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A counterexample on tropical linear spaces

Yaroslav Shitov

National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia

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ABSTRACT

Let K be a field with non-Archimedean valuation v, and assume A is a matrix of size $m \times n$ and rank k over K. Richter-Gebert, Sturmfels, and Theobald proved that the rows of A are a tropical basis of the corresponding linear space whenever $m = k$ and any submatrix formed by k columns of $v(A)$ has tropical rank k. We show that this result is no longer true without assumption $m = k$, refuting the conjecture proposed by Yu and Yuster in 2007.

The basic concept of tropical geometry is the *tropical semiring*, which is the set $\mathbb{R} \cup \{\infty\}$ equipped with operations $\oplus : (a, b) \to \min\{a, b\}$ and $\odot : (a, b) \to a + b$. Various problems of tropical geometry arise from classical algebraic geometry over a field K with non-Archimedean valuation. A standard example of such a field K is the set of generalized Puiseux series with operations of formal addition and multiplication [1]. Recall that the elements of K are formal sums $a(t) = \sum_{e \in \mathbb{R}} a_e t^e$ with coefficients a_e in $\mathbb{C}$ and well-ordered *support* $\operatorname{Supp}(a) = \{e \in \mathbb{R} : a_e \neq 0\}$. The *tropicalization* mapping $v : K^* \to \mathbb{R}$ sends a series a to the exponent of its *leading term*; in other words, we define $v(a) = \min \operatorname{Supp}(a)$ and $v(0) = \infty$.

E-mail address: yaroslav-shitov@yandex.ru.

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Let $U \in K^n$ be an affine variety arising as a locus of zeros of polynomials from a set $S \subset K[x_1, \ldots, x_n]$. The *tropical variety* of S is defined as the tropicalization $v(U)$ and denoted by $\mathcal{T}(S)$. If S consists of a single polynomial f, the corresponding tropical variety is referred to as a *tropical hypersurface*. The *fundamental theorem of tropical varieties* [3] states that $\mathcal{T}(S) = \bigcap_{f \in I} \mathcal{T}(f)$, where I is the ideal generated by S. If S is a finite subset and the corresponding tropical variety is cut out by hypersurfaces of polynomials from S, then S is called a *tropical basis* of I. In other words, S is a tropical basis whenever $\mathcal{T}(S) = \bigcap_{f \in S} \mathcal{T}(f)$. Also, we say that the variety $\mathcal{T}(S)$ is a *tropical linear space* if all the polynomials from S are linear forms. Note that Speyer [2] gives a combinatorial definition of this concept, which turns out to be more general; the object of our study is actually what Speyer refers to as a *realizable* linear space.

Richter-Gebert, Sturmfels, and Theobald [1] gave a sufficient condition for a set of linear forms to be a tropical basis. In order to recall their result, we need to define the concept of the tropical rank of a matrix. Let X be an $m \times n$ matrix whose entries are variables, and denote by J_k the set of determinants of all $k \times k$ submatrices of X. The *tropical rank* of an $m \times n$ tropical matrix C is the largest integer r such that $C \notin \bigcap_{f \in J_r} \mathcal{T}(f)$. In the following theorem, we think of a vector $(w_1, \ldots, w_n) \in K^n$ as a linear form $w_1 x_1 + \ldots + w_n x_n \in K[x_1, \ldots, x_n]$. The following result is a part of Theorem 5.3 from [1].

Theorem 1. *[1] Let A be an $m \times n$ matrix of rank k over K, assume that any submatrix formed by k columns of $v(A)$ has tropical rank k. If $m = k$, then the rows of A form a tropical basis.*

Is this result true without the assumption $m = k$? Yu and Yuster conjectured a positive answer to this question.

Conjecture 2. *[4, Conjecture 13] Let L be an $(n-k)$-dimensional linear subspace in K^n all of whose Plücker coordinates are non-zero. Let $M \in K^{m \times n}$ be a matrix whose rows are non-zero elements in the orthogonal complement of L. If any k columns of $v(M)$ have tropical rank k, then the rows of M form a tropical basis.*

We construct the counterexample as follows. Let a linear space $L \subset K^5$ consist of all vectors $(x_1, \ldots, x_5)$ satisfying $x_1 = \ldots = x_5$. Therefore, the tropicalization $v(L)$ also consists of vectors with all coordinates equal. It is clear from its definition that the Plücker coordinates of L are non-zero. We define the matrix M as

$$\begin{pmatrix} -t & 1 & -1 & t & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 1-t^2 & -1 & 0 & t^2 & 0 \end{pmatrix},$$

and we note that sum of the entries in every row is zero. In other words, the rows of M belong to the orthogonal complement of L. The matrix $C = v(M)$ equals

$$\begin{pmatrix} 1 & 0 & 0 & 1 & \infty \\ \infty & 0 & 0 & \infty & \infty \\ 0 & \infty & \infty & 0 & 0 \\ \infty & \infty & \infty & 0 & 0 \\ 0 & 0 & \infty & 2 & \infty \end{pmatrix},$$

and we note that the cofactors C_{15}, C_{14}, C_{33}, C_{32}, C_{31} have full tropical rank. Let us check that the rows of M are not a tropical basis; we do it by checking that the vector $\alpha = (1, 1, 1, 0, 0) \notin v(L)$ belongs to the tropical hypersurface of every row of M. Indeed, we define $\beta_1 = (t, t + t^2, 2t, 1, 1)$, $\beta_2 = \beta_4 = (t, t, t, 1, 1)$, $\beta_3 = (t, t, t, t - 1, t + 1)$, $\beta_5 = (t, t + t^2, t, 1 + t, 1)$, and we can check that every β_i is orthogonal to the ith row of M and satisfies $v(\beta_i) = \alpha$.

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