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The quaternion group has ghost number three

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A B S T R A C T

We prove that the group algebra of the quaternion group Q_8 over any field of characteristic two has ghost number three.

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1. Introduction

The study of ghost maps in stable categories originated with Freyd's generating hypothesis in homotopy theory [6], which is still an open question. In this paper we are concerned with ghosts in modular representation theory. Let G be a group and K a field of characteristic p. A map $f\colon M \to N$ in the stable category $\operatorname{stmod}(KG)$ of finitely generated KG-modules is called a *ghost* if it vanishes under Tate cohomology, that is if $f_*\colon \hat{H}^*(G,M) \to \hat{H}^*(G,N)$ is zero. The ghost maps then form an ideal in $\operatorname{stmod}(KG)$, and Chebolu, Christensen and Mináč [3] define the *ghost number* of KG to be the nilpotency degree of this ideal.

Determining the exact value of the ghost number is hard in all but the simplest cases. In [4], Christensen and Wang studied ghost numbers for p-group algebras. They gave conjectural upper and lower bounds for the ghost number of an arbitrary p-group, and also showed that the ghost number (over a field of characteristic two) of the quaternion group Q_8 is either three or four. In our earlier paper [1], we established most cases of their conjectural bounds. In this paper, we shall prove the following theorem.

Theorem 1.1. *Let K be any field of characteristic two. Then the group algebra KQ_8 has ghost number three.*

Let us call a composition of n ghost maps an *n-fold ghost*. Given the result of Christensen and Wang on Q_8, our Theorem 1.1 is equivalent to the statement that every threefold ghost map $M \xrightarrow{f} N$ is stably trivial. To show this, we take any embedding $M \rightarrowtail I$ of M in a finitely generated KQ_8-module and show that f factors through I.

In Section 2, we recall Dade's presentation of the group algebra KQ_8 and derive some properties of ghost maps, including the crucial Lemma 2.5. In Section 3, we recall a theorem of Kronecker which classifies the linear relations on a vector space V. This leads us to the construction of the lift in Section 4: We have $I = KQ_8 \otimes_K V$ for some K-vector space V. As we may assume M to be projective-free, we have $M \subseteq J \otimes_K V$ for J the Jacobson radical $J = J(KQ_8)$. Since a threefold ghost kills $\operatorname{soc}^3(M)$, it follows that f factors through $M/\operatorname{soc}^3(M)$, which is a subspace of $(J/J^2) \otimes_K V \cong V^2$. That is, $M/\operatorname{soc}^3(M)$ is a linear relation on V; and using Lemma 2.5 we are able to construct a lift for each indecomposable summand in its Kronecker decomposition, thus proving the theorem.

2. Ghost maps and Dade's generators

We only need the following property of ghost maps.

Lemma 2.1 *([3], Proposition 2.1). Let G be a p-group, K a field of characteristic p, and $M \xrightarrow{f} N$ a ghost map between projective-free KG-modules. Then $\operatorname{Im}(f) \subseteq \operatorname{rad}(N)$ and $\operatorname{soc}(M) \subseteq \ker(f)$.* $\square$

The next result is presumably well-known.

Lemma 2.2. *Let G be a finite group, K/k a finite field extension, and $M \xrightarrow{f} N$ a map in* $\mathrm{stmod}(kG)$*. If $K \otimes_k M \xrightarrow{\mathrm{Id}_K \otimes f} K \otimes_k N$ is trivial in* $\mathrm{stmod}(KG)$*, then f is trivial in* $\mathrm{stmod}(kG)$*. Hence* $\text{ghost number}(kG) \leq \text{ghost number}(KG)$.

Proof. As a map of k-vector spaces, inclusion $k \overset{i}{\hookrightarrow} K$ is a split monomorphism; let $K \overset{\pi}{\twoheadrightarrow} k$ be a splitting. Suppose that $\mathrm{Id}_K \otimes f$ factors through a finitely generated KG-projective module P. Then $f = (\pi \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_K \otimes f) \circ (i \otimes \mathrm{Id}_M)$ also factors through P, which is also a finitely generated kG-projective module. The last part follows, since extending scalars preserves ghost maps. □

Remark 2.3. Consider now the quaternion group $Q_8 = \langle i, j \rangle$. Let K be a field of characteristic 2 which contains $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$. In [5, (1.2)], Dade defines $x, y \in J(KQ_8)$ by

$$x = \omega i + \bar{\omega} j + ij \qquad y = \bar{\omega} i + \omega j + ij\,.$$

He then shows that KQ_8 is the K-algebra generated by x, y with relations

$$x^2 = yxy \qquad y^2 = xyx \qquad xy^2 = y^2x = x^2y = yx^2 = 0\,.$$

Hence $1, x, y, xy, yx, xyx, yxy, xyxy = yxyx$ is a K-basis of KQ_8. Moreover, the terms in the radical and socle series of KQ_8 are as follows:

$$\begin{array}{llll}
\mathrm{rad}^0(KQ_8) & = \mathrm{soc}^5(KQ_8) & \text{is } KQ_8 & \\
\mathrm{rad}(KQ_8) & = \mathrm{soc}^4(KQ_8) & \text{has basis} & x, y, xy, yx, xyx, yxy, xyxy \\
\mathrm{rad}^2(KQ_8) & = \mathrm{soc}^3(KQ_8) & \text{has basis} & xy, yx, xyx, yxy, xyxy \\
\mathrm{rad}^3(KQ_8) & = \mathrm{soc}^2(KQ_8) & \text{has basis} & xyx, yxy, xyxy \\
\mathrm{rad}^4(KQ_8) & = \mathrm{soc}(KQ_8) & \text{has basis} & xyxy \\
\mathrm{rad}^5(KQ_8) & = \mathrm{soc}^0(KQ_8) & \text{is } 0. &
\end{array}$$

Notation. From now on, we write $R = KQ_8$ and $J = J(R) = \mathrm{rad}(R) = (x, y) \trianglelefteq R$. Hence $J^n = \mathrm{rad}^n(KQ_8) = \mathrm{soc}^{5-n}(KQ_8)$.

Lemma 2.4. *Suppose that $[t + J^2(R)] \in \mathbb{P}(J/J^2)$ is neither $[x + J^2]$ nor $[y + J^2]$. Then for all R-modules M, the map* $\mathrm{rad}(M) \to \mathrm{rad}^2(M)$*, $m \mapsto tm$ is surjective.*

Proof. It is enough to prove the case $M = R$; and by Nakayama it suffices to prove that the map $J/J^2 \to J^2/J^3$, $r + J^2 \mapsto tr + J^3$ is surjective. As J/J^2 and J^2/J^3 are both two-dimensional, $r \mapsto tr$ is surjective if and only if it is injective.

If $t \in \alpha x + \beta y + J^2(R)$ and $r \in \lambda x + \mu y + J^2(R)$ and then $tr \in \alpha\mu xy + \beta\lambda yx + J^3(R)$. So if $tr \in J^3$ then $\alpha\mu = 0 = \beta\lambda$. But the assumption on t means that α, β are both non-zero: so $r \in 0 + J^2$. □

Lemma 2.5. *Suppose that $M \xrightarrow{f} N$ is a threefold ghost for KQ_8, with M, N projective-free. Embedding M in an injective module $R \otimes_K V$ for some K-vector space V, we have $M \subseteq J \otimes_K V$. Suppose further that $m \in M$ satisfies $m \in t \otimes v + J^2 \otimes_K V$ with $v \in V$ and $t \in \{x, y\}$. Then there is an $n \in N$ such that*

$$f(m) = \begin{cases} xyxn & t = x \\ yxyn & t = y \end{cases} .$$

Proof. We treat the case $t = x$; the other case is analogous. Hence $m = x \otimes v + xyu + yxw$ for some $u, w \in R \otimes_K V$, and so $yxm = xyxyw \in \operatorname{soc}(M)$. Let

$$M = N_0 \xrightarrow{f_1} N_1 \xrightarrow{f_2} N_2 \xrightarrow{f_3} N_3 = N$$

be a realisation of f as a threefold ghost, with N_1 and N_2 projective-free. Recall from Lemma 2.1 that $\operatorname{soc}(N_{i-1}) \subseteq \ker(f_i)$ and $\operatorname{Im}(f_i) \subseteq \operatorname{rad}(N_i)$.

Since $\operatorname{soc}(M) \subseteq \ker(f_1)$ it follows that $yxf_1(m) = 0$. As $\operatorname{Im}(f_1) \subseteq \operatorname{rad}(N_1)$ there are $\alpha, \beta \in N_1$ with $f_1(m) = x\alpha + y\beta$. Since $yxf_1(m) = 0$, we deduce that $yxy\beta = 0$ and hence $xy\beta \in \operatorname{soc}(N_1) \subseteq \ker(f_2)$.

Therefore $xyf_2(\beta) = 0$. But $\operatorname{Im}(f_2) \subseteq \operatorname{rad}(N_2)$, and so $f_2(\beta) = x\gamma + y\delta$ with $\gamma, \delta \in N_2$. From $xyf_2(\beta) = 0$ it follows that $xyx\gamma = 0$, hence $yx\gamma \in \operatorname{soc}(N_2) \subseteq \ker(f_3)$ and $yxf_3(\gamma) = 0$. It follows that

$$f(m) = xf_3f_2(\alpha) + yxf_3(\gamma) + xyxf_3(\delta) = xf_3f_2(\alpha) ,$$

since $f_3(\delta) \in \operatorname{rad}(N)$ and therefore $xyxf_3(\delta) \in \operatorname{rad}^4(N) = 0$. So $f(m) = xn'$ for $n' = f_3f_2(\alpha) \in \operatorname{rad}^2(N)$. But then $n' = xyn'_1 + yxn'_2$ for some $n'_1, n'_2 \in N$, and so $f(m) = xyxn'_2$. □

3. Kronecker's Theorem

Theorem 3.1 *(Kronecker). Let K be a field, V a finite-dimensional K-vector space, and $L \subseteq V^2$ a subspace. Suppose further that the pair (V, L) is indecomposable, in the following sense: $V \neq 0$, and there is no proper direct sum decomposition $V = V_1 \oplus V_2$ such that $L = (L \cap V_1^2) \oplus (L \cap V_2^2)$. Then there is a basis $e_1, \dots, e_n$ of V such that one of the following cases holds:*

1. *L has basis $(e_1, 0), (e_2, e_1), (e_3, e_2), \dots, (e_n, e_{n-1}), (0, e_n)$.*
2. *L either has basis $(e_1, 0)$, (e_2, e_1), (e_3, e_2), $\dots$, (e_n, e_{n-1}) or it has basis $(0, e_1)$, (e_1, e_2), (e_2, e_3), $\dots$, (e_{n-1}, e_n).*
3. *L has basis $(e_2, e_1), (e_3, e_2), \dots, (e_n, e_{n-1})$.*
4. *$L = \{(v, F(v)) \mid v \in V\}$ for an automorphism F of V which has indecomposable rational canonical form with respect to the basis $e_1, \dots, e_n$. A rational canonical form*

is indecomposable if it consists of only one block, whose characteristic polynomial is moreover a power of an irreducible element of $K[X]$.

Proof. In the language of [2, p. 112], the assumptions say that L is an indecomposable linear relation on V, which is the same thing as an indecomposable representation of the Kronecker quiver with $\ker(a) \cap \ker(b) \neq 0$. So the result can be read off from Kronecker's Theorem (Theorem 4.3.2 of [2]): note that Case (i) in [2] corresponds to our cases (2) and (4). $\square$

Corollary 3.2. *For every subspace $L \subseteq V^2$ there is a direct sum decomposition $V = \bigoplus_{i=1}^r V_i$ such that*

(1) $L = \bigoplus_{i=1}^r L_i$ *for* $L_i = L \cap V_i^2$.
(2) *For each $1 \leq i \leq r$ the pair (V_i, L_i) is indecomposable in the sense of Theorem 3.1.*

We write $(V, L) = \bigoplus_{i=1}^r (V_i, L_i)$. $\square$

4. Constructing the lift

Recall from Remark 2.3 that $x + J^2$, $y + J^2$ is a basis of J/J^2. Let V be a finite dimensional K-vector space. Then any submodule $M \subseteq J \otimes_K V$ defines a subspace of V^2:

$$L_{x,y}(M) := \{(u, w) \in V^2 \mid x \otimes u + y \otimes w \in M + J^2 \otimes_K V\}.$$

The following result is an immediate consequence of the description of the radical and socle series in Remark 2.3.

Lemma 4.1. *Let $M \subseteq J \otimes_K V$. Then*

(1) $\operatorname{soc}^3(M) = M \cap (J^2 \otimes_K V)$.
(2) *Set $L = L_{x,y}(M)$, and let $(V, L) = \bigoplus_{i=1}^r (V_i, L_i)$ be the direct sum decomposition of Corollary 3.2. If each L_i has basis $(u_{i1}, w_{i1}), \dots, (u_{i,d_i}, w_{i,d_i})$, then for any choice of elements*

$$m_{ij} \in M \cap (x \otimes u_{ij} + y \otimes w_{ij} + J^2 \otimes_K V),$$

we have $M = \operatorname{soc}^3(M) + \sum_{i=1}^N M_i$, where $M_i = \sum_{j=1}^{d_i} R m_{ij}$. $\square$

Proposition 4.2. *For $M \subseteq J \otimes_K V$ set $L = L_{x,y}(M)$. Let $(V, L) = \bigoplus_{i=1}^r (V_i, L_i)$ be a decomposition into indecomposables. Suppose additionally that for each indecomposable pair (V_i, L_i) which satisfies Case (4) of Theorem 3.1, the roots of the characteristic polynomial of the automorphism F all lie in K.*

Suppose further that N is projective-free. Then every threefold ghost $M \xrightarrow{f} N$ extends to a map $R \otimes_K V \xrightarrow{\bar{f}} \mathrm{rad}^2(N)$.

Proof. Suppose first that the indecomposable (V_i, L_i) satisfies Case (1) of Theorem 3.1. Then V_i has a basis $e_1, \dots, e_n$ such that L_i has basis $(0, e_1)$, (e_1, e_2), (e_2, e_3), $\dots$, $(e_{n-1}, e_n), (e_n, 0)$. By construction of L, there are $m_0, \dots, m_n \in M$ such that $m_j \in x \otimes e_j + y \otimes e_{j+1} + J^2 \otimes_K V$, where $e_0 = e_{n+1} = 0$. Since $\mathrm{Im}(f) \subseteq \mathrm{rad}^3(N)$ there are $a_j, b_j \in N$ for $0 \leq j \leq n$ such that

$$f(m_j) = xyxa_j + yxyb_j \, ;$$

and by Lemma 2.5, we may take $a_0 = b_n = 0$. We then define $\bar{f}$ on $R \otimes_K V_i$ by $\bar{f}(1 \otimes e_j) = xyb_{j-1} + yxa_j$. It follows that

$$\bar{f}(y \otimes e_1) = f(m_0) \qquad \bar{f}(x \otimes e_j + y \otimes e_{j+1}) = f(m_j) \qquad \bar{f}(x \otimes e_n) = f(m_n) \, .$$

The two subcases of Case (2) are analogous to each other, so we only consider the case where L_i has basis $(0, e_1)$, (e_1, e_2), (e_2, e_3), $\dots$, (e_{n-1}, e_n). This corresponds to the case $f(m_n) = 0$ of Case (1) above, where we may take $a_n = 0$.

Case (3) is even simpler: this time we have $f(m_0) = f(m_n) = 0$ and therefore $b_0 = a_n = 0$.

Case (4): By assumption, the matrix of F with respect to the basis $e_1, \dots, e_n$ of V_i is a rational canonical form which has only one block, and the minimal polynomial of this block is $(X - \lambda)^n$ for some $\lambda \in K^\times$. It follows that there is a basis $e'_1, \dots, e'_n$ of V_i with respect to which the matrix of F is the $(n \times n)$ Jordan block for the eigenvalue λ. Consequently, L_i has basis

$$(e'_1, \lambda e'_1) \, , \qquad (e'_j, e'_{j-1} + \lambda e'_j) \text{ for } 2 \leq j \leq n.$$

We may therefore pick elements $m_1, \dots, m_n \in M$ such that

$$\begin{aligned} & m_1 \in (x + \lambda y) \otimes e'_1 + J^2 \otimes_K V \\ & m_j \in y \otimes e'_{j-1} + (x + \lambda y) \otimes e'_j + J^2 \otimes_K V \quad \text{for } 2 \leq j \leq n. \end{aligned}$$

So since $f(m_j) \in \mathrm{rad}^3(N)$ for all j, and since $[x + \lambda y + J^2]$ is neither $[x + J^2]$ nor $[y + J^2]$, Lemma 2.4 tells us that we can inductively pick $\bar{f}(1 \otimes e'_1)$, $\dots$, $\bar{f}(1 \otimes e'_n) \in \mathrm{rad}^2(N)$ such that

$$\begin{aligned} & \bar{f}((x + \lambda y) \otimes e'_1) = f(m_1) \\ & \bar{f}((x + \lambda y) \otimes e'_j) = f(m_j) + \bar{f}(y \otimes e'_{j-1}) \quad \text{for } 2 \leq j \leq n. \end{aligned}$$

Treating each summand (V_i, L_i) in this way we obtain a map $\bar{f}\colon R \otimes_K V \to \mathrm{rad}^2(N)$, which therefore satisfies $\bar{f}(J^2 \otimes_K V) = 0$. It follows that all the equations above such as $\bar{f}(x \otimes e_j + y \otimes e_{j+1}) = f(m_j)$ can be simplified to $\bar{f}(m_j) = f(m_j)$. As f and $\bar{f}$ are also both zero on $\mathrm{soc}^3(M) \subseteq J^2 \otimes_K V$, it follows by Lemma 4.1 that $\bar{f}|_M = f$. $\square$

Proof of Theorem 1.1. By [4], the ghost number is at least three. So we have to show that every threefold ghost $M \xrightarrow{f} N$ is stably trivial. Stripping projective summands if necessary, we may assume that M, N are projective free. Taking an injective hull, we see that M embeds in $R \otimes_K V$ for some finite-dimensional K-vector space V. Since M is projective free, we actually have $M \subseteq J \otimes_K V$.

By Lemma 2.2, we may replace K by a finite extension field: so we may assume that $\mathbb{F}_4 \subseteq K$. Set $L = L_{x,y}(M)$. Corollary 3.2 says that (V, L) is a direct sum of indecomposables. Replacing K by a finite extension field again if necessary, we may assume in Case (4) of Theorem 3.1 that the characteristic polynomial of the automorphism F always splits over K. By Proposition 4.2, it follows that f extends to a map $\bar{f}\colon R \otimes_K V \to \mathrm{rad}^2(N)$. As $R \otimes_K V$ is free and hence projective, this factorisation

$$M \xrightarrow{\text{inclusion}} R \otimes_K V \xrightarrow{\bar{f}} \mathrm{rad}^2(N) \hookrightarrow N$$

of f demonstrates that f is stably trivial. $\square$

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