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Fractional Calculus on Fractal Interpolation for a Sequence of Data with Countable Iterated Function System

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Abstract. In recent years, the concept of fractal analysis is the best nonlinear tool towards understanding the complexities in nature. Especially, fractal interpolation has flexibility for approximation of nonlinear data obtained from the engineering and scientific experiments. Random fractals and attractors of some iterated function systems are more appropriate examples of the continuous *everywhere* and *nowhere* differentiable (highly irregular) functions, hence fractional calculus is a mathematical operator which best suits for analyzing such a function. The present study deals the existence of fractal interpolation function (FIF) for a sequence of data $\{(x_n, y_n) : n \geq 2\}$ with countable iterated function system, where x_n is a monotone and bounded sequence, y_n is a bounded sequence. The integer order integral of FIF for sequence of data is revealed if the value of the integral is known at the initial endpoint or final endpoint. Besides, Riemann–Liouville fractional calculus of fractal interpolation function had been investigated with numerical examples for analyzing the results.

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Keywords. Attractor, countable iterated function system, fractal interpolation function, fractional calculus.

1. Introduction

The interpolation theory deals with the existence and construction of a continuous function which interpolates the given data set. Let $P = \{x_0, x_1, x_2, \ldots, x_N\}$ be a partition of the closed interval $I = [x_0, x_N] \subset \mathbb{R}$ such that $x_0 < x_1 < x_2 < \cdots < x_N$, where $N \in \mathbb{N}$. Let $J_0 = \{0, 1, 2, \ldots, N : N \in \mathbb{N}\}$ and $\{(x_i, y_i) \in I \times \mathbb{R} : i \in J_0\}$ be a given data set, and the interpolation

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function f is a continuous function $f : I \longrightarrow \mathbb{R}$ that satisfies $f(x_i) = y_i$ for $i \in J_0$. The traditional interpolation techniques generate smooth or piecewise differentiable interpolation function, despite the data points being irregular. Euclidean geometry which deals with regular objects and functions is used to approximate the data obtained from the realistic environment as well as the experimental data. But most of the natural objects such as lightning, clouds, mountain ranges, wall cracks, etc., have irregular and complex structure in which Euclidean geometry cannot be applied successfully to describe their structures. Hence there is a need of a nonlinear tool in approximation theory to resolve the above said problems which helps in the better understanding of complexities that prevail in nature.

The geometrical structures and the properties of such irregular objects were first addressed by Mandelbrot and he coined it as fractal theory [1]. Later, Hutchinson [2] introduced the conventional explanation of deterministic fractals through the theory of Iterated Function System (IFS). Meanwhile, Barnsley [3] formulated the theory of IFS called Hutchinson–Barnsley theory (HB theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the Banach fixed point theorem. Based on the theory of IFS, Barnsley constructed the fractal interpolation function (FIF) such that the attractor of the IFS is the graph of a FIF [4].

The attractor of the IFS is the graph of a FIF that interpolates given set of data points. Generally, approximation is done by smooth and rough curves, but fractal interpolations are sophisticated in approximation theory, because FIF may not be necessarily differentiable, and hence it could be sophisticated for rough curve approximation. Barnsley and Harrington [5] presented the construction of p-times frequently differentiable FIF when the derivative values reach up to pth order are available at the initial end point of the interval. These will be used for smooth curve approximation. Chand et al. [6,7] have developed the calculus of C^p FIF and described the principle of construction of C^p FIF that interpolates the given data. The advantage of FIF is that it is more flexible for interpolating a highly irregular data such as signals, waveforms, cell growth and so on.

Due to the sophisticated usage of fractals in nonlinear approximation, there are sequel efforts to extend Hutchinson–Barnsley classical framework for fractals to more general spaces and countable IFS or, more generally, to an infinite IFS. Secelean [8] has attempted to generalize the results presented in [2] by constructing deterministic fractals through countable iterated function system [9]. Moreover, he has proved the existence of countable iterated function system whose attractor is the graph of a fractal interpolation function of countable set of data $\{(x_n, y_n) : n \geq 2\}$, where x_n is taken as a strictly increasing and bounded sequence, y_n is a convergent sequence. As a consequence of these results, the present study provides the existence of fractal interpolation function with countable iterated function system while taking x_n as monotone and bounded sequence, y_n as bounded sequence. Further, fractional order integral, integer order integral of FIF for sequence of data are described, if the value of the integral of FIF is known at the initial endpoint or final endpoint. Finally, the fractional derivatives of FIF and its fractional integrals are given due importance as they are more precise and suitable for FIFs which are nowhere differentiable but continuous at all points. Hence, fractional calculus is a mathematical operator which is most suitable for analyzing such a FIF and which may also change the fractal dimension, when applied to fractal objects or functions.

The paper is organized as follows: Sect. 2 elicits the notations and preliminary facts of the objects to be investigated by the present study. Section 3 provides the existence of fractal interpolation function for sequence of data and existence of countable iterated function system if the FIF is given. Section 4 reveals the integer order integral of FIF for sequence of data, if the value of the integral is known at the initial endpoint or final endpoint. After that, these results are implemented into fractional order integral. In order to validate the results, numerical examples are given in Sects. 3 and 4.

2. Preliminaries

The concept of fractal analysis has encountered significant attention in recent years for understanding the irregularity [6,7,10–14]. Fractal analysis and fractional calculus [15,16] when combined together produce the enhancing results in nonlinear analysis. Tatom [17] applied the fractional calculus to random fractal functions and has shown that the fractal dimension of the function is a linear function of the order of fractional integro-differentiation. Liang and Su [18] investigated the relationship between the Hausdorff dimension of a type of fractal functions and the order of their Riemann–Liouville fractional calculus. In order to estimate the variance ratio between given data and interpolation functions, the Riemann–Liouville fractional integrals are studied in [19]. Initially, let us review some basic features concerning the fractal interpolation theory for finite data set.

Let $I_n = [x_{n-1}, x_n]$, $J_1 = \{1, 2, \ldots, N\}$, and $L_n : I \longrightarrow I_n, n \in J_1$ be contractive homeomorphisms satisfying

$$L_n(x_0) = x_{n-1}, \quad L_n(x_N) = x_n, \tag{2.1}$$

$$|L_n(x_1) - L_n(x_2)| \le l_n |x_1 - x_2|, \quad \forall\ x_1, x_2 \in I, \text{ for some } l_n \in [0, 1). \tag{2.2}$$

Let $[a, b]$ be a closed interval in $\mathbb{R}$ satisfying $a \le \min_i\ y_i \le \max_i\ y_i \le b$. Let $K = I \times [a, b]$. For some $r_n \in [0, 1)$, define a continuous mappings $R_n : K \longrightarrow [a, b]$ such that

$$R_n(x_0, y_0) = y_{n-1}, \quad R_n(x_N, y_N) = y_n, \tag{2.3}$$

$$|R_n(t, y_1) - R_n(t, y_2)| \le r_n |y_1 - y_2|, \quad \text{for all } t \in I, y_1, y_2 \in [a, b] \text{ and } n \in J_1. \tag{2.4}$$

Define $w_n(x, y) : K \longrightarrow K$, for $n \in J_1$, by

$$w_n(x, y) = (L_n(x), R_n(x, y)), \tag{2.5}$$

then $\{K; w_n : n \in J_1\}$ is called an iterated function system (IFS).

Theorem 2.1. *The IFS $\{K; w_n : n \in J_1\}$ defined above has a unique attractor G, i.e., $G = \bigcup_{n\in J_1} w_n(G)$. G is the graph of a continuous function $f : I \longrightarrow \mathbb{R}$ which interpolates the data set $\{(x_i, y_i) : i \in J_0\}$, i.e., $G = \{(x, f(x)) : x \in I\}$ and $f(x_i) = y_i$ for $i \in J_0$.*

A continuous function mentioned in theorem (2.1) is called a Fractal Interpolation Function (FIF) with respect to the IFS $\{K; w_n : n \in J_1\}$ and an unique f enjoying the functional equation,

$$f(L_n(x)) = R_n(x, f(x)) \quad \text{(or)} \quad f(t) = R_n(L_n^{-1}(x), f \circ L_n^{-1}(x)), \\ n \in J_1, x \in I. \tag{2.6}$$

More complete and rigorous treatments are given by Barnsley [4].

Normally interpolation theory deals with finite data set $\{(x_i, y_i) : i \in J_0\}$. Suppose $\{(x_i, y_i) : i \in \mathbb{N}\}$ is a sequence of data set. Whether it is possible to find a continuous function f which interpolates the given sequence of data will be analyzed in the next section after discussing the countable iterated function system.

Let (X, d) be a complete metric space and $\mathscr{K}(X)$ be the collection of all non-empty compact subsets of X. Define, $d(x, B) = \inf_{y\in B} d(x, y)$ and $d(A, B) = \sup_{x\in A} d(x, B)$ for all $x \in X$ and $A, B \in \mathscr{K}(X)$. Define the Hausdorff metric H_d on $\mathscr{K}(X)$ by $H_d(A, B) = \max\{d(A, B), d(B, A)\}$. The space $(\mathscr{K}(X), H_d)$ is a complete metric space.

Consider a finite set of contractions $f_1, f_2, \ldots, f_N$ for $N \in \mathbb{N}$, on X, simply $(f_n)_{n=1}^N$, having contraction ratios $r_n, n = 1, 2, \ldots, N$, respectively. Define $F : \mathscr{K}(X) \longrightarrow \mathscr{K}(X)$, by $F(B) = \bigcup_{n=1}^N f_n(B)$ for all $B \in \mathscr{K}(X)$, then F is a Banach contraction on $\mathscr{K}(X)$ with contraction ratio $r = \max_n r_n$. According to Banach contraction principle, F has a unique fixed point A^* in $\mathscr{K}(X)$, that is $A^* = F(A^*) = \bigcup_{n=1}^N f_n(A^*)$. Moreover, for any $B \in \mathscr{K}(X)$ $\lim_{n\to\infty} F^{\circ n}(B) = A^*$. The non-empty compact set A^* is called an invariant or self-referential or attractor of IFS $\{X; f_n : n = 1, 2, \ldots, N\}$.

Suppose $(f_n)_{n\geq 1}$ is sequence of contractions on X with ratios $r_n \in [0, 1)$. Then $\{X; f_n : n \geq 1\}$ is called a countable iterated function system (CIFS). Define a self-map $\mathscr{F} : \mathscr{K}(X) \longrightarrow \mathscr{K}(X)$ by $\mathscr{F}(B) = \overline{\bigcup_{n=1}^\infty} f_n(B)$, for all $B \in \mathscr{K}(X)$, then $\mathscr{F}$ is a contraction mapping with ratio $r \leq \sup_n r_n$. Again by Banach contraction principle $\mathscr{F}$ has a unique fixed point $\mathscr{A}^*$ such that $\mathscr{A}^* = \mathscr{F}(\mathscr{A}^*) = \overline{\bigcup_{n=1}^\infty} f_n(\mathscr{A}^*)$. Furthermore, for any $B \in \mathscr{K}(X)$, $\lim_{n\to\infty} \mathscr{F}^{\circ n}(B) = \mathscr{A}^*$. The non-empty compact invariant set $\mathscr{A}^*$ is called the attractor of the CIFS $\{X; f_n : n \geq 1\}$.

The attractor A^* depends on the corresponding IFS. Suppose A_N^* is an attractor of the IFS $\{X; f_n : n = 1, 2, \ldots, N\}$ and F_N is the associated HB operator for $N \geq 1$, then Theorem 2.2 reveals the relation between the attractors of IFS and the attractor of CIFS of Banach contractions on a complete metric space X.

Theorem 2.2 [20]. *If $B \in \mathscr{K}(X)$, then*

$$\mathscr{F}(B) = \lim_{N\to\infty} F_N(B) = \lim_{N\to\infty} \bigcup_{n=1}^N f_n(B).$$

In particular, if $\mathscr{A}$ *is the attractor of CIFS* $\{X; f_n : n \geq 1\}$, *then*

$$\mathscr{A} = \mathscr{F}(\mathscr{A}) = \lim_{N\to\infty} A_N^* = \lim_{N\to\infty} \lim_{n\to\infty} F_N^{\circ n}(B)$$

also, $\mathscr{A} = \lim_{N\to\infty} \lim_{n\to\infty} F_N^{\circ n}(B)$.

The above theorem shows that the attractor of CIFS $\{X; f_n : n \geq 1\}$ is approximated by the attractors of IFS $\{X; f_n : n = 1, 2, \dots, N\}, N \geq 1$, with respect to Hausdorff metric.

3. Fractal Interpolation Function for CIFS

Let $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ be increasing and bounded, $(y_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $\sup_n x_n = x^*$, then the sequence $x_1 < x_2 < x_3 < \cdots$ partitions the closed interval $I = [x_1, x^*]$. Let $\sup_n y_n = y^*, \inf_n y_n = y_1$. Consider the sequence of data $\{(x_n, y_n) : n \geq 1\}$, where $y_n \in [y_1, y^*]$ and intend to construct a continuous function which passes through the given data. That is, $f : I \longrightarrow \mathbb{R}$ such that $f(x_n) = y_n$, for all $n \in \mathbb{N}$ and the graph of f is attractor of the CIFS.

Let $K = [x_1, x^*] \times [y_1, y^*]$. Define $I_n = [x_{n-1}, x_n], L_n : I \longrightarrow I_n$ for $n \geq 2$ be contractive homeomorphisms such that

$$\begin{gathered} L_n(x_1) = x_{n-1}, \quad L_n(x^*) = x_n, \\ |L_n(s) - L_n(t)| \leq l_n \, |s - t| \, , \quad \text{for all } s, t \in I, \end{gathered}$$

for some $l_n \in [0, 1)$.

Let $R_n : K \longrightarrow [y_1, y^*]$ be continuous with, for some $r_n \in [1, 0)$,

$$\begin{gathered} R_n(x_1, y_1) = y_{n-1}, \quad R_n(x^*, y^*) = y_n, \\ |R_n(s, y_1) - R_n(s, y_2)| \leq r_n \, |y_1 - y_2| \, , \\ \text{for all } s \in I, y_1, y_2 \in [y_1, y^*] \text{ and } n \geq 2. \end{gathered}$$

For all $n \geq 2$, define $w_n : K \longrightarrow K$ by $w_n(x, y) = (L_n(x), R_n(x, y))$, then $\{K; w_n : n \geq 2\}$ is a CIFS. Define $\mathscr{F} : \mathscr{K}(K) \longrightarrow \mathscr{K}(K)$ by $\mathscr{F}(B) = \overline{\bigcup_{n=1}^{\infty}} w_n(B)$, for all $B \in \mathscr{K}(K)$.

Theorem 3.1. *The function* $\mathscr{F}$, *defined above, has a unique fixed point* $\mathscr{A}$ *which is the graph of a continuous interpolation function for the sequence of data* $\{(x_n, y_n) : n \geq 1\}$.

Proof. Let $\mathscr{A}$ be any attractor of the CIFS $\{K; w_n : n \geq 2\}$, that is

$$\mathscr{A} = \mathscr{F}(\mathscr{A}) = \overline{\bigcup_{2}^{\infty}} w_n(\mathscr{A}).$$

For each $N \in \mathbb{N}$, $T_N = \{K; w_n : n = 2, 3, \dots, N\}$ is partial IFS, by theorem (2.1) F has a unique attractor, denoted as A_N.

$$\begin{aligned}\mathscr{A} &= \overline{\bigcup_{N=2}^{\infty} A_N} = \overline{\bigcup_{N=2}^{\infty} F(A_N)} = \overline{\bigcup_{N=2}^{\infty} \bigcup_{n=2}^{N} w_n(A_N)} \\ &= \overline{\bigcup_{N=2}^{\infty} \bigcup_{n=2}^{N}} \left(L_n(x), R_n(x,y)\right), \quad \text{for all } (x,y) \in A_N.\end{aligned}$$

Let $\tilde{I} = \{x \in I : (x,y) \in G$ for some $y \in [y_1, y^*]\}$ by Theorem 2.2 $\tilde{I} = \overline{\bigcup_{N=2}^{\infty} \bigcup_{n=2}^{N}} L_n(\tilde{I})$, Meanwhile L_n is contraction for each n, on I and $\bigcup_{n=2}^{N} L_n(I) = I$. Hence $\tilde{I} = I$.

Let $S_i = \{(x_i, y) \in G : i \in \mathbb{N}\}$. Consider $S_1 = \{(x_1, y) \in G : y \in [y_1, y^*]\}$, clearly $S_1 \subset G$, hence $S \subset \overline{\bigcup_{N=2}^{\infty} \bigcup_{n=2}^{N}} w_n(G)$,

$$\begin{aligned}\Rightarrow S_1 &\subseteq \overline{\bigcup_{N=2}^{\infty}} (L_2(x_1), R_2(x_1,y)) \cup (L_3(x_1), \\ &\qquad R_3(x_1,y)) \cup \cdots \cup (L_N(x_1), \\ R_N(x_1,y)) &= \overline{\bigcup_{N=2}^{\infty}} (x_1, R_2(x_1,y)) \cup (x_2, R_3(x_1,y)) \cup \cdots \cup (x_{N-1}, R_N(x_1,y)) \\ \Rightarrow S_1 &\subseteq (x_1, R_2(x_1,y)) = w_2(x_1, R_2(x_1,y))\end{aligned}$$

and $w_2(x_1, R_2(x_1,y)) \subseteq S_1 \Rightarrow S_1 = w_2(L_2(x_1), R_2(x_1,y))$. Since w_2 is strict contraction, $S_1 = (x_1, y_1)$. Now, consider $S_* = \{(x^*, y) \in G : y \in [y_1, y^*]\}$ and $w_* = \lim_{N\to\infty} \bigcup_{n=2}^{N} w_n(x^*, R_n(x^*, y))$, the same argument yields, $S_* = w_*(x^*, R_n(x^*, y)) = (x^*, y^*)$. Hence, for each $i \geq 1, (x_{i+1}, y_{i+1}) = w_i(S_1) \cup w_{i+1}(S_*)$. Consider $\delta = \sup\{|t_1 - t_2| : (x, t_1) \in G, (x, t_2) \in G,$ for some $x \in I\}$, G is compact, hence G has supremum. Suppose supremum is achieved at $(\tilde{x}, t_1), (\tilde{x}, t_2)$, then there exists two points $(L_n^{-1}(\tilde{x}), u_1), (L_n^{-1}(\tilde{x}), u_2) \in G$ such that $t_1 = F_n(L_n^{-1}(\tilde{x}, u_1)), t_2 = F_n(L_n^{-1}(\tilde{x}, u_2))$, hence $\delta = |t_1 - t_2| = |F_n(L_n^{-1}(\tilde{x}, u_1)) - F_n(L_n^{-1}(\tilde{x}, u_2))| \leq q.|u_1 - u_2| \leq q.\delta$, with $q \in [0,1)$, therefore $\delta = 0$. Hence, only one y is associated to each $x \in I$. Therefore G is the graph of a function $f : I \longrightarrow [y_1, y^*]$ and $f(x_n) = y_n$, for each $n \in \mathbb{N}$. Define $C_0(I) = \{f : f : I \longrightarrow [y_1, y^*], f$ is continuous, $g(x_1) = y_1, g(x^*) = y^*\}$, clearly $C_0(I)$ is a closed subspace of the Banach space of continuous real-valued functions $g : I \longrightarrow \mathbb{R}$, with supremum norm $\|g\|_\infty = \sup\{|g(x)| : x \in I\}$, denoted as $C(I)$. Hence $C_0(I)$ is a complete metric space, define $T : C_0(I) \longrightarrow C_0(I)$ by

$$Tg(x) = F_n(L_n^{-1}(x), g(L_n^{-1}(x)))$$

when $x \in I_n$. Meanwhile, T is a contraction mapping with contraction ratio q and T is induced by the CIFS.

$$\begin{aligned}\|Th - Tg\|_\infty &= \sup\{\left\|F_n(L_n^{-1}(x), h(L_n^{-1}(x))) - F_n(L_n^{-1}(x), g(L_n^{-1}(x)))\right\| : \\ &\qquad x \in I_n, n \geq 2\} \\ &\leq \sup\{q. \left\|h(L_n^{-1}(x)) - g(L_n^{-1}(x))\right\| : x \in I_n, n \geq 2\} \\ &\leq q. \|h - g\|_\infty .\end{aligned}$$

Hence T has a unique fixed point $\tilde{h} \in C_0(I)$. The graph of $\tilde{h}$ is an attractor of the CIFS, it gives that $\tilde{h} = f$, hence f is a continuous interpolation function of the sequence of data $\{(x_n, y_n) : n \geq 1\}$.

Remark 3.2.

1. If $(x_n)_{n=1}^{\infty}$ is a decreasing and bounded sequence, then the sequence $x_1 > x_2 > x_3 > \cdots$ partitions the closed interval $I = [x^*, x_1]$, where $x^* = \inf_n x_n$. Hence, the above arguments give Theorem 3.1.
2. Let $(x_n)_{n=1}^{\infty}$ be an increasing and bounded sequence, if $(y_n)_{n=1}^{\infty}$ is a convergent sequence then the above theorem is stated as "there exists an interpolation function f corresponding to the given sequence of data such that the graph of f is the attractor of the associated CIFS". Hence Theorem 3.1 is generalization of theorem (2) presented in [8].

Theorem 3.1 presents the existence of fractal interpolation function for a given sequence of data, while the following corollary assures the existence of countable iterated function system if fractal interpolation function is given.

Corollary 3.3. *For any $g \in \mathscr{K}(K)$ and let $\epsilon > 0$ be given. Choose the CIFS $\{K; w_n : n \geq 2\}$ with contractive ratio $r \in [0, 1)$ such that $H_d(g, \mathscr{F}(g)) \leq \epsilon$. Then $H_d(g, \mathscr{A}) \leq \frac{\epsilon}{1-r}$, where $\mathscr{A}$ is the attractor of CIFS.*

Proof.

$$\begin{aligned}
H_d(g, \mathscr{A}) &= H_d\left(g, \lim_{m\to\infty} \mathscr{F}^{\circ m}(g)\right)\\
&= \lim_{m\to\infty} H_d(g, \mathscr{F}^{\circ m}(g))\\
&= \lim_{m\to\infty} H_d\left(g, \overline{\bigcup_{n=2}^{\infty} w_n^{\circ m}(g)}\right)\\
&= \lim_{m\to\infty} H_d\left(g, \lim_{k\to\infty} \bigcup_{n=2}^{k} w_n^{\circ m}(g)\right)\\
&= \lim_{m\to\infty} \lim_{k\to\infty} H_d(g, F^{\circ m}(g))\\
&\leq \lim_{m\to\infty} \lim_{k\to\infty} \sum_{t=1}^{m} H_d(F^{\circ(t-1)}(g), F^{\circ t}(g))\\
&\leq \lim_{m\to\infty} \lim_{k\to\infty} \sum_{t=1}^{m} r_t^{m-1} H_d(g, F(g))\\
&\leq \lim_{m\to\infty} \sum_{t=1}^{m} max(r_t^{m-1}) H_d(g, \mathscr{F}(g))\\
&\leq \sum_{t=1}^{\infty} r^{t-1} H_d(g, \mathscr{F}(g)) \leq \frac{\epsilon}{1-r}.
\end{aligned}$$

Remark 3.4.

1. For any $\tilde{g} \in C_0(I)$ and let $\epsilon > 0$ be given. Choose the CIFS $\{K; w_n : n \geq 2\}$ associated with FIF $\tilde{f}$ such that $\|\tilde{g} - T(\tilde{g})\|_{\infty} \leq \epsilon$,

where $T : C_0(I) \longrightarrow C_0(I)$ is defined in the proof of Theorem 3.1. Then by above Corollary $\left\|\tilde{g} - \tilde{f}\right\|_\infty \leq \frac{\epsilon}{1-r}$.

Let $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ be an increasing and bounded sequence. Let y_n be a bounded sequence of real numbers. $I_n = [x_{n-1}, x_n], n \geq 2$, $L_n : I \longrightarrow I_n$ defined by $L_n(x) = a_n x + e_n$, where $a_n = \frac{x_n - x_{n-1}}{x^* - x_1}, e_n = \frac{x^* x_{n-1} - x_1 x_n}{x^* - x_1}$. Clearly L_n defines a homeomorphic between I and I_n, for any $n \geq 2$, such that $L_n(x_1) = x_{n-1}, L_n(x^*) = x_n$. Now consider $R_n(x, y) = c_n x + \lambda_n y + f_n$, if R_n obey the end point condition $R_n(x_1, y_1) = y_{n-1}, R_n(x^*, y^*) = y_n$ and $r_n \in [0, 1)$, then the equations

$$R_n(x_1, y_1) = y_{n-1} = c_n x_1 + r_n y_1 + f_n,$$
$$R_n(x^*, y^*) = y_n = c_n x^* + r_n y^* + f_n,$$

are giving $c_n = \frac{(y_n - y_{n-1}) - r_n(y^* - y_1)}{x^* - x_1}, f_n = y_{n-1} - r_n y_1 - c_n x_1$, for $n \geq 2$. If r_n is given, then the above discussion generates the CIFS of the form

$$w_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_n & 0 \\ c_n & r_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_n \\ f_n \end{bmatrix},$$

where $a_n = \frac{x_n - x_{n-1}}{x^* - x_1}, c_n = \frac{(y_n - y_{n-1}) - r_n(y^* - y_1)}{x^* - x_1}, e_n = \frac{x^* x_{n-1} - x_1 x_n}{x^* - x_1}, f_n = y_{n-1} - r_n y_1 - c_n x_1$ and w_ns obey the end point conditions.

Example 3.5. If $x_n = (\frac{2n-1}{n})_{n=1}^{\infty}, y_n = (\frac{-1}{n})_{n=1}^{\infty}$ and $r_n = 1/2$ for all $n \in \mathbb{N}$, then the CIFS interpolating the data set $\{(x_n, y_n), n \in \mathbb{N}\}$ is

$$w_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{n^2 - n} & 0 \\ \frac{1}{(n^2 - n)} - \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2n^2 - 3n - 1}{n^2 - n} \\ \frac{n^2 - 2n - 1}{n^2 - n} \end{bmatrix}.$$

Hence, $L_n(x) = \frac{1}{n^2 - n} x + \frac{2n^2 - 3n - 1}{n^2 - n}, R_n(x, y) = (\frac{1}{(n^2 - n)} - \frac{1}{2})x + \frac{1}{2} y + \frac{n^2 - 2n - 1}{n^2 - n}$. The pictorial representation of attractors of countable iterated function system approximation process are given as in Fig. 1.

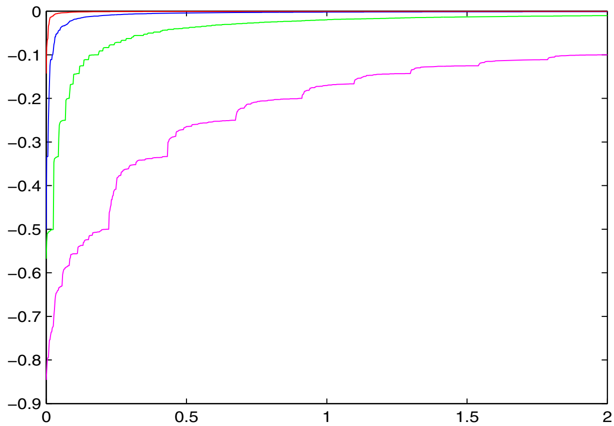
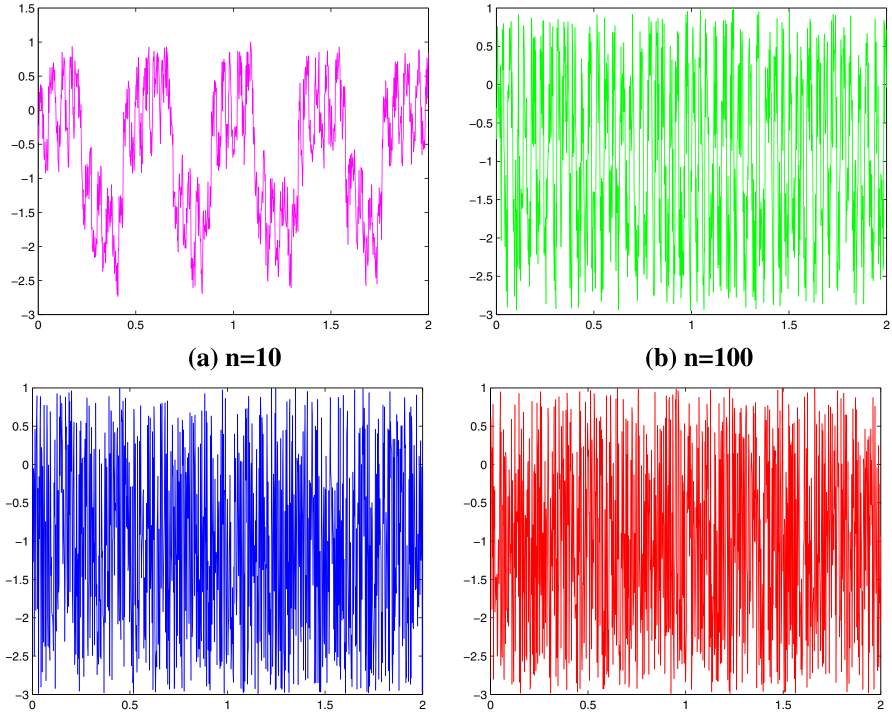


Figure 1. FIF for CIFS with $r_n = 1/2$



(a) n=10 (b) n=100

(c) n=1000 (d) n=10000

Figure 2. FIF for CIFS with $r_n = 1/2$

Example 3.6. If $y_n = ((-1)^n)_{n=1}^{\infty}$, x_n remains the same in the above example and $r_n = 1/2$ for all $n \in \mathbb{N}$, then CIFS of the form

$$w_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{n^2-n} & 0 \\ -2(-1)^{n-1} - 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2n^2-3n-1}{n^2-n} \\ 3(-1)^{n-1} + \frac{3}{2} \end{bmatrix}.$$

Hence, $L_n(x) = \frac{1}{n^2-n}x + \frac{2n^2-3n-1}{n^2-n}$, $R_n(x,y) = (-2(-1)^{n-1} - 1)x + \frac{1}{2}y + 3(-1)^{n-1} + \frac{3}{2}$ (Fig. 2).

4. Fractional Calculus on Interpolation Function of Sequence of Data

For the rest of this section, CIFS of the form

$$L_n(x) = a_n x + e_n, R_n(x,y) = r_n y + q_n(x), \quad n \geq 2, \tag{4.1}$$

has been considered, where $-1 < r_n < 1$, and $q_n : I \longrightarrow \mathbb{R}, n \geq 2$, are continuous function satisfies

$$q_n(x_1) = y_n - r_n y_1, \quad q_n(x^*) = y_{n+1} - r_n y^*.$$

The interpolation function f of the above CIFS, with respect to the sequence of data $\{(x_n, y_n) : n \geq 1\}$ is known as linear FIF.

Theorem 4.1. *If* $\hat{f}(x) = \hat{y}_1 + \int_{x_1}^{x} f(t)\mathrm{d}t$, *then* $\hat{f}$ *is the FIF associated with* $\{(L_n(x), \hat{R}_n(x, y)) : n \geq 2\}$, *where*

$$\begin{aligned}
\hat{R}_n(x, y) &= a_n r_n y + \hat{q}_n(x),\\
a_n &= \frac{x_n - x_{n-1}}{x^* - x_1},\\
\hat{q}_n(x) &= \hat{y}_{n-1} - a_n r_n \hat{y}_1 + a_n \int_{x_1}^{x} q_n(t)\mathrm{d}t,\\
\hat{y}_n &= \hat{f}(x_n) = \hat{y}_1 + \sum_{i=2}^{n} a_i \left[r_i(\hat{y}^* - \hat{y}_1) + \int_{x_1}^{x^*} q_i(t)\mathrm{d}t \right],\\
\hat{y}^* &= \hat{y}_1 + \frac{\sum_{n=2}^{\infty} a_n \int_{x_1}^{x^*} q_n(t)\mathrm{d}t}{1 - \sum_{n=2}^{\infty} a_n r_n}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\hat{f}(L_n(x)) &= \hat{y}_1 + \int_{x_1}^{L_n(x)} f(t)\mathrm{d}t\\
&= \hat{y}_{n-1} + a_n \int_{x_1}^{x} (r_n f(t) + q_n(t))\mathrm{d}t\\
&= \hat{y}_{n-1} + a_n r_n \int_{x_1}^{x} f(t)\mathrm{d}t + a_n \int_{x_1}^{x} q_n(t))\mathrm{d}t\\
&= \hat{y}_{n-1} + a_n r_n(\hat{f}(x) - \hat{y}_1) + a_n \int_{x_1}^{x} q_n(t))\mathrm{d}t,
\end{aligned}$$

here the existence of $\hat{y}_{n-1}$, for all $n \geq 2$, follows from the continuity of f on $[x_1, x^*]$.

If $x = x^*$, $\hat{f}(L_n(x^*)) = \hat{y}_n - \hat{y}_{n-1} = a_n[r_n(\hat{y}^* - \hat{y}_1) + \int_{x_1}^{x^*} q_n(t)\mathrm{d}t]$. But $\hat{y}_n = \hat{y}_1 + \sum_{i=1}^{n}(\hat{y}_i - \hat{y}_{i-1}) = \hat{y}_1 + \sum_{i=1}^{n} a_i[r_i(\hat{y}^* - \hat{y}_1) + \int_{x_1}^{x^*} q_i(t)\mathrm{d}t]$.

Theorem 4.1 reveals that $\hat{f}$ is FIF which interpolates the sequence of data $\{(x_n, \hat{y}_n) : n \geq 2\}$, if the integral values of FIF f is known at the initial endpoint. Simultaneously, the following corollary ensures the similar result when the integral values of FIF f are known at the final endpoint.

Corollary 4.2. *If* $\hat{f}(x) = \hat{y}^* - \int_{x}^{x^*} f(t)\mathrm{d}t$, *then* $\hat{f}$ *is the FIF associated with* $\{(L_n(x), \hat{R}_n(x, y)) : n \geq 2\}$, *where*

$$\begin{aligned}
\hat{R}_n(x, y) &= a_n r_n y + \hat{q}_n(x),\\
a_n &= \frac{x_n - x_{n-1}}{x^* - x_1},\\
\hat{q}_n(x) &= \hat{y}_n - a_n r_n \hat{y}^* - a_n \int_{x}^{x^*} q_n(t)\mathrm{d}t,
\end{aligned}$$

$$\hat{y}_n = \hat{f}(x_n) = \hat{y}^* - \sum_{i=2}^{n} a_i \left[r_i(\hat{y}^* - \hat{y}_1) + \int_{x_1}^{x^*} q_i(t)\mathrm{d}t \right],$$

$$\hat{y}_1 = \hat{y}^* - \frac{\sum_{n=2}^{\infty} a_n \int_{x_1}^{x^*} q_n(t)\mathrm{d}t}{1 - \sum_{n=2}^{\infty} a_n r_n}.$$

Proof. Similar arguments of above theorem yields

$$\begin{aligned}\hat{f}(L_n(x)) &= \hat{y}^* - \int_{L_n(x)}^{x^*} f(t)\mathrm{d}t \\ &= \hat{y}_n - a_n \int_x^{x^*} (r_n f(t) + q_n(t))\mathrm{d}t \\ &= \hat{y}_n + a_n r_n \int_x^{x^*} f(t)\mathrm{d}t - a_n \int_x^{x^*} q_n(t))\mathrm{d}t \\ &= \hat{y}_n - a_n r_n(\hat{y}^* - \hat{f}(x)) - a_n \int_x^{x^*} q_n(t))\mathrm{d}t,\end{aligned}$$

if $x = x_1$, $\hat{f}(L_n(x_1)) = \hat{y}_{n-1} - \hat{y}_n = -a_n[r_n(\hat{y}^* - \hat{y}_1) - \int_x^{x^*} q_n(t))\mathrm{d}t]$. But $\hat{y}_n = \hat{y}^* - \sum_{i=2}^{n}(\hat{y}_i - \hat{y}_{i-1}) = \hat{y}^* - \sum_{i=1}^{n} a_i[r_i(\hat{y}^* - \hat{y}_1) + \int_{x_1}^{x^*} q_i(t)\mathrm{d}t]$.

If $f(x) \in \mathcal{C}([a,b])$ and $a < x < b$, then *the Riemann–Liouville fractional integral* of order $\beta > 0$ is defined by

$$I_a^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (t-x)^{\beta-1} f(t)\mathrm{d}t$$

and *the Riemann–Liouville fractional derivative* of order $\beta > 0$ is defined by

$$D_a^\beta f(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{f(t)\mathrm{d}t}{(x-t)^{\beta-n+1}},$$

where $n - 1 \le \beta < n$ (n is the greatest integer of β). $\mathcal{C}^p(I)$ denotes the space of all continuous and p-times differentiable functions. Now, let us define the fractional integral of FIF at the initial endpoint as $I_{x_1}^\beta f^{(k)}(x_1) = 0$ and $I_{x_1}^\beta f^{(k)}(x^*) = \frac{1}{\Gamma(\beta)} \int_{x_1}^{x^*} (x-t)^{\beta-1} f^{(k)}(t)\mathrm{d}t$.

Theorem 4.3. *Let f be the $\mathcal{C}^p$-linear FIF determined by the CIFS defined in* (4.1), *then $I_{x_1}^\beta f^{(k)}(x)$ is the $\mathcal{C}^p$-linear FIF associated with $\{K; w_n = (L_n(x), \hat{R}_{n,\beta}^{(k)}(x,y), n \ge 2)\}$, where $\hat{y}_{1,\beta} = 0$ and for each $n \ge 2$*

$$\begin{aligned}\hat{R}_{n,\beta}^{(k)}(x,y) &= a_n^\beta r_n y + \hat{q}_{n,\beta}^{(k)}(x), \\ a_n &= \frac{(x_n - x_{n-1})}{(x^* - x_1)}, \\ \hat{q}_{n,\beta}^{(k)}(x) &= \hat{y}_{n-1,\beta}^{(k)} + f_{n-1,\beta}^{(k)}(x) + a_n^\beta I_{x_1}^\beta q_n^{(k)}(x), \\ \hat{y}_{n,\beta}^{(k)} &= I_{x_1}^\beta f^{(k)}(x_n), \\ f_{n,\beta}^{(k)}(x) &= \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_{n-1}} [(L_n(x) - t)^{\beta-1} - (x-t)^{\beta-1}] f^{(k)}(t)\mathrm{d}t,\end{aligned}$$

for all $k = 1, 2, \ldots, p$.

Proof. As f is a $\mathcal{C}^p$-linear FIF determined by the CIFS of the form (4.1), f satisfies the functional equation $f^{(k)}(L_n(x)) = R\left(x, f^{(k)}(x)\right)$. Hence $f^{(k)}(L_n(x)) = r_n f^{(k)}(x) + q_n^{(k)}(x)$ for all $x \in I, n \geq 2$.

$$
\begin{aligned}
&I_{x_1}^{\beta} f^{(k)}(L_n(x))\\
&\quad= \frac{1}{\Gamma(\beta)} \int_{x_1}^{L_n(x)} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t\\
&\quad= \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_{n-1}} (x_{n-1} - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t - \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_{n-1}} (x_{n-1} - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t\\
&\qquad+ \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_{n-1}} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t + \frac{1}{\Gamma(\beta)} \int_{x_{n-1}}^{L_n(x)} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t\\
&\quad= \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_{n-1}} (x_{n-1} - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t + \frac{1}{\Gamma(\beta)} \int_{x_{n-1}}^{L_n(x)} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t\\
&\qquad+ \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_{n-1}} [(L_n(x) - t)^{\beta-1} - (x_{n-1} - t)^{\beta-1}] f^{(k)}(t)\mathrm{d}t\\
&\quad= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x) + \frac{1}{\Gamma(\beta)} \int_{x_{n-1}}^{L_n(x)} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t\\
&\quad= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x) + \frac{1}{\Gamma(\beta)} \int_{x_1}^{x} (L_n(x) - L_n(u))^{\beta-1} f^{(k)}(L_n(u)) a_n \mathrm{d}u\\
&\quad= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x) + \frac{a_n}{\Gamma(\beta)} \int_{x_1}^{x} (a_n(x-u))^{\beta-1} \{r_n f^{(k)}(u) + q_n^{(k)}(u)\}\mathrm{d}u\\
&\quad= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x) + \frac{a_n^{\beta}}{\Gamma(\beta)} \int_{x_1}^{x} (x-u)^{\beta-1} r_n f^{(k)}(u)\mathrm{d}u\\
&\qquad+ \frac{a_n^{\beta}}{\Gamma(\beta)} \int_{x_1}^{x} (x-u)^{\beta-1} q_n^{(k)}(u)\mathrm{d}u\\
&\quad= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x) + a_n^{\beta} r_n I_{x_1}^{\beta} f^{(k)}(x) + a_n^{\beta} I_{x_1}^{\beta} q_n^{(k)}(x)\\
&\quad= \hat{R}_{n,\beta}^{(k)}\left(x, I_{x_1}^{\beta} f^{(k)}(x)\right).
\end{aligned}
$$

Moreover, for each k, $f^{(k)}$ is continuous on $[x_1, x^*]$, hence $I_{x_1}^{\beta} f^{(k)}(x)$ is continuous on $[x_1, x^*]$. Further, $\lim_{n\to\infty} x_n = x^*$, it must give $\lim_{n\to\infty} I_{x_1}^{\beta} f^{(k)}(x_n) = I_{x_1}^{\beta} f^{(k)}(x^*)$. Therefore $\hat{y}_{n,\beta}^{(k)}$ exist for all n.

$$
\begin{aligned}
\hat{R}_{n,\beta}^{(k)}\left(x_1, \hat{y}_{1,\beta}\right) &= \hat{R}_{n,\beta}^{(k)}\left(x_1, 0\right)\\
&= \hat{q}_{n,\beta}^{(k)}(x_1) + a_n^{\beta} r_n I_{x_1}^{\beta} f^{(k)}(x_1)\\
&= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x_1) + a_n^{\beta} I_{x_1}^{\beta} q_n^{(k)}(x_1) + a_n^{\beta} r_n I_{x_1}^{\beta} f^{(k)}(x_1)\\
&= \hat{y}_{n-1,\beta}^{(k)},\\
\hat{R}_{n,\beta}^{(k)}\left(x^*, \hat{y}_{\beta}^*\right) &= \hat{q}_{n,\beta}^{(k)}(x^*) + a_n^{\beta} r_n I_{x_1}^{\beta} f^{(k)}(x^*)\\
&= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x^*) + a_n^{\beta} I_{x_1}^{\beta} q_n^{(k)}(x^*) + a_n^{\beta} r_n I_{x_1}^{\beta} f^{(k)}(x^*)\\
&= I_{x_1}^{\beta} f^{(k)}(L_n(x^*))\\
&= \hat{y}_{n,\beta}^{(k)}.
\end{aligned}
$$

Hence, $I_{x_1}^{\beta} f^{(k)}(x)$ is $\mathcal{C}^p$-linear FIF associated with $\{K; w_n = (L_n(x), \hat{R}_{n,\beta}^{(k)}(x,y), n \geq 2)\}$.

Consequence of Theorem 4.3 defines the fractional integral of FIF at the terminal endpoint as $I_{x^*}^{\beta} f^{(k)}(x^*) = 0$ and $I_{x^*}^{\beta} f^{(k)}(x_1) = \frac{-1}{\Gamma(\beta)} \int_{x_1}^{x^*} (x - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t$, then the above results are stated as follows.

Corollary 4.4. *Let f be the $\mathcal{C}^p$-linear FIF determined by the CIFS defined in (4.1), then $I_{x^*}^{\beta} f^{(k)}(x)$ is the $\mathcal{C}^p$-linear FIF associated with $\{K; w_n = (L_n(x), \hat{R}_{n,\beta}^{(k)}(x,y), n \geq 2)\}$, where $\hat{y}_{\beta}^* = 0$ and for each $n \geq 2$*

$$\begin{aligned}
\hat{R}_{n,\beta}^{(k)}(x,y) &= a_n^{\beta} r_n y + \hat{q}_{n,\beta}^{(k)}(x), \\
a_n &= \frac{(x_n - x_{n-1})}{(x^* - x_1)}, \\
\hat{q}_{n,\beta}^{(k)}(x) &= \hat{y}_{n-1,\beta}^{(k)} + f_{n-1,\beta}^{(k)}(x) + a_n^{\beta} I_{x^*}^{\beta} q_n^{(k)}(x), \\
\hat{y}_{n,\beta}^{(k)} &= I_{x^*}^{\beta} f^{(k)}(x_n), \\
f_{n,\beta}^{(k)}(x) &= \frac{-1}{\Gamma(\beta)} \int_{x_{n-1}}^{x^*} [(L_n(x) - t)^{\beta-1} - (x-t)^{\beta-1}] f^{(k)}(t)\mathrm{d}t,
\end{aligned}$$

for all $k = 1, 2, \ldots, p$.

Proof. Similar arguments of the theorem (4.3) yields

$$\begin{aligned}
I_{x^*}^{\beta} f^{(k)}(L_n(x)) &= \frac{-1}{\Gamma(\beta)} \int_{L_n(x)}^{x^*} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t \\
&= \frac{1}{\Gamma(\beta)} \int_{x_{n-1}}^{x^*} (x_{n-1} - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t \\
&\quad - \frac{1}{\Gamma(\beta)} \int_{x_{n-1}}^{x^*} (x_{n-1} - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t \\
&\quad - \frac{1}{\Gamma(\beta)} \int_{L_n(x)}^{x_{n-1}} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t \\
&\quad - \frac{1}{\Gamma(\beta)} \int_{x_{n-1}}^{x^*} (L_n(x) - t)^{\beta-1} f^{(k)}(t)\mathrm{d}t \\
&= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x) + a_n^{\beta} r_n I_{x^*}^{\beta} f^{(k)}(x) + a_n^{\beta} I_{x^*}^{\beta} q_n^{(k)}(x) \\
&= \hat{R}_{n,\beta}^{(k)}\left(x, I_{x^*}^{\beta} f^{(k)}(x)\right).
\end{aligned}$$

Moreover, for each k, $f^{(k)}$ is continuous on $[x_1, x^*]$, hence $I_{x^*}^{\beta} f^{(k)}(x)$ is continuous on $[x_1, x^*]$. Further, $\lim_{n\to\infty} x_n = x^*$, it must give $\lim_{n\to\infty} I_{x^*}^{\beta} f^{(k)}(x_n) = I_{x^*}^{\beta} f^{(k)}(x^*) = 0$. Therefore, $\hat{y}_{n,\beta}^{(k)}$ exist for all n.

$$\begin{aligned}
\hat{R}_{n,\beta}^{(k)}(x_1, \hat{y}_{1,\beta}) &= \hat{q}_{n,\beta}^{(k)}(x_1) + a_n^{\beta} r_n I_{x^*}^{\beta} f^{(k)}(x_1) \\
&= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x_1) + a_n^{\beta} I_{x^*}^{\beta} q_n^{(k)}(x_1) + a_n^{\beta} r_n I_{x_1}^{\beta} f^{(k)}(x_1) \\
&= I_{x^*}^{\beta} f^{(k)}(L_n(x_1))
\end{aligned}$$

$$
\begin{aligned}
&= \hat{y}_{n,\beta}^{(k)}, \\
\hat{R}_{n,\beta}^{(k)}\left(x^*, \hat{y}_\beta^*\right) &= \hat{q}_{n,\beta}^{(k)}(x^*) + a_n^\beta r_n I_{x^*}^\beta f^{(k)}(x^*) \\
&= \hat{y}_{n-1,\beta}^{(k)} + f_{n,\beta}^{(k)}(x^*) + a_n^\beta I_{x^*}^\beta q_n^{(k)}(x^*) \\
&= \hat{y}_{n-1,\beta}^{(k)}.
\end{aligned}
$$

Hence, $I_{x_1}^\beta f^{(k)}(x)$ is $\mathcal{C}^p$-linear FIF associated with $\{K; w_n = (L_n(x), \hat{R}_{n,\beta}^{(k)}(x,y), n \geq 2)\}$.

Corollary 4.5. *Let f be the $\mathcal{C}^p$-linear FIF determined by the CIFS (4.1), $D_{x_1}^\alpha(I_{x_1}^\beta f^{(k)}(x)) = D_{x_1}^{\alpha-\beta} f^{(k)}(x), \alpha \geq \beta \geq 0$, if and only if $I_{x_1}^\beta f^{(k)}(x)$ is the $\mathcal{C}^p$-linear FIF associated with $\{K; w_n = (L_n(x), \hat{R}_{n,\beta}^{(k)}(x,y), n \geq 2)\}$, where $\hat{R}_{n,\beta}^{(k)}(x,y) = a_n^\beta \hat{r}_n y + a_n^\beta \hat{q}_{n,\beta}^{(k)}(x)$, $\hat{r}_n = r_n a_n^\beta$, $D_{x_1}^\alpha(\hat{r}_{n,\beta}^{(k)}(x)) = a_n^\beta D_{x_1}^{\alpha-\beta} q_n^{(k)}(x)$.*

Proof. Consider the CIFS $\{K; w_n = (L_n(x), F_n(x,y), n \geq 2)\}$, where $L_n(x) = a_n x + b_n$, $R_n(x,y) = \alpha_n y + q_n(x)$. Here $|\alpha_n| < 1$ and $q_n(x) \in \mathcal{C}^p(I)$. Let f be the $\mathcal{C}^p$-linear FIF. Assume that $D_{x_1}^\alpha(I_{x_1}^\beta f^{(k)}(x)) = D_{x_1}^{\alpha-\beta} f^{(k)}(x)$, $\alpha \geq \beta \geq 0$. $I_{x_1}^\beta f^{(k)}(x)$ is the $\mathcal{C}^p$-linear FIF associated with

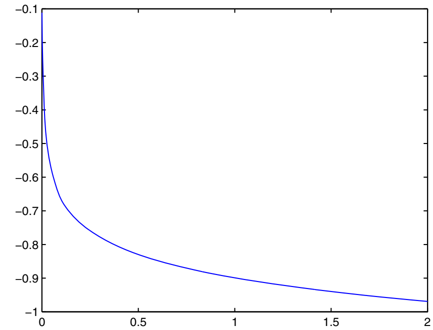
$$
\begin{aligned}
\hat{R}_{n,\beta}^{(k)}(x,y) &= a_n^\beta r_n y + \hat{q}_{n,\beta}^{(k)}(x), \\
a_n &= \frac{(x_n - x_{n-1})}{(x^* - x_1)} \\
\hat{q}_{n,\beta}^{(k)}(x) &= \hat{y}_{n-1,\beta}^{(k)} + f_{n-1,\beta}^{(k)}(x) + a_n^\beta I_{x_1}^\beta q_n^{(k)}(x) \\
D_{x_1}^\alpha(\hat{R}_{n,\beta}^{(k)}(x,y)) &= D_{x_1}^\alpha\{a_n^\beta r_n y + I_{x_1}^\beta f^{(k)}(x_{n-1})\} \\
&\quad + D_{x_1}^\alpha\left\{\frac{1}{\Gamma(\beta)}\int_{x_1}^{x_{n-2}}[(L_n(x)-t)^{\beta-1} - (x-t)^{\beta-1}] f^{(k)}(t)\mathrm{d}t\right\} \\
&\quad + \{D_{x_1}^\alpha a_n^\beta I_{x_1}^\beta q_n^{(k)}(x)\},
\end{aligned}
$$

but $D_{x_1}^\alpha(\hat{R}_{n,\beta}^{(k)}(x,y)) = R_n(x,y) = r_n y + D_{x_1}^{\alpha-\beta} q_n^{(k)}(x)$, therefore it must occur when $\hat{R}_{n,\beta}^{(k)}(x,y) = a_n^\beta \hat{r}_n y + a_n^\beta \hat{q}_{n,\beta}^{(k)}(x)$.

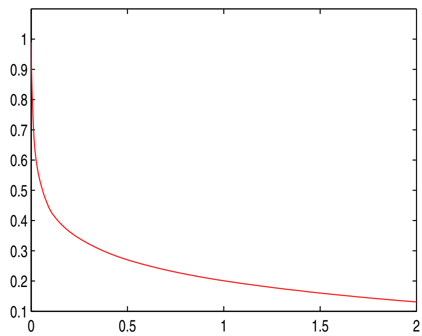
Conversely, if $I_{x_1}^\beta f^{(k)}(t)$ is $\mathcal{C}^p$-linear CFIF associated with $\hat{R}_{n,\beta}^{(k)}(x,y) = a_n^\beta \hat{r}_n y + a_n^\beta \hat{q}_{n,\beta}^{(k)}(x)$, $D_{x_1}^\alpha(\hat{R}_{n,\beta}^{(k)}(x,y)) = r_n y + D_{x_1}^{\alpha-\beta} q_n^{(k)}(x)$, it gives that $D_{x_1}^\alpha(I_{x_1}^\beta f^{(k)}(x)) = D_{x_1}^{\alpha-\beta} f^{(k)}(x)$.

Example 4.6. If FIF f generated by the CIFS of the form $L_n(x) = \frac{1}{n^2-n}x + \frac{2n^2-3n-1}{n^2-n}$, $R_n(x,y) = (\frac{1}{(n^2-n)} - \frac{1}{2})x + \frac{1}{2}y + \frac{n^2-2n-1}{n^2-n}$ and f passing through the data set $\{(x_n, y_n), n \in \mathbb{N}\}$, where $x_n = (\frac{2n-1}{n})_{n=1}^\infty, y_n = (\frac{-1}{n})_{n=1}^\infty$. If we choose $\hat{y}_1 = y_1$, then FIF $\hat{f}$ generated by the CIFS of the form

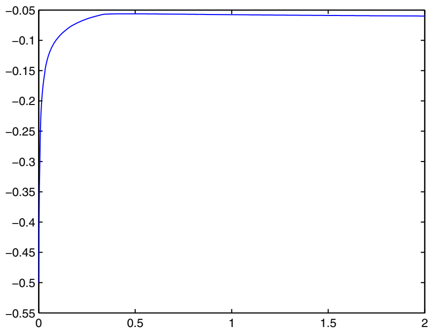
$$
\begin{aligned}
\hat{R}_n(x,y) = \frac{y}{2(n^2-n)} + \hat{y}_{n-1} + \frac{1}{2(n^2-n)} \\
+ \frac{1}{n^2-n}\left\{\left(\frac{1}{2(n^2-n)} - \frac{1}{4}\right)(x^2-1) + \left(\frac{n^2-2n-1}{n^2-n}\right)(x-1)\right\},
\end{aligned}
$$



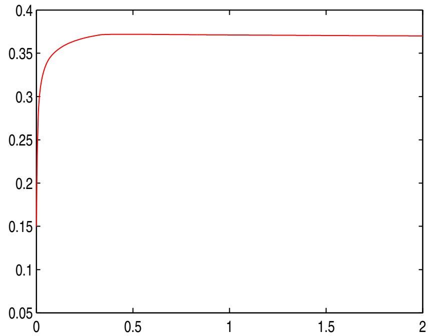
(a) $\hat{f}$ with initial endpoint, $\hat{y}_1 = -1, r_n = 1/2$



(b) $\hat{f}$ with final endpoint, $\hat{y}^* = 0, r_n = 1/2$



(c) $I^{\beta}_{x_1} f^{(k)}(x)$ with $\hat{y}_{1,\beta} = 0, \beta = 0.5, r_n = 1/2$



(d) $I^{\beta}_{x^*} f^{(k)}(x)$ with $\hat{y}^*_{\beta} = 0, \beta = 0.5, r_n = 1/2$

Figure 3. Fractional and ordinary integral of FIF for CIFS

where $\hat{y}_{n-1} = -1 + \sum_{i=2}^{n-1} \frac{24i^2 - 1(13+2\pi^2)-2}{8(i^2-i)}$.

$\hat{f}$ interpolates the data $\{(x_n, \hat{y}_{n-1}), n \in \mathbb{N}\}$, when $\hat{f}$ value chosen at the initial endpoint $\hat{y}_1 = -1$ and graph of $\hat{f}$ is presented in Fig. 4a, c reveals the fractional integral of order 0.5 of $\hat{f}$ with $\hat{y}_{1,\beta} = 0$, contraction ratio $r_n = 1/2$. If we choose $\hat{y}^* = y^*$, then FIF $\hat{f}$ generated by the CIFS of the form

$$\hat{R}_n(x,y) = \frac{y}{2(n^2-n)} + \hat{y}_n - \frac{1}{n^2-n}\left\{\left(\frac{1}{2(n^2-n)} - \frac{1}{4}\right)(4-x^2) + \left(\frac{n^2-2n-1}{n^2-n}\right)(2-x)\right\},$$

where $\hat{y}_n = -\sum_{i=2}^{n} \frac{i^2(10+2\pi^2) - i(28+2\pi^2)-2}{8(i^2-i)}$.

Here $\hat{f}$ interpolates the data $\{(x_n, \hat{y}_n), n \in \mathbb{N}\}$, when $\hat{f}$ value chosen at the terminal endpoint as $\hat{y}_1 = 0$ and graph of $\hat{f}$ is presented in Fig. 4b; further Fig. 4d reveals the fractional integral of order 0.5 of $\hat{f}$ with $\hat{y}^*_{\beta} = 0$ and contraction ratio $r_n = 1/2$ (Fig. 3).

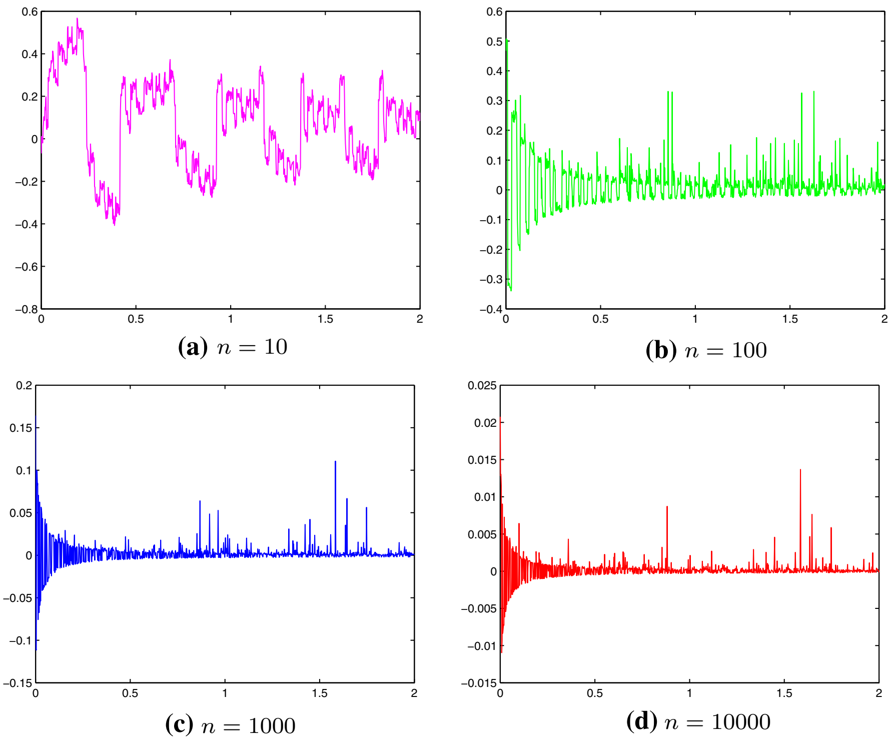
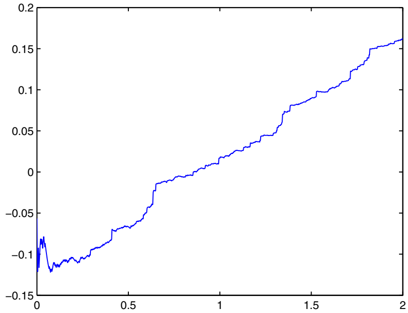


Figure 4. FIF of CIFS with $r_n = 1/3$

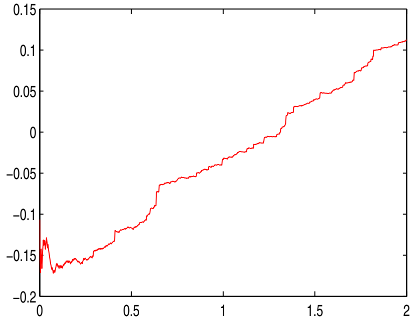
Example 4.7. If FIF f generated by the CIFS of the form $L_n(x) = \frac{1}{n^2-n}x + \frac{2n^2-3n-1}{n^2-n}$, $R_n(x,y) = (\frac{(-1)^{n-1}}{(n^2-n)} - \frac{1}{2})x + \frac{1}{3}y + \frac{(-1)^{n-1}(3n+1)}{n^2-n} + \frac{5}{6}$ and f passing through the data set $\{(x_n, y_n), n \in \mathbb{N}\}$, where $x_n = (\frac{2n-1}{n})_{n=1}^{\infty}, y_n = (\frac{(-1)^n}{n})_{n=1}^{\infty}$. Pictorial representation of attractors of CIFS approximation process is given as in Fig. 4. If we choose $\hat{y}_1 = -1$, then FIF $\hat{f}$ generated by the CIFS of the form $\hat{R}_n(x,y) = \frac{y}{3(n^2-n)} + \hat{y}_{n-1} + \frac{1}{3(n^2-n)} + \frac{1}{n^2-n}\{(\frac{(-1)^{n-1}}{n^2-n} - \frac{1}{2})(\frac{x^2-1}{2}) + (\frac{(-1)^{n-1}(3n+1)}{n^2-n} + \frac{5}{6})(x-1)\}$, where $\hat{y}_{n-1} = -1 + \sum_{i=2}^{n-1} \frac{1}{i^2-i}(\frac{(-1)^{n-1}(1-6i)}{2(i^2-i)} + \frac{1788}{1327})$.

$\hat{f}$ interpolates the data $\{(x_n, \hat{y}_{n-1}), n \in \mathbb{N}\}$, when $\hat{f}$ value chosen at the initial endpoint $\hat{y}_1 = -1$ and graph of $\hat{f}$ is presented in Fig. 5a, c reveals the fractional integral of order 0.5 with $r_n = 1/3, \hat{y}_{1,\beta} = 0$. If we choose $\hat{y}^* = 1/2$, then FIF $\hat{f}$ generated by the CIFS of the form $L_n(x)$ remains the same as found above and

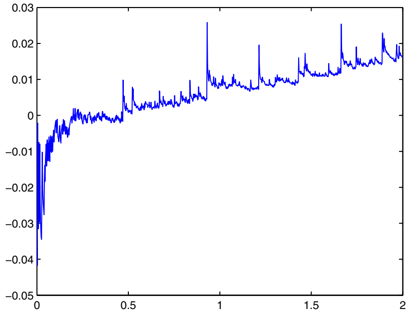
$$\hat{R}_n(x,y) = \frac{y}{3(n^2-n)} + \hat{y}_n - \frac{1}{6(n^2-n)} - \frac{1}{n^2-n}\left\{\left(\frac{(-1)^{n-1}}{n^2-n} - \frac{1}{2}\right)\left(\frac{4-x^2}{2}\right) + \left(\frac{(-1)^{n-1}(3n+1)}{n^2-n} + \frac{5}{6}\right)(2-x)\right\},$$



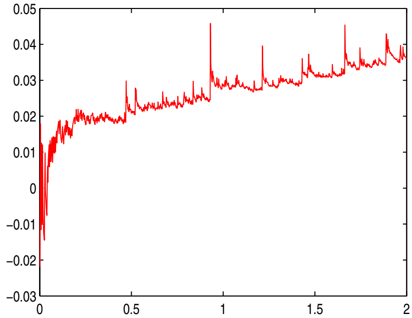
(a) $\hat{f}$ with $\hat{y}_1 = -1$, $r_n = 1/3$



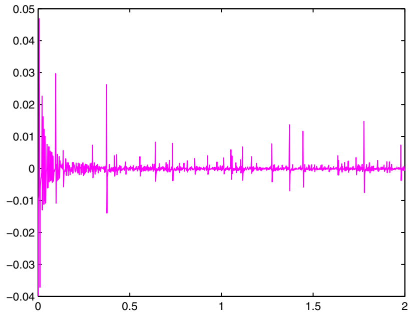
(b) $\hat{f}$ with $\hat{y}^* = 0$, $r_n = 1/3$



(c) $I^{\beta}_{x_1} f^{(k)}(x)$ with $\hat{y}_{1,\beta} = 0, \beta = 0.5, r_n = 1/3$



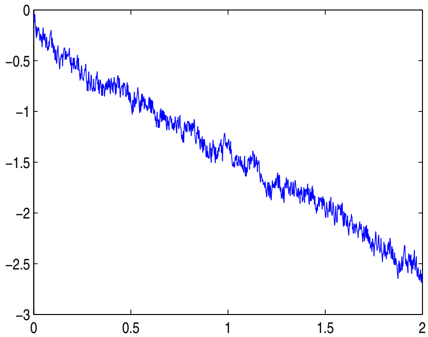
(d) $I^{\beta}_{x^*} f^{(k)}(x)$ with $\hat{y}^*_{\beta} = 0, \beta = 0.5, r_n = 1/3$



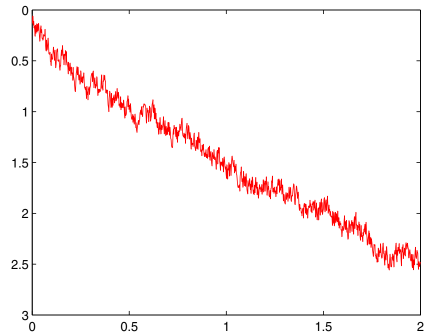
(e) $D^{\beta}_{x_1} f^{(k)}(x)$ with $\beta = 0.5, r_n = 1/3$

Figure 5. Fractional, ordinary integral and fractional derivative of FIF for CIFS

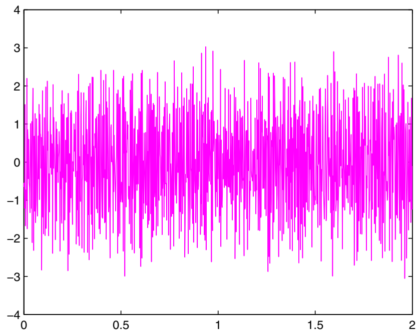
where $\hat{y}_n = \frac{1}{2} - \sum_{i=2}^{n} \frac{1}{i^2-i}\left(\frac{(-1)^{n-1}(1-6i)}{2(i^2-i)} + \frac{1788}{1327}\right)$. Here $\hat{f}$ interpolates the data $\{(x_n, \hat{y}_n), n \in \mathbb{N}\}$, when $\hat{f}$ value chosen at the terminal endpoint as $\hat{y}^* = 0$ and graph of $\hat{f}$ is presented in Fig. 5b, further Fig. 5d reveals the fractional integral of order 0.5 of $\hat{f}$, when $\hat{y}^*_{\beta} = 0$ and Fig. 5e provides the fractional derivative of order 0.5 with $r_n = 1/3$ of FIF f. Finally, Fig. 6a, b presents,



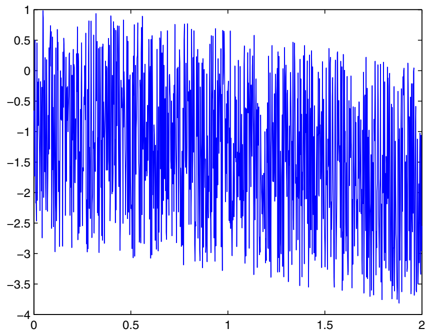
(a) $I^{\beta}_{x_1} f^{(k)}(x)$ with $\hat{y}_1 = -1$, $\beta = 0.5, r_n = 1/2$



(b) $I^{\beta}_{x^*} f^{(k)}(x)$ with $\hat{y}^* = 0$, $\beta = 0.5, r_n = 1/2$



(c) $D^{\beta}_{x_1} f^{(k)}(x)$ with $\beta = 0.5, r_n = 1/2$



(d) fractional derivative of order 0.8 of $I^{\beta}_{x_1} f^{(k)}(x)$ with, $\beta = 0.5, r_n = 1/2$

Figure 6. Fractional integral and fractional derivative of FIF for CIFS

respectively, the fractional integral of order $0.5, r_n = 1/3$, with initial and final endpoint values of $I^{\beta}_{a} f^{(k)}(x)$ are defined as $\hat{y}_{1,\beta} = \hat{y}^*_{\beta} = 0$, where FIF is considered as in Example 3.6. Figure 6c presents the fractional derivative of FIF f and Fig. 6d reveals the fractional derivative of order 0.8 of $I^{\beta}_{x_1} f^{(k)}(x)$ with $\beta = 0.5, r_n = 1/2$.

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