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Research paper

Complete group classification of systems of two nonlinear second-Order ordinary differential equations of the form $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$

G.F. Oguis[a], S. Moyo[b], S.V. Meleshko[a,*]

[a] Suranaree University of Technology, School of Mathematics, Nakhon Ratchasima 30000, Thailand
[b] Durban University of Technology, Research and Postgraduate Support & Institute for Systems Science, P O Box 1334, Steve Biko Campus, Durban 4000, South Africa

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ABSTRACT

Extensive work has been done on the group classification of systems of equations in the literature. This paper identifies the gap in the literature which concerns the group classification of systems of two nonlinear second-order ordinary differential equations. We provide a complete group classification of systems of two ordinary differential equations of the form, $\mathbf{y}'' = \mathbf{F}(\mathbf{y})$, which occur in many physical applications using two approaches which form the essence of this paper.

1. Introduction

Systems of autonomous nonlinear second-order ordinary differential equations, where the independent variable, usually assumed to be time, does not appear on the right hand side of the system, arise in various physical problems. This effectively assumes that the laws of nature which hold true in the present are presumably applicable in the past and the future. Hence, like all other systems of differential equations, the study of their symmetry structure poses an important role as their presence in a system allows one to reduce the order of the studied equations and also to find general solutions in quadratures.

Group classification studies, dating more than a century back, were first initiated by the founder of symmetry analysis, Sophus Lie [1–4]. These studies were long forgotten until Ovsiannikov [5,6] revived the work five decades ago. Lie's works put emphasis on tackling the group classification in two ways: 1) the direct way and 2) the indirect way also known as the algebraic approach. The direct way involves directly finding solutions of the determining equations and allows one to study all possible admitted Lie algebras without omission. On the other hand the indirect way involves solving the determining equations up to finding relations between constants defining admitted generators. The algebraic approach, as in the studies [7–10][1], takes into account the algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of the admitted Lie algebras in order to allow significant simplification of the group classification. It is also worth mentioning

* Corresponding author. Fax: +6644224185.
E-mail addresses: gfoguis@gmail.com (G.F. Oguis), moyos@dut.ac.za (S. Moyo), sergey@math.sut.ac.th (S.V. Meleshko).

[1] See also references therein.

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here the alternative method proposed in [11,12], where the combination of the algebraic approach and the equivalence Lie group is applied to the group classification problem.

In one of Lie's works [1], he gave a complete group classification of a single second-order ordinary differential equation of the form $y'' = f(x, y)$. Later on Ovsiannikov [13] did this group classification in a different way. The method he used, now also known as the direct approach, involved a two-step technique where the determining equations were first simplified through exploiting equivalence transformations and later on solved for the reduced cases of the generators. The same technique was used in a study conducted in [14] to classify a more general case of equations of the form $y'' = P_3(x, y; y')$, where $P_3(x, y; y')$ is a polynomial of a third degree with respect to the first-order derivative y'. Observe that sometimes difficulties arise in using the direct approach. Sometimes it is difficult to select or tease out equivalent cases with respect to equivalence transformations. As is observed in the classification of a general scalar second-order ordinary differential equation of the form $y'' = f(x, y; y')$, the application of the direct technique gives rise to overwhelming difficulties. For this paper, both the direct and indirect techniques are employed.

Apart from dealing with classification problems there is a significant amount of research that deals with the dimension and structure of symmetry algebras of linearizable ordinary differential equations [7,15–19]. This is also of importance since some nonlinear equations appear in disguised forms.

In addition to extensive studies on properties of scalar second-order ordinary differential equations, there are also several researchers committed to studying systems of two linear second-order ordinary differential equations [[19–26], [27]]. Surprisingly, the group classification of systems of two nonlinear second-order ordinary differential equations has not yet been exhausted, in particular, the group classification of systems of two autonomous nonlinear second-order ordinary differential equations is not yet complete. Hence this paper considers the group classification of systems of two autonomous nonlinear second-order ordinary differential equations of the form

$$\mathbf{y}'' = \mathbf{F}(\mathbf{y}). \tag{1}$$

The system studied here is a generalization of Lie's study [2]. Studied cases such as systems of two linear second-order ordinary differential equations and the degenerate case which is equivalent to the following

$$y'' = F(x, y, z), \qquad z'' = 0 \tag{2}$$

are omitted from this paper. We call systems that are equivalent to these cases as reducible systems and irreducible otherwise.

The paper is organized as follows. A preliminary study of systems of two nonlinear second-order ordinary differential equations is tackled first and is followed by the subsequent group classification applied to autonomous systems (1) of two second-order ordinary differential equations. The group classification is divided into two parts depending on the coefficient of the infinitesimal generator. The direct approach is applied on one case while a combination of the optimal system of subalgebras and direct approach is applied to the other case. The latter part of the paper lists the different cases with their respective results and is then followed by the conclusion.

2. Background study of systems of the form $\mathbf{y}'' = \mathbf{F}(x, \mathbf{y})$

This section focuses on systems of two nonlinear second-order ordinary differential equations of the form [24,27]

$$\mathbf{y}'' = \mathbf{F}(x, \mathbf{y}), \tag{3}$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} F(x, y, z) \\ G(x, y, z) \end{pmatrix}.$$

2.1. Equivalence transformations

System (3) has the following equivalence transformations:

(1) a linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with constant nonsingular 2×2 matrix P;
(2) the change $\tilde{y} = y + \phi(x)$ and $\tilde{z} = z + \psi(x)$; and
(3) the transformation related with the change $\tilde{x} = \phi(x)$, $\tilde{y} = y\psi(x)$, $\tilde{z} = z\psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ satisfy the condition $\dfrac{\phi''}{\phi'} = 2\dfrac{\psi'}{\psi}$.

2.2. Determining equations

The determining equations in matrix form for irreducible systems of the form (3) are given by

$$2\xi \mathbf{F}_x + 3\xi' \mathbf{F} + (((A + \xi' E)\mathbf{y} + \zeta) \cdot \nabla)\mathbf{F} - A\mathbf{F} = \xi''' \mathbf{y} + \zeta'', \tag{4}$$

Table 1
Systems of 2 nonlinear second-order ordinary differential equations belong to one of these 10 types. For all the cases, h_1, h_2, f and g are arbitrary functions of their arguments.

	F and G	Relations and conditions	Admitted generator
1.	$F = e^{ax}f(u,v)$, $G = e^{bx}g(u,v)$	$u = ye^{-ax},\ v = ze^{-bx}$, a, b are constant	$\partial_x + ay\partial_y + bz\partial_z$
2.	$F = e^{ax}(\cos(cx)f(u,v) + \sin(cx)g(u,v))$, $G = e^{ax}(-\sin(cx)f(u,v) + \cos(cx)g(u,v))$	$u = e^{-ax}(y\cos(cx) - z\sin(cx))$, $v = e^{-ax}(y\sin(cx) + z\cos(cx))$, $a, c \neq 0$ are constant	$\partial_x + (ay + cz)\partial_y + (-cy + az)\partial_z$
3.	$F = e^{ax}(f(u,v) + xg(u,v))$, $G = e^{ax}g(u,v)$	$u = e^{-ax}(y - zx),\ v = ze^{-ax}$, a is constant	$\partial_x + (ay + z)\partial_y + az\partial_z$
4.	$F = (y + h_1(x))f(x,v) - h_1''$, $G = (z + h_2(x))g(x,v) - h_2''$	$v = (z + h_2(x))(y + h_1(x))^{\alpha}$, $\alpha \neq 0$ is constant	$(ay + h_1)\partial_y + (bz + h_2)\partial_z$
5.	$F = (y + h_1(x))f(x,v) - h_1''(x)$, $G = h_2''(x)\ln(y + h_1(x)) + g(x,v)$	$v = z - h_2(x)\ln(y + h_1(x))$	$(ay + h_1)\partial_y + h_2\partial_z$
6.	$F = \dfrac{h_1''(x)}{h_1(x)}y + f(x,v)$, $G = \dfrac{h_2''(x)}{h_1(x)}y + g(x,v)$	$v = z - \dfrac{h_2(x)}{h_1(x)}y,\ h_1(x) \neq 0$	$h_1\partial_y + h_2\partial_z$
7.	$F = e^{au}(\cos(cu)f(x,v) + \sin(cu)g(x,v))$, $G = e^{au}(-\sin(cu)f(x,v) + \cos(cu)g(x,v))$	$y = ve^{au}\sin(cu),\ z = e^{au}\cos(cu)$, $a, c \neq 0$ are constant	$(ay + cz + h_1)\partial_y + (-cy + az + h_2)\partial_z$
8.	$F = \dfrac{y}{z + h_1(x)}f(x,v) + g(x,v)$, $G = -h_1''(x) + f(x,v)$	$v = z + h_1(x)$	$(z + h_1)\partial_y$
9.	$F = \dfrac{h_2''(x)}{2}u^2 + uf(x,v) + g(x,v)$, $G = -h_1''(x) + h_2''(x)u + f(x,v)$	$u = \dfrac{z + h_1(x)}{h_2(x)},\ v = y - \dfrac{(z + h_1(x))^2}{h_2(x)}$, $h_2(x) \neq 0$	$(z + h_1)\partial_y + h_2\partial_z$
10.	$F = e^u(uf(x,v) + g(x,v))$, $G = e^u f(x,v)$	$y = uve^u,\ z = ve^u$	$(ay + z + h_1)\partial_y + (az + h_2)\partial_z$

where the matrix $A = (a_{ij})$ is constant. The associated infinitesimal generator is

$$X = 2\xi(x)\partial_x + (A\mathbf{y} + \zeta(x)) \cdot \nabla,$$

where $\nabla = (\partial_y, \partial_z)^t$ and " $\cdot$ " means the scalar product $\mathbf{b} \cdot \nabla = b_i\partial_{y_i}$, where the summation with respect to the repeated index is used [24].

The equivalence transformation (1) with linear change $\tilde{\mathbf{y}} = P\mathbf{y}$, when applied to Eq. (3), reduces Eq. (4) and its associated infinitesimal generator to the same form with the matrix A and the vector ζ changed. Eq. (3) become

$$\tilde{\mathbf{y}}'' = \tilde{\mathbf{F}}(x, \tilde{\mathbf{y}})$$

with

$$\tilde{\mathbf{F}}(x, \tilde{\mathbf{y}}) = P\mathbf{F}(x, P\tilde{\mathbf{y}}),$$

and the partial derivatives with respect to the variables $\mathbf{y}$ are also changed as follows

$$\mathbf{b} \cdot \nabla = (P\mathbf{b}) \cdot \tilde{\nabla}.$$

Consequently, the determining Eq. (4) become

$$2\xi\tilde{\mathbf{F}}_x + 3\xi'\tilde{\mathbf{F}} + (((\tilde{A} + \xi'E)\tilde{\mathbf{y}} + \tilde{\zeta}) \cdot \tilde{\nabla})\tilde{\mathbf{F}} - \tilde{A}\tilde{\mathbf{F}} - \xi'''\tilde{\mathbf{y}} - \tilde{\zeta}'' = 0,$$

where

$$\tilde{A} = PAP^{-1}, \qquad \tilde{\zeta} = P\zeta$$

and the associated infinitesimal generator is also changed as follows

$$X = 2\xi(x)\partial_x + (\tilde{A}\tilde{\mathbf{y}} + \tilde{\zeta}(x)) \cdot \tilde{\nabla}.$$

As in [24,27] and in the succeeding pages, this transformation places a very important role in the group classification process.

From the study [24], the systems of two nonlinear second-order ordinary differential equations are equivalent to one of the following 10 types listed in Table 1. Looking closely at these systems, there is a necessity to conduct an initial study where the systems of two equations do not depend on x. This forms the core of this paper.

3. Autonomous systems (1) of two nonlinear second-order ordinary differential equations and their group classification

Since for autonomous systems $\mathbf{F}_x = 0$, the determining equations for autonomous systems have the form

$$3\xi'\mathbf{F} + (((A + \xi'E)\mathbf{y} + \zeta) \cdot \nabla)\mathbf{F} - A\mathbf{F} - \xi'''\mathbf{y} - \zeta'' = 0. \quad (5)$$

This also implies that the generator ∂_x is admitted by system (1).

Differentiating the determining Eq. (5) with respect to x, the group classification study is reduced into two cases, namely,

(1) the case with at least one admitted generator with $\xi'' \neq 0$; and
(2) the case where all admitted generators have $\xi'' = 0$.

For the first case, the direct approach by Lie is utilized, whereas for the second case, a combination of the optimal system of subalgebras of the Lie algebra and the direct method is used.

3.1. Systems admitting at least one admitted generator with $\xi'' \neq 0$

For the case with at least one generator with $\xi'' \neq 0$, we initially consider the differentiated determining Eq. (5) with respect to x and divide them by ξ''. The determining equations become

$$3\mathbf{F} + \left(\left(\mathbf{y} + \frac{\zeta'}{\xi''}\right)\cdot\nabla\right)\mathbf{F} - \frac{\xi^{(4)}}{\xi''}\mathbf{y} - \frac{\zeta'''}{\xi''} = 0. \tag{6}$$

Fixing x, and shifting y and z, Eq. (6) are reduced to

$$3\mathbf{F} + (\mathbf{y}\cdot\nabla)\mathbf{F} - a\mathbf{y} - \mathbf{b} = 0$$

where vector $\mathbf{b} = (b, c)^t$, with a, b, c constant.

The general solution of these equations is

$$F = \frac{b}{3} + \frac{ay}{4} + y^{-3}f(u), \quad G = \frac{c}{3} + \frac{az}{4} + z^{-3}g(u), \tag{7}$$

where $u = z/y$ and $f'g' \neq 0$. It is easy to see that if $f'g' = 0$, the studied system is equivalent to a reducible case. The functions F and G are then substituted into the determining Eq. (5). The determining equations are then solved directly in order to find generators admitted by Eq. (1). The first part of the determining equations is given as follows:

$$\xi''' - a\xi' = 0, \tag{8}$$

$$(\zeta_1 u - \zeta_2)f_u + 3\zeta_1 f = 0, \quad (u^2\zeta_1 - u\zeta_2)g_u + 3\zeta_2\, g = 0, \tag{9}$$

$$12\zeta_1'' - 12b\xi' - 3a\zeta_1 + 4a_{11}b + 4a_{12}c = 0, \quad 12\zeta_2'' - 12c\xi' - 3a\zeta_2 + 4a_{21}b + 4a_{22}c = 0, \tag{10}$$

$$(a_{11}u^4 + a_{12}u^5 - a_{21}u^3 - a_{22}u^4)f_u + (4a_{11}u^3 + 3a_{12}u^4)f + a_{12}g = 0, \quad (a_{11}u^2 + a_{12}u^3 - a_{21}u - a_{22}u^2)g_u + a_{21}u^4 f + (3a_{21} + 4a_{22}u)g = 0. \tag{11}$$

From Eq. (8), it can be seen that the general solution of ξ depends on three values of a, i.e., $a = 0$, $a = -p^2$ and $a = p^2$, where $p \neq 0$. For $a = 0$, the general solution of ξ is

$$\xi = \xi_2 x^2 + \xi_1 x + \xi_0,$$

where $\xi_2 \neq 0, \xi_1, \xi_0$ are constant. For $a = -p^2$, the general solution of ξ is

$$\xi = \xi_1 \cos(px) + \xi_2 \sin(px) + \xi_0,$$

where $\xi_2 \neq 0, \xi_1 \neq 0, \xi_0$ are constant. Lastly, for $a = p^2$, the general solution of ξ is

$$\xi = \xi_1 e^{-px} + \xi_2 e^{px} + \xi_0,$$

where $\xi_2 \neq 0, \xi_1 \neq 0, \xi_0$ are constant. Subsequently the determining Eq. (9) lead to the study of two cases where: (1) there exists a generator with $\zeta_1 \neq 0$ and (2) all generators have $\zeta_1 = 0$.

Considering the case where there exists a generator with $\zeta_1 \neq 0$, we divide by ζ_1 and differentiate Eq. (9) with respect to x to obtain, $\zeta_2 = k\zeta_1$, where k is a constant. Substituting this back to Eq. (9), one obtains $f = f_0(u-k)^{-3}$ and $g = g_0u^3(u-k)^{-3}$. Also, differentiating Eq. (10) with respect to x, it follows that $c = kb$. From here, one can verify that this is a reducible case.

Consider that all generators have $\zeta_1 = 0$. From Eq. (9), it follows that $\zeta_2 = 0$. Differentiating Eq. (10) with respect to x, it immediately follows that $b = c = 0$. From Eq. (11), the equivalence transformation $\tilde{y} = Py$, where P is a constant nonsingular

Table 2
Group classification of systems admitting at least one generator with $\xi'' \neq 0$.

F	G	κ	Extension of Kernel
$\kappa y + \frac{f_0 y}{z^4}\left(\frac{z}{y}\right)^{-4/(\gamma-1)}$	$\kappa z + \frac{f_1}{z^3}\left(\frac{z}{y}\right)^{-4/(\gamma-1)}$	0	Y_2, Y_3, Y_4
		−1	Y_7, Y_8, Y_4
		1	Y_9, Y_{10}, Y_4
$\kappa y + (f_0 y - f_1 z)\tau(y,z)$	$\kappa z + (f_0 z + f_1 y)\tau(y,z)$	0	Y_2, Y_3, Y_5
		−1	Y_7, Y_8, Y_5
		1	Y_9, Y_{10}, Y_5
$\kappa y + e^{\frac{y}{z}} z^{-4}(f_0 y + f_1 z)$	$\kappa z + f_0 z^{-3} e^{\frac{y}{z}}$	0	Y_2, Y_3, Y_6
		−1	Y_7, Y_8, Y_6
		1	Y_9, Y_{10}, Y_6

2×2 matrix, is utilized in order to obtain the general solution of f and g. Note that the constant matrix A is reduced to one of the following real-valued Jordan forms

$$J_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad J_2 = \begin{pmatrix} a_{11} & 1 \\ -1 & a_{11} \end{pmatrix}, \quad J_3 = \begin{pmatrix} a_{11} & 1 \\ 0 & a_{11} \end{pmatrix}. \tag{12}$$

The general solutions for f and g are listed as follows:

Jordan form	f	g
J_1	$f_0 u^{-(4+\frac{4}{\gamma-1})}$	$f_1 u^{-\frac{4}{\gamma-1}}$
J_2	$(f_0 y - f_1 z)\tau(y,z)$	$(f_0 z + f_1 y)\tau(y,z)$
J_3	$e^{-u}(f_0 u^{-4} + f_1 u^{-3})$	$f_0 e^{-u}$.

In this instance, $\tau(y,z) = e^{4\alpha \arctan\frac{z}{y}}(y^2+z^2)^{-2}$, and $f_0 \neq 0$, $f_1 \neq 0$, $\gamma \neq 1$, $\alpha \neq 1$ are constant.

Excluding reducible systems, the classes of functions F and G of Eq. (5) admitting a corresponding Lie group and the extension of the kernel of the admitted Lie algebras are obtained as seen in Table 2. The kernel of the admitted Lie algebras consists of the generator $X_1 = \partial_x$, which is omitted on the list. The extension of the kernel is listed as follows:

$$\begin{array}{ll} Y_2 = 2x\partial_x + y\partial_y + z\partial_z, & Y_3 = x(x\partial_x + y\partial_y + z\partial_z), \\ Y_4 = \gamma y\partial_y + z\partial_z, & Y_5 = (\alpha y - z)\partial_y + (y + \alpha z)\partial_z, \\ Y_6 = (y + 4z)\partial_y + z\partial_z, & Y_7 = \cos 2x\partial_x - \sin 2x(y\partial_y + z\partial_z), \\ Y_8 = \sin 2x\partial_x + \cos 2x(y\partial_y + z\partial_z), & Y_9 = e^{-2x}(\partial_x - (y\partial_y + z\partial_z)), \\ Y_{10} = e^{2x}(\partial_x + y\partial_y + z\partial_z). & \end{array}$$

The Lie algebras Y_2, Y_3, Y_7, Y_8, Y_9, and Y_{10} are associated with the coefficient ξ and the Lie algebras Y_4, Y_5, and Y_6 are related to the type of Jordan form of matrix A.

3.2. Systems where all admitted generators have $\xi'' = 0$

Note that the action of equivalence transformations coincides with the action of group automorphisms. For the direct approach, sometimes it is difficult to select out equivalent cases with respect to equivalence transformations. Fortunately, if the algebraic structure of the admitted Lie algebra is known, then using the algebraic approach aids in simplifying the group classification problem. Thus, for finding the group classification of systems of two autonomous nonlinear second-order ordinary differential equations with all admitted generators satisfying $\xi'' = 0$, the one-dimensional optimal system of one parameter subgroups is utilized and is proceeded by the direct approach. Firstly, the two-step algorithm of Ovsiannikov [28] is employed here, for which the optimal systems of subgroup and group invariant solutions are reconstructed. Invariant solutions are then substituted back into the determining Eq. (5) where the direct method is used to find all possible admitted Lie algebras.

Firstly returning to the analysis of the determining Eq. (5), since $\xi'' = 0$, it follows that $\xi = \xi_0 + \xi_1 x$, where ξ_0 and ξ_1 are constant. This property of the coefficient forces ζ to become constant[2]. The determining Eq. (5) are then reduced to

$$3\xi_1 \mathbf{F} + (((A + \xi_1 E)\mathbf{y} + \mathbf{k}) \cdot \nabla)\mathbf{F} - A\mathbf{F} = 0 \tag{13}$$

with the following admitted generator

$$X = 2(\xi_0 + \xi_1 x)\partial_x + (A\mathbf{y} + \mathbf{k}) \cdot \nabla \tag{14}$$

[2] See in Appendix.

where ξ_0, ξ_1, the matrix A and the vector $\mathbf{k}$ are constant. By rewriting (14), the generator can be represented as

$$X = \sum_{i=1}^{8} c_i X_i \tag{15}$$

where $c_i (i = 1, \ldots 8)$ are constant. Corresponding to the constants $c_i (i = 1, \ldots, 8)$, the basis operators of the Lie algebra are as follows:

$$\begin{aligned} &X_1 = \partial_x \quad X_2 = x\partial_x \quad X_3 = \partial_y \quad X_4 = \partial_z \\ &X_5 = y\partial_y \quad X_6 = z\partial_z \quad X_7 = z\partial_y \quad X_8 = y\partial_z. \end{aligned} \tag{16}$$

From here, the one-dimensional optimal system of one parameter subgroups of the main group of system (1) with $\xi'' = 0$ is constructed. The commutators of the basis operators are

$$\begin{array}{ll} [X_1, X_2] = X_1, & [X_5, X_7] = -X_7, \\ [X_3, X_5] = X_3, & [X_5, X_8] = X_8, \\ [X_3, X_8] = X_4, & [X_6, X_7] = X_7, \\ [X_4, X_6] = X_4, & [X_6, X_8] = -X_8, \\ [X_4, X_7] = X_3, & [X_7, X_8] = X_6 - X_5. \end{array} \tag{17}$$

The following inner automorphisms $A_i (i = 1, \ldots, 8)$ of the above Lie algebra are found without difficulties:

$$\begin{aligned} &A_1 : \hat{c}_1 = c_1 - a_1 c_2, \\ &A_2 : \hat{c}_1 = e^{a_2} c_1, \\ &A_3 : \hat{c}_3 = c_3 - a_3 c_5, \quad \hat{c}_4 = c_4 - a_3 c_8, \\ &A_4 : \hat{c}_3 = c_3 - a_4 c_7, \quad \hat{c}_4 = c_4 - a_4 c_6, \\ &A_5 : \hat{c}_3 = e^{a_5} c_3, \quad \hat{c}_7 = e^{a_5} c_7 \quad \hat{c}_8 = e^{-a_5} c_8, \\ &A_6 : \hat{c}_4 = e^{a_6} c_4, \quad \hat{c}_7 = e^{-a_6} c_7 \quad \hat{c}_8 = e^{a_6} c_8, \\ &A_7 : \hat{c}_3 = c_3 + a_7 c_4, \ \hat{c}_5 = c_5 + a_7 c_8, \ \hat{c}_6 = c_6 - a_7 c_8, \\ &\qquad \hat{c}_7 = c_7 - a_7^2 c_8 + a_7 c_6 - a_7 c_5, \\ &A_8 : \hat{c}_4 = c_4 + a_8 c_3, \ \hat{c}_5 = c_5 - a_8 c_7, \ \hat{c}_6 = c_6 + a_8 c_7, \\ &\qquad \hat{c}_8 = c_8 - a_8^2 c_7 - a_8 c_6 + a_8 c_5. \end{aligned} \tag{18}$$

Note that a_i $(i = 1, \ldots, 8)$ are the parameters on which the transformations of the group depend on. Apart from these automorphisms, we have the following involutions:

$$\begin{aligned} &E_1 : \bar{z} = -z| \quad \bar{c_4} = -c_4, \ \bar{c_7} = -c_7, \ \bar{c_8} = -c_8; \\ &E_2 : \bar{y} = -y| \quad \bar{c_3} = -c_3, \ \bar{c_7} = -c_7, \ \bar{c_8} = -c_8; \\ &E_3 : \bar{x} = -x| \quad \bar{c_1} = -c_1; \\ &E_4 : \bar{y} = z, \ \bar{z} = y| \quad \bar{c_3} = c_4, \ \bar{c_4} = c_3, \ \bar{c_5} = c_6, \ \bar{c_5} = c_5, \ \bar{c_7} = c_8, \ \bar{c_8} = c_7. \end{aligned}$$

We study the way in which the coefficients of Eq. (15) are changed under the action of inner automorphisms of the group above. Here and further on, only changeable coordinates of the generator are presented. Looking closely at the commutators, the Lie algebra L_8, which is composed of the generators X_i $(i = 1, \ldots, 8)$, can be split into 2 subalgebras $L_2 \oplus L_6 = \{X_1, X_2\} \oplus \{X_3, X_4, X_5, X_6, X_7, X_8\}$. Note also that L_6 can be decomposed further to $L_4 \oplus I_2 = \{X_5, X_6, X_7, X_8\} \oplus \{X_3, X_4\}$, where L_4 makes up a 4-dimensional subalgebra and I_2 is ideal.

Let us first study the 4-dimensional subalgebra $L_4 = \{X_5, X_6, X_7, X_8\}$. We consider this study here due to a misprint found in the classification of this Lie algebra in [29]. Now consider the operator X of a one parameter subgroup of the form

$$X = c_5 X_5 + c_6 X_6 + c_7 X_7 + c_8 X_8. \tag{19}$$

Automorphisms A_5 up to A_8 are made use of in order to find the one-dimensional optimal system of subalgebras of this Lie algebra. From the automorphisms A_5 and A_6, one can find the invariant $\bar{c_7}\bar{c_8} = c_7 c_8$, which leads one to consider the following cases:

$$\begin{array}{ll} (a) & c_7 c_8 > 0 \\ (b) & c_7 c_8 < 0 \\ (c) & c_7 c_8 = 0. \end{array}$$

Then utilizing the invariant of A_7 and A_8, which is $\bar{c_5} + \bar{c_6} = c_5 + c_6$, one can obtain relations of c_5 and c_6. Upon further computations using automorphisms, it can be verified that for case (a), the coefficients of Eq. (19) satisfy $c_5 - c_6 \neq 0$, $c_7 = 0$ and $c_8 = 0$. For case (b), it follows that $c_5 = c_6$, $c_7 = -1$ and $c_8 = 1$. For case (c) if $c_5 \neq c_6$ then $c_7 = 0$ and $c_8 = 0$. If $c_5 = c_6$, then $c_7 = 1$ and $c_8 = 0$. The involutions are also utilized. Hence, the following one-dimensional optimal system of subalgebras of the Lie algebra L_4 is obtained:

$$\begin{aligned}
&1.\quad X_5+\alpha X_6 \quad \text{where } -1\le\alpha\le 1\\
&2.\quad \alpha(X_5+X_6)+X_8-X_7 \quad \text{where } \alpha\ge 0\\
&3.\quad \beta(X_5+X_6)+X_7 \text{ where } \beta=0,1\\
&4.\quad 0.
\end{aligned}\tag{20}$$

Note that the 0 element is considered on this list [28]. There is a necessity to include this element on the list as when the direct sum $L_4 \oplus I_2$ is applied, more subalgebras of the Lie algebra L_6 may appear on the list.

Remark 1. As the action of the above automorphisms coincides with the action of the equivalence transformations, it is possible to get the optimal system of one-dimensional subalgebras of the Lie algebra L_4 using the latter. From the determining Eq. (13) and the utilization of the equivalence transformation $\tilde{y}=Py$, where P is a nonsingular 2×2 matrix with constant entries, the matrix of coefficients of (19)

$$\begin{pmatrix} c_5 & c_7\\ c_8 & c_6\end{pmatrix}$$

is reduced to one of its real-valued Jordan forms (12). Looking closely at (20), subalgebra 1. coincides with Jordan matrix J_1, subalgebra 2. coincides with Jordan matrix J_2, and subalgebra 3. coincides with Jordan matrix J_3.

3.2.1. Optimal system of subalgebras of the algebra $L_6=\{X_3,X_4,X_5,X_6,X_7,X_8\}$

After obtaining the one-dimensional optimal system (20) of subalgebras of the Lie algebra $L_4=\{X_5,X_6,X_7,X_8\}$, the next step is to combine L_4 with the ideal $I_2=\{X_3,X_4\}$. Here, again Ovsiannikov's two-step method [28] is applied. Hence, for the study of the one-dimensional subalgebras of the Lie algebra L_6, the study is reduced to analyzing the following elements:

$$\begin{aligned}
&1.\quad c_3X_3+c_4X_4+X_5+\alpha X_6 \quad \text{where } -1\le\alpha\le 1\\
&2.\quad c_3X_3+c_4X_4+\alpha(X_5+X_6)+X_8-X_7 \quad \text{where } \alpha\ge 0\\
&3.\quad c_3X_3+c_4X_4+\beta(X_5+X_6)+X_7 \quad \text{where } \beta=0,1\\
&4.\quad c_3X_3+c_4X_4.
\end{aligned}\tag{21}$$

Using automorphisms A_3 and A_4 and the involutions, the list of one-dimensional subalgebras of the Lie algebra $L_6=\{X_3,X_4,X_5,X_6,X_7,X_8\}$ is obtained as follows:

$$\begin{aligned}
&1.\quad X_5+\alpha X_6 \quad \text{where } -1\le\alpha\le 1\\
&2.\quad X_4+X_5\\
&3.\quad X_8-X_7\\
&4.\quad \beta X_3+\alpha(X_5+X_6)+X_8-X_7 \quad \text{where } \beta=-1,0,1,\ \alpha>0\\
&5.\quad \beta X_4+X_7 \quad \text{where } \beta=0,1\\
&6.\quad X_5+X_6+X_7\\
&7.\quad X_3\\
&8.\quad 0.
\end{aligned}\tag{22}$$

Again, it is necessary to study the element 0 of the subalgebras of the Lie algebra L_6 as this may generate additional elements when L_6 is combined with L_2.

3.2.2. Optimal system of subalgebras of the algebra $L_8=\{X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8\}$

Combining L_6 with L_2 and keeping in mind that for autonomous systems X_1 is already admitted, the following elements comprise the list of one-dimensional subalgebras of the Lie algebra L_8:

$$\begin{aligned}
&1.\quad \gamma X_2+X_5+\alpha X_6 \quad \text{where } -1\le\alpha\le 1\\
&2.\quad \gamma X_2+X_4+X_5\\
&3.\quad \gamma X_2+X_8-X_7\\
&4.\quad \gamma X_2+\beta X_3+\alpha(X_5+X_6)+X_8-X_7 \quad \text{where } \beta=-1,0,1,\ \alpha>0\\
&5.\quad \gamma X_2+\beta X_4+X_7 \quad \text{where } \beta=0,1\\
&6.\quad \gamma X_2+X_5+X_6+X_7\\
&7.\quad \gamma X_2+X_3\\
&8.\quad X_2.
\end{aligned}\tag{23}$$

Using this list of subalgebras, the next step is to obtain invariant solutions F and G of the determining Eq. (13). These functions are substituted into the determining Eq. (5), which are solved completely in order to find all other generators admitting Eq. (1).

3.2.3. Representations of systems of two nonlinear second-order ordinary differential equations with all generators having $\xi''=0$

From (15), c_i $(i=1,\dots,8)$ are the coefficients of the generator chosen from the above list of subalgebras. Only one subalgebra is presented in this paper as computations for the other subalgebras are done in a similar way.

3.2.3.1. Subalgebra 1. with the generator $\gamma X_2+X_5+\alpha X_6$ where $-1\le\alpha\le 1$. For this case, the determining Eq. (13) become

$$\begin{aligned}&yF_y+\alpha zF_z-(2\gamma-1)F=0,\\&yG_y+\alpha zG_z-(2\gamma-\alpha)G=0.\end{aligned}$$

The general solution of these equations is

$$F(y,z)=f(u)y^{1-2\gamma}\quad\text{and}\quad G(y,z)=g(u)y^{\alpha-2\gamma},\tag{24}$$

where $u=y^\alpha/z$ and $gf'\neq 0$. If $gf'=0$, then system (1) is equivalent to a reducible case. Substituting these functions to the determining Eq. (5), the following initial determining equations are obtained

$$\begin{aligned}&y^{2\alpha}a_{12}(\alpha uf'+(1-2\gamma)f-ug)\\&\quad+y^{\alpha+1}u((\alpha a_{11}+(\alpha-1)\xi_1-a_{22})uf'-2(\gamma a_{11}+(\gamma-2)\xi_1)f)\\&\quad+y^\alpha\zeta_1u(\alpha uf'+(1-2\gamma)f)-y^2a_{21}u^3f'-y\zeta_2u^3f'=0,\\&y^{2\alpha}a_{12}(\alpha ug'+(\alpha-2\gamma)g)\\&\quad+y^{\alpha+1}u((\alpha a_{11}+(\alpha-1)\xi_1-a_{22})ug'-((\alpha-2\gamma)a_{11}+(\alpha-2\gamma+3)\xi_1-a_{22})g)\\&\quad+y^\alpha u\zeta_1(\alpha ug'+(\alpha-2\gamma)g)-y^2a_{21}u(u^2g'+f)-y\zeta_2u^3g'=0.\end{aligned}\tag{25}$$

Determining Eq. (25) can be split with respect to y, where the powers of y depend on the values of α. Thus, upon further analysis, one needs to evaluate the following cases: (1) $\alpha=0$, (2) $\alpha=\frac{1}{2}$, (3) $\alpha=1$ and (4) $\alpha\neq 0,\ \frac{1}{2},\ 1$.

(1) **Case** $\alpha=0$.
After splitting Eq. (25) with respect to y, it can be verified that $a_{21}=0$ and one is left with the following determining equations

$$((1-2\gamma)(\zeta_1u+a_{12}))f-a_{12}ug=0,\tag{26a}$$

$$2(\gamma a_{11}+(\gamma-2)\xi_1)f+(a_{22}+\xi_1+\zeta_2u)uf'=0,\tag{26b}$$

$$\gamma(a_{12}+\zeta_1u)g=0,\tag{26c}$$

$$(2\gamma a_{11}+(2\gamma-3)\xi_1+a_{22})g+(a_{22}+\xi_1+\zeta_2u)ug'=0.\tag{26d}$$

From Eq. (26c), notice that if $\gamma=0$, G becomes a function solely of z and hence, this case is reducible. Thus, it follows that $a_{12}=0$ and $\zeta_1=0$. These conditions also satisfy Eq. (26a). Dividing Eq. (26b) by f' and u, and differentiating it with respect to u 2 times, one can study the following cases: a. $\left(\frac{f}{uf'}\right)''\neq 0$ and b. $\left(\frac{f}{uf'}\right)''=0$.

1.a Case $\left(\frac{f}{uf'}\right)''\neq 0$.

For this case, it follows that $a_{11}=\xi_1\frac{(2-\gamma)}{\gamma}$. Consequently, $\zeta_2=0$ and $a_{22}=-\xi_1$. These conditions also satisfy Eq. (26d). These conditions give no other extensions of the generator apart from the studied subalgebra.

1.b Case $\left(\frac{f}{uf'}\right)''=0$.

For this case, it follows that $\frac{f}{uf'}=\kappa u+\beta$. Furthermore, the general solution of this depends on β. Thus, one needs to study whether i. $\beta\neq 0$ or ii. $\beta=0$.

1.b.i Case $\beta\neq 0$.

For this case, the general solution for f (with a possible shift) is $f_0\left(\frac{1}{u}\right)^\beta$, $f_0\neq 0$. Substituting this into the determining Eq. (26b), one gets $a_{22}=\frac{(2\gamma-\beta-4)\xi_1+2\gamma a_{11}}{\beta}$ and $\zeta_2=0$. Consequently, the general solution for g is $g_0\left(\frac{1}{u}\right)^{\beta+1}$, $g_0\neq 0$. From here, the extension

$$\beta X_5+2\gamma X_6$$

is obtained along with the studied subalgebra.

1.b.ii Case $\beta = 0$.
For this case, it follows that $\kappa \neq 0$. Hence, the general equation for f is $f_0 e^{\kappa/u}$, $f_0 \neq 0$. Substituting this into the determining Eq. (26b), one obtains $\zeta_2 = \dfrac{2(\gamma a_{11} + (\gamma - 2)\xi_1)}{\kappa}$ and $a_{22} = -\xi_1$. Consequently, the general solution for g is $g_0 e^{\kappa/u}$, $g_0 \neq 0$. The extension

$$2\gamma X_4 + \kappa X_5$$

is obtained aside from the studied subalgebra.

(2) **Case** $\alpha = \frac{1}{2}$.
After splitting Eq. (25) with respect to y, it follows that $a_{21} = 0$. Also, since $(1-4\gamma)g + ug' = 0$ leads to a reducible case it then follows that $\zeta_1 = 0$. The remaining determining equations are

$$2a_{12}(1-2\gamma)f - 2a_{12}ug + (a_{12} - 2\zeta_2 u^2)uf' = 0, \tag{27a}$$

$$4((2-\gamma)\xi_1 - \gamma a_{11})f + (a_{11} - 2a_{22} - \xi_1)uf' = 0, \tag{27b}$$

$$(1-4\gamma)a_{12}g + (a_{12} - 2\zeta_2 u^2)ug' = 0, \tag{27c}$$

$$((1-4\gamma)a_{11} + (7-4\gamma)\xi_1 - 2a_{22})g + (a_{11} - 2a_{22} - \xi_1)ug' = 0. \tag{27d}$$

Dividing Eq. (27d) by g (as it is nonzero) and differentiating it with respect to u, one is left to study the following cases: a. $\left(\dfrac{ug'}{g}\right)' \neq 0$ and b. $\left(\dfrac{ug'}{g}\right)' = 0$.

2.a Case $\left(\dfrac{ug'}{g}\right)' \neq 0$.

For this case, it follows that $a_{11} = 2a_{22} + \xi_1$. If $\gamma = 0$ then $\xi_1 = 0$, but if $\gamma \neq 0$ then $a_{22} = \xi_1\left(\dfrac{1-\gamma}{\gamma}\right)$. These conditions also satisfy Eq. (27b). From Eq. (27c), one needs to study the following cases: i. there exists a generator with $a_{12} \neq 0$, and ii. all generators have $a_{12} = 0$.

2.a.i Case there exists a generator with $a_{12} \neq 0$.
If there exists a generator with $a_{12} \neq 0$, then g satisfies the form $(1-4\gamma)g + (1-\beta u^2)ug' = 0$. Notice that $\beta = 0$ leads to a reducible case. Hence, $\beta \neq 0$. Without loss of generality, one can assume that $\beta = 1$. Then the general solution of g is $g_0\left(1 - \dfrac{1}{u^2}\right)^{\tilde{\gamma}}$, where $g_0 \neq 0$ and $\tilde{\gamma} = \dfrac{1-4\gamma}{2} \neq 0$ (if $\tilde{\gamma} = 0$, the case is reducible). Substituting this into the determining Eq. (27c), we obtain that $\zeta_2 = \dfrac{a_{12}}{2}$. From Eq. (27a), it follows that $f = \phi(u)\left(1 - \dfrac{1}{u^2}\right)^{\tilde{\gamma}+(1/2)}$, where $\phi = f_0 - 2g_0\left(\dfrac{1}{(u^2-1)^{(1/2)}}\right)$. Here, the extension

$$X_4 + 2X_7$$

is obtained besides the studied subalgebra.

2.a.ii Case all generators have $a_{12} = 0$.
For the case where all generators have $a_{12} = 0$, it follows that $\zeta_2 = 0$. All remaining equations are satisfied, and no other extensions are obtained.

2.b Case $\left(\dfrac{ug'}{g}\right)' = 0$.

For this case, the general solution is $g = g_0 u^{\kappa}$, where $g_0 \neq 0$. Substituting this into the determining Eq. (27c), further analysis leads one to obtain that $a_{12} = 0$ and $\zeta_2 = 0$. These conditions also satisfy (27a). From Eq. (27b), the form of f satisfies $(\kappa+1)f - uf' = 0$. The general solution is $f = f_0 u^{\kappa+1}$, where $f_0 \neq 0$ and $\kappa \neq -1$. Moreover, this leads to $a_{22} = (\kappa - 4\gamma + 1)(a_{11} - \xi_1) + 8\xi_1$. Here, the extension

$$(\kappa + 1)X_2 + 2X_6$$

is obtained apart from the studied subalgebra.

(3) **Case** $\alpha = 1$.
The determining equations after splitting Eq. (25) with respect to y are as follows

$$(1-2\gamma)\zeta_1 f + (\zeta_1 - \zeta_2 u)uf' = 0, \tag{28a}$$

$$((1-2\gamma)a_{12}+((4-2\gamma)\xi_1 u-2\gamma a_{11}u))f-a_{12}ug$$
$$+((a_{11}-a_{22})u+a_{12}-a_{21}u^2)uf'=0, \tag{28b}$$

$$(1-2\gamma)\zeta_1\, g+(\zeta_1-\zeta_2 u)ug'=0, \tag{28c}$$

$$-a_{21}uf+g((1-2\gamma)a_{11}u+(1-2\gamma)a_{12}+(4-2\gamma)\xi_1 u-a_{22}u)$$
$$+g'u((a_{11}-a_{22})u+a_{12}-a_{21}u^2)=0. \tag{28d}$$

From Eqs. (28a) and (28c), one can study the following 2 cases: a. $fg'-gf'=0$, and b. $fg'-gf'\neq 0$.

3.a Case $fg'-gf'=0$.

For this case, we obtain the relation $g=g_0 f$ where g_0 is a constant. Analysis shows that this is a reducible case.

3.b Case $fg'-gf'\neq 0$.

It follows from Eqs. (28a) and (28c) that $\zeta_1=\zeta_2=0$. From here, one can assume that $g=\phi(u)f$ (as f is nonzero), where $\phi'\neq 0$. If it is assumed further that $\phi=\psi(u)+1/u$, then the remaining determining Eqs. (28b) and (28d) are reduced as follows:

$$(2(-\gamma a_{11}u+(2-\gamma)\xi_1 u)-(\gamma+\psi u)a_{12})f$$
$$+((a_{11}-a_{22})u+a_{12}-a_{21}u^2)uf'=0, \tag{29a}$$

$$(a_{11}-a_{22})u+a_{12}-a_{21}u^2)\psi'+a_{12}\psi^2+(a_{11}+2a_{12}u^{-1}-a_{22})\psi=0. \tag{29b}$$

These equations lead one to study the following two cases:
(1) there exists at least one generator with $a_{12}\neq 0$, and
(ii) all generators have $a_{12}=0$.

3.b.i Case there exists at least one generator with $a_{12}\neq 0$.

For this case, it follows that $\psi(u)=-\dfrac{\kappa u^2+\lambda u+\beta}{u(\beta-\psi_0 u)}$, where $\beta\neq 0, \psi_0\neq 0$, λ,κ are constant. Without loss of generality, it is assumed further that $\beta=1$. Consequently, we obtain $a_{11}=\lambda a_{12}+a_{22}$ and $a_{21}=-\kappa a_{12}$. Substituting this into determining Eq. (29a), the solution for f appears, which depends on the following three cases: A. $4\kappa-\lambda^2>0$, B. $4\kappa-\lambda^2<0$, and C. $4\kappa-\lambda^2=0$.

3.b.i.A Case $4\kappa-\lambda^2>0$.

For this case, it is assumed that $4\kappa-\lambda^2=p^2$, $p\neq 0$. The solution for f is

$$f_0\frac{(1-\psi_0 u)u^{2\gamma-1}}{(\kappa u^2+\lambda u+1)^\gamma}e^{\left(\frac{(2\lambda\gamma-4\mu)}{p}\arctan\left(\frac{\lambda+2\kappa u}{p}\right)\right)}$$

where μ is constant.

3.b.i.B Case $4\kappa-\lambda^2<0$.

For this case, it is assumed that $4\kappa-\lambda^2=-p^2$, $p\neq 0$. The solution for f is

$$f_0\frac{(1-\psi_0 u)u^{2\gamma-1}}{(\kappa u^2+\lambda u+1)^\gamma}\left(\frac{2\kappa u+\lambda-p}{2\beta\kappa u+\lambda+p}\right)^{\frac{\lambda\gamma-2\mu}{p}}$$

where μ is constant.

3.b.i.C Case $4\kappa-\lambda^2=0$.

For this case, it follows that

$$f=f_0\frac{(1-\psi_0 u)u^{2\gamma-1}}{(\kappa u^2+\lambda u+1)^\gamma}e^{\left(-\frac{4(\gamma+\mu u)}{\lambda u+2}\right)}$$

where μ is constant.

All three cases yield the same results for a_{22}, ξ_1 and the extension of the generator. If $\gamma\neq 0$, then $a_{22}=\dfrac{(2-\gamma)\xi_1-\mu a_{12}}{\gamma}$. If $\gamma=0$, then $\xi_1=\dfrac{\mu a_{12}}{2}$. The extension

$$\mu X_2+\lambda X_5+X_7-\kappa X_8$$

is obtained apart from the studied subalgebra.

3.b.ii Case all generators have $a_{12}=0$.

For this case, the determining Eq. (29) are reduced to

$$2((2-\gamma)\xi_1-\gamma a_{11})f+(a_{11}-a_{22}-a_{21}u)uf'=0, \tag{30a}$$

Table 3
Group classification of systems admitting all generator with $\xi'' = 0$. Here we have $\theta_1(u, v) = (\cos(u)f(v) + \sin(u)g(v))$, $\theta_2(u, v) = \sin(u)f(v) - \cos(u)g(v)$, $\chi_1(\alpha) = \dfrac{\alpha}{\alpha^2+1}$ and $\chi_2(\alpha) = \dfrac{1}{\alpha^2+1}$.

F	G	Relations	Extension of Kernel
$f(u)y^{(1-2\gamma)}$	$g(u)y^{(\alpha-2\gamma)}$	$u = \frac{y^\alpha}{z}$, $-1 \le \alpha \le 1$, $f'g \neq 0$	$\gamma X_2 + X_5 + \alpha X_6$
$f(u)y^{(1-2\gamma)}$	$g(u)y^{(-2\gamma)}$	$u = ye^{-z}$, $f'g \neq 0$	$\gamma X_2 + X_4 + X_5$
$e^{-2\gamma u}\theta_1(u, v)$	$-e^{-2\gamma u}\theta_2(u, v)$	$y = v\cos(u)$, $z = v\sin(u)$, $f^2 + g^2 \neq 0$	$\gamma X_2 - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u, v)$	$e^{(\alpha-2\gamma)u}\theta_2(u, v)$	$y = ve^{\alpha u}\cos(u) + \chi_1(\alpha)$ $z = ve^{\alpha u}\sin(u) - \chi_2(\alpha)$, $\alpha > 0$, $f^2 + g^2 \neq 0$	$\gamma X_2 - X_3 + \alpha(X_5 + X_6) - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u, v)$	$e^{(\alpha-2\gamma)u}\theta_2(u, v)$	$y = ve^{\alpha u}\cos(u) - \chi_1(\alpha)$ $z = ve^{\alpha u}\sin(u) + \chi_2(\alpha)$, $\alpha > 0$, $f^2 + g^2 \neq 0$	$\gamma X_2 + X_3 + \alpha(X_5 + X_6) - X_7 + X_8$
$e^{(\alpha-2\gamma)u}\theta_1(u, v)$	$e^{(\alpha-2\gamma)u}\theta_2(u, v)$	$y = ve^{\alpha u}\cos(u)$ $z = ve^{\alpha u}\sin(u)$, $\alpha > 0$, $f^2 + g^2 \neq 0$	$\gamma X_2 + \alpha(X_5 + X_6) - X_7 + X_8$
$(g(v)u + f(v))e^{(-2\gamma u)}$	$g(v)e^{(-2\gamma u)}$	$y = uv$, $z = v$, $g \neq 0$	$\gamma X_2 + X_7$
$(g(u)z + f(u))e^{(-2\gamma z)}$	$g(u)e^{(-2\gamma z)}$	$u = z^2 - 2y$, $g' \neq 0$	$\gamma X_2 + X_4 + X_7$
$((y/z)g(u) + f(u))e^{((1-2\gamma)(y/z))}$	$g(u)e^{((1-2\gamma)(y/z))}$	$u = ze^{-y/z}$, $g \neq 0$	$\gamma X_2 + X_5 + X_6 + X_7$
$f(z)e^{-2\gamma y}$	$g(z)e^{-2\gamma y}$	$f'g \neq 0$	$\gamma X_2 + X_3$

$$((4-2\gamma)\xi_1 - a_{22} + (1-2\gamma)a_{11})g - a_{21}f \\ +(a_{11} - a_{22} - a_{21}u)ug' = 0. \tag{30b}$$

Dividing Eq. (30a) by uf' and differentiating this equation with respect to u twice, this leads to the study of the following sub-cases: A. $\left(\dfrac{f}{uf'}\right)'' \neq 0$, and B. $\left(\dfrac{f}{uf'}\right)'' = 0$.

3.b.ii.A Case $\left(\dfrac{f}{uf'}\right)'' \neq 0$, If $\left(\dfrac{f}{uf'}\right)'' \neq 0$, then it follows that if $\gamma \neq 0$ then $a_{11} = \xi_1\dfrac{2-\gamma}{\gamma}$, $a_{22} = \xi_1\dfrac{2-\gamma}{\gamma}$ and $a_{21} = 0$. If $\gamma = 0$ then $\xi_1 = 0$, $a_{22} = a_{11}$ and $a_{21} = 0$. For both cases, no extensions are obtained apart from the studied subalgebra.

3.b.ii.B Case $\left(\dfrac{f}{uf'}\right)'' = 0$,

If $\left(\dfrac{f}{f'u}\right)'' = 0$, then the general solution for f is $f_0\left(\dfrac{u}{1+u}\right)^{\kappa}$, where $\kappa \neq 0$ (else it is reducible) and $f_0 \neq 0$. Substituting this into the determining Eq. (30a), one obtains that $a_{21} = 2\left(\dfrac{-\gamma a_{11} + (2-\gamma)\xi_1}{\kappa}\right)$ and $a_{22} = \dfrac{(\kappa - 2\gamma)a_{11} + (4-2\gamma)\xi_1}{\kappa}$. Substituting this into Eq. (30b), one finds that g satisfies $g'u(1+u) + (1-\kappa)g + f = 0$. The general solution of this is $g = \left(g_0 - f_0\dfrac{u}{u+1}\right)\left(\dfrac{u}{u+1}\right)^{\kappa-1}$. The extension

$$\kappa X_2 + 2(X_6 + X_8)$$

is obtained aside from the studied subalgebra.

(4) **Case** $\alpha \neq 0, \frac{1}{2}, 1$.
For the case where $\alpha \neq 0, \frac{1}{2}, 1$, the determining Eq. (25) are split with respect to y. Since $f' \neq 0$, it follows that $\zeta_2 = 0$ and $a_{21} = 0$. Notice also that since $\alpha ug' + (\alpha - 2\gamma)g = 0$ leads to a reducible case, then $\zeta_1 = 0$ and $a_{12} = 0$. Substituting these conditions, the determining Eq. (25) become

$$(\alpha a_{11} + (\alpha-1)\xi_1 - a_{22})uf' + (-2\gamma a_{11} + (4-2\gamma)\xi_1)f = 0, \tag{31a}$$

$$(\alpha a_{11} + (\alpha-1)\xi_1 - a_{22})ug' + ((\alpha-2\gamma)a_{11} + (\alpha-2\gamma+3)\xi_1 - a_{22})g = 0. \tag{31b}$$

Dividing Eq. (31a) by f (as it is nonzero) and differentiating with respect to u, it can be observed that there is a need to study the following cases: a. $\left(\dfrac{uf'}{f}\right)' \neq 0$ and b. $\left(\dfrac{uf'}{f}\right)' = 0$.

4.a **Case** $\left(\dfrac{uf'}{f}\right)' \neq 0$.

For this case, it follows from Eq. (31a) that $a_{22} = \alpha a_{11} + (\alpha-1)\xi_1$. Consequently, if $\gamma \neq 0$ then $a_{11} = \dfrac{2-\gamma}{\gamma}\xi_1$, and if $\gamma = 0$ then $\xi_1 = 0$. These conditions also satisfy Eq. (31b). No other extensions of the generator were found other than the studied subalgebra.

Table 4
Group classification of systems admitting all generator with $\xi'' = 0$. Here, $f_0, g_0, \phi_0, \phi_1, \alpha, \beta, \kappa, \mu_0, \lambda$ and γ are constant.

Subalgebra 1. $\gamma X_2 + X_5$

F	G	Relations	Additional extension of Kernel
$f_0 z^{\beta} y^{1+\tilde{\gamma}}$	$g_0 z^{\beta+1} y^{\tilde{\gamma}}$	$\tilde{\gamma} = -2\gamma \neq 0,\ \beta \neq 0,\ f_0 g_0 \neq 0$	$\beta X_5 - \tilde{\gamma} X_6$
$f_0 y^{1+\tilde{\gamma}} e^{\kappa z}$	$g_0 y^{\tilde{\gamma}} e^{\kappa z}$	$\tilde{\gamma} = -2\gamma \neq 0,\ \kappa \neq 0,\ f_0 g_0 \neq 0$	$\kappa X_5 - \tilde{\gamma} X_4$

Subalgebra 1. $\gamma X_2 + X_5 + \frac{1}{2} X_6$

F	G	Relations	Additional extension of Kernel
$\phi(u)(y - z^2)^{\tilde{\gamma}}$	$g_0 (y - z^2)^{\tilde{\gamma}}$	$\tilde{\gamma} = \dfrac{1 - 4\gamma}{2} \neq 0$	
		$\phi = f_0 (y - z^2)^{1/2} + 2 g_0 z,\ g_0 \neq 0$	$X_4 + 2X_7$
$f_0 z^{-\tilde{\kappa}} y^{\tilde{\gamma}+1}$	$g_0 z^{1-\tilde{\kappa}} y^{\tilde{\gamma}}$	$f_0 g_0 \neq 0, \tilde{\kappa} = \kappa + 1 \neq 0,\ \tilde{\gamma} = \dfrac{\tilde{\kappa} - 4\gamma}{2} \neq 0$	$\tilde{\kappa} X_2 + 2X_6$

Subalgebra 1. $\gamma X_2 + X_5 + X_6$

F	G	Relations	Additional extension of Kernel
$f_0 \dfrac{z - \alpha y}{(z^2 + \lambda yz + \kappa y^2)^{\gamma}} \psi_i(y, z)$	$-f_0 \dfrac{(\kappa y + (\lambda + \alpha) z)}{(z^2 + \lambda yz + \kappa y^2)^{\gamma}} \psi_i(y, z)$	$i = 1, 2, 3,\ \alpha \neq 0,\ f_0 \neq 0$	$\mu X_2 + \lambda X_5 + X_7 - \kappa X_8$
Here, $\psi_1(y, z) = e^{\dfrac{2\lambda\gamma - 4\mu}{p} \arctan \dfrac{\lambda z + 2\kappa y}{pz}}$ with $4\kappa - \lambda^2 = p^2,\ p \neq 0$; $\psi_2(y, z) = \left(\dfrac{2\kappa y + (\lambda + p) z}{2\kappa y + (\lambda - p) z} \right)^{\frac{2\mu - \lambda\gamma}{p}}$ with $4\kappa - \lambda^2 = -p^2,\ p \neq 0$; and $\psi_3(y, z) = e^{-\dfrac{4(\mu y + \gamma z)}{\lambda y + 2z}}$ with $4\kappa - \lambda^2 = 0$			
$f_0 \left(\dfrac{y}{y + z} \right)^{\kappa} y^{1 - 2\gamma}$	$\left(g_0 - f_0 \dfrac{y}{y + z} \right) \left(\dfrac{y}{y + z} \right)^{\kappa - 1} y^{1 - 2\gamma}$	$\gamma\ \kappa \neq 0, f_0 \neq 0$	$\kappa X_2 + 2(X_6 + X_8)$

Subalgebra 1. $\gamma X_2 + X_5 + \alpha X_6,\ -1 \leq \alpha \leq 1$

F	G	Relations	Additional extension of Kernel
$f_0 z^{-\kappa} y^{\tilde{\gamma}+1}$	$g_0 z^{1-\kappa} y^{\tilde{\gamma}}$	$\tilde{\gamma} = \alpha\kappa - 2\gamma,\ \alpha \neq 0,\ 1/2,\ 1,\ \kappa \neq 0,\ f_0 g_0 \neq 0$	$\kappa X_2 + 2X_6$

Subalgebra 2. $\gamma X_2 + X_4 + X_5$

F	G	Relations	Additional extension of Kernel
$f_0 y^{(\kappa+1)} e^{-\alpha z}$	$g_0 y^{(\kappa)} e^{-\alpha z}$	$\gamma = \dfrac{\alpha - \kappa}{2},\ f_0 g_0 \neq 0,\ \kappa\alpha \neq 0$	$\alpha X_5 + \kappa X_4$

Subalgebra 3. $-X_7 + X_8$

F	G	Relations	Additiona extension of Kernel
$(f_0 \cos(u) + g_0 \sin(u)) v^{\kappa}$	$(f_0 \sin(u) - g_0 \cos(u)) v^{\kappa}$	${f_0}^2 + {g_0}^2 \neq 0,\ \kappa \neq 1$	
		$u = \arctan(z/y),\ v^2 = y^2 + z^2$	$\dfrac{1 - \kappa}{2} X_2 + X_5 + X_6$

Subalgebra 3. $\gamma X_2 - X_7 + X_8,\ \gamma \neq 0$

F	G	Relations	Additional extension of Kernel
$e^{\tilde{\gamma} u}(f_0 \cos(u) + g_0 \sin(u)) v^{\kappa}$	$e^{\tilde{\gamma} u}(f_0 \sin(u) - g_0 \cos(u)) v^{\kappa}$	${f_0}^2 + {g_0}^2 \neq 0,\ \kappa \neq 1$	
		$\tilde{\gamma} = -2\gamma \neq 0,\ u = \arctan(z/y),\ v^2 = y^2 + z^2$	$\tilde{\gamma}(X_5 + X_6) + (1 - \kappa)(X_8 - X_7)$

Subalgebra 4. $\gamma X_2 - X_3 + \alpha(X_5 + X_6) - X_7 + X_8,\ \alpha > 0$

F	G	Relations	Additional extension of Kernel
$e^{(\alpha - 2\gamma) u}(f_0 \cos(u) + g_0 \sin(u)) v^{\kappa}$	$e^{(\alpha - 2\gamma) u}(f_0 \sin(u) - g_0 \cos(u)) v^{\kappa}$	${f_0}^2 + {g_0}^2 \neq 0,\ \kappa \neq 1$	
		$u = \arctan \left(\dfrac{z + \chi_2(\alpha)}{y - \chi_1(\alpha)} \right)$,	
		$v^2 = e^{-2\alpha u}((y - \chi_1(\alpha))^2 + (z + \chi_2(\alpha))^2)$,	
		$\chi_1(\alpha) = \dfrac{\alpha}{\alpha^2 + 1},\ \chi_2(\alpha) = \dfrac{1}{\alpha^2 + 1}$	$\dfrac{1 - \kappa}{2} X_2 + X_5 + X_6 - \chi_1 X_3 + \chi_2 X_4$

(continued on next page)

Table 4 (*continued*)

Subalgebra 4. $\gamma X_2+X_3+\alpha(X_5+X_6)-X_7+X_8,\ \alpha>0$			
F	G	Relations	Additional extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0\cos(u)+g_0\sin(u))v^{\kappa}$	$e^{(\alpha-2\gamma)u}(f_0\sin(u)-g_0\cos(u))v^{\kappa}$	$f_0{}^2+g_0{}^2\neq 0,\ \kappa\neq 1$ $u=\arctan\left(\dfrac{z-\chi_2(\alpha)}{y+\chi_1(\alpha)}\right),$ $v^2=e^{-2\alpha u}((y+\chi_1(\alpha))^2+(z-\chi_2(\alpha))^2),$ $\chi_1(\alpha)=\dfrac{\alpha}{\alpha^2+1},\ \chi_2(\alpha)=\dfrac{1}{\alpha^2+1}$	$\dfrac{1-\kappa}{2}X_2+X_5+X_6+\chi_1X_3-\chi_2X_4$
Subalgebra 4. $\gamma X_2+\alpha(X_5+X_6)-X_7+X_8,\ \alpha>0$			
F	G	Relations	Additional Extension of Kernel
$e^{(\alpha-2\gamma)u}(f_0\cos(u)+g_0\sin(u))v^{\kappa}$	$e^{(\alpha-2\gamma)u}(f_0\sin(u)-g_0\cos(u))v^{\kappa}$	$f_0{}^2+g_0{}^2\neq 0,\ \kappa\neq 1$ $u=\arctan(z/y),\ v^2=e^{-2\alpha u}(y^2+z^2)$	$\dfrac{1-\kappa}{2}X_2+X_5+X_6$
Subalgebra 5. γX_2+X_7			
F	G	Relations	Additional Extension of Kernel
$g_0z^{\beta-1}e^{-y/z}(y+\kappa\tilde{\gamma}z)$	$g_0z^{\beta}e^{-y/z}$	$\tilde{\gamma}=2\gamma,\ g_0\neq 0$	$X_5+\tilde{\gamma}X_6+(\beta-1)X_7$
Subalgebra 5. $\gamma X_2+X_4+X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0z+f_0)e^{(\beta u-2\gamma z)}$	$g_0e^{(\beta u-2\gamma z)}$	$u=z^2-2y,\ \beta\neq 0,\ g_0\neq 0$	βX_2+X_3
Subalgebra 5. X_4+X_7			
F	G	Relations	Additional Extension of Kernel
$(g_0z+f_0(\beta+u)^{1/2})(\beta+u)^{\kappa}$	$g_0(\beta+u)^{\kappa}$	$u=z^2-2y,\ \kappa=\frac{1-2\tilde{\kappa}}{2}\neq 0,\ g_0\neq 0$	$\tilde{\kappa}X_2-\beta X_3+2X_5+X_6$
Subalgebra 6. $\gamma X_2+X_5+X_6+X_7$			
F	G	Relations	Additional Extension of Kernel
$(g_0y+f_0z)z^{\kappa-1}e^{-\tilde{\gamma}(y/z)}$	$g_0z^{\kappa}e^{-\tilde{\gamma}(y/z)}$	$\tilde{\gamma}=2\gamma+\kappa-1\neq 0,\ g_0\neq 0$	$\tilde{\gamma}X_2+2X_7$
Subalgebra 7. $\gamma X_2+X_3,\ \gamma\neq 0$			
F	G	Relations	Additional Extension of Kernel
$(f_0z^{\beta-1}e^{\kappa z-\tilde{\gamma}y})(\kappa z+\tilde{\gamma}\phi_1)$	$g_0z^{\beta}e^{\kappa z-\tilde{\gamma}y}$	$\tilde{\gamma}=2\gamma,\ f_0=g_0/\tilde{\gamma}\neq 0$	$(\beta-1)X_3+\kappa X_7+\tilde{\gamma}X_6$
$(g_0e^{\beta z+\kappa z^2-\tilde{\gamma}y})\phi(z)$	$g_0e^{\beta z+\kappa z^2-\tilde{\gamma}y}$	$\tilde{\gamma}=2\gamma,\ g_0\neq 0$ $\phi=\phi_0z+\phi_1,\ \phi_0\neq 0,\ \kappa=(\tilde{\gamma}\phi_0)/2$	$\beta X_3+2\kappa X_7+\tilde{\gamma}X_4$

4.b **Case** $\left(\dfrac{uf'}{f}\right)'=0.$

For this case, the general solution for f is f_0u^{κ}, where $f_0\kappa\neq 0$. Substituting this function into the determining Eq. (31a), it follows that $a_{22}=\dfrac{(\kappa\alpha-2\gamma)a_{11}+(\kappa\alpha-\kappa-2\gamma+4)\xi_1}{\kappa}$. From (31b), extensions of the generator can only be found if g satisfies the condition $g'u-g(\kappa-1)=0$, where the general solution for g is $g_0u^{\kappa-1}$, $g_0\neq 0$. Another extension of the generator apart from the studied subalgebra is

$$\kappa X_2+2X_6.$$

The complete representative classes for the autonomous system Eq. (1) with all admitted generators having $\xi''=0$ is listed in Tables 3 and 4.

4. Conclusion

A complete group classification of the systems of two autonomous nonlinear second-order ordinary differential equations of the form $\mathbf{y}''=\mathbf{F}(\mathbf{y})$ excluding the systems which are equivalent to linear systems and the degenerate case were presented using both the direct and algebraic approach. The important thing in this study is that the analysis of the determining equations where split into two cases: 1) the case where at least one admitted generator has $\xi''\neq 0$ and 2) the case where all admitted generators have $\xi''=0$. The first was analyzed through the direct approach while the latter was analyzed using the one-dimensional optimal system of subalgebras followed by the direct approach. For the direct approach, all possible Lie algebras were found with the aid of the equivalence transformations applied on the determining equations. As for the algebraic approach, the study was reduced to the analysis of relations between constants of the generator with its corresponding basis operators. The obtained classification is summarized on Tables 2-4. It is highly likely that the same methods shown in this paper are applicable to the group classification of systems of two nonlinear second-order ordinary differential equations, which will be next goal for further studies. In addition, it is also believed that this can be extended to systems in more general cases.

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Appendix A

For $\xi''=0$, the determining equations have the form

$$\begin{cases} \zeta_1''=F_y\zeta_1+F_z\zeta_2+q_1, \\ \zeta_2''=G_y\zeta_1+G_z\zeta_2+q_2, \end{cases}$$

where q_1 and q_2 are functions of y and z. Differentiating them with respect to x, one has

$$\begin{cases} \zeta_1'''=F_y\zeta_1'+F_z\zeta_2', \\ \zeta_2'''=G_y\zeta_1'+G_z\zeta_2', \end{cases}$$

Differentiating the latter equations with respect to y and z

$$\begin{cases} F_{yy}\zeta_1'+F_{yz}\zeta_2'=0, \\ F_{yz}\zeta_1'+F_{zz}\zeta_2'=0, \end{cases}$$

$$\begin{cases} G_{yy}\zeta_1'+G_{yz}\zeta_2'=0, \\ G_{yz}\zeta_1'+G_{zz}\zeta_2'=0, \end{cases}$$

Case 1. Let $F_{zz}\neq 0$, then

$$\zeta_2'=-\frac{F_{yz}}{F_{zz}}\zeta_1',\ \ F_{yy}-\frac{F_{yz}^2}{F_{zz}}=0,\ \ G_{yy}-G_{yz}\frac{F_{yz}}{F_{zz}}=0,\ \ G_{yz}-G_{zz}\frac{F_{yz}}{F_{zz}}=0.$$

Thus,

$$\frac{F_{yz}}{F_{zz}}=k,$$

and

$$F_{yy}-kF_{yz}=0,\ \ G_{yy}-kG_{yz}=0,\ \ G_{yz}-kG_{zz}=0$$

or

$$(F_y-kF_z)_y=0,\ \ (F_y-kF_z)_z=0,\ \ (G_y-kG_z)_y=0,\ \ (G_y-kG_z)_z=0,$$

One has

$$F_y-kF_z=k_1,\ \ G_y-kG_z=k_2,$$

$$\frac{dy}{1}=\frac{dz}{-k}=\frac{dF}{k_1},\ \ F=\Phi(z+ky)+k_1y,$$

$$\frac{dy}{1}=\frac{dz}{-k}=\frac{dG}{k_2},\ \ G=\Psi(z+ky)+k_2y.$$

Changing the variables

$$\bar{y}=y,\ \ \bar{z}=z+ky,$$

the original system

$$y''=F(y,z),\quad z''=G(y,z),$$

becomes

$$y''=\Phi(\bar{z})+k_1y,\ \ \bar{z}''=(k\Phi(\bar{z})+\Psi(\bar{z}))+(kk_1+k_2)y$$

Thus one needs to study the equations

$$y''=k_1y+F(z),\ \ z''=k_2y+G(z),\ \ k_2F''\neq 0.$$

$$\begin{cases} \zeta_1'' = k_1\zeta_1 + F'\zeta_2 + q_1, \\ \zeta_2'' = k_2\zeta_1 + G'\zeta_2 + q_2, \end{cases}$$

Notice that because of $F'' \neq 0$, then $\zeta_2' = 0$ from the second equation

$$0 = k_2\zeta_1' \Rightarrow \zeta_1' = 0.$$

Case 2. Let $F_{zz} = 0$, then by symmetry $G_{yy} = 0$. Hence

$$F_{yy}\zeta_1' + F_{yz}\zeta_2' = 0, \quad F_{yz}\zeta_1' = 0, \quad G_{yz}\zeta_2' = 0, \quad G_{yz}\zeta_1' + G_{zz}\zeta_2' = 0,$$

If

$$F_{yz} \neq 0 \Rightarrow \zeta_1' = 0, \quad \zeta_2' = 0.$$

Hence,

$$F_{yz} = 0, \quad F_{zz} = 0, \quad G_{yy} = 0, \quad G_{yz} = 0$$

and

$$F_{yy}\zeta_1' = 0, \quad G_{zz}\zeta_2' = 0.$$

Thus

$$y'' = k_1 z + F(y), \quad z'' = k_2 y + G(z), \quad k_1 k_2 (F''^2 + G''^2) \neq 0.$$

$$\begin{cases} \zeta_1'' = F'\zeta_1 + k_1\zeta_2 + q_1, \\ \zeta_2'' = k_2\zeta_1 + G'\zeta_2 + q_2, \end{cases}$$

Let $F'' \neq 0$, then $\zeta_1' = 0$ and because

$$\zeta_1''' = F'\zeta_1' + k_1\zeta_2' \Rightarrow k_1\zeta_2' = 0 \Rightarrow \zeta_2' = 0$$

Let $G'' \neq 0$, then $\zeta_2' = 0$ and because

$$\zeta_2''' = k_2\zeta_1' + G'\zeta_2' \Rightarrow k_2\zeta_1' = 0 \Rightarrow \zeta_1' = 0$$

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