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Value of the distant future: Model-independent results

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HIGHLIGHTS

- Model-independent estimates of the long-run discount factor are obtained.
- The universal origin of declining long-term tails of discount curves is uncovered.
- Non-Markovian generalization of the Ramsey discounting formula is derived.
- Obtained analytical results allowing simple calibration.

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ABSTRACT

This paper shows that the model-independent account of correlations in an interest rate process or a log-consumption growth process leads to declining long-term tails of discount curves. Under the assumption of an exponentially decaying memory in fluctuations of risk-free real interest rates, I derive the analytical expression for an apt value of the long run discount factor and provide a detailed comparison of the obtained result with the outcome of the benchmark risk-free interest rate models. Utilizing the standard consumption-based model with an isoelastic power utility of the representative economic agent, I derive the non-Markovian generalization of the Ramsey discounting formula. Obtained analytical results allowing simple calibration, may augment the rigorous cost-benefit and regulatory impact analysis of long-term environmental and infrastructure projects.

1. Introduction

Suppose we wish to estimate the present value of $1 received in T years from now. In the investment-based approach, as long as the annualized rate of return on a default-free investment r is constant in time, the net present value of the future cash flow is determined by the simple discounting formula, $d(r, T) = \exp(-rT)$. The exponential discounting of future benefits is the direct consequence of the first order kinetics that governs the cumulative growth of a default-free investment: $1 invested today will yield $\exp(+rT)$ at time T in the future. Traditionally, the account of temporal fluctuations of r in a long-run discounting has been done by plugging the historical average rate of return on investment $m = \langle r \rangle$ into the exponential discounting factor, $d(m, T)$. Clearly, the exponential discounting factor can significantly diminish the present monetary value of future benefits. For instance, the US Office of Management and Budget requires to use $d(m, T)$ with $m = 7\%$ per year in valuation costs and benefits of intra-generational projects [1]. In this case, the net present value of $1 received in 30 years is only $0.12. In fact, at really long-time horizons that are typical for inter-generational environmental projects, exponential discounting can lead to a negligible present monetary value of future benefits. For example, the today's value of $1 received in 200 years from now estimated with the average discount rate, say, $m = 3\%$ is only $0.003. Unsurprisingly, in evaluating long-run public projects, governments of France and the United Kingdom use *declining* discount curve schedules [1].

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The irregular temporal behavior of interest rates leads to fundamental uncertainty and complexity of valuation of the relevant stochastic discount factor (SDF). This problem has acquired a renewed interest in the context of climate finance. Starting with pioneering works of Weitzman [2,3] it becomes clear that an account of the probability distribution of future interest rates makes the effective "certainty equivalent" discount rate decline significantly over time. Weitzman [2] clearly pointed out that 'what should be averaged over states of the world is not discount *rates* at various times, but the discount *factors*':

$$D\left(T\right)=E\left[\exp\left(-\int_{0}^{T}r(t')\mathrm{d}t'\right)\right]=\exp[-y\left(T\right)T]. \tag{1}$$

Here $D\left(T\right)$ denotes the present value of SDF for the future date T, the operator E denotes an average over all possible interest rate paths $\{r\left(t\right)\}$ that begin at present time 0 and terminate at time T under the appropriate probability measure, and $y\left(T\right)$ is an effective risk-free continuously-compounded discount rate.

Empirically real, i.e., adjusted for inflation interest rates, which play the central role in the cost-benefit analysis of large public projects with benefits differed to a distant future, exhibit considerable persistence and mean-reversion [4]. In the investment-based approach, epitomized by Eq. (1), this behavior is reflected in the benchmark models of instantaneous interest rates. In the Vasicek model $r\left(t\right)$ follows the Ornstein–Uhlenbeck (OU) stochastic process [5,6]. The model proposed by Cox, Ingersoll and Ross (CIR) [7] assumes that $r(t)$ follows the Feller process [8]. Both Vasicek and CIR models predict the *declining* in time long-term tail of a discount curve. In a long run $y\left(T\right)$ converges towards its time-invariant asymptotic that can be substantially smaller than the mean of realized interest rates [9–13].

Similar conclusions have been made within the consumption-based approach to discounting. This framework is rooted in the seminal work of Ramsey [14], connecting discount rates to the underlying growth of consumption, for details see, e.g., Arrow et al. [15] and references therein. The canonical consumption-based model is based on a power utility of the representative economic agent and assumes that the log-consumption growth rate follows the i.i.d. Gaussian process. It has been shown that an uncertainty in the future growth leads to a *time-invariant* effective long-term discount rate that is smaller than m, see Refs. [16–21] and references therein. Thus, both investment- and consumption-based models predict that the present value of benefits expected in a distant future can be significantly higher than it is forecasted by the traditional methodology utilizing an exponential discounting factor, $d(m, T)$. It is hard to overestimate an importance of these results for a long-term pension planning, infrastructure projects, and climate finance.

Motivated by these studies, the present paper begins by noting that the approximate *model-independent* description of a long-term behavior of SDF is possible in terms of cumulants of a stationary risk-free interest or consumption growth rate process. The general method of cumulant expansion is closely related to the "linked diagram" expansion of many-body problems and is broadly used in modern statistical physics [22]. Recently, it has been applied by Martin [23] to the problem of long-term discounting. Account of higher cumulants of the log-consumption growth process allows Martin to extend the consumption-based theory of SDF beyond the assumption of normality of this process. Unfortunately, Martin has considered only the i.i.d. process for the log consumption growth. Since all cumulants of any i.i.d. process – Gaussian or not – are linearly growing with time, his model leads to a completely flat discount curve. On the other hand, it has been shown by Gollier [24,25] that the method of cumulant expansion yields a declining term-structure of a discount curve in the auto-regressive model accounting for memory effects in a consumption growth process [26].

Here, following both investment- and consumption-based approaches, I employ the method of cumulant expansion to derive approximate model-independent expressions for a long-term tail of a discount curve and SDF. From the mathematical point of view, the key insight comes from the close analogy between the general formulations of the stochastic discounting problem and the spectral line-shape problem [27–30]. If we replace the factor -1 in Eq. (1) with imaginary unit and substitute the instantaneous interest rate r with the stochastically modulated frequency of a spectroscopic transition, the analogy becomes obvious. Of course, it does not go beyond the formal mathematical similarity. However, if the relevant conditions of validity are met, the Kubo's form of the cumulant expansion that has been successfully used to describe spectral line-shapes in dense media is applicable to calculations of the discount curve. The cumulant expansion can be truncated at the second term if (i) the action of stochastic perturbation is weak; (ii) the process is assumed to be the Gaussian white-noise. In either case, for a stationary stochastic interest rate process or log-consumption growth process the non-Markovian character of obtained results (approximate model-independent or exact model-dependent) accounts for a declining term structure of the discount curve at times comparable with a correlation time of the relevant stochastic process. With a growth of time $y\left(T\right)$ converges towards its time-invariant asymptotic, $y^{(M)}=y\left(T\rightarrow\infty\right)<m$, which corresponds to the conventional Markovian exponential decay of SDF over time.

It has been shown by Farmer et al. [12] that for countries with stable economy the autocorrelation function of long-term real interest rates is exponentially decaying over time. This paper shows that in this case the apt value of the long-term SDF is obtained by multiplication of the customary exponential discount factor $d(m, T)$ by the time-dependent coefficient derived in this paper. I provide a detailed comparison of obtained results with the outcome of the benchmark risk-free interest rate models. Furthermore, utilizing the standard consumption-based model with an isoelastic power utility, I derive the non-Markovian generalization of the Ramsey discounting formula. Obtained analytical expressions allowing simple calibration to data, may augment the rigorous cost-benefit and regulatory impact analysis of long-term environmental and infrastructure projects.

2. Truncated cumulant expansion: the investment-based approach

Notice that average over all stochastic realizations of an instantaneous risk-free interest rate r between the present date 0 and the future date T must be performed on the general *solution*, Eq. (1), rather than on the stochastic differential equation of motion for SDF. It has the advantage of being free of any assumptions about the statistical properties of the process $r(t)$. The averaged exponential functional on the RHS of Eq. (1) can be formally expanded into the cumulant series that can be used to derive the system's response to a stochastic perturbation [27–30]. In the problem under consideration, SDF is responding to fluctuations of instantaneous interest rates. Hence, an effective risk-free continuously-compounded zero-coupon yield to maturity (discount curve) can be formally represented by the Kubo's cumulant expansion as

$$\begin{aligned} y(T) &= -T^{-1} \ln E\left[\exp\left(-\int_0^T r(t')\mathrm{d}t'\right)\right] \\ &= -T^{-1}\sum_{n=1}^{\infty}(-1)^n \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n\, E_c\, r(t_1)r(t_2)\, \ldots\, r(t_n). \end{aligned} \tag{2}$$

Here the operator E_c denotes the *cumulant* average [27–30]:

$$E_c\, x = E\, x, \qquad E_c\, xx = E\, xx - E\, x E\, x, \qquad E_c\, xxx = E\, xxx - 3\, E\, xx\, E\, x + 2E\, x E\, x E\, x. \tag{3}$$

We are interested in the behavior of SDF for a distant future when, by assumption, a stochastic interest rate process converges into an ergodic stationary process with some non-zero autocorrelation time:

$$\tau_r = \int_0^{\infty} K_r(\tau)\mathrm{d}\tau / K_r(0). \tag{4}$$

Here $K_r(\tau)$ is the autocorrelation function of a stationary (time-homogeneous) interest rate process

$$K_r(\tau) = E\{[r(\tau) - m][r(0) - m]\} = \rho^2\varphi(\tau), \tag{5}$$

where m is the stationary mean value of instantaneous risk-free interest rates, ρ denotes the measure of fluctuations' strength, $\varphi(\tau)$ decays to zero with the characteristic time scale τ_r and $\varphi(0) = 1$. The decay of correlations should be sufficiently fast to insure the finite autocorrelation time of a stochastic process. To the second order in $\rho\tau_r$ one can keep only first two cumulants in the expansion (2), which yields:

$$y\,(T) = m - T^{-1}\int_0^T \mathrm{d}t_1 \int_0^{t_1} K_r(t_2)\mathrm{d}t_2 \; = m - T^{-1}\int_0^T (T-\tau)K_r(\tau)\, \mathrm{d}\tau, \tag{6}$$

where $\tau = t_1 - t_2$. This approximate model-independent result is valid for *any* stationary process $r(t)$ as long as the action of a stochastic perturbation is weak, $\rho^2\tau_r^2 \ll 1$. Notice that for weak stochastic perturbations the discount curve $y(T)$ is always positive, even if $\rho \approx m > 0$. According to Farmer et al. [12], for long-term risk-free real interest rates the annualized measure of fluctuations strength $\rho \approx 3 \div 4\%$, whereas $\tau_r = 5 \div 10$ years. Therefore, empirically the action of random shocks in long-term real interest rates is very weak, $\rho\tau_r \approx 0.15 \div 0.4$, and the second order approximation leading to Eq. (6) is not very restrictive. Obviously, calculation of higher order cumulants will improve an accuracy of the theory. Explicit expressions for higher cumulants can be obtained, e.g., by employing analytical capabilities of the Wolfram Matematica tool.

The important point to make here is that the key formula (6) describes the declining discount curve leading to the non-exponential term-structure of SDF, which is non-Markovian in this sense:

$$D\,(T) = \exp\left[-m\; T + \int_0^T (T-t')K_r(t')\, \mathrm{d}t'\right]. \tag{7}$$

Formally, at short times $T \ll \tau_r$, the discount curve, Eq. (6), is linearly declining with a growth of time, $y\,(T) \approx m - \frac{K_r(0)}{T}\int_0^T (T-t')\,\mathrm{d}t' = m - 0.5\rho^2 T$. Note, however, that T must be long enough to allow for the interest rate process to become stationary. Hence, Eqs. (6) and (7) are applicable only for description of the long-term evolution of a discount curve and SDF, respectively.

The complex kinetic of SDF, described by Eq. (7), is always preceding the simple Markovian asymptotic. At sufficiently long times, $T \gg \tau_r$, the integration over time in Eq. (7) can be extended to infinity leading to the exponential Markovian long-term asymptotic of SDF with the time-invariant effective discount rate:

$$y^{(M)} = m - \int_0^{\infty} K_r(t')\,\mathrm{d}t' = m - \rho^2\tau_r. \tag{8}$$

This behavior is inevitable on a coarse grained time scale. It is consistent with flattening of the nominal interest rate curve of US Treasury bonds with long maturities, $T > 10$ years, and with flat long-term discount curves inferred from the real estate data by Giglio et al. [31].

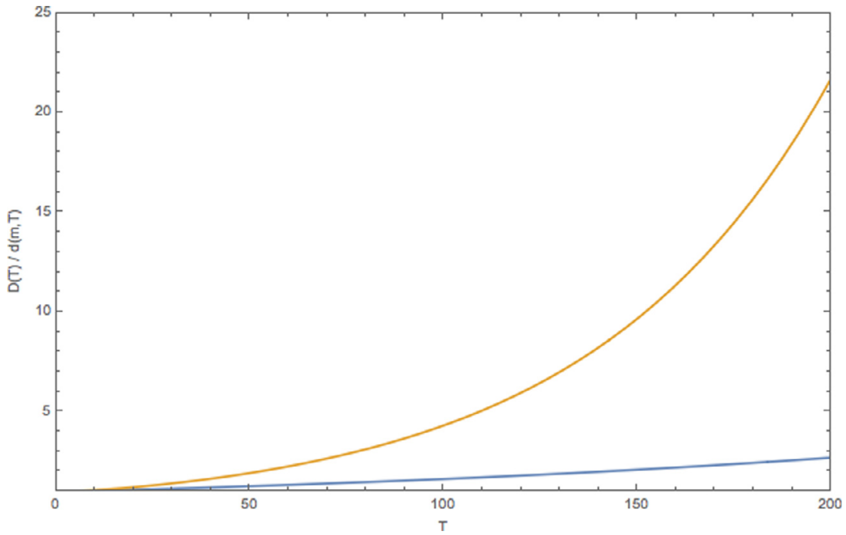

Fig. 1. (Color online) The temporal behavior of the universal multiplication factor, Eq. (12). Values of the parameters: $\tau_r = 5.6$ years and $\rho = 3\%$ per year for the lower curve; $\tau_r = 10$ years and $\rho = 4\%$ per year for the upper curve. T is measured in years.

To proceed further, consider the empirically relevant [12] model of the stochastic process $r(t)$ with an exponentially decaying autocorrelation function:

$$K_r(\tau) = \rho^2 \exp(-\tau/\tau_r). \tag{9}$$

In this case, substituting Eq. (9) into Eq. (6) and taking into account Eq. (8), we obtain the following non-Markovian correction to the long-run discount rate:

$$y(T) = m - \rho^2\tau_r^2 T^{-1}[e^{-T/\tau_r} + T/\tau_r - 1] = y^{(M)} - \rho^2\tau_r^2 T^{-1}[e^{-T/\tau_r} - 1]. \tag{10}$$

The time-independent term in Eq. (10) determines the long-term *universal* Markovian asymptotic of a yield curve, whereas the transient terms describe its model-dependent negative slope. Now it is easy to calculate the long-term tail of a forward curve

$$f(T) = -\partial_T[\ln D(T)] = y^{(M)} + \rho^2\tau_r e^{-T/\tau_r} \tag{11}$$

and the multiplication factor that can be used to obtain the apt present value of benefits expected in a distant future:

$$k(T) = D(T)/d(m, T) = \exp\{\rho^2\tau_r^2\,[T/\tau_r + e^{-T/\tau_r} - 1]\}. \tag{12}$$

Empirical analysis of the time series of risk-free bonds with 10 years maturities shows an exponential decay of the autocorrelation function of real interest rates with estimated $\tau_r = 5.3$ years for UK and 5.6 years for USA [12]. Taking the average of historical long-term default-free real interest rates, Farmer et al. [12] have also estimated the annualized $m = 3.3\%$ and 2.6% for UK (1694–2012) and USA (1820–2012), respectively. Parameter $\rho^2 = K_r(0)$ can be easily recovered from the values of the volatility in the Vasicek model of interest rate process [5] reported in Ref. [12] (for details see the next section). This leads to the substantial decrease of the long-term asymptotic of the discount curve, $\rho^2\tau_r \approx 0.5\%$ in Eq. (8), for both UK and USA. Note that for countries with less stable economy, the decrease of the long-term discount rate forecasted by Eq. (8) can be more substantial. Consequently, the estimated net present value of \$1 received in, say, 200 years from now could be more than twenty times larger than the one estimated by the conventional discounting methodology, see Fig. 1. Moreover, even for weak stochastic perturbations, $\rho^2\tau_r^2 \ll 1$, the decline of the discount curve with time described by Eq. (10) can be very pronounced, see Fig. 2. This plot shows the declining term-structure of $y(T)$, which is approaching its long-term asymptotic, Eq. (8), as time goes by. It is clearly seen that for the annualized $\rho = 4\%$ and $\tau_r = 10$ years, the schedule of declining discount rates is spanning a century.

3. Comparison with the benchmark investment-based models of SDF

It is instructive to compare results derived in the previous section with the outcome of the benchmark investment-based models of SDF leading to stationary distributions of instantaneous interest rates with a growth of time. In the concise form both Vasicek and CIR models can be represented by the single Ito stochastic differential equation:

$$dr(t) = \mu(r)dt + s(r)dW(t). \tag{13}$$

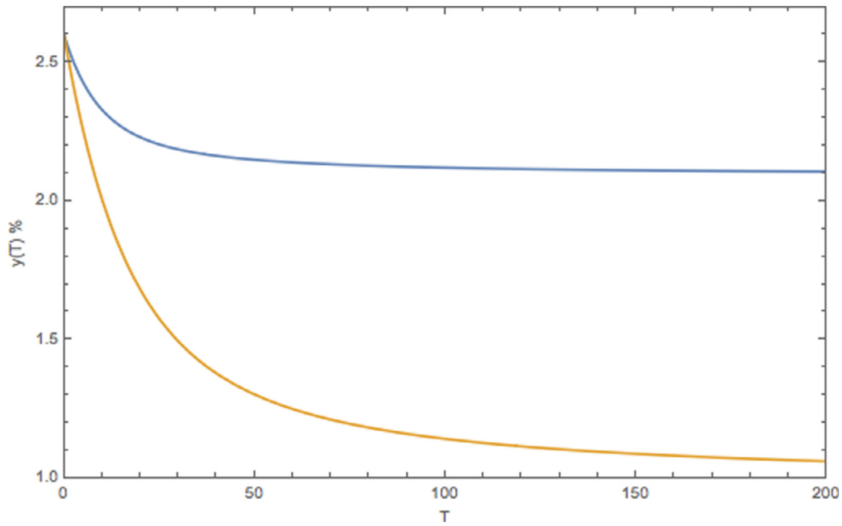

Fig. 2. (Color online) The term-structure of the discount rate, see Eq. (10). Values of the parameters: $m = 2.6\%$ per year, $\tau_r = 5.6$ years and $\rho = 3\%$ per year for the upper curve; $\tau_r = 10$ years and $\rho = 4\%$ per year for the lower curve. T is measured in years.

Here $\mu(r) = -\alpha\ [r(t) - m]$ denotes the instantaneous drift, where $\alpha > 0$ characterizes the speed of reversion to the mean value (normal level) $m > 0$; the instantaneous standard deviation $s(r)$ is assumed to be independent of time, and $W(t)$ is the Wiener process. The main difference between these models is related to the assumed functional form of $s(r)$. The model with constant amplitude of fluctuations of short-term interest rates $s(r) = \sigma_V$ was proposed by Vasicek [5]. On the other hand, in the CIR model [7] the noise intensity is state-dependent, $s(r) = \sigma_{CIR}\sqrt{r}$. By design, when r goes to zero, the noise vanishes and the drift drags r toward the positive level m. This feature of the CIR model always keeps interest rates positive. Notice the difference in dimensionality between σ_V and σ_{CIR}, which reflects the different assumptions regarding the functional forms of $s(r)$.

In the long-run the Vasicek (OU process) model leads to the Gaussian stationary distribution, whereas the CIR (Feller process) model leads to the Gamma stationary distribution of instantaneous interest rates. Both models yield the same stationary mean value of the process, m, and the stationary autocorrelation functions that are exponentially decreasing with time:

$$K_r^{(V)}(T) = \frac{\sigma_V^2}{2\alpha}\exp(-\alpha T) \tag{14a}$$

$$K_r^{(CIR)}(T) = \frac{m\sigma_{CIR}^2}{2\alpha}\exp(-\alpha T). \tag{14b}$$

Comparison with Eq. (9) shows that in both cases the autocorrelation time of the interest rate process is characterized by the inverse speed of its mean-reversion, $\tau_r = 1/\alpha$, whereas the magnitude of random shocks is characterized by the model-dependent second moment of the relevant stationary distribution: $\rho_V^2 = 0.5\alpha^{-1}\sigma_V^2$ and $\rho_{CIR}^2 = 0.5m\alpha^{-1}\sigma_{CIR}^2$. Substitution of these expressions into Eq. (10) yields

$$y_V(T) = m - \frac{\sigma_V^2}{2\alpha^2} - \frac{\sigma_V^2}{2\alpha^3 T}(e^{-\alpha T} - 1) \tag{15a}$$

$$y_{CIR}(T) = m\left[1 - \frac{\sigma_{CIR}^2}{2\alpha^2} - \frac{\sigma_{CIR}^2}{2\alpha^3 T}(e^{-\alpha T} - 1)\right]. \tag{15b}$$

It is well known that for the Gaussian distribution all cumulants higher than the second order vanish identically. In this model, all results obtained in the previous section are valid for an arbitrary strength of the stochastic perturbation, $\rho\tau_r$. Therefore, the long-term time-invariant Markovian asymptotic $y_V^{(M)} = m - 0.5\alpha^{-2}\sigma_V^2$ as well as the transient terms in Eq. (15a) exactly coincides with the long-term asymptotic of the Vasicek analytic result [5]. However, the CIR model with a state-dependent volatility does not lead to the Gaussian stationary distribution. Keeping only first two cumulants in the expansion can be justified only if the action of the stochastic perturbation is small, $\rho_{CIR}^2\alpha^{-2} \ll 1$. Therefore, the Markovian asymptotic of Eq. (15b), $y_{CIR}^{(M)} = m(1 - 0.5\alpha^{-2}\sigma_{CIR}^2)$, represents the second order approximation in σ_{CIR}/α of the long-term asymptotic of the exact analytical result obtained in the CIR model [7,13].

It is important to stress here that the exact expressions for the temporal evolution of SDF and a discount curve obtained in both Vasicek and CIR models forecast $D(T)$ and $y(T)$ that depend on the initial value of the instantaneous interest rate at time horizons $T \ll 1/\alpha$, thereby revealing a dynamic nature of the underlining OU and Feller processes at short times.

On the other hand, Eqs. (15a) and (15b) are valid only in a relatively long-term, $T \geq 1/\alpha$, when a stochastic interest rate process converges to a stationary one.

4. Truncated cumulant expansion and the consumption-based approach

The standard consumption-based model assumes that the utility function of the representative agent in the economy $U(C)$, where C denotes the level of consumption, is isoelastic with the constant coefficient of agent's relative risk aversion (CCRRA), $\gamma > 1$:

$$U(C) = C^{1-\gamma}/(1-\gamma). \tag{16}$$

In equilibrium the CCRRA model yields the following optimal effective default-free rate of discount associated with the time horizon T, see, e.g., Ref. [17]:

$$y_g(T) = \delta - T^{-1} \ln E\left[\exp\left(-\gamma \int_0^T g(t')\mathrm{d}t'\right)\right]. \tag{17}$$

Here δrepresents the rate of pure preference for the present (the "utility discount rate") of the representative economic agent. The operator E in Eq. (17) denotes an average over all possible paths of the log-consumption growth rate $g(t) = \ln C(t)/C(0)$ that begin at present time 0 and terminate at time T under the appropriate probability measure, $C(t)$denotes the instantaneous consumption level at date t. Notice that by the fundamental assumption of the consumption capital asset pricing model, the stochastic process $g(t)$ is stationary [32].

Thus, similarly to Eq. (1), the average in Eq. (17) can be formally represented by the Kubo's cumulant expansion as

$$\ln E\left[\exp\left(-\gamma \int_0^T g(t')\mathrm{d}t'\right)\right] = \sum_{n=1}^{\infty}(-\gamma)^n \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n\, E_c\, g(t_1)g(t_2)\ \ldots\ g(t_n). \tag{18}$$

Hence under similar technical conditions leading to Eq. (6) – weak action of a stochastic perturbation, $\gamma^2\rho_g^2\tau_g^2 \ll 1$, where ρ_g denotes the measure of $g(t)$ fluctuations' strength, $\tau_g = \int_0^\infty K_g(\tau)\mathrm{d}\tau/K_g(0)$ is the autocorrelation time of the log-consumption growth process, and $K_g(\tau)$ is the relevant autocorrelation function – truncation of the cumulant expansion Eq. (18) at the second term can be approximately valid for *any* stationary process $g(t)$. This approximation yields the following non-Markovian generalization of the seminal Ramsey formula:

$$y_g(T) = \delta + \gamma\, m_g - \gamma^2\, T^{-1} \int_0^T (T-\tau_g)K_g(\tau)\ \mathrm{d}\tau. \tag{19}$$

Here m_g denotes the stationary mean value of an annualized log-consumption growth rate. In the absence of fluctuations, $\rho_g = 0$, this expression reproduces the Ramsey discounting formula. The negative third term on the RHS of Eq. (19) represents the so-called 'precautionary effect' that drives agents facing uncertainty in the future consumption growth to save more today, see, e.g., Refs. [1,15,17]. Contrary to the standard consumption-based model, Eq. (19) accounts for a finite correlation time of $g(t)$, leading to a declining term-structure of $y_g(T)$. Similarly to Eq. (6), at sufficiently long times, $T \gg \tau_g$, this expression leads to the time-invariant long-term Markovian asymptotic of the effective discount rate, corresponding to its lowest possible value in equilibrium:

$$y_g^{(M)} = \delta + \gamma\, m_g - \gamma^2\rho_g^2\tau_g. \tag{20}$$

It is useful to see that the model with an exponentially decreasing autocorrelation function of the log-consumption growth process, $K_g(\tau) = \rho_g^2 \exp(-\tau/\tau_g)$, yields

$$y_g(T) = y_g^{(M)} - \gamma^2\rho_g^2\,\tau_g^2 T^{-1}(\mathrm{e}^{-T/\tau_g} - 1). \tag{21}$$

The transient terms on the RHS of this formula might be significant at times comparable with τ_g. If the consumption growth rate process can be modeled by the OU stochastic process, the log-term tail of the discount curve is described as follows

$$y_g^{(OU)}(T) = \delta + \gamma\, m_g - \frac{\gamma^2\sigma_g^2}{2\alpha_g^2} - \frac{\gamma^2\sigma_g^2}{2\alpha_g^3 T}(\mathrm{e}^{-\alpha_g T} - 1). \tag{22}$$

Here, similarly to Eq. (15a), $\alpha_g = 1/\tau_g > 0$ characterizes the speed of reversion to the normal level of the average consumption growth rate m_g and σ_g is the volatility of the stationary log-consumption growth process. With few exceptions known to the author [17,20,24,25], conventional consumption-based models of SDF ignore the persistence of a log-consumption growth rate. The standard model assumes that fluctuations of $g(t)$ follow the arithmetic Brownian motion with constant drift m_g and volatility σ_{Bg}:

$$\mathrm{d}g(t) = m_g\mathrm{d}t + \sigma_{Bg}\mathrm{d}W(t). \tag{23}$$

This assumption leads to a completely flat discount curve, see for example Ref. [17] and references therein:

$$y_g^{(B)} = \delta + \gamma m_g - 0.5\gamma^2\sigma_{Bg}^2. \tag{24}$$

This result follows from the i.i.d. model of fluctuations of $g(t)$ described by the Gaussian white noise $dW(t)$ in Eq. (23). In fact, even if the assumption of normality is dropped and higher cumulants of log-consumption process are taken into account [23], the standard i.i.d. CCRRA model leads to a time-invariant effective discount rate.

Empirically, the assumption that the consumption growth rate follows a pure random walk with no serial correlations is inconsistent with data in most countries [33,34]. Comparison of Eqs. (21) and (22) with Eq. (24) shows that the account of correlations in log-consumption growth, $\tau_g \neq 0$, tends to magnify the effect of uncertainty in the consumption growth rate and reduce a long-term effective discount rate. Analysis of the real per capita consumption in the USA over the period 1889–1978 leads to the following estimates of the parameters [35]: $m_g \approx 2\%$ and $\sigma_{Bg} \approx 4\%$. If we use these values, fix the unobservable CCRRA coefficient $\gamma = 2$, and use the "ethical" value of $\delta = 0$, Eq. (24) gives $y_g^{(B)} \approx 3.7\%$ per annum, which is probably too high. On the other hand, for $\gamma = 2$, $\delta = 0$ and the plausible values of the autocorrelation time $\tau_g = 5 - 10$ years and $\rho_g = 3\%$, Eq. (20) yields more reasonable $y_g^{(M)} \approx 2.2\%$–0.4%. Importantly, this calibration keeps the perturbation parameter, $(\gamma\rho_g\tau_g)^2 \approx 0.1$–$0.36$, sufficiently small to guarantee a fast convergence of the Kubo's cumulant expansion and, hence, a good accuracy of estimates based on Eq. (20).

5. Summary and discussion

The curse of the exponential discounting can drastically diminish the present value of long-term environmental and infrastructure projects. Therefore, the cost-benefit analysis of projects with long time horizons, especially climate finance, is extremely sensitive to uncertainties about future interest rates. Different models of a stochastic risk-free interest rate process lead to a declining discount curve with a growth of time. However, despite the obvious importance of the rigorous discounting policy, currently there is neither a consensus about the valuation methodology nor even a clear definition of the time horizon that might be considered as a distant future.

Three "stylized facts" are fundamental for our consideration: (i) observations on financial and real estate markets show that the long-term tail of a discount curve is declining and flattening with a growth of time; (ii) fluctuations of real interest rates and a log-consumption growth are persistent and mean reverting; (iii) stochastic perturbations of a long-term SDF caused by random shocks of interest rates or log-consumption growth rates are small. Overall, it seems clear that the observed decline and flattening of the discount curve (real or nominal) for time horizons beyond a decade from the present time is consistent, at least in principle, with a key assumptions and results of this paper.

The powerful Kubo's truncated cumulant expansion method makes explicit that a negative slope of the discount curve and a non-exponential decrease of SDF naturally appear at time horizons comparable to the correlation time of an interest rate or log-consumption-return process. This transient regime preceding the universal long-term Markovian asymptotic is absent in the standard consumption-based model of SDF, which customary ignores the persistence of shocks in the log-consumption growth rate, assuming the "i.i.d. log-normal economy". The account of positive serial correlations in log-consumption-returns tends to magnify the effect of uncertainty in the consumption growth rate and can substantially reduce a long-term discount rate. The key derived formulas (6) and (19) describe the transient decline of the long-term tail of the discount curve and do not depend on the model of noise. Note that Eq. (19) provides the non-Markovian generalization of the seminal Ramsey discounting formula. The latter is central for consumption-based asset pricing and macroeconomics [32]. It has been shown that as time goes by, both $y(T)$ and $y_g(T)$ are surely approaching their stationary Markovian asymptotic, corresponding to their lowest possible values in equilibrium.

Under the assumption of an exponentially decaying memory in fluctuations of long-term real interest rates, the long-term asymptotic and the transient behavior of the discount curve are consistent with the exact results of the benchmark investment-based discounting models. Simple calibration to data shows that, even in countries with stable economy, the magnitude of decline can be substantial, e.g., 0.5% for USA and UK. Due to the long memory in real interest rates the schedule of declining discount curve could span a century. To obtain an apt value of the long-term SDF, the customary exponential discount factor should be multiplied by the generic time-dependent coefficient derived in this paper. Notice that if real interest rates or log-consumption growth rates follow the OU stochastic process, Eqs. (10) and (21) describing the long-term tail of the relevant discount curves are exact.

Words of caution are due here. First, all obtained results are applicable only for description of the long-term evolution of a discount curve and SDF, when the relevant stochastic process is assumed to become stationary. Second, statistical tests of stationarity of long-term real interest rates are not conclusive [4]. Moreover, even the assumption of an exponentially decaying memory in a real interest rate process and a log-consumption growth process requires further empirical verification. Third, an account of catastrophic shocks to interest rates or consumption levels with long memory could also invalidate the applicability of the truncated cumulant approach employed in this paper. Finally, as Weitzman [2] pointed out, due to unforeseeable future economic and non-economic events, the stationary and ergodic regime of interest rate process may fail to hold for 'far-far-distant' millennia. I look forward to explore these scenarios in future theoretical and empirical work.

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