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Analysis and approximation of a nonlocal obstacle problem[☆]

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ABSTRACT

An obstacle problem for a nonlocal operator is considered; the operator is a nonlocal integral analogue of the Laplacian operator and, as a special case, reduces to the fractional Laplacian. In the analysis of classical obstacle problems for the Laplacian, the obstacle is taken to be a smooth function. For the nonlocal obstacle problem considered here, obstacles are allowed to have jump discontinuities. We cast the nonlocal obstacle problem as a minimization problem wherein the solution is constrained to lie above the obstacle. We prove the existence and uniqueness of a solution in an appropriate function space. Then, the well posedness and convergence of finite element approximations are demonstrated. The results of numerical experiments are provided that illustrate the theoretical results and the differences between solutions of local, i.e., partial differential equation, and nonlocal obstacle problems.

1. Introduction

A class of obstacle problems can be cast in the form of an elliptic variational inequality as follows. Given an open domain $\Omega \in \mathbb{R}^n$ with boundary $\partial\Omega$, $f \in L^2(\Omega)$, and $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\psi \leq 0$ on $\partial\Omega$, find u belonging to the closed convex set $\mathcal{K} := \left\{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e.in } \Omega\right\}$ such that

$$a(u, v-u) \geq (f, v-u) \quad \text{for all } v \in \mathcal{K}, \tag{1}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ and $(f,v) = \int_\Omega f v \, dx$. The variational inequality (1) is equivalent to the minimization problem

$$\min_{u\in\mathcal{K}} \left(\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega uf \, dx \right). \tag{2}$$

Obstacle problems of the type (1) or (2) have many applications such as membrane deformation in elasticity theory and nonparametric minimal and capillary surfaces as geometrical problems [1–4]. In general, ψ is assumed to be a smooth (at least continuous) function, in which case it is known that the solution of (1) exists, is unique, is continuous, and possesses Lipschitz continuous first derivatives [1,2,5]. To our knowledge, there are few results about the well-posedness and regularity for obstacle problem with discontinuous obstacle function. Numerical methods for determining approximations of the solution of (1) have also been developed; see, e.g., [1,2,6–8].

Nonlocal obstacle problems arise, e.g., in settings modeled by fractional partial differential equations such as those involving the fractional Laplacian operator [9–13]. In this paper, we treat more general nonlocal problems, with fractional

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Laplacian and other fractional derivative problems being special cases. Nonlocal operators in the peridynamics theory of solid mechanics [14–16] and anomalous diffusion problems [17–20] also fall within the purview of our study. For nonlocal obstacle problems, one can choose ψ to be a less regular, even discontinuous, function. In addition, because the local, partial differential equation problems are, in a precise sense, the limits of the nonlocal problems we study [19,18,21], one could glean some information about the former for discontinuous ψ by studying the latter.

We define the action of the nonlocal operator $\mathcal{L}$ on a function $u(x) : \Omega \to \mathbb{R}$ by

$$\mathcal{L}u(x) := 2\int_{\mathbb{R}^n} \gamma(x,y)\big(u(x)-u(y)\big)\,dy \quad \forall\, x \in \Omega \subseteq \mathbb{R}^n. \tag{3}$$

The operator $\mathcal{L}$ is deemed *nonlocal* because the value of $\mathcal{L}u$ at a point x requires information about u at points y separated from x by a finite distance; this should be contrasted with, e.g., the *local* Laplacian operator for which the value of Δu at a point x requires information about u only in an infinitesimal neighborhood of x.

The operator $\mathcal{L}$ has a special case the fractional Laplacian operator which is the pseudo-differential operator with Fourier symbol $\mathcal{F}$ given by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\widehat{u}(\xi), \quad 0 < s < 1,$$

where $\widehat{u}$ denotes the Fourier transform of u. Suppose that $u \in L^2(\mathbb{R}^n)$ and that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{n+2s}}dydx < \infty, \quad 0 < s < 1.$$

The vector space of such functions defines the fractional Sobolev space $H^s(\mathbb{R}^n)$. The Fourier transform can be used to show that an equivalent characterization of the fractional Laplacian is

$$(-\Delta)^s u = c_{n,s}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \quad 0 < s < 1,$$

for some normalizing constant $c_{n,s}$, see [22,9,23,10,11]. When $\Omega = \mathbb{R}^n$, the fractional Laplacian is the special case of the operator $\mathcal{L}$ defined above for the choice of $\gamma(x,y)$ proportional to $1/|x-y|^{n+2s}$.

When Ω has finite volume we have that the volume constrained minimization problem

$$\min_u\Big(\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dxdy - \int_{\mathbb{R}^n} ufdxdy\Big) \quad \text{subject to } u = 0 \text{ on } \mathbb{R}^n/\Omega \tag{4}$$

is well posed for $0 \le s < 1$, see [18,17]. Note that the volume constraint $u = 0$ on $\mathbb{R}^n/\Omega$ appearing in (4) is needed for well posedness. In fact, the boundary value problem

$$(-\Delta)^s u = g \quad \text{on } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \tag{5}$$

is not well posed. For $1/2 < s < 1$, solutions are not uniquely defined and for $s \le 1/2$, existence is not, in general, guaranteed. To formulate a well-posed problem, the boundary condition in (5) must be replaced by the volume constraint $u = 0$ on $\mathbb{R}^n/\Omega$; see, e.g., [18]. To differentiate between the two types of constraints, we naturally refer to the constraint $u = 0$ on $\partial\Omega$ in (5) as a boundary condition and refer to the constraint $u = 0$ on $\mathbb{R}^n/\Omega$ in (4) as a *volume constraint*. We also use the latter terminology to refer to constraints of the type $u = 0$ on $\Omega_B \subset \mathbb{R}^n/\Omega$ whenever Ω_B has nonzero volume in $\mathbb{R}^n$.

Here, we treat two types of kernels $\gamma(x,y)$ in (3).

Case 1. For $s \in (0,1)$, $\delta > 0$, $c_{n,s} > 0$, and $x, y \in \mathbb{R}^n$,

$$\gamma_s(x,y) = \begin{cases} \dfrac{c_{n,s}}{|y-x|^{n+2s}} & \text{if } |y-x| \le \delta \\ 0 & \text{if } |y-x| > \delta. \end{cases}$$

Note that $\gamma_s(x,y)$ is non-integrable in Riemann sense.

Case 2. For $l \in (0,n)$, $\delta > 0$, $c_{n,l} > 0$, and $x, y \in \mathbb{R}^n$

$$\gamma_l(x,y) = \begin{cases} \dfrac{c_{n,l}}{|y-x|^{n-l}} & \text{if } |y-x| \le \delta \\ 0 & \text{if } |y-x| > \delta. \end{cases}$$

Note that $\gamma_l(x,y)$ is integrable. Also, note that for both cases the parameter δ, sometimes referred to as the *horizon*, limits the extent of the nonlocal interactions at a point x to the ball of radius δ centered at x.

Let $\Omega_I \subset \mathbb{R}^n$ denote a bounded open domain with piecewise smooth boundary that satisfies the interior cone condition. The corresponding "boundary" domain is defined by

$$\Omega_B := \{y \in \mathbb{R}^n \setminus \Omega_I \text{ such that } \exists\, x \in \Omega_I \text{ such that } \gamma(x,y) \ne 0\},$$

where γ stands for either γ_s or γ_I. Note that, in general, Ω_B has nonzero volume in $\mathbb{R}^n$. We set the extension domain $\Omega := \Omega_I \cup \Omega_B$, and define

$$I[u] := \frac{1}{2}\int_\Omega\int_\Omega \gamma(x,y)(u(x)-u(y))^2\,dydx - \int_\Omega uf\,dx. \tag{6}$$

In this paper, we consider the constrained minimization problem

$$\min_u I[u];\ u \geq \psi \quad \text{and} \quad u = 0 \quad \text{on } \Omega_B \tag{7}$$

where, if $s < 1/2$, ψ can be a function having jump discontinuities. In Section 2, we consider the existence and uniqueness of the solution and the convergence of numerical approximations for Case 1. In Section 3, we do the same for Case 2. In Section 4, we provide some numerical results that illustrate our theoretical results and the differences between solutions of local and nonlocal obstacle problems. In Section 5, we provide some concluding remarks.

2. Nonlocal obstacle problems for the non-integrable kernels

We first study nonlocal obstacle problems for the non-integrable kernels of Case 1. As demonstrated in [18], the minimization problem (7) with $\psi = -\infty$ is well posed. Specifically, it is well posed for all $0 < s < 1$.

From [18,19,21], we know that with an appropriate choice of $c_{n,s}$ (that depends on δ and s) and for sufficiently smooth u, following works like [24,21], one should be able to prove that the problem (7) converges to the problem (2) as $\delta \to 0$. On the other hand, from [17], we know that the problem (7) converges to the problem (4) as $\psi = -\infty$ and $\delta \to \infty$.

2.1. Notation and preliminary results

As in [18], we define an "energy" norm, an energy space, and a constrained energy space by

$$|||u||| := \left(\int_\Omega\int_\Omega \gamma_s(x,y)(u(x)-u(y))^2\,dydx\right)^{1/2}$$

$$V(\Omega) := \{u \in L^2(\Omega) : |||u||| < \infty\}$$

$$V_c^s(\Omega) := \{u \in V(\Omega) : u(x) = 0,\ x \in \Omega_B\},$$

respectively and Ω is the extension domain in all following context. For $s \in (0,1)$, the standard fractional order Sobolev space is defined as

$$H^s(\Omega) := \{u \in L^2(\Omega) : \|u\|_{L^2(\Omega)} + |u|_{H^s(\Omega)} < \infty\},$$

where

$$|u|_{H^s(\Omega)} := \left(\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dydx\right)^{1/2}.$$

Moreover, we define the subspace

$$H_c^s(\Omega) := \{u \in H^s(\Omega) : u(x) = 0 \text{ for } x \in \Omega_B\}.$$

For these spaces and norms, we have the following results.

Lemma 1 ([18]). *The spaces $V_c^s(\Omega)$ and $H_c^s(\Omega)$ are equivalent; in particular, for some constants $0 < c_* \leq c^* < \infty$,*

$$c_*\|u\|_{H^s(\Omega)} \leq |||u||| \leq c^*\|u\|_{H^s(\Omega)} \quad \forall\, u \in V_c^s(\Omega).$$

Lemma 2 ([18]). *There exists a positive constant C such that*

$$\|u\|_{L^2(\Omega)}^2 \leq C|||u|||^2 \quad \forall\, u \in V_c^s(\Omega).$$

Lemma 3 ([24]). *For all $v \in H^1(\Omega)$, there exists a positive constant C such that $|||v||| \leq C|v|_{H^1}$.*

Lemma 4 ([24]). *$V_c^s(\Omega)$ is a Hilbert space equipped with the inner product*

$$(u,v)_s = \int_\Omega\int_\Omega \gamma_s(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,dydx.$$

We define $V_{c,0}^s(\Omega)$ to be the closure of $C_0^\infty(\Omega_I)$ in $V_c^s(\Omega)$. $V_{c,0}^s(\Omega)$ is also a Hilbert space. Let $\psi \in V_c^s(\Omega)$ and define the subset

$$\mathcal{A} := \{u \in V_{c,0}^s(\Omega) : u \geq \psi\}, \tag{8}$$

where Ω is the extension domain. With this definition, it is easy to get the following lemma.

Lemma 5. *$\mathcal{A}$ is closed and convex.*

2.1.1. The nonlocal energy functional and its properties

We have the following result.

Lemma 6 ([25]). *Let $s \in (0,1)$, $p \in [1,+\infty)$, $q \in [1,p]$, $\Omega \subset \mathbb{R}^n$ denote a bounded extension domain for $W^{s,p}(\Omega)$, and $\mathcal{J}$ denote a bounded subset of $L^p(\Omega)$. Suppose that*

$$\sup_{v\in\mathcal{J}} \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}dxdy < +\infty.$$

Then, $\mathcal{J}$ is pre-compact in $L^q(\Omega)$.

From Lemmas 1, 2, and 6 with $p = q = 2$, one can easily prove the result:

Theorem 1. *$V_c^s(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Consider the energy functional

$$I[v] := \frac{1}{2}\int_\Omega\int_\Omega \gamma_s(x,y)\big(v(x)-v(y)\big)^2 dydx - \int_\Omega f v dx \quad \forall\, v \in V_{c,0}^s(\Omega). \tag{9}$$

Then, we state the following easily obtained results without proof.

Lemma 7. *The functional $I[v]$ is strictly convex.*

Lemma 8. *$I[\cdot]$ is weakly lower semicontinuous on $V_{c,0}^s(\Omega)$, i.e., $I[v] \le \liminf_{k\to\infty} I[v_k]$ whenever $v_k \rightharpoonup v$ weakly in $V_{c,0}^s(\Omega)$.*

2.2. Existence and uniqueness of solutions of a nonlocal obstacle problem

The *nonlocal obstacle problem* is defined by:

$$\textit{seek } u \in \mathcal{A} \textit{ such that} \quad I[u] \le I[v]\ \forall\, v \in \mathcal{A}, \tag{10}$$

where the energy functional $I[v]$ is defined by (9) and the admissibility set $\mathcal{A}$ is defined by (8).

Theorem 2. *There exists a unique function $u \in \mathcal{A}$ that solves the problem (10).*

Proof. By Theorem 1 and Lemmas 5, 7, and 8, the results are easily obtained.

2.3. Variational inequality form of the nonlocal obstacle problem

Let

$$a(u,v) = \int_\Omega\int_\Omega \gamma_s(x,y)\big(u(x)-u(y))(v(x)-v(y)\big)dxdy \quad \text{and} \quad (f,v) = \int_\Omega f v dx.$$

Consider the problem:

$$\textit{seek } u \in \mathcal{A} \textit{ such that} \quad a(u, v-u) \ge (f, v-u)\ \forall\, v \in \mathcal{A}. \tag{11}$$

For the problems (10) and (11), we have the following standard equivalence result.

Theorem 3. *$u \in \mathcal{A}$ is the solution of (10) if and only if it is the solution of (11).*

2.4. The approximate problem

Suppose we are given a family $\{V_h\}_h$ of closed subspaces of $V_{c,0}^s$ parameterized by $h \to 0$. Also, suppose there is a family $\{\mathcal{A}_h\}_h$ with $\mathcal{A}_h \subset V_h$ of closed convex nonempty subsets of $V_{c,0}^s$ that satisfies the following conditions.

Assumption 1. (i) If $v_h \in \mathcal{A}_h$ and $\{v_h\}_h$ are bounded in $V_{c,0}^s$, then the weak cluster points of $\{v_h\}_h$ belong to $\mathcal{A}$.

(ii) There exist $\theta \subset V_{c,0}^s$ for which $\overline{\theta} = \mathcal{A}$ and an operator $r_h : \theta \to \mathcal{A}_h$ such that, for all $v \in \theta$, $\lim_{h\to 0} r_h v = v$ strongly in $V_{c,0}^s$.

In Section 2.4.1, we provide a useful setting for which these assumptions are verified to hold.

We define a discretization of the problem (11) by the following problem:

$$\textit{seek } u_h \in \mathcal{A}_h \textit{ such that} \quad a(u_h, v_h-u_h) \ge (f, v_h-u_h)\ \forall\, v_h \in \mathcal{A}_h. \tag{12}$$

Theorem 4. *The problem (12) has a unique solution.*

Proof. The proof is as for Theorem 3 with $V^s_{c,0}$ replaced by V_h and $\mathcal{A}$ replaced by $\mathcal{A}_h$.

Theorem 5. *Let u_h denote the solution of (12) and u the solution of (11). Then, with the assumptions made about $\mathcal{A}$ and $\{\mathcal{A}_h\}_h$, we have*

$$\lim_{h\to 0} |||u_h - u||| = 0. \tag{13}$$

Proof. Same as the proof of in [2, Theorem 5.2, Chapter I].

We next prove the main result Lemma 9 of this paper which plays the key role in proving Theorem 6. Then suppose Assumption 2 is valid, with Theorem 6 and the continuous piecewise linear finite element approximations in Section 2.4.1, one can prove that Assumption 1 made about $\mathcal{A}$ and $\{\mathcal{A}_h\}_h$ is valid which means Theorem 5 is true.

Lemma 9. *Suppose $v \in V^s_c$ and $\psi \in V^s_c$ with $v \geq \psi$, $0 < s < 1$. Also, suppose we have sequences $\{\psi_n\}$ and $\{v_n\}$ in V^s_c such that $\lim_{n\to\infty} \psi_n \to \psi$ and $\lim_{n\to\infty} v_n \to v$ strongly in V^s_c. Then, $\lim_{n\to\infty} \max(v_n, \psi_n) \to v$ strongly in V^s_c.*

Proof. (1) We prove that

$$\lim_{n\to\infty} \max(v_n, \psi_n) \to v \quad \text{strongly in } L^2(\Omega). \tag{14}$$

From the assumptions, we know $v_n \to v$ and $\psi_n \to \psi$ strongly in $L^2(\Omega)$. Let $\|\cdot\|$ denote the $L^2(\Omega)$ norm, let $m(\cdot)$ be the Lebesgue measure, and define

$$\phi_n = \max(\psi_n, v_n)$$

so that

$$\phi_n = \frac{\psi_n + v_n + |\psi_n - v_n|}{2}.$$

Then, the goal is to prove that $\phi_n \to v$ strongly in $L^2(\Omega)$ as $n \to \infty$.

We need to prove that

$$\lim_{n\to\infty} \phi_n = \frac{\psi + v + |\psi - v|}{2}.$$

Define $\phi = \frac{1}{2}(\psi + v + |\psi - v|)$ so that $\phi = \max(v, \psi) = v$. Then,

$$\begin{aligned}\|\phi_n - \phi\| &= \frac{1}{2}\|\psi_n - \psi + v_n - v + |\psi_n - v_n| - |\psi - v|\,\| \\ &\leq \|\psi_n - \psi\| + \|v_n - v\| + \|\,|\psi_n - v_n| - |\psi - v|\,\|.\end{aligned}$$

Because $\psi_n \to \psi$ strongly and $v_n \to v$ strongly, we only need to prove $|\psi_n - v_n| \to |\psi - v|$ strongly.

To do so, we define, for all $\epsilon > 0$, the set $s_n = \{x \mid v_n(x) - \psi_n(x) \leq -\epsilon\}$ and let $e_n = m(s_n)$. We assert that $\lim_{n\to\infty} e_n = 0$ because otherwise there exists $\epsilon_0 > 0$ such that $e_n \geq e_0 > 0$ as $n \to \infty$ (a subsequence). This implies that $v_n - \psi_n \leq -\epsilon_0$ on the set s_n which in turn implies that $v_n - \psi \leq \psi_n - \psi - \epsilon_0$. However, as $n \to \infty$, the measure of $s_{n,0} = \{x \mid |\psi_n(x) - \psi(x)| \geq \epsilon_0/2\}$ goes to 0 so that we can choose $n > N$ such that $m(s_n/s_{n,0}) > e_0/2$. Thus, on the set $s_n/s_{n,0}$, $v_n - v \leq -\epsilon_0/2$ which cannot be true because $v_n \to v$ strongly.

Now, define the set $s^-_n = \{x| -\epsilon < v_n(x) - \psi_n(x) < 0\}$. If $m(s^-_n) \not\to 0$ as $n \to \infty$, we have $e_0 > 0$ such that $m(s^-_n) > e_0$ as $n \to \infty$ (a subsequence). We define $\sigma_n = \{x \mid v(x) - \psi_n(x) > \psi(x) - \psi_n(x) + \epsilon\} \cap s^-_n$. We assert that $m(\sigma_n) \to 0$ as $n \to \infty$. Otherwise, if $m(\sigma_n) \not\to 0$, there exists $0 < e_1 \leq e_0$ such that $m(\sigma_n) \geq e_1$. However, as $n \to \infty$, the measure of $s_{n,1} = \{x \mid |\psi_n(x) - \psi(x)| \geq \epsilon/2\}$ goes to 0 so that we can choose $n > N$ such that $m(\sigma_n/s_{n,1}) > e_1/2$. This implies that on $\sigma_n/s_{n,1}$, $v - v_n \geq \epsilon/2$ as $n \to \infty$; because $v_n \to v$ strongly, this cannot be true. Thus, $m(\sigma_n) \to 0$ as $n \to \infty$.

We can now take up the task of proving that $|\psi_n - v_n| \to |\psi - v|$ strongly, starting with

$$\begin{aligned}\|\,|\psi_n - v_n| - |\psi - v|\,\|^2 &= \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{\Omega/\{s_n \cup s^-_n\}} + \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{s_n} \\ &\quad + \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{s^-_n/\sigma_n} + \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{\sigma_n} \\ &= \|v_n - v + \psi - \psi_n\|^2|_{\Omega/\{s_n \cup s^-_n\}} + \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{s_n} \\ &\quad + \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{s^-_n/\sigma_n} + \|\,|\psi_n - v_n| - |\psi - v|\,\|^2|_{\sigma_n}\end{aligned}$$

so that, for all $\epsilon > 0$, we have

$$\lim_{n\to\infty} \|\,|\psi_n - v_n| - |\psi - v|\,\|^2 \leq 4\epsilon^2 m(\Omega).$$

Then, again because $\psi_n \to \psi$ strongly and $v_n \to v$ strongly, we have

$$\lim_{n\to\infty} \|\phi_n - \phi\| = 0$$

which, from the definitions of ϕ_n and ϕ, yields (14).

(2) We next prove that

$$\lim_{n\to\infty} \max(v_n, \psi_n) \to v \quad \text{strongly in } V_c^s.$$

Here we use the norm $|||\cdot|||$ as before. As in step (1), we just need to prove

$$\lim_{n\to\infty} |||\, |\psi_n - v_n| - |\psi - v|\, ||| = 0.$$

We have

$$|||\, |\psi_n - v_n| - |\psi - v|\, |||^2 = \int_\Omega \int_\Omega \gamma_s(x,y)\big((|\xi_n| - |\xi|)(x) - (|\xi_n| - |\xi|)(y)\big)^2 dydx = I_1(\delta') + I_2(\delta'),$$

where $\xi_n = v_n - \psi_n$ and $\xi = v - \psi$. Letting χ denote the characteristic function, we have

$$I_1(\delta') = \int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\ge\delta'}(|x-y|)\big((|\xi_n| - |\xi|)(x) - (|\xi_n| - |\xi|)(y)\big)^2 dydx$$

and

$$I_2(\delta') = \int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)\big((|\xi_n| - |\xi|)(x) - (|\xi_n| - |\xi|)(y)\big)^2 dydx.$$

Because the kernel function $\gamma_s(x,y)\chi_{|x-y|\ge\delta'}(|x-y|) \in L^1_{loc}(\mathbb{R}^n)$, we have the estimate

$$I_1(\delta') \le C(\delta')\big\| |\xi_n| - |\xi| \big\|^2.$$

On the other hand,

$$\begin{aligned} I_2(\delta') \;&\le 2\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)(|\xi_n(x)| - |\xi_n(y)|)^2 dydx \\ &+ \;\; 2\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)(|\xi(x)| - |\xi(y)|)^2 dydx. \end{aligned}$$

As $\|a| - |b\| \le |a-b|$, we estimate the first term of $I_2(\delta')$ as follows:

$$\begin{aligned} &\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)(|\xi_n(x)| - |\xi_n(y)|)^2 dydx \\ &\quad \le 2\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)\big(v_n(x) - v_n(y)\big)^2 dydx \\ &\quad\quad + 2\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)\big(\psi_n(x) - \psi_n(y)\big)^2 dydx. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\lim_{n\to\infty} I_1(\delta') = 0 \\ &\lim_{n\to\infty} I_2(\delta') \;\le\; 9\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)\big(v(x) - v(y)\big)^2 dydx \\ &\qquad\qquad\qquad + 9\int_\Omega \int_\Omega \gamma_s(x,y)\chi_{|x-y|\le\delta'}(|x-y|)\big(\psi(x) - \psi(y)\big)^2 dydx. \end{aligned}$$

For all $\epsilon > 0$, we have $\delta' > 0$ such that $\lim_{n\to\infty} I_2(\delta') < \epsilon$ so that

$$\lim_{n\to\infty} |||\, |\psi_n - v_n| - |\psi - v|\, |||^2 = \lim_{n\to\infty}\big(I_1(\delta') + I_2(\delta')\big) \le \epsilon.$$

Thus, we have the result $\lim_{n\to\infty} \max(v_n, \psi_n) \to v$ strongly in V_c^s.

We define

$$\mathfrak{D} := \{v|_{\Omega_I} \in C_0^\infty(\Omega_I) \mid v|_{\Omega_B} = 0\}.$$

Theorem 6. *Let $\psi \in V_c^s(\Omega)$, $0 < s < 1$. If there exist functions $\{\psi_n\}_n \in V_c^s$ such that $\psi_n|_{\Omega_I} \in C(\Omega_I)$, $\psi_n \le 0$ in a neighborhood of $\partial\Omega_I$, $\psi_n \ge \psi$, and $\psi_n \to \psi$ as $n \to \infty$ strongly in V_c^s, then $\overline{\mathfrak{D} \cap \mathcal{A}} = \mathcal{A}$.*

Proof. We first show that

$$\mathfrak{A} = \{v \in \mathcal{A} \cap C_0^0(\Omega_I)\} \quad \text{is dense in } \mathcal{A}. \tag{15}$$

Letting $v \in \mathcal{A} \subset V_{c,0}^s(\Omega)$, there exists a sequence $\{v_n\}_n$ in $\mathfrak{D}$ such that

$$\lim_{n\to\infty} v_n = v \text{ strongly in } V_{c,0}^s.$$

Define ϕ_n by $\phi_n = \max(\psi_n, v_n)$ so that $\phi_n \in C_0^0(\Omega_I)$. From Lemma 9, we have $\phi_n \in V_c^s(\Omega)$. Using a similar method as in the proof of [24, Lemma 2.5], we can obtain $\phi_n \in V_{c,0}^s(\Omega)$. Then, by Lemma 9, we have

$$\lim_{n\to\infty} \phi_n = v \text{ strongly in } V_{c,0}^s$$

so that $\mathfrak{A}$ is dense in $\mathcal{A}$.

Next, as in the proof in [2, Lemma 2.4, Chapter II], it is easy to show that, for every $v \in \mathfrak{A}$, there exists a sequence $\{v_m\}_m$ such that

$$v_m \in \mathfrak{D} \cap \mathcal{A} \quad \text{and} \quad \lim_{m\to\infty} \|\!|\!| v_m - v |\!|\!\| = 0. \tag{16}$$

Then, together with (15), we have that $\mathfrak{D} \cap \mathcal{A}$ is dense in $\mathcal{A}$.

2.4.1. *Verification of Assumption 1*

As in Section 1, Ω_I is a bounded open domain with piecewise smooth boundary that satisfies the interior cone condition. We verify Assumption 1 in the setting of continuous piecewise linear finite element approximations in two dimensions, in three or more dimensions, the analysis are similar. Let $\mathcal{T}_h$ denote a regular triangulation of Ω_I and Σ_h denote the set of vertices of the triangles in $\mathcal{T}_h$. For each triangle $T \in \mathcal{T}_h$, let M_{iT}, $i = 1, 2, 3$, denote its vertices and G_T its centroid. Also, let P_1 denote the space of polynomials in x_1 and x_2 of degree less than or equal to one.

The space $V_{c,0}^s$ is approximated by the family of subspaces $\{V_h\}_h$, where

$$V_h = \{v_h \in C^0(\Omega),\ v_h|_{\Omega_B} = 0,\ \text{and } v_h|_T \in P_1\ \forall\, T \in \mathcal{T}_h\}.$$

Note that V_h is finite dimensional. Then, there exists a basis $\{\phi_p\}$ for V_h such that any $v_h \in V_h$ can be written as

$$v_h = \sum v_h(p)\phi_p,$$

where $\{v_h(p)\}$ denotes the set of values of $v_h(x_1, x_2)$ at the points $p \in \Sigma_h$.

We define the approximation ψ_h of the obstacle function ψ by

$$\psi_h \in C^0(\Omega), \qquad \psi_h|_T \in P_1, \quad \text{and} \quad \psi_h(p) = \psi(p) \quad \text{for all } p \in \Sigma_h. \tag{17}$$

For the functions ψ and ψ_h, we assume the following Assumption 2 is valid.

Assumption 2. (i) ψ satisfies the conditions in Theorem 6.
(ii) $\lim_{h\to 0} \psi_h = \psi$ strongly in $L^2(\Omega)$.

We approximate $\mathcal{A}$ by

$$\mathcal{A}_h = \{v_h \in V_h \text{ and } v_h(p) \ge \psi_h(p)\ \forall\, p \in \Sigma_h\}.$$

Verification of (i) *of* Assumption 1. Similarly as in [2], consider $\eta \in \mathfrak{D}$ with $\eta \ge 0$ and let η_h denote the piecewise constant function $\eta_h = \sum_T \eta(G_T)\chi_T$, where χ_T denotes the characteristic function of T. It is easy to see from the uniform continuity of η that

$$\lim_{h\to 0} \eta_h = \eta \text{ strongly in } L^\infty(\Omega). \tag{18}$$

Now, consider a sequence $\{v_h\}_h$ with $v_h \in \mathcal{A}_h$ for all h such that

$$\lim_{h\to 0} v_h = v \text{ weakly in } V_{c,0}^s. \tag{19}$$

Then, from (18) and (19), we have

$$\lim_{h\to 0} \int_\Omega (v_h - \psi_h)\eta_h dx = \int_\Omega (v - \psi)\eta dx \quad \forall\, \eta \in \mathfrak{D}$$

or, using the definition of η_h,

$$\int_\Omega (v_h - \psi_h)\eta_h dx = \sum_T \eta(G_T) \int_T (v_h - \psi_h)dx \quad \forall\, \eta \in \mathfrak{D}. \tag{20}$$

From the definitions of v_h and ψ_h, we obtain

$$\int_T (v_h - \psi_h)dx = \frac{\text{meas}(T)}{3} \sum_{i=1}^{3} [v_h(M_{iT}) - \psi_h(M_{iT})] \quad \forall\, T \in \mathcal{T}_h. \tag{21}$$

Using the fact that $\eta_h \ge 0$ whenever $\eta \ge 0$, the definition of $\mathcal{A}_h$, and (21), it follows from (20) that

$$\int_\Omega (v_h - \psi_h)\eta_h dx \ge 0 \quad \forall\, \eta \in \mathfrak{D},\ \eta \ge 0,$$

so that, taking the limit as $h \to 0$,

$$\int_\Omega (v - \psi)\eta\, dx \ge 0 \quad \forall\, \eta \in \mathfrak{D},\ \eta \ge 0$$

which implies that $v \ge \psi$ a.e. in Ω. Hence (i) is verified.

Verification of (ii) *of* Assumption 1. Let $\chi = \mathfrak{D} \cap \mathcal{A}$ and $r_h v = v_h \in V_h$, where $v_h(p) = v(p)$ for all $p \in \Sigma_h$. By Lemma 3, it is easy to see $v_h \in \mathcal{A}_h$ and $\lim_{h\to 0} r_h v = v$ strongly in $V^s_{c,0}$.

The verifications of Assumption 1 are similar to the local case if Assumption 2 is valid, however, Assumption 2 is totally new, and we need to answer the question: if a ψ satisfying Assumption 2 exists? Here, we provide some examples satisfying Assumption 2.

(i) First, it is easy to verify that if

$$\psi \in \{\psi \mid \psi|_{\Omega_I} \in H^1_0(\Omega_I) \cap C^0_0(\Omega_I),\ \psi|_{\Omega_B} = 0\}, \tag{22}$$

then Assumption 2 holds which means Assumption 1 holds.

(ii) For $0 < s < 1/2$ and in one dimension, we have an example of a class of functions with a finite number of jump discontinuities that satisfies Assumption 2. Specifically, let $\Omega_I = (a, b)$, and define the sub-intervals $I_1 = (a, a_1)$, $I_2 = [a_1, a_2), \ldots, I_m = [a_{m-1}, b)$, where m is a finite number. Then, if

$$\psi \in \{\psi|_{\bar{I}_i} \in C^1(\bar{I}_i),\ \text{and } \psi \le 0 \text{ in a neighborhood of } \partial\Omega_I\}, \tag{23}$$

where $i = 1, \ldots, m$, then the Assumption 2 holds. Note that, in one dimension, for $0 < s < 1/2$, all bounded functions $\psi \in V^s_c$ are in $V^s_{c,0}$.

(iii) We also have a specific example for $0 < s < 1/2$ in two dimensions of a function with a jump discontinuity. Let $\Omega_1 \subset \Omega_I$ and $\Omega_2 \subset \Omega_I$ denote two circular regions such that $d(\Omega_1, \Omega_2) > 0$. Then, if

$$\psi = 1 \quad \forall\, x \in \Omega_1 \cup \Omega_2 \quad \text{and} \quad \psi = 0 \text{ elsewhere}, \tag{24}$$

the Assumption 2 holds.

3. Nonlocal obstacle problems for the integrable kernels

We now study nonlocal obstacle problems for the integrable kernels of Case 2. As demonstrated in [18], the minimization problem

$$\min_u \frac{1}{2} \int_\Omega \int_\Omega \gamma_I(x, y)(u(x) - u(y))^2\, dydx - \int_\Omega uf\, dx;\ u = 0 \quad \text{on } \Omega_B \tag{25}$$

is well posed.

3.1. Notation and preliminary results

As in [18], we define an "energy" norm, an energy space, and a constrained energy space by

$$|||u||| = \left(\int_\Omega \int_\Omega \gamma_I(x, y)(u(x) - u(y))^2 dydx\right)^{1/2}$$

$$W^I(\Omega) = \{u \in L^2(\Omega) : |||u|||^2 < \infty\}$$

$$W_c(\Omega) := \{u \in W^I(\Omega) : u(x) = 0, x \in \Omega_B\},$$

respectively. We also define the subspace

$$L^2_c(\Omega) := \{u \in L^2(\Omega) : u(x) = 0, x \in \Omega_B\}.$$

We then have the following result.

Lemma 10 ([18]). *The space $W_c(\Omega)$ is equivalent to $L^2_c(\Omega)$, i.e., there exist constant $0 < c_* < c^* < \infty$ such that*

$$c_*\|u\|_{L^2(\Omega)} \le |||u||| \le c^*\|u\|_{L^2(\Omega)} \quad \forall\, u \in W_c(\Omega).$$

Letting $\psi \in W_c(\Omega)$, we define the set

$$\mathcal{B} := \{u \in W_c(\Omega) : u \ge \psi\}.$$

$\mathcal{B} \subset W_c(\Omega)$ is closed and convex.

3.2. Existence and uniqueness of solutions of the nonlocal obstacle problem

Similar to Section 2.3, we define

$$b(u, v) = \int_\Omega \int_\Omega \gamma_l(x, y)(u(x) - u(y))(v(x) - v(y))dxdy \quad \text{and} \quad (f, v) = \int_\Omega f v dx.$$

Alternatively, we seek $u \in \mathcal{B}$ such that

$$b(u, v - u) \ge (f, v - u) \quad \forall\, v \in \mathcal{B}. \tag{26}$$

We then can show that the problem (26) has a unique solution.

Theorem 7. *The problem (26) has a unique solution.*

Proof. Similar to that in [2, Theorem 3.1, Chapter I].

3.3. The approximate problem

Suppose we are given a family $\{W_h\}_h$ of closed subspaces of W_c parameterized by $h \to 0$. Also, suppose there is a family $\{\mathcal{B}_h\}_h$ with $\mathcal{B}_h \subset W_h$ for all h (but not, in general, $\mathcal{B}_h \subset \mathcal{B}$) of closed convex nonempty subsets of W_c that satisfies the following two conditions.

Assumption 3. If $\{v_h\}_h$ is such that $v_h \in \mathcal{B}_h$ for all h and $\{v_h\}_h$ is bounded in W_c, then the weak cluster points of $\{v_h\}_h$ belong to $\mathcal{B}$.

Assumption 4. There exists a set $K \subset W_c$ with $\bar{K} = \mathcal{B}$ and a mapping $r_h : K \to \mathcal{B}_h$ such that, for all $v \in K$, $\lim_{h\to 0} r_h v = v$ strongly in W_c.

Note that if $\mathcal{B}_h \subset \mathcal{B}$ for all h, then Assumption 3 is trivially satisfied because $\mathcal{B}$ is weakly closed.

The problem (26) is approximated by:

$$\text{find } u_h \in \mathcal{B}_h \text{ such that } b(u_h, v_h - u_h) \ge (f, v_h - u_h) \quad \forall\, v_h \in \mathcal{B}_h. \tag{27}$$

Theorem 8. *The problem (27) has a unique solution.*

Proof. The result follows by replacing, in the proof of Theorem 7, W_c by W_h and $\mathcal{B}$ by $\mathcal{B}_h$.

Theorem 9. *Assume that Assumptions 3 and 4 hold and let u and u_h denote the solutions of (26) and (27), respectively. Then,*

$$\lim_{h\to 0} \|u_h - u\| = 0.$$

Proof. As in the proof of [2, Theorem 5.2, Chapter I], we first obtain $\|u_h\| \le C$ for all h.

We next consider the weak convergence of $\{u_h\}_h$. Because $\|u_h\|$ is uniformly bounded, there exists a subsequence, say $\{u_{h_i}\}$, that converges to u^* weakly in W_c. By Assumption 3 about $\{\mathcal{B}_h\}_h$, we have $u^* \in \mathcal{B}$. We prove that u^* is a solution of (26). We have

$$b(u_{h_i}, u_{h_i}) \le b(u_{h_i}, v_{h_i}) - (f, v_{h_i} - u_{h_i}) \quad \forall\, v_{h_i} \in \mathcal{B}_{h_i}. \tag{28}$$

Let $v \in K$ and $v_{h_i} = r_{h_i} v$. Substituting into (28) results in

$$b(u_{h_i}, u_{h_i}) \le b(u_{h_i}, v) + b(u_{h_i}, r_{h_i} v - v) - (f, r_{h_i} v - u_{h_i}). \tag{29}$$

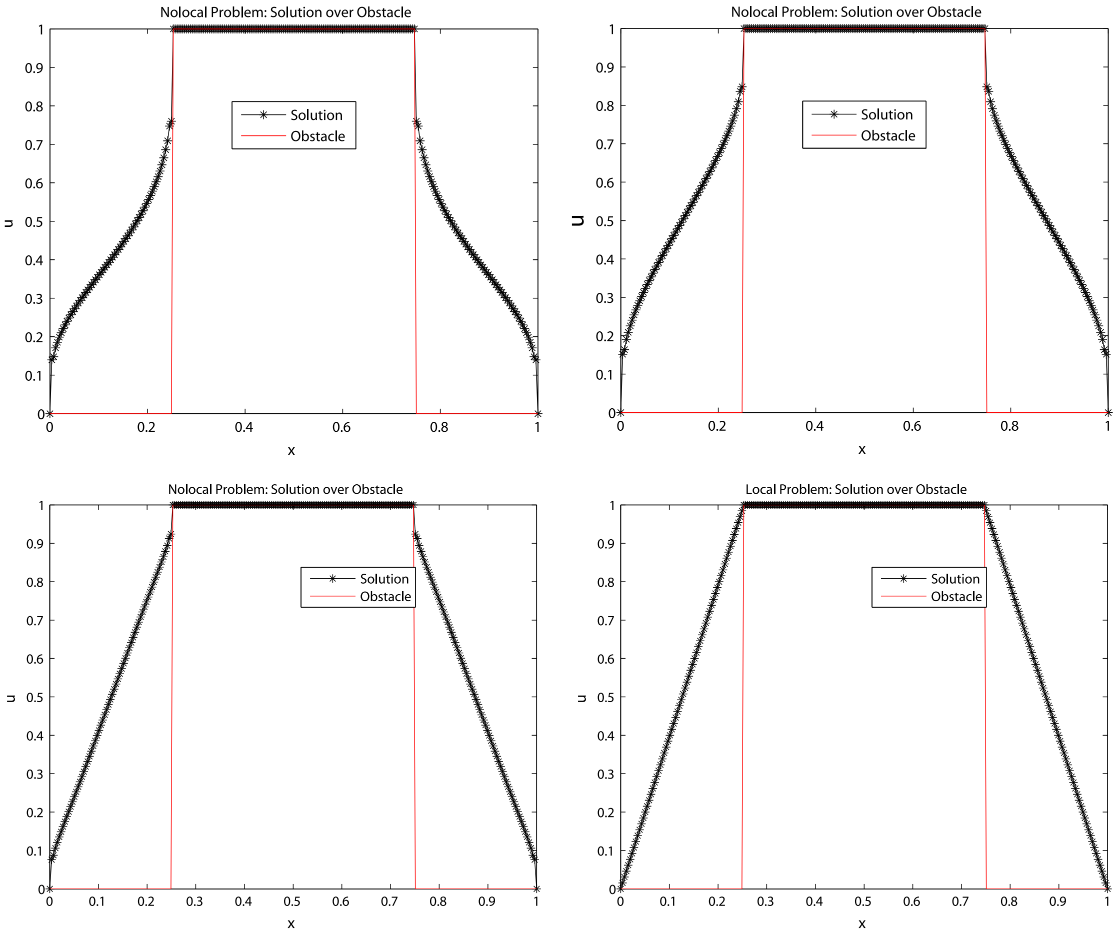

Fig. 1. For the discontinuous obstacle function (31) and for $s = 1/4$ and $h = 1/257$, approximate solutions of the nonlocal obstacle problem for $\delta = 2$ (top-left), $\delta = 0.5$ (top-right), and $\delta = 0.1$ (bottom-left) and the approximate solution of the local obstacle problem (bottom-right).

Because, as $h_i \to 0$, $\{r_{h_i} v\}$ converges strongly to v and $\{u_{h_i}\}$ converges to u^* weakly, taking the limit in (29) and applying the Riesz representation theorem, we have

$$\begin{aligned} b(u, v) &= (Bu, v) = (u, Bv) \\ b(u_{h_i}, v) &= (u_{h_i}, Bv) \\ |b(u_{h_i}, v_{h_i} - v)| &\le C\|u_{h_i}\|\,\|v_{h_i} - v\|, \end{aligned}$$

and then, taking the limit in (29), we obtain

$$\liminf_{h_i \to 0} b(u_{h_i}, u_{h_i}) \le b(u^*, v) - (f, v - u^*) \quad \forall\, v \in K. \tag{30}$$

Similarly to the proof of in [2, Theorem 5.2, Chapter I], we have that the whole sequence $\{u_h\}_h$ converges to u weakly and then strong convergence can be easily obtained.

We again define $\mathfrak{D} = \{v|_{\Omega_I} \in C_0^\infty(\Omega_I)$ and $v|_{\Omega_B} = 0\}$ and choose $K = \mathfrak{D} \cap \mathcal{B}$.

Theorem 10. *Let $\psi \in W_c(\Omega)$. If there exist functions $\{\psi_n\}_n \in W_c$ such that $\psi_n|_{\Omega_I} \in C(\Omega_I)$, $\psi_n \le 0$ in a neighborhood of $\partial\Omega_I$, $\psi_n \ge \psi$, and $\psi_n \to \psi$ as $n \to \infty$ strongly in W_c, then $\overline{\mathfrak{D} \cap \mathcal{B}} = \mathcal{B}$.*

Proof. Combined with the first step of Lemma 9, the proof is similar to that of Theorem 6, using the fact that $C_0^\infty(\Omega_I)$ is dense in $L^2(\Omega_I)$.

To provide examples, we choose W_h as V_h in Section 2.4.1 to again be the space of continuous piecewise linear polynomials constructed with respect to a triangulation of Ω_I. Then, it is natural to approximate $\mathcal{B}$ by

$$\mathcal{B}_h = \{v_h \in W_h \text{ such that } v_h(P) \ge \psi(P)\}.$$

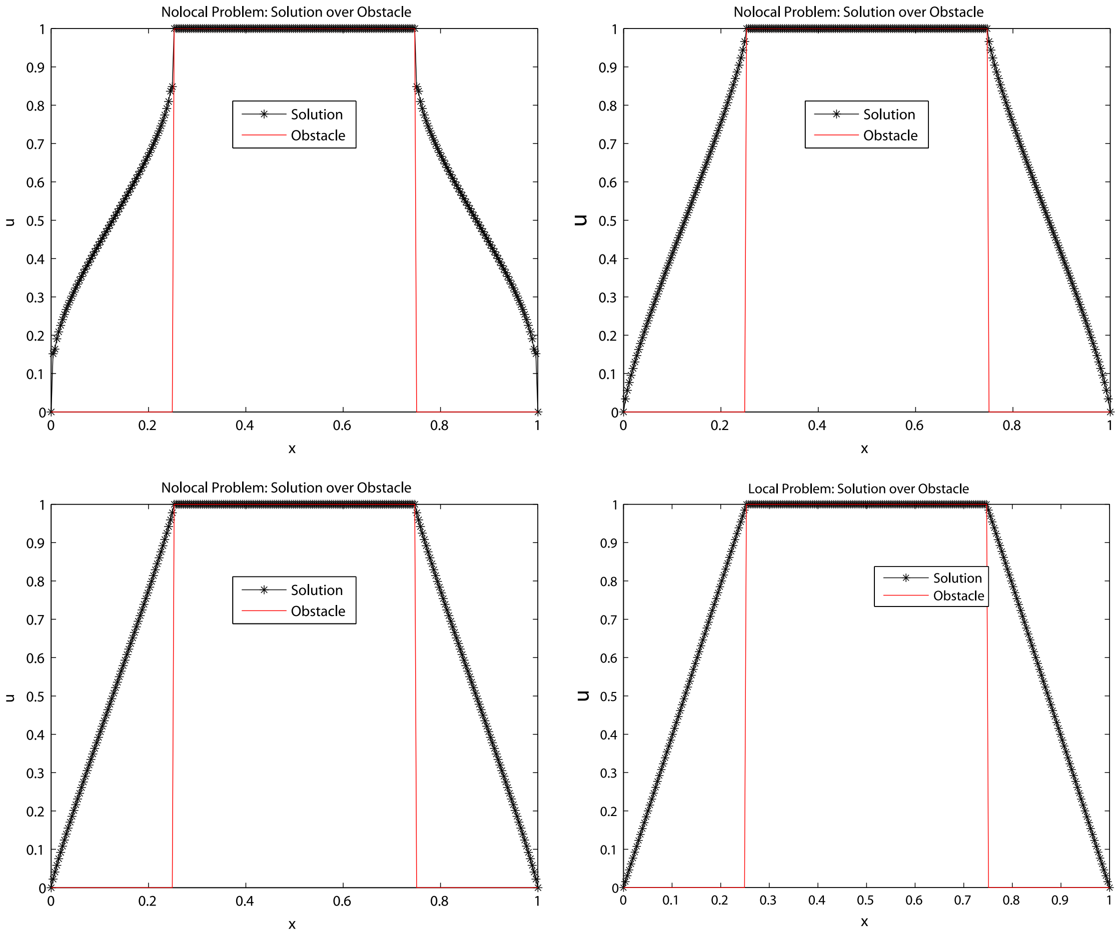

Fig. 2. For the discontinuous obstacle function (31) and for $\delta = 0.5$ and $h = 1/257$, approximate solutions of the nonlocal obstacle problem for $s = 0.25$ (top-left), $s = 0.5$ (top-right), and $s = 0.9$ (bottom-left) and the approximate solution of the local obstacle problem (bottom-right).

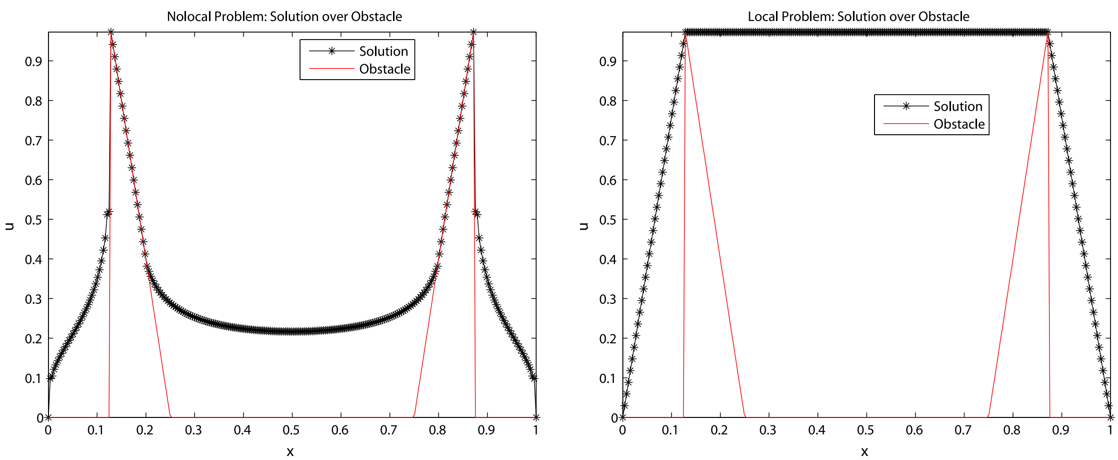

Fig. 3. For the discontinuous obstacle function (32) and for $h = 1/257$, approximate solutions of the nonlocal obstacle problem with $\delta = 2$ and $s = 0.25$ (left) and of the local obstacle problem (right).

Then $\mathcal{B}_h$ is a closed convex nonempty subset of W_h. We also approximate the obstacle function ψ by ψ_h as defined in (17) and we assume that ψ satisfies the assumptions of Theorem 10 and that $\lim_{h\to 0} \psi_h = \psi$ strongly $L^2(\Omega)$. We can verify that Assumptions 3 and 4 are satisfied and that any of (22), (23), and (24) satisfy the assumptions made about ψ.

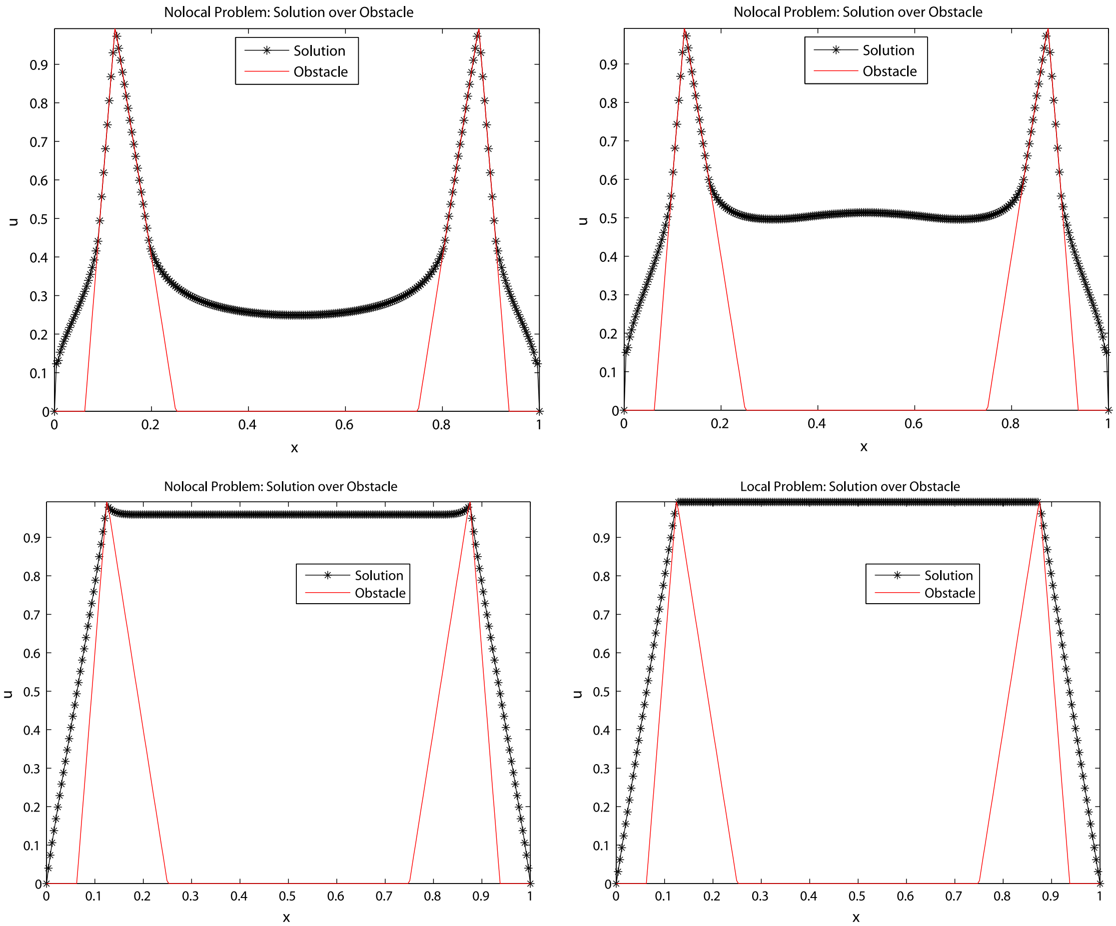

Fig. 4. For the continuous obstacle function (33) and for $h = 1/257$, approximate solutions of the nonlocal obstacle problem for $\delta = 0.5$ and $s = 0.25$ (top-left), $\delta = 0.5$ and $s = 0.5$ (top-right), and $\delta = 0.1$ and $s = 0.9$ (bottom-left) and the approximate solution of the local obstacle problem (bottom-right).

4. Numerical results

We now provide some one-dimensional illustrations of solutions of the nonlocal obstacle problems (10) and (11) to observe the dependence on the parameters appearing in the problems and also to compare with solutions of the local obstacle problem (1). For nonlocal obstacle problems, we use the constrained point over-relaxation method to solve the discrete minimization problem

$$I[u_h] = \min_{v_h} I[v_h], \quad v_h \in \mathcal{A}_h \text{ or } v_h \in \mathcal{B}_h,$$

which is equivalent to (12) or (27), respectively; for details about the method, see [2,1]. All experiments are done with continuous piece-wise linear element.

In all cases, we set the forcing function $f = 0$ and the domains $\Omega_I = (0, 1)$ and $\Omega_B = (-\delta, 0] \cup [1, 1 + \delta)$.

4.1. Non-integrable kernels

We have $c_s = \frac{2-2s}{\delta^{2-2s}}$ throughout this subsection. This scaling choice ensures that, as $\delta \to 0$, the nonlocal obstacle problem reduces to the local one.

The first obstacle we consider is the discontinuous Heaviside function

$$\psi = \begin{cases} 1 & \text{if } 1/4 < x \le 3/4 \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

In Fig. 1, we illustrate the dependence on δ of the approximate solutions of the corresponding nonlocal obstacle problems (10) for fixed values of s and h and also illustrate the solution of the local obstacle problem (1). Note that the discontinuous obstacle (31) does not fall within standard theories for the local (elliptic partial differential equation) obstacle problem.

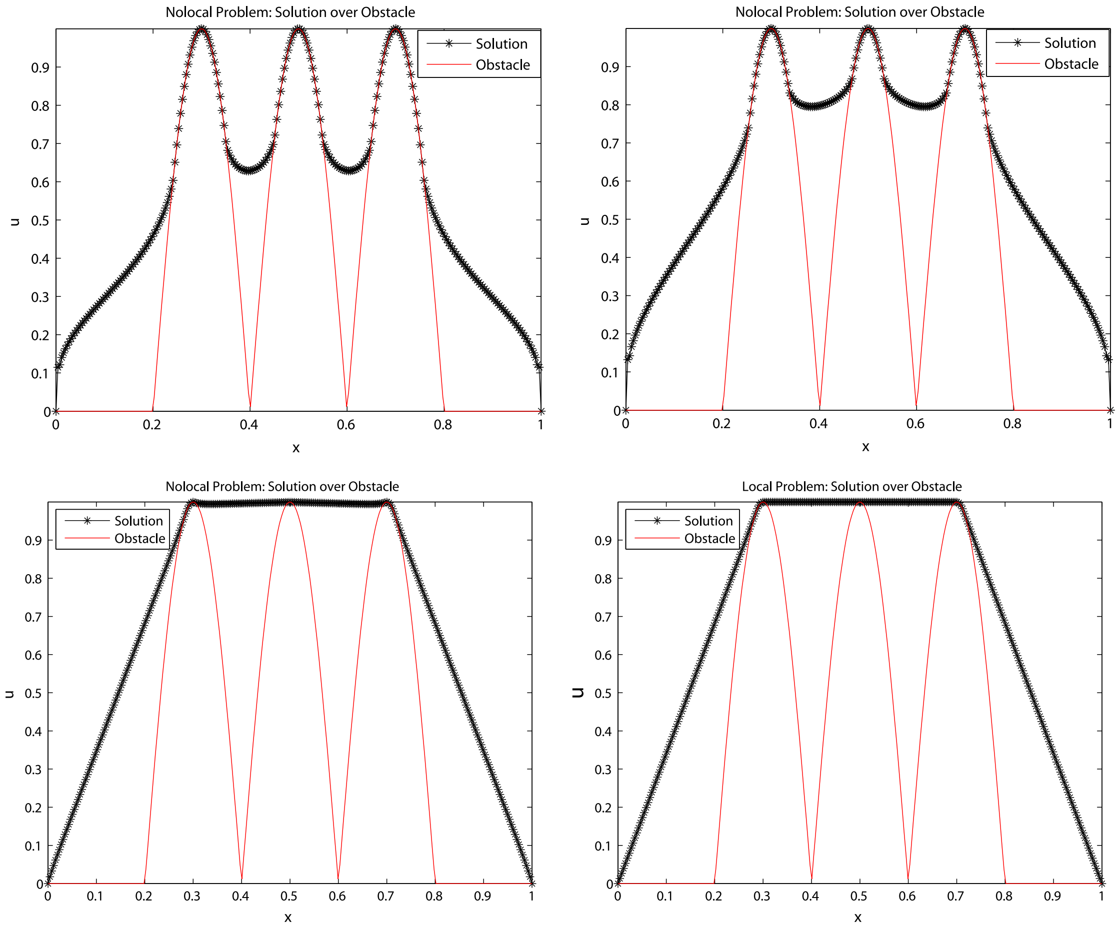

Fig. 5. For the continuous obstacle function (34) and for $h = 1/257$, approximate solutions of the nonlocal obstacle problem for $\delta = 0.5$ and $s = 0.25$ (top-left), $\delta = 0.5$ and $s = 0.5$ (top-right), and $\delta = 0.1$ and $s = 0.9$ (bottom-left) and the approximate solution of the local obstacle problem (bottom-right).

Nonetheless, we can determine approximate solutions as illustrated in the bottom right plot in Fig. 1. Note also that because the local obstacle problem does not admit discontinuous solutions, the approximate solution given in that plot approximates a continuous function so that the best one can hope for is for the "membrane" to stretch over the obstacle. On the other hand, for the nonlocal obstacle problem, discontinuous obstacles do fit into the theory and, for the value $s = \frac{1}{4}$, the nonlocal problem admits discontinuous solutions. As a result, as illustrated in the other plots in Fig. 1, now the "membrane" can better conform to the obstacle. We also observe that the larger the value of δ, i.e., the larger the extent of nonlocal interactions, the better the "membrane" conforms to the obstacle; correspondingly, as δ becomes smaller and smaller, the solution of the nonlocal obstacle problem approaches that of the local obstacle problem.

For the same discontinuous obstacle (31), we illustrate, in Fig. 2, the dependence of the solution of the nonlocal obstacle problem on s. As already mentioned, for $s = \frac{1}{4} < \frac{1}{2}$, the nonlocal obstacle problem admits discontinuous solutions as illustrated in the top-left plot of Fig. 2. However, for $s \geq \frac{1}{2}$, solutions of the obstacle problem are continuous. Thus, even though we have nonlocal interactions, it is not surprising that the top-right and bottom-left plots of that figure look very much like the bottom-right plot which is for the solution of the local obstacle problem.

To further illustrate the differences, for discontinuous obstacle functions, in the solutions of the local and nonlocal obstacle problems, we next consider the obstacle function

$$\psi = \begin{cases} 2 - 8x & \text{if } \dfrac{1}{8} < x \leq \dfrac{1}{4} \\[2ex] 8x - 6 & \text{if } \dfrac{3}{4} < x \leq \dfrac{7}{8} \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{32}$$

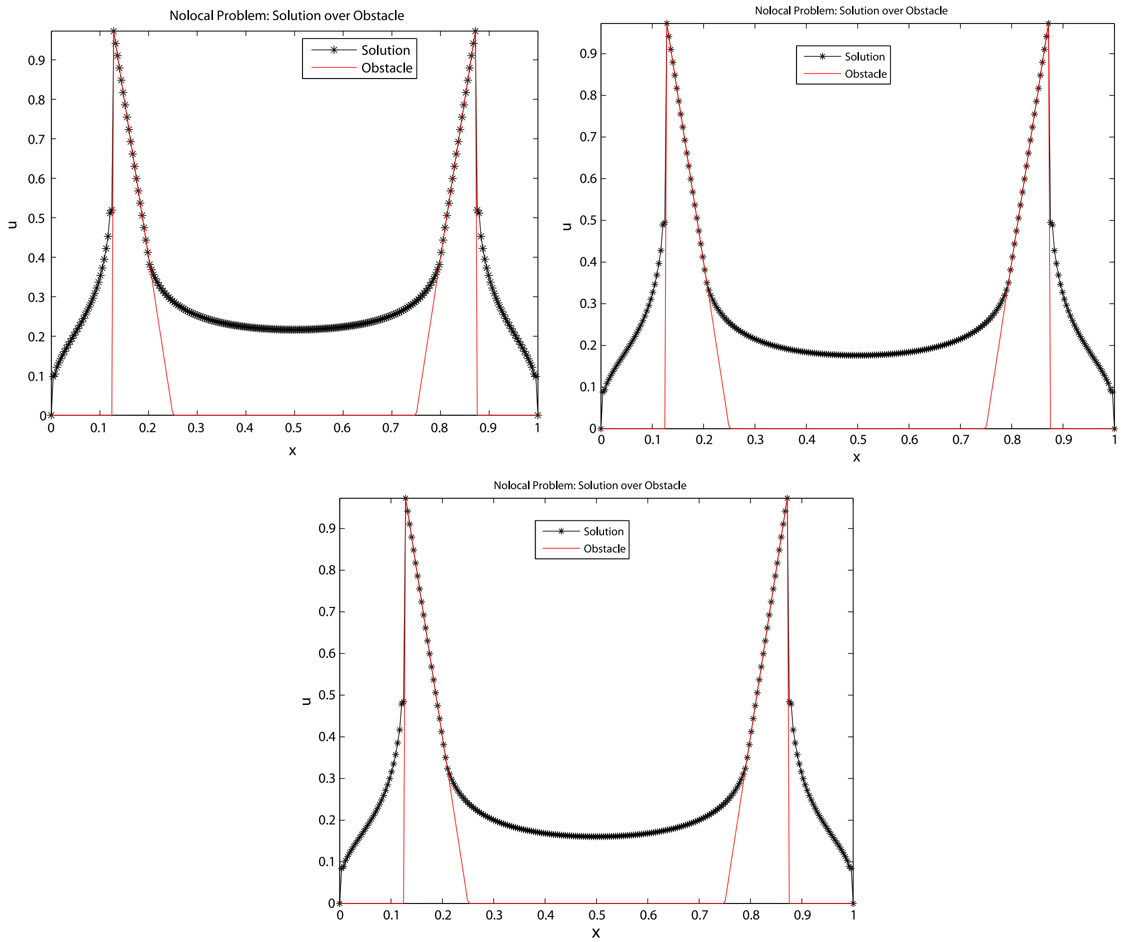

Fig. 6. For the discontinuous obstacle function (32) with $s = 0.25$ and $\delta = 2, 5,$ and 10 (left to right), approximate solutions of the nonlocal obstacle problem for $h = 1/257$.

From Fig. 3, we again see that the best the local obstacle problem can do is the stretch the "membrane" over the top of the obstacle, whereas for $s = \frac{1}{4} < \frac{1}{2}$, the solution of the nonlocal obstacle problem can better conform to the sides of the obstacle.

We next consider, in Fig. 4, the *continuous* obstacle function

$$\psi = \begin{cases} 16x - 1 & \text{if } \dfrac{1}{16} < x \le \dfrac{1}{8} \\ 2 - 8x & \text{if } \dfrac{1}{8} < x \le \dfrac{1}{4} \\ 8x - 6 & \text{if } \dfrac{3}{4} < x \le \dfrac{7}{8} \\ 15 - 16x & \text{if } \dfrac{7}{8} < x \le \dfrac{15}{16} \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

For this case, we can make the same observations which were made above for Figs. 1–3. Note that even though the obstacle function is now continuous, the solution of local obstacle problem still stretches over the peaks of the obstacle function and is not able to conform to the sides of the obstacle. On the other hand, solutions of the nonlocal obstacle problem conform better with the shape of the obstacle, especially for larger values of δ and values of s not close to 1.

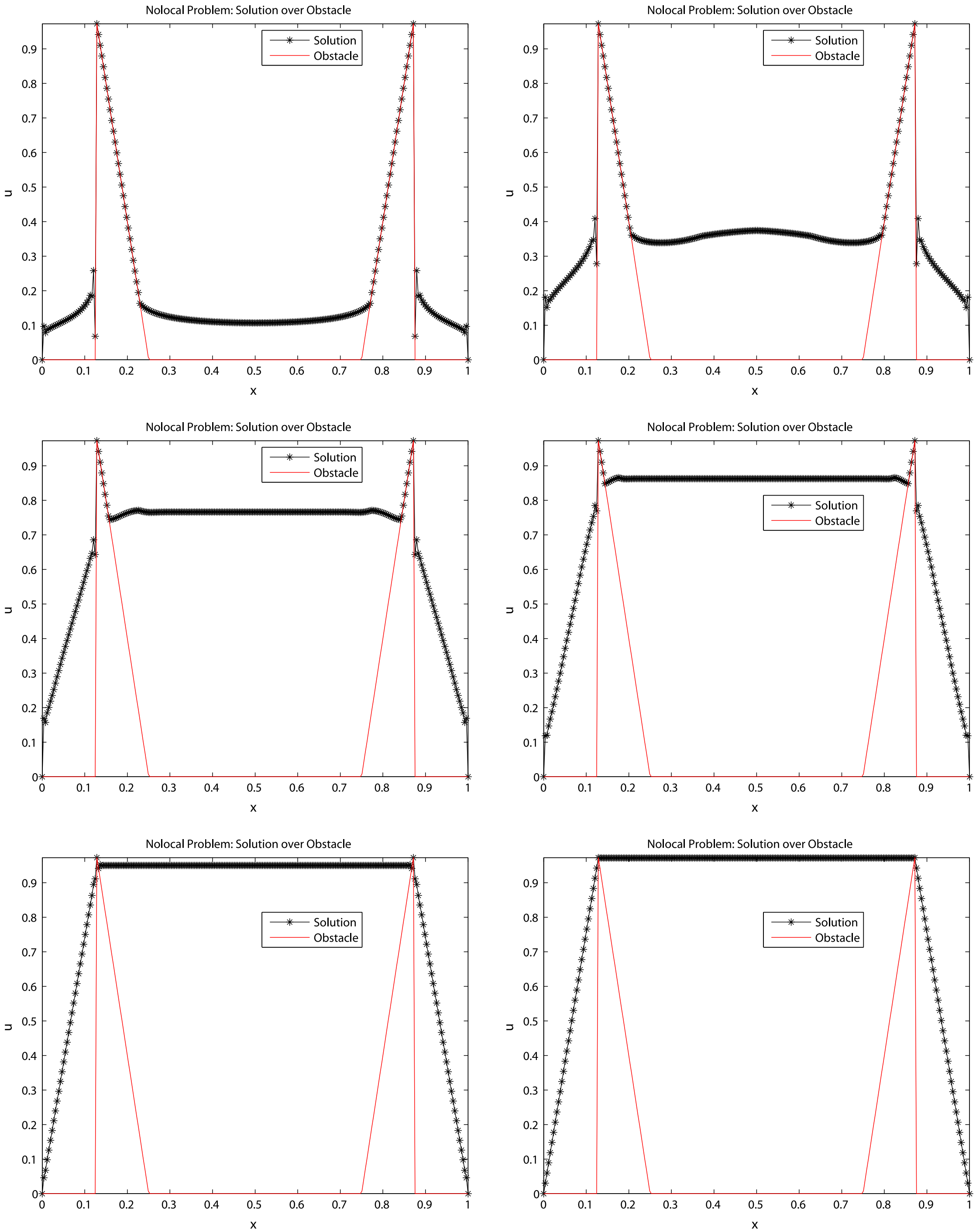

Fig. 7. For the obstacle function (32) and for $l = 0.1$ and $h = 1/257$, approximate solutions of the nonlocal obstacle problem for $\delta = 2$ (top-left), $\delta = 0.5$ (top-right), $\delta = 0.1$ (middle-left), $\delta = 0.05$ (middle-right), $\delta = 0.01$ (bottom-left), and $\delta = 0.001$(bottom-right).

The same observations as those made for Fig. 4 can be made for Fig. 5 which corresponds to the continuous obstacle function

$$\psi = \begin{cases} |\sin(5\pi x)| & \text{if } \frac{1}{5} < x \leq \frac{4}{5} \\ 0 & \text{where else.} \end{cases} \tag{34}$$

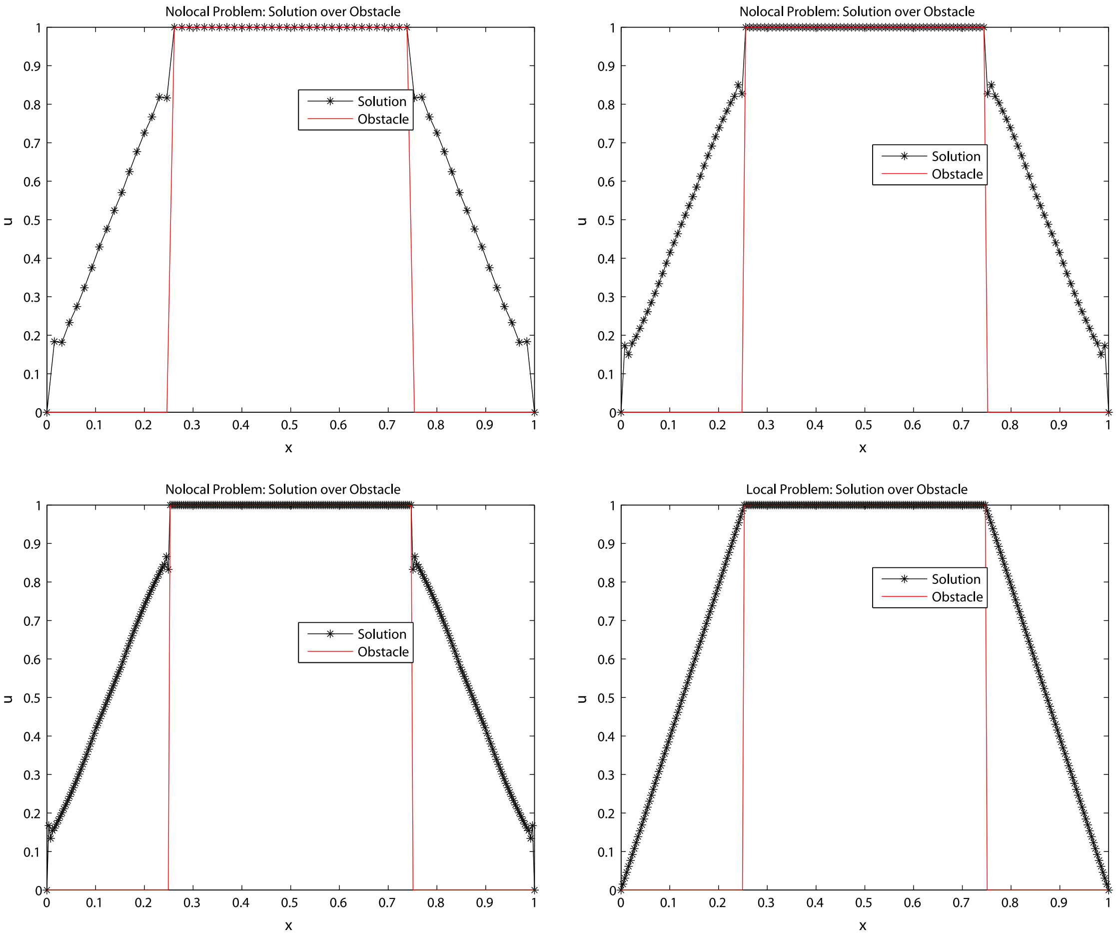

Fig. 8. For the obstacle function (31) and for $l = 0.9$ and $\delta = 0.1$, approximate solutions of the nonlocal obstacle problem for $h = 1/65$ (top-left), $h = 1/129$ (top-right), and $h = 1/257$ (bottom-left) and for the approximate solution of the local obstacle problem for $h = 1/257$ (bottom-right).

For fixed s, as $\delta \to \infty$, the nonlocal operator $\mathcal{L}$ converges to the fractional Laplacian operator $(-\Delta)^s$. For finite but large δ, the nonlocal problem approximates the fractional Laplacian problem on a bounded domain D; see [17] for details. Presumably, the nonlocal obstacle problem with large δ would be a good approximation to the obstacle problem for the fractional Laplacian. The convergence of the solution of the obstacle problems δ increases is observed in Fig. 6.

4.2. Integrable kernels

We next consider the case of the integrable kernel with $c_l = \frac{2+l}{\delta^{2+l}}$. Again, this scaling choice ensures that, as $\delta \to 0$, the nonlocal obstacle problem reduces to the local one.

In Fig. 7, we consider the obstacle function (32). We again see that for the larger values of δ, we obtain much better conformity to the obstacle compared to that for the local solution (see the plot on the right in Fig. 3). We also see that as δ approaches zero, the solution of the nonlocal obstacle problem approaches that of the local obstacle problem.

Our final figure is meant to illustrate convergence with respect to the spatial grid size. For Fig. 8, we consider the discontinuous obstacle function (31) and fix the values of l and δ and examine solutions of the nonlocal obstacle problem for three values of h as well as the solution of local obstacle problem for the finest value of h.

5. Concluding remarks

In this work, we considered two cases of constrained minimization problems. For both cases, if we replace $\mathcal{A}$ by $V^s_{c,0}$ and $\mathcal{B}$ by W_c, then we can obtain the same convergence results for non-constrained problems as in [18]. Compared with Case 1, we can see that we have more freedom to choose the obstacle function ψ for Case 2. One reason is because $C_0^\infty(\Omega_I)$ is naturally dense in W_c, however we are not sure if this is true in V^s_c. In future, we will apply our methods here to obstacle problems for other fractional derivative problems and also try to obtain convergence rates.

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