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Rayleigh waves with impedance boundary condition: Formula for the velocity, existence and uniqueness

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A B S T R A C T

The propagation of Rayleigh waves in an isotropic elastic half-space with impedance boundary conditions was investigated recently by Godoy et al. [Wave Motion 49 (2012), 585–594]. The authors have proved the existence and uniqueness of the wave. However, they were not successful in obtaining an analytical exact formula for the wave velocity. The main purpose of this paper is to find such a formula. By using the complex function method, an analytical exact formula for the velocity of Rayleigh waves has been derived. Furthermore, from the obtained formula, the existence and uniqueness of the wave has been established easily.

1. Introduction

Elastic surface waves, discovered by Rayleigh (1885) more than 120 years ago for compressible isotropic elastic solids, have been studied extensively and exploited in a wide range of applications in seismology, acoustics, geophysics, telecommunications industry and materials science, for example. For the Rayleigh wave, its speed is a fundamental quantity which is of great interest to researchers in various fields of science. It is discussed in almost every survey and monograph on the subject of surface acoustic waves in solids. It involves Green's function for many elastodynamic problems for a half-space and it is a convenient tool for evaluating nondestructively pre-stresses of structures before and during loading. Explicit formulas for the Rayleigh wave speed are clearly of practical as well as theoretical interest.

Although the existence and uniqueness theorems for the secular equations of Rayleigh waves were proved, they remained unsolved for more than 100 years because of their complicated and transcendent nature, as mentioned in Voloshin (2010).

In 1995, the first formula for the Rayleigh wave speed in compressible isotropic elastic solids has been obtained by Rahman and Barber (1995). As this formula is defined by two different expressions depending on the sign of the discriminant of the cubic Rayleigh equation, it gives a big inconvenience when applying it to inverse problems.

Employing Riemann problem theory, Nkemzi (1997) derived a formula for the velocity of Rayleigh waves that is expressed as a continuous function of Poisson's ratio. It is rather cumbersome (Destrade, 2003), and the final result as printed in his paper is incorrect (Malischewsky, 2000). Malischewsky (2000) obtained a formula, given by one expression, for the speed of Rayleigh waves by using Cardan's formula together with trigonometric formulas for the roots of a cubic equation and MATHEMATICA. In Malischewsky (2000), it is not shown, however, how Cardan's formula together with the Trigonometric formulas for the roots of the cubic equation are used with MATHEMATICA to obtain the formula.

Vinh and Ogden (2004a) gave a detailed derivation of this formula together with an alternative formula by using the method of cubic equations. Following this method, these authors derived the Rayleigh wave velocity formulas for the orthotropic materials (Ogden and Vinh, 2004; Vinh and Ogden, 2004b, 2005), for the pre-stressed materials (Vinh, 2010, 2011; Vinh and Giang, 2010).

In all works mentioned above, it is assumed that the surface of half-spaces is free of the traction, and the Rayleigh waves are called "Rayleigh waves with traction-free condition". As mentioned in Godoy et al. (2012), in many fields of physics such as acoustics and electromagnetism, it is common to use impedance boundary conditions, that is, when a linear combination of the unknown function and their derivatives is prescribed on the boundary. See, for examples, Antipov (2002), Zakharov (2006), Yla-Oijala and Jarvenppa

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(2006), Mathews and Jeans (2007), Castro and Kapanadze (2008), Qin and Colton (2012) for the acoustics case and Senior (1960), Asghar and Zahid (1986), Stupfel and Poget (2011), Hiptmair et al. (2014) for the electromagnetism one, and the references therein. The Rayleigh waves propagating in half-spaces subjected to impedance boundary conditions are called "Rayleigh waves with impedance boundary condition". It is clear that the Rayleigh waves with traction-free condition is a special class of the Rayleigh waves with impedance boundary condition (see also Godoy et al., 2012). On the other hand, when studying the propagation of Rayleigh waves in a half-space coated by a thin layer, the researchers often replace the effect of the thin layer on the half-space by boundary conditions on the surface of the half-space (see Tiersten, 1969; Bovik, 1996; Dai et al., 2010; Vinh and Linh, 2012; Vinh and Anh, 2014). They are called the effective boundary conditions and are of impedance boundary conditions. As addressed in Makarov et al. (1995), Niklasson et al. (2000), a thin layer on a half-space is a model finding a broad range of applications, including: the Earth's crust in seismology, the foundation/soil interaction in geotechnical engineering, thermal barrier coatings, tissue structures in biomechanics, coated solids in material science, and micro-electro-mechanical systems. The "Rayleigh waves with impedance boundary condition" are therefore significant in many fields of science and technology. The exact analytical expressions for their velocity need to be found.

It should be noted that there are three kinds of Rayleigh waves, namely, subsonic, transonic and supersonic Rayleigh waves (see Barnett and Lothe, 1985) whose velocity is smaller than, equal to and bigger than the limiting velocity $\hat{v}$, respectively. For compressible isotropic half-spaces $\hat{v} = c_2$, where $c_2 = \sqrt{\mu/\rho}$ is the velocity of the transverse wave. Therefore, the velocity c of subsonic Rayleigh waves propagating in these half-spaces satisfies $0 < c < c_2$. The Rayleigh waves mentioned above are all subsonic Rayleigh waves. Since the rest of paper concerns only the subsonic Rayleigh waves, we call them "Rayleigh waves" for seeking the simplicity.

Recently, Godoy et al. (2012) investigated the propagation of Rayleigh waves with impedance boundary condition in an isotropic elastic half-space. The authors have proved the existence and uniqueness of the wave. However, they were not successful in obtaining an analytical exact formula for the wave velocity.

The main purpose of this paper is to find such a formula. By using the complex function method, an analytical exact formula for the velocity of Rayleigh waves has been derived. Furthermore, based on the obtained formula it has been easily shown that there always exists a unique Rayleigh wave.

2. Secular equation

In this section, we present briefly the derivation of secular equation of Rayleigh waves propagating in a compressible isotropic half-space subjected to impedance boundary conditions. For more details, the reader is referred to the Godoy et al. (2012), Malischewsky (1987).

Let us consider a compressible isotropic elastic half-space occupying the domain $x_2 \geq 0$. We are interested in planar motion in the (x_1x_2)-plane with the displacement components u_1, u_2, u_3 such that:

$$u_i = u_i(x_1, x_2, t),\ i = 1, 2,\ u_3 \equiv 0 \tag{1}$$

where t is the time. The equations of motion are of the form (Achenbach, 1973):

$$c_1^2 u_{1,11} + c_2^2 u_{1,22} + \left(c_1^2 - c_2^2\right) u_{2,12} = \ddot{u}_1,$$

$$\left(c_1^2 - c_2^2\right) u_{1,12} + c_1^2 u_{2,22} + c_2^2 u_{2,11} = \ddot{u}_2 \tag{2}$$

in which a superposed dot denotes differentiation with respect to t, commas indicate differentiation with respect to spatial variables x_i, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the speed of the longitudinal wave and of the transverse wave, respectively, ρ is the mass density, λ and μ are the Lame constants. The stress components on the planes $x_2 = \text{const}$ are related to the displacement gradients by:

$$\sigma_{12} = \mu\left(u_{1,2} + u_{2,1}\right),\ \sigma_{22} = \lambda\left(u_{1,1} + u_{2,2}\right) + 2\mu u_{2,2} \tag{3}$$

Suppose that the surface $x_2 = 0$ is subjected to the impedance boundary conditions (Godoy et al., 2012):

$$\sigma_{12} + \omega Z u_1 = 0,\ \sigma_{22} = 0 \text{ at } x_2 = 0 \tag{4}$$

where ω is the wave circular frequency, $Z\ (\in \mathbb{R})$ is the impedance parameter whose dimension is of stess/velocity (see Godoy et al., 2012; Malischewsky, 1987). In addition to Eq. (2) and the boundary condition Eq. (4), the decay condition is required, i. e.:

$$u_i = 0\ (i = 1, 2) \text{ as } x_2 \to +\infty \tag{5}$$

Now we consider the propagation of a Rayleigh wave with the velocity $c\,(>0)$ and the wave number $k\,(=\omega/c>0)$ traveling in the x_1-direction and decaying away from the surface $x_2 = 0$. It is not difficult to verify that the displacement components u_1, u_2 of Rayleigh waves satisfying the equations of motion Eq. (2) and the decay condition Eq. (5) are (Achenbach, 1973):

$$u_1 = \left(A_1 e^{-kb_1x_2} + A_2 e^{-kb_2x_2}\right) e^{ik(x_1 - ct)},$$

$$u_2 = \left(-\frac{b_1}{i} A_1 e^{-kb_1x_2} + \frac{i}{b_2} A_2 e^{-kb_2x_2}\right) e^{ik(x_1 - ct)} \tag{6}$$

where A_1, A_2 are constants to be determined, b_1 and b_2 are given by:

$$b_1 = \sqrt{1 - \frac{c^2}{c_1^2}},\quad b_2 = \sqrt{1 - \frac{c^2}{c_2^2}} \tag{7}$$

Note that both b_1 and b_2 are positive real numbers due to the fact: $0 < c < c_2 < c_1$. Using Eqs. (3) and (6) into the impedance boundary conditions Eq. (4) yields a system of two homogeneous linear equations for A_1, A_2/b_2, namely:

$$\begin{aligned} &(\delta_1 - 2b_1)A_1 + (\delta_1 b_2 + x - 2)\frac{A_2}{b_2} = 0, \\ &(x - 2)A_1 - 2b_2\frac{A_2}{b_2} = 0 \end{aligned} \tag{8}$$

where $x = c^2/c_2^2\ (0 < x < 1)$ is the dimensionless squared velocity of Rayleigh waves, $\gamma = c_2^2/c_1^2\ (0 < \gamma < 1)$ is the dimensionless material parameter and $\delta_1 = (Z\omega)/(\mu k)$. Vanishing the determinant of coefficients of the system Eq. (8) gives:

$$f(x) := (x - 2)^2 - 4\sqrt{1 - x}\sqrt{1 - \gamma x} + \delta x\sqrt{x}\sqrt{1 - x} = 0 \tag{9}$$

where $\delta = Z/\sqrt{\rho\mu}\ (\in \mathbb{R})$ is the dimensionless impedance parameter. This is the secular equation of Rayleigh waves propagating in a compressible isotropic elastic subjected to the impedance boundary condition (4). It coincides with Eq. (15) in Godoy et al. (2012) in other notations. Note that the explicit secular equations of Rayleigh waves propagating in anisotropic half-spaces subjected to impedance boundary conditions were derived recently by Vinh and Hue (2014a, 2014b) and Eq. (9) is a special case of Eq. (25) in Vinh and Hue (2014a).

Remark 1: If a Rayleigh wave exists, then Eq. (9) has a solution x_r so that $0<x_r<1$ and x_r is the dimensionless squared velocity of the Rayleigh wave. Inversely, if Eq. (9) has a solution x_r lying in the interval $(0,1)$, then a Rayleigh wave is possible.

In the next section we will find an exact analytical formula of x_r by employing the complex function method. The existence and uniqueness of Rayleigh waves will be established in Section 4 by using the obtained formula.

3. Exact formula for the Rayleigh wave velocity

3.1. Complex form of secular equation

We introduce the transformation:

$$x=\frac{w+1}{2w}\Leftrightarrow w=\frac{1}{2x-1},\ |w|>1 \tag{10}$$

In terms of the new variable w, Eq. (9) becomes:

$$F(w)=0,\ |w|>1 \tag{11}$$

where:

$$\begin{aligned}F(w):&=(3w-1)^2-8w\sqrt{w-1}\sqrt{(2-\gamma)w-\gamma}\\&+\delta(w+1)\sqrt{w+1}\sqrt{w-1}\end{aligned} \tag{12}$$

Note that the transformation Eq. (10) is a 1–1 mapping from $0<x<1$ to $|w|>1$. Now we consider a complex equation:

$$\begin{aligned}F(z):&=(3z-1)^2-8z\sqrt{z-1}\sqrt{(2-\gamma)z-\gamma}\\&+\delta(z+1)\sqrt{z+1}\sqrt{z-1}=0,\ z\in\mathbb{C}\end{aligned} \tag{13}$$

where $\sqrt{z+1}$, $\sqrt{z-1}$ and $\sqrt{(2-\gamma)z-\gamma}$ are chosen as the principal branches of the corresponding square roots. When $z\in\mathbb{R},\ |z|>1$ Eq. (13) coincides with Eq. (11), therefore Eq. (13) is called the complex form of Eq. (11). In order to find x_r we find a real solution z_r of Eq. (13) so that $|z_r|>1$.

3.2. Properties of function F(z)

Denote $L=L_1\cup L_2$, $L_1=[-1,\gamma/(2-\gamma)]$, $L_2=[\gamma/(2-\gamma),1]$, $S=\{z\in\mathbb{C},z\notin L\}$. $N(z_0)=\{z\in S:|z-z_0|<\varepsilon\}$, ε is a sufficiently small positive number, z_0 is some point of the complex plane $\mathbb{C}$. If a function $\varphi(z)$ is holomorphic in $\Omega\subset\mathbb{C}$ we write $\varphi(z)\in H(\Omega)$. Note that from $0<\gamma<1$ it follows $0<\gamma/(2-\gamma)<1$. Using Eq. (13) it is not difficult to prove that:

Proposition 1. (f_1) $F(z)\in H(S)$.
(f_2) $F(z)=O(z^2)$ as $|z|\to\infty$.
(f_3) $F(z)$ is bounded in $N(-1)$ and $N(1)$.
(f_4) $F(-1)=0$.
(f_5) $F(z)$ is continuous on L from the left and from the right (see Muskhelishvili, 1953) with the boundary values $F^+(t)$ (the right boundary value of $F(z)$), $F^-(t)$ (the left boundary value of $F(z)$) defined as follows:

$$F^{\pm}(t)=\begin{cases}F_1^{\pm}(t),\ t\in L_1\\F_2^{\pm}(t),\ t\in L_2\end{cases} \tag{14}$$

where:

$$\begin{aligned}F_1^+(t)&=(3t-1)^2+8t\sqrt{1-t}\sqrt{\gamma-(2-\gamma)t}\\&+i\delta(t+1)\sqrt{t+1}\sqrt{1-t},\\F_1^-(t)&=(3t-1)^2+8t\sqrt{1-t}\sqrt{\gamma-(2-\gamma)t}\\&-i\delta(t+1)\sqrt{t+1}\sqrt{1-t},\end{aligned} \tag{15}$$

$$\begin{aligned}F_2^+(t)&=(3t-1)^2+i\sqrt{1-t}\Big[\delta(t+1)\sqrt{t+1}\\&-8t\sqrt{(2-\gamma)t-\gamma}\Big],\\F_2^-(t)&=(3t-1)^2-i\sqrt{1-t}\Big[\delta(t+1)\sqrt{t+1}\\&-8t\sqrt{(2-\gamma)t-\gamma}\Big]\end{aligned} \tag{16}$$

3.3. Properties of function $\Gamma(z)$

Now we introduce the function $g(t)$ $(t\in L)$ defined as follows:

$$g(t)=\begin{cases}\dfrac{F_1^+(t)}{F_1^-(t)},\ t\in L_1\\[2ex]\dfrac{F_2^+(t)}{F_2^-(t)},\ t\in L_2\end{cases} \tag{17}$$

From Eqs. (14) and (17) it is clear that:

$$F^+(t)=g(t)F^-(t),\ t\in L \tag{18}$$

Consider the function $\Gamma(z)$ (Cauchy-type integral) defined as:

$$\Gamma(z)=\frac{1}{2\pi i}\int_L\frac{\log g(t)}{t-z}dt \tag{19}$$

It is not difficult to prove the followings (see Muskhelishvili, 1953):

Proposition 2. (γ_1) $\Gamma(z)\in H(S)$.
(γ_2) $\Gamma(\infty)=0$.
(γ_3) $\Gamma(z)=\Omega_0(z),\ z\in N(-1),\ \Gamma(z)=\Omega_1(z),\ z\in N(1)$, where $\Omega_0(z)$ $[\Omega_1(z)]$ is bounded in $N(-1)$ $[N(1)]$ and takes a defined value at $z=-1$ $[z=1]$.

It is noted that (γ_3) comes from the fact (see Muskhelishvili, 1953):

$$\log g(-1)=\log g(1)=0 \tag{20}$$

3.4. Properties of $\Phi(z)$

Introduce the function $\Phi(z)$ given by:

$$\Phi(z)=\exp\Gamma(z) \tag{21}$$

It is implied from $(\gamma_1)-(\gamma_3)$:

Proposition 3. (ϕ_1) $\Phi(z)\in H(S)$.
(ϕ_2) $\Phi(z)=O(1)$ as $|z|\to\infty$.
(ϕ_3) $\Phi(z)$ is bounded in $N(-1)$ and $N(1)$ and takes (non-zero)

defined values at $z=-1$ and $z=1$.

(ϕ_4) $\Phi^+(t)=g(t)\Phi^-(t),\ t\in L$.

Note that for proving (ϕ_4) the Plemelj formula (see Muskhelishvili, 1953) is employed.

3.5. Properties of Y(z)

We now consider the function $Y(z)$ defined as follows:

$$Y(z)=F(z)/\Phi(z) \tag{22}$$

From (f_1)-(f_3), Eq. (18), (ϕ_1)-(ϕ_4) and Eq. (22), it follows that:

Proposition 4. (y_1) $Y(z)\in H(S)$.

(y_2) $Y(z)=O(z^2)$ as $|z|\to\infty$.

(y_3) $Y(z)$ is bounded in $N(-1)$ and $N(1)$.

(y_4) $Y^+(t)=Y^-(t),\ t\in L$.

Proposition 5. $Y(z)$ is a second-order polynomial.

Proof: Properties (y_1) and (y_4) of the function $Y(z)$ show that $Y(z)$ is holomorphic in the entire complex plane **C**, with the possible exception of points: $z=-1$ and $z=1$. By (y_3) these points are removable singularity points and it may be assumed that the function $Y(z)$ is holomorphic in the entire complex plane **C** (see Muskhelishvili, 1963). Thus, by the generalized Liouville theorem (Muskhelishvili, 1963), $Y(z)$ is a polynomial, and according to (y_2), $Y(z)$ is a second-order polynomial:

$$Y(z)=P_2(z):=A_2z^2+A_1z+A_0\ ,\ A_2\neq 0 \tag{23}$$

3.6. Equation F(z)=0 equivalent to a quadratic equation

Proposition 6. Equation $F(z)=0\Leftrightarrow P_2(z)=0$ in the domain $S\cup\{-1\}\cup\{1\}$.

Proof: From Eq. (22) and $Y(z)=P_2(z)$ we have:

$$F(z)=\Phi(z).P_2(z) \tag{24}$$

From (ϕ_1) and (ϕ_3) it follows that $\Phi(z)\neq 0\ \forall\ z\in S\cup\{-1\}\cup\{1\}$. Proposition 6 is proved by this fact and the equality Eq. (24).

Remark 2:

(i) Equation $F(z)=0$ has no solutions in the interval $(-1,1)$ due to the discontinuity of $F(z)$ in this interval, according to (f_5). This means that all solutions of $F(z)=0$ fall in the domain $S\cup\{-1\}\cup\{1\}$.

(ii) As $0<\left|\Phi^{\pm}(t)\right|<\infty\ \forall\ t\in(-1,\ 1)$, therefore by (i) and the equality Eq. (24) two roots of the quadratic equation $P_2(z)=0$ also fall in the domain $S\cup\{-1\}\cup\{1\}$.

(iii) According to Proposition 6, instead of finding the analytical solution of the transcendent equation $F(z)=0$ we look for the one of a much simpler equation, namely the quadratic equation $P_2(z)=0$, in the domain $S\cup\{-1\}\cup\{1\}$.

Proposition 7. Equation $F(z)=0$ has exactly two roots, namely $z_1=-1$ and $z_2=1-A_1/A_2$.

Proof:

- By (f_4), $z_1=-1$ is a solution of the equation $F(z)=0$.

- From Proposition 6 and this fact it follows that $z_1=-1$ is also a root of the equation $P_2(z)=0$. According to Vieta's formulas, the second root of the quadratic equation $P_2(z)=0$ is $z_2=1-A_1/A_2$ and it lies in the domain $S\cup\{-1\}\cup\{1\}$ due to Remark 2 (ii). Again according to Proposition 6, z_2 is a solution of the equation $F(z)=0$.

3.7. Determination of coefficients A_1 and A_2 of P(z)

From Eqs. (21) and (24) we have:

$$P(z)=F(z)e^{-\Gamma(z)}\ . \tag{25}$$

From Eqs. (14)–(18) it follows:

$$\log g(t)=i\theta(t),\ \theta(t):=Arg g(t) \tag{26}$$

where:

$$\theta(t)=\begin{cases}\theta_1(t),\ t\in L_1\\ \theta_2(t),\ t\in L_2\end{cases} \tag{27}$$

in which the functions $\theta_k(t)$ are determined as:

(i) For $\gamma\in(0,\ 1)$ and $\delta>0$:

$$\theta_1(t)=\pi-2\mathrm{atan}\varphi_1(t),\ \theta_2(t)=2\mathrm{atan}\varphi_2(t) \tag{28}$$

(ii) For $\gamma\in(0,\ 1)$ and $\delta<0$:

$$\theta_1(t)=-\pi-2\mathrm{atan}\varphi_1(t),\ \theta_2(t)=2\mathrm{atan}\varphi_2(t) \tag{29}$$

(iii) For $\gamma\in(0,\ 1)$ and $\delta=0$:

$$\theta_1(t)\equiv 0,\ \theta_2(t)=-2\mathrm{atan}\varphi_3(t) \tag{30}$$

where:

$$\begin{aligned}\varphi_1(t)&=\frac{(3t-1)^2+8t\sqrt{1-t}\sqrt{\gamma-(2-\gamma)t}}{\delta(t+1)\sqrt{t+1}\sqrt{1-t}},\\ \varphi_2(t)&=\frac{\sqrt{1-t}\left[\delta(t+1)\sqrt{t+1}-8t\sqrt{(2-\gamma)t-\gamma}\right]}{(3t-1)^2},\\ \varphi_3(t)&=\frac{8t\sqrt{1-t}\sqrt{(2-\gamma)t-\gamma}}{(3t-1)^2}\end{aligned} \tag{31}$$

Using Eq. (19) and the identity:

$$\left(e^{-\Gamma(z)}\right)'=(-\Gamma(z))'e^{-\Gamma(z)} \tag{32}$$

one can see that $e^{-\Gamma(z)}$ can be expanded asymptotically at the infinity as (see also Nkemzi, 1997; Vinh, 2013):

$$e^{-\Gamma(z)}=1+\frac{a_1}{z}+\frac{a_2}{z^2}+O\left(z^{-3}\right) \tag{33}$$

in which:

$$a_1=I_0=\frac{1}{2\pi}\left(\int_{-1}^{\gamma/(2-\gamma)}\theta_1(t)dt+\int_{\gamma/(2-\gamma)}^{1}\theta_2(t)dt\right) \tag{34}$$

Expanding $\sqrt{1-\frac{1}{z}}$, $\sqrt{1+\frac{1}{z}}$ and $\sqrt{1-\frac{\gamma}{(2-\gamma)z}}$ into Laurent series at infinity and then introducing the resulting results in to the expression (13) of $F(z)$ yield:

$$F(z) = B_2 z^2 + B_1 z + B_0 + O\left(z^{-1}\right) \tag{35}$$

where:

$$\begin{cases} B_2 = 9 - 8\sqrt{2-\gamma} + \delta \\ B_1 = -6 + \dfrac{8}{\sqrt{2-\gamma}} + \delta \\ B_0 = 1 + \dfrac{4(\gamma-1)^2}{(2-\gamma)^{3/2}} - \dfrac{1}{2} \end{cases} \tag{36}$$

Introducing Eqs. (33) and (35) into Eq. (25) and taking into account Eqs. (34) and (36) we have:

$$A_2 = 9 - 8\sqrt{2-\gamma} + \delta,\ A_1 = \frac{8}{\sqrt{2-\gamma}} + \delta - 6 + A_2 I_0 \tag{37}$$

I_0 is given by Eq. (34) in which the functions $\theta_k(t)$ are calculated by Eqs. (28)–(31).

Remark 3: Since γ and δ are all real numbers, it implies from Eq. (37), Eqs. (34) and (28)–(31) that the root $z_2 = 1 - A_1/A_2$ of Eq. $F(z) = 0$ is a real number.

3.8. Formula for the wave velocity

Theorem 1: If a Rayleigh wave exists, its dimensionless velocity x_r is given by:

$$x_r = \frac{1+z_2}{2z_2} \tag{38}$$

in which z_2 is expressed in terms of γ and δ as follows:

(i) For $\gamma \in (0,\ 1)$ and $\delta \in \mathbb{R}, \delta > 0$:

$$z_2 = \frac{8 + (\delta-6)\sqrt{2-\gamma}}{8(2-\gamma) - (9+\delta)\sqrt{2-\gamma}} + \frac{1-\gamma}{2-\gamma} + \widehat{I}_0 \tag{39}$$

(ii) For $\gamma \in (0,\ 1)$ and $\delta \in \mathbb{R}, \delta < 0$:

$$z_2 = \frac{8 + (\delta-6)\sqrt{2-\gamma}}{8(2-\gamma) - (9+\delta)\sqrt{2-\gamma}} + \frac{3-\gamma}{2-\gamma} + \widehat{I}_0 \tag{40}$$

(iii) For $\gamma \in (0,\ 1)$ and $\delta = 0$:

$$z_2 = \frac{8(3-\gamma) - 15\sqrt{2-\gamma}}{8(2-\gamma) - 9\sqrt{2-\gamma}} + \frac{8}{\pi}\int_{\gamma/(2-\gamma)}^{1} \operatorname{atan}\varphi_3(t)dt \tag{41}$$

where:

$$\widehat{I}_0 = \frac{1}{\pi}\left(\int_{-1}^{\gamma/(2-\gamma)} \operatorname{atan}\varphi_1(t)dt - \int_{\gamma/(2-\gamma)}^{1} \operatorname{atan}\varphi_2(t)dt\right) \tag{42}$$

and the functions $\varphi_k(t)$ are given by Eq. (31).

Proof:

Suppose that a Rayleigh wave exists. According to Remark 1, the secular equation (9) has a solution x_r lying in the interval (0,1) and it is the dimensionless squared Rayleigh wave velocity. This leads to that the equation $F(w)$=0 has a (real) solution w_r lying in the domain $|w| > 1$ and $x_r = (1+w_r)/2w_r$, according to Eq. (10). Consequently, the (complex) equation $F(z)$=0 has a real root z_r: $|z_r| > 1$. By Proposition 7, Eq. $F(z)$=0 has two solutions $z_1 = -1$ and $z_2 = 1 - A_1/A_2$. As $|z_1| = 1$ it implies that z_2 is a real number and $z_2 = z_r = w_r$. Therefore x_r is given by Eq. (38). From Eqs. (28)–(31), (34), (37) and $z_2 = 1 - A_1/A_2$ we immediately arrive at Eqs. (39)–(41). The proof of Theorem 1 is completed.

Remark 4: The case $\delta = 0$ corresponds to the Rayleigh waves with traction-free boundary condition whose velocity formula has been obtained by Malischewsky (2000), Vinh and Ogden (2004a). Since these formulas are algebraic expressions of γ, they are much more convenient in use than the integral formula {(38), (41)}.

As a checking example, a number of numerical values of x_r are calculated by using the formulas (38)–(42) (denoted by $x_r^{(1)}$) and by directly solving the secular equation (9) in the domain $0 < x < 1$ (denoted by $x_r^{(2)}$). It is seen from Table 1 that they are the same.

Table 1
Some values of the Rayleigh wave velocity that are computed by using the formulas (38)–(42) (denoted by $x_r^{(1)}$) and by directly solving the secular equation (9) in the domain $0 < x < 1$ (denoted by $x_r^{(2)}$). They are the same.

δ	γ	z_2	$x_r^{(1)}$	$x_r^{(2)}$
1	1/4	2.7520	0.6817	0.6817
0	1/4	1.3528	0.8696	0.8696
−1	1/4	1.1541	0.9332	0.9332
2	1/2	−1.6085	0.1892	0.1892
0	1/2	1.8944	0.7639	0.7639
−2	1/2	1.1239	0.9449	0.9449
3	2/3	−1.0980	0.0446	0.0446
0	2/3	5.0278	0.5994	0.5994
−3	2/3	1.0956	0.9564	0.9564

4. Existence and uniqueness of Rayleigh waves

The existence and uniqueness of Rayleigh waves are stated by the following theorem.

Theorem 2: Suppose $\gamma \in (0,\ 1)$ and $\delta \in \mathbb{R}$, then:

(i) A Rayleigh wave is always possible.
(ii) If a Rayleigh wave exists, then it is unique.

Proof:

(i) From Remark 1 and Proposition 7 it implies that the necessary and sufficient conditions for a Rayleigh wave to exist are: $z_2 \in \mathbb{R}$ and $z_2 \notin [-1,\ 1]$.

The fact z_2 is a real number is already stated in Remark 3. As $F(1)$=4 it follows $z_2 \neq 1$. According to Remark 2 (ii) we have $z_2 \notin (-1,\ 1)$.

To finish the proof of the statement (i) we need prove that $z_2 \neq -1$. Suppose $z_2 = -1 (= z_1)$. From Eq. (24), Proposition 6 and (γ_3) it follows $\lim\limits_{z\to-1}\frac{F(z)}{(z+1)^2} = m$, $|m| < \infty$. This leads to $\lim\limits_{x\to 0}\frac{f(x)}{x^2} = n$, $|n| < \infty$. This contradicts the fact $\lim\limits_{x\to 0}\frac{f(x)}{x^2} = \infty$ that is easily proved by using Eq. (9).

(ii) Suppose that there exist two different Rayleigh waves with corresponding velocities $x_r^{(1)}$, $x_r^{(2)}$ ($x_r^{(1)} \neq x_r^{(2)}$). Then $x_r^{(1)}$ and $x_r^{(2)}$ are two different roots of Eq. $f(x)$=0 and $0 < x_r^{(1)}, x_r^{(2)} < 1$ according to Remark 1. Since the transformation Eq. (10) is a 1–1 mapping from $0 < x < 1$ to $|w| > 1$, it follows that Eq. $F(w)$=0 has two different (real) roots lying in the domain $|w| > 1$, so does Eq. $F(z)$=0. From this fact, the Proposition 6 and $F(-1)=0$ it implies that the quadratic equation $P(z)=0$ has three different roots. It is impossible and the proof of the statement (ii) is completed.

Remark 5: Since Eq. (9) is a linear equation for δ, it gives a unique value of δ for a given $x\in(0\ 1)$. From this fact and Theorem 2, it follows immediately that the dimensionless Rayleigh wave velocity $x(\delta)$ is a monotonic function of $\delta\in(-\infty\ +\infty)$ as proved by Godoy et al. (2012).

5. Conclusions

In this paper, an exact analytical formula for the velocity of Rayleigh waves propagating in a compressible isotropic half-space subjected to impedance boundary conditions has been derived by using the complex function method. Based on the obtained formula, the existence and uniqueness of Rayleigh waves have been established immediately. Since the obtained formula is exact and totally explicit, it is of theoretical as well as practical interest.

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