

Lamplighter Random Walks on Fractals

Takashi Kumagai¹ · Chikara Nakamura¹

Received: 4 May 2015 / Revised: 28 September 2016
© Springer Science+Business Media New York 2016

Abstract We consider on-diagonal heat kernel estimates and the laws of the iterated logarithm for a switch-walk-switch random walk on a lamplighter graph under the condition that the random walk on the underlying graph enjoys sub-Gaussian heat kernel estimates.

Keywords Random walks · Wreath products · Fractals · Heat kernels · Sub-Gaussian estimates · Laws of the iterated logarithm (LILs)

Mathematics Subject Classification (2010) 60J10 · 60J35 · 60J55

1 Introduction

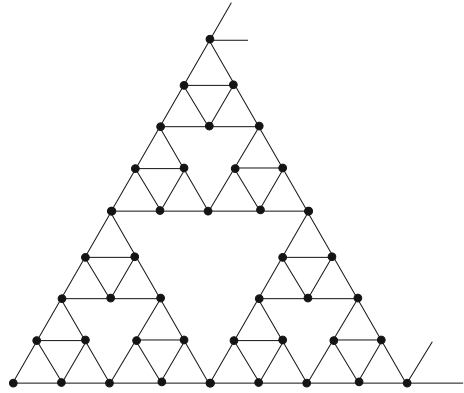
Let G be a connected infinite graph and consider the situation that on each vertex of G there is a lamp. Consider a lamplighter on the graph that makes the following random movements; first, the lamplighter turns on or off the lamp on the site with equal probability, and then, he/she moves to the nearest neighbor of G with equal probability, and turns on or off the lamp on the new site with equal probability. The lamplighter repeats this random movement. Such a movement can be considered as a random walk on the wreath product of graphs $\mathbb{Z}_2 \wr G$ which is roughly a graph putting $\mathbb{Z}_2 = \{0, 1\}$ on each vertex of G (see Definition 2.1 for precise definition), and it is called a “switch-walk-switch walk” or “lamplighter walk” on $\mathbb{Z}_2 \wr G$. We are interested

✉ Takashi Kumagai
kumagai@kurims.kyoto-u.ac.jp

Chikara Nakamura
chikaran@kurims.kyoto-u.ac.jp

¹ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

Fig. 1 The Sierpinski gasket graph



in the long-time behavior of the walk. Some results are known when G is a specific graph. Pittet and Saloff-Coste [15] established on-diagonal heat kernel asymptotics of the random walk on $\mathbb{Z}_2 \wr \mathbb{Z}^d$. More precisely, they obtained the following estimates; there exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 \exp \left[-c_2 n^{\frac{d}{d+2}} \right] \leq h_{2n}(g, g) \leq c_3 \exp \left[-c_4 n^{\frac{d}{d+2}} \right] \quad (1.1)$$

for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}^d$, where $h_n(\cdot, \cdot)$ is the heat kernel (see [19, 20] for earlier results for the case of $G = \mathbb{Z}$ and [6] for the case that G is a finitely generated group with polynomial volume growth). Revelle [18] considered the lamplighter walk on the wreath product $H \wr \mathbb{Z}$ when H is either a finite set or in a class of groups, and obtained some relations between the rate of escape of random walks on H and the law of the iterated logarithm (LIL in short) on $H \wr \mathbb{Z}$. In particular, when $H = \mathbb{Z}_2$, he proved that there exist (non-random) constants $c_1, c_2, c_3, c_4 > 0$ such that the following hold for all $x \in \mathbb{Z}_2 \wr \mathbb{Z}$:

$$\begin{aligned} c_1 &\leq \limsup_{n \rightarrow \infty} \frac{d(Y_0, Y_n)}{n^{1/2}(\log \log n)^{1/2}} \leq c_2, \\ c_3 &\leq \liminf_{n \rightarrow \infty} \frac{d(Y_0, Y_n)}{n^{1/2}(\log \log n)^{-1/2}} \leq c_4, \quad P_x - \text{a.s.}, \end{aligned} \quad (1.2)$$

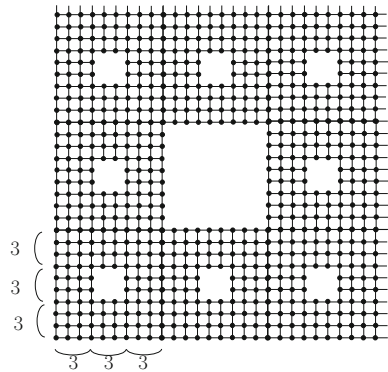
where $\{Y_n\}$ is the lamplighter random walk and $d(\cdot, \cdot)$ is the graph distance on $\mathbb{Z}_2 \wr \mathbb{Z}$.

We are interested in the following question:

(Question) How do the exponents in (1.1), (1.2) change when the graph G is more general?

In this paper, we will consider the question when G is typically a fractal graph. Figures 1 and 2 illustrate concrete examples of fractal graphs. It is known that the random walk on such a fractal graph behaves anomalously in that it diffuses slower than a simple random walk on \mathbb{Z}^d . It has been proved that the heat kernel $h_n(x, y)$ of the random walk $\{X_n\}_{n \geq 0}$ enjoys the following sub-Gaussian estimates; there exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that

Fig. 2 The Sierpinski carpet graph



$$\begin{aligned} \frac{c_1}{n^{d_f/d_w}} \exp\left(-c_2 \left(\frac{d(x, y)^{d_w}}{n}\right)^{1/(d_w-1)}\right) &\leq h_{2n}(x, y) \\ &\leq \frac{c_3}{n^{d_f/d_w}} \exp\left(-c_4 \left(\frac{d(x, y)^{d_w}}{n}\right)^{1/(d_w-1)}\right) \end{aligned} \quad (1.3)$$

for all $d(x, y) \leq 2n$ (note that $h_{2n}(x, y) = 0$ when $d(x, y) > 2n$), where $d(\cdot, \cdot)$ is the graph distance, d_f is the volume growth exponent of the fractal graph and d_w is called the walk dimension, which expresses how fast the random walk on the fractal spreads out. Indeed, by integrating (1.3), one can obtain the following estimates; there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 n^{1/d_w} \leq E d(X_{2n}, X_0) \leq c_2 n^{1/d_w}$$

for all $n > 0$. For more details on diffusions on fractals and random walks on fractal graphs, see [1, 11, 13]. As we see, properties of random walks on graphs are related to the geometric properties of the graphs. The volume growth is one of such properties. For the graphs with polynomial volume growth, there are well-established general methods to analyze the properties of random walks on them. But for the graphs with exponential volume growth, these methods are not applicable. In this sense, the graphs with exponential volume growth give us interesting research subject. The wreath product $\mathbb{Z}_2 \wr G$ belongs to this category, and this is another reason why we are interested in the lamplighter random walks on fractal graphs.

We consider the random walk on $\mathbb{Z}_2 \wr G$, where the random walk on G enjoys the sub-Gaussian heat kernel estimates (1.3). The main results of this paper are the following;

- (1) Sharp on-diagonal heat kernel estimates for the random walk on $\mathbb{Z}_2 \wr G$ (Theorem 2.3),
- (2) LILs for the random walk on $\mathbb{Z}_2 \wr G$ (Theorem 2.4).

The on-diagonal heat kernel estimates are heavily related to the asymptotic properties of the spectrum of the corresponding discrete operator. We can obtain the exponent $d_f/(d_f + d_w)$ in our framework as the generalization of $d/(d + 2)$.

For the LILs, we establish the LIL for $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$, where $\{Y_n\}_{n \geq 0}$ is the random walk on $\mathbb{Z}_2 \wr G$, and the so-called another law of the iterated logarithm that gives the almost sure asymptotic behavior of the liminf of $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$. Note that in (1.2), various properties that are specific to \mathbb{Z} were used, so the generalization to other graphs G is highly non-trivial. We have overcome the difficulty by finding some relationship between the range of the random walk on G and $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$. To our knowledge, these are the first results on the LILs for the wreath product beyond the case of $G = \mathbb{Z}$.

The outline of this paper is as follows. In Sect. 2, we explain the framework and the main results of this paper. In Sect. 3, we give some consequences of sub-Gaussian heat kernel estimates. These are preliminary results for Sects. 4 and 5, where we mainly treat the lamplighter random walks on fractal graphs. In Sect. 4, we prove the on-diagonal heat kernel estimates. Section 5 has three subsections. In Sect. 5.1, we give a relation between the range of random walk on G and $d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)$. Here, one of the keys is to prove the existence of a path that covers a subgraph of G with the length of the path being (uniformly) comparable to the volume of the subgraph (Lemma 5.3). In Sect. 5.2, we deduce the LILs for the random walk on $\mathbb{Z}_2 \wr G$ from the LILs for the range of the random walk on G (Theorem 5.5) when G is a strongly recurrent graph. In Sect. 5.3, we prove the LILs for the random walk on $\mathbb{Z}_2 \wr G$ when G is a transient graph. In the Appendix 1, we give an outline of the proof of Theorem 5.5, on which the proof in Sect. 5.2 is based.

Throughout this paper, we use the following notation.

- Notation** (1) For two nonnegative sequences $\{a(n)\}_{n \geq 0}$ and $\{b(n)\}_{n \geq 0}$, we write
- $a(n) \asymp b(n)$ if there exist positive constants $c_1, c_2 > 0$ such that $c_1 a(n) \leq b(n) \leq c_2 a(n)$ for all n .
 - $a(n) \approx b(n)$ if there exist positive constants $c_1, c_2, c_3, c_4 > 0$ such that $c_1 a(c_2 n) \leq b(n) \leq c_3 a(c_4 n)$ for all n .
- (2) We use c, C, c_1, c_2, \dots to denote deterministic positive finite constants whose values are insignificant. These constants do not depend on time parameters n, k, \dots , distance parameters r, \dots , or vertices of graphs.

2 Framework and Main Results

In this section, we introduce the framework and the main results of this paper.

2.1 Framework

Let $G = (V(G), E(G))$ be an infinite, locally finite, connected graph. We assume $V(G)$ is a countable set. We say that G is a graph of bounded degree if

$$M = \sup_{v \in V(G)} \deg v < \infty. \quad (2.1)$$

We denote $d(x, y)$ the graph distance of x, y in G , i.e., the shortest length of paths between x and y . When we want to emphasize the graph G , we write $d_G(x, y)$ instead of $d(x, y)$.

Next, we introduce the wreath product of graphs.

Definition 2.1 (*Wreath product*) Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. We define the *wreath product of G and H* (denoted by $H \wr G$) in the following way. We define the vertex set of the wreath product as

$$V(H \wr G) = \left\{ (f, v) \in V(H)^{V(G)} \times V(G) \mid \# \text{Supp } f < \infty \right\},$$

where $\text{Supp } f = \{x \in V(G) \mid f(x) \neq 0\}$ for a fixed element $0 \in V(H)$. For $(f, u), (g, v) \in V(H \wr G)$, $((f, u), (g, v)) \in E(H \wr G)$ if either (a) or (b) is satisfied:

- (a) $f = g$ and $(u, v) \in E(G)$,
- (b) $u = v$, $f(x) = g(x)$ ($\forall x \in V(G) \setminus \{u\}$) and $(f(u), g(u)) \in E(H)$.

We call G the underlying graph of $H \wr G$ and H the fiber graph of $H \wr G$.

Throughout the paper, we will only consider the case $H = \mathbb{Z}_2$ that consists of two vertices $\{0, 1\}$, say, and one edge that connects the two vertices (As noted in Remark 2.5(2), the results in this paper hold when H is a finite graph). We denote the elements of $V(\mathbb{Z}_2 \wr G)$ by bold alphabets $\mathbf{x}, \mathbf{y} \dots$ and the elements of $V(G)$ by standard alphabets x, y, \dots .

Next, we introduce the notion of weighted graphs. Let $\mu : V(G) \times V(G) \rightarrow [0, \infty)$ be a symmetric function such that $\mu_{xy} = \mu(x, y) > 0$ if and only if $(x, y) \in E(G)$. We call the pair (G, μ) a weighted graph. For a weighted graph (G, μ) , we define a measure $m = m_G$ on $V(G)$ by $m(A) = \sum_{x \in A} m(x)$ for $A \subset V(G)$ where $m(x) = \sum_{y: y \sim x} \mu_{xy}$. We will write $V(x, r) = V_G(x, r) = m(B(x, r))$, where $B(x, r) = \{y \in V(G) \mid d(x, y) \leq r\}$.

Let $\{X_n\}_{n \geq 0}$ be the (time-homogeneous) random walk on G whose transition probability is $P = (p(x, y))_{x, y \in V(G)}$, where $p(x, y) = \mu_{xy}/m(x)$. We call $\{X_n\}_{n \geq 0}$ the random walk associated with the weighted graph (G, μ) . $\{X_n\}_{n \geq 0}$ is reversible w.r.t. m , i.e., $m(x)p(x, y) = m(y)p(y, x)$ for all $x, y \in V(G)$. Define

$$p_n(x, y) := P_x(X_n = y), \quad \forall x, y \in V(G).$$

$p_n(x, y)/m(y)$ is called the heat kernel of the random walk.

We next give a set of conditions for the graph and the random walks.

Assumption 2.2 Let (G, μ) be a weighted graph. We consider the following assumptions for (G, μ) .

- (1) (p_0 -condition): (G, μ) satisfies p_0 -condition, i.e., there exists $p_0 > 0$ such that $\mu_{xy}/m(x) \geq p_0$ for all $x, y \in V(G)$ with $\{x, y\} \in E(G)$.
- (2) (d_f -set condition): There exist positive constants $d_f, c_1, c_2 > 0$ such that

$$c_1 r^{d_f} \leq V(x, r) \leq c_2 r^{d_f} \tag{2.2}$$

for all $x \in V(G), r \in \mathbb{N} \cup \{0\}$. Here, we regard 0^{d_f} as 1.

- (3) (On-diagonal heat kernel upper bound): There exists $d_s > 0$ such that the heat kernel $\frac{p_n(x, y)}{m(y)}$ of $\{X_n\}_{n \geq 0}$ satisfies the following estimate for all $x, y \in V(G)$, $n \geq 1$:

$$\frac{p_n(x, y)}{m(y)} \leq c_1 n^{-d_s/2}. \quad (2.3)$$

- (4) (Sub-Gaussian heat kernel estimates): There exists $d_w > 1$ such that the heat kernel $\frac{p_n(x, y)}{m(y)}$ of $\{X_n\}_{n \geq 0}$ satisfies the following estimates:

$$\frac{p_n(x, y)}{m(y)} \leq \frac{c_1}{V(x, n^{\frac{1}{d_w}})} \exp \left(-c_2 \left(\frac{d(x, y)^{d_w}}{n} \right)^{\frac{1}{d_w-1}} \right) \quad (2.4)$$

for all $x, y \in V(G)$, $n \geq 1$, and

$$\frac{p_n(x, y)}{m(y)} + \frac{p_{n+1}(x, y)}{m(y)} \geq \frac{c_3}{V(x, n^{\frac{1}{d_w}})} \exp \left(-c_4 \left(\frac{d(x, y)^{d_w}}{n} \right)^{\frac{1}{d_w-1}} \right) \quad (2.5)$$

for $x, y \in V(G)$, $n \geq 1$ with $d(x, y) \leq n$.

We will clarify which of the conditions above are assumed in each statement. As one can easily see, Assumptions 2.2 (2) and (2.4) imply Assumption 2.2 (3) with

$$d_s/2 = d_f/d_w.$$

d_s is called the spectral dimension.

The fractal graphs such as the Sierpinski gasket graph and the Sierpinski carpet graph given in Sect. 1 satisfy Assumption 2.2 (see [3, 10]). Note that under Assumption 2.2 (1), G satisfies (2.1) with $M = 1/p_0$. Also note that, under Assumption 2.2 (2), we have $c_1 \leq m_G(x) \leq c_2$ for all $x \in V(G)$ and hence

$$c_1 \sharp A \leq m(A) \leq c_2 \sharp A, \quad \forall A \subset V(G), \quad (2.6)$$

where $\sharp A$ is the cardinal number of A . Finally, under Assumption 2.2 (1) (2) we have

$$0 < \inf_{x, y \in V(G), x \sim y} \mu_{xy} \leq \sup_{x, y \in V(G), x \sim y} \mu_{xy} < \infty. \quad (2.7)$$

Next, we define the lamplighter walk on $\mathbb{Z}_2 \wr G$. We denote the transition probability on \mathbb{Z}_2 by $P^{(\mathbb{Z}_2)} = (p^{(\mathbb{Z}_2)}(a, b))_{a, b \in \mathbb{Z}_2}$, where $P^{(\mathbb{Z}_2)}$ is given by

$$p^{(\mathbb{Z}_2)}(a, b) = \frac{1}{2}, \quad \text{for all } a, b \in \mathbb{Z}_2.$$

We can lift $P = (p(x, y))_{x, y \in G}$ and $P^{(\mathbb{Z}_2)} = (p^{(\mathbb{Z}_2)}(a, b))_{a, b \in \mathbb{Z}_2}$ on $\mathbb{Z}_2 \wr G$, by

$$\begin{aligned}\tilde{p}^{(G)}((f, x), (g, y)) &= \begin{cases} p(x, y) & \text{if } f = g \\ 0 & \text{otherwise} \end{cases}, \\ \tilde{p}^{(\mathbb{Z}_2)}((f, x), (g, y)) &= \begin{cases} \frac{1}{2} & \text{if } x = y \text{ and } f(v) = g(v) \text{ for all } v \neq x \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

Let $Y_n = \{(\eta_n, X_n)\}_{n \geq 0}$ be a random walk on $\mathbb{Z}_2 \wr G$ whose transition probability \tilde{p} is given by

$$\begin{aligned}\tilde{p}((f, x), (g, y)) &= \tilde{p}^{(\mathbb{Z}_2)} * \tilde{p}^{(G)} * \tilde{p}^{(\mathbb{Z}_2)}((f, x), (g, y)) \\ &= \sum_{(h_1, w_1), (h_2, w_2)} \tilde{p}^{(\mathbb{Z}_2)}((f, x), (h_1, w_1)) \tilde{p}^{(G)}((h_1, w_1), (h_2, w_2)) \tilde{p}^{(\mathbb{Z}_2)}((h_2, w_2), (g, y)).\end{aligned}$$

Note that if $(f, x), (g, y) \in V(\mathbb{Z}_2 \wr G)$ satisfy $x \sim y$ and $f(z) = g(z)$ for all $z \neq x, y$ then

$$P(Y_{n+1} = (g, y) \mid Y_n = (f, x)) = \frac{1}{4} p(x, y),$$

and otherwise it is zero.

This random walk moves in the following way. Let X_n be the site of lamplighter at time n and η_n be the on-lamp state at time n . The lamplighter changes the lamp at X_n with probability $1/2$, moves on G according to the transition probability $P = (p(x, y))_{x, y \in G}$, and then changes the lamp at X_{n+1} with probability $1/2$. The lamplighter repeats this procedure (In the first paragraph of Sect. 1, we discussed the case when $\{X_n\}$ is a simple random walk on G).

Note that $\{Y_n\}_{n \geq 0}$ is reversible w.r.t. $m_{\mathbb{Z}_2 \wr G}$, where

$$m_{\mathbb{Z}_2 \wr G}((\eta, x)) = m(x).$$

We denote the transition probability of $\{Y_n\}_{n \geq 0}$ as $p(\mathbf{x}, \mathbf{y})$ (cf. $p(x, y)$ is the transition probability of $\{X_n\}_{n \geq 0}$). We sometimes write $m(\mathbf{x})$ instead of $m_{\mathbb{Z}_2 \wr G}(\mathbf{x})$.

2.2 Main Results

In this subsection, we state the main results of this paper.

Theorem 2.3 *Suppose that Assumption 2.2 (1), (2) and (4) hold. Then, the following holds;*

$$\frac{p_{2n}(\mathbf{x}, \mathbf{x})}{m_{\mathbb{Z}_2 \wr G}(\mathbf{x})} \approx \exp[-n^{\frac{d_f}{d_f + d_w}}], \quad \forall \mathbf{x} \in V(\mathbb{Z}_2 \wr G).$$

Next we state the results on the LILs when $d_s/2 < 1$ and $d_s/2 > 1$, respectively.

Theorem 2.4 *Let (G, μ) be a weighted graph.*

(I) *Assume that Assumption 2.2 (1), (2), (4) and $d_s/2 < 1$ hold. Then, there exist (non-random) constants $c_1, c_2, c_3, c_4 > 0$ such that the following hold for all $\mathbf{x} \in V(\mathbb{Z}_2 \wr G)$:*

$$c_1 \leq \limsup_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n^{d_s/2} (\log \log n)^{1-d_s/2}} \leq c_2, \quad P_{\mathbf{x}}\text{-a.s.} \quad (2.8)$$

$$c_3 \leq \liminf_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n^{d_s/2} (\log \log n)^{-d_s/2}} \leq c_4, \quad P_{\mathbf{x}}\text{-a.s.} \quad (2.9)$$

(II) *Assume that Assumption 2.2 (1), (2), (3) and $d_s/2 > 1$ hold. Then, there exist (non-random) positive constants $c_1, c_2 > 0$ such that the following hold for all $\mathbf{x} \in V(\mathbb{Z}_2 \wr G)$:*

$$c_1 \leq \liminf_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n} \leq \limsup_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n} \leq c_2, \quad P_{\mathbf{x}}\text{-a.s.} \quad (2.10)$$

Remark 2.5 (1) Note that in Theorem 2.4(II), we do not need Assumption 2.2(4) but only need the on-diagonal upper bound (2.3). Since the transient case is discussed under a general framework in [14] (see Sect. 5.3), we do not pursue the minimum assumption for (2.10) to hold.

(2) We can obtain the same results (by the same proof) if we replace \mathbb{Z}_2 by a finite graph H with $\sharp H \geq 2$ and $p^{(\mathbb{Z}_2)}$ by $p^{(H)}$, where $p^{(H)}$ is the transition probability on H given by

$$p^{(H)}(a, b) = \frac{1}{\sharp H}, \quad \text{for all } a, b \in V(H).$$

(3) For each $0 < \alpha < 1$, Rau [17, Proposition 1.2] constructed a graph G_α such that the random walk on G_α satisfies the following heat kernel estimates:

$$p_{2n}(x, x) \approx \exp(-n^\alpha). \quad (2.11)$$

For the case $1/3 \leq \alpha < 1$, the graphs constructed by Rau are the wreath product on \mathbb{Z} , but the fiber graphs are different site by site (The definition of wreath product given by Rau is more general than ours). On the other hand, for each d_f, d_w such that $2 \leq d_w \leq 1 + d_f$ and $d_f \geq 1$, Barlow [2, Theorem 2] constructed weighted graphs that satisfy Assumption 2.2. Combining this and Theorem 2.3, for any given $1/3 \leq \alpha < 1$ we can give an alternative example where the heat kernel enjoys (2.11).

(4) For the case of $d_s/2 = 1$, we have not been able to obtain the LIL in general. However, one can obtain the LIL for the case of \mathbb{Z}^2 as follows (Note that $d_s/2 = 1$ in this case since $d_f = d_w = 2$). Define $R_n = \sharp\{X_0, \dots, X_n\}$. Dvoretzky and

Erdős [8, Theorem 1,4] proved the following law of large numbers of R_n :

$$\lim_{n \rightarrow \infty} \frac{R_n}{\pi n / \log n} = 1, \quad P\text{-a.s.}$$

In Propositions 5.1 and 5.2, we will show that $\frac{1}{4}R_n \leq d_{\mathbb{Z}_2 \wr \mathbb{Z}^2}(Y_0, Y_n)$ for all but finitely many n and that there exists a positive constant $C > 0$ such that $d_{\mathbb{Z}_2 \wr \mathbb{Z}^2}(Y_0, Y_n) \leq CR_n$ for all n . Using these facts, we see that there exist positive constants $c_1, c_2 > 0$ such that for all $x \in V(\mathbb{Z}_2 \wr G)$

$$c_1 \leq \liminf_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr \mathbb{Z}^2}(Y_0, Y_n)}{n / \log n} \leq \limsup_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr \mathbb{Z}^2}(Y_0, Y_n)}{n / \log n} \leq c_2, \quad P_x\text{-a.s.}$$

As we see, the exponents differ from those in the cases of $d_s/2 < 1$ and $d_s/2 > 1$.

3 Consequences of Heat Kernel Estimates

In this section, we give preliminary results obtained from the sub-Gaussian heat kernel estimates (2.3), (2.4) and (2.5). Throughout this section, we assume that Assumption 2.2 (1), (2) hold.

First, the following can be obtained by a simple modification of [1, Lemma 3.9] (Note that (2.5) is not needed here).

Lemma 3.1 *Suppose (2.4). Then, there exist positive constants $c_1, c_2 > 0$ such that*

$$P_y \left(\max_{0 \leq j \leq n} d(x, X_j) \geq 3r \right) \leq c_1 \exp \left(-c_2 \left(\frac{r^{d_w}}{n} \right)^{\frac{1}{d_w-1}} \right)$$

for all $n \geq 1, r \geq 1, x, y \in V(G)$ with $d(x, y) \leq r$.

The following lemma will be used in Sect. 5.1. Note that unlike Lemma 3.1, we need only weaker condition (2.3) here.

Lemma 3.2 *Suppose (2.3). Then, there exist positive constants $c_1, c_2 > 0$ such that*

$$P_x \left(\max_{0 \leq j \leq n} d(x, X_j) \leq r \right) \leq c_1 \exp \left(-c_2 \frac{n}{r^{d_w}} \right) \quad (3.1)$$

for all $x \in V(G), n, r \geq 1$.

Proof We first show that there exists a positive constant $c_1 > 0$ and a positive integer R such that

$$P_x \left(\max_{0 \leq j \leq [r^{d_w}]} d(x, X_j) \leq 2c_1 r \right) \leq \frac{1}{2} \quad (3.2)$$

for all $x \in V(G)$ and for all $r \geq R$, where $[r^{d_w}]$ means the greatest integer which is less than or equal to r^{d_w} . Using (2.3), for $r \geq 1$ we have

$$\begin{aligned} P_x \left(\max_{0 \leq j \leq [r^{d_w}]} d(x, X_j) \leq 2c_1 r \right) &\leq P_x(d(x, X_{[r^{d_w}]}) \leq 2c_1 r) \\ &\leq c_2 \frac{1}{([r^{d_w}])^{d_f/d_w}} \sum_{y \in B(x, 2c_1 r)} m(y) \leq c_3 \frac{1}{r^{d_f}} m(B(x, 2c_1 r)). \end{aligned}$$

Recall that $c_4 r^{d_f} \leq m(B(x, r)) \leq c_5 r^{d_f}$ for $r \in \mathbb{N} \cup \{0\}$ by (2.2). Take c_1 as a positive constant which satisfies $c_3(2c_1)^{d_f} \leq \frac{1}{2c_5}$. We first consider the case of $2c_1 r \geq 1$. In this case, by the above estimate we have

$$P_x \left(\max_{0 \leq j \leq [r^{d_w}]} d(x, X_j) \leq 2c_1 r \right) \leq c_3 \frac{1}{r^{d_f}} m(B(x, 2c_1 r)) \leq c_3 c_5 (2c_1)^{d_f} \leq \frac{1}{2}.$$

Next, we consider the case of $2c_1 r < 1$. In this case, $m(B(x, 2c_1 r)) = m(\{x\}) \leq c_5$. Take R such that $R > (2c_3 c_5)^{1/d_f}$. By the above estimate, we have

$$P_x \left(\max_{0 \leq j \leq [r^{d_w}]} d(x, X_j) \leq 2c_1 r \right) \leq c_3 \frac{1}{r^{d_f}} m(B(x, 2c_1 r)) \leq c_3 \frac{c_5}{r^{d_f}} \leq \frac{1}{2}$$

for $r \geq R$. We thus obtain (3.2).

We now prove (3.1). Let $r \geq R$. It is enough to consider the case $n \geq [r^{d_w}]$ since otherwise (3.1) is immediate by adjusting the constants. Let $k \geq 1$ be such that $k[r^{d_w}] \leq n < (k+1)[r^{d_w}]$ and let $t_i = i[r^{d_w}]$. Then, by the Markov property and (3.2) we have

$$\begin{aligned} P_x \left(\max_{0 \leq j \leq n} d(x, X_j) \leq c_1 r \right) &\leq P_x \left(\bigcap_{0 \leq i \leq k-1} \left\{ \max_{t_i \leq j \leq t_{i+1}} d(X_{t_i}, X_j) \leq 2c_1 r \right\} \right) \\ &\leq \left\{ \sup_y P_y \left(\max_{0 \leq j \leq [r^{d_w}]} d(y, X_j) \leq 2c_1 r \right) \right\}^k \\ &\leq \left(\frac{1}{2} \right)^k \leq c_6 \exp(-c_7 k) \leq c_8 \exp(-c_9 n r^{-d_w}). \end{aligned}$$

It is immediate for the case of $1 \leq r \leq R$ from the above estimate. Hence, we obtain (3.1) by adjusting the constants. \square

In the next proposition, we show that Lemma 3.2 is sharp up to constants if we assume both (2.4) and (2.5). The idea of the proof is based on [16, Lemma 7.4.3], where a similar estimate was given for a class of random walks on \mathbb{Z}^d .

Proposition 3.3 *Suppose (2.4) and (2.5). Then, there exist $N_0 \geq 1$ and positive constants $c_1, c_2 > 0$ such that*

$$P_x \left(\max_{0 \leq j \leq n} d(x, X_j) \leq r \right) \geq c_1 \exp \left(-c_2 \frac{n}{r^{d_w}} \right)$$

for all $r \geq N_0$ and $n \geq 4^{\frac{d_w}{d_w-1}}$.

The proof consists of the following two lemmas.

Lemma 3.4 Under (2.5) there exists $\epsilon \in (0, 1)$ such that

$$P_y(d(x, X_n) > r) \leq 1 - \epsilon$$

for all $r \geq 1$, $n \geq 4^{\frac{d_w}{d_w-1}}$ with $n < r^{d_w}$ and $x, y \in V(G)$ with $d(x, y) \leq r$.

Proof We follow the argument in [16, Lemma 7.4.7]. Let $\gamma = 1/(d_w - 1)$. Let $\ell_{x,y}$ be a geodesic path from x to y in G . Let $x_n \in \ell_{x,y}$ be the $(\lfloor n^{1/d_w} \rfloor + 1)$ -th vertex from y . Then, $B(x_n, \lfloor n^{1/d_w} \rfloor) \subset B(x, r - 1)$ since for all $z \in B(x_n, \lfloor n^{1/d_w} \rfloor)$ we have $d(x, z) \leq d(x, x_n) + d(x_n, z) \leq (d(x, y) - \lfloor n^{1/d_w} \rfloor - 1) + \lfloor n^{1/d_w} \rfloor \leq r - 1$. Also for all $z \in B(x_n, \lfloor n^{1/d_w} \rfloor)$, we have $d(y, z) \leq d(y, x_n) + d(x_n, z) \leq 2\lfloor n^{1/d_w} \rfloor + 1$. Hence, by (2.5) and (2.2) we have

$$\begin{aligned} 2P_y(d(x, X_n) \leq r) &\geq P_y(d(x, X_{n-1}) \leq r - 1) + P_y(d(x, X_n) \leq r - 1) \\ &\geq c_1 \sum_{z \in B(x_n, \lfloor n^{1/d_w} \rfloor)} \frac{m(z)}{V(y, (n - 1)^{1/d_w})} \exp \left[-c_2 \left(\frac{d(y, z)^{d_w}}{n - 1} \right)^\gamma \right] \\ &\geq c_3 \left(\sum_{z \in B(x_n, \lfloor n^{1/d_w} \rfloor)} \frac{m(z)}{n^{d_f/d_w}} \right) \exp \left[-c_2 (2 \cdot 3^{d_w})^\gamma \right] \geq c_4 \exp \left[-c_2 (2 \cdot 3^{d_w})^\gamma \right], \end{aligned}$$

provided $n \geq 4^{\frac{d_w}{d_w-1}}$ so that $d(y, z) \leq 2n^{1/d_w} + 1 \leq 3n^{1/d_w} \leq n - 1$ for any $z \in B(x_n, \lfloor n^{1/d_w} \rfloor)$. The proof completes by taking $\epsilon = \frac{1}{2}c_4 \exp \left[-c_2 (2 \cdot 3^{d_w})^\gamma \right]$ (note that we may take $c_4 < 1$). \square

Lemma 3.5 Let ϵ be as in Lemma 3.4. Then, under (2.4) and (2.5), there exists $\eta \geq 1$ such that for all $x, y \in V(G)$, positive integers $r \geq 8^{\frac{1}{d_w-1}}$ with $d(x, y) \leq 2r$ and for all integers $k \geq 0$ and $\ell \geq 4^{\frac{d_w}{d_w-1}}$ with $k[r^{d_w}] \leq \ell \leq (k + 1)[r^{d_w}]$, we have

$$P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r \right) \geq \left(\frac{\epsilon}{2} \right)^k \wedge \frac{\epsilon}{2}.$$

Proof We follow the argument in the proof of [16, Lemma 7.4.3]. We prove the assertion by induction in k .

Step I: We first prove the case $k = 0, 1$. Let $\gamma = 1/(d_w - 1)$. In general, $1 \leq P(A) + P(B) + P((A \cup B)^c)$ holds for any events A, B . As A, B take

$A = \{\max_{0 \leq j \leq \ell} d(x, X_j) > 3\eta r\}$, $B = \{d(x, X_\ell) > 2r\}$. Let $4^{\frac{d_w}{d_w-1}} \leq \ell \leq 2[r^{d_w}]$. By Lemmas 3.1 and 3.4, we have

$$\begin{aligned} 1 &\leq P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) > 3\eta r \right) + P_y(d(x, X_\ell) > 2r) \\ &\quad + P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r \right) \\ &\leq c_1 \exp \left[-c_2 \left(\frac{(\eta r)^{d_w}}{\ell} \right)^\gamma \right] + (1 - \epsilon) \\ &\quad + P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r \right). \end{aligned}$$

From above and using $\ell \leq 2[r^{d_w}]$ we have

$$\begin{aligned} &P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r \right) \\ &\geq \epsilon - c_1 \exp \left[-c_2 \left(\frac{(\eta r)^{d_w}}{\ell} \right)^\gamma \right] \geq \epsilon - c_1 \exp \left[-c_2 \left(\frac{\eta^{d_w}}{2} \right)^\gamma \right]. \end{aligned}$$

Taking $\eta > \left\{ \frac{2^\gamma}{c_2} \log \left(\frac{2c_1}{\epsilon} \right) \right\}^{1/(\gamma d_w)} \vee 1$, we obtain

$$P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r \right) \geq \frac{\epsilon}{2}$$

for $4^{\frac{d_w}{d_w-1}} \leq \ell \leq 2[r^{d_w}]$.

Step II: Let $k \geq 1$ and assume that the result holds up to k . Let ℓ satisfy $(k+1)[r^{d_w}] \leq \ell \leq (k+2)[r^{d_w}]$. Define $\ell' = k[r^{d_w}]$. Then, since $\ell' \wedge (\ell - \ell') \geq [r^{d_w}] \geq \frac{1}{2}r^{d_w} \geq 4^{\frac{d_w}{d_w-1}}$ by $r \geq 8^{\frac{1}{d_w-1}}$, using the Markov property and induction hypothesis, we have

$$\begin{aligned} &P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r \right) \\ &\geq P_y \left(\max_{0 \leq j \leq \ell} d(x, X_j) \leq 3\eta r, d(x, X_\ell) \leq 2r, d(x, X_{\ell'}) \leq 2r \right) \\ &= E_y \left[1_{\{\max_{0 \leq j \leq \ell'} d(x, X_j) \leq 3\eta r, d(x, X_{\ell'}) \leq 2r\}} P_{X_{\ell'}} \left(d(x, X_{\ell-\ell'}) \right. \right. \\ &\quad \left. \left. \leq 2r, \max_{0 \leq j \leq \ell-\ell'} d(x, X_j) \leq 3\eta r \right) \right] \\ &\geq \frac{\epsilon}{2} P_y \left(\max_{0 \leq j \leq \ell'} d(x, X_j) \leq 3\eta r, d(x, X_{\ell'}) \leq 2r \right) \geq \left(\frac{\epsilon}{2} \right)^{k+1}. \end{aligned}$$

We thus complete the proof. \square

Given Lemma 3.5, it is straightforward to obtain Proposition 3.3.

4 On-Diagonal Heat Kernel

In this section, we give the proof of Theorem 2.3.

The lower bound follows by the same approach as in [16, Section 7] (cf. [15, Section 7] and [21, Section 15.D]). We use Proposition 3.3 for the proof.

Proof of the lower bound of Theorem 2.3 Let $\mathbf{x} = (\eta, x) \in V(\mathbb{Z}_2 \wr G)$. As we said before we write $m_{\mathbb{Z}_2 \wr G}$ as m . For any finite subset $A \subset V(\mathbb{Z}_2 \wr G)$, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{p_{2n}(\mathbf{x}, \mathbf{x})}{m(\mathbf{x})} &= \sum_{\mathbf{y} \in V(\mathbb{Z}_2 \wr G)} \frac{p_n(\mathbf{x}, \mathbf{y})p_n(\mathbf{y}, \mathbf{x})}{m(\mathbf{x})} \\ &= \sum_{\mathbf{y} \in V(\mathbb{Z}_2 \wr G)} \frac{p_n(\mathbf{x}, \mathbf{y})^2}{m(\mathbf{y})} \geq \sum_{\mathbf{y} \in A} \frac{p_n(\mathbf{x}, \mathbf{y})^2}{m(\mathbf{y})} \geq \frac{1}{m(A)} P_{\mathbf{x}}(Y_n \in A)^2. \end{aligned} \quad (4.1)$$

Set $A := \{\mathbf{y} = (f, y) \in V(\mathbb{Z}_2 \wr G) \mid y \in B_G(x, r), f(z) = 0 \text{ for all } z \in V(G) \text{ such that } d(x, z) > r\}$. Using (2.2) and (2.6), we have

$$m_{\mathbb{Z}_2 \wr G}(A) = \sum_{y \in B_G(x, r)} m_G(y) 2^{\sharp B_G(x, r)} \leq c_1 r^{d_f} 2^{c_2 r^{d_f}},$$

and using Proposition 3.3 we have

$$P_{\mathbf{x}}(Y_n \in A) \geq P_{\mathbf{x}}\left(\max_{0 \leq j \leq n} d(x, X_j) \leq r\right) \geq c_3 \exp\left[-c_4 \frac{n}{r^{d_w}}\right]$$

provided $n \geq 4 \frac{d_w}{d_w-1}$ and $r \geq N_0$. Hence, by (4.1), we have

$$\frac{p_{2n}(\mathbf{x}, \mathbf{x})}{m(\mathbf{x})} \geq c_5 \exp\left[-c_6 \left(d_f \log r + r^{d_f} + \frac{n}{r^{d_w}}\right)\right].$$

Optimize the right-hand side (take $r = n^{1/(d_f+d_w)}$), then we obtain

$$\frac{p_{2n}(\mathbf{x}, \mathbf{x})}{m(\mathbf{x})} \geq c \exp\left[-C n^{\frac{d_f}{d_f+d_w}}\right]$$

provided $n \geq 4 \frac{d_w}{d_w-1} \vee N_0^{d_f+d_w}$. The lower bound for $1 \leq n \leq 4 \frac{d_w}{d_w-1} \vee N_0^{d_f+d_w}$ is obvious, and we thus complete the proof. \square

We next prove the upper bound of Theorem 2.3 (cf. [15, Section 8] and [21, Section 15.D]).

Proof of the upper bound of Theorem 2.3 Without loss of generality, we may and do assume η_0 is identically 0. For the switch-walk-switch random walk $\{Y_n = (\eta_n, X_n)\}_{n \geq 0}$ on $\mathbb{Z}_2 \wr G$, η_n is equi-distributed on $\{f \in \prod_{z \in V(G)} \mathbb{Z}_2 \mid \text{Supp}(f - \eta_0) \subset \bar{R}_n\}$, where $\bar{R}_n = \{X_0, X_1, \dots, X_n\}$. Hence, setting $R_n = \sharp \bar{R}_n$, we have

$$P_x(Y_n = \mathbf{x}) = \sum_{k=0}^{n+1} P_x(Y_n = \mathbf{x}, R_n = k) \leq \sum_{k=0}^{n+1} E_x \left[1_{\{R_n=k\}} 2^{-k} \right] \leq E_x \left[2^{-R_n} \right].$$

In [9, Theorem 1.2], Gibson showed the following Donsker–Varadhan type range estimate: for any $\nu > 0$ and any $x \in V(G)$,

$$-\log E_x \left[\exp \left\{ -\nu m(\bar{R}_{[n^{d_f+d_w} V(x,n)]}) \right\} \right] \asymp V(x, n).$$

Note that $V(x, n) \asymp n^{d_f}$. Replacing n with $n^{1/(d_f+d_w)}$ we have

$$E_x \left[\exp \left\{ -\nu m(\bar{R}_n) \right\} \right] \approx \exp \left[-n^{d_f/(d_f+d_w)} \right].$$

Since $cm(\bar{R}_n) \leq R_n$ (due to (2.6)), by the above estimates, we obtain the upper estimate, i.e.,

$$\frac{p_n(\mathbf{x}, \mathbf{x})}{m(\mathbf{x})} \leq c \exp \left[-C n^{\frac{d_f}{d_f+d_w}} \right].$$

We thus complete the proof. \square

5 Laws of the Iterated Logarithm

In this section, we will prove Theorem 2.4.

We first explain the idea of the proof. Let (G, μ) be a weighted graph such that G is of bounded degree. For notational simplicity, let $o \in V(G)$ be a distinguished point and $\mathbf{0}$ be the element of $(\mathbb{Z}_2)^{V(G)}$ such that $\mathbf{0}(v) = 0$ for all $v \in V(G)$. In order to realize a given lamp state $(\eta, x) \in V(\mathbb{Z}_2 \wr G)$ from the initial lamp state $(\mathbf{0}, o) \in V(\mathbb{Z}_2 \wr G)$, we need to visit all the “on-lamp vertices”. So,

$$\begin{aligned} \sum_{i \in V(G)} \eta(i) &\leq d_{\mathbb{Z}_2 \wr G}((\mathbf{0}, o), (\eta, x)) \\ &\leq (\text{the minimum number of steps to visit all the “on-lamp vertices” from } o \text{ to } x) \\ &\quad + \sum_{i \in V(G)} \eta(i). \end{aligned} \tag{5.1}$$

We apply the above observation to the lamplighter walk $\{Y_n = (\eta_n, X_n)\}_{n \geq 0}$ on $\mathbb{Z}_{2\wr} G$. Note that the lamp at a certain vertex of G (say z) cannot be changed without making the lamplighter visit at z . From this and (5.1), we see that $d_{\mathbb{Z}_{2\wr} G}(Y_0, Y_n)$ is heavily related to the range of random walk $\{X_n\}_{n \geq 0}$ on G . Set $R_n = \sharp\{X_0, X_1, \dots, X_n\}$. Intuitively, $\sum_{i \in V(G)} \eta_n(i)$ is close to $\frac{1}{2}R_n$. Indeed, we will show the following in Propositions 5.1 and 5.2:

$$\frac{1}{4}R_n \leq d_{\mathbb{Z}_{2\wr} G}(Y_0, Y_n) \quad \text{for all but finitely many } n, \quad a.s., \quad (5.2)$$

$$d_{\mathbb{Z}_{2\wr} G}(Y_0, Y_n) \leq (2M + 1)R_n, \quad \text{for all } n, \quad (5.3)$$

where M is defined by (2.1). We will prove (5.2) and (5.3) in Sect. 5.1. The behavior of R_n differs for $d_s/2 < 1$ and $d_s/2 > 1$. In Sect. 5.2 (resp. 5.3), we prove the LILs of R_n and $d_{\mathbb{Z}_{2\wr} G}(Y_0, Y_n)$ for $d_s/2 < 1$ (resp. for $d_s/2 > 1$).

5.1 Relations Between the Distance and the Range

The main goal of this subsection is to prove (5.2) and (5.3).

Proposition 5.1 *Suppose that Assumption 2.2 (1), (2) and (3) hold. Then, the following holds;*

$$\frac{1}{4}R_n \leq \sum_{i \in \{X_0, X_1, \dots, X_n\}} \eta_n(i) \quad \text{for all but finitely many } n, \quad P_{(\mathbf{0}, x)}\text{-a.s., for all } x \in V(G).$$

Proof We fix $x \in V(G)$ and write P instead of $P_{(\mathbf{0}, x)}$. Define $S_n = \sum_{i \in \{X_0, X_1, \dots, X_n\}} \eta_n(i)$. It is easy to see that

$$P(S_n = l \mid R_n = k) = \left(\frac{1}{2}\right)^k \binom{k}{l}$$

for $0 \leq l \leq k$. Then, we have

$$\begin{aligned} P\left(S_n \leq \frac{1}{4}R_n\right) &= \sum_{l=0}^n P\left(S_n \leq \frac{1}{4}l, R_n = l\right) = \sum_{l=0}^n \sum_{m=0}^{\left[\frac{1}{4}l\right]} \left(\frac{1}{2}\right)^l \binom{l}{m} P(R_n = l) \\ &\leq \sum_{l=0}^n \exp\left(-\frac{1}{16}l\right) P(R_n = l) \quad \text{by the Chernoff bound} \\ &\leq P\left(R_n \leq n^{\frac{1}{2dw}}\right) + \exp\left(-\frac{1}{16}n^{\frac{1}{2dw}}\right) P\left(R_n \geq n^{\frac{1}{2dw}}\right) \\ &\leq P\left(\sup_{0 \leq l \leq n} d(X_0, X_l) \leq n^{\frac{1}{2dw}}\right) + \exp\left(-\frac{1}{16}n^{\frac{1}{2dw}}\right) \end{aligned}$$

$$\leq c_1 \exp(-c_2 n^{\frac{1}{2}}) + \exp\left(-\frac{1}{16} n^{\frac{1}{2dw}}\right), \quad \text{by Lemma 3.2.}$$

Using the Borel–Cantelli lemma, we complete the proof. \square

Proposition 5.2 *Let (G, μ) be a weighted graph such that G is of bounded degree, and set $M = \sup_{v \in V(G)} \deg v (< \infty)$. Then, for all realization of $\{Y_n\}_{n=0}^\infty$ and all $n \geq 0$,*

$$d_{\mathbb{Z}_2^d G}(Y_0, Y_n) \leq (2M + 1)R_n.$$

To prove the above proposition, we need the following lemma. To exclude ambiguity, we first introduce some terminology. Let H be a connected subgraph of G .

- A *path* γ in H is a sequence of vertices $v_0 v_1 \dots v_k$ such that $v_i \in V(H)$ and $v_i v_{i+1} \in E(H)$ for all i . For a path γ , we set $V(\gamma) = \{v_0, v_1, \dots, v_k\}$, and define the length of γ as $|\gamma| = k$. $e_j = v_j v_{j+1}$ ($j = 0, 1, \dots, k-1$) are said to be the *edges* of γ .
- For the path γ given above and a given edge $e \in E(G)$, we define $F(\gamma, e) = \{l \in \{0, 1, \dots, k-1\} \mid e_l = e\}$.
- We denote $\vec{e} = \overrightarrow{uv}$ if the edge e is directed from u to v and call \vec{e} a *directed edge*. For two directed edges $\vec{e}_1 = \overrightarrow{u_1 v_1}$ and $\vec{e}_2 = \overrightarrow{u_2 v_2}$, \vec{e}_1 and \vec{e}_2 are equal if and only if $u_1 = u_2, v_1 = v_2$.

Lemma 5.3 *Let G be a graph of bounded degree, $M = \sup_{v \in V(G)} \deg v (< \infty)$ and H be a finite connected subgraph of G .*

- (1) *Let $x, y \in V(H)$. Then, there exists a path $\gamma = w_0 w_1 \dots w_k$ in H such that the following hold.*
 - (a) $\{w_0, w_1, \dots, w_k\} = V(H)$.
 - (b) *Define $\tilde{e}_j = w_j w_{j+1}$ for $j = 0, 1, \dots, k-1$. Then, $\sharp\{j \in \{0, 1, \dots, k-1\} \mid \tilde{e}_j = \tilde{e}_s\} \leq 2$ for all $s = 0, 1, \dots, k-1$.*
 - (c) $w_0 = x$ and $w_k = y$.
- (2) *Let γ be as in (1). Then, $|\gamma| \leq 2M \sharp V(H)$.*

Proof (1) Take a path $\eta = u_0 u_1 \dots u_n$ in H such that $u_0 = x, u_n = y$ and $\{u_0, u_1, \dots, u_n\} = V(H)$. Define $f_j = u_j u_{j+1}$. If each edge f_j satisfies $\sharp\{l \in \{0, 1, \dots, n-1\} \mid f_l = f_j\} \leq 2$ for $j = 0, 1, \dots, n-1$, then η satisfies the conditions (a), (b) and (c). So, we may assume that there exists f_j such that $\sharp\{l \in \{0, 1, \dots, n-1\} \mid f_l = f_j\} \geq 3$. For such an edge f_j , there exist at least two elements $s, t \in F(\eta, f_j)$ such that $\overrightarrow{u_s u_{s+1}} = \overrightarrow{u_t u_{t+1}}$. Let $s < t$. Define $\eta_{st} = u_s u_{s+1} u_{s+2} \dots u_{t-1} u_t$ (see Fig. 3). Replace $\eta = u_0 \dots u_{s-1} \eta_{st} u_{t+1} \dots u_n$ by $\tilde{\eta} = \tilde{u}_0 \tilde{u}_1 \dots \tilde{u}_n = u_0 \dots u_{s-1} \tilde{\eta}_{st} u_{t+1} \dots u_n$ where $\tilde{\eta}_{st} = u_s u_{t-1} u_{t-2} \dots u_{s+2}$. $\tilde{\eta}$ is again a path, $V(\tilde{\eta}) = V(H)$ and $\sharp F(\tilde{\eta}, f_j) = \sharp F(\eta, f_j) - 2$ (see Fig. 4). Repeat this operation to f_0, f_1, \dots, f_{n-1} inductively until obtaining a path satisfying (a), (b) and (c).

(2) Note that w_j appears in $V(\gamma)$ at most $2 \deg(w_j)$ times for each vertex $w_j \in V(H)$. The conclusion can be verified easily. \square

Fig. 3 An example of the path η

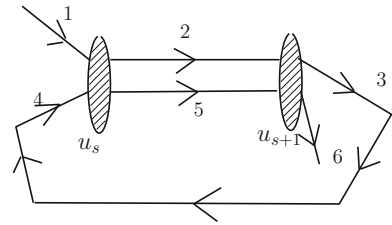
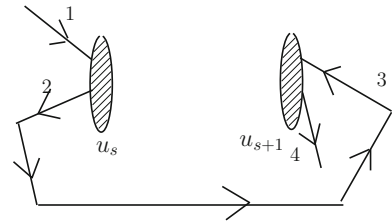


Fig. 4 An example of the surgery of η



Remark 5.4 While the revision of this paper was under review, Pierre Mathieu pointed us out a simpler proof of this lemma (therefore, a simpler proof of Proposition 5.2). Take a spanning tree of $V(H)$ and let γ' be an exploration path of the spanning tree from x to x (i.e., a path that crosses each bond of the tree exactly twice). Then, one can produce a desired path γ by an easy modification of γ' .

Proof of Proposition 5.2 The graph $G' = (V(G'), E(G'))$, where $V(G') = \{X_0, X_1, \dots, X_n\}$ and $E(G') = \{X_i X_{i+1} \in E(G) \mid 0 \leq i \leq n-1\}$, is itself a connected subgraph of G . So, applying Lemma 5.3 to G' , we have

$$\min\{|\gamma| \mid \gamma \text{ is a path starting at } X_0, V(\gamma) = \{X_0, X_1, \dots, X_n\}\} \leq 2MR_n.$$

By this and (5.1), we obtain $d_{\mathbb{Z}_2, G}(Y_0, Y_n) \leq (2M + 1)R_n$. \square

5.2 Proof of Theorem 2.4(I)

In this subsection, we prove the LILs for $\{Y_n\}_{n \geq 0}$ when $d_s/2 < 1$.

Theorem 5.5 Assume that Assumption 2.2 (1), (2), (4) and $d_s/2 < 1$ hold. Then, there exist (non-random) constants $c_1, c_2 > 0$ such that the following hold:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{R_n}{n^{d_s/2} (\log \log n)^{1-d_s/2}} &= c_1, & P_x\text{-a.s. } \forall x \in V(G), \\ \liminf_{n \rightarrow \infty} \frac{R_n}{n^{d_s/2} (\log \log n)^{-d_s/2}} &= c_2, & P_x\text{-a.s. } \forall x \in V(G). \end{aligned}$$

This is a discrete analog of [5, Propositions 4.9 and 4.10]. Note that the proof of these propositions relies on the self-similarity of the process. Since our random walk does not satisfy this property, we need non-trivial modifications for the proof. Quite

recently, Kim et al. [12, Theorem 4.14] proved the LIL of the range for jump processes without using self-similarity of the process. By easy modifications, we can apply their argument to our random walk. The proof of Theorem 5.5 will be given in Appendix 1.

Proof of Theorem 2.4(I) Note that Assumption 2.2 (1) implies that G is of bounded degree. Thus, by (5.1), Propositions 5.1, 5.2 and Theorem 5.5, we obtain (2.8) and (2.9). \square

5.3 Proof of Theorem 2.4(II)

In this subsection, we prove the LILs for $\{Y_n\}_{n \geq 0}$ when $d_s/2 > 1$.

First, we explain the notion of “uniform condition” defined in [14]. We define the Dirichlet form \mathcal{E} on the weighted graph (G, μ) by

$$\mathcal{E}(f, f) = \sum_{x, y \in V(G)} (f(x) - f(y))^2 \mu_{xy},$$

for $f : V(G) \rightarrow \mathbb{R}$, and the effective resistance $R_{\text{eff}}(\cdot, \cdot)$ as

$$R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f); f|_A = 1, f|_B = 0\} \quad (5.4)$$

for $A, B \subset V(G)$ with $A \cap B = \emptyset$. Denote $\rho(x, n) = R_{\text{eff}}(\{x\}, B(x, n)^c)$ for any $x \in V(G)$, $n \in \mathbb{N}$ and $\rho(x) = \lim_{n \rightarrow \infty} \rho(x, n)$.

Definition 5.6 (Okamura [14]) We say that a weighted graph (G, μ) satisfies the uniform condition (U) if $\rho(x, n)$ converges uniformly in $x \in V(G)$ to $\rho(x)$ as $n \rightarrow \infty$.

For $A \subset G$, define

$$T_A^+ = \inf\{n \geq 1 \mid X_n \in A\}.$$

We write T_x^+ instead of $T_{\{x\}}^+$.

The following is an improvement of [14, Corollary 2.3].

Proposition 5.7 Let (G, μ) be a weighted graph satisfying (U) and (2.7) and assume that G is of bounded degree. If $\sup_{x \in V(G)} P_x(M < T_x^+ < \infty) = O(M^{-\delta})$ for some $\delta > 0$, then

$$1 - F_2 \leq \liminf_{n \rightarrow \infty} \frac{R_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{R_n}{n} \leq 1 - F_1, \quad P_x\text{-a.s.}$$

for all $x \in V(G)$, where $F_1 = \inf_{x \in V(G)} P_x(T_x^+ < \infty)$ and $F_2 = \sup_{x \in V(G)} P_x(T_x^+ < \infty)$.

Remark 5.8 In [14, Corollary 2.3], a stronger condition $\sup_x P_x(M < T_x^+ < \infty) = O(M^{-1-\delta})$ for some $\delta > 0$ is imposed to prove $1 - F_2 \leq \liminf_{n \rightarrow \infty} \frac{R_n}{n}$. As we prove below, it is enough to assume $\sup_x P_x(M < T_x^+ < \infty) = O(M^{-\delta})$.

Proof of Proposition 5.7 For the upper bound $\limsup_{n \rightarrow \infty} \frac{R_n}{n} \leq 1 - F_1$, [14, Proof of Corollary 2.3] goes through without any modifications.

Hence, we prove $1 - F_2 \leq \liminf_{n \rightarrow \infty} \frac{R_n}{n}$ under our assumption. Fix $\epsilon > 0$. By [14, (2.5), (2.6), (2.7)] there exists $a \in (0, 1)$ such that for any n and M we have

$$P_x \left(\frac{R_n}{n} \leq 1 - F_2 - \epsilon \right) \leq \frac{2}{\epsilon} \sup_{x \in V(G)} P_x(M < T_x^+ < \infty) + (M + 1)a^{n/(M+1)}. \quad (5.5)$$

Choose $k > 2/\delta$. Replacing n by n^k in (5.5), we have

$$\begin{aligned} P_x \left(\frac{R_{n^k}}{n^k} \leq 1 - F_2 - \epsilon \right) &\leq \frac{2}{\epsilon} \sup_{x \in V(G)} P_x(M < T_x^+ < \infty) + (M + 1)a^{n^k/(M+1)} \\ &= \frac{2}{\epsilon} O(M^{-\delta}) + (M + 1)a^{n^k/(M+1)}. \end{aligned}$$

Take $M = M(n) = n^{k/2} - 1$ and we have

$$P_x \left(\frac{R_{n^k}}{n^k} \leq 1 - F_2 - \epsilon \right) \leq \frac{2}{\epsilon} O \left(\frac{1}{n^{k\delta/2}} \right) + n^{k/2} a^{n^{k/2}}.$$

Since $k\delta/2 > 1$, we can apply the Borel–Cantelli lemma and we obtain

$$1 - F_2 \leq \liminf_{n \rightarrow \infty} \frac{R_{n^k}}{n^k}, \quad P_x\text{-a.s.}$$

For any m , choose n as $n^k \leq m < (n + 1)^k$, and we then have

$$\frac{R_m}{m} \geq \frac{n^k}{m} \frac{R_{n^k}}{n^k} = \left(\frac{n}{n+1} \right)^k \frac{(n+1)^k}{m} \frac{R_{n^k}}{n^k} \geq \left(\frac{n}{n+1} \right)^k \frac{R_{n^k}}{n^k}.$$

Take $\liminf_{m \rightarrow \infty}$ and we obtain $1 - F_2 \leq \liminf_{n \rightarrow \infty} \frac{R_n}{n}$. □

Proof of Theorem 2.4(II) Note that the uniform condition (U) is satisfied in our framework by [14, Proposition 4.6].

Since $d_s/2 > 1$, we have

$$P_x(M < T_x^+ < \infty) \leq \sum_{n=M+1}^{\infty} p_n(x, x) \leq c \sum_{n=M+1}^{\infty} n^{-d_s/2} = O(M^{1-d_s/2}). \quad (5.6)$$

Note that Assumption 2.2 (1), (2) imply (2.1) and (2.7). By (5.6) and Proposition 5.7, we have

$$1 - F_2 \leq \liminf_{n \rightarrow \infty} \frac{R_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{R_n}{n} \leq 1 - F_1, \quad P_x\text{-a.s.} \quad (5.7)$$

Define $G(x, x) = \sum_{n=0}^{\infty} p_n(x, x)$, and $F(x, x) = \sum_{n=1}^{\infty} P_x(T_x^+ = n) = P_x(T_x^+ < \infty)$. It is well known that

$$G(x, x) - 1 = F(x, x)G(x, x). \quad (5.8)$$

Since $d_s/2 > 1$, we have

$$\sup_{x \in V(G)} \sum_{n=0}^{\infty} p_n(x, x) \leq c \sum_{n=1}^{\infty} \frac{1}{n^{d_s/2}} < \infty.$$

By this and (5.8), we have

$$F_2 = \sup_x F(x, x) < 1. \quad (5.9)$$

Thus, by (5.1), Propositions 5.1, 5.2, (5.7) and (5.9), we conclude that for all $x \in V(\mathbb{Z}_2 \wr G)$,

$$\begin{aligned} 0 < \frac{1}{4}(1 - F_2) &\leq \liminf_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{d_{\mathbb{Z}_2 \wr G}(Y_0, Y_n)}{n} \leq (2M + 1)(1 - F_1) < \infty, \quad P_x\text{-a.s.} \end{aligned}$$

Hence, we complete the proof. \square

Acknowledgments We would like to thank Martin T. Barlow for stimulating discussions when this project was initiated. We also thank the anonymous referee for detailed comments and careful corrections. Especially, we are grateful to the referee for pointing out to us that Theorem 2.4(II) and Lemma 3.2 require only the on-diagonal heat kernel upper bound (2.3) rather than the full upper bound (2.4). The first author was partially supported by JSPS KAKENHI Grant Number 25247007. The second author was partially supported by JSPS KAKENHI Grant Number 15J02838.

Appendix 1: Proof of Theorem 5.5

In this section, we will explain briefly the essential part of the proof of Theorem 5.5, which is a discrete analog of [5, Propositions 4.9 and 4.10]. Note that the results in [5] are for the range of Brownian motion on fractals, and the proof heavily relies on the self-similarity of Brownian motion. Quite recently, Kim et al. [12, Theorem 4.16] have obtained the LIL for the range of jump processes on metric measure spaces without using scaling law of the process. We employ the results and techniques in [5, 7, 12], and prove the LIL for the range of the random walk without using scaling law of the process or of the heat kernel.

The key to prove the LILs for the range of the process is to establish those for the maximum of local times. We assume $d_f < d_w$ and define the local times at $x \in V(G)$ up to the time n as

$$L_n(x) = \begin{cases} \frac{1}{m(x)} \sum_{k=0}^{n-1} 1_{\{X_k=x\}} & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

and the maximum of the local times up to the time n as

$$L_n^* = \sup_{x \in V(G)} L_n(x).$$

Let $\theta = (d_w - d_f)/2$. Recall (5.4) for the definition of the effective resistance, and write $R(x, y) := R_{\text{eff}}(\{x\}, \{y\})$. Also, let $R^{(i)}(x, y) := i^{-(d_w - d_f)} R(x, y)$ for all $x, y \in V(G)$ and $i > 0$. We first cite a result from [7].

Lemma 5.9 ([7, Lemma 6.3 (a)]) *Suppose Assumption 2.2 (1), (2), (4) and $d_f < d_w$. Then, there exist constants $c_0, c_1 > 0$ such that*

$$\begin{aligned} & \sup_{i \in [1, \infty)} \sup_{\substack{x, y, z \in V(G) \\ d(x, y) \leq i}} P_z \left(\max_{0 \leq k \leq i^{d_w}} i^{-(d_w - d_f)} |L_k(x) - L_k(y)| \geq \lambda \sqrt{R^{(i)}(x, y)} \right) \\ & \leq c_1 \exp(-c_0 \lambda) \end{aligned}$$

for all $\lambda \geq 0$. In particular, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} & \sup_{i \in [1, \infty)} \sup_{x, y, z \in V(G) \atop d(x, y) \leq i} P_z \left(\max_{0 \leq k \leq i^{d_w}} |L_k(x) - L_k(y)| \geq \lambda (i d(x, y))^\theta \right) \\ & \leq c_1 \exp(-c_2 \lambda) \end{aligned} \quad (5.10)$$

for all $\lambda \geq 0$.

Proof Note that by [4, Theorem 1.3], we have the following relation between the resistance metric and the graph distance

$$R(x, y) \asymp d(x, y)^{d_w - d_f}, \quad \forall x, y \in V(G),$$

which is a consequence of Assumption 2.2 (1), (2), (4) and $d_f < d_w$. The first statement is the result of [7, Lemma 6.3 (a)] for the case of $\kappa = T = 1$. The second statement can be proved by applying the above relation between the resistance metric and the graph distance. \square

The next theorem is an analog of [12, Proposition 4.8]. Since our proof is different from that of [12, Proposition 4.8] which uses a scaling argument, we give the proof below.

Theorem 5.10 (Moduli of continuity of local times) *Suppose Assumption 2.2 (1), (2), (4) and $d_f < d_w$. Then, there exist constants $c, C > 0$ such that*

$$P_o \left(\max_{\substack{x, y \in B_d(o, \kappa u^{1/d_w}) \\ d(x, y) \leq L}} \max_{0 \leq t \leq u} |L_t(x) - L_t(y)| \geq A \right) \\ \leq c \frac{(u^{1/d_w} \kappa)^{2d_f}}{L^{2d_f}} \exp \left(-\frac{CA}{(\kappa u^{1/d_w} L)^\theta} \right)$$

for all $o \in V(G)$, $u \geq 1$, $\kappa \geq 1$, $0 < L \leq 2\kappa u^{1/d_w}$ and $A > 0$.

Proof Let c_1, c_2 be as in Lemma 5.9. Let $G^{(i)}$ be a graph with $V(G^{(i)}) = B_d(o, 6i)$ and $E(G^{(i)}) = \{(x, y) \in E(G) \mid x, y \in V(G^{(i)})\}$. We denote by $m_i(\cdot) = m(\cdot \cap V(G^{(i)}))$ the measure on $G^{(i)}$. Then, the following holds by the proof of [7, Theorem 6.1]; There exists a positive constant c_3 (not depending on i) such that

$$m_i(B_d(x, r)) \geq c_3 r^{d_f} \quad (5.11)$$

for all $i \in [\frac{1}{6}, \infty)$, $x \in V(G^{(i)})$ and $r \in [1, 12i]$ (In fact, [7, Proof of Theorem 6.1] discusses the case where $i \geq 1$ is an integer, and we can obtain (5.11) for $i \in [\frac{1}{6}, \infty)$ by adjusting the constant). Set $6i = \kappa u^{1/d_w}$. By (5.11), we have

$$\min_{x \in V(G^{(i)})} m_i(B_{d_i}(x, r)) = \min_{x \in V(G^{(i)})} m_i(B_d(x, ir)) \geq c_3 i^{d_f} r^{d_f},$$

where $d_i = \frac{1}{i} d_i$. We now apply a discrete version of Garsia's Lemma (see [7, Proposition 3.1, Remark 3.2]) to the graph $G^{(i)}$ with distance $d_i = \frac{1}{i} d$, $p(x) = x^\theta$, $\psi(x) = \exp(c_* |x|) - 1$, and the function $f(x) = \frac{1}{i^{2\theta}} L_t(x)$ on $V(G^{(i)})$ for $0 \leq t \leq u$, where $c_* = 12^{-\theta} c_2/2$. For $x, y \in V(G^{(i)}) = B_d(o, 6i)$ with $d(x, y) \leq L$ and $t \in [0, u]$, we have

$$\frac{1}{i^{2\theta}} |L_t(x) - L_t(y)| \leq \frac{4}{c_*} \int_0^{2d_i(x, y)} 4^\theta s^{\theta-1} \log \left(\frac{\Gamma\left(\frac{1}{i^{2\theta}} L_t\right)}{c_3^2 i^{2d_f} s^{2d_f} / 2^{2d_f}} + 1 \right) ds \\ \leq \frac{4^{\theta+1}}{c_*} \int_0^{2L/i} s^{\theta-1} \log \left(\frac{\tilde{\Gamma}\left(\frac{1}{i^{2\theta}} L_u\right)}{c_4 i^{2d_f} s^{2d_f}} + 1 \right) ds, \quad (5.12)$$

where $c_4 = c_3^2/2^{2d_f}$ and

$$\Gamma\left(\frac{1}{i^{2\theta}} L_t\right) := \sum_{x, y \in V(G^{(i)})} \exp\left(c_* \frac{|L_t(x) - L_t(y)|}{(id(x, y))^\theta}\right) m(x)m(y), \\ \tilde{\Gamma}\left(\frac{1}{i^{2\theta}} L_u\right) := \sum_{x, y \in V(G^{(i)})} \exp\left(c_* \frac{\sup_{0 \leq t \leq u} |L_t(x) - L_t(y)|}{(id(x, y))^\theta}\right) m(x)m(y).$$

Define $v = \frac{\tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right)}{c_4 i^{2d_f} s^{2d_f}}$. Then, by (5.12), we have

$$\begin{aligned} & \frac{1}{i^{2\theta}} |L_t(x) - L_t(y)| \\ & \leq \frac{4^{\theta+1}}{c_*} \frac{1}{2d_f} \frac{1}{c_4} \frac{1}{i^\theta} \tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right)^{\frac{\theta}{2d_f}} \int_b^\infty \left(\frac{1}{v}\right)^{\theta/(2d_f)+1} \log(v+1) dv \\ & = c_5 \frac{1}{i^\theta} \tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right)^{\frac{\theta}{2d_f}} \int_b^\infty \left(\frac{1}{v}\right)^{\theta/(2d_f)+1} \log(v+1) dv, \end{aligned}$$

where $c_5 = \frac{4^{\theta+1}}{c_*} \frac{1}{2d_f} \frac{1}{c_4}$, $b = \frac{\tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right)}{c_6 L^{2d_f}}$, $c_6 = 2^{2d_f} c_4$. By easy calculus we have

$$\int_b^\infty \left(\frac{1}{v}\right)^{\theta/(2d_f)+1} \log(v+1) dv \leq \frac{\log(b+1) + 2d_f/\theta}{\frac{\theta}{2d_f} \cdot b^{\theta/2d_f}}.$$

Thus, we have

$$\frac{1}{i^{2\theta}} |L_t(x) - L_t(y)| \leq c_7 \left(\frac{L}{i}\right)^\theta \left\{ \log(b+1) + \frac{2d_f}{\theta} \right\}.$$

where $c_7 = c_5(2d_f/\theta)c_6^{\frac{\theta}{2d_f}}$, so

$$\begin{aligned} & P_o \left(\max_{\substack{x,y \in B_d(o, \kappa u^{1/d_w}) \\ d(x,y) \leq L}} \max_{0 \leq t \leq u} |L_t(x) - L_t(y)| \geq A \right) \\ & \leq P_o \left(\log(b+1) \geq \frac{A}{c_7(iL)^\theta} - \frac{2d_f}{\theta} \right) \\ & \leq c_8 \left(\frac{E_o \left[\tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right) \right]}{c_6 L^{2d_f}} + 1 \right) \exp \left(-c_9 \frac{A}{(iL)^\theta} \right). \end{aligned}$$

By (5.10), noting $c_* = 12^{-\theta} c_2/2$, $\kappa \geq 1$ and $6i = \kappa u^{1/d_w}$ (in particular $u \leq (6i)^{d_w}$), we have

$$\begin{aligned} E_o \left[\tilde{\Gamma}\left(\frac{1}{i^{2\theta}}L_u\right) \right] &= \sum_{x,y \in V(G^{(i)})} E_o \left[\exp \left(c_* \frac{\sup_{0 \leq t \leq u} |L_t(x) - L_t(y)|}{(id(x,y))^\theta} \right) \right] m(x)m(y) \\ &\leq \sum_{x,y \in V(G^{(i)})} \sum_n \exp(c_*(n+1)) P_o \left(\frac{\sup_{0 \leq t \leq u} |L_t(x) - L_t(y)|}{(id(x,y))^\theta} \geq n \right) m(x)m(y) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{x,y \in V(G^{(i)})} \sum_n \left\{ \sup_{\substack{x',y' \in V(G) \\ d(x',y') \leq 12i}} P_o \left(\sup_{0 \leq t \leq (12i)^{d_w}} |L_t(x') - L_t(y')| \geq \frac{n}{12^\theta} (12i d(x', y'))^\theta \right) \right\} \\
&\quad \times \exp(c_*(n+1)) m(x)m(y) \\
&\leq \sum_{x,y \in V(G^{(i)})} \sum_n \left\{ c_1 \exp\left(-c_2 \frac{n}{12^\theta}\right) \right\} \exp(c_*(n+1)) m(x)m(y) \\
&\leq c_{10} i^{2d_f} \leq c_{11} u^{2d_f/d_w} \kappa^{2d_f}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&P_o \left(\max_{\substack{x,y \in B_d(o, \kappa u^{1/d_w}) \\ d(x,y) \leq L}} \max_{0 \leq t \leq u} |L_t(x) - L_t(y)| \geq A \right) \\
&\leq c_{12} \left(\frac{u^{2d_f/d_w} \kappa^{2d_f}}{L^{2d_f}} + 1 \right) \exp \left(-c_{13} \frac{A}{(\kappa u^{1/d_w} L)^\theta} \right).
\end{aligned}$$

Since we are assuming $L \leq 2\kappa u^{1/d_w}$, we complete the proof. \square

Given Theorem 5.10, the following theorem can be proved similarly to [12, Proof of Theorems 4.11 and 4.15]. (See also [5, Propositions 4.7 and 4.8]).

Theorem 5.11 (LILs for the local times) *Suppose Assumption 2.2 (1), (2), (4) and $d_f < d_w$. Then, there exist positive constants c_1, c_2 such that the following hold.*

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \frac{L_n^*}{n^{1-d_s/2} (\log \log n)^{d_s/2}} &= c_1, \quad P_x\text{-a.s. for } \forall x \in V(G), \\
\liminf_{n \rightarrow \infty} \frac{L_n^*}{n^{1-d_s/2} (\log \log n)^{d_s/2-1}} &= c_2, \quad P_x\text{-a.s. for } \forall x \in V(G).
\end{aligned}$$

Given Theorem 5.11, the proof of Theorem 5.5 can be done similarly as in [12, Theorem 4.16] by using the relation $n = \sum_{x \in R_n} L_n(x) \leq R_n L_n^*$ (See also [5, Propositions 4.9 and 4.10]).

References

1. Barlow, M.T.: Diffusions on fractals. In: Lecture Notes in Mathematics, vol. 1690, Ecole d'été de probabilités de Saint-Flour XXV, 1995. Springer, New York (1998)
2. Barlow, M.T.: Which values of the volume growth and escape time exponent are possible for a graph? Rev. Mat. Iberoamericana **20**(1), 1–31 (2004)
3. Barlow, M.T., Bass, R.F.: Random walks on graphical Sierpinski carpets. In: Random Walks and Discrete Potential Theory (Cortona, 1997), 26–55. Sympos. Math., XXXIX, Cambridge University Press, Cambridge (1997)
4. Barlow, M.T., Coulhon, T., Kumagai, T.: Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. Commun. Pure Appl. Math. **58**(12), 1642–1677 (2005)
5. Bass, R.F., Kumagai, T.: Laws of the iterated logarithm for some symmetric diffusion processes. Osaka J. Math. **37**(3), 625–650 (2000)

6. Coulhon, T., Grigor'yan, A., Pittet, C.: A geometric approach to on-diagonal heat kernel lower bounds on groups. *Ann. Inst. Fourier (Grenoble)* **51**(6), 1763–1827 (2001)
7. Croydon, D.A.: Moduli of continuity of local times of random walks on graphs in terms of the resistance metric. *Trans. Lond. Math. Soc.* **2**(1), 57–79 (2015)
8. Dvoretzky, A., Erdős, P.: Some problems on random walk in space. In: *Proceedings of Second Berkeley Symposium on Mathematical Statistics and Probability*. University of California Press (1951), pp. 353–367
9. Gibson, L.R.: The mass of sites visited by a random walk on an infinite graph. *Electron. J. Probab.* **13**, 1257–1282 (2008)
10. Jones, O.D.: Transition probabilities for the simple random walk on the Sierpiński graph. *Stoch. Process. Appl.* **61**(1), 45–69 (1996)
11. Kigami, J.: *Analysis on fractals*, Cambridge Tracts in Mathematics, vol. 143. Cambridge University Press, Cambridge (2001)
12. Kim, P., Kumagai, T., Wang, J.: Laws of the Iterated Logarithm for Symmetric Jump Processes. *Bernoulli* (to appear). [arXiv:1504.06210v2](https://arxiv.org/abs/1504.06210v2)
13. Kumagai, T.: Random walks on disordered media and their scaling limits. In: *Lecture Notes in Mathematics*, vol. 2101, *Ecole d'été de probabilités de Saint-Flour XL*, 2010. Springer, New York (2014)
14. Okamura, K.: On the range of random walk on graphs satisfying a uniform condition. *ALEA Lat. Am. J. Probab. Math. Stat.* **11**(1), 341–357 (2014)
15. Pittet, C., Saloff-Coste, L.: Amenable groups, isoperimetric profiles and random walks. In: *Geometric Group Theory Down Under* (Canberra, 1996), 293–316. de Gruyter, Berlin (1999)
16. Pittet, C., Saloff-Coste, L.: A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples. Preprint 2001. <http://www.math.cornell.edu/~lsc/articles.html>
17. Rau, C.: Existence of graphs with sub exponential transitions probability decay and applications. *Bull. Soc. Math. Fr.* **138**(4), 491–542 (2010)
18. Revelle, D.: Rate of escape of random walks on wreath products and related groups. *Ann. Probab.* **31**(4), 1917–1934 (2003)
19. Varopoulos, N.: Random walks on soluble groups. *Bull. Sci. Math.* **107**, 337–344 (1983)
20. Varopoulos, N.: A potential theoretic property of soluble groups. *Bull. Sci. Math.* **108**, 263–273 (1984)
21. Woess, W.: *Random walks on infinite graphs and groups*. Cambridge Tracts in Mathematics, vol. 138. Cambridge University Press, Cambridge (2000)