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Steady-state and Hopf bifurcations in the Langford ODE and PDE systems[☆]

Gaihui Guo[a], Xiaoni Wang[a], Xiaolin Lin[a], Meihua Wei[b,*]

[a] *School of Arts and Science, Shaanxi University of Science and Technology, Xi'an 710021, China*
[b] *Department of Mathematics and Statistics, Yulin University, Yulin 719000, China*

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ABSTRACT

This paper is concerned with the Langford ODE and PDE systems. For the Langford ODE system, the existence of steady-state solutions is firstly obtained by Lyapunov–Schmidt method, and the stability and bifurcation direction of periodic solutions are established. Then for the Langford PDE system, the steady-state bifurcations from simple and double eigenvalues are intensively studied. The techniques of space decomposition and implicit function theorem are adopted to deal with the case of double eigenvalue. Finally, by the center manifold theory and the normal form method, the direction of Hopf bifurcation and the stability of spatially homogeneous and inhomogeneous periodic solutions for the PDE system are investigated.

1. Introduction

From the theoretical and practical points of view, one of the most concerned and important problems is the analysis on the bifurcation of the dynamical system by changing its parameters of the system. It is known that such bifurcation can lead to the appearance of steady-state and periodic solutions, which induces the complex spatiotemporal pattern formation [1–16].

The Langford system was originally proposed by Hopf [17] as a possible third order ODE system for describing fluid dynamic turbulence, which has the form

$$\begin{cases} u_t = (\mu-1)u - v + uw, & t>0, \\ v_t = u + (\mu-1)v + vw, & t>0, \\ w_t = \mu w - (u^2+v^2+w^2), & t>0. \end{cases} \tag{1.1}$$

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* Corresponding author.
E-mail addresses: guogaihui@sust.edu.cn (G. Guo), wangxiaoni@163.com (X. Wang), linxl@sust.edu.cn (X. Lin), wei_meihua@163.com (M. Wei).

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A good qualitative analysis of the regular behavior in the Langford system (1.1) was given in [18] and the chaotic solution for specific values of bifurcation parameter was studied in [19]. The localization problem of compact invariant sets of (1.1) was examined by using the iterative localization method of periodic orbits in [20]. For a general form of the Langford system (1.1), the control of Hopf bifurcation by a linear or nonlinear controller was investigated in [21–23].

Motivated by the papers [4–6,10–12,15] mentioned above, we will focus on the steady-state and periodic solutions of the Langford PDE system, that is, the system (1.1) with space distribution and Neumann boundary conditions such as

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 u_{xx} + (\mu - 1)u - v + uw, & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 v_{xx} + u + (\mu - 1)v + vw, & x \in (0,\pi),\ t > 0, \\ \dfrac{\partial w}{\partial t} = d_3 w_{xx} + \mu w - (u^2 + v^2 + w^2), & x \in (0,\pi),\ t > 0, \\ u_x = v_x = w_x = 0, & x = 0,\pi,\ t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad w(x,0) = w_0(x), & x \in (0,\pi), \end{cases} \tag{1.2}$$

where the constants d_1, d_2, d_3 are positive. On the other hand, we will also discuss the solutions for the Langford ODE system (1.1). For the Hopf bifurcations of both ODE and PDE systems, the existence, stability and bifurcation direction of spatially homogeneous and inhomogeneous periodic solutions are discussed by Hopf bifurcation theory [18]. For the steady-state bifurcation of ODE system we obtained the stationary solutions by the Lyapunov–Schmidt method, and for the steady-state bifurcation of PDE system we mainly focus on the bifurcation from double eigenvalue where multiple bifurcation could occur. In the case of double eigenvalue, the Crandall–Rabinowitz bifurcation theorem [24] cannot be directly applied, and then our proof is based on the techniques of space decomposition and implicit function theorem which have been used in [25]. But, up to now, in the case of more than double eigenvalue there is no ready-made theory to our knowledge to obtain the existence of steady-state solutions for reaction–diffusion equations. In fact, there are lots of steady-state bifurcation results for many different reaction–diffusion models, but in which the assumption of simple eigenvalue was always imposed (see [4,10,11,15]). The main contributions of this article are as follows: (a) the Hopf bifurcation analysis of the third order Langford PDE system is different from the previous works for other models [5,6,10–12,15]; (b) the steady-state bifurcation of ODE system and the double bifurcation of PDE system are detailed discussed by the Lyapunov–Schmidt technique and the methods of space decomposition and implicit function theorem, respectively. The studies indicate that the simpler the system is, the richer bifurcation behaviors it will have.

The rest of the paper is arranged as follows. In Section 2, the existence of steady-state solutions, the direction of Hopf bifurcation and the stability of periodic solutions for the Langford ODE system are established. Section 3 is devoted to considering the steady-state bifurcations from simple and double eigenvalues and the existence and stability of Hopf bifurcating periodic solutions for the Langford PDE system.

2. Steady-state and Hopf bifurcations in the ODE system

It is obvious that $E_0 = (0,0,0)$ and $E_1 = (0,0,\mu)$ are the equilibrium points of (1.1). In this section, we first give the stability of the equilibrium solutions $E_0 = (0,0,0)$ and $E_1 = (0,0,\mu)$, and then discuss the steady-state and periodic solutions of (1.1) by Lyapunov–Schmidt method and Hopf bifurcation theory for the bifurcation parameter $\mu (\geq 0)$.

The Jacobian matrix of system (1.1) at the equilibrium point E_0 is

$$J_0(\mu) = \begin{pmatrix} \mu - 1 & -1 & 0 \\ 1 & \mu - 1 & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

which has a real eigenvalue $\lambda_1^0 = \mu$ and a pair of complex conjugate eigenvalues $\lambda_{2,3}^0 = \mu - 1 \pm i$. The Jacobian matrix of system (1.1) at E_1 is

$$J_1(\mu) = \begin{pmatrix} 2\mu - 1 & -1 & 0 \\ 1 & 2\mu - 1 & 0 \\ 0 & 0 & -\mu \end{pmatrix}$$

with eigenvalues $\lambda_1^1 = -\mu$, $\lambda_{2,3}^1 = 2\mu - 1 \pm i$. Thus E_0 is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$, and E_1 is asymptotically stable for $0 < \mu < \frac{1}{2}$ and unstable for $\mu < 0$ or $\mu > \frac{1}{2}$.

Therefore, when μ is increased past 0 the equilibrium point E_0 loses stability while the equilibrium point E_1 gains stability. However, when μ is increased past $\frac{1}{2}$ the equilibrium point E_1 also loses stability.

2.1. Steady-state bifurcation in the ODE system

Define the map $F : R^+ \times X \to Y$ by

$$F(\mu, u, v, w) = \begin{pmatrix} (\mu - 1)u - v + uw \\ u + (\mu - 1)v + vw \\ \mu w - (u^2 + v^2 + w^2) \end{pmatrix}$$

and the inner product in $X = Y = R^3$ by

$$\langle \bar{U}, \bar{V} \rangle = \sum_{i=1}^{3} u_i v_i, \qquad \bar{U} = (u_1, u_2, u_3)^T, \qquad \bar{V} = (v_1, v_2, v_3)^T \in R^3.$$

Then the steady-state solutions of (1.1) are exactly the zeros of the equation

$$F(\mu, u, v, w) = 0. \tag{2.1}$$

Note that $F(\mu, 0, 0, 0) = 0$ and $F(\mu, 0, 0, \mu) = 0$. Next, we look for the solution bifurcating from $E_0 = (0, 0, 0)$ by Lyapunov–Schmidt method.

Theorem 2.1. *For the system (2.1), the steady-state bifurcation from $E_0 = (0, 0, 0)$ at $\mu = 0$ is transcritical, and the bifurcating solution is asymptotically stable for $0 < \mu < \frac{1}{2}$ and unstable for $\mu < 0$ or $\mu > \frac{1}{2}$.*

Proof. The derivative of F with respect to (u, v, w) at $(0, 0, 0, 0)$ can be expressed by

$$L_0 = F_{(u,v,w)}(0, 0, 0, 0) = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whose eigenvalues are 0 and $-1 \pm i$. It is readily found that the kernel space and the range space are, respectively,

$$\ker L_0 = \{(u, v, w) \mid u = v = 0\},$$
$$R(L_0) = \{(u, v, w) \mid w = 0\}.$$

Clearly, $R(L_0) = (\ker L_0)^{\perp}$, then X and Y have the same decomposition

$$X = Y = \ker L_0 \oplus R(L_0). \tag{2.2}$$

Let $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$, then it is clear that $\ker L_0 = \text{span}\{e_3\}$, $R(L_0) = \text{span}\{e_1, e_2\}$. For the latter use, we define the operator P on Y by

$$P\bar{U} = \langle e_3, \bar{U}\rangle e_3, \quad \bar{U} \in Y,$$

so that P is the projection from Y to $\ker L_0$. Then, we denote the projection from Y to $R(L_0)$ as

$$(I - P)\bar{U} = \langle e_1, \bar{U}\rangle e_1 + \langle e_2, \bar{U}\rangle e_2.$$

Thus, split (2.1) into an equivalent pair of equations,

$$PF(\mu, u, v, w) = 0, \tag{2.3}$$

$$(I - P)F(\mu, u, v, w) = 0. \tag{2.4}$$

For any $U = (u, v, w)^T \in X$, we use (2.2) to write

$$U = U_k + U_R, \tag{2.5}$$

where $U_k = we_3 \in \ker L_0$, $U_R = ue_1 + ve_2 \in R(L_0)$. Then Eq. (2.4) can be written as

$$\langle e_1, F(\mu, U_k + U_R)\rangle e_1 + \langle e_2, F(\mu, U_k + U_R)\rangle e_2 = 0,$$

that is,

$$\begin{pmatrix} (\mu - 1)u - v + uw \\ u + (\mu - 1)v + vw \\ 0 \end{pmatrix} = 0.$$

Simple calculations lead to $u = 0$ and $v = 0$. Substituting $u = 0, v = 0$ into (2.3) yields

$$\mu w - w^2 = 0,$$

which has only two solutions $w = 0$ and $w = \mu$. It readily follows from (2.5) that Eq. (2.1) has two solutions $(0,0,0)$ and $(0,0,\mu)$, and the bifurcation from $(0,0,0)$ at $\mu = 0$ is transcritical. From the stability analysis above, it is easy to see that the bifurcating solution $(0,0,\mu)$ is asymptotically stable for $0 < \mu < \frac{1}{2}$ and unstable for $\mu < 0$ or $\mu > \frac{1}{2}$. The proof is completed. □

Remark 2.2. The steady-state bifurcation from $E_1 = (0,0,\mu)$ at $\mu = 0$ leads to get the solution $(0,0,0)$, the analysis of which is similar to Theorem 2.1 and omitted here.

2.2. Hopf bifurcation in the ODE system

In this subsection, we discuss the Hopf bifurcation from the constant solutions $E_0 = (0,0,0)$ and $E_1 = (0,0,\mu)$.

Theorem 2.3. *The system (1.1) undergoes a Hopf bifurcation at $(0,0,0)$ when $\mu = 1$. The direction of the Hopf bifurcation is subcritical, and the bifurcating periodic solution is unstable and approximated by*

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + (1 - \mu)^{\frac{1}{2}} \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + o(\mu - 1). \tag{2.6}$$

Proof. According to the stability analysis of $(0,0,0)$, we know that the Jacobian matrix $J_0(\mu)$ has the eigenvalues $\lambda_0^1 = \mu,\ \lambda_{2,3}^1 = \mu - 1 \pm i \triangleq \alpha(\mu) \pm i\omega(\mu)$, that is, $\alpha(\mu) = \mu - 1,\ \omega(\mu) = 1$. Then when $\mu = 1$, we have $\lambda_0^1 = 1,\ \alpha_0 = \alpha(1) = 0,\ \omega_0 = \omega(1) = 1,\ \alpha'(1) = 1 > 0$ and $\omega'(1) = 0$. From [18], the system (1.1) undergoes a Hopf bifurcation at $\mu = 1$.

From [18] we further consider the direction of Hopf bifurcation and the stability of bifurcating periodic solution.

The system (1.1) can be written as

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = J_0(\mu) \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix}, \quad t > 0,$$

where

$$F^1 = uw, \qquad F^2 = vw, \qquad F^3 = -(u^2 + v^2 + w^2). \tag{2.7}$$

Then, we can obtain the following derivatives at $(\mu, u, v, w) = (1, 0, 0, 0)$ as follows:

$$\begin{aligned}
g_{11} &= \frac{1}{4}(F_{uu}^1 + F_{vv}^1 + i(F_{uu}^2 + F_{vv}^2)) = 0, \\
g_{02} &= \frac{1}{4}(F_{uu}^1 - F_{vv}^1 - 2F_{uv}^2 + i(F_{uu}^2 - F_{vv}^2 + 2F_{uv}^1)) = 0, \\
g_{20} &= \frac{1}{4}(F_{uu}^1 - F_{vv}^1 + 2F_{uv}^2 + i(F_{uu}^2 - F_{vv}^2 - 2F_{uv}^1)) = 0, \\
G_{21} &= \frac{1}{8}(F_{uuu}^1 + F_{uvv}^1 + F_{uuv}^2 + F_{vvv}^2 + i(F_{uuu}^2 + F_{uvv}^2 - F_{uuv}^1 - F_{vvv}^1)) = 0, \\
h_{11} &= \frac{1}{4}(F_{uu}^3 + F_{vv}^3) = \frac{1}{4}(-2 - 2) = -1, \\
h_{20} &= \frac{1}{4}(F_{uu}^3 - F_{vv}^3 - 2iF_{uv}^3) = \frac{1}{4}(-2 + 2 - 2i \cdot 0) = 0.
\end{aligned}$$

From the Jacobian matrix

$$J_0(1) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & D \end{pmatrix},$$

we obtain $D = 1$. Then solve the linear system

$$D\chi_{11} = -h_{11}, \qquad (D - 2i\omega_0)\chi_{20} = -h_{20}$$

for χ_{11} and χ_{20}. Further calculations lead to $\chi_{11} = 1,\ \chi_{20} = 0$. Thus, from

$$\begin{aligned}
G_{110} &= \frac{1}{2}(F_{uw}^1 + F_{vw}^2 + i(F_{uw}^2 - F_{vw}^1)) = \frac{1}{2}(1 + 1 + i(0 - 0)) = 1, \\
G_{101} &= \frac{1}{2}(F_{uw}^1 - F_{vw}^2 + i(F_{uw}^2 + F_{vw}^1)) = \frac{1}{2}(1 - 1 + i(0 + 0)) = 0,
\end{aligned}$$

we get

$$g_{21} = G_{21} + 2G_{110}\chi_{11} + G_{101}\chi_{20} = 2.$$

By calculations above, we have

$$c_1(1) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2} = 1 > 0.$$

Hence, $\mu_2 = -\mathrm{Re}(c_1(1))/\alpha'(1) = -1 < 0$, and then the direction of the Hopf bifurcation is subcritical. The bifurcating periodic solution is clearly unstable since $(0,0,0)$ is unstable for $\mu = 1 > 0$. For $\tau_2 = -(\mathrm{Im}(c_1(1)) + \mu_2\omega'(1))/\omega_0 = 0$, the periodic solution is approximated by (2.6) from [18]. □

Theorem 2.4. *The system (1.1) undergoes a Hopf bifurcation at $(0,0,\mu)$ when $\mu = \frac{1}{2}$. The direction of the Hopf bifurcation is supercritical, and the bifurcating periodic solution is orbitally asymptotically stable and approximated by*

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \left(\mu - \frac{1}{2}\right)^{\frac{1}{2}} \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + o\left(\mu - \frac{1}{2}\right). \tag{2.8}$$

Proof. For $(0,0,\mu)$, we have that $\alpha(\mu) = 2\mu - 1,\ \omega(\mu) = 1,\ \omega_0 = \omega(\frac{1}{2}) = 1,\ \alpha'(\frac{1}{2}) = 2 > 0,\ \omega'(\frac{1}{2}) = 0$, and $J_1(\mu)$ has eigenvalues $\lambda_1^1 = -\frac{1}{2},\ \lambda_{2,3}^1 = \pm i$ for $\mu = \frac{1}{2}$. From [18], the system (1.1) undergoes a Hopf bifurcation at $\mu = \frac{1}{2}$.

Similar to the analysis in Theorem 2.3 and [18], we give the following analysis for the bifurcation from $(0,0,\mu)$.

Set the translation $\tilde{u} = u, \tilde{v} = v, \tilde{w} = w - \mu$, and still denote $\tilde{u}, \tilde{v}, \tilde{w}$ as u, v, w. Then the system (1.1) can be written as

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = J_1(\mu) \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix}, \quad t > 0,$$

where F^1, F^2, F^3 are defined in (2.7). Thus, we know that $g_{11} = 0,\ g_{02} = 0,\ g_{20} = 0,\ G_{21} = 0,\ h_{11} = -1,\ h_{20} = 0$ at $(\mu, u, v, w) = (\frac{1}{2}, 0, 0, \frac{1}{2})$. Due to $D = -\frac{1}{2}$, we solve the linear system

$$D\chi_{11} = -h_{11}, \qquad (D - 2i\omega_0)\chi_{20} = -h_{20}$$

to get $\chi_{11} = -2,\ \chi_{20} = 0$. And then $G_{110} = 1,\ G_{101} = 0,\ g_{21} = -4$ and $c_1(\frac{1}{2}) = -2$. For $\mu = \frac{1}{2}$, we can obtain $\mu_2 = 1 > 0$ and $\tau_2 = 0$. Hence, the direction of the Hopf bifurcation is supercritical, and the bifurcating periodic solution is orbitally asymptotically stable according to $\mathrm{Re}(c_1(\frac{1}{2})) = -2 < 0$ and $\lambda_1^1 = -\frac{1}{2}$, which is approximated by (2.8). □

Remark 2.5. Based on the discussions in this section and [18], the steady-state and Hopf bifurcations of Langford ODE system (1.1) are shown in Fig. 1, where the bifurcating curve from $(\frac{1}{2}, 0, 0, \frac{1}{2})$ joins up with the one from $(1, 0, 0, 0)$.

Remark 2.6. For the Hopf bifurcation from $(\frac{1}{2}, 0, 0, \frac{1}{2})$, the calculated result of g_{21} in page 108 of [18] was $g_{21} = 4$, which should be $g_{21} = -4$ obtained in Theorem 2.4.

3. Steady-state and Hopf bifurcations in the PDE system

It is obvious that $S_0 = (0,0,0)$ and $S_1 = (0,0,\mu)$ are the constant solutions of (1.2). In this section, we first give the stability of solutions $S_0 = (0,0,0)$ and $S_1 = (0,0,\mu)$, and then establish the steady-state and periodic solutions of (1.2).

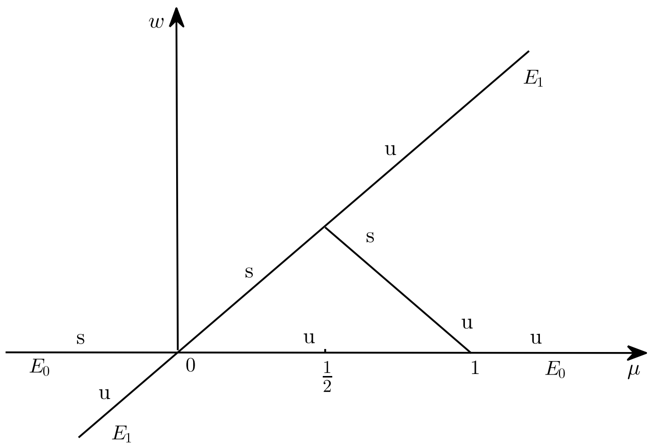

Fig. 1. The bifurcation diagram for the Langford ODE system (1.1) (s = stable, u = unstable).

It is well known that the operator $u \to -u_{xx}$ with no-flux boundary conditions has eigenvalues and eigenfunctions as follows:

$$\lambda_0 = 0, \qquad \phi_0(x) = \sqrt{\frac{1}{\pi}}, \qquad \lambda_k = k^2, \qquad \phi_k(x) = \sqrt{\frac{2}{\pi}} \cos kx$$

for $k = 1, 2, 3, \ldots$.

The linearized system of (1.2) at $(0, 0, \mu)$ has the form

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = L_1(\mu) \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$L_1(\mu) = \begin{pmatrix} d_1 \Delta + 2\mu - 1 & -1 & 0 \\ 1 & d_2 \Delta + 2\mu - 1 & 0 \\ 0 & 0 & d_3 \Delta - \mu \end{pmatrix}, \quad \Delta = \frac{\partial^2}{\partial x^2}. \tag{3.1}$$

Consider the characteristic problem $L_1(\mu)(\phi, \psi, \varphi)^T = \sigma(\phi, \psi, \varphi)^T$ and let $(\phi, \psi, \varphi)^T = \sum_{k=0}^{\infty} (a_k, b_k, c_k)^T \cos kx$. Then we get

$$\sum_{k=0}^{\infty} (J_k - \sigma I)(a_k, b_k, c_k)^T \cos kx = 0,$$

where

$$J_k = \begin{pmatrix} 2\mu - 1 - d_1 k^2 & -1 & 0 \\ 1 & 2\mu - 1 - d_2 k^2 & 0 \\ 0 & 0 & -\mu - d_3 k^2 \end{pmatrix}.$$

It is clear to see that all the eigenvalues of $L_1(\mu)$ are given by the eigenvalues of J_k for $k = 0, 1, 2, \ldots$. Note that the characteristic equation of J_k is

$$(\sigma^2 - T_k(\mu)\sigma + D_k(\mu))(M_k(\mu) - \sigma) = 0, \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

where

$$T_k(\mu) = 4\mu - 2 - (d_1 + d_2)k^2, \qquad D_k(\mu) = (2\mu - 1 - d_1 k^2)(2\mu - 1 - d_2 k^2) + 1, \qquad M_k(\mu) = -\mu - d_3 k^2.$$

For $0 < \mu < \frac{1}{2}$, we have $T_k(\mu) < 0,\ D_k(\mu) > 0,\ M_k(\mu) < 0,\ k = 0, 1, 2, \ldots$. But, we get $M_0(\mu) > 0$ for $\mu < 0$, and $T_0(\mu) > 0,\ D_0(\mu) > 0$ for $\mu > \frac{1}{2}$.

Similar to the arguments above, we analyze the stability of the solution $(0, 0, 0)$. The linearized system of (1.2) at $(0, 0, 0)$ has the form

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = L_0(\mu) \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$L_0(\mu) = \begin{pmatrix} d_1\Delta + \mu - 1 & -1 & 0 \\ 1 & d_2\Delta + \mu - 1 & 0 \\ 0 & 0 & d_3\Delta + \mu \end{pmatrix}, \quad \Delta = \frac{\partial^2}{\partial x^2}. \tag{3.3}$$

We also get the characteristic equation

$$(\sigma^2 - P_k(\mu)\sigma + Q_k(\mu))(N_k(\mu) - \sigma) = 0, \quad k = 0, 1, 2, \ldots, \tag{3.4}$$

where

$$P_k(\mu) = 2\mu - 2 - (d_1 + d_2)k^2, \qquad Q_k(\mu) = (\mu - 1 - d_1k^2)(\mu - 1 - d_2k^2) + 1, \qquad N_k(\mu) = \mu - d_3k^2.$$

By similar calculations, we can obtain $P_k(\mu) < 0,\ Q_k(\mu) > 0,\ N_k(\mu) < 0,\ k = 0, 1, 2, \ldots$ for $\mu < 0$ and $N_0(\mu) > 0$ for $\mu > 0$. Then we have the following results.

Theorem 3.1. (i) $(0, 0, 0)$ *is asymptotically stable for* (1.2) *when* $\mu < 0$ *and unstable when* $\mu > 0$.
(ii) $(0, 0, \mu)$ *is asymptotically stable for* (1.2) *when* $0 < \mu < \frac{1}{2}$ *and unstable when* $\mu < 0$ *or* $\mu > \frac{1}{2}$.

3.1. Steady-state bifurcation in the PDE system

In this section, we focus on the following elliptic system:

$$\begin{cases} -d_1u_{xx} = (\mu - 1)u - v + uw, & x \in (0, \pi), \\ -d_2v_{xx} = u + (\mu - 1)v + vw, & x \in (0, \pi), \\ -d_3w_{xx} = \mu w - (u^2 + v^2 + w^2), & x \in (0, \pi), \\ u_x = v_x = w_x = 0, & x = 0, \pi. \end{cases} \tag{3.5}$$

We find that there exists multiple steady-state bifurcation for the Langford system (3.5). The Crandall–Rabinowitz bifurcation theorem can be used to derive bifurcation from simple eigenvalue. For the case of double eigenvalue, the techniques of space decomposition and implicit function theorem are applied, but for the case of more than double eigenvalue, there is no further discussion.

Let $X = \{(u, v, w) : u, v, w \in W^{2,p}(0, \pi),\ u_x = v_x = w_x = 0 \text{ at } x = 0, \pi\}, Y = L^p(0, \pi) \times L^p(0, \pi) \times L^p(0, \pi)$, and define the inner product of Y by

$$\langle U, V\rangle = \int_0^\pi (u_1v_1 + u_2v_2 + u_3v_3)dx, \quad U = (u_1, u_2, u_3)^T,\ V = (v_1, v_2, v_3)^T \in Y$$

and the map $G : R^+ \times X \to Y$ by

$$G(\mu, U) = \begin{pmatrix} d_1u_{xx} + (\mu - 1)u - v + uw \\ d_2v_{xx} + u + (\mu - 1)v + vw \\ d_3w_{xx} + \mu w - (u^2 + v^2 + w^2) \end{pmatrix}, \quad U = (u, v, w).$$

Thus the solutions of (3.5) are exactly the zeros of this map. Note that $(0, 0, 0)$ and $(0, 0, \mu)$ are the constant solutions of (3.5), that is, $G(\mu, 0, 0, 0) = 0,\ G(\mu, 0, 0, \mu) = 0$.

Firstly, we consider the bifurcation from $(0,0,0)$. For $\sigma = 0$ in (3.4), we set

$$\mu_{k3}^0 = d_3 k^2$$

for $k \geq 0$ or

$$\mu_{k\pm}^0 = \frac{2 + (d_1 + d_2)k^2 \pm \sqrt{(d_2 - d_1)^2 k^4 - 4}}{2}$$

for

$$k^4 > \frac{4}{(d_2 - d_1)^2}. \tag{3.6}$$

Based on monotonicity analysis, we find that $\mu_{m3}^0 \neq \mu_{n3}^0$ and $\mu_{m+}^0 \neq \mu_{n+}^0$ for any $m \neq n, \mu_{m+}^0 > \mu_{n-}^0$ for any $m \geq n$, but $\mu_{m-}^0 = \mu_{n-}^0$ for some $m \neq n$. Then for the simple and double bifurcations from $(0,0,0)$ at μ_{k3}^0 or $\mu_{k\pm}^0$ we set the following cases.

$$E_1 = \{\mu_{i3}^0,\ \mu_{j+}^0,\ \mu_{m-}^0 :\ \mu_{i3}^0 \neq \mu_{j+}^0,\ \mu_{i3}^0 \neq \mu_{m-}^0,\ \mu_{j+}^0 \neq \mu_{m-}^0 (j < m) \text{ and } \mu_{m-}^0 \neq \mu_{n-}^0 (m \neq n)$$
$$\text{for any } i, j, m, n \geq 1\},$$
$$E_2 = \{\mu_{i3}^0,\ \mu_{j+}^0,\ \mu_{m-}^0 :\ \text{only if } \mu_{i3}^0 = \mu_{j+}^0,\ \text{or } \mu_{i3}^0 = \mu_{m-}^0,\ \text{or } \mu_{j+}^0 = \mu_{m-}^0 (j < m),\ \text{or } \mu_{m-}^0$$
$$= \mu_{n-}^0 (m \neq n) \text{ for some } i, j, m, n \geq 1\},$$
$$E_3 = \{\mu_{i3}^0,\ \mu_{j+}^0,\ \mu_{m-}^0 :\ \text{only if } \mu_{i3}^0 = \mu_{j+}^0 = \mu_{m-}^0 (j < m),\ \text{or } \mu_{i3}^0 = \mu_{m-}^0 = \mu_{n-}^0 (m \neq n),\ \text{or}$$
$$\mu_{j+}^0 = \mu_{m-}^0 = \mu_{n-}^0 (j < m, n \text{ and } m \neq n),\ \text{or } \mu_{i3}^0 = \mu_{j+}^0 = \mu_{m-}^0 = \mu_{n-}^0 (j < m, n \text{ and}$$
$$m \neq n) \text{ for some } i, j, m, n \geq 1\}.$$

Theorem 3.2. (i) *If μ_{k3}^0 $(k \geq 1)$ belongs to E_1, then $(\mu_{k3}^0, 0, 0, 0)$ $(k \geq 0)$ is a bifurcation point of $G = 0$. There is a curve of solutions $(\mu(s), u(s), v(s), w(s))$ of $G = 0$ for $|s|$ sufficiently small, satisfying $\mu(0) = \mu_{k3}^0$, $(u(0), v(0), w(0)) = (0,0,0)$, $u(s) = o(s^2)$, $v(s) = o(s^2)$, $w(s) = s\phi_k(x) + o(s^2)$, where $\mu(s), u(s), v(s), w(s)$ are continuous functions. Moreover, the bifurcating steady-state solution from $\mu = \mu_{03}^0$ coincides with the steady-state solution of the corresponding ODE system.*

(ii) *If $\mu_{k\pm}^0$ satisfies (3.6) and E_1, then $(\mu_{k\pm}^0, 0, 0, 0)$ is a bifurcation point of $G = 0$. There is a curve of solutions $(\mu(s), (u(s), v(s), w(s)))$ of $G = 0$ for $|s|$ sufficiently small, satisfying $\mu(0) = \mu_{k\pm}^0$, $(u(0), v(0), w(0)) = (0,0,0), u(s) = s\phi_k(x) + o(s^2), v(s) = s\rho_{k\pm}\phi_k(x) + o(s^2), w(s) = o(s^2)$, where $\mu(s), u(s), v(s), w(s)$ are continuous functions and $\rho_{k\pm} = \frac{(d_2-d_1)k^2 \pm \sqrt{(d_2-d_1)^2 k^4 - 4}}{2}$.*

Proof. (i) Recall that the operator

$$L_0(\mu_{k3}^0) = G_U(\mu_{k3}^0, 0) = \begin{pmatrix} d_1 \Delta + \mu_{k3}^0 - 1 & -1 & 0 \\ 1 & d_2 \Delta + \mu_{k3}^0 - 1 & 0 \\ 0 & 0 & d_3 \Delta + \mu_{k3}^0 \end{pmatrix}.$$

It is clear that the linear operators G_U, G_μ and $G_{\mu U}$ are continuous. Then we have

$$\ker(L_0(\mu_{k3}^0)) = \text{span}\{\Phi_k\}, \quad \Phi_k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \phi_k.$$

The adjoint operator is defined by

$$L_0^*(\mu_{k3}^0) = \begin{pmatrix} d_1 \Delta + \mu_{k3}^0 - 1 & 1 & 0 \\ -1 & d_2 \Delta + \mu_{k3}^0 - 1 & 0 \\ 0 & 0 & d_3 \Delta + \mu_{k3}^0 \end{pmatrix}.$$

Similarly,

$$\ker(L_0^*(\mu_{k3}^0)) = \operatorname{span}\{\Phi_k^*\}, \quad \Phi_k^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \phi_k.$$

Because of $R(L) = (\ker L^*)^{\perp}$, we get

$$\operatorname{codim} R(L_0(\mu_{k3}^0)) = \dim \ker(L_0^*(\mu_{k3}^0)) = 1.$$

Finally, since

$$G_{\mu U}(\mu_{k3}^0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\langle G_{\mu U}(\mu_{k3}^0, 0)\Phi_k, \Phi_k^* \rangle = \int_0^{\pi} \phi_k^2(x)dx = 1 \neq 0,$$

we obtain $G_{\mu U}(\mu_{k3}^0, 0)\Phi_k \notin R(L_0(\mu_{k3}^0))$. Thus, the proof of (i) is completed by the Crandall–Rabinowitz theorem [24] for the bifurcation from simple eigenvalue.

(ii) In the same way, we have

$$L_0(\mu_{k\pm}^0) = G_U(\mu_{k\pm}^0, 0) = \begin{pmatrix} d_1\Delta + \mu_{k\pm}^0 - 1 & -1 & 0 \\ 1 & d_2\Delta + \mu_{k\pm}^0 - 1 & 0 \\ 0 & 0 & d_3\Delta + \mu_{k\pm}^0 \end{pmatrix},$$

$$L_0^*(\mu_{k\pm}^0) = \begin{pmatrix} d_1\Delta + \mu_{k\pm}^0 - 1 & 1 & 0 \\ -1 & d_2\Delta + \mu_{k\pm}^0 - 1 & 0 \\ 0 & 0 & d_3\Delta + \mu_{k\pm}^0 \end{pmatrix}.$$

Then we obtain

$$\ker(L_0(\mu_{k\pm}^0)) = \operatorname{span}\{\Phi_{k\pm}\}, \quad \Phi_{k\pm} = \begin{pmatrix} 1 \\ \rho_{k\pm} \\ 0 \end{pmatrix} \phi_k,$$

$$\ker(L_0^*(\mu_{k\pm}^0)) = \operatorname{span}\{\Phi_{k\pm}^*\}, \quad \Phi_{k\pm}^* = \begin{pmatrix} 1 \\ -\rho_{k\pm} \\ 0 \end{pmatrix} \phi_k,$$

where $\rho_{k\pm} = \mu_{k\pm}^0 - 1 - d_1k^2 = \frac{(d_2-d_1)k^2 \pm \sqrt{(d_1-d_2)^2k^4-4}}{2} \neq 0$. Thus it is obvious that $\operatorname{codim} R(L_0(\mu_{k\pm}^0)) = \dim \ker(L_0^*(\mu_{k\pm}^0)) = 1$.

Furthermore, we have

$$G_{\mu U}(\mu_{k\pm}^0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and then

$$\langle G_{\mu U}(\mu_{k\pm}^0, 0)\Phi_{k\pm}, \Phi_{k\pm}^* \rangle = (1 - \rho_{k\pm}^2) \int_0^{\pi} \phi_k^2(x)dx = \mp\rho_{k\pm}\sqrt{(d_2-d_1)^2k^4 - 4} \neq 0,$$

which leads to $G_{\mu U}(\mu_{k\pm}^0, 0)\Phi_k \notin R(L_0(\mu_{k\pm}^0))$. Hence, we complete the proof of (ii). $\square$

Remark 3.3. When $\mu_{i3}^0,\ \mu_{j+}^0,\ \mu_{m-}^0 \in E_2$ for any i, j, m, there exists the double bifurcation. Without loss of generality, we will only discuss $\mu_{k+}^0 = \mu_{m-}^0$ for some $k < m$. However, for $\mu_{i3}^0,\ \mu_{j+}^0,\ \mu_{m-}^0 \in E_3$, the involved triple and quadruple bifurcations have not been discussed.

Theorem 3.4. *Suppose that there exists a positive integer $k < m$ satisfying $\mu_{k+}^0 = \mu_{m-}^0 \in E_2$ and (3.6). Then $(\mu_{k+}^0, 0, 0, 0)$ is a bifurcation point of $G = 0$. There is a curve of nontrivial solutions $(\mu(s), s(\cos\tau\Phi_{k+} + \sin\tau\Phi_{m-} + W(s)))$ of $G = 0$ for sufficiently small $|s|$ with $\mu(0) = \tilde{\mu}$ and $W(0) = (0,0,0)^T$.*

Proof. For $\mu_{k+}^0 = \mu_{m-}^0 \triangleq \tilde{\mu}, k < m$, we have

$$\ker(L_0(\tilde{\mu})) = \operatorname{span}\{\Phi_{k+}, \Phi_{m-}\}, \qquad \ker(L_0^*(\tilde{\mu})) = \operatorname{span}\{\Phi_{k+}^*, \Phi_{m-}^*\},$$

and

$$R(L_0(\tilde{\mu})) = \left\{(u,v,w) \in Y : \int_0^\pi (u - \rho_{k+}v)\phi_k dx = \int_0^\pi (u - \rho_{m-}v)\phi_m dx = 0\right\},$$

which lead to $\dim\ker(L_0(\tilde{\mu})) = \operatorname{codim} R(L_0(\tilde{\mu})) = 2$.

Obviously, the condition of the Crandall–Rabinowitz bifurcation theorem is not satisfied and the theorem does not work. Now, we resort to the techniques of space decomposition and implicit function theorem to deal with the case of double bifurcation.

Firstly, we define

$$P\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{1-\rho_{k+}^2}\int_0^\pi (u - \rho_{k+}v)\phi_k dx\,\Phi_{k+} + \frac{1}{1-\rho_{m-}^2}\int_0^\pi (u - \rho_{m-}v)\phi_m dx\,\Phi_{m-},$$

so that P is the projection from Y to $\ker(L_0(\tilde{\mu}))$, and then decompose Y as $Y = Y_1 \oplus Y_2$ with $Y_1 = R(P) = \ker(L_0(\tilde{\mu}))$ and $Y_2 = \ker P = R(L_0(\tilde{\mu}))$. Similarly, $X = X_1 \oplus X_2$ with $X_1 = \ker(L_0(\tilde{\mu}))$ and $X_2 = R(L_0(\tilde{\mu})) \cap X$.

Set $\alpha = \mu - \tilde{\mu}$, and rewrite $G(\mu, U)$ as

$$G(\alpha + \tilde{\mu}, U) = L_0(\alpha + \tilde{\mu})\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} uw \\ vw \\ -(u^2 + v^2 + w^2) \end{pmatrix}.$$

Then we look for the solutions of $G(\mu, U) = 0$ in the form

$$(u,v,w)^T = s(\cos\tau\Phi_{k+} + \sin\tau\Phi_{m-} + W),\ W = (W_1, W_2, W_3)^T \in X_2,$$

where $s, \tau \in R$ are parameters. Define a nonlinear mapping $K(\alpha, W; s) : R \times X_2 \times R \to Y$ by

$$\begin{aligned} K(\alpha, W; s) &= s^{-1}G(\alpha + \tilde{\mu}, s(\cos\tau\Phi_{k+} + \sin\tau\Phi_{m-} + W)) \\ &= L_0(\alpha + \tilde{\mu})(\cos\tau\Phi_{k+} + \sin\tau\Phi_{m-} + W) + s(g_1, g_2, g_3)^T \\ &= L_0(\tilde{\mu})W + \alpha(\cos\tau\Phi_{k+} + \sin\tau\Phi_{m-} + W) + s(g_1, g_2, g_3)^T, \end{aligned}$$

where $g_1 = W_3(\cos\tau\phi_k + \sin\tau\phi_m + W_1)$, $g_2 = W_3(\rho_{k+}\cos\tau\phi_k + \rho_{m-}\sin\tau\phi_m + W_2)$, $g_3 = -[(\cos\tau\phi_k + \sin\tau\phi_m + W_1)^2 + (\rho_{k+}\cos\tau\phi_k + \rho_{m-}\sin\tau\phi_m + W_2)^2 + W_3^2]$.

Clearly, $K(0,0;0) = 0$. The Fréchet derivative of $K(\alpha, W; s)$ with respect to (α, W) at $(\alpha, W; s) = (0,0;0)$ is the linear mapping

$$(\hat{\alpha}, \hat{W}) \to L_0(\tilde{\mu})\hat{W} + \hat{\alpha}\cos\tau\Phi_{k+} + \hat{\alpha}\sin\tau\Phi_{m-},$$

which is an isomorphism from $R \times X_2 \to Y$. Using the implicit function theorem for

$$K(\alpha, W; s) = 0,$$

there exist the continuously differentiable functions $(\alpha(s), W(s))$ defined for sufficiently small $|s|$, which satisfy $\alpha(0) = 0,\ W(0) = (0,0,0)^T$ and $K(\alpha(s), W(s); s) = 0$. Hence,

$$(u, v, w)^T = s(\cos\tau\Phi_k + \sin\tau\Phi_m + W(s))$$

are the nontrivial solutions of $G = 0$. $\square$

For the bifurcation from $(0, 0, \mu)$, to get $\sigma = 0$ in (3.2), we set

$$\mu^1_{k3} = -d_3k^2$$

when $k \ge 0$ or

$$\mu^1_{k\pm} = \frac{2 + (d_1 + d_2)k^2 \pm \sqrt{(d_2 - d_1)^2k^4 - 4}}{4} \tag{3.7}$$

when (3.6) holds. Then the bifurcation from $(0, 0, \mu)$ at nonnegative μ^1_{03} or $\mu^1_{k\pm}$ can be discussed in the same way as above, which are omitted here.

3.2. Hopf bifurcation in the PDE system

Let $X = \{(u, v, w) : u, v, w \in H^2[(0, \pi)],\ u_x = v_x = w_x = 0 \text{ at } x = 0, \pi\}$, and $X_{\mathbb{C}} = X + iX$, where $H^2[(0, \pi)]$ is the standard Sobolev space. Let $\langle\cdot,\cdot\rangle$ be the complex-valued L^2 inner product on Hilbert space $X_{\mathbb{C}}$, which is defined as

$$\langle U, V\rangle = \int_0^\pi (\overline{u_1}v_1 + \overline{u_2}v_2 + \overline{u_3}v_3)dx, \quad U = (u_1, u_2, u_3)^T,\ V = (v_1, v_2, v_3)^T \in X_{\mathbb{C}}.$$

Now, we look for periodic solutions bifurcating from $(0, 0, \mu)$ for (1.2). Set the translation $\tilde{u} = u, \tilde{v} = v, \tilde{w} = w - \mu$, and still denote $\tilde{u}, \tilde{v}, \tilde{w}$ as u, v, w. Then we rewrite system (1.2) in the following form

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = L_1(\mu)\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

where

$$f = uw, \qquad g = vw, \qquad h = -(u^2 + v^2 + w^2). \tag{3.8}$$

Due to the real $M_k(\mu)$, we let $\alpha_k(\mu) \pm i\omega_k(\mu)$ be the roots of $\sigma^2 - T_k(\mu)\sigma + D_k(\mu) = 0, k = 0, 1, 2, \ldots$. Then $\alpha_k(\mu) = T_k(\mu)/2 = 2\mu - 1 - (d_1 + d_2)k^2/2,\ \omega_k(\mu) = \sqrt{D_k(\mu) - \alpha_k^2(\mu)}$ and $\alpha'_k(\mu) = 2 > 0$. Letting

$$\mu^1_k = (2 + (d_1 + d_2)k^2)/4$$

be the roots of $T_k(\mu) = 0$, then we have

$$\alpha_k(\mu^1_k) = 0, \qquad \alpha'_k(\mu^1_k) > 0, \qquad T_k(\mu^1_k) = 0 \quad \text{and} \quad T_j(\mu^1_k) \ne 0 \text{ for } j \ne k.$$

The transverse condition is always satisfied. We only need to verify $D_k(\mu^1_k) > 0,\ D_j(\mu^1_k) \ne 0$ for $j = 0, 1, 2, \ldots$ and $j \ne k$. Here, we know that $D_k(\mu^1_k) = d_1d_2k^4 - (2\mu^1_k - 1)(d_1 + d_2)k^2 + (2\mu^1_k - 1)^2 + 1 > 0$ is equivalent to

$$0 \le k^4 < \frac{4}{(d_1 - d_2)^2}. \tag{3.9}$$

Now we derive a condition on the parameters so that $D_j(\mu_k^1) = d_1 d_2 j^4 - (2\mu_k^1 - 1)(d_1 + d_2)j^2 + (2\mu_k^1 - 1)^2 + 1 \neq 0$ for $j = 0, 1, 2, \dots$ and $j \neq k$. Denote

$$D_j(\mu) = (2\mu - 1)^2 - (2\mu - 1)(d_1 + d_2)j^2 + d_1 d_2 j^4 + 1.$$

Then when $\triangle = (d_1 - d_2)^2 j^4 - 4 < 0$ we know that $D_j(\mu) > 0$, and when $(d_1 - d_2)^2 j^4 - 4 = 0$, $D_j(\mu_k^1) > 0$ for $j \neq k$. When $(d_1 - d_2)^2 j^4 - 4 > 0$, we only need $\mu_k^1 \neq \mu_{j\pm}^1$ for $D_j(\mu_k^1) \neq 0$, where $\mu_{j\pm}^1$ are defined in (3.7). This leads to

$$k^2 \neq j^2 \pm \frac{\sqrt{(d_1 - d_2)^2 j^4 - 4}}{d_1 + d_2} \quad \text{when } (d_1 - d_2)^2 j^4 - 4 > 0. \tag{3.10}$$

Thus, when (3.9) and (3.10) are valid, $D_k(\mu_k^1) > 0$ and $D_j(\mu_k^1) \neq 0$ for $j = 0, 1, 2, \dots$ and $j \neq k$, which imply that (1.2) undergoes a Hopf bifurcation at $\mu = \mu_k^1$. Apparently, $\mu = \mu_0^1$ is always the unique value for the Hopf bifurcation of spatially homogeneous periodic solutions to (1.2).

Theorem 3.5. *If there exists some $k = 0, 1, 2, \dots$ such that (3.9) and (3.10) hold, then the system (1.2) undergoes a Hopf bifurcation from $(0, 0, \mu)$ at $\mu = \mu_k^1 := (2 + (d_1 + d_2)k^2)/4$. Moreover, the bifurcating periodic solution from $\mu = \mu_0^1$ is spatially homogeneous, which coincides with the periodic solution of the corresponding* ODE *system, and the bifurcating periodic solutions from $\mu = \mu_k^1 (k \geq 1)$ are spatially inhomogeneous.*

Next we consider the direction of Hopf bifurcation and the stability of bifurcating periodic solutions from $\mu = \mu_0^1$ and $\mu = \mu_k^1 (k \geq 1)$.

Theorem 3.6. *For the system (1.2), the Hopf bifurcation from $(0, 0, \mu)$ at $\mu = \mu_0^1$ is supercritical, and the spatially homogeneous bifurcating periodic solution is asymptotically stable.*

Proof. From [11,18], the bifurcation is supercritical (resp. subcritical) if

$$-\frac{1}{\alpha'(\mu_0^1)} \mathrm{Re}(c_1(\mu_0^1)) > 0 \quad (\text{resp.} < 0)$$

and in addition, if all other eigenvalues of $L_1(\mu_0^1)$ have negative real parts, then the bifurcating periodic solution is stable (resp. unstable) if $\mathrm{Re}(c_1(\mu_0^1)) < 0$ (resp. > 0). Let $L_1^*(\mu)$ be the conjugate operator of $L_1(\mu)$ defined in (3.1),

$$L_1^*(\mu) = \begin{pmatrix} d_1 \varDelta + 2\mu - 1 & 1 & 0 \\ -1 & d_2 \varDelta + 2\mu - 1 & 0 \\ 0 & 0 & d_3 \varDelta - \mu \end{pmatrix}, \quad \varDelta = \frac{\partial^2}{\partial x^2}.$$

When $\mu = \mu_0^1$, we put

$$q := \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \qquad q^* := \begin{pmatrix} a_0^* \\ b_0^* \\ c_0^* \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{3.11}$$

satisfying $\langle q^*, q \rangle = 1$, $\langle q^*, \bar{q} \rangle = 0$, $L_1(\mu_0^1)q = i\omega_0 q$ and $L_1^*(\mu_0^1)q^* = -i\omega_0 q^*$, where $\omega_0 = \omega(\mu_0^1) = 1$. We decompose $X = X_c \oplus X_s$ with $X_c := \{zq + \bar{z}\bar{q} | z \in \mathbb{C}\}$ and $X_s := \{U \in X | \langle q^*, U \rangle = 0\}$. For any $(u, v, w)^T \in X$, there exist $z \in \mathbb{C}$ and $\chi = (\chi^1, \chi^2, \chi^3) \in X_s$ such that

$$(u, v, w)^T = zq + \bar{z}\bar{q} + \chi.$$

Recall that $f = uw,\ g = vw,\ h = -(u^2 + v^2 + w^2)$. Then we can get the derivatives with respect to u, v and w at $(\mu, u, v, w) = (\mu_0^1, 0, 0, 0)$ as follows:

$$f_{uw} = g_{vw} = 1, \qquad h_{uu} = h_{vv} = h_{ww} = -2,$$

and the other second derivatives and third derivatives of f, g, h are all zero.

For the later uses, we calculate $Q_{qq},\ Q_{q\bar{q}}$ and $Q_{\chi_{11}q}$ at $(\mu_0^1, 0, 0, 0)$ as follows:

$$Q_{qq} = \begin{pmatrix} d_{1k} \\ d_{2k} \\ d_{3k} \end{pmatrix} \cos^2 kx, \qquad Q_{q\bar{q}} = \begin{pmatrix} e_{1k} \\ e_{2k} \\ e_{3k} \end{pmatrix} \cos^2 kx, \qquad Q_{\chi_{11}q} = \begin{pmatrix} f_{1k} \\ f_{2k} \\ f_{3k} \end{pmatrix} \cos kx,$$

where $\chi_{11} = (\chi_{11}^1, \chi_{11}^2, \chi_{11}^3)^T$ and

$$\begin{aligned}
d_{1k} &= f_{uu}a_k^2 + f_{vv}b_k^2 + f_{ww}c_k^2 + 2f_{uv}a_kb_k + 2f_{uw}a_kc_k + 2f_{vw}b_kc_k, \\
d_{2k} &= g_{uu}a_k^2 + g_{vv}b_k^2 + g_{ww}c_k^2 + 2g_{uv}a_kb_k + 2g_{uw}a_kc_k + 2g_{vw}b_kc_k, \\
d_{3k} &= h_{uu}a_k^2 + h_{vv}b_k^2 + h_{ww}c_k^2 + 2h_{uv}a_kb_k + 2h_{uw}a_kc_k + 2h_{vw}b_kc_k, \\
e_{1k} &= f_{uu}|a_k|^2 + f_{vv}|b_k|^2 + f_{ww}|c_k|^2 + f_{uv}(a_k\overline{b_k} + \overline{a_k}b_k) + f_{uw}(a_k\overline{c_k} + \overline{a_k}c_k) + f_{vw}(b_k\overline{c_k} + \overline{b_k}c_k), \\
e_{2k} &= g_{uu}|a_k|^2 + g_{vv}|b_k|^2 + g_{ww}|c_k|^2 + g_{uv}(a_k\overline{b_k} + \overline{a_k}b_k) + g_{uw}(a_k\overline{c_k} + \overline{a_k}c_k) + g_{vw}(b_k\overline{c_k} + \overline{b_k}c_k), \\
e_{3k} &= h_{uu}|a_k|^2 + h_{vv}|b_k|^2 + h_{ww}|c_k|^2 + h_{uv}(a_k\overline{b_k} + \overline{a_k}b_k) + h_{uw}(a_k\overline{c_k} + \overline{a_k}c_k) + h_{vw}(b_k\overline{c_k} + \overline{b_k}c_k), \\
f_{1k} &= 2[(f_{uu}a_k + f_{uv}b_k + f_{uw}c_k)\chi_{11}^1 + (f_{uv}a_k + f_{vv}b_k + f_{vw}c_k)\chi_{11}^2 + (f_{uw}a_k + f_{vw}b_k + f_{ww}c_k)\chi_{11}^3], \\
f_{2k} &= 2[(g_{uu}a_k + g_{uv}b_k + g_{uw}c_k)\chi_{11}^1 + (g_{uv}a_k + g_{vv}b_k + g_{vw}c_k)\chi_{11}^2 + (g_{uw}a_k + g_{vw}b_k + g_{ww}c_k)\chi_{11}^3], \\
f_{3k} &= 2[(h_{uu}a_k + h_{uv}b_k + h_{uw}c_k)\chi_{11}^1 + (h_{uv}a_k + h_{vv}b_k + h_{vw}c_k)\chi_{11}^2 + (h_{uw}a_k + h_{vw}b_k + h_{ww}c_k)\chi_{11}^3].
\end{aligned}$$

For $k = 0$,

$$\begin{aligned}
&d_{10} = 0, \qquad d_{20} = 0, \qquad d_{30} = 0, \\
&e_{10} = 0, \qquad e_{20} = 0, \qquad e_{30} = -4.
\end{aligned}$$

Because of the third derivatives of f, g, h are all zero, it is easy to get

$$C_{qq\bar{q}} = (0, 0, 0)^T.$$

Further calculations lead to

$$\langle q^*, Q_{qq} \rangle = 0, \qquad \langle q^*, Q_{q\bar{q}} \rangle = 0, \qquad \langle \bar{q}^*, Q_{qq} \rangle = 0, \qquad \langle \bar{q}^*, Q_{q\bar{q}} \rangle = 0.$$

Hence it is straightforward to calculate

$$\begin{aligned}
H_{20} &= Q_{qq} - \langle q^*, Q_{qq} \rangle q - \langle \bar{q}^*, Q_{qq} \rangle \bar{q} = (0, 0, 0)^T, \\
H_{11} &= Q_{q\bar{q}} - \langle q^*, Q_{q\bar{q}} \rangle q - \langle \bar{q}^*, Q_{q\bar{q}} \rangle \bar{q} = (0, 0, -4)^T.
\end{aligned}$$

From $\chi_{20} = [2i\omega_0 I - L_1(\mu_0^1)]^{-1} H_{20}$ and $\chi_{11} = -[L_1(\mu_0^1)]^{-1} H_{11}$, we can get $\chi_{20} = (0, 0, 0)^T$ and solve $L_1(\mu_0^1)\chi_{11} = -H_{11}$ to gain $\chi_{11} = (0, 0, -8)^T$, which lead to $Q_{\chi_{20}\bar{q}=0}$ and $f_{10} = -16,\ f_{20} = 16i,\ f_{30} = 0$. Then we obtain $g_{20} = \langle q^*, Q_{qq} \rangle = 0,\ g_{11} = \langle q^*, Q_{q\bar{q}} \rangle = 0$ and

$$\begin{aligned}
g_{21} &= \langle q^*, C_{qq\bar{q}} \rangle + \langle q^*, Q_{\chi_{20}\bar{q}} \rangle + \langle q^*, Q_{\chi_{11}q} \rangle \\
&= \langle q^*, Q_{\chi_{11}q} \rangle = \frac{1}{2\pi} \left\langle \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \begin{pmatrix} -16 \\ 16i \\ 0 \end{pmatrix} \right\rangle \\
&= -16.
\end{aligned}$$

Thus,

$$\mathrm{Re}(c_1(\mu_0^1)) = \mathrm{Re}\left(\frac{i}{2\omega_0}g_{20}g_{11} + \frac{g_{21}}{2}\right) = -8 < 0.$$

Recall that $\alpha'(\mu) = 2 > 0$ and all other eigenvalues of $L_1(\mu_0^1)$ have negative real parts. Hence, the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solution is asymptotically stable. □

Theorem 3.7. *For the system (1.2), the Hopf bifurcation from $(0, 0, \mu)$ at $\mu = \mu_k^1, k \geq 1$ satisfying (3.9) and (3.10) is subcritical (supercritical) if $Re(c_1(\mu_k^1)) > 0$ (<0), where $Re(c_1(\mu_k^1))$ is defined in (3.15), and the spatially inhomogeneous bifurcating periodic solutions are unstable.*

Proof. When $\mu = \mu_k^1,\ k \geq 1$, we have

$$q := \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \cos kx = \begin{pmatrix} 1 \\ \dfrac{(d_2 - d_1)k^2}{2} - i\omega_k \\ 0 \end{pmatrix} \cos kx, \tag{3.12}$$

$$q^* := \begin{pmatrix} a_k^* \\ b_k^* \\ c_k^* \end{pmatrix} \cos kx = \frac{1}{\pi\omega_k}\begin{pmatrix} \dfrac{(d_2 - d_1)k^2}{2} i + \omega_k \\ -i \\ 0 \end{pmatrix} \cos kx \tag{3.13}$$

with $\omega_k = \omega_k(\mu_k^1) = \sqrt{4 - (d_1 - d_2)^2 k^4}/2$, which lead to $\langle q^*, q\rangle = 1,\ \langle q^*, \bar{q}\rangle = 0,\ L_1(\mu_k^1)q = i\omega_k q$ and $L_1^*(\mu_k^1)q^* = -i\omega_k q^*$.

Due to $\int_0^\pi \cos^3 kx dx = 0$ for $k \geq 1$, it follows that $\langle q^*, Q_{qq}\rangle = \langle q^*, Q_{q\bar{q}}\rangle = \langle \bar{q}^*, Q_{qq}\rangle = \langle \bar{q}^*, Q_{q\bar{q}}\rangle = 0$. We know that $f = uw,\ g = vw,\ h = -(u^2 + v^2 + w^2),\ f_{uw} = g_{vw} = 1,\ h_{uu} = h_{vv} = h_{ww} = -2$, and the other second derivatives and third derivatives of f, g, h with respect to u, v and w are all zero, which result in $\langle q^*, C_{qq\bar{q}}\rangle = 0$. For $k \geq 1$, we can get the derivatives at $(\mu_k^1, 0, 0, 0)$ as follows:

$$\begin{aligned} &d_{1k} = 0, \qquad d_{2k} = 0, \qquad d_{3k} = -2 - 2b_k^2 = -(d_2 - d_1)^2 k^4 + 2(d_2 - d_1)k^2\omega_k i \triangleq g, \\ &e_{1k} = 0, \qquad e_{2k} = 0, \qquad e_{3k} = -2 - 2|b_k|^2 = -4. \end{aligned}$$

Then we obtain $H_{20} = Q_{qq} = \cos^2 kx(0, 0, g)^T,\ H_{11} = Q_{q\bar{q}} = \cos^2 kx(0, 0, -4)^T$. In order to calculate $\mathrm{Re}(c_1(\mu_k^1))$, we only need to consider

$$\langle q^*, Q_{\chi_{20}\bar{q}}\rangle, \qquad \langle q^*, Q_{\chi_{11}q}\rangle.$$

For the later discussions, we set

$$Q_{\chi_{20}\bar{q}} = \begin{pmatrix} l_{1k} \\ l_{2k} \\ l_{3k} \end{pmatrix} \cos kx$$

with

$$\begin{aligned} l_{1k} &= (f_{uu}\overline{a_k} + f_{uv}\overline{b_k} + f_{uw}\overline{c_k})\chi_{20}^1 + (f_{uv}\overline{a_k} + f_{vv}\overline{b_k} + f_{vw}\overline{c_k})\chi_{20}^2 + (f_{uw}\overline{a_k} + f_{vw}\overline{b_k} + f_{ww}\overline{c_k})\chi_{20}^3, \\ l_{2k} &= (g_{uu}\overline{a_k} + g_{uv}\overline{b_k} + g_{uw}\overline{c_k})\chi_{20}^1 + (g_{uv}\overline{a_k} + g_{vv}\overline{b_k} + g_{vw}\overline{c_k})\chi_{20}^2 + (g_{uw}\overline{a_k} + g_{vw}\overline{b_k} + g_{ww}\overline{c_k})\chi_{20}^3, \\ l_{3k} &= (h_{uu}\overline{a_k} + h_{uv}\overline{b_k} + h_{uw}\overline{c_k})\chi_{20}^1 + (h_{uv}\overline{a_k} + h_{vv}\overline{b_k} + h_{vw}\overline{c_k})\chi_{20}^2 + (h_{uw}\overline{a_k} + h_{vw}\overline{b_k} + h_{ww}\overline{c_k})\chi_{20}^3. \end{aligned}$$

According to

$$\chi_{20} = [2i\omega_k I - L_1(\mu_k^1)]^{-1}\frac{\cos 2kx + 1}{2}\begin{pmatrix}0\\0\\g\end{pmatrix},$$

$$\chi_{11} = -[L_1(\mu_k^1)]^{-1}\frac{\cos 2kx + 1}{2}\begin{pmatrix}0\\0\\-4\end{pmatrix},$$

we can obtain

$$\chi_{20} = \begin{pmatrix}0\\0\\l_1\end{pmatrix}\cos 2kx + \begin{pmatrix}0\\0\\l_2\end{pmatrix},$$
$$\chi_{11} = \begin{pmatrix}0\\0\\f_1\end{pmatrix}\cos 2kx + \begin{pmatrix}0\\0\\f_2\end{pmatrix},$$

where

$$l_1 = \frac{g}{2(4d_3k^2 + \mu_k^1 + 2\omega_k i)}, \qquad l_2 = \frac{g}{2(\mu_k^1 + 2\omega_k i)}, \qquad f_1 = \frac{-2}{4d_3k^2 + \mu_k^1}, \qquad f_2 = \frac{-2}{\mu_k^1}. \tag{3.14}$$

Further calculations lead to

$$l_{1k} = l_1\cos 2kx + l_2, \qquad l_{2k} = \overline{b_k}l_{1k}, \qquad l_{3k} = 0,$$
$$f_{1k} = 2(f_1\cos 2kx + f_2), \qquad f_{2k} = b_k f_{1k}, \qquad f_{3k} = 0.$$

For $k \geq 1$, we notice that

$$\int_0^\pi \cos^2 kx dx = \frac{\pi}{2}, \qquad \int_0^\pi \cos 2kx\cos^2 kx dx = \frac{\pi}{4}.$$

Thus we can get

$$\begin{aligned}
g_{21} &= \langle q^*, C_{qq\bar{q}}\rangle + \langle q^*, Q_{\chi_{20}\bar{q}}\rangle + \langle q^*, Q_{\chi_{11}q}\rangle \\
&= \langle q^*, Q_{\chi_{20}\bar{q}}\rangle + \langle q^*, Q_{\chi_{11}q}\rangle \\
&= \frac{1}{\pi\omega_k}\left\langle \begin{pmatrix}\frac{(d_2-d_1)k^2}{2}i + \omega_k\\ -i\\ 0\end{pmatrix}\cos kx, \begin{pmatrix}l_{1k}+f_{1k}\\ l_{2k}+f_{2k}\\ 0\end{pmatrix}\cos kx\right\rangle \\
&= \frac{1}{\pi\omega_k}\left\langle \begin{pmatrix}\frac{(d_2-d_1)k^2}{2}i + \omega_k\\ -i\\ 0\end{pmatrix}\cos kx, \begin{pmatrix}l_1+2f_1\\ \overline{b_k}l_1+2b_kf_1\\ 0\end{pmatrix}\cos 2kx\cos kx + \begin{pmatrix}l_2+2f_2\\ \overline{b_k}l_2+2b_kf_2\\ 0\end{pmatrix}\cos kx\right\rangle \\
&= \frac{1}{\omega_k}\left[\left(\omega_k - \frac{(d_2-d_1)k^2}{2}i\right)\left(\frac{l_1+2f_1}{4} + \frac{l_2+2f_2}{2}\right) + i\left(\frac{\overline{b_k}l_1+2b_kf_1}{4} + \frac{\overline{b_k}l_2+2b_kf_2}{2}\right)\right].
\end{aligned}$$

Hence,

$$\mathrm{Re}(c_1(\mu_k^1)) = \mathrm{Re}\left[\left(\omega_k - \frac{(d_2-d_1)k^2}{2}i\right)\left(\frac{l_1+2f_1}{4} + \frac{l_2+2f_2}{2}\right) + i\left(\frac{\overline{b_k}l_1+2b_kf_1}{4} + \frac{\overline{b_k}l_2+2b_kf_2}{2}\right)\right], \tag{3.15}$$

where $l_i, f_i, i = 1, 2$ are defined in (3.14). Recall that $\alpha'(\mu) = 2 > 0$, so the direction of the Hopf bifurcation is subcritical (supercritical) if $\mathrm{Re}(c_1(\mu_k^1)) > 0$ (<0) with $\mathrm{Re}(c_1(\mu_k^1))$ defined in (3.15). The bifurcating periodic solutions are clearly unstable since the $(0, 0, \mu)$ is unstable for $\mu = \mu_k^1 > \frac{1}{2}, k \geq 1$. □

Remark 3.8. From [11,18], the periodic solutions have the form

$$\begin{pmatrix} u(s)(x,t) \\ v(s)(x,t) \\ w(s)(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu_k^1 \end{pmatrix} + s\left(\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} e^{2\pi it/T(s)} + \begin{pmatrix} \overline{a_k} \\ \overline{b_k} \\ \overline{c_k} \end{pmatrix} e^{-2\pi it/T(s)}\right)\cos kx + \begin{pmatrix} o(s^2) \\ o(s^2) \\ o(s^2) \end{pmatrix},$$

where

$$T(s) = \frac{2\pi}{\omega_0}(1 + \tau_2 s^2) + o(s^4).$$

Here, we further consider periodic solutions of (1.2) bifurcating from $(0, 0, 0)$. Similarly, we can have the following discussions.

Due to the real N_k, we let $\beta_k(\mu) \pm i\gamma_k(\mu)$ be the roots of $\sigma^2 - P_k(\mu)\sigma + Q_k(\mu) = 0, k = 0, 1, 2, \ldots$. Then $\beta_k(\mu) = P_k(\mu)/2 = \mu - 1 - (d_1 + d_2)k^2/2,\ \gamma_k(\mu) = \sqrt{Q_k(\mu) - \beta_k^2(\mu)}$ and $\beta_k'(\mu) = 1 > 0$. Denote

$$\mu_k^0 = 1 + (d_1 + d_2)k^2/2$$

as the roots of $P_k(\mu) = 0$, then we have

$$\beta_k(\mu_k^0) = 0, \qquad \beta_k'(\mu_k^0) > 0, \qquad P_k(\mu_k^0) = 0 \quad \text{and} \quad P_j(\mu_k^0) \neq 0 \text{ for } j \neq k.$$

Similar calculations lead to $Q_k(\mu_k^0) > 0$ and $Q_j(\mu_k^0) \neq 0$ for $j = 0, 1, 2, \ldots$ and $j \neq k$ when (3.9) and (3.10) are valid, respectively.

Theorem 3.9. *If there exists some $k = 0, 1, 2, \ldots$ such that (3.9) and (3.10) hold, then the system (1.2) undergoes a Hopf bifurcation from $(0, 0, 0)$ at $\mu = \mu_k^0 := 1 + (d_1 + d_2)k^2/2$. Moreover, the bifurcating periodic solution from $\mu = \mu_0^0$ is spatially homogeneous, which coincides with the periodic solution of the corresponding* ODE *system, and the bifurcating periodic solutions from $\mu = \mu_k^0 (k \geq 1)$ are spatially inhomogeneous.*

Theorem 3.10. *For the system (1.2), the Hopf bifurcation from $(0, 0, 0)$ at $\mu = \mu_0^0$ is subcritical, and the spatially homogeneous bifurcating periodic solution is unstable.*

Proof. We rewrite system (1.2) in the form

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = L_0(\mu)\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

where f, g, h are defined in (3.8). The conjugate operator of $L_0(\mu)$ defined in (3.3) is given by

$$L_0^*(\mu) = \begin{pmatrix} d_1\Delta + \mu - 1 & 1 & 0 \\ -1 & d_2\Delta + \mu - 1 & 0 \\ 0 & 0 & d_3\Delta + \mu \end{pmatrix}, \quad \Delta = \frac{\partial^2}{\partial x^2}.$$

For $\mu = \mu_0^0$ and $\gamma_0 = \gamma(\mu_0^0) = 1 = \omega_0$, we also get $q = (1, -i, 0)^T$ and $q^* = \frac{1}{2\pi}(1, -i, 0)^T$ defined in (3.11) with $\langle q^*, q\rangle = 1,\ \langle q^*, \bar{q}\rangle = 0,\ L_0(\mu_0^0)q = i\gamma_0 q$ and $L_0^*(\mu_0^0)q^* = -i\gamma_0 q^*$. Recall that

$$f_{uw} = g_{vw} = 1, \qquad h_{uu} = h_{vv} = h_{ww} = -2,$$

and the other second derivatives and third derivatives of f, g, h are all zero. According to Theorem 3.6, we know

$$C_{qq\bar{q}} = (0,0,0)^T, \qquad H_{20} = (0,0,0)^T, \qquad H_{11} = (0,0,-4)^T, \qquad g_{20} = 0, \qquad g_{11} = 0.$$

Then we get $\chi_{20} = (0,0,0)^T$ and $\chi_{11} = (0,0,4)^T$ from $\chi_{20} = [2i\gamma_0 I - L_0(\mu_0^0)]^{-1}H_{20}$ and $\chi_{11} = -[L_0(\mu_0^0)]^{-1}H_{11}$, respectively. Thus, $Q_{\chi_{20}\bar{q}=0}$ and

$$g_{21} = \langle q^*, Q_{\chi_{11}q}\rangle = \frac{1}{2\pi}\left\langle \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ -8i \\ 0 \end{pmatrix} \right\rangle = 8.$$

Therefore,

$$\mathrm{Re}(c_1(\mu_0^0)) = \mathrm{Re}\left(\frac{i}{2\gamma_0}g_{20}g_{11} + \frac{g_{21}}{2}\right) = 4 > 0.$$

Recall that $\beta'(\mu) = 1 > 0$. Hence, the direction of the Hopf bifurcation is subcritical. The bifurcating periodic solution is unstable because $(0,0,0)$ is unstable for $\mu = \mu_0^0 = 1 > 0$. $\square$

Theorem 3.11. *For the system (1.2), the Hopf bifurcation from $(0,0,0)$ at $\mu = \mu_k^0, k \geq 1$ satisfying (3.9) and (3.10) is subcritical (supercritical) if $Re(c_1(\mu_k^0)) > 0$ (<0), where $Re(c_1(\mu_k^0))$ is defined in (3.17), and the spatially inhomogeneous bifurcating periodic solutions are unstable.*

Proof. For $\mu = \mu_k^0,\ k \geq 1$ and $\gamma_k = \gamma_k(\mu_k^0) = \sqrt{4 - (d_1 - d_2)^2 k^4}/2 = \omega_k$, we get q and q^* as in (3.12) and (3.13), respectively, which lead to $\langle q^*, q\rangle = 1,\ \langle q^*, \bar{q}\rangle = 0,\ L_1(\mu_k^0)q = i\gamma_k q$ and $L_1^*(\mu_k^0)q^* = -i\gamma_k q^*$.

Similarly, $\langle q^*, Q_{qq}\rangle = \langle q^*, Q_{q\bar{q}}\rangle = \langle \bar{q}^*, Q_{qq}\rangle = \langle \bar{q}^*, Q_{q\bar{q}}\rangle = \langle q^*, C_{qq\bar{q}}\rangle = 0$, and $H_{20} = \cos^2 kx(0,0,g)^T$, $H_{11} = \cos^2 kx(0,0,-4)^T$. Then in order to calculate $\mathrm{Re}(c_1(\mu_k^0))$, we only need to consider

$$\langle q^*, Q_{\chi_{20}\bar{q}}\rangle,\ \langle q^*, Q_{\chi_{11}q}\rangle.$$

According to

$$\chi_{20} = [2i\omega_k I - L_0(\mu_k^0)]^{-1}\frac{\cos 2kx + 1}{2}\begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix},$$

$$\chi_{11} = -[L_0(\mu_k^0)]^{-1}\frac{\cos 2kx + 1}{2}\begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix},$$

we can obtain

$$\chi_{20} = \begin{pmatrix} 0 \\ 0 \\ l_3 \end{pmatrix}\cos 2kx + \begin{pmatrix} 0 \\ 0 \\ l_4 \end{pmatrix},$$

$$\chi_{11} = \begin{pmatrix} 0 \\ 0 \\ f_3 \end{pmatrix}\cos 2kx + \begin{pmatrix} 0 \\ 0 \\ f_4 \end{pmatrix}$$

with

$$l_3 = \frac{g}{2(4d_3k^2 - \mu_k^0 + 2\omega_k i)}, \qquad l_4 = \frac{g}{2(2\omega_k i - \mu_k^0)}, \qquad f_3 = \frac{-2}{4d_3k^2 - \mu_k^0}, \qquad f_4 = \frac{2}{\mu_k^0}. \tag{3.16}$$

Further calculations lead to

$$\begin{aligned}
&l_{1k} = l_3 \cos 2kx + l_4, \qquad l_{2k} = \overline{b_k} l_{1k}, \qquad l_{3k} = 0,\\
&f_{1k} = 2(f_3 \cos 2kx + f_4), \qquad f_{2k} = b_k f_{1k}, \qquad f_{3k} = 0.
\end{aligned}$$

Since $\int_0^\pi \cos^2 kxdx = \frac{\pi}{2}$ and $\int_0^\pi \cos 2kx \cos^2 kxdx = \frac{\pi}{4}$ for $k \geq 1$, we obtain

$$\begin{aligned}
g_{21} &= \langle q^*, Q_{\chi_{20}\bar{q}}\rangle + \langle q^*, Q_{\chi_{11}q}\rangle\\
&= \frac{1}{\omega_k}\left[\left(\omega_k - \frac{(d_2-d_1)k^2}{2}i\right)\left(\frac{l_3+2f_3}{4} + \frac{l_4+2f_4}{2}\right) + i\left(\frac{\overline{b_k}l_3 + 2b_kf_3}{4} + \frac{\overline{b_k}l_4 + 2b_kf_4}{2}\right)\right].
\end{aligned}$$

Hence,

$$\mathrm{Re}(c_1(\mu_k^0)) = \mathrm{Re}\left[\left(\omega_k - \frac{(d_2-d_1)k^2}{2}i\right)\left(\frac{l_3+2f_3}{4} + \frac{l_4+2f_4}{2}\right) + i\left(\frac{\overline{b_k}l_3 + 2b_kf_3}{4} + \frac{\overline{b_k}l_4 + 2b_kf_4}{2}\right)\right], \tag{3.17}$$

where $l_i, f_i, i = 3, 4$ are defined in (3.16). From $\beta'(\mu) = 1 > 0$, the direction of the Hopf bifurcation is subcritical (supercritical) if $\mathrm{Re}(c_1(\mu_k^0)) > 0$ (<0). The bifurcating periodic solutions are clearly unstable since the $(0, 0, 0)$ is unstable for $\mu = \mu_k^0 > 0, k \geq 1$. □

Remark 3.12. The calculations of the Hopf bifurcation in this section for the third order Langford PDE system are different from [5,6,10–12,15].

Remark 3.13. When d_1 closes to d_2 and they are small enough, it readily follows from (3.6) and (3.9) that the steady-state bifurcation points are less and the Hopf bifurcation points are more. Moreover, there is an open problem for the bifurcation analysis when $k^4 = \frac{4}{(d_1-d_2)^2}$, which leads to $\mu_{k+}^0 = \mu_{k-}^0 = \mu_k^0 \triangleq \tilde{\mu}^0$ and $P_k(\tilde{\mu}^0) = Q_k(\tilde{\mu}^0) = 0$, and $\mu_{k+}^1 = \mu_{k-}^1 = \mu_k^1 \triangleq \tilde{\mu}^1$ and $T_k(\tilde{\mu}^1) = D_k(\tilde{\mu}^1) = 0$, but $L_0(\tilde{\mu}^0)$ or $L_1(\tilde{\mu}^1)$ corresponds to one eigenvector with eigenvalue 0 for $\mu_{k+}^0 = \mu_{k-}^0 \in E_2$ or $\mu_{k+}^1 = \mu_{k-}^1 \in E_2$, respectively.

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