Journal of Computational and Applied Mathematics 312 (2017) 174–180

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Statistical analysis of solution accuracy for inverse problems in electrodynamics

Alexander M. Vishnevsky, Anna A. Firsova *

Department of ship electrodynamics and hydrophysics, Krylov State Research Centre, St. Petersburg, Russian Federation

ARTICLE INFO

Article history:
Received 30 September 2015
Received in revised form 29 January 2016

Keywords:
Inverse problem
Functional calculation
Error estimates
Statistical analysis
Monte-Carlo simulation

ABSTRACT

Calculation of electric/magnetic field parameters and source intensity based on measurements is discussed. It is required to assess the accuracy of obtained solution. Upper estimates of the measurement error limits and analytical approaches to error calculations give considerable overestimations and prove to be inefficient for practical applications. The approach based on statistical estimate of solution error is proposed. The obtained error estimate is in good agreement with experimental data.

1. Introduction

Constant magnetic field and static electric field of various engineering objects are considered. Let us describe the field sources by function u considered as an element of normalized space $U \quad (u \in U)$, the measured field values (input data f) are considered as an element of the normed space $F \quad (f \in F)$. The relation between u and f is governed by the operator equation

$$Au = f, \tag{1}$$

where operator A continuously represents U in F.

Determination of sources u based on the known values of field f necessitates solution of an inverse problem. In some cases, it is useful to determine functionals based on distribution of sources: the problem for calculation of values is formulated

$$g = Cu, \tag{2}$$

where C is an operator acting from the normed space U to the space G.

For the magnetic field the function u may represent magnetization $\vec{M} = (M_X, M_Y, M_Z)^T$, f—magnetic flux density in measurement points $\vec{B} = (B_X, B_Y, B_Z)^T$, and g—multipole moments or magnetic flux density in locations other than measurement points (field extrapolation). In this case, the relation between the values of the field and the sources is given by the equation

$$\int_V K(x, y, z, x', y', z')\vec{M}(x', y', z')\, dV = \vec{B}(x, y, z), \tag{3}$$

* Corresponding author.
E-mail address: nurka100@mail.ru (A.A. Firsova).

http://dx.doi.org/10.1016/j.cam.2016.03.015

where K is an equation kernel that can be presented as a matrix function

$$K = \begin{pmatrix} K_{XX} & K_{XY} & K_{XZ} \\ K_{YX} & K_{YY} & K_{YZ} \\ K_{ZX} & K_{ZY} & K_{ZZ} \end{pmatrix},$$

x, y, z—coordinates of measurement point, x', y', z'—coordinates of volume point V.

Magnetic dipole moment $\vec{P} = (P_X, P_Y, P_Z)^T$ is calculated by the formula

$$\int_V \vec{M}(x', y', z')\, dV = \vec{P}. \tag{4}$$

In its turn, the function u may represent current source strength I for static electric field, f—the values of potential at measurement points φ, and g—multipole moments or values of the potential in other conditions than during the measurement. When the sources of current are located at the surface of object S, the values of the field are related to the sources by the equation

$$\int_S \Phi(x, y, z, x', y', z')\, I(x', y', z')\, dS = \varphi(x, y, z), \tag{5}$$

where Φ is an equation kernel.

Dipole electric moment is calculated as

$$\int_S I(x', y', z')\vec{r}\, dS = \vec{P}, \tag{6}$$

where $\vec{r} = (x', y', z')^T$.

This paper considers the class of problems with known source locations or known boundaries of source distribution but unknown strength. It is assumed that a priori information about the strength of field sources is not available.

Eqs. (3) and (5) are Fredholm equations of the 1st kind. The solution of these equations is an ill-posed problem.

The problem of magnetic field extrapolation based on equivalent surface sources is investigated in [1]. Directly, recovery of magnetic field sources, i.e. solution to Problem (1), is covered in [2–4]. The purpose of these papers is to develop algorithms for solving the inverse problem. No accuracy assessment for obtained solution to the inverse problem, or extrapolation error assessment for the general case is performed.

The purpose of this study is to estimate calculation error for the functionals of kind (2).

2. On the choice of solution method

In accordance with general notion regarding the solution of ill-posed problems, a regularization algorithm is formulated taking into account the information about the error of input data and operator. In case of Tikhonov method, a regularization parameter is selected based on residual method or generalized residual method and directly depends on the level of error. In iteration algorithms, the number of iterations depends on the error of input data. It has been theoretically proved that in general case it is impossible to obtain the solution to an incorrect problem if error of input data and operator are unknown [5,6]. However, in certain practical applications, a number of algorithms [7,8] that do not use error data might provide an acceptable solution.

Regularization error can be estimated for some classes of problems. In this case, a priori information is required for such estimation (for example, belonging of solution to some set) as well as the data on the error of input data and operator. In the considered statement of the problem, normal pseudosolution obtained on the basis of accurate input data is selected as an accurate solution for estimation of regularization error. However, the total error of problem solution cannot be found based on obtained estimations due to discrepancy between true (physical) solution and normal pseudosolution.

In classical theory of ill-posed problems, the obtained estimations of accuracy appear to be efficient at relatively low level of input data error but it is not always the case for solution of some practical problems.

When the field is measured in specially equipped laboratories, it is possible to obtain sufficiently accurate measurements and estimate the error of measurements. However, for engineering problems it may well be required to take measurements outside the laboratories when it is not feasible to move an object or it is too large in size. In such conditions, the external influence can be unavoidable and rather strong causing a large measurement error. When the actual error of input data is great, some methods of ill-posed problems solution directly relating the solution to measurement error (for example, residual method) are inefficient. In this case, it is difficult to estimate the accuracy of solution. Even if full-scale measurements are performed in well-controlled conditions (under moderate external influence), the error of measurement is governed by many factors and external effects and it is difficult to determine the actual error of measurement. In some cases, only upper error estimates can be obtained. However, if the upper estimates of measurement error are used for solution of inverse problem, it results in excessively smoothed solution and overestimation of obtained values in case of solution accuracy assessment.

Another feature of the problem is a small number of measurement points and limited space of measurement. If the number of measurements is small, the discrepancy between normal pseudosolution and true solution is the most significant. In this case, when the normal pseudosolution is calculated, the strength of sources at maximum distance from measurement points tends to minimum resulting in discrepancy with physical solution. In such conditions the error due to discrepancy between normal pseudosolution and physical solution is rather large and should be taken into consideration in total error of inverse problem solution along with regularization error.

Statistical method is proposed for estimation of solution error in case of problems related to determination of functionals based on the inverse problem solution.

The methods indirectly relating the solution u with the error of input data were used to solve Eq. (1): Tikhonov method [9] or truncated singular value decomposition (SVD) [8] with quasi-optimal algorithm for choosing the regularization parameter.

In a context where the final objective is to determine g functional, Tikhonov regularization parameter was chosen by means of a modified quasi-optimal method [10]. In accordance with this method the value of parameter α is selected in order to achieve a minimum of function $\psi(\alpha)$ determined by the expression

$$\psi(\alpha) = \frac{\left\| \alpha \frac{dg_\alpha}{d\alpha} \right\|}{\|g_\alpha\|}, \tag{7}$$

where g_α—the value of the functional obtained by solving the problem with parameter α.

This method for the selection of regularization parameter can also be applied for truncated SVD. In this case parameter α is related to the number of zeroed singular values.

Modified quasi-optimal method (7) is an empirical one. Its efficiency is confirmed by numerical experiments.

3. Error estimation based on statistical method

In order to estimate the error of solution, the sources u and the functional g were calculated using a set of test problems. Monte-Carlo method based on obtaining of a great number of random-process realizations was used to form such set.

The test problems were solved using the following procedure. Forward problem on calculation of characteristics of a given source system was formulated, and then it was numerically solved. Calculation results were used as input data to solve the inverse problem and as reference values. The error was introduced in the input data if it was required. The inverse problem was solved and the functionals were calculated. Calculation results were compared with the reference values. The error in determination of functionals was calculated

$$\delta^g = \frac{\left| g - g^{Exact} \right|}{\left\| g^{Exact} \right\|}, \tag{8}$$

where g^{Exact}—reference values obtained in solving of direct problem, g—the values obtained in solving of inverse problem.

When a great number of test problems are solved, the error δ^g is considered as a random value.

The analysis of numerical simulation results proved that random value δ^g has normal distribution. The parameters of normal distribution are estimated: mathematical expectation $\bar{\delta}^g$ and standard deviation $S(\delta^g)$.

The value as per the required probability of error span determination is considered as final error estimation, for example the three-sigma rule can be used.

Statistical method makes it possible to estimate the influence of various factors on the results (separately or jointly). The possible variants of random processes simulation considered for stating of test problems are given below.

Let us consider the estimation of method error when the input data are accurate. It depends on the method of problem solution, mesh of measurement points and complexity of field sources system. To solve practical problems, it is usually required to estimate the error of solution for fixed mesh of measurement points. Therefore, random process is represented by the field sources, i.e. their distribution and strength. One, two and three-dimensional models can be used for simulation of field sources. In this case, the model of sources used for accuracy estimation should not coincide with the model of sources used for solution of inverse problem. Simple models such as distribution of sources over some length, distribution of sources over the surface of simple geometry (for example, cylindrical surface) or distribution of sources in the volume of a simple geometrical figure (for example, parallelepiped) provide good results in estimation of error.

The elements of a system of sources might be point magnetic dipoles, and their number is assumed to be fixed. Element intensity is a random value, it has uniform distribution within specified limits $[-M_{\max}; M_{\max}]$. Another random value is position of field sources, element coordinates being distributed uniformly within the specified part of space.

The measurement error is simulated to estimate the error due to the error of input data and regularization error. For each sensor the error of measurement is considered as a sum of random measurement error (noise) and the error due to deterministic factors.

Random error is presented as white Gaussian noise—random value distributed according to the normal law with mathematical expectation equal to zero and pre-specified standard deviation. The value of standard deviation can be set equal to instrumental error of sensors. In case of multiple measurements the value of standard deviation is obtained

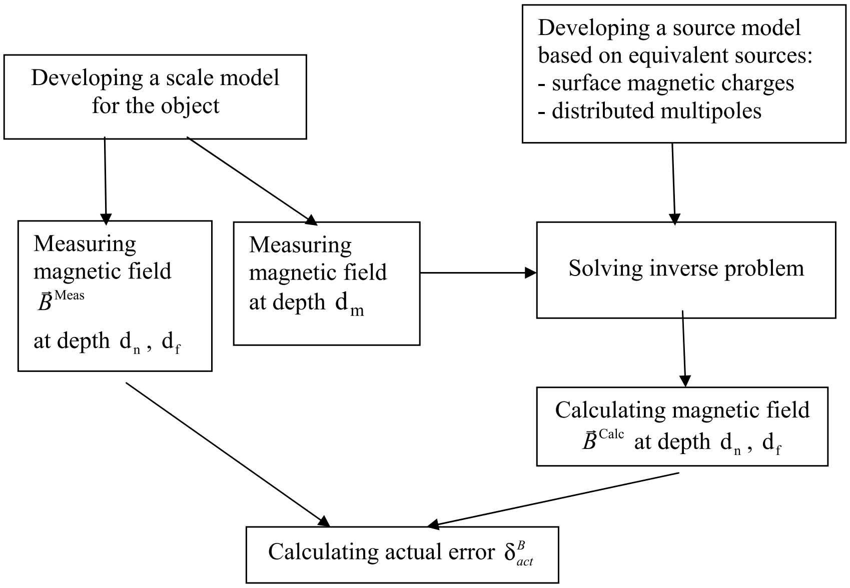

Fig. 1. Flowchart for calculation of actual extrapolation error.

by measurement data processing. It should be noted that the processing of multiple measurements makes it possible to determine only random measurement error but does not include systematic error.

The simulation of deterministic factors resulting in measurement error depends on the specific application task. Generally the measurement error in case of magnetic or electric field measurement is caused by

– error of sensor installation: deviation of the actual sensor position from the expected location, tilt of sensor axes,
– external noise-generating field sources.

The error of sensor installation can be considered as a random process for the system of measurement points (in case of a great number of sensors or multiple changes of one sensor position). The deviations of sensor position along the axes X, Y, Z appear to be a random value uniformly distributed at specified interval. Sensor tilt angle also appears to be a random value with uniform distribution. It should be noted that statistical method enables direct simulating the factors that govern the measurement error and setting the values of deviations in measurement units of this value.

Due to the lack of information about external noise-generating sources and possibility of source intensity changing during the measurement, appropriate component of error is also considered as a random process. The external field sources are simulated by distribution of local sources relatively close to measuring sensors. The random values are a number of sources, location of sources (X, Y, Z coordinates), dimensions and strength of sources. The given values have uniform distribution.

The total estimation of problem solution error is calculated based on the results of concurrent simulation of factors taken into consideration. The analysis of the effect of each individual factor makes it possible to reveal the main components of functional g calculation error.

4. Comparison of statistical estimates with laboratory data

The statistical estimations obtained were compared versus actual error in solution of inverse problems in order to check how adequate these estimations were. Actual error was calculated from the laboratory test data. Laboratory experiments included magnetic field measurements with subsequent extrapolation or calculation of dipole magnetic moment.

4.1. Extrapolation of magnetic field

For procedure of laboratory experiments and calculation of actual extrapolation error, see Fig. 1. Layout of measurement and extrapolation planes is shown in Fig. 2.

Actual extrapolation error was calculated according to the formula:

$$\delta^B_{act} = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\sum_{\xi=x,y,z}\left(B^{\text{Meas}}_{\xi i} - B^{\text{Calc}}_{\xi i}\right)^2}}{\max\limits_{i=1\ldots N}\left|B^{\text{Meas}}_{i}\right|} \cdot 100\%. \tag{9}$$

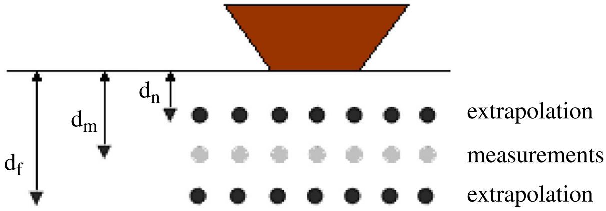

Fig. 2. Arrangement of measurement and extrapolation planes.

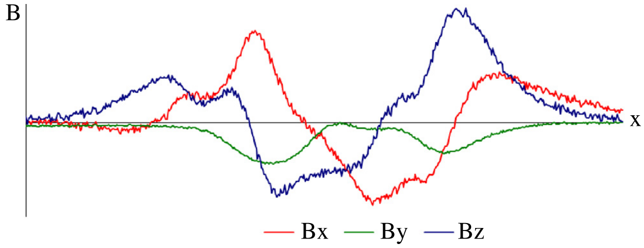

Fig. 3. Measured magnetic field of the physical model.

Table 1
Actual-versus-estimated extrapolation errors.

Model of sources in the inverse problem	Estimated error $S\left(\delta^B\right)$, %		Actual error δ^B_{act}, %	
	Depth d_n	Depth d_f	Depth d_n	Depth d_f
Surface magnetic charges	8	3	9	4
Distributed multipoles	12	3	15	5

Example of magnetic field measurement data at depth d_m is provided in Fig. 3.

For measurements of magnetic field executed under laboratory conditions, it is possible to determine the measurement error. For the measurements shown in Fig. 3, the main source of error was the white noise generating relative measurement error $\Delta_{lab} = 5\%$.

Statistical estimation of extrapolation error δ^B was performed for the following parameters:

- System of magnetic field sources consists of 50 magnetic dipoles distributed in the volume corresponding to the size of the scale model;
- Input data error was simulated by the white noise with level $\Delta_{lab} = 5\%$.

Statistical analysis results show that mathematical expectation of error $\bar{\delta}^B$ is close to zero, so further estimation of error will only imply standard deviation $S\left(\delta^B\right)$. Table 1 compares actual and estimated errors.

4.2. Determination of dipole moment

For procedure of laboratory experiments and calculation of actual error for dipole moment, see Fig. 4. A setup of sensors is shown in Fig. 5.

Actual error in determination of dipole moment was calculated as per the formula

$$\delta^{P_\xi}_{act} = \frac{\left|P^{Calc}_\xi - P^{Ref}_\xi\right|}{\left\|P^{Ref}\right\|} \cdot 100\%, \quad \xi = x, y, z. \tag{10}$$

Example of measurement data is given in Fig. 6.

Statistical estimation of error in the determination of dipole moment was performed for the following parameters:

- System of magnetic field sources consists of 20 magnetic dipoles distributed along a horizontal stretch;
- Input data error was simulated by the white noise with level $\Delta_{lab} = 5\%$.

Table 2 compares actual error and error estimation represented as standard deviation $S(\delta^{P_\xi})$.

The results contained in Tables 1 and 2 indicate that the algorithm proposed to estimate the functional calculation error provides results adequate to the actual error and can be used for engineering applications.

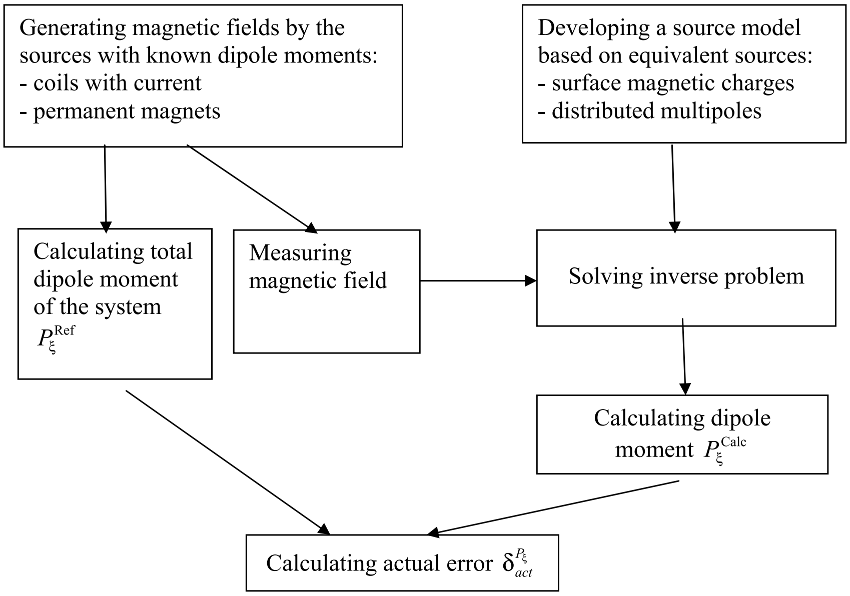

Fig. 4. Flowchart for calculation of actual error of dipole moment.

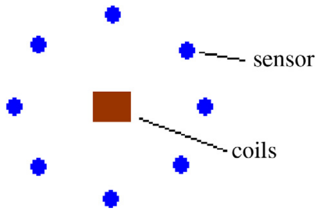

Fig. 5. Arrangement of sensors for magnetic moment calculation.

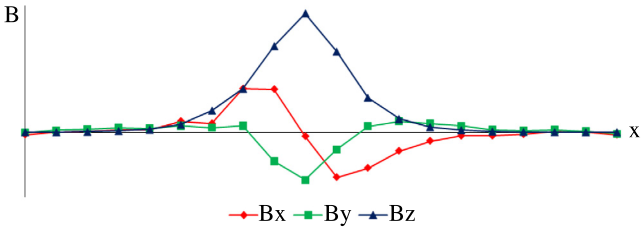

Fig. 6. Measured magnetic field for a system of coils with current.

Table 2
Actual-versus-estimated errors in determination of dipole moment.

Model of sources in the inverse problem	Estimated error, %			Actual error, %		
	$S(\delta^{P_X})$	$S(\delta^{P_Y})$	$S(\delta^{P_Z})$	$\delta_{act}^{P_X}$	$\delta_{act}^{P_Y}$	$\delta_{act}^{P_Z}$
Surface magnetic charges	15	5	5	10	7	5
Distributed multipoles	18	5	5	8	6	4

5. Conclusion

The paper addressed a statistical approach to error estimation of functional calculation based on inverse problem solutions. The main advantages of this approach are the following:

– easy to implement;
– makes it possible to assess the influence of various factors on the result and evaluate the effect of each individual factor;
– provides total error estimates;
– makes it possible to simulate factors of measurement errors in original values;

– estimates the solution accuracy at large measurement errors;
– gives no overestimation of error;
– statistical error estimates are in good agreement with experimental data.

References

[1] F.M. Duthoit, L. Krahenbuhl, A. Nicolas, The boundary integral equation method for the extrapolation of field measurement, IEEE Trans. Magn. 21 (1985) 2439–2442.
[2] D.V. Lukyanenko, A.G. Yagola, N.A. Evdokimova, Application of inversion methods in solving ill-posed problems for magnetic parameter identification of steel hull vessel, J. Inverse Ill-Posed Probl. 18 (2011) 1013–1029.
[3] O. Chadebec, J.-L. Coulomb, G. Cauffet, J.-P. Bongiraud, How to well pose a magnetization identification problem, IEEE Trans. Magn. 39 (2003) 1634–1637.
[4] G. Jeung, C. Yang, H. Chung, K. Lee, D. Kim, Magnetic dipole modeling combined with material sensitivity analysis for solving an inverse problem of thin ferromagnetic sheet, IEEE Trans. Magn. 45 (2009) 4169–4172.
[5] A.G. Yagola, A.S. Leonov, V.N. Titarenko, Data errors and an error estimation for ill-posed problems, Inverse Probl. Eng. 10 (2002) 117–129.
[6] A.B. Bakushinskii, Remark about the choice of the regularization parameter by the quasioptimality criterion and the propertion criterion, Comput. Math. Math. Phys. 24 (1984) 1258–1259.
[7] G.H. Golub, M. Heath, G. Wahba, Generalized cross-validation as a method for choosing a good ridge parameter, Technometrics 21 (1979) 215–222.
[8] P.C. Hansen, Rank-Deficient and Discrete Ill-Posed Problem, SIAM, Philadelphia, PA, 1998.
[9] A.N. Tikhonov, V.Y. Arsenin, Solutions of Ill-Posed Problems, Wiley, New York, 1977.
[10] A.M. Vishnevsky, A.A. Firsova, Choice of regularization parameter in the problem of magnetic field extrapolation, Trans. Krylov Shipbuild. Res. Inst. 36 (2008) 107–121 (in Russian).