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Characterizing the creep of viscoelastic materials by fractal derivative models

Wei Cai, Wen Chen*, Wenxiang Xu*

State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Institute of Soft Matter Mechanics, College of Mechanics and Materials, Hohai University, Nanjing 210098, China.

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ABSTRACT

In this paper, we make the first attempt to apply the fractal derivative to modeling viscoelastic behavior. The methodology of scaling transformation is utilized to obtain the creep modulus and relaxation compliance for the proposed fractal Maxwell and Kelvin models. Comparing with the fractional derivatives reported in the literature, the fractal derivative as a local operator has lower calculation costs and memory storage requirements. Moreover, numerical results show that the proposed fractal models require fewer parameters, have simpler mathematical expression and result in higher accuracy than the classical integer-order derivative models. Results further confirm that the proposed fractal models can characterize the creep behavior of viscoelastic materials.

1. Introduction

There are a variety of time-dependent viscoelastic materials in nature, such as soft soil, asphalt, colloid and brain tissue, just to mention a few. Many attempts have been devoted to the theory of viscoelastic mechanics in the past decades. The viscoelastic behavior can be characterized by integral or differential constitutive relationship equations:

$$\sum_{k=0}^{m} p_k \frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k} = \sum_{k=0}^{n} q_k \frac{\mathrm{d}^k\varepsilon}{\mathrm{d}t^k},\ n \geq m \text{ or } \varepsilon(t) = \int_{-\infty}^{t} J(t-\zeta)\dot{\sigma}(\zeta)\mathrm{d}\zeta, \tag{1}$$

where p_k and q_k are the materials constants, and $J(t)$ denotes creep modulus. There have been well-known viscoelastic models for describing the viscoelastic behavior on the basis of the combination of springs and dashpots, such as the Maxwell, the Kelvin and the standard linear solid models [1,2]. In order to achieve a better fitting accuracy against the experimental data, more springs or dashpots are often combined in the classical models [1,2]. Such methodology, however, results in more complex models with some cumbersome parameters in physics.

In recent decades, fractional calculus has been successfully employed to model the viscoelastic behavior. Reports claim that this methodology is suitable for establishing the constitutive relationship of the time-dependent viscoelastic behavior [3–10]. It is a remarkable fact that the fractional calculus has succeeded in modeling power-law rheology phenomena (see for instance Ref. [11], and references therein). The use of fractional calculus is motivated in large part by the fact that fewer parameters are required in the fractional model to achieve the accurate description of experimental data compared with the classical models of integer-order derivative [1,4,12,13]. Nevertheless, the global property of the fractional calculus requires considerably more computational costs and memory requirements for its numerical simulation.

As an alternative modeling formalism to the global fractional derivative, the fractal derivative has been proposed as a local derivative to model complex behavior of fractal materials by one of the co-authors of this paper [14–18]. The concept of the Hausdorff fractal derivative was introduced to analyze the given function with respect to a fractal measure t^p [14]. In recent years, the fractal derivative has been successfully applied to investigating anomalous diffusion [15–17], oscillation [18] and heat generation [19]. It is observed that compared with the fractional derivative, the fractal derivative is a local operator and computationally far efficient to calculate.

In this study, we attempt to employ the fractal derivative to characterize the behavior of viscoelastic materials. We propose the fractal Maxwell and Kelvin models to depict the corresponding creep modulus and relaxation compliance, respectively. Subsequently, we compare the proposed fractal models with the existing classical and fractional models.

The rest of this paper is structured as follows. In Section 2, a brief definition of the fractal derivative is presented. The fractal Maxwell and Kelvin models are also proposed versus the fractional derivative models. Section 3 provides numerical results for the comparisons

* Corresponding authors.
E-mail addresses: chenwen@hhu.edu.cn (W. Chen), xuwenxiang@hhu.edu.cn (W. Xu).

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between the proposed and existing models. Conclusions are drawn in Section 4.

2. Fractal viscoelastic models

2.1. The definition of fractal derivative

By transforming standard integer space-time into fractal space-time fabric, the definition of the fractal derivative can be formulated as [14]

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t^p} = \lim_{t' \to t} \frac{u(t) - u(t')}{t^p - t'^p}$$
$$\frac{\mathrm{d}u(x)}{\mathrm{d}x^q} = \lim_{x' \to x} \frac{u(x) - u(x')}{x^q - x'^q}. \tag{2}$$

where p and q represent the fractals in time and space and the orders of the fractal derivative, respectively. It is easily seen from the above expression that the fractal derivative is a local operator without convolution integral. This is quite different from the integral definition of the fractional derivative.

Under the hypotheses of fractal invariance and fractal equivalence [14], the fractal derivative can be reduced to normal derivative by using the scaling transformation:

$$\begin{cases} \hat{t} = t^p \\ \hat{x} = x^q \end{cases}. \tag{3}$$

Such methodology has been successfully applied in obtaining the fundamental solution of fractal diffusion equation, which is in the form of a stretched Gaussian distribution [15]. Similarly, the non-Boltzmann scaling phenomenon for water transport in porous media has also been well described by the fractal Richard's equation [16].

2.2. Fractal Maxwell and Kelvin models

Owing to the merits of fractional models requiring relatively fewer parameters and simpler forms, the fractional calculus has been used to describe viscoelastic behavior. The well-known fractional viscoelastic models, such as the fractional Maxwell and Kelvin models, can be achieved by replacing the dashpot in the traditional viscoelastic models with a Scott-Blair element [20], which is characterized as $\sigma_f = \eta \mathrm{d}^p \varepsilon_f / \mathrm{d}t^p$. The studies on the creep modulus of such fractional models have widely been discussed and applied to a variety of problems [21–24].

It is well known that the definition of fractional calculus contains the convolution operator resulting in its global property. Take the popular Caputo definition as an example, which can be formulated as [25]:

$$\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau)\mathrm{d}\tau,\ 0 < \alpha < 1, \tag{4}$$

where α is the order of the fractional derivative. Due to such non-local property, in general the numerical solution of the fractional derivative model is computationally expensive.

Based on the above consideration and noting from Eq. (2) that the fractal derivative is a local operator, we use a fractal dashpot $\sigma_f = \eta \mathrm{d}\varepsilon_f / \mathrm{d}t^p$ to characterize the rheology property of viscoelastic materials, instead of a Scott-Blair element of fractional derivative. The schematic diagrams of the fractal dashpot and the Scott-Blair element are shown in Fig. 1. The comparisons of creep modulus between the fractal dashpot and the fractional Scott-Blair element are

Fig. 1. The schematic diagram of a Scott-Blair element (left) and a fractal dashpot (right).

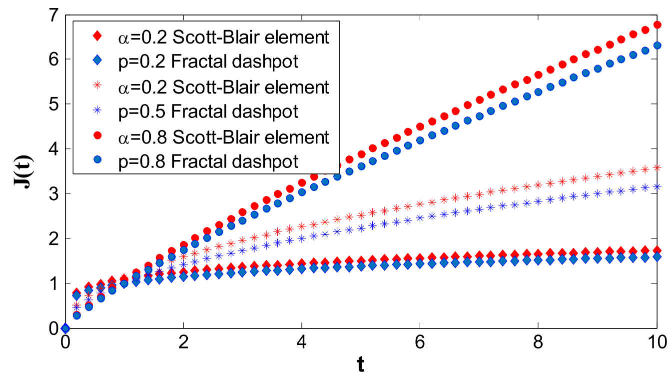

Fig. 2. The comparison of creep modulus between a fractal dashpot and a Scott-Blair element with varying fractal or fractional orders (η=1).

displayed in Fig. 2. It can be concluded from Fig. 2 that the creep modulus increases with increasing fractal orders. The tendency of the curves of such two elements is similar, which indicates the capability of the fractal dashpot to describe the power-law behavior...

By connecting a fractal dashpot in series with a spring element, the fractal Maxwell model is obtained, of which the basic equations can be given by

$$\begin{cases} \sigma_e = E\varepsilon_e,\ \sigma_f = \eta \frac{\mathrm{d}\varepsilon_f}{\mathrm{d}t^p} \\ \varepsilon = \varepsilon_e + \varepsilon_f,\ \sigma = \sigma_e = \sigma_f \end{cases}, \tag{5}$$

where the subscripts e and f represent respectively the elastic and the fractal elements, E and η are Young's modulus and viscous coefficient, respectively. The constitutive equation of the fractal model is formulated by

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t^p} = \frac{1}{E}\frac{\mathrm{d}\sigma}{\mathrm{d}t^p} + \frac{\sigma}{\eta} \tag{6}$$

Similarly, on the basis of the above derivation, it is convenient to obtain the constitutive relationship equation of the fractal Kelvin model in the same manner, that is

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t^p} + \frac{E}{\eta}\varepsilon = \frac{\sigma}{\eta}. \tag{7}$$

2.3. Derivation of creep modulus and relaxation compliance

In this section, we provide two means to derive the creep modulus and relaxation compliance for the proposed fractal models and compare them with those for classical and fractional models.

Firstly, the methodology of the scaling transformation $\hat{t} = t^p$ is applied to Eq. (6) [15] to derive the creep modulus and relaxation compliance of the fractal Maxwell model. Consequently, Eq. (6) is transformed into the form of the classical Maxwell model. In doing so, the creep modulus and relaxation compliance for the fractal Maxwell model are easily achieved. The creep modulus is formulated by

$$J(t) = \frac{1}{E} + \frac{t^p}{\eta}, \tag{8}$$

as well as the relaxation compliance is as follows

$$G(t) = Ee^{-t^p E/\eta}. \tag{9}$$

Similarly, the creep modulus and relaxation compliance for the fractal Kelvin model are written as

$$J(t) = \frac{1}{E}(1 - e^{-Et^p/\eta}), \tag{10}$$

$$G(t) = E + \eta\delta(t^p). \tag{11}$$

To verify the above derivation, we employ the following relationship

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t^p} = \frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t^p} = \frac{\dot{f}(t)}{pt^{p-1}}, \tag{12}$$

for fractal Maxwell and Kelvin constitutive relationships. Considering the fractal Maxwell model, Eq. (6) can be reformulated as

$$\frac{\dot{\varepsilon}}{pt^{p-1}} = \frac{1}{E}\frac{\dot{\sigma}}{pt^{p-1}} + \frac{\sigma}{\eta}. \tag{13}$$

To derive the creep modulus, the stress is set to be a constant. So Eq. (13) can be rewritten as

$$\frac{\dot{\varepsilon}}{pt^{p-1}} = \frac{\sigma}{\eta}. \tag{14}$$

Then

$$\int \dot{\varepsilon}dt = \int \frac{\sigma}{\eta} pt^{p-1}dt. \tag{15}$$

Finally, by incorporating the initial value $\varepsilon_0 = \sigma/E$, we can obtain the same formulation of creep modulus for fractal Maxwell model as Eq. (8). The rest of the formulations shown in Eqs. (9–11) can also be verified in the same way.

The above derived creep modulus and relaxation compliance are compared with those of the existing classical and fractional models, as displayed in Table 1. It is observed that the fractal and fractional Maxwell models have almost the same creep modulus. It can also be seen in Table 1 that the creep modulus of the classical Kelvin model is a special case of the fractal Kelvin model. The comparisons between the classical and the fractal Kelvin model are presented in Fig. 3. The achieved relaxation compliance of the fractal Maxwell model is also found to be similar to the well-known stretched exponential expression [26]..

It can be concluded from above that the creep modulus and relaxation compliance of the fractal models can be achieved from those of the classical models via replacing t with t^p. To validate such point of view, we consider another fundamental viscoelastic model, the Zener model, as displayed in Fig. 4. It is more appropriate to represent responses of relaxation and creep of viscoelastic solids. The corresponding fractal constitutive relationship is

$$E_1E_2\varepsilon + E_2\eta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t^p} = (E_1 + E_2)\sigma + \eta\frac{\mathrm{d}\sigma}{\mathrm{d}t^p}. \tag{16}$$

.

Adopting Eq. (12), we can easily obtain the relaxation compliance

$$G(t) = \frac{E_1E_2}{E_1 + E_2} + \frac{E_2^2}{E_1 + E_2}\exp(-(E_1 + E_2)t^\alpha/\eta), \tag{17}$$

and the creep modulus

$$J(t) = \frac{1}{E_2} + \frac{1}{E_1}(1 - \exp(-E_1t^\alpha/\eta)). \tag{18}$$

Comparing Eqs. (17) and (18) with those from the classical Zener model and recalling the aforementioned conclusions, the fractal models are found to be feasible to characterize the power-law rheological phenomena with the scaling transformation t^p.

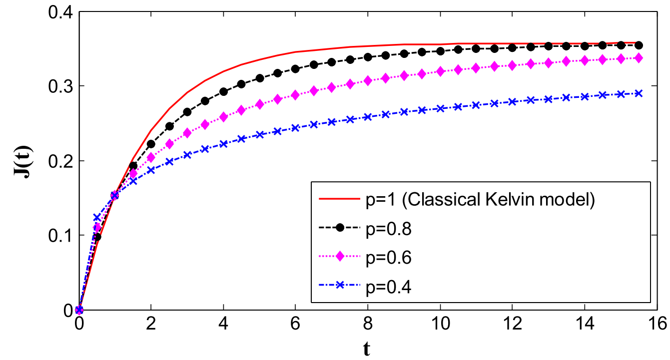

Fig. 3. The comparison between the classical and fractal Kelvin models with the same parameters (E=2.8, η=5).

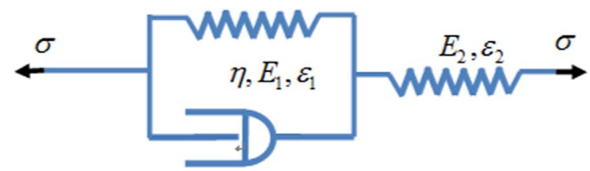

Fig. 4. The schematic diagram for the fractal Zener model.

3. Numerical simulations and analysis

Geomaterials are considered as typical kinds of viscoelastic media with various constituents, and have attracted much attention thanks to their extremely importance in sciences and engineering. Bagley and Torivk [27] assumed salt rock to be viscoelastic. Yin et al. [28,29] investigated the time-dependent property of muddy clay and soft soils. In this section, numerical results comparing with experimental data of geomaterials are displayed below to test the efficacy of the proposed fractal derivative model.

As a widely used laboratory testing method, the triaxial test has been successfully used to obtain strength parameters for a variety of geomaterials. Assuming that the vertical strain ε only results from deviatoric stress σ_1–σ_3, the formula for creep of geomaterials can be obtained through substituting σ_1–σ_3 and ε for σ and ε in the constitutive equation derived above [30]. A series of triaxial creep experiments on geomaterials have been completed by Zhang et al. [31], Yang et al. [32] and Cristescu [33], and. In order to validate the fractal derivative formulation derived above, we first fit these creep tests by using Eqs. (8) and (10). Fig. 5(a)–(c) show that the derived creep moduli are in a good simulation of the measured stress-time relationship for creep..

Subsequently, Eqs. (8) and (10) are utilized to compare with fractional derivative models, as displayed in Fig. 6 [34]. From analytical analysis and the comparisons of the simulation results, it can be concluded from Fig. 6 that the fractal Maxwell model can reach the same accuracy as the fractional Maxwell model, with the same

Table 1
The comparison between fractal, classical integer-order and fractional derivative models.

		Integer-order derivative model	Fractal derivative model	Fractional derivative model
Maxwell	Creep	$J(t) = \frac{1}{E} + \frac{t}{\eta}$	$J(t) = \frac{1}{E} + \frac{t^p}{\eta}$	$J(t) = \frac{1}{E} + \frac{t^\alpha}{\eta\Gamma(1+\alpha)}$
	Relaxation	$G(t) = Ee^{-tE/\eta}$	$G(t) = Ee^{-t^pE/\eta}$	$G(t) = EE_{\alpha,1}(-\frac{E}{\eta}t^\alpha)$
Kelvin	Creep	$J(t) = \frac{1}{E}(1 - e^{-Et/\eta})$	$J(t) = \frac{1}{E}(1 - e^{-Et^p/\eta})$	$J(t) = \frac{1}{\eta}t^\alpha E_{\alpha,\alpha+1}(-\frac{E}{\eta}t^\alpha)$
	Relaxation	$G(t) = E + \eta\delta(t)$	$G(t) = E + \eta\delta(t^p)$	$G(t) = E + \eta\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$

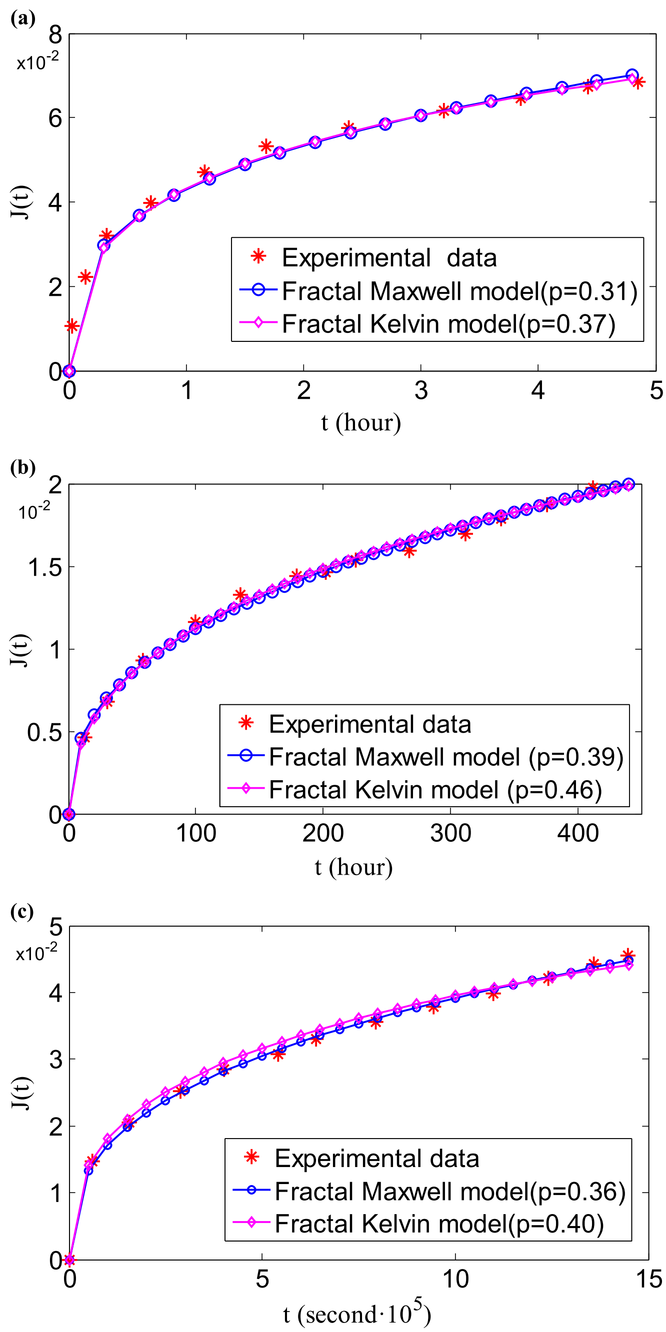

Fig. 5. (a) Frozen soil with σ_1=6.5 MPa and σ_3=2.75 MPa (Experimental data adopted from Ref. [31]). (b) Salt rock with σ_1=28.7 MPa and σ_3=7.18 MPa (Experimental data adopted from Ref. [32]). (c) Rock salt with σ_1=21.4 MPa and σ_3=0.7 MPa (Experimental data adopted from Ref. [33]).

number of parameters. It can also be observed from Figs. 5 and 6 that the parameters p for both the fractal Maxwell and Kelvin models are almost the same, while such conclusion is not suitable for the fractional models as seen in Fig. 6. The parameter p is suspected to be an inherent property related to the fractal of a given material, as the fractal derivative is proposed based on the scaling transformation. It deserves further explorations..

Finally, the fractal Zener model, Eq. (18), is employed to compare with classical Zener model and five-element model [35]. The creep modulus for five-element model is formulated as

$$J(t) = \frac{1}{E_1} + \frac{1}{E_2}(1 - e^{-\frac{E_2}{\eta_1}t}) + \frac{1}{E_3}(1 - e^{-\frac{E_3}{\eta_2}t}), \tag{19}$$

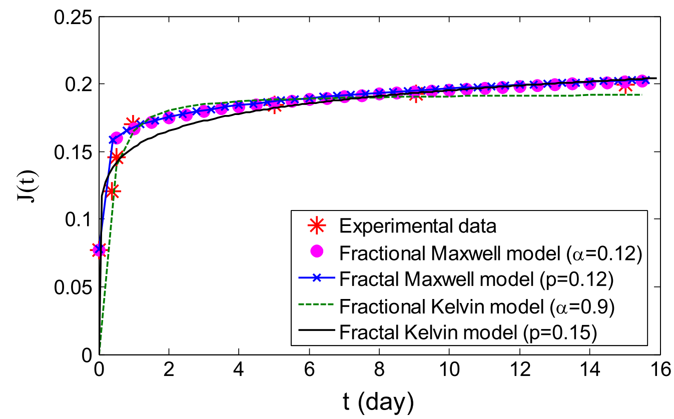

Fig. 6. The comparison between the fractal models and fractional models for experimental data of soft soil (Experimental data adopted from Ref. [34]).

Table 2
Best-fitting parameters and relative error of classical and fractal creep modulus for the experimental test of concrete B 300.

E_1	E_2	η_1	E_1	η_2	p	Relative error
0.68	5.9	6.22			0.5	0.0296
0.78	5.9	36.92				0.1287
5.7	5.7	0.75	0.35	17.50		0.0609

where E_1, E_2 and E_3 are Young's modulus, η_1 and η_2 are viscous coefficient. The fitting parameters are displayed in Table 2. To estimate the difference between the experimental data and the fitting curve, the relative error had better be given, which is defined as

$$\text{Rerr}=\sqrt{\frac{\sum_{k=1}^{n}(J_{0k} - J_k)^2}{\sum_{k=1}^{n}J_k^2}}, \tag{20}$$

where J_{Ok} and J_k represent the numerical and experimental data at different points k, and n denotes the total number of these points. From Fig. 7 and Table 2, it can be concluded that the fractal model enjoy particular advantages of fewer parameters, simpler mathematical expression and higher accuracy. To achieve the same accuracy as the proposed model, more elements are required for the classical models of integer-order derivative, such as five-element model..

The adopted experimental data vary from 5 h (Fig. 5(a)) to 1200 days (Fig. 7), which validates the applicability of the proposed models for characterizing the rheology behavior of viscoelastic materials in a large time scale.

As an alternative modeling formalism, the fractal derivative is a local operator and demands far less amount of calculation and memory

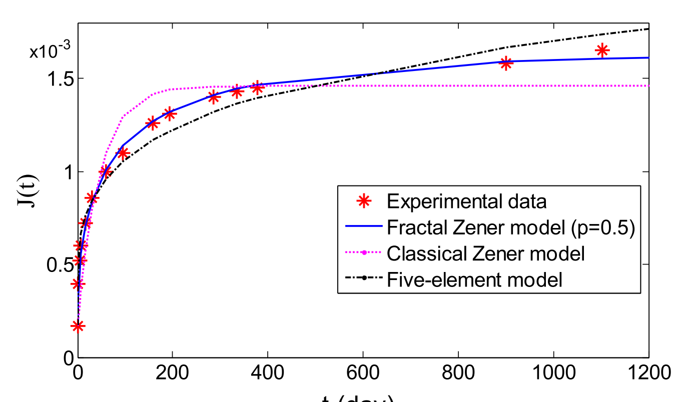

Fig. 7. The comparison between fractal and integer-order models for creep of concrete B 300 (Experimental data adopted from [35]).

Table 3
The computational time for the fractal and fractional Maxwell models (E=1, η=1, T=8). (Intel Core i5, RAM 4 GB, 64 bit windows 7 and MATLAB 2014a).

Maxwell model	order α=p	Time (s)		
		Δt=0.01	Δt=0.001	Δt=0.0001
Fractal	1	0.024	0.035	0.135
Fractional		0.041	1.448	135.666
Fractal	0.9	0.023	0.038	0.149
Fractional		0.141	11.286	1118.027

Table 4
The computational time for the fractal and fractional Kelvin models. (E=1, η=1, T=8) (Intel Core i5, RAM 4 GB, 64 bit windows 7 and MATLAB 2014a).

Kelvin model	order α=p	Time (s)		
		Δt=0.01	Δt=0.001	Δt=0.0001
Fractal	1	0.023	0.031	0.096
Fractional		0.043	1.609	155.105
Fractal	0.9	0.026	0.033	0.113
Fractional		0.137	11.478	1148.414

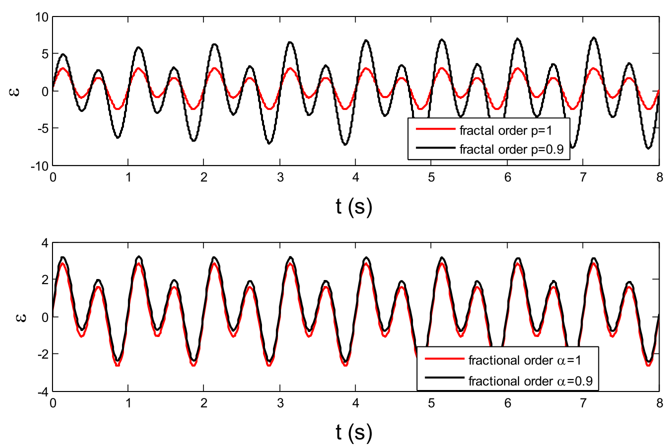

Fig. 8. The response of fractal and fractional Maxwell models subjected to complex load.

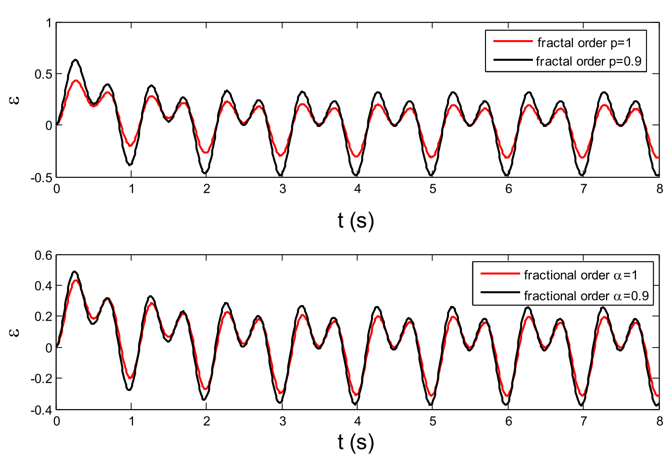

Fig. 9. The response of fractal and fractional Kelvin models subjected to complex load.

storage. To highlight the numerical advantages of the proposed model, the fractal Maxwell or Kelvin type of viscoelastic material is assumed to be subjected to the following complex loading

$$\sigma(t) = \sigma_1 \sin(2\pi f_1 t) + 2\sigma_1 \sin(4\pi f_1 t). \tag{21}$$

For the sake of convenience, σ_1 and f_1 in Eq. (21) are set to be 1. Thus, the fractal derivative of the loading is

$$\frac{d\sigma}{dt^p} = \frac{2\pi\cos(2\pi t) + 8\pi\cos(4\pi t)}{pt^{p-1}}, \tag{22}$$

and the fractional derivative of the loading in the sense of Riemann-Liouville can be formulated as [25]

$$\frac{d^\alpha\sigma}{dt^\alpha} = (2\pi)^\alpha \sin\left(2\pi t + \frac{\pi\alpha}{2}\right) + 2(4\pi)^\alpha \sin\left(4\pi t + \frac{\pi\alpha}{2}\right). \tag{23}$$

It can be observed from Tables 3 and 4 that the fractal models require less calculation time than the fractional models, especially when the time step is really small. The corresponding dynamic responses for such two different models are displayed in Figs. 8 and 9...

4. Conclusions

This paper provides an alternative modeling formalism, the fractal derivative, to characterize the creep of viscoelastic materials. The fractal dashpot was introduced to characterize viscoelastic behavior. It was found that the creep modulus of the fractal dashpot increased with increasing fractal orders. Also, comparisons of creep modulus between a fractional derivative Scott-Blair element and a fractal derivative dashpot showed the similar tendency of the curve. It indicates that the fractal dashpot can be implemented to describe the power-law behavior.

The fractal Maxwell and Kelvin models were achieved with the combination of a fractal dashpot and a spring element. By means of the methodology of scaling transformation, we derived the corresponding creep modulus and relaxation compliance. It is worthy to note that, for fractal Maxwell model, the creep modulus and relaxation compliance are similar to the creep modulus of fractional Maxwell model and the well-known stretched exponential expression, respectively. The creep modulus of the classical Kelvin model was also found to be a special case of the fractal Kelvin model. It seems that the fractal model could serve as a bridge between the classical model and the fractional model by means of scaling transformation.

The good data fitting of the fractal Maxwell and Kelvin models showed the efficacy of the proposed models for characterizing time-dependent viscoelastic materials. Simulation results presented that the fractal and fractional Maxwell models seemed to reach the same accuracy. It deserves to be mentioned that the fitting parameters p for both the fractal Maxwell and Kelvin models are almost the same, which is supposed to be an inherent property of the given material. However, the fractal derivative is a local operator and requires far lower computation time than the fractional models in numerical simulation. The proposed models were also found to have the edges over the traditional models of integer-order derivative thanks to the fewer parameters, simpler mathematical expression and higher accuracy.

Acknowledgements

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