Physica A 465 (2017) 562–572

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Space–time fractional diffusion equation using a derivative with nonsingular and regular kernel

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HIGHLIGHTS

- Atangana–Baleanu fractional operators are applied to fractional diffusion equations.
- The generalization of the equations in space–time exhibits anomalous behavior.
- To keep the dimensionality an auxiliary parameter is introduced.
- Based on the Mittag-Leffler function new behaviors for concentration were obtained.

ARTICLE INFO

Article history:
Received 11 April 2016
Received in revised form 1 August 2016
Available online 31 August 2016

Keywords:
Integral transform operator
Fractional diffusion equation
Atangana–Baleanu fractional derivative
Anomalous diffusion
Subdiffusion

ABSTRACT

In this paper, using the fractional operators with Mittag-Leffler kernel in Caputo and Riemann–Liouville sense the space–time fractional diffusion equation is modified, the fractional equation will be examined separately; with fractional spatial derivative and fractional temporal derivative. For the study cases, the order considered is $0 < \beta, \gamma \leq 1$ respectively. In this alternative representation we introduce the appropriate fractional dimensional parameters which characterize consistently the existence of the fractional space–time derivatives into the fractional diffusion equation, these parameters related to equation results in a fractal space–time geometry provide a new family of solutions for the diffusive processes. The proposed mathematical representation can be useful to understand electrochemical phenomena, propagation of energy in dissipative systems, viscoelastic materials, material heterogeneities and media with different scales.

1. Introduction

Fractional calculus (FC) has attracted the interest of researchers in recent decades, FC involving derivatives and integrals of arbitrary order [1,2]. The fractional representation of the standard diffusion equation has been studied in Refs. [3–6]. Gorenflo and Mainardi studied the time fractional diffusion equation obtained from a fractional Fick law, the fundamental obtained solution is related to a phenomenon of slow anomalous diffusion [6]. The authors in Ref. [7] studied subdiffusive transport equations in the Caputo and Riemann–Liouville sense. Anomalous diffusion and relaxation behaviors are often described in terms of fractional equations and generalized stochastic equations [8]. Examples for subdiffusion include charge carrier motion in amorphous semiconductors [9], diffusion on fractals [10], nuclear magnetic resonance, glassy materials and transport on fractal geometries [11]. Tadjeran in Ref. [12] obtained temporally and spatially second-order accurate numerical solutions of the fractional order diffusion equations, the authors derived the solutions based on the classical Crank–Nicholson method. Based on kernel-based approximation technique, the authors of Ref. [13] proposed an efficient

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http://dx.doi.org/10.1016/j.physa.2016.08.072
0378-4371/

and accurate numerical scheme for solving a backward space–time fractional diffusion problem. Sun in Ref. [14] developed a variable-order fractional diffusion equation, the model characterizes diffusion process in inhomogeneous porous media. In recent papers of Luchko [15–17], the generalized time-fractional diffusion equation with variable coefficients was considered. The author shows the existence and uniqueness of the solution for the initial boundary value problem; the Fourier method was used to construct a formal solution. Other applications of fractional calculus in anomalous diffusion are given in Refs. [18–28].

The Riemann–Liouville and Caputo representations have the disadvantage that their kernel had singularity, this kernel includes memory effects and therefore both definitions cannot accurately describe these effects [29]. In Ref. [30] the authors present the Caputo–Fabrizio fractional derivative, this derivative possesses very interesting properties, for instance, the possibility to describe fluctuations and structures with different scales. The novelty in this operator is that, the derivative has regular kernel, nevertheless, due to these properties some researchers have concluded that this operator can be viewed as filter regulator. Properties and applications of this new fractional derivative are reviewed in detail in the papers [31–35]. Recently Atangana and Baleanu proposed a new kernel based on the Mittag-Leffler function [36–38]. This kernel is nonlocal, nonsingular and has all the benefits of Riemann–Liouville, Caputo and Caputo–Fabrizio fractional derivatives.

The main aim of this work is to obtain analytical solutions for the diffusion equation applying the Atangana–Baleanu fractional derivatives in Caputo and Riemann–Liouville sense, in these representations the dimensionality of the ordinary derivative operator was analyzed in order to define a dimensionally correct fractional derivative operator [39].

The paper is structured as follows, in Section 2 we explain the basic definitions of the fractional calculus, in Section 3 we present the fractional diffusion equation and give conclusions in Section 4.

2. Basic definitions

The Atangana–Baleanu fractional derivative in Caputo sense is defined as follows [36–38]

$$ {}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \dot{f}(\theta) E_\alpha\left[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\right] \mathrm{d}\theta, \quad 0<\alpha\leq 1 \tag{1} $$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} = {}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{t}$ is an Atangana–Baleanu fractional derivative in Caputo sense with respect to t, $B(\alpha)$ is a normalization function and has the same properties as in Caputo and Caputo–Fabrizio case.

The Laplace transform of (1) is defined as follows

$$ \begin{aligned} \mathscr{L}[{}^{ABC}_{\ a}\mathcal{D}^{\alpha}_{t} f(t)](s) &= \frac{B(\alpha)}{1-\alpha} \mathscr{L}\left[\int_a^t \dot{f}(\theta) E_\alpha\left[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\right] \mathrm{d}\theta\right] \\ &= \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha \mathscr{L}[f(t)](s) - s^{\alpha-1} f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \end{aligned} \tag{2} $$

The definition of the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is defined as follows [36–38]

$$ {}^{ABR}_{\ a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_b^t f(\theta) E_\alpha\left[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\right] \mathrm{d}\theta, \quad 0<\alpha\leq 1 \tag{3} $$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} = {}^{ABR}_{\ a}\mathcal{D}^{\alpha}_{t}$ is an Atangana–Baleanu fractional derivative in Riemann–Liouville sense with respect to t, $B(\alpha)$ is a normalization function as in the above definition.

The Laplace transform of (3) is defined as follows

$$ \begin{aligned} \mathscr{L}[{}^{ABR}_{\ a}\mathcal{D}^{\alpha}_{t} f(t)](s) &= \frac{B(\alpha)}{1-\alpha} \mathscr{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\theta) E_\alpha\left[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\right] \mathrm{d}\theta\right] \\ &= \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha \mathscr{L}[f(t)](s)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \end{aligned} \tag{4} $$

Properties of these new fractional derivatives are reviewed in detail in Atangana and Koca [38].

3. Fractional diffusion equation

The local diffusion equation is represented in (5)

$$ D\frac{\partial^2 C(x,t)}{\partial x^2} - \frac{\partial C(x,t)}{\partial t} = E(x,t), \tag{5} $$

Eq. (5) describes the ordinary diffusion, D represents the reciprocal of the time constant of the system or diffusion coefficient.

To keep the dimensionality of the differential equation a new parameter σ_x (dimensions of length) and σ_t (dimensions of time) was introduced [40]. For the Atangana–Baleanu fractional derivative in the Caputo sense we have

$$\frac{d^2}{dx^2} \rightarrow \frac{1}{\sigma_x^{2(1-\beta)}} \cdot {}_{a}^{ABC}\mathcal{D}_x^{2\beta}, \quad 0 < \beta \leq 1 \tag{6}$$

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma_t^{1-\gamma}} \cdot {}_{a}^{ABC}\mathcal{D}_t^{\gamma}, \quad 0 < \gamma \leq 1 \tag{7}$$

and for the Atangana–Baleanu fractional derivative in the Riemann–Liouville sense we have

$$\frac{d^2}{dx^2} \rightarrow \frac{1}{\sigma_x^{2(1-\beta)}} \cdot {}_{a}^{ABR}\mathcal{D}_x^{2\beta}, \quad 0 < \beta \leq 1 \tag{8}$$

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma_t^{1-\gamma}} \cdot {}_{a}^{ABR}\mathcal{D}_t^{\gamma}, \quad 0 < \gamma \leq 1 \tag{9}$$

when $\beta = 1$ and $\gamma = 1$, the expressions (6)–(9) become ordinary operators. The existence of fractional structures (components that show an intermediate behavior between a system conservative and dissipative) in the system is characterized by the parameters σ_x and σ_t, these parameters related to equation results in a fractal space–time geometry presented a new family of solutions for the diffusion equation [40]. In the temporal operator case, the authors of Ref. [41] used the Planck time, $t_p = 5.39106 \times 10^{-44}$ s, with the finality to preserve the dimensional compatibility. Following Ref. [41] the σ parameter corresponds to the t_p in our calculations, and for the spatial case, we used the Planck length, $l_p = 1.616199 \times 10^{-35}$ m, the σ_x parameter corresponds to the l_p. From now on, we will apply this idea to generalize the case of the fractional diffusion equation.

Consider equations (6)–(7) and (8)–(9), the fractional representation of (5) is

$$\frac{D}{\sigma_x^{2(1-\beta)}} \cdot {}_{a}^{ABC}\mathcal{D}_x^{2\beta} - \frac{1}{\sigma_t^{1-\gamma}} \cdot {}_{a}^{ABC}\mathcal{D}_t^{\gamma} = E(x, t), \tag{10}$$

$$\frac{D}{\sigma_x^{2(1-\beta)}} \cdot {}_{a}^{ABR}\mathcal{D}_x^{2\beta} - \frac{1}{\sigma_t^{1-\gamma}} \cdot {}_{a}^{ABR}\mathcal{D}_t^{\gamma} = E(x, t), \tag{11}$$

the order of the derivative is $0 < \beta, \gamma \leq 1$, $C(x, t)$ represents the concentration. The fractional diffusion equation (10) or (11) might be useful for investigating the mechanism of anomalous diffusion in electrochemical phenomena, diffusion on fractals, models of porous electrodes and transport on fractal geometries. Now two cases will be analyzed: the first case occurs when the temporal operator is ordinary and the spatial operator is fractional; the second case takes place when the spatial operator is ordinary and the temporal operator is fractional.

3.1. Fractional space diffusion equation with Atangana–Baleanu derivatives

Considering Eq. (10), for the first case, the spatial fractional equation is

$${}_{a}^{ABC}\mathcal{D}_x^{2\beta} - \frac{\sigma_x^{2(1-\beta)}}{D} \frac{\partial C(x, t)}{\partial t} = E(x, t) \quad 0 < \beta \leq 1 \tag{12}$$

Eq. (12) represents a random walk type Lévy flight [42]. A particular solution of Eq. (12) may be found in the form

$$C(x, t) = C_0 \cdot e^{-\omega t} u(x), \tag{13}$$

substituting (13) into (12) we obtain

$${}_{a}^{ABC}\mathcal{D}_x^{2\beta} u(x) + \tilde{k}^2 u(x) = E(x), \tag{14}$$

where, $E(x) = 1$ for $x \geq 0$ and $E(x) = 0$ for $x < 0$, $\tilde{k}^2 = k^2 \sigma_x^{2(1-\beta)}$ is the fractional wave number and $k = \frac{\omega}{D}$ is the classical wave number.

Applying the Laplace transform (2) to (14) and considering $u(0) = u_0$ and $\dot{u}(0) = 0$, yields the following expression

$$U(s) = \frac{1}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \left[\frac{(1-\beta)^2 s^{2\beta-1}}{\left(s^\beta + \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta - \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)} \right.$$
$$\left. + \frac{2\beta(1-\beta)s^{\beta-1}}{\left(s^\beta + \frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta - \frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)} \right.$$

$$+\frac{\beta^2 s^{-1}}{\left(s^\beta+\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta-\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\Bigg]$$
$$+\frac{B(\beta)^2 u_0}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\left[\frac{s^{2\beta-1}}{\left(s^\beta+\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta-\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\right]. \tag{15}$$

Taking the inverse Laplace transform of (15), we obtain the following particular solution of Eq. (13)

$$\begin{aligned} C(x,t) = C_0\cdot \mathrm{e}^{-\omega t}\Bigg[&\frac{(1-\beta)^2}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,1}(ax^\beta)+bE_{\beta,1}(-bx^\beta)}{a+b}\\ &+\frac{2\beta(1-\beta)}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,\beta+1}(ax^\beta)+bE_{\beta,\beta+1}(-bx^\beta)}{a+b}\cdot x^\beta\\ &+\frac{\beta^2}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,2\beta+1}(ax^\beta)+bE_{\beta,2\beta+1}(-bx^\beta)}{a+b}\cdot x^{2\beta}\\ &+\frac{B(\beta)u_0}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,1}(ax^\beta)+bE_{\beta,1}(-bx^\beta)}{a+b}\Bigg] \end{aligned} \tag{16}$$

where, $a=\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}$, $b=\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}$ and E_β is the Mittag-Leffler function.

Fig. 1(a) and (b) show numerical simulations of Eq. (16) for different values of the fractional order $\beta \in (0.6; 1]$. In these cases the non-Markovian Lévy flights [11] are described.

Now, considering Eq. (11), for the first case, the spatial fractional equation is

$${}^{ABR}_{\ a}\mathcal{D}_x^{2\beta} - \frac{\sigma_x^{2(1-\beta)}}{D}\frac{\partial C(x,t)}{\partial t} = E(x,t) \quad 0<\beta\le 1 \tag{17}$$

Eq. (17) represents a random walk type Lévy flight [42]. A particular solution of Eq. (17) may be found in the form

$$C(x,t) = C_0\cdot \mathrm{e}^{-\omega t}u(x), \tag{18}$$

substituting (18) into (17) we obtain

$${}^{ABR}_{\ a}\mathcal{D}_x^{2\beta}u(x) + \tilde{k}^2 u(x) = E(x), \tag{19}$$

where, $E(x)=1$ for $x\ge 0$ and $E(x)=0$ for $x<0$, $\tilde{k}^2=k^2\sigma_x^{2(1-\beta)}$ is the fractional wave number and $k=\frac{\omega}{D}$ is the classical wave number.

Applying the Laplace transform (2) to (19) yields the following expression

$$\begin{aligned} U(s) = \frac{1}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\Bigg[&\frac{(1-\beta)^2 s^{2\beta-1}}{\left(s^\beta+\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta-\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\\ &+\frac{2\beta(1-\beta)s^{\beta-1}}{\left(s^\beta+\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta-\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}+\frac{\beta^2 s^{-1}}{\left(s^\beta+\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}\right)\left(s^\beta-\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}\right)}\Bigg]. \end{aligned} \tag{20}$$

Taking the inverse Laplace transform of (20), we obtain the following particular solution of Eq. (18)

$$\begin{aligned} C(x,t) = C_0\cdot \mathrm{e}^{-\omega t}\Bigg[&\frac{(1-\beta)^2}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,1}(ax^\beta)+bE_{\beta,1}(-bx^\beta)}{a+b}\\ &+\frac{2\beta(1-\beta)}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,\beta+1}(ax^\beta)+bE_{\beta,\beta+1}(-bx^\beta)}{a+b}\cdot x^\beta\\ &+\frac{\beta^2}{B(\beta)^2+\tilde{k}^2(1-\beta)^2}\cdot\frac{aE_{\beta,2\beta+1}(ax^\beta)+bE_{\beta,2\beta+1}(-bx^\beta)}{a+b}\cdot x^{2\beta}\Bigg] \end{aligned} \tag{21}$$

where, $a=\frac{i\tilde{k}\beta}{B(\beta)+i\tilde{k}(1-\beta)}$, $b=\frac{i\tilde{k}\beta}{B(\beta)-i\tilde{k}(1-\beta)}$ and E_β is the Mittag-Leffler function.

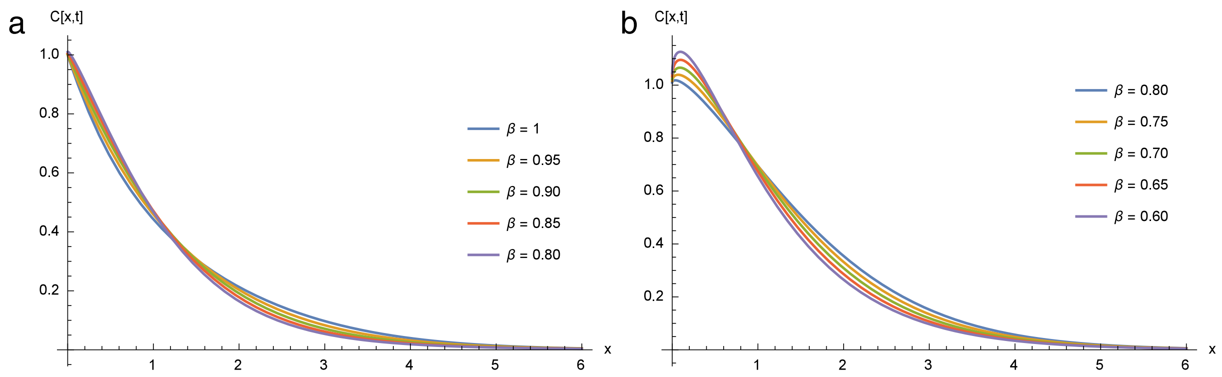

Fig. 1. Concentration for $\beta \in (0.6; 1]$.

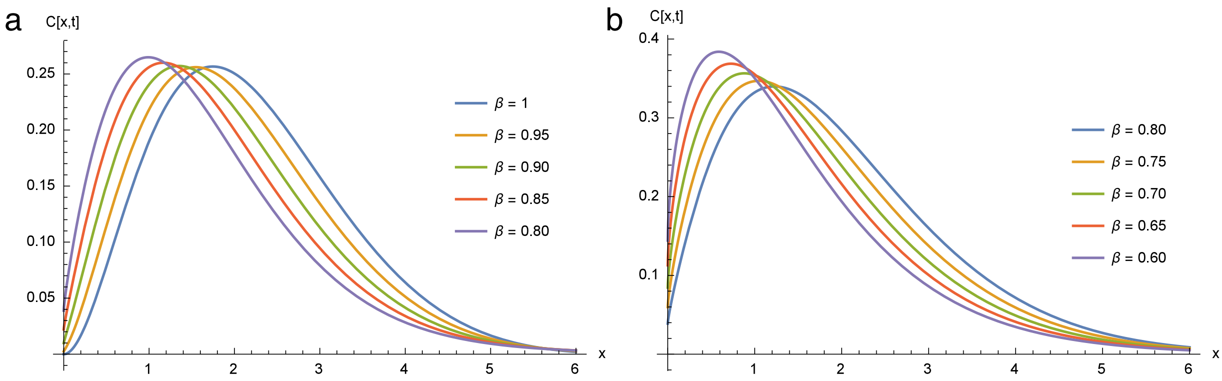

Fig. 2. Concentration for $\beta \in [0.6; 1]$.

Fig. 2(a) and (b) show numerical simulations of Eq. (21) for different values of the fractional order $\beta \in [0.6; 1]$. In these cases the non-Markovian Lévy flights [11] are described.

3.2. Fractional time diffusion equation with Atangana–Baleanu derivatives

Considering Eq. (10), for the second case, the temporal fractional equation is

$$ {}^{ABC}_{a}\mathcal{D}^{\gamma}_{t} - D\sigma_t^{1-\gamma}\frac{\partial^2 C(x,t)}{\partial x^2} = E(x,t), \quad 0 < \gamma \le 1 \tag{22}$$

Eq. (22) represents a stochastic scheme (continuous time random walk) [28]. A particular solution of Eq. (22) may be found in the form

$$ C(x,t) = C_0 \cdot \mathrm{e}^{-ikx} u(t), \tag{23}$$

substituting (23) into (22) we obtain

$$ {}^{ABC}_{a}\mathcal{D}^{\gamma}_{t} u(t) + \tilde{\omega} u(t) = E(t), \tag{24}$$

where, $E(t) = 1$ for $t \ge 0$ and $E(t) = 0$ for $t < 0$, $\tilde{\omega} = \omega\sigma_t^{1-\gamma}$ and $\omega = Dk^2$.

Applying the Laplace transform (2) to (24) and considering $u(0) = u_0$, yields the following expression

$$ \begin{aligned} U(s) = {} & \frac{1}{B(\gamma) + \tilde{\omega}(1-\gamma)} \left[\frac{(1-\gamma)s^{\gamma-1}}{s^{\gamma} + \frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}} + \frac{\gamma}{s^{\gamma} + \frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}} \cdot \frac{1}{s} \right] \\ & + \frac{B(\gamma)U_0}{B(\gamma)+\tilde{\omega}(1-\gamma)} \cdot \frac{s^{\gamma-1}}{s^{\gamma} + \frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}}. \end{aligned} \tag{25}$$

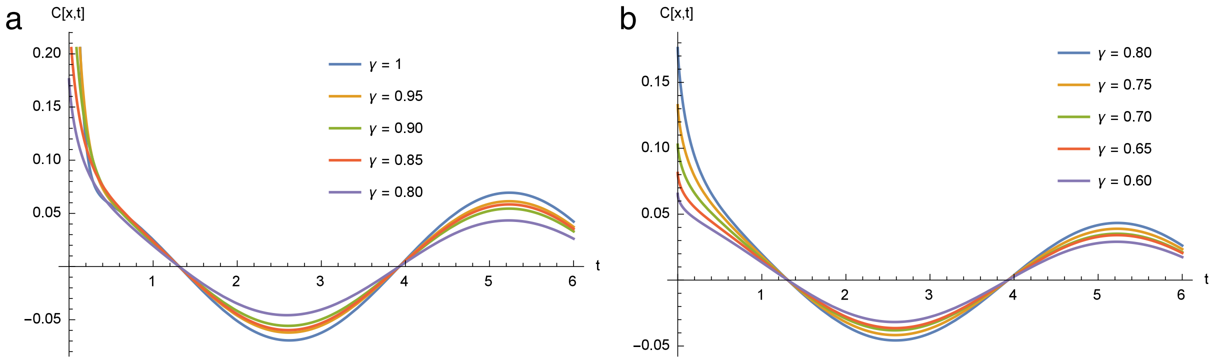

Fig. 3. Concentration for $\gamma \in [0.6; 1]$.

Taking the inverse Laplace transform of (25), we obtain the following particular solution of (23)

$$C(x,t) = C_0 \cdot e^{-ikx}\Bigg[\frac{1-\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}\right)t^{\gamma}\right] + \frac{B(\gamma)U_0}{B(\gamma)+\tilde{\omega}(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}\right)t^{\gamma}\right] + \frac{\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}\int_0^t \tau^{\gamma-1}E_{\gamma,\gamma}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}\right)\tau^{\gamma}\right]\mathrm{d}\tau\Bigg] \tag{26}$$

where E_β is the Mittag-Leffler function.

Fig. 3(a) and (b) show numerical simulations of Eq. (26) for different values of the fractional order $\gamma \in [0.6; 1]$. For the cases $0 < \gamma \leq 1$, the subdiffusion phenomena is described and when $\gamma = 1$ the phenomena of Brownian diffusion is presented [11].

Consider $\tilde{\omega} = Dk^2\sigma_t^{1-\gamma}$ and $1/D$ the reciprocal of the time constant or diffusion coefficient, we have

$$\tilde{\omega} = k^2\left(\frac{\sigma_t^{1-\gamma}}{D}\right) = k^2\left(\frac{1}{\tau_\gamma}\right), \tag{27}$$

where, $\tau_\gamma = D\sigma_t^{\gamma-1}$, it can be called fractional time constant, when $\gamma = 1$, we have the classical time constant, $\tilde{\omega}$ is the angular frequency in the medium in the presence of fractional time components, k is the wave number and $1/D$ is the time constant of the system. Substituting (27) into (26) we obtain

$$C(x,t) = C_0 \cdot e^{-ikx}\Bigg[\frac{1-\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\left(\frac{k^2}{\tau_\gamma}\right)\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)}\right)t^{\gamma}\right] + \frac{B(\gamma)U_0}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\left(\frac{k^2}{\tau_\gamma}\right)\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)}\right)t^{\gamma}\right] + \frac{\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)}\int_0^t \tau^{\gamma-1}E_{\gamma,\gamma}\left[-\left(\frac{\left(\frac{k^2}{\tau_\gamma}\right)\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)}\right)\tau^{\gamma}\right]\mathrm{d}\tau\Bigg]. \tag{28}$$

Fig. 4 show the simulation of Eq. (28) for the fractional exponents $\gamma = 1$, $\gamma = 0.95$, $\gamma = 0.9$, $\gamma = 0.85$ and $\gamma = 0.80$, respectively.

Table 1 shows the different values of the concentration when γ changes from $\gamma = 1$, $\gamma = 0.95$, $\gamma = 0.90$, $\gamma = 0.85$ to $\gamma = 0.80$, respectively, when $\gamma < 1$ the diffusion occurs in more time than the ordinary diffusion. This phenomenon indicates the change of the medium properties and the system presents dissipative effects that correspond to the nonlinear situation of the physical process (realistic behavior that is non-local in time), namely anomalous diffusion (subdiffusion).

Now, considering Eq. (11), for the second case, the temporal fractional equation is

$${}_{b}^{ABR}\mathcal{D}_t^{\gamma} - D\sigma_t^{1-\gamma}\frac{\partial^2 C(x,t)}{\partial x^2} = E(x,t), \quad 0 < \gamma \leq 1 \tag{29}$$

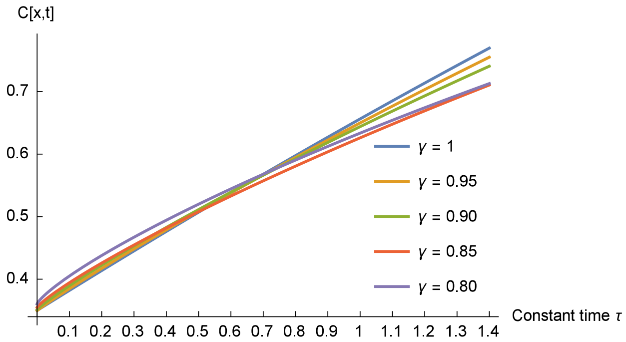

Fig. 4. Diffusion vs. Constant time, exponent $\gamma = 1$, $\tau = 0.63$ located in $t = 1$ s, $\gamma = 0.95$, $\tau = 0.63$ located in $t = 1.033$ s, $\gamma = 0.90$, $\tau = 0.63$ located in $t = 1.037$ s, $\gamma = 0.85$, $\tau = 0.63$ located in $t = 1.041$ s and $\gamma = 0.80$, $\tau = 0.63$ located in $t = 1.044$ s.

Table 1
Diffusion vs. Constant time.

γ	Constant time (τ)	Diffusion (C)
1	1	0.63
0.95	1.033	0.63
0.90	1.059	0.63
0.85	1.092	0.63
0.80	1.145	0.63

Eq. (29) represents a stochastic scheme (continuous time random walk) [28]. A particular solution of Eq. (29) may be found in the form

$$C(x,t) = C_0 \cdot \mathrm{e}^{-ikx} u(t), \tag{30}$$

substituting (30) into (29) we obtain

$${}^{ABR}_{\ \ b}\mathcal{D}_t^{\gamma} u(t) + \tilde{\omega} u(t) = E(t), \tag{31}$$

where, $E(t) = 1$ for $t \geq 0$ and $E(t) = 0$ for $t < 0$, $\tilde{\omega} = \omega \sigma_t^{1-\gamma}$ and $\omega = Dk^2$.

Applying the Laplace transform (2) to (31) yields the following expression

$$U(s) = \frac{1}{B(\gamma) + \tilde{\omega}(1-\gamma)} \left[\frac{(1-\gamma)s^{\gamma-1}}{s^{\gamma} + \frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}} + \frac{\gamma}{s^{\gamma} + \frac{\tilde{\omega}\gamma}{B(\gamma)+\tilde{\omega}(1-\gamma)}} \cdot \frac{1}{s} \right]. \tag{32}$$

Taking the inverse Laplace transform of (32), we obtain the following particular solution of Eq. (30)

$$\begin{aligned} C(x,t) &= C_0 \cdot \mathrm{e}^{-ikx} \left[\frac{(1-\gamma)}{B(\gamma) + \tilde{\omega}(1-\gamma)} \cdot E_{\gamma,1} \left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) t^{\gamma} \right] \right. \\ &\quad \left. + \frac{\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \int_0^t \tau^{\gamma-1} E_{\gamma,\gamma} \left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) \tau^{\gamma} \right] \mathrm{d}\tau \right] \end{aligned} \tag{33}$$

where E_{β} is the Mittag-Leffler function.

Fig. 5(a) and (b) show numerical simulations of Eq. (33) for different values of the fractional order $\gamma \in [0.6; 1]$. For the cases $0 < \gamma \leq 1$, the subdiffusion phenomenon is described and when $\gamma = 1$ the phenomenon of Brownian diffusion is presented [11].

Consider $\tilde{\omega} = Dk^2 \sigma_t^{1-\gamma}$ and $1/D$ the reciprocal of the time constant or diffusion coefficient, we have

$$\tilde{\omega} = k^2 \left(\frac{\sigma_t^{1-\gamma}}{D} \right) = k^2 \left(\frac{1}{\tau_{\gamma}} \right), \tag{34}$$

where, $\tau_{\gamma} = D\sigma_t^{\gamma-1}$, it can be called fractional time constant, when $\gamma = 1$, we have the classical time constant, $\tilde{\omega}$ is the angular frequency in the medium in the presence of fractional time components, k is the wave number and $1/D$ is the time

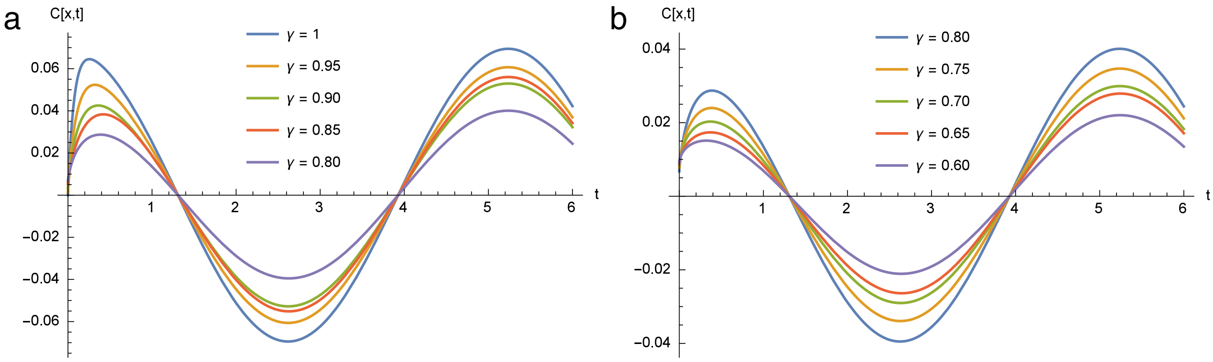

Fig. 5. Concentration for $\gamma \in [0.6; 1]$.

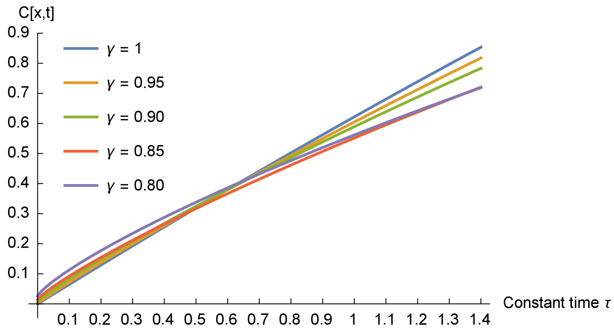

Fig. 6. Diffusion vs. Constant time, exponent $\gamma = 1$, $\tau = 0.63$ located in $t = 1$ s, $\gamma = 0.95$, $\tau = 0.63$ located in $t = 1.033$ s, $\gamma = 0.90$, $\tau = 0.63$ located in $t = 1.037$ s, $\gamma = 0.85$, $\tau = 0.63$ located in $t = 1.041$ s and $\gamma = 0.80$, $\tau = 0.63$ located in $t = 1.044$ s.

Table 2
Diffusion vs. Constant time.

γ	Constant time (τ)	Diffusion (C)
1	1	0.63
0.95	1.039	0.63
0.9	1.088	0.63
0.85	1.182	0.63
0.80	1.204	0.63

constant of the system. Substituting (34) into (33) we obtain

$$C(x,t) = C_0 \cdot \mathrm{e}^{-ikx}\left[\frac{1-\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\left(\frac{k^2}{\tau_\gamma}\right)\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)}\right)t^\gamma\right]\right.$$
$$\left. + \frac{\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)} \int_0^t \tau^{\gamma-1} E_{\gamma,\gamma}\left[-\left(\frac{\left(\frac{k^2}{\tau_\gamma}\right)\gamma}{B(\gamma)+\left(\frac{k^2}{\tau_\gamma}\right)(1-\gamma)}\right)\tau^\gamma\right]\mathrm{d}\tau\right]. \tag{35}$$

Fig. 6 show the simulation of Eq. (35) for the fractional exponents $\gamma = 1$, $\gamma = 0.95$, $\gamma = 0.90$, $\gamma = 0.85$ and $\gamma = 0.80$, respectively.

Table 2 shows the different values of the concentration when γ changes from $\gamma = 1$, $\gamma = 0.95$, $\gamma = 0.90$, $\gamma = 0.85$ to $\gamma = 0.80$, respectively, when $\gamma < 1$ the diffusion occurs in more time than the ordinary diffusion. This phenomenon indicates the change of the medium properties and the system presents dissipative effects that corresponds to the nonlinear situation of the physical process (realistic behavior that is non-local in time), namely anomalous diffusion (subdiffusion).

3.3. Fractional space–time diffusion equation with Atangana–Baleanu derivatives

Now considering Eqs. (10) and (11) and assuming that the space and time derivatives are fractional, the order of the space–time fractional differential equation is $0 < \beta, \gamma \leq 1$, the full solution of Eq. (10) in Atangana–Baleanu Caputo sense and (11) in Atangana–Baleanu Riemann–Liouville sense is given by (37) and (38), respectively

$$C(x,t) = C_0 \cdot \Bigg\{ \Bigg[\frac{(1-\beta)^2}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,1}(ax^{\beta}) + bE_{\beta,1}(-bx^{\beta})}{a+b} + \frac{2\beta(1-\beta)}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,\beta+1}(ax^{\beta}) + bE_{\beta,\beta+1}(-bx^{\beta})}{a+b} \cdot x^{\beta} + \frac{\beta^2}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,2\beta+1}(ax^{\beta}) + bE_{\beta,2\beta+1}(-bx^{\beta})}{a+b} \cdot x^{2\beta} + \frac{B(\beta)u_0}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,1}(ax^{\beta}) + bE_{\beta,1}(-bx^{\beta})}{a+b} \Bigg] \Bigg\} \tag{36}$$

$$\cdot \Bigg\{ \Bigg[\frac{1-\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) t^{\gamma} \right] + \frac{B(\gamma)U_0}{B(\gamma) + \tilde{\omega}(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) t^{\gamma} \right] + \frac{\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \int_0^t \tau^{\gamma-1} E_{\gamma,\gamma}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) \tau^{\gamma} \right] d\tau \Bigg] \Bigg\}, \tag{37}$$

and

$$C(x,t) = C_0 \cdot \Bigg\{ \Bigg[\frac{(1-\beta)^2}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,1}(ax^{\beta}) + bE_{\beta,1}(-bx^{\beta})}{a+b} + \frac{2\beta(1-\beta)}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,\beta+1}(ax^{\beta}) + bE_{\beta,\beta+1}(-bx^{\beta})}{a+b} \cdot x^{\beta} + \frac{\beta^2}{B(\beta)^2 + \tilde{k}^2(1-\beta)^2} \cdot \frac{aE_{\beta,2\beta+1}(ax^{\beta}) + bE_{\beta,2\beta+1}(-bx^{\beta})}{a+b} \cdot x^{2\beta} \Bigg] \Bigg\}$$

$$\cdot \Bigg\{ \Bigg[\frac{(1-\gamma)}{B(\gamma) + \tilde{\omega}(1-\gamma)} \cdot E_{\gamma,1}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) t^{\gamma} \right] + \frac{\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \int_0^t \tau^{\gamma-1} E_{\gamma,\gamma}\left[-\left(\frac{\tilde{\omega}\gamma}{B(\gamma) + \tilde{\omega}(1-\gamma)} \right) \tau^{\gamma} \right] d\tau \Bigg] \Bigg\}. \tag{38}$$

3.4. *Numerical results*

Fig. 7(a)–(d) show numerical simulations for Eqs. (37) and (38), respectively, the fractional time and spatial fractional derivative are taken at the same time for β and γ equal to 0.90.

In the range $0 < \beta \leq 1$, we observe the non-Markovian Lévy flights and in the range $0 < \gamma < 1$, the subdiffusion phenomena, this subdiffusion occurs in many physical systems, porus media [43], NMR diffusometry in disordered materials [44,45], dynamics of a bead in a polymer network [46,47] or gel solvents [48]. The general solutions of the fractional differential equations are given in terms of the Mittag-Leffler functions depending only on a small number of parameters and related to equation results in a fractal space–time geometry preserving the physical units of the system for any value taken by the exponent of the fractional derivative.

4. Conclusions

This paper presents the analysis of the diffusion equation applying the novel fractional definition based on the Mittag-Leffler function recently introduced by Abdon Atangana and Dumitru Baleanu with non-local and non-singular kernel, these characteristics allow better description of systems with memory, which typically are dissipative and complex.

The new concept of fractional derivatives was therefore used in this work to present alternative solutions to the space–time fractional wave equations in dielectric media. The order of the fractional derivatives considered was $0 < \beta, \gamma \leq 1$. Two auxiliary parameters α_x and α_t are introduced characterizing the existence of the fractional space and

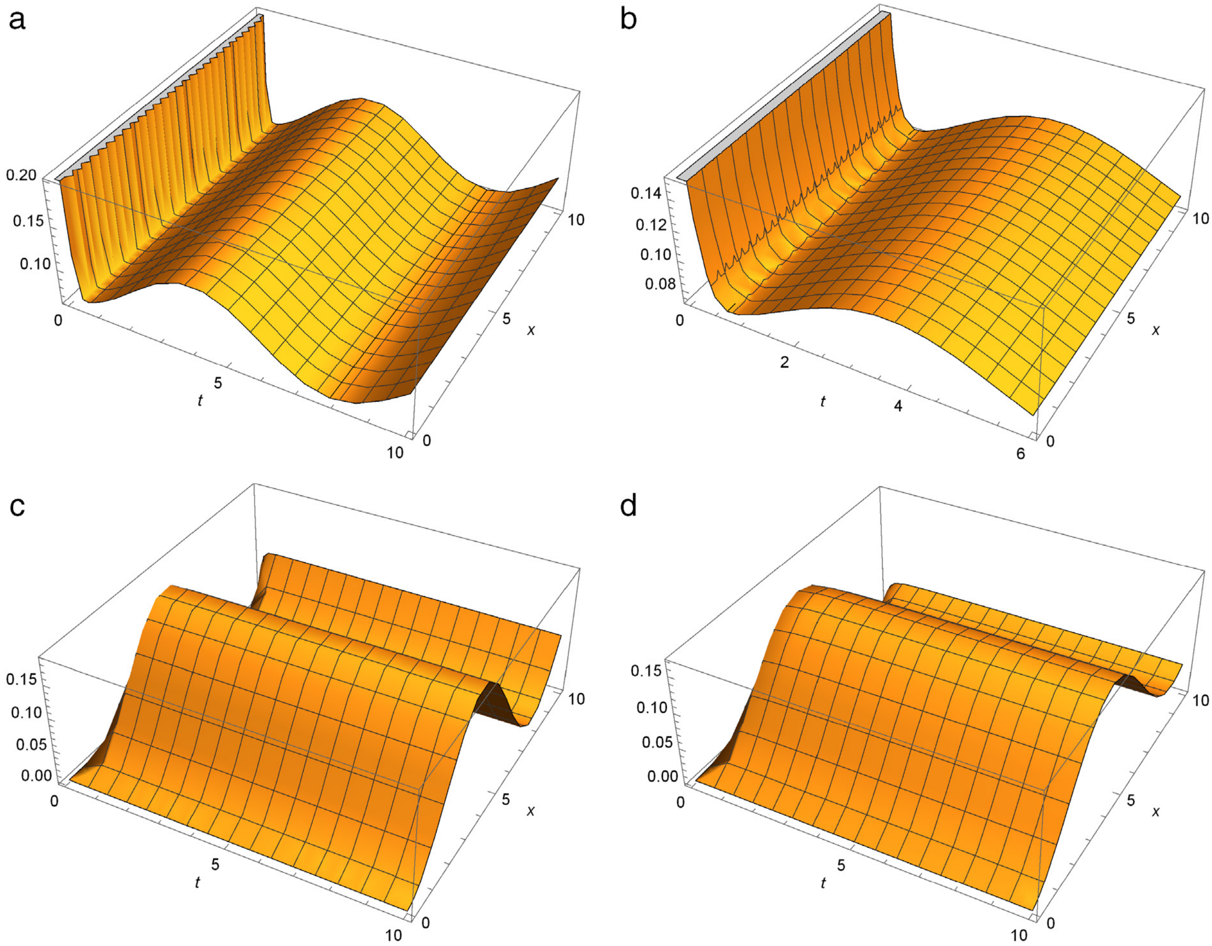

Fig. 7. Numerical simulation of Eq. (37), in (a) $\beta = \gamma = 1$; in (b) $\beta = \gamma = 0.90$. In (c) and (d), numerical simulation of Eq. (38) for $\beta = \gamma = 1$ and $\beta = \gamma = 0.90$, respectively.

time components respectively; these parameters show that the system presents dissipative effects that correspond to the nonlinear situation of the physical process (realistic behavior that is non-local in time). These solutions represent a non-local behavior interpreted as the existence of memory effects, which correspond to intrinsic dissipation characterized by the exponent of the fractional derivatives β and γ in the system; and related to diffusion in a fractal space–time geometry presented in an entire new family of solutions.

For the fractional constant time, the change from $\gamma = 1$, $\gamma = 0.95$, $\gamma = 0.90$, $\gamma = 0.85$ to $\gamma = 0.80$, respectively, indicates that the constant time tends to move forward in time as this exponent is varying, that is, the diffusion occurs in more time than it would take the entire order of exponent (ordinary diffusion). This phenomenon indicates the change of the medium properties, different from the ideal properties (classical case) showing fractional structures, this change presents anomalous diffusion (subdiffusion) in the system. In both models, Eqs. (28) and (35), the anomalous diffusion phenomena occur, this model of anomalous diffusion does not introduce additional elements characterizing the change of the medium properties.

We consider that these results can be useful to gain understanding of the electrochemical phenomena, non-Fourier processes, models of porous electrodes and the description of anomalous complex processes.

Acknowledgments

We would like to thank Mayra Martínez for the interesting discussions. José Francisco Gómez Aguilar acknowledges the support provided by CONACYT: cátedras CONACYT para jovenes investigadores 2014.

References

[1] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[2] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[3] D. Ben-Avraham, S. Havlin, Cambridge University Press, 2000.
[4] S. Kumar, J.J. Yao, A. Kumar, Adv. Mech. Eng. 7 (12) (2015) 1–11.
[5] R. Gorenflo, F. Mainardi, J. Anal. Appl. 18 (1999) 231–246.
[6] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, Nonlinear Dynam. 29 (2002) 129–143.
[7] P. Straka, S. Fedotov, J. Theoret. Biol. 366 (2015) 71–83.
[8] A.D. Viñales, K.G. Wang, M.A. Despósito, Phys. Rev. E 80 (2009) 011101.
[9] H. Scher, E.W. Montroll, Phys. Rev. B 12 (1975) 2455.
[10] A. Klemm, R. Metzler, R. Kimmich, Phys. Rev. E 65 (2002) 021112.
[11] R. Metzler, J. Klafter, Phys. Rep. 339 (2000) 1–77.
[12] C. Tadjeran, M.M. Meerschaert, H.P. Scheffler, J. Comput. Phys. 213 (1) (2006) 205–213.
[13] F.F. Dou, Y.C. Hon, Comput. Math. Appl. 71 (1) (2016) 356–367.
[14] H. Sun, W. Chen, C. Li, Y. Chen, Int. J. Bifurcation Chaos 22 (04) (2012) 1250085.
[15] Y. Luchko, J. Math. Anal. Appl. 351 (2009) 218–223.
[16] Y. Luchko, Comput. Math. Appl. 59 (2010) 1766–1772.
[17] Y. Luchko, J. Math. Anal. Appl. 374 (2011) 538–548.
[18] R. Metzler, A. Compte, J. Phys. Chem. B 104 (2000) 3858–3865.
[19] H. Guang Sun, W. Chen, Y. Quan Chen, Physica A 388 (21) (2009) 4586–4592.
[20] J. Bisquert, A. Compte, J. Electroanal. Chem. 499 (2001) 112–120.
[21] X.B. Yin, S. Kumar, D. Kumar, Adv. Mech. Eng. 7 (12) (2015) 1–8.
[22] X.J. Yang, D. Baleanu, W.P. Zhong, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. 14 (2) (2013) 127–133.
[23] S. Kumar, A. Kumar, D. Baleanu, Nonlinear Dynam. 1 (2016) 1–17.
[24] O.S. Iyiola, Progr. Fract. Differ. Appl. 2 (1) (2016) 21–30.
[25] Z. Sheng, G. Yuan, J. Comput. Phys. 315 (2016) 182–193.
[26] Y. Liang, W. Chen, R.L. Magin, Physica A 453 (2016) 327–335.
[27] S. Kumar, Alexandria Eng. J. 53 (1) (2014) 225–231.
[28] A. de Pablo, F. Quirós, A. Rodríguez, J.L. Vázquez, J. Math. Pures Appl. 101 (6) (2014) 901–924.
[29] A. Atangana, B.S.T. Alkahtani, Entropy 17 (2015) 4439.
[30] M. Caputo, M. Fabricio, Progr. Fract. Differ. Appl. 1 (2015) 73.
[31] J. Lozada, J.J. Nieto, Progr. Fract. Differ. Appl. 1 (2015) 87.
[32] J.F. Gómez-Aguilar, Physica A 447 (2016) 467–481.
[33] A. Atangana, B.S.T. Alkahtani, Arab. J. Geosci. 9 (1) (2016) 1–6.
[34] A. Atangana, Appl. Math. Comput. 273 (2016) 948–956.
[35] J.F. Gómez Aguilar, H. Yépez Martínez, C. Calderón Ramón, I. Cruz Orduña, R.F. Escobar Jiménez, V.H. Olivares Peregrino, Entropy 17 (2015) 6289.
[36] A. Atangana, D. Baleanu, Therm. Sci. 18 (2016) http://dx.doi.org/10.2298/TSCI160111018A.
[37] A. Coronel-Escamilla, J.F. Gómez-Aguilar, M.G. López-López, V.M. Alvarado-Martínez, G.V. Guerrero-Ramírez, Chaos Solitons Fractals 91 (2016) 248–261.
[38] A. Atangana, I. Koca, Chaos Solitons Fractals (2016).
[39] J.F. Gómez Aguilar, R. Razo Hernández, D. Granados Lieberman, Rev. Mexicana Fís. 60 (1) (2014) 32–38.
[40] J.F. Gómez Aguilar, M. Miranda Hernández, M.G. López López, V.M. Alvarado Martínez, D. Baleanu, Commun. Nonlinear Sci. Numer. Simul. 30 (2016) 115–127.
[41] A.E. Calik, Rev. Mexicana Fís. 61 (2015) 58–63.
[42] A.S. Chaves, Phys. Lett. A 239 (1998) 13.
[43] T. Kosztolowicz, J. Phys. A: Math. Gen. 37 (2004) 10779–10789.
[44] W. Einicke, D. Enke, M. Dvoyashkin, R. Valiullin, R. Gläser, Materials 6 (2013) 3688–3709.
[45] D. Mehlhorn, R. Valiullin, J. Kärger, K. Cho, R. Ryoo, Materials 5 (4) (2012) 699–720.
[46] M.D. Graham, Annu. Rev. Fluid Mech. 43 (2011) 273–298.
[47] A. Kutvonen, G. Rossi, T. Ala-Nissila, Phys. Rev. E 85 (2012) 041803.
[48] K. Dworecki, S. Wasik, T. Kosztolowicz, Acta Phys. Polon. B 34 (2003) 3695.