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Leonard pairs and quantum algebra $U_q(sl_2)$

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ABSTRACT

Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. A *Leonard pair* on V is an ordered pair of linear transformations in $\mathrm{End}(V)$ such that for each of these transformations there exists a basis for V with respect to which the matrix representing that transformation is diagonal and the matrix representing the other transformation is irreducible tridiagonal. Fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity. Consider the quantum algebra $U_q(sl_2)$ with equitable generators $x^{\pm 1}$, y, z. Let d denote a nonnegative integer and let $V_{d,1}$ denote an irreducible $U_q(sl_2)$-module of dimension $d+1$ and of type 1. In this paper, we determine all linear transformations A in $\mathrm{End}(V_{d,1})$ such that on $V_{d,1}$, the pair A, x^{-1}, the pair A, y and the pair A, z are all Leonard pairs.

1. Introduction

Leonard pairs were introduced by Terwilliger [10] to extend the algebraic approach of Bannai and Ito [4] to a result of D. Leonard concerning the sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal

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polynomials. Because these polynomials frequently arise in connection with the finite-dimensional representations of good Lie algebras and quantum groups, it is natural to find Leonard pairs associated with these algebraic objects. Leonard pairs of Krawtchouk type have been described in [8,11] using split basis and normalized semisimple generators of sl_2. Leonard pairs of q-Krawtchouk type have been described in [11] using split basis of $U_q(sl_2)$. Recently, Alnajjar and Curtin [1] gave general construction of Leonard pairs of Racah, Hahn, dual Hahn and Krawtchouk type using equitable basis of sl_2. Alnajjar [2,3] gave general construction of Leonard pairs of q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, and affine q-Krawtchouk type using equitable generators of $U_q(sl_2)$. Equitable presentations for sl_2 and $U_q(sl_2)$ were introduced in [5] and [7], respectively.

In this paper we describe a relationship between Leonard pairs and quantum algebra $U_q(sl_2)$ that appears to be new. Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity. Consider the quantum algebra $U_q(sl_2)$ with equitable generators $x^{\pm 1}$, y, z. Let d denote a nonnegative integer and let $V_{d,1}$ denote an irreducible $U_q(sl_2)$-module of dimension $d+1$ and of type 1. We determine all linear transformations A in $\mathrm{End}(V_{d,1})$ such that on $V_{d,1}$, the pair A, x^{-1}, the pair A, y and the pair A, z are all Leonard pairs.

2. Preliminaries

In this section we recall the definitions and some related facts concerning Leonard pairs and the quantum algebra $U_q(sl_2)$.

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field of characteristic zero.

2.1. Leonard pairs

In this subsection we recall some terms of Leonard pairs.

Let d be a nonnegative integer. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of the column vectors of length $d+1$, and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of the $(d+1) \times (d+1)$ matrices. The algebra $\mathrm{Mat}_{d+1}(\mathbb{K})$ acts on $\mathbb{K}^{d+1}$ by left multiplication.

Let V denote a $\mathbb{K}$-vector space of dimension $d+1$. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. Let $\{v_i\}_{i=0}^d$ denote a basis for V. For $A \in \mathrm{End}(V)$ and $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say X *represents* A *with respect to* $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d X_{ij} v_i$ for $0 \leq j \leq d$.

Let X be a square matrix. X is said to be *upper* (resp. *lower*) *bidiagonal* whenever every nonzero entry appears on or immediately above (resp. below) the main diagonal. X is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero.

Definition 2.1. [10, Definition 1.1] By a *Leonard pair* on V, we mean an ordered pair $A : V \to V$ and $A^* : V \to V$ that satisfy both (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal, and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal, and the matrix representing A is diagonal.

Let A, A^* be a Leonard pair on V. Obviously, A^*, A is also a Leonard pair on V.

Let A, A^* be a Leonard pair on V. By [10, Lemma 1.3] each of A, A^* has mutually distinct $d + 1$ eigenvalues. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A. For $0 \leq i \leq d$ pick an eigenvector $v_i \in V$ of A associated with θ_i. Then the ordering $\{\theta_i\}_{i=0}^d$ is said to be *standard* whenever the basis $\{v_i\}_{i=0}^d$ satisfies Definition 2.1(ii). A standard ordering of A^* is similarly defined. Note that for a standard ordering $\{\theta_i\}_{i=0}^d$ of the eigenvalues of A, the ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result applies to A^*. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) be a standard ordering of the eigenvalues of A (resp. A^*). By [10, Theorem 1.9] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{1}$$

are equal and independent of i for $2 \leq i \leq d - 1$. Let β be one less the common value of (1). We call β the *fundamental parameter* of A, A^*. Let q be a nonzero scalar such that $\beta = q^2 + q^{-2}$. We call q a *quantum parameter* of A, A^* [9].

Lemma 2.2. *Let A, A^* be a Leonard pair on V with quantum parameter q that is not a root of unity. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) be a standard ordering of the eigenvalues of A (resp. A^*). Then there exist scalars α, α^*, a, a', b, b' such that*

$$\theta_i = \alpha + aq^{2i-d} + a'q^{d-2i} \qquad (0 \leq i \leq d), \tag{2}$$

$$\theta^*_i = \alpha^* + bq^{2i-d} + b'q^{d-2i} \qquad (0 \leq i \leq d). \tag{3}$$

Proof. Immediate from [10, Lemma 9.2]. □

Definition 2.3. By a *Leonard pair* in $\mathrm{Mat}_{d+1}(\mathbb{K})$, we mean an ordered pair A, A^* in $\mathrm{Mat}_{d+1}(\mathbb{K})$ that acts on $\mathbb{K}^{d+1}$ as a Leonard pair.

Definition 2.4. [9, Definition 1.4] An ordered pair of matrices A, A^* in $\mathrm{Mat}_{d+1}(\mathbb{K})$ is said to be *LB-TD* whenever A is lower bidiagonal with subdiagonal entries all 1 and A^* is irreducible tridiagonal.

Definition 2.5. [9, Definition 1.5] A Leonard pair on V is said to have *LB-TD form* whenever there exists a basis for V with respect to which the matrices representing A, A^* form an LB-TD pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

Consider the following LB-TD pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$:

$$A = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & \theta_{d-1} & \\ 0 & & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} x_0 & y_1 & & & & \\ z_1 & x_1 & y_2 & & & \\ & z_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x_{d-1} & y_d \\ & & & & z_d & x_d \end{pmatrix}. \tag{4}$$

Lemma 2.6. *[9, Proposition 1.7] Fix a nonzero scalar q that is not a root of unity. Let α, α^*, a, a', b, b', c be scalars with $c \neq 0$. Define scalars $\{\theta_i\}_{i=0}^d$, $\{x_i\}_{i=0}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ by*

$$\theta_i = \alpha + aq^{2i-d} + a'q^{d-2i}, \tag{5}$$

$$x_i = \alpha^* + (b+b')q^{d-2i} + a'cq^{d-2i}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1}), \tag{6}$$

$$y_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(b - a'cq^{d-2i+1})(b' - a'cq^{d-2i+1})c^{-1}, \tag{7}$$

$$z_i = -cq^{d-2i+1}. \tag{8}$$

Then the matrices A, A^* in (4) form an LB-TD Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ if and only if the scalars a, a', b, b', c satisfy the following inequalities:

$$a \notin \{a'q^{2d-2}, a'q^{2d-4}, \ldots, a'q^{2-2d}\}, \tag{9}$$

$$b \notin \{b'q^{2d-2}, b'q^{2d-4}, \ldots, b'q^{2-2d}\}, \tag{10}$$

$$bc^{-1}, b'c^{-1} \notin \{aq^{d-1}, aq^{d-3}, \ldots, aq^{1-d}\} \cup \{a'q^{d-1}, a'q^{d-3}, \ldots, a'q^{1-d}\}. \tag{11}$$

Lemma 2.7. *[9, Theorem 1.10] Consider sequences of scalars $\{\theta_i\}_{i=0}^d$, $\{x_i\}_{i=0}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ such that $y_i z_i \neq 0$ for $1 \le i \le d$, and consider the matrices A, A^* in (4). Assume that A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ with quantum parameter q that is not a root of unity. Then, after replacing q with q^{-1} if necessary, there exist scalars α, α^*, a, a', b, b', c with $c \neq 0$ that satisfy (5)–(11).*

Definition 2.8. [13, Definition 22.1] By a *parameter array* over $\mathbb{K}$ of diameter d, we mean a sequence of scalars

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$$

taken from $\mathbb{K}$ that satisfies the following conditions:

(PA1) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leq i, j \leq d)$.
(PA2) $\varphi_i \neq 0$, $\phi_i \neq 0$ $(1 \leq i \leq d)$.
(PA3) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$ $(1 \leq i \leq d)$.
(PA4) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0)$ $(1 \leq i \leq d)$.
(PA5) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Lemma 2.9. *[10, Corollary 14.2] Let B and B^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$ of the form*

$$B = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & 0 \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ 0 & & & & \theta_d^* \end{pmatrix}.$$

Then the following are equivalent:

(i) *B, B^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.*
(ii) *There exist scalars $\phi_1, \phi_2, \cdots, \phi_d$ in $\mathbb{K}$ such that the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ is a parameter array.*

Lemma 2.10. *[12, Theorem 1.5] Let A, A^* denote a Leonard pair on V. Then there exists a sequence of scalars β, γ, γ^*, ϱ, ϱ^*, ω, η, η^* taken from $\mathbb{K}$ such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I, \tag{12}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I. \tag{13}$$

The sequence is uniquely determined by the pair A, A^* provided the dimension of V is at least 4.

We refer to (12) and (13) as the *Askey–Wilson relations*. We call the sequence β, γ, γ^*, ϱ, ϱ^*, ω, η, η^* satisfying (12) and (13) an *Askey–Wilson parameter sequence* for A, A^*.

Lemma 2.11. *[9, Lemma 5.3] With reference to Lemma 2.2, assume $\alpha = 0$ and $\alpha^* = 0$. Then there exists some scalar ξ in $\mathbb{K}$ such that the Askey–Wilson parameter sequence for A, A^* is as follows.*

$$\gamma = 0, \qquad \gamma^* = 0,$$
$$\varrho = -aa'(q^2 - q^{-2})^2, \qquad \varrho^* = -bb'(q^2 - q^{-2})^2,$$
$$\omega = (q - q^{-1})^2((q^{d+1} + q^{-d-1})\xi - (a + a')(b + b')),$$
$$\eta = -(q - q^{-1})(q^2 - q^{-2})((a + a')\xi - aa'(b + b')(q^{d+1} + q^{-d-1})),$$
$$\eta^* = -(q - q^{-1})(q^2 - q^{-2})((b + b')\xi - bb'(a + a')(q^{d+1} + q^{-d-1})).$$

2.2. Quantum algebra $U_q(sl_2)$

For the rest of this paper fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity. In this subsection we recall some facts concerning irreducible finite-dimensional $U_q(sl_2)$-modules.

Definition 2.12. [6] The quantum algebra $U_q(sl_2)$ is the $\mathbb{K}$-algebra with generators e, f, $k^{\pm 1}$ satisfying the following conditions:

$$kk^{-1} = k^{-1}k = 1,\ ke = q^2 ek,\ kf = q^{-2} fk,\ ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

We call e, f, $k^{\pm 1}$ the *Chevalley generators* for $U_q(sl_2)$.

Lemma 2.13. *[7, Theorem 2.1] The quantum algebra $U_q(sl_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $x^{\pm 1}$, y, z and the relations $xx^{-1} = 1$, $x^{-1}x = 1$,*

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We call $x^{\pm 1}$, y, z the *equitable generators* for $U_q(sl_2)$.

Lemma 2.14. *[7, Lemma 4.2] For each $\epsilon \in \{1, -1\}$, there is an irreducible finite-dimensional $U_q(sl_2)$-module $V_{d,\epsilon}$ with basis $v_0, v_1, \cdots, v_d$ and action*

$$\epsilon x v_i = q^{d-2i} v_i \quad (0 \le i \le d),$$
$$\epsilon y v_i = q^{2i-d} v_i + (q^{-d} - q^{2i+2-d}) v_{i+1} \quad (0 \le i \le d-1),$$
$$\epsilon y v_d = q^d v_d,$$
$$\epsilon z v_i = (q^d - q^{2i-2-d}) v_{i-1} + q^{2i-d} v_i \quad (1 \le i \le d),$$
$$\epsilon z v_0 = q^{-d} v_0.$$

We call ϵ the *type* of the module. Since the module $V_{d,-1}$ can be treated similar to module $V_{d,1}$, we shall prove our results only for the module $V_{d,1}$.

Definition 2.15. Let $v_0, v_1, \cdots, v_d$ be the basis for $V_{d,1}$ from Lemma 2.14. Denote the basis $v_0, v_1, \cdots, v_d$ by $[x]_{row}$.

Definition 2.16. [14, Definition 10.2] Let K_q denote the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ with (i,i)-entry q^{d-2i} for $0 \leq i \leq d$.

Definition 2.17. [14, Definition 10.7] Let E_q denote the upper bidiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ with (i,i)-entry q^{2i-d} for $0 \leq i \leq d$ and $(i-1,i)$-entry $q^d - q^{2i-2-d}$ for $1 \leq i \leq d$.

Definition 2.18. [14, Definition 10.4] We define a matrix $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$ as follows. For $0 \leq i, j \leq d$ the (i,j)-entry is $\delta_{i+j,d}$.

Lemma 2.19. *Consider the elements x, y, z of $U_q(sl_2)$. Then the matrices that represent these elements with respect to $[x]_{row}$ are K_q, $SE_{q^{-1}}S$, E_q, respectively, where K_q is from Definition 2.16, E_q is from Definition 2.17 and S is from Definition 2.18.*

Proof. Immediate from Lemma 2.14. □

3. Main result

In this section we give the main result in this paper. The main result is Theorem 3.10. We start with the following assumption.

Assumption 3.1. Let $V_{d,1}$ be an irreducible module of $U_q(sl_2)$ of dimension $d+1$. Let $[x]_{row}$ be the basis from Definition 2.15. Recall that with respect to $[x]_{row}$, the matrices respecting the actions of x, y, z on $V_{d,1}$ are K_q, $SE_{q^{-1}}S$, E_q, respectively. Let A be a linear transformation on $V_{d,1}$, and let B denote the matrix representing A with respect to $[x]_{row}$. Assume that on $V_{d,1}$ the pair A, x^{-1}, the pair A, y and the pair A, z are all Leonard pairs.

Note 3.2. With reference to Assumption 3.1, for any scalars ζ, ξ in $\mathbb{K}$ with $\zeta \neq 0$, on $V_{d,1}$ the pair $\zeta A + \xi I, x^{-1}$, the pair $\zeta A + \xi I, y$ and the pair $\zeta A + \xi I, z$ are all Leonard pairs. Here I denotes the identity.

Lemma 3.3. *With reference to Assumption 3.1, the matrix B is of irreducible tridiagonal. Moreover, the eigenvalues of A are of the form*

$$\alpha^* + bq^{2i-d} + b'q^{d-2i} \qquad (0 \leq i \leq d),$$

where α^*, b, b' are some scalars in $\mathbb{K}$.

Proof. Consider the Leonard pair x^{-1}, A on $V_{d,1}$. Recall that K_q^{-1} is the matrix representing the action of x^{-1} on $V_{d,1}$ with respect to the basis $[x]_{row}$. Denote the diagonal entries $(K_q^{-1})_{i,i}$ of K_q^{-1} by θ_i for $0 \leq i \leq d$. Note that K_q^{-1} is a diagonal matrix and its diagonal entries θ_i are of the form

$$\theta_i = q^{2i-d} \qquad (0 \leq i \leq d). \tag{14}$$

Comparing (2) and (14), we find $\{\theta_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of the action of x^{-1} on $V_{d,1}$. By this and the comments below Definition 2.1, the basis $[x]_{row}$ satisfies Definition 2.1(ii). So, the matrix B is of irreducible tridiagonal. By Lemma 2.2, there exist some scalars α^*, b, b' such that the ordering of the eigenvalues of A is the same as in (3), and hence the results hold. $\square$

Definition 3.4. With reference to Assumption 3.1 and Lemma 3.3, write

$$B = \begin{pmatrix} x_0 & y_1 & & & & \\ z_1 & x_1 & y_2 & & & \\ & z_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & y_d \\ & & & & z_d & x_d \end{pmatrix},$$

where x_i, y_i, z_i are some scalars in $\mathbb{K}$ with $y_i z_i \neq 0$ for $1 \leq i \leq d$.

Lemma 3.5. *With reference to Assumption 3.1 and Definition 3.4, the Leonard pairs y, A and z, A on $V_{d,1}$ have LB-TD form. Moreover, there exist some scalars b, b', c, α^* in $\mathbb{K}$ such that the following hold.*

$$x_i = \alpha^* + (b + b')q^{d-2i} \qquad (0 \leq i \leq d), \tag{15}$$

$$y_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(q^{-d} - q^{2i-d})^{-1} bb'c^{-1} \qquad (1 \leq i \leq d), \tag{16}$$

$$z_i = -c(q^{-2i+1} - q) \qquad (1 \leq i \leq d), \tag{17}$$

where, b, b', c satisfy the following conditions.

$$b \notin \{b'q^{2d-2}, b'q^{2d-4}, \ldots, b'q^{2-2d}\}, \tag{18}$$

$$bc^{-1}, b'c^{-1} \notin \{q^{d-1}, q^{d-3}, \ldots, q^{1-d}\}. \tag{19}$$

Proof. We first prove the Leonard pair y, A on $V_{d,1}$ has LB-TD form. Recall that $v_0, v_1, \cdots, v_d$ is the basis $[x]_{row}$ for $V_{d,1}$ from Definition 2.15, with respect to which the matrix representing the action of y is $SE_{q^{-1}}S$ and the matrix representing A is B. Set $\alpha_i = k_0 k_1 \cdots k_i$ for $0 \leq i \leq d$, where $k_0 = 1$, $k_i = q^{-d} - q^{2i-d}$ $(1 \leq i \leq d)$. Then with respect to the basis $\alpha_0 v_0, \alpha_1 v_1, \cdots, \alpha_d v_d$, the matrices representing the actions of y and A are

$$\begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \theta_d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_0 & k_1 y_1 & & & & \\ z_1/k_1 & x_1 & k_2 y_2 & & & \\ & z_2/k_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & k_d y_d \\ & & & & z_d/k_d & x_d \end{pmatrix}, \tag{20}$$

respectively, where $\theta_i = q^{2i-d}$ $(0 \le i \le d)$. By the above arguments and Definition 2.4, we find the matrices in (20) form a LB-TD Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$, and hence the Leonard pair y, A on $V_{d,1}$ has LB-TD form.

Next, we show the entries in the matrix B are as in (15)–(17). By Lemma 2.6 and since $\theta_i = q^{2i-d}$ $(0 \le i \le d)$ and the matrices in (20) form a LB-TD Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$, there exist some scalars α^*, b, b', c with $c \neq 0$ such that

$$x_i = \alpha^* + (b + b')q^{d-2i} \qquad (0 \le i \le d), \tag{21}$$

$$k_i y_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})bb'c^{-1} \qquad (1 \le i \le d), \tag{22}$$

$$z_i/k_i = -cq^{d-2i+1} \qquad (1 \le i \le d), \tag{23}$$

where the scalars b, b', c satisfy the conditions (18), (19). Obviously, (15)–(17) follow from (21)–(23).

Finally, we prove the Leonard pair z, A on $V_{d,1}$ has LB-TD form. Recall that E_q is the matrix representing the action of z with respect to the basis $v_0, v_1, \cdots, v_d$. Set $\beta_i = l_0 l_1 \cdots l_i$ for $0 \le i \le d$, where $l_0 = 1$, $l_i = q^d - q^{d-2i}$ $(1 \le i \le d)$. Then with respect to the basis $\beta_0 v_d, \beta_1 v_{d-1}, \cdots, \beta_d v_0$, the matrices representing the actions of z and A are

$$\begin{pmatrix} \theta_d & & & & 0 \\ 1 & \theta_{d-1} & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \theta_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_d & l_1 z_d & & & & \\ y_d/l_1 & x_{d-1} & l_2 z_{d-1} & & & \\ & y_{d-1}/l_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & l_d z_1 \\ & & & & y_1/l_d & x_0 \end{pmatrix}, \tag{24}$$

respectively, where $\theta_{d-i} = q^{d-2i}$ $(0 \le i \le d)$. By the above arguments and Definition 2.4, we find the matrices in (24) form a LB-TD Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$, and hence the Leonard pair z, A on $V_{d,1}$ has LB-TD form. $\square$

Definition 3.6. Define the matrix $C \in \mathrm{Mat}_{d+1}(\mathbb{K})$ to be

$$C = \begin{pmatrix} x_0 & y_1 & & & & \\ z_1 & x_1 & y_2 & & & \\ & z_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & y_d \\ & & & & z_d & x_d \end{pmatrix},$$

where

$$x_i = (b + b')q^{d-2i} \qquad (0 \le i \le d), \tag{25}$$

$$y_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(q^{-d} - q^{2i-d})^{-1} bb'c^{-1} \qquad (1 \le i \le d), \tag{26}$$

$$z_i = -c(q^{-2i+1} - q) \qquad (1 \le i \le d), \tag{27}$$

and b, b', c are some scalars in $\mathbb{K}$ satisfying the following conditions.

$$b \notin \{b'q^{2d-2}, b'q^{2d-4}, \dots, b'q^{2-2d}\}, \tag{28}$$

$$bc^{-1}, b'c^{-1} \notin \{q^{d-1}, q^{d-3}, \dots, q^{1-d}\}. \tag{29}$$

Proposition 3.7. *With reference to Definitions 2.17, 2.18 and 3.6, the pair of matrix $SE_{q^{-1}}S, C$ forms a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.*

Proof. Define the matrix $P_1 \in \mathrm{Mat}_{d+1}(\mathbb{K})$ to be

$$P_1 = \begin{pmatrix} k_0 & & & & 0 \\ & k_0k_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & k_0k_1\cdots k_d \end{pmatrix},$$

where $k_0 = 1$, $k_i = q^{-d} - q^{2i-d}$ $(1 \le i \le d)$. Then the matrices $P_1^{-1}SE_{q^{-1}}SP_1$ and $P_1^{-1}CP_1$ are of the form

$$\begin{pmatrix} \theta_0 & & & & & 0 \\ 1 & \theta_1 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_d \end{pmatrix}, \quad \begin{pmatrix} x_0 & k_1y_1 & & & & \\ z_1/k_1 & x_1 & k_2y_2 & & & \\ & z_2/k_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & k_dy_d \\ & & & & z_d/k_d & x_d \end{pmatrix},$$

respectively, where $\theta_i = q^{2i-d}$ $(0 \le i \le d)$. Moreover, using (25)–(27), we find the entries of matrix $P_1^{-1}CP_1$ are as follows.

$$\begin{aligned} x_i &= (b+b')q^{d-2i} \qquad (0 \le i \le d), \\ k_i y_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})bb'c^{-1} \qquad (1 \le i \le d), \\ z_i/k_i &= -cq^{d-2i+1} \qquad (1 \le i \le d). \end{aligned}$$

Then by (28), (29) and Lemma 2.6, the LB-TD pair $P_1^{-1}SE_{q^{-1}}SP_1, P_1^{-1}CP_1$ forms a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$, and hence $SE_{q^{-1}}S, C$ is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. □

Proposition 3.8. *With reference to Definitions 2.17 and 3.6, the pair of matrix* E_q, C *forms a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$.

Proof. Define the matrix P_2 in $\mathrm{Mat}_{d+1}(\mathbb{K})$ to be

$$P_2 = \begin{pmatrix} l_0 & & & & & 0 \\ & l_0l_1 & & & & \\ & & l_0l_1l_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & l_0l_1\cdots l_d \end{pmatrix},$$

where $l_0 = 1$, $l_i = q^d - q^{d-2i}$ $(1 \le i \le d)$. Then the matrices $P_2^{-1}SE_qSP_2$ and $P_2^{-1}SCSP_2$ are, respectively, of the form

$$\begin{pmatrix} \theta_d & & & & & 0 \\ 1 & \theta_{d-1} & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} x_d & l_1z_d & & & & \\ y_d/l_1 & x_{d-1} & l_2z_{d-1} & & & \\ & y_{d-1}/l_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & l_dz_1 \\ & & & & y_1/l_d & x_0 \end{pmatrix},$$

where $\theta_{d-i} = q^{d-2i}$ $(0 \le i \le d)$. Moreover, using (25)–(27), we find the entries of matrix $P_2^{-1}SCSP_2$ are as follows.

$$x_{d-i} = (b+b')q^{2i-d} \qquad (0 \le i \le d), \tag{30}$$

$$z_{d-i+1}l_i = -c(q^{-d+i-1} - q^{d-i+1})(q^i - q^{-i}) \qquad (1 \le i \le d), \tag{31}$$

$$y_{d-i+1}/l_i = -q^{-d+2i-1}bb'c^{-1} \qquad (1 \le i \le d). \tag{32}$$

Replace q with q^{-1} and $bb'c^{-1}$ with c_1 in (30)–(32). Then $\theta_{d-i} = q^{2i-d}$ $(0 \le i \le d)$ and (30)–(32) become

$$x_{d-i} = (b + b')q^{d-2i} \qquad (0 \le i \le d),$$
$$z_{d-i+1}l_i = -(q^{-d+i-1} - q^{d-i+1})(q^i - q^{-i})bb'c_1^{-1} \qquad (1 \le i \le d),$$
$$y_{d-i+1}/l_i = -c_1 q^{d-2i+1} \qquad (1 \le i \le d),$$

respectively. Note that $bc_1^{-1} = (b'c^{-1})^{-1}$ and $b'c_1^{-1} = (bc^{-1})^{-1}$. Then by (28), (29) and Lemma 2.6, the LB-TD pair $P_2^{-1}SE_qSP_2, P_2^{-1}SCSP_2$ forms a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$, and hence E_q, C is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. $\square$

Proposition 3.9. *With reference to Definitions 2.16 and 3.6, the pair of matrix K_q^{-1}, C forms a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.*

Proof. Define a lower triangular matrix P in $\mathrm{Mat}_{d+1}(\mathbb{K})$ as follows:

$$P_{i,0} = q^{(2-i)(d-1)}b'^{\,2-i}c^{i-2}$$

for $0 \le i \le d$ and

$$P_{i,j} = q^{(2-i)(d-1)}b'^{\,2-i}c^{i-2}\prod_{k=1}^{j}(q^{2i-d} - q^{2k-2-d})$$

for $1 \le j \le i \le d$, where b', c are from Definition 3.6. Then the matrices $P^{-1}K_q^{-1}P$ and $P^{-1}CP$ are of the form

$$\begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta_0^* & \varphi_1 & & & 0 \\ & \theta_1^* & \varphi_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varphi_d \\ 0 & & & & \theta_d^* \end{pmatrix},$$

respectively, where $\theta_i = q^{2i-d}$, $\theta_i^* = bq^{2i-d} + b'q^{d-2i}$ for $0 \le i \le d$ and $\varphi_i = (q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})bq^{2i-d-1}$ for $1 \le i \le d$. Let ϕ_i be the scalar $(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})b'q^{d-2i+1}$ for $1 \le i \le d$. By calculation, we find the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ satisfies Conditions (PA1)–(PA5) in Definition 2.8, and hence it is a parameter array. So the pair of matrix $P^{-1}K_q^{-1}P, P^{-1}CP$ forms a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ by Lemma 2.9, which implies that the pair of matrix K_q^{-1}, C also forms a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. $\square$

Theorem 3.10. *Recall $V_{d,1}$ is an irreducible $U_q(sl_2)$-module of dimension $d+1$ and $[x]_{row}$ is the basis for $V_{d,1}$ from Definition 2.15. Let $A \in \mathrm{End}(V_{d,1})$. Then on $V_{d,1}$ the pair A, x^{-1}, the pair A, y and the pair A, z are all Leonard pairs if and only if with respect to $[x]_{row}$ the matrix representing A is of the form*

$$\begin{pmatrix} x_0 & y_1 & & & & \\ z_1 & x_1 & y_2 & & & \\ & z_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & y_d \\ & & & & z_d & x_d \end{pmatrix},$$

where

$$x_i = \alpha^* + (b + b')q^{d-2i} \qquad (0 \le i \le d),$$
$$y_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(q^{-d} - q^{2i-d})^{-1}bb'c^{-1} \qquad (1 \le i \le d),$$
$$z_i = -c(q^{-2i+1} - q) \qquad (1 \le i \le d),$$

for some scalars α^*, b, b' and nonzero scalar c in $\mathbb{K}$, which satisfy the following conditions

$$b \notin \{b'q^{2d-2}, b'q^{2d-4}, \ldots, b'q^{2-2d}\},$$
$$bc^{-1}, b'c^{-1} \notin \{q^{d-1}, q^{d-3}, \ldots, q^{1-d}\}.$$

Proof. The only if part is from Lemma 3.5 and the if part is from Propositions 3.7, 3.8, 3.9 and Note 3.2. □

We finish this paper with some comments.

Lemma 3.11. *With reference to Assumption 3.1, on $V_{d,1}$, the pair A, x is not a Leonard pair.*

Proof. Assume that on $V_{d,1}$, the pair A, x is a Leonard pair for a contradiction. Then by Theorem 3.10, the matrix representing A with respect to the basis $[x]_{row}$ is of the form in Theorem 3.10. By Note 3.2, we assume that $\alpha^* = 0$. Namely, C is the matrix representing A with respect to the basis $[x]_{row}$. Recall that K_q is the matrix representing the action of x on $V_{d,1}$ with respect to the basis $[x]_{row}$. Note that $\{bq^{2i-d} + b'q^{d-2i}\}_{i=0}^{d}$ (resp. $\{q^{d-2i}\}_{i=0}^{d}$) is a standard ordering of the eigenvalues of A (resp. the action of x). Then by Lemmas 2.10, 2.11, we obtain

$$K_q^2C - (q^2 + q^{-2})K_qCK_q + CK_q^2 = \omega K_q + \eta I, \tag{33}$$
$$C^2K_q - (q^2 + q^{-2})CK_qC + K_qC^2 - \varrho^* K_q = \omega C + \eta^* I, \tag{34}$$

where

$$\varrho^* = -bb'(q^2 - q^{-2})^2, \tag{35}$$
$$\omega = (q - q^{-1})^2((q^{d+1} + q^{-d-1})\xi - (b + b')), \tag{36}$$

$$\eta = -(q - q^{-1})(q^2 - q^{-2})\xi, \tag{37}$$

$$\eta^* = -(q - q^{-1})(q^2 - q^{-2})((b + b')\xi - bb'(q^{d+1} + q^{-d-1})) \tag{38}$$

and ξ is some scalar in $\mathbb{K}$.

For $1 \leq i \leq d$, comparing the $(i-1, i)$-entry of both sides of (34) and simplifying the result by using (36), we obtain

$$\begin{aligned}
&(q - q^{-1})^2(q^{d+1} + q^{-d-1})\xi \\
&= (b + b')(q^{2d-4i+4} - q^{2d-4i+6} + q^{2d-4i} - q^{2d-4i-2} + q^2 + q^{-2} - 2),
\end{aligned}$$

which forces $b + b' = 0$ and $\xi = 0$. So $\omega = 0$ by (36) and $x_i = 0$ for $0 \leq i \leq d$ by (25).

For $1 \leq i \leq d$, comparing the (i, i)-entry of both sides of (34) by using the results $\omega = 0$ and $x_i = 0$ for $0 \leq i \leq d$, we obtain

$$z_i y_i (2q^{d-2i} - (q^2 + q^{-2})q^{d-2i+2}) + y_{i+1}z_{i+1}(2q^{d-2i} - (q^2 + q^{-2})q^{d-2i-2}) = \varrho^* q^{d-2i} + \eta^*. \tag{39}$$

Substituting y_i, z_i, ϱ^*, η^* in (39) using (26), (27), (35) and (38), respectively, and simplifying the result, we obtain

$$\begin{aligned}
&bb'(q - q^{-1})(q^2 - q^{-2})(q^{d+1} + q^{-d-1}) \\
&= -bb'q^{d-2i}(q^{2d-4i+6} - q^{2d-4i+2} - q^{2d-4i-2} + q^{2d-4i-6} - q^{2d-2i+6} + q^{2d-2i+2} \\
&\quad + q^{2d-2i} - q^{2d-2i-4} - q^{-2i+4} + q^{-2i} + q^{-2i-2} - q^{-2i-6}),
\end{aligned}$$

which forces $bb' = 0$. By the above arguments, we find that $b = b' = 0$, which contradicts to the assumption that A, x is a Leonard pair on $V_{d,1}$. □

Lemma 3.12. *With reference to Assumption 3.1 and Theorem 3.10, on $V_{d,1}$,*

$$A = \alpha^* 1 + (b + b')x + \frac{c}{q - q^{-1}}[x, y] + \frac{bb'c^{-1}}{q - q^{-1}}[z, x]. \tag{40}$$

Proof. Let $[x]_{row}$ be the basis for $V_{d,1}$ from Definition 2.15. Compare the matrix representing the action of each side of (40) with respect to the basis $[x]_{row}$. We find both sides are equal to the matrix $\alpha^* I + C$, where I denotes the identity and C is from Definition 3.6. □

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