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Dissipative operators, symplectic geometry, and limit-circle solutions

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ABSTRACT

For a general symmetric ordinary differential expression M and a positive weight function w, the authors characterized the dissipative and strictly dissipative extensions of the minimal operator generated by M in terms of subspaces of a symplectic geometry space. Here we characterize these subspaces in terms of LC solutions, namely, we characterize the boundary conditions of the dissipative and strictly dissipative extensions by LC solutions in a Hilbert space.

1. Introduction

Given a symmetric (formally self-adjoint) differential expression M of even or odd order and a positive weight function w the self-adjoint realizations of the equation

$$My = \lambda w\, y \text{ on } J = (a,b),\ -\infty \le a < b \le \infty \tag{1.1}$$

are generally studied in the framework of the Hilbert space $H = L^2(J,w)$. The expression M generates minimal and maximal operators $S_{\min}$ and $S_{\max}$ in H with domains $D_{\min} = D(S_{\min})$, $D_{\max} = D(S_{\max})$ and the self-adjoint extensions S of $S_{\min}$ satisfy

$$S_{\min} \subset S = S^* \subset S_{\max} \tag{1.2}$$

Thus these operators S differ from each other only by their domains and the characterization of these domains is of considerable interest. Although these operators S are generally described as extensions of the minimal operator $S_{\min}$ it is clear from (1.2) that they are also restrictions of the maximal operator $S_{\max}$.

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In 1999 Everitt and Markus (EM) [1,2] characterized the self-adjoint domains for even and odd order expressions M in terms of Lagrangian subspaces of complex symplectic spaces. Given a general symmetric expression M whose minimal operator has equal deficiency indices, there exists a natural one-to-one correspondence between the set of all self-adjoint operators generated by M and the set of all complete Lagrangian subspaces in the complex symplectic space $\tilde{S} = D_{max}/D_{min}$ with symplectic product [:] given by

$$[\tilde{f} : \tilde{g}] = [f + D_{min} : g + D_{min}] := [f : g] = [f, g]_a^b.$$

For the even order case with real coefficients, Wang, Sun and Zettl [11] constructed limit-circle (LC) solutions of (1.1) and Hao, Sun, Wang and Zettl [4] characterized the domains of self-adjoint extensions of $S_{\min}$ in terms of LC solutions in the Hilbert space H. These LC solutions are higher order analogues of the celebrated Titchmarsh–Weyl limit-circle solutions in the second order case but, in contrast to the second order case, only some of the solutions in H are LC solutions in general. Recently, this LC characterization has been used to obtain information about the spectrum of these operators S [6,5,10].

In 2013 [13] the authors found a 1–1 correspondence between the characterizations in these two very different spaces and thereby added the methods of symplectic geometry to the investigation of the spectrum of self-adjoint differential operators in Hilbert space.

In 2014 [14] the authors identified the subspaces of the complex symplectic space $\tilde{S}$ which characterize the dissipative and strictly dissipative extensions of $S_{\min}$ in H and named these subspaces Dissipative and strictly Dissipative subspaces of $\tilde{S}$, respectively.

In this paper we characterize these Dissipative and strictly Dissipative subspaces of $\tilde{S}$ in terms of boundary conditions of equation (1.1) determined by LC solutions and identify three classes of subspaces: separated, coupled and mixed as in the self-adjoint case.

The organization of this paper is as follows: This Introduction is followed by a brief summary of symplectic geometry, symmetric differential expressions M and LC solutions in Section 2. Section 3 contains the characterization of the dissipative and strictly dissipative extensions of the minimal operator $S_{\min}$ in terms of LC solutions and the construction of a new basis for the EM space. In Section 4 the boundary conditions of the dissipative extensions are classified into three mutually exclusive classes: separated, coupled and mixed.

2. Maximal and minimal operators, LC solutions, symplectic geometry

In the first subsection we recall the maximal and minimal operators $S_{\max}$ and $S_{\min}$ and the Lagrange identity; the second subsection contains the construction of LC solutions, the third some basic definitions for general symplectic spaces and the introduction of the symplectic space used here.

2.1. Maximal and minimal operators

For a general symmetric quasi-differential expression M of order $n = 2k$ with real coefficients and a positive weight function w, see [16] for a detailed definition and historical comments, the minimal and maximal operators $S_{\min}$, $S_{\max}$ with domains, $D_{\min}$, $D_{\max}$, respectively, can be defined as follows:

$$\begin{aligned} D_{\max} &= \{y \in L^2(J, w) : y^{[r]} \in AC_{loc}(J),\ w^{-1}My \in L^2(J, w)\},\ r = 1, \cdots, n-1\}. \\ S_{\max}\, y &= w^{-1}My,\ y \in D_{\max}. \\ S_{\min} &= S_{\max}^*,\ \ S_{\max} = S_{\min}^*. \\ D_{\min} &= D(S_{\min}). \end{aligned}$$

Here $y^{[r]}$ denotes the r-th quasi-derivative. Let d denote the deficiency index of $S_{\min}$ in H. See [16] for a definition of $y^{[r]}$ and d. It is well known that $S_{\min}$ and $S_{\max}$ are densely defined closed operators in H, and $S_{\min}$ is symmetric and $S_{\max} = S^*_{\min}$.

The symmetric quasi-differential expressions M considered here are much more general than those studied by Naimark in [8] and assume no smoothness conditions on the coefficients. In particular the coefficients may be piece wise continuous.

Fundamental to the study of boundary value problems is the Lagrange identity:

Lemma 1. *For any y, z in $D_{\max}$ we have*

$$\int_a^b \{\bar{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a). \tag{2.1}$$

Here $[y,z]$ is the Lagrange bracket and $[y,z](b)$, $[y,z](a)$ exist as finite limits for all y, z in $D_{\max}$.

Proof. See [3] or [7]. □

2.2. LC solutions

In this subsection we briefly recall the Wang–Sun–Zettl [11] and Hao et al. [4] construction of LC solutions in H.

Theorem 1. *Let M be a symmetric differential expression and w a weight function. Consider the equation*

$$My = \lambda wy. \tag{2.2}$$

Let $c \in (a,b)$ and let d_a and d_b denote the deficiency indices of (2.2) on (a,c) and (c,b), respectively. Assume that for some $\lambda = \lambda_a \in \mathbb{R}$, the equation (2.2) has d_a linearly independent solutions u_i on (a,c) which lie in $L^2((a,c),w)$ and for some $\lambda = \lambda_b$ the equation (2.2) has d_b linearly independent solutions v_j on (c,b) which lie in $L^2((c,b),w)$, $i = 1,\cdots,d_a$, $j = 1,\cdots,d_b$. Then

(1) *For $m_a = 2d_a - 2k$ the solutions u_i, $i = 1,\cdots,d_a$ can be ordered such that the $m_a \times m_a$ matrix $U = ([u_i,u_j](c))$, $1 \le i,j \le m_a$, is given by*

$$U = (-1)^{k+1}E_{m_a},\ E_{m_a} = ((-1)^r\delta_{r,m_1+1-s})^{m_a}_{r,s}. \tag{2.3}$$

(2) *For $m_b = 2d_b - 2k$ the solutions v_j, $j = 1,\cdots,d_b$ on (c,b) can be ordered such that the $m_b \times m_b$ matrix $V = ([v_i,v_j](c))$, $1 \le i,j \le m_b$, is given by*

$$V = (-1)^{k+1}E_{m_b},\ E_{m_b} = ((-1)^r\delta_{r,m_2+1-s})^{m_b}_{r,s}. \tag{2.4}$$

(3) *For every $y \in D_{\max}(a,b)$ we have*

$$[y,u_j](a) = 0,\ \text{for } j = m_a+1,\cdots,d_a, \tag{2.5}$$

$$[y,v_j](b) = 0,\ \text{for } j = m_b+1,\cdots,d_b. \tag{2.6}$$

(4) *For $1 \le i,j \le d_a$, $1 \le r,s \le d_b$, we have*

$$[u_i,u_j](a) = [u_i,u_j](c). \tag{2.7}$$

$$[v_r,v_s](b) = [v_r,v_s](c). \tag{2.8}$$

(5) *The solutions* u_i *can be extended to* (a,b) *such that the extended functions, also denoted by* u_i*, satisfy* $u_i \in D_{\max}(a,b)$ *and* u_i *is identically zero in a left neighborhood of* b*,* $j = 1, \cdots, d_a$*.*
(6) *The solutions* v_j *can be extended to* (a,b) *such that the extended functions, also denoted by* v_j*, satisfy* $v_j \in D_{\max}(a,b)$ *and* v_j *is identically zero in a right neighborhood of* a*,* $j = 1, \cdots, d_b$*.*

Proof. See Theorem 4.1 in [4]. □

It is well known [12,16] that $d = d_a + d_b - 2k$ and hence $2d = m_a + m_b$.

Remark 1 *(LC and LP solutions).* The solutions $u_1, \cdots, u_{m_a}$ and $v_1, \cdots, v_{m_b}$ are LC solutions at the endpoints a and b, respectively. The solutions $u_{m_a+1}, \cdots, u_{d_a}$ and $v_{m_b+1}, \cdots, v_{d_b}$ are LP solutions at a and b, respectively. For example if $n = 4$ and $d_a = 3 = d_b$, then $m_a = 2 = m_b$ and thus for $\lambda = \lambda_a$ two of the three solutions in H are LC solutions, the third is an LP solution. This terminology was introduced by Wang–Sun–Zettl in analogy with the celebrated Weyl limit-circle, limit-point terminology in the second order case. But in the second order case, at each endpoint, either all solutions are LC or none are (for real λ) and it is well known, see [15], that the *LC* solutions can be used to characterize the self-adjoint domains whereas the LP solutions play no role in this characterization. There is no boundary condition required or allowed at an LP endpoint in the second order case. Thus the identification of LC and LP solutions by Wang–Sun–Zettl in the higher order cases is critical for the characterization of the self-adjoint domains. Also in the second order case all solutions are in H for some $\lambda \in \mathbb{C}$ if and only if this is true for all $\lambda \in \mathbb{C}$. The solutions not in H also play no role in the characterization of the self-adjoint domains.

Next we state a theorem of Hao–Sun–Wang–Zettl which gives a decomposition of the maximal domain; it is proven in [4] using a method of Sun [9]. We believe it is of independent interest.

Theorem 2. *Let the notation and hypotheses of Theorem 1 hold. Then*

$$D_{\max}(a,b) = D_{\min}(a,b) \oplus span\{u_1, \cdots, u_{m_a}\} \oplus span\{v_1, \cdots, v_{m_b}\}. \tag{2.9}$$

2.3. General complex symplectic space

In this subsection we recall the definition of complex symplectic space and define the space $\tilde{S} = D_{max}/D_{min}$ which is used below.

Definition 1. [1] A complex linear space S, together with a complex-valued function on the product space $S \times S$,

$$u, v \to [u : v], \quad S \times S \to \mathbb{C}$$

is a pre-symplectic space if this function

(i) is sesquilinear, i.e.

$$[c_1 u + c_2 v : w] = c_1[u : w] + c_2[v : w], [u : c_3 v + c_4 w] = \bar{c}_3[u : v] + \bar{c}_4[u : w],$$

for all $u, v, w \in S$, and $c_j \in \mathbb{C}$, $j = 1, \cdots, 4$; and

(ii) skew-hermitian

$$(ii) \quad [u : v] = -\overline{[v : u]}, \quad \text{for all } u, v \in S.$$

If, in addition to properties (i) and (ii), it is
(iii) non-degenerate

$$[u : S] = 0 \text{ implies } u = 0,$$

then S, together with the non-degenerate, skew-Hermitian, sesquilinear form $[:]$, is a complex symplectic space.

Definition 2. [1] A linear subspace L in the complex symplectic space S is called Lagrangian in case $[L : L] = 0$, that is,

$$[u : v] = 0 \quad \text{for all vectors } u, v \in L.$$

Furthermore, a Lagrangian manifold $L \subset S$ is said to be complete in case

$$u \in S \text{ and } [u : L] = 0 \text{ imply } u \in L.$$

Definition 3. [1] Let S be a complex symplectic space with symplectic form $[:]$. Then linear subspaces S_- and S_+ are symplectic ortho-complements in S, written as

$$S = S_- \oplus S_+,$$

in case

$$(i)\ S = span\{S_-, S_+\}, \quad (ii)\ [S_- : S_+] = 0.$$

In this case $S_- \cap S_+ = 0$, so S is the direct sum of S_- and S_+.

Definition 4. [14] A linear subspace $\mathfrak{D}$ in the complex symplectic space S is called Dissipative in case

$$\Im[u : u] \geq 0 \quad \text{for all vectors } u \in \mathfrak{D}.$$

A linear subspace $\mathfrak{A}$ in the complex symplectic space S is called Accumulative in case

$$\Im[u : u] \leq 0 \quad \text{for all vectors } u \in \mathfrak{A}.$$

Definition 5. [14] A Dissipative subspace $\mathbb{D} \subset S$ is said to be maximal, if for any Dissipative subspace $\bar{\mathfrak{D}}$ such that $\mathfrak{D} \subseteq \mathbb{D}$ we have $\mathbb{D} = \bar{\mathfrak{D}}$.

An Accumulative subspace $\mathfrak{A} \subset S$ is said to be maximal, if for any Accumulative subspace $\bar{\mathfrak{A}}$ such that $\mathfrak{A} \subseteq \bar{\mathfrak{A}}$, we have $\mathfrak{A} = \bar{\mathfrak{A}}$.

Definition 6. [14] A Dissipative subspace $\mathfrak{D}$ is called strictly Dissipative in case

$$\Im[u, u] > 0, \quad \text{for } \forall u \in \mathfrak{D},\ u \neq 0.$$

An Accumulative subspace $\mathfrak{A}$ is called strictly Accumulative in case

$$\Im[v, v] < 0, \quad \text{for } \forall v \in \mathfrak{A},\ v \neq 0.$$

We denote the strictly Dissipative (Accumulative) subspaces as $\mathfrak{D}_s$ $(\mathfrak{A}_s)$.

2.3.1. The complex symplectic space $\tilde{S} = D_{max}/D_{min}$

Here we briefly discuss the complex symplectic space $\tilde{S} = D_{max}/D_{min}$ with the symplectic product [:] inherited from D_{max} defined by

$$[\tilde{f} : \tilde{g}] = [f + D_{min} : g + D_{min}] := [f : g],$$

where the skew-Hermitian form $[f : g]$ is defined for $f, g \in D_{max}$ by

$$[f : g] =< M(f), g > - < f, M(g) >= [f, g]_a^b.$$

Here $<, >$ is the inner product of Hilbert space H, and $[f, g]$ is the Lagrange bracket (see Lemma 1).

Hence

$$D_{min} = \{f \in D_{max} : [f : D_{max}] = 0\}.$$

Theorem 3. *[13,14] Let $\tilde{S}$ be the symplectic space defined above, then*

(1) $p = q = d$, $\dim \tilde{S} = 2d$, *and* $Ex = 0$,
(2) *there exist complete Lagrangian subspaces $\tilde{L}$ of $\tilde{S}$,*
(3) *there are more then one maximal (strictly) Dissipative subspaces.*

Proof. Item (1) follows from Lemma 7 in [13], (2) from Theorem 2 in [2] and (3) from Corollary 1 in [14]. □

Theorem 4. *[13] Let the notation and hypotheses of Theorem 1 hold, and $\tilde{S} = D_{max}/D_{min}$, then*

(1) $\tilde{S} = span\{\tilde{u}_1, \cdots, \tilde{u}_{m_a}, \ \tilde{v}_1, \cdots, \tilde{v}_{m_b}\}$.
(2) *$\tilde{S}$ is the complexification of the unique real symplectic space $\mathbb{R}^{2d}$.*
(3) *$\tilde{S}$ is symplectic isomorphic to a complex symplectic space $\mathbb{C}^{2d}$.*
(4) *For the basis $\{\tilde{u}_1, \cdots, \tilde{u}_{m_a}, \ \tilde{v}_1, \cdots, \tilde{v}_{m_b}\}$, let*

$$H = \begin{pmatrix} -U_{m_a \times m_a} & 0 \\ 0 & V_{m_b \times m_b} \end{pmatrix},$$

then H is a skew-Hermitian matrix, and for every $\tilde{f} = (f_1, \cdots, f_{m_a}, \dot{f}_1, \cdots, \dot{f}_{m_b})$ and $\tilde{g} = (g_1, \cdots, g_{m_a}, \dot{g}_1, \cdots, \dot{g}_{m_b})$ in $\tilde{S}$, we have

$$[\tilde{f} : \tilde{g}] = (f_1, \cdots, f_{m_a}, \dot{f}_1, \cdots, \dot{f}_{m_b}) H (g_1, \cdots, g_{m_a}, \dot{g}_1, \cdots, \dot{g}_{m_b})^*.$$

Here U is defined by (2.3) and V is defined by (2.4) in Theorem 1.

Proof. See [13]. □

3. A representation of domains of dissipative extensions

There exists a natural bi-unique correspondence between the set of all dissipative extensions T_D of S_{min} and the set of all Dissipative subspaces $\tilde{\mathfrak{D}}$ in $\tilde{S}$, and studying the properties of the Dissipative subspaces is easier then studying the dissipative extension domains directly.

In this section we represent the dissipative extension domains $D(T_D)$ with Limit-Circle solutions after carefully investigating Dissipative subspaces. We start by recalling some lemmas from [14].

Lemma 2. *Consider a complex symplectic space S, with symplectic form $[:]$, and finite dimension $D \geq 1$. Then $\mathfrak{D} \subseteq S$ is a Dissipative but not strictly Dissipative subspace if and only if there exists a Lagrangian subspace $\mathfrak{D}_L \subseteq \mathfrak{D}$ and a strictly Dissipative subspace $\mathfrak{D}_s \subseteq \mathfrak{D}$ such that*

$$\mathfrak{D} = \mathfrak{D}_L \oplus \mathfrak{D}_s. \tag{3.1}$$

Proof. See [14] for the proof. □

Lemma 3. *Let M be a general symmetric differential expression and $\tilde{S} = D_{\max}/D_{\min}$, then*

(1) *there exists a natural one-to-one correspondence between the set of all dissipative operators T_D generated by M and the set of all Dissipative subspaces $\mathfrak{D}$ in the complex symplectic space $\tilde{S} = D_{\max}/D_{\min}$. Namely, for each such dissipative extension T_D with domain $D(T_D) \subset D_{\max}$, the corresponding Dissipative subspace $\mathfrak{D}$ is defined by*

$$\mathfrak{D} = D(T_D)/D_{\min}.$$

(2) *there exists a natural one-to-one correspondence between the set of all strictly dissipative operators T_s generated by M and the set of all strictly Dissipative subspaces $\mathfrak{D}_s$ in the complex symplectic space $\tilde{S} = D_{\max}/D_{\min}$.*

Proof. See [14] for a proof. □

Given the natural one-to-one correspondence between the dissipative operator extensions and the Dissipative subspaces in Lemma 3, we now study the dissipative subspaces of $\tilde{S}$ instead of the dissipative operator extensions in H directly. This allows us to study dissipative operators in the Hilbert space H by using methods of symplectic geometry to study dissipative subspaces in the symplectic space $\tilde{S}$.

Theorem 5. *Let the notation and hypotheses of Theorem 1 and Theorem 4 hold. A linear subspace $D(T_D)$ of $D_{\max}$ is the domain of a dissipative but not strictly dissipative extension T_D of $S_{\min}$ if and only if there exist*

$$\gamma_k,\ \alpha_j \in \mathbb{C}^{2d},\ k = 1, \cdots, r_L,\ j = 1, \cdots, r_s,\ \textit{are linearly independent},$$

and satisfying

$$[\gamma_i : \gamma_k] = 0, \quad i, k = 1, \cdots, r_L, \tag{3.2}$$

$$\Im[\alpha_j : \alpha_j] > 0, \quad j = 1, \cdots, r_s, \tag{3.3}$$

$$[\gamma_k : \alpha_j] = 0, \quad k = 1, \cdots, r_L,\ j = 1, \cdots, r_s, \tag{3.4}$$

$$\Im[\sum_{i=1}^{r_s} c_j\alpha_j : \sum_{i=1}^{r_s} c_j\alpha_j] > 0, \quad j = 1, \cdots, r_s, \tag{3.5}$$

such that

$$D(T_D) = D_{\min} \oplus span\{w_1, w_2, \cdots w_{r_L}\} \oplus span\{\chi_1, \chi_2, \cdots, \chi_{r_s}\},$$

where

$$w_k = \gamma_k W,\ \chi_j = \alpha_j W,\ k = 1, \cdots, r_L,\ j = 1, \cdots, r_s,$$
$$W = (u_1, \cdots, u_{m_a}, v_1, \cdots, v_{m_b})^T.$$

Proof. ($\Rightarrow$) Let $D(T_D)$ be the domain of a dissipative but not strictly dissipative extension T_D. By Lemma 3, $D(T_D)/D_{\min}$ is a dissipative but not a strictly dissipative subspace of $\tilde{S}$, then from Lemma 2 there exists a Lagrangian subspace $\mathfrak{D}_L \subset D(T_D)$ and a strictly dissipative subspace $\mathfrak{D}_s \subset D(T_D)$ such that

$$D(T_D) = \mathfrak{D}_L \oplus \mathfrak{D}_s.$$

Denote $r_L = \dim \mathfrak{D}_L$, $r_s = \dim \mathfrak{D}_s$, and let

$$w_k = \gamma_k W,\ k = 1, \cdots, r_L, \text{ and } \chi_j = \alpha_j W,\ j = 1, \cdots, r_s$$

be a basis for $\mathfrak{D}_L$ and $\mathfrak{D}_s$, where $W = (u_1, \cdots, u_{m_a}, v_1, \cdots, v_{m_b})^T$, then

$$\gamma_k, \alpha_j \in \mathbb{C}^{2d}\ k = 1, \cdots, r_L,\ j = 1, \cdots, r_s$$

are linearly independent.

From the definition of a Lagrangian subspace we have $\gamma_k \in \mathbb{C}^{2d}$, $k = 1, \cdots, r_L$ satisfy (3.2), and from the definition of the strictly Dissipative subspace we get $\alpha_j \in \mathbb{C}^{2d}$, $j = 1, \cdots, r_s$ satisfy (3.3) and (3.5), and from Lemma 2, we get (3.4).

($\Leftarrow$) From (3.2), we conclude that

$$w_k = \gamma_k W,\ k = 1, \cdots, r_L,$$

construct a Lagrangian subspace $\mathfrak{D}_L \subset \tilde{S}$. From (3.3) and (3.5), we conclude that

$$\chi_j = \alpha_j W, j = 1, \cdots, r_s$$

construct a strictly dissipative subspace $\mathfrak{D}_s \subset \tilde{S}$. Then from (3.4) and Lemma 2, $\mathfrak{D}_L \oplus \mathfrak{D}_s$ is a Dissipative but not a strictly Dissipative subspace of $\tilde{S}$. So from Lemma 3, we conclude that

$$D(T_D) = D_{\min} \oplus span\{w_1, w_2, \cdots w_{r_L}\} \oplus span\{\chi_1, \chi_2, \cdots, \chi_{r_s}\},$$

is the domain of a dissipative extension of $S_{\min}$. $\square$

Note that $[\gamma_i : \gamma_k]$, $[\alpha_j : \alpha_j]$ etc. which appear in Theorem 5 and Corollary 1 and Theorem 6 below are the symplectic product in $\mathbb{C}^{2d}$ as $[\gamma_i : \gamma_k] = \gamma_i H \gamma_j^*$, where H is the skew-Hermitian matrix defined in item (4) of Theorem 4. Since $\tilde{S}$ is symplectic isomorphic to $\mathbb{C}^{2d}$, we don't distinguish it from its corresponding element in $\tilde{S}$.

Corollary 1. *Let the notation and hypotheses of Theorem 1 and Theorem 4 hold. A linear subspace $D(T_sD)$ of $D_{\max}$ is the domain of an r_s strictly dissipative extension T_sD of $S_{\min}$ if and only if there exist*

$$\alpha_j \in \mathbb{C}^{2d},\ j = 1, \cdots, r_s \textit{ are linearly independent,}$$

satisfying

$$\Im[\alpha_j : \alpha_j] > 0, \quad j = 1, \cdots, r_s, \tag{3.6}$$

$$\Im[\sum_{i=1}^{r_s} c_j \alpha_j : \sum_{i=1}^{r_s} c_j \alpha_j] > 0, \quad j = 1, \cdots, r_s, \tag{3.7}$$

such that

$$D(T_sD) = D_{\min} \oplus span\{\chi_1, \chi_2, \cdots, \chi_{r_s}\},$$

where $\chi_j = \alpha_j W$, $j = 1, \cdots, r_s$, *and* $W = (u_1, \cdots, u_{m_a}, v_1, \cdots, v_{m_b})^T$.

Proof. Since $D(T_s)$ is the domain of a strictly dissipative extension T_s of $S_{\min}$, there are no nontrivial Lagrangian elements in the strictly Dissipative subspace $D(T_s)/D_{\min}$ and thus this Corollary follows from Theorem 5. □

Remark 2. Note that conditions (3.5) and (3.7) are not easy to check. Below we will find alternative conditions which are easier to check.

Example 1. Consider the complex symplectic space $S = span\{e_1, e_2, a_1, a_2\}$ with customary basis vectors

$$[e_j : e_j] = i, \ [a_j : a_j] = -i, \ j = 1, 2,$$

and all other symplectic products are zero. That is, we use the skew-Hermitian matrix $H = diag\{i, i, -i, -i\}$ to define the symplectic structure on $S = \mathbb{C}^4$.

Define

$$\mathfrak{D}_1 = span\{2e_1 + a_1, 2e_2 + a_1\},$$

in $\mathfrak{D}_1$, $2e_1 + a_1$ and $2e_2 + a_1$ are dissipative elements, and $[2e_1 + a_1 : 2e_2 + a_1] = -i$, i.e. $2e_1 + a_1$ and $2e_2 + a_1$ are not symplecticly orthogonal. But it is easy to verify that $\alpha(2e_1 + a_1) + \beta(2e_2 + a_1)$, $\forall \alpha, \beta \in \mathbb{C}$ is a dissipative element, therefore $\mathfrak{D}_1$ is a Dissipative subspace of $\mathbb{C}^4$.

Define

$$\mathfrak{D}_2 = span\{e_1, e_2\},$$

in $\mathfrak{D}_2$, since e_1 and e_2 are dissipative elements and $[e_1 : e_2] = 0$, clearly $\alpha e_1 + \beta e_2$, for $\alpha, \beta \in \mathbb{C}$ are dissipative elements, therefore $\mathfrak{D}_2$ is a Dissipative subspace of $\mathbb{C}^4$.

Define

$$\mathfrak{D}_3 = span\{5e_1 + 4a_1, 6e_2 + 5a_1\},$$

in $\mathfrak{D}_3$, $5e_1 + 4a_1$ and $6e_2 + 5a_1$ are dissipative elements, but $(5e_1 + 4a_1) + (6e_2 + 5a_1) = 5e_1 + 6e_2 + 9a_1$, is not a dissipative element, therefore $\mathfrak{D}_3$ is not a Dissipative subspace of $\mathbb{C}^4$.

Now we extend condition (3.5) to get a sufficient condition for $D(T_D)$ to be the domain of a dissipative extension.

Theorem 6. *Let the notation and hypotheses of Theorem 1 and Theorem 4 hold, assume that there exist*

$$\gamma_k, \alpha_j \in \mathbb{C}^{2d} \ k = 1, \cdots, r_L, \ j = 1, \cdots, r_s \ \textit{are linearly independent},$$

satisfying

$$[\gamma_i : \gamma_k] = 0, \quad i, k = 1, \cdots, r_L. \tag{3.8}$$

$$\Im[\alpha_j : \alpha_j] > 0, \quad j = 1, \cdots, r_s. \tag{3.9}$$

$$[\gamma_k : \alpha_j] = 0, \quad k = 1, \cdots, r_L, \ j = 1, \cdots, r_s. \tag{3.10}$$

$$[\alpha_i : \alpha_j] = 0, \ i \neq j, \ i, j = 1, \cdots, r_s. \tag{3.11}$$

Then

$$D(T_D) = D_{\min} \oplus span\{w_1, w_2, \cdots w_{r_L}\} \oplus span\{\chi_1, \chi_2, \cdots, \chi_{r_s}\} \tag{3.12}$$

is the domain of a dissipative extension T_D. Here $w_k = \gamma_k W$, $\chi_j = \alpha_j W$, $k = 1, \cdots, r_L$, $j = 1, \cdots, r_s$, $W = (u_1, \cdots, u_{m_a}, v_1, \cdots, v_{m_b})^T$.

Proof. We know that $[\alpha_i : \alpha_j] = 0$, $i \neq j$, $i, j = 1, \cdots, r_s$ is a sufficient but not necessary condition for

$$\Im[\sum_{i=1}^{r_s} c_j\alpha_j : \sum_{i=1}^{r_s} c_j\alpha_j] > 0, \quad j = 1, \cdots, r_s.$$

So γ_k, $\alpha_j \in \mathbb{C}^{2d}$ $k = 1, \cdots, r_L$, $j = 1, \cdots, r_s$ satisfying (3.8)–(3.11) define a domain of a dissipative extension of $S_{\min}$. $\square$

Remark 3. Although we obtain the complete characterization of the domains of dissipative extensions in terms of LC solutions in Theorem 5, Corollary 1 and Theorem 6, it may be complicated to check a specific dissipative extension, since the characterization of a Dissipative subspace under the basis $\tilde{u}_1, \cdots, \tilde{u}_{m_a}$, $\tilde{v}_1, \cdots, \tilde{v}_{m_b}$ is complicated. Next we construct a new basis for $\tilde{S}$ in terms of LC solutions which makes this easier.

Lemma 4. *There exists a nonsingular $m_a \times m_a$ matrix Q_a and an $m_b \times m_b$ matrix Q_b such that*

$$Q_a(-U_{m_a \times m_a})Q_a^* = diag\{\underbrace{i, \cdots, i}_{\frac{m}{2}}, \underbrace{-i, \cdots, -i}_{\frac{m_1}{2}}\},$$

$$Q_2(V_{m_2 \times m_2})Q_2^* = diag\{\underbrace{i, \cdots, i}_{\frac{m_2}{2}}, \underbrace{-i, \cdots, -i}_{\frac{m_2}{2}}\}.$$

Proof. (1) If k is even and $\frac{m_a}{2}$ is even, let

$$Q_a = \left(\begin{array}{cccccc|cccccc}
1 & & & & & & & & & & & i \\
 & i & & & & & & & & & 1 & \\
 & & 1 & & & & & & & i & & \\
 & & & i & & & & & 1 & & & \\
 & & & & \ddots & & & \vdots & & & & \\
 & & & & & i & 1 & & & & & \\
\hline
i & & & & & & & & & & & 1 \\
 & 1 & & & & & & & & & i & \\
 & & i & & & & & & & 1 & & \\
 & & & 1 & & & & & i & & & \\
 & & & & \ddots & & & \vdots & & & & \\
 & & & & & 1 & i & & & & &
\end{array}\right)$$

(2) If k is even and $\frac{m_a}{2}$ is odd, let

$$Q_a = \left(\begin{array}{cccccc|cccccc}
1 &&&&&&&&&&& i \\
& i &&&&&&&&& 1 & \\
&& 1 &&&&&&& i && \\
&&& i &&&&& 1 &&& \\
&&&& \ddots &&& \vdots &&&& \\
&&&&& 1 & i &&&&& \\
\hline
i &&&&&&&&&&& 1 \\
& 1 &&&&&&&&& i & \\
&& i &&&&&&& 1 && \\
&&& 1 &&&&& i &&& \\
&&&& \ddots &&& \vdots &&&& \\
&&&&& i & 1 &&&&&
\end{array}\right)$$

(3) If k is odd and $\frac{m_a}{2}$ is even, let

$$Q_a = \left(\begin{array}{cccccc|cccccc}
i &&&&&&&&&&& 1 \\
& 1 &&&&&&&&& i & \\
&& i &&&&&&& 1 && \\
&&& 1 &&&&& i &&& \\
&&&& \ddots &&& \vdots &&&& \\
&&&&& 1 & i &&&&& \\
\hline
1 &&&&&&&&&&& i \\
& i &&&&&&&&& 1 & \\
&& 1 &&&&&&& i && \\
&&& i &&&&& 1 &&& \\
&&&& \ddots &&& \vdots &&&& \\
&&&&& i & 1 &&&&&
\end{array}\right)$$

(4) If k is odd and $\frac{m_a}{2}$ is odd, let

$$Q_a = \left(\begin{array}{cccccc|cccccc}
i &&&&&&&&&&& 1 \\
& 1 &&&&&&&&& i & \\
&& i &&&&&&& 1 && \\
&&& 1 &&&&& i &&& \\
&&&& \ddots &&& \vdots &&&& \\
&&&&& i & 1 &&&&& \\
\hline
1 &&&&&&&&&&& i \\
& i &&&&&&&&& 1 & \\
&& 1 &&&&&&& i && \\
&&& i &&&&& 1 &&& \\
&&&& \ddots &&& \vdots &&&& \\
&&&&& i & i &&&&&
\end{array}\right),$$

then it is easy to check that

$$Q_a(-U_{m_a \times m_a})Q_a^* = diag\{\underbrace{i, \cdots, i}_{\frac{m_a}{2}}, \underbrace{-i, \cdots, -i}_{\frac{m_a}{2}}\}.$$

Similarly Q_b can be constructed. $\square$

Theorem 7. *Let the notation and hypotheses of Theorem 1, Lemma 4 hold, let*

$$(\tilde{x}_1, \cdots, \tilde{x}_{\frac{m_a}{2}}, \tilde{x}_{\frac{m_a}{2}+1}, \cdots, \tilde{x}_{m_a})^T = Q_a(\tilde{u}_1, \cdots, \tilde{u}_{m_a})^T,$$

$$(\tilde{z}_1, \cdots, \tilde{z}_{\frac{m_b}{2}}, \tilde{z}_{\frac{m_b}{2}+1}, \cdots, \tilde{z}_{m_b})^T = Q_b(\tilde{v}_1, \cdots, \tilde{v}_{mb})^T.$$

Then

(1) *span*$\{\tilde{x}_1, \cdots, \tilde{x}_{\frac{m_a}{2}}\}$ *and span*$\{\tilde{z}_1, \cdots, \tilde{z}_{\frac{m_b}{2}}\}$ *are strictly Dissipative subspaces of* $\tilde{S}$.
(2) *span*$\{\tilde{x}_{\frac{m_a}{2}+1}, \cdots, \tilde{x}_{m_a}\}$ *and span*$\{\tilde{z}_{\frac{m_b}{2}+1}, \cdots, \tilde{z}_{m_b}\}$ *are strictly Accumulative subspaces of* $\tilde{S}$.

Proof. From (4) of Theorem 4, $([\tilde{u}_i : \tilde{u}_j]) = -U_{m_a \times m_a}$, $([\tilde{v}_i : \tilde{v}_j]) = V_{m_b \times mb}$, and from Lemma 4,

$$Q_a(-U_{m_a \times m_a})Q_a^* = diag\{\underbrace{i, \cdots, i}_{\frac{m_a}{2}}, \underbrace{-i, \cdots, -i}_{\frac{m_a}{2}}\},$$
$$Q_b(V_{m_b \times m_b})Q_b^* = diag\{\underbrace{i, \cdots, i}_{\frac{m_b}{2}}, \underbrace{-i, \cdots, -i}_{\frac{mb}{2}}\},$$

so we have

$$([\tilde{x}_i : \tilde{x}_j]) = Q_a([\tilde{u}_i : \tilde{u}_j])Q_a^* = diag\{\underbrace{i, \cdots, i}_{\frac{m_a}{2}}, \underbrace{-i, \cdots, -i}_{\frac{m_a}{2}}\},$$

and from Corollary 1 and Theorem 6, item (1) is obtained. Similarly item (2) can be proven. □

Theorem 8. *Let the notation and hypotheses of Theorem 1 and Theorem 7 hold. Let*

$$\tilde{D} = span\{\tilde{x}_1, \cdots, \tilde{x}_{\frac{m_a}{2}}, \tilde{z}_1, \cdots, \tilde{z}_{\frac{m_b}{2}}\},$$
$$\tilde{A} = span\{\tilde{x}_{\frac{m_a}{2}+1}, \cdots, \tilde{x}_{m_a}, \tilde{z}_{\frac{m_b}{2}+1}, \cdots, \tilde{z}_{m_b}\}.$$

Then

$\tilde{D}$ *is a maximal Dissipative subspace of* $\tilde{S}$.
$\tilde{A}$ *is a maximal Accumulative subspace of* $\tilde{S}$. *And*

$$\tilde{S} = \tilde{D} \oplus \tilde{A}, \quad [\tilde{D} : \tilde{A}] = 0.$$

Proof. From the constructions of the new basis of $\tilde{S}$: $\tilde{x}_1, \cdots, \tilde{x}_{\frac{m_a}{2}}, \tilde{x}_{\frac{m_a}{2}+1}, \cdots, \tilde{x}_{m_a}, \tilde{z}_1, \cdots, \tilde{z}_{\frac{m_b}{2}}, \tilde{z}_{\frac{m_b}{2}+1}, \cdots,$ $\tilde{z}_{m_b}$ and Theorem 6, $\tilde{D}$ is a maximal Dissipative subspace of $\tilde{S}$, and $\tilde{A}$ is a maximal Accumulative subspace of $\tilde{S}$. Furthermore, from the definition of symplectic ortho-complement subspace, we have

$$\tilde{S} = \tilde{D} \oplus \tilde{A}, \quad [\tilde{D} : \tilde{A}] = 0. \quad \square$$

4. Classification of boundary conditions of dissipative extensions

In this section, we classify the boundary conditions of the dissipative extensions into separated, coupled and mixed and we illustrate these three cases.

Lemma 5. *[13] Let the notation and hypotheses of Theorem 1 hold, and let* $\tilde{S} = D_{max}/D_{min}$ *be the complex vector space defined above. Assume that*

$$\tilde{S}_l = \{\tilde{f} \in \tilde{S} : [f, v_1](b) = \cdots = [f, v_{m_b}](b) = 0\},$$
$$\tilde{S}_r = \{\tilde{f} \in \tilde{S} : [f, u_1](a) = \cdots = [f, u_{ma}](a) = 0\}.$$

Then

$$\tilde{S}_l = span\{\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_{m_a}\},$$
$$\tilde{S}_r = span\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_{m_b}\},$$

and $\tilde{S} = \tilde{S}_l \oplus \tilde{S}_r$.

Proof. See [13] for a proof. □

Lemma 6. *Each of $\tilde{S}_l$ and $\tilde{S}_r$ is itself a complex symplectic space. Specifically, $\tilde{S}_l$ is symplectic isomorphic to $\mathbb{C}^{m_a}$, with $[:]$ defined by the skew-Hermitian matrix $-U$, and $\tilde{S}_r$ is symplectic isomorphic to $\mathbb{C}^{m_b}$, with $[:]$ defined by the skew-Hermitian matrix V. Furthermore, let*

$$D_l,\ p_l,\ q_l,\ \triangle_l,\ Ex_l \text{ and } D_r,\ p_r,\ q_r,\ \triangle_r,\ Ex_r$$

denote the corresponding symplectic invariants for $\tilde{S}_l$, and $\tilde{S}_r$ respectively, then

$$D_l = m_a,\ p_l = q_l = \triangle_l = \frac{m_a}{2},\ Ex_l = 0, \tag{4.1}$$
$$D_r = m_b,\ p_r = q_r = \triangle_r = \frac{m_b}{2},\ Ex_r = 0. \tag{4.2}$$

Proof. See [13] for the proof. □

Definition 7. Let $\tilde{S} = D_{max}/D_{min}$ be the complex vector space defined in subsection 2.3 having a direct sum decomposition given by Lemma 5

$$\tilde{S} = \tilde{S}_l \oplus \tilde{S}_r \quad \text{with } [\tilde{S}_l : \tilde{S}_r] = 0.$$

(1) A non-zero vector $v \in \tilde{S}$ is called separated at the left (right) in case $v \in \tilde{S}_l$ ($v \in \tilde{S}_r$), and v is coupled otherwise.
(2) A Dissipative subspace $D \in \tilde{S}$ is called separated if for every $v \in D$ it is separated at the left and at the right.
(3) A Dissipative subspace $D \in \tilde{S}$ is called coupled if for every $v \in D$ it is coupled.
(4) A Dissipative subspace $D \in \tilde{S}$ is called mixed if D is neither separated nor coupled.

Definition 8. Let M be a symmetric differential expression studied in subsection 2.1, then

(1) the boundary condition of a dissipative extension T_D of $S_{\min}$ is called separated if the Dissipative subspace $D(T_D)/D_{\min}$ is separated,
(2) the boundary condition of a dissipative extension T_D of $S_{\min}$ is called coupled if the Dissipative subspace $D(T_D)/D_{\min}$ is coupled,
(3) the boundary condition of a dissipative extension T_D of $S_{\min}$ is called mixed if the Dissipative subspace $D(T_D)/D_{\min}$ is mixed.

Theorem 9. *Let the notation and hypotheses of Theorem 1 hold. Let $\tilde{D}$ defined as in Theorem 8:*

$$\tilde{D} = span\{\tilde{x}_1, \cdots, \tilde{x}_{\frac{m_a}{2}}, \tilde{z}_1, \cdots, \tilde{z}_{\frac{m_b}{2}}\}.$$

Then

(1) *The dissipative subspace $\tilde{D}$ is separated.*

(2) *The dissipative subspace* $\tilde{D}$ *has the following property:*

$$D_l - dim\tilde{D}\bigcap\tilde{S}_l = \frac{m_a}{2},$$
$$D_r - dim\tilde{D}\bigcap\tilde{S}_r = \frac{m_b}{2},$$

and $D_l - dim\tilde{D}\bigcap\tilde{S}_l = D_r - dim\tilde{D}\bigcap\tilde{S}_r$ *if and only if* $m_a = m_b$.

Proof. From the construction of $\tilde{x}_1, \cdots, \tilde{x}_{\frac{m_a}{2}}, \tilde{z}_1, \cdots, \tilde{z}_{\frac{m_b}{2}}$, we have $\tilde{x}_i \in \tilde{S}_l$, $i = 1, \cdots, \frac{ma}{2}$, and $\tilde{z}_j \in \tilde{S}_r$, $j = 1, \cdots, \frac{m_b}{2}$. So the dissipative subspace $\tilde{D}$ is separated.

It is obvious that

$$dim\tilde{D}\bigcap\tilde{S}_l = \frac{m_a}{2},\ dim\tilde{D}\bigcap\tilde{S}_r = \frac{m_b}{2},$$

then $D_l - dim\tilde{D}\bigcap\tilde{S}_l = D_r - dim\tilde{D}\bigcap\tilde{S}_r$ if and only if $m_a = m_b$. □

Remark 4. Note that every coupled Dissipative subspace $\mathfrak{D}_c$ has the balanced intersection property with the direct sum decomposition $\tilde{S} = \tilde{S}_l \oplus \tilde{S}_r$ when $m_a = m_b$, as

$$D_l - dim\mathfrak{D}_c\bigcap\tilde{S}_l = m_a = m_b = D_r - dim\mathfrak{D}_c\bigcap\tilde{S}_r.$$

Although the maximal Dissipative subspace $\tilde{D}$ defined in Theorem 8, and every coupled Dissipative subspace $\mathfrak{D}_c$ have some balanced intersection property with the direct sum decomposition $\tilde{S} = \tilde{S}_l \oplus \tilde{S}_r$ when $m_a = m_b$, we find that other dissipative subspaces may not have the balanced intersection property even if $m_a = m_b$.

Next we give an example which has separated coupled and mixed subspaces. Of course, the corresponding differential equations (1.1) have self-adjoint realizations with all three types of boundary conditions: separated, coupled and mixed.

Example 2. Let the notation and hypotheses of Theorem 1 and Theorem 7 hold, and, for convenience, assume that $m_a = m_b = 6$. Then $\tilde{S}$ has a basis

$$x_1, x_2, x_3, x_4, x_5, x_6, z_1, z_2, z_3, z_4, z_5, z_6,$$

such that the associated skew-Hermitian matrix H is given by

$$H = \begin{pmatrix} \mathcal{U}_{6\times 6} & 0 \\ 0 & \mathcal{V}_{6\times 6} \end{pmatrix},$$

where

$$\mathcal{U} = \begin{pmatrix} i & & & & & \\ & i & & & & \\ & & i & & & \\ & & & -i & & \\ & & & & -i & \\ & & & & & -i \end{pmatrix}, \mathcal{V} = \begin{pmatrix} i & & & & & \\ & i & & & & \\ & & i & & & \\ & & & -i & & \\ & & & & -i & \\ & & & & & -i \end{pmatrix}.$$

Define

$$\mathfrak{D}_4 = span\{x_1, x_2, x_3, z_1, z_2, z_3\},$$

then from Theorem 9, $\mathfrak{D}_4$ is a maximal strictly Dissipative subspace and it is strictly separated with

$$D_l - dim\mathfrak{D}_4 \bigcap \tilde{S}_l = D_r - dim\mathfrak{D}_4 \bigcap \tilde{S}_r = 3.$$

Define

$$\mathfrak{D}_5 = span\{2x_1 + z_4, 2x_2 + z_5\},$$

then $\mathfrak{D}_5$ is a coupled Dissipative subspace with

$$D_l - dim\mathfrak{D}_5 \bigcap \tilde{S}_l = D_r - dim\mathfrak{D}_5 \bigcap \tilde{S}_r = 6.$$

Define

$$\mathfrak{D}_6 = span\{2x_1 + z_4, 2z_1 + z_5\},$$

then $\mathfrak{D}_6$ is a mixed Dissipative subspace with

$$D_l - dim\mathfrak{D}_5 \bigcap \tilde{S}_l = 6, \; D_r - dim\mathfrak{D}_5 \bigcap \tilde{S}_r = 5.$$

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