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Exceptional sets of the Oppenheim expansions over the field of formal Laurent series

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ABSTRACT

Let $\mathbb{F}_q$ be a finite field with q elements, $\mathbb{F}_q((z^{-1}))$ denote the field of all formal Laurent series with coefficients in $\mathbb{F}_q$ and I be the valuation ideal of $\mathbb{F}_q((z^{-1}))$. For any formal Laurent series $x = \sum_{n=\nu}^{\infty} c_n z^{-n} \in I$, the series $\frac{1}{a_1(x)} + \sum_{n=1}^{\infty} \frac{r_1(a_1(x))\cdots r_n(a_n(x))}{s_1(a_1(x))\cdots s_n(a_n(x))} \frac{1}{a_{n+1}(x)}$ is the Oppenheim expansion of x. Suppose $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. In this paper, we quantify the size, in the sense of Hausdorff dimension, of the set

$$E(\phi) = \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\phi(n)} = 1 \right\},$$

where $\Delta_0(x) = \deg a_1(x)$ and $\Delta_n(x) = \deg a_{n+1}(x) - 2\deg a_n(x) - \deg r_n(a_n(x)) + \deg s_n(a_n(x))$ for all $n \geq 1$. As applications, we investigate the cases when $\phi(n)$ are the given polynomial or exponential functions. At the end of the article, we list some special cases (including Lüroth, Engel, Sylvester

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expansions of Laurent series and Cantor infinite products of Laurent series) to which we apply the conclusions above.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and $\mathbb{F}_q((z^{-1}))$ denote the field of all formal Laurent series with coefficients in $\mathbb{F}_q$. Recall that $\mathbb{F}_q[z]$ denote the ring of polynomials in z with coefficients in $\mathbb{F}_q$.

For each $x = \sum_{n=\nu}^{\infty} c_n z^{-n} \in \mathbb{F}_q((z^{-1}))$, call $[x] = \sum_{\nu \le n \le 0} c_n z^{-n} \in \mathbb{F}_q[z]$ the integral part of x and $\deg x = -\inf\{n \in \mathbb{Z} : c_n \neq 0\}$ the degree of x, with the convention that $\deg 0 = -\infty$. Define the absolute value on $\mathbb{F}_q((z^{-1}))$ as $\|x\| = q^{\deg x}$ which is a non-Archimedean absolute value. The field $\mathbb{F}_q((z^{-1}))$ is locally compact and complete under the metric $\rho(x, y) = \|x - y\|$.

Denote $I = \{x \in \mathbb{F}_q((z^{-1})) \colon \|x\| < 1\} = \{x = \sum_{n=1}^{\infty} c_n z^{-n} \colon c_n \in \mathbb{F}_q\}$, which is the valuation ideal of $\mathbb{F}_q((z^{-1}))$. Let P be the Haar measure on $\mathbb{F}_q((z^{-1}))$ normalized to 1 on I.

Now we recall the Oppenheim expansions of Laurent series which is introduced by A. Knopfmacher and J. Knopfmacher [5,6].

Let $\{r_n\}_{n\ge1}$ and $\{s_n\}_{n\ge1}$ be two sequences of nonzero polynomials over $\mathbb{F}_q$ satisfying the following hypothesis

$$\deg s_n - \deg r_n \le 2, \quad \text{for all } n \ge 1. \tag{1.1}$$

Given $x \in \mathbb{F}_q((z^{-1}))$, put $a_0(x) = [x]$, then define $A_1(x) = x - a_0(x)$. Suppose $A_n(x)$ $(n \ge 1)$ is defined. If $A_n(x) \neq 0$, then let $a_n(x) = \left[\frac{1}{A_n(x)}\right]$ and define

$$A_{n+1}(x) = \left(A_n(x) - \frac{1}{a_n(x)}\right) \frac{s_n(a_n(x))}{r_n(a_n(x))}, \tag{1.2}$$

where $s_n(a_n(x))$ and $r_n(a_n(x))$ denote the composition of polynomials. If $A_n(x) = 0$, this recursive process stops. We call $\{a_n(x)\}$ the digits of x. It was shown in [5,6] that the above algorithm leads to an unique finite or infinite series which converges to x (respect to ρ). This series (see below) is called the Oppenheim expansion of Laurent series of x.

Theorem 1.1 *([5,6]). Every $x \in \mathbb{F}_q((z^{-1}))$ has an unique finite or infinite convergent (respect to ρ) expansion of the form*

$$x = a_0(x) + \frac{1}{a_1(x)} + \sum_{n=1}^{\infty} \frac{r_1(a_1(x)) \cdots r_n(a_n(x))}{s_1(a_1(x)) \cdots s_n(a_n(x))} \frac{1}{a_{n+1}(x)},$$

where $a_n(x) \in \mathbb{F}_q[z]$, $a_0(x) = [x]$, *and* $\deg a_1(x) \geq 1$, *for any* $n \geq 1$,

$$\deg a_{n+1}(x) \geq 2 \deg a_n(x) + 1 + \deg r_n(a_n(x)) - \deg s_n(a_n(x)).$$

Here are some special cases which were extensively studied:

- Lüroth expansion: $s_n(z) = z(z-1)$, $r_n(z) = 1$;
- Engel expansion: $s_n(z) = z$, $r_n(z) = 1$;
- Sylvester expansion: $s_n(z) = 1$, $r_n(z) = 1$;
- Cantor infinite product: $s_n(z) = z$, $r_n(z) = z + 1$.

Denote

$$\Delta_0(x) := \deg a_1(x),$$
$$\Delta_n(x) := \deg a_{n+1}(x) - 2 \deg a_n(x) - \deg r_n(a_n(x)) + \deg s_n(a_n(x)) \quad (n \geq 1),$$

then $\{\Delta_n(x)\}_{n=0}^{\infty}$ is strictly increasing sequence (see Lemma 4 in [2]).

Ai-Hua Fan and Jun Wu in [1] proved that

Theorem 1.2 *([1]). Suppose that the hypothesis (1.1) is satisfied. For P-almost all $x \in I$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Delta_j(x) = \frac{q}{q-1}.$$

Theorem 1.2 implies that almost surely, the sum of $\Delta_j(x)$ grows linearly.

By Theorem 1.2 and noting that $\Delta_j(x) \geq 1$, Ai-Hua Fan and Jun Wu also consider the following exceptional sets

$$A(\alpha) = \left\{ x \in I : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Delta_j(x) = \alpha \right\} \quad (\alpha \geq 1).$$

Suppose that there exists an integer $L \leq 2$ such that

$$\deg s_n - \deg r_n = L, \quad \text{for any } n \geq 1. \tag{1.3}$$

Under the hypothesis (1.3), Ai-Hua Fan and Jun Wu calculated the Hausdorff dimensions of $A(\alpha)$.

Theorem 1.3 *([1]). Suppose that the hypothesis (1.3) is satisfied:*

(1) *If $L \leq 1$, we have $\dim_{\mathrm{H}} A(\alpha) = 1$ for all $\alpha \geq 1$.*

(2) *If* $L = 2$, *we have* $\dim_{\mathrm{H}} A(\alpha) = f(q,\alpha)$ *with*

$$f(q,\alpha) = \frac{\alpha \log(\alpha q(q-1)) + (1-\alpha)\log((\alpha-1)(q-1))}{2\alpha \log q}.$$

Here $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension.*

In this paper, we would like to know what happens when the sum of $\Delta_j(x)$ grows with a general functional rate, namely, we consider the size of the following set

$$E(\phi) = \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\phi(n)} = 1 \right\}$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. We prove that

Theorem 1.4. *Suppose that the hypothesis (1.3) is satisfied and* $E(\phi) \neq \emptyset$.

(1) *If* $L \leq 1$, *we have*

$$\dim_{\mathrm{H}} E(\phi) \geq \frac{1}{1+\gamma}, \quad \text{with } \gamma = \limsup_{n\to\infty} \frac{\phi(n+1)}{\sum_{j=1}^{n} (2-L)^{n-j} \phi(j)}.$$

(2) *If* $L = 2$, *we have*

$$\dim_{\mathrm{H}} E(\phi) = \frac{1}{1+b}, \quad \text{with } b = \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(n)}.$$

As applications, we obtain the following theorem.

Theorem 1.5. *Suppose that the hypothesis (1.3) is satisfied and the following sets are nonempty.*

(1) *If* $L \leq 0$, *for any* $\tau > 2 - L$ *and* $\xi > 0$, *we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\xi \tau^n} = 1 \right\} \geq \frac{1}{L + \tau - 1};$$

(2) *If* $L = 1$, *for any* $\beta > 1$ *and* $\alpha > 0$, *we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\alpha n^{\beta}} = 1 \right\} = 1;$$

for any $\tau > 1$ *and* $\xi > 0$, *we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\xi \tau^n} = 1 \right\} \geq \frac{1}{\tau};$$

(3) *If $L = 2$, for any $\beta > 1$ and $\alpha > 0$, we have*

$$\dim_{\mathrm{H}}\left\{x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\alpha n^{\beta}} = 1\right\} = \frac{1}{2};$$

for any $\tau > 1$ and $\xi > 0$, we have

$$\dim_{\mathrm{H}}\left\{x \in I : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta_j(x)}{\xi \tau^{n}} = 1\right\} = \frac{1}{\tau + 1}.$$

At the end of this section, we give a remark on ϕ. Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be a function such that $\phi(n)/n \to \infty$ as $n \to \infty$. If $E(\phi) \neq \emptyset$, then there exists an $x_0 \in E(\phi)$, define $\bar{\phi}(n) = \sum_{j=0}^{n-1} \Delta_j(x_0)$ for all $n \geq 1$. Obviously, we have $\bar{\phi}(n)/\phi(n) \to 1$ as $n \to \infty$, so $E(\phi) = E(\bar{\phi})$. Hence, in what follows, we can always assume that $\phi\colon \mathbb{N} \to \mathbb{N}$ and $\phi(n+1) - \phi(n) \geq 1$ once $E(\phi)$ is non-empty.

2. Preliminary

We first collect some basic properties possessed by the Oppenheim expansions of Laurent series, see [1,2].

A finite sequence of polynomials $\{k_1, k_2, \ldots, k_n\} \subset \mathbb{F}_q[z]$ is said to be *admissible* if it satisfies the *admissibility condition*

$$\deg k_1 \geq 1, \quad \deg k_{j+1} \geq 2\deg k_j + 1 + \deg r_j(k_j) - \deg s_j(k_j) \quad (1 \leq j \leq n-1).$$

We similarly define the admissibility for an infinite sequence of polynomials. It is clear that the digital sequences $\{a_n(x)\}_{n\geq 1}$ of the formal Laurent series are just all possible admissible sequences.

We use $|\cdot|$ to denote the diameter of a set.

Lemma 2.1 *([1,2]). Suppose that $\{k_1, k_2, \cdots, k_n\} \subset \mathbb{F}_q[z]$ $(n \geq 1)$ is an admissible sequence. Call*

$$J(k_1, k_2, \cdots, k_n) = \{x \in I : a_1(x) = k_1, a_2(x) = k_2, \cdots, a_n(x) = k_n\}$$

an n-th order cylinder, then the set $J(k_1, k_2, \cdots, k_n)$ is a disk with center $\frac{1}{k_1} + \sum\limits_{j=2}^{n} \frac{r_1(k_1)\cdots r_{j-1}(k_{j-1})}{s_1(k_1)\cdots s_{j-1}(k_{j-1})} \frac{1}{k_j}$, and diameter

$$|J(k_1, k_2, \cdots, k_n)| = q^{\sum_{j=1}^{n-1}(\deg r_j(k_j) - \deg s_j(k_j)) - 2\deg k_n - 1}.$$

Remark. Since the metric ρ is non-Archimedean, each point of a disk may be considered as the center of the disk. Thus if two disks intersect, then one must contain the other.

The lower bound of $\dim_H E(\phi)$ is obtained by estimating the Hausdorff dimension of a homogeneous Moran subset of $E(\phi)$. Now we recall the definition and a basic dimensional result of the homogeneous Moran set, see [3,4,7,8] for details.

Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $\{c_k\}_{k\geq 1}$ be a sequence of positive numbers satisfying $n_k \geq 2$, $0 < c_k < 1$, $n_1c_1 \leq \delta$ and $n_kc_k \leq 1$ $(k \geq 2)$, where δ is some positive number.

Let

$$D = \bigcup_{k\geq 0} D_k, D_0 = \{\emptyset\}, D_k = \{(i_1, \cdots, i_k)\colon 1 \leq i_j \leq n_j, 1 \leq j \leq k\}.$$

If $\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k$, $\tau = (\tau_1, \cdots, \tau_m) \in D_m$, we define the concatenation of σ and τ as

$$\sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m).$$

Let (X, d) be a metric space, suppose that $J \subset X$ is a closed subset, with diameter $\delta > 0$. A collection $\mathfrak{F} = \{J_\sigma : \sigma \in D\}$ of closed subsets of J is said to have a homogeneous Moran structure if it satisfies:

(1) $J_\emptyset = J$;
(2) For any $k \geq 1$ and $\sigma \in D_{k-1}, J_{\sigma*1}, J_{\sigma*2}, \cdots, J_{\sigma*n_k}$ are subsets of J_σ and $\mathrm{int}(J_{\sigma*i}) \bigcap \mathrm{int}(J_{\sigma*j}) = \emptyset$ $(i \neq j)$, where $\mathrm{int}A$ denotes the interior of A;
(3) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $1 \leq j \leq n_k$, we have

$$\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_k.$$

If $\mathfrak{F}$ is such a collection, $E := \bigcap_{k\geq 1} \bigcup_{\sigma\in D_k} J_\sigma$ is called a homogeneous Moran set determined by $\mathfrak{F}$.

Lemma 2.2 *([7]). For the above defined homogeneous Moran set, we have*

$$\dim_H E \geq \liminf_{k\to\infty} \frac{\log n_1n_2\cdots n_k}{-\log c_1c_2\cdots c_{k+1}n_{k+1}}.$$

3. Proof of theorem

Proof of Theorem 1.4. We divide the proof into two parts.

Part I: $L \leq 1$:

We define a sequence of integers $\{q_n, n \geq 1\}$ as follows:

$$q_1 = \phi(1), \quad q_{n+1} = (2 - L)q_n + \phi(n + 1) - \phi(n) \quad (\forall n \geq 1).$$

Then let

$$D_n = \{\sigma\colon \sigma = (b_1, b_2, \cdots, b_n) \in \mathbb{F}_q[z]^n,\ \deg b_k = q_k,\ 1 \le k \le n\}.$$

We first show that for any $(b_1, b_2, \cdots, b_n) \in D_n$, the cylinder $J(b_1, b_2, \cdots, b_n)$ is a nonempty disk.

By the condition (1.3), for any $a, b \in \mathbb{F}_q[z]$ with positive degree, we have

$$\deg b - 2\deg a + \deg s_n(a) - \deg r_n(a) = \deg b - (2 - L)\deg a. \tag{3.1}$$

By (3.1), we can check that $\{b_1, b_2, \cdots, b_n\}$ is admissible. In fact,

$$\begin{aligned}
\deg b_1 &= \phi(1) \ge 1,\\
\deg b_{j+1} &= q_{j+1} = (2 - L)q_j + \phi(j + 1) - \phi(j)\\
&\ge (2 - L)q_j + 1\\
&= 2\deg b_j + \deg r_j(b_j) - \deg s_j(b_j) + 1 \quad (1 \le j \le n).
\end{aligned}$$

Thus, by Lemma 2.1, $J(b_1, b_2, \cdots, b_n)$ is the disk with center $\frac{1}{b_1} + \sum\limits_{j=2}^{n} \frac{r_1(b_1)\cdots r_{j-1}(b_{j-1})}{s_1(b_1)\cdots s_{j-1}(b_{j-1})}\frac{1}{b_j}$, and diameter

$$q^{\sum\limits_{j=1}^{n-1}(\deg r_j(b_j) - \deg s_j(b_j)) - 2\deg b_n - 1} = q^{-L\sum\limits_{j=1}^{n-1} q_j - 2q_n - 1}.$$

Put

$$E_0 = I, E_n = \bigcup_{(b_1, b_2, \cdots, b_n)\in D_n} J(b_1, b_2, \cdots, b_n), \quad \forall\ n \ge 1.$$

Define

$$E = \bigcap_{n=1}^{+\infty} E_n.$$

From the definitions of D_n and q_n, it is easy to check that

$$E_n = \{x \in I : \Delta_0(x) = \phi(1), \cdots, \Delta_{n-1}(x) = \phi(n) - \phi(n - 1)\},$$

so that

$$E = \big\{x \in I : \Delta_0(x) = \phi(1), \Delta_n(x) = \phi(n + 1) - \phi(n) \text{ for all } n \ge 1\big\}.$$

Thus $E \subset E(\phi)$.

Take $n_k = (q-1)q^{q_k}$ and $c_1 = q^{-2q_1-1}$, $c_k = q^{(2-L)q_{k-1}-2q_k}$ for all $k \geq 2$. From the above structure, it follows that each component $J(b_1, b_2, \cdots, b_{k-1})$ in E_{k-1} contains n_k many elements $J(b_1, b_2, \cdots, b_k)$ in E_k with the same ratio c_k. Thus E is a standard homogeneous Moran set.

By the definition of q_n, we have

$$\sum_{j=1}^{n} q_j = (2-L)\sum_{j=1}^{n-1} q_j + \phi(n).$$

This implies

$$q_n = (1-L)\left(\sum_{j=1}^{n-1} q_j\right) + \phi(n), \tag{3.2}$$

and

$$\sum_{j=1}^{n} q_j = \sum_{j=1}^{n} (2-L)^{n-j}\phi(j). \tag{3.3}$$

Thus, by Lemma 2.2, notice that $\phi(n)/n \to \infty$ as $n \to \infty$, and together with (3.2) and (3.3), we have

$$\begin{aligned}
\dim_{\mathrm{H}} E &\geq \liminf_{k\to\infty} \frac{k\log(q-1) + (q_1 + \cdots + q_k)\log q}{-\log(q-1) + L(q_1 + \cdots + q_k)\log q + q_{k+1}\log q} \\
&\geq \liminf_{k\to\infty} \frac{1}{L + \dfrac{q_{k+1}}{q_1 + \cdots + q_k}} \\
&\geq \frac{1}{1 + \limsup\limits_{k\to\infty} \frac{\phi(k+1)}{\sum_{j=1}^{k}(2-L)^{k-j}\phi(j)}} \\
&\geq \frac{1}{1+\gamma}.
\end{aligned}$$

Since $E \subset E(\phi)$, we have

$$\dim_{\mathrm{H}} E(\phi) \geq \frac{1}{1+\gamma}. \qquad \square$$

Part II: $L = 2$:

In this case, we have

$$\Delta_n(x) = \deg a_{n+1}(x), \qquad \Delta_0(x) = \deg a_1(x),$$

and every sequence $\{b_1, b_2, \cdots, b_n\} \subset \mathbb{F}_q[z]$ such that $\deg b_j \geq 1$ $(1 \leq j \leq n)$ is admissible.

Lower bound We first give the lower bound of the Hausdorff dimension. We define a sequence of integer $\{q_n, n \geq 1\}$ as follows:

$$q_1 = \phi(1), \quad q_{n+1} = \phi(n+1) - \phi(n) \quad (\forall n \geq 1).$$

Let

$$D_n = \{\sigma \colon \sigma = (b_1, b_2, \cdots, b_n) \in \mathbb{F}_q[z]^n,\ \deg b_k = q_k,\ 1 \leq k \leq n\}.$$

We know that for any $(b_1, b_2, \cdots, b_n) \in D_n$, the cylinder $J(b_1, b_2, \cdots, b_n)$ is a nonempty disk with center $\frac{1}{b_1} + \sum\limits_{j=2}^{n} \frac{r_1(b_1)\cdots r_{j-1}(b_{j-1})}{s_1(b_1)\cdots s_{j-1}(b_{j-1})} \frac{1}{b_j}$, and diameter

$$q^{\sum\limits_{j=1}^{n-1}(\deg r_j(b_j) - \deg s_j(b_j)) - 2\deg b_n - 1} = q^{-2\sum\limits_{j=1}^{n} q_j - 1}.$$

Put

$$E_0 = I, E_n = \bigcup_{(b_1, b_2, \cdots, b_n) \in D_n} J(b_1, b_2, \cdots, b_n), \quad \forall\ n \geq 1.$$

Define

$$E = \bigcap_{n=1}^{+\infty} E_n.$$

From the definitions of D_n and q_n, it is easy to check that

$$E_n = \{x \in I : \Delta_0(x) = \phi(1), \cdots, \Delta_{n-1}(x) = \phi(n) - \phi(n-1)\},$$

so that

$$E = \big\{x \in I : \Delta_0(x) = \phi(1), \Delta_n(x) = \phi(n+1) - \phi(n) \text{ for all } n \geq 1\big\}.$$

Thus $E \subset E(\phi)$.

Take $n_k = (q-1)q^{q_k}$ and $c_1 = q^{-2q_1 - 1}$, $c_k = q^{-2q_k}$ for all $k \geq 2$. From the above structure, it follows that each component $J(b_1, b_2, \cdots, b_{k-1})$ in E_{k-1} contains n_k many elements $J(b_1, b_2, \cdots, b_k)$ in E_k with the same ratio c_k. Thus E is a standard homogeneous Moran set. By Lemma 2.2, and noticing that $\phi(n)/n \to \infty$ as $n \to \infty$, we have

$$
\begin{aligned}
\dim_{\mathrm{H}} E &\geq \liminf_{k\to\infty} \frac{k\log(q-1)+\phi(k)\log q}{-\log(q-1)+(\phi(k+1)+\phi(k))\log q} \\
&\geq \liminf_{k\to\infty} \frac{1}{1+\frac{\phi(k+1)}{\phi(k)}} \\
&\geq \frac{1}{1+b}.
\end{aligned}
$$

Since $E \subset E(\phi)$, we have

$$\dim_{\mathrm{H}} E(\phi) \geq \frac{1}{1+b}.$$

Upper bound Now we give the upper bound of the Hausdorff dimension. Since $\phi(n)$ is monotonic increasing, we have $b \geq 1$, the following proof is distinguished into three cases according to $1 < b < \infty$, $b = 1$ and $b = \infty$.

Case I: $1 < b < \infty$.

For any $\{b_1, b_2, \cdots, b_n\} \subset \mathbb{F}_q[z]$ such that $\deg b_j \geq 1$ $(1 \leq j \leq n)$, $J(b_1, b_2, \cdots, b_n)$ is a disk with diameter

$$|J(b_1, b_2, \cdots, b_n)| = q^{-2\sum\limits_{j=1}^{n}\deg b_j - 1}.$$

For each $t > 1$, we define a probability measure μ_t on I by letting

$$\mu_t(J(b_1, b_2, \cdots, b_n)) = q^{(-t\sum\limits_{j=1}^{n}\deg b_j - nP(t))}$$

where

$$P(t) = \log_q(q(q-1)) - \log_q(q^t - q).$$

By using the equalities

$$\sum_{b:\deg b\geq 1} q^{-t\deg b} = \sum_{n=1}^{\infty}\sum_{\deg b=n} q^{-tn} = \sum_{n=1}^{\infty} q^{-tn}(q-1)q^n = \frac{q(q-1)}{q^t - q} = q^{p(t)},$$

it is easy to check that

$$\sum_{b_{n+1}:\deg b_{n+1}\geq 1} \mu_t(J(b_1, b_2, \cdots, b_{n+1})) = \mu_t(J(b_1, b_2, \cdots, b_n));$$

$$\sum_{b_1, b_2, \cdots, b_n} \mu_t(J(b_1, b_2, \cdots, b_n)) = 1,$$

where the sum is taken over all b_j such that $\deg b_j \geq 1$. So the measure μ_t is well defined.

Fix $t > 1$, $\epsilon > 0$ such that $b > \frac{1+\epsilon}{(1-\epsilon)^2}$. By the given conditions $\phi(n)/n \to \infty$ as $n \to \infty$, we have

- There exists $N(t,\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(t,\varepsilon)$,

$$nP(t) \leq \varepsilon(1-\varepsilon)\phi(n). \tag{3.4}$$

- We can choose a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ with $n_k \geq N(t,\varepsilon)$ for all $k \geq 1$ and

$$\phi(n_k+1) \geq \phi(n_k)b(1-\varepsilon). \tag{3.5}$$

Now we give a cover of the set $E(\phi)$. For any $n \geq 1$, let

$$\mathfrak{C}_n(\epsilon) = \left\{(b_1, b_2, \cdots, b_n) \in \mathbb{F}_q^n[z], (1-\epsilon) \leq \frac{1}{\phi(n)} \sum_{j=1}^{n} \deg b_j \leq (1+\epsilon)\right\}.$$

For any $(b_1, b_2, \cdots, b_n) \in \mathfrak{C}_n(\epsilon)$, let

$$D_{n+1}(\epsilon, (b_1, b_2, \cdots, b_n)) = \left\{b_{n+1} \in \mathbb{F}_q[z], (b_1, b_2, \cdots, b_{n+1}) \in \mathfrak{C}_{n+1}(\epsilon)\right\},$$

and

$$\mathcal{I}(b_1, b_2, \cdots, b_n) = \bigcup_{b_{n+1} \in D_{n+1}(\epsilon,(b_1,b_2,\cdots,b_n))} J(b_1, b_2, \cdots, b_{n+1}).$$

Then

$$E(\phi) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{(b_1,b_2,\cdots,b_n) \in \mathfrak{C}_n(\epsilon)} \mathcal{I}(b_1, b_2, \cdots, b_n).$$

For each $N \geq 1$, $(b_1, b_2, \cdots, b_{n_k}) \in \mathfrak{C}_{n_k}(\epsilon)$ with $n_k \geq N$, we will estimate the length of $\mathcal{I}(b_1, b_2, \cdots, b_{n_k})$.

For any $b_{n_k+1} \in D_{n_k+1}(\epsilon, (b_1, b_2, \cdots, b_{n_k}))$, by the definition of $D_{n_k+1}(\epsilon, (b_1, b_2, \cdots, b_{n_k}))$ and $\mathfrak{C}_{n_k}(\epsilon)$, together with (3.5), we have

$$\sum_{j=1}^{n_k+1} \deg b_j \geq \phi(n_k+1)(1-\epsilon) \geq \phi(n_k)b(1-\epsilon)^2 \geq \frac{b(1-\epsilon)^2}{1+\epsilon} \sum_{j=1}^{n_k} \deg b_j.$$

Thus

$$\deg b_{n_k+1} \geq \left(\frac{b(1-\epsilon)^2}{1+\epsilon} - 1\right) \sum_{j=1}^{n_k} \deg b_j.$$

Writing $\beta = \frac{b(1-\epsilon)^2}{1+\epsilon} - 1$, by Lemma 2.1, we have

$$
\begin{aligned}
|\mathcal{I}(b_1, b_2, \cdots, b_{n_k})| &\le \sum_{b_{n_k+1}:\deg b_{n_k+1} \ge \beta \sum_{j=1}^{n_k} \deg b_j} |J(b_1, b_2, \cdots, b_{n_k+1})| \\
&\le \sum_{k=\lfloor \beta \sum_{j=1}^{n_k} \deg b_j \rfloor}^{\infty} (q-1) q^{-k} q^{-2\sum\limits_{j=1}^{n_k} \deg b_j - 1} \\
&\le q^2 q^{-(2+\beta)\sum\limits_{j=1}^{n_k} \deg b_j} \\
&\ll |J(b_1, b_2, \cdots, b_{n_k})|^{\frac{2+\beta}{2}}.
\end{aligned}
$$

Subsequently, for each $(b_1, \cdots, b_{n_k}) \in \mathfrak{C}_{n_k}(\epsilon)$, by the definition of $\mathfrak{C}_n(\epsilon)$ and (3.4), we have

$$
\begin{aligned}
|J(b_1, b_2, \cdots, b_{n_k})|^{\frac{t+\epsilon}{2}} &\ll (q^{-2\sum\limits_{j=1}^{n_k} \deg b_j})^{\frac{t+\epsilon}{2}} \\
&\le q^{-t\sum\limits_{j=1}^{n_k} \deg b_j - n_k P(t)} = \mu_t(J(b_1, b_2, \cdots, b_{n_k})).
\end{aligned}
$$

After these preliminaries, we will estimate the $\frac{t+\epsilon}{2+\beta}$-dimensional Hausdorff measure of $E(\phi)$.

$$
\begin{aligned}
\mathcal{H}^{\frac{t+\epsilon}{2+\beta}}(E(\phi)) &\le \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} |\mathcal{I}(b_1, b_2, \cdots, b_{n_k})|^{\frac{t+\epsilon}{2+\beta}} \\
&\ll \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} |J(b_1, b_2, \cdots, b_{n_k})|^{\frac{t+\epsilon}{2}} \\
&\ll \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} \mu_t(J(b_1, b_2, \cdots, b_{n_k})) \\
&\le 1.
\end{aligned}
$$

So, we have $\dim_{\mathrm{H}} E(\phi) \le \frac{t+\epsilon}{2+\beta}$. Letting $\epsilon \to 0$ and $t \to 1$, we obtain

$$\dim_{\mathrm{H}} E(\phi) \le \frac{1}{1+b}.$$

Case II: $b = 1$.

In this case, it can be proved by just applying the natural covering system

$$E(\phi) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{(b_1,b_2,\cdots,b_n)\in\mathfrak{C}_n(\epsilon)} J(b_1, b_2, \cdots, b_n).$$

Then, take the same steps, by the definition of $\mathfrak{C}_{n_k}(\epsilon)$ and (3.4), we have

$$
\begin{aligned}
\mathcal{H}^{\frac{t+\epsilon}{2}}(E(\phi)) &\le \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} |J(b_1,b_2,\cdots,b_{n_k})|^{\frac{t+\epsilon}{2}} \\
&\ll \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} q^{-(t+\epsilon)\sum_{j=1}^{n_k}\deg b_j} \\
&\ll \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} q^{-t\sum_{j=1}^{n_k}\deg b_j - n_k p(t)} \\
&\ll \liminf_{k\to\infty} \sum_{(b_1,b_2,\cdots,b_{n_k})\in\mathfrak{C}_{n_k}(\epsilon)} \mu_t(J(b_1,b_2,\cdots,b_{n_k})) \\
&\le 1.
\end{aligned}
$$

It follows that $\dim_{\mathrm{H}} E(\phi) \le \frac{t+\epsilon}{2}$. Letting $\epsilon \to 0$ and $t \to 1$, we have

$$\dim_{\mathrm{H}} E(\phi) \le \frac{1}{2}.$$

Case III: $b = \infty$.

In this case, we replace b in case I by arbitrary large number m, then we have $\dim_{\mathrm{H}} E(\phi) \le \frac{1}{1+m}$. Letting $m \to \infty$, we get $\dim_{\mathrm{H}} E(\phi) = 0$. □

Proof of Theorem 1.5. Let $\phi(n) = \alpha n^\beta$ and $\xi\tau^n$ respectively. By Theorem 1.4, we get the results of Theorem 1.5. □

We now list some special cases to which we apply Theorem 1.4 and Theorem 1.5. The case of Engel expansions has been studied by Meiying Lü [7]. However, the following results on Lüroh expansions, Sylvester expansions and Cantor infinite products are original.

Example 1. Let $s_n(z) = z(z-1)$, $r_n(z) = 1$ for all $n \ge 1$. Then the algorithm (1.2) leads to a Lüroth expansion of Laurent series of $x \in I$,

$$x = \frac{1}{a_1(x)} + \sum_{n=2}^{\infty} \frac{1}{a_1(x)(a_1(x)-1)\cdots a_{n-1}(x)(a_{n-1}(x)-1)a_n(x)}.$$

In this case, $L = 2$ and $\Delta_n(x) = \deg a_{n+1}(x)$ for all $n \ge 0$. By Theorem 1.4 and Theorem 1.5, we have

Corollary 3.1. *For Lüroth expansion of Laurent series, set*

$$E(\phi) = \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=1}^{n} \deg a_j(x)}{\phi(n)} = 1 \right\}$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. If $E(\phi) \neq \emptyset$, we have

$$\dim_{\mathrm{H}} E(\phi) = \frac{1}{1+b}, \quad \text{with } b = \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(n)}.$$

Corollary 3.2. *Suppose the following sets are nonempty, then:*

- *For any $\beta > 1$ and $\alpha > 0$, we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=1}^{n} \deg a_j(x)}{\alpha n^{\beta}} = 1 \right\} = \frac{1}{2};$$

- *For any $\tau > 1$ and $\xi > 0$, we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\sum_{j=1}^{n} \deg a_j(x)}{\xi \tau^{n}} = 1 \right\} = \frac{1}{\tau+1}. \qquad \square$$

Example 2. Let $s_n(z) = z$, $r_n(z) = 1$ for all $n \geq 1$. Then the algorithm (1.2) leads to a Engel expansion of Laurent series of $x \in I$,

$$x = \sum_{n=1}^{\infty} \frac{1}{a_1(x)a_2(x)\cdots a_n(x)}.$$

In this case, $L = 1$ and $\Delta_0(x) = \deg a_1(x)$, $\Delta_n(x) = \deg a_{n+1}(x) - \deg a_n(x)$ for all $n \geq 1$. By Theorem 1.4 and Theorem 1.5, we have

Corollary 3.3 *([7]). For Engel expansion of Laurent series, set*

$$E(\phi) = \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x)}{\phi(n)} = 1 \right\}$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. If $E(\phi) \neq \emptyset$, we have

$$\dim_{\mathrm{H}} E(\phi) \geq \frac{1}{1+\gamma}, \quad \text{with } \gamma = \limsup_{n\to\infty} \frac{\phi(n+1)}{\sum_{j=1}^{n} \phi(j)}.$$

Corollary 3.4 *([7]). Suppose the following sets are nonempty, then:*

- *For any $\beta > 1$ and $\alpha > 0$, we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x)}{\alpha n^{\beta}} = 1 \right\} = 1;$$

- *For any $\tau > 1$ and $\xi > 0$, we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x)}{\xi \tau^n} = 1 \right\} \geq \frac{1}{\tau}. \qquad \square$$

Example 3. Let $s_n(z) = 1$, $r_n(z) = 1$ for all $n \geq 1$. Then the algorithm (1.2) leads to a Sylvester expansion of Laurent series of $x \in I$,

$$x = \sum_{n=1}^{\infty} \frac{1}{a_n(x)}.$$

In this case, $L = 0$ and $\Delta_0(x) = \deg a_1(x)$, $\Delta_n(x) = \deg a_{n+1}(x) - 2\deg a_n(x)$ for all $n \geq 1$. By Theorem 1.4 and Theorem 1.5, we have

Corollary 3.5. *For Sylvester expansion of Laurent series, set*

$$E(\phi) = \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x) - \sum_{j=1}^{n-1} \deg a_j(x)}{\phi(n)} = 1 \right\}$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. If $E(\phi) \neq \emptyset$, we have

$$\dim_{\mathrm{H}} E(\phi) \geq \frac{1}{1+\gamma}, \quad \text{with } \gamma = \limsup_{n\to\infty} \frac{\phi(n+1)}{\sum_{j=1}^{n} 2^{n-j}\phi(j)}.$$

Corollary 3.6. *Suppose the following set is nonempty, for any $\tau > 2$ and $\xi > 0$, we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x) - \sum_{j=1}^{n-1} \deg a_j(x)}{\xi \tau^n} = 1 \right\} \geq \frac{1}{\tau - 1}. \qquad \square$$

Example 4. Let $s_n(z) = z$, $r_n(z) = z + 1$ for all $n \geq 1$. Then the algorithm (1.2) leads to a Cantor infinite product of Laurent series of $x \in I$,

$$1 + x = \prod_{n=1}^{\infty} \left(1 + \frac{1}{a_n(x)} \right).$$

In this case, $L = 0$ and $\Delta_0(x) = \deg a_1(x)$, $\Delta_n(x) = \deg a_{n+1}(x) - 2\deg a_n(x)$ for all $n \geq 1$. By Theorem 1.4 and Theorem 1.5, we have

Corollary 3.7. *For the Cantor infinite products of Laurent series, set*

$$E(\phi) = \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x) - \sum_{j=1}^{n-1} \deg a_j(x)}{\phi(n)} = 1 \right\}$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. If $E(\phi) \neq \emptyset$, we have

$$\dim_{\mathrm{H}} E(\phi) \geq \frac{1}{1+\gamma}, \quad \textit{with } \gamma = \limsup_{n\to\infty} \frac{\phi(n+1)}{\Sigma_{j=1}^{n} 2^{n-j}\phi(j)}.$$

Corollary 3.8. *Suppose the following set is nonempty, for any $\tau > 2$ and $\xi > 0$, we have*

$$\dim_{\mathrm{H}} \left\{ x \in I : \lim_{n\to\infty} \frac{\deg a_n(x) - \sum_{j=1}^{n-1} \deg a_j(x)}{\xi \tau^n} = 1 \right\} \geq \frac{1}{\tau - 1}. \qquad \square$$

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