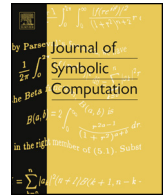




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Extreme rays of Hankel spectrahedra for ternary forms



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ABSTRACT

The cone of sums of squares is one of the central objects in convex algebraic geometry. Its defining linear inequalities correspond to the extreme rays of the dual convex cone. This dual cone is a spectrahedron, which can be explicitly realized as a section of the cone of positive semidefinite matrices with the linear subspace of Hankel (or middle catalecticant) matrices. In this paper we initiate a systematic study of the extreme rays of Hankel spectrahedra for ternary forms. We show that the Zariski closure of the union of extreme rays is the variety of all Hankel matrices of corank at least 4, an irreducible variety of codimension 10 and we determine its degree. We explicitly construct an extreme ray of maximal rank using the Cayley–Bacharach Theorem for plane curves. We apply our results to the study of the algebraic boundary of the cone of sums of squares. Its irreducible components are dual to varieties of Gorenstein ideals with certain Hilbert functions. We determine these Hilbert functions for some cases of small degree. We also observe surprising gaps in the ranks of Hankel matrices of the extreme rays.

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0. Introduction

The following convex cones are fundamental objects in convex algebraic geometry: the cone $P_{n,2d}$ of homogeneous polynomials (forms) of degree $2d$ in $\mathbb{R}[x_1, \dots, x_n]$ that are nonnegative on \mathbb{R}^n , and the cone $\Sigma_{n,2d}$ consisting of sums of squares of degree $2d$. Hilbert showed that only in the following

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three cases every nonnegative form is a sum of squares of forms: bivariate forms, quadratic forms, and ternary forms of degree 4. In all other cases Hilbert showed the existence of nonnegative polynomials that are not sums of squares (Hilbert, 1888).

The dual cones $P_{n,2d}^\vee$ and $\Sigma_{n,2d}^\vee$ consist of all linear functionals nonnegative on the corresponding primal cone. The *extreme rays* of the dual cones provide the defining linear inequalities of the primal cones. Therefore, understanding extreme rays of $\Sigma_{n,2d}^\vee$ is crucial in understanding the boundary of the cone $\Sigma_{n,2d}$, as well as the difference between the cones $P_{n,2d}$ and $\Sigma_{n,2d}$. In the cases where there exist nonnegative polynomials that are not sums of squares, $\Sigma_{n,2d}^\vee$ must contain extreme rays that do not belong to $P_{n,2d}^\vee$. In recent years there has been considerable progress in understanding the extreme rays of $\Sigma_{n,2d}^\vee$ and the *algebraic boundary* of $\Sigma_{n,2d}$, i.e. the Zariski closure of its Euclidean boundary, in the two smallest cases where nonnegative polynomials are not equal to sums of squares: $n = 3, 2d = 6$ and $n = 4, 2d = 4$ (Blekherman, 2012; Blekherman et al., 2012). In Blekherman (2012), extreme rays of $\Sigma_{3,6}^\vee$ and $\Sigma_{4,4}^\vee$ were described using the Cayley–Bacharach theorem. In Blekherman et al. (2012), this description led to a quite surprising connection between the algebraic boundaries of $\Sigma_{3,6}$ and $\Sigma_{4,4}$ and moduli spaces of K3 surfaces. In Blekherman (2015), the first author related the study of extreme rays of $\Sigma_{n,2d}^\vee$ to the associated Gorenstein ideals.

Taking these results as a point of departure, we begin a systematic study of extreme rays of the cone $\Sigma_{n,2d}^\vee$ for ternary forms, i.e. $n = 3$. We will denote the associated cones simply by Σ_{2d} and Σ_{2d}^\vee . Our main technical tool will be the Buchsbaum–Eisenbud structure theorem for ternary Gorenstein ideals, and its refined analysis by Diesel (1996). We will see that irreducible components of the algebraic boundary of Σ_{2d} are dual varieties to varieties of Gorenstein ideals with certain Hilbert functions. This gives us a beautiful melding of convex geometry, commutative algebra, and algebraic geometry.

The case of $2d = 6$ was completely described in Blekherman (2012), Blekherman et al. (2012) and therefore we restrict our attention to $2d \geq 8$. Our first main result deals with the Zariski closure of the set of all extreme rays of Σ_{2d}^\vee and tells us that extreme rays of Σ_{2d}^\vee are plentiful, when compared to extreme rays of P_{2d}^\vee .

Theorem (Theorem 2.15). *For any $d \geq 4$, the Zariski closure of the set of extreme rays of Σ_{2d}^\vee is the variety of Hankel matrices of corank at least 4. It is irreducible, has codimension 10, and degree $\prod_{\alpha=0}^3 \binom{N+\alpha}{4-\alpha} / \binom{2\alpha+1}{\alpha}$, where $N = \binom{d+2}{2}$.*

By contrast, the Zariski closure of the extreme rays of P_{2d}^\vee is the $2d$ -th Veronese embedding of \mathbb{P}^2 and has dimension 2 (Blekherman et al., 2013, Chapter 4). Note that for Σ_6^\vee , it follows from results of Blekherman (2012), Blekherman et al. (2012) that the Zariski closure of the set of extreme rays is the variety of Hankel matrices of corank at least 3. It has dimension 21, codimension 6 and degree 2640. Existence of extreme rays of co-rank 4 is shown via an intricate explicit construction, which makes heavy use of Cayley–Bacharach theorem for plane curves. The details are given in Section 2.

The extreme rays of the dual cone Σ_{2d}^\vee are stratified by the rank of the associated Hankel (middle catalecticant) matrix. This intricate stratification characterizes the algebraic boundary of the sums of squares cone via projective duality theory. We show the following theorem in section 2.

Theorem (Theorem 2.17). *Let X be an irreducible component of the algebraic boundary of Σ_{2d} . Then its dual projective variety X^* is a subvariety of the Zariski closure of the union of extreme rays of Σ_{2d}^\vee , i.e. the variety of Hankel matrices of corank ≥ 4 . Moreover, there is a Hilbert function T such that the quasiprojective variety $\text{Gor}(T)$ of all Gorenstein ideals with Hilbert function T is Zariski dense in X^* .*

We work out the first three nontrivial cases $d = 3, 4, 5$ in Section 3, extending the study of the algebraic boundary of the sums of squares cones for ternary sextics and quaternary quartics in Blekherman et al. (2012). More specifically we show in section 3:

Proposition. *The Hankel spectrahedron Σ_8^\vee has extreme rays of rank 1, 10, and 11. We construct extreme rays of rank 10 and 11 such that the Hilbert function of the corresponding Gorenstein ideal is*

$$T_{10} = (1, 3, 6, 9, 10, 9, 6, 3, 1) \text{ and}$$

$$T_{11} = (1, 3, 6, 10, 11, 10, 6, 3, 1),$$

respectively. The dual varieties to $\text{Gor}(T_{10})$ and $\text{Gor}(T_{11})$ are irreducible components of the algebraic boundary of Σ_8 .

It is possible to show using refined analysis of Diesel (1996, Proposition 3.9) that these are the only Hilbert functions of Gorenstein ideals corresponding to extreme rays of Σ_8^\vee , and thus the algebraic boundary of Σ_8 has 3 irreducible components: the discriminant, which is dual to rank 1 extreme rays, and the dual varieties to $\text{Gor}(T_{10})$ and $\text{Gor}(T_{11})$.

Theorem (Theorem 3.1 for $d = 5$). For every $r \in \{13, \dots, 17\}$, there is an extreme ray $\mathbb{R}_+\ell_r$ of Σ_{10}^\vee such that the rank of the Hankel matrix B_{ℓ_r} is r . We construct extreme rays such that the Hilbert function T_r of the associated Gorenstein ideal $I(\ell_r)$ is:

$$T_{13} = (1, 3, 6, 9, 12, 13, 12, 9, 6, 3, 1),$$

$$T_{14} = (1, 3, 6, 10, 13, 14, 13, 10, 6, 3, 1),$$

$$T_{15} = (1, 3, 6, 10, 14, 15, 14, 10, 6, 3, 1),$$

$$T_{16} = (1, 3, 6, 10, 14, 16, 14, 10, 6, 3, 1),$$

$$T_{17} = (1, 3, 6, 10, 15, 17, 15, 10, 6, 3, 1).$$

The dual varieties to $\text{Gor}(T_r)$ form irreducible components of the algebraic boundary of the sums of squares cone Σ_{10} for all $r \in \{13, \dots, 17\}$.

It follows from the above the theorem that the algebraic boundary of Σ_{10} has at least 6 irreducible components. We conjecture that this list is complete.

We saw previously that for $d \geq 4$ the minimal co-rank of an extreme ray is 4. It was shown in Blekherman (2015) that Σ_{2d}^\vee has no extreme rays of rank r with $1 < r < 3d - 2$. We also see from the above results that this is the only gap in rank of extreme rays for Σ_8^\vee and Σ_{10}^\vee . Surprisingly, the cone Σ_{12}^\vee has another gap in possible ranks of extreme rays. We show the following theorem in section 3.

Theorem. The cones Σ_{2d}^\vee for $d = 4, 5$ have extreme rays of rank r for $r = 1$ and all r such that $3d - 2 \leq r \leq \binom{d+2}{2} - 4$. The cone Σ_{12}^\vee has no extreme ray of rank 17, but has extreme rays of rank r for all $16 \leq r \leq 24$, $r \neq 17$.

We leave the reader with the following open questions:

Open questions.

- (1) What are the possible ranks of Hankel matrices of extreme rays of Σ_{2d}^\vee ?
- (2) Given the rank of a Hankel matrix of an extreme ray of Σ_{2d}^\vee , what are the possible Hilbert functions of the associated Gorenstein ideal? In all observed examples, the rank uniquely determines the Hilbert function for an extreme ray of Σ_{2d}^\vee .
- (3) If there exists an extreme ray of Σ_{2d}^\vee with Gorenstein ideal with Hilbert function T , then is variety $\text{Gor}(T)$ necessarily dual to an irreducible component of $\partial_a \Sigma_{2d}$? We conjecture that this is the case.

1. Hankel matrices and Gorenstein ideals

Let us fix the following notations: We denote by $k[\underline{x}] = k[x, y, z]$ the polynomial ring over a field k generated by 3 variables. We consider it with the standard total degree grading and denote by $k[\underline{x}]_m$ the k -vector space of homogeneous polynomials of degree m , which has dimension $\binom{m+2}{2}$.

A linear functional ℓ on the real vector space of ternary forms of degree $2d$ is non-negative on every square if and only if the bilinear form

$$B_\ell: \begin{cases} \mathbb{R}[x, y, z]_d \times \mathbb{R}[x, y, z]_d \rightarrow \mathbb{R} \\ (f, g) \mapsto \ell(f \cdot g) \end{cases}$$

is positive semi-definite. The representing matrix of this bilinear form with respect to the monomial basis is the Hankel matrix associated with ℓ . Therefore, the convex cone dual to the cone Σ_{2d} of sums of squares of polynomials is the Hankel spectrahedron:

$$\Sigma_{2d}^\vee = \{\ell \in \mathbb{R}[x, y, z]_{2d}^* : (\ell(x^{\alpha+\beta}))_{\alpha, \beta} \text{ is positive semi-definite}\}.$$

Every real point evaluation $\text{ev}_x: \mathbb{R}[x, y, z]_{2d} \rightarrow \mathbb{R}$, $p \mapsto p(x)$, at $x \in \mathbb{R}^3$ is an extreme ray of Σ_{2d}^\vee . In fact, by the Veronese embedding of \mathbb{P}^2 of degree $2d$, they are exactly the positive semi-definite rank 1 Hankel matrices. We are interested in extreme rays of higher rank. These correspond to supporting hyperplanes of Σ_{2d} which expose a face whose relative interior consists of strictly positive polynomials. Conversely, for every non-negative polynomial p that is not a sum of squares, there exists an extreme ray $\mathbb{R}_+ \ell$ of Σ_{2d}^\vee such that $\ell(p) < 0$.

1.1. Gorenstein ideals

Let $\ell \in \mathbb{C}[\underline{x}]_m^*$ be a linear functional on ternary forms of degree m . To ℓ and every pair of positive integers $u, v \in \mathbb{N}$ with $u + v = m$, we associate the bilinear form

$$B_{\ell, u, v}: \begin{cases} \mathbb{C}[\underline{x}]_u \times \mathbb{C}[\underline{x}]_v \rightarrow \mathbb{C} \\ (p, q) \mapsto \ell(pq). \end{cases}$$

The representing matrices of these bilinear forms with respect to the monomial bases are called the *Catalecticant matrices* of ℓ .

Definition 1.1. Let $\ell \in \mathbb{C}[\underline{x}]_m^*$ be a linear functional. We call the homogeneous ideal $I(\ell)$ of $\mathbb{C}[\underline{x}]$ generated by

$$\{p \in \mathbb{C}[\underline{x}]_k : k > m \text{ or } \ell(pq) = 0 \text{ for all } q \in \mathbb{C}[\underline{x}]_{m-k}\}$$

the *Gorenstein ideal with socle* ℓ . We call m the *socle degree* of the ideal.

These ideals were studied extensively in the literature, see [Iarrobino and Kanev \(1999\)](#). Our definition is probably the most direct for 0-dimensional Gorenstein ideals, see [Eisenbud \(1995, Theorem 21.6 and Exercise 21.7\)](#).

Remark 1.2. The degree u part of the ideal is the left-kernel of the bilinear form $B_{\ell, u, v}$ for $u \leq m$. In particular, the Hilbert function of a Gorenstein ideal I with even socle degree $2d$ is symmetric around d , i.e. $\text{Hilb}(I, i) = \text{Hilb}(I, 2d - i)$ for all $0 \leq i \leq 2d$.

We can consider the set of all Gorenstein ideals with a fixed socle degree m as a projective space by identifying an ideal with its socle, which is uniquely determined by the ideal up to scaling. In this projective space, we consider the set $\text{Gor}(T)$ of all Gorenstein ideals with a given Hilbert function T .

Proposition 1.3. The set $\text{Gor}(T)$ of all Gorenstein ideals with socle degree m and Hilbert function T is a quasiprojective subvariety of the projective space of all Gorenstein ideals with socle degree m .

Proof. The condition to have a given Hilbert function can be expressed as rank conditions on the Catalecticant matrices, namely

$$\text{rk}(B_{\ell, u, v}) = T(u). \quad \square$$

Remark 1.4. (a) The quasiprojective variety $\text{Gor}(T)$ is defined over \mathbb{Q} , because the minors of the Catalecticant matrices are polynomials with coefficients in \mathbb{Z} .

(b) Note that a k -rational point $\ell \in \text{Gor}(T)$ for a subfield $k \subset \mathbb{C}$ is a linear functional $\ell = \ell \otimes 1 \in k[\underline{x}]_m \otimes \mathbb{C}$.

Definition 1.5. We call a Hilbert function T *permissible* if there is a Gorenstein ideal $I \subset \mathbb{C}[\underline{x}]$ with Hilbert function T .

Using the Buchsbaum–Eisenbud Structure Theorem for height 3 Gorenstein ideals (see [Buchsbaum and Eisenbud, 1977](#)), Diesel proved the following.

Theorem 1.6 (see [Diesel, 1996, Theorem 1.1 and 2.7](#)). *For every permissible Hilbert function T , the variety $\text{Gor}(T)$ is an irreducible unirational variety.*

We will use the fact that $\text{Gor}(T)$ is unirational to determine the dimension of $\text{Gor}(T)$ for special Hilbert functions T . In order to do this, we need the more precise information on the unirationality of $\text{Gor}(T)$ given by Diesel. The information we need is spread out over the paper [Diesel \(1996\)](#). We will give a short summary with references, using her notation and setup.

Remark 1.7. Diesel proves that for a given permissible Hilbert function T there is a minimal set (with respect to inclusion) $D_{\min} = (Q, P)$ of degrees of generators $Q = \{q_1, \dots, q_u\}$ and relations $P = \{p_1, \dots, p_u\}$ for a Gorenstein ideal with Hilbert function T . We assume $q_1 \leq q_2 \leq \dots \leq q_u$ and $p_1 \geq p_2 \geq \dots \geq p_u$. The set $\text{Gor}_{D_{\min}}$ of all Gorenstein ideals with generators of degree as specified by Q is a dense subset of $\text{Gor}(T)$, see the proof of [Diesel \(1996, Theorem 2.7 and Theorem 3.8\)](#). Given D_{\min} , the affine space $\mathbb{A}^{h(E_M)}$ of skew-symmetric matrices with entries in $\mathbb{C}[\underline{x}]$, where the (i, j) -th entry is homogeneous of degree $p_j - q_i$ ($i \neq j$), has dimension $h(E_M)$, which is equal to the sum of the dimensions of the vector space of forms of degree $p_j - q_i$ for $j > i$. We consider the rational map $\pi: \mathbb{A}^{h(E_M)} \dashrightarrow \text{Gor}_{D_{\min}}$ that takes a matrix to the Gorenstein ideal generated by its Pfaffians. This statement uses the Buchsbaum–Eisenbud Structure Theorem, see [Diesel \(1996\)](#), p. 367 and p. 369. Note that this rational map is defined over \mathbb{Q} .

Given a Hilbert function T , the set D_{\min} of degrees of generators and relations for T is determined in a combinatorial way: Given the socle degree m and the minimal degree k of a generator of the ideal, there is a one-to-one correspondence between permissible Hilbert functions of order k and self-complementary partitions of $2k$ by $m - 2k + 2$ blocks, see [Diesel \(1996, Proposition 3.9\)](#). These partitions give the maximum number of generators, which is $2k + 1$, see [Diesel \(1996, Theorem 3.3\)](#). To refine these sequences to D_{\min} , we iteratively delete pairs (q_i, q_j) from Q and (p_i, p_j) from P whenever they satisfy $r_i + r_j = p_i + p_j - q_i - q_j = 0$, see [Diesel \(1996\)](#), p. 380. Here, $r_i = p_i - q_i$ as in [Diesel \(1996\)](#).

We are particularly interested in Gorenstein ideals with socle in even degree $2d$ with the property that the middle Catalecticant has corank 4, i.e. $\text{rank} \binom{d+2}{2} - 4$. The proof of the following statement is analogous to the proof of [Diesel \(1996, Theorem 4.4\)](#).

Lemma 1.8. *Let $d \geq 4$ be an integer. The projective variety X_{-4} of middle Catalecticant matrices of corank at least 4, i.e. of rank at most $\binom{d+2}{2} - 4$, is irreducible of codimension 10 in the space of middle Catalecticant matrices. In particular, it is defined by the $(\binom{d+2}{2} - 3)$ -minors of the generic middle Catalecticant matrix.*

Proof. Let $N = \binom{d+2}{2}$. The quasiprojective variety S_{-4} of symmetric $N \times N$ -matrices of rank $N - 4$ has codimension 10 in the projective space of the vector space of symmetric $N \times N$ -matrices. Therefore the intersection X_{-4} of S_{-4} with the subspace of middle Catalecticant matrices has codimension at most 10 in this linear space. We will show, that it has codimension exactly 10 by counting dimensions

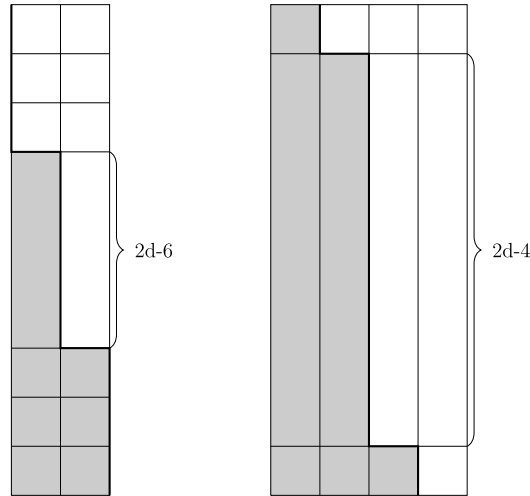


Fig. 1. The partition on the right of $2d \times 2$ blocks corresponds to T_1 , the partition on the left of $(2d - 2) \times 4$ blocks to T_2 .

of the possible $\text{Gor}(T)$ using their unirationality. We will use the setup and notation used by Diesel (1996), see also Remark 1.7.

There are only two possible Hilbert functions for a Gorenstein ideal I with socle degree $2d$ and $\text{Hilb}(I, d) = \binom{d+2}{2} - 4$ by their symmetry, namely

$$T_1 = (1, 3, 6, \dots, \binom{d+1}{2}, \binom{d+2}{2} - 4, \dots),$$

which corresponds to the case of four generators in degree d and no generators of lower degree, and

$$T_2 = (1, 3, 6, \dots, \binom{d+1}{2} - 1, \binom{d+2}{2} - 4, \dots),$$

which corresponds to the case of one generator of degree $d - 1$ and one generator of degree d . More precisely, these two Hilbert functions correspond to the self-complementary partitions of $2 \times 2d$ resp. $4 \times (2d - 2)$ blocks shown in Fig. 1 by the correspondence explained in Diesel (1996, section 3.4, in particular Proposition 3.9).

We first consider T_1 . The sequence of degrees of the generators for the minimal set D_{\min} is in this case different for $d = 4$ and $d \geq 5$, namely $(4, 4, 4, 4, 6)$ for $d = 4$ and $(d, d, d, d, d + 1, \dots, d + 1)$ with $(2d - 9)$ many generators of degree $d + 1$ for $d \geq 5$, see Remark 1.7. Since $q_i + p_i = 2d + 3$, the degree matrices are

$$\begin{pmatrix} 0 & 3 & 3 & 3 & 1 \\ & 0 & 3 & 3 & 1 \\ & & 0 & 3 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 3 & 3 & 2 & \dots & \dots & 2 \\ & 0 & 3 & 3 & \vdots & & & \vdots \\ & & 0 & 3 & \vdots & & & \vdots \\ & & & 0 & 2 & \dots & \dots & 2 \\ & & & & 0 & 1 & \dots & 1 \\ & & & & & 0 & \ddots & \vdots \\ & & & & & & \ddots & 1 \\ & & & & & & & 0 \end{pmatrix}$$

where the right one is of size $(2d - 5) \times (2d - 5)$. Every entry of the matrix can be generically chosen among the forms of the indicated degree and its Pfaffians will generate a Gorenstein ideal with Hilbert

function T_1 . Therefore, for $d = 4$, we have $h(E_M) = 6 \dim(\mathbb{C}[\underline{x}]_3) + 4 \dim(\mathbb{C}[\underline{x}]_1) = 72$ and for $d \geq 5$ we have

$$\begin{aligned} h(E_M) &= 6 \dim(\mathbb{C}[\underline{x}]_3) + 4(2d - 9) \dim(\mathbb{C}[\underline{x}]_2) \\ &\quad + \binom{2d - 9}{2} \dim(\mathbb{C}[\underline{x}]_1) \\ &= 6d^2 - 9d - 21. \end{aligned}$$

This overcounts the dimension of $\text{Gor}_{D_{\min}}$ because for every choice of generators of a given ideal we get a matrix with these generators as Pfaffians. So for $d = 4$, we choose a basis of a 4-dimensional subspace of forms of degree 4 and one generator of degree 6 from a $\dim(\mathbb{C}[\underline{x}]_6) - T_1(2) = 22$ -dimensional space. Therefore we overcount the dimension of $\text{Gor}_{D_{\min}}$ by at least $4^2 + 22 = 38$ and the dimension of $\text{Gor}(T)$ is at most 34. Since $\dim(\mathbb{P}(\mathbb{C}[\underline{x}]_3^*)) = 44$, its codimension is at least 10. For $d \geq 5$, we choose a basis of a 4-dimensional subspace of forms of degree d and $2d - 9$ linearly independent generators from a space of dimension $\dim(\mathbb{C}[\underline{x}]_{d+1}) - T_1(d - 1) = 2d + 3$. In this case we overcount by at least $4^2 + (2d - 9)(2d + 3)$ and the dimension of $\text{Gor}_{D_{\min}}$ is at most $2d^2 + 3d - 10$. The projective dimension of the space of middle Catalecticant matrices is $\dim(\mathbb{C}[\underline{x}]_{2d}^*) - 1 = 2d^2 + 3d$, which again implies that the codimension of $\text{Gor}(T_1)$ is at least 10. From the fact that it can be at most 10, it follows that it is exactly 10.

We now repeat the count for the Hilbert function T_2 . In this case, $D_{\min} = \{Q_{\min}, P_{\min}\} = \{(d - 1, d, d + 1, d + 1, \dots, d + 1), (d + 4, d + 3, d + 2, d + 2, \dots, d + 2)\}$ with $(2d - 5)$ times the entry $d + 1$ in Q_{\min} and $d + 2$ in P_{\min} , see Fig. 1. Therefore, the degree matrix is

$$\begin{pmatrix} 0 & 4 & 3 & \cdots & \cdots & 3 \\ & 0 & 2 & \cdots & \cdots & 2 \\ & & 0 & 1 & \cdots & 1 \\ & & & 0 & \ddots & \vdots \\ & & & & & 1 \\ & & & & & 0 \end{pmatrix},$$

which is of size $(2d - 3) \times (2d - 3)$. We compute

$$\begin{aligned} h(E_M) &= \dim(\mathbb{C}[\underline{x}]_4) + (2d - 5) \dim(\mathbb{C}[\underline{x}]_3) + (2d - 5) \dim(\mathbb{C}[\underline{x}]_2) \\ &\quad + \binom{2d - 5}{2} \dim(\mathbb{C}[\underline{x}]_1) \\ &= 6d^2 - d - 20. \end{aligned}$$

Here we choose one generator of degree $d - 1$, one generator of degree d from a 4-dimensional space and $(2d - 5)$ generators from a $\dim(\mathbb{C}[\underline{x}]_{d+1}) - T_2(d - 1) = (2d + 4)$ -dimensional space. Therefore the dimension of $\text{Gor}(T_2)$ is at most $6d^2 - d - 20 - 4 - (2d - 5)(2d + 4) = 2d^2 + d - 4$. The codimension is at least $2d + 4 \geq 12$. So $\text{Gor}(T_2)$ cannot be an irreducible component of X_{-4} and we conclude that $\text{Gor}(T_2) \subset \text{cl}(\text{Gor}(T_1))$.

In summary, $\text{Gor}(T_1)$ is a dense subset of X_{-4} and X_{-4} is irreducible (see Diesel, 1996, Theorem 2.7) and has the expected codimension 10 in the space of middle Catalecticant matrices. \square

The tangent space to the quasiprojective variety $\text{Gor}(T)$ for a permissible Hilbert function T at a Gorenstein ideal I can be described in terms of the ideal. We identify $\mathbb{C}[\underline{x}]_m$ with its dual space by using the apolar bilinear form, i.e. we identify a monomial $x^\alpha \in \mathbb{C}[\underline{x}]_m$ with the linear form $p \mapsto \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha} p$ that takes a polynomial $p = \sum p_\beta x^\beta$ to p_α . Using this identification, we can state a characterization of the tangent space to $\text{Gor}(T)$ at an ideal I in terms of this ideal.

Theorem 1.9 (Iarrobino and Kanev, 1999, Theorem 3.9 and 4.21). Let T be a permissible Hilbert function. The quasiprojective variety $\text{Gor}(T)$ is smooth. Let $\ell \in \mathbb{C}[\underline{x}]_m^*$ be a linear functional such that the corresponding Gorenstein ideal $I = I(\ell)$ has Hilbert function T . Then the tangent space to $\text{Gor}(T)$ at ℓ is

$$((I^2)_m)^\perp \subset \mathbb{C}[\underline{x}]_m.$$

The irreducible variety X_{-4} of middle Catalecticant matrices of corank at least 4 is defined over \mathbb{Q} and we will later show that it has a smooth rational point, i.e. a point with rational coordinates. Therefore, the real points of X_{-4} are Zariski-dense in it and the above statement of Theorem 1.9 also applies to real points of X_{-4} , see Bochnak et al. (1998, Section 2.8).

2. Extreme rays of maximal rank and positive Gorenstein ideals

In this section, we recapitulate bounds on the rank of Hankel matrices of extreme rays of Σ_{2d}^\vee which are not point evaluations. The lower bound and its tightness are proved in Blekherman (2015, Theorem 2.1). We constructively establish tightness of the upper bound. We show that the Zariski closure of the set of extreme rays is the variety of Hankel matrices of corank at least 4, which is irreducible; in particular, it is (at least set-theoretically) defined by the symmetric $r \times r$ minors of the generic Hankel matrix, where $r = \binom{d+2}{2} - 3$.

To a linear functional $\ell \in \mathbb{R}[\underline{x}]_m^*$, we associate the bilinear form

$$B_\ell: \begin{cases} \mathbb{R}[\underline{x}]_d \times \mathbb{R}[\underline{x}]_d \rightarrow \mathbb{R} \\ (p, q) \mapsto \ell(pq), \end{cases}$$

whose representing matrix with respect to the monomial bases is called the *Hankel matrix* of ℓ .

One of the main results of Blekherman is a characterization of extreme rays of Σ_{2d}^\vee by the associated Gorenstein ideals.

Proposition 2.1 (Blekherman, 2012, Lemma 2.2 and Blekherman, 2015, Proposition 4.2).

- (a) A linear functional $\ell \in \mathbb{R}[\underline{x}]_{2d}^*$ spans an extreme ray of Σ_{2d}^\vee if and only if the bilinear form B_ℓ is positive semi-definite and the degree d part $I(\ell)_d$ of the Gorenstein ideal $I(\ell)$ is maximal with respect to inclusion over all Gorenstein ideals with socle degree $2d$.
- (b) Let I be a Gorenstein ideal with socle degree $2d$. Then I_d is maximal with respect to inclusion over all Gorenstein ideals with socle degree $2d$ if and only if the degree $2d$ part of the ideal generated by I_d is a hyperplane in $\mathbb{R}[\underline{x}]_{2d}$. In this case, it is equal to I_{2d} .

Lower bounds on the ranks for extreme rays were established by Blekherman.

Theorem 2.2 (Blekherman, 2015, Theorem 2.1). Let $d \geq 3$ and $\ell \in \Sigma_{2d}^\vee$ and suppose $\mathbb{R}_+\ell$ is an extreme ray. Then the rank r of B_ℓ is 1, in which case ℓ is a point evaluation, or its rank is at least $3d - 2$. These bounds are tight and extreme rays Σ_{2d}^\vee of rank $3d - 2$ can be explicitly constructed.

From Blekherman's work, we can easily deduce an upper bound.

Theorem 2.3. Let $\ell \in \Sigma_{2d}^\vee$, $d \geq 4$ and suppose $\mathbb{R}_+\ell$ is an extreme ray. The rank of B_ℓ is at most $\binom{d+2}{2} - 4$, i.e. the corank is at least 4.

Proof. Since $\mathbb{R}_+\ell$ is an extreme ray, we know that the degree $2d$ part of the ideal generated by $I(\ell)_d$ is a hyperplane in the space of forms of degree $2d$. The dimension of the space $\mathbb{R}[\underline{x}]_d I(\ell)_d$ is bounded by $\dim(\mathbb{R}[\underline{x}]_d) \dim(I(\ell)_d) = \binom{d+2}{2} \text{crk}(B_\ell)$. In case $\text{crk}(B_\ell) \leq 3$ and $d \geq 5$, this bound is smaller than the dimension $\binom{2d+2}{2} - 1$ of a hyperplane in $\mathbb{R}[\underline{x}]_{2d}$. The case $\text{crk}(B_\ell) \leq 3$ and $d = 4$ needs a more precise count: Suppose that $\text{crk}(B_\ell) = 3$ and the kernel of B_ℓ is generated by f_1, f_2, f_3 . Then the dimension

of the space $\mathbb{R}[\underline{x}]_4 I(\ell)_4$ is bounded by $3 \dim(\mathbb{R}[\underline{x}]_4) - 3 = 42 < 45 - 1 = \dim(\mathbb{R}[\underline{x}]_8) - 1$ because there are the 3 obvious relations, namely $f_i f_j - f_j f_i = 0$ for $i \neq j$. \square

Remark 2.4. The upper bound in the case $d = 3$ is corank 3, which agrees with the lower bound.

A main tool in this section is the Cayley–Bacharach Theorem.

Theorem 2.5 (Cayley–Bacharach, see [Eisenbud et al., 1996, CB5](#)). Let $X_1, X_2 \subset \mathbb{P}^2$ be plane curves defined over \mathbb{R} of degree d and e intersecting in $d \cdot e$ points. Set $s = d + e - 3$ and decompose $X_1 \cap X_2 = \Gamma_1 \cup \Gamma_2$ into two disjoint sets defined over \mathbb{R} . Then for all $k \leq s$, the following equality holds

$$\dim(\mathcal{I}(\Gamma_1)_k) - \dim(\mathcal{I}(X_1 \cap X_2)_k) = |\Gamma_2| - \dim \text{span}\{\text{Re } \text{ev}_x, \text{Im } \text{ev}_x \in \mathbb{R}[\underline{x}]_{s-k}^* : x \in \Gamma_2\}.$$

The left hand side is the dimension of the space of forms of degree k vanishing on Γ_1 modulo the subspace of forms vanishing in every point of $X_1 \cap X_2$. The right hand side is the linear defect of point evaluations on forms of dual degree $s - k$ at points of Γ_2 .

Probably the most famous instance of this theorem is the following application to the complete intersection of two cubic curves, stated here for a totally real intersection.

Example 2.6. Suppose $X_1, X_2 \subset \mathbb{P}^2$ are plane cubic curves intersecting in 9 points. Then $d = e = 3$ and so $s = 3$. Pick $\Gamma_2 = \{P\}$ for any intersection point P and put $\Gamma_1 = (X_1 \cap X_2) \setminus \{P\}$. Let us consider $k = 3$ and compute the right hand side of the Cayley–Bacharach equality: Since $\dim \text{span}\{\text{ev}_x \in \mathbb{R}[\underline{x}]_0^* : x \in \Gamma_2\} = 1$, we conclude

$$\dim(\mathcal{I}(\Gamma_1)_3) - \dim(\mathcal{I}(X_1 \cap X_2)_3) = 0,$$

which means that every cubic form that vanishes in the 8 points of Γ_1 also vanishes at the ninth point P of the intersection. In other words, the point evaluation $\text{ev}_P \in \mathbb{R}[\underline{x}]_3^*$ lies in the subspace U_{Γ_1} spanned by the eight point evaluations $\{\text{ev}_x \in \mathbb{R}[\underline{x}]_3^* : x \in \Gamma_1\}$. The annihilator of U_{Γ_1} is the 2-dimensional subspace of $\mathbb{R}[\underline{x}]_3$ spanned by the defining equations of X_1 and X_2 . Since this is true for any point $P \in X_1 \cap X_2$, we conclude, that there is a unique linear relation among the point evaluations $\{\text{ev}_x \in \mathbb{R}[\underline{x}]_3^* : x \in X_1 \cap X_2\}$ and all coefficients of this relation are non-zero.

Using the Cayley–Bacharach Theorem, we will first show that there are extreme rays of corank 4 under the following constraint on the degree. We will get rid of this constraint in [Lemma 2.13](#).

Constraint 2.7. Let $d \geq 4$. There is a unique conic C going through the following six points in the plane: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(d - 1, d - 1)$, $(d - 2, d - 1)$, $(d - 1, d - 2)$; its equation is given by $C = \mathcal{V}(x^2 + y^2 - \frac{2(d-2)}{d-1}xy - x - y)$. From now on, we assume that this conic does not go through any other integer point. The only exceptional cases in the interval $\{4, 5, \dots, 100\}$ are: 9, 19, 21, 29, 33, 34, 36, 40, 49, 51, 57, 61, 73, 78, 79, 81, 89, 99.

Proposition 2.8. Set $L_1 = \prod_{j=0}^{d-1} (x - jz)$ and $L_2 = \prod_{j=0}^{d-1} (y - jz)$ and let $\Gamma = \mathcal{V}(L_1, L_2) = \{(j : k : 1) : j, k = 0, \dots, d - 1\}$ be the intersection of their zero sets in \mathbb{P}^2 . Split these points into

$$\Gamma_2 = \{(x : y : 1) : x + y = 2\} \cup \bigcup_{j=1}^{d-4} \{(x : y : 1) : x + y = d + j\}$$

and $\Gamma_1 = \Gamma \setminus \Gamma_2$. Then there is a unique linear relation $\sum_{v \in \Gamma_1} u_v \text{ev}_v = 0$ among the point evaluations on forms of degree d at points of Γ_1 and all coefficients $u_v \in \mathbb{R}$ in this relation are non-zero. The set of all forms of degree d vanishing on Γ_1 is a 3-dimensional space spanned by L_1, L_2 and a form p which is non-zero at any point of Γ_2 .

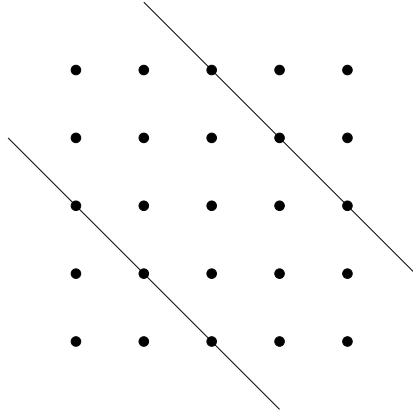


Fig. 2. The construction of an extreme ray of Σ_{2d}^\vee of corank 4 for $d = 5$.

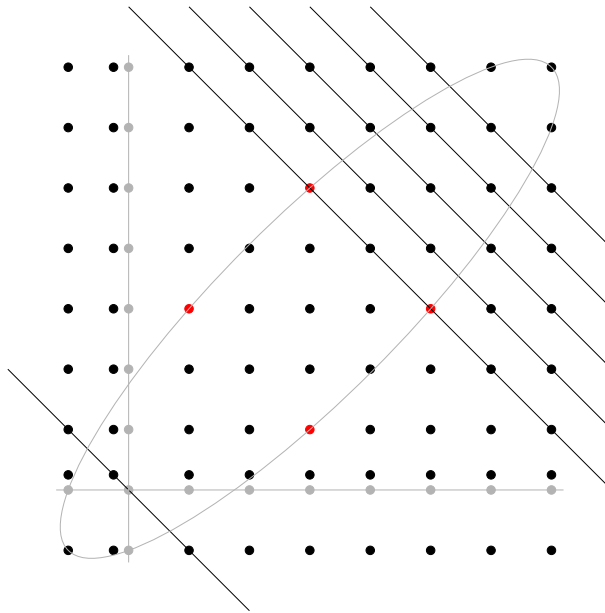


Fig. 3. A picture of the perturbation for general $d \geq 4$ shown for the first critical case $d = 9$: The black points and the four red points are the perturbed point configuration for which our construction works. The four red points are the additional points through which the grey ellipse goes. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

See Fig. 2 for the case $d = 5$ and Fig. 3 for the case $d = 9$.

Proof. First observe that there is a unique (up to scaling) form of degree $d - 3$ vanishing on Γ_2 , namely $(x + y - 2z) \prod_{j=1}^{d-4} (x + y - (d + j)z)$, the product of diagonals defining Γ_2 : Indeed, suppose f is a form of degree $d - 3$ vanishing on Γ_2 , then it intersects the line $x + y = d + 1$ in $d - 2$ integer points. Therefore it vanishes identically on it and we can divide f by this linear polynomial and get a form of degree $d - 4$ vanishing on $d - 3$ points on the line $x + y = d + 2$. Inductively, we conclude that f is (again up to scaling) the claimed product of linear forms. Therefore, by the Cayley-Bacharach Theorem, the space of forms of degree d vanishing on Γ_1 is 3-dimensional, so it is spanned by L_1, L_2

and a third form p . We will explicitly construct this form: Let p be the product of the linear forms $x + y - jz$ for $j = 3, \dots, d$ and of the ellipse $\mathcal{V}(x^2 + y^2 - \frac{2(d-2)}{(d-1)}xy - x - y)$ passing through the six points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(d-2, d-1)$, $(d-1, d-1)$ and $(d-1, d-2)$. By construction, p vanishes on Γ_1 , is of degree d and does not vanish on all of Γ . Therefore $\{L_1, L_2, p\}$ is a basis of the space of forms of degree d vanishing on Γ_1 . By assumption on d , the form p does not vanish on any point of Γ other than the six mentioned above.

Note that $|\Gamma_1| = \binom{d+2}{2} - 2$, because $|\Gamma_1| = d^2 - |\Gamma_2| = d^2 - (3 + \sum_{j=1}^{d-4} (d-1-j)) = d^2 - \binom{d-1}{2}$. So the fact that the space of forms of degree d vanishing on Γ_1 is 3-dimensional implies that there is a unique linear relation among the point evaluations on forms of degree d at points of Γ_1 . To see that all coefficients u_v in the relation $\sum_{v \in \Gamma_1} u_v \text{ev}_v = 0$ are non-zero, note that the unique form f of degree $d-3$ vanishing on Γ_2 does not vanish on any point of Γ_1 . Therefore, there is no form of degree $d-3$ vanishing on $\Gamma_2 \cup \{v_0\}$ for any $v_0 \in \Gamma_1$ and Cayley–Bacharach implies that the point evaluations $\{\text{ev}_v : v \in \Gamma_1\} \setminus \{\text{ev}_{v_0}\}$ are linearly independent. \square

Lemma 2.9. *There is an extreme ray $\mathbb{R}_+\ell$ of Σ_{2d}^\vee such that B_ℓ has corank 4. The Hilbert function of the ideal $I(\ell)$ is $\text{Hilb}(I(\ell), j) = \binom{j+2}{2} = \text{Hilb}(I(\ell), 2d-j)$ for $0 \leq j < d$ and $\text{Hilb}(I(\ell), d) = \binom{d+2}{2} - 4$.*

Proof. Let L_1, L_2, p be as in Proposition 2.8 and consider the splitting $\mathcal{V}(L_1, L_2) = \Gamma = \Gamma_1 \cup \Gamma_2$ of the points defined there. Pick a point $P \in \Gamma_1$ and set $\Lambda = \Gamma_1 \setminus \{P\}$. We claim that the linear functional

$$\ell = \sum_{v \in \Lambda} \text{ev}_v - \frac{u_P^2}{\sum_{v \in \Lambda} u_v^2} \text{ev}_P,$$

where u_v are the coefficients of the Cayley–Bacharach relation as in Proposition 2.8, is an extreme ray of Σ_{2d}^\vee and that the corresponding Hankel matrix B_ℓ has corank 4.

First note that B_ℓ is positive semi-definite because

$$\begin{aligned} \ell(f^2) &= \sum_{v \in \Lambda} f(v)^2 - \frac{u_P^2}{\sum_{v \in \Lambda} u_v^2} f(P)^2 \\ &= \sum_{v \in \Lambda} f(v)^2 - \frac{u_P^2}{\sum_{v \in \Lambda} u_v^2} \frac{1}{u_P^2} \left(\sum_{v \in \Lambda} u_v f(v) \right)^2 \\ &= \|(f(v))_{v \in \Lambda}\|^2 - \left\| \frac{1}{\|(u_v)_{v \in \Lambda}\|} (u_v)_{v \in \Lambda}, (f(v))_{v \in \Lambda} \right\|^2 \\ &\geq 0 \end{aligned}$$

by the Cauchy–Schwarz inequality for all polynomials $f \in \mathbb{R}[\mathbf{x}]_d$. More precisely, $\ell(f^2)$ is zero for a form f not identically zero on Γ if and only if $f(v) = \alpha u_v$ for all $v \in \Lambda$ and some $\alpha \in \mathbb{R}^*$. Therefore, the degeneration space of the Hankel matrix is spanned by L_1, L_2, p and the form uniquely determined (modulo L_1, L_2, p) by $f(v) = u_v$ for all $v \in \Lambda$; it has dimension 4 as desired. Indeed, the form f is uniquely determined because $\{\text{ev}_x \in (\mathbb{R}[\mathbf{x}]_d / \text{span}(L_1, L_2, p))^* : x \in \Lambda\}$ is a basis.

We now prove extremality of ℓ in Σ_{2d}^\vee by checking the characterization that $I(\ell)_d$ generates a hyperplane in the vector space of forms of degree $2d$, see Blekherman (2015, Proposition 4.2). As a first step, we show that $\langle L_1, L_2, p \rangle_{2d-3}$ has codimension $|\mathcal{V}(L_1, L_2, p)| = |\Gamma_1|$. So suppose $a_1 L_1 + a_2 L_2 + bp = 0$, where $a_1, a_2, b \in \mathbb{R}[\mathbf{x}]_{d-3}$ are forms of degree $d-3$. By evaluating at points of Γ_2 , we conclude that b is the uniquely determined form of degree $d-3$ vanishing on Γ_2 , see proof of Proposition 2.8. Since L_1 and L_2 are coprime, this is a unique syzygy and we conclude

$$\dim(\langle L_1, L_2, p \rangle_{2d-3}) = 3 \binom{d-1}{2} - 1 = \frac{3}{2}(d^2 - 3d + 2) - 1,$$

which means codimension $|\Gamma_1| = d^2 - \binom{d-1}{2}$ in $\mathbb{R}[\mathbf{x}]_{2d-3}$. In particular, the codimension of $\langle L_1, L_2, p \rangle_{2d}$ in $\mathbb{R}[\mathbf{x}]_{2d}$ is also $|\Gamma_1|$ because the point evaluations $\{\text{ev}_v : v \in \Gamma_1\}$ are linearly independent on forms of degree $2d-3$ and consequently also on forms of degree $2d$.

Now suppose $a_1 L_1 + a_2 L_2 + bp + cf = 0$ for forms $a_1, a_2, b, c \in \mathbb{R}[x]_d$ of degree d . Evaluation at points of Γ_1 implies that c lies in the span of L_1, L_2, p . So we have three syzygies and the codimension of $\langle L_1, L_2, p, f \rangle_{2d}$ is $|\Gamma_1| - \binom{d+2}{2} + 3 = 1$, as desired. \square

Example 2.10. We follow the construction in [Proposition 2.8](#) and [Lemma 2.9](#) in the case $d = 5$. Then $\Gamma = \mathcal{V}(L_1, L_2)$ consists of the 25 points $(i : j : 1) \in \mathbb{P}^2$ where $i, j = 0, \dots, 4$, see [Fig. 2](#). The six points on the two lines $x + y = 2$ and $x + y = 6$ are the points of Γ_2 . Indeed, the point evaluations at the 19 points of $\Gamma_1 = \Gamma \setminus \Gamma_2$ on forms of degree 5 satisfy a unique linear relation, namely

$$\begin{pmatrix} -1 & 3 & 0 & -5 & 3 \\ 3 & -16 & 18 & 0 & -5 \\ 0 & 18 & -36 & 18 & 0 \\ -5 & 0 & 18 & -16 & 3 \\ 3 & -5 & 0 & 3 & -1 \end{pmatrix},$$

where the (i, j) -th entry of this matrix is the coefficient of the point evaluation at $(5 - i : j - 1 : 1)$ in the linear relation, i.e. visually, it is the coefficient corresponding to the points in the 5×5 -grid seen in [Fig. 2](#). The 21×21 Hankel matrix can be exactly computed using a computer algebra system. In Mathematica, the following code will do the job:

```
d = 5;
m = MonomialList[(x+y+z)^d];
q1 = x (x - z) (x - 2 z) (x - 3 z) (x - 4 z);
q2 = y (y - z) (y - 2 z) (y - 3 z) (y - 4 z);
Pevalall = Solve[{q1 == 0, q2 == 0, z == 1}, {x, y, z}];
Pevalfoo = Select[Pevalall, ({y + x - 2 z} /. #) != {0} &];
Peval = Select[Pevalfoo, ({y + x - 6 z} /. #) != {0} &];
Peval0 = Drop[Peval, -1];

evals = m/.Peval;
CBrel = NullSpace[Transpose[evals]];
CB = Transpose[{Drop[CBrel[[1]], -1]}];

H = Transpose[{m}].{m};
P = Peval0;
Q = Sum[H /. P[[i]], {i, 1, Length[P]}];
Qp = H /. Peval[[-1]];
l = Norm[CB]^2;
Hankel = Q - 1/l (rel[[1]][[-1]])^2 Qp;
```

We set up the monomial basis m and the totally real complete intersection of 25 points, where $q1 = L_1$ and $q2 = L_2$. The two lines using the `Select`-command remove the points on the two diagonals $x + y = 2$ and $x + y = 6$, so $\Gamma_1 = \text{Peval}$. With the `Drop`-command, we remove one of the points from the list. The next three lines compute the unique Cayley-Bacharach relation CBrel on the point evaluations at Peval . We need CB when we solve the linear relation for the point evaluation at $\text{Peval}[[-1]]$. The matrix H is the general Hankel matrix and Q is the Hankel matrix of the linear functional $\sum_{v \in \Gamma_1 \setminus \text{Peval}[[-1]]} \text{ev}_v$ and Qp the Hankel matrix of the point evaluation at $\text{Peval}[[-1]]$. So Hankel is the Hankel matrix of the extreme ray that we constructed.

Remark 2.11. Note that the proof of the [Lemma 2.9](#) shows that the face of the cone Σ_{2d} of sums of squares exposed by the constructed extreme ray consists of the sums of squares of polynomials in $I(\ell)_d$.

The fact that the conic vanishes in additional integer points on the $d \times d$ grid defined by the products of linear forms L_1 and L_2 in [Proposition 2.8](#) destroys the extremality of the constructed linear

functional because we get additional syzygies among the generators of the corresponding Gorenstein ideal. In order to deal with this problem, we will make a perturbation of our point arrangement. First, we want to observe the following fact, which motivates why we should be able to get around this obstacle by perturbation:

Remark 2.12. Consider the setup in Proposition 2.8 and suppose the conic C vanishes in additional integer points in the $d \times d$ integer grid $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. Pick such a point $P \in \Gamma$. Then every form of degree d vanishing on Γ_1 will also vanish at P because L_1 , L_2 and the third form p , which is the product of lines and the conic, form a basis of this space. By the Theorem of Cayley–Bacharach applied to $\Gamma = \Gamma'_1 \cup \Gamma'_2$ for $\Gamma'_1 = \Gamma_1 \cup \{P\}$ and $\Gamma'_2 = \Gamma_2 \setminus \{P\}$, there is a unique linear relation among the point evaluations at points of Γ'_2 on forms of degree $d - 3$. In particular, the coefficient of the point evaluation at P in the unique linear relation among point evaluations at Γ_2 on forms of degree $d - 3$ is zero. The converse is also true by Cayley–Bacharach, so we have:

The conic C vanishes in a point in $P \in \Gamma_2$ if and only if the coefficient of the point evaluation at P in the unique linear relation among $\{\text{ev}_v \in \mathbb{R}[\underline{x}]_{d-3}^* : v \in \Gamma_2\}$ is zero. This seems to be a non-generic property and we will indeed show that we can make all coefficients in the linear relation among these point evaluations non-zero by a careful perturbation of L_1 and L_2 .

We now drop the assumptions on d made in 2.7 and prove Lemma 2.9 for all $d \geq 4$:

Lemma 2.13. For any $d \geq 4$, there is an extreme ray $\mathbb{R}_+ \ell$ of Σ_{2d}^\vee such that B_ℓ has corank 4. The Hilbert function of the ideal $I(\ell)$ is $\text{Hilb}(I(\ell), j) = \binom{j+2}{2}$ for $0 \leq j < d$ and $\text{Hilb}(I(\ell), d) = \binom{d+2}{2} - 4$.

Proof. We start as above with the products of linear forms $L_1 = \prod_{j=0}^{d-1} (x - jz)$ and $L_2 = \prod_{j=0}^{d-1} (y - jz)$ and denote by Γ the complete intersection $\mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. Split Γ into

$$\Gamma_2 = \{(x : y : 1) : x + y = 2\} \cup \bigcup_{j=1}^{d-4} \{(x : y : 1) : x + y = d + j\}$$

and $\Gamma_1 = \Gamma \setminus \Gamma_2$. Then the space of forms of degree d vanishing on Γ_1 has dimension 3. Let p be the uniquely determined form of degree d such that L_1, L_2, p is a basis of this space. By Cayley–Bacharach, we know that there is a unique relation among the point evaluations $\{\text{ev}_x \in \mathbb{R}[\underline{x}]_{d-3}^* : x \in \Gamma_2\}$, say

$$\sum_{x \in \Gamma_2} w_x \text{ev}_x = 0.$$

Note that by the preceding Remark 2.12, the coefficient of $\text{ev}_{(1:1:1)}$ is non-zero. Set $\Gamma'_1 = \Gamma_1 \cup \{(1 : 1 : 1)\}$ and $\Gamma'_2 = \Gamma'_2 \setminus \{(1 : 1 : 1)\}$. Then the point evaluations $\{\text{ev}_x \in \mathbb{R}[\underline{x}]_{d-3}^* : x \in \Gamma'_2\}$ are linearly independent and span a hyperplane H in $\mathbb{R}[\underline{x}]_{d-3}^*$. So there is a unique form q of degree $d - 3$ vanishing on Γ'_2 , namely the one vanishing on all of Γ_2 , i.e. $q = (x + y - 2z) \prod_{j=1}^{d-4} (x + y - (d + j)z)$.

We will now perturb the point $(1 : 1 : 1)$ along the line $x + y = 2$, see Fig. 3 for a visualization in case $d = 9$: Let $v_t := (t, 2 - t)$. Of course, $q(v_t) = 0$ for every $t \in \mathbb{R}$, i.e. the point evaluation $\text{ev}_{v_t} \in \mathbb{R}[\underline{x}]_{d-3}^*$ lies in the hyperplane spanned by the point evaluations at Γ'_2 ; write

$$\text{ev}_{v_t} = \sum_{x \in \Gamma'_2} \alpha_x(t) \text{ev}_x,$$

where the coefficients $\alpha_x(t)$ are rational functions of the parameter t .

Suppose there is a point $P \in \Gamma'_2$ such that $\alpha_P(t) = 0$ for all $t \in \mathbb{R}$. Then $\text{ev}_{v_t} \in \text{span}(\text{ev}_v : v \in \Gamma'_2 \setminus \{P\})$. Dually this means, that there is a form f_P of degree $d - 3$, uniquely determined modulo q , such that $f_P(P) = 1$, $f_P(v) = 0$ for all $v \in \Gamma'_2 \setminus \{P\}$ and consequently $f_P(v_t) = 0$ for all $t \in \mathbb{R}$. Such a form cannot exist: Since v_t ranges over the whole line defined by $x + y = 2$, the form f_P vanishes identically on this line; so we can factor it out. Furthermore, f_P vanishes identically on every diagonal

defining Γ_2 to the left of P , i.e. $f_P(x, j - x) = 0$ for all $d < j < P_1 + P_2$ because it has too many zeros on these lines from Γ'_2 .

Now $\Gamma'_2 \cap \{x + y = P_1 + P_2\}$ consists of $2d - 1 - P_1 - P_2$ many points. We have already established $P_1 + P_2 - d$ linear factors of f_P , so the remaining cofactor has degree $2d - P_1 - P_2 - 3$. Therefore, f_P vanishes identically on this line, which is a contradiction because it contains P .

So there is an $\epsilon > 0$ such that for all $t \in (1 - \epsilon, 1)$, all coefficients of the linear relation

$$\text{ev}_{v_t} = \sum_{v \in \Gamma'_2} \alpha_v(t) \text{ev}_v$$

are non-zero. Pick a t_0 in this interval and consider the totally real complete intersection $\Gamma = \mathcal{V}(L'_1) \cap \mathcal{V}(L'_2)$ for $L_1 = x(x - t_0 z) \prod_{j=2}^{d-1} (x - jz)$ and $L_2 = y(y - (2 - t_0)z) \prod_{j=2}^{d-1} (x - jz)$ and argue as above: we split the points into Γ_1 and Γ_2 , where Γ_2 is the same union of diagonals as above. The Theorem of Cayley–Bacharach then implies the existence of a form of degree d vanishing on Γ_1 and not identically on Γ . In fact, by Remark 2.12, this form does not vanish in any point of Γ_2 , so we can now complete the proof as in Lemma 2.9. \square

Remark 2.14. In particular, the union of all extreme rays of Σ_{2d}^\vee need not be closed, e.g. for $d = 9$, extremality fails in our original construction but a perturbation gives an extreme ray.

Theorem 2.15. For any $d \geq 4$, the Zariski closure of the set of extreme rays of Σ_{2d}^\vee is the variety of Hankel matrices of corank at least 4. It is irreducible, has codimension 10, and degree $\prod_{\alpha=0}^3 \binom{N+\alpha}{4-\alpha} / \binom{2\alpha+1}{\alpha}$, where $N = \binom{d+2}{2}$.

Proof. We have shown in the proof of Lemma 1.8 that the quasi-projective variety $\text{Gor}(T)$ of all Gorenstein ideals with Hilbert function $T(j) = \binom{j+2}{2}$ for $0 \leq j < d$ and $T(d) = \binom{d+2}{2} - 4$ is dense in X_{-4} . It is also smooth, see Theorem 1.9 or Iarrobino and Kanev (1999, Theorem 4.21). Therefore, the intersection X_{-4} of the variety S_{-4} of symmetric $N \times N$ matrices of corank at least 4 and the linear space of Hankel matrices is generically transversal and hence preserves the degree, i.e. $\deg(X_{-4}) = \deg(S_{-4})$. The degree of S_{-4} was computed in Harris and Tu (1984, Proposition 12(b)) and is equal to

$$\prod_{\alpha=0}^3 \binom{N+\alpha}{4-\alpha} / \binom{2\alpha+1}{\alpha}.$$

We now turn to the claim about the density in the Zariski topology: Lemma 2.9 shows that there is an extreme ray $\mathbb{R}_+ \ell_0$ of Σ_{2d}^\vee with $I(\ell_0) \in \text{Gor}(T)$. We will now show that every linear functional in an open neighbourhood of ℓ_0 in $\text{Gor}(T)(\mathbb{R})$ spans an extreme ray of Σ_{2d}^\vee . Since $I(\ell) \in \text{Gor}(T)$ implies that the corank of the Hankel matrix B_ℓ is 4, there is an open neighbourhood of ℓ_0 such that B_ℓ is positive semi-definite for all ℓ in this neighbourhood, because the eigenvalues of a symmetric matrix depend continuously on its entries. Therefore, a linear functional ℓ in this neighbourhood spans an extreme ray of Σ_{2d}^\vee if and only if $I(\ell)_d$ generates a hyperplane in $\mathbb{R}[\underline{x}]_{2d}$, i.e. $\langle I(\ell)_d \rangle_{2d} = I(\ell)_{2d}$. By Gauss' algorithm (column echelon form), we can write a basis (b_1, b_2, b_3, b_4) of the kernel of B_ℓ in terms of rational functions in the entries of B_ℓ . We consider the linear map

$$\mathbb{R}[\underline{x}]_d^4 \rightarrow \mathbb{R}[\underline{x}]_{2d}, (f_1, f_2, f_3, f_4) \mapsto f_1 b_1 + f_2 b_2 + f_3 b_3 + f_4 b_4.$$

The rank of this map is at least $\binom{2d+2}{2} - 1$ (i.e. the image is a hyperplane) because $\ell \in \text{Gor}(T)$. The image is a hyperplane for $\ell = \ell_0$. So the same is true for every ℓ in a neighbourhood of ℓ_0 in $\text{Gor}(T)$, which shows that these ℓ are extreme rays of Σ_{2d}^\vee . \square

Remark 2.16. In the proof of the above Theorem, we see that if T is a Hilbert function occurring for a Gorenstein ideal corresponding to an extreme ray of Σ_{2d}^\vee , then there is an open subset of extreme

rays in a connected component of $\text{Gor}(T)(\mathbb{R})$ because $\text{Gor}(T)$ is smooth. As we remarked above, it might not be the entire connected component.

In fact, this gives an interesting connection to irreducible components of the algebraic boundary of the cone Σ_{2d} of sums of squares, the Zariski closure of its boundary in the Euclidean topology.

Theorem 2.17. *Let $X \subset \partial_a \Sigma_{2d}$ be an irreducible component. Then its dual projective variety X^* is a subvariety of the Zariski closure of the union of extreme rays of Σ_{2d}^\vee , i.e. the variety of Hankel matrices of corank ≥ 4 . Moreover, there is a Hilbert function T such that $\text{Gor}(T)$ is Zariski dense in X^* .*

Proof. We rely on the results of Sinn (2015) for the proof: By Sinn (2015, Proposition 3.1), the projective dual variety of X is contained in $\text{Exr}_a(\Sigma_{2d}^\vee)$, the Zariski closure of the union of extreme rays of Σ_{2d}^\vee and $X^* \cap \text{Exr}(\Sigma_{2d}^\vee)$ is Zariski dense in X^* . So let ℓ be an extreme ray of Σ_{2d}^\vee and a general point of X^* . Let T_ℓ be the Hilbert function of the corresponding Gorenstein ideal $I(\ell)$. Since $\text{Gor}(T_\ell)$ is smooth and every point in a neighbourhood of ℓ in $\text{Gor}(T_\ell)$ is also an extreme ray of Σ_{2d}^\vee , the quasiprojective variety $\text{Gor}(T_\ell)$ is Zariski dense in X^* . Indeed, the variety $\text{Gor}(T)$ is irreducible for any permissible Hilbert function T and the irreducible variety X^* is the union of some of these varieties. So one of them must be Zariski dense in X^* and for a general $\ell \in X^*$, the Hilbert function of the corresponding Gorenstein ideal $I(\ell)$ identifies this variety $\text{Gor}(T)$. \square

Our construction of an extreme ray of maximal rank also gives a base-point free special linear system with a totally real representative on a smooth curve of degree $d \geq 4$, which might be interesting in itself.

Proposition 2.18. *Let $d \geq 4$. There is a smooth real curve $X \subset \mathbb{P}^2$ of degree d and an effective divisor D of degree $g = \binom{d-1}{2}$ supported on $X(\mathbb{R})$ such that $|D|$ has dimension 1 and is base-point free.*

Proof. Start with a complete intersection $\mathcal{V}(L_1) \cap \mathcal{V}(L_2)$ of products of d linear forms and a choice of $\binom{d-1}{2}$ points $\Gamma_2 \subset \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$ such that there is a unique curve of degree $d-3$ passing through these points. Moreover, assume that all coefficients in the linear relation among the point evaluations $\{\text{ev}_v \in \mathbb{R}[X]_{d-3}^* : v \in \Gamma_2\}$ are non-zero. This situation is established in the proof of Lemma 2.13. By Bertini's Theorem (Beltrametti et al., 2009, Theorem 6.2.11) or (Jouanolou, 1983, Théorème 6.6.2), there is a smooth curve $\mathcal{V}(f)$ of degree d passing through Γ_2 such that f is a perturbation of L_1 ; more precisely, we want $\Gamma = \mathcal{V}(f) \cap \mathcal{V}(L_2)$ to be a totally real transversal intersection. Then the complete linear system $|\Gamma_2| \subset \text{Div}(\mathcal{V}(f))$ is cut out by forms of degree d through $\Gamma \setminus \Gamma_2$, i.e. $|\Gamma_2|$ is the set of all effective divisors in

$$\{C \cdot \mathcal{V}(f) - (\Gamma - \Gamma_2) : C \subset \mathbb{P}^2 \text{ of degree } d\},$$

see Eisenbud et al. (1996, Corollary 5 (to Brill–Noether's Restsatz)). We have argued in Remark 2.12 that this linear system is base-point free. We compute its dimension with the help of the Cayley–Bacharach Theorem, more precisely Eisenbud et al. (1996, Corollary 6):

$$1 = |\Gamma_2| - (\ell((d-3)H) - \ell((d-3)H - \Gamma_2)) = g - (g - \ell((d-3)H - \Gamma_2)),$$

where $H \subset \mathbb{P}^2$ is a line and $\ell(D)$ is the dimension of the Riemann–Roch space of the divisor D . This implies

$$\ell(\Gamma_2) = \deg(\Gamma_2) + 1 - g + \ell((d-3)H - \Gamma_2) = 2. \quad \square$$

Remark 2.19. Conversely, given such a linear system on a smooth curve $X \subset \mathbb{P}^2$, we can apply the construction in the proof of Lemma 2.9 to construct an extreme ray of Σ_{2d}^\vee of maximal rank, at least if there is a totally real transversal intersection $C \cap X$ with $C \cdot X - D \geq 0$. The fact, that the linear system has dimension 1 gives the unique linear relation among the point evaluations at $C \cdot X - D$ on

forms of degree d . Extremality then follows from the fact that $|D|$ is base-point free by the count of dimensions as in the proof of [Lemma 2.9](#).

3. The case $d = 5$ or ternary decics

For $d = 3$, a complete characterization of extreme rays of Σ_6^\vee was given by [Blekherman \(2012\)](#). It led to a complete description of the algebraic boundary of the sums of squares cone Σ_6 by Blekherman, Hauenstein, Ottem, Ranestad and Sturmfels, see [Blekherman et al. \(2012\)](#).

For $d = 4$, there are only two possible ranks (> 1) for extreme rays of Σ_8^\vee , namely 10 and 11; in particular, we know how to construct one of each rank. For rank 10, we use a complete intersection of a cubic and a quartic and the unique linear relation among the corresponding point evaluations on quartics to construct an extreme ray as above, see also [Blekherman \(2015\)](#). For rank 11, which is the maximal rank, we use the construction from [section 2](#). The Hilbert functions of the extreme rays constructed in this way are

$$T_{10} = (1, 3, 6, 9, 10, 9, 6, 3, 1) \text{ and}$$

$$T_{11} = (1, 3, 6, 10, 11, 10, 6, 3, 1).$$

It is possible to prove, similarly to the cases below, that both these ranks give rise to irreducible components of the algebraic boundary of Σ_8 by projective duality.

So the first new case from this point of view is $d = 5$: In fact, we can construct an extreme ray of Σ_{10}^\vee of every rank in the interval $\{13, \dots, 17\}$ between the lower and upper bound using the Cayley–Bacharach Theorem. Moreover, using the results of [Sinn \(2015\)](#), we can prove by projective duality that there is an irreducible component of the algebraic boundary of Σ_{10} for every one of these ranks; in particular, $\partial_a \Sigma_{10}$ has at least 6 irreducible components. In the following propositions in this section, we will prove the following theorem.

Theorem 3.1. *For every $r \in \{13, \dots, 17\}$, there is an extreme ray $\mathbb{R}_+ \ell_r$ of Σ_{10}^\vee such that the rank of the Hankel matrix B_{ℓ_r} is r . The Hilbert function T_r of $I(\ell_r)$ is*

$$T_{13} = (1, 3, 6, 9, 12, 13, 12, 9, 6, 3, 1),$$

$$T_{14} = (1, 3, 6, 10, 13, 14, 13, 10, 6, 3, 1),$$

$$T_{15} = (1, 3, 6, 10, 14, 15, 14, 10, 6, 3, 1),$$

$$T_{16} = (1, 3, 6, 10, 14, 16, 14, 10, 6, 3, 1),$$

$$T_{17} = (1, 3, 6, 10, 15, 17, 15, 10, 6, 3, 1).$$

The dual varieties to $\text{Gor}(T_r)$ are irreducible components of the algebraic boundary of the sums of squares cone Σ_{10} for all $r \in \{13, \dots, 17\}$. The variety $\text{Gor}(T_{17})$ is Zariski dense in the Zariski closure of the union of all extreme rays. It has dimension 55 and degree 53300016.

The construction given in the preceding section for extreme rays of maximal rank $\binom{d+2}{2} - 4$ leads to an extreme ray $\mathbb{R}_+ \ell$ of Σ_{10}^\vee such that the Hilbert function of the corresponding Gorenstein ideal $I(\ell)$ is

$$T_{17} = (1, 3, 6, 10, 15, 17, 15, 10, 6, 3, 1).$$

By [Theorem 2.15](#), the Zariski closure of the set of extreme rays of Σ_{10}^\vee is $\text{cl}(\text{Gor}(T_{17}))$, a unirational variety of codimension 10 in \mathbb{P}^{65} . So [Sinn \(2015, Theorem 3.3\)](#) implies that its dual variety is an irreducible component of the algebraic boundary of Σ_{10} .

We now work our way up beginning with the lowest rank 13, following the construction in [Blekherman \(2015\)](#):

Proposition 3.2. *There is an extreme ray $\mathbb{R}_+\ell$ of Σ_{10}^\vee of rank 13. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{13} = (1, 3, 6, 9, 12, 13, 12, 9, 6, 3, 1)$$

and the variety dual to $\text{cl}(\text{Gor}(T_{13}))$ is an irreducible component of the algebraic boundary of Σ_{10} .

Proof. Let $L_1 = x(x-z)(x-2z)(x-3z)(x-4z)$ and $L_2 = y(y-z)(y-2z)$ and $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. By construction, there is a unique linear relation among $\{\text{ev}_v \in \mathbb{R}[\underline{x}]_5^* : v \in \Gamma\}$, say $\sum_{v \in \Gamma} u_v \text{ev}_v = 0$, and all coefficients in this relation are non-zero. The linear functional

$$\ell = \sum_{v \in \Gamma \setminus \{P\}} \text{ev}_v - \frac{u_P^2}{\sum_{v \in \Gamma \setminus \{P\}} u_v^2} \text{ev}_P$$

is positive semi-definite of rank 13 for any $P \in \Gamma$ by the Cauchy–Schwarz inequality by the same argument as in the proof of [Lemma 2.9](#). By a Hilbert function computation using `Macaulay2` ([Grayson and Stillman](#)), we verify, that the degree 5 part of the corresponding Gorenstein ideal $I(\ell)$ generates a hyperplane in degree 10. To prove that the dual variety to $\text{Gor}(T_{13})$ is an irreducible component of $\partial_a \Sigma_{10}$, we use [Sinn \(2015, Theorem 3.7\)](#). The condition given there is equivalent to

$$(T_\ell \text{Gor}(T_{13}))^\perp = (I(\ell)_5)^2$$

because the face of Σ_{10} supported by ℓ is the set of sums of squares of polynomials in $I(\ell)_5$, which spans the vector space $(I(\ell)_5)^2$. By the description of the tangent space to $\text{Gor}(T_{13})$ at ℓ (see [Theorem 1.9](#)), this is equivalent to

$$(I(\ell)^2)_{10} = (I(\ell)_5)^2,$$

which we also check using `Macaulay2` ([Grayson and Stillman](#)). \square

Proposition 3.3. *There is an extreme ray $\mathbb{R}_+\ell$ of Σ_{10}^\vee of rank 14. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{14} = (1, 3, 6, 10, 13, 14, 13, 10, 6, 3, 1)$$

and the variety dual to $\text{cl}(\text{Gor}(T_{14}))$ is an irreducible component of the algebraic boundary of Σ_{10} .

Proof. In this case, take $L_1 = x(x-z)(x-2z)(x-3z)$ and $L_2 = y(y-z)(y-2z)(y-3z)$ and set $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. There is a unique linear relation among $\{\text{ev}_v \in \mathbb{R}[\underline{x}]_5^* : v \in \Gamma\}$, say $\sum_{v \in \Gamma} u_v \text{ev}_v = 0$, and all its coefficients are non-zero. As above, the linear functional

$$\ell = \sum_{v \in \Gamma \setminus \{P\}} \text{ev}_v - \frac{u_P^2}{\sum_{v \in \Gamma \setminus \{P\}} u_v^2} \text{ev}_P$$

is positive semi-definite of rank 14 for any $P \in \Gamma$. Again, using `Macaulay2` ([Grayson and Stillman](#)), we verify, that the degree 5 part of the corresponding Gorenstein ideal $I(\ell)$ generates a hyperplane in degree 10 and that

$$(I(\ell)^2)_{10} = (I(\ell)_5)^2. \quad \square$$

Proposition 3.4. *There is an extreme ray $\mathbb{R}_+\ell$ of Σ_{10}^\vee of rank 15. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{15} = (1, 3, 6, 10, 14, 15, 14, 10, 6, 3, 1)$$

and the variety dual to $\text{cl}(\text{Gor}(T_{15}))$ is an irreducible component of the algebraic boundary of Σ_{10} .

Proof. In this case, we start with a complete intersection of a quartic and a quintic, $L_1 = x(x-z)(x-2z)(x-3z)(x-4z)$, $L_2 = y(y-z)(y-2z)(y-3z)$ and $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. Choose $\Gamma_2 = \{(0:2:1), (1:1:1), (2:0:1)\}$ and set $\Gamma_1 = \Gamma \setminus \Gamma_2$. By Cayley–Bacharach, there is a unique linear relation among the 17 points of Γ_1 . Using `Macaulay2` (Grayson and Stillman), we complete the proof as above. \square

Proposition 3.5. *There is an extreme ray $\mathbb{R}_+\ell$ of Σ_{10}^\vee of rank 16. The Hilbert function of the Gorenstein ideal $I(\ell)$ is*

$$T_{16} = (1, 3, 6, 10, 14, 16, 14, 10, 6, 3, 1)$$

and the variety dual to $\text{cl}(\text{Gor}(T_{16}))$ is an irreducible component of the algebraic boundary of Σ_{10} .

Proof. Choose $L_1 = x(x-z)(x-2z)(x-3z)(x-4z)$, $L_2 = y(y-z)(y-2z)(y-3z)(y-4z)$ and $\Gamma = \mathcal{V}(L_1) \cap \mathcal{V}(L_2)$. This time, $\Gamma_2 = \{(0:2:1), (1:1:1), (2:0:1), (1:4:1), (2:3:1), (3:2:1), (4:1:1)\}$ and $\Gamma_1 = \Gamma \setminus \Gamma_2$ do the job: Cayley–Bacharach gives a unique linear relation among the 18 points of Γ_1 . Using `Macaulay2` (Grayson and Stillman), we complete the proof as above. \square

For general $d > 5$, our constructive method using the Cayley–Bacharach Theorem cannot construct an extreme ray of every rank in the interval $\{3d-2, \dots, \binom{d+2}{2}\}$ given by the lower and upper bound. The first failure occurs for $d = 6$ and rank $17 \in \{16, \dots, 24\}$. In fact, Σ_{12}^\vee does not have an extreme ray of rank 17, as we will see below, see Lemma 3.7. Let us first argue why we cannot construct an extreme ray of rank 17 of Σ_{12}^\vee .

Remark 3.6. Our construction starts with a totally real intersection of two curves X_1 and X_2 with $\deg(X_1) + \deg(X_2) \geq d+3$; we then need 19 intersection points such that the corresponding point evaluations on forms of degree 6 satisfy a unique linear relation in which all coefficients are non-zero. This configuration would lead to a positive linear functional such that the Hankel matrix has the desired rank 17 (of course we would still need to prove extremality). We will see that this is not possible:

The following tuples are permissible choices for the degrees of the curves $(3, 6)$, $(4, 5)$, $(4, 6)$, $(5, 5)$, $(5, 6)$ and $(6, 6)$. For $(\deg(X_1), \deg(X_2)) = (3, 6)$, the transversal intersection has only 18 points. In the case $(4, 5)$, there is a unique linear relation among point evaluations at the 20 intersection points such that all coefficients are non-zero; in particular, whatever point we remove, the remaining 19 point evaluations are linearly independent on forms of degree 6. In the cases $(4, 6)$, $(5, 5)$ and $(5, 6)$, we cannot have the desired number of points on a curve of dual degree $s-d$: For example, in order to apply the duality of the Cayley–Bacharach Theorem to the 24 intersection points in the case $(4, 6)$, we would need to have 5 of the intersection points on a line, which intersects the quartic in only 4 points. The last case $(6, 6)$ is more subtle: We would like to find exactly 17 intersection points on a cubic, which is impossible, because there is a unique linear relation among the corresponding point evaluations on forms of degree 6 on the complete intersection of a cubic and a sextic, see Eisenbud et al. (1996, CB4).

This is not a defect of our construction in this case. In fact, there are no extreme rays of Σ_{12}^\vee of rank 17.

Lemma 3.7. *There is no Gorenstein ideal $I \subset \mathbb{C}[x, y, z]$ with socle in degree 12 such that $\text{Hilb}(I, 6) = 17$ and I_6 is maximal with respect to inclusion among J_6 , where J runs over all Gorenstein ideals with socle in degree 12.*

In the proof of this lemma, we will use the following fact multiple times.

Remark 3.8. The complete intersection of three ternary forms of degree d_1 , d_2 , and d_3 respectively is a Gorenstein ideal in $\mathbb{C}[x, y, z]$ with socle in degree $d_1 + d_2 + d_3 - 3$, see Eisenbud et al. (1996,

Theorem CB8). The socle degree follows using an elementary count of dimensions and the fact that the generators must be relatively prime.

Proof. We exclude possibilities arguing by the lowest degree k of a generator of I . First note that maximality of I_6 implies that $\langle I_6 \rangle_{12} = I_{12}$ and $\mathcal{V}(I_6) = \emptyset$. Indeed, suppose $\langle I_6 \rangle_{12} \subsetneq I_{12}$. Then there are two linearly independent linear forms ℓ_1 and ℓ_2 in $\mathbb{C}[x, y, z]_{12}^*$ that vanish on the linear space $\langle I_6 \rangle_{12}$, but $I(\ell_1)_6 = I_6 = I(\ell_2)_6$ by maximality of I_6 . Since the middle Catalecticant matrices of ℓ_1 and ℓ_2 have the same kernel, we can find a linear combination $\ell = \alpha_1 \ell_1 + \alpha_2 \ell_2$ such that the kernel $I(\ell)_6$ of the middle Catalecticant of ℓ has higher dimension than I_6 , which contradicts the maximality of I_6 . The fact that $\mathcal{V}(I_6) = \emptyset$ now follows because if there was a point p in $\mathcal{V}(I_6)$, then ev_p on $\mathbb{C}[x, y, z]_{12}$ would be the socle of I , which is impossible because of the constraint $\text{Hilb}(I, 6) = 17$ on the Hilbert function.

So in particular, we can always choose a complete intersection of three forms in I_6 , one of which can be chosen to be a suitable multiple of a generator of minimal degree of I . Let k be the minimal degree of a generator of I . The ideal I cannot contain a quadric generator, because the linear function defining I would then be supported on 12 points by the apolarity lemma (see [Iarrobino and Kanev, 1999, Chapter 1](#)), which implies $\text{Hilb}(I, 6) \leq 12$. In case $k = 3$, the Gorenstein ideal is actually generated by a cubic and two sextics that are a complete intersection, see [Stanley \(1978\)](#), so $\text{Hilb}(I, 6) = 16 = 28 - (10 + 2)$. The case $k = 6$ is also easily excluded because Hilbert functions of Gorenstein ideals in $\mathbb{C}[x, y, z]$ are unimodal by [Stanley \(1978, Theorem 4.2\)](#), which implies in this case that $\text{Hilb}(I, 5) \leq \text{Hilb}(I, 6) = 17$, or equivalently $\dim(I_5) \geq 4$. This leaves the two cases $k = 5$ and $k = 4$:

Suppose $k = 5$, then the Hilbert functions of Gorenstein ideals with socle in degree 12 and order 5 are in 1–1 correspondence with self-complimentary partitions of 10×4 blocks, see [Diesel \(1996, Proposition 3.9\)](#). By unimodality of Hilbert functions, we have $\dim(I_5) \geq 4$. This determines the first three rows of the blocks, so we can choose two more generators of degree ≤ 6 . Since a block of degree 5 forces a relation in degree 6 by the self-complimentarity of the partition, the 4 generators of degree 5 generate a $4 \cdot 3 - 3 = 9$ -dimensional subspace of $\mathbb{C}[x, y, z]_6$. The only two possible degrees for the other two generators to achieve $\dim(I_6) = 28 - 17 = 11$ are therefore one more generator of degree 5 and one generator of degree 7 or two generators in degree 6. In the first case, the degrees of generators are $q = (5, 5, 5, 5, 5, 7, 7, 9, 9, 9, 9)$ and the corresponding relation degrees are $p = (10, 10, 10, 10, 10, 8, 8, 6, 6, 6, 6)$. Since there is no generator of degree 6 and $\mathcal{V}(I_6) = \emptyset$, we find a complete intersection of three forms of degree 5 contained in I . These generate a Gorenstein ideal with socle in degree $5 + 5 + 5 - 3 = 12$. This is impossible, because I contains further generators. In the second case, the degrees of generators are $q = (5, 5, 5, 5, 6, 6, 8, 8, 9, 9, 9)$ and the corresponding relation degrees are $p = (10, 10, 10, 10, 10, 9, 9, 7, 7, 6, 6)$. The quasiprojective variety $\text{Gor}(T)$, where T is given by these generator and relation degrees, contains a dense subset of Gorenstein ideals generated by polynomials of degree $q_{\min} = (5, 5, 5, 5, 8, 8, 9)$ by [Diesel \(1996, Section 3.3\)](#). So the same argument as in the first case excludes this possibility, too.

So now we are left with the case $k = 4$: In this case, Hilbert functions of Gorenstein ideals with socle in degree 12 and order 4 correspond to self-complimentary partitions of 8×6 blocks. If $\dim(I_4) = 2$, then these would generate a $2 \cdot 6 = 12$ -dimensional subspace in $\mathbb{C}[x, y, z]_6$ because they correspond to relations in degree 7. So $\dim(I_4) = 1$. We can choose 4 more degrees of generators ≤ 6 . A generator of degree 5 comes with a relation in degree 5 and therefore, the only two possible choices of degrees for the generators with $\text{Hilb}(I, 6) = 17$ are one generator of degree 5 and three generators of degree 6 or two generators of degree 5 and one generator of degree 6. Let's first consider $q = (4, 5, 5, 6, 7, 7, 8, 9, 9)$, which corresponds to relation degrees $p = (11, 10, 10, 9, 8, 8, 7, 6, 6)$. Again, $\text{Gor}(T)$, where T is given by these generator and relation degrees, contains a dense subset of Gorenstein ideals generated by polynomials of degree $q_{\min} = (4, 5, 5, 7, 9)$. There is no generator of degree 6 anymore, so $\mathcal{V}(I_6) = \emptyset$ implies that we find a complete intersection of a quartic and two quintics, which generate a Gorenstein ideal with socle in degree $4 + 5 + 5 - 3 = 11$, which is impossible. The last remaining case is $q = (4, 5, 6, 6, 6, 8, 8, 8, 9)$ with corresponding relation degrees $p = (11, 10, 9, 9, 9, 7, 7, 7, 6)$. Here, $\text{Gor}(T)$ contains a dense subset of Gorenstein ideals with generators of degree $q_{\min} = (4, 5, 6, 6, 8, 8, 8)$ and correspondingly $p_{\min} = (11, 10, 9, 9, 7, 7, 7)$. The

assumption $\mathcal{V}(I_6) = \emptyset$ implies only that we can find a complete intersection of a quartic and two sextics. They generate a Gorenstein ideal with socle in degree $4 + 6 + 6 - 1 = 13$. This Gorenstein ideal has Hilbert function 3 in degree 12 because Hilbert functions of Gorenstein ideals are symmetric, see [Remark 1.2](#). This is impossible because this complete intersection together with the other 4 generators would then fill $\mathbb{C}[x, y, z]_{12}$. This concludes the case study. \square

We can use similar ideas as above for $d = 5$ to show that Σ_{12}^\vee has extreme rays of ranks $18, \dots, 23$. We start with the complete intersection of a sextic with a quartic, quintic, or sextic in 24, 30, and 36 points respectively and remove the desired number of points on the appropriate number of lines to get a unique linear relation among the point evaluations on sextics at the remaining points. We can then use Macaulay2 ([Grayson and Stillman](#)) to check extremality as before. Again, for all these ranks, the projective dual variety to $\text{Gor}(T_7)$ will be an irreducible component of the algebraic boundary of Σ_{12} .

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