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Neighborhood radius estimation for Arnold's miniversal deformations of complex and p-adic matrices

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V.I. Arnold (1971) constructed a simple normal form to which all complex matrices B in a neighborhood U of a given square matrix A can be reduced by similarity transformations that smoothly depend on the entries of B. We calculate the radius of the neighborhood U. A.A. Mailybaev (1999, 2001) constructed a reducing similarity transformation in the form of Taylor series; we construct this transformation by another method. We extend Arnold's normal form to matrices over the field $\mathbb{Q}_p$ of p-adic numbers and the field $\mathbb{F}((T))$ of Laurent series over a field $\mathbb{F}$.

1. Introduction

The reduction of a complex matrix to its Jordan form is an unstable operation: both the Jordan form and a reduction transformation depend discontinuously on the entries of

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the original matrix. Arnold [1] (see also [2,3]) constructed a *miniversal deformation* of a square complex matrix A; i.e., a simple normal form B_{arn} to which all complex matrices B close to A can be reduced by similarity transformations that smoothly depend on the entries of B.

More precisely: Arnold supposes without restriction that A is a Jordan canonical matrix and reduces all matrices B in a neighborhood U of A to the form B_{arn} by a smooth similarity transformation that acts identically on A. Klimenko and Sergeichuk [14] described this reduction in detail.

Many applications of Arnold's normal form in different areas of mathematics are given in more than 120 articles from the Mathematical Reviews Citation Database that cite [1]. We mention only Mailybaev's articles [16–18] in which applications of miniversal deformations are based on the fact that the spectrum of $B \in U$ and B_{arn} coincide but B_{arn} has a simple form. Mailybaev also constructed a smooth similarity transformation (in the form of Taylor series) that transforms all $B \in U$ to B_{arn}.

Galin [9] (see also [3, § 30E]) obtained miniversal deformations of real matrices by realification of Arnold's miniversal deformations of complex matrices. Simpler miniversal deformations of real matrices were given by Garcia-Planas and Sergeichuk [11].

The main results of our paper are formulated in Theorem 7:

- *We extend Arnold's normal form of complex matrices to matrices over any field that is complete with respect to a nontrivial absolute value* (in particular, over *the field* $\mathbb{Q}_p$ *of* p*-adic numbers* and *the field* $\mathbb{F}((T))$ *of Laurent series over a field* $\mathbb{F}$; see Example 6). We use the Frobenius canonical form for similarity over an arbitrary field instead of the Jordan canonical form.
- *Over such a field, we construct a smooth similarity transformation that transforms all* $B \in U$ *to* B_{arn}. Our method differs from the method developed by Mailybaev [16–18].
- *We give the neighborhood* U *in an explicit form*, which is important for applications. As far as we know, the estimate of the radius of U was unknown even for complex matrices.

Miniversal deformations, reducing transformations, and neighborhood radius estimations for complex matrices under congruence and *congruence were given by Dmytryshyn, Futorny, and Sergeichuk [6,7].

2. Preliminaries

In this section, we recall some definitions and known facts.

2.1. Arnold's miniversal deformations of Jordan matrices

The similarity class of an $n \times n$ complex A in a small neighborhood of A can be obtained by a very small deformation of the affine matrix space

$$\{A + XA - AX \mid X \in \mathbb{C}^{n\times n}\}$$

since for each sufficiently small matrix $X \in \mathbb{C}^{n\times n}$,

$$\begin{aligned}(I_n - X)^{-1}A(I_n - X) &= (I_n + X + X^2 + \cdots)A(I_n - X)\\ &= A + (XA - AX) + X(XA - AX) + X^2(XA - AX) + \cdots\\ &= A + (XA - AX) + X(I_n + X + X^2 + \cdots)(XA - AX)\\ &= A + \underbrace{(XA - AX)}_{\text{small}} + \underbrace{X(I_n - X)^{-1}(XA - AX)}_{\text{very small}}.\end{aligned} \tag{1}$$

(X can be taken small due to the Lipschitz property [22]: if A and B are $n \times n$ complex matrices close to each other and $B = S^{-1}AS$ with a nonsingular S, then S can be taken near I_n.) The vector space

$$T(A) := \{XA - AX \mid X \in \mathbb{C}^{n\times n}\} \tag{2}$$

is the *tangent space* to the similarity class of A at the point A. The numbers

$$\dim T(A), \qquad n^2 - \dim T(A)$$

are called the *dimension* and *codimension*, respectively, of the similarity class of A.

We use the *matrix norm* $\|M\| := \sqrt{\sum |m_{ij}|^2}$ of $M = [m_{ij}]$, the Jordan blocks

$$J_m(\lambda) := \begin{bmatrix}\lambda & 1 & & 0\\ & \lambda & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{bmatrix} \qquad (m\text{-by-}m,\ \lambda \in \mathbb{C}), \tag{3}$$

and the matrices

$$0^{\downarrow} := \begin{bmatrix} & 0 & \\ * & \cdots & *\end{bmatrix}, \qquad 0^{\leftarrow} := \begin{bmatrix} * & \\ \vdots & 0\\ * & \end{bmatrix}$$

in which all entries are zeros except the last row of $0^{\downarrow}$ and the first column of $0^{\leftarrow}$ that consist of stars.

The following theorem was proved by V.I. Arnold.

Theorem 1 *([1, Theorem 4.4]). Let*

$$J := \bigoplus_{i=1}^{t} \left(J_{m_{i1}}(\lambda_i) \oplus J_{m_{i2}}(\lambda_i) \oplus \cdots \oplus J_{m_{ik_i}}(\lambda_i)\right) \tag{4}$$

be a Jordan matrix of size $n \times n$, in which

$$m_{i1} \geq m_{i2} \geq \ldots \geq m_{ik_i} \quad \textit{for all } i \tag{5}$$

and $\lambda_1, \ldots, \lambda_t$ are distinct complex numbers. Then

(a) *all complex matrices $J + X$ that are sufficiently close to J can be simultaneously reduced by some transformation*

$$J + X \mapsto \mathcal{S}(X)^{-1}(J + X)\mathcal{S}(X), \quad \begin{array}{c}\mathcal{S}(X) \textit{ is nonsingular and} \\ \textit{analytic at zero, } \mathcal{S}(0) = I_n\end{array} \tag{6}$$

to the form

$$J + \mathcal{D} := \bigoplus_{i=1}^{t} \begin{bmatrix} J_{n_{i1}}(\lambda_i) + 0^{\downarrow} & 0^{\downarrow} & \ldots & 0^{\downarrow} \\ 0^{\leftarrow} & J_{n_{i2}}(\lambda_i) + 0^{\downarrow} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0^{\downarrow} \\ 0^{\leftarrow} & \ldots & 0^{\leftarrow} & J_{n_{ik_i}}(\lambda_i) + 0^{\downarrow} \end{bmatrix} \tag{7}$$

in which the stars of $\mathcal{D}$ represent elements that depend analytically on the entries of X;

(b) *the number of stars in $\mathcal{D}$ is minimal that can be achieved by transformations of the form (6); this number of stars is equal to the codimension of the similarity class of the matrix J.*

Example 2. If $J = J_3(5) \oplus J_2(5)$, then (7) takes the form

$$J + \mathcal{D} = \left[\begin{array}{ccc|cc} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] + \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * \\ \hline * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * \end{array}\right].$$

Remark 3. Garcia-Planas and Sergeichuk [11] proved that the statements (a) and (b) are also true for matrices over $\mathbb{R}$ if the real Jordan canonical form is taken instead of (4), which simplifies the miniversal deformations of real matrices constructed by Galin [9] (see also [3, § 30E]).

Remark 4. Belitskii [4] proved that for each Jordan matrix J there exists a permutation of rows and the same permutation of columns such that the obtained matrix $J' = S^{-1}JS$ (S is a permutation matrix) possesses the property: all commuting with J' matrices are upper block triangular. Klimenko and Sergeichuk [13] showed that the same permutations of rows and columns of (7) transform it to $J' + \mathcal{D}' = S^{-1}(J + \mathcal{D})S$, which is lower block triangular. The matrix J' was called the *Weyr canonical matrix* by Sergeichuk [23].

Remark 5. Let $\mathcal{D}$ be the matrix in (7). Denote by $\mathcal{D}(\mathbb{C})$ the vector space of all matrices obtained from $\mathcal{D}$ by replacing its stars by complex numbers. Arnold [1] states that this vector space is a direct complement of the tangent space $T(J)$; that is,

$$\mathbb{C}^{n\times n} = T(J) \oplus \mathcal{D}(\mathbb{C}). \tag{8}$$

(Thus, the number of stars in $\mathcal{D}$ is equal to the codimension of the similarity class of J.) Moreover, *if $\mathcal{D}$ is any matrix consisting of 0's and $*$'s that satisfies (8), then $\mathcal{D}$ can be used in (7)*. This result about the similarity action of the group of nonsingular complex matrices was generalized by Tannenbaum [24, Part V, Theorem 1.2] to a Lie group acting on a complex manifold. Simplest miniversal deformations of matrix pencils and contragredient matrix pencils [8,11], and matrices under congruence and *congruence [6,7] were constructed by methods that are based on direct sum decompositions analogous to (8).

2.2. An absolute value

We extend Theorem 1 to matrices over a field $\mathbb{F}$ with topology given by an *absolute value*, which is a real valued function $|.| : \mathbb{F} \to \mathbb{R}$ such that

- $|x| \geq 0$, $|x| = 0 \Leftrightarrow x = 0$,
- $|xy| = |x||y|$, and $|x + y| \leq |x| + |y|$

for all $x, y \in \mathbb{F}$; see [15, Chapter XII]. Then $|1| = 1$, $|-x| = |x|$, and $|x^{-1}| = |x|^{-1}$.

A sequence $a_1, a_2, \dots$ in $\mathbb{F}$ *converges* to $a \in \mathbb{F}$ if for every $\varepsilon > 0$ there exists a natural number N such that $|a_n - a| < \varepsilon$ for all $n \geq N$. A sequence $a_1, a_2, \dots$ in $\mathbb{F}$ is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists N such that $|a_n - a_m| < \varepsilon$ for all $m, n > N$. Every convergent sequence is Cauchy. If every Cauchy sequence converges, then $\mathbb{F}$ is called *complete*.

Each field $\mathbb{F}$ possesses the trivial absolute value defined by $|0| = 0$ and $|x| = 1$ for all $0 \neq x \in \mathbb{F}$. By [15, Proposition 2.1], each field with nontrivial absolute value is a dense subfield of a complete field, which is unique up to $\mathbb{F}$-isomorphism.

Example 6. The most known complete fields with nontrivial absolute value are

- the fields $\mathbb{R}$ and $\mathbb{C}$ with the usual absolute value,
- the field $\mathbb{Q}_p$ of p-adic numbers for any prime p with the absolute value

$$|x|_p := p^{-z}, \qquad x = a_z p^z + a_{z+1}p^{z+1} + \dots \in \mathbb{Q}_p \tag{9}$$

in which $z \in \mathbb{Z}$, $a_z, a_{z+1}, \dots \in \{0, 1, \dots, p-1\}$ and $a_z \neq 0$,

- the field $\mathbb{F}((T))$ of Laurent series over a field $\mathbb{F}$ with the absolute value

$$|x| := 2^{-z}, \qquad x = a_z T^z + a_{z+1}T^{z+1} + a_{z+2}T^{z+2} + \cdots \tag{10}$$

in which $z \in \mathbb{Z}$, $a_z, a_{z+1}, a_{z+2}, \ldots \in \mathbb{F}$ and $a_z \neq 0$ (see [5, p. 316]).

An absolute value is *non-Archimedean* if $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{F}$ (then $|x + y| = \max\{|x|, |y|\}$ if $|x| \neq |y|$). Its geometric properties are unaccustomed: every triangle is isosceles (since $|x - z| = |(x - y) + (y - z)| = \max\{|x - y|, |y - z|\}$ if $|x - y| \neq |y - z|$); every point of the open sphere $\mathrm{S}_\varepsilon(x) := \{y \in \mathbb{F} \,|\, |y - x| < \varepsilon\}$ may serve as a center; any two spheres are either disjoint or one is contained inside the other; see [20]. The absolute values (9) and (10) are non-Archimedean.

2.3. Real Jordan and Frobenius canonical matrices

Define the $m \times m$ matrix

$$C_m(a, b) := \begin{bmatrix} C(a,b) & I_2 & & 0 \\ & C(a,b) & \ddots & \\ & & \ddots & I_2 \\ 0 & & & C(a,b) \end{bmatrix} \quad \text{with } C(a,b) := \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

for each $a, b \in \mathbb{R}$ and $m = 2, 4, 6, \ldots$. We use the *real Jordan canonical form* (see [12, Theorem 3.4.1.5]): each square real matrix is similar over $\mathbb{R}$ to a direct sum

$$\begin{aligned} C := &\left[\bigoplus_{i=1}^{t'} \Big(J_{m_{i1}}(\lambda_i) \oplus J_{m_{i2}}(\lambda_i) \oplus \cdots \oplus J_{m_{ik_i}}(\lambda_i) \Big) \right] \\ &\oplus \left[\bigoplus_{i=t'+1}^{t} \Big(C_{m_{i1}}(a_i, b_i) \oplus C_{m_{i2}}(a_i, b_i) \oplus \cdots \oplus C_{m_{ik_i}}(a_i, b_i) \Big) \right] \end{aligned} \tag{11}$$

with $\lambda_i, a_i, b_i \in \mathbb{R}$ and $b_i > 0$; this direct sum is uniquely determined up to permutation of summands. We suppose that

$$m_{i1} \geq m_{i2} \geq \ldots \geq m_{ik_i} \qquad \text{for all } i. \tag{12}$$

We also use the *Frobenius canonical form* for similarity (see [21, Section 14]): each square matrix over an arbitrary field $\mathbb{F}$ is similar to a direct sum, determined uniquely up to permutation of summands, of matrices of the form

$$\Phi_m(p) := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -c_m & \ldots & -c_2 & -c_1 \end{bmatrix} \qquad (m\text{-by-}m) \tag{13}$$

whose characteristic polynomial $x^m + c_1x^{m-1} + \cdots + c_m \in \mathbb{F}[x]$ is an integer power of a polynomial $p(x)$ that is irreducible over $\mathbb{F}$.

3. The main result

Let us fix a field $\mathbb{F}$ that is complete with respect to a nontrivial absolute value, and a matrix $A \in \mathbb{F}^{n\times n}$. In this section, we formulate Theorem 7 about miniversal deformations of A over $\mathbb{F}$.

Let $\mathcal{D}$ be an $n \times n$ matrix consisting of 0's and $*$'s such that

$$\mathbb{F}^{n\times n} = T(A) \oplus \mathcal{D}(\mathbb{F}), \tag{14}$$

in which $T(A)$ is defined by (2) with $\mathbb{F}$ instead of $\mathbb{C}$, and $\mathcal{D}(\mathbb{F})$ is the vector space of all matrices obtained from $\mathcal{D}$ by replacing its stars with elements of $\mathbb{F}$. Such a matrix $\mathcal{D}$ always exists; to construct it we can take the set of matrix units lexicographically arranged:

$$E_{11},\ E_{12},\ \ldots,\ E_{1n};\ E_{21},\ E_{22},\ \ldots,\ E_{2n};\ \ldots;\ E_{n1},\ E_{n2},\ \ldots,\ E_{nn},$$

and delete those of them that are linear combinations of the preceding units and elements of $T(A)$. The remaining units generate a vector space that can be used as $\mathcal{D}(\mathbb{F})$.

Let us fix n^2 matrices $F_{ij} \in \mathbb{F}^{n\times n}$, $i, j = 1, \ldots, n$, such that

$$E_{ij} + F_{ij}A - AF_{ij} \in \mathcal{D}(\mathbb{F}) \tag{15}$$

for each $n \times n$ matrix unit E_{ij} (F_{ij} exists by (14)). We can and will take

$$F_{ij} = 0_n \qquad \text{if } E_{ij} \in \mathcal{D}(\mathbb{F}). \tag{16}$$

Define the neighborhood of zero

$$U := \left\{ X \in \mathbb{F}^{n\times n} \;\middle|\; \|X\| < \frac{1}{48\sqrt{n}(a+1)f^2} \right\}, \tag{17}$$

in which

$$a := \|A\|, \qquad f := \max\Big\{ \sum_{i,j} \|F_{ij}\|, \tfrac{1}{3} \Big\}, \tag{18}$$

and

$$\|M\| := \sqrt{\sum |m_{ij}|^2} \qquad \text{for all } M = [m_{ij}] \in \mathbb{F}^{n\times n}.$$

For each $X \in U$, we construct a sequence

$$M_1 := X,\ M_2,\ M_3, \ldots \tag{19}$$

of $n \times n$ matrices as follows: if

$$M_k = [m_{ij}^{(k)}] \tag{20}$$

has been constructed, then M_{k+1} is defined by

$$A + M_{k+1} := (I_n - C_k)^{-1}(A + M_k)(I_n - C_k), \quad C_k := \sum_{i,j} m_{ij}^{(k)} F_{ij}. \tag{21}$$

Our main result is the following theorem.

Theorem 7. *Let $\mathbb{F}$ be a complete field with respect to a nontrivial absolute value, and let $A \in \mathbb{F}^{n\times n}$. Let $\mathcal{D}$ be an $n \times n$ matrix consisting of 0's and $*$'s that satisfies (14). Then the following statements hold:*

(a) *Let $C_1, C_2, \ldots$ be formed by (21). For each matrix X from the set U defined in (17), the infinite product*

$$\mathcal{S}(X) := (I_n - C_1)(I_n - C_2)(I_n - C_3)\cdots \tag{22}$$

is convergent and nonsingular. The related matrix function $\mathcal{S} : U \to \mathbb{F}^{n\times n}$ is continuous and $\mathcal{S}(0_n) = I_n$; this function is analytic if $\mathbb{F} = \mathbb{C}$.

(b) *If $X \in U$, then*

$$D(X) := \mathcal{S}(X)^{-1}(A + X)\mathcal{S}(X) - A \in \mathcal{D}(\mathbb{F}), \tag{23}$$

for which $\mathcal{S}(X)$ is defined in (22). This means that all matrices $A + X$ with $X \in U$ are reduced by the similarity transformation $\mathcal{S}(X)^{-1}(A+X)\mathcal{S}(X)$ to the form $A+\mathcal{D}$. The stars in $\mathcal{D}$ represent the entries that depend continuously on the entries of X. The number of stars in $\mathcal{D}$ is equal to the codimension of the similarity class of A. In particular, for each ε satisfying $0 < \varepsilon \le 1/2$ and for each $X \in 2\varepsilon U$, we have

$$\|\mathcal{S}(X) - I_n\| < -1 + (1 + \varepsilon)(1 + \varepsilon^2)(1 + \varepsilon^3)\cdots, \tag{24}$$

$$\|D(X)\| \le \varepsilon/(2f), \tag{25}$$

in which f is defined in (18).

(c) *Let one of the following conditions hold:*

(i) *A is a Frobenius canonical matrix*

$$\bigoplus_{i=1}^{t} \Big(\Phi_{m_{i1}}(p_i) \oplus \Phi_{m_{i2}}(p_i) \oplus \cdots \oplus \Phi_{m_{ik_i}}(p_i)\Big), \quad m_{i1} \ge \ldots \ge m_{ik_i}$$

(see (13)) in which $p_1, p_2, \ldots, p_t$ are distinct irreducible polynomials over $\mathbb{F}$,

(ii) $\mathbb{F} = \mathbb{C}$ *and* A *is a Jordan canonical matrix* (4) *satisfying* (5),
(iii) $\mathbb{F} = \mathbb{R}$ *and* A *is a real Jordan canonical matrix* (11) *satisfying* (12).
Then the matrix $\mathcal{D}$ *satisfying* (14) *can be taken as follows:*

$$\mathcal{D} := \bigoplus_{i=1}^{t} \begin{bmatrix} 0^{\downarrow}_{m_{i1}} & 0^{\downarrow} & \dots & 0^{\downarrow} \\ 0^{\leftarrow} & 0^{\downarrow}_{m_{i2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0^{\downarrow} \\ 0^{\leftarrow} & \dots & 0^{\leftarrow} & 0^{\downarrow}_{m_{ik_i}} \end{bmatrix},$$

in which $0^{\downarrow}_m$ *denotes the matrix* $0^{\downarrow}$ *of size* $m \times m$.

Note that Theorem 7(a) constructs a matrix $\mathcal{S}(X)$ that transforms a family of all matrices $A + X$ with $X \in U$ to the form $A + \mathcal{D}$. Garcia-Planas and Mailybaev [10,17,18] construct analogous matrices (in the form of Taylor series) that transform families of complex matrices under similarity and families of complex matrix pencils under strict equivalence to their miniversal deformations. They also give numerous applications.

4. Proof of the main result

In the remainder of this article we prove parts (a)–(c) of Theorem 7.

4.1. Proof of part (a)

By (15), (20), and (21),

$$\sum_{i,j} m_{ij}^{(k)} E_{ij} + \sum_{i,j} m_{ij}^{(k)} F_{ij} A - \sum_{i,j} m_{ij}^{(k)} A F_{ij} \in \mathcal{D}(\mathbb{F}),$$

$$M_k + C_k A - A C_k \in \mathcal{D}(\mathbb{F}).$$

For each $P = [p_{ij}] \in \mathbb{F}^{n \times n}$, we write

$$\|P\|_{\mathcal{D}} := \sqrt{\sum_{(i,j) \notin \mathcal{I}(\mathcal{D})} |p_{ij}|^2},$$

in which $\mathcal{I}(\mathcal{D}) \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$ is the set of indices of the stars in $\mathcal{D}$.

Let us fix $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon \leq 1/2$. Define a sequence

$$\delta_1,\ \tau_1,\ \delta_2,\ \tau_2,\ \delta_3,\ \tau_3,\ \dots$$

of positive real numbers by induction:

$$\begin{aligned} \tau_1 = \delta_1 &:= \varepsilon/(8fv) = \varepsilon/(24\sqrt{n}(a+1)f^2), \\ \tau_{k+1} := \tau_k + \delta_k v, \quad \delta_{k+1} &:= \delta_k \varepsilon = \delta_1 \varepsilon^k \quad (k = 1, 2, \dots), \end{aligned} \tag{26}$$

in which a and f were defined in (18) and

$$v := 3\sqrt{n}(a+1)f.$$

Lemma 8. *If $\|M_1\| < \tau_1$ in (19), then*

$$\|M_\ell\| < \tau_\ell, \quad \|M_\ell\|_{\mathcal{D}} < \delta_\ell, \quad \|C_\ell\| \le \delta_\ell f \quad (\ell = 1, 2, \dots). \tag{27}$$

Proof. If the first inequality in (27) holds, then the third holds too since

$$\begin{aligned} \|C_\ell\| &= \|\sum_{(i,j)\notin \mathcal{I}(\mathcal{D})} m_{ij}^{(\ell)} F_{ij}\| \quad \text{(by (16) and (21))} \\ &\le \sum_{(i,j)\notin \mathcal{I}(\mathcal{D})} |m_{ij}^{(\ell)}| \cdot \|F_{ij}\| \le \sum_{(i,j)\notin \mathcal{I}(\mathcal{D})} \delta_\ell \cdot \|F_{ij}\| \le \delta_\ell f < 1/10. \end{aligned}$$

The series

$$(I - C_\ell)^{-1} = I + C_\ell + C_\ell^2 + \cdots$$

is convergent since the sequence of its partial sums I, $I + C_\ell$, $I + C_\ell + C_\ell^2$, $\dots$ is a Cauchy sequence. Moreover,

$$\begin{aligned} \|(I - C_\ell)^{-1}\| &= \|I + C_\ell + C_\ell^2 + \cdots\| \le \|I\| + \|C_\ell\| + \|C_\ell^2\| + \cdots \\ &\le \sqrt{n}(1 + \|C_\ell\| + \|C_\ell\|^2 + \cdots) = \sqrt{n}/(1 - \|C_\ell\|) < 1.5\sqrt{n}. \end{aligned}$$

Suppose that (27) holds for all $\ell \le k$; let us prove it for $\ell = k+1$. Write $A_k := A + M_k$, then

$$\|A_k\| = \|A + M_k\| \le \|A\| + \|M_k\| \le a + \delta_k < a + 1.$$

By (21),

$$A_{k+1} = (I + C_k + C_k^2 + \cdots)A_k(I - C_k) = A_k + (I - C_k)^{-1}(C_k A_k - A_k C_k).$$

Subtracting A from these equations and taking the absolute value, we get

$$\|M_{k+1}\| \le \|M_k\| + 1.5\sqrt{n} \cdot 2\|C_k\| \cdot \|A_k\| \le \tau_k + 3\sqrt{n} \cdot \delta_k f \cdot (a+1) = \tau_{k+1},$$

which proves the first inequality in (27) for $\ell = k + 1$.

Let us prove the second inequality in (27). By (1),

$$A + M_{k+1} = A + M_k + C_k(A + M_k) - (A + M_k)C_k + C_k(I - C_k)^{-1}(C_kA_k - A_kC_k),$$
$$M_{k+1} = \underbrace{M_k + C_kA - A_kC_k}_{\text{belongs to } \mathcal{D}(\mathbb{F})} + C_kM_k - M_kC_k + C_k(I - C_k)^{-1}(C_kA_k - A_kC_k),$$

$$\begin{aligned}\|M_{k+1}\|_{\mathcal{D}} &\le 2\|C_k\| \cdot \|M_k\| + \|C_k\| \cdot 1.5\sqrt{n} \cdot 2\|C_k\| \cdot \|A_k\| \\ &\le 2 \cdot \delta_k f \cdot (\tau_k + 1.5\sqrt{n} \cdot \delta_k f \cdot (a + \delta_k)) \le \delta_k f(2\tau_k + v\delta_k).\end{aligned}$$

This reduces the request inequality $\|M_{k+1}\|_{\mathcal{D}} < \delta_{k+1} (= \delta_k\varepsilon)$ to the inequality $2f\tau_k + fv\delta_k \le \varepsilon$.

Since

$$\begin{aligned}\tau_k &= \tau_{k-1} + v\delta_{k-1} = \tau_{k-2} + v(\delta_{k-2} + \delta_{k-1}) \\ &\le \tau_1 + v(\delta_1 + \delta_2 + \delta_3 + \cdots) = \delta_1 + v(\delta_1 + \delta_1\varepsilon + \delta_1\varepsilon^2 + \cdots) \\ &= \delta_1(1 + v/(1 - \varepsilon)) \le \delta_1(1 + 2v) \le 3\delta_1 v,\end{aligned} \tag{28}$$

we obtain that

$$2f\tau_k + fv\delta_k \le 6f\delta_1 v + fv\delta_k \le 7f\delta_1 v \le 7\varepsilon/8 \le \varepsilon. \qquad \square$$

Proof of Theorem 7(a). Write

$$\mathcal{S}_{k,l}(X) := \prod_{i=k}^{l}(I - C_i), \quad \mathcal{S}_l(X) := \mathcal{S}_{1,l}(X) \qquad \text{for all } 1 \le k \le l \le \infty. \tag{29}$$

If $\|X\| \le \tau_1 = \varepsilon/(8fv)$ and $l < \infty$, then by Lemma 8

$$\begin{aligned}\|\mathcal{S}_{k,l}(X) - I\| &= \|(I - C_k)(I - C_{k+1}) \cdots (I - C_l) - I\| \\ &\le \sum_{k \le i \le l} \|C_i\| + \sum_{k \le i < j \le l} \|C_i\| \, \|C_j\| + \cdots + \|C_k\| \cdots \|C_l\| \\ &\le -1 + (1 + \|C_k\|)(1 + \|C_{k+1}\|) \cdots (1 + \|C_l\|) \\ &\le -1 + (1 + \varepsilon^{k-1}\delta_1 f)(1 + \varepsilon^k\delta_1 f)(1 + \varepsilon^{k+1}\delta_1 f) \cdots \\ &\le -1 + \left(1 + \tfrac{\varepsilon^k}{8v}\right)\left(1 + \tfrac{\varepsilon^{k+1}}{8v}\right)\left(1 + \tfrac{\varepsilon^{k+2}}{8v}\right) \cdots \\ &\le -1 + (1 + \varepsilon^k)(1 + \varepsilon^{k+1})(1 + \varepsilon^{k+2}) \cdots\end{aligned} \tag{30}$$

The infinite product $\omega_k(\varepsilon) := \prod_{i=k}^{\infty}(I + \varepsilon^i)$ is convergent since any product $\prod_{i=1}^{\infty}(1 + a_i)$ with positive real a_i's converges if and only if the infinite series $\sum_{i=1}^{\infty} a_i$ converges (see [19, Theorem 15.14]). Write $\omega_k := \omega_k(1/2)$.

The infinite product $\mathcal{S}(X) = \prod_{i=1}^{\infty}(I - C_i)$ defined in (22) is convergent for each $X \in U$. Indeed, the sequence $\mathcal{S}_1(X)$, $\mathcal{S}_2(X)$, $\mathcal{S}_3(X), \dots$ is a Cauchy sequence since $1 \le k \le l$ implies

$$\begin{aligned}\|\mathcal{S}_l(X) - \mathcal{S}_k(X)\| &= \|\mathcal{S}_k(X)(\mathcal{S}_{k+1,l}(X) - I)\| \\ &\le \|(\mathcal{S}_k(X) - I) + I\| \cdot \|\mathcal{S}_{k+1,l}(X) - I)\| \\ &\le ((\omega_1 - 1) + \sqrt{n})(\omega_{k+1} - 1) \quad \text{by (30)}.\end{aligned}$$

Let us prove that the matrix function $\mathcal{S} : U \to \mathbb{F}^{n\times n}$ is continuous. By (21), the entries of each C_i are polynomials in the entries of X. Thus, the entries of $\mathcal{S}_k(X)$ are polynomials in the entries of X for each k. Since the operations $x + y$, $x - y$, xy, x^{-1} are continuous, the matrices $\mathcal{S}_k(X)$ are continuous functions. Let ε' be a small positive number. For any $X, Y \in U$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{S}(Y) - \mathcal{S}(X) &= \mathcal{S}_k(Y)\mathcal{S}_{k+1,\infty}(Y) - \mathcal{S}_k(X)\mathcal{S}_{k+1,\infty}(X) \\ &= (\mathcal{S}_k(Y) - \mathcal{S}_k(X))\mathcal{S}_{k+1,\infty}(Y) + \mathcal{S}_k(X)(\mathcal{S}_{k+1,\infty}(Y) - \mathcal{S}_{k+1,\infty}(X)),\end{aligned}$$

hence

$$\begin{aligned}&\|\mathcal{S}(Y) - \mathcal{S}(X)\| \\ &\le \|\mathcal{S}_k(Y) - \mathcal{S}_k(X)\| \cdot \|\mathcal{S}_{k+1,\infty}(Y)\| + \|\mathcal{S}_k(X)\| \cdot \|\mathcal{S}_{k+1,\infty}(Y) - \mathcal{S}_{k+1,\infty}(X)\| \\ &\le \|\mathcal{S}_k(Y) - \mathcal{S}_k(X)\|(\omega_1 + \sqrt{n}) + \omega_1\|\mathcal{S}_{k+1,\infty}(Y) - \mathcal{S}_{k+1,\infty}(X)\|.\end{aligned}$$

Because $\mathcal{S}_k(X)$ converges uniformly, there exists $\delta' > 0$ such that

$$\|\mathcal{S}_k(Y) - \mathcal{S}_k(X)\| \cdot (\omega_1 + \sqrt{n}) < \varepsilon'/2 \qquad \text{if } \|Y - X\| < \delta'.$$

Since $\omega_t \to 1$, there exists k such that

$$\begin{aligned}\omega_1\|\mathcal{S}_{k+1,\infty}(Y) - \mathcal{S}_{k+1,\infty}(X)\| &\le \omega_1\big(\|\mathcal{S}_{k+1,\infty}(Y) - I\| + \|\mathcal{S}_{k+1,\infty}(X) - I\|\big) \\ &\le \omega_1 \cdot 2(\omega_{k+1} - 1) < \varepsilon'/2.\end{aligned}$$

Thus, $\|\mathcal{S}(Y) - \mathcal{S}(X)\| < \varepsilon'$ if $\|Y - X\| < \delta'$ and so $\mathcal{S} : U \to \mathbb{F}^{n\times n}$ is a continuous function.

The matrix $\mathcal{S}(X)$ is nonsingular for each $X \in U$. Indeed, $\mathcal{S}(X) = \mathcal{S}_k(X)\mathcal{S}_{k+1,\infty}(X)$, in which $\mathcal{S}_k(X)$ is nonsingular for each k, and $\mathcal{S}_{k+1,\infty}(X)$ is certainly nonsingular for a sufficiently large k since $\|\mathcal{S}_{k+1,\infty}(X) - I\| \to 0$ by (30).

If $X = 0_n$, then all $M_i = C_i = 0_n$, and so $\mathcal{S}(0_n) = I_n$.

Let now $\mathbb{F} = \mathbb{C}$; we should prove that $\mathcal{S} : U \to \mathbb{F}^{n\times n}$ is analytic. By (21), the entries of each C_i are polynomials in the entries of X. Hence the entries of each $\mathcal{S}_\ell(X)$ (see (29)) are polynomials in the entries of X. Since $\mathcal{S}_\ell(X) \to \mathcal{S}(X)$, the Weierstrass theorem on uniformly convergent sequences of analytic functions (see [19, Theorem 15.8]) ensures that the entries of $\mathcal{S}(X)$ are analytic functions in the entries of X. $\square$

4.2. Proof of part (b)

By (21) and (29),

$$A + M_k(X) = \mathcal{S}_{k-1}(X)^{-1}(A+X)\mathcal{S}_{k-1}(X) \quad \text{for each } X \in U.$$

Hence, $M_k(X) \to D(X)$ as $k \to \infty$, where $D(X)$ is defined in (23). By (27) and (26), $\|M_k(X)\|_{\mathcal{D}} < \delta_k = \delta_1\varepsilon^{k-1} \to 0$ as $k \to \infty$, and so $\|D(X)\|_{\mathcal{D}} = 0$, which proves (23). The inequality (24) follows from (30) and the inequality (25) follows from

$$\begin{aligned} \|M_k\| &\le \tau_k \le 3\delta_1 v \quad \text{(by (28))} \\ &\le 3\varepsilon/(8f) \le \varepsilon/(2f). \end{aligned}$$

Since (14) is a direct sum, the number of stars in $\mathcal{D}$ is equal to $n^2 - \dim T(A)$, which is the codimension of the similarity class of A.

4.3. Proof of part (c)

Let us prove the statement (i) in (c). Write

$$T(P,Q) := \{XQ - PX \mid X \in \mathbb{F}^{m\times n}\}, \qquad P \in \mathbb{F}^{m\times m},\ Q \in \mathbb{F}^{n\times n}.$$

Let $A = A_1 \oplus \cdots \oplus A_l \in \mathbb{F}^{n\times n}$, in which every A_i is of size $n_i \times n_i$. Let $\mathcal{D} = [\mathcal{D}_{ij}]_{i,j=1}^{l}$ be a block matrix, in which every $\mathcal{D}_{ij}$ is a block of size $n_i \times n_j$ whose entries are 0's and $*$'s. Clearly, $\mathbb{F}^{n\times n} = T(A) \oplus \mathcal{D}(\mathbb{F})$ if and only if

$$\mathbb{F}^{n_i\times n_j} = T(A_i, A_j) \oplus \mathcal{D}_{ij}(\mathbb{F}) \qquad \text{for all } i,j = 1,\dots,l.$$

Thus, the statement (i) is implied by the following lemma.

Lemma 9. *Let $\Phi = \Phi(p^r)$ and $\Psi = \Phi(q^s)$ be $m\times m$ and $n\times n$ Frobenius blocks (see (13)) in which $p, q \in \mathbb{F}[x]$ are irreducible polynomials with leading coefficient 1. The following hold:*

(α) *If $p \neq q$, then $\mathbb{F}^{m\times n} = T(\Phi,\Psi)$.*
(β) *If $p = q$ and $r \ge s$, then $\mathbb{F}^{m\times n} = T(\Phi,\Psi) \oplus 0^{\downarrow}(\mathbb{F})$.*
(γ) *If $p = q$ and $r \le s$, then $\mathbb{F}^{m\times n} = T(\Phi,\Psi) \oplus 0^{\leftarrow}(\mathbb{F})$.*

Proof. Let us compute the dimension of $T(\Phi,\Psi)$. The space $T(\Phi,\Psi)$ is the image of the linear operator

$$\xi : \mathbb{F}^{m\times n} \to \mathbb{F}^{m\times n}, \quad X \mapsto X\Psi - \Phi X; \tag{31}$$

hence

$$\dim T(\Phi,\Psi) = mn - \operatorname{Ker}(\xi), \quad \operatorname{Ker}(\xi) = \{H \in \mathbb{F}^{m\times n} \mid H\Psi = \Phi H\}. \tag{32}$$

For each matrix $A \in \mathbb{F}^{t\times t}$, denote by $M(A) = (\mathbb{F}^t, A)$ the module over the polynomial ring $\mathbb{F}[x]$ that is the vector space $\mathbb{F}^t$ with multiplication $f(x)v := f(A)v$ for all $f \in \mathbb{F}[x]$ and $v \in \mathbb{F}^t$. Each matrix $H \in \operatorname{Ker}(\xi)$ defines the $\mathbb{F}[x]$-homomorphism

$$\varphi_H : M(\Psi) \to M(\Phi), \quad v \mapsto Hv;$$

moreover, $H \mapsto \varphi_H$ is a linear bijection from $\operatorname{Ker}(\xi)$ to the $\mathbb{F}$-space of homomorphisms $\operatorname{Hom}_{\mathbb{F}[x]}(M(\Psi), M(\Phi))$. Thus,

$$\dim_{\mathbb{F}} \operatorname{Ker}(\xi) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}[x]}(M(\Psi), M(\Phi)).$$

The module $M(\Psi)$ is cyclic because

$$M(\Psi) = \mathbb{F}[x]e_n \simeq \mathbb{F}[x]/q(x)^n\mathbb{F}[x], \qquad \text{where } e_n := (0,\dots,0,1)^T \in \mathbb{F}^n.$$

Hence, each homomorphism $\varphi : M(\Psi) \to M(\Phi)$ is fully determined by $\varphi(e_n)$.

- If $p \neq q$, then $\varphi(e_n) = 0$ since
$$q(x)^s\varphi(e_n) = \varphi(q(x)^s e_n) = \varphi(0) = 0, \tag{33}$$
$p(x)^r\varphi(e_n) \in p(x)^r M(\Phi) = 0$, and $fp^r + gq^r = 1$ for some $f, g \in \mathbb{F}[x]$.
- If $p = q$ and $r \geq s$, then $\varphi(e_n) \in p(x)^{r-s}M(\Phi)$ by (33). Moreover, $\varphi(e_n)$ is an arbitrary element of the submodule $p(x)^{r-s}M(\Phi) \simeq M(\Psi)$ of dimension n.
- If $p = q$ and $r \leq s$, then $\varphi(e_n)$ is an arbitrary element of $M(\Phi)$ of dimension m.

Therefore,

$$\dim_{\mathbb{F}} \operatorname{Ker}(\xi) = \begin{cases} 0 & \text{if } p \neq q \\ n & \text{if } p = q \text{ and } r \geq s \\ m & \text{if } p = q \text{ and } r \leq s \end{cases}$$

for the linear operator (31). By (32),

$$\dim_{\mathbb{F}} T(\Phi,\Psi) = \begin{cases} mn & \text{if } p \neq q \\ (m-1)n & \text{if } p = q \text{ and } r \geq s \\ m(n-1) & \text{if } p = q \text{ and } r \leq s. \end{cases} \tag{34}$$

(α) If $p \neq q$, then $\mathbb{F}^{m\times n} = T(\Phi, \Psi)$ by (34).

(β) Let $p = q$ and $r \geq s$. Denote by E_{ij} the (i, j) matrix unit in $\mathbb{F}^{m\times n}$. For each $i = 1, \dots, m-1$, the matrices $\xi(-E_{i1})$, $\xi(-E_{i2})$, $\dots$, $\xi(-E_{in})$ have the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \hdotsfor{4} \\ * & * & \cdots & * \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ * & * & \cdots & * \\ \hdotsfor{4} \\ * & * & \cdots & * \end{bmatrix}, \quad \dots, \quad \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \dots & 1 \\ * & * & \cdots & * \\ \hdotsfor{4} \\ * & * & \cdots & * \end{bmatrix},$$

respectively, with units in the ith row. Since $T(\Phi, \Psi) = \operatorname{Im} \xi$, we have $\mathbb{F}^{m\times n} = T(\Phi, \Psi) + 0^{\downarrow}(\mathbb{F})$. This sum is direct by (34) and because $\dim 0^{\downarrow}(\mathbb{F}) = n$.

(γ) Let $p = q$ and $r \leq s$. For each $j = 2, \dots, n$, the matrices $\xi(E_{1j})$, $\xi(E_{2j})$, $\dots$, $\xi(E_{mj})$ have the form

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots 0 \\ * & \cdots & * & 0 & 0 & \cdots 0 \\ \hdotsfor{6} \\ * & \cdots & * & 0 & 0 & \cdots 0 \end{bmatrix}, \quad \dots, \quad \begin{bmatrix} * & \cdots & * & 0 & 0 & \cdots 0 \\ * & \cdots & * & 0 & 0 & \cdots 0 \\ \hdotsfor{6} \\ * & \cdots & * & 1 & 0 & \cdots 0 \end{bmatrix}$$

with units in the jth column. Since $T(\Phi, \Psi) = \operatorname{Im} \xi$, we have $\mathbb{F}^{m\times n} = T(\Phi, \Psi) + 0^{\leftarrow}(\mathbb{F})$. This sum is direct by (34) and because $\dim 0^{\leftarrow}(\mathbb{F}) = m$.

We have proved the statement (i) of part (c). The statements (ii) and (iii) were proved by Arnold [1, Theorem 4.4] and by Garcia-Planas and Sergeichuk [11, Theorem 3.1], respectively. □

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