Linear Algebra and its Applications 510 (2016) 373–394

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Recovery of eigenvectors of rational matrix functions from Fiedler-like linearizations

Rafikul Alam *, Namita Behera

Department of Mathematics, IIT Guwahati, Guwahati 781039, India

ARTICLE INFO

Article history:
Received 10 June 2016
Accepted 8 September 2016
Available online 12 September 2016
Submitted by F. Dopico

MSC:
65F15
15A57
15A18
65F35

Keywords:
Rational matrix function
Zeros
Poles
Matrix polynomial
Eigenvalue
Eigenvector
Minimal realization
Matrix pencil
Linearization
Fiedler pencil

ABSTRACT

Linearization is a standard method often used when dealing with matrix polynomials. Recently, the concept of linearization has been extended to rational matrix functions and Fiedler-like matrix pencils for rational matrix functions have been constructed. A linearization $\mathbb{L}(\lambda)$ of a rational matrix function $G(\lambda)$ does not necessarily guarantee a simple way of recovering eigenvectors of $G(\lambda)$ from those of $\mathbb{L}(\lambda)$. We show that Fiedler-like pencils of $G(\lambda)$ allow an easy operation free recovery of eigenvectors of $G(\lambda)$, that is, eigenvectors of $G(\lambda)$ are recovered from eigenvectors of Fiedler-like pencils of $G(\lambda)$ without performing any arithmetic operations. We also consider Fiedler-like pencils of the Rosenbrock system polynomial $\mathcal{S}(\lambda)$ associated with an LTI system Σ in state-space form (SSF) and show that the Fiedler-like pencils allow operation free recovery of eigenvectors of $\mathcal{S}(\lambda)$. The eigenvectors of $\mathcal{S}(\lambda)$ are the invariant zero directions of the LTI system Σ.

* Corresponding author. Fax: +91 361 2690762; +91 361 2582649.
E-mail addresses: rafik@iitg.ernet.in, rafikul68@gmail.com (R. Alam), niku.namita@gmail.com (N. Behera).

http://dx.doi.org/10.1016/j.laa.2016.09.009

1. Introduction

Let $P(\lambda)$ be an $n \times n$ matrix polynomial (regular or singular) of degree m. Then an $mn \times mn$ matrix pencil $L(\lambda) := A + \lambda B$ is said to be a *linearization* [5,7] of $P(\lambda)$ if there are $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $U(\lambda)L(\lambda)V(\lambda) = \mathrm{diag}(I_{(m-1)n}, P(\lambda))$ for all $\lambda \in \mathbb{C}$, where I_k denotes the $k \times k$ identity matrix. Linearization is a standard technique often used when dealing with matrix polynomials especially for solving polynomial eigenvalue problems, see [5,7,2,4] and references therein.

Zeros (eigenvalues) and poles of rational matrix functions play an important role in Linear Systems Theory [6,9,11] as well as in many other applications such as in acoustic emissions of high speed trains, calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid–solid structures, to name only a few, see [8,12,13,10].

Linearizations of rational matrix functions have been introduced recently in [1,3] via matrix-fraction descriptions (MFD) of rational matrix functions. Let $G(\lambda)$ be an $n \times n$ rational matrix function and let $G(\lambda) = N(\lambda)D(\lambda)^{-1}$ be a *right coprime* MFD of $G(\lambda)$, where $N(\lambda)$ and $D(\lambda)$ are matrix polynomials with $D(\lambda)$ being regular. Then the *zero structure* of $G(\lambda)$ is the same as the *eigenstructure* of $N(\lambda)$ and the *pole structure* of $G(\lambda)$ is the same as the *eigenstructure* of $D(\lambda)$, see [6].

Definition 1.1 *(Linearization, [1])*. Let $G(\lambda)$ be an $n \times n$ rational matrix function (regular or singular) and let $G(\lambda) = N(\lambda)D(\lambda)^{-1}$ be a *right coprime* MFD of $G(\lambda)$. Set $r := \deg(\det(D(\lambda)))$. Then a matrix pencil $\mathbb{L}(\lambda)$ of the form

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} X + \lambda Y & \mathcal{C} \\ \hline \mathcal{B} & A + \lambda E \end{array}\right] \tag{1.1}$$

is said to be a linearization of $G(\lambda)$ provided that $\mathbb{L}(\lambda)$ is a linearization of $N(\lambda)$ and $A + \lambda E$ is a linearization of $D(\lambda)$, where E is an $r \times r$ nonsingular matrix and the pencil $X + \lambda Y$ and the matrices $\mathcal{B}$ and $\mathcal{C}$ are of appropriate dimensions.

Thus the zeros and the poles of $G(\lambda)$ can be computed by solving the twin generalized eigenvalue problems $\mathbb{L}(\lambda)u = 0$ and $(A + \lambda E)v = 0$. Our main aim in this paper is to recover left and right eigenvectors of $G(\lambda)$ from those of $\mathbb{L}(\lambda)$ when $G(\lambda)$ is regular. The nonzero vectors u and v are said to be left and right eigenvectors of $G(\lambda)$ corresponding to an eigenvalue λ provided that $u^T G(\lambda) = 0$ and $G(\lambda)v = 0$.

The Fiedler-like pencils of $G(\lambda)$ have been constructed in [1,3] by considering a *realization* [6] of $G(\lambda)$ of the form

$$G(\lambda) = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1}B =: P(\lambda) + C(\lambda E - A)^{-1}B, \tag{1.2}$$

where $A - \lambda E$ is an $r \times r$ pencil with E being nonsingular. Then, for any permutation $\sigma := (i_0, i_1, \ldots, i_{m-1})$ of $(0, 1, \ldots, m-1)$, the pencil [1]

$$\mathbb{L}_\sigma(\lambda) := \lambda \mathbb{M}_m - \mathbb{M}_{i_0}\mathbb{M}_{i_1} \cdots \mathbb{M}_{i_{m-1}} =: \lambda \mathbb{M}_m - \mathbb{M}_\sigma$$

is called a Fiedler pencil of $G(\lambda)$, where $\mathbb{M}_0, \mathbb{M}_1, \ldots, \mathbb{M}_m$ are Fiedler matrices associated with the realization (1.2). The Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ is a linearization of $G(\lambda)$ when the realization (1.2) is minimal, see [1]. For example, the companion pencil

$$\mathcal{C}(\lambda) := \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] + \left[\begin{array}{cccc|c} A_{m-1} & A_{m-2} & \cdots & A_0 & C \\ -I_n & 0 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & -I_n & 0 & \\ \hline & & & B & A \end{array}\right], \tag{1.3}$$

is a Fiedler pencil associated with the permutation $\sigma = (m-1, \ldots, 1, 0)$, that is, $\mathcal{C}(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_{m-1} \cdots \mathbb{M}_1 \mathbb{M}_0$. If $\begin{bmatrix} u^T, & v^T \end{bmatrix}^T$ with $u \in \mathbb{C}^{mn}$ and $v \in \mathbb{C}^r$ is a right eigenvector of $\mathcal{C}(\lambda)$ corresponding to an eigenvalue λ then it turns out that $(e_m^T \otimes I_n)u$ is a right eigenvector of $G(\lambda)$ corresponding to λ. On the other hand, if $\begin{bmatrix} w^T, & z^T \end{bmatrix}^T$ with $w \in \mathbb{C}^{mn}$ and $z \in \mathbb{C}^r$ is a left eigenvector of $\mathcal{C}(\lambda)$ corresponding to λ then $(e_1^T \otimes I_n)w$ is a left eigenvector of $G(\lambda)$ corresponding to λ. We show that a Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ allows a similar operation free recovery of eigenvectors of $G(\lambda)$ and the recovery is determined by the consecution–inversion property of the permutation σ.

A realization of $G(\lambda)$ of the form (1.2) is associated with a linear time-invariant (LTI) system Σ in *state-space-form* (SSF) given by [9,11]

$$\Sigma : \quad \begin{array}{l} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + P(\frac{d}{dt})u(t) \end{array} \tag{1.4}$$

for which $G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B$ is the *transfer function*, where $x(t)$ is the state vector and $u(t)$ is the control vector of the system. The *Rosenbrock system polynomial* (also referred to as the Rosenbrock system matrix) associated with the LTI system Σ in (1.4) is an $(n + r) \times (n + r)$ matrix polynomial $\mathcal{S}(\lambda)$ given by [9,6]

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right]. \tag{1.5}$$

The eigenvalues of $\mathcal{S}(\lambda)$ are called *invariant zeros* of the LTI system Σ and the associated eigenvectors are called *invariant zero directions* [9,6]. The invariant zeros of LTI systems play an important role in Linear Systems Theory [9,6,11].

Definition 1.2 *(Rosenbrock linearization, [1]).* Let $\mathcal{S}(\lambda)$ be a system matrix given by (1.5) and $\deg(P) = m \geq 1$. Then an $(mn + r) \times (mn + r)$ matrix pencil $\mathbb{L}(\lambda)$ of the form

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} X + \lambda Y & \mathcal{C} \\ \hline \mathcal{B} & A - \lambda E \end{array}\right],$$

where $X + \lambda Y$ is an $mn \times mn$ pencil and the matrices $\mathcal{B}$ and $\mathcal{C}$ are of appropriate dimensions, is said to be a Rosenbrock linearization of $\mathcal{S}(\lambda)$ provided that there are $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \mathbb{L}(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] \tag{1.6}$$

for all $\lambda \in \mathbb{C}$.

It is shown in [1] that a Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ of $P(\lambda)+C(\lambda E - A)^{-1}B$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$. We, therefore, compute eigenvectors of $\mathcal{S}(\lambda)$ from those of $\mathbb{L}_\sigma(\lambda)$. We show that the Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ allows an easy operation free recovery of eigenvectors of $\mathcal{S}(\lambda)$ and the recovery is completely determined by the consecution–inversion property of the permutation σ.

The rest of the paper is organized as follows. Section 2 presents Fiedler pencils associated with the Rosenbrock system matrix $\mathcal{S}(\lambda)$. Section 3 describes recovery of eigenvectors of $\mathcal{S}(\lambda)$ from those of the Fiedler pencils of $\mathcal{S}(\lambda)$ and shows that the recovery is operation free. Section 4 describes recovery of eigenvectors of $G(\lambda)$ from those of the Fiedler pencils associated with a minimal realization of $G(\lambda)$.

Notation. We denote by $\mathbb{C}[\lambda]$ the polynomial ring over the complex field $\mathbb{C}$. Further, we denote by $\mathbb{C}^{m\times n}$ and $\mathbb{C}[\lambda]^{m\times n}$, respectively, the vector spaces of $m \times n$ matrices and matrix polynomials over $\mathbb{C}$. An $m \times n$ rational matrix function $G(\lambda)$ is an $m \times n$ matrix whose entries are rational functions of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials in $\mathbb{C}[\lambda]$. An $n \times n$ rational matrix function $G(\lambda)$ is said to be *regular* if $\mathrm{rank}(G(\lambda)) = n$ for some $\lambda \in \mathbb{C}$. If $G(\lambda)$ is regular then $\mu \in \mathbb{C}$ is said to be an **eigenvalue** of $G(\lambda)$ if $\mathrm{rank}(G(\mu)) < n$. An $n \times n$ matrix polynomial $U(\lambda)$ is said to be *unimodular* if $\det(U(\lambda))$ is a nonzero constant independent of λ. We denote the j-th column of the $n \times n$ identity matrix I_n by e_j and the transpose of a matrix A by A^T. We denote the Kronecker product of matrices A and B by $A \otimes B$. The right and the left null spaces of an $m \times n$ matrix A are given by $\mathcal{N}_r(A) := \{x \in \mathbb{C}^n : Ax = 0\}$ and $\mathcal{N}_l(A) := \{y \in \mathbb{C}^m : y^T A = 0\}$.

2. Fielder pencils of rational matrix functions

For the rest of the paper, we consider the realization of $G(\lambda)$ given in (1.2) and the associated Rosenbrock system matrix $\mathcal{S}(\lambda)$ given in (1.5). It is well known [6,9,11] that the realization of $G(\lambda)$ given in (1.2) is *minimal* if and only if

$$\mathrm{rank}([A - \lambda E \quad B]) = r = \mathrm{rank}\left(\begin{bmatrix} A - \lambda E \\ C \end{bmatrix}\right) \text{ for } \lambda \in \mathbb{C}.$$

Also, for the rest of the paper, we assume that $P(\lambda) := \sum_{j=0}^{m} A_j\lambda^j$, the matrix polynomial that appears in the realization (1.2) and the system matrix in (1.5).

We consider the $(nm+r) \times (nm+r)$ Fiedler-like matrices $\mathbb{M}_0, \ldots, \mathbb{M}_m$ of $G(\lambda)$ associated with the realization (1.2) given by [1]

$$\mathbb{M}_0 := \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \ \mathbb{M}_m := \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & -E \end{array}\right], \ \mathbb{M}_i := \left[\begin{array}{c|c} M_i & 0 \\ \hline 0 & I_r \end{array}\right],$$

where $M_0, \ldots, M_m$ are Fiedler matrices associated with the matrix polynomial $P(\lambda)$ given by [4]

$$M_m := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix},$$

$$M_i := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, i = 1, \ldots, m-1.$$

We refer to the matrices $\mathbb{M}_0, \ldots, \mathbb{M}_m$ as the *Fiedler matrices* associated with $\mathcal{S}(\lambda)$ or the realization $G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B$.

Definition 2.1 *(Fiedler pencil, [1]).* Let $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ be a bijection. Set $\mathbb{M}_\sigma := \mathbb{M}_{\sigma^{-1}(1)}\mathbb{M}_{\sigma^{-1}(2)} \cdots \mathbb{M}_{\sigma^{-1}(m)}$. Then the $(mn + r) \times (mn + r)$ matrix pencil $\mathbb{L}_\sigma(\lambda)$ given by

$$\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_m - \mathbb{M}_{\sigma^{-1}(1)}\mathbb{M}_{\sigma^{-1}(2)} \cdots \mathbb{M}_{\sigma^{-1}(m)} = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$$

is said to be the Fiedler pencil of the realization $G(\lambda)$ associated with σ. The pencil $\mathbb{L}_\sigma(\lambda)$ is also referred to as the Fiedler pencil of the system matrix $\mathcal{S}(\lambda)$ associated with σ.

The consecution and inversion of a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$, which will pay an important role in the subsequent development, are defined as follows.

Definition 2.2. [4] Let $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ be a bijection.

(1) For $d = 0, \ldots, m-2$, we say that σ has a consecution at d if $\sigma(d) < \sigma(d+1)$ and σ has an inversion at d if $\sigma(d) > \sigma(d+1)$.

(2) The tuple $\mathrm{CISS}(\sigma) := (c_1, i_1, c_2, i_2, \ldots, c_l, i_l)$ is called the consecution–inversion structure sequence of σ, where σ has c_1 consecutive consecutions at $0, 1, \ldots, c_1 - 1$; i_1 consecutive inversions at $c_1, c_1 + 1, \ldots, c_1 + i_1 - 1$ and so on, up to i_l inversions at $m - 1 - i_l, \ldots, m-2$.

(3) We denote the total number of consecutions and inversions in σ by $c(\sigma)$ and $i(\sigma)$, respectively. Then $c(\sigma) = \sum_{j=1}^{l} c_j$, $i(\sigma) = \sum_{j=1}^{l} i_j$, and $c(\sigma) + i(\sigma) = m - 1$.

Said differently, if $\sigma : \{0, 1, \dots, m-1\} \to \{1, 2, \dots, m\}$ is a bijection then σ has a consecution at d if and only if $\mathbb{M}_d$ is to the left of $\mathbb{M}_{d+1}$ in $\mathbb{M}_\sigma$, while σ has an inversion at d if and only if $\mathbb{M}_d$ is to the right of $\mathbb{M}_{d+1}$ in $\mathbb{M}_\sigma$.

Theorem 2.3 *(Linearization, [1]). Let $\sigma : \{0, 1, \dots, m-1\} \to \{1, 2, \dots, m\}$ be a bijection. Let $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ and $L_\sigma(\lambda) := \lambda M_m - M_\sigma$, respectively, be the Fiedler pencil of $\mathcal{S}(\lambda)$ and $P(\lambda)$ associated with σ. If CISS$(\sigma) := (c_1, i_1, c_2, i_2, \dots, c_l, i_l)$ then we have*

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & C_\sigma \\ \hline B_\sigma & A - \lambda E \end{array}\right], \tag{2.1}$$

where $B_\sigma = e_{m-c_1}^T \otimes B$ and, $C_\sigma = e_m \otimes C$ when $c_1 > 0$ and $C_\sigma = e_{m-i_1} \otimes C$ when $c_1 = 0$. Further, $\mathbb{L}_\sigma(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$. Furthermore, $\mathbb{L}_\sigma(\lambda)$ is a linearization of $G(\lambda)$ (in the sense of Definition 1.1) whenever the realization of $G(\lambda)$ in (1.2) is minimal.

3. Eigenvector recovery for Rosenbrock system polynomials

We now describe recovery of eigenvectors of $\mathcal{S}(\lambda)$ from eigenvectors of an associated Fiedler pencil $\mathbb{L}_\sigma(\lambda)$. We proceed as follows.

Definition 3.1. Let $E : \mathbb{C}^n \to \mathbb{C}^m$, $m > n$, be linear. Then E is said to be *left invertible* if there exists a linear map $E_\ell : \mathbb{C}^m \to \mathbb{C}^n$ such that $E_\ell E = I_n$. In such a case, E_ℓ is called a left inverse of E.

We have the following elementary result which will play an important role in recovering eigenvectors from linearizations.

Lemma 3.2. *Let $E : \mathbb{C}^n \to \mathbb{C}^m$ be linear and left invertible with a left inverse E_ℓ. Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$. If $E : \mathcal{N}(A) \to \mathcal{N}(B)$ is an isomorphism then $E_\ell : \mathcal{N}(B) \to \mathcal{N}(A)$ is an isomorphism.*

Proof. Let $y \in \mathcal{N}(B)$. Then $y = Ex$ for some $x \in \mathcal{N}(A) \Rightarrow E_\ell y = E_\ell E x = x \in \mathcal{N}(A) \Rightarrow E_\ell y \in \mathcal{N}(A)$. This shows that $E_\ell : \mathcal{N}(B) \to \mathcal{N}(A)$. Let $y \in \mathcal{N}(B)$ and suppose that $E_\ell y = 0$. Then $E_\ell y = E_\ell E x = 0$ for some $x \in \mathcal{N}(A) \Rightarrow x = 0 \Rightarrow y = Ex = 0 \Rightarrow E_\ell \restriction_{\mathcal{N}(B)}$ is one to one. Next, let $x \in \mathcal{N}(A)$. Let $y = Ex$. Then $E_\ell y = E_\ell E x = x \Rightarrow E_\ell \restriction_{\mathcal{N}(B)}$ is onto. Hence E_ℓ is an isomorphism. $\square$

Next, we consider a Rosenbrock linearization $\mathbb{L}(\lambda)$ of $\mathcal{S}(\lambda)$ and determine an isomorphism between $\mathcal{N}_r(\mathcal{S}(\lambda))$ and $\mathcal{N}_r(\mathbb{L}(\lambda))$.

Theorem 3.3 *(Eigenvector recovery maps). Let $\mathcal{S}(\lambda)$ be a regular $(n+r)\times(n+r)$ Rosenbrock system matrix and $\mathbb{L}(\lambda)$ be an $(mn+r)\times(mn+r)$ matrix pencil given by*

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A-\lambda E) \end{array}\right] \text{ and } \mathbb{L}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & \mathcal{C} \\ \hline \mathcal{B} & (A-\lambda E) \end{array}\right], \tag{3.1}$$

where $P(\lambda)$ is an $n\times n$ matrix polynomial of degree m and $L(\lambda) := X+\lambda Y$ is an $mn\times mn$ matrix pencil. Let $U(\lambda)$ and $V(\lambda)$ be $mn\times mn$ unimodular matrix polynomials such that, for all $\lambda\in\mathbb{C}$, we have

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \mathbb{L}(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} \pm I_{n(m-1)} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right]. \tag{3.2}$$

(a) *Let $\lambda\in\mathbb{C}$. Define $\mathbb{E}(\mathcal{S}) : \mathbb{C}^{n+r}\to\mathbb{C}^{nm+r}$ and $\mathbb{F}(\mathcal{S}) : \mathbb{C}^{nm+r}\to\mathbb{C}^{n+r}$ by*

$$\mathbb{E}(\mathcal{S}) := \left[\begin{array}{c|c} V(\lambda)(e_m\otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right] \text{ and } \mathbb{F}(\mathcal{S}) := \left[\begin{array}{c|c} (e_m^T\otimes I_n)V(\lambda)^{-1} & 0 \\ \hline 0 & I_r \end{array}\right].$$

Then $\mathbb{E}(\mathcal{S})$ is left invertible and $\mathbb{F}(\mathcal{S})$ is a left inverse of $\mathbb{E}(\mathcal{S})$. Further, the restricted maps $\mathbb{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda))\to\mathcal{N}_r(\mathbb{L}(\lambda))$ and $\mathbb{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}(\lambda))\to\mathcal{N}_r(\mathcal{S}(\lambda))$ are isomorphisms.

(b) *Let $\lambda\in\mathbb{C}$. Define $\mathbb{H}(\mathcal{S}) : \mathbb{C}^{n+r}\to\mathbb{C}^{nm+r}$ and $\mathbb{K}(\mathcal{S}) : \mathbb{C}^{nm+r}\to\mathbb{C}^{n+r}$ by*

$$\mathbb{H}(\mathcal{S}) := \left[\begin{array}{c|c} U(\lambda)^T(e_m\otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right] \text{ and } \mathbb{K}(\mathcal{S}) := \left[\begin{array}{c|c} (e_m^T\otimes I_n)U(\lambda)^{-T} & 0 \\ \hline 0 & I_r \end{array}\right].$$

Then $\mathbb{H}(\mathcal{S})$ is left invertible and $\mathbb{K}(\mathcal{S})$ is a left inverse of $\mathbb{H}(\mathcal{S})$. Further, the restricted maps $\mathbb{H}(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda))\to\mathcal{N}_l(\mathbb{L}(\lambda))$ and $\mathbb{K}(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}(\lambda))\to\mathcal{N}_l(\mathcal{S}(\lambda))$ are isomorphisms.

Proof. (a) We have $\mathbb{F}(\mathcal{S})\mathbb{E}(\mathcal{S}) = I_{n+r}$ which shows that $\mathbb{F}$ is a left inverse of $\mathbb{E}$ and that $\mathbb{E}$ is injective. Let $u\in\mathbb{C}^n$ and $v\in\mathbb{C}^r$ be such that $\left[u^T,\ v^T\right]^T\in\mathcal{N}_r(\mathcal{S}(\lambda))$. Then we have

$$\begin{aligned} \mathbb{L}(\lambda)\mathbb{E}(\mathcal{S})\begin{bmatrix} u \\ v \end{bmatrix} &= \mathbb{L}(\lambda)\left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right]\begin{bmatrix} e_m\otimes u \\ v \end{bmatrix} \\ &= \left[\begin{array}{c|c} U(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} \pm I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\begin{bmatrix} e_m\otimes u \\ v \end{bmatrix} = 0 \end{aligned}$$

showing that $\mathbb{E}(\mathcal{S})\left[u^T,\ v^T\right]^T \in \mathcal{N}_r(\mathbb{L}(\lambda))$ and that $\mathbb{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}(\lambda))$ is well defined. By (3.2) we have $\dim \mathcal{N}_r(\mathbb{L}(\lambda)) = \dim \mathcal{N}_r(\mathcal{S}(\lambda))$ which shows that $\mathbb{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}(\lambda))$ is an isomorphism.

Since $\mathbb{F}(\mathcal{S})$ is a left inverse of $\mathbb{E}(\mathcal{S})$, by Lemma 3.2 we conclude that $\mathbb{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is also an isomorphism. This completes the proof of (a).

The proof of part (b) is similar and follows from the fact that $\mathcal{N}_l(\mathcal{S}) = \mathcal{N}_r(\mathcal{S}^T)$. Indeed, in view of (3.2), we have $\mathbb{H}(\mathcal{S}) = \mathbb{E}(\mathcal{S}^T)$ and $\mathbb{K}(\mathcal{S}) = \mathbb{F}(\mathcal{S}^T)$. $\square$

Remark 3.4. We refer to $\mathbb{F}(\mathcal{S})$ and $\mathbb{K}(\mathcal{S})$, respectively, as the right and the left eigenvector recovery maps for the system polynomial $\mathcal{S}(\lambda)$.

We mention that a similar result holds when $\mathcal{S}(\lambda)$ is singular. In such a case, the right null space $\mathcal{N}_r(\mathcal{S})$ and the left null space $\mathcal{N}_l(\mathcal{S})$ are given by

$$\mathcal{N}_r(\mathcal{S}) := \left\{ v(\lambda) \in \mathbb{C}^{(n+r)}[\lambda] : \mathcal{S}(\lambda)v(\lambda) = 0 \right\},$$
$$\mathcal{N}_l(\mathcal{S}) := \left\{ u(\lambda) \in \mathbb{C}^{(n+r)}[\lambda] : u(\lambda)^T\mathcal{S}(\lambda) = 0 \right\}.$$

Then the isomorphisms between the null spaces of $\mathcal{S}(\lambda)$ and $\mathbb{L}(\lambda)$ are given by Theorem 3.5 whose proof is similar to that of Theorem 3.3.

Theorem 3.5. *Let $\mathcal{S}(\lambda), \mathbb{L}(\lambda), U(\lambda)$ and $V(\lambda)$ be as in Theorem 3.3. Suppose that $\mathcal{S}(\lambda)$ singular.*

(a) *Define $\mathbb{E}(\mathcal{S}) : \mathbb{C}^{n+r}[\lambda] \to \mathbb{C}^{nm+r}[\lambda]$ and $\mathbb{F}(\mathcal{S}) : \mathbb{C}^{nm+r}[\lambda] \to \mathbb{C}^{n+r}[\lambda]$ by*

$$\mathbb{E}(\mathcal{S})\begin{bmatrix} u(\lambda) \\ v(\lambda) \end{bmatrix} := \begin{bmatrix} V(\lambda)(e_m \otimes u(\lambda)) \\ v(\lambda) \end{bmatrix},\ \mathbb{F}(\mathcal{S})\begin{bmatrix} x(\lambda) \\ y(\lambda) \end{bmatrix} := \begin{bmatrix} (e_m^T \otimes I_n)V(\lambda)^{-1}x(\lambda) \\ y(\lambda) \end{bmatrix}.$$

Then $\mathbb{E}(\mathcal{S})$ is left invertible and $\mathbb{F}(\mathcal{S})$ is a left inverse of $\mathbb{E}(\mathcal{S})$. Further, the maps $\mathbb{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \to \mathcal{N}_r(\mathbb{L})$ and $\mathbb{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}) \to \mathcal{N}_r(\mathcal{S})$ are isomorphisms.

(b) *Define $\mathbb{H}(\mathcal{S}) : \mathbb{C}^{n+r}[\lambda] \to \mathbb{C}^{nm+r}[\lambda]$ and $\mathbb{K}(\mathcal{S}) : \mathbb{C}^{nm+r}[\lambda] \to \mathbb{C}^{n+r}[\lambda]$ by*

$$\mathbb{H}(\mathcal{S})\begin{bmatrix} u(\lambda) \\ v(\lambda) \end{bmatrix} := \begin{bmatrix} U(\lambda)^T(e_m \otimes u(\lambda)) \\ v(\lambda) \end{bmatrix},\ \mathbb{K}(\mathcal{S})\begin{bmatrix} x(\lambda) \\ y(\lambda) \end{bmatrix} := \begin{bmatrix} (e_m^T \otimes I_n)U(\lambda)^{-T}x(\lambda) \\ y(\lambda) \end{bmatrix}.$$

Then $\mathbb{H}(\mathcal{S})$ is left invertible and $\mathbb{K}(\mathcal{S})$ is a left inverse of $\mathbb{H}(\mathcal{S})$. Further, the maps $\mathbb{H}(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}) \to \mathcal{N}_l(\mathbb{L})$ and $\mathbb{K}(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathcal{S})$ are isomorphisms.

Our main goal is to determine the eigenvector recovery maps $\mathbb{F}(\mathcal{S})$ and $\mathbb{K}(\mathcal{S})$ for Fiedler pencils of $\mathcal{S}(\lambda)$. We proceed as follows.

Definition 3.6 *(Horner shift, [4])*. Let $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ be a matrix polynomial of degree m. For $k = 0, \ldots m$, the degree k Horner shift of $P(\lambda)$ is the matrix polynomial $P_k(\lambda) := A_{m-k} + \lambda A_{m-k+1} + \cdots + \lambda^k A_m$. These Horner shifts satisfy the following:

$$P_0(\lambda) = A_m,\; P_{k+1}(\lambda) = \lambda P_k(\lambda) + A_{m-k-1},\; \text{for } 0 \le k \le m-1, P_m(\lambda) = P(\lambda).$$

The block transpose of a block matrix is defined as follows.

Definition 3.7 *(Block transpose, [4]).* Let $\mathcal{H} := (\mathcal{H}_{ij})$ be a block $m \times n$ matrix, where each block $\mathcal{H}_{ij}$ is a $p \times q$ matrix. Then the *block transpose* of $\mathcal{H}$ is the block $n \times m$ matrix $\mathcal{H}^{\mathcal{B}}$ given by $(\mathcal{H}^{\mathcal{B}})_{ij} = \mathcal{H}_{ji}$.

We frequently use the following auxiliary matrices associated with a matrix polynomial for determining the eigenvector recovery maps.

Definition 3.8 *([4], Definition 4.2).* Let $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ be an $n \times n$ matrix polynomial and let $P_i(\lambda)$ be the degree i Horner shift of $P(\lambda)$. For $1 \le i \le m-1$, define the following $nm \times nm$ matrix polynomials:

$$Q_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & \lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix},$$

$$R_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0_n & I_n & \\ & I_n & P_i(\lambda) & \\ & & & I_{(m-i-1)n} \end{bmatrix},$$

$$T_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & \lambda P_{i-1}(\lambda) & \\ & \lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(m-i-1)n} \end{bmatrix},$$

$$D_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & P_{i-1}(\lambda) & 0_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix},$$

and $D_m(\lambda) := \operatorname{diag}\left[0_{(m-1)n}, P_{m-1}(\lambda)\right]$. For simplicity, we often write Q_i, R_i, T_i, D_i in place of $Q_i(\lambda), R_i(\lambda), T_i(\lambda), D_i(\lambda)$. Note that $D_1(\lambda) = M_m$, and $Q_i(\lambda)$, $R_i(\lambda)$ are unimodular for all $i = 1, \ldots, m-1$. Also note that $R_i^{\mathcal{B}}(\lambda) = R_i(\lambda)$.

We have

$$Q_i^{\mathcal{B}}(\lambda) = \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & 0_n & \\ & \lambda I_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix}, \tag{3.3}$$

$$Q_i(\lambda)^{-1} = \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & -\lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix}, \tag{3.4}$$

$$(Q_i^{\mathcal{B}}(\lambda))^{-1} = \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & 0_n & \\ & -\lambda I_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix} = (Q_i(\lambda)^{-1})^{\mathcal{B}}, \tag{3.5}$$

$$R_i(\lambda)^{-1} = \begin{bmatrix} I_{(i-1)n} & & & \\ & -P_i(\lambda) & I_n & \\ & I_n & 0_n & \\ & & & I_{(m-i-1)n} \end{bmatrix} = (R_i^{\mathcal{B}}(\lambda))^{-1}, \tag{3.6}$$

Lemma 3.9. *Let $P_d(\lambda)$ be the Horner shifts of the matrix polynomial $P(\lambda)$ for $d = 0, \ldots, m$. Also, let $(Q_i^{\mathcal{B}})^{-1}$ and R_i^{-1} be as in (3.5) and (3.6) for $i = 1, \ldots, m-1$. Then for each $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-i$, we have*

$$Q(i,j) := (Q_i^{\mathcal{B}})^{-1}(Q_{i+1}^{\mathcal{B}})^{-1} \cdots (Q_{i+j-1}^{\mathcal{B}})^{-1}$$

$$= \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & I_n & 0 & 0 & \ldots & 0 & \\ & -\lambda I_n & I_n & 0 & & 0 & \\ & & -\lambda I_n & I_n & \ddots & \vdots & \\ & & & \ddots & \ddots & 0 & \\ & & & & -\lambda I_n & I_n & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right],$$

$$R(i,j) := R_i^{-1} R_{i+1}^{-1} \cdots R_{i+j-1}^{-1}$$

$$= \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & -P_i(\lambda) & -P_{i+1}(\lambda) & \ldots & -P_{i+j-1}(\lambda) & I & \\ & I_n & 0 & \ldots & 0 & 0 & \\ & & \ddots & \ddots & \vdots & \vdots & \\ & & & \ddots & 0 & 0 & \\ & & & & I_n & 0 & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right].$$

Proof. We have $R_i^{-1}R_{i+1}^{-1} =$

$$\left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & -P_i & I_n & & \\ & I_n & 0 & & \\ & & & I_n & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right] \cdot \left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & I_n & & & \\ & & -P_{i+1} & I_n & \\ & & I_n & 0 & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right]$$

$$= \left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & -P_i & -P_{i+1} & I_n & \\ & I_n & 0 & 0 & \\ & 0 & I_n & 0 & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right]$$

and $R_i^{-1}R_{i+1}^{-1}R_{i+2}^{-1} =$

$$\left[\begin{array}{c|cccc|c} I_{(i-1)n} & & & & & \\ \hline & -P_i & -P_{i+1} & I_n & & \\ & I_n & 0 & 0 & & \\ & 0 & I_n & 0 & & \\ & & & & I_n & \\ \hline & & & & & I_{(m-i-3)n} \end{array}\right] \cdot \left[\begin{array}{c|cccc|c} I_{(i-1)n} & & & & & \\ \hline & I_n & & & & \\ & & I_n & & & \\ & & & -P_{i+2} & I_n & \\ & & & I_n & 0 & \\ \hline & & & & & I_{(m-i-3)n} \end{array}\right]$$

$$= \left[\begin{array}{c|cccc|c} I_{(i-1)n} & & & & & \\ \hline & -P_i & -P_{i+1} & -P_{i+2} & I_n & \\ & I_n & 0 & 0 & 0 & \\ & 0 & I_n & 0 & 0 & \\ & 0 & 0 & I_n & 0 & \\ \hline & & & & & I_{(m-i-3)n} \end{array}\right].$$

Similarly, we have $(Q_i^{\mathcal{B}})^{-1}(Q_{i+1}^{\mathcal{B}})^{-1} =$

$$\left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & I_n & 0 & & \\ & -\lambda I_n & I_n & & \\ & & & I_n & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right] \cdot \left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & I_n & & & \\ & & I_n & 0 & \\ & & -\lambda I_n & I_n & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right]$$

$$= \left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & I_n & 0 & & \\ & -\lambda I_n & I_n & & \\ & & -\lambda I_n & I_n & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right].$$

Thus the proof is a straight forward induction on the number of factors. $\square$

The next result provides us with unimodular matrices $U(\lambda)$ and $V(\lambda)$ associated with a Fiedler pencil $L_\sigma(\lambda)$ which will play an important role in the sequel.

Theorem 3.10 *(Fiedler linearization, [4]). Let $P(\lambda)$ be a matrix polynomial (regular or singular) of degree $m \geq 1$ and $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Let Q_i, R_i for $i = 1, \ldots, m-1$, be the matrices introduced in Definition 3.8. Then*

$$U(\lambda)L_\sigma(\lambda)V(\lambda) = \begin{bmatrix} -I_{(m-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

where $U(\lambda)$ and $V(\lambda)$ are the following $nm \times nm$ unimodular matrix polynomials:

$$U(\lambda) = U_0U_1 \cdots U_{m-3}U_{m-2} \text{ with } U_i = \begin{cases} Q^{\mathcal{B}}_{m-(i+1)} & \text{if } \sigma \text{ has a consecution at } i \\ R^{\mathcal{B}}_{m-(i+1)} & \text{if } \sigma \text{ has an inversion at } i, \end{cases}$$

$$V(\lambda) = V_{m-2}V_{m-3} \cdots V_1V_0 \text{ with } V_i = \begin{cases} R_{m-(i+1)} & \text{if } \sigma \text{ has a consecution at } i \\ Q_{m-(i+1)} & \text{if } \sigma \text{ has an inversion at } i. \end{cases}$$

Thus $L_\sigma(\lambda)$ is a linearization of $P(\lambda)$.

Given a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ with $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, we define $s_0 := 0$, $s_j := \sum_{k=1}^{j}(c_k + i_k)$, for $j = 1, \ldots, l$. Note that $s_l = m - 1$.

Theorem 3.11. *Let $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ and let $U(\lambda)$ be as in Theorem 3.10. If $CISS(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$ then*

$$U(\lambda)^{-1}(e_m \otimes I_n) = \begin{cases} e_m \otimes I_n, & \text{if } c_1 > 0, \\ e_{m-i_1} \otimes I_n, & \text{if } c_1 = 0. \end{cases}$$

Proof. Note that

$$U(\lambda)^{-1} = U^{-1}_{m-2}U^{-1}_{m-3} \cdots U^{-1}_1U^{-1}_0 \text{ with}$$
$$U^{-1}_i = \begin{cases} (Q^{\mathcal{B}}_{m-(i+1)})^{-1} & \text{if } \sigma \text{ has a consecution at } i \\ (R^{\mathcal{B}}_{m-(i+1)})^{-1} & \text{if } \sigma \text{ has an inversion at } i, \end{cases} \tag{3.7}$$

where $(Q^{\mathcal{B}}_i)^{-1}$ and $(R^{\mathcal{B}}_i)^{-1}$ are given in (3.5) and (3.6). As $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, the factors defining $U^{-1}(\lambda)$ in (3.7) can be sorted together into the form $U(\lambda)^{-1} = \widetilde{U}_l\widetilde{U}_{l-1} \cdots \widetilde{U}_2\widetilde{U}_1$, where

$$\widetilde{U}_j = R(m - s_j, i_j)\, Q(m - s_j + i_j, c_j)$$

is associated with the pair (c_j, i_j) from CISS(σ) and consists of i_j consecutive R^{-1}-factors and c_j consecutive $(Q^{\mathcal{B}})^{-1}$-factors defined in Lemma 3.9. Recall that $s_0 = 0, s_j = \sum_{k=1}^{j}(c_k + i_k)$, for $j = 1, \ldots, l$, and $s_l = m - 1$. A direct multiplication shows that $\widetilde{U}_j =$

$$
\left[\begin{array}{c|cccc|c|ccccc|c}
I_{(m-s_j-1)n} & & & & & & & & & & & \\ \hline
 & -P_{m-s_j} & \cdots & \cdots & -P_{m-s_j+i_j-1} & I_n & & & & & & \\
 & I_n & 0 & \cdots & 0 & 0 & & & & & & \\
 & & \ddots & \ddots & \vdots & \vdots & & & & & & \\
 & & & I_n & 0 & 0 & & & & & & \\ \hline
 & & & & I_n & 0 & & & & & & \\ \hline
 & & & & & -\lambda I_n & I_n & & & & & \\
 & & & & & & -\lambda I_n & I_n & & & & \\
 & & & & & & & -\lambda I_n & I_n & & & \\
 & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & -\lambda I_n & I_n & \\ \hline
 & & & & & & & & & & & I_{(s_j-1)n}
\end{array}\right]
$$

The central part of $\widetilde{U}_j$ is a $(i_j + 1 + c_j) \times (i_j + 1 + c_j)$ block matrix with $n \times n$ blocks, i.e., the order of the second diagonal block matrix in $\widetilde{U}_j$ is $(i_j \times i_j)n$ and the order of the fourth diagonal block matrix is $(c_j \times c_j)n$. To find out the last block column of $U(\lambda)^{-1}$, we calculate $\widetilde{U}_l\widetilde{U}_{l-1}\cdots\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n)$. If $c_1 > 0$ then for $j = 1$ we have $\widetilde{U}_1(e_m \otimes I_n) = e_m \otimes I_n$, since for $j = 1$ we have $s_0 = 0$. Now for $j = 2$, the order of the last diagonal block is $s_1 n$. So $\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_m \otimes I_n$. Applying induction on $j = 1, \ldots, l$ we have

$$U(\lambda)^{-1}(e_m \otimes I_n) = \widetilde{U}_j\widetilde{U}_{j-1}\cdots\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_m \otimes I_n.$$

If $c_1 = 0$ then in $\widetilde{U}_1$ the fourth diagonal block matrix is not there. So for $j = 1$ we have $s_0 = 0$, and hence $\widetilde{U}_1(e_m \otimes I_n) = e_{m-i_1} \otimes I_n$. Similarly, $\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_{m-i_1} \otimes I_n$. Applying induction on $j = 1, \ldots, l$, we have

$$U(\lambda)^{-1}(e_m \otimes I_n) = \widetilde{U}_j\widetilde{U}_{j-1}\cdots\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_{m-i_1} \otimes I_n.$$

Hence the result follows. □

We have seen that the last block column of $U(\lambda)^{-1}$ is independent of λ. Now we show that the last block row of $V(\lambda)^{-1}$ is independent of λ.

Lemma 3.12. *Let $P_k(\lambda)$ for $k = 0, \ldots, m$ be the Horner shifts of the matrix polynomial $P(\lambda)$. Also, let Q_i^{-1} and R_i^{-1} for $i = 1, \ldots, m-1$ be as in (3.4) and (3.6), respectively. Then for each $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-i$, we have*

$$\mathbf{Q}^{-1}(i,j) := Q_{i+j-1}^{-1} \cdots Q_{i+1}^{-1} Q_i^{-1}$$

$$= \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & I_n & -\lambda I_n & 0 & \dots & 0 & \\ & & I_n & -\lambda I_n & & 0 & \\ & & & I_n & \ddots & \vdots & \\ & & & & \ddots & -\lambda I_n & \\ & & & & & I_n & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right],$$

$$\mathbf{R}^{-1}(i,j) := R_{i+j-1}^{-1} \cdots R_{i+1}^{-1} R_i^{-1}$$

$$= \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & -P_i(\lambda) & I_n & & & & \\ & -P_{i+1}(\lambda) & 0 & I_n & & & \\ & \vdots & \ddots & \ddots & \ddots & & \\ & -P_{i+j-1}(\lambda) & & \ddots & 0 & I_n & \\ & I & \dots & \dots & 0 & 0 & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right].$$

Proof. Proof is similar to that of Lemma 3.9 and follows by induction on the number of factors. $\square$

Theorem 3.13. *Let $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ and let $V(\lambda)$ be as in Theorem 3.10. If CISS(σ) $= (c_1, i_1, \dots, c_l, i_l)$ then $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_{m-c_1}^T \otimes I_n$.*

Proof. We have

$$V(\lambda)^{-1} = V_0^{-1} V_1^{-1} \cdots V_{m-3}^{-1} V_{m-2}^{-1} \text{ with}$$

$$V_i^{-1} = \begin{cases} R_{m-(i+1)}^{-1} & \text{if } \sigma \text{ has a consecution at } i \\ Q_{m-(i+1)}^{-1} & \text{if } \sigma \text{ has an inversion at } i, \end{cases} \tag{3.8}$$

where Q_i^{-1} and R_i^{-1} are given in (3.4) and (3.6). Since CISS(σ) $= (c_1, i_1, \dots, c_l, i_l)$, the factors of $V(\lambda)^{-1}$ given in (3.8) can be grouped together so that $V(\lambda)^{-1} = \widetilde{V}_1 \widetilde{V}_2 \cdots \widetilde{V}_l$, where

$$\widetilde{V}_j = \mathbf{R}^{-1}(m - s_j + i_j, c_j)\, \mathbf{Q}^{-1}(m - s_j, i_j)$$

is associated with the pair (c_j, i_j) from CISS(σ), and consists of c_j consecutive R^{-1}-factors and i_j consecutive Q^{-1}-factors defined in Lemma 3.12. A direct multiplication shows that

$$
\widetilde{V}_j = \left[\begin{array}{c|cccc|cccccc|c}
I_{(m-s_j-1)n} & & & & & & & & & & & \\
\hline
 & I_n & -\lambda I_n & \dots & 0 & 0 & & & & & & \\
 & & I_n & \ddots & \vdots & \vdots & & & & & & \\
 & & & \ddots & -\lambda I_n & 0 & & & & & & \\
 & & & & I_n & -\lambda I_n & & & & & & \\
\hline
 & & & & & -P_{m-s_j+i_j} & I_n & & & & & \\
 & & & & & -P_{m-s_j+i_j+1} & 0 & I_n & & & & \\
 & & & & & \vdots & & \ddots & \ddots & & & \\
 & & & & & \vdots & \ddots & & \ddots & \ddots & & \\
 & & & & & -P_{m-s_j+i_j+c_j-1} & & \ddots & & \ddots & I_n & \\
 & & & & & I_n & \dots & & \dots & 0 & 0 & \\
\hline
 & & & & & & & & & & & I_{(s_j-1)n}
\end{array}\right]
$$

The central four blocks of $\widetilde{V}_j$ are a $(i_j+c_j+1)\times(i_j+c_j+1)$ block matrix with $n\times n$ blocks. To find out the last block row of $V^{-1}(\lambda)$, we calculate $(e_m^T \otimes I_n)\widetilde{V}_1\widetilde{V}_2\cdots\widetilde{V}_l$. Applying the induction on $j=1,\dots,l$, we have $(e_m^T\otimes I_n)\widetilde{V}_1\widetilde{V}_2\cdots\widetilde{V}_j = e_{m-c_1}^T\otimes I_n$. Hence the result follows. $\square$

As a byproduct of Theorems 3.11 and 3.13, we obtain an alternative proof of the fact that a Fiedler pencil of $\mathcal{S}(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$; see [1] for an alternative approach.

Theorem 3.14 *(Linearization). Let $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be a Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection $\sigma:\{0,1,\dots,m-1\}\to\{1,2,\dots,m\}$. Then $\mathbb{L}_\sigma(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$, that is, we have*

$$
\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} -I_{n(m-1)} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] \tag{3.9}
$$

for all $\lambda\in\mathbb{C}$, where $U(\lambda)$ and $V(\lambda)$ are given in Theorem 3.10.

Proof. Suppose that $\mathrm{CISS}(\sigma) = (c_1, i_1, c_2, i_2, \dots, c_l, i_l)$. Then by (2.1) we have

$$
\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & C_\sigma \\ \hline B_\sigma & A-\lambda E \end{array}\right],
$$

where $B_\sigma = e_{m-c_1}^T\otimes B$, $C_\sigma = e_m\otimes C$ when $c_1>0$ and $C_\sigma = e_{m-i_1}\otimes C$ when $c_1 = 0$.

By Theorem 3.10, we have

$$
\begin{bmatrix} -I_{(m-1)n} & \\ & P(\lambda) \end{bmatrix} = U(\lambda)L_\sigma(\lambda)V(\lambda).
$$

Consequently, we have

$$\left[\begin{array}{c|c} -I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -I_{(m-1)n} & & 0 \\ & P(\lambda) & C \\ \hline 0 & B & A-\lambda E \end{array}\right] = \left[\begin{array}{c|c} U(\lambda)L_\sigma(\lambda)V(\lambda) & e_m \otimes C \\ \hline e_m^T \otimes B & (A-\lambda E) \end{array}\right]$$

$$= \left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} L_\sigma(\lambda) & U(\lambda)^{-1}(e_m \otimes C) \\ \hline (e_m^T \otimes B)V(\lambda)^{-1} & (A-\lambda E) \end{array}\right] \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right].$$

The last equality follows from the fact that by Theorems 3.11 and 3.13, we have

$$U(\lambda)^{-1}(e_m \otimes I_n) = \begin{cases} e_m \otimes I_n, & \text{if } c_1 > 0, \\ e_{m-i_1} \otimes I_n, & \text{if } c_1 = 0, \end{cases}$$

and $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_{m-c_1}^T \otimes I_n$. This completes the proof. □

We are now ready to recover eigenvectors of $\mathcal{S}(\lambda)$ from those of the Fiedler pencils of $\mathcal{S}(\lambda)$.

Theorem 3.15 *(Eigenvector from Fiedler pencil). Let $\mathcal{S}(\lambda)$ given in (1.5) be regular. Let $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Let $\mathbb{F}_\sigma(\mathcal{S})$ and $\mathbb{K}_\sigma(\mathcal{S})$, respectively, denote the eigenvector recovery maps corresponding to $\mathbb{L}_\sigma(\lambda)$ as defined in Theorem 3.3. If CISS$(\sigma) := (c_1, i_1, c_2, i_2, \ldots, c_l, i_l)$ then*

$$\mathbb{F}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} (e_{m-c_1}^T \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right] \quad \text{and} \quad \mathbb{K}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} K_\sigma & 0 \\ \hline 0 & I_r \end{array}\right],$$

where $K_\sigma = e_m^T \otimes I_n$ when $c_1 > 0$ and $K_\sigma = e_{m-i_1}^T \otimes I_n$ when $c_1 = 0$. Thus we have the following:

(a) *If $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ then $x := \begin{bmatrix} (e_{(m-c_1)}^T \otimes I_n)u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$, where $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$.*

(b) *Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *be a basis of* $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ *with* $\mathbf{v}_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$, *where* $u_j \in \mathbb{C}^{nm}$ *and* $v_j \in \mathbb{C}^r$. *Define* $x_j := \begin{bmatrix} (e^T_{(m-c_1)} \otimes I_n)u_j \\ v_j \end{bmatrix}$ *for* $j = 1, \ldots, p$. *Then* $\{x_1, \ldots, x_p\}$ *is a basis of* $\mathcal{N}_r(\mathcal{S}(\lambda))$.

(c) *Let* $\begin{bmatrix} u \\ w \end{bmatrix} \in \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ *with* $u \in \mathbb{C}^{nm}$ *and* $w \in \mathbb{C}^r$. *Define* $y := \begin{bmatrix} (e^T_m \otimes I_n)u \\ w \end{bmatrix}$ *if* $c_1 > 0$, *and* $y := \begin{bmatrix} (e^T_{(m-i_1)} \otimes I_n)u \\ w \end{bmatrix}$ *if* $c_1 = 0$. *Then* $y \in \mathcal{N}_l(\mathcal{S}(\lambda))$.

(d) *Let* $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ *be a basis of* $\mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ *with* $\mathbf{w}_j = \begin{bmatrix} u_j \\ w_j \end{bmatrix}$, *where* $u_j \in \mathbb{C}^{nm}$ *and* $w_j \in \mathbb{C}^r$. *For* $j = 1, \ldots, p$, *define* $y_j := \begin{bmatrix} (e^T_{(m-i_1)} \otimes I_n)u_j \\ w_j \end{bmatrix}$ *if* $c_1 = 0$, *and* $y_j := \begin{bmatrix} (e^T_m \otimes I_n)u_j \\ w_j \end{bmatrix}$ *if* $c_1 > 0$. *Then* $\{y_1, \ldots, y_p\}$ *is a basis of* $\mathcal{N}_l(\mathcal{S}(\lambda))$.

Proof. In view of Theorems 3.3 and 3.14, the block entries of the matrices $\mathbb{F}_\sigma(\mathcal{S})$ and $\mathbb{K}_\sigma(\mathcal{S})$ immediately follow from Theorem 3.13 and Theorem 3.11, respectively.

(a) Since $x = \mathbb{F}_\sigma(\mathcal{S}) \begin{bmatrix} u \\ v \end{bmatrix}$ and $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism, it follows that $x \in \mathcal{N}_r(\mathcal{S}(\lambda))$.
(b) Let $\mathcal{B}_r := \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and $\mathcal{E}_r := \{x_1, \ldots, x_p\}$. Since $\mathbb{F}_\sigma(\mathcal{S})$ is an isomorphism, the set $\mathcal{E}_r = \mathbb{F}_\sigma(\mathcal{S})(\mathcal{B}_r)$ is a basis for $\mathcal{N}_r(\mathcal{S}(\lambda))$.
(c) Since $y = \mathbb{K}_\sigma(\mathcal{S}) \begin{bmatrix} u \\ w \end{bmatrix}$ and $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S})) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism, it follows that $y \in \mathcal{N}_l(\mathcal{S}(\lambda))$.
(d) Let $C_l := \{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ and $D_l := \{y_1, \ldots, y_p\}$. Since $\mathbb{K}_\sigma(\mathcal{S})$ is an isomorphism, the set $D_l = \mathbb{K}_\sigma(\mathcal{S})(C_l)$ is a basis for $\mathcal{N}_l(\mathcal{S}(\lambda))$.

This completes the proof. $\square$

Remark 3.16. Observe that the eigenvector recovery maps $\mathbb{F}_\sigma(\mathcal{S})$ and $\mathbb{K}_\sigma(\mathcal{S})$ are operation free maps in the sense that the computation of $\mathbb{F}_\sigma(\mathcal{S})\mathbf{x}$ and $\mathbb{K}_\sigma(\mathcal{S})\mathbf{y}$ do not require any arithmetic operations to be performed on the vectors $\mathbf{x}$ and $\mathbf{y}$.

We now illustrate eigenvector recovery from the first companion form of $\mathcal{S}(\lambda)$ given in (1.3)

$$\mathcal{C}(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2} \cdots \mathbb{M}_1\mathbb{M}_0 = \lambda \mathbb{M}_m - \mathbb{M}_\sigma.$$

Then we have $\mathrm{CISS}(\sigma) = (c_1, i_1) = (0, m-1)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$ and let $\begin{bmatrix} u^T, & v^T \end{bmatrix}^T$ and $\begin{bmatrix} w^T, & z^T \end{bmatrix}^T$ be corresponding right and left eigenvectors of $\mathcal{C}(\lambda)$, where $u, w \in \mathbb{C}^{mn}$ and $v, z \in \mathbb{C}^r$. Then by Theorem 3.15,

$$
\begin{aligned}
x &= \mathbb{F}_\sigma(\mathcal{S}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (e_{m-c_1}^T \otimes I_n)u \\ v \end{bmatrix} = \begin{bmatrix} (e_m^T \otimes I_n)u \\ v \end{bmatrix}, \\
y &= \mathbb{K}_\sigma(\mathcal{S}) \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} (e_{m-i_1}^T \otimes I_n)w \\ z \end{bmatrix} = \begin{bmatrix} (e_1^T \otimes I_n)w \\ z \end{bmatrix}
\end{aligned}
$$

are right and left eigenvectors of $\mathcal{S}(\lambda)$.

Next, consider the second companion form of $\mathcal{S}(\lambda)$ given by [1]

$$
\mathcal{C}_2(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_0 \mathbb{M}_1 \cdots \mathbb{M}_{m-1} = \lambda \mathbb{M}_m - \mathbb{M}_\sigma.
$$

Then we have

$$
\mathcal{C}_2(\lambda) = \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array} \right] - \left[\begin{array}{ccccc|c} -A_{m-1} & I_n & & & & \\ -A_{m-2} & 0 & \ddots & & & \\ \vdots & & \ddots & & I_n & \\ -A_0 & \cdots & & 0 & 0 & -C \\ \hline -B & & & & & -A \end{array} \right]. \tag{3.10}
$$

In this case, $\mathrm{CISS}(\sigma) = (c_1, i_1) = (m-1, 0)$. Consequently, if $\begin{bmatrix} u^T, & v^T \end{bmatrix}^T$ and $\begin{bmatrix} w^T, & z^T \end{bmatrix}^T$ are right and left eigenvectors of $\mathcal{C}_2(\lambda)$ then by Theorem 3.15,

$$
x = \mathbb{F}_\sigma(\mathcal{S}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (e_1^T \otimes I_n)u \\ v \end{bmatrix} \text{ and } y = \mathbb{K}_\sigma(\mathcal{S}) \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} (e_m^T \otimes I_n)w \\ z \end{bmatrix}
$$

are right and left eigenvectors of $\mathcal{S}(\lambda)$.

4. Eigenvector recovery for rational matrix functions

We now describe recovery of eigenvectors of rational matrix functions from their linearizations. Recall that $\lambda_0 \in \mathbb{C}$ is an eigenvalue of a regular $n \times n$ rational matrix function $G(\lambda)$ provided that $\mathrm{rank}(G(\lambda_0)) < n$.

Theorem 4.1 *(Eigenvector recovery maps). Let $\mathcal{S}(\lambda)$, $\mathbb{L}(\lambda)$, $U(\lambda)$ and $V(\lambda)$ be as in Theorem 3.3. Let $G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B$ be regular and λ_0 be an eigenvalue of $G(\lambda)$.*

(a) *Define* $\mathcal{E}(G) : \mathbb{C}^n \to \mathbb{C}^{nm+r}$ *and* $\mathcal{F}(G) : \mathbb{C}^{nm+r} \to \mathbb{C}^n$ *by*

$$\mathcal{E}(G) := \begin{bmatrix} V(\lambda_0)(e_m \otimes I_n) \\ \\ (\lambda_0 E - A)^{-1} B \end{bmatrix} \text{ and } \mathcal{F}(G) := \left[(e_m^T \otimes I_n) V(\lambda_0)^{-1} \;\middle|\; 0_{n\times r} \right].$$

Then $\mathcal{E}(G)$ *is left invertible and* $\mathcal{F}(G)$ *is a left inverse of* $\mathcal{E}(G)$*. Further, the restricted maps* $\mathcal{E}(G) : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathbb{L}(\lambda_0))$ *and* $\mathcal{F}(G) : \mathcal{N}_r(\mathbb{L}(\lambda_0)) \to \mathcal{N}_r(G(\lambda_0))$ *are isomorphisms.*

(b) *Define* $\mathcal{H}(G) : \mathbb{C}^n \to \mathbb{C}^{nm+r}$ *and* $\mathcal{K}(G) : \mathbb{C}^{nm+r} \to \mathbb{C}^n$ *by*

$$\mathcal{H}(G) := \begin{bmatrix} U(\lambda_0)^T (e_m \otimes I_n) \\ \\ C(\lambda_0 E - A)^{-T} \end{bmatrix} \text{ and } \mathcal{K}(G) := \left[(e_m^T \otimes I_n) U(\lambda_0)^{-T} \;\middle|\; 0_{n\times r} \right].$$

Then $\mathcal{H}(G)$ *is left invertible and* $\mathcal{K}(G)$ *is a left inverse of* $\mathcal{H}(G)$*. Further, the restricted maps* $\mathcal{H}(G) : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathbb{L}(\lambda_0))$ *and* $\mathcal{K}(G) : \mathcal{N}_l(\mathbb{L}(\lambda_0)) \to \mathcal{N}_l(G(\lambda_0))$ *are isomorphisms.*

Proof. Obviously $\mathcal{F}(G)$ is a left inverse of $\mathcal{E}(G)$ and $\mathcal{K}(G)$ is a left inverse of $\mathcal{H}(G)$, that is, $\mathcal{F}(G)\mathcal{E}(G) = I_n$ and $\mathcal{K}(G)\mathcal{H}(G) = I_n$.

Now define $f : \mathbb{C}^n \to \mathbb{C}^{n+r}$ and $g : \mathbb{C}^n \to \mathbb{C}^{n+r}$ by

$$f(x) := \begin{bmatrix} x \\ (\lambda_0 E - A)^{-1} B x \end{bmatrix} \text{ and } g(x) := \begin{bmatrix} x \\ (C(\lambda_0 E - A)^{-1})^T x \end{bmatrix}.$$

Then the maps $f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathcal{S}(\lambda_0))$ and $g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathcal{S}(\lambda_0))$ are isomorphisms. Indeed, let $x \in \mathcal{N}_r(G(\lambda_0))$. Then $G(\lambda_0)x = 0$. Now

$$\begin{aligned} \mathcal{S}(\lambda_0) \begin{bmatrix} x \\ (\lambda_0 E - A)^{-1} B x \end{bmatrix} &= \left[\begin{array}{c|c} P(\lambda_0) & C \\ \hline B & A - \lambda_0 E \end{array}\right] \begin{bmatrix} x \\ (\lambda_0 E - A)^{-1} B x \end{bmatrix} \\ &= \begin{bmatrix} G(\lambda_0)x \\ Bx - Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

which shows that $f(x) \in \mathcal{N}_r(\mathcal{S}(\lambda_0))$ and $f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathcal{S}(\lambda_0))$. Obviously f is injective. On the other hand, if $\left[u^T,\ v^T\right]^T \in \mathcal{N}_r(\mathcal{S}(\lambda_0))$ then it is easy to see that $G(\lambda_0)u = 0$ and $v = (\lambda_0 E - A)^{-1} B u$. Hence $\left[u^T,\ v^T\right]^T = f(u)$ and $u \in \mathcal{N}_r(G(\lambda_0))$. This shows that $f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathcal{S}(\lambda_0))$ is an isomorphism. Similarly, the map $g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathcal{S}(\lambda_0))$ is an isomorphism.

Since $f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathcal{S}(\lambda_0))$ and $g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathcal{S}(\lambda_0))$ are isomorphisms and, by Theorem 3.3, $\mathbb{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda_0)) \to \mathcal{N}_r(\mathbb{L}(\lambda_0))$ and $\mathbb{H}(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda_0)) \to$

$\mathcal{N}_l(\mathbb{L}(\lambda_0))$ are isomorphisms, it follows that $\mathcal{E}(G) = \mathbb{E}(\mathcal{S}) \circ f : \mathcal{N}_r(G(\lambda_0)) \to \mathcal{N}_r(\mathbb{L}(\lambda_0))$ and $\mathcal{H}(G) = \mathbb{H}(\mathcal{S}) \circ g : \mathcal{N}_l(G(\lambda_0)) \to \mathcal{N}_l(\mathbb{L}(\lambda_0))$ are isomorphisms. This completes the proof. □

Remark 4.2. We refer to $\mathcal{F}(G)$ and $\mathcal{K}(G)$, respectively, as the right and the left eigenvector recovery maps for $G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B$.

Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of $G(\lambda)$. Then in view of Theorem 4.1 and Theorem 3.15, we have the following result whose proof is immediate.

Corollary 4.3. *Let $G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B$ be regular and $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Let $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be a Fiedler pencil of $G(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Let $\mathcal{F}_\sigma(G)$ and $\mathcal{K}_\sigma(G)$, respectively, denote the right and the left eigenvector recovery maps corresponding to $\mathbb{L}_\sigma(\lambda)$ as defined in Theorem 4.1. If CISS$(\sigma) := (c_1, i_1, c_2, i_2, \ldots, c_l, i_l)$ then*

$$\mathcal{F}_\sigma(G) := \left[\, (e^T_{m-c_1} \otimes I_n) \;\middle|\; 0_{n\times r} \,\right] \quad \text{and} \quad \mathcal{K}_\sigma(G) := \left[\, K_\sigma \;\middle|\; 0_{n\times r} \,\right],$$

where $K_\sigma = e^T_m \otimes I_n$ when $c_1 > 0$ and $K_\sigma = e^T_{m-i_1} \otimes I_n$ when $c_1 = 0$. Thus we have the following.

(a) *If $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda_0))$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$ then $x := (e^T_{m-c_1} \otimes I_n)u \in \mathcal{N}_r(G(\lambda_0))$.*

(b) *Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a basis of $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda_0))$ with $\mathbf{v}_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$, where $u_j \in \mathbb{C}^{nm}$ and $v_j \in \mathbb{C}^r$. Define $x_j := (e^T_{m-c_1} \otimes I_n)u_j$ for $j = 1, \ldots, p$. Then $\{x_1, \ldots, x_p\}$ is a basis of $\mathcal{N}_r(G(\lambda_0))$.*

(c) *Let $\begin{bmatrix} u \\ w \end{bmatrix} \in \mathcal{N}_l(\mathbb{L}_\sigma(\lambda_0))$ with $u \in \mathbb{C}^{nm}$ and $w \in \mathbb{C}^r$. Define $y := (e^T_m \otimes I_n)u$ if $c_1 > 0$, and $y := (e^T_{m-i_1} \otimes I_n)u$ if $c_1 = 0$. Then $y \in \mathcal{N}_l(G(\lambda_0))$.*

(d) *Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ be a basis of $\mathcal{N}_l(\mathbb{L}_\sigma(\lambda_0))$ with $\mathbf{w}_j = \begin{bmatrix} u_j \\ w_j \end{bmatrix}$, where $u_j \in \mathbb{C}^{nm}$ and $w_j \in \mathbb{C}^r$. For $j = 1, \ldots, p$, define $y_j := (e^T_{m-i_1} \otimes I_n)u_j$ if $c_1 = 0$, and $y_j := (e^T_m \otimes I_n)u_j$ if $c_1 > 0$. Then $\{y_1, \ldots, y_p\}$ is a basis of $\mathcal{N}_l(G(\lambda_0))$.*

Consider the first and the second companion forms of $G(\lambda)$ given in (1.3) and (3.10). For the first companion form $\mathcal{C}(\lambda)$, we have CISS$(\sigma) = (c_1, i_1) = (0, m-1)$, and for the second companion form $\mathcal{C}_2(\lambda)$, we have CISS$(\sigma) = (c_1, i_1) = (m-1, 0)$. Suppose that $\mathcal{C}(\lambda) \begin{bmatrix} u \\ v \end{bmatrix} = 0$ for some nonzero $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^{nm+r}$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$.

By Corollary 4.3, we have $x := (e_{m-c_1}^T \otimes I_n)u = (e_m^T \otimes I_n)u$ is a right eigenvector of $G(\lambda)$. On the other hand, if $\begin{bmatrix} u \\ v \end{bmatrix}^T \mathcal{C}(\lambda) = 0$ then $y := (e_{m-i_1}^T \otimes I_n)u = ((e_1^T \otimes I_n)u$ is a left eigenvector of $G(\lambda)$. Similarly, if $\mathcal{C}_2(\lambda) \begin{bmatrix} v \\ w \end{bmatrix} = 0$ and $\begin{bmatrix} u \\ z \end{bmatrix}^T \mathcal{C}_2(\lambda) = 0$ then $x := (e_{m-c_1}^T \otimes I_n)v = ((e_1^T \otimes I_n)v$ is a right eigenvector of $G(\lambda)$ and $y := (e_m^T \otimes I_n)u$ is a left eigenvector of $G(\lambda)$. Indeed, it is easy to see that

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} x \\ (\lambda A_m + A_{m-1})x \\ \vdots \\ (\lambda^{m-1}A_m + \lambda^{m-2}A_{m-1} + \cdots + A_1)x \\ (\lambda E - A)^{-1}Bx \end{bmatrix} \text{ and } \begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} \lambda^{m-1}y \\ \lambda^{m-2}y \\ \vdots \\ y \\ (C(\lambda E - A)^{-1})^T y \end{bmatrix}.$$

We end this section by considering a Fiedler pencil which is different from companion forms of $G(\lambda)$ and recover eigenvectors of $G(\lambda)$.

Example 4.4. Consider $G(\lambda) = A_4\lambda^4 + \cdots + A_0 + C(\lambda E - A)^{-1}B$ and the Fiedler pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_3 = \lambda\mathbb{M}_4 - \mathbb{M}_\sigma$ of $G(\lambda)$. Then $\mathrm{CISS}(\sigma) = (c_1, i_1, c_2, i_2) = (1, 1, 1, 0)$. Thus if $\mathbb{L}_\sigma(\lambda) \begin{bmatrix} v \\ w \end{bmatrix} = 0$ and $\begin{bmatrix} u \\ z \end{bmatrix}^T \mathbb{L}_\sigma(\lambda) = 0$ then $x := (e_{m-c_1}^T \otimes I_n)v = ((e_3^T \otimes I_n)v$ is a right eigenvector of $G(\lambda)$ and $y := (e_m^T \otimes I_n)u = (e_4^T \otimes I_n)u$ is a left eigenvector of $G(\lambda)$. In fact, it can be shown that

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda P_1(\lambda)x \\ x \\ P_3(\lambda)x \\ (\lambda E - A)^{-1}Bx \end{bmatrix} \text{ and } \begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} \lambda^2 y \\ \lambda y \\ \lambda P_2(\lambda)^T y \\ y \\ (C(\lambda E - A)^{-1})^T y \end{bmatrix},$$

where $P_j(\lambda)$ are Horner shifts. □

5. Conclusions

We have considered Fiedler pencils of rational matrix functions and have shown that the Fiedler pencils allow operation free recovery of eigenvectors of the rational matrix functions, that is, the eigenvectors of the rational matrix functions are recovered from the eigenvectors of the Fiedler pencils without performing any arithmetic operations. In fact, we have explicitly determined the right and the left eigenvector recovery maps which provide isomorphisms between null spaces of the rational matrix functions and the associated Fiedler pencils. We have shown that the recovery maps are operation free, that is, the evaluation of the maps on vectors do no require arithmetic operations. As a byproduct, we have shown that the recovery maps also recover eigenvectors of the Rosenbrock system matrix associated with an LTI state-space system in SSF from the Fiedler pencils of the system matrix.

References

[1] R. Alam, N. Behera, Linearizations for rational matrix functions and Rosenbrock system polynomials, SIAM J. Matrix Anal. Appl. 37 (2016) 354–380.
[2] E.N. Antoniou, S. Vologiannidis, A new family of companion forms of polynomial matrices, Electron. J. Linear Algebra 11 (2004) 78–87.
[3] N. Behera, Fiedler linearizations for LTI state-space systems and for rational eigenvalue problems, PhD Thesis, IIT Guwahati, 2014.
[4] F. De Terán, F.M. Dopico, D.S. Mackey, Fiedler companion linearizations and the recovery of minimal indices, SIAM J. Matrix Anal. Appl. 31 (2009–2010) 2181–2204.
[5] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press Inc., New York, London, 1982.
[6] T. Kailath, Linear Systems, Prentice-Hall Inc., Englewood Cliffs, N.J., 1980.
[7] D.S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Vector spaces of linearizations for matrix polynomials, SIAM J. Matrix Anal. Appl. 28 (2006) 971–1004.
[8] V. Mehrmann, H. Voss, Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods, GAMM-Mitt. 27 (2004) 121–152.
[9] H.H. Rosenbrock, State-Space and Multivariable Theory, John Wiley & Sons, Inc., New York, 1970.
[10] Y. Su, Z. Bai, Solving rational eigenvalue problems via linearization, SIAM J. Matrix Anal. Appl. 32 (2011) 201–216.
[11] A.I.G. Vardulakis, Linear Multivariable Control, John Wiley & Sons Ltd., 1991.
[12] H. Voss, A rational spectral problem in fluid–solid vibration, Electron. Trans. Numer. Anal. 16 (2003) 93–105.
[13] H. Voss, Iterative projection methods for computing relevant energy states of a quantum dot, J. Comput. Phys. 217 (2006) 824–833.