k}{b(t)}.$$

Integrating the last inequality, one gets

$$w'(t) > \frac{k}{a(t)}\int_{t_1}^t \frac{1}{b(s)}\,\mathrm{d}s.$$

Integrating once more, we obtain

$$w(t) \geq k\int_{t_1}^t \frac{1}{a(u)}\int_{t_1}^u \frac{1}{b(s)}\,\mathrm{d}s\,\mathrm{d}u.$$

Setting the last estimate into (2.6), we have

$$\ell > \int_{t_1}^\infty q(s)\,h\left(k\int_{t_1}^{\sigma(s)} \frac{1}{a(u)}\int_{t_1}^u \frac{1}{b(v)}\,\mathrm{d}v\,\mathrm{d}u\right)\mathrm{d}s,$$

which contradicts to (2.1) and we conclude that $\lim_{t\to\infty} b(t)\big(a(t)w'(t)\big)' = \infty$. Using the L'Hospital rule, we see that

$$\lim_{t\to\infty} \frac{w(t)}{\displaystyle\int_{t_1}^{t} \frac{1}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\, du\, ds} = \lim_{t\to\infty} b(t)\big(a(t)w'(t)\big)' = \infty.$$

Since $x(t) > w(t)$, we conclude that (2.2) holds true.

The last one possibility is $w(t) < 0$ which yields

$$\begin{aligned} x(t) &< \int_t^\infty \frac{1}{a(s)} \int_s^\infty \frac{1}{b(u)} \int_u^\infty p(v) f(x(\tau(v)))\, dv\, du\, ds \\ &< k \int_t^\infty \frac{1}{a(s)} \int_s^\infty \frac{1}{b(u)} \int_u^\infty p(v)\, dv\, du\, ds. \end{aligned}$$

Condition $(H_5)$ implies that $\lim_{t\to\infty} x(t) = 0$ and the proof is complete. □

For partial case of (E), we obtain the following easily verifiable criterion.

**Corollary 1.** *Assume that (2.1) holds and*

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} \frac{1}{a(v)} \int_v^\infty \frac{1}{b(u)} \int_u^\infty q(s)\, ds\, du dv > \frac{1}{e}. \tag{$P_1$}$$

*Then every nonoscillatory solution of the trinomial differential equation*

$$\Big(b(t)\big(a(t)x'(t)\big)'\Big)' + p(t) f(x(\tau(t))) - q(t)x(\sigma(t)) = 0, \tag{$E_L$}$$

*satisfies either (2.2) or (2.3).*

**Proof.** Theorem 2.4.1 in [18] implies that $(P_1)$ guarantees oscillation $(E_0)$ with $h(u) = u$. The assertion of the corollary now follows from Theorem 1. □

As a matter of fact we are able to provide general criterion for studied property of (E).

**Corollary 2.** *Assume that (2.1) holds and*

$$\limsup_{t\to\infty} \int_t^{\sigma(t)} \frac{1}{a(v)} \int_v^\infty \frac{1}{b(u)} \int_u^\infty q(s)\, ds\, du\, dv > \limsup_{u\to\infty} \frac{u}{h(u)}. \tag{$P_2$}$$

*Then every nonoscillatory solution of (E) satisfies either (2.2) or (2.3).*

**Proof.** By Theorem 1, it is sufficient to show that $(E_0)$ is oscillatory. Assume on the contrary that $(E_0)$ possesses a nonoscillatory, let us say positive solution $y(t)$. It follows from $(E_0)$ that $y'(t) > 0$. Thus, there exists $\lim_{t\to\infty} y(t) = c > 0$. An integration of $(E_0)$ from $t$ to $\sigma(t)$ provides

$$\begin{aligned} y(\sigma(t)) &= y(t) + \int_t^{\sigma(t)} \frac{1}{a(v)} \int_v^\infty \frac{1}{b(u)} \int_u^\infty q(s) h(y(\sigma(s)))\, ds\, du\, dv \\ &\geq y(t) + h(y(\sigma(t))) \int_t^{\sigma(t)} \frac{1}{a(v)} \int_v^\infty \frac{1}{b(u)} \int_u^\infty q(s)\, ds\, du\, dv. \end{aligned}$$

The last inequality together with $(P_2)$ implies that $c = \infty$ and what is more,

$$\frac{y(\sigma(t))}{h\big(y(\sigma(t))\big)} \geq \int_t^{\sigma(t)} \frac{1}{a(v)} \int_v^\infty \frac{1}{b(u)} \int_u^\infty q(s)\, ds\, du\, dv$$

Taking lim sup on both sides, we get a contradiction with $(P_2)$. □

For function $h(u) = u^\beta$, $\beta$ quotient of odd positive integers we immediately get:

**Corollary 3.** *Let $\beta > 1$. Assume that (2.1) holds and*

$$\limsup_{t\to\infty} \int_t^{\sigma(t)} \frac{1}{a(v)} \int_v^\infty \frac{1}{b(u)} \int_u^\infty q(s)\, ds\, du\, dv > 0. \tag{$P_3$}$$

*Then every solution of*

$$\Big(b(t)\big(a(t)x'(t)\big)'\Big)' + p(t) f(x(\tau(t))) - q(t)x^\beta(\sigma(t)) = 0 \tag{$E_S$}$$

*either oscillates or satisfies (2.2).*

**Example 1.** Consider the third order trinomial differential equation

$$\left(t^{1/2}\left(t^{1/3}x'(t)\right)'\right)' + \frac{p}{t^3}\arctan\left(y(\tau(t))\right) - \frac{q}{t^{13/6}}\,y\big(\lambda t\big) = 0, \tag{$E_x$}$$

where $p > 0$, $q > 0$, $\lambda > 1$ and $f(u) = \arctan(u)$ is bounded. It is easy to verify that condition (2.1) holds true and ($P_1$) takes the form

$$q\ln\lambda > \frac{7}{9\,\mathrm{e}}, \tag{2.7}$$

which implies that every nonoscillatory solution $x(t)$ of ($E_x$) satisfies either (2.2) or (2.3), which means that either

$$\lim_{t\to\infty}\frac{|x(t)|}{t^{7/6}} = \infty$$

or

$$|x(t)| \le \frac{K}{t^{5/6}}, \quad K > 0.$$

If we replace condition (2.1) by request for oscillation of another suitable first order equation, we are able to eliminate nonoscillatory solutions of (E) satisfying (2.2). We use the auxiliary function $\xi(t) \in C^1([t_0, \infty))$ satisfying

$$\xi'(t) > 0, \quad \xi(t) < t, \quad \eta(t) = \sigma(\xi(\xi(t))) > t. \tag{2.8}$$

**Theorem 2.** *Let (2.8) hold and ($E_0$) be oscillatory. If the first order advanced differential equation*

$$z'(t) - \left[\frac{1}{a(t)}\int_{\xi(t)}^{t}\frac{1}{b(s)}\int_{\xi(s)}^{s} q(u)\,\mathrm{d}u\mathrm{d}s\right] h(z(\eta(t))) = 0 \tag{$E_1$}$$

*is oscillatory, then every nonoscillatory solution of (E) satisfies (2.3).*

**Proof.** Assume that (E) has a positive solution $x(t)$. Let $w(t)$ be defined by (2.4). Proceeding exactly as in the proof of Theorem 1, we see that oscillation of ($E_0$) ensures that $w(t) \notin \mathcal{N}_1$. Now, we admit that $w(t) \in \mathcal{N}_3$. It follows from ($E_p$) and $x(t) > w(t)$ that

$$\left(b(t)\left(a(t)w'(t)\right)'\right)' \ge q(t)h(w(\sigma(t))).$$

An integration of this inequality leads to

$$b(t)\left(a(t)w'(t)\right)' \ge \int_{\xi(t)}^{t} q(s)h(w(\sigma(s)))\,\mathrm{d}s \ge h(w(\sigma(\xi(t))))\int_{\xi(t)}^{t} q(s)\,\mathrm{d}s.$$

Integrating once more, we get

$$a(t)w'(t) \ge h(w(\sigma(\xi(\xi(t)))))\int_{\xi(t)}^{t}\frac{1}{b(s)}\int_{\xi(s)}^{s} q(u)\,\mathrm{d}u\mathrm{d}s.$$

That is $w(t)$ is a positive solution of the differential inequality

$$w'(t) - \left[\frac{1}{a(t)}\int_{\xi(t)}^{t}\frac{1}{b(s)}\int_{\xi(s)}^{s} q(u)\,\mathrm{d}u\mathrm{d}s\right] h(y(\eta(t))) \ge 0.$$

It follows from Lemma 2.3 in [7] that the corresponding differential equation ($E_1$) also has a positive solution. A contradiction and thus, the case $w(t) \in \mathcal{N}_3$ is impossible and we conclude that every nonoscillatory solution of (E) satisfies (2.3). □

**Remark 1.** Employing sufficient conditions for oscillation of ($E_1$) together with those for ($E_0$), we obtain oscillatory criteria for (E).

The following result is obvious.

**Corollary 4.** *Assume that (2.8) and ($P_1$) holds. If*

$$\liminf_{t\to\infty}\int_{t}^{\eta(t)}\left[\frac{1}{a(v)}\int_{\xi(v)}^{v}\frac{1}{b(s)}\int_{\xi(s)}^{s} q(u)\,\mathrm{d}u\mathrm{d}s\right]\mathrm{d}v > \frac{1}{\mathrm{e}}. \tag{$P_4$}$$

*then every nonoscillatory solution of ($E_L$) satisfies (2.3).*

**Example 2.** Consider once more the differential equation

$$\left(t^{1/2}\left(t^{1/3}x'(t)\right)'\right)' + \frac{p}{t^3}\arctan(y(\tau(t))) - \frac{q}{t^{13/6}}\,y(\lambda t)) = 0, \tag{$E_x$}$$

We set $\xi(t) = \alpha t$, where $\alpha = \frac{1+\sqrt{\lambda}}{2\sqrt{\lambda}}$. Then $\eta(t) = \frac{\lambda+2\sqrt{\lambda}+1}{4}$ and a simple computation reveal that ($P_4$) takes the form

$$\frac{36q}{28}\left(\alpha^{-7/6} - 1\right)\left(\alpha^{-4/6} - 1\right)\ln\left(\lambda\alpha^2\right) > \frac{1}{\mathrm{e}}. \tag{2.9}$$

By Corollary 4, then every nonoscillatory solution of ($E_x$) satisfies (2.3) if both conditions (2.7) and (2.9) are satisfied.

## 3. Comparison with existing results

The results obtained provide new technique for studying asymptotic properties of functional trinomial third order differential equation with quasi-derivatives with positive and negative terms via oscillation of suitable first order equations. Our results complement papers [1–21].

## References

[1] R.P. Agarwal, M. Bohner, T. Li, C. Zhang, Oscillation of third-order nonlinear delay differential equations, Taiwan. J. Math. 17 (2013) 545–558.
[2] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Marcel Dekker, Kluwer Academic, Dordrecht, 2000.
[3] B. Baculíková, J. Džurina, Oscillation of third-order neutral differential equations, Math. Comput. Model. 52 (2010) 215–226.
[4] B. Baculíková, J. Graef, J. Džurina, On the oscillation of higher order delay differential equations, Nonlinear Oscil. 15 (2012) 13–24.
[5] B. Baculíková, J. Džurina, Oscillation of third-order nonlinear differential equations, Appl. Math. Lett. 24 (2011) 466–470.
[6] B. Baculíková, J. Džurina, Oscillation of third-order functional differential equations, EJQTDE 43 (2010) 1–10.
[7] B. Baculíková, Properties of third order nonlinear functional differential equations with mixed arguments, Abstr. Appl. Anal. 2011 (2011) 1–15.
[8] B. Baculíková, J. Džurina, Y. Rogovchenko, Oscillation of third order trinomial differential equations, Appl. Math. Comput. 218 (2012) 7023–7033.
[9] B. Baculíková, J. Džurina, Property (a) and oscillation of third order differential equations with mixed arguments, Funkcialaj Ekvacioj 55 (2012) 239–253.
[10] T. Candan, R. S.Dahiya, Oscillation of third-order functional differential equations with delay, EJDE 2010 (2003) 79–88.
[11] J.D. zurina, Comparison theorems for nonlinear ODE's, Math. Slovaca 42 (1992) 299–315.
[12] J.D. zurina, R. Kotorova, Zero points of the solutions of a differential equation, Acta Electrotech. Inf. 7 (2007) 26–29.
[13] L. Erbe, Q. Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.
[14] S.R. Grace, R.P. Agarwal, R. Pavani, E. Thandapani, On the oscillation of certain third - order nonlinear functional differential equations, Appl. Math. Comput. 202 (2008) 102–112.
[15] I.T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)|u|^n sign\, u = 0$, Mat. Sb. 65 (1964) 172–187. Russian.
[16] I.T. Kiguradze, T.A. Chanturia, Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations, Kluwer Acad. Publ., Dordrecht, 1993.
[17] T. Kusano, M. Naito, Comparison theorems for functional differential equations with deviating arguments, J. Math. Soc. Jpn. 3 (1981) 509–533.
[18] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.
[19] S. Panigrahi, R. Basu, Oscillations results for third order nonlinear mixed neutral differential equations, Math. Slovaca, in press.
[20] E. Thandapani, T. Li, On the oscillation of third - order quasi - linear neutral functional differential equations, Arch. Math. 47 (2011) 181–199.
[21] A. Tiryaki, M.F. Aktas, Oscillation criteria of certain class of third - order nonlinear delay differential equations with damping, J. Math. Anal. Appl. 325 (2007) 54–68.