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The heat flux identification problem for a nonlinear parabolic equation in 2D

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ABSTRACT

We consider the heat flux identification problem (HFIP) based on the boundary measurements for a nonlinear parabolic equation in 2-dimensional space. The standard linearization algorithm is applied to the nonlinear direct problem. The method of Conjugate Gradient Algorithm, based on the gradient formula for the cost functional, is then proposed for numerical solution of the inverse heat flux problem. Numerical analysis of the algorithm applied to the inverse problem in typical classes of flux functions is presented. Computational results, obtained for random noisy output data, indicate how the iteration number of the Conjugate Gradient Algorithm can be estimated. Numerical results illustrate bounds of applicability of proposed algorithm, as well as its efficiency and accuracy.

1. Introduction

We study the following inverse problems of determining the unknown flux terms $f := (f_1(x,t), f_2(x,t))$, in the following heat conduction problem

$$\begin{cases} u_t = \nabla(k(|\nabla u|^2)\nabla u) + F(x,t), \quad (x,t) \in \Omega_T \\ -k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = f_1(x,t), \quad (x,t) \in \Gamma_1^T, \\ -k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = f_2(x,t), \quad (x,t) \in \Gamma_2^T \\ u(x,t) = 0, \quad (x,t) \in \Gamma_3^T \cup \Gamma_4^T, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases} \tag{1.1}$$

from the supplementary boundary measurements $h := (h_1(x,t), h_2(x,t))$:

$$\begin{aligned} h_1(x,t) &= u(x,t), \quad (x,t) \in \Gamma_1^T; \\ h_2(x,t) &= u(x,t), \quad (x,t) \in \Gamma_2^T; \end{aligned} \tag{1.2}$$

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where $\Gamma_i^T := \Gamma_i \times (0,T]$, $i = 1,2,3,4$, $x := (x_1,x_2)$, $x \in \Omega$, $\Omega_T := \Omega \times (0,T]$ and $\Omega := (0,\ell_{x_1}) \times (0,\ell_{x_2})$ is assumed to be a bounded simply connected domain with a piecewise smooth boundary $\Gamma := \bigcup_i \Gamma_i$, $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, meas $\Gamma_i \neq 0$, $i = 1,2,3,4$ and $0 < T < \infty$ and $\ell_{x_1}, \ell_{x_2} > 0$.

$$\Gamma_1 := \{0\} \times (0,\ell_{x_2}), \qquad \Gamma_2 := (0,\ell_{x_1}) \times \{0\},$$
$$\Gamma_3 := \{\ell_{x_1}\} \times (0,\ell_{x_2}), \qquad \Gamma_4 := (0,\ell_{x_1}) \times \{\ell_{x_2}\}.$$

The initial temperature $u_0(x)$ and the boundary data $h_1(x,t)$ and $h_2(x,t)$ satisfy the consistency conditions $u_0|_{\Gamma_1} = h_1(x,0)$ and $u_0|_{\Gamma_2} = h_2(x,0)$, respectively. For a given flux terms $f_1(x,t)$, $f_2(x,t)$ the problem (1.1) is defined to be the *direct problem*. When the flux term $f := (f_1,f_2)$ needs to be defined, the problem of identifying the unknown f using (1.1)–(1.2) is defined as to be *heat flux identification problem* (HFIP).

Note that in practice the input/output data are obtained from physical experiments and may not be smooth functions. Hence methods based on classical solution of the direct problem cannot be applied for large class of inverse problems. The proposed approach uses the energy method [1] for the solvability analysis of the direct problem. Thus, we are interested in weak (generalized) solution of the parabolic problem (1.1) in $\mathring{V}^{1,0}(\Omega_T) := \{v \in V^{1,0}(\Omega_T) : v(x,t)|_{(\Gamma_3^T \cup \Gamma_4^T)} = 0,\ \forall t \in (0,T]\}$. Here $\mathring{V}^{1,0}$ is the Sobolev space of functions with square integrable gradient ∇u with the norm (see, [2]):

$$\|u\|_{V^{1,0}(\Omega_T)} := \max_{t\in[0,T]} \|u\|_{H^0(\Omega)} + \|\nabla u\|_{H^0(\Omega_T)}.$$

This weak solution $u \in \mathring{V}^{1,0}(\Omega_T)$, with $u(x,0) = u_0(x)$, satisfies the following integral identity:

$$\frac{1}{2}\int_\Omega [u^2(x,t)]dx + \int_{\Omega_t} k(|\nabla u|^2)|\nabla u|^2\, dx d\tau = \int_{\Omega_t} F(x,\tau)u\, dx d\tau + \frac{1}{2}\int_\Omega [u_0^2(x)]dx + \int_0^T\int_{\Gamma_1} f_1(x,t)u(x,t)dxdt + \int_0^T\int_{\Gamma_2} f_2(x,t)u(x,t)dxdt.$$

For the existence of the unique solution $u(x,t) \in V^{1,0}(\Omega_T)$ we require that the functions $k(\xi)$, $F(x,t)$, $u_0(x)$, satisfy the following conditions [1]:

$$\begin{cases} F(x,t) \in L_2(\Omega_T), \quad f_1(x,t) \in L_2(\Gamma_1^T), \quad f_2(x,t) \in L_2(\Gamma_2^T), \\ u_0(x) \in L_2(\Omega), \quad k(\xi) \in L_\infty(0,\xi^*). \end{cases} \tag{1.3}$$

To investigate solvability conditions for the problem (1.1) in $\mathring{V}^{1,0}$ we add following conditions to function $k(\xi)$:

$$\begin{cases} k(\xi) + 2\xi k'(\xi) \geq \gamma_0 > 0, \quad \xi \in [0,\xi^*] \\ k'(\xi) < 0. \end{cases} \tag{1.4}$$

This study presents a systematic analysis of inverse flux problems aims to estimate as accurately as possible the f, under the overspecified data h at the boundary, given by (1.2). The analysis is based on the proposed variational approach which permits to derive explicitly gradient of the cost functional:

$$J(f) = \int_0^T\int_{\Gamma_1} [u(x,t;f) - h_1(x,t)]^2 dxdt + \int_0^T\int_{\Gamma_2} [u(x,t;f) - h_2(x,t)]^2 dxdt \tag{1.5}$$

corresponding to the above defined problem HFIP. The conjugate gradient method (CGM) with the derived explicit formula for the gradient of the cost functional $J(f)$ is then applied for numerical solution of HFIP.

The paper is organized as follows. In Section 2, we defined quasi solution of the inverse flux problem, introducing the admissible unknown fluxes. In Section 3, we linearize the nonlinear problem to linear one and derived the integral relationship between solutions of the introduced adjoint problems and direct problem. Using these identities we prove the Fréchet differentiability of the cost functional and unicity of the solution in Section 4. In Section 5 the numerical results for the CGM applied to the HFIP are presented for various noise free and noisy output data.

2. The quasi-solution approach of the inverse problems

Let us define the set $\mathcal{W} := \mathcal{F}_1 \times \mathcal{F}_2 \subseteq L_2(\Gamma_1^T) \times L_2(\Gamma_2^T)$ of admissible unknown fluxes $f_1 \in \mathcal{F}_1 \subset L_2(\Gamma_1^T)$ and $f_2 \in \mathcal{F}_1 \subset L_2(\Gamma_2^T)$, which satisfy the following conditions:

$$\begin{cases} -\infty < \underline{F1} < f_1(x,t) < \overline{F1} < \infty, \quad a.e.\ \forall (x,t) \in \Gamma_1^T \\ -\infty < \underline{F2} < f_2(x,t) < \overline{F2} < \infty, \quad a.e.\ \forall (x,t) \in \Gamma_2^T. \end{cases} \tag{2.1}$$

Evidently, $\mathcal{W}$ is a closed and convex subset in $L_2(\Gamma_1^T) \times L_2(\Gamma_2^T)$. For a given element $f \in \mathcal{W}$ we denote by $u(x,t;f) \in \mathring{V}^{1,0}(\Omega_T)$, with $u(x,0;f) = u_0(x)$, the weak solution of the direct problem (1.1). If the function $u(x,t;f)$ satisfies also the additional condition (1.2), then it will be defined as a strict solution of the problem HFIP, accordingly. In this case, one can introduce the input–output maps $\Phi f := u(x,t;f)|_{\Gamma_1^T \times \Gamma_2^T} := (u(x,t;f)|_{\Gamma_1^T}, u(x,t;f)|_{\Gamma_2^T})$, $\Phi : \mathcal{W} \to \mathcal{H}$, $\mathcal{H} \subset L_2(\Gamma_1^T) \times L_2(\Gamma_2^T)$, and reformulate the problem as the following functional equation:

$$\Phi f = h, \quad h := (h_1(x,t), h_2(x,t)) \in \mathcal{H}. \tag{2.2}$$

Therefore the inverse problems with the given measured output data $h(x,t) \in \mathcal{H}$ can be reduced to the solution of the functional Eq. (2.2) or to inverting the input–output mapping. On the other hand, due to measurement error in $h(x,t)$, the exact fulfilment of condition (1.2) is almost impossible in practice. For this reason we define a quasisolution of the considered inverse problems, according to [3,4], as a solution of the minimization problem for the cost functionals $J(f)$, defined by (1.5):

$$J(f^*) = inf\, J(f), \quad f \in \mathcal{W}.$$

Evidently, if $J(f^*) = 0$ then the quasi-solution $f^* \in \mathcal{W}$ is also a strict solution of HFIP.

3. Adjoint problem approach for linearized inverse problem and Fréchet differentiability of the cost functional

In this section, to use the adjoint problem approach technique, we linearize the nonlinear problem (1.1). By $u^{(n)}(x,t)$, we indicate the solution of the linearized direct problem defined as follows:

$$\begin{cases} u_t^{(n)} = \nabla(k(|\nabla u^{(n-1)}|^2)\nabla u^{(n)}) + F(x,t), & (x,t) \in \Omega_T \\ -k(|\nabla u^{(n-1)}|^2)\dfrac{\partial u^{(n)}}{\partial n} = f_1(x,t), & (x,t) \in \Gamma_1^T, \\ -k(|\nabla u^{(n-1)}|^2)\dfrac{\partial u^{(n)}}{\partial n} = f_2(x,t), & (x,t) \in \Gamma_2^T \\ u^{(n)}(x,t) = 0, & (x,t) \in \Gamma_3^T \cup \Gamma_4^T \\ u^{(n)}(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

here the initial function $u^{(0)}(x,t)$ is chosen as the zero function for all $(x,t) \in \Omega_T$. Under the conditions (1.3) we assume that there exists a function $\tilde{u}(x,t) \in \mathring{V}^{1,0}(\Omega_T)$ and

$$\|u^{(n)} - \tilde{u}\|_{\mathring{V}^{1,0}(\Omega_T)} \to 0 \quad \text{whenever } n \to \infty.$$

Then the linearized direct problem (3.1) is transformed into linear problem defining $\tilde{k}(x,t) := k(|\nabla\tilde{u}|^2)$ as follows:

$$\begin{cases} \tilde{u}_t = \nabla(\tilde{k}(x,t)\nabla\tilde{u}) + F(x,t), & (x,t) \in \Omega_T \\ -\tilde{k}(x,t)\dfrac{\partial \tilde{u}}{\partial n} = f_1(x,t), & (x,t) \in \Gamma_1^T, \\ -\tilde{k}(x,t)\dfrac{\partial \tilde{u}}{\partial n} = f_2(x,t), & (x,t) \in \Gamma_2^T \\ \tilde{u}(x,t) = 0, & (x,t) \in \Gamma_3^T \cup \Gamma_4^T \\ \tilde{u}(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.2}$$

In this case the additional conditions (1.2) are defined as follows:

$$\begin{aligned} h_1(x,t) &= \tilde{u}(x,t), \quad (x,t) \in \Gamma_1^T; \\ h_2(x,t) &= \tilde{u}(x,t), \quad (x,t) \in \Gamma_2^T. \end{aligned} \tag{3.3}$$

Thus the cost $J(f)$ is defined as

$$\tilde{J}(f) := \int_0^T \int_{\Gamma_1} [\tilde{u}(x,t;f) - h_1(x,t)]^2 dxdt + \int_0^T \int_{\Gamma_2} [\tilde{u}(x,t;f) - h_2(x,t)]^2 dxdt. \tag{3.4}$$

We define $\tilde{J}(f)$ as the cost functional of the linearized inverse problem (3.2)–(3.3) and we consider the first variation of the cost functional $\tilde{J}(f)$.

$$\begin{aligned} \delta\tilde{J}(f) &= \tilde{J}(f+\delta f) - \tilde{J}(f) \\ &= 2\int_{\Gamma_1^T} [\delta\tilde{u}(x,t;\delta f) - h_1(x,t)]\delta\tilde{u}(x,t;\delta f)dtdx + 2\int_{\Gamma_2^T} [\delta\tilde{u}(x,t;\delta f) - h_2(x,t)]\delta\tilde{u}(x,t;\delta f)dxdt \\ &\quad + \int_{\Gamma_2^T} [\delta\tilde{u}(x,t;\delta f)]^2 dxdt + \int_{\Gamma_1^T} [\delta\tilde{u}(x,t;\delta f)]^2 dxdt, \end{aligned} \tag{3.5}$$

where $f+\delta f := (f_1+\delta f_1, f_2+\delta f_2) \in \mathcal{W}$, $\delta\tilde{u} := \tilde{u}(x,t;f+\delta f) - \tilde{u}(x,t;f) \in \mathring{V}^{1,0}(\Omega_T)$. Evidently the function $\delta\tilde{u}(x,t)$ is the solution of the following parabolic problem

$$\begin{cases} \delta\tilde{u}_t = \nabla(\tilde{k}(x,t)\nabla\delta\tilde{u}), \quad (x,t)\in\Omega_T \\ -\tilde{k}(x,t)\dfrac{\partial\delta\tilde{u}}{\partial n} = \delta f_1(x,t), \quad (x,t)\in\Gamma_1^T, \\ -\tilde{k}(x,t)\dfrac{\partial\delta\tilde{u}}{\partial n} = \delta f_2(x,t), \quad (x,t)\in\Gamma_2^T \\ \delta\tilde{u}(x,t) = 0, \quad (x,t)\in\Gamma_3^T\cup\Gamma_4^T \\ \delta\tilde{u}(x,0)=0, \quad x\in\Omega. \end{cases} \tag{3.6}$$

Lemma 3.1. *Assume $\tilde{u}(x,t;f) \in \mathring{V}^{1,0}(\Omega_T)$ and $\tilde{u}(x,t;f+\delta f) \in \mathring{V}^{1,0}(\Omega_T)$ are solutions of the direct problem (3.2) corresponding to the $f\in\mathcal{W}$ and $f+\delta f\in\mathcal{W}$, and let $\tilde{\varphi}(x,t)\in\mathring{V}^{1,0}(\Omega_T)$ is the solution of the following backward parabolic problem:*

$$\begin{cases} \tilde{\varphi}_t = -\nabla(\tilde{k}(x,t)\nabla\tilde{\varphi}), \quad (x,t)\in\Omega_T \\ -\tilde{k}(x,t)\dfrac{\partial\tilde{\varphi}}{\partial n} = 2[\tilde{u}(x,t;f)-h_1(x,t)], \quad (x,t)\in\Gamma_1^T, \\ -\tilde{k}(x,t)\dfrac{\partial\tilde{\varphi}}{\partial n} = 2[\tilde{u}(x,t;f)-h_2(x,t)], \quad (x,t)\in\Gamma_2^T \\ \tilde{\varphi}(x,t)=0, \quad (x,t)\in\Gamma_3^T\cup\Gamma_4^T \\ \tilde{\varphi}(x,T)=0, \quad x\in\Omega. \end{cases} \tag{3.7}$$

Then for all $f(x,t)\in\mathcal{W}$ the following integral identity holds:

$$\begin{aligned} &2\int_0^T\int_{\Gamma_1}\left(\tilde{u}(x,t;f)-h_1(x,t)\right)\delta\tilde{u}(x,t;\delta f)dxdt + 2\int_0^T\int_{\Gamma_2}\left(\tilde{u}(x,t;f)-h_2(x,t)\right)\delta\tilde{u}(x,t;\delta f)dxdt \\ &= \int_0^T\int_{\Gamma_1}\tilde{\varphi}(x,t;f)\delta f_1(x,t)dxdt + \int_0^T\int_{\Gamma_2}\tilde{\varphi}(x,t;f)\delta f_2(x,t)dxdt, \end{aligned} \tag{3.8}$$

where $\tilde{\varphi}(x,t) := \varphi^{(n)}(x,t)$.

Proof. Multiplying both sides of Eq. (3.6) by the arbitrary function $\tilde{\varphi}(x,t)$ for each linearized problem we get:

$$\int_{\Omega_T}\delta\tilde{u}_t(x,t;f)\tilde{\varphi}(x,t)dxdt = \int_{\Omega_T}\nabla(\tilde{k}(x,t)\nabla\delta\tilde{u})\tilde{\varphi}(x,t)dxdt.$$

Applying the integration by parts formula to the both left and right hand side above, we obtain:

$$\begin{aligned} \int_{\Omega_T}\nabla(\tilde{k}(x,t)\nabla\delta\tilde{u})\tilde{\varphi}(x,t)dxdt &= \int_{\Omega_T}\nabla(\tilde{k}(x,t)\nabla\tilde{\varphi}(x,t))\delta\tilde{u}(x,t;f)dxdt \\ &- \int_{\Gamma_1^T}\tilde{k}(x,t)\varphi_{x_1}(x,t;f)\delta\tilde{u}(x,t;f)|_0^{l_{x_1}}dxdt + \int_{\Gamma_1^T}(\tilde{k}(x,t)\varphi(x,t;f)\delta\tilde{u}_{x_1}(x,t;f))|_0^{l_{x_1}}dxdt \\ &- \int_{\Gamma_2^T}\tilde{k}(x,t)\varphi_{x_2}(x,t;f)\delta\tilde{u}(x,t;f)|_0^{l_{x_2}}dxdt + \int_{\Gamma_2^T}(\tilde{k}(x,t)\varphi(x,t;f)\delta\tilde{u}_{x_2}(x,t;f))|_0^{l_{x_2}}dxdt. \end{aligned} \tag{3.9}$$

Now we claim that $\tilde{\varphi}(x,t)$ is the solution of the backward problem (3.7) and taking into account here the boundary conditions in (3.6) and (3.7) finally we get the proof.

Remark 3.1. We will define the parabolic problem (3.7) as an adjoint problem, corresponding to the inverse problem (1.1)–(1.2). The parabolic equation (3.7) is a backward one, and due to the "final condition" at $t=T$ it is a well-posed initial boundary-value problem under a reversal of time.

Now we use the integral identity (3.8) on the right-hand side of the formula (3.5) for the first variation of the cost functional $\tilde{J}(f)$. Then we have

$$\delta\tilde{J}(f) = \int_{\Gamma_1^T}\tilde{\varphi}(x,t;f)\delta f_1(x,t)dxdt + \int_{\Gamma_2^T}\tilde{\varphi}(x,t;f)\delta f_2(x,t)dxdt + o(\|\delta\tilde{u}(x,t;\delta f)\|^2), \tag{3.10}$$

where $o(\|\delta\tilde{u}(x,t;\delta f)\|^2) := \int_{\Gamma_1^T}[\delta\tilde{u}(x,t;\delta f)]^2dxdt + \int_{\Gamma_2^T}[\delta\tilde{u}(x,t;\delta f)]^2dxdt$. According to the definition of the Fréchet-differential we need to transform the right-hand side of (3.10) into the following form:

$$\delta\tilde{J}(f) = (\nabla\tilde{J}(f),\delta f)_{L_2(\Gamma_1^T)\times L_2(\Gamma_2^T)} + \int_{\Gamma_1^T}[\delta\tilde{u}(x,t;\delta f)]^2dxdt + \int_{\Gamma_2^T}[\delta\tilde{u}(x,t;\delta f)]^2dxdt.$$

This formula provides further insight into the gradient of the functional $\tilde{J}(f)$ via the solution of the adjoint parabolic problem (3.7). Due to the definition of the Fréchet-differential, we need to show that the last term on the right-hand side of (3.10) is of order $o(\|\delta f\|^p)$, with $p \geq 1$. The following result precisely shows an estimate for the H^0-norm $\|\delta\tilde{u}(x,t;\delta f)\|_0$ of the increment $\delta\tilde{u}(x,t;\delta f)$ on the $W := \Gamma_1^T \times \Gamma_2^T$, in terms of $o(\|\delta f\|^2)$.

For all $\delta f \in \mathcal{W}$ by the definition $J(f+\delta f) - J(f) = (\nabla J, \delta f)_{L_2(\Gamma_1^T)\times L_2(\Gamma_2^T)} + o(\|\delta f_1\|)^2_{L_2(\Gamma_1^T)} + o(\|\delta f_2\|)^2_{L_2(\Gamma_2^T)}$ of the Fréchet differential, we need to prove that the last integral term in (3.10) is of order $o(\|\delta f_1\|)^2_{L_2(\Gamma_1^T)} + o(\|\delta f_2\|)^2_{L_2(\Gamma_2^T)}$.

Lemma 3.2. *Let $\mathcal{W} \subset H^0$ and conditions (1.3) hold. Then for the solution $\delta\tilde{u} := \tilde{u}(x,t;\delta f) \in \mathring{V}^{1,0}(\Omega_T)$ of the parabolic problem (3.6), corresponding to a given increment $\delta f \in \mathcal{W}$, the following estimates hold:*

$$\|\delta\tilde{u}(x,t;f)_{|_{\Gamma_1^T}}\|^2 + \|\delta\tilde{u}(x,t;f)_{|_{\Gamma_2^T}}\|^2 \leq L\left(\|\delta f_1\|^2_{L_2(\Gamma_1^T)} + \|\delta f_2\|^2_{L_2(\Gamma_2^T)}\right), \quad \forall \delta f \in \mathcal{W},$$

where

$$L = \max\left\{\frac{\varepsilon l_{x_1}}{2\sigma_{\varepsilon_{x_1}}}, \frac{\varepsilon l_{x_2}}{2\sigma_{\varepsilon_{x_2}}}\right\}, \qquad \sigma_{\varepsilon_{x_1}} = \left[k_* - \frac{l_{x_1}}{2\varepsilon}\right]^{1/2}, \qquad \sigma_{\varepsilon_{x_2}} = \left[k_* - \frac{l_{x_2}}{2\varepsilon}\right]^{1/2}, \quad k_* = \min_{\xi\in[0,\xi^*]} k(\xi).$$

Proof. Multiplying both sides of the parabolic equation (3.6) by $\delta\tilde{u}$, integrating on Ω_T and using the initial and boundary conditions we obtain the following energy identity

$$\frac{1}{2}\int_\Omega \delta\tilde{u}^2(x,t)dx + \int_{\Omega_t} \tilde{k}(x,t)|\nabla\delta\tilde{u}|^2\, dxd\tau$$
$$= \int_0^T \int_{\Gamma_1} \delta f_1(x,t)\delta\tilde{u}(x,t)dxdt + \int_0^T\int_{\Gamma_2} \delta f_2(x,t)\delta\tilde{u}(x,t)dxdt.$$

Using here the Cauchy-ε-inequality and after some calculation we conclude

$$[2\varepsilon k_* - l_{x_1}]\int_{\Gamma_1^T}[\delta\tilde{u}(x,t;f)]^2 dxdt + [2\varepsilon k_* - l_{x_2}]\int_{\Gamma_2^T}[\delta\tilde{u}(x,t;f)]^2 dxdt$$
$$\leq \int_{\Gamma_1^T}[\delta f_1(x,t)]^2 dxdt + \int_{\Gamma_2^T}[\delta f_2(x,t)]^2 dxdt.$$

The required estimate follows from this inequality by choosing the constant $L > 0$, which completes the proof.

Theorem 3.1. *Let the conditions of Lemma 3.2 hold. Then the costfunctional $\tilde{J}(f)$ corresponding to the (HFIP), is Fréchet-differentiable: $\tilde{J}(f) \in C^{1,1}(\mathcal{W})$. Moreover, the Fréchet derivative at $f \in \mathcal{W}$ of the cost functional $\tilde{J}(f)$ can be defined via the solution $\tilde{\varphi}(x,t;f) \in \mathring{V}^{1,0}(\Omega_T)$ of the well-posed adjoint problem (3.7) as follows:*

$$\nabla\tilde{J}(f) := (\tilde{\varphi}(x,t;f)_{|_{\Gamma_1^T}}, \tilde{\varphi}(x,t;f)_{|_{\Gamma_2^T}}). \tag{3.11}$$

4. Lipschitz continuity of the gradient $\nabla\tilde{J}(f)$ and convergence of the gradient method

The minimum value $f_* \in \mathcal{W}$ of the cost functional $\tilde{J}(f)$ defines a quasisolution of (HFIP). On the other hand, the gradient $\nabla\tilde{J}(f)$ of the cost functional $\tilde{J}(f)$ can be determined via the solution $\tilde{\varphi}(x,t;f) \in \mathring{V}^{1,0}(\Omega_T)$ of the well-posed adjoint problem (3.7), which contains the measured output data h. This result, with the gradient formula, suggests a use of gradient-type iterative methods for approximate solution of (HFIP). However, any gradient method for the minimization problem requires an estimation of the iteration parameter $\alpha_n > 0$ in the iteration process

$$f_{n+1} = f_n - \alpha_n\nabla\tilde{J}(f_n), \quad n = 0, 1, 2, \ldots,$$

with the given initial iteration $f_0 \in \mathcal{W}$. In the case of Lipschitz continuity of the gradient $\nabla\tilde{J}(f)$, the parameter α_n can be estimated via the Lipschitz constant L_1, i.e. $0 < \delta_0 \leq \alpha_n \leq 2/(L_1 + 2\delta_1)$. Here δ_0, δ_1 are arbitrary parameters. The following result shows the continuity of the gradient $\nabla\tilde{J}(f)$.

Theorem 4.1. *Let the conditions of Lemma 3.1 hold. Then the functional $\tilde{J}(f)$ is of Holder class $C^{1,1}(\mathcal{W})$ and*

$$\|\nabla\tilde{J}(f+\delta f) - \nabla\tilde{J}(f)\| \leq L_1\|f\|_{\Gamma_1^T\times\Gamma_2^T}, \quad \forall f, f+\delta f \in \mathcal{W}.$$

Here $L_1 = (2\varepsilon_1\gamma_\varepsilon/\sigma_{\varepsilon_1})^{1/2}\, l > 0$. Here $\sigma_{\varepsilon_1} = k_* - l/\varepsilon_1 > 0$, the parameter $\gamma_\varepsilon > 0$ and the arbitrary parameter $\varepsilon_1 > 0$ satisfies the condition: $\varepsilon_1 > l/k_*$.

Proof. $\delta\tilde{\varphi}(x,t,f) := \tilde{\varphi}(x,t,f+\delta f) - \tilde{\varphi}(x,t,f)$ and $\tilde{\varphi} \in V^{1,0}(\Omega_T)$ and solution of the problem below.

$$\begin{cases} \delta\tilde{\varphi}_t = -\nabla(\tilde{k}(x,t)\nabla\delta\tilde{\varphi}), & (x,t) \in \Omega_T \\ -\tilde{k}(x,t)\dfrac{\partial\delta\tilde{\varphi}}{\partial n} = 2\delta\tilde{u}(x,t;\delta f), & (x,t) \in \Gamma_1^T, \\ -\tilde{k}(x,t)\dfrac{\partial\delta\tilde{\varphi}}{\partial n} = 2\delta\tilde{u}(x,t;\delta f), & (x,t) \in \Gamma_2^T \\ \delta\tilde{\varphi}(x,t) = 0, & (x,t) \in \Gamma_3^T \cup \Gamma_4^T \\ \delta\tilde{\varphi}(x,T) = 0, & x \in \Omega. \end{cases}$$

Multiplying both sides of the parabolic equation $\delta\tilde{\varphi}(x,t,f)$, integrating on Ω_T and using the initial and boundary conditions we obtain the following energy identity

$$\int_{\Omega_T} \tilde{k}(x,t)|\nabla\delta\tilde{\varphi}(x,t,f)|^2 dxdt + \int_{\Omega} [\delta\tilde{\varphi}(x,0;f)]^2 dx$$
$$= 2\int_{\Gamma_1^T} \delta\tilde{u}(x,t;\delta f)\delta\tilde{\varphi}(x,t;f)dxdt + 2\int_{\Gamma_2^T} \delta\tilde{u}(x,t;\delta f)\delta\tilde{\varphi}(x,t;f)dxdt.$$

Using here the energy inequality, we conclude:

$$k_* \int_{\Omega_T} |\nabla\delta\tilde{\varphi}(x,t,f)|^2 dxdt \le 2\int_{\Gamma_1^T} \delta u(x,t;f)\delta\tilde{\varphi}(x,t;f)dxdt + 2\int_{\Gamma_2^T} \delta u(x,t;f)\delta\tilde{\varphi}(x,t;f)dxdt.$$

By using Poincare inequality, we prove the theorem.

Theorem 4.2. *For any initial data $f_0(x,t) \in \mathcal{W}$ the sequence of iterations $\{f_n(x,t)\} \subset \mathcal{W}$, given by $f_{n+1} = f_n - \alpha_n \nabla\tilde{J}(f_n)$, converges to a quasisolution $\{f_*\} \in \mathcal{W}$ of (HFIP) in the norm of $H^0(\Gamma_1^T \times \Gamma_2^T)$. The sequence of functionals $(\tilde{J}(f_n))$ is a monotone decreasing and convergent one. Moreover, for the rate of convergence the following estimate holds* [4]*:*

$$0 \le \tilde{J}(f^{(n)}) - \tilde{J}(f_*) \le 2L_1 d^2 n^{-1}, \quad d > 0,\ n = 0,1,2,\ldots,$$

where $L_1 > 0$ is the Lipschitz constant defined in Theorem 4.1. Indeed, if $\tilde{J}(f) \in C^{1,1}(\mathcal{W})$ and $\mathcal{W}$ is a closed convex set, then $f_* \in \mathcal{W}_*$ iff the variational inequality holds

$$\langle \nabla\tilde{J}(f_*), f - f_* \rangle \ge 0, \quad \forall f \in \mathcal{W},$$

where $\mathcal{W}_* \subset \mathcal{W}$ is the set of solutions of the minimization problem $\tilde{J}(f_*) = \min_{f\in\mathcal{W}} \tilde{J}(f)$. On the other hand, if the functional $\tilde{J}(f) \in C^{1,1}(\mathcal{W})$ is strictly convex, then the operator $\nabla\tilde{J} : \mathcal{W} \mapsto H^0$ is strictly monotone, i.e.

$$\langle \nabla\tilde{J}(f+\delta f) - \nabla\tilde{J}(f), \delta f \rangle > 0, \quad \forall f,\ f+\delta f \in \mathcal{W}.$$

In this case the above minimization problem has a unique solution.

Theorem 4.3. *The cost functionals $\tilde{J}(f)$, is strictly convex iff the condition,*

$$\int_{\Gamma_1^T} [\tilde{u}(x,t;f_{1i}) - \tilde{u}(x,t;f_{1j})]^2 dxdt + \int_{\Gamma_2^T} [\tilde{u}(x,t;f_{2i}) - \tilde{u}(x,t;f_{2j})]^2 dxdt > 0$$

holds for all $f_i := (f_{1i}, f_{2i}) \in \mathcal{W}, f_{1i} \ne f_{2i}$ where the functions $u(x,t;f_m)$ are the solutions of corresponding direct problem for the given flux functions $f_m := (f_{1m}, f_{2m}),\ f_m \in \mathcal{W},\ m = 1,2$. These inequalities can be used as unicity criterions as well as for degrees of ill-posedness of the considered (HFIP).

Lemma 4.1. *Let the coefficient $k = k(\xi)$ satisfies condition (1.4). Then the approximate solution $u^{(n)}(x,t) \in \mathring{V}^{1,0}(\Omega_T)$ defined by the iteration scheme (3.1) of the nonlinear problem (1.1) converges to the unique exact solution in $\mathring{V}^{1,0}(\Omega_T)$ norm. Thus cost functional of the linearized inverse problem $\tilde{J}(f)$ converges to cost functional of the nonlinear inverse problem $J(f)$.*

Proof.

$$\|\tilde{J}(f) - J(f)\| = \int_{\Gamma_1^T} [\tilde{u}(x,t;f) - h_1(x,t)]^2 - [u^{(n)}(x,t;f) - h_1(x,t)]^2 dxdt$$
$$+ \int_{\Gamma_2^T} [\tilde{u}(x,t;f) - h_2(x,t)]^2 - [u^{(n)}(x,t;f) - h_2(x,t)]^2 dxdt$$

$$= \int_{\Gamma_1^T} (\tilde{u}(x,t;f) - u^{(n)}(x,t;f))(\tilde{u}(x,t;f) + u^{(n)}(x,t;f) - 2h_1(x,t))dxdt$$
$$+ \int_{\Gamma_2^T} (\tilde{u}(x,t;f) - u^{(n)}(x,t;f))(\tilde{u}(x,t;f) + u^{(n)}(x,t;f) - 2h_2(x,t))dxdt. \quad (4.1)$$

By using the $u^n, u \in \mathring{V}^{1,0}(\Omega_T)$ and $h \in L_2(\Gamma_1^T \times \Gamma_2^T)$ we prove the lemma.

5. Numerical analysis of CGA and optimal choice of the stopping parameter

Any numerical method for an inverse problem related to differential equations requires, first of all, construction of an optimal computational mesh for the corresponding direct problem. This mesh needs to be fine enough in order to obtain an accurate numerical solution of the direct problem, on the one hand. On the other hand, an implementation of any iterative method for inverse problems requires solving the forward problem efficiently in each iteration step. Hence minimum mesh size restrictions need to be taken into account in numerical solution of the direct problem. In particular, the conjugate gradient method (CGM) will be used for numerical solution of (HFIP). This method assumes numerical solution of the direct and the corresponding adjoint problems at each nth step of the (CGM) iteration process. The standard monotone finite-difference scheme is used here for the numerical solution of the linearized direct problem (3.2) [5]:

The number n of iterations for which the stopping condition $e(n;\gamma;f) := \|\Phi(f) - h\|_{L_2(\mathcal{W})} < \varepsilon_J$ holds, will be defined below as *the optimal number of iterations* (n_{opt}) corresponding to the stopping parameter $\varepsilon_J > 0$ of CGA. This parameter is defined from the well-known Morozov's discrepancy principle [6]:

$$\mu_1\delta \le e(n;\alpha;\gamma) \le \mu_2\delta.$$

Here $\mu_2 \ge \mu_1 > 1$, and $\delta > 0$ is the parameter in $\|h^\gamma - h\|_{L^2(\mathcal{H})} < \delta$, where h and h^γ are the noise free and noisy measured output data, γ is noise level. In the computational examples the parameters are taken as $\mu_1 = \mu_2 = 1.1$.

In this section we will demonstrate a performance comparison of the CGM, described in [7] for the inverse problems. Then we will estimate an optimal value $\varepsilon_J^{opt} > 0$ of the stopping parameter $\varepsilon_J > 0$ in the stopping condition $e(n;\gamma;f) < \varepsilon_J$. This estimate will be based on an analysis of the dependence on the iteration number n of the *convergence error* $e(n;\cdot;\gamma)$ and the *accuracy error* $E(n;\cdot;\gamma)$, defined as follows:

$$e(n;\gamma;f) := \|\Phi(f) - h\|_{L_2(\mathcal{W})}, \qquad E(n;\gamma;f) := \|f - f_n\|_{L_2(\mathcal{W})}.$$

Based on this analysis we will estimate an optimal number of iterations n_{opt} in CGM.

Assuming $u(x,t)$ and $u_h(x,t)$ the exact and numerical solutions of the direct problem (1.1) for a given flux functions $(f_1(x,t), f_2(x,t)) \in \mathcal{W}$, we define the parameter

$$\varepsilon_u^{(1)} = \|u(x,t)|_{\Gamma_1^T} - u_h(x,t)|_{\Gamma_1^T}\|_{L_{2h}(\Gamma_1^T)},$$
$$\varepsilon_u^{(2)} = \|u(x,t)|_{\Gamma_2^T} - u_h(x,t)|_{\Gamma_2^T}\|_{L_{2h}(\Gamma_2^T)}$$

to be as the *computational noise levels* for the problem (HFIP). Here $\|\cdot\|_{L_{2h}}$ is the discrete analogue of the L_2-norm $\|\cdot\|_{L_2}$. To estimate the computational noise levels for the continuous conductivity case we consider the following example. In the computational experiments here and below $l_{x_1} = l_{x_2} = 1,\ T = 1$.

Example 5.1. The function $u(x_1,x_2,t) = \exp(-t)\sin(\pi x_1)\sin(\pi x_2)$, $x = (x_1,x_2) \in [0,l_{x_1}] \times [0,l_{x_2}]$, $t \in [0,T]$, is the exact solution of the direct problem (1.1), with the thermal conductivity $k(\xi) = 1/(1+\xi)$, $\xi = |\nabla u|^2$, and the initial data $u_0(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2)$. The corresponding flux functions are obtained from the formulas $f_1(x,t) = -k(\xi)\frac{\partial u}{\partial n}(x,t),\ (x,t) \in \Gamma_1^T, f_2(x,t) = -k(\xi)\frac{\partial u}{\partial n}(x,t),\ (x,t) \in \Gamma_2^T$ using the exact solution $u(x,t)$. For continuous thermal conductivity $k(\xi)$ the computational noise levels, corresponding to (HFIP), obtained by the standard finite-difference scheme and corresponding to various mesh parameters, are given in the fourth and fifth columns of the Table 1. The minimal mesh size when these levels are of order 10^{-1}, is $N_{x_1} \times N_{x_2} \times N_t = 21 \times 21 \times 151$. For this reason, for the continuous thermal conductivity this mesh will be defined as an *optimal mesh*, and will be used in subsequent inversion algorithms.

Example 5.2. We consider following synthetic data for estimate numerical solution of (HFIP).

$$\begin{cases} k(\xi) = 1/(1+\xi), \quad \xi = |\nabla u|^2 \\ u_0(x_1,x_2) = 0, \qquad F(x_1,x_2,t) = \sin(3\pi x_1), \quad (x_1,x_2) \in (0,l_{x_1}) \times (0,l_{x_2}),\ l_{x_1} = l_{x_2} = 1 \\ f_1(x_1,t) = 1.5\pi e^{-t}\sin(3\pi x_1)(t-t^2), \quad (x_1,t) \in \Gamma_1^T = (0,l_{x_1}) \times (0,T),\ T = 1, \\ f_2(x_2,t) = 1.5\pi e^{-t}\sin(2\pi x_2)(t-t^2), \quad (x_2,t) \in \Gamma_2^T = (0,l_{x_2}) \times (0,T). \end{cases}$$

The synthetic noise free output data $h(x,t) := (\tilde{u}_h(x,t;f)|_{\Gamma_1^T}, \tilde{u}_h(x,t;f)|_{\Gamma_2^T})$ is generated from the numerical solution $\tilde{u}_h(x,t;f)$ of the direct problem (1.1). Then the random noisy output data $h^\gamma(x,t) := h(x,t) + \gamma\ rand(h)\|h\|_\infty$is obtained

Table 1
Computational noise levels for continuous thermal conductivity $k(\xi) = 1/(1+\xi)$.

$N_{x_1} \times N_{x_2} \times N_t$	$h_{x_1} \times h_{x_2}$	τ	$\varepsilon_u^{(1)}$	$\varepsilon_u^{(2)}$
$21 \times 21 \times 21$	0.05×0.05	0.05	0.455×10^{-1}	0.421×10^{-1}
$21 \times 21 \times 51$	0.05×0.05	0.02	0.386×10^{-1}	0.375×10^{-1}
$21 \times 21 \times 101$	0.05×0.05	0.01	0.294×10^{-1}	0.266×10^{-1}
$21 \times 21 \times 151$	0.05×0.05	0.0067	0.269×10^{-1}	0.187×10^{-1}

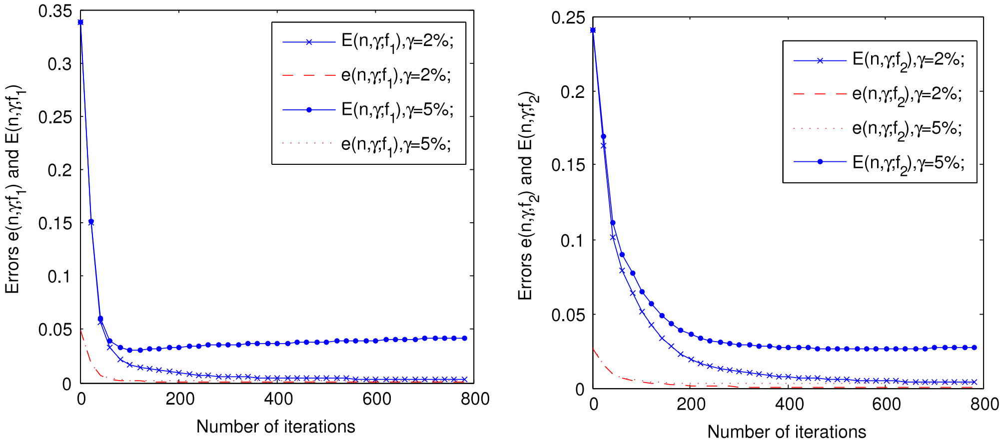

Fig. 1. The convergence error $e(n; \gamma; f)$ and the accuracy error $E(n; \gamma; f)$, depending on the *iteration number* n of CGA.

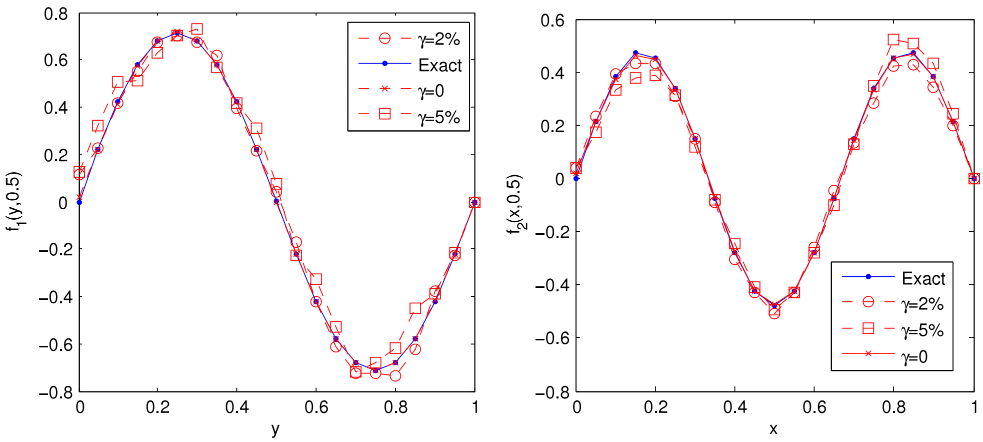

Fig. 2. Exact, noise-free and noisy numerical solutions of (HFIP) for noisy data $\gamma = 2\%$, $\gamma = 5\%$.

for (HFIP), using the MATLAB "randn" function, which generates arrays of random numbers whose elements are normally distributed with mean 0 and standard deviation $\sigma = 1$. Here $\gamma > 0$ is the noise level with the noise levels $\gamma = 2\%$ and $\gamma = 5\%$. The numerical results obtained by CGA are plotted in Figs. 1 and 2. The initial iteration $f_0(x, t)$ is taken as the zero function for all $(x, t) \in \Gamma_1^T \times \Gamma_2^T$. The accuracy errors corresponding to these noise levels are obtained as $E(n; \gamma; f) = 0.0754 - 0.086$ and $E(n; \gamma; f) = 0.1669 - 0.1817$ after $n_{opt} = 800$ iterations, respectively. The performance characteristics of CGA in the reconstruction of the fluxes $f_1(x, t)$ and $f_2(x, t)$ are given in Table 2. These results show that the reconstructions obtained by the algorithm of CGM is acceptable.

6. Conclusions

The aim of this article was to demonstrate an implementation of the adjoint problem approach in the mathematical analysis of heat flux identification problems for nonlinear parabolic equations. The presented approach permits one to prove monotonicity, Lipschitz continuity, and hence invertibility of input–output mapping of inverse problem. Note that such type of result cannot be obtained either from the output least-squares approach or from equation error techniques.

Table 2
The performance characteristics of CGA in the reconstruction of the fluxes $f_1(x,t)$ and $f_2(x,t)$.

Noise level	$\|f_1-f_{1h}\|_\infty$	$\frac{\|f_1-f_{1h}\|_\infty}{\|f_1\|_\infty}$	$\|f_1-f_{1h}\|_{L_2}$	$\frac{\|f_1-f_{1h}\|_{L_2}}{\|f_1\|_{L_2}}$
Noise free	0.1268×10^{-1}	0.1152×10^{0}	0.237×10^{-1}	0.93×10^{-1}
2%	0.875×10^{-1}	0.2506×10^{0}	0.285×10^{-1}	0.754×10^{-1}
5%	0.2310×10^{0}	0.3043×10^{0}	0.63×10^{-1}	0.1669×10^{0}
Noise level	$\|f_2-f_{2h}\|_\infty$	$\frac{\|f_2-f_{2h}\|_\infty}{\|f_2\|_\infty}$	$\|f_2-f_{2h}\|_{L_2}$	$\frac{\|f_2-f_{2h}\|_{L_2}}{\|f_2\|_{L_2}}$
Noise free	0.264×10^{-1}	0.463×10^{-1}	0.91×10^{-2}	0.366×10^{-1}
2%	0.796×10^{-1}	0.1572×10^{0}	0.214×10^{-1}	0.861×10^{-1}
5%	0.1769×10^{0}	0.3494×10^{0}	0.451×10^{-1}	0.1817×10^{0}

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