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Research paper

A generalized super AKNS hierarchy associated with Lie superalgebra $sl(2|1)$ and its super bi-Hamiltonian structure

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ABSTRACT

Starting from a 3×3 matrix-valued spectral problem associated with a Lie superalgebra $sl(2|1)$, a generalized super Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy is derived. The resulting super AKNS hierarchy has a super bi-Hamiltonian structure by the supertrace identity.

1. Introduction

Searching for new soliton hierarchies plays an important role in the soliton and integrable systems. Matrix spectral problem or Lax pair is a crucial key to construct soliton hierarchies. And the trace identity provides a powerful method to construct Hamiltonian structures of the resulting soliton hierarchies. In what follows, let us recall the standard procedure for constructing soliton hierarchies. Suppose a given spatial spectral problem as

$$\phi_x = U\phi, \quad U = U(u,\lambda) \in \tilde{g}, \tag{1}$$

where $\tilde{g}$ is a matrix loop algebra, u is a potential and λ is a spectral parameter. We solve the stationary equation

$$V_x = [U, V],$$

where

$$V = V(u,\lambda) = \sum_{i\geq 0} V_i\lambda^{-i}, \quad V_i \in \tilde{g}, \quad i \geq 0.$$

Then, we formulate the temporal spectral problems:

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u,\lambda)\phi, \tag{2}$$

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where

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n, \quad \Delta_n \in \tilde{g}, \quad n \geq 0,$$

with P_+ means the polynomial part of P in λ. The compatibility conditions of (1) and (2), i. e. the zero curvature equations, are given by

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geq 0, \tag{3}$$

which will engender a hierarchy of soliton equations

$$u_{t_n} = K_n(u), \quad n \geq 0. \tag{4}$$

With the aid of the trace identity [1]:

$$\frac{\delta}{\delta u} \int tr\left(\frac{\partial U}{\partial \lambda} V\right) dx = \lambda^{-s} \frac{\partial}{\partial \lambda} \lambda^s tr\left(\frac{\partial U}{\partial u} V\right), \tag{5}$$

the obtained soliton hierarchy (4) has the following Hamiltonian structure:

$$u_{t_n} = K_n(u) = J \frac{\delta H_n}{\delta u}, \quad n \geq 0, \tag{6}$$

where J is a Hamiltonian operator and all of the H_n are Hamiltonian functions. The Hamiltonian structures of some famous soliton hierarchies (such as the AKNS hierarchy [2], the Kaup–Newell (KN) hierarchy [3], the Wadati–Konno–Ichikawa (WKI) hierarchy [4], the Boiti–Pempinelli–Tu (BPT) hierarchy [5], and so on) are constructed in Ref. [1].

This method is usually called the Tu scheme, which has successfully applied to the super spectral problems. Successful examples include the super AKNS hierarchy [6,7], the super Dirac hierarchy [6,8], the super coupled Korteweg–de Vires (cKdV) hierarchy [9], the super KN hierarchy [10,11], etc. [12–14]. And in these references, their super Hamiltonian structures are respectively furnished by the supertrace identity.

In recent years, generalized hierarchies of the classical soliton equations have been widely studied by many researchers. For example, the generalized AKNS hierarchy [15–17], the generalized KN hierarchy [18], the generalized WKI hierarchy [19] and so forth. In very recent years, Grahovski and Mikhailov proposed a new super soliton equation (s-cNLS) with two boson variables and two fermi variables [20]. Zhou has showed that the s-cNLS equation is actually a member of the $sl(2|1)$ super AKNS hierarchy [21]. Here we shall consider a generalization of the super AKNS hierarchy related to a Lie superalgebra $sl(2|1)$.

The paper is organized as follows. In the next section, we shall derive a generalized super AKNS hierarchy associated with a Lie superalgebra $sl(2|1)$. Then in Section 3, the resulting generalized super AKNS hierarchy can be written as the super bi-Hamiltonian structure by making use of the supertrace identity. Some conclusions and discussions are listed in the last section.

2. A generalized $sl(2|1)$ super AKNS hierarchy

Let us start with the following matrix-valued spectral problem associated with a Lie superalgebra $sl(2|1)$:

$$\phi_x = U(u,\lambda)\phi, \quad U(u,\lambda) = \begin{pmatrix} \lambda + r & p & \alpha \\ q & -\lambda - r & \beta \\ \gamma & \zeta & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{7}$$

where $u = (p, q, \alpha, \beta, \gamma, \zeta)^T$ is a potential, λ is a spectral parameter, p and q are even potentials, and α, β, γ and ζ are odd potentials. Here we note that $r = \varepsilon(pq + \alpha\gamma + \beta\zeta)$ with ε is an arbitrary even constant.

Remark 1.

(1) When $\varepsilon = 0$ in (7), the spectral matrix U is exactly one of the spectral problem (1) in Ref. [21] by proper variable substitutions.
(2) When $\varepsilon = 0$, $\gamma = \beta$ and $\zeta = -\alpha$ in (7), the spectral matrix U is exactly one of the super AKNS case. For detail, we can refer to the references [6,7,22–24].

Therefore, the spatial spectral problem (7) is regarded as an extension of the $sl(2|1)$ super AKNS spectral problem. To derive the generalized hierarchies of equations, we solve the stationary zero curvature equation

$$V_x = [U, V], \tag{8}$$

where

$$V = \begin{pmatrix} A & B & \rho \\ C & E - A & \delta \\ \xi & \eta & E \end{pmatrix}, \tag{9}$$

with A, B, C, E are even variables, ρ, δ, ξ, η are odd variables. A direct calculation leads to

$$\begin{cases} A_x = pC - qB + \alpha\xi + \gamma\rho, \\ B_x = 2(\lambda + r)B + p(E - 2A) + \alpha\eta + \zeta\rho, \\ C_x = -2(\lambda + r)C - q(E - 2A) + \beta\xi + \gamma\delta, \\ E_x = \alpha\xi + \beta\eta + \gamma\rho + \zeta\delta, \\ \rho_x = (\lambda + r)\rho + p\delta + \alpha(E - A) - \beta B, \\ \delta_x = -(\lambda + r)\delta + \beta A + q\rho - \alpha C, \\ \xi_x = -(\lambda + r)\xi - \gamma(E - A) - q\eta + \zeta C, \\ \eta_x = (\lambda + r)\eta - \zeta A + \gamma B - p\xi. \end{cases} \tag{10}$$

Upon setting

$$\begin{aligned} &A = \sum_{j\geq 0} a_j\lambda^{-j}, \quad B = \sum_{j\geq 0} b_j\lambda^{-j}, \quad C = \sum_{j\geq 0} c_j\lambda^{-j}, \quad E = \sum_{j\geq 0} e_j\lambda^{-j}, \\ &\rho = \sum_{j\geq 0} \rho_j\lambda^{-j}, \quad \delta = \sum_{j\geq 0} \delta_j\lambda^{-j}, \quad \xi = \sum_{j\geq 0} \xi_j\lambda^{-j}, \quad \eta = \sum_{j\geq 0} \eta_j\lambda^{-j}, \end{aligned} \tag{11}$$

and balancing the coefficients of the same powers of λ in Eq. (10), we obtain

$$\begin{cases} b_0 = c_0 = \rho_0 = \delta_0 = \xi_0 = \eta_0 = 0, \\ a_{j,x} = pc_j - qb_j + \alpha\xi_j + \gamma\rho_j, \quad j \geq 0, \\ b_{j+1} = \frac{1}{2}b_{j,x} - rb_j - \frac{1}{2}pe_j + pa_j - \frac{1}{2}\alpha\eta_j - \frac{1}{2}\zeta\rho_j, \quad j \geq 0, \\ c_{j+1} = -\frac{1}{2}c_{j,x} - rc_j + \frac{1}{2}\beta\xi_j - \frac{1}{2}qe_j + qa_j + \frac{1}{2}\gamma\delta_j, \quad j \geq 0, \\ e_{j,x} = \alpha\xi_j + \beta\eta_j + \gamma\rho_j + \zeta\delta_j, \quad j \geq 0, \\ \rho_{j+1} = \rho_{j,x} - r\rho_j - p\delta_j - \alpha(e_j - a_j) + \beta b_j, \quad j \geq 0, \\ \delta_{j+1} = -\delta_{j,x} - r\delta_j + q\rho_j + \beta a_j - \alpha c_j, \quad j \geq 0, \\ \xi_{j+1} = -\xi_{j,x} - r\xi_j + \gamma(a_j - e_j) + \zeta c_j - q\eta_j, \quad j \geq 0, \\ \eta_{j+1} = \eta_{j,x} - r\eta_j - \gamma b_j + \zeta a_j + p\xi_j, \quad j \geq 0, \end{cases} \tag{12}$$

which gives rise to a recursive relationship

$$\begin{pmatrix} c_{j+1} \\ b_{j+1} \\ \xi_{j+1} \\ \eta_{j+1} \\ -\rho_{j+1} \\ -\delta_{j+1} \end{pmatrix} = L_1 \begin{pmatrix} c_j \\ b_j \\ \xi_j \\ \eta_j \\ -\rho_j \\ -\delta_j \end{pmatrix}, \quad j \geq 0, \tag{13}$$

where the recursive operator L_1 is given by

$$L_1 = \begin{pmatrix} -\frac{1}{2}\partial - r + q\partial^{-1}p & -q\partial^{-1}q & \frac{1}{2}\beta + \frac{1}{2}q\partial^{-1}\alpha & -\frac{1}{2}q\partial^{-1}\beta & -\frac{1}{2}q\partial^{-1}\gamma & -\frac{1}{2}\gamma + \frac{1}{2}q\partial^{-1}\zeta \\ p\partial^{-1}p & \frac{1}{2}\partial - r - p\partial^{-1}q & \frac{1}{2}p\partial^{-1}\alpha & -\frac{1}{2}\alpha - \frac{1}{2}p\partial^{-1}\beta & \frac{1}{2}\zeta - \frac{1}{2}p\partial^{-1}\gamma & \frac{1}{2}p\partial^{-1}\zeta \\ \zeta + \gamma\partial^{-1}p & -\gamma\partial^{-1}q & -\partial - r & -q - \gamma\partial^{-1}\beta & 0 & \gamma\partial^{-1}\zeta \\ \zeta\partial^{-1}p & -\gamma - \zeta\partial^{-1}q & p + \zeta\partial^{-1}\alpha & \partial - r & -\zeta\partial^{-1}\gamma & 0 \\ -\alpha\partial^{-1}p & -\beta + \alpha\partial^{-1}q & 0 & \alpha\partial^{-1}\beta & \partial - r & -p - \alpha\partial^{-1}\zeta \\ \alpha - \beta\partial^{-1}p & \beta\partial^{-1}q & -\beta\partial^{-1}\alpha & 0 & q + \beta\partial^{-1}\gamma & -\partial - r \end{pmatrix},$$

with $\partial = \frac{\partial}{\partial x}$ and ∂^{-1} satisfies the equality $\partial\partial^{-1} = \partial^{-1}\partial = 1$. It is easy to find that $a_{0,x} = e_{0,x} = 0$. Explicitly, by taking the initial value conditions $a_0 = 1$ and $e_0 = 0$, and all constants of integration are chosen as zero (i. e. $a_j|_{u=0} = b_j|_{u=0} = c_j|_{u=0} = e_j|_{u=0} = \rho_j|_{u=0} = \delta_j|_{u=0} = \xi_j|_{u=0} = \eta_j|_{u=0} = 0$ $(j \geq 1)$), the first three sets can be computed as follows:

$$\begin{aligned} b_1 &= p, \quad c_1 = q, \quad \rho_1 = \alpha, \quad \delta_1 = \beta, \quad \xi_1 = \gamma, \quad \eta_1 = \zeta, \quad a_1 = e_1 = 0, \\ b_2 &= \frac{1}{2}p_x - rp, \quad c_2 = -\frac{1}{2}q_x - rq, \quad \rho_2 = \alpha_x - r\alpha, \quad \delta_2 = -\beta_x - r\beta, \\ \xi_2 &= -\gamma_x - r\gamma, \quad \eta_2 = \zeta_x - r\zeta, \quad a_2 = -\frac{1}{2}pq - \alpha\gamma, \quad e_2 = \gamma\alpha + \beta\zeta, \\ b_3 &= \frac{1}{4}p_{xx} - \frac{1}{2}r_xp - rp_x + r^2p + \frac{1}{2}p\gamma\alpha - \frac{1}{2}p\beta\zeta - \frac{1}{2}p^2q - \frac{1}{2}\alpha\zeta_x - \frac{1}{2}\zeta\alpha_x, \\ c_3 &= \frac{1}{4}q_{xx} + \frac{1}{2}r_xq + rq_x + r^2q - \frac{1}{2}\beta\gamma_x + \frac{1}{2}q\gamma\alpha - \frac{1}{2}q\beta\zeta - \frac{1}{2}pq^2 - \frac{1}{2}\gamma\beta_x, \\ \rho_3 &= \alpha_{xx} - r_x\alpha - 2r\alpha_x + r^2\alpha + p\beta_x - \alpha\beta\zeta - \frac{1}{2}pq\alpha + \frac{1}{2}p_x\beta, \\ \delta_3 &= \beta_{xx} + r_x\beta + 2r\beta_x + r^2\beta + q\alpha_x - \frac{1}{2}pq\beta + \alpha\beta\gamma + \frac{1}{2}q_x\alpha, \\ \xi_3 &= \gamma_{xx} + r_x\gamma + 2r\gamma_x + r^2\gamma - \frac{1}{2}pq\gamma - \frac{1}{2}q_x\zeta - q\zeta_x + \beta\gamma\zeta, \end{aligned}$$

$$\eta_3 = \zeta_{xx} - r_x\zeta - 2r\zeta_x + r^2\zeta - \frac{1}{2}p_x\gamma - p\gamma_x - \frac{1}{2}pq\zeta - \alpha\gamma\zeta,$$
$$a_3 = \frac{1}{4}(pq_x - p_xq) + r(pq + 2\alpha\gamma) + \frac{1}{2}p\gamma\beta + \frac{1}{2}q\zeta\alpha + \alpha\gamma_x - \alpha_x\gamma,$$
$$e_3 = \alpha\gamma_x - \alpha_x\gamma + 2r(\alpha\gamma + \zeta\beta) + p\gamma\beta + q\zeta\alpha + \beta\zeta_x - \beta_x\zeta.$$

Then, introduce the temporal spectral problems as

$$\phi_{t_n} = V^{[n]}\phi, \tag{14}$$

where

$$V^{[n]} = \sum_{j=0}^{n}\begin{pmatrix} a_j & b_j & \rho_j \\ c_j & e_j - a_j & \delta_j \\ \xi_j & \eta_j & e_j \end{pmatrix}\lambda^{n-j} + \Delta_n, \quad n \ge 0,$$

with Δ_n being the modification terms, which didn't appear in the $sl(2|1)$ super AKNS case (compare with (19) in Ref. [21]). Assuming $\Delta_n = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & k \end{pmatrix}$, the compatibility conditions of the spectral problems (7) and (14) yield to the following zero curvature equations

$$U_{t_n} - V_x^{[n]} + [U, V^{[n]}] = 0, \quad n \ge 0. \tag{15}$$

Making use of (12), we arrive at

$$\begin{cases} r_{t_n} = a_x = -d_x, \quad b = c = e = f = g = h = 0, \quad k_x = 0, \\ p_{t_n} = b_{n,x} - 2rb_n - p(e_n - 2a_n) - \alpha\eta_n - \zeta\rho_n + p(a-d) = 2b_{n+1} + p(a-d), \\ q_{t_n} = c_{n,x} + 2rc_n + q(e_n - 2a_n) - \beta\xi_n - \gamma\delta_n - q(a-d) = -2c_{n+1} - q(a-d), \\ \alpha_{t_n} = \rho_{n,x} - r\rho_n - p\delta_n + \alpha(a_n - e_n) + \beta b_n + \alpha(a-k) = \rho_{n+1} + \alpha(a-k), \\ \beta_{t_n} = \delta_{n,x} - q\rho_n + r\delta_n + \alpha c_n - \beta a_n + \beta(d-k) = -\delta_{n+1} + \beta(d-k), \\ \gamma_{t_n} = \xi_{n,x} - \gamma(a_n - e_n) - \zeta c_n + r\xi_n + q\eta_n - \gamma(a-k) = -\xi_{n+1} - \gamma(a-k), \\ \zeta_{t_n} = \eta_{n,x} - \gamma b_n + \zeta a_n + p\xi_n - r\eta_n - \zeta(d-k) = \eta_{n+1} - \zeta(d-k). \end{cases} \tag{16}$$

After choosing $d = -a$ and $k = 0$, we obtain the following identity:

$$(pq + \alpha\gamma + \beta\zeta)_{t_n} = 2qb_{n+1} - 2pc_{n+1} - \gamma\rho_{n+1} - \alpha\xi_{n+1} + \zeta\delta_{n+1} + \beta\eta_{n+1} = (-2a_{n+1} + e_{n+1})_x.$$

So, we take $a = \varepsilon(-2a_{n+1} + e_{n+1})$. And the generalized hierarchy of equations is given by as follows:

$$u_{t_n} = \begin{pmatrix} p_{t_n} \\ q_{t_n} \\ \alpha_{t_n} \\ \beta_{t_n} \\ \gamma_{t_n} \\ \zeta_{t_n} \end{pmatrix} = \begin{pmatrix} 2b_{n+1} + 2\varepsilon p(-2a_{n+1} + e_{n+1}) \\ -2c_{n+1} - 2\varepsilon q(-2a_{n+1} + e_{n+1}) \\ \rho_{n+1} + \varepsilon\alpha(-2a_{n+1} + e_{n+1}) \\ -\delta_{n+1} - \varepsilon\beta(-2a_{n+1} + e_{n+1}) \\ -\xi_{n+1} - \varepsilon\gamma(-2a_{n+1} + e_{n+1}) \\ \eta_{n+1} + \varepsilon\zeta(-2a_{n+1} + e_{n+1}) \end{pmatrix}, \quad n \ge 0. \tag{17}$$

By proper variable substitutions, the case of Eq. (17) with $\varepsilon = 0$ is exactly the $sl(2|1)$ super AKNS hierarchy [21]. Therefore, Eq. (17) is called the generalized hierarchy of the $sl(2|1)$ super AKNS equations.

When $n = 2$ in Eq. (17), the first non-trivial flow is given by as follows:

$$\begin{cases} p_{t_2} = \frac{1}{2}p_{xx} - r_xp - 2rp_x - \frac{pr}{\varepsilon} + \alpha\zeta_x - \zeta\alpha_x + \varepsilon p(p_xq - pq_x - 2\alpha\gamma_x + 2\alpha_x\gamma + 2\beta\zeta_x - 2\beta_x\zeta) - 2r^2p, \\ q_{t_2} = -\frac{1}{2}q_{xx} - r_xq - 2rq_x + \beta\gamma_x + \gamma\beta_x + \frac{qr}{\varepsilon} - \varepsilon q(p_xq - pq_x - 2\alpha\gamma_x + 2\alpha_x\gamma + 2\beta\zeta_x - 2\beta_x\zeta) + 2r^2q, \\ \alpha_{t_2} = \alpha_{xx} - r_x\alpha - 2r\alpha_x + p\beta_x + \frac{1}{2}p_x\beta - \alpha\beta\zeta - \frac{1}{2}pq\alpha + \frac{1}{2}\varepsilon\alpha(p_xq - pq_x + 2\alpha_x\gamma + 2\beta\zeta_x - 2\beta_x\zeta) - r^2\alpha, \\ \beta_{t_2} = -\beta_{xx} - r_x\beta - 2r\beta_x - q\alpha_x - \frac{1}{2}q_x\alpha + \frac{1}{2}pq\beta - \alpha\beta\gamma - \frac{1}{2}\varepsilon\beta(p_xq - pq_x - 2\alpha\gamma_x + 2\alpha_x\gamma - 2\beta_x\zeta) + r^2\beta, \\ \gamma_{t_2} = -\gamma_{xx} - r_x\gamma - 2r\gamma_x + \frac{1}{2}pq\gamma + \frac{1}{2}q_x\zeta + q\zeta_x - \beta\gamma\zeta - \frac{1}{2}\varepsilon\gamma(p_xq - pq_x - 2\alpha\gamma_x + 2\beta\zeta_x - 2\beta_x\zeta) + r^2\gamma, \\ \zeta_{t_2} = \zeta_{xx} - r_x\zeta - 2r\zeta_x - \frac{1}{2}p_x\gamma - p\gamma_x - \frac{1}{2}pq\zeta - \alpha\gamma\zeta + \frac{1}{2}\varepsilon\zeta(p_xq - pq_x - 2\alpha\gamma_x + 2\alpha_x\gamma + 2\beta\zeta_x) - r^2\zeta, \end{cases} \tag{18}$$

whose Lax pairs are determined by U in (7) and $V^{[2]}$, given by

$$V^{[2]} = \begin{pmatrix} V_{11}^{[2]} & V_{12}^{[2]} & V_{13}^{[2]} \\ V_{21}^{[2]} & V_{22}^{[2]} & V_{23}^{[2]} \\ V_{31}^{[2]} & V_{32}^{[2]} & V_{33}^{[2]} \end{pmatrix},$$

with

$$\begin{cases} V_{11}^{[2]} = \lambda^2 - \frac{1}{2}pq - \alpha\gamma + \frac{\varepsilon}{2}(p_x q - pq_x - 2\alpha\gamma_x + 2\alpha_x\gamma + 2\beta\zeta_x - 2\beta_x\zeta) - 2r^2, \\ V_{12}^{[2]} = p\lambda + \frac{1}{2}p_x - rp, \\ V_{13}^{[2]} = \alpha\lambda + \alpha_x - r\alpha, \\ V_{21}^{[2]} = q\lambda - \frac{1}{2}q_x - rq, \\ V_{22}^{[2]} = -\lambda^2 + \frac{1}{2}pq + \beta\zeta - \frac{\varepsilon}{2}(p_x q - pq_x - 2\alpha\gamma_x + 2\alpha_x\gamma + 2\beta\zeta_x - 2\beta_x\zeta) + 2r^2, \\ V_{23}^{[2]} = \beta\lambda - \beta_x - r\beta, \\ V_{31}^{[2]} = \gamma\lambda - \gamma_x - r\gamma, \\ V_{32}^{[2]} = \zeta\lambda + \zeta_x - r\zeta, \\ V_{33}^{[2]} = \gamma\alpha + \beta\zeta. \end{cases}$$

By proper variable substitutions, the case of Eq. (18) with $\varepsilon = 0$ is just the super cNLS equation (12) [21]. For the super cNLS equation with four fermi variables, its elementary Darboux transformations and integrable discretisations have been studied by Grahovski and Mikhailov in Ref. [20].

3. Super bi-Hamiltonian structure

In this section, we shall exhibit super bi-Hamiltonian structure of the generalized hierarchy of the $sl(2|1)$ super AKNS equations (17). To this end, we shall apply the supertrace identity, which was discussed in Refs. [25,26] and rigorously proved by Ma et al. in Ref. [6]:

$$\frac{\delta}{\delta u}\int Str\left(V\frac{\partial U}{\partial\lambda}\right)dx = \left(\lambda^{-s}\frac{\partial}{\partial\lambda}\lambda^{s}\right)Str\left(\frac{\partial U}{\partial u}V\right), \tag{19}$$

where *Str* is the abbreviation of the supertrace. After an easy calculation, we have

$$\begin{aligned} &Str\left(V\frac{\partial U}{\partial\lambda}\right) = 2A - E, \quad Str\left(\frac{\partial U}{\partial p}V\right) = C + q\varepsilon(2A-E), \quad Str\left(\frac{\partial U}{\partial q}V\right) = B + p\varepsilon(2A-E), \\ &Str\left(\frac{\partial U}{\partial\alpha}V\right) = \xi + \gamma\varepsilon(2A-E), \quad Str\left(\frac{\partial U}{\partial\beta}V\right) = \eta + \zeta\varepsilon(2A-E), \\ &Str\left(\frac{\partial U}{\partial\gamma}V\right) = -\rho - \alpha\varepsilon(2A-E), \quad Str\left(\frac{\partial U}{\partial\zeta}V\right) = -\delta - \beta\varepsilon(2A-E). \end{aligned} \tag{20}$$

Substituting (20) into (19), and balancing the coefficients of λ^{-n-1}, we get

$$\frac{\delta}{\delta u}\int(2a_{n+1} - e_{n+1})dx = (s-n)\begin{pmatrix} c_n + q\varepsilon(2a_n - e_n) \\ b_n + p\varepsilon(2a_n - e_n) \\ \xi_n + \gamma\varepsilon(2a_n - e_n) \\ \eta_n + \zeta\varepsilon(2a_n - e_n), \\ -\rho_n - \alpha\varepsilon(2a_n - e_n), \\ -\delta_n - \beta\varepsilon(2a_n - e_n) \end{pmatrix}, \quad n \geq 0.$$

The identity with $n = 1$ tells $s = 0$. Thus, we have

$$\begin{pmatrix} c_n + q\varepsilon(2a_n - e_n) \\ b_n + p\varepsilon(2a_n - e_n) \\ \xi_n + \gamma\varepsilon(2a_n - e_n) \\ \eta_n + \zeta\varepsilon(2a_n - e_n), \\ -\rho_n - \alpha\varepsilon(2a_n - e_n), \\ -\delta_n - \beta\varepsilon(2a_n - e_n) \end{pmatrix} = \frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 0, \tag{21}$$

where $\mathcal{H}_n = \int \frac{e_{n+1}-2a_{n+1}}{n} dx$. Moreover, a direct calculation yields to the following recursive relationship

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \\ \xi_{n+1} \\ \eta_{n+1} \\ -\rho_{n+1} \\ -\delta_{n+1} \end{pmatrix} = R_1 \begin{pmatrix} c_{n+1} + q\varepsilon(2a_{n+1} - e_{n+1}) \\ b_{n+1} + p\varepsilon(2a_{n+1} - e_{n+1}) \\ \xi_{n+1} + \gamma\varepsilon(2a_{n+1} - e_{n+1}) \\ \eta_{n+1} + \zeta\varepsilon(2a_{n+1} - e_{n+1}), \\ -\rho_{n+1} - \alpha\varepsilon(2a_{n+1} - e_{n+1}), \\ -\delta_{n+1} - \beta\varepsilon(2a_{n+1} - e_{n+1}) \end{pmatrix}, \quad n \geq 0, \tag{22}$$

where R_1 is defined by

$$R_1 = \begin{pmatrix} 1-2\varepsilon q\partial^{-1}p & 2\varepsilon q\partial^{-1}q & -\varepsilon q\partial^{-1}\alpha & \varepsilon q\partial^{-1}\beta & \varepsilon q\partial^{-1}\gamma & -\varepsilon q\partial^{-1}\zeta \\ -2\varepsilon p\partial^{-1}p & 1+2\varepsilon p\partial^{-1}q & -\varepsilon p\partial^{-1}\alpha & \varepsilon p\partial^{-1}\beta & \varepsilon p\partial^{-1}\gamma & -\varepsilon p\partial^{-1}\zeta \\ -2\varepsilon\gamma\partial^{-1}p & 2\varepsilon\gamma\partial^{-1}q & 1-\varepsilon\gamma\partial^{-1}\alpha & \varepsilon\gamma\partial^{-1}\beta & \varepsilon\gamma\partial^{-1}\gamma & -\varepsilon\gamma\partial^{-1}\zeta \\ -2\varepsilon\zeta\partial^{-1}p & 2\varepsilon\zeta\partial^{-1}q & -\varepsilon\zeta\partial^{-1}\alpha & 1+\varepsilon\zeta\partial^{-1}\beta & \varepsilon\zeta\partial^{-1}\gamma & -\varepsilon\zeta\partial^{-1}\zeta \\ 2\varepsilon\alpha\partial^{-1}p & -2\varepsilon\alpha\partial^{-1}q & \varepsilon\alpha\partial^{-1}\alpha & -\varepsilon\alpha\partial^{-1}\beta & 1-\varepsilon\alpha\partial^{-1}\gamma & \varepsilon\alpha\partial^{-1}\zeta \\ 2\varepsilon\beta\partial^{-1}p & -2\varepsilon\beta\partial^{-1}q & \varepsilon\beta\partial^{-1}\alpha & -\varepsilon\beta\partial^{-1}\beta & -\varepsilon\beta\partial^{-1}\gamma & 1+\varepsilon\beta\partial^{-1}\zeta \end{pmatrix}.$$

Hence, on the one hand, the generalized hierarchy of the $sl(2|1)$ super AKNS equations (17) has the following super Hamiltonian structure:

$$u_{t_n} = R_2 \begin{pmatrix} c_{n+1} \\ b_{n+1} \\ \xi_{n+1} \\ \eta_{n+1} \\ -\rho_{n+1} \\ -\delta_{n+1} \end{pmatrix} = R_2 R_1 \begin{pmatrix} c_{n+1} + q\varepsilon(2a_{n+1} - e_{n+1}) \\ b_{n+1} + p\varepsilon(2a_{n+1} - e_{n+1}) \\ \xi_{n+1} + \gamma\varepsilon(2a_{n+1} - e_{n+1}) \\ \eta_{n+1} + \zeta\varepsilon(2a_{n+1} - e_{n+1}), \\ -\rho_{n+1} - \alpha\varepsilon(2a_{n+1} - e_{n+1}), \\ -\delta_{n+1} - \beta\varepsilon(2a_{n+1} - e_{n+1}) \end{pmatrix} = J\frac{\delta\mathcal{H}_{n+1}}{\delta u}, \quad n \geq 0, \tag{23}$$

where

$$R_2 = \begin{pmatrix} -4\varepsilon p\partial^{-1}p & 2+4\varepsilon p\partial^{-1}q & -2\varepsilon p\partial^{-1}\alpha & 2\varepsilon p\partial^{-1}\beta & 2\varepsilon p\partial^{-1}\gamma & -2\varepsilon p\partial^{-1}\zeta \\ -2+4\varepsilon q\partial^{-1}p & -4\varepsilon q\partial^{-1}q & 2\varepsilon q\partial^{-1}\alpha & -2\varepsilon q\partial^{-1}\beta & -2\varepsilon q\partial^{-1}\gamma & 2\varepsilon q\partial^{-1}\zeta \\ -2\varepsilon\alpha\partial^{-1}p & 2\varepsilon\alpha\partial^{-1}q & -\varepsilon\alpha\partial^{-1}\alpha & \varepsilon\alpha\partial^{-1}\beta & -1+\varepsilon\alpha\partial^{-1}\gamma & -\varepsilon\alpha\partial^{-1}\zeta \\ 2\varepsilon\beta\partial^{-1}p & -2\varepsilon\beta\partial^{-1}q & \varepsilon\beta\partial^{-1}\alpha & -\varepsilon\beta\partial^{-1}\beta & -\varepsilon\beta\partial^{-1}\gamma & 1+\varepsilon\beta\partial^{-1}\zeta \\ 2\varepsilon\gamma\partial^{-1}p & -2\varepsilon\gamma\partial^{-1}q & -1+\varepsilon\gamma\partial^{-1}\alpha & -\varepsilon\gamma\partial^{-1}\beta & -\varepsilon\gamma\partial^{-1}\gamma & \varepsilon\gamma\partial^{-1}\zeta \\ -2\varepsilon\zeta\partial^{-1}p & 2\varepsilon\zeta\partial^{-1}q & -\varepsilon\zeta\partial^{-1}\alpha & 1+\varepsilon\zeta\partial^{-1}\beta & \varepsilon\zeta\partial^{-1}\gamma & -\varepsilon\zeta\partial^{-1}\zeta \end{pmatrix},$$

and the super Hamiltonian operator J is given by

$$J = \begin{pmatrix} -8\varepsilon p\partial^{-1}p & 2+8\varepsilon p\partial^{-1}p & -4\varepsilon p\partial^{-1}\alpha & 4\varepsilon p\partial^{-1}\beta & 4\varepsilon p\partial^{-1}\gamma & -4\varepsilon p\partial^{-1}\zeta \\ -2+8\varepsilon q\partial^{-1}p & -8\varepsilon q\partial^{-1}q & 4\varepsilon q\partial^{-1}\alpha & -4\varepsilon q\partial^{-1}\beta & -4\varepsilon q\partial^{-1}\gamma & 4\varepsilon q\partial^{-1}\zeta \\ -4\varepsilon\alpha\partial^{-1}p & 4\varepsilon\alpha\partial^{-1}q & -2\varepsilon\alpha\partial^{-1}\alpha & 2\varepsilon\alpha\partial^{-1}\beta & -1+2\varepsilon\alpha\partial^{-1}\gamma & -2\varepsilon\alpha\partial^{-1}\zeta \\ 4\varepsilon\beta\partial^{-1}p & -4\varepsilon\beta\partial^{-1}q & 2\varepsilon\beta\partial^{-1}\alpha & -2\varepsilon\beta\partial^{-1}\beta & -2\varepsilon\beta\partial^{-1}\gamma & 1+2\varepsilon\beta\partial^{-1}\zeta \\ 4\varepsilon\gamma\partial^{-1}p & -4\varepsilon\gamma\partial^{-1}q & -1+2\varepsilon\gamma\partial^{-1}\alpha & -2\varepsilon\gamma\partial^{-1}\beta & -2\varepsilon\gamma\partial^{-1}\gamma & 2\varepsilon\gamma\partial^{-1}\zeta \\ -4\varepsilon\zeta\partial^{-1}p & 4\varepsilon\zeta\partial^{-1}q & -2\varepsilon\zeta\partial^{-1}\alpha & 1+2\varepsilon\zeta\partial^{-1}\beta & 2\varepsilon\zeta\partial^{-1}\gamma & -2\varepsilon\zeta\partial^{-1}\zeta \end{pmatrix}.$$

On the other hand, with the help of the recursive relationship (13), the generalized hierarchy of the $sl(2|1)$ super AKNS equations (17) has another super Hamiltonian structure:

$$u_{t_n} = R_2 L_1 \begin{pmatrix} c_n \\ b_n \\ \xi_n \\ \eta_n \\ -\rho_n \\ -\delta_n \end{pmatrix} = R_2 L_1 R_1 \begin{pmatrix} c_n + q\varepsilon(2a_n - e_n) \\ b_n + p\varepsilon(2a_n - e_n) \\ \xi_n + \gamma\varepsilon(2a_n - e_n) \\ \eta_n + \zeta\varepsilon(2a_n - e_n), \\ -\rho_n - \alpha\varepsilon(2a_n - e_n), \\ -\delta_n - \beta\varepsilon(2a_n - e_n) \end{pmatrix} = M\frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 0, \tag{24}$$

where $M = R_2 L_1 R_1 = (M_{ij})_{6\times 6}$, presented by as follows:

$$\begin{aligned}
M_{11} &= 2p\partial^{-1}p + 2\varepsilon(p\partial^{-1}p\partial - \partial p\partial^{-1}p + 2p\partial^{-1}pr + 2pr\partial^{-1}p) - 4\varepsilon^2 p\partial^{-1}\Omega\partial^{-1}p, \\
M_{12} &= \partial - 2r - 2p\partial^{-1}q + 2\varepsilon(\partial p\partial^{-1}q - 2pr\partial^{-1}q) + 4\varepsilon^2 p\partial^{-1}\Omega\partial^{-1}q, \\
M_{13} &= p\partial^{-1}\alpha + \varepsilon(2p\partial^{-1}\alpha\partial + 2p\partial^{-1}r\alpha - \partial p\partial^{-1}\alpha + 2pr\partial^{-1}\alpha) - 2\varepsilon^2 p\partial^{-1}\Omega\partial^{-1}\alpha, \\
M_{14} &= -\alpha - p\partial^{-1}\beta + \varepsilon(2p\partial^{-1}\beta\partial - 2p\partial^{-1}r\beta + \partial p\partial^{-1}\beta - 2rp\partial^{-1}\beta) + 2\varepsilon^2 p\partial^{-1}\Omega\partial^{-1}\beta, \\
M_{15} &= \zeta - p\partial^{-1}\gamma + \varepsilon(2p\partial^{-1}\gamma\partial - 2p\partial^{-1}r\gamma + \partial p\partial^{-1}\gamma - 2pr\partial^{-1}\gamma) + 2\varepsilon^2 p\partial^{-1}\Omega\partial^{-1}\gamma, \\
M_{16} &= p\partial^{-1}\zeta + \varepsilon(2p\partial^{-1}\zeta\partial + 2p\partial^{-1}r\zeta - \partial p\partial^{-1}\zeta + 2pr\partial^{-1}\zeta) - 2\varepsilon^2 p\partial^{-1}\Omega\partial^{-1}\zeta, \\
M_{21} &= \partial + 2r - 2q\partial^{-1}p - 2\varepsilon(q\partial^{-1}p\partial + \partial q\partial^{-1}p + 2q\partial^{-1}pr + 2qr\partial^{-1}p) + 4\varepsilon^2 q\partial^{-1}\Omega\partial^{-1}p, \\
M_{22} &= 2q\partial^{-1}q + 2\varepsilon(\partial q\partial^{-1}q - q\partial^{-1}q\partial + 2q\partial^{-1}qr + 2qr\partial^{-1}q) - 4\varepsilon^2 q\partial^{-1}\Omega\partial^{-1}q,
\end{aligned}$$

$$
\begin{aligned}
M_{23} &= -\beta - q\partial^{-1}\alpha - \varepsilon(\partial q\partial^{-1}\alpha + 2q\partial^{-1}\alpha\partial + 2q\partial^{-1}r\alpha + 2qr\partial^{-1}\alpha) + 2\varepsilon^2 q\partial^{-1}\Omega\partial^{-1}\alpha,\\
M_{24} &= q\partial^{-1}\beta + \varepsilon(\partial q\partial^{-1}\beta + 2qr\partial^{-1}\beta - 2q\partial^{-1}\beta\partial + 2q\partial^{-1}r\beta) - 2\varepsilon^2 q\partial^{-1}\Omega\partial^{-1}\beta,\\
M_{25} &= q\partial^{-1}\gamma + \varepsilon(\partial q\partial^{-1}\gamma - 2q\partial^{-1}\gamma\partial + 2q\partial^{-1}r\gamma + 2qr\partial^{-1}\gamma) - 2\varepsilon^2 q\partial^{-1}\Omega\partial^{-1}\gamma,\\
M_{26} &= \gamma - q\partial^{-1}\zeta - \varepsilon(\partial q\partial^{-1}\zeta + 2q\partial^{-1}\zeta\partial + 2q\partial^{-1}r\zeta + 2qr\partial^{-1}\zeta) + 2\varepsilon^2 q\partial^{-1}\Omega\partial^{-1}\zeta,\\
M_{31} &= \alpha\partial^{-1}p + \varepsilon(\alpha\partial^{-1}p\partial + 2\alpha\partial^{-1}pr - 2\partial\alpha\partial^{-1}p + 2r\alpha\partial^{-1}p) - 2\varepsilon^2\alpha\partial^{-1}\Omega\partial^{-1}p,\\
M_{32} &= \beta - \alpha\partial^{-1}q + \varepsilon(\alpha\partial^{-1}q\partial - 2\alpha\partial^{-1}qr + 2\partial\alpha\partial^{-1}q - 2r\alpha\partial^{-1}q) + 2\varepsilon^2\alpha\partial^{-1}\Omega\partial^{-1}q,\\
M_{33} &= \varepsilon(\alpha\partial^{-1}\alpha\partial + \alpha\partial^{-1}r\alpha - \partial\alpha\partial^{-1}\alpha + r\alpha\partial^{-1}\alpha) - \varepsilon^2\alpha\partial^{-1}\Omega\partial^{-1}\alpha,\\
M_{34} &= -\alpha\partial^{-1}\beta + \varepsilon(\alpha\partial^{-1}\beta\partial - \alpha\partial^{-1}r\beta + \partial\alpha\partial^{-1}\beta - r\alpha\partial^{-1}\beta) + \varepsilon^2\alpha\partial^{-1}\Omega\partial^{-1}\beta,\\
M_{35} &= -\partial + r + \varepsilon(\alpha\partial^{-1}\gamma\partial - \alpha\partial^{-1}r\gamma + \partial\alpha\partial^{-1}\gamma - r\alpha\partial^{-1}\gamma) + \varepsilon^2\alpha\partial^{-1}\Omega\partial^{-1}\gamma,\\
M_{36} &= p + \alpha\partial^{-1}\zeta + \varepsilon(\alpha\partial^{-1}\zeta\partial + \alpha\partial^{-1}r\zeta - \partial\alpha\partial^{-1}\zeta + r\alpha\partial^{-1}\zeta) - \varepsilon^2\alpha\partial^{-1}\Omega\partial^{-1}\zeta,\\
M_{41} &= \alpha - \beta\partial^{-1}p - \varepsilon(\beta\partial^{-1}p\partial + 2\beta\partial^{-1}pr + 2\partial\beta\partial^{-1}p + 2r\beta\partial^{-1}p) + 2\varepsilon^2\beta\partial^{-1}\Omega\partial^{-1}p,\\
M_{42} &= \beta\partial^{-1}q - \varepsilon(\beta\partial^{-1}q\partial - 2\beta\partial^{-1}qr - 2\partial\beta\partial^{-1}q - 2r\beta\partial^{-1}q) - 2\varepsilon^2\beta\partial^{-1}\Omega\partial^{-1}q,\\
M_{43} &= -\beta\partial^{-1}\alpha - \varepsilon(\beta\partial^{-1}\alpha\partial + \beta\partial^{-1}r\alpha + \partial\beta\partial^{-1}\alpha + r\beta\partial^{-1}\alpha) + \varepsilon^2\beta\partial^{-1}\Omega\partial^{-1}\alpha,\\
M_{44} &= \varepsilon(\beta\partial^{-1}r\beta - \beta\partial^{-1}\beta\partial + \partial\beta\partial^{-1}\beta + r\beta\partial^{-1}\beta) - \varepsilon^2\beta\partial^{-1}\Omega\partial^{-1}\beta,\\
M_{45} &= q + \beta\partial^{-1}\gamma + \varepsilon(p\partial^{-1}r\gamma - p\partial^{-1}\gamma\partial + \partial\beta\partial^{-1}\gamma + r\beta\partial^{-1}\gamma) - \varepsilon^2\beta\partial^{-1}\Omega\partial^{-1}\gamma,\\
M_{46} &= -\partial - r - \varepsilon(\beta\partial^{-1}\zeta\partial + \beta\partial^{-1}r\zeta + \partial\beta\partial^{-1}\zeta + r\beta\partial^{-1}\zeta) + \varepsilon^2\beta\partial^{-1}\Omega\partial^{-1}\zeta,\\
M_{51} &= -\zeta - \gamma\partial^{-1}p - \varepsilon(\gamma\partial^{-1}p\partial + 2\gamma\partial^{-1}pr + 2\partial\gamma\partial^{-1}p + 2r\gamma\partial^{-1}p) + 2\varepsilon^2\gamma\partial^{-1}\Omega\partial^{-1}p,\\
M_{52} &= \gamma\partial^{-1}q + \varepsilon(2\gamma\partial^{-1}qr - \gamma\partial^{-1}q\partial + 2\partial\gamma\partial^{-1}q + 2r\gamma\partial^{-1}q) - 2\varepsilon^2\gamma\partial^{-1}\Omega\partial^{-1}q,\\
M_{53} &= \partial + r - \varepsilon(\gamma\partial^{-1}\alpha\partial + \gamma\partial^{-1}r\alpha + \partial\gamma\partial^{-1}\alpha + r\gamma\partial^{-1}\alpha) + \varepsilon^2\gamma\partial^{-1}\Omega\partial^{-1}\alpha,\\
M_{54} &= q + \gamma\partial^{-1}\beta + \varepsilon(\gamma\partial^{-1}r\beta - \gamma\partial^{-1}\beta\partial + \partial\gamma\partial^{-1}\beta + r\gamma\partial^{-1}\beta) - \varepsilon^2\gamma\partial^{-1}\Omega\partial^{-1}\beta,\\
M_{55} &= \varepsilon(\gamma\partial^{-1}r\gamma - \gamma\partial^{-1}\gamma\partial + \partial\gamma\partial^{-1}\gamma + r\gamma\partial^{-1}\gamma) - \varepsilon^2\gamma\partial^{-1}\Omega\partial^{-1}\gamma,\\
M_{56} &= -\gamma\partial^{-1}\zeta - \varepsilon(\gamma\partial^{-1}\zeta\partial + \gamma\partial^{-1}r\zeta + \partial\gamma\partial^{-1}\zeta + r\gamma\partial^{-1}\zeta) + \varepsilon^2\gamma\partial^{-1}\Omega\partial^{-1}\zeta,\\
M_{61} &= \zeta\partial^{-1}p + \varepsilon(\zeta\partial^{-1}p\partial + 2\zeta\partial^{-1}pr - 2\partial\zeta\partial^{-1}p + 2r\zeta\partial^{-1}p) - 2\varepsilon^2\zeta\partial^{-1}\Omega\partial^{-1}p,\\
M_{62} &= -\gamma - \zeta\partial^{-1}q + \varepsilon(\zeta\partial^{-1}q\partial - 2\zeta\partial^{-1}qr + 2\partial\zeta\partial^{-1}q - 2r\zeta\partial^{-1}q) + 2\varepsilon^2\zeta\partial^{-1}\Omega\partial^{-1}q,\\
M_{63} &= p + \zeta\partial^{-1}\alpha + \varepsilon(\zeta\partial^{-1}\alpha\partial + \zeta\partial^{-1}r\alpha - \partial\zeta\partial^{-1}\alpha + r\zeta\partial^{-1}\alpha) - \varepsilon^2\zeta\partial^{-1}\Omega\partial^{-1}\alpha,\\
M_{64} &= \partial - r + \varepsilon(\zeta\partial^{-1}\beta\partial - \zeta\partial^{-1}r\beta + \partial\zeta\partial^{-1}\beta - r\zeta\partial^{-1}\beta) + \varepsilon^2\zeta\partial^{-1}\Omega\partial^{-1}\beta,\\
M_{65} &= -\zeta\partial^{-1}\gamma + \varepsilon(\zeta\partial^{-1}\gamma\partial - \zeta\partial^{-1}r\gamma + \partial\zeta\partial^{-1}\gamma - r\zeta\partial^{-1}\gamma) + \varepsilon^2\zeta\partial^{-1}\Omega\partial^{-1}\gamma,\\
M_{66} &= \varepsilon(\zeta\partial^{-1}\zeta\partial + \zeta\partial^{-1}r\zeta - \partial\zeta\partial^{-1}\zeta + r\zeta\partial^{-1}\zeta) - \varepsilon^2\zeta\partial^{-1}\Omega\partial^{-1}\zeta.
\end{aligned}
$$

with $\Omega = p\partial q + q\partial p + \alpha\partial\gamma + \beta\partial\zeta - \gamma\partial\alpha - \zeta\partial\beta$. Here M is the second super Hamiltonian operator.

4. Conclusions and discussions

In this paper, starting from a given spatial spectral problem related to a Lie superalgebra $sl(2|1)$ (7), we obtained a generalized hierarchy of equations (17). Comparing the new generalized hierarchy (17) with Eq. (11) in Ref. [21], we found that Eq. (17) is actually an extension of the $sl(2|1)$ super AKNS hierarchy. And comparing the first non-trivial equation (18) of the generalized hierarchy (17) with Eq. (2. 3) in Ref. [20], we knew that Eq. (18) with $\varepsilon = 0$ is just the super cNLS equation after some replacement between variables. Therefore, we believe that the resulting generalized hierarchy (17) is very important in the field of soliton and integrable system, especially in the super integrable system. Moreover, by making use of the supertrace identity (19), the generalized $sl(2|1)$ super AKNS hierarchy can be written as the super bi-Hamiltonian structures (23) and (24). Is this method applied to the other super soliton hierarchies? If we start from a spatial spectral problem associated with the other Lie superalgebras, can we obtain more meaningful hierarchies of equations? All of these questions will be answered in our future paper.

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