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On the asymptotic expansions of products related to the Wallis, Weierstrass, and Wilf formulas

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ABSTRACT

For all integers $n \geq 1$, let

$$W_n(p,q) = \prod_{j=1}^{n} \left\{ e^{-p/j} \left(1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\}$$

and

$$R_n(p,q) = \prod_{j=1}^{n} \left\{ e^{-p/(2j-1)} \left(1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\},$$

where p, q are complex parameters. The infinite product $W_\infty(p, q)$ includes the Wallis and Wilf formulas, and also the infinite product definition of Weierstrass for the gamma function, as special cases. In this paper, we present asymptotic expansions of $W_n(p, q)$ and $R_n(p, q)$ as $n \to \infty$. In addition, we also establish asymptotic expansions for the Wallis sequence.

1. Introduction

The famous Wallis sequence W_n, defined by

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \qquad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

has the limiting value

$$W_\infty = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \frac{\pi}{2} \tag{1.2}$$

established by Wallis in 1655; see [5, p. 68]. Several elementary proofs of this well-known result can be found in [3,23,37]. An interesting geometric construction that produces the above limiting value can be found in Myerson [30]. Many formulas exist for the representation of π, and a collection of these formulas is listed [33,34]. For more history of π see [2,4,5,14].

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The following infinite product definition for the gamma function is due to Weierstrass (see, for example, [1, p. 255, Entry (6.1.3)]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left\{e^{-z/n}\left(1+\frac{z}{n}\right)\right\}, \tag{1.3}$$

where γ denotes the Euler–Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.5772156649\ldots.$$

In 1997, Wilf [39] posed the following elegant infinite product formula as a problem:

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{1}{2j^2}\right)\right\} = \frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}}, \tag{1.4}$$

which contains three of the most important mathematical constants, namely π, e and γ. Subsequently, Choi and Seo [12] proved (1.4), together with three other similar product formulas, by making use of well-known infinite product formulas for the circular and hyperbolic functions and the familiar Stirling formula for the factorial function.

In 2003, Choi et al. [11] presented the following two general infinite product formulas, which include Wilf's formula (1.4) and other similar formulas in Choi and Seo [12] as special cases:

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{\alpha^2+1/4}{j^2}\right)\right\} = \frac{2(e^{\pi\alpha}+e^{-\pi\alpha})}{(4\alpha^2+1)\pi e^{\gamma}} \qquad \left(\alpha\in\mathbb{C};\ \alpha\neq\pm\frac{1}{2}i\right) \tag{1.5}$$

and

$$\prod_{j=1}^{\infty}\left\{e^{-2/j}\left(1+\frac{2}{j}+\frac{\beta^2+1}{j^2}\right)\right\} = \frac{e^{\pi\beta}-e^{-\pi\beta}}{2\beta(\beta^2+1)\pi e^{2\gamma}} \qquad (\beta\in\mathbb{C}\setminus\{0\};\ \beta\neq\pm i), \tag{1.6}$$

where $i=\sqrt{-1}$ and $\mathbb{C}$ denotes the set of complex numbers. In 2013, Chen and Choi [7] presented a more general infinite product formula that included the formulas (1.5) and (1.6) as special cases:

$$\prod_{j=1}^{\infty}\left\{e^{-p/j}\left(1+\frac{p}{j}+\frac{q}{j^2}\right)\right\} = \frac{e^{-p\gamma}}{\Gamma\left(1+\frac{1}{2}p+\frac{1}{2}\Delta\right)\Gamma\left(1+\frac{1}{2}p-\frac{1}{2}\Delta\right)} \tag{1.7}$$

and also another interesting infinite product formula:

$$\prod_{j=1}^{\infty}\left\{e^{-p/(2j-1)}\left(1+\frac{p}{2j-1}+\frac{q}{(2j-1)^2}\right)\right\} = \frac{2^{-p}\pi e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2}+\frac{1}{4}p+\frac{1}{4}\Delta\right)\Gamma\left(\frac{1}{2}+\frac{1}{4}p-\frac{1}{4}\Delta\right)}, \tag{1.8}$$

where $p,\ q\in\mathbb{C}$ and $\Delta := \sqrt{p^2-4q}$.

The formula (1.7) can be seen to include the formulas (1.2)–(1.6) as special cases. By setting $(p,q)=(0,-1/4)$ in (1.7), we have

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{4j^2}\right) = \frac{2}{\pi}, \tag{1.9}$$

whose reciprocal becomes the Wallis product (1.2). Also setting $q=0$ in (1.7), we obtain

$$\prod_{j=1}^{\infty}\left\{e^{-p/j}\left(1+\frac{p}{j}\right)\right\} = \frac{e^{-p\gamma}}{\Gamma(p+1)}. \tag{1.10}$$

Noting that $\Gamma(z+1)=z\Gamma(z)$ and replacing p by z in (1.10) we recover the Weierstrass formula (1.3). Setting $(p,q)=(1,1/2)$ in (1.7) yields the Wilf formula (1.4) and setting

$$(p,q)=\left(1,\alpha^2+\frac{1}{4}\right) \quad \text{and} \quad (p,q)=\left(2,\beta^2+1\right) \tag{1.11}$$

in (1.7) yields the formulas (1.5) and (1.6), respectively.

With $(p,q)=(-1,1/4)$ in (1.7), we obtain the beautiful infinite product formula expressed in terms of the most important constants π, e and γ, namely

$$\prod_{j=1}^{\infty}\left\{e^{1/j}\left(1-\frac{1}{2j}\right)^2\right\} = \frac{e^{\gamma}}{\pi}. \tag{1.12}$$

Also worthy of note are the infinite products that result from setting $(p, q) = (-2, 0)$ and $(p, q) = (2, 0)$ in (1.8) to yield respectively

$$\prod_{j=1}^{\infty} \left\{ e^{2/(2j-1)} \left(1 - \frac{2}{2j-1} \right) \right\} = -2e^{\gamma} \tag{1.13}$$

and

$$\prod_{j=1}^{\infty} \left\{ e^{-2/(2j-1)} \left(1 + \frac{2}{2j-1} \right) \right\} = \frac{1}{2e^{\gamma}}. \tag{1.14}$$

Remark 1.1. The constant e^{γ} is important in number theory and equals the following limit, where p_n is the nth prime number:

$$e^{\gamma} = \lim_{n\to\infty} \frac{1}{\ln p_n} \prod_{j=1}^{n} \frac{p_j}{p_j - 1}.$$

This restates the third of Mertens' theorems (see [38]). The numerical value of e^{γ} is:

$$e^{\gamma} = 1.7810724179\ldots.$$

There is the curious radical representation

$$e^{\gamma} = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1\cdot 3}\right)^{1/3} \left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{1/4} \left(\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}\right)^{1/5} \cdots, \tag{1.15}$$

where the nth factor is

$$\left(\prod_{k=0}^{n} (k+1)^{(-1)^{k+1}\binom{n}{k}} \right)^{1/(n+1)}.$$

The product (1.15), first discovered in 1926 by Ser [32], was rediscovered in [16,35,36].

Recently, Chen and Paris [9] generalized the formula (1.7) to include m parameters $(p_1, \ldots, p_m)$. Subsequently, Chen and Paris [10] considered the asymptotic expansion of products related to generalization of the Wilf problem. However, these authors did not give a general formula for the coefficients in their expansions.

For $n \in \mathbb{N}$, let

$$W_n(p, q) = \prod_{j=1}^{n} \left\{ e^{-p/j} \left(1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\} \tag{1.16}$$

and

$$R_n(p, q) = \prod_{j=1}^{n} \left\{ e^{-p/(2j-1)} \left(1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\}, \tag{1.17}$$

where p and q are complex parameters. In this paper, we present asymptotic expansions of $W_n(p, q)$ and $R_n(p, q)$ as $n \to \infty$, including recurrence relations for the coefficients in these expansions. Furthermore, we establish asymptotic expansions for the Wallis sequence W_n.

2. Asymptotic expansions of $W_n(p, q)$ and $R_n(r, s)$

It was established by Chen and Choi [7] that the finite products $W_n(p, q)$ and $R_n(p, q)$ defined in (1.16) and (1.17) can be expressed in the following closed form

$$W_n(p, q) = \frac{e^{-p(\psi(n+1)+\gamma)}\Gamma\left(n+1+\frac{1}{2}p+\frac{1}{2}\Delta\right)\Gamma\left(n+1+\frac{1}{2}p-\frac{1}{2}\Delta\right)}{\left(\Gamma(n+1)\right)^2\Gamma\left(1+\frac{1}{2}p+\frac{1}{2}\Delta\right)\Gamma\left(1+\frac{1}{2}p-\frac{1}{2}\Delta\right)} \tag{2.1}$$

and

$$R_n(p, q) = \frac{e^{-\frac{p}{2}\left(\psi(n+\frac{1}{2})+\gamma+2\ln 2\right)}\Gamma\left(n+\frac{1}{2}+\frac{1}{4}p+\frac{1}{4}\Delta\right)\Gamma\left(n+\frac{1}{2}+\frac{1}{4}p-\frac{1}{4}\Delta\right)\pi}{\left(\Gamma(n+\frac{1}{2})\right)^2\Gamma\left(\frac{1}{2}+\frac{1}{4}p+\frac{1}{4}\Delta\right)\Gamma\left(\frac{1}{2}+\frac{1}{4}p-\frac{1}{4}\Delta\right)}, \tag{2.2}$$

where $\psi(z)$ denotes the psi (or digamma) function, defined by

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\{\ln\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We observe that allowing $n \to \infty$ in (2.1) and (2.2), respectively, yields (1.7) and (1.8).

Define the function $f(z)$ by

$$f(z) := \frac{e^{-\lambda\psi(z)}\Gamma(z+\mu)\Gamma(z+\nu)}{\big(\Gamma(z)\big)^2}, \tag{2.3}$$

where $\lambda, \mu, \nu \in \mathbb{C}$. It is well known that the logarithm of the gamma function has the asymptotic expansion (see [22, p. 32]):

$$\ln\Gamma(z+a) \sim \left(z+a-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(a)}{n(n+1)}\frac{1}{z^n} \tag{2.4}$$

for $z \to \infty$ in $|\arg z| < \pi$, where $B_n(t)$ denote the Bernoulli polynomials defined by the following generating function:

$$\frac{ze^{tz}}{e^z-1} = \sum_{n=0}^{\infty} B_n(t)\frac{z^n}{n!}.$$

Note that the Bernoulli numbers B_n $(n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\})$ are defined by $B_n := B_n(0)$. The psi function has the asymptotic expansion (see [22, p. 33]):

$$\psi(z) \sim \ln z - \frac{1}{z} - \sum_{j=1}^{\infty}\frac{B_j}{jz^j} \qquad (z\to\infty;\ |\arg z| < \pi). \tag{2.5}$$

Using (2.4) and (2.5), we then find that

$$\ln f(z) \sim (\mu+\nu-\lambda)\ln z + \sum_{j=1}^{\infty}\frac{a_j}{z^j}$$

or

$$f(z) \sim z^{\mu+\nu-\lambda}\exp\left(\sum_{j=1}^{\infty}\frac{a_j}{z^j}\right) \tag{2.6}$$

for $z \to \infty$ in $|\arg z| < \pi$, where the coefficients $a_j \equiv a_j(\lambda, \mu, \nu)$ are given by

$$a_1 = \frac{\lambda + B_2(\mu) + B_2(\nu) - 2B_2}{2}, \quad a_j = \frac{\lambda B_j}{j} + \frac{(-1)^{j+1}\Big(B_{j+1}(\mu) + B_{j+1}(\nu) - 2B_{j+1}\Big)}{j(j+1)} \quad (j \ge 2). \tag{2.7}$$

The choice

$$(\lambda, \mu, \nu) = \left(p,\ \frac{1}{2}p + \frac{1}{2}\Delta,\ \frac{1}{2}p - \frac{1}{2}\Delta\right),$$

where $\mu + \nu - \lambda = 0$, leads to the first few coefficients $a_j(p, q)$ given by:

$$a_1(p,q) = \frac{1}{2}p^2 - q,$$
$$a_2(p,q) = -\frac{1}{6}p^3 + \frac{1}{2}pq + \frac{1}{4}p^2 - \frac{1}{2}q,$$
$$a_3(p,q) = \frac{1}{12}p^4 - \frac{1}{3}p^2q + \frac{1}{6}q^2 - \frac{1}{6}p^3 + \frac{1}{2}pq + \frac{1}{12}p^2 - \frac{1}{6}q.$$

From (2.1) and (2.6), we obtain the following

Theorem 2.1. *As $n \to \infty$, we have*

$$W_n(p,q) \sim \frac{e^{-p\gamma}}{\Gamma\big(1+\frac{1}{2}p+\frac{1}{2}\Delta\big)\Gamma\big(1+\frac{1}{2}p-\frac{1}{2}\Delta\big)}\exp\left(\sum_{j=1}^{\infty}\frac{a_j(p,q)}{(n+1)^j}\right), \tag{2.8}$$

where the coefficients $a_j(p, q)$ are given by

$$a_1(p,q) = \frac{1}{2}p^2 - q \quad \text{and}$$

$$a_j(p,q)=\frac{pB_j}{j}+\frac{(-1)^{j+1}\Big(B_{j+1}(\tfrac12 p+\tfrac12\Delta)+B_{j+1}(\tfrac12 p-\tfrac12\Delta)-2B_{j+1}\Big)}{j(j+1)}\qquad (j\geq 2). \tag{2.9}$$

Thus we have the expansion

$$W_n(p,q)\sim\frac{e^{-p\gamma}}{\Gamma\big(1+\frac12 p+\frac12\Delta\big)\Gamma\big(1+\frac12 p-\frac12\Delta\big)}\times\exp\Bigg(\frac{\frac12 p^2-q}{n+1}+\frac{-\frac16 p^3+\frac12 pq+\frac14 p^2-\frac12 q}{(n+1)^2}$$
$$+\frac{\frac{1}{12}p^4-\frac13 p^2q+\frac16 q^2-\frac16 p^3+\frac12 pq+\frac{1}{12}p^2-\frac16 q}{(n+1)^3}+\cdots\Bigg) \tag{2.10}$$

as $n\to\infty$.

Remark 2.1. Note that since $W_n=1/W_n(0,-\frac14)$, it follows by setting $(p,q)=(0,-\frac14)$ in (2.10) that

$$W_n\sim\frac{\pi}{2}\exp\left(-\frac{1}{4(n+1)}-\frac{1}{8(n+1)^2}-\frac{5}{96(n+1)^3}-\cdots\right) \tag{2.11}$$

as $n\to\infty$.

The same procedure with the choice

$$(\lambda,\mu,\nu)=\left(\frac12 p,\ \frac14 p+\frac14\Delta,\ \frac14 p+\frac14\Delta\right)$$

in (2.2) and (2.6) leads to the following

Theorem 2.2. *As* $n\to\infty$, *we have*

$$R_n(p,q)\sim\frac{2^{-p}\pi e^{-p\gamma/2}}{\Gamma\big(\frac12+\frac14 p+\frac14\Delta\big)\Gamma\big(\frac12+\frac14 p-\frac14\Delta\big)}\exp\left(\sum_{j=1}^{\infty}\frac{b_j(p,q)}{(n+\frac12)^j}\right), \tag{2.12}$$

where the coefficients $b_j(p,q)$ *are given by*

$$b_1(p,q)=\frac18 p^2-\frac14 q\quad\text{and}$$
$$b_j(p,q)=\frac{pB_j}{2j}+\frac{(-1)^{j+1}\Big(B_{j+1}(\tfrac14 p+\tfrac14\Delta)+B_{j+1}(\tfrac14 p-\tfrac14\Delta)-2B_{j+1}\Big)}{j(j+1)}\qquad (j\geq 2). \tag{2.13}$$

Thus we have the expansion

$$R_n(p,q)\sim\frac{2^{-p}\pi e^{-p\gamma/2}}{\Gamma\big(\frac12+\frac14 p+\frac14\Delta\big)\Gamma\big(\frac12+\frac14 p-\frac14\Delta\big)}\times\exp\Bigg(\frac{\frac18 p^2-\frac14 q}{n+\frac12}+\frac{-\frac{1}{48}p^3+\frac{1}{16}pq+\frac{1}{16}p^2-\frac18 q}{(n+\frac12)^2}$$
$$+\frac{\frac{1}{192}p^4-\frac{1}{48}p^2q+\frac{1}{96}q^2-\frac{1}{48}p^3+\frac{1}{16}pq+\frac{1}{48}p^2-\frac{1}{24}q}{(n+\frac12)^3}+\cdots\Bigg) \tag{2.14}$$

as $n\to\infty$.

The first two terms in the expansions (2.10) and (2.14) can be shown to agree with the expansions in inverse powers of n obtained in [10, Eqs. (4.3) and (4.4)].

3. Asymptotic series expansions of the Wallis sequence

Some inequalities and asymptotic formulas associated with the Wallis sequence W_n can be found in [6,13,15,17–21,24–29,31]. For example, Elezović et al. [15] showed that the following asymptotic expansion holds:

$$W_n\sim\frac{\pi}{2}\left(1-\frac{\frac14}{n+\frac58}+\frac{\frac{3}{256}}{(n+\frac58)^3}+\frac{\frac{3}{2048}}{(n+\frac58)^4}-\frac{\frac{51}{16384}}{(n+\frac58)^5}-\frac{\frac{75}{65536}}{(n+\frac58)^6}+\frac{\frac{2253}{1048576}}{(n+\frac58)^7}+\cdots\right) \tag{3.1}$$

as $n\to\infty$. Deng et al. [13] proved that for all $n\in\mathbb{N}$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\alpha}\right)<W_n\leq\frac{\pi}{2}\left(1-\frac{1}{4n+\beta}\right) \tag{3.2}$$

with the best possible constants

$$\alpha=\frac52\quad\text{and}\quad\beta=\frac{32-9\pi}{3\pi-8}=2.614909986\ldots.$$

In fact, Elezović et al. [15] have previously shown that $\frac{5}{2}$ is the best possible constant for a lower bound of W_n of the type $\frac{\pi}{2}\left(1-\frac{1}{4n+\alpha}\right)$. Moreover, the authors pointed out that

$$W_n = \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right) + O\left(\frac{1}{n^3}\right) \qquad (n\to\infty).$$

Here, we will establish two more accurate asymptotic expansions for W_n (see Theorems 3.1 and 3.2) by making use of the fact that

$$W_n = \frac{\pi}{2}\cdot\frac{1}{n+\frac{1}{2}}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 = \frac{\pi}{2}\cdot\frac{\Gamma(n+1)^2}{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}. \tag{3.3}$$

The following lemma is required in our present investigation.

Lemma 3.1 (See [8])**.** *Let*

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n} \qquad (x\to\infty)$$

be a given asymptotical expansion. Then the composition $\exp(A(x))$ *has asymptotic expansion of the following form*

$$\exp(A(x)) \sim \sum_{n=0}^{\infty} b_n x^{-n} \qquad (x\to\infty), \tag{3.4}$$

where

$$b_0 = 1, \quad b_n = \frac{1}{n}\sum_{k=1}^{n} k a_k b_{n-k} \qquad (n\ge 1). \tag{3.5}$$

From (2.4) and (3.3), we find as $n\to\infty$

$$W_n \sim \frac{\pi}{2}\exp\left(\sum_{j=1}^{\infty}\frac{\nu_j}{n^j}\right), \tag{3.6}$$

where the coefficients ν_j are given by

$$\nu_j = \frac{(-1)^{j+1}\left(2B_{j+1} - B_{j+1}(\frac{1}{2}) - B_{j+1}(\frac{3}{2})\right)}{j(j+1)} \qquad (j\ge 1). \tag{3.7}$$

Noting that (See [1, pp. 805–804])

$$B_n(1-x) = (-1)^n B_n(x), \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1} \qquad (n\in\mathbb{N}_0)$$

and

$$B_n(\tfrac{1}{2}) = -(1-2^{1-n})B_n \qquad (n\in\mathbb{N}_0),$$

we find that (3.7) can be written as

$$\nu_j = \frac{(-1)^{j+1}\left((4-2^{1-j})B_{j+1} - (j+1)\cdot 2^{-j}\right)}{j(j+1)} \qquad (j\ge 1). \tag{3.8}$$

Thus, we obtain the expansion

$$W_n \sim \frac{\pi}{2}\exp\left(-\frac{1}{4n}+\frac{1}{8n^2}-\frac{5}{96n^3}+\frac{1}{64n^4}-\frac{1}{320n^5}+\frac{1}{384n^6}-\frac{25}{7168n^7}\right.$$
$$\left. + \frac{1}{2048n^8}+\frac{29}{9216n^9}+\frac{1}{10240n^{10}}-\frac{695}{90112n^{11}}+\cdots\right). \tag{3.9}$$

By Lemma 3.1, we then obtain from (3.6)

$$W_n \sim \frac{\pi}{2}\sum_{j=0}^{\infty}\frac{\mu_j}{n^j}, \tag{3.10}$$

where the coefficients μ_j are given by the recurrence relation

$$\mu_0 = 1, \quad \mu_j = \frac{1}{j}\sum_{k=1}^{j} k\nu_k\mu_{j-k} \qquad (j\ge 1). \tag{3.11}$$

and the ν_j are given in (3.8). This produces the expansion in inverse powers of n given by

$$W_n \sim \frac{\pi}{2}\Big(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \frac{625}{65536n^6} - \frac{1843}{262144n^7} + \frac{24323}{8388608n^8} + \frac{61477}{33554432n^9} - \frac{14165}{268435456n^{10}} - \frac{8084893}{1073741824n^{11}} + \cdots\Big) \tag{3.12}$$

as $n \to \infty$.

Theorem 3.1. *The Wallis sequence has the following asymptotic expansion:*

$$W_n \sim \frac{\pi}{2}\left(1 + \sum_{\ell=1}^{\infty} \frac{\alpha_\ell}{(n+\beta_\ell)^{2\ell-1}}\right) \qquad (n \to \infty), \tag{3.13}$$

where α_ℓ and β_ℓ are given by the pair of recurrence relations

$$\alpha_\ell = \mu_{2\ell-1} - \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \qquad (\ell \ge 2) \tag{3.14}$$

and

$$\beta_\ell = -\frac{1}{(2\ell-1)\alpha_\ell}\left\{\mu_{2\ell} + \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}\right\} \qquad (\ell \ge 2), \tag{3.15}$$

with $\alpha_1 = -\frac{1}{4}$ and $\beta_1 = \frac{5}{8}$. Here μ_j are given by the recurrence relation (3.11).

Proof. Let

$$W_n \sim \frac{\pi}{2}\left(1 + \sum_{\ell=1}^{\infty} \frac{\alpha_\ell}{(n+\beta_\ell)^{2\ell-1}}\right) \qquad (n \to \infty),$$

where α_ℓ and β_ℓ are real numbers to be determined. This can be written as

$$\frac{2}{\pi} W_n \sim 1 + \sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \left(1 + \frac{\beta_j}{n}\right)^{-2j+1}. \tag{3.16}$$

Direct computation yields

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \left(1 + \frac{\beta_j}{n}\right)^{-2j+1} &\sim \sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \sum_{k=0}^{\infty} \binom{-2j+1}{k} \frac{\beta_j^k}{n^k} \\ &\sim \sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-2}{k} \frac{\beta_j^k}{n^k} \\ &\sim \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \alpha_{k+1} \beta_{k+1}^{j-k-1} (-1)^{j-k-1} \binom{j+k-1}{j-k-1} \frac{1}{n^{j+k}}.\end{aligned}$$

The last expression has been obtained using the transformation $(j,k) \mapsto (k+1, j-k-1)$. Consequently, using also the transformation $(j,k) \mapsto (j+k, k+1)$,

$$\sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \left(1 + \frac{\beta_j}{n}\right)^{-2j+1} \sim \sum_{j=1}^{\infty} \left\{\sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1}\right\} \frac{1}{n^j} \tag{3.17}$$

follows. It then follows from (3.16) and (3.17) that

$$\frac{2}{\pi} W_n \sim 1 + \sum_{j=1}^{\infty} \left\{\sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1}\right\} \frac{1}{n^j}. \tag{3.18}$$

On the other hand, we have from (3.10) that

$$\frac{2}{\pi} W_n \sim 1 + \sum_{j=1}^{\infty} \frac{\mu_j}{n^j}. \tag{3.19}$$

Equating coefficients of n^{-j} on the right-hand sides of (3.18) and (3.19), we obtain

$$\mu_j = \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \qquad (j \in \mathbb{N}). \tag{3.20}$$

Setting $j = 2\ell - 1$ and $j = 2\ell$ in (3.20), respectively, we find

$$\mu_{2\ell-1} = \sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \tag{3.21}$$

and

$$\mu_{2\ell} = -\sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}. \tag{3.22}$$

For $\ell = 1$, we obtain from (3.21) and (3.22)

$$\alpha_1 = \mu_1 = -\frac{1}{4} \quad \text{and} \quad \beta_1 = -\frac{\mu_2}{\alpha_1} = \frac{5}{8},$$

and for $\ell \geq 2$ we have

$$\mu_{2\ell-1} = \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \alpha_\ell$$

and

$$\mu_{2\ell} = -\sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - (2\ell-1)\alpha_\ell \beta_\ell.$$

We then obtain the recurrence relations (3.14) and (3.15). The proof of Theorem 3.1 is complete. □

We now give explicit numerical values of the first few α_ℓ and β_ℓ by using the recurrence relations (3.14) and (3.15). This demonstrates the ease with which the constants α_ℓ and β_ℓ in (3) can be determined. We find

$$\alpha_1 = -\frac{1}{4}, \quad \beta_1 = \frac{5}{8},$$

$$\alpha_2 = \mu_3 - \alpha_1 \beta_1^2 = -\frac{11}{128} - \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^2 = \frac{3}{256},$$

$$\beta_2 = -\frac{\mu_4 + \alpha_1 \beta_1^3}{3\alpha_2} = -\frac{\frac{83}{2048} + \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^3}{3 \cdot \left(\frac{3}{256}\right)} = \frac{7}{12},$$

$$\alpha_3 = \mu_5 - \alpha_1 \beta_1^4 - 6\alpha_2 \beta_2^2 = -\frac{143}{8192} - \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^4 - 6 \cdot \left(\frac{3}{256}\right) \cdot \left(\frac{7}{12}\right)^2 = -\frac{53}{16384},$$

$$\beta_3 = -\frac{\mu_6 + \alpha_1 \beta_1^5 + 10\alpha_2 \beta_2^3}{5\alpha_3} = -\frac{\frac{625}{65536} + \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^5 + 10 \cdot \left(\frac{3}{256}\right) \cdot \left(\frac{7}{12}\right)^3}{5 \cdot \left(-\frac{53}{16384}\right)} = \frac{2113}{3816}.$$

Continuation of this procedure then enables the following coefficients to be derived:

$$\alpha_4 = \frac{224573}{93782016}, \quad \beta_4 = \frac{22119189899}{41134587264},$$

$$\alpha_5 = -\frac{5962972409837457969 31}{176651089583152098705408}, \quad \beta_5 = \frac{38909478384301921254232134966821}{73585322683584986068354328660352}.$$

We then obtain the following explicit asymptotic expansion:

$$W_n \sim \frac{\pi}{2} \left(1 - \frac{\frac{1}{4}}{n + \frac{5}{8}} + \frac{\frac{3}{256}}{(n + \frac{7}{12})^3} - \frac{\frac{53}{16384}}{(n + \frac{2113}{3816})^5} + \frac{\frac{224573}{93782016}}{(n + \frac{22119189899}{41134587264})^7} - \frac{\frac{596297240983745796931}{176651089583152098705408}}{(n + \frac{38909478384301921254232134966821}{73585322683584986068354328660352})^9} + \cdots \right). \tag{3.23}$$

Thus, we would appear to obtain an alternating odd-type asymptotic expansion for W_n. From a computational viewpoint, (3.23) is an improvement on the formulas (3.1) and (3.12).

Theorem 3.2. *The Wallis sequence has the following asymptotic expansion:*

$$W_n \sim \frac{\pi}{2} \exp\left(\sum_{\ell=1}^{\infty} \frac{\omega_\ell}{(n + \frac{1}{2})^{2\ell-1}} \right) \qquad (n \to \infty) \tag{3.24}$$

with the coefficients ω_ℓ given by the recurrence relation

$$\omega_1 = -\frac{1}{4} \quad \text{and} \quad \omega_\ell = \nu_{2\ell-1} - \sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \qquad (\ell \geq 2), \tag{3.25}$$

where the ν_j are given in (3.8).

Proof. Let

$$W_n \sim \frac{\pi}{2} \exp\left(\sum_{\ell=1}^{\infty} \frac{\omega_\ell}{(n+\frac{1}{2})^{2\ell-1}}\right) \qquad (n \to \infty),$$

where ω_ℓ are real numbers to be determined. This can be written as

$$\ln\left(\frac{2}{\pi} W_n\right) \sim \sum_{j=1}^{\infty} \frac{\omega_j}{n^{2j-1}} \left(1 + \frac{1}{2n}\right)^{-2j+1}.$$

The choice $\beta_j = \frac{1}{2}$ in (3.17), with α_j replaced by ω_j, yields

$$\sum_{j=1}^{\infty} \frac{\omega_j}{n^{2j-1}} \left(1 + \frac{1}{2n}\right)^{-2j+1} \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \omega_k \left(\frac{1}{2}\right)^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{n^j}.$$

We then obtain

$$\ln\left(\frac{2}{\pi} W_n\right) \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \omega_k \left(\frac{1}{2}\right)^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{n^j}. \tag{3.26}$$

On the other hand, we have from (3.6) that

$$\ln\left(\frac{2}{\pi} W_n\right) \sim \sum_{j=1}^{\infty} \frac{\nu_j}{n^j}. \tag{3.27}$$

Equating coefficients of n^{-j} on the right-hand sides of (3.26) and (3.27), we obtain

$$\nu_j = \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \omega_k \left(\frac{1}{2}\right)^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \qquad (j \in \mathbb{N}). \tag{3.28}$$

Setting $j = 2\ell - 1$ in (3.28), we find

$$\nu_{2\ell-1} = \sum_{k=1}^{\ell} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k} \binom{2\ell-2}{2\ell-2k}. \tag{3.29}$$

Substitution of $\ell = 1$ in (3.29) yields $\omega_1 = \nu_1 = -\frac{1}{4}$, and for $\ell \geq 2$ we have

$$\nu_{2\ell-1} = \sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \omega_\ell.$$

We then obtain the recurrence relation (3.25). The proof of Theorem 3.2 is complete. □

Remark 3.1. Setting $j = 2\ell$ in (3.28), we find

$$\nu_{2\ell} = -\sum_{k=1}^{\ell} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}. \tag{3.30}$$

For $\ell = 1$ in (3.30) this yields $\omega_1 = -2\nu_2 = -\frac{1}{4}$, and for $\ell \geq 2$ we have

$$\nu_{2\ell} = -\sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - \left(\ell - \frac{1}{2}\right) \omega_\ell.$$

We then obtain the alternative recurrence relation for the coefficients ω_j in (3.24) in terms of the even coefficients ν_j:

$$\omega_1 = -\frac{1}{4} \quad \text{and} \quad \omega_\ell = -\frac{2}{2\ell-1} \left\{ \nu_{2\ell} + \sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \right\} \qquad (\ell \geq 2). \tag{3.31}$$

Hence, from (3.24), we obtain the following explicit asymptotic expansion:

$$W_n \sim \frac{\pi}{2} \exp\left(-\frac{\frac{1}{4}}{n+\frac{1}{2}} + \frac{\frac{1}{96}}{(n+\frac{1}{2})^3} - \frac{\frac{1}{320}}{(n+\frac{1}{2})^5} + \frac{\frac{17}{7168}}{(n+\frac{1}{2})^7} - \frac{\frac{31}{9216}}{(n+\frac{1}{2})^9} + \cdots \right). \tag{3.32}$$

This would appear to be an alternating odd-type expansion for W_n. From a computational viewpoint, (3.32) is an improvement on the formulas (2.11) and (3.9).

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