Physica A 463 (2016) 437–444

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Consensus analysis of switching multi-agent systems with fixed topology and time-delay[☆]

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HIGHLIGHTS

- The multi-agent systems are Markov switching.
- We consider the time-delay in the feedback controller.
- The controller can be calculated by our results.

ARTICLE INFO

Article history:
Received 7 March 2016
Received in revised form 11 May 2016
Available online 29 July 2016

Keywords:
Multi-agent systems
Average consensus
Discrete-time system
Markov switching system
Time-delay
Lyapunov functional

ABSTRACT

This paper investigates the average consensus problems of the discrete-time Markov switching linear multi-agent systems (**LMAS**) with fixed topology and time-delay. Firstly, we introduce a concept of the average consensus to adapt the stochastic systems. Secondly, a time-delay switching consensus protocol is proposed. By developing a new signal mode, the switching signal of the systems and the time-delay signal of the controller can be merged into one signal. Thirdly, by Lyapunov technique, two LMIs criteria of average consensus are provided, and they reveal that the consensus of the multi-agent systems relates to the spectral radius of the Laplacian matrix. Furthermore, by our results and **CCL**-type algorithms, we can get the gain matrices. Finally, a numerical example is given to illustrate the efficiency of our results.

1. Introduction

Distributed multi-agent systems have attracted significant attention in the last decade. The consensus problems of the multi-agent networks are one of the most focused research areas due to its broad applications in many fields including formation control [1], synchronization [2], flocking [3], sensor networks [4] and so on.

The multi-agent systems consist of the agents which can represent robots, humans or animals. Most research of the consensus problems assumes that the dynamics of agent is the controller. For this model, there are many results about continuous-time dynamics [5], discrete-time dynamics [6], first-order dynamics [5], second-order dynamics [7], high-order dynamics [8] and leader-following consensus [9].

In recent years, the linear multi-agent systems (**LMAS**) have attracted several researchers, in which the dynamics of agent is modeled by a linear system. They considered the communication data rate for consensusability [10], the leader-following consensus [11–14], the observer-based protocols [15], the robust consensus control [16] and the event-triggered control [17]. Most works of the consensus problems in the **LMAS** focused on the continuous-time dynamics [12–20]. Results

☆ This work was supported by the National Natural Science Foundation of China (61174039).
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http://dx.doi.org/10.1016/j.physa.2016.07.039
0378-4371/

about the discrete-time dynamics are less [10,11]. In this paper, we investigate the consensus problems of the discrete-time **LMAS**. The dynamics of each agent is discrete-time linear system.

Switching phenomenon widely exists in the real world. Many papers investigated the consensus problems of the multi-agent systems with switching topology [5,21]. Refs. [11,20,22–25] studied the consensus problems of **LMAS** with switching topology. However, to the best of our knowledge, no one studied the consensus problems of Markov switching **LMAS**. In this paper, the **LMAS** is Markov switching, and the controller is also Markov switching. Adapted for Markov switching **LMAS**, we introduce a concept of the average consensus, which is generalized from the similar concepts in Ref. [26]. We design the controller by the state feedback and the switching signal feedback, and assume they have various delays.

In many papers about consensus problems, it is proved that the consensus of the multi-agent systems relates to the second smallest eigenvalue and the eigenratio (the ratio of the second smallest eigenvalue to the spectral radius) of the Laplacian matrix. Refs. [5,27] presented the convergence rate of the average consensus over an undirected graph is determined by the second smallest eigenvalue of the Laplacian matrix. The eigenratio is an important factor for the consensusability of the multi-agent systems [10]. A larger eigenratio corresponds to better consensusability. By aforementioned results, we can associate with that the spectral radius of the Laplacian matrix is also an important factor for consensus. This is confirmed by our result.

Time-delays are frequently encountered in many practical systems such as engineering, communications and biological systems. For multi-agent systems, time-delay often occurs in information communication. Therefore, communication delay is an inevitable problem, and was considered in many articles [5,7,9,19–21,25]. In this paper, the time-delay is also considered.

Motivated by above, we will solve the consensus problems of Markov switching **LMAS** with fixed topology in this paper. The rest of this paper is organized as follows. Section 2 introduces some graph knowledge and property of Kronecker product. Section 3 presents the consensus problems of discrete-time Markov switching **LMAS** with fixed topology, and defines the average consensus of the stochastic systems. In Section 4, we give two sufficient conditions of consensus, and analyze the relation between consensus and the spectral radius of the Laplacian matrix. Section 5 gives a numerical example to illustrate the efficiency of our results. Concluding remarks are finally stated in Section 6.

Notation: The following notation will be used throughout this paper. $\mathbf{1}$ ($\mathbf{0}$) is a compatible dimension vector with all elements to be one (zero). I_N is the $N \times N$-dimensional identity matrix, and I is the identity matrix of compatible dimensions. The notation $*$ always denotes the symmetric block in one symmetric matrix. The transpose of matrix A is denoted by A^T. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum eigenvalue and the minimum eigenvalue of A respectively. The shorthand $diag\{\cdots\}$ denotes the block diagonal matrix. $\|\cdot\|$ refers to the Euclidean norm for vectors. $E(\cdot)$ stands for the mathematical expectation operator. $\otimes$ denotes the Kronecker product of matrices. Some properties of Kronecker product are useful in this paper: $(A\otimes B)^T = A^T \otimes B^T$, $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$, $A\otimes B + A\otimes C = A\otimes(B+C)$. The Kronecker product of two positive definite matrices is positive definite.

2. Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a *graph* of order N, where $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_N\}$ is the set of nodes, $\mathcal{E} \in \mathcal{V}\times\mathcal{V}$ is the set of edges, and $\mathcal{A} = (a_{ij})_{N\times N}$ is the *weighted adjacency matrix*. The node indexes belong to a finite index set $\mathcal{I} = \{1, 2, \ldots, N\}$. $(i, j) \in \mathcal{E}$ denotes there is an edge connect ν_i and ν_j, and ν_i can receive information from ν_j. In the following, it is stipulated that $(i, j) \in \mathcal{E}$ if and only if $a_{ij} > 0$ and $a_{ii} = 0$ for $0 \le i \le N$. If $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$, $\mathcal{G}$ is called an *undirected graph*. If there is a sequence of edges $(i, i_1), (i_1, i_2), \ldots, (i_k, j) \in \mathcal{E}$ for any two agents $i, j \in \mathcal{V}$, $\mathcal{G}$ is called a *connected graph*.

The matrix $\mathcal{L} = (l_{ij})_{N\times N}$ is the *Laplacian matrix* of $\mathcal{G}$, where

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \displaystyle\sum_{k=1,k\neq i}^{N} a_{ik} & i = j. \end{cases}$$

Lemma 1 (*[28]*). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted undirected graph with the Laplacian $\mathcal{L}$, $\lambda_1 \le \cdots \le \lambda_N$ be the eigenvalues of $\mathcal{L}$. If $\mathcal{G}$ is connected, $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$.*

Assumption 1. In this paper, we assume the communication topology is undirected and connected. Then the eigenvalues of $\mathcal{L}$ are $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$.

In a multi-agent networks with N agents, the information flow between agents can be described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. The node ν_i in graph $\mathcal{G}$ corresponds to agent i in the networks. $(i, j) \in \mathcal{E}$ expresses that the information of the agent j can be spread to agent i.

Let $\{r(k), k \in \mathbb{Z}_+\}$ be discrete-time Markov chain, with finite state space $\Upsilon = \{0, 1, \ldots, d-1\}$. The state transition matrices of $\{r(k)\}$ is $P = (p_{ij})$, where $p_{ij} = \Pr\{r(k+1) = j | r(k) = i\} \ge 0$, for $i, j \in \Upsilon$, denotes the transition probability from i to j. In this paper, all systems are defined on a complete probability space (Ω, F, P).

Lemma 2. *For any matrices $A, B, C \in \mathbb{R}^{n\times n}$ and symmetric nonsingular matrix $D \in \mathbb{R}^{N\times N}$,*

$$\begin{pmatrix} I\otimes A & D\otimes B \\ D\otimes B^T & D^2\otimes C \end{pmatrix} < 0, \tag{1}$$

$$\begin{pmatrix} I\otimes A & I\otimes B \\ I\otimes B^T & I\otimes C \end{pmatrix} < 0 \tag{2}$$

and

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} < 0 \tag{3}$$

are equivalent.

Proof. (3) $\Leftrightarrow$ (2): If (3) holds, for any nonzero vector $\theta = (\alpha_1^T, \dots, \alpha_N^T, \beta_1^T, \dots, \beta_N^T)^T \in \mathbb{R}^{2Nn}$, we have

$$\theta^T \begin{pmatrix} I\otimes A & I\otimes B \\ I\otimes B^T & I\otimes C \end{pmatrix} \theta = \sum_{i=1}^{N} (\alpha_i^T, \beta_i^T) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} < 0.$$

If (2) holds, for any nonzero vector $(\alpha^T, \beta^T)^T \in \mathbb{R}^{2n}$, we have

$$(\alpha^T, \beta^T) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \eta^T \begin{pmatrix} I\otimes A & I\otimes B \\ I\otimes B^T & I\otimes C \end{pmatrix} \eta < 0,$$

where $\eta = (\alpha^T, \mathbf{0}^T, \dots, \mathbf{0}^T, \beta, \mathbf{0}^T, \dots, \mathbf{0}^T)^T$.

(2) $\Leftrightarrow$ (1): If (2) holds, for any nonzero vector $\alpha \in \mathbb{R}^{2Nn}$, let $\tilde{\alpha} = \begin{pmatrix} I\otimes I & \\ & D\otimes I \end{pmatrix} \alpha \neq 0$. Then

$$\alpha^T \begin{pmatrix} I\otimes A & D\otimes B \\ D\otimes B^T & D^2\otimes C \end{pmatrix} \alpha = \tilde{\alpha}^T \begin{pmatrix} I\otimes A & I\otimes B \\ I\otimes B^T & I\otimes C \end{pmatrix} \tilde{\alpha} < 0.$$

If (1) holds, for any nonzero vector $\beta \in \mathbb{R}^{2Nn}$, let $\tilde{\beta} = \begin{pmatrix} I\otimes I & \\ & D^{-1}\otimes I \end{pmatrix} \beta \neq 0$. Then

$$\beta^T \begin{pmatrix} I\otimes A & I\otimes B \\ I\otimes B^T & I\otimes C \end{pmatrix} \beta = \tilde{\beta}^T \begin{pmatrix} I\otimes A & D\otimes B \\ D\otimes B^T & D^2\otimes C \end{pmatrix} \tilde{\beta} < 0.$$

This completes the proof. □

3. Consensus problems for Markov switching LMAS

This paper considers the consensus problems of discrete-time Markov switching **LMAS** with fixed topology $\mathcal{G}$. The eigenvalues of the Laplacian matrix $\mathcal{L}$ are $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$. The dynamics of all agents are Markov switching linear systems:

$$x_i(k+1) = A(r(k))x_i(k) + B(r(k))u_i(k), \quad i \in \mathcal{I}, \tag{4}$$

where $x_i(k) \in \mathbb{R}^n$ and $u_i(k) \in \mathbb{R}^m$ represent respectively the state and the control input of agent i at time k. $A(r(k)) \in \mathbb{R}^{n\times n}$ and $B(r(k)) \in \mathbb{R}^{n\times m}$ are state and input matrices.

The time-delayed average consensus protocol is designed as

$$u_i(k) = K(r(k-\tau)) \sum_{j=1}^{N} a_{ij}(x_j(k-\sigma) - x_i(k-\sigma)) = 0, \quad i \in \mathcal{I}, \tag{5}$$

where $K(\upsilon) \in \mathbb{R}^{m\times n}$, for all $\upsilon \in \Upsilon$, is the gain matrix to be designed. $\tau, \sigma \in \mathbb{N}$ are constant delays occurring in the mode signal $r(k)$ and state feedback respectively.

Let $x(k) = [x_1^T(k), \dots, x_N^T(k)]^T$. Under the protocol (5), systems (4) can be written as

$$x(k+1) = [I_N \otimes A(r(k))]x(k) - [\mathcal{L} \otimes B(r(k))K(r(k-\tau))]x(k-\sigma). \tag{6}$$

We define the *center* of $x(k)$ as $\bar{x}(k) \triangleq \frac{1}{N}\sum_{i=1}^{N} x_i(k)$. Since $\mathbf{1}^T\mathcal{L} = \mathbf{0}^T$, the following property of the center in systems (6) is in force:

$$\bar{x}(k+1) = \frac{1}{N}(\mathbf{1}^T \otimes I_n)x(k+1) = A(r(k))\bar{x}(k). \tag{7}$$

The objective of this paper is to get the determinate conditions of the average consensus in stochastic multi-agent systems (6). Inspired by Ref. [29], we give the definition of the average consensus as follows.

Definition 1. Multi-agent systems (4) are said to reach a Mean Square Average Consensus (**MSAC**) under the protocol (5), if $\bar{x}(k)$ and $x_i(k)$ satisfy the following properties: $\lim_{k\to\infty} E\{\|x_i(k)-\bar{x}(k)\|^2|x_0, r_0\} = 0$, for all $i \in \mathcal{V}$; the initial conditions $x_0 = \{x(-\sigma), x(-\sigma+1), \ldots, x(0)\}$ and $r_0 = \{r(-\tau), r(-\tau+1), \ldots, r(0)\}$.

4. The main results

Inspired by Refs. [30,31], we define a *vector switching signal* $s(k) \triangleq [r(k), r(k-1), \ldots, r(k-\tau)]^T$ with finite state space $\Upsilon^{\tau+1} \triangleq \underbrace{\Upsilon \times \Upsilon \times \cdots \times \Upsilon}_{\tau+1}$. The transition probability from ν to η denotes as $\tilde{p}_{\nu\eta}$, for all $\nu, \eta \in \Upsilon^{\tau+1}$. For any $\nu = [\nu_0, \nu_{-1}, \ldots, \nu_{-\tau}]^T$ and $\eta = [\eta_0, \eta_{-1}, \ldots, \eta_{-\tau}]^T \in \Upsilon^{\tau+1}$, we can get $\tilde{p}_{\nu\eta} = p_{\nu_0\eta_0}$, if $[\nu_0, \ldots, \nu_{-\tau+1}]^T = [\eta_{-1}, \ldots, \eta_{-\tau}]^T$, otherwise $\tilde{p}_{\nu\eta} = 0$.

Furthermore, let $\tilde{A}(s(k)) \triangleq A(r(k))$, $\tilde{B}(s(k)) \triangleq B(r(k))$ and $\tilde{K}(s(k)) \triangleq K(r(k-\tau))$. Then system (6) can be written as

$$x(k+1) = [I_N \otimes \tilde{A}(s(k))]x(k) - [\mathcal{L} \otimes \tilde{B}(s(k))\tilde{K}(s(k))]x(k-\sigma). \tag{8}$$

Lemma 3. *System (8) reaches a **MSAC** if and only if system (9) satisfies* $\lim_{k\to\infty} E\{\|y(k)\|^2|x_0, r_0\} = 0$ *for any initial conditions* x_0, r_0*, where* $\tilde{\mathcal{L}} = diag\{\lambda_2, \ldots, \lambda_N\}$.

$$y(k+1) = [I_{N-1} \otimes \tilde{A}(s(k))]y(k) - [\tilde{\mathcal{L}} \otimes \tilde{B}(s(k))\tilde{K}(s(k))]y(k-\sigma). \tag{9}$$

Proof. We define $\delta_i(k) \triangleq x_i(k) - \bar{x}(k)$, $i \in \mathcal{I}$, and $\delta(k) = [\delta_1^T(k), \ldots, \delta_N^T(k)]^T$. So multi-agent systems (8) reach a **MSAC**, if and only if

$$\lim_{k\to\infty} E\{\|\delta(k)\|^2|x_0, r_0\} = 0,$$

for any initial conditions x_0, r_0. By (7) and (8), we can get

$$\delta(k+1) = [I_N \otimes \tilde{A}(s(k))]\delta(k) - [\mathcal{L} \otimes \tilde{B}(s(k))\tilde{K}(s(k))]\delta(k-\sigma). \tag{10}$$

Let c_i^T be a unit left eigenvector associated with the eigenvalue λ_i of $\mathcal{L}$, and $c_1 \triangleq \mathbf{1}/\sqrt{N}$. Denote $C = [c_1, c_2, \ldots, c_N]^T$. Then $CLC^{-1} = diag\{0, \lambda_2, \ldots, \lambda_N\}$. Noting that

$$\mathbf{1}^T\delta(k) = \sum_{i=1}^{N} \delta_i^T(k) = \sum_{i=1}^{N} x_i^T(k) - N\bar{x}(k) = \mathbf{0}^T,$$

we denote $(\mathbf{0}^T, y_2^T(k), \ldots, y_N^T(k)) = C \otimes I_n\delta(k)$. Pre-multiplying (10) by $C \otimes I_n$ immediately leads to that

$$\begin{pmatrix} 0 \\ y(k+1) \end{pmatrix} = [I_N \otimes \tilde{A}(s(k))] \begin{pmatrix} 0 \\ y(k) \end{pmatrix} - \left[\begin{pmatrix} 0 & \\ & \tilde{\mathcal{L}} \end{pmatrix} \otimes \tilde{B}(s(k))\tilde{K}(s(k))\right] \begin{pmatrix} 0 \\ y(k-\sigma) \end{pmatrix},$$

where $y(k) = (y_2^T(k), \ldots, y_N^T(k))^T$. And this system can be simplified to (9).

Since $C \otimes I_N$ is nonsingular, $\lim_{k\to\infty} E\{\|\delta(k)\|^2|x_0, r_0\} = 0$ is equivalent to that $\lim_{k\to\infty} E\{\|y(k)\|^2|x_0, r_0\} = 0$. This completes the proof. □

Through the above analysis, we can solve the average consensus problems of system (6) by Lyapunov technique.

Theorem 1. *For an undirected connected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$*, the discrete-time Markovian switching **LMAS** (4) can reach a **MSAC** under the protocol (5), if there exist symmetric positive definite matrices* $R_\nu, Q \in \mathbb{R}^{n\times n}$, $\nu \in \Upsilon^{\tau+1}$*, satisfying the LMIs*

$$\Xi_\nu = \begin{pmatrix} \Xi_{\nu 11} & \Xi_{\nu 12} \\ * & \Xi_{\nu 22} \end{pmatrix} < 0,$$

for all $\nu = [\nu_0, \nu_{-1}, \ldots, \nu_{-\tau}]^T$, $\eta = [\eta_0, \eta_{-1}, \ldots, \eta_{-\tau}]^T \in \Upsilon^{\tau+1}$*, where*

$$\Xi_{\nu 11} = I_{N-1} \otimes \left(A^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0}R_\mu\right)A(\nu_0) - R_\nu + Q\right),$$

$$\Xi_{\nu 12} = -\tilde{\mathcal{L}} \otimes \left(A^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0}R_\mu\right)B(\nu_0)K(\nu_{-\tau})\right),$$

$$\Xi_{\nu 22} = \tilde{\mathcal{L}}^2 \otimes \left(K^T(\nu_{-\tau})B^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0}R_\mu\right)B(\nu_0)K(\nu_{-\tau})\right) - I_{N-1} \otimes Q,$$

$$\mu = [\eta_0, \nu_0, \nu_{-1}, \ldots, \nu_{-\tau+1}]^T \in \Upsilon^{\tau+1}.$$

Proof. Let $\tilde{y}(k) = [y^T(k), y^T(k-1), \ldots, y^T(k-\sigma)]^T$. Take the stochastic Lyapunov function

$$V(\tilde{y}(k), s(k), k) = y^T(k)(I \otimes R_{s(k)})y(k) + \sum_{g=k-\sigma}^{k-1} y^T(g)(I \otimes Q)y(g).$$

For simplicity, set $V(k) = V(\tilde{y}(k), s(k), k)$. By the property of the signal $s(k)$, we have $\sum_{\eta \in \Upsilon^{\tau+1}} \tilde{p}_{\nu\eta} R_\eta = \sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0} R_\mu$.

Then, we can get

$$E\{V(k+1)|\tilde{y}(k), s(k), k\} - V(k) = \xi^T(k)\Xi_\nu \xi(k),$$

where $\xi(k) = [y^T(k), y^T(k-\sigma)]^T$. Let $\gamma = \inf_{\nu \in \Upsilon^{\tau+1}}(\lambda_{\min}(-\Xi_\nu))$. Since $\Xi_\nu < 0$, we have

$$E\{V(k+1)|\tilde{y}(k), s(k), k\} - V(k) \le -\gamma \|y^T(k)\|^2, \tag{11}$$

for all $y^T(k) \ne 0$. From (11), we obtain that for any $k \ge 0$

$$E\{V(k+1)\} - E\{V(0)\} \le -\gamma \sum_{t=0}^{k} E\{\|y^T(t)\|^2\}.$$

Thus

$$\sum_{t=0}^{k} E\{\|y^T(t)\|^2\} \le \frac{1}{\gamma}(E\{V(0)\} - E\{V(k+1)\}) \le \frac{1}{\gamma} E\{V(y_0, s(0), 0)\},$$

implying

$$\sum_{t=0}^{\infty} E\{\|y^T(t)\|^2\} \le \frac{1}{\gamma} E\{V(0)\} < \infty.$$

As a result $\lim_{k\to\infty} E\{\|y(k)\|^2 | x_0, r_0\} = 0$. By Lemma 3, this leads to systems (4) reach a **MSAC**. This completes the proof. □

Theorem 2. *For an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, discrete-time Markovian switching linear multi-agent systems (4) can reach a **MSAC** under the protocol (5), if there exist symmetric positive definite matrices R_ν, $Q \in \mathbb{R}^{n\times n}$, satisfying the LMIs*

$$\hat{\Xi}_\nu = \begin{pmatrix} \hat{\Xi}_{\nu 11} & \hat{\Xi}_{\nu 12} \\ * & \hat{\Xi}_{\nu 22} \end{pmatrix} < 0, \tag{12}$$

for all $\nu = [\nu_0, \nu_{-1}, \ldots, \nu_{-\tau}]^T$, $\eta = [\eta_0, \eta_{-1}, \ldots, \eta_{-\tau}]^T \in \Upsilon^{\tau+1}$, where

$$\hat{\Xi}_{\nu 11} = A^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0} R_\mu\right) A(\nu_0) - R_\nu + Q,$$

$$\hat{\Xi}_{\nu 12} = -A^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0} R_\mu\right) B(\nu_0)K(\nu_{-\tau}),$$

$$\hat{\Xi}_{\nu 22} = K^T(\nu_{-\tau})B^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0} R_\mu\right) B(\nu_0)K(\nu_{-\tau}) - \frac{1}{\lambda_N^2} Q,$$

$$\mu = [\eta_0, \nu_0, \nu_{-1}, \ldots, \nu_{-\tau+1}]^T \in \Upsilon^{\tau+1}.$$

Proof. Since $(I_{N-1} - \frac{1}{\lambda_N^2}\tilde{\mathcal{L}}^2) \ge 0$ and $Q > 0$, we have $(I_{N-1} - \frac{1}{\lambda_N^2}\tilde{\mathcal{L}}^2) \otimes Q \ge 0$. Then

$$\Xi_\nu \le \begin{pmatrix} \Xi_{\nu 11} & \Xi_{\nu 12} \\ * & \Theta \end{pmatrix},$$

where

$$\Theta = \tilde{\mathcal{L}}^2 \otimes \left(K^T(\nu_{-\tau})B^T(\nu_0)\left(\sum_{\eta_0=0}^{d-1} p_{\nu_0\eta_0} R_\mu\right) B(\nu_0)K(\nu_{-\tau}) - \frac{1}{\lambda_N^2} Q\right).$$

By Lemma 2, $\hat{\Xi}_\nu < 0$ is equivalent to $\begin{pmatrix} \Xi_{\nu 11} & \Xi_{\nu 12} \\ * & \Theta \end{pmatrix} < 0$. Then $\hat{\Xi}_\nu < 0$, which means systems (4) reach a **MSAC**. This completes the proof of Theorem 2. □

Remark 1. In Theorem 1, the determination on the consensus of systems (4) relates to the system matrices, gain matrices $K(i)$, $i \in \Upsilon$, and Laplacian matrix $\tilde{\mathcal{L}}$. However, in Theorem 2, instead of the Laplacian matrix $\tilde{\mathcal{L}}$, it relates to λ_N. We can see that the determinate condition in Theorem 1 is more relaxed than that in Theorem 2, but Theorem 2 needs less information of the network.

Corollary 1. *For an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, discrete-time Markovian switching linear multi-agent systems (4) can reach a* ***MSAC*** *under the protocol (5), if there exist $K(\upsilon) \in \mathbb{R}^{m\times n}$ and symmetric positive definite matrices $R_\nu, U_\nu, Q \in \mathbb{R}^{n\times n}$, satisfying the LMIs*

$$\begin{pmatrix} -R_\nu + Q & 0 & M_{1\nu} \\ * & -\dfrac{1}{\lambda_N^2}Q & M_{2\nu} \\ * & * & -\Lambda_\nu \end{pmatrix} < 0, \tag{13}$$

with equality constraints $R_\nu U_\nu = I$, for all $\nu = [\nu_0, \nu_{-1}, \ldots, \nu_{-\tau}]^T \in \Upsilon^{\tau+1}$, where

$$\begin{aligned} M_{1\nu} &= [\sqrt{p_{\nu_0 0}}, \sqrt{p_{\nu_0 1}}, \ldots, \sqrt{p_{\nu_0 d-1}}] \otimes A^T(\nu_0), \\ M_{2\nu} &= -[\sqrt{p_{\nu_0 0}}, \sqrt{p_{\nu_0 1}}, \ldots, \sqrt{p_{\nu_0 d-1}}] \otimes K^T(\nu_{-\tau})B^T(\nu_0), \\ \Lambda_\nu &= diag\{U_{\mu_0}, U_{\mu_1}, \ldots, U_{\mu_{d-1}}\}, \\ \mu_0 &= [0, \nu_0, \nu_{-1}, \ldots, \nu_{-\tau+1}]^T, \ldots, \mu_{d-1} = [d-1, \nu_0, \nu_{-1}, \ldots, \nu_{-\tau+1}]^T. \end{aligned}$$

Proof. By Schur complement and Theorem 2, this corollary can be proofed. □

Remark 2. Although the solution set of (13) is not convex due to equality constraints $R_\nu U_\nu = I$, the cone complementarity linearization type (**CCL**-type) algorithms [32] can be employed to solve such problems effectively. By Corollary 2 and **CCL**-type algorithms, we can get the gain matrices $K(\upsilon)$.

Corollary 2. *For an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, discrete-time Markovian switching linear multi-agent systems (4) can reach a* ***MSAC*** *under the protocol (5), if there exist $K(\upsilon) \in \mathbb{R}^{m\times n}$ and symmetric positive definite matrices $R_\nu, U_\nu, Q \in \mathbb{R}^{n\times n}$, satisfying the LMIs*

$$\begin{pmatrix} I\otimes(-R_\nu + Q) & 0 & M_{1\nu} \\ * & -I\otimes Q & M_{2\nu} \\ * & * & -\Lambda_\nu \end{pmatrix} < 0, \tag{14}$$

with equality constraints $R_\nu U_\nu = I$, for all $\nu = [\nu_0, \nu_{-1}, \ldots, \nu_{-\tau}]^T \in \Upsilon^{\tau+1}$, where

$$\begin{aligned} M_{1\nu} &= [\sqrt{p_{\nu_0 0}}, \sqrt{p_{\nu_0 1}}, \ldots, \sqrt{p_{\nu_0 d-1}}] \otimes (I \otimes A^T(\nu_0)), \\ M_{2\nu} &= -[\sqrt{p_{\nu_0 0}}, \sqrt{p_{\nu_0 1}}, \ldots, \sqrt{p_{\nu_0 d-1}}] \otimes (\tilde{\mathcal{L}} \otimes K^T(\nu_{-\tau})B^T(\nu_0)), \\ \Lambda_\nu &= diag\{I\otimes U_{\mu_0}, I\otimes U_{\mu_1}, \ldots, I\otimes U_{\mu_{d-1}}\}, \\ \mu_0 &= [0, \nu_0, \nu_{-1}, \ldots, \nu_{-\tau+1}]^T, \ldots, \mu_{d-1} = [d-1, \nu_0, \nu_{-1}, \ldots, \nu_{-\tau+1}]^T. \end{aligned}$$

Proof. By Schur complement and Theorem 1, this corollary can be proofed. □

5. Numerical example

In this section, we give a numerical example to illustrate the efficiency of our results.

Consider a 5-agents connected network which is shown in Fig. 1. The Laplacian matrix

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & -1/3 & -1/3 & -1/3 \\ 0 & 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & -1/3 & 0 & 2/3 & 0 \\ -1/3 & 0 & -1/3 & 0 & 2/3 \end{pmatrix}.$$

The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$, $\lambda_2 = 0.4607$, $\lambda_3 = 0.7940$ and $\lambda_4 = 1.5393$. The state space $\Upsilon = \{0, 1\}$, the signal mode delay $\tau = 1$, and the system state delay $\sigma = 2$. The state transition matrices of $r(k)$ is

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{pmatrix}.$$

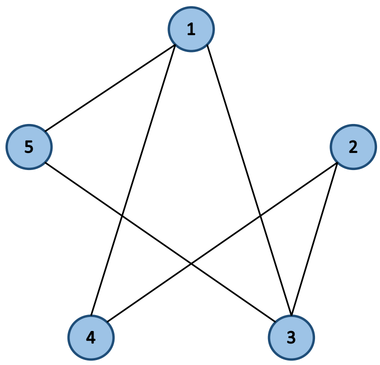

Fig. 1. 5-agents networks.

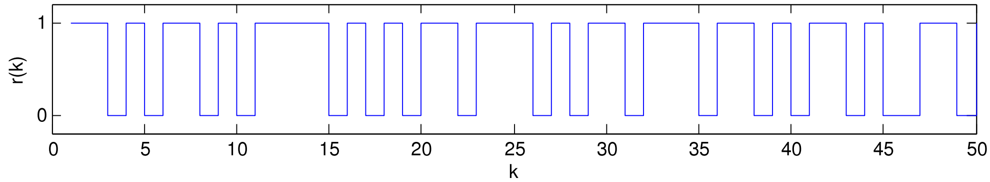

Fig. 2. Markov switching signal.

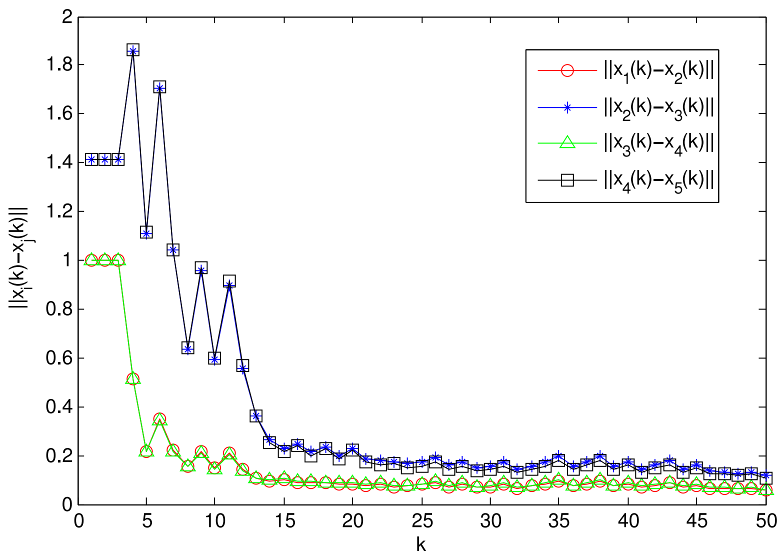

Fig. 3. Error of states.

The state and input matrices of systems (4) are:

$$A(0) = \begin{pmatrix} 1.5 & 0.3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}, \qquad A(1) = \begin{pmatrix} 0.6 & 0 & 0 \\ 0.1 & 0.2 & 0.3 \\ 0 & 0.1 & 1.1 \end{pmatrix}, \qquad B(0) = B(1) = I.$$

By Corollary 2 and **CCL**-type algorithms, we can get the gain matrices

$$K(0) = \begin{pmatrix} 0.0045 & -0.0156 & 0.0004 \\ -0.0144 & 0.1055 & -0.0140 \\ 0.0000 & -0.0142 & 0.0076 \end{pmatrix},$$

$$K(1) = \begin{pmatrix} 0.0077 & -0.0193 & 0.0007 \\ -0.0181 & 0.1321 & -0.0233 \\ 0.0008 & -0.0229 & 0.0084 \end{pmatrix}.$$

The initial conditions are $x_1(-2) = x_1(-1) = x_1(0) = [1, -1, 0]^T$, $x_2(-2) = x_2(-1) = x_2(0) = [1, 0, 1]^T$, $x_3(-2) = x_3(-1) = x_3(0) = [0, -1, 0]^T$, $x_4(-2) = x_4(-1) = x_4(0) = [0, 0, 0]^T$, $x_5(-2) = x_5(-1) = x_5(0) = [-1, -1, 0]^T$ and $r(-1) = r(0) = 1$. Fig. 2 depicts the Markov switching signal. And Fig. 3 shows that the systems reach a consensus under the time-delayed average consensus protocol (5).

6. Conclusions

In this paper, we have provided the consensus analysis of the discrete-time Markov switching **LMAS** with fixed topology under time-delay average consensus protocol. Firstly, **MSAC** has been defined to adapt for this stochastic systems. Since there is time-delay switching signal, we have defined a new switching signal, and merged the switching signal of the systems and the time-delay signal of the controller into one signal. Then we have used the discrete-time stochastic Lyapunov techniques to analyze the consensus of the **LMAS**. By this method, we have solved the average consensus problems, and presented two sufficient criteria of the discrete-time Markov switching **LMAS** respectively. These results demonstrate that the consensus of the multi-agent systems relates to the spectral radius of the Laplacian matrix, a smaller spectral radius corresponds to a more relaxed determinate condition. And a numerical example has been given to illustrate the efficiency of our results. The defect of the paper is that we assumed the topology is fixed. We will study this problem under switching topology in the future.

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