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Holomorphic functions on non-Runge domains and related problems

L. Bernal-González [a], M.C. Calderón-Moreno [a], J. López-Salazar [b,*], J.B. Seoane-Sepúlveda [c]

[a] *Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, Avenida Reina Mercedes, Sevilla, 41080, Spain*
[b] *Departamento de Matemática Aplicada a las Tecnologías de la Información y las Comunicaciones, Escuela Técnica Superior de Ingeniería y Sistemas de Telecomunicación, Universidad Politécnica de Madrid, Carretera de Valencia, Km. 7, Madrid, 28031, Spain*
[c] *ICMAT and Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Plaza de Ciencias 3, Universidad Complutense de Madrid, Madrid, 28040, Spain*

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ABSTRACT

Let U be a domain in $\mathbb{C}^N$ that is not a Runge domain. We study the topological and algebraic properties of the family of holomorphic functions on U which cannot be approximated by polynomials.

* Corresponding author.
E-mail addresses: lbernal@us.es (L. Bernal-González), mccm@us.es (M.C. Calderón-Moreno), jlsalazar@euitt.upm.es (J. López-Salazar), jseoane@ucm.es (J.B. Seoane-Sepúlveda).

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1. Introduction and preliminaries

Let U be a domain in the N-dimensional complex space $\mathbb{C}^N$ (where $N \in \mathbb{N}$, the set of positive integers); that is, U is a nonempty connected open subset of $\mathbb{C}^N$. In this paper, we consider the space $H(U)$ of all holomorphic functions from U into $\mathbb{C}$, endowed with the topology of uniform convergence on compacta. It is well known that, under this topology, $H(U)$ becomes a Fréchet space, meaning a complete metrizable locally convex topological vector space. Not always a holomorphic function on U can be approximated by polynomials with respect to this topology, as the example of the function $f(z) = 1/z$ on $U = \mathbb{C} \setminus \{0\}$ shows. Then the question arises naturally: how many holomorphic functions (in both topological and algebraic senses) cannot be approximated in this way, assuming that there is at least one of such functions? The aim of this paper is to shed some light on this matter.

Let us fix some terminology and notation, most of them standard. The symbol $\mathbb{N}_0$ stands for the set $\mathbb{N} \cup \{0\}$. If A is a set, then $\mathrm{card}(A)$ represents its cardinality; in particular, $\aleph_0 := \mathrm{card}(\mathbb{N})$ and $\mathfrak{c} := \mathrm{card}(\mathbb{R})$, the cardinality of continuum. Turning to polynomial approximation, it is more natural to consider domains for which not every holomorphic functions can be extended beyond the boundary.

A domain $U \subset \mathbb{C}^N$ is called a domain of holomorphy (or domain of existence) whenever there do not exist two domains G_1 and G_2 in $\mathbb{C}^N$ with the following properties:

(1) $G_2 \subset U \cap G_1$ and $G_1 \not\subset U$.
(2) For every $f \in H(U)$ there is $\widetilde{f} \in H(G_1)$ such that $\widetilde{f} = f$ on G_2.

If $N = 1$, then any domain is of holomorphy (see [15]). For the general theory of complex analysis on finite dimensional spaces we refer the reader to [16] or [19].

A domain of holomorphy $U \subset \mathbb{C}^N$ is said to be a Runge domain if for every holomorphic function $f : U \to \mathbb{C}$, every compact set $K \subset U$ and every $\varepsilon > 0$ there exists a polynomial P on $\mathbb{C}^N$ such that $|f(z) - P(z)| < \varepsilon$ for all $z \in K$. For instance, every open ball and every polydisc is a Runge domain, since for those domains the Taylor series of a holomorphic function f converges to f uniformly on the compact subsets of U. And by a *non-Runge domain* we mean a domain of holomorphy $U \subset \mathbb{C}^N$ such that the last property does not hold or, in other words, such that

$$H_{\mathrm{nap}}(U) := H(U) \setminus \overline{\mathcal{P}(\mathbb{C}^N)} \neq \emptyset,$$

where $\overline{\mathcal{P}(\mathbb{C}^N)}$ denotes the closure in $H(U)$ of the set of (restrictions to U) of holomorphic polynomials in N variables.

Our aim in this paper is to study topological and algebraic properties of the family $H_{\mathrm{nap}}(U)$, provided that U is a non-Runge domain. It will be shown that this family contains, except for zero, large vector spaces or algebras. This will be carried out in

Section 2. In order to establish our results in a precise way, we devote the rest of this section to fix some terminology coming from the new theory of lineability (see [1,2,9, 11–13,20]).

Assume that X is a vector space. A subset $A \subset X$ is said to be *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$. If X is, in addition, a topological vector space, then A is called *dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ (hence dense-lineability implies lineability as soon as $\dim(X) = \infty$). The set A is *spaceable* in X if there is a closed infinite dimensional vector subspace $M \subset X$ such that $M \setminus \{0\} \subset A$. When X is also a (linear) algebra, then A is called *algebrable* provided that there exists an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite. Similarly, A is *closely algebrable* if there is an infinitely generated closed algebra $M \subset X$ such that $M \setminus \{0\} \subset A$. Finally, A is *densely algebrable* if there is an infinitely generated dense algebra $M \subset X$ such that $M \setminus \{0\} \subset A$.

2. Spaces of holomorphic functions on non-Runge domains

We start with the topological size of our family $H_{\text{nap}}(U)$. This is easy: it happens to be huge, as expected.

Theorem 2.1. *If U is a non-Runge domain in $\mathbb{C}^N$, then $H_{\text{nap}}(U)$ is a dense open subset of $H(U)$. In particular, it is a residual subset of the Baire space $H(U)$.*

Proof. The set $H_{\text{nap}}(U)$ is the complement in $H(U)$ of the closed subspace $\overline{\mathcal{P}(\mathbb{C}^N)}$ (as the closure of a subspace in a topological vector space). Hence is open and, of course, nonempty. Moreover, it is well known that a proper vector subspace of a topological vector space has always empty interior, so its complement is dense. □

Firstly, our setting will be the complex plane $\mathbb{C}$. Let us recall that a domain $U \subset \mathbb{C}$ is said to be simply connected provided that it lacks holes, that is, its complement $\mathbb{C}_\infty \setminus U$ with respect the one-point compactification $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is connected. The proof of the following auxiliary results can be found, respectively, in [10, page 202], [8, Theorem 2.5] (see also [1, Theorem 7.3.3]) and [17, Theorem 2.2] (see also [1, Theorem 7.4.1]).

Lemma 2.2. *A domain $U \subset \mathbb{C}$ is a Runge domain if and only if U is simply connected.*

Lemma 2.3. *Let X be a metrizable separable topological vector space and Y be a vector subspace of X. If Y has infinite codimension, then $X \setminus Y$ is dense-lineable.*

Lemma 2.4. *Let Y be a closed vector subspace of a Fréchet space X. If Y has infinite codimension, then $X \setminus Y$ is spaceable.*

Regarding the hypotheses of Lemmas 2.3 and 2.4, note that Y has infinite codimension if and only if $X \setminus Y$ is lineable. Then the converse of Lemma 2.4 is always true, while the converse of Lemma 2.3 holds as soon as X is infinite dimensional. It is worth mentioning that in [17] Kitson and Timoney ascribe the content of Lemma 2.4 to Kalton, which in turn extended to Fréchet spaces a general criterium discovered by Wilanski [21] for Banach spaces.

We are now ready to establish our "linear" assertions.

Theorem 2.5. *If U is a non-Runge domain in $\mathbb{C}$, then $H_{\text{nap}}(U)$ is dense-lineable and spaceable.*

Proof. Take $X = H(U)$ and $Y = \overline{\mathcal{P}(\mathbb{C})}$, so that $H_{\text{nap}}(U) = X \setminus Y$. It is known that $H(U)$ is a separable space (see [18, page 370]). Hence, according to Lemmas 2.3 and 2.4, it is enough to prove that $X \setminus Y$ is lineable, that is, $H_{\text{nap}}(U)$ contains, except for zero, an infinite dimensional vector space.

To this end, observe that thanks to Lemma 2.2 there is a closed rectifiable Jordan curve γ in U whose (geometrical) interior is not totally contained in U. Without loss of generality, we will assume that 0 belongs to the interior of γ but $0 \notin U$. Then the functions

$$f_m(z) := \frac{1}{z^m} \quad (m \in \mathbb{N})$$

are holomorphic on U. The sequence $\{f_m : m \in \mathbb{N}\}$ is linearly independent. Indeed, if $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $\lambda_1 f_1 + \cdots + \lambda_m f_m = 0$, then for all $z \in U$ we have

$$\frac{\lambda_1}{z} + \frac{\lambda_2}{z^2} + \cdots + \frac{\lambda_m}{z^m} = 0.$$

Hence $\lambda_1 z^{m-1} + \lambda_2 z^{m-2} + \cdots + \lambda_m = 0$ for all $z \in U$. Since a nonzero polynomial can have only finitely many zeros, we can conclude that $\lambda_1 = 0, \ldots, \lambda_m = 0$, from which the linear independence of $\{f_m : m \in \mathbb{N}\}$ follows.

It remains to show that any nonzero linear combination $f := \lambda_1 f_1 + \cdots + \lambda_p f_p$ belongs to $H_{\text{nap}}(U)$. Note that we can assume $\lambda_p \neq 0$. Suppose, by way of contradiction, that $f \notin H_{\text{nap}}(U)$. Then f can be approximated by polynomials on compacta in U. Hence there exists a polynomial P satisfying

$$|f(z) - P(z)| < \frac{|\lambda_p|}{(\text{dist}(0, \gamma))^p} \quad \text{for all } z \in \gamma$$

because γ is a compact subset of U. Multiplying by $|\lambda_p|^{-1}|z|^p$ leads us to

$$|1 + Q(z)| < 1 \quad \text{for all } z \in \gamma,$$

where

$$Q(z) := -\lambda_p^{-1} z^p P(z) + \sum_{j=1}^{p-1} \frac{\lambda_j}{\lambda_p} z^{p-j}.$$

Note that Q is a polynomial satisfying $Q(0) = 0$. As 0 belongs to the interior of the curve γ, an application of the Maximum Modulus Principle yields $|1 + Q(0)| < 1$, which entails $1 < 1$. This the sought-after contradiction. $\square$

Concerning the existence of algebras consisting of functions which cannot be approximated by polynomials, observe that if $f \in H_{\mathrm{nap}}(U)$, then we do not necessarily infer that $f^2 \in H_{\mathrm{nap}}(U)$. For instance, let $U := \mathbb{C} \setminus [-1,1]$. It is easy to see that there is a holomorphic branch of $\sqrt{z^2-1}$ in U, say Φ. Then $\Phi \in H_{\mathrm{nap}}(U)$. Indeed, assume that this is not true, that is, there is a sequence $(P_n)_{n=1}^{\infty}$ of polynomials which converges to Φ uniformly on compacta in U. Then $P_n \to \Phi$ uniformly on $K := \{z \in \mathbb{C} : |z| = 2\}$, the circle with radius 2 centered at 0. By the triangle inequality, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|P_m(z) - P_n(z)| < \varepsilon$ for all $m, n \geq n_0$ and all $z \in K$. By the Maximum Modulus Principle, the last inequality holds for all $z \in D(0,2)$, the open disc with center 0 and radius 2. Therefore, $(P_n)_{n=1}^{\infty}$ is a Cauchy sequence in $H(D(0,2))$. By completeness, there exists $\Psi \in H(D(0,2))$ such that $P_n \to \Psi$ uniformly on compacta in $D(0,2)$. The uniqueness of the limit together with the fact that compact convergence implies pointwise convergence yields $\Phi(z) = \Psi(z)$ for all $z \in D(0,2) \setminus [-1,1]$. Hence the branch Φ would extend holomorphically to a neighborhood of $[-1,1]$, which is false. Thus $\Phi \in H_{\mathrm{nap}}(U)$ but, obviously, $\Phi^2 \notin H_{\mathrm{nap}}(U)$ because it is a polynomial. Despite this, we can obtain a high level of algebrability.

Theorem 2.6. *If U is a non-Runge domain in $\mathbb{C}$, then $H_{\mathrm{nap}}(U)$ is closely algebrable.*

Proof. As in the proof of Theorem 2.5, we may assume that $0 \notin U$ and that there is a closed rectifiable Jordan curve γ in U whose geometrical interior contains 0. The vector space $M := \mathrm{span}\{f_m : m \in \mathbb{N}\}$ (with $f_m(z) = \frac{1}{z^m}$) considered there is in fact an algebra, because it is nothing but the set of all functions of the form $P(1/z)$, where P runs over all polynomials without constant term. Let $\mathcal{A}$ be its closure in $H(U)$, that is, $\mathcal{A} = \overline{\mathrm{span}}\{f_m : m \in \mathbb{N}\}$. It is well known that the closure of an algebra inside a topological algebra is also an algebra. Consequently, we have obtained a closed algebra $\mathcal{A}$.

Assume that $\mathcal{A}$ is finitely generated and that $\{g_1, \ldots, g_s\}$ is a set of generators. This means that $\mathcal{A}$, considered as a vector space, is linearly generated by the finite products $g_1^{m_1} \cdots g_s^{m_s}$, where $(m_1, \ldots, m_s) \in \mathbb{N}_0^s \setminus \{(0, \ldots, 0)\}$. This entails that $\dim(\mathcal{A}) \leq \aleph_0$, which is impossible, since a standard application of the Baire category theorem yields that the dimension of an infinite dimensional complete metrizable topological vector space cannot be countable. Observe that $\mathcal{A}$ fulfills these conditions because it contains the independent vectors f_m $(m = 1, 2, \ldots)$ and it becomes a Fréchet space as a closed subspace of $H(U)$. Therefore, $\mathcal{A}$ is an infinitely generated closed algebra in $H(U)$.

Our unique task is to prove that $\mathcal{A} \setminus \{0\} \subset H_{\mathrm{nap}}(U)$ or, equivalently,

$$\overline{\operatorname{span}}\,\{f_m : m \in \mathbb{N}\} \cap \overline{\mathcal{P}\,(\mathbb{C})} = \{0\}\,. \tag{2.1}$$

A bit of preparation is needed. Let $a \in \gamma$ and $R > 0$ be such that

$$|a| = R = \max\{|z| : z \in \gamma\}\,.$$

Since $a \in \gamma \subset U$, there is $r > 0$ such that $D(a, r) \subset U$. Let

$$0 < t < \frac{r}{|a|} \qquad \text{and} \qquad b := (1 + t)\,a.$$

Then $|b - a| = |ta| < r$, so $b \in U$.

Let now $g \in \overline{\operatorname{span}}\,\{f_m : m \in \mathbb{N}\} \cap \overline{\mathcal{P}\,(\mathbb{C})}$. Then there is a sequence of polynomials $(P_n)_{n=1}^{\infty}$ which converges to g uniformly on γ. Given $k \in \mathbb{N}_0$, we have also that $\lim_{n\to\infty} z^k P_n\,(z) = z^k g\,(z)$ uniformly on γ. Therefore,

$$\lim_{n\to\infty} \oint_\gamma z^k P_n\,(z)\,dz = \oint_\gamma z^k g\,(z)\,dz.$$

By the Cauchy Theorem, we know that

$$\oint_\gamma z^k P_n\,(z)\,dz = 0$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, so

$$\oint_\gamma z^k g\,(z)\,dz = 0 \tag{2.2}$$

for every $k \in \mathbb{N}_0$.

Since $g \in \overline{\operatorname{span}}\,\{f_m : m \in \mathbb{N}\}$, for every $\varepsilon > 0$ there is $f \in \operatorname{span}\{f_m : m \in \mathbb{N}\}$ such that

$$\sup_{z\in\gamma\cup\{b\}} |f\,(z) - g\,(z)| < \frac{\varepsilon}{L}, \tag{2.3}$$

where L denotes the length of γ. The function f can be written as

$$f\,(z) = \frac{\lambda_1}{z} + \frac{\lambda_2}{z^2} + \cdots + \frac{\lambda_m}{z^m}$$

for some $m \in \mathbb{N}$ and some $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

Let $k \in \{0, \ldots, m-1\}$. By the residue theorem,

$$\oint_\gamma \frac{z^k}{R^k} f(z)\, dz = \frac{1}{R^k} \oint_\gamma \left(\lambda_1 z^{k-1} + \cdots + \frac{\lambda_{k+1}}{z} + \cdots + \frac{\lambda_m}{z^{m-k}}\right) dz = \frac{2\pi i \lambda_{k+1}}{R^k}.$$

By (2.2), we have

$$\begin{aligned} \frac{2\pi |\lambda_{k+1}|}{R^k} &= \left| \oint_\gamma \frac{z^k}{R^k} f(z)\, dz \right| \le \left| \oint_\gamma \frac{z^k}{R^k} f(z) - \frac{z^k}{R^k} g(z)\, dz \right| + \left| \frac{1}{R^k} \oint_\gamma z^k g(z)\, dz \right| \\ &\le L \cdot \sup_{z\in\gamma} \left| \frac{z^k}{R^k} f(z) - \frac{z^k}{R^k} g(z) \right|. \end{aligned}$$

Since $R = \max\{|z| : z \in \gamma\}$, it follows that

$$\frac{2\pi |\lambda_{k+1}|}{R^k} \le L \cdot \sup_{z\in\gamma} |f(z) - g(z)| < \varepsilon$$

for all $k \in \{0, \ldots, m-1\}$. That is,

$$|\lambda_1| < \frac{\varepsilon}{2\pi}, \qquad |\lambda_2| < \frac{\varepsilon R}{2\pi}, \qquad \ldots \qquad |\lambda_m| < \frac{\varepsilon R^{m-1}}{2\pi}. \tag{2.4}$$

We now estimate the value of $g(b)$. By (2.3) and (2.4),

$$\begin{aligned} |g(b)| &\le |f(b)| + \frac{\varepsilon}{L} \le \frac{|\lambda_1|}{|b|} + \frac{|\lambda_2|}{|b|^2} + \cdots + \frac{|\lambda_m|}{|b|^m} + \frac{\varepsilon}{L} \\ &< \frac{\varepsilon}{2\pi |b|} + \frac{\varepsilon R}{2\pi |b|^2} + \cdots + \frac{\varepsilon R^{m-1}}{2\pi |b|^m} + \frac{\varepsilon}{L} \\ &= \frac{\varepsilon}{2\pi |b|} \left(1 + \frac{R}{|b|} + \cdots + \frac{R^{m-1}}{|b|^{m-1}}\right) + \frac{\varepsilon}{L} \\ &< \frac{\varepsilon}{2\pi |b|} \cdot \sum_{k=0}^{\infty} \left(\frac{R}{|b|}\right)^k + \frac{\varepsilon}{L} = \frac{\varepsilon}{2\pi (|b| - R)} + \frac{\varepsilon}{L}. \end{aligned}$$

Note that $|b| = (1+t)|a| > |a| = R$ and then $0 < R/|b| < 1$. We have obtained that

$$|g(b)| < \frac{\varepsilon}{2\pi (|b| - R)} + \frac{\varepsilon}{L}.$$

That inequality holds for every $\varepsilon > 0$, so $g(b) = 0$. Thus, we have proved that $g((1+t)a) = 0$ whenever $0 < t < r/|a|$. By the Identity Theorem, we deduce that $g = 0$ on U. This proves (2.1), as required. □

Remark 2.7. The last theorem contains, obviously, the spaceability part of Theorem 2.5. Nevertheless, we have decided to keep the very short proof of this part because it might be of independent interest.

We have not been able to prove the *dense* algebrability of $H_{\text{nap}}(U)$, which we pose here as an open problem. Nevertheless, if one does not demand any topological property to the required subalgebra, then it is possible to get a subalgebra with richer properties (see Theorem 2.9 below). If X is a vector space contained in some commutative algebra, then a subset $B \subset X$ is said to generate a *free algebra* A whenever A is the algebra generated by B and, if $b_1, \ldots, b_p$ are distinct elements of B and P is a polynomial of p variables without constant term such that $P(b_1, \ldots, b_p) = 0$, then $P = 0$. If $\operatorname{card}(B) = \alpha$, then A is said to be an α-generated free algebra. The study of the property of existence of large free algebras within nonlinear sets was initiated by Bartoszewicz and Glab in [5]. The property was named strong algebrability, and it is in fact strictly stronger than the mere algebrability.

In order to establish our statement on strong algebrability, we are going to construct superpositions of a fixed function of $H_{\text{nap}}(U)$ with the representatives of some well chosen algebra of functions; this approach was already used in [14]. The proof of the following lemma can be found in [7, Proposition 2.3], being a complex version of an algebrability criterium given in [3, Proposition 7] (see also [4, Theorem 1.5] and [6]). By $\mathcal{E}$ we denote the algebra of entire functions of the form

$$\varphi(z) = \sum_{j=1}^{m} a_j e^{b_j z}, \tag{2.5}$$

where $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{C} \setminus \{0\}$ and $b_1, \ldots, b_m$ are distinct numbers in $\mathbb{C} \setminus \{0\}$.

Lemma 2.8. *Let Ω be a nonempty set and $\mathcal{F}$ be a family of functions from Ω into $\mathbb{C}$. If there exists a function $f \in \mathcal{F}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{E}$, then $\mathcal{F}$ contains, except for zero, a $\mathfrak{c}$-generated free algebra.*

Theorem 2.9. *If U is a non-Runge domain in $\mathbb{C}$, then $H_{\text{nap}}(U)$ contains, except for zero, a $\mathfrak{c}$-generated free algebra.*

Proof. The domain U is not simply connected, so there is a closed rectifiable Jordan curve γ in U whose interior is not totally contained in U. We will again assume that 0 belongs to the interior of γ but $0 \notin U$. Then the function $f(z) = 1/z$ is holomorphic in U and, obviously, the set $f(U)$ is uncountable. We will prove that $\varphi \circ f \in H_{\text{nap}}(U)$ for every $\varphi \in \mathcal{E}$.

Fix such a function φ, which has an expression as in (2.5), and suppose, by way of contradiction, that $\varphi \circ f \in \overline{\mathcal{P}(\mathbb{C})}$. Then there is a sequence of polynomials $(P_n)_{n=1}^{\infty}$ which converges to $\varphi \circ f$ uniformly on γ. Given $k \in \mathbb{N}_0$, we also have that $\lim_{n\to\infty} z^k P_n(z) = z^k(\varphi \circ f)(z)$ uniformly on γ. Therefore,

$$\lim_{n\to\infty} \oint_\gamma z^k P_n(z)\, dz = \oint_\gamma z^k (\varphi \circ f)(z)\, dz = \oint_\gamma z^k \sum_{j=1}^m a_j e^{b_j/z} dz = \sum_{j=1}^m a_j \oint_\gamma z^k e^{b_j/z} dz.$$

By the Cauchy Theorem, we know that

$$\oint_\gamma z^k P_n(z)\, dz = 0$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, so

$$\sum_{j=1}^m a_j \oint_\gamma z^k e^{b_j/z} dz = 0 \tag{2.6}$$

for every $k \in \mathbb{N}_0$. If $z \neq 0$, then

$$z^k e^{b_j/z} = z^k \left(1 + \frac{b_j}{z} + \frac{b_j^2}{2!z^2} + \frac{b_j^3}{3!z^3} + \cdots\right) = z^k + b_j z^{k-1} + \frac{b_j^2}{2!} z^{k-2} + \frac{b_j^3}{3!} z^{k-3} + \cdots$$

It follows from the residue theorem that

$$\oint_\gamma z^k e^{b_j/z} dz = 2\pi i \frac{b_j^{k+1}}{(k+1)!} \tag{2.7}$$

for every $j \in \{1, \ldots, m\}$ and every $k \in \mathbb{N}_0$. By (2.6) and (2.7), if $k \in \mathbb{N}_0$, then

$$\frac{2\pi i}{(k+1)!} \sum_{j=1}^m a_j \cdot b_j^{k+1} = 0.$$

If we just consider $k \in \{0, 1, \ldots, m-1\}$, then

$$\begin{cases} a_1 b_1 + a_2 b_2 + \cdots + a_m b_m = 0 \\ a_1 b_1^2 + a_2 b_2^2 + \cdots + a_m b_m^2 = 0 \\ \vdots \\ a_1 b_1^m + a_2 b_2^m + \cdots + a_m b_m^m = 0. \end{cases}$$

These equations are equivalent to the following one:

$$\begin{pmatrix} b_1 & b_2 & \cdots & b_m \\ b_1^2 & b_2^2 & \cdots & b_m^2 \\ \vdots & & & \\ b_1^m & b_2^m & \cdots & b_m^m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.8}$$

We note that

$$\det \begin{pmatrix} b_1 & b_2 & \cdots & b_m \\ b_1^2 & b_2^2 & \cdots & b_m^2 \\ \vdots & & & \\ b_1^m & b_2^m & \cdots & b_m^m \end{pmatrix} = b_1 b_2 \cdots b_m \cdot \det \begin{pmatrix} 1 & b_1 & \cdots & b_1^{m-1} \\ 1 & b_2 & \cdots & b_2^{m-1} \\ \vdots & & & \\ 1 & b_m & \cdots & b_m^{m-1} \end{pmatrix}.$$

The last one is a Vandermonde determinant, so it is different from 0 because the coefficients $b_1, \ldots, b_m$ are distinct and nonzero. Then the equation (2.8) implies $a_1 = 0, \ldots, a_m = 0$. This contradicts the definition of the function φ. Hence, $\varphi \circ f \in H_{\mathrm{nap}}(U)$ for every $\varphi \in \mathcal{E}$. By Lemma 2.8, we have that $H_{\mathrm{nap}}(U)$ contains a $\mathfrak{c}$-generated algebra. $\square$

Remark 2.10. Observe that the conclusion of the last theorem is optimal with respect to the cardinality of the generating set of the free algebra.

We would like to extend the previous results to the case of *several variables.* If U is a non-Runge domain in $\mathbb{C}^N$ and $f \in H_{\mathrm{nap}}(U)$ then the set $\{f^n : n \in \mathbb{N}\}$ is linearly independent. Indeed, if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\lambda_1 f + \lambda_2 f^2 + \cdots + \lambda_n f^n = 0$, then for all $z \in U$ we have

$$\lambda_1 f(z) + \lambda_2 (f(z))^2 + \cdots + \lambda_n (f(z))^n = 0.$$

Since f is not constant, it takes infinitely many values, so the polynomial $\lambda_1 z + \lambda_2 z^2 + \cdots + \lambda_n z^n = 0$ has infinitely many roots. This implies that $\lambda_1 = 0, \ldots, \lambda_n = 0$. Hence $\operatorname{span}\{f^n : n \in \mathbb{N}\}$ is infinite dimensional. Unfortunately, even in the case $N = 1$ one cannot guarantee for a power of f to belong to $H_{\mathrm{nap}}(U)$, as the example given before Theorem 2.6 shows. Then our research on lineability for several variables should explore new ways.

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