Journal of Computational and Applied Mathematics 312 (2017) 13–26

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New approach for the Fornberg–Whitham type equations

Bariza Boutarfa [a], Ali Akgül [b,*], Mustafa Inc [c]

[a] Guelma University, Department of Material Sciences, 24000 Guelma, Algeria

[b] Siirt University, Art and Science Faculty, Department of Mathematics, TR-56100 Siirt, Turkey

[c] Fırat University, Science Faculty, Department of Mathematics, 23119 Elazığ, Turkey

ARTICLE INFO

Article history:
Received 18 August 2015

MSC:
47B32
46E22
74S30

Keywords:
Reproducing kernel method
Series solutions
Fornberg–Whitham equations

ABSTRACT

This paper applies the reproducing kernel Hilbert space method to the solutions of three types of Fornberg–Whitham equations: original, modified and time fractional. Comparison with Adomian decomposition method, homotopy analysis method and the variational iteration method shows the validity and applicability of the technique.

1. Introduction

The Fornberg–Whitham equation has caught much attention in recent years. Recently, there has been much work focusing on finding travelling wave solutions for this equation [1]. Eq. (1.1) was used to study the qualitative behaviour of wave-breaking [2]. Modifying the nonlinear term $u\frac{\partial u}{\partial x}$ in Eq. (1.1) to $u^2\frac{\partial u}{\partial x}$, He et al. proposed in [3] the modified Fornberg–Whitham equation. Possible travelling wave solutions of Eq. (1.1) were given in [4] and were classified in [5]. A linear dispersive Fornberg–Whitham equation was investigated in [6] where smooth and non-smooth travelling wave solutions were acquired. The global existence of solution to the viscous Fornberg–Whitham equation was shown in [7]. The boundary control of it was investigated in [8]. The variational iteration method (VIM) for the time-fractional Fornberg–Whitham equation was investigated by Saka et al. in [9].

In this paper, we consider the Fornberg–Whitham equation [10], modified Fornberg–Whitham equation [3] and time-fractional Fornberg–Whitham equation [9] of the forms:

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial u}{\partial x}=u\frac{\partial^3 u}{\partial x^3}-u\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial u}{\partial x}=u\frac{\partial^3 u}{\partial x^3}-u^2\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

and

$$\frac{\partial u^\alpha}{\partial t^\alpha}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial u}{\partial x}=u\frac{\partial^3 u}{\partial x^3}-u\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

* Corresponding author.
E-mail addresses: b.ellagoune@yahoo.fr (B. Boutarfa), aliakgul00727@gmail.com (A. Akgül), minc@firat.edu.tr (M. Inc).

http://dx.doi.org/10.1016/j.cam.2015.09.016
0377-0427/

The reproducing kernel Hilbert space method (RKHSM) [11] is implemented for solving these equations. Numerical experiments are shown to verify the efficiency of the RKHSM. The method is implemented for three numerical examples. The linear and nonlinear problems are solved by using RKHSM.

The theory of reproducing kernels [12] was used for the first time at the beginning of the 20th century by Zaremba. Some researchers investigate much problems by RKHSM recently [11]. For more details see [13–19].

This paper is prepared as follows. Section 2 gives useful reproducing kernel functions. The representation in $W(\Omega)$ and a related linear operator are presented in Section 3. Section 4 presents the main results. The exact and approximate solutions of Eq. (1.2) and an iterative method are developed for the kind of problems in the reproducing kernel Hilbert space. We prove that the approximate solution converges uniformly to the exact solution. Numerical experiments are shown in Section 5. Applied examples are given in Section 6. Some conclusions are given in the final section.

2. Reproducing kernel functions

We give some useful reproducing kernel functions in this space.

Definition 2.1. Let $P \neq \emptyset$. A function $Z : P \times P \to \mathbb{C}$ is called a *reproducing kernel function* of the Hilbert space H if and only if

(a) $Z(\cdot, x) \in H$ for all $x \in P$,
(b) $\langle \varrho, Z(\cdot, x)\rangle = \varrho(x)$ for all $x \in P$ and all $\varrho \in H$.

Definition 2.2. $H_2^1[0,1]$ is defined by:

$$H_2^1[0,1] = \{v \in AC[0,1] : \ v' \in L^2[0,1]\}.$$

$$\langle v, g\rangle_{H_2^1} = v(0)g(0) + \int_0^1 v'(t)g'(t)\mathrm{d}t, \quad v, g \in H_2^1[0,1],$$

and

$$\|v\|_{H_2^1} = \sqrt{\langle v, v\rangle_{H_2^1}}, \quad v \in H_2^1[0,1],$$

are the inner product and the norm for $H_2^1[0,1]$, respectively.

Lemma 2.3. *Reproducing kernel function q_s of reproducing kernel space $H_2^1[0,1]$ is presented by* [11, *page* 123]*:*

$$q_s(t) = \begin{cases} 1+t, & t \le s, \\ 1+s, & t > s. \end{cases}$$

Definition 2.4. $F_2^2[0,1]$ is given as:

$$F_2^2[0,1] = \{v \in AC[0,1] : \ v' \in AC[0,1], \ v'' \in L^2[0,1], \ v(0) = 0\}.$$

$$\langle v, g\rangle_{F_2^2} = \sum_{i=0}^{1} v^{(i)}(0)g^{(i)}(0) + \int_0^1 v''(t)g''(t)\mathrm{d}t, \quad v, g \in F_2^2[0,1],$$

and

$$\|v\|_{F_2^2} = \sqrt{\langle v, v\rangle_{F_2^2}}, \quad v \in F_2^2[0,1],$$

are the inner product and the norm of $F_2^2[0,1]$, respectively.

Lemma 2.5. *Reproducing kernel function r_s of $F_2^2[0,1]$ is given by* [11, *page* 148]*:*

$$r_s(t) = \begin{cases} st + \dfrac{1}{2}st^2 - \dfrac{1}{6}t^3, & t \le s, \\[2ex] ts + \dfrac{1}{2}ts^2 - \dfrac{1}{6}s^3, & t > s. \end{cases}$$

Definition 2.6. We define the space $W_2^4[0,1]$ by

$$W_2^4[0,1] = \{v \in AC[0,1] : \ v', v'', v^{(3)} \in AC[0,1], \ v^{(4)} \in L^2[0,1]\}.$$

$$\langle v, g\rangle_{W_2^4} = \sum_{i=0}^{3} v^{(i)}(0)g^{(i)}(0) + \int_0^1 v^{(4)}(x)g^{(4)}(x)\mathrm{d}x, \quad v, g \in W_2^4[0,1],$$

and

$$\|v\|_{W_2^4} = \sqrt{\langle v, v\rangle_{W_2^4}}, \quad v \in W_2^4[0, 1],$$

are the inner product and the norm of $W_2^4[0, 1]$, respectively.

Theorem 2.7. *Reproducing kernel function B_y of the reproducing kernel space $W_2^4[0, 1]$ is acquired as:*

$$B_y(x) = \begin{cases} \sum_{i=1}^{8} c_i(y)x^{i-1}, & x \le y, \\ \sum_{i=1}^{8} d_i(y)x^{i-1}, & x > y, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} &c_1(y) = 1, \quad c_2(y) = y, \quad c_3(y) = \frac{1}{4}y^2, \quad c_4(y) = \frac{1}{36}y^3, \\ &c_5(y) = \frac{1}{144}y^3, \quad c_6(y) = -\frac{1}{240}y^2, \quad c_7(y) = \frac{1}{720}y, \\ &c_8(y) = -\frac{1}{5040}, \quad d_1(y) = 1 - \frac{1}{5040}y^7, \quad d_2(y) = y + \frac{1}{720}y^6, \\ &d_3(y) = \frac{1}{4}y^2 - \frac{1}{240}y^5, \quad d_4(y) = \frac{1}{36}y^3 + \frac{1}{144}y^4, \\ &d_5(y) = 0, \quad d_6(y) = 0, \quad d_7(y) = 0, \quad d_8(y) = 0. \end{aligned}$$

Proof. Let $v \in W_2^4[0, 1]$ and $0 \le y \le 1$. We get

$$\begin{aligned} \langle v, B_y\rangle_{W_2^4} &= \sum_{i=0}^{3} v^{(i)}(0)B_y^{(i)}(0) + \int_0^1 v^{(4)}(x)B_y^{(4)}(x)\mathrm{d}x \\ &= v(0)B_y(0) + v'(0)B_y'(0) + v''(0)B_y''(0) + v^{(3)}(0)B_y^{(3)}(0) \\ &\quad + v^{(3)}(1)B_y^{(4)}(1) - v^{(3)}(0)B_y^{(4)}(0) - v''(1)B_y^{(5)}(1) + v''(0)B_y^{(5)}(0) \\ &\quad + v'(1)B_y^{(6)}(1) - v'(0)B_y^{(6)}(0) - \int_0^1 v'(x)B_y^{(7)}(x)\mathrm{d}x, \end{aligned}$$

by Definition 2.6 and integrating by parts three times. After substituting the values of $B_y(0)$, $B_y'(0)$, $B_y''(0)$, $B_y^{(3)}(0)$, $B_y^{(4)}(0)$, $B_y^{(5)}(0)$, $B_y^{(6)}(0)$, $B_y^{(4)}(1)$, $B_y^{(5)}(1)$, and $B_y^{(6)}(1)$ into the above equation we obtain

$$\begin{aligned} \langle v, B_y\rangle_{W_2^4} &= v(0)1 + v'(0)y + v''(0)\left(\frac{y^2}{2}\right) + v^{(3)}(0)\left(\frac{y^3}{6}\right) + v^{(3)}(1)0 - v^{(3)}(0)\left(\frac{y^3}{6}\right) - v''(1)0 + v''(0)\left(-\frac{y^2}{2}\right) \\ &\quad + v'(1)0 - v'(0)y - \int_0^1 v'(x)B_y^{(7)}(x)\mathrm{d}x \end{aligned}$$

thus we obtain

$$\begin{aligned} \langle v, B_y\rangle_{W_2^4} &= v(0) - \int_0^1 v'(x)B_y^{(7)}(x)\mathrm{d}x \\ &= v(0) - \int_0^y v'(x)B_y^{(7)}(x)\mathrm{d}x - \int_y^1 v'(x)B_y^{(7)}(x)\mathrm{d}x \\ &= v(0) - \int_0^y v'(x)(-1)\mathrm{d}x - \int_y^1 v'(x)0\mathrm{d}x \\ &= v(0) + v(y) - v(0) \\ &= v(y). \end{aligned}$$

This completes the proof. □

Definition 2.8. The binary space $W(\Omega)$ is defined by

$$W(\Omega)=\left\{v:\ \frac{\partial^4 v}{\partial x^3\partial t}\text{ is completely continuous in }\Omega=[0,1]\times[0,1],\ \frac{\partial^6 v}{\partial x^4\partial t^2}\in L^2(\Omega),\ v(x,0)=0\right\}.$$

$$\begin{aligned}\langle v,g\rangle_W &= \sum_{i=0}^{3}\int_0^1\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}v(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}g(0,t)\right]\mathrm{d}t+\sum_{j=0}^{1}\left\langle\frac{\partial^j}{\partial t^j}v(x,0),\ \frac{\partial^j}{\partial t^j}g(x,0)\right\rangle_{W_2^4}\\ &\quad+\int_0^1\int_0^1\left[\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial t^2}v(x,t)\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial t^2}g(x,t)\right]\mathrm{d}t\mathrm{d}x\end{aligned}$$

and

$$\|v\|_W=\sqrt{\langle v,v\rangle_W},\quad v\in W(\Omega),$$

are the inner product and the norm of $W(\Omega)$, respectively.

Lemma 2.9. *Reproducing kernel function $K_{(y,s)}$ of $W(\Omega)$ is given by* [11, *page* 148]*:*

$$K_{(y,s)}=B_y r_s.$$

Definition 2.10. The binary space $\widehat{W}(\Omega)$ is defined by

$$\widehat{W}(\Omega)=\left\{v:\ \frac{\partial^3 v}{\partial x^3}\text{ is completely continuous in }\Omega=[0,1]\times[0,1],\ \frac{\partial^5 v}{\partial x^4\partial t}\in L^2(\Omega),\right\}.$$

$$\begin{aligned}\langle v,g\rangle_{\widehat{W}} &= \sum_{i=0}^{3}\int_0^1\left[\frac{\partial}{\partial t}\frac{\partial^i}{\partial x^i}v(0,t)\frac{\partial}{\partial t}\frac{\partial^i}{\partial x^i}g(0,t)\right]\mathrm{d}t+\langle v(x,0),g(x,0)\rangle_{W_2^4}\\ &\quad+\int_0^1\int_0^1\left[\frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}v(x,t)\frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}g(x,t)\right]\mathrm{d}t\mathrm{d}x,\end{aligned}$$

and

$$\|v\|_{\widehat{W}}=\sqrt{\langle v,v\rangle_{\widehat{W}}},\quad v\in\widehat{W}(\Omega),$$

are the inner product and the norm of $\widehat{W}(\Omega)$, respectively.

Lemma 2.11. *Reproducing kernel function $G_{(y,s)}$ of $\widehat{W}(\Omega)$ is given as* [11, *page 148*]*:*

$$G_{(y,s)}=B_y q_s.$$

3. Exact and approximate solutions in $W(\Omega)$

Solutions of Eqs. (1.1)–(1.3) are presented in the $W(\Omega)$ in this section. $L:W(\Omega)\to\widehat{W}(\Omega)$ is defined by

$$\begin{aligned}Lv &= \frac{\partial v}{\partial t}(x,t)-\frac{\partial^3 v}{\partial x^2\partial t}(x,t)+\frac{\partial v}{\partial x}(x,t)\left(1+A^2\frac{1}{\cosh^4(cx)}+\frac{6Ac^2}{\cosh^2(cx)}-\frac{18Ac^2\sinh^2(cx)}{\cosh^4(cx)}\right)\\ &\quad+v(x,t)\left(-\frac{4A^2c\sinh(cx)}{\cosh^5(cx)}-\frac{16Ac^3\sinh(cx)}{\cosh^3(cx)}+\frac{24Ac^3\sinh^3(cx)}{\cosh^5(cx)}\right).\end{aligned}$$

After homogenizing the initial condition, (1.2) changes to the

$$\begin{cases}Lv=M(x,t,v(x,t),v_x(x,t),v_{xx}(x,t),v_{xxx}(x,t)), & (x,t)\in\Omega,\\ v(x,0)=0,\end{cases}\tag{3.1}$$

where

$$v(x,t)=u(x,t)-u(x,0)=u(x,t)-A\frac{1}{\cosh^2(cx)},$$

$$A=\frac{3}{4}\left(\sqrt{15}-5\right),\qquad c=\frac{1}{20}\left(\sqrt{10(5-\sqrt{15})}\right),$$

$$\begin{aligned}&M(x,t,v(x,t),v_x(x,t),v_{xx}(x,t),v_{xxx}(x,t))\\ &\quad=-v^2(x,t)\left(v_x(x,t)+\frac{2Ac\sinh(cx)}{\cosh^3(cx)}\right)-2A\frac{1}{\cosh^2(cx)}v(x,t)v_x(x,t)+v(x,t)v_{xxx}(x,t)\\ &\qquad+3v_x(x,t)v_{xx}(x,t)+\frac{2A^3c\sinh(cx)}{\cosh^7(cx)}+\frac{28A^2c^3\sinh(cx)}{\cosh^5(cx)}-\frac{60A^2c^3\sinh^3(cx)}{\cosh^7(cx)}+\frac{2Ac\sinh(cx)}{\cosh^3(cx)},\end{aligned}$$

for convenience, we again write u instead of v in (3.1). In a similar way same things can be shown for model problem (1.1) and (1.3).

Lemma 3.1. *L is a bounded linear operator.*

Proof. Let $u \in W(\Omega)$ and $(x,t) \in \Omega$. By Lemma 2.9, we get

$$u(x,t) = \left\langle u, K_{(x,t)} \right\rangle_W,$$

and thus

$$\begin{aligned}
&Lu(x,t) = \left\langle u, LK_{(x,t)} \right\rangle_W, \qquad \frac{\partial}{\partial x}Lu(x,t) = \left\langle u, \frac{\partial}{\partial x}LK_{(x,t)} \right\rangle_W, \\
&\frac{\partial^2}{\partial x^2}Lu(x,t) = \left\langle u, \frac{\partial^2}{\partial x^2}LK_{(x,t)} \right\rangle_W, \qquad \frac{\partial^3}{\partial x^3}Lu(x,t) = \left\langle u, \frac{\partial^3}{\partial x^3}LK_{(x,t)} \right\rangle_W, \\
&\frac{\partial^4}{\partial x^4}Lu(x,t) = \left\langle u, \frac{\partial^3}{\partial x^3}LK_{(x,t)} \right\rangle_W, \qquad \frac{\partial}{\partial t}Lu(x,t) = \left\langle u, \frac{\partial}{\partial t}LK_{(x,t)} \right\rangle_W, \\
&\frac{\partial}{\partial t}\frac{\partial}{\partial x}Lu(x,t) = \left\langle u, \frac{\partial}{\partial t}\frac{\partial}{\partial x}LK_{(x,t)} \right\rangle_W, \qquad \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}Lu(x,t) = \left\langle u, \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}LK_{(x,t)} \right\rangle_W, \\
&\frac{\partial}{\partial t}\frac{\partial^3}{\partial x^3}Lu(x,t) = \left\langle u, \frac{\partial}{\partial t}\frac{\partial^3}{\partial x^3}LK_{(x,t)} \right\rangle_W, \qquad \frac{\partial}{\partial t}\frac{\partial^4}{\partial x^4}Lu(x,t) = \left\langle u, \frac{\partial}{\partial t}\frac{\partial^4}{\partial x^4}LK_{(x,t)} \right\rangle_W.
\end{aligned}$$

Hence there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} > 0$ such that

$$\begin{aligned}
&|Lu(x,t)| \le a_1 \|u\|_W, \qquad \left|\frac{\partial}{\partial x}Lu(x,t)\right| \le a_2 \|u\|_W, \\
&\left|\frac{\partial^2}{\partial x^2}Lu(x,t)\right| \le a_3 \|u\|_W, \qquad \left|\frac{\partial^3}{\partial x^3}Lu(x,t)\right| \le a_4 \|u\|_W, \\
&\left|\frac{\partial^4}{\partial x^4}Lu(x,t)\right| \le a_5 \|u\|_W, \qquad \left|\frac{\partial}{\partial t}Lu(x,t)\right| \le a_6 \|u\|_W, \\
&\left|\frac{\partial}{\partial t}\frac{\partial}{\partial x}Lu(x,t)\right| \le a_7 \|u\|_W, \qquad \left|\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}Lu(x,t)\right| \le a_8 \|u\|_W, \\
&\left|\frac{\partial}{\partial t}\frac{\partial^3}{\partial x^3}Lu(x,t)\right| \le a_9 \|u\|_W, \qquad \left|\frac{\partial}{\partial t}\frac{\partial^4}{\partial x^4}Lu(x,t)\right| \le a_{10} \|u\|_W.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle Lu, Lu\rangle_{\widehat{W}} &= \sum_{i=0}^{3}\int_0^1 \left[\frac{\partial}{\partial t}\frac{\partial^i}{\partial x^i}Lu(0,t)\frac{\partial}{\partial t}\frac{\partial^i}{\partial x^i}Lu(0,t)\right]dt + \langle Lu(x,0), Lu(x,0)\rangle_{W_2^4} \\
&\quad + \int_0^1\int_0^1 \left[\frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}Lu(x,t)\frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}Lu(x,t)\right]dt dx \\
&\le N \|u\|_W^2,
\end{aligned}$$

where

$$N = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 + a_{10}^2.$$

This completes the proof. □

Now, choose a countable dense subset $\{(x_1,t_1), (x_2,t_2), \ldots\}$ in Ω and define

$$\varphi_i = G_{(x_i,t_i)}, \qquad \Psi_i = L^*\varphi_i,$$

where L^* is the adjoint operator of L. The orthonormal system $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$ of $W(\Omega)$ can be derived from the process of Gram–Schmidt orthogonalization of $\{\Psi_i\}_{i=1}^{\infty}$ as

$$\widehat{\Psi}_i = \sum_{k=1}^{i} \beta_{ik}\Psi_k.$$

Theorem 3.2. *Assume that $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω. Then $\{\Psi_i(x,t)\}_{i=1}^{\infty}$ is a complete system in $W(\Omega)$, and*

$$\Psi_i = LK_{(x_i,t_i)}(x,t).$$

Proof. We have

$$\begin{aligned}\Psi_i &= L^*\varphi_i = \left\langle L^*\varphi_i, K_{(x,t)}\right\rangle_W = \left\langle \varphi_i, LK_{(x,t)}\right\rangle_{\widehat{W}} \\ &= \left\langle LK_{(x,t)}, G_{(x_i,t_i)}\right\rangle_{\widehat{W}} = LK_{(x,t)}(x_i, t_i) \\ &= LK_{(x_i,t_i)}(x, t).\end{aligned}$$

Clearly $\Psi_i \in W(\Omega)$. If

$$\langle u, \Psi_i\rangle_W = 0, \quad i = 1, 2, \ldots,$$

then

$$\begin{aligned}0 &= \langle u, \Psi_i\rangle_W = \left\langle u, L^*\varphi_i\right\rangle_W \\ &= \langle Lu, \varphi_i\rangle_{\widehat{W}} = Lu(x_i, t_i), \quad i = 1, 2, \ldots.\end{aligned}$$

Note that $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in Ω. Hence, $Lu = 0$. From the existence of L^{-1}, it follows that $u = 0$. This completes the proof. $\square$

Theorem 3.3. *If $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in Ω, then the solution of (3.1) is given by*

$$u = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} M(x, t, v(x, t), v_x(x, t), v_{xx}(x, t), v_{xxx}(x, t))\widehat{\Psi}_i. \tag{3.2}$$

Proof. By Theorem 3.2 $\{\Psi_i(x, t)\}_{i=1}^{\infty}$ is a complete system in $W(\Omega)$. Thus

$$\begin{aligned}u &= \sum_{i=1}^{\infty}\left\langle u, \widehat{\Psi}_i\right\rangle_W \widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\, \langle u, \Psi_k\rangle_W\, \widehat{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left\langle u, L^*\varphi_k\right\rangle_W \widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\, \langle Lu, \varphi_k\rangle_{\widehat{W}}\, \widehat{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left\langle Lu, G_{(x_k,t_k)}\right\rangle_{\widehat{W}} \widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} Lu(x_k, t_k)\widehat{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, v(x_k, t_k), v_x(x_k, t_k), v_{xx}(x_k, t_k), v_{xxx}(x_k, t_k))\widehat{\Psi}_i. \quad \square\end{aligned}$$

Approximate solution u_n can be acquired as:

$$u_n = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, v(x_k, t_k), v_x(x_k, t_k), v_{xx}(x_k, t_k), v_{xxx}(x_k, t_k))\widehat{\Psi}_i. \tag{3.3}$$

Obviously

$$\|u_n(x, t) - u(x, t)\|_W \to 0, \quad n \to \infty.$$

Theorem 3.4. *If $u \in W(\Omega)$ then*

$$\|u_n(x, t) - u(x, t)\|_W \to 0, \quad n \to \infty.$$

Moreover a sequence $\|u_n(x, t) - u(x, t)\|_W$ is monotonically decreasing in n.

Proof. We have

$$\|u_n(x, t) - u(x, t)\|_W = \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, v, v_x, v_{xx}, v_{xxx})\widehat{\Psi}_i(x, t)\right\|_W,$$

by (3.2) and (3.3). Therefore

$$\|u_n(x, t) - u(x, t)\|_W \to 0, \quad n \to \infty.$$

In addition

$$\|u_n(x,t)-u(x,t)\|_W^2 = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, v, v_x, v_{xx}, v_{xxx}) \widehat{\Psi}_i(x,t) \right\|_W^2$$
$$= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} M(x_k, t_k, v, v_x, v_{xx}, v_{xxx}) \widehat{\Psi}_i(x,t) \right)^2 .$$

Clearly, $\|u_n(x,t)-u(x,t)\|_W$ is monotonically decreasing in n. □

4. The method implementation

(i) If (3.1) is linear then the analytical solution of (3.1) can be obtained directly by (3.2).
(ii) If (3.1) is nonlinear then the solution of (3.1) can be acquired by the following iterative method.

We construct an iterative sequence u_n, putting,

$$\begin{cases} \text{any fixed } u_0 \in W, \\ u_n = \sum_{i=1}^{n} A_i \widehat{\Psi}_i, \end{cases} \tag{4.1}$$

where

$$\begin{cases} A_1 = \beta_{11} M(x_k, t_k, v(x_k,t_k), v'(x_k,t_k), v''(x_k,t_k), v^{(3)}(x_k,t_k)) \\ A_2 = \sum_{k=1}^{2} \beta_{2k} M(x_k, t_k, v_{k-1}(x_k,t_k), v'_{k-1}(x_k,t_k), v''_{k-1}(x_k,t_k), v^{(3)}_{k-1}(x_k,t_k)), \\ \dots \\ A_n = \sum_{k=1}^{n} \beta_{nk} M(x_k, t_k, v_{k-1}(x_k,t_k), v'_{k-1}(x_k,t_k), v''_{k-1}(x_k,t_k), v^{(3)}_{k-1}(x_k,t_k)). \end{cases} \tag{4.2}$$

Theorem 4.1. *Suppose that $\|u_n\|$ defined by (4.1) is bounded and (3.1) has a unique solution. If $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω, then u_n converges to the analytical solution u of (3.1), and*

$$u = \sum_{i=1}^{\infty} A_i \widehat{\Psi}_i,$$

where A_i is given by (4.2).

Proof. First, we prove the convergence of u_n. From (4.1), and the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, we infer that

$$\|u_{n+1}\|^2 = \sum_{i=1}^{n+1} A_i^2 = \sum_{i=1}^{n} A_i^2 + A_{n+1}^2 = \|u_n\|^2 + A_{n+1}^2 \geq \|u_n\| . \tag{4.3}$$

By (4.3), $\|u_n\|$ is nondecreasing, and by the assumption, $\|u_n\|$ is bounded. Thus $\|u_n\|$ is convergent. By (4.3) there exists a constant c such that

$$\sum_{i=1}^{\infty} A_i^2 = c.$$

This implies that

$$\{A_i\}_{i=1}^{\infty} \in \ell^2.$$

If $m > n$, then

$$\|u_m - u_n\|^2 = \left\| \sum_{k=n}^{m-1} u_{k+1} - u_k \right\|^2 \leq \sum_{k=n}^{m-1} \|u_{k+1} - u_k\|^2$$
$$= \sum_{k=n}^{m-1} A_{k+1}^2 \to 0, \quad n \to \infty.$$

The completeness of $W(\Omega)$ shows that there exists $\widehat{u} \in W(\Omega)$ such that $u_n \to \widehat{u}$ as $n \to \infty$. Now, we prove that $\widehat{u}$ solves (3.1). Taking limits in (3.3), we get

$$\widehat{u} = \sum_{i=1}^{\infty} A_i \widehat{\Psi}_i.$$

Note that

$$L\widehat{u} = \sum_{i=1}^{\infty} A_i L\widehat{\Psi}_i$$

and

$$\begin{aligned}(L\widehat{u})(x_k, t_k) &= \sum_{i=1}^{\infty} A_i L\widehat{\Psi}_i(x_k, t_k) = \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\Psi}_i, G_{(x_k,t_k)}\right\rangle_{\widehat{W}} \\ &= \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\Psi}_i, \varphi_k\right\rangle_{\widehat{W}} = \sum_{i=1}^{\infty} A_i \left\langle \widehat{\Psi}_i, L^*\varphi_k\right\rangle_{W} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\Psi}_i, \Psi_k\right\rangle_{W}.\end{aligned}$$

In view of (4.2), we get

$$L\widehat{u}(x_l, t_l) = M(x_l, t_l, v_{l-1}(x_l, t_l), v'_{l-1}(x_l, t_l), v''_{l-1}(x_l, t_l), v^{(3)}_{l-1}(x_l, t_l)).$$

Since $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense in Ω, for each $(y, s) \in \Omega$, there exists a subsequence $\{(x_{n_j}, t_{n_j})\}_{j=1}^{\infty}$ such that

$$(x_{n_j}, t_{n_j}) \to (y, s), \quad j \to \infty.$$

We know that

$$L\widehat{u}(x_{n_j}, t_{n_j}) = M(x_{n_j}, t_{n_j}, v_{n_j-1}(x_{n_j}, t_{n_j}), v'_{n_j-1}(x_{n_j}, t_{n_j}), v''_{n_j-1}(x_{n_j}, t_{n_j}), v^{(3)}_{n_j-1}(x_{n_j}, t_{n_j})).$$

Let $j \to \infty$. By the continuity of M, we have

$$(L\widehat{u})(y, s) = M(y, s, \widehat{u}(y, s), \widehat{u}'(y, s), \widehat{u}''(y, s), \widehat{u}^{(3)}(y, s))$$

which indicates that $\widehat{u}$ satisfies (3.1). This completes the proof. □

5. Numerical results

We implement RKHSM to solve the Fornberg–Whitham type equations in this section.

Example 5.1. Regard the Fornberg–Whitham equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} = u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{5.1}$$

with the initial condition

$$u(x, 0) = \exp\left(\frac{x}{2}\right). \tag{5.2}$$

The exact solution of (5.1)–(5.2) is given as [1]

$$u(x, t) = \exp\left(\frac{x}{2} - \frac{2t}{3}\right).$$

After homogenizing the initial condition and using the above method we obtain Tables 6.2–6.7 and Figs. 6.1–6.3.

Example 5.2. Consider the modified Fornberg–Whitham equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} = u\frac{\partial^3 u}{\partial x^3} - u^2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{5.3}$$

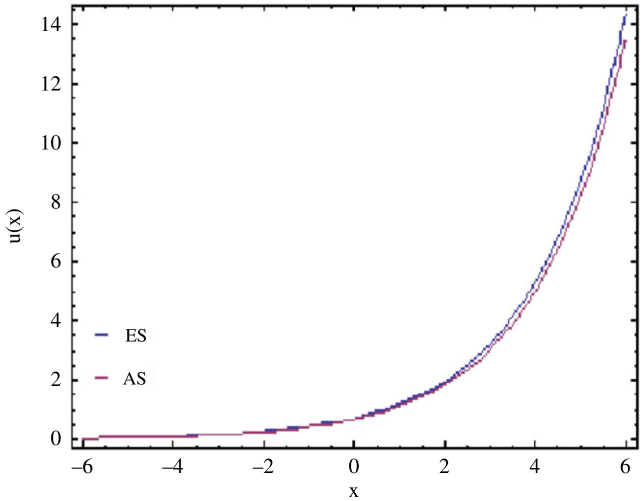

Fig. 6.1. Graph of numerical results for Example 5.1 for $t = 0.5$.

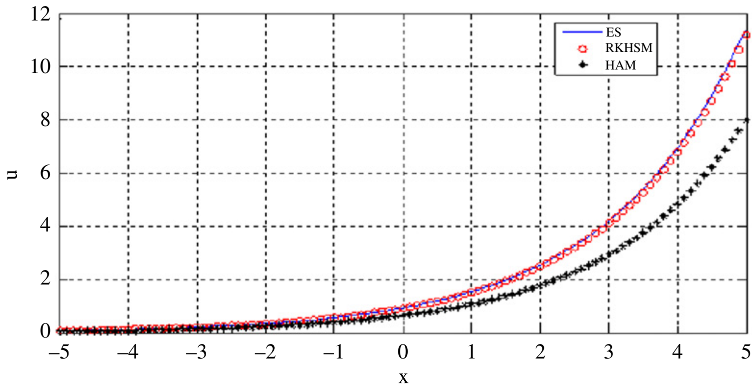

Fig. 6.2. Comparing Exact solution, RKHSM solution and HAM solution for Example 5.1.

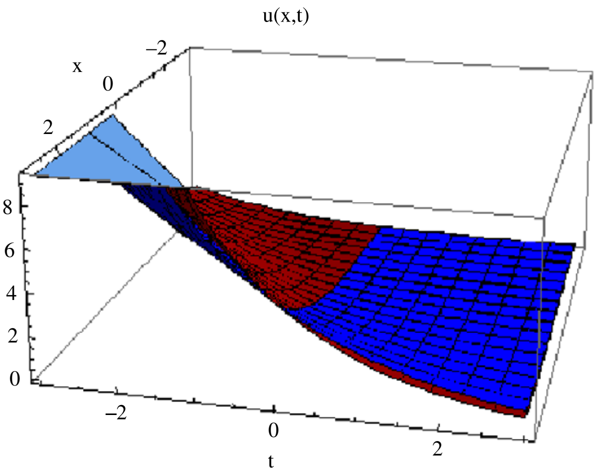

Fig. 6.3. Exact solution (red) and approximate solution (blue) for Example 5.1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

with the following initial condition

$$u(x,0) = \frac{3}{4}\left(\sqrt{15}-5\right)\frac{1}{\cosh^2(cx)}, \tag{5.4}$$

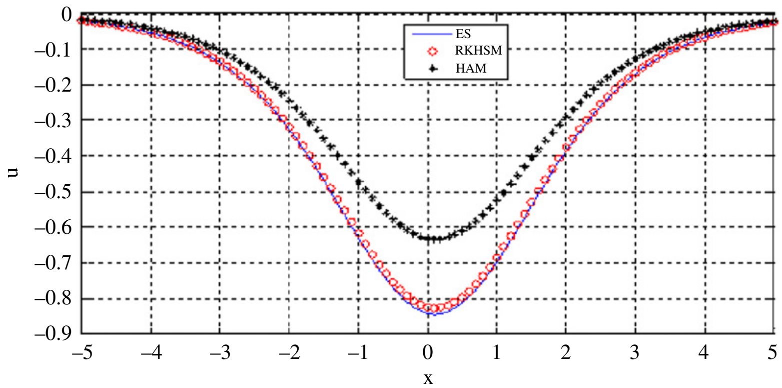

Fig. 6.4. Comparing Exact solution, RKHSM solution and HAM solution for Example 5.2.

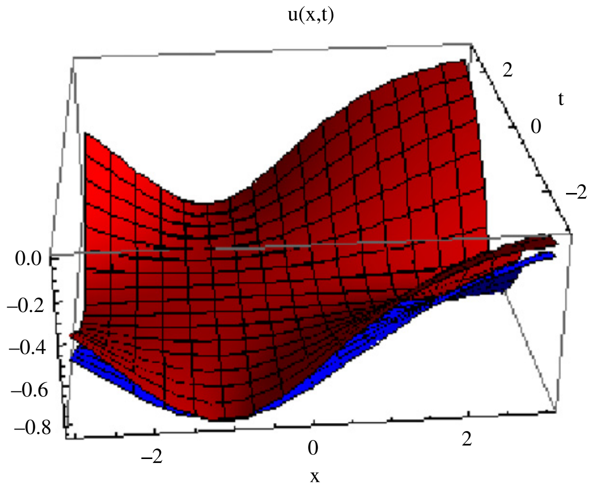

Fig. 6.5. Exact solution (red) and approximate solution (blue) for Example 5.2 for $c = \frac{1}{20}\sqrt{10(5-\sqrt{15})}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

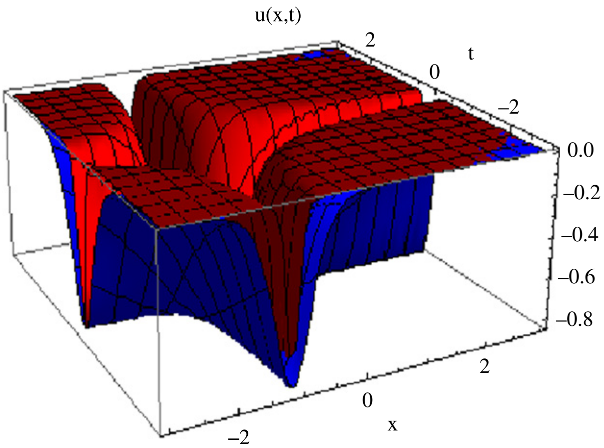

Fig. 6.6. Exact solution (red) and approximate solution (blue) for Example 5.2 for $c = 3$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

with a constant

$$c = \frac{1}{20}\sqrt{10(5-\sqrt{15})}.$$

Table 6.1
AS of time-fractional Fornberg–Whitham equation for different values of α.

x	t	ES	AS ($\alpha = 1$)	AS ($\alpha = 0.8$)
0.5	0.5	1.226725886172430997	1.226205721561058784 3	1.109445557308519372 4
0.5	1.0	0.878987506933924995	0.8768593429364480377	0.8398155516046179367
0.5	1.5	0.629822070321352942	0.6297200724277173582	0.7320367199169766663
0.5	2.0	0.451287233475656278	0.4553945611874197522	0.6920988617168632507
1.0	0.5	1.575147217154194640	1.5741302021100485542	1.4726243142711092894
1.0	1.0	1.128642299854152098	1.1252665169645424226	1.0915991091061473071
1.0	1.5	0.808707546283511231	0.8080724206913721037	0.9140411806502195614
1.0	2.0	0.579464278009437630	0.5875111670008122211	0.8999273234729346186
1.5	0.5	2.022529061850951194	2.0224283957894948425	1.8986915152626067478
1.5	1.0	1.449205399361638518	1.4474489401188910281	1.3874001034878765475
1.5	1.5	1.038401044095206491	1.0381243170021222153	1.1734666922694023940
1.5	2.0	0.744046861026729428	0.7560475355569940629	1.0509899370798853388

Table 6.2
AS (first line) and ES (second line) for Example 5.1.

x	t				
	0.2	0.4	0.6	0.8	1
−4	0.1184418296 0.1184418291	0.1036571291 0.1036571286	0.09071795752 0.09071795329	0.0793939229 0.0793939323	0.06948345158 0.06948345120
−2	0.3219582712 0.3219582716	0.2817692908 0.2817692890	0.2465969625 0.2465969639	0.2158150882 0.2158150835	0.1888756014 0.1888756028
0	0.8751733265 0.8751733191	0.7659283443 0.7659283383	0.6703200419 0.6703200460	0.5866462264 0.5866462195	0.5134171179 0.5134171190
2	2.378967729 2.378967730	2.082009080 2.082009084	1.822118795 1.822118800	1.594669790 1.594669758	1.395612434 1.395612425
4	6.466704758 6.466704753	5.659487451 5.659487458	4.95303249 4.95303242	4.3347619 4.33476182	3.79366789 3.79366789

The exact solitary wave solution of (5.3)–(5.4) is given as [3]

$$u(x,t) = \frac{3}{4}(\sqrt{15}-5)\frac{1}{\cosh^2(c(x-(5-\sqrt{15})t))}.$$

After homogenizing the initial condition and using the above method, we obtain Tables 6.8–6.11 and Figs. 6.4–6.6.

Remark 5.3. In Tables 6.2–6.1, we abbreviate the exact solution and the approximate solution with ES and AS, respectively. AE stands for the absolute error, i.e., the absolute value of the difference of the exact solution and the approximate solution, while RE indicates the relative error, i.e., the absolute error divided by the absolute value of the exact solution.

6. Applied examples

In this section, we research the nonlinear time-fractional Fornberg–Whitham equation [9] by the reproducing kernel method.

$$\frac{\partial u^{\alpha}}{\partial t^{\alpha}} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} = u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < \alpha \le 1, \tag{6.1}$$

subject to initial condition

$$u(x,0) = \frac{4}{3}\exp\left(\frac{x}{2}\right), \tag{6.2}$$

where $u(x,t)$ is the fluid velocity, α is constant and lies in the interval $(0, 1]$, t is the time and x is the spatial coordinate. After homogenizing the initial condition and using the above method, we obtain approximate solutions of (6.1)–(6.2) for different values of α and show them by Table 6.1.

7. Conclusion

In this study, Fornberg–Whitham, modified Fornberg–Whitham and time-fractional Fornberg–Whitham equations were investigated by RKHSM. The obtained results show that this technique can solve the problem impressively. The results that we acquired were compared with the results that were obtained by ADM, HAM and VIM. Numerical experiments on test examples show that our proposed method is of high accuracy, supporting the theoretical results. It has been shown that the obtained results are uniform convergent and the operator that was used is a bounded linear operator.

Table 6.3
AE (first line) and RE (second line) for Example 5.1 for different values of x and t.

x	t				
	0.2	0.4	0.6	0.8	1
−4	5×10^{-10} 4.221×10^{-9}	5×10^{-10} 4.82×10^{-9}	4.23×10^{-9} 4.66×10^{-8}	9.40×10^{-9} 1.18×10^{-7}	3.8×10^{-10} 5.46×10^{-9}
−2	4×10^{-10} 1.242×10^{-9}	1.8×10^{-9} 6.38×10^{-9}	1.4×10^{-9} 5.6772×10^{-9}	4.7×10^{-9} 2.177×10^{-8}	1.4×10^{-9} 7.412×10^{-9}
0	7.4×10^{-9} 8.455×10^{-9}	6.0×10^{-9} 7.833×10^{-9}	4.1×10^{-9} 6.116×10^{-9}	6.9×10^{-9} 1.1761×10^{-8}	1.1×10^{-9} 2.142×10^{-9}
2	1×10^{-9} 4.203×10^{-10}	4×10^{-9} 1.921×10^{-9}	5×10^{-9} 2.74×10^{-9}	3.2×10^{-8} 2.006×10^{-8}	9×10^{-9} 6.4487×10^{-9}
4	5×10^{-9} 7.7319×10^{-10}	7×10^{-9} 1.236×10^{-9}	7.2×10^{-8} 1.453×10^{-8}	8.0×10^{-8} 1.845×10^{-8}	4×10^{-9} 1.054×10^{-9}

Table 6.4
CPU time(s) for Example 5.1.

x	t				
	0.2	0.4	0.6	0.8	1
−4	0.390	0.359	0.453	0.390	0.374
−2	0.406	0.437	0.468	0.500	0.484
0	0.561	0.422	0.437	0.421	0.453
2	0.359	0.374	0.359	0.358	0.358
4	0.344	0.374	0.406	0.375	0.437

Table 6.5
A comparison between RKHSM (first line) and HAM [1] (second line) for Example 5.1.

x	t				
	0.2	0.4	0.6	0.8	1
−4	5E−10 2.22193E−5	5E−10 9.47416E−6	4.23E−9 4.83886E−5	9.40E−9 6.71560E−5	3.8E−10 5.36314E−5
−2	4E−10 6.03987E−5	1.8E−9 2.57532E−5	1.4E−9 1.31533E−4	4.7E−9 1.82549E−4	1.4E−9 1.45785E−4
0	7.4E−9 1.64180E−5	6.0E−9 7.00049E−5	4.1E−9 3.57546E−4	6.9E−9 4.96219E−4	1.1E−9 3.96285E−4
2	1E−9 4.46289E−4	4E−9 1.90293E−4	5E−9 9.71910E−4	3.2E−8 1.34886E−3	9E−9 1.07721E−3
4	5E−9 1.21314E−4	7E−9 5.17269E−4	7E−8 2.64192E−3	8.0E−8 3.66659E−3	4E−9 2.92817E−3

Table 6.6
ES and AS for ADM, HAM and RKHSM for Example 5.1 when $t = 5$.

x	u_{exact}	u_{ADM}	u_{HAM}	u_{RKHSM}
−4	0.004827949995	0.0031719207	0.0048752608	0.0048279514
−2	0.01312372874	0.0086221743	0.0132523327	0.0131237255
0	0.03567399336	0.0234374990	0.0360235773	0.0356740033
2	0.09697196790	0.0637097231	0.0979222379	0.096971949
4	0.2635971382	0.1731809521	0.2661802682	0.263597291

Table 6.7
AE for ADM, HAM and RKHSM for Example 5.1 when $t = 5$.

x	$\lvert u_{exact} - u_{ADM}\rvert$	$\lvert u_{exact} - u_{HAM}\rvert$	$\lvert u_{exact} - u_{RKHSM}\rvert$
−4	1.65602E−3	4.73109E−5	1.405E−9
−2	4.50155E−3	1.28604E−4	3.24E−9
0	1.22364E−2	3.49584E−4	9.94E−9
2	3.32622E−2	9.50270E−4	1.89E−8
4	9.04161E−2	2.58313E−3	1.528E−7

Table 6.8
AS (first line) and ES (second line) for Example 5.2 for different values of x and t.

t	$x = 2.5$	$x = 5$	$x = 7.5$	$x = 10$
0.02	−0.7144249843 −0.7144249598	−0.4504779212 −0.4504780048	−0.2349678714 −0.2349678528	−0.1107643868 −0.1107643186
0.04	−0.7165464263 −0.7165463773	−0.4528101955 −0.4528101891	−0.2364825099 −0.2364825808	−0.1115481917 −0.1115481838
0.06	−0.7186566715 −0.7186566519	−0.4551474661 −0.4551474665	−0.2380051612 −0.2380051672	−0.1123371196 0.1123371717
0.08	−0.7207556337 −0.7207555967	−0.4574897077 −0.4574897553	−0.2395356954 −0.2395356248	−0.1131312117 −0.1131313100
0.1	−0.7228430209 −0.7228430251	−0.4598370439 −0.4598369728	−0.2410739190 −0.2410739688	−0.1139306377 −0.1139306254

Table 6.9
AE (first line) and RE (second line) for Example 5.2.

t	$x = 2.5$	$x = 5$	$x = 7.5$	$x = 10$
0.02	2.45×10^{-8} 3.429331×10^{-8}	8.36×10^{-8} 1.855806×10^{-7}	1.86×10^{-8} 7.915976×10^{-8}	6.82×10^{-8} 6.157217×10^{-7}
0.04	4.90×10^{-8} 6.838357×10^{-8}	6.4×10^{-9} 1.413395×10^{-8}	7.09×10^{-8} 2.998106×10^{-7}	7.9×10^{-9} 7.082141×10^{-8}
0.06	1.96×10^{-8} 2.72731×10^{-8}	$4. \times 10^{-10}$ 8.78836×10^{-10}	6.0×10^{-9} 2.520953×10^{-8}	5.21×10^{-8} 4.637823×10^{-7}
0.08	3.70×10^{-8} 5.133501×10^{-8}	4.76×10^{-8} 1.04046×10^{-7}	7.06×10^{-8} 2.947369×10^{-7}	9.83×10^{-8} 8.689018×10^{-7}
0.1	4.2×10^{-9} 5.81039×10^{-9}	7.11×10^{-8} 1.5462×10^{-7}	4.98×10^{-8} 2.065756×10^{-7}	1.23×10^{-8} 1.079604×10^{-7}

Table 6.10
CPU time(s) for Example 5.2.

t	$x = 2.5$	$x = 5$	$x = 7.5$	$x = 10$
0.02	0.842	0.874	0.718	0.780
0.04	0.858	0.796	0.842	0.748
0.06	1.030	0.780	0.718	0.734
0.08	0.842	0.873	0.764	0.765
0.1	0.827	0.811	0.765	0.718

Table 6.11
A comparison between RKHSM (first line) and variational iteration method [20] (second line) for Example 5.2.

t	$x = 2.5$	$x = 5$	$x = 7.5$	$x = 10$
0.02	2.45e−8 1.180e−4	8.36e−8 2.124e−5	1.86e−8 2.805e−5	6.82e−8 5.528e−6
0.04	4.90E−8 2.363e−4	6.4e−9 4.797e−5	7.09e−8 5.772e−5	7.9e−9 1.084e−5
0.06	1.96e−8 3.547e−4	4.e−10 8.029e−5	6.0e−9 8.902e−5	5.21e−8 1.591e−5
0.08	3.70e−8 4.731e−4	4.76e−8 1.183e−4	7.06e−8 1.220e−4	9.83e−8 2.071e−5
0.1	4.2e−9 5.914e−4	7.11e−8 1.622e−4	4.98e−8 1.565e−4	1.23e−8 2.524e−5

References

[1] Fayçal Abidi, Khaled Omrani, The homotopy analysis method for solving the Fornberg–Whitham equation and comparison with Adomian's decomposition method, Comput. Math. Appl. 59 (8) (2010) 2743–2750.
[2] G.B. Whitham, Non-linear dispersion of water waves, J. Fluid Mech. 27 (1967) 399–412.
[3] Bin He, Qing Meng, Shaolin Li, Explicit peakon and solitary wave solutions for the modified Fornberg–Whitham equation, Appl. Math. Comput. 217 (5) (2010) 1976–1982.
[4] Aiyong Chen, Jibin Li, Xijun Deng, Wentao Huang, Travelling wave solutions of the Fornberg–Whitham equation, Appl. Math. Comput. 215 (8) (2009) 3068–3075.
[5] Jiuli Yin, Lixin Tian, Xinghua Fan, Classification of travelling waves in the Fornberg–Whitham equation, J. Math. Anal. Appl. 368 (1) (2010) 133–143.
[6] Bo Jiang, Qinsheng Bi, Smooth and non-smooth traveling wave solutions of the Fornberg–Whitham equation with linear dispersion term, Appl. Math. Comput. 216 (7) (2010) 2155–2162.

[7] Lixin Tian, Ying Gao, The global attractor of the viscous Fornberg–Whitham equation, Nonlinear Anal. 71 (11) (2009) 5176–5186.
[8] Yiping Meng, Lixin Tian, Boundary control on the viscous Fornberg–Whitham equation, Nonlinear Anal. RWA 11 (2) (2010) 827–837.
[9] Mehmet Giyas Sakar, Fevzi Erdogan, Ahmet Yıldırım, Variational iteration method for the time-fractional Fornberg–Whitham equation, Comput. Math. Appl. 63 (9) (2012) 1382–1388.
[10] Jiangbo Zhou, Lixin Tian, A type of bounded traveling wave solutions for the Fornberg–Whitham equation, J. Math. Anal. Appl. 346 (1) (2008) 255–261.
[11] Minggen Cui, Yingzhen Lin, Nonlinear Numerical Analysis in the Reproducing Kernel Space, Nova Science Publishers Inc., New York, 2009.
[12] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950) 337–404.
[13] Ali Akgül, A new method for approximate solutions of fractional order boundary value problems, Neural Parallel Sci. Comput. 22 (1–2) (2014) 223–237.
[14] Ali Akgül, Adem Kiliçman, Solving delay differential equations by an accurate method with interpolation, Abstr. Appl. Anal. (2015) 7. Art. ID 676939.
[15] Ali Akgül, New reproducing kernel functions, Math. Probl. Eng. (2015) 10. Art. ID 158134.
[16] Ali Akgül, Mustafa Inc, Esra Karatas, Dumitru Baleanu, Numerical solutions of fractional differential equations of Lane–Emden type by an accurate technique, Adv. Differential Equations 220 (2015) 2015.
[17] Mustafa Inc, Ali Akgül, Adem Kiliçman, Explicit solution of telegraph equation based on reproducing kernel method, J. Funct. Spaces Appl. (2012) 23. Art. ID 984682.
[18] X.Y. Li, B.Y. Wu, R.T. Wang, Reproducing kernel method for fractional Riccati differential equations, Abstr. Appl. Anal. (2014) 6. Art. ID 970967.
[19] Yu-Lan Wang, Hao Yu, Fu-Gui Tan, Shanshan Qu, Solving a class of singularly perturbed partial differential equation by using the perturbation method and reproducing kernel method, Abstr. Appl. Anal. (2014) 5. Art. ID 615840.
[20] Junfeng Lu, An analytical approach to the Fornberg–Whitham type equations by using the variational iteration method, Comput. Math. Appl. 61 (8) (2011) 2010–2013.