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Dolbeault dga and L_∞-algebroid of the formal neighborhood ☆

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ABSTRACT

We continue the study the Dolbeault dga of the formal neighborhood of an arbitrary closed embedding of complex manifolds previously defined by the author in [14]. The special case of the diagonal embedding has been analyzed in [13]. We describe here the Dolbeault dga of a general embedding explicitly in terms of the formal differential geometry of the embedding. Moreover, we show that the Dolbeault dga is the completed Chevalley–Eilenberg dga of an L_∞-algebroid structure on the shifted normal bundle of the submanifold. This generalizes the result of Kapranov on the diagonal embedding and Atiyah class.

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1. Introduction

This paper is the continuation of [14] and [13]. In [14] we introduced the notion of *Dolbeault differential graded algebra (dga)* of the formal neighborhood of a closed embedding of complex manifolds, which contains all the formal geometric information of the embedding. Then in [13] we studied the Dolbeault dga of the diagonal embedding and recovered Kapranov's description of the formal neighborhood of the diagonal in terms of the Atiyah class [7]. In the current paper, we will generalize the results in [13] to arbitrary closed embeddings and show how to describe the Dolbeault dga explicitly in terms of the differential geometry of the submanifold, especially when the ambient manifold has a Kähler metric.

The Dolbeault dga $A = (\mathcal{A}^\bullet(\hat{Y}), \overline{\partial})$ for the formal neighborhood $\hat{Y}$ of any closed embedding $i : X \hookrightarrow Y$ of complex manifolds is defined in a canonical way, independently of any auxiliary geometric structures of the manifolds. A certain dg category $\mathcal{P}_A$ of certain dg-modules over $\mathcal{A}$ was built in [14] following the work of Block [2] and was shown to be a dg-enhancement of the derived category of coherent sheaves over $\hat{Y}$. We are interested in explicit construction of objects in $\mathcal{P}_A$, among which the most important one for us is the derived direct image of $\mathscr{O}_X$ on $\hat{Y}$, which will be the main content of our upcoming work [15]. For this purpose, we need a geometric description of the Dolbeault dga, which reflects how the submanifold 'curls' in the ambient manifold.

The precedent paper [13] provides such a description in the case of the diagonal embedding $\Delta : X \hookrightarrow X \times X$. It was shown that there exist isomorphisms between the Dolbeault dga of the formal neighborhood $X^{(\infty)}_{X\times X}$ and the dga $(\mathcal{A}^{0,\bullet}_X(\hat{S}(T^*X)), D_\sigma)$ of the Dolbeault resolution of the completed symmetric algebra of the cotangent bundle of X, where the isomorphisms and the differentials D_σ depend on sections σ of a certain jet bundle with infinite dimensional fibers and related to the Atiyah class of X. In the current paper, we generalize this result to the case of a general embedding $i : X \hookrightarrow Y$, i.e., we show that there are isomorphisms between the dgas $(\mathcal{A}^\bullet(\hat{Y}), \overline{\partial})$ and $(\mathcal{A}^{0,\bullet}_X(\hat{S}(N^\vee)), \mathfrak{D})$, where $N^\vee$ is the conormal bundle of the submanifold and the differential $\mathfrak{D}$ again depends on sections of some jet bundle. The main idea is to consider the graph $\widetilde{i} : X \hookrightarrow X \times Y$ of i, which is also an embedding, and the natural map between the pairs $(X, X\times Y) \to (X, Y)$.

The formal neighborhood $\hat{Y}$ of X inside Y can then be studied by studying the formal neighborhood $X^{(\infty)}_{X\times Y}$ of X inside $X \times Y$. The latter has a similar description as that of the diagonal embedding (Theorem 3.4). Namely, there is an isomorphism between dgas

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X^{(\infty)}_{X\times Y}), \overline{\partial}) \xrightarrow{\simeq} (\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(f^*T^*Y)), D_\sigma)$$

for some differential D_σ. In the case when Y is Kähler, D_σ can be chosen as

$$D_\sigma = \overline{\partial} + \sum_{n\geq 2} \widetilde{R}_n,$$

where R_n is the $(n-2)$-th covariant derivative of the curvature $R_2 = R$ of Y and $\widetilde{R}_n$ is the derivation of degree $+1$ induced by the pullback of R_n to X.

Kapranov's original result was formulated as an L_∞-algebra structure on the shifted tangent bundle $TX[-1]$, whose binary bracket is given by the Atiyah class. In our language, the completed Chevalley–Eilenberg dga of $TX[-1]$ is the Dolbeault dga of $X^{(\infty)}_{X\times X}$. In particular, it induces an Lie algebra structure on $TX[-1]$ as an object in the derived category of X. Our result on general embeddings can also be reformulated as an L_∞-structure on the shifted normal bundle $N[-1]$. The novel discovery here is an extra ∞-anchor map $N[-1] \to TX$ which makes $N[-1]$ into an *L_∞-algebroid*. In the case of the diagonal embedding, it recovers Kapranov's L_∞-algebra, where the anchor map vanishes. To our knowledge, the notion of L_∞-algebroid was first defined in the work of Kjeseth [9,8] under the name of strong homotopy Lie–Rinehart algebras. The notion of L_∞-algebroid also appears in other context, such as string theory [10] and the study of foliations [11].

We want to mention that Calaque, Căldăraru and Tu have established similar results in the algebraic setting [4] from the viewpoint of derived algebraic geometry. They built a dg-Lie algebroid on some dg-sheaf which is quasi-isomorphic to $i_*N[-1]$ in the derived category of Y yet much bigger. Instead of a dg-Lie algebroid, our construction here gives an L_∞-algebroid structure on (the Dolbeault resolution of) the normal bundle whose higher brackets encode the formal geometry of the embedded submanifold directly. This will allow us to derive explicitly formulas for homological invariants of the embedding, such as the quantized cycle class defined by Grivaux [6]. The quantized cycle class specializes to the usual Todd class for a diagonal embedding and hence not surprisingly has a close relation with the Hirzebruch–Riemann–Roch formula. Our new approach to the Todd class will lead to a new derivation of the Hirzebruch–Riemann–Roch formula from the geometry of jet bundles [15].

The paper is organized as follows. In § 2 we recall the general definition of the Dolbeault dga $\mathcal{A}^\bullet(\hat{Y})$ for the formal neighborhood $\hat{Y}$ of an arbitrary closed holomorphic embedding $i : X \hookrightarrow Y$ of complex manifolds. In § 3 we briefly review our reformulation [13] of Kapranov's result of the diagonal embedding. We recall various infinite dimensional fiber bundles arising from formal geometry, which we already used heavily in [13] to derive Kapranov's results. We then apply them in § 4 to get a description of the Dolbeault dga of an arbitrary embedding, in which other differential geometric quantities other than the curvature, such as shape operator, come into the picture. The main result is Theorem 4.5. For convenience we only discuss the Kähler case, yet the formulas make sense in broader context (see Remark 4.8). We construct an isomorphism from our canonical yet abstractly defined Dolbeault dga $\mathcal{A}^\bullet(\hat{Y})$ to a concrete dga, namely the completed symmetric algebra $\mathcal{A}^{0,\bullet}(\hat{S}(N^\vee))$ of the conormal bundle of X in Y, and compute the differential on it. The main result is Theorem 4.5. Finally, § 5 is contributed to the equivalent L_∞-description. We will recall the basic definitions of L_∞-algebroids from [11]. For convenience, we will mainly use $L_\infty[1]$-algebroid, which is a shifted version of L_∞-algebroid, since the signs involved in the formulas are enormously simplified.

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2. Dolbeault dga of formal neighborhoods

2.1. Definitions and notations

We review the notations and definitions from [14]. Let $i : (X, \mathscr{O}_X) \hookrightarrow (Y, \mathscr{O}_Y)$ be a closed embedding of complex manifolds where $\mathscr{O}_X$ and $\mathscr{O}_Y$ are the structure sheaves of germs of holomorphic functions over the complex manifolds X and Y respectively. Let $\mathscr{I}$ the ideal sheaf of $\mathscr{O}_Y$ of holomorphic functions vanishing along X. The main objects studied by this paper are *the r-th formal neighborhood* $\hat{Y}^{(r)}$ of X in Y, which is defined as the ringed space $(X, \mathscr{O}_{\hat{Y}^{(r)}})$ whose the structure sheaf is

$$\mathscr{O}_{\hat{Y}^{(r)}} = \mathscr{O}_Y / \mathscr{I}^{r+1},$$

and *the (complete) formal neighborhood* $\hat{Y} = \hat{Y}^{(\infty)}$, which is defined to be the ringed space $(X, \mathscr{O}_{\hat{Y}})$ where

$$\mathscr{O}_{\hat{Y}} = \varprojlim_r \mathscr{O}_{\hat{Y}^{(r)}} = \varprojlim_r \mathscr{O}_X / \mathscr{I}^{r+1}.$$

We will also use the notations $X_Y^{(\infty)} = \hat{Y}$ and $X_Y^{(r)} = \hat{Y}^{(r)}$ when we need to emphasize the submanifolds.

In [14] *the Dolbeault differential graded algebra (dga)* of the embedding $i : X \hookrightarrow Y$ is defined as follows. Let $(\mathcal{A}^{0,\bullet}(Y), \overline{\partial}) = (\wedge^\bullet \Omega_Y^{0,1}, \overline{\partial})$ be the Dolbeault complex of Y, thought of as a dga. For each nonnegative integer r, $\mathfrak{a}_r^\bullet$ is set to be the graded ideal of $\mathcal{A}^{0,\bullet}(Y)$ consisting of those forms $\omega \in \mathcal{A}^{0,\bullet}(Y)$ satisfying

$$i^*(\mathcal{L}_{V_1}\mathcal{L}_{V_2}\cdots\mathcal{L}_{V_l}\omega) = 0, \quad \forall\ 1 \le j \le l, \tag{2.1}$$

for any collection of smooth (1,0)-vector fields $V_1, V_2, \ldots, V_l$ over Y, where $0 \le l \le r$. By Proposition 2.1, [14], $\mathfrak{a}_r^\bullet$ is invariant under the action of $\overline{\partial}$ and hence is a dg-ideal of $(\mathcal{A}^{0,\bullet}(Y), \overline{\partial})$.

Definition 2.1 *(Definition 2.3, [14]).* The *Dolbeault dga of the r-th formal neighborhood* $\hat{Y}^{(r)}$ is the quotient dga

$$\mathcal{A}^\bullet(\hat{Y}^{(r)}) := \mathcal{A}^{0,\bullet}(Y)/\mathfrak{a}_r^\bullet.$$

The *Dolbeault dga of the complete formal neighborhood* $\hat{Y}$ is defined to be the inverse limit

$$\mathcal{A}^\bullet(\hat{Y}) = \mathcal{A}^\bullet(\hat{Y}^{(\infty)}) := \varprojlim_r \mathcal{A}^\bullet(\hat{Y}^{(r)}).$$

We will write $\mathcal{A}(\hat{Y}) = \mathcal{A}^0(\hat{Y})$ and $\mathcal{A}(\hat{Y}^{(r)}) = \mathcal{A}^0(\hat{Y}^{(r)})$ for the zeroth components of the Dolbeault dgas.

The Dolbeault dga $\mathcal{A}^\bullet(\hat{Y}^{(r)})$ can be sheafified to a soft sheaf of dgas $\mathscr{A}^\bullet(\hat{Y}^{(r)})$ over X for $r \in \mathbb{N}$ or $r = \infty$ (see [14] for details). Moreover, there are natural inclusions of sheaves of algebras $\mathscr{O}_{\hat{Y}^{(r)}} \hookrightarrow \mathscr{A}(\hat{Y}^{(r)})$ The following result was proved in [14].

Theorem 2.2 *(Prop. 2.8., [14]). For any nonnegative integer r or $r = \infty$, the complex of sheaves*

$$0 \to \mathscr{O}_{\hat{Y}^{(r)}} \to \mathscr{A}^0_{\hat{Y}^{(r)}} \xrightarrow{\overline{\partial}} \mathscr{A}^1_{\hat{Y}^{(r)}} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{A}^m_{\hat{Y}^{(r)}} \to 0$$

is exact, where $m = \dim X$. In other words, $(\mathscr{A}^\bullet_{\hat{Y}^{(r)}}, \overline{\partial})$ is a soft resolution of $\mathscr{O}_{\hat{Y}^{(r)}}$.

As the completion of $\mathcal{A}^{0,\bullet}(Y)$ with respect to the filtration $\mathfrak{a}_r^\bullet$, the dga $\mathcal{A}^\bullet(\hat{Y})$ is itself filtered and its associated graded dga is

$$\operatorname{gr} \mathcal{A}^\bullet(\hat{Y}) \simeq (\mathcal{A}^{0,\bullet}(Y)/\mathfrak{a}_0^\bullet) \oplus \bigoplus_{r=0}^{\infty} \mathfrak{a}_r^\bullet/\mathfrak{a}_{r+1}^\bullet.$$

Note that $\mathcal{A}^{0,\bullet}(X) \simeq \mathcal{A}^{0,\bullet}(Y)/\mathfrak{a}_0^\bullet$ and $\mathfrak{a}_r^\bullet/\mathfrak{a}_{r+1}^\bullet$ are dg-modules over $(\mathcal{A}^{0,\bullet}(X), \overline{\partial})$. We define, for each $r \ge 0$, a 'cosymbol map' of complexes

$$\tau_r : (\mathfrak{a}_r^\bullet/\mathfrak{a}_{r+1}^\bullet) \to \mathcal{A}_X^{0,\bullet}(S^{r+1}N^\vee), \tag{2.2}$$

where $S^{r+1}N^\vee$ is the $(r+1)$-fold symmetric tensor of the conormal bundle $N^\vee$ of the embedding. Given any $(r+1)$-tuple of smooth sections $\mu_1, \ldots, \mu_{r+1}$ of N, we lift them to smooth sections of $TY|_X$ and extend to smooth (1,0)-tangent vector fields $\tilde{\mu}_1, \ldots, \tilde{\mu}_{r+1}$

on Y (defined near X). We then define the image of $\omega + \mathfrak{a}^\bullet_{r+1} \in \mathfrak{a}^\bullet_r/\mathfrak{a}^\bullet_{r+1}$ under τ for any $\omega \in \mathfrak{a}^\bullet_r$, thought of as linear functionals on $(N)^{\otimes(r+1)}$, by the formula

$$\tau_r(\omega + \mathfrak{a}^\bullet_{r+1})(\mu_1 \otimes \cdots \otimes \mu_{r+1}) = i^* \mathcal{L}_{\tilde{\mu}_1} \mathcal{L}_{\tilde{\mu}_2} \cdots \mathcal{L}_{\tilde{\mu}_{r+1}} \omega. \tag{2.3}$$

The map τ_r is well-defined and is independent of the choice of the representative ω and $\tilde{\mu}_j$'s. Moreover, the tensor part of $\tau_r(\omega + \mathfrak{a}^\bullet_{r+1})$ is indeed symmetric.

Proposition 2.3. *The map τ_r in (2.2) is an isomorphism of dg-modules over $(\mathcal{A}^{0,\bullet}(X), \overline{\partial})$.*

Corollary 2.4. *We have a natural isomorphism of dgas*

$$\operatorname{gr} \mathcal{A}^\bullet(\hat{Y}) \simeq \bigoplus_{n=0}^{\infty} \mathcal{A}^{0,\bullet}_X(S^n N^\vee).$$

3. Dolbeault dgas of diagonal embeddings

Among all important and interesting examples is the diagonal embedding $\Delta : X \hookrightarrow X \times X$ of a complex manifold X into product of two copies of itself. We then have the formal neighborhood of the diagonal, $X^{(\infty)}_{X\times X}$, which is understood as the dga $(\mathcal{A}^{0,\bullet}(X^{(\infty)}_{X\times X}), \overline{\partial})$ constructed in the previous section. It is isomorphic to the Dolbeault resolution of the infinite holomorphic jet bundle $\mathcal{J}^\infty_X$. In this case one can write down the $\overline{\partial}$-derivation explicitly (at least when X is Kähler) due to a theorem below by Kapranov [7], of which we will reproduce the Kähler case in a slightly different way and discuss the general situation later.

Intuitively one would expect that there is an isomorphism

$$\mathcal{A}^{0,\bullet}(X^{(\infty)}_{X\times X}) \simeq \mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(T^*X)) \tag{3.1}$$

by taking 'Taylor expansions', where $\hat{S}^\bullet(T^*X)$ is the bundle of complete symmetric tensor algebra generated by the cotangent bundle of X (which is natural identified with the conormal bundle of the diagonal embedding). Such an isomorphism does exist, but there is no canonical way to define it since one need to first choose some local coordinates to get Taylor expansions. Indeed we will see that the isomorphism depends (in a more or less 1–1 manner) on a smooth choice of formal (holomorphic) coordinates on X.

3.1. Diagonal embeddings and jet bundles

We consider the case of diagonal embeddings. Let X be a complex manifold and let $\Delta : X \hookrightarrow X \times X$ be the diagonal map. For convenience, we write $\Delta^{(r)} = X^{(r)}_{X\times X}$ for $r \in \mathbb{N}$ or $r = \infty$ throughout this section. Denote by $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$ the projections onto the first and second component of $X \times X$ respectively. The jet bundle $\mathcal{J}^r_X$ of order r ($r \in \mathbb{N}$ or $r = \infty$) can be viewed as the sheaf of algebras

$$\mathcal{J}_X^r = \mathrm{pr}_{1*}\,\mathscr{O}_{\Delta^{(r)}},$$

which is a sheaf of $\mathscr{O}_X$-modules where the $\mathscr{O}_X$-actions are induced from the projection pr_1. The sheaf $(\mathscr{A}^{0,\bullet}(\mathcal{J}_X^r), \overline{\partial})$ of Dolbeault complexes of $\mathcal{J}_X^r$ is a sheaf of dgas and its global sections forms a dga

$$\mathcal{A}^{0,\bullet}(\mathcal{J}_X^r) = \Gamma(X, \mathscr{A}^{0,\bullet}(\mathcal{J}_X^r)).$$

Since $\mathcal{J}_X^r$ is a sheaf of $\mathscr{O}_X$-modules, $\mathcal{A}^{0,\bullet}(\mathcal{J}_X^r)$ is a dga over the dga $(\mathcal{A}^{0,\bullet}(X), \overline{\partial})$.

The Dolbeault dga $(\mathcal{A}^\bullet(\Delta^{(r)}), \overline{\partial})$ of the formal neighborhood $(\mathcal{A}^\bullet(\Delta^{(r)}), \overline{\partial})$ ($r \in \mathbb{N}$ or $r = \infty$) is also an $(\mathcal{A}^{0,\bullet}(X), \overline{\partial})$-dga via the compositions of homomorphisms of dgas

$$\mathcal{A}^{0,\bullet}(X) \xrightarrow{\mathrm{pr}_1^*} \mathcal{A}^{0,\bullet}(X \times X) \to \mathcal{A}^\bullet(\Delta^{(r)}).$$

Proposition 3.1 *(Prop. 2.8., [13]). The natural inclusion $\mathscr{O}_{\Delta^{(r)}} \hookrightarrow \mathscr{A}_{\Delta^{(r)}}$ determines an $(\mathcal{A}^{0,\bullet}(X), \overline{\partial})$-linear quasi-isomorphism of dgas*

$$I_r : \mathcal{A}^{0,\bullet}(\mathcal{J}_X^r) \xrightarrow{\simeq} \mathcal{A}^\bullet(\Delta^{(r)})$$

either when $r \in \mathbb{N}$ or $r = \infty$. Similar results hold for the corresponding sheaves.

3.2. Formal geometry

Formal geometry is a technique of study differential geometry by solving them first universally over a formal disc and equivariantly with respect to the Lie algebra of vector fields on the polydisc. Among the few early references are the work [5] by Gelfand and Kazhdan and [3] by R. Bott. More recent formulations relevant to our context can be found in the work [1] by Bezrukavnikov and Kaledin and [7] by Kapranov.

3.2.1. Differential geometry on formal discs

Fix a complex vector space V of dimension n. The formal disc $\widehat{V}$ is the formal neighborhood of 0 in V. Its function algebra is the formal power series algebra

$$\mathcal{F} = \mathbb{C}[\![V^*]\!] = \hat{S}(V^*) = \prod_{i \geq 0} S^i V^*.$$

It is a complete regular local algebra with a unique maximal ideal $\mathfrak{m}$ consisting of formal power series with vanishing constant term. The associated graded algebra with respect to the usual $\mathfrak{m}$-filtration is the symmetric algebra

$$\mathrm{gr}\,\mathcal{F} = S(V^*) = \bigoplus_{i \geq 0} S^i V^*.$$

Since we are in the complex analytic situation, we endow $\mathcal{F}$ with the canonical Fréchet topology. In algebraic setting, one need to use the $\mathfrak{m}$-adic topology on $\mathcal{F}$. However, the associated groups and spaces in question remain the same, though the topologies on them will be different. Since our arguments work for both Fréchet and $\mathfrak{m}$-adic settings, the topology will not be mentioned explicitly unless necessary. We also use $\widehat{V} = \operatorname{Spf} \mathcal{F}$ to denote the formal polydisc, either as a formal analytic space or a formal scheme.

We recall several definitions in §4, [13]:

$$\begin{aligned}
G^{(\infty)}(V) &= \text{the proalgebraic group of automorphisms of the formal space } \widehat{V}, \\
J^{(\infty)}(V) &= \operatorname{Ker}(d_0 : G^{(\infty)} \to \mathbf{GL}_n(V)), \text{ where } d_0(\phi) \text{ is the tangent map of } \phi \in G^{(\infty)} \text{ at } 0, \\
\mathbf{g}^{(\infty)}(V) &= \text{Lie algebra of } G^{(\infty)}(V) \\
&= \text{Lie algebra of formal vector fields on } \widehat{V} \text{ vanishing at } 0 \\
&= \prod_{i \geq 1} V \otimes S^i V^* \\
&= \prod_{i \geq 1} \operatorname{Hom}(V^*, S^i V^*), \qquad (3.2) \\
\mathbf{j}^{(\infty)}(V) &= \text{Lie algebra of } J^{(\infty)}(V) \\
&= \text{Formal vector fields on } \widehat{V} \text{ with vanishing constant and linear terms} \\
&= \prod_{i \geq 2} V \otimes S^i V^* \\
&= \prod_{i \geq 2} \operatorname{Hom}(V^*, S^i V^*),
\end{aligned}$$

The Lie algebras $\mathbf{g}^{(\infty)}(V)$ and $\mathbf{j}^{(\infty)}(V)$ act on $\mathcal{F} = \prod_{i \geq 0} S^i V^*$ as derivations in the obvious way. For convenience we will write $G^{(\infty)} = G^{(\infty)}(V)$ and so on.

The short exact sequence

$$1 \to J^{(\infty)} \to G^{(\infty)} \to \mathbf{GL}_n(V) \to 1 \tag{3.3}$$

splits canonically by identifying elements of $\mathbf{GL}_n = \mathbf{GL}_n(V)$ as jets of linear transformations on $\widehat{V}$. Hence $G^{(\infty)}$ is the semidirect product $G^{(\infty)} = J^{(\infty)} \rtimes \mathbf{GL}_n$. The canonical bijection of sets

$$q : J^{(\infty)} \xrightarrow{\simeq} G^{(\infty)}/\mathbf{GL}_n \tag{3.4}$$

is $G^{(\infty)}$-equivariant if we endow $G^{(\infty)}/\mathbf{GL}_n$ with the usual left $G^{(\infty)}$-action and the left $G^{(\infty)}$-action on $J^{(\infty)}$ is given by

$$\psi \cdot \varphi = \psi \circ \varphi \circ (d_0\psi)^{-1}, \quad \forall\, \psi \in G^{(\infty)},\ \forall\, \varphi \in J^{(\infty)}. \tag{3.5}$$

In view of (3.5), it is natural to interpret $J^{(\infty)}$ as the set of *formal exponential maps* $\varphi : \widehat{T_0V} \xrightarrow{\simeq} \widehat{V}$, where $\widehat{T_0V}$ is the completion of the tangent space T_0V at the origin, which is canonically identified with $\widehat{V}$ itself. Such φ induces the identity map on the tangent spaces of the two formal spaces at the origins. Moreover, giving a formal exponential map φ is equivalent to giving an isomorphism between filtered algebras $\varphi^* : \mathcal{F} \to \mathcal{F}_T$, where

$$\mathcal{F}_T := \hat{S}((T_0V)^*) = \hat{S}(V^*)$$

is the algebra of functions on $\widehat{T_0V}$, such that the induced isomorphism between the associated graded algebras, which are both $S(V^*)$, is the identity map. Notice that $\mathcal{F}_T$ carries a $G^{(\infty)}$-action via the natural action of $\mathbf{GL}_n$ precomposed by the map $G^{(\infty)} \to \mathbf{GL}_n$, so it is different from the $G^{(\infty)}$-action on $\mathcal{F}$ even though both $\mathcal{F}$ and $\mathcal{F}_T$ are identified with $\hat{S}(V^*)$. The two actions induce a conjugation action of $G^{(\infty)}$ on $J^{(\infty)}$, which is exactly the one defined by (3.5).

In [13] we introduced the set $\mathfrak{Conn}$ of all flat torsion-free connections on $\widehat{V}$, i.e., each element of $\mathfrak{Conn}$ is a $\mathbb{C}$-linear map

$$\nabla : T\widehat{V} \to T\widehat{V} \otimes_{\mathscr{O}_{\widehat{V}}} T^*\widehat{V}$$

satisfying the Leibniz rule and the flatness and torsion-freeness conditions. By abuse of notation, we also use ∇ to denote the induced connection on the cotangent bundle of $\widehat{V}$ and its associated tensor bundles:

$$\nabla : T^*\widehat{V} \to T^*\widehat{V} \otimes_{\mathscr{O}_{\widehat{V}}} T^*\widehat{V}.$$

There is a canonical bijection

$$\exp : \mathfrak{Conn} \xrightarrow{\simeq} J^{(\infty)}, \quad \nabla \mapsto \exp_{\nabla}, \tag{3.6}$$

which assigns to each connection ∇ a formal exponential map $\exp_{\nabla} : \widehat{T_0V} \to \widehat{V}$, which is completely determined by the way it pulls back functions $f \in \mathcal{F}$,

$$\exp^*_{\nabla}(f) = (\nabla^i f|_0)_{i\geq 0} = (f(0), \nabla f|_0, \nabla^2 f|_0, \cdots) \in \prod_{i\geq 0} S^i(T_0V)^* = \mathcal{F}_T, \tag{3.7}$$

where $\nabla f = df$ and $\nabla^i f = \nabla^{i-1} df$ for $i \geq 2$. The torsion-freeness and flatness of ∇ guarantee that the terms in the expression are symmetric tensors.

Moreover, $G^{(\infty)}$ naturally acts on $\mathfrak{Conn}$ from left by pushing forward connections via automorphisms of $\widehat{V}$. By Lemma 4.1, [13], the bijection $\exp : \mathfrak{Conn} \to J^{(\infty)}$ is $G^{(\infty)}$-equivariant.

3.2.2. Bundle of formal coordinates and connections

We introduce the bundle of formal coordinate systems X_{coord} of a smooth complex manifold X from §4.4., [7]. At each point $x \in X$ the fiber $X_{coord,x}$ is the space of infinite jets of local biholomorphisms near the origin $\varphi : V \simeq \mathbb{C}^n \to X$ with $\varphi(0) = x$. X_{coord} is naturally a holomorphic principal $G^{(\infty)}$-bundle.

We can apply the associated bundle construction to the principal $G^{(\infty)}$-bundle X_{coord} to globalize various objects defined in §3.2.1. There is a canonical isomorphism between bundles of algebras

$$X_{coord} \times_{G^{(\infty)}} \mathcal{F} \simeq \mathcal{J}_X^\infty$$

and hence we have a tautological trivialization of the jet bundle $\mathcal{J}_X^\infty$ over X_{coord}

$$X_{coord} \times_X \mathcal{J}_X^\infty \simeq X_{coord} \times \mathcal{F}. \tag{3.8}$$

Other related jet bundles, such as $\mathcal{J}^\infty T_X$ ($\mathcal{J}^\infty T_X^*$, resp.), the jet bundle of the tangent bundle (cotangent bundle, resp.) as a bundle on X, can be obtained in a similar way by the associated bundle construction.

Another related bundle $\pi : X_{exp} \to X$ is the bundle of formal exponential maps introduced in [7], which we denote by X_{exp}. At each $x \in X$ the fiber $X_{exp,x}$ is the space of jets of holomorphic maps $\phi : T_xX \to X$ such that $\phi(0) = x$, $d_0\phi = \mathrm{Id}_{T_xX}$. We have a canonical isomorphism

$$X_{exp} \to X_{coord} \times_{G^{(\infty)}} J^{(\infty)} \simeq X_{coord} \times_{G^{(\infty)}} (G^{(\infty)}/\mathbf{GL}_n)$$

which hence induces a biholomorphism

$$X_{coord}/\mathbf{GL}_n \simeq X_{exp} \tag{3.9}$$

We also defined in [13] the bundle of jets of flat torsion-free connection

$$X_{conn} = X_{coord} \times_{G^{(\infty)}} \mathfrak{Conn}$$

whose fiber at a each point $x \in X$ consists of all flat torsion-free connections on the formal neighborhood of x. The $G^{(\infty)}$-equivariant bijection $\exp : \mathfrak{Conn} \xrightarrow{\simeq} J^{(\infty)}$ induces an identification between the X_{conn} and X_{exp}. We regard them as the same bundle with different descriptions. From now on we will use either X_{conn} or X_{exp} depending on the context and we will also denote the projection $X_{conn} \to X$ by π.

There is a tautological flat and torsion-free connection over X_{conn},

$$\nabla_{tau} : \pi^*\mathcal{J}^\infty T^*X \to \pi^*\mathcal{J}^\infty T^*X \otimes_{\pi^*\mathcal{J}_X^\infty} \pi^*\mathcal{J}^\infty T^*X,$$

which is $\mathscr{O}_{X_{conn}}$-linear and satisfies the Leibniz rule with respect to the differential

$$\widetilde{d}^{(\infty)} : \pi^*\mathcal{J}_X^\infty \to \pi^*\mathcal{J}^\infty T^*X$$

that is the pullback of

$$d^{(\infty)} : \mathcal{J}_X^\infty \to \mathcal{J}^\infty T^* X.$$

Here $d^{(\infty)}$ is a $\mathscr{O}_X$-linear differential obtained by apply the associated bundle construction with X_{coord} and the de Rham differential $d : \mathscr{O}_{\widehat{V}} \to T^*\widehat{V}$ on the formal disc $\widehat{V}$.

On the other hand, since X_{conn} can also be interpreted as the bundle X_{exp} of formal exponential maps, we have a tautological isomorphism between sheaves of algebras over $X_{conn} = X_{exp}$,

$$Exp^* : \pi^*(X_{coord} \times_{G^{(\infty)}} \mathcal{F}) \to \pi^*(X_{coord} \times_{G^{(\infty)}} \mathcal{F}_T).$$

The domain is identified as $\pi^*\mathcal{J}_X^\infty$ or $\pi^*\mathscr{O}_{X^{(\infty)}_{X\times X}}$, the pullback via π of the structure sheaf of the formal neighborhood of the diagonal in $X \times X$, while for the codomain we have

$$X_{coord} \times_{G^{(\infty)}} \mathcal{F}_T \simeq X_{coord} \times_{G^{(\infty)}} \mathbf{GL}_n \times_{\mathbf{GL}_n} \mathcal{F}_T \simeq X_{coord}/J^{(\infty)} \times_{\mathbf{GL}_n} \mathcal{F}_T$$

by our definition of the $G^{(\infty)}$-action on $\mathcal{F}_T$. But the principal $\mathbf{GL}_n$-bundle $X_{coord}/J^{(\infty)}$ is exactly the bundle of (0th-order) frames on X, so

$$X_{coord}/J^{(\infty)} \times_{\mathbf{GL}_n} V \simeq X_{coord}/J^{(\infty)} \times_{\mathbf{GL}_n} T_0V \simeq TX,$$

and similarly

$$X_{coord}/J^{(\infty)} \times_{\mathbf{GL}_n} V^* \simeq T^*X.$$

Since the $\mathbf{GL}_n$-action respects the decomposition $\mathcal{F}_T = \prod_{i\geq 0} S^i V^*$, we get

$$X_{coord}/J^{(\infty)} \times_{\mathbf{GL}_n} \mathcal{F}_T \simeq \prod_{i\geq 0} S^i T^* X = \hat{S}(T^*X),$$

which is the structure sheaf of $X^{(\infty)}_{TX}$, the formal neighborhood of the zero section of TX. In short, we have a *tautological exponential map*

$$Exp : \pi^* X^{(\infty)}_{TX} \to \pi^* X^{(\infty)}_{X\times X}$$

or equivalently, a *tautological Taylor expansion map*

$$Exp^* : \pi^*\mathscr{O}_{X^{(\infty)}_{X\times X}} \to \pi^*\mathscr{O}_{X^{(\infty)}_{TX}},$$

which is an isomorphism of bundles of topological algebras. The induced map between associated bundle of graded algebras

$$\operatorname{gr} Exp^* : \pi^* \operatorname{gr} \mathscr{O}_{X^{(\infty)}_{X\times X}} = \pi^* S(T^*X) \to \pi^* S(T^*X)$$

is the identity map. By (3.7) we can write Exp^* explicitly in terms of ∇_{tau},

$$Exp^*(f) = (\nabla^i_{tau} f|_0)_{i\geq 0} = (f(0), \nabla_{tau} f|_0, \nabla^2_{tau} f|_0, \cdots) \in \pi^* \hat{S}(T^*X), \tag{3.10}$$

where $|_0$ stands for the 'restriction to the origin' map $\pi^* S^i \mathcal{J}^\infty T^*X \to \pi^* S^i T^*X$. It is the globalization of the natural restriction map $T^*\widehat{V} \to T^*_0\widehat{V} = V^*$ by applying the associated bundle construction with X_{coord} and then pulling back onto X_{conn} via π. Again $\nabla_{tau} f$ means $d^{(\infty)} f$ and so on.

The Taylor expansion map Exp^* induces a natural bijection between global smooth sections of X_{conn} and all possible smooth isomorphisms between $\mathcal{J}^\infty_X$ and $\hat{S}(T^*X)$ which are the identity map on the level of associated graded algebras. Given any smooth section σ of X_{conn}, we denote by

$$\exp^*_\sigma : \mathcal{J}^\infty_X \to \hat{S}(T^*X)$$

the corresponding smooth homomorphism of bundles of algebras over X. By abuse of notation, we also denote the $\mathcal{A}^{0,\bullet}(X)$-linear extension of $\exp^*_\sigma$ by

$$\exp^*_\sigma : \mathcal{A}^\bullet(X^{(\infty)}_{X\times X}) \to \mathcal{A}^{0,\bullet}_X(\hat{S}(T^*X)) \tag{3.11}$$

between graded algebras. It is in general not a homomorphism of dgas since σ may not be holomorphic. In general, X_{conn} carries a flat principal $(0,1)$-connection $\overline{d}$, such that for any given smooth section σ of X_{conn}, its anti-holomorphic differential

$$\omega_\sigma := \overline{d}\sigma \in \mathcal{A}^{0,1}(\mathbf{j}^{(\infty)}(TX)) \tag{3.12}$$

is well-defined and it satisfies a Maurer–Cartan type equation

$$\overline{\partial}\omega_\sigma + \frac{1}{2}[\omega_\sigma, \omega_\sigma] = 0. \tag{3.13}$$

We denote by

$$\alpha^n_\sigma \in \mathcal{A}^{0,1}_X(\mathrm{Hom}(S^n TX, TX)) = \mathcal{A}^{0,1}_X(\mathrm{Hom}(T^*X, S^n T^*X)),$$

the n-th graded component of ω in the decomposition (3.2). Then we have

$$\omega_\sigma = \overline{\partial}\exp^*_\sigma \circ (\exp^*_\sigma)^{-1}, \tag{3.14}$$

where $\overline{\partial}\exp^*_\sigma = [\overline{\partial}, \exp^*_\sigma]$. Define a new differential $D_\sigma = \overline{\partial} - \sum_{n\geq 2} \widetilde{\alpha}^n_\sigma$ on $\mathcal{A}^{0,\bullet}_X(\hat{S}(T^*X))$, where $\widetilde{\alpha}^n_\sigma$ is the odd derivation of the graded algebra $\mathcal{A}^{0,\bullet}(\hat{S}(T^*X))$ induced by α^n_σ. Then $D^2_\sigma = 0$ by (3.13) and we have

Proposition 3.2 *([13], Prop. 4.4.). For any given smooth section σ of X_{conn}, the map*

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X^{(\infty)}_{X\times X}), \overline{\partial}) \to (\mathcal{A}^{0,\bullet}(\hat{S}(T^*X)), D_\sigma)$$

is an isomorphism of dgas.

3.3. Kapranov's result for Kähler manifolds

Now suppose that X is equipped with a Kähler metric h. Let ∇ be the canonical $(1,0)$-connection in TX associated with h, so that

$$[\nabla, \nabla] = 0 \text{ in } \mathcal{A}^{2,0}_X(\mathrm{End}(TX)), \tag{3.15}$$

and it is torsion-free, which is equivalent to the condition that the metric h is Kähler.

Set $\widetilde{\nabla} = \nabla + \overline{\partial}$, where $\overline{\partial}$ is the $(0,1)$-connection defining the complex structure. The curvature of $\widetilde{\nabla}$ is just

$$R = [\overline{\partial}, \nabla] \in \mathcal{A}^{1,1}_X(\mathrm{End}(TX)) = \mathcal{A}^{0,1}_X(\mathrm{Hom}(TX \otimes TX, TX)) \tag{3.16}$$

which is a Dolbeault representative of the Atiyah class α_{TX} of the tangent bundle. In particular one has the Bianchi identity:

$$\overline{\partial} R = 0 \text{ in } \mathcal{A}^{0,2}_X(\mathrm{Hom}(TX \otimes TX, TX))$$

Actually, by the torsion-freeness we have

$$R \in \mathcal{A}^{0,1}_X(\mathrm{Hom}(S^2TX, TX))$$

Now define tensor fields R_n, $n \geq 2$, as higher covariant derivatives of the curvature:

$$R_n \in \mathcal{A}^{0,1}_X(\mathrm{Hom}(S^2TX \otimes TX^{\otimes(n-2)}, TX)), \quad R_2 := R, \quad R_{i+1} = \nabla R_i \tag{3.17}$$

In fact R_n is totally symmetric, i.e.,

$$R_n \in \mathcal{A}^{0,1}_X(\mathrm{Hom}(S^nTX, TX)) = \mathcal{A}^{0,1}_X(\mathrm{Hom}(T^*X, S^nT^*X))$$

by the flatness of ∇ (Eq. (3.15)). Note that if we think of ∇ as the induced connection on the cotangent bundle, the same formulas (3.16) and (3.17) give $-R_n$.

The connection ∇ determines a smooth section of X_{conn}, which we write as $\sigma = [\nabla]_\infty$, by assigning to each point $x \in X$ the holomorphic jets of ∇. This has been done implicitly in the proof of Lemma 2.9.1., [7]. One can check that the induced Taylor expansion map

$$\exp_\sigma^* : \mathcal{A}^\bullet(X^{(\infty)}_{X\times X}) \xrightarrow{\simeq} \mathcal{A}^{0,\bullet}_X(\hat{S}(T^*X))$$

by

$$\exp_\sigma^*([\eta]_\infty) = (\Delta^*\eta, \Delta^*\nabla\eta, \Delta^*\nabla^2\eta, \cdots, \Delta^*\nabla^n\eta, \cdots) \in \mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X)) \tag{3.18}$$

for any $[\eta]_\infty \in \mathcal{A}^\bullet(X^{(\infty)}_{X\times X})$. Here ∇ is understood as the pullback of ∇ (on the cotangent bundle) via pr_2, which is a constant family of connections along fibers of pr_1, instead of jets of ∇. The key observation is that the right hand side of the formula (3.18) only depends on the class $[\eta]_\infty \in \mathcal{A}^\bullet(X^{(\infty)}_{X\times X})$ and the holomorphic jets of the ∇. In [13], we reformulated and reproved Theorem 2.8.2., [7] in our language.

Theorem 3.3 *(Thm 3.2., [13]). Assume X is Kähler. With the notations from §3.2.2 and above, we have $\alpha_\sigma^n = -R_n$, i.e., $\omega_\sigma = -\sum_{n\geq 2} R_n$. Thus there is an isomorphism between dgas*

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X^{(\infty)}_{X\times X}), \overline{\partial}) \to (\mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X)), D_\sigma)$$

The derivation $D_\sigma = \overline{\partial} + \sum_{n\geq 2} \widetilde{R}_n$, where $\widetilde{R}_n$ is the odd derivation of $\mathcal{A}^{0,\bullet}(\hat{S}(T^*X))$ induced by R_n.

We conclude this section by a slightly more generalized version of Theorem 3.3, which will be used later in § 4.2. Suppose $f : X \to Y$ is an arbitrary holomorphic map. We consider the graph of f

$$\widetilde{f} = (\mathrm{Id}, f) : X \to X \times Y$$

which is a closed embedding. So we can consider the formal neighborhood $X^{(\infty)}_{X\times Y}$. All the constructions above can be carried out in almost the same way with only slight adjustment and give us a description of the Dolbeault dga $\mathcal{A}^\bullet(X^{(\infty)}_{X\times Y})$. Namely, consider the pullback bundle f^*Y_{conn} over X. Each smooth section σ of f^*Y_{conn} naturally corresponds to an isomorphism

$$\eta_\sigma : \mathcal{A}^\bullet(X^{(\infty)}_{X\times Y}) \hookrightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}(f^*T^*Y))$$

of graded algebras. One can also view sections of f^*Y_{conn} as jets of flat torsion-free connections on $X^{(\infty)}_{X\times Y}$ along Y-fibers. In particular, when Y carries a Kähler metric and the canonical $(1,0)$-connection ∇, we can pullback ∇ via the projection $X \times Y \to Y$, which determines a smooth section σ of f^*Y_{conn} and a Taylor expansion map

$$\exp_\sigma^* : \mathcal{A}^\bullet(X^{(\infty)}_{X\times Y}) \to \mathcal{A}_X^{0,\bullet}(\hat{S}(f^*T^*Y)), \quad [\zeta]_\infty \mapsto (\widetilde{f}^*\zeta, \widetilde{f}^*\nabla\zeta, \widetilde{f}^*\nabla^2\zeta, \cdots),$$

for any $[\zeta]_\infty \in \mathcal{A}^\bullet(X^{(\infty)}_{X\times Y})$. By abuse of notations, we still write

$$\alpha_\sigma^n \in \mathcal{A}_Z^{0,1}(\mathrm{Hom}(f^*TY, S^n(f^*TY)))$$

as the pullback of tensors α_σ^n of Y in Proposition 3.2 and its covariant derivatives via f. Then we have the following theorem,

Theorem 3.4. *We have an isomorphism between dgas*

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X^{(\infty)}_{X\times Y}), \overline{\partial}) \xrightarrow{\simeq} (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(f^*T^*Y)), D_\sigma)$$

where $D_\sigma = \overline{\partial} - \sum_{n\geq 2} \widetilde{\alpha}_\sigma^n$ and $\widetilde{\alpha}_\sigma^n$ is the derivation of degree $+1$ induced by α_σ^n. When Y is Kähler, α_σ^n can be chosen as the pullback of $-R_n$ on X.

4. Case of general embeddings

Let $i : X \hookrightarrow Y$ be an arbitrary embedding and $\mathcal{A}^\bullet(\hat{Y})$ the Dolbeault dga associated to the formal neighborhood of X inside Y as in § 3.2.2. The goal is to build some appropriate isomorphism $\mathcal{A}^\bullet(\hat{Y}) \simeq \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee))$ and write down the $\overline{\partial}$-derivation explicitly under this identification. We will show that this can be derived from the special yet universal case considered in § 3.

4.1. Differential geometry of complex submanifolds

4.1.1. Splitting of normal exact sequence and Kodaira–Spencer class

Over X we have the short exact sequence of holomorphic vector bundles defining the normal bundle N of X in Y

$$0 \to TX \xrightarrow{\iota} i^*TY \xrightarrow{p} N \to 0 \tag{4.1}$$

and its dual

$$0 \to N^\vee \xrightarrow{p^\vee} i^*T^*Y \xrightarrow{\iota^\vee} T^*X \to 0 \tag{4.2}$$

We fix a choice of C^∞-splitting of the normal exact sequence (4.1), i.e., two smooth homomorphisms of vector bundles $\tau : i^*TY \to TX$ and $\rho : N \to i^*TY$ satisfying

$$\tau \circ \iota = \mathrm{Id}_{TX}, \quad p \circ \rho = \mathrm{Id}_N, \quad \iota \circ \tau + \rho \circ p = \mathrm{Id}_{i^*TY}$$

and denote the corresponding dual splitting on the conormal exact sequence (4.2) by $\tau^\vee : T^*X \to i^*T^*Y$ and $\rho^\vee : i^*T^*Y \to N^\vee$. We can choose the splittings as the orthonormal decomposition induced by a Kähler metric on Y if there is one, but again we will never need a metric in our general discussion.

Think of $\tau^\vee$ as a C^∞-section of the holomorphic vector bundle $\mathrm{Hom}(T^*X, i^*T^*Y)$, we can form

$$\beta = \beta_{X/Y} := \overline{\partial}\tau^\vee \in \mathcal{A}_X^{0,1}(\mathrm{Hom}(T^*X, T^*Y)).$$

In fact

$$\beta \in \mathcal{A}_X^{0,1}(\mathrm{Hom}(T^*X, N^\vee)).$$

To see this, just apply $\overline{\partial}$ on both sides of equality $\iota^\vee \circ \tau^\vee = \mathrm{Id}_{T^*X}$ and note that $\iota^\vee$ is holomorphic and hence $\overline{\partial}\iota^\vee = 0$. By definition $\overline{\partial}\beta = 0$, thus β defines a cohomology class $[\beta] \in \mathrm{Ext}_X^1(T^*X, N^\vee)$, which is the obstruction class for the existence of a holomorphic splitting of the exact sequence (4.2) or (4.1). We call it the *Kodaira–Spencer class*. Also note that

$$\beta = -\overline{\partial}\rho = -\overline{\partial}\rho^\vee \in \mathcal{A}^{0,1}(\mathrm{Hom}(N, TX)) = \mathcal{A}^{0,1}(\mathrm{Hom}(T^*X, N^\vee)). \tag{4.3}$$

4.1.2. Shape operator

Suppose ∇ is an arbitrary $(1,0)$-connection on TY without any additional assumption. We use the same notation for the pullback connection on i^*TY or i^*T^*Y over X. The induced connection on the normal bundle via the chosen splitting is denoted by $\nabla^\perp$:

$$\nabla_V^\perp \mu := p(\nabla_V \rho(\mu)) \in C_X^\infty(N), \quad \forall\ \mu \in C_X^\infty(N),\ V \in C^\infty(TX)$$

(here we identify $T^{1,0}X$ with TX). Analogous to the shape operator in Riemannian geometry, we also define a linear operator $S_N : TX \otimes N \to TX$ by

$$S_N^\mu(V) = -\tau(\nabla_V \rho(\mu)), \quad \forall\ \mu \in C_X^\infty(N),\ V \in C^\infty(TX). \tag{4.4}$$

That is, we first lift a smooth section μ of the normal bundle to a section of i^*TY via the splitting, then take its derivatives with respect to the induced connection on i^*TY and finally project the output into TX. Note that S_N is in general not a holomorphic map between vector bundles.

4.2. Taylor expansions in normal direction

4.2.1. General discussions

Let $i : X \hookrightarrow Y$ be a closed embedding, where Y is not necessarily Kähler. Similar to what has been done in § 3.2.2, we can consider all isomorphisms

$$\mathcal{A}^\bullet(\hat{Y}) \xrightarrow{\simeq} \mathcal{A}_X^{0,\bullet}(\hat{S}(N^\vee))$$

which induces identity on the associated graded bundle $S(N^\vee)$ and there is a infinite dimensional bundle $\Psi_{X/Y} \to X$ whose smooth sections correspond exactly to such isomorphisms. Indeed, for each $x \in X$, the fiber $\Psi_{X/Y,x}$ is the space of jets of holomorphic maps $\psi : N_x \to Y$ with $\psi(0) = x$ and $p \circ d_0\psi = \mathrm{Id}_{N_x}$, where N_x is the fiber of the normal bundle at point $x \in X$ and $p : i^*TY \to N$ is the natural projection.

Similarly, we can define another bundle $\Theta_{X/Y}$ over X whose fiber at x is the space of jets of maps $\theta : N_x \to T_xY$ with $\theta(0) = x$ and $p \circ d_0\theta = \mathrm{Id}_{N_x}$. Then $\Theta_{X/Y}$ admits a natural left action of $J^{(\infty)}(T^*Y)$ (or more precisely, $J^{(\infty)}(T^*Y|_X)$). In fact, we have a canonical isomorphism

$$\Psi_{X/Y} \simeq Y_{exp}|_X \times_{J^{(\infty)}(T^*Y)} \Theta_{X/Y} \simeq Y_{conn}|_X \times_{J^{(\infty)}(T^*Y)} \Theta_{X/Y}.$$

For our purpose here, however, it is not convenient to deal with $\Psi_{X/Y}$ since even we have already understood Y_{conn} in various geometric ways, general sections of the bundle $\Theta_{X/Y}$ are difficult to handle. So instead we only look at linear liftings $N_x \to TY_x$ which form a subbundle $\Theta^{(1)}_{X/Y} \subset \Theta_{X/Y}$. Moreover, there is a canonical retraction $\Theta_{X/Y} \to \Theta^{(1)}_{X/Y}$ sending jets of maps $\psi : N_x \hookrightarrow Y$ to their linearizations $d_0\psi$. Thus we have a fiberwise surjection of bundles

$$\kappa : Y_{conn}|_X \times_X \Theta^{(1)}_{X/Y} \to Y_{conn}|_X \times_{J^{(\infty)}(T^*Y)} \Theta_{X/Y} = \Psi_{X/Y}$$

As an affine bundle over the vector bundle $N^\vee \otimes TX$, $\Theta^{(1)}_{X/Y}$ admits smooth sections which are nothing but C^∞-liftings $\rho : N \to TY$. Such a lifting ρ and any section σ of $Y_{conn}|_X$ together determine a section Ξ of $\Psi_{X/Y}$ via κ, and hence an isomorphism of graded algebras

$$\exp^*_{X/Y,\Xi} : \mathcal{A}^\bullet(\hat{Y}) \xrightarrow{\simeq} \mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(N^\vee))$$

The rest of the job is to determine which differential we should put on the codomain of this isomorphism in terms of ρ and σ to make it into an isomorphism of dgas.

As in Theorem 3.4, denote the graph of i by

$$\widetilde{i} := (\mathrm{Id}, i) : X \to X \times Y.$$

We regard X as a submanifold of $X \times Y$ via $\widetilde{i}$, then by Theorem 3.4 a section σ of $Y_{conn}|_X$ induces

$$\exp^*_\sigma : (\mathcal{A}^\bullet(X^{(\infty)}_{X\times Y}), \overline{\partial}) \xrightarrow{\simeq} (\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(T^*Y)), D_\sigma)$$

where T^*Y is understood as the pullback i^*T^*Y (we will omit i^* as long as there is no confusion). The derivation

$$D_\sigma = \overline{\partial} + \sum_{n\geq 2} \widetilde{R}_n \tag{4.5}$$

and $\widetilde{R}_n$ is induced by (the pullback of) the covariant derivatives of curvature forms of Y.

Note that we have a commutative diagram of holomorphic maps

$$\begin{tikzcd} X \arrow[r, hook, "\widetilde{i}"] \arrow[d, equal] & X \times Y \arrow[d, "\pi"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

where $\pi : X \times Y \to Y$ is the natural projection. Thus by functoriality, π induces an injective homomorphism of dgas

$$\pi^* : (\mathcal{A}^\bullet(\hat{Y}), \overline{\partial}) \to (\mathcal{A}^\bullet(X^{(\infty)}_{X\times Y}), \overline{\partial}), \quad [\eta]_\infty \mapsto [\pi^*\eta]_\infty. \tag{4.6}$$

We then compose π^* with the isomorphism $\exp_\sigma^*$ to get

$$\exp_\sigma^* \circ \pi^* : (\mathcal{A}^\bullet(\hat{Y}), \overline{\partial}) \to (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y), D_\sigma)$$

We then extend $\rho^\vee : T^*Y \to N^\vee$ to obtain a homomorphism of graded algebras

$$\rho^\vee : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \to \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)). \tag{4.7}$$

Composing with $\exp_\sigma^* \circ \pi^*$ in (4.11) we get a homomorphism of graded algebras

$$\rho^\vee \circ \exp_\sigma^* \circ \pi^* \ : \ \mathcal{A}^\bullet(\hat{Y}) \to \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)).$$

Lemma 4.1. *With all the notations above, we have*

$$\exp^*_{X/Y,\Xi} = \rho^\vee \circ \exp_\sigma^* \circ \pi^*. \tag{4.8}$$

Proof. Follows immediately from the definition of κ. $\square$

Via the isomorphism $\exp^*_{X/Y,\Xi}$ we can transfer the $\overline{\partial}$-derivation on $\mathcal{A}^\bullet(\hat{Y})$ to a derivation on $\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee))$, denoted as $\mathfrak{D}$, i.e.,

$$\mathfrak{D} = \exp^*_{X/Y,\Xi} \circ \overline{\partial} \circ (\exp^*_{X/Y,\Xi})^{-1}.$$

Hence $\exp^*_{X/Y,\Xi}$ becomes an isomorphism of dgas

$$\exp^*_{X/Y,\Xi} : (\mathcal{A}^\bullet(\hat{Y}), \overline{\partial}) \xrightarrow{\simeq} (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)), \mathfrak{D}).$$

Thus we can also transfer the homomorphism π^* in (4.6) to a homomorphism

$$\widetilde{\pi}^* : (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)), \mathfrak{D}) \to (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y), D_\sigma),$$

by setting

$$\widetilde{\pi}^* = \exp_\sigma^* \circ \pi^* \circ (\exp^*_{X/Y,\Xi})^{-1}. \tag{4.9}$$

We then get the following commutative diagram:

$$\begin{tikzcd}
(\mathcal{A}^\bullet(\hat{Y}),\overline{\partial}) \arrow[r, "\pi^*"] \arrow[d, "\simeq"', "\exp^*_{X/Y,\Xi}"] & (\mathcal{A}^{0,\bullet}(X^{(\infty)}_{X\times Y}),\overline{\partial}) \arrow[d, "\simeq"', "\exp^*_\sigma"] \\
(\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(N^\vee)),\mathfrak{D}) \arrow[r, "\widetilde{\pi}^*"] & (\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(T^*Y),D)
\end{tikzcd}$$

Note that by (4.8) and (4.9) we have

$$\rho^\vee \circ \widetilde{\pi}^* = \mathrm{Id} : (\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(N^\vee)),\mathfrak{D}) \to (\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(N^\vee)),\mathfrak{D}) \tag{4.10}$$

even though $\rho^\vee$ is not a homomorphism of dgas.

4.2.2. Description of $\widetilde{\pi}^*$

For simplicity, we assume Y is Kähler and the section σ of $Y_{conn}|_X$ is determined by the associated $(1,0)$-connection ∇. However, we want to keep the reader aware that all the arguments and computations below will still work even if the Kähler condition is dropped and the section σ is arbitrary. All the ∇ appearing in the formulae can be interpreted as formal connections determined by σ without any change just as in Proposition 3.2. See Remark 4.8. The Kähler assumption just makes sure that those terms in the final formula (4.22) of $\mathfrak{D}$ have clearer differential geometric meanings.

By the discussion at the end of §3.3, the homomorphism

$$\exp^*_\sigma \circ \pi^* : (\mathcal{A}^\bullet(\hat{Y}),\overline{\partial}) \to (\mathcal{A}^{0,\bullet}_X(\hat{S}^\bullet(T^*Y),D_\sigma)$$

is given by

$$[\eta]_\infty \mapsto (i^*\eta, i^*\nabla\eta, i^*\nabla^2\eta, \cdots) \tag{4.11}$$

since $\widetilde{i}^*\nabla^n\pi^*\eta = i^*\nabla^n\eta$, where the ∇ on the right hand side is the original $(1,0)$-connection on Y while the one on the left hand side is the pullback one along Y-fibers of $X \times Y$.

Before we give a description of the homomorphism $\widetilde{\pi}^*$, we make some conventions on notations. We abuse the notations and write $TY = TX \oplus N$ and $T^*Y = T^*X \oplus N^\vee$ induced by the fixed splittings. The decompositions extend to tensor products, i.e., tensor product $TY^{\otimes n}$ can be decomposed into direct sum of mixed tensors of TX and N components and similarly for $T^*Y^{\otimes n}$. The same happens for symmetric tensor products:

$$S^n T^*Y = \bigoplus_{p+q=n} S^p T^*X \cdot S^q N^\vee, \tag{4.12}$$

where the dot stands for the commutative multiplication in the symmetric algebras.

We define a derivation

$$\overline{\nabla} : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet T^*Y) \to \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet+1} T^*Y) \tag{4.13}$$

of degree 0 for the grading from $\mathcal{A}_X^{0,\bullet}$ as the composition of the operators

$$\overline{\nabla} := \overline{\mathrm{Sym}} \circ \nabla^{TX}$$

Namely, for any $\eta \in \mathcal{A}_X^{0,\bullet}(S^n T^*Y)$, first apply the connection ∇^{TX} on $i^* S^n T^*Y$ induced by ∇ to get a $T^*X \otimes S^n T^*Y$-valued form

$$\nabla^{TX}\eta \in \mathcal{A}_X^{0,\bullet}(T^*X \otimes S^n T^*Y),$$

then apply a variation of the usual symmetrization map

$$\overline{\mathrm{Sym}} : T^*X \otimes S^n T^*Y \to S^{n+1} T^*Y,$$

which we define on each component of the decomposition (4.12) as

$$\overline{\mathrm{Sym}}_{m,n} : T^*X \otimes (S^{m-1} T^*X \cdot S^{n-m+1} N^\vee) \to S^m T^*X \cdot S^{n-m+1} N^\vee$$

by the formula

$$\overline{\mathrm{Sym}}_{m,n}(v_0 \otimes (v_1 \cdot v_2 \cdots v_n)) = \frac{1}{m} v_0 \cdot v_1 \cdots v_n,$$
$$\forall\ v_0 \otimes (v_1 \cdot v_2 \cdots v_n) \in T^*X \otimes (S^{m-1} T^*X \cdot S^{n-m+1} N^\vee) \tag{4.14}$$

and we finally get

$$\overline{\nabla}\eta \in \mathcal{A}_X^{0,\bullet}(S^{n+1} T^*Y)$$

When $\eta \in \mathcal{A}_X^{0,\bullet}(S^0(T^*Y)) = \mathcal{A}_X^{0,\bullet}$, $\overline{\nabla}\eta = \tau^\vee(\partial\eta) \in \mathcal{A}_X^{0,\bullet}(T^*Y)$ where ∂ is the $(1,0)$-derivation of forms on X. We can inductively apply $\overline{\nabla}$ and get

$$\overline{\nabla}^k : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \to \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet+k}(T^*Y))$$

In particular, since $N^\vee$ is naturally identified as a subbundle of T^*Y via $p^\vee : N^\vee \to T^*Y$, we can form the restriction of $\overline{\nabla}^k$ on $\hat{S}^\bullet(N^\vee)$:

$$\overline{\nabla}^k : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)) \to \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet+k}(T^*Y))$$

Note that here $\overline{\nabla}^0$ is the natural inclusion $\hat{S}^\bullet(N^\vee) \hookrightarrow \hat{S}^\bullet(T^*Y)$.

Proposition 4.2. *We have*

$$\widetilde{\pi}^* = \sum_{k=0}^{\infty} \overline{\nabla}^k$$

where $\widetilde{\pi}^* : \mathcal{A}_X^{0,\bullet}(\hat{S}(N^\vee)) \to \mathcal{A}_X^{0,\bullet}(\hat{S}(T^*Y))$ *is as in* (4.9). *That is, given* $\mu = (\mu_k)_{k=0}^\infty \in \mathcal{A}_X^{0,\bullet}(\hat{S}(N^\vee))$, *the* n*-th component of its image* $\nu = \widetilde{\pi}^*(\mu)$ *is*

$$\nu_n = \sum_{k=0}^{n} \overline{\nabla}^k \mu_{n-k} \in \mathcal{A}_X^{0,\bullet}(S^n N^\vee).$$

Proof. Assume that $\exp^*_{X/Y,\Xi}([\eta]_\infty) = \mu$ where $[\eta]_\infty \in \mathcal{A}^\bullet(\hat{Y})$. By (4.8) and (4.11), this means

$$\mu_k = \rho^\vee(i^* \nabla^k \eta)$$

where $\rho^\vee$ is the projection

$$\rho^\vee : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \to \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee))$$

as in (4.7). Moreover, by the definition of $\widetilde{\pi}^*$ (4.9), we have

$$\widetilde{\pi}^*(\mu) = \exp^*_\sigma \circ \pi^* \circ (\exp^*_{X/Y,\Xi})^{-1}(\mu) = \exp^*_\sigma \circ \pi^*([\eta]_\infty) = (i^*\nabla^k \eta)_{k=0}^\infty.$$

Thus all we need to show is that

$$i^*\nabla^n \eta = \sum_{k=0}^{n} \overline{\nabla}^k (\rho^\vee(i^*\nabla^{n-k}\eta))$$

for all $n \geq 0$. We prove it by induction on n. The $n = 0$ case is trivial. For $n \geq 1$, note that we can write

$$i^*\nabla^n \eta = \rho^\vee(i^*\nabla^n \eta) + \text{component in } T^*X \cdot S^{n-1}T^*Y \subset S^n T^*Y$$

via the decomposition (4.12). The second term on the right hand side is nothing but $\overline{\nabla}(i^*\nabla^{n-1}\eta)$. To see this, consider what happens when we evaluate $i^*\nabla^n\eta$ at some section s of $TY^{\otimes n}$, which lies in a mixed tensor of m copies of TX ($m \geq 1$) and $n-m$ copies of N (of arbitrary order). One can permute any of the TX-factors to the first place and plug it into the first ∇ in $i^*\nabla^n\eta$, since $i^*\nabla^n\eta$ is a symmetric tensor. This implies that $i^*\nabla^n\eta$

should be a symmetrization of $\nabla^{TX}(i^*\nabla^{n-1}\eta)$. The value of the latter at s, however, is m times what we need since s contains m TX-factors. This explains the fractional factor $1/m$ in the formula (4.14). Finally we end the proof by applying the inductive assumption. □

4.2.3. Description of the derivation $\mathfrak{D}$

To determine the derivation $\mathfrak{D}$, note that by (4.10) we have

$$\mathfrak{D} = \rho^\vee \circ \widetilde{\pi}^* \circ \mathfrak{D} = \rho^\vee \circ D \circ \widetilde{\pi}^*$$

where the last equality is by the definition of $\mathfrak{D}$. Thus for given $\mu = (\mu_k)_{k=0}^\infty \in \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee))$, the n-th component of $\mathfrak{D}\mu$ is

$$(\mathfrak{D}\mu)_n = \sum_{s+t=n} \rho^\vee \circ \overline{\partial} \circ \overline{\nabla}^s \mu_t + \sum_{\substack{r+s+t=n \\ r\geq 1}} \rho^\vee \circ \widetilde{R}_{r+1} \circ \overline{\nabla}^s \mu_t \tag{4.15}$$

by Proposition 4.2 and (4.5). To simplify the right hand side further, first observe that all we need are the components of $\overline{\nabla}^s \mu_t$ lying in $\hat{S}^\bullet(N^\vee)$ and $T^*X \cdot \hat{S}^\bullet(N^\vee)$ and we can ignore the remaining ones in $S^2T^*X \cdot \hat{S}^\bullet(T^*Y)$. The reason is that if we apply the derivations $\overline{\partial}$ and $\widetilde{R}_n$ on any section from $S^2T^*X \cdot \hat{S}^\bullet(T^*Y)$, the outcomes must lie in $T^*X \cdot \hat{S}^\bullet(T^*Y)$, which will then be eliminated by the projection $\rho^\vee$.

Thus we first denote the projections onto the only two 'effective' components respectively by

$$P_0 = p^\vee \circ \rho^\vee : \hat{S}^\bullet(T^*Y) \to \hat{S}^\bullet(N^\vee) \subset \hat{S}^\bullet(T^*Y)$$

and

$$P_1 : \hat{S}^\bullet(T^*Y) \to T^*X \cdot \hat{S}^\bullet(N^\vee) \subset \hat{S}^\bullet(T^*Y).$$

Secondly, we define the derivation of degree +1

$$\widetilde{\beta} : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \to \mathcal{A}_X^{0,\bullet+1}(\hat{S}^\bullet(T^*Y))$$

induced by

$$\beta \in \mathcal{A}_X^{0,1}(\mathrm{Hom}(T^*X, N^\vee)) \subset \mathcal{A}_X^{0,1}(\mathrm{Hom}(T^*Y, T^*Y)),$$

where the last inclusion comes again from the splitting of T^Y. Note that $\widetilde{\beta}$ acts on $\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)) \subset \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y))$ as the zero map. So we can also think of $\widetilde{\beta}$ as the operator

$$\widetilde{\beta} : \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(N^\vee)) \to \mathcal{A}_X^{0,\bullet+1}(\hat{S}^{\bullet+1}(N^\vee))$$

The following lemma is immediate from (4.3).

Lemma 4.3. *As derivations* $\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \to \mathcal{A}_X^{0,\bullet+1}(\hat{S}^\bullet(N^\vee))$,

$$[\rho^\vee, \overline{\partial}] = \widetilde{\beta} \circ P_1.$$

Hence the first sum on the right hand side of (4.15) can be rewritten as

$$\begin{aligned}\sum_{s+t=n} \rho^\vee \circ \overline{\partial} \circ \overline{\nabla}^s \mu_t &= \sum_{s+t=n} \overline{\partial} \circ \rho^\vee \circ \overline{\nabla}^s \mu_t + \sum_{s+t=n} \widetilde{\beta}_X \circ P_1 \circ \overline{\nabla}^s \mu_t \\ &= \overline{\partial} \mu_n + \sum_{s+t=n} \widetilde{\beta}_X \circ P_1 \circ \overline{\nabla}^s \mu_t\end{aligned} \tag{4.16}$$

The last equality is because that $\overline{\nabla}^s \mu_t \in \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(T^*Y))$ unless $s = 0$.

Thirdly, we split $R_n \in \mathcal{A}_X^{0,1}(\mathrm{Hom}(T^*Y, S^n T^*Y))$ into two components,

$$R_n^\perp := \rho^\vee \circ R_n \circ p^\vee \in \mathcal{A}_X^{0,1}(\mathrm{Hom}(N^\vee, S^n N^\vee)) \tag{4.17}$$

and

$$R_n^\top := \rho^\vee \circ R_n \circ \tau^\vee \in \mathcal{A}_X^{0,1}(\mathrm{Hom}(T^*X, S^n N^\vee)) \tag{4.18}$$

and denote the induced operators by

$$\widetilde{R}_n^\perp : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N^\vee)) \to \mathcal{A}_X^{0,\bullet+1}(\hat{S}^{\bullet+n-1}(N^\vee))$$

and

$$\widetilde{R}_n^\top : \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(N^\vee)) \to \mathcal{A}_X^{0,\bullet+1}(\hat{S}^{\bullet+n}(N^\vee))$$

respectively. To unify notations, we also write $R_1^\top := \beta$ and $\widetilde{R}_1^\top := \widetilde{\beta}$.

We can then split the second term on the right hand side of (4.15):

$$\begin{aligned}&\sum_{\substack{r+s+t=n \\ r\geq 1}} \rho^\vee \circ \widetilde{R}_{r+1} \circ \overline{\nabla}^s \mu_t \\ &= \sum_{\substack{r+s+t=n \\ r\geq 1}} \rho^\vee \circ \widetilde{R}_{r+1} \circ P_0 \circ \overline{\nabla}^s \mu_t + \sum_{\substack{r+s+t=n \\ r\geq 1}} \rho^\vee \circ \widetilde{R}_{r+1} \circ P_1 \circ \overline{\nabla}^s \mu_t \\ &= \sum_{\substack{r+s+t=n \\ r\geq 1}} \widetilde{R}_{r+1}^\perp \circ P_0 \circ \overline{\nabla}^s \mu_t + \sum_{\substack{r+s+t=n \\ r\geq 1}} \widetilde{R}_{r+1}^\top \circ P_1 \circ \overline{\nabla}^s \mu_t \\ &= \sum_{k=2}^n \widetilde{R}_k^\perp \circ \mu_{n-k+1} + \sum_{\substack{r+s+t=n \\ r\geq 1}} \widetilde{R}_{r+1}^\top \circ P_1 \circ \overline{\nabla}^s \mu_t\end{aligned} \tag{4.19}$$

Combine (4.15), (4.16) and (4.19) we get

$$(\mathfrak{D}\mu)_n = \bar{\partial}\mu_n + \sum_{k=2}^{n} \widetilde{R}_k^{\perp} \circ \mu_{n-k+1} + \sum_{\substack{r+s+t=n \\ r,t\geq 0,\ s\geq 1}} \widetilde{R}_{r+1}^{\top} \circ P_1 \circ \overline{\nabla}^s \mu_t \tag{4.20}$$

Finally, to compute the terms $P_1 \circ \overline{\nabla}^s \mu_t$, we define two derivations of degree 0 with respect to the grading from $\mathcal{A}_X^{0,\bullet}$:

$$\overline{\nabla}^{\perp} : \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet}(N^{\vee})) \to \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^{\bullet}(N^{\vee}))$$

induced by the connection $\nabla^{\perp}$ on $N^{\vee}$ the same way as we define $\overline{\nabla}$ in (4.13). $\nabla^{\perp} f$ of a function f is again understood as ∂f, the $(1,0)$ differential.

The second one

$$\widetilde{S_N} : \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^{\bullet}(N^{\vee})) \to \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^{\bullet+1}(N^{\vee}))$$

induced by the shape operator $S_N : T^*X \to T^*X \otimes N^{\vee}$ as in (4.4) yet again with the images symmetrized.

Lemma 4.4. *With the notations above, we have*

$$P_1 \circ \overline{\nabla} = \overline{\nabla}^{\perp} \circ P_0 + \widetilde{S_N} \circ P_1,$$

thus $\forall\ s \geq 1$, $\forall\ \mu \in \hat{S}^{\bullet}(N^{\vee})$,

$$P_1 \circ \overline{\nabla}^s \mu = (\widetilde{S_N})^{s-1} \circ \overline{\nabla}^{\perp} \mu. \tag{4.21}$$

Applying the equality (4.21) to (4.20), we eventually get

Theorem 4.5. *Given* $\mu = (\mu_k)_{k=0}^{\infty} \in \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet}(N^{\vee}))$, *the* n*-th component of* $\mathfrak{D}\mu$ *is*

$$(\mathfrak{D}\mu)_n = \bar{\partial}\mu_n + \sum_{k=2}^{n} \widetilde{R}_k^{\perp} \circ \mu_{n-k+1} + \sum_{\substack{r+s+t=n \\ r,t\geq 0,\ s\geq 1}} \widetilde{R}_{r+1}^{\top} \circ \widetilde{S_N}^{s-1} \circ \overline{\nabla}^{\perp} \mu_t.$$

In other words,

$$\mathfrak{D} = \bar{\partial} + \sum_{k\geq 2} \widetilde{R}_k^{\perp} + \sum_{p\geq 1,\ q\geq 0} \widetilde{R}_p^{\top} \circ \widetilde{S_N}^{q} \circ \overline{\nabla}^{\perp}. \tag{4.22}$$

Remark 4.6. From Theorem 4.5 we see that, even when $\mathfrak{D}$ acts on a function f (or a form), higher term would be produced in general. Namely, by (4.22)

$$\mathfrak{D} f=\overline{\partial} f+\sum_{p \geq 1,\ q \geq 0} \widetilde{R}_p^{\top} \circ \widetilde{S_N}^q \circ \partial f \tag{4.23}$$

This is a huge difference between the general situation and the case of diagonal embedding.

Remark 4.7. Although we get $\mathfrak{D}^2=0$ for free from how we construct it, it is still an interesting (yet tedious) exercise to verify it by hands and one will observe the Gauss–Codazzi–Ricci equations in classical differential geometry (see [12]). We leave the details to interested readers.

Remark 4.8. In non-Kähler case, we can consider the bundle $X_{conn}^{(1)}$ of first order jets of flat torsion free connections on X and it is the same as the affine bundle $\mathrm{Conn}_{tf}(TX)$ of torsion free connections as in §2.2, [7] (since flatness is a condition on second jets). We can fix a smooth torsion free connection ∇ on X, regard it as a section of $X_{conn}^{(1)}$ and lift it to a section σ of X_{conn} via the natural projection $X_{conn} \to X_{conn}^{(1)}$. Let $\nabla^{\perp}$ be the connection on N induced by ∇, the shape operator S_N be the usual one for the connection ∇, $R_n^{\perp}$ and $R_n^{\top}$ be defined by the same equations (4.17) and (4.18), where $R_n=-\alpha_\sigma^n$ is determined by σ as in Theorem 3.4, then Eq. (4.22) still holds. In particular, if we have chosen a holomorphic splitting of the normal short exact sequence (4.1), then $R_1^{\top}=\beta=0$ and the lower order terms of $\mathfrak{D}$ is given by

$$\mathfrak{D}=\overline{\partial}+\widetilde{R}_2^{\perp}+\widetilde{R}_2^{\top} \circ \overline{\nabla}^{\perp}+\cdots,$$

where $R_2^{\perp}$ and $R_2^{\top}$ are $\overline{\partial}$-closed and $R_2^{\perp}$ is a representative of the Atiyah class of the normal bundle.

5. L_∞-algebroid of the formal neighborhood

In this section we repackage the results in § 4 in terms of a L_∞-algebroid structure on the shifted cotangent bundle $N[-1]$, or more precisely, on the complex $\mathcal{A}_X^{0,\bullet}(N[-1])=\mathcal{A}_X^{0,\bullet-1}(N)$, whose Chevalley–Eilenberg complex is exactly the Dolbeault dga $\mathcal{A}^\bullet(Y_X^{(\infty)})$. However, to keep the signs simple, we will work with the $L_\infty[1]$-algebroid structure on the unshifted normal bundle rather than the L_∞-algebroid.

5.1. Conventions and notations

We follow the notations and sign conventions in [11]. For any positive integers $k_1, \ldots, k_l$, let $\mathrm{Sh}(k_1, \ldots k_l)$ be the set of $(k_1, \ldots, k_l)$-*unshuffles*, i.e., permutations σ of set of integers $\{1,2, \ldots, k_1+k_2+\cdots+k_l\}$ satisfying

$$\sigma(k_1+\cdots+k_{i-1}+m)<\sigma(k_1+\cdots+k_{i-1}+n), \quad \forall\, 1 \leq m<n \leq k_i,\ i=1, \ldots, l.$$

Suppose $V = \oplus_i V^i$ is a graded vector space over a field $\mathbb{K}$ of zero characteristic. Given a list of homogeneous vectors in $\mathbf{v} = (v_1, \dots, v_n)$ in V and a permutation $\sigma \in S_n$, we denote by $\alpha(\sigma, \mathbf{v})$ (resp., $\chi(\sigma, \mathbf{v})$) the sign determined by

$$v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)} = \alpha(\sigma, \mathbf{v}) v_1 \odot \cdots \odot v_n \quad (\text{resp. } v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma, \mathbf{v}) v_1 \wedge \cdots \wedge v_n)$$

where $\odot$ (resp., $\wedge$) is the graded symmetric (resp., graded skew-symmetric) product in the graded symmetric algebra $S(V)$ (resp., graded exterior algebra $\wedge V$).

5.2. L_∞- and $L_\infty[1]$-algebroids

We recall the definition of L_∞-algebras and $L_\infty[1]$-algebras from [11].

Definition 5.1. An L_∞-algebra is a graded vector space $L^\bullet$ equipped with a family of n-ary multilinear operations (n-brackets)

$$l_n : (L^\bullet)^{\times n} \to L^\bullet, \qquad (x_1, \cdots, x_n) \mapsto [x_1, \dots, x_n]_n, \quad n \in \mathbb{N}$$

of degree $2 - n$, such that

(1) $[\cdot, \dots, \cdot]_n$ is graded skew-symmetric, i.e., for any permutation $\sigma \in S_n$ and vector $\mathbf{v} = (v_1, \dots, v_n)$ of homogeneous elements in $L^\bullet$,

$$[v_{\sigma(1)}, \dots, v_{\sigma(n)}]_n = \chi(\sigma, \mathbf{v})[v_1, \dots, v_n]_n,$$

and

(2) the higher Jacobi identity is satisfied:

$$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,j)} (-1)^{ij} \chi(\sigma, \mathbf{v})[[v_{\sigma(1)}, \dots v_{\sigma(i)}]_i, v_{\sigma(i+1)}, \dots, v_{\sigma(n)}]_{j+1} = 0, \tag{5.1}$$

for any $n \in \mathbb{N}$. In particular, $(L^\bullet, d = [\ \cdot\ ]_1)$ is a cochain complex.

Definition 5.2. An $L_\infty[1]$-algebra is a graded vector space $L^\bullet$ equipped with a family of n-ary multilinear operations (n-brackets)

$$\ell_n : (L^\bullet)^{\times n} \to L^\bullet, \qquad (x_1, \cdots, x_n) \mapsto \{x_1, \dots, x_n\}_n, \quad n \in \mathbb{N}$$

of degree 1, such that

(1) $\{\cdot, \dots, \cdot\}_n$ is graded symmetric, i.e., for any permutation $\sigma \in S_n$ and homogeneous vectors $\mathbf{v} = (v_1, \dots, v_n)$ in $L^\bullet$,

$$\{v_{\sigma(1)}, \dots, v_{\sigma(n)}\}_n = \alpha(\sigma, \mathbf{v})\{v_1, \dots, v_n\}_n,$$

and

(2) the higher Jacobi identity is satisfied:

$$\sum_{i+j=n}\sum_{\sigma\in \mathrm{Sh}(i,j)} \alpha(\sigma,\mathbf{v})\{\{v_{\sigma(1)},\dots v_{\sigma(i)}\}_i, v_{\sigma(i+1)},\dots,v_{\sigma(n)}\}_{j+1}=0, \tag{5.2}$$

for any $n\in\mathbb{N}$. In particular, $(L^\bullet, d=\{\ \cdot\ \}_1)$ is a cochain complex.

In the case when $l_n=0$ for all $n\geq 2$, we call $L^\bullet$ as a *shifted differential graded Lie algebra* or simply a *shifted DGLA*.

There is a one-to-one correspondence between L_∞-algebra structures $\{[\cdot,\cdots,\cdot]_n|\ n\in\mathbb{N}\}$ on a graded vector space L, and $L_\infty[1]$-algebra structures $\{\{\cdot,\cdots,\cdot\}_n|\ n\in\mathbb{N}\}$ on $L[1]$, the shifted graded vector space of L, given by

$$\{v_1,\dots,v_n\}=(-1)^{(k-1)|v_1|+(k-2)|v_2|+\cdots+|v_{k-1}|}[v_1,\cdots,v_n],\ \forall\ v_1,\cdots,v_n\in L,\ \forall\ k\in\mathbb{N}.$$

Definition 5.3. A morphism $f: L\to L'$ between the $L_\infty[1]$-algebras $(L,\{\cdot,\cdots,\cdot\}_k)$ and $(L',\{\cdot,\cdots,\cdot\}'_k)$ is a collection $f=(f_n,\ n\in\mathbb{N})$ of n-ary, multilinear, graded symmetric maps of degree 0,

$$f_n: L^{\times n}\to L',\ n\in\mathbb{N},$$

such that

$$\begin{aligned}&\sum_{i+j=n}\sum_{\sigma\in S_{i,j}} \alpha(\sigma,\mathbf{v}) f_{i+j+1}(\{v_{\sigma(1)},\dots,v_{\sigma(i)}\}, v_{\sigma(i+1)},\dots,v_{\sigma(i+j)})\\ &=\sum_{l=1}^{k}\sum_{\substack{n_1+\cdots+n_l=n\\ 1\leq n_1\leq\cdots\leq n_l}}\sum_{\sigma\in \mathrm{Sh}(n_1,\dots,n_l)} \alpha(\sigma,\mathbf{v})\{f_{n_1}(v_{\sigma(1)},\dots,v_{\sigma(n_1)}),\dots,\\ &\qquad f_{k_l}(v_{\sigma(k-k_l+1)},\dots,v_{\sigma(k)})\}',\end{aligned} \tag{5.3}$$

for any $\mathbf{v}\in L^{\times k}$.

Definition 5.4. An $L_\infty[1]$-*algebroid* or a *strong homotopy (SH) Lie–Rinehart algebra* ($LR_\infty[1]$-*algebra*) is a pair $(L^\bullet,\mathcal{A}^\bullet)$, where $\mathcal{A}^\bullet$ is a unital graded commutative $\mathbb{K}$-algebra and $L^\bullet$ is an $L_\infty[1]$-algebra with **$\mathbb{K}$-multilinear** n-brackets $\{\cdot,\cdots,\cdot\}_n$, $n\in\mathbb{N}$, which also possesses the structure of a graded $\mathcal{A}^\bullet$-module. Moreover, there is a family of n-ary $\mathbb{K}$-multilinear operations

$$\{\cdot,\cdots,\cdot|\text{-}\}_n: L^{\times(n-1)}\times\mathcal{A}^\bullet\to\mathcal{A}^\bullet,\ n\in\mathbb{N}$$

of degree 1 such that

(1) Each $\{\cdot, \cdots, \cdot|\text{-}\}_n$ is $\mathcal{A}^\bullet$-multilinear subject to the Koszul sign rules and graded symmetric in the first $n-1$ entries and a derivation in the last entry. When $n=1$, the operation $\{|\text{-}\}_1 : \mathcal{A}^\bullet \to \mathcal{A}^\bullet$ does not depend on $L^\bullet$ and determines a derivation d_A on $\mathcal{A}^\bullet$ of degree one. Hence $A = (\mathcal{A}^\bullet, d_A)$ is a unital commutative dga. When $n \geq 1$, the induced maps (of degree 0)

$$\alpha_n : L^{\times n} \to \mathcal{D}er^\bullet(A)[1],\ (v_1, \ldots, v_n) \mapsto \{v_1, \ldots, v_n|\text{-}\}_{n+1}, \tag{5.4}$$

form a morphism α of $L_\infty[1]$-algebras between $L^\bullet$ and shifted DGLA $\mathcal{D}er^\bullet(A)[1]$ of derivations of A. We call α as the ∞-*anchor map* (or simply *anchor map*) and α_n the nth anchor map of the $L_\infty[1]$-algebroid L. Moreover, α_n is A-multilinear.

(2) The brackets of L and the anchor map satisfies the equality

$$\begin{aligned}\{v_1, \ldots, v_{n-1}, a \cdot v_n\}_n &= \{v_1, \ldots, v_{n-1}|a\}_n \cdot v_n \\ &\quad + (-1)^{|a|(|v_1|+\cdots+|v_{n-1}|+1)} a \cdot \{v_1, \ldots, v_n\}_n\end{aligned} \tag{5.5}$$

for any $n \in \mathbb{N}$. Note that when $n=1$, this means $(L, d_L = \{\cdot\}_1)$ is a dg-module over the dga $A = (A^\bullet, d_A)$.

Remark 5.5. Unlike [11], we always write explicitly the underlying dga $A = (\mathcal{A}^\bullet, d_A)$ and we also call $(L^\bullet, d_L)$ as an $L_\infty[1]$-algebroid over the dga A if the operation $\{|-\}_1$ equals d_A.

We now describe the (completed) Chevalley–Eilenberg dga of an $L_\infty[1]$-algebroid. Instead of the multi-differential algebra structure on the uncompleted symmetric algebra used in [11], we use the completed symmetric algebra. The difference is minor.

Let $\mathcal{A}^\bullet$ be a unital graded commutative $\mathbb{K}$-algebra and $L^\bullet$ a graded $\mathcal{A}^\bullet$-module. Let $S^r_{\mathcal{A}}(L, \mathcal{A})$ be the graded $\mathcal{A}^\bullet$ module of graded symmetric, $\mathcal{A}^\bullet$-multilinear maps with r entries. A homogeneous element $\eta \in S^r_{\mathcal{A}}(L, \mathcal{A})$ is a homogeneous, graded symmetric, $\mathbb{K}$-multilinear map

$$\eta : (L^\bullet)^{\times r} \to \mathcal{A}^\bullet$$

such that

$$\eta(av_1, v_2, \ldots, v_r) = (-1)^{|a||\eta|} a\eta(v_1, v_2, \ldots, v_r),\ a \in \mathcal{A}^\bullet, v_1, \ldots, v_r \in L^\bullet.$$

In particular, $S^0_{\mathcal{A}}(L, \mathcal{A}) = \mathcal{A}^\bullet$ and $S^1_{\mathcal{A}}(L, \mathcal{A}) = L^{\vee\bullet} := \mathrm{Hom}_{\mathcal{A}^\bullet}(L^\bullet, \mathcal{A}^\bullet)$. Define the completed symmetric algebra

$$\hat{S}_{\mathcal{A}}(L, \mathcal{A}) = \prod_{r \geq 0} S^r_{\mathcal{A}}(L, \mathcal{A}),$$

where the product of two homogeneous elements $\eta \in S^r_{\mathcal{A}}(L,\mathcal{A})$ and $\eta' \in S^{r'}_{\mathcal{A}^\bullet}(L^\bullet,\mathcal{A})$ is given by the formula

$$(\eta\eta')(v_1,\dots,v_{r+r'})$$
$$= \sum_{\sigma\in S_{r,r'}} (-1)^{|\eta'|(|v_{\sigma(1)}|+\cdots+v_{\sigma(r)})}\alpha(\sigma,\mathbf{v})\eta(v_{\sigma(1)},\dots,v_{\sigma(r)})\eta'(v_{\sigma(r+1)},\dots,v_{\sigma(r+r')}),$$

for any $\mathbf{v} = (v_1,\dots,v_{r+r'}) \in (L^\bullet)^{\times(r+r')}$. $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$ is a graded commutative unital algebra.

Remark 5.6. Suppose $L^\bullet = \mathcal{A}^\bullet \otimes_{\mathcal{A}^0} L^0$, where $\mathcal{A}^0$ and L^0 are the zeroth component of $\mathcal{A}^\bullet$ and $L^\bullet$ respectively, and L^0 is a projective and finitely generated A^0-module. Then $\hat{S}_{\mathcal{A}}(L,\mathcal{A}) \simeq \mathcal{A}^\bullet\hat{\otimes}_{\mathcal{A}^0}\hat{S}_{\mathcal{A}^0}((L^0)^\vee)$ as $\mathcal{A}^\bullet$-modules, where $\hat{\otimes}$ is the complete tensor product with respect to the projective topology on $\hat{S}_{\mathcal{A}^0}((L^0)^\vee)$.

The following theorem is an analogue of Theorem 12, [11].

Theorem 5.7. *Let $\mathcal{A}^\bullet$ be an graded commutative unital $\mathbb{K}$-algebra and $L^\bullet$ be a projective and finitely generated $\mathcal{A}^\bullet$-module. Then an $LR_\infty[1]$-algebra structure on $(L^\bullet,\mathcal{A}^\bullet)$ is equivalent to a degree one derivation on $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$.*

Sketch of proof. We only recall the construction of the derivation from [11], which will be used later. For the detailed proof see the Appendix A of [11].

Since $L^\bullet$ is projective and finitely generated, $\hat{S}_{\mathcal{A}}(L,\mathcal{A}) \simeq \hat{S}_{\mathcal{A}}(L^\vee)$, the completed symmetric algebra generated by the $\mathcal{A}^\bullet$-module $L^{\vee\bullet}$, and any derivation is hence determined by its action on $\mathcal{A}^\bullet$ and $L^{\vee\bullet}$. Define degree one derivation D_n on $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$ as follows. For any $a \in \mathcal{A}^\bullet$, set

$$(D_n a)(v_1,\dots,v_n) := (-1)^{|a|(|v_1|+\cdots+|v_n|)}\{v_1,\dots,v_n|a\}_{n+1},\ \forall\ v_1,\dots,v_n \in L^\bullet,\ n\geq 0.$$

We have $D_n a \in S^n_{\mathcal{A}}(L^\vee)$. Note that $D_0 a = d_A a$. For any $\eta \in L^{\vee\bullet}$, set

$$(D_n\eta)(v_1,\dots,v_{n+1}) := \sum_{i=1}^{n+1}(-1)^\theta\{v_1,\dots,\widehat{v_i},\dots,v_{n+1}|\eta(v_i)\}_{n+1}$$
$$+ (-1)^{|\eta|}\eta(\{v_1,\dots,v_{n+1}\}),$$

where $\theta := |\eta|(|v_1| + \cdots + \widehat{|v_i|} + \cdots + |v_{n+1}|) + |v_i|(|v_{i+1}| + \cdots + |v_{n+1}|)$, $v_1,\dots,v_{n+1} \in L^\bullet$, $n \geq 0$, and a hat $\widehat{\cdot}$ stands for omission. We have $D_n\eta \in S^{n+1}(L^\vee)$. Note that D_0 restricted on $L^{\vee\bullet}$ is the differential $d_{L^\vee}$ induced from d_L on $L^\bullet$.

The unique extension of D_n as a degree one derivation on $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$ satisfies the higher Chevalley–Eilenberg formula:

$$\begin{aligned}
&(D_n\eta)(v_1,\ldots,v_{n+r}) \\
:= &\sum_{\sigma\in \mathrm{Sh}(n,r)} (-1)^{|\eta|(|v_{\sigma(1)}|+\cdots+|v_{\sigma(n)}|)}\alpha(\sigma,\mathbf{v})\{v_{\sigma(1)},\ldots,v_{\sigma(n)}|\eta(v_{\sigma(n+1)},\ldots,v_{\sigma(n+r)})\}_{n+1} \\
&- \sum_{\tau\in \mathrm{Sh}(n+1,r-1)} (-1)^{|\eta|}\alpha(\tau,\mathbf{v})\eta(\{v_{\tau(1)},\ldots,v_{\tau(n+1)}\}_{n+1},v_{\tau(n+2)},\ldots,v_{\tau(n+r)}),
\end{aligned} \tag{5.6}$$

for any $\eta\in S^r_{\mathcal{A}}(L,\mathcal{A})$, $\mathbf{v}=(v_1,\ldots,v_{n+k})\in (L^\bullet)^{\times(n+r)}$.

It is proved in [11] that $\sum_{j+k=n} D_jD_k = 0$ for all $n\geq 0$. Hence we can define the degree 1 derivation $D=\sum_{n\geq 0} D_n$ on $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$ and $D^2=0$.

Conversely, any degree 1 derivation D on $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$ can be written as $D=\sum_{n\geq 0} D_n$, where D_n maps $S^r_{\mathcal{A}}(L,\mathcal{A})$ to $S^{r+n}_{\mathcal{A}}(L,\mathcal{A})$. For any $a\in\mathcal{A}^\bullet$ and $v_1,\ldots,v_n\in L^\bullet$, set

$$\{v_1,\ldots,v_{n-1}|a\}_n := (-1)^{|a|(|v_1|+\cdots+|v_{n-1}|)}(D_{n-1}a)(v_1,\ldots,v_{n-1})\in\mathcal{A}^\bullet \tag{5.7}$$

and let $\{v_1,\ldots,v_n\}_n$ be the unique element in $L^\bullet$ satisfying

$$\begin{aligned}
\eta(\{v_1,\ldots,v_n\}_n) := (-1)^{|\eta|}\sum_{i=1}^n\sum_{i=1}^n (-1)^{|v_i|(|v_1|+\cdots+|v_{i-1}|)} D_{n-1}(\eta(v_i))(v_1,\ldots,\widehat{v_i},\ldots,v_n) \\
- (-1)^{|\eta|}(D_{n-1}\eta)(v_1,\ldots,v_n),
\end{aligned} \tag{5.8}$$

for any $\eta\in L^{\vee\bullet}$ (here we use the projectivity and finiteness of $L^\bullet$). $\square$

Definition 5.8. The dga $(\hat{S}_{\mathcal{A}}(L,\mathcal{A}),D)$ determined by an $L_\infty[1]$-algebroid $(L^\bullet,\mathcal{A}^\bullet)$ in Theorem 5.7 is called the *(completed) Chevalley–Eilenberg dga* of $L^\bullet$.

Remark 5.9. If the dga structure $A=(\mathcal{A}^\bullet,d_A)$ is fixed in advance, an $L_\infty[1]$-algebroid structure over A on $L^\bullet$ is equivalent to a derivation D on $\hat{S}_{\mathcal{A}}(L,\mathcal{A})$, whose zeroth component D_0 acts on $\mathcal{A}^\bullet$ as d_A and acts on $L^{\vee\bullet}$ as $d_{L^\vee}$.

5.3. $L_\infty[1]$-algebroid of the formal neighborhood

We now repackage the differential $\mathfrak{D}$ in the formula (4.22) in the language of $L_\infty[1]$-algebroids. The underlying dga A is the Dolbeault dga $(\mathcal{A}^{0,\bullet}(X),\overline{\partial})$, hence the shifted DGLA $\mathcal{A}^{0,\bullet+1}_X(TX)$ of shifted Dolbeault complex of the tangent bundle TX equipped with the usual Lie bracket is a shifted dg-Lie subalgebra inside $\mathcal{D}er^\bullet(A)[1]$. The underlying A-module of the $L_\infty[1]$-algebroid is the Dolbeault complex $(L^\bullet,d_L)=(\mathcal{A}^{0,\bullet}_X(N),\overline{\partial})$ of the normal bundle N of X inside Y. The structure maps are given as follows. First of all, we compute the values of the anchors on $S^r_A(L,A) = \mathcal{A}^{0,0}_X(S^r N^\vee)$, using (4.23), (5.4) and (5.7), and get the recursive formulas,

$$\alpha_1 = R_1^\top = \beta : \mathcal{A}^{0,0}_X(N) \to \mathcal{A}^{0,1}_X(TX) \subset \mathcal{D}er^1(A), \tag{5.9}$$

$$\begin{aligned} \alpha_n = R_n^\top + \sum_{\sigma \in \mathrm{Sh}(n-1,1)} S_N \circ (\alpha_{n-1} \times 1) \circ \sigma : \mathcal{A}^{0,0}_X(N)^{\times n} \\ \to \mathcal{A}^{0,1}_X(TX) \subset \mathcal{D}er^1(A), \quad n \geq 2, \end{aligned} \tag{5.10}$$

where σ acts as permutation on the n $\mathcal{A}^{0,0}_X(N)$-factors and $S_N : TX \otimes N \to TX$ is the shape operator.) Then we extend α_n to an A-multilinear map from $\mathcal{A}^{0,\bullet}_X(N)^{\times n}$ to $\mathcal{A}^{0,\bullet+1}_X(TX)$ subject to the Koszul sign rules.

Similarly, by (5.8), (4.22) and the formulas above for α, we get the formulas for the n-ary brackets,

$$\ell_n = R_n^\perp + \sum_{\sigma \in \mathrm{Sh}(n-1,1)} \nabla^\perp \circ (\alpha_{n-1} \times 1) \circ \sigma : \mathcal{A}^{0,0}_X(N)^{\times n} \to \mathcal{A}^{0,1}_X(N), \quad n \geq 2, \tag{5.11}$$

where the connection $\nabla^\perp$ on the normal bundle is considered as a map

$$\nabla^\perp : \mathcal{A}^{0,\bullet}_X(TX) \otimes_{\mathbb{C}} \mathcal{A}^{0,\bullet}_X(N) \to \mathcal{A}^{0,\bullet}_X(N)$$

of degree zero. Then we can extend the brackets uniquely such that it satisfies the condition (1) of Definition 5.4.

References

[1] R. Bezrukavnikov, D. Kaledin, Fedosov quantization in algebraic context, Mosc. Math. J. 4 (3) (2004) 559–592, 782.

[2] J. Block, Duality and equivalence of module categories in noncommutative geometry, in: A Celebration of the Mathematical Legacy of Raoul Bott, in: CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 311–339.

[3] R. Bott, Lectures on characteristic classes and foliations, in: Lectures on Algebraic and Differential Topology, Second Latin American School in Math., Mexico City, 1971, in: Lecture Notes in Math., vol. 279, Springer, Berlin, 1972, pp. 1–94, notes by Lawrence Conlon, with two appendices by J. Stasheff.

[4] D. Calaque, A. Căldăraru, J. Tu, On the Lie algebroid of a derived self-intersection, Adv. Math. 262 (2014) 751–783.

[5] I.M. Gelfand, D.A. Kazhdan, Some problems of the differential geometry and the calculation of cohomologies of Lie algebras of vector fields, Dokl. Akad. Nauk Ser. Fiz. 200 (1971) 269–272.

[6] J. Grivaux, Chern classes in Deligne cohomology for coherent analytic sheaves, Math. Ann. 347 (2) (2010) 249–284.

[7] M. Kapranov, Rozansky–Witten invariants via Atiyah classes, Compos. Math. 115 (1) (1999) 71–113.

[8] L. Kjeseth, A homotopy Lie–Rinehart resolution and classical BRST cohomology, Homology, Homotopy Appl. 3 (1) (2001) 165–192.

[9] L. Kjeseth, Homotopy Rinehart cohomology of homotopy Lie–Rinehart pairs, Homology, Homotopy Appl. 3 (1) (2001) 139–163.

[10] H. Sati, U. Schreiber, J. Stasheff, Twisted differential string and fivebrane structures, Comm. Math. Phys. 315 (1) (2012) 169–213.

[11] L. Vitagliano, On the strong homotopy Lie–Rinehart algebra of a foliation, Commun. Contemp. Math. 16 (6) (2014) 1450007, 49 pp.

[12] Y. Xin, Minimal Submanifolds and Related Topics, Nankai Tracts Math., vol. 8, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[13] S. Yu, The Dolbeault dga of the formal neighborhood of the diagonal, J. Noncommut. Geom. 9 (1) (2015) 161–184.

[14] S. Yu, Dolbeault dga of a formal neighborhood, Trans. Amer. Math. Soc. 368 (11) (2016) 7809–7843.

[15] S. Yu, Todd class via homotopy perturbation theory, submitted, arXiv:1510.07936 [math.AG].