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The uniform normal form of a linear mapping

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This paper gives a normal form for a linear mapping of a finite dimensional vector space over a field of characteristic 0 into itself, which yields a better description of its structure than the classical companion matrix. Finding this normal form does not use any factorization of the characteristic polynomial of the linear mapping and requires only a finite number of operations in the field to compute.

Let V be a finite dimensional vector space over a field k of characteristic 0. Let $A : V \to V$ be a linear mapping of V into itself with characteristic polynomial χ_A. The goal of this paper is to determine a normal form for A, which describes its structure better than the classical companion matrix. Finding this normal form does not require knowing a factorization of χ_A and uses only a finite number of operations in the field k to compute.

The main result of [2] gives an algorithm, involving no factorization of χ_A and only a finite number of operations in the field k, which yields the Jordan decomposition of A, namely, writes A as a sum of commuting semisimple and nilpotent S and N parts,

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respectively. For more details see [4]. In what follows we will assume that S and N are known.

1. Nilpotent normal form

In this section we describe the well known Jordan normal form for a nilpotent linear transformation N.

A linear transformation $N : V \to V$ is said to be *nilpotent of index n* if there is an integer $n \geq 1$ such that $N^{n-1} \neq 0$ but $N^n = 0$. Suppose that for some positive integer ≥ 1 there is a nonzero vector v, which lies in $\ker N^\ell \setminus \ker N^{\ell-1}$. The set $\{v, Nv, \ldots, N^{\ell-1}v\}$ is a *Jordan chain* of *length* ℓ with *generating vector* v. The space V^ℓ spanned by the vectors in a given Jordan chain of length ℓ is a N-*cyclic subspace* of V. Because $N^\ell v = 0$, the subspace V^ℓ is N-invariant. Since $\ker N|V^\ell = \mathrm{span}\{N^{\ell-1}v\}$, the mapping $N|V^\ell$ has exactly one eigenvector corresponding to the eigenvalue 0.

Fact 1.1. Vectors in a Jordan chain of length ℓ are linearly independent.

With respect to the *standard basis* $\{v, Nv, \ldots N^{\ell-2}v, N^{\ell-1}v\}$ of V^ℓ the matrix of $N|V^\ell$ is the $\ell \times \ell$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

which is a *Jordan block* of size ℓ. The Jordan normal form theorem [1, pp. 270–274] states

Fact 1.2. V is a direct sum of N-cyclic subspaces.

A suitable reordering of the basis giving the Jordan normal form of N is a basis of V, realizes the Young diagram of N. The elements of the Young diagram are given by a dark dot $\bullet$ or an open dot $\circ$ in Fig. 1.1 and the arrows give the action of N on the basis vectors. The columns of the Young diagram of N are Jordan chains with generating vector given by an open dot. The black dots form a basis for the image $\mathrm{im}\, N$ of N. The open dots form a basis for a complementary subspace of $\mathrm{im}\, N$ in V. The dots on or above the jth row of the Young diagram form a basis for $\ker N^j$ and the black dots in the first row form a basis for $\ker N \cap \mathrm{im}\, N$. Let r_j be the number of dots in the jth row. Then $r_j = \dim\ker N^j - \dim\ker N^{j-1}$. Thus the Young diagram of N is unique.

We note that finding the generating vectors of the Young diagram of N or equivalently the Jordan normal form of N, involves solving linear equations with coefficients in the field k and thus requires only a finite number of operations in the field k to be determined.

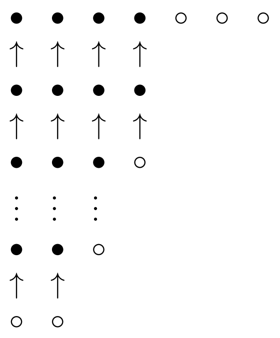

Fig. 1.1. The Young diagram of N.

2. Some facts about S

We now study the semisimple part S of the linear map A.

Lemma 2.1. *Let W be an S-invariant proper subspace of V. Then the characteristic polynomial $\chi_{S|W}$ of $S|W$ is a factor of the characteristic polynomial χ_S on V.*

Proof. Since S is a semisimple linear map and W is S-invariant, there is an S-invariant subspace U of V such that $V = W \oplus U$. Consequently, $\chi_S = \chi_{S|W} \chi_{S|U}$. □

Lemma 2.2. $V = \ker S \oplus \operatorname{im} S$. *Moreover the characteristic polynomial $\chi_S(\lambda)$ of S can be written as a product of λ^n, where $n = \dim \ker S$, and $\chi_{S|\operatorname{im} S}$, the characteristic polynomial of $S|\operatorname{im} S$. Note that $\chi_{S|\operatorname{im} S}(0) \neq 0$.*

Proof. $\ker S$ is an S-invariant subspace of V. Since $Sv = 0$ for every $v \in \ker S$, the characteristic polynomial of $S|\ker S$ is λ^n.

Because S is semisimple, there is an S-invariant subspace Y of V such that $V = \ker S \oplus Y$. The linear mapping $S|Y : Y \to Y$ is invertible, for if $Sy = 0$ for some $y \in Y$, then $S(y + u) = 0$ for every $u \in \ker S$. Therefore $y + u \in \ker S$, which implies that $y \in \ker S \cap Y = \{0\}$, that is, $y = 0$. So $S|Y$ is invertible. Suppose that $y \in Y$, then $y = S\big((S|Y)^{-1} y\big) \in \operatorname{im} S$. Thus $Y \subseteq \operatorname{im} S$. But $\dim \operatorname{im} S = \dim V - \dim \ker S = \dim Y$. So $Y = \operatorname{im} S$.

Since $\ker S \cap \operatorname{im} S = \{0\}$, we see that λ does not divide the polynomial $\chi_{S|\operatorname{im} S}(\lambda)$. Consequently, $\chi_{S|\operatorname{im} S}(0) \neq 0$. From Lemma 2.1 we obtain

$$\chi_S(\lambda) = \chi_{S|\ker S}(\lambda)\, \chi_{S|\operatorname{im} S}(\lambda) = \lambda^n \chi_{S|\operatorname{im} S}(\lambda). \qquad \square$$

Lemma 2.3. *The subspaces* $\ker S$ *and* $\operatorname{im} S$ *are N-invariant and hence A-invariant.*

Proof. Suppose that $x \in \operatorname{im} S$. Then there is a vector $v \in V$ such that $x = Sv$. So $Nx = N(Sv) = S(Nv) \in \operatorname{im} S$. In other words, $\operatorname{im} S$ is an N-invariant subspace of V.

Because $\operatorname{im} S$ is also S-invariant and $A = S + N$, it follows that $\operatorname{im} S$ is an A-invariant subspace of V. Suppose that $x \in \ker S$, that is, $Sx = 0$. Then $S(Nx) = N(Sx) = 0$. So $Nx \in \ker S$. Therefore $\ker S$ is an N-invariant and hence A-invariant subspace of V. □

3. Description of uniform normal form

We now describe the uniform normal form of the linear mapping $A : V \to V$, using its Jordan decomposition into commuting semisimple and nilpotent summands, S and N, respectively.

Determine the Jordan normal form for the nilpotent linear maps $N|\ker S$ and $N|\operatorname{im} S$. Since $\ker S$ and $\operatorname{im} S$ are N-invariant and $V = \ker S \oplus \operatorname{im} S$, this determines the Jordan normal form of N. For $1 \le \ell \le p$ let F_{q_ℓ} be the q_ℓ-dimensional space spanned by the generating vectors of Jordan chains of N of length m_ℓ, where for $1 \le \ell \le r$ the subspace F_{q_ℓ} lies in $\ker S$ and for $r+1 \le \ell \le p$ it lies in $\operatorname{im} S$.

Now we prove

Claim 3.1. *For each $1 \le \ell \le p$ the space F_{q_ℓ} is S-invariant.*

Proof. Let $v^\ell \in F_{q_\ell}$. Then $\{v^\ell, Nv^\ell, \ldots, N^{m_\ell-1}v^\ell\}$ is a Jordan chain of length m_ℓ with generating vector v^ℓ. For each $1 \le \ell \le r$ we have $F_{q_\ell} \subseteq \ker S$. So trivially F_{q_ℓ} is S-invariant, because $S = 0$ on F_{q_ℓ}. Now suppose that $r+1 \le \ell \le p$. Then $F_{q_\ell} \subseteq \operatorname{im} S$. Consider the Jordan chain $\{Sv^\ell, N(Sv^\ell), \ldots, N^{m_\ell-1}(Sv^\ell)\}$, which lies in $\operatorname{im} S$, since $v^\ell \in \operatorname{im} S$ and $\operatorname{im} S$ is S and N invariant. We now show that this Jordan chain has length m_ℓ. Suppose that for some $\alpha_j \in \mathrm{k}$ with $0 \le j \le m_\ell - 1$ we have $0 = \sum_{j=0}^{m_\ell-1} \alpha_j N^j(Sv^\ell)$. Then $0 = S\big(\sum_{j=0}^{m_\ell-1} \alpha_j N^j v^\ell\big)$, because on $\operatorname{im} S$ the maps S and N commute. Since $S|\operatorname{im} S$ is invertible, the preceding equality implies $0 = \sum_{j=0}^{m_\ell-1} \alpha_j N^j v^\ell$. Applying Fact 1.1 to the Jordan chain $\{v^\ell, Nv^\ell, \ldots, N^{m_\ell-1}v^\ell\}$ of length m_ℓ, it follows that $\alpha_j = 0$ for every $0 \le j \le m_\ell - 1$. So the Jordan chain $\{Sv^\ell, N(Sv^\ell), \ldots, N^{m_\ell-1}(Sv^\ell)\}$ with generating vector Sv^ℓ has length m_ℓ, that is, $Sv^\ell \in F_{q_\ell}$. Hence F_{q_ℓ} is an S-invariant subspace of $\operatorname{im} S$ and thus one of V. □

Following [3] we say that an A-invariant subspace U of V is *uniform* of *height* $m-1$ if $N^{m-1}U \ne \{0\}$ but $N^m U = \{0\}$ and $\ker N^{m-1} \cap U = NU$. The concept of a uniform subspace is essential in the classification of indecomposable types (and hence of conjugacy classes) for the classical groups over the real numbers.

For each $1 \le \ell \le p$ let U^{q_ℓ} be the space spanned by the vectors in the Jordan chains of N of length m_ℓ with generating vectors in F_{q_ℓ}. Because $\ker S$ and $\operatorname{im} S$ are N-invariant, for $1 \le \ell \le r$ the subspaces U^{q_ℓ} lie in $\ker S$, while for $r+1 \le \ell \le p$ they lie in $\operatorname{im} S$. Since U^{q_ℓ} is S and N invariant, U^{q_ℓ} is A-invariant.

Claim 3.2. *For $1 \le \ell \le p$ the subspace U^{q_ℓ} is uniform of height $m_\ell - 1$.*

Proof. From the Young diagram of N and the definition of U^{q_ℓ} we see $U^{q_\ell} = F_{q_\ell} \oplus NF_{q_\ell} \oplus \cdots \oplus N^{m_\ell -1}F_{q_\ell}$. Since $N^{m_\ell}F_{q_\ell} = \{0\}$ but $N^{m_\ell -1}F_{q_\ell} \neq \{0\}$, the subspace U^{q_ℓ} is A-invariant and of height $m_\ell - 1$. To show that U^{q_ℓ} is uniform we need only show that $\ker N^{m_\ell -1} \cap U^{q_\ell} \subseteq NU^{q_\ell}$, since the inclusion of NU^{q_ℓ} in $\ker N^{m_\ell -1} \cap U^{q_\ell}$ follows from the fact that $N^{m_\ell}U^{q_\ell} = 0$. Suppose that $u \in \ker N^{m_\ell -1} \cap U^{q_\ell}$, then for every $0 \le i \le m_\ell - 1$ there are vectors $f_i \in F_{q_\ell}$ such that $u = f_0 + Nf_1 + \cdots + N^{m_\ell -1}f_{m_\ell -1}$. Since $u \in \ker N^{m_\ell -1}$ we get $0 = N^{m_\ell -1}u = N^{m_\ell -1}f_0$. If $f_0 \neq 0$, then the preceding equality contradicts the fact that f_0 is a generating vector of a Jordan chain of N of length m_ℓ. Therefore $f_0 = 0$, which means that $u = N(f_1 + \cdots + N^{m_\ell -2}f_{m_\ell -1}) \in NU^{q_\ell}$. Thus $\ker N^{m_\ell -1} \cap U^{q_\ell} \subseteq NU^{q_\ell}$. Hence $\ker N^{m_\ell -1} \cap U^{q_\ell} = NU^{q_\ell}$, that is, the subspace U^{q_ℓ} is uniform of height $m_\ell - 1$. □

Now we give an explicit description of the uniform normal form of the linear mapping A. For each $1 \le \ell \le p$ let $\chi_{S|F_{q_\ell}}$ be the characteristic polynomial of S on F_{q_ℓ}. Note that when $1 \le \ell \le r$, then $\chi_{S|F_{q_\ell}} = 0$, since $S|F_{q^\ell} = 0$. Choose a basis $\{u^\ell_j\}_{j=1}^{q_\ell}$ of F_{q_ℓ} so that the matrix of $S|F_{q^\ell}$ is the $q_\ell \times q_\ell$ companion matrix C_{q_ℓ}, which is 0, when $1 \le \ell \le r$, or

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & -a_{q_\ell -2} \\ 0 & \cdots & \cdots & 1 & -a_{q_\ell -1} \end{pmatrix},$$

which is associated to the characteristic polynomial

$$\chi_{S|F_{q_\ell}} = a_0 + a_1\lambda + \cdots + a_{q_\ell -1}\lambda^{q_\ell -1} + \lambda^{q_\ell}$$

of $S|F_{q^\ell}$, when $r+1 \le \ell \le p$. Using the basis $\{u^\ell_j, Nu^\ell_j, \ldots, N^{m_\ell -1}u^\ell_j\}_{j=1}^{q_\ell}$ for U^{q_ℓ}, the matrix of $A|U^{q_\ell}$ is the $m_\ell q_\ell \times m_\ell q_\ell$ matrix

$$D_{m_\ell q_\ell} = \begin{pmatrix} C_{q_\ell} & 0 & 0 & \cdots & \cdots & 0 \\ I & C_{q_\ell} & 0 & \cdots & \vdots & 0 \\ 0 & I & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & I & C_{q_\ell} & 0 \\ 0 & \cdots & \cdots & 0 & I & C_{q_\ell} \end{pmatrix}.$$

Since $V = \sum_{\ell=1}^{p} \oplus U^{q_\ell}$, the matrix of A is $\mathrm{diag}(D_{m_1q_1}, \ldots, D_{m_pq_p})$ with respect to the basis $\{u^\ell_j, Nu^\ell_j, \ldots, N^{m_\ell -1}u^\ell_j\}_{(j,\ell)=(1,1)}^{(q_\ell,p)}$. We call preceding matrix the *uniform normal form* for the linear map A of V into itself. We note that this normal form can be computed using only a finite number of operations in the field k.

We obtain a factorization of the characteristic polynomial of A, whose factors are not necessarily irreducible.

Corollary 3.3. *We have*

$$\chi_A(\lambda) = \prod_{\ell=1}^{p} \chi^{m_\ell}_{S|F_{q_\ell}}(\lambda) = \lambda^n \prod_{\ell=r+1}^{p} \chi^{m_\ell}_{S|F_{q_\ell}}(\lambda), \tag{1}$$

where $n = \sum_{\ell=1}^{r} m_\ell = \dim \ker S$. *The polynomials* $\chi_{S|F_{q_\ell}}$, $r+1 \le \ell \le p$, *are pairwise relatively prime and* $\chi_{S|\,\mathrm{im}\, S} = \prod_{\ell=r+1}^{p} \chi_{S|F_{q_\ell}}$.

Proof. Equation (1) follows immediately from the uniform normal form of A. To prove the assertion about the factors of χ_A we argue as follows. Because for each $r+1 \le \ell \le p$ the subspace F_{q_ℓ} of $\mathrm{im}\, S$ is S-invariant, from Lemma 2.1 it follows that $\chi_{S|F_{q_\ell}}$ is a factor of $\chi_{S|\,\mathrm{im}\, S}$ for $r+1 \le \ell \le p$. For some ℓ and ℓ' between $r+1$ and p suppose that the polynomials $\chi_{S|F_{q_\ell}}$ and $\chi_{S|F_{q_{\ell'}}}$ have a nonconstant factor u. Then u^2 is a factor of $\chi_{S|\,\mathrm{im}\, S}$, which contradicts the fact that $\chi_{S|\,\mathrm{im}\, S}$ is square free, since $S|\,\mathrm{im}\, S$ is semisimple. Hence, the factors $\chi_{S|F_{q_\ell}}$, $r+1 \le \ell \le p$ are pairwise relatively prime.

From equation (1) it follows that $\chi_{A|\,\mathrm{im}\, S}(\lambda) = \prod_{\ell=r+1}^{p} \chi^{m_\ell}_{S|F_{q_\ell}}(\lambda)$. Because the factors $\chi_{S|F_{q_\ell}}$ for $r+1 \le \ell \le p$ are pairwise relatively prime, the polynomial $\prod_{\ell=r+1}^{p} \chi_{S|F_{q_\ell}}$ is a square free factorization of $\chi_{A|\,\mathrm{im}\, S}$, that is, the quotient of $\chi_{A|\,\mathrm{im}\, S}$ and the greatest common divisor of $\chi_{A|\,\mathrm{im}\, S}$ and its derivative. Thus $\prod_{\ell=r+1}^{p} \chi_{S|F_{q_\ell}}$ is the characteristic polynomial $\chi_{S|\,\mathrm{im}\, S}$ of the semisimple part $S|\,\mathrm{im}\, S$ of $A|\,\mathrm{im}\, S$. □

Remark. Using the notation of Claim 3.1 and the discussion after Claim 3.2, let $F = F_{q_\ell}$ for some $r+1 \le \ell \le p$ and let C be the $q_\ell \times q_\ell$ companion matrix of $S|F = S|F_{q_\ell}$. If we could write F as a direct sum of a finite number n of proper S-invariant subspaces F_i, then using a suitable basis, we could write $C = \mathrm{diag}(C_1, \ldots, C_n)$. This would give a factorization $\chi_{S|F} = \prod_{i=1}^{n} \chi_{S|F_i}$ of the characteristic polynomial of $S|F$ into distinct relatively prime nonconstant factors. Conversely, knowing such a factorization of $\chi_{S|F}$ would give rise to a direct sum decomposition of F into S-invariant subspaces. (The summands in the S-invariant direct sum decomposition of F are of minimal positive dimension if and only if each of the distinct factors of $\chi_{S|F}$ is irreducible.) Thus without additional hypotheses on the factors $\chi_{S|F_{q_\ell}}$ of $\chi_{S|\,\mathrm{im}\, S}$, the dimension q_ℓ of F_{q_ℓ} for $r+1 \le \ell \le p$ is minimal. Hence the diagonal block sizes in the uniform normal form of A are minimal.

4. Note added in proof

The author would like to thank Dr. Vladimir Sergeichuk for pointing out [5] to him. Robinson's generalized Jordan canonical form is the same as our uniform normal form, although he used a factorization to obtain it.

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