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A numerical solution for variable order fractional functional differential equation

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ABSTRACT

This paper is concerned with an efficient numerical scheme for variable order fractional functional boundary value problems. The algorithm relies on the simplified reproducing kernel method (SRKM). The convergence of the method is proven, followed by estimates on approximate solution. A numerical example with exact solutions is studied to demonstrate the performance of the method. Results obtained by the method indicate the algorithm performs extremely well in terms of accuracy, efficiency and simplicity.

1. Introduction

A great quantity of nature phenomena can be modeled by variable fractional differential equations. The study of such problems has attracted much attention recently. Results about existence and uniqueness results for boundary value problems of fractional differential equations have been obtained in [1,2]. On the other side, several numerical methods, including finite difference method, matrix approach and reproducing kernel method, for solving variable-order fractional differential equations were proposed in [3–8]. At the mean time, Sun and Chen described some valuable applications of variable fractional derivative in [9–11]. Functional differential equations arose in a variety of applications, such as electrodynamics, astrophysics, nonlinear dynamical systems, and probability theory on algebraic structure. Consequently, several research papers were issued to investigate the theory and solutions of some functional differential equations (see [12–14] and references therein).

However, limited work has been done in the study on variable order fractional functional equation. In this letter, based on previous work [15–20], we focus on providing a numerical scheme to solve the following

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variable fractional functional boundary value problems in reproducing kernel space.

$$\begin{cases} D^{\alpha(x)}u(x) + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x)) = f(x), & x \in [0,1], \\ u(0) = \lambda_0, \qquad u(1) = \lambda_1 \end{cases} \tag{1}$$

where $a(x), b(x), c(x) \in C^2[0,1]$, $\alpha(x), \tau(x), f(x) \in C[0,1]$, $1 \le \alpha(x) < 2$, $0 \le \tau(x) \le 1$ and $D^{\alpha(x)}$ denotes the variable order Caputo derivative.

We organize the rest of paper as follows. In Section 2, we present some reproducing kernel space. In Section 3, the SRKM is applied to solve Eq. (1) and the uniform convergence analysis is carried out as well. A numerical experiment is studied to demonstrate the efficiency, reliability and simplicity of the algorithm in Section 4. Finally, we summarize this paper in Section 5.

2. Several reproducing kernel spaces

Definition 2.1. The variable fractional order $\alpha(x)$ of function $u(x)$ in Caputo's sense, is defined as

$$D^{\alpha(x)}u(x) = \frac{1}{\Gamma(2-\alpha(x))}\int_0^x (x-t)^{1-\alpha(x)}u''(t)dt,$$

where $\Gamma(\cdot)$ is Gamma function.

Definition 2.2. $W_2^m[0,1] = \{u(x)|u^{m-1}(x)$ *is an absolutely continuous real value function in* $[0,1], u^{(m)}(x) \in L^2[0,1]\}$ with the inner products and norms as follows:

$$\langle u(x), v(x)\rangle_m = \sum_{k=0}^{m-1} u^{(k)}(0)v^{(k)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)dx, \quad \forall u, v \in W_2^m[0,1],$$
$$\|u\|_m^2 = \sqrt{\langle u, u\rangle_m}, \quad \forall u \in W_2^m[0,1]. \tag{2}$$

Clearly, the inner spaces $W_2^3[0,1]$ and $W_2^1[0,1]$ are reproducing kernel spaces with kernel functions $R_x(y)$ and $r_x(y)$, respectively, (see [21]), and the expression of $R_x(y)$ is

$$R_x(y) = \begin{cases} \dfrac{1}{120}(120 - 5xy^4 + y^5 + 10x^2y^2(3+y) + 120xy), & y > x, \\ \dfrac{1}{120}(120 + x^5 - 5x^4y + 10x^2y^2(3+x) + 120xy), & y \le x. \end{cases}$$

By Eq. (1), we define a linear operator $\mathcal{L} : W_2^3 \to W_2^1$, for all $u \in W_2^3$ as follows

$$\mathcal{L}u(x) = D^{\alpha(x)}u(x) + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x)), \tag{3}$$

then Eq. (1) can be converted into the following equivalent operator equation

$$\begin{cases} \mathcal{L}u = f(x), \\ u(0) = \lambda_0, \qquad u(1) = \lambda_1. \end{cases} \tag{4}$$

Lemma 2.3. *$\mathcal{L}$ is a bounded linear operator.*

Proof. By the reproducing property of $R_x(y)$, we have,

$$u^{(i)}(x) = \langle u, \partial_x^i R_x\rangle_3, \quad i = 0,1,2,3, \quad u^{(j)}(\tau(x)) = \langle u, \partial_x^j R_{\tau(x)}\rangle_3, \quad j = 0,1. \tag{5}$$

A direct application of the *Cauchy–Schwarz* inequality to Eq. (5) yields that

$$|u^{(i)}(x)| \leq \| u\|_3 \cdot \| \partial_x^i R_x\|_3, \qquad |u^{(j)}(\tau(x))| \leq \| u\|_3 \cdot \| \partial_x^j R_{\tau(x)}\|_3.$$

By the fact that $\|\partial_x^i R_x\|_3$ and $\| \partial_x^j R_{\tau(x)}\|_3$ are boundedness on [0, 1], there exist positive constants M_i, N_j such that

$$|u^{(i)}(x)| \leq M_i \| u\|_3, \qquad |u^{(j)}(\tau(x))| \leq N_j \| u\|_3. \tag{6}$$

A simple calculation together with Eq. (6) yields that

$$\begin{aligned}\int_0^1 (b(x)u(x))'^2 dx &= \int_0^1 [b'^2(x)u^2(x) + b^2(x)u'^2(x) + 2b(x)b'(x)u(x)u'(x)]dx \\ &\leq \widetilde{M}\left[\int_0^1 u^2(x)dx + \int_0^1 u'^2(x)dx + \int_0^1 u(x)u'(x)dx\right] \\ &\leq \widetilde{A_0} \| u\|_3^2.\end{aligned}$$

Therefore, there holds

$$\|b(x)u(x)\|_1^2 = b^2(0)u^2(0) + \int_0^1 (b(x)u(x))'^2 dx \leq A_0 \| u\|_3^2, \tag{7}$$

and by the same argument,

$$\|a(x)u'(x)\|_1^2 \leq A_1 \| u\|_3^2, \qquad \|c(x)u(\tau(x))\|_1^2 \leq A_2 \| u\|_3^2. \tag{8}$$

Next, the use of integration by parts gives

$$D^{\alpha(x)}u(x) = \frac{x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))}u''(0) + \frac{1}{\Gamma(3-\alpha(x))}\int_0^x (x-t)^{2-\alpha(x)}u'''(t)dt.$$

Since

$$\begin{aligned}\left\|\int_0^x (x-t)^{2-\alpha(x)}u'''(t)dt\right\|_1^2 &= \int_0^1 \left[\int_0^x (x-t)^{2-\alpha(x)}u'''(t)dt\right]'^2 dx \\ &= \int_0^1 \left[\int_0^x (x-t)^{2-\alpha(x)}\left(-\alpha'(x)\ln(x-t) + \frac{2-\alpha(x)}{x-t}\right)u'''(t)dt\right]^2 dx \\ &\leq \| u\|_3^2 \int_0^1 \left[\frac{\alpha'(x)}{(3-\alpha(x))^2}x^{3-\alpha(x)} - \frac{\alpha'(x)}{3-\alpha(x)}x^{3-\alpha(x)}\ln(x) + x^{2-\alpha(x)}\right]^2 dx \\ &\leq \widetilde{A_3} \| u\|_3^2,\end{aligned}$$

we immediately have

$$\|D^{\alpha(x)}u(x)\|_1^2 \leq A_3 \| u\|_3^2. \tag{9}$$

Summarizing the estimate Eqs. (7)–(9), there exists a positive constant A such that

$$\|\mathcal{L}u(x)\|_1 \leq A \| u\|_3.$$

Namely, $\mathcal{L}$ is a bounded linear operator. $\square$

3. SRKM and convergence analysis

3.1. The approximate solution of Eq. (4)

In view of Lemma 2.3, it is clear that $\mathcal{L} : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator, put $\psi_i(x) = \mathcal{L}^* r_x(x_i)$, where $\mathcal{L}^*$ is the conjugate operator of $\mathcal{L}$ and $\{x_i\}_1^\infty$ is a dense subset of $[0,1]$. Considering the boundary conditions, we skillfully construct $\varphi_1(x)$ and $\varphi_2(x)$ like collocation method in [22,23] as follows:

$$\varphi_1(x) = R_x(0), \qquad \varphi_2(x) = R_x(1).$$

Theorem 3.1. *For each fixed n, $\{\varphi_1, \varphi_2\} \bigcup \{\psi_j\}_1^n$ are linearly independent in $W_2^3[0,1]$.*

The proof can be found in [19].

Let

$$S = \overline{span\{\{\varphi_1, \varphi_2\} \bigcup \{\psi_j\}_1^\infty\}}, \qquad S_n = \overline{span\{\{\varphi_1, \varphi_2\} \bigcup \{\psi_j\}_1^n\}}$$

and we denote $\mathcal{P}$ and $\mathcal{P}_n$ as the projection operators from W_2^3 to S and S_n, respectively.

Theorem 3.2. *If $u(x)$ is the solution of Eq. (4), then $y = \mathcal{P}u$ is also the solution of Eq. (4) in S.*

Proof.

$$\begin{aligned} \mathcal{L}(\mathcal{P}u)(x_i) &= \langle \mathcal{P}u, \mathcal{L}^* r_{x_i} \rangle_3 = \langle \mathcal{P}u, \psi_i \rangle_3 = \langle u, \mathcal{P}\psi_i \rangle_3 = \langle u, \psi_i \rangle_3 \\ &= \langle u, \mathcal{L}^* r_{x_i} \rangle_3 = \langle \mathcal{L}u, r_{x_i} \rangle_1 = \mathcal{L}u(x_i) = f(x_i). \end{aligned}$$

Since $\{x_i\}_{i=1}^\infty$ is a dense subset of $[0,1]$, we have $\mathcal{L}(\mathcal{P}u) = f$. On the other hand,

$$\begin{aligned} \mathcal{P}u(0) &= \langle \mathcal{P}u, \varphi_1(x) \rangle = \langle u, \mathcal{P}R_x(0) \rangle = \langle u, R_x(0) \rangle = u(0) = \lambda_0, \\ \mathcal{P}u(1) &= \langle \mathcal{P}u, \varphi_2(x) \rangle = \langle u, \mathcal{P}R_x(1) \rangle = \langle u, R_x(1) \rangle = u(1) = \lambda_1. \end{aligned}$$

Namely, $\mathcal{P}u$ is the solution of Eq. (4). $\square$

Considering numerical computation, we define the n-term approximation to $y(x)$ by

$$y_n(x) = (\mathcal{P}_n y)(x).$$

Therefore, the simplified representation of the reproducing kernel method reads as: find $y_n \in S_n$ such that

$$\begin{cases} \langle y_n, \psi_i \rangle = f(x_i), & i = 1, 2, \ldots, n, \\ \langle y_n, \varphi_1 \rangle = \lambda_0, & \langle y_n, \varphi_2 \rangle = \lambda_1. \end{cases} \tag{10}$$

As $y_n \in S_n$, y_n has the representation

$$y_n(x) = \sum_{i=1}^{2} \beta_i \varphi_i(x) + \sum_{j=1}^{n} \alpha_j \psi_j(x), \tag{11}$$

where $\beta_1, \beta_2, \alpha_j, j = 1, 2, \ldots, n$ are constants to be determined.

Substituting Eq. (11) into Eq. (10), we derive a linear system. Thereupon, the approximate solution $y_n(x)$ given in Eq. (11) can be obtained by solving the linear system.

Table 1
Comparison of absolute errors for Example 4.1.

x	$E_{10}(x)$ in [8]	Our method	$E_{20}(x)$ in [8]	Our method
0.1	1.43×10^{-6}	1.05×10^{-7}	1.92×10^{-7}	1.17×10^{-8}
0.2	2.28×10^{-6}	1.80×10^{-7}	3.19×10^{-7}	1.77×10^{-8}
0.3	2.84×10^{-6}	2.32×10^{-7}	4.01×10^{-7}	2.17×10^{-8}
0.4	3.11×10^{-6}	2.60×10^{-7}	4.44×10^{-7}	2.39×10^{-8}
0.5	3.13×10^{-6}	2.66×10^{-7}	4.53×10^{-7}	2.45×10^{-8}
0.6	2.94×10^{-6}	2.51×10^{-7}	4.30×10^{-7}	2.34×10^{-8}
0.7	2.53×10^{-6}	2.24×10^{-7}	3.80×10^{-6}	2.07×10^{-8}
0.8	1.88×10^{-6}	1.60×10^{-7}	3.11×10^{-7}	1.59×10^{-8}
0.9	9.97×10^{-7}	6.54×10^{-8}	2.08×10^{-7}	1.11×10^{-8}

Table 2
Maximum absolute errors for Example 4.1.

E'_{10}	$D^{3/2}E_{10}$	E'_{20}	$D^{3/2}E_{20}$
1.15×10^{-6}	3.50×10^{-6}	1.01×10^{-8}	2.83×10^{-8}

3.2. Convergence analysis

Theorem 3.3. *If $y_n(x) \longrightarrow y(x)$ in S, then $y_n \to y$ uniformly on $[0,1]$. Furthermore, $D^{\alpha(x)}y_n \to D^{\alpha(x)}y$ uniformly on $[0,1]$.*

Proof. Note that

$$\begin{aligned} y_n(x) &= \langle y_n, R_x\rangle_3, & D^{\alpha(x)}y_n(x) &= \langle y_n, D^{\alpha(x)}R_x\rangle_3,\\ y(x) &= \langle y, R_x\rangle_3, & D^{\alpha(x)}y(x) &= \langle y, D^{\alpha(x)}R_x\rangle_3. \end{aligned}$$

By applying *Cauchy-Schwarz* inequality and the continuity of $\|D^{\alpha(x)}R_x\|_3$, there holds

$$|y_n(x) - y(x)| = |\langle y_n - y, R_x\rangle| \le \|y_n - y\|_3 \cdot \|R_x\|_3 \le M\|y_n - y\|_3 \longrightarrow 0.$$
$$|D^{\alpha(x)}(y_n(x) - y(x))| = |\langle y_n - y, D^{\alpha(x)}R_x\rangle| \le \|y_n - y\|_3 \cdot \|D^{\alpha(x)}R_x\|_3 \le N\|y_n - y\|_3 \longrightarrow 0.$$

Namely, y_n and $D^{\alpha(x)}y_n$ uniformly convergent to y and $D^{\alpha(x)}y$, respectively. □

The following corollary is a direct consequence of Theorems 3.2 and 3.3.

Corollary 3.4. *Let $u(x)$ be the solution of Eq. (4), then $y_n = \mathcal{P}_n y \to y = \mathcal{P}u$, moreover, $y_n \to y$ uniformly.*

Theorem 3.5. *The approximate solution $y_n(x)$ for Eq. (4) has second order convergence, that is, there exists a positive constant C such that*

$$|y_n(x) - y(x)| \le \frac{C}{n^2}\|f\|_1.$$

For the proof, one may refer to [20].

4. Numerical examples

Example 4.1. Consider the following variable fractional BVPs [8] of the form

$$\begin{cases} D^{\alpha(x)}u(x) + \cos x\; u'(x) + 4u(x) + 5u(x^2) = f(x), & x \in [0,1],\\ u(0) = 0, \qquad u(1) = 1, \end{cases}$$

where $\alpha(x) = \frac{5+\sin(x)}{4}$, $f(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 5x^4 + 4x^2 + 2x\cos(x)$. The exact solution is $u(x) = x^2$. Applying SRKM, we take $x_i = \frac{i}{n}, i = 1, 2, \ldots, n$, $n = 10, 20$. The absolute error $E_n(x) = |y_n(x) - u(x)|$ obtained is compared with [8] in Table 1. Moreover, we report maximum errors $E'_n = |y'_n(x) - u'(x)|_{max}$ and $D^{3/2}E_n = |D^{3/2}y_n(x) - D^{3/2}u(x)|_{max}$ in Table 2.

5. Conclusion

In this paper, an effective numerical algorithm based on SRKM is proposed for variable order fractional differential equations with boundary conditions. Numerical example shows that our algorithm is simple, accurate and effective.

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