Applied Mathematics and Computation 295 (2017) 53–64

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A finite frequency approach to control of Markov jump linear systems with incomplete transition probabilities

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ARTICLE INFO

Keywords:
Markov jump systems
Finite frequency domain
Linear matrix inequality

ABSTRACT

This paper is concerned with the state feedback control of continuous Markov jump linear systems with incomplete transition probabilities in finite frequency domain. By developing a new technique to handle the coupling among Lyapunov variable, system matrix and controller parameter, new sufficient conditions for the closed-loop system to be stochastically stable with the required finite frequency performance are established in terms of linear matrix inequalities. Meanwhile, the finite frequency state feedback controller is also obtained by the proposed conditions directly. The validity of the proposed method is demonstrated by a numerical example.

1. Introduction

The Markov jump system is categorized as a class of stochastic hybrid system. Due to its widespread application in the fields of economics, robotics, and wireless communications, much effort has been devoted to this field [1]. On the hypothesis of known transition probabilities, fruitful results on stability, stabilization, sliding mode control, H_2 and H_∞ control and filtering are referred to [2–15]. Although the hypothesis is apt to do analysis and synthesis in theory, it is difficult to employ the obtained results to practical engineering problems. To shorten the gap between theory and practical applications, uncertain transition probabilities are assumed to be norm bounded or polytope and handled by the robust methodology [16,17]. In this situation, the structure and nominal terms of these uncertainties should be known in advance [18]. To approximate the realistic situation, transition probabilities are allowed to partly known [18]. With the partly known transition probabilities presented in [18], a multiple integral approach is employed to cast the impulsive synchronization of Markovian jumping neural networks in [19] and stochastic sampled-date based exponential synchronization of Markovian jumping neural networks is discussed in [20]. To reduce the possible conservativeness incurred by unknown transition probabilities, a free weighting matrix method is built in [21] and the property of transition probabilities are made thoroughly in [22].

On the other hand, the frequency characteristic of disturbance may be known beforehand, especially in mechanical systems. Consequently, in the course of controller design for mechanical systems, integrating the frequency characteristic could enhance the closed-loop system performance [23,24]. Taking into account this point, H_∞ control, fault detection and model reduction in finite frequency domain for linear systems are given in [25–29]. What's more, finite frequency fuzzy filtering

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http://dx.doi.org/10.1016/j.amc.2016.09.024
0096-3003/

of nonlinear systems is addressed in [30] and H_∞ state feedback control of fuzzy system in finite frequency is presented in [31]. Regarding to Markov jump system, only few results are available in the literature [32,33]. Specifically, modeling the accessability of each node to the shared channel as a Markov chain, a geometric scheme based fault detection and isolation for the resultant discrete Markov jump system in finite frequency domain is proposed in [32]. Based on the generalized Kalman–Yakubovich–Popov lemma, stochastic consensus control of continuous Markov jump system with middle frequency specification is discussed in [33]. Nevertheless, transition probabilities in these results are required to be known. Once they are unknown, the proposed approaches are out of use due to nonlinearities induced by them. Moreover, the technique to deal with finite frequency still has room to be further improved.

Stimulated by the above mentioned points, this paper is devoted to state feedback control of Markov jump linear systems with finite frequency disturbances. Incomplete transition probabilities contain known, uncertain and unknown. Instead of using the generalized Kalman–Yakubovich–Popov lemma directly, enlightening by the time domain inequalities proposed in [24], a finite frequency performance for the considered Markov jump linear system is defined firstly. Then, nonlinearities induced by unknown transition probabilities are conquered by the transition matrix property. Based on the definition and a new decoupling measurement, sufficient conditions for the closed-loop system to be stochastically stable with the required finite frequency performance are established in the framework of linear matrix inequalities. With the help of Finsler lemma, the parameter dependent Lyapunov function approach is adopted to ensure the closed-loop system to be stochastically stable and meet the required finite frequency performance. Compared with the existing method to get the controller gain, the proposed method has no need to pre- and post-multiply an inverse matrix. A single-link robot arm is simulated to show the fact that the finite frequency performance is less conservativeness than that of the full frequency.

This article is organized as follows. System model, definition and technical lemma are introduced in Section 2. An effective finite frequency controller design method is proposed in Section 3. A single-link robot arm is given in Section 4 to verify the validity of the proposed approach. Lastly, Section 5 concludes the paper.

Notation: Throughout the paper, the notation $R > 0$ (< 0) means that R is symmetric and positive (negative) definite. $\mathbb{R}^n$ indicates the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. $\mathbb{L}_2$ means the space of square integrable vector functions over $[0\ \ \infty)$ with norm $||x||_2 = \{\int_0^\infty x^T(t)x(t)dt\}^{\frac{1}{2}}$. The transpose of M is denoted by M^T. $\star$ stands for the entries of matrices implied by symmetry. Then, symbols $sym(X)$ and $He(X)$ are employed to represent $X + X^T$ and $\frac{(X+X^T)}{2}$ respectively. What is more, matrices, if not explicitly stated, are assumed to have compatible dimensions. $\mathbb{E}$ denotes the expectation operator. $\mathcal{H}^\perp$ is the kernel of $\mathcal{H}$. j is the imaginary unit.

2. Problem statement and preliminaries

Consider the following continuous Markov jump linear system as

$$\begin{cases}\dot{x}(t) = A(r(t))x(t) + B_1(r(t))u(t) + B_2(r(t))w(t)\\ z(t) = C(r(t))x(t) + D_1(r(t))u(t) + D_2(r(t))w(t)\end{cases} \tag{1}$$

where $x(t)$, $u(t)$ and $w(t)$ are continuous system state vector, control input and the energy bounded noise with known frequency respectively. $z(t)$ is the measured output. $A(r(t))$, $B_1(r(t))$, $B_2(r(t))$, $C(r(t))$, $D_1(r(t))$, $D_2(r(t))$ are system matrices with approximate dimension. $r(t)(t \geq 0)$ is continuous Markov process belonging to a finite set $\mathcal{I} = \{1, 2, \ldots, N\}$. The mode transition probabilities of continuous case $r(t)$ satisfies

$$Pr(r(t+h) = l|r(t) = i) = \begin{cases}\pi_{il}h + o(h), & if \quad l \neq i\\ 1 + \pi_{ii}h + o(h), & if \quad l = i\end{cases}$$

where $h > 0$ and $\lim_{h\to 0}\frac{o(h)}{h} = 0$, $\pi_{il} \geq 0$ $(i, l \in \mathcal{I}, l \neq i)$ represents the transition probability from mode i to mode l and $\pi_{ii} = -\sum_{l\neq i}^N \pi_{il}$.

As discussed in the existing result [22], transition probabilities in this paper are also deemed to be incomplete. For instance, the incomplete transition probability matrix for system (1) with four modes is

$$\begin{bmatrix}\pi_{11} & ? & \alpha_{13} & ?\\ ? & ? & ? & ?\\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34}\\ \pi_{41} & ? & ? & \pi_{44}\end{bmatrix} \tag{2}$$

where ? means that the corresponding elements are inaccessible and α_{ij} is uncertain with known lower and upper bounds $(\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$. To formulate the accessability of transition probability concisely, two sets are utilized to cover all above information

$$\begin{cases}\mathcal{I}_k^i = \{l|\pi_{il} \ \ is \ \ known \ \ or \ \ uncertain, \quad l \in \mathcal{I}\}\\ \mathcal{I}_{uk}^i = \{l|\pi_{il} \ \ is \ \ unknown, l \in \mathcal{I}\}\end{cases} \tag{3}$$

Remark 1. In [18], uncertain transition probabilities are treated as unknown. Obviously, this treatment could be conservative when all transition probabilities are uncertain with known bounds. To fill up this deficiency, the above set $\mathcal{I}_k^i$ includes uncertain ones.

Referring to [24], we introduce the following finite frequency range

$$\Theta_{\varpi} := \{\varpi \in \mathbb{R} | \tau(\varpi - \varpi_1)(\varpi - \varpi_2) \le 0\} \tag{4}$$

where $\tau = \pm 1$, ϖ_1 and ϖ_2 are known real scalars, and ϖ is a frequency variable.

Remark 2. Actually, the above frequency interval contains low-, middle-, high- and full frequencies by choosing τ individually. Namely, selecting $\tau = 1$, $\varpi_1 = -\varpi_2$ and $\varpi_1 \neq \varpi_2$ are corresponding to low- and middle cases respectively. The high- and full frequencies are reduced by taking $\varpi_1 = -\varpi_2$ and $\varpi_1 = \varpi_2 = 0$ when $\tau = -1$.

We consider the following state feedback controller

$$u(t) = K(r(t))x(t) \tag{5}$$

where $K(r(t))$ is the mode-dependent controller gain to be designed.

To facilitate the following presentation, for $r(t) = i$, $A(r(t))$, $B_1(r(t))$, $B_2(r(t))$, $C(r(t))$, $D_1(r(t))$, $D_2(r(t))$ and $K(r(t))$ are abbreviated as A_i, B_{1i}, B_{2i}, C_i, D_{1i}, D_{2i} and K_i.

Integrating (1) and (5), the resulted closed-loop system is

$$\begin{cases} \dot{x}(t) = \bar{A}_i x(t) + B_{2i} w(t) \\ z(t) = \bar{C}_i x(t) + D_{2i} w(t) \end{cases} \tag{6}$$

where

$$\bar{A}_i = A_i + B_{1i}K_i, \bar{C}_i = C_i + D_{1i}K_i.$$

Before proceeding further, a finite frequency performance for (6) is defined below

Definition 1. The system (1) has a finite frequency performance γ if the inequality

$$\mathbb{E}\left\{\int_0^{\infty} z^T(t)z(t)dt\right\} \le \gamma^2 \mathbb{E}\left\{\int_0^{\infty} w^T(t)w(t)dt\right\} \tag{7}$$

holds for zero initial condition with $w(t) \in \mathbb{L}_2$ satisfying

$$\mathbb{E}\left\{\int_0^{\infty} \tau(\varpi_1 x(t) + j\dot{x}(t))(\varpi_2 x(t) + j\dot{x}(t))^T dt\right\} \le 0 \tag{8}$$

Remark 3. The definition is inspired by de Farias et al. [2,24] for the full frequency. Namely, the defined finite frequency performance index γ is reduced to the standard H_∞ performance in [2] when $\tau = -1$ and $\varpi_1 = \varpi_2 = 0$.

On the basis of Definition 1, the aim of this paper is to propose a feasible approach to design the controller (5) such that the closed-loop system (6) with incomplete transition probabilities is stochastically stable and satisfies the finite frequency performance level γ.

To get the main results, a useful technical lemma is introduced before ending this section.

Lemma 1. *Finsler Lemma [37]. Letting $\upsilon \in R^n$, $\mathcal{P} = \mathcal{P}^T \in R^{n\times n}$, and $\mathcal{H} \in R^{m\times n}$ such that rank$(\mathcal{H}) = r < n$, then the following statements are equivalent:*

1) $\upsilon^T \mathcal{P} \upsilon < 0$, for all $\upsilon \neq 0$, $\mathcal{H}\upsilon = 0$;

2) $\mathcal{H}^{\perp T} \mathcal{P} \mathcal{H}^{\perp} < 0$;

3) $\exists \mathcal{X} \in R^{n\times m}$ such that $\mathcal{P} + He(\mathcal{X}\mathcal{H}) < 0$.

3. Main results

In this section, finite frequency performance analysis for the closed-loop system (6) is presented in Theorem 1. Based on Theorem 1, a new finite frequency synthesis method is established in Theorem 2 by means of linear matrix inequalities. Conditions for the closed-loop system (6) to be stochastically stable is given in Theorem 3.

Theorem 1. *Given a scalar γ, if there exist matrices $P_i > 0$, symmetric matrices M_i, F_i, and Q_i ($i \in \mathcal{I}, u \in \mathcal{I}^i_{uk}$) with approximate dimension such that*

$$\Phi_i = \begin{bmatrix} \Phi_i(1,1) & \Phi_i(1,2) & 0 & M_iB_{2i} \\ \star & \Phi_i(2,2) & \bar{C}_i^T & F_iB_{2i} \\ \star & \star & -I & D_{2i} \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{9}$$

$$P_u \le P_i \tag{10}$$

where

$$\Phi_i(1,1) = sym(-M_i) - Q_i$$
$$\Phi_i(1,2) = M_i\bar{A}_i - F_i^T + P_i + j\varpi_{12}Q_i \quad \left(\varpi_{12} = \frac{\varpi_1 + \varpi_2}{2}\right)$$
$$\Phi_i(2,2) = \begin{cases} sym(F_iA_i) + \sum_{l\in\mathcal{I}_k^i}^N \bar{\pi}_{il}P_l - \sum_{l\in\mathcal{I}_k^i}^N \underline{\pi}_{il}P_u - \varpi_1\varpi_2Q_i & (i\in\mathcal{I}_k^i) \\ sym(F_iA_i) + \sum_{l\in\mathcal{I}_k^i}^N \bar{\pi}_{il}P_l - \sum_{l\in\mathcal{I}_k^i}^N \underline{\pi}_{il}P_i - \varpi_1\varpi_2Q_i & (i\in\mathcal{I}_{uk}^i) \end{cases}$$

then the closed-loop system (6) with incomplete transition probabilities in the finite frequency domain (4) meets the required finite frequency performance index γ.

Proof. Choose a candidate stochastic Lyapunov function as

$$V(x,i) = x^T(t)P_ix(t)$$

where $P_i > 0$.

Calculating $\mathbb{E}\{\dot{V}(x,i)\}$ gives

$$\begin{aligned}\mathbb{E}\{\dot{V}(x,i)\} &= \dot{x}^T(t)P_ix(t) + x^T(t)P_i\dot{x}(t) + x^T(t)\sum_{l=1}^N \pi_{il}P_lx(t) \\ &= \zeta^T(t)\mathscr{A}_i^T\Phi_{i1}\mathscr{A}_i\zeta(t)\end{aligned} \tag{11}$$

where

$$\zeta(t) = \begin{bmatrix} x(t) \\ w(t)\end{bmatrix}, \mathscr{A}_i = \begin{bmatrix}\bar{A}_i & B_{2i} \\ I & 0\end{bmatrix}, \Phi_{i1} = \begin{bmatrix} 0 & P_i \\ \star & \sum_{l=1}^N \pi_{il}P_l\end{bmatrix}$$

On the other hand, let

$$\mathbb{E}\{\Upsilon(t,i)\} = \zeta^T(t)\mathscr{A}_i^T\Phi_{i2}\mathscr{A}_i\zeta(t) \tag{12}$$

where

$$\Phi_{i2} = \begin{bmatrix} -Q_i & j\varpi_{12}Q_i \\ -j\varpi_{12}Q_i & -\varpi_1\varpi_2Q_i\end{bmatrix}$$

Then we have

$$\begin{aligned}\mathbb{E}\{\Upsilon(t,i)\} = &-\dot{x}^T(t)Q_i\dot{x}(t) + j\varpi_{12}\dot{x}^T(t)Q_ix(t) \\ &- j\varpi_{12}x^T(t)Q_i\dot{x}(t) - \varpi_1\varpi_2x^T(t)Q_ix(t)\end{aligned} \tag{13}$$

Using the fact $\xi^TQ\varkappa = tr(\varkappa\xi^TQ)$ (ξ and $\varkappa$ are vectors), (13) is rewritten as

$$\mathbb{E}\{\Upsilon(t,i)\} = -tr[He((\varpi_1x(t) + j\dot{x}(t))(\varpi_2x(t) + j\dot{x}(t))^T)Q_i] \tag{14}$$

Integrating (14) from 0 to ∞ yields

$$\mathbb{E}\left\{\int_0^\infty \Upsilon(t,i)dt\right\} = -tr[He(S)Q_i] \tag{15}$$

where $S = \mathbb{E}\left\{\int_0^\infty(\varpi_1x(t) + j\dot{x}(t))(\varpi_2x(t) + j\dot{x}(t))^Tdt\right\}$

Resorting to the Parseval's theorem, the Fourier form of S is

$$S = \frac{1}{2\pi}\int_0^\infty(\varpi_1 - \varpi)(\varpi_2 - \varpi)\mathbb{E}\left\{\hat{x}\hat{x}^T\right\}d\varpi \tag{16}$$

where $\hat{x}$ is the Fourier transform of x.

According to (16), S is Hermitian and $-tr[He(S)Q_i] = -tr[SQ_i]$. Combining $\tau Q_i \geq 0$ and (8), it is not difficult to get $\mathbb{E}\{\int_0^\infty \Upsilon(t,i)dt\} \geq 0$.

Take into account the following index

$$\mathscr{T} = \mathbb{E}\left\{\int_0^\infty \left(z^T(t)z(t) - \gamma^2w^T(t)w(t)\right)dt\right\} \tag{17}$$

Adding $\mathbb{E}\{\dot{V}(t,i)\}$ and $\mathbb{E}\{\int_0^\infty \Upsilon(t,i)dt\}$ under zero initial states to (17) produces

$$\begin{aligned}\mathscr{T} &\le \mathbb{E}\left\{\int_0^\infty \left(z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t,i) + \Upsilon(t,i)\right)dt\right\} - \mathbb{E}\{V(\infty)\} \\ &\le \mathbb{E}\left\{\int_0^\infty \left(z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t,i) + \Upsilon(t,i)\right)dt\right\} \\ &= \int_0^\infty \zeta^T(t)\Pi_{i1}\zeta(t)dt \end{aligned} \tag{18}$$

where

$$\Pi_{i1} = \begin{bmatrix} \bar{A}_i & B_{2i} \\ I & 0 \\ \bar{C}_i & D_i \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -Q_i & P_i + j\varpi_{12}Q_i & 0 & 0 \\ \star & \sum_{l=1}^N \pi_{il}P_l - \varpi_1\varpi_2 Q_i & 0 & 0 \\ \star & \star & I & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{A}_i & B_{2i} \\ I & 0 \\ \bar{C}_i & D_i \\ 0 & I \end{bmatrix}$$

To ensure the required performance (7), one just needs to get $\Pi_{i1} < 0$. Alternatively, applying Lemma 1 to Π_{i1}, $\Pi_{i1} < 0$ is equivalent to the following inequality

$$\begin{bmatrix} -Q_i & P_i + j\varpi_{12}Q_i & 0 & 0 \\ \star & \sum_{l=1}^N \pi_{il}P_l - \varpi_1\varpi_2 Q_i & 0 & 0 \\ \star & \star & I & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} + He\left(\begin{bmatrix} M_i & 0 \\ F_i & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -I & 0 \\ \bar{A}_i^T & \bar{C}_i^T \\ 0 & -I \\ B_{2i}^T & D_{2i}^T \end{bmatrix}^T\right) < 0 \tag{19}$$

which is just

$$\mathcal{M}_i = \begin{bmatrix} sym(-M_i) - Q_i & M_i\bar{A}_i - F_i^T + P_i + j\varpi_{12}Q_i & 0 & M_iB_{2i} \\ \star & He(F_i\bar{A}_i) + \sum_{l=1}^N \pi_{il}P_l - \varpi_1\varpi_2 Q_i & \bar{C}_i^T & F_iB_{2i} \\ \star & \star & -I & D_{2i} \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0$$

Since $\sum_{l=1}^N \pi_{il} = \sum_{l\in\mathcal{I}_k^i}^N \pi_{il} + \sum_{u\in\mathcal{I}_{uk}^i}^N \pi_{iu}$, $\mathcal{M}_i$ is reformulated as

$$\mathcal{M}_i = M_k^i + \sum_{u\in\mathcal{I}_{uk}^i}^N \pi_{iu}M_{iu} \tag{20}$$

where

$$M_k^i = \begin{bmatrix} sym(-M_i) - Q_i & M_i\bar{A}_i - F_i^T + P_i + j\varpi_{12}Q_i & 0 & M_iB_{2i} \\ \star & He(F_i\bar{A}_i) + \sum_{l\in\mathcal{I}_k^i} \pi_{il}P_l - \varpi_1\varpi_2 Q_i & \bar{C}_i^T & F_iB_{2i} \\ \star & \star & -I & D_{2i} \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

$$M_u^i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \star & P_u & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix}$$

We first consider $i \in \mathcal{I}_k^i$. In this case, $\sum_{u\in\mathcal{I}_{uk}^i}^N \pi_{iu} = -\sum_{l\in\mathcal{I}_k^i}^N \pi_{il}$. If $\pi_{ii} = 0$, $\pi_{il} = 0$ $(l \in \mathcal{I})$, then $\mathcal{M}_i < 0$ is ensured from (9). Therefore, our attention is captured to $\pi_{ii} \neq 0$.

Using the fact $\frac{\sum_{u\in\mathcal{I}_{uk}^i}\pi_{iu}}{-\sum_{l\in\mathcal{I}_k^i}\pi_{il}} = 1$, (20) is rewritten as

$$\begin{aligned}\mathcal{M}_i &= \frac{\sum_{u\in\mathcal{I}_{uk}^i}\pi_{iu}}{-\sum_{l\in\mathcal{I}_k^i}\pi_{il}} M_k^i + \sum_{u\in\mathcal{I}_{uk}^i}\pi_{iu}M_{iu} \\ &= \frac{\sum_{l\in\mathcal{I}_{uk}^i}\pi_{il}}{-\sum_{l\in\mathcal{I}_k^i}\pi_{il}}\left\{M_k^i + \left(-\sum_{l\in\mathcal{I}_k^i}\pi_{il}\right)M_{il}\right\}\end{aligned} \tag{21}$$

According to (9), $\mathcal{M}_i < 0$ holds.

For $i \in \mathcal{I}_{uk}^i$, using the fact $\pi_{ii} = -\sum_{l\in\mathcal{I}_k^i} \pi_{il} - \sum_{u\neq i, u\in\mathcal{I}_{uk}^i} \pi_{iu}$ to (20) gives

$$\mathcal{M}_i = M_k^i + \sum_{l\neq i, l\in\mathcal{I}_{uk}^i} \pi_{iu}\{M_{iu} - M_{ii}\} \tag{22}$$

Referring to (9) and (10) for $i \in \mathcal{I}_{uk}^i$, we get $\mathcal{M}_i < 0$.

Summarizing (21) and (22), conditions (9) and (10) could ensure $\Pi_{i1} < 0$. Therefore, the closed-loop system meets the required finite frequency performance index γ. □

Remark 4. When considering the full frequency domain and known transition probabilities, just put $Q_i = 0$ in Theorem 1. Under this situation, the analysis condition is same as that of [3]. Moreover, if $N = 1$, the obtained bounded real lemma is reduced to [34].

Once transition probabilities are known or uncertain, conditions given in Theorem 1 are reduced to the following corollary.

Corollary 1. *Given a scalar γ, if there exist matrices $P_i > 0$, M_i, F_i, and $Q_i = Q_i^T$ $(i \in \mathcal{I})$ with approximate dimension such that*

$$\bar{\bar{\Phi}}_i = \begin{bmatrix} \bar{\bar{\Phi}}_i(1,1) & \bar{\bar{\Phi}}_i(1,2) & 0 & M_iB_{2i} \\ \star & \bar{\bar{\Phi}}_i(2,2) & \bar{C}_i^T & F_iB_{2i} \\ \star & \star & -I & D_i \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{23}$$

where

$$\bar{\bar{\Phi}}_i(1,1) = sym(-M_i) - Q_i$$
$$\bar{\bar{\Phi}}_i(1,2) = M_i\bar{A}_i - F_i^T + P_i + j\varpi_{12}Q_i$$
$$\bar{\bar{\Phi}}_i(2,2) = sym(F_i\bar{A}_i) + \sum_{l\neq i, l\in\mathcal{I}_k^i} \bar{\pi}_{il}P_l - \sum_{l\neq i, l\in\mathcal{I}_k^i} \underline{\pi}_{il}P_i - \varpi_1\varpi_2Q_i$$

then the closed-loop system (6) meets the required finite frequency performance index γ.

Noting that there are two slack variables M_i and F_i in Theorem 1, to get the controller gain K_i, under $M_i = \mu_iF_i$ (μ_i is a constant), a routine method is to pre- and post-multiply an inverse matrix [34]. Instead of the routine pre- and post-multiplying, a new method based on Lemma 1 is proposed in the following Theorem.

Theorem 2. *Given scalars τ_{1i}, τ_{2i}, τ_{3i}, τ_{4i} and γ $(\gamma > 0)$, if there exist matrices $P_i > 0$, M_i, F_i, W_i, N_i and Q_i $(i \in \mathcal{I}, u \in \mathcal{I}_{uk}^i)$ with approximate dimension such that*

$$\bar{\Phi}_i = \begin{bmatrix} \bar{\Phi}_i(1,1) & \bar{\Phi}_i(1,2) & 0 & M_iB_{2i} & \bar{\Phi}_i(1,5) \\ \star & \bar{\Phi}_i(2,2) & \bar{\Phi}_i(2,3) & F_iB_{2i} & \bar{\Phi}_i(2,5) \\ \star & \star & -I & D_{2i} & \bar{\Phi}_i(3,5) \\ \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & \bar{\Phi}_i(5,5) \end{bmatrix} < 0 \tag{24}$$

$$P_u \leq P_i \tag{25}$$

where

$$\bar{\Phi}_i(1,1) = sym(-M_i) - Q_i,\ \bar{\Phi}_i(1,2) = M_iA_i + \tau_{1i}B_{1i}N_i - F_i^T + P_i + j\varpi_{12}Q_i$$
$$\bar{\Phi}_i(1,5) = M_iB_{1i} - \tau_{1i}B_{1i}W_i,\ \bar{\Phi}_i(2,5) = F_iB_{1i} - \tau_{2i}B_{1i}W_i + \tau_{4i}N_i^T$$
$$\bar{\Phi}_i(2,2) = \begin{cases} sym(F_iA_i + \tau_{2i}B_{1i}N_i) + \sum_{l\in\mathcal{I}_k^i} \bar{\pi}_{il}P_l - \sum_{l\in\mathcal{I}_k^i} \underline{\pi}_{il}P_u - \varpi_1\varpi_2Q_i & (i \in \mathcal{I}_k^i) \\ sym(F_iA_i + \tau_{2i}B_{1i}N_i) + \sum_{l\in\mathcal{I}_k^i} \bar{\pi}_{il}P_l - \sum_{l\in\mathcal{I}_k^i} \underline{\pi}_{il}P_i - \varpi_1\varpi_2Q_i & (i \in \mathcal{I}_{uk}^i) \end{cases}$$
$$\bar{\Phi}_i(2,3) = (C_{1i} + \tau_{3i}D_{1i}N_i)^T,\ \bar{\Phi}_i(4,5) = D_{1i} - \tau_{3i}D_{1i}W_i,\ \bar{\Phi}_i(5,5) = He(-\tau_{4i}W_i)$$

then the closed-loop system (6) with incomplete transition probabilities meets the prescribed finite frequency performance level γ and $K_i = N_iW_i^{-1}$.

Proof. Recalling $\bar{A}_i = A_i + B_{1i}K_i$ and $\bar{C}_i = C_{1i} + D_{1i}K_i$ yields

$$\Phi_i^c = \begin{bmatrix} \Phi_i(1,1) & \Phi_i^c(1,2) & 0 & M_iB_{2i} \\ \star & \Phi_i^c(2,2) & C_i^{cT} & F_iB_{2i} \\ \star & \star & -I & D_i \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0 \tag{26}$$

where

$$\Phi_i^c(1,2) = M_i(A_i + B_{1i}K_i) - F_i^T + P_i + j\varpi_{12}Q_i, C_i^c = C_i + D_iK_i$$

$$\Phi_i^c(2,2) = \begin{cases} sym(F_iA_i + B_{1i}K_i) + \sum\limits_{l\in\mathcal{I}_k^i}^{N} \bar{\pi}_{il}P_l - \sum\limits_{l\in\mathcal{I}_k^i}^{N} \underline{\pi}_{il}P_u - \varpi_1\varpi_2Q_i & (i \in \mathcal{I}_k^i) \\ sym(F_iA_i + B_{1i}K_i) + \sum\limits_{j\in\mathcal{I}_k^i}^{N} \bar{\pi}_{ij}P_j - \sum\limits_{j\in\mathcal{I}_k^i}^{N} \underline{\pi}_{ij}P_i - \varpi_1\varpi_2Q_i & (i \in \mathcal{I}_{uk}^i) \end{cases}$$

On the other hand, (26) is rewritten as

$$\underbrace{\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ K_i & 0 & 0 & 0 \end{bmatrix}^T}_{\mathcal{H}^{\perp T}} \underbrace{\begin{bmatrix} \Phi_i(1,1) & \Phi_i^c(1,2) & 0 & M_iB_{2i} & 0 \\ \star & \Phi_i^c(2,2) & C_i^{cT} & F_iB_{2i} & 0 \\ \star & \star & -I & D_i & 0 \\ \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix}}_{\mathcal{P}} \underbrace{\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ K_i & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{H}^{\perp}} < 0 \tag{27}$$

Adopting Lemma 1 to (27) yields

$$\underbrace{\begin{bmatrix} \Phi_i(1,1) & \Phi_i^c(1,2) & 0 & M_iB_{2i} & 0 \\ \star & \Phi_i^c(2,2) & C_i^{cT} & F_iB_{2i} & 0 \\ \star & \star & -I & D_i & 0 \\ \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix}}_{\mathcal{P}} + He\left(\underbrace{\begin{bmatrix} \tau_{1i}B_iW_i - G_iB_i \\ \tau_{2i}B_iW_i - M_iB_i \\ \tau_{3i}D_{1i}W_i - D_{1i} \\ 0 \\ \tau_{4i}W_i \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} K_i^T \\ 0 \\ 0 \\ 0 \\ -I \end{bmatrix}^T}_{\mathcal{H}} \right) < 0 \tag{28}$$

Let $N_i = W_iK_i$, then (28) is equal to (24). □

Remark 5. Motivated by the structure of $K_i = W_i^{-1}N_i$, a constructive strategy is develop to handle $P_iB_iK_i$ as $P_iB_iK_i = P_iB_iW_i^{-1}N_i = (P_iB_i - B_iW_i)W_i^{-1}N_i + B_iN_i$. With this separated structure and Lemma 1 (Finsler lemma), there is no need to pre- and post-multiply an inverse matrix to get controller gain K_i. Additionally, three sets of slack variables without specific structures are introduced.

Although the finite frequency performance could be guaranteed by Theorem 2, conditions for the stochastic stability could be further developed. Along with the procedures in Theorem 1 and 2, the corresponding conditions for stochastically stable is given in Theorem 3.

Theorem 3. *Given scalars τ_{1i}^s, τ_{2i}^s, and τ_{3i}^s, if there exist matrices $P_i^s > 0$, M_i^s, F_i^s, W_i, and N_i $(i \in \mathcal{I}, u \in \mathcal{I}_{uk}^i)$ with approximate dimension such that*

$$\bar{\Phi}_i^s = \begin{bmatrix} \Phi_i^s(1,1) & \bar{\Phi}_i^s(1,2) & \bar{\Phi}_i^s(1,3) \\ \star & \bar{\Phi}_i^s(2,2) & \bar{\Phi}_i(2,3) \\ \star & \star & \bar{\Phi}_i^s(3,3) \end{bmatrix} < 0 \tag{29}$$

$$P_u^s \leq P_i^s \tag{30}$$

where

$$\Phi_i^s(1,1) = sym(-M_i^s), \bar{\Phi}_i^s(1,2) = M_i^sA_i + \tau_{1i}B_{1i}N_i - (F_i^s)^T + P_i^s,$$
$$\bar{\Phi}_i(1,3) = M_i^sB_{1i} - \tau_{1i}B_{1i}W_i, \bar{\Phi}_i(2,3) = F_i^sB_{1i} - \tau_{2i}B_{1i}W_i + \tau_{3i}^sN_i^T,$$

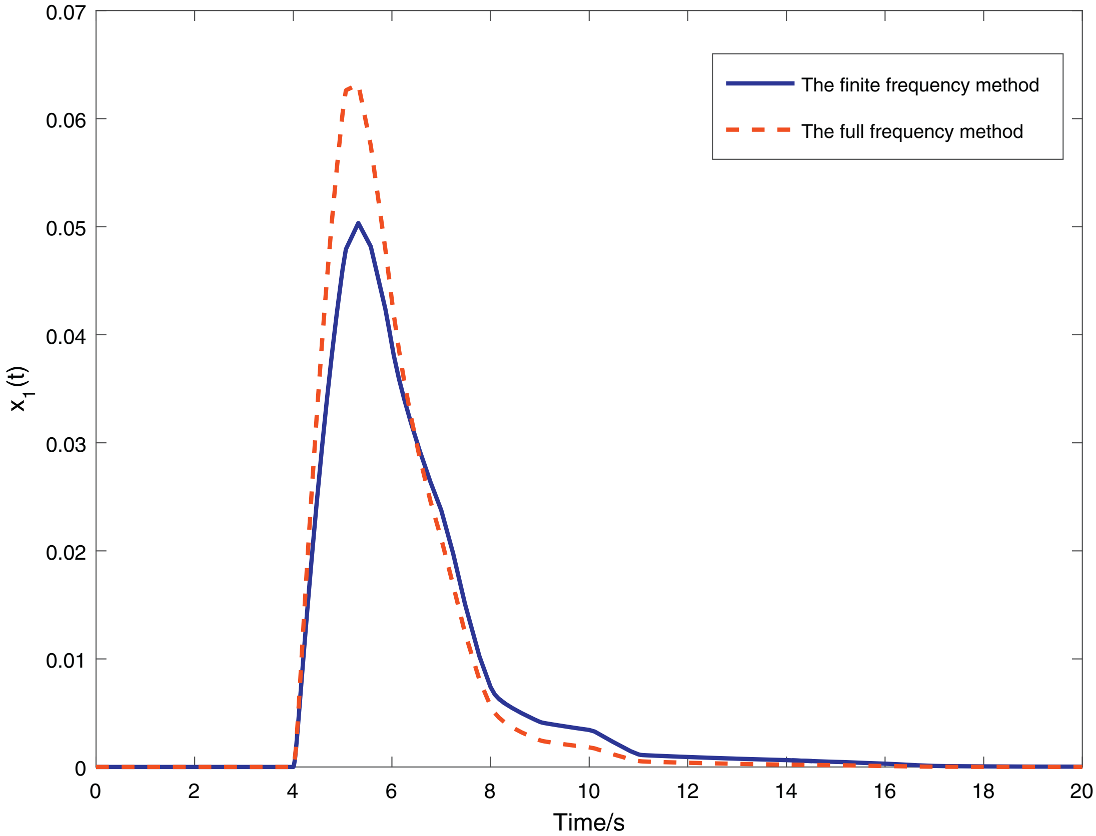

Fig. 1. The comparison of the state responses of $x_1(t)$ for $|\varpi| \leq 1$(blue solid) and $\varpi \in (-\infty, +\infty)$ (red dot). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\bar{\Phi}_i(2,2) = \begin{cases} sym(F_i^s A_i + \tau_{2i} B_{1i} N_i) + \sum\limits_{l \in \mathcal{I}_k^i} \bar{\pi}_{il} P_l - \sum\limits_{l \in \mathcal{I}_k^i} \underline{\pi}_{il} P_u & (i \in \mathcal{I}_k^i) \\ sym(F_i^s A_i + \tau_{2i} B_{1i} N_i) + \sum\limits_{l \in \mathcal{I}_k^i} \bar{\pi}_{il} P_l - \sum\limits_{l \in \mathcal{I}_k^i} \underline{\pi}_{il} P_i & (i \in \mathcal{I}_{uk}^i) \end{cases}$$

$$\bar{\Phi}_i^s(3,3) = He(-\tau_{3i}^s W_i)$$

then the closed-loop system (6) with incomplete transition probabilities is stochastically stable. Moreover, the controller gain is $K_i = W_i^{-1} N_i$.

Remark 6. Combining conditions given in Theorems 2 and 3, the closed-loop system (6) is stochastically stable with the prescribed finite frequency performance index γ.

Remark 7. For given scalars τ_{1i}, τ_{2i}, τ_{3i}, τ_{4i}, τ_{1i}^s, τ_{2i}^s and τ_{3i}^s, Conditions given in Theorems 2 and 3 are in terms of linear matrix inequalities. In this case, convex optimization algorithm is adopted to minimize γ as

$$\min_{(24),(25),(29),(30)} \beta$$

where $\beta = \gamma^2$. Otherwise, conditions in these theorems are nonconvex. Fortunately, one may resort to a line search algorithm proposed in [35].

4. Numerical example

In this section, a single-link robot arm system borrowed from [9,36] is given to show the effectiveness of the proposed method. The dynamic equation of the robot arm is depicted by

$$d^2\theta(t) = \left[-\frac{MgL}{J} sin(\theta(t)) - \frac{D(t)}{J}\dot{\theta}(t) + \frac{1}{J}u(t) + \frac{L}{J}w(t) \right] \tag{31}$$

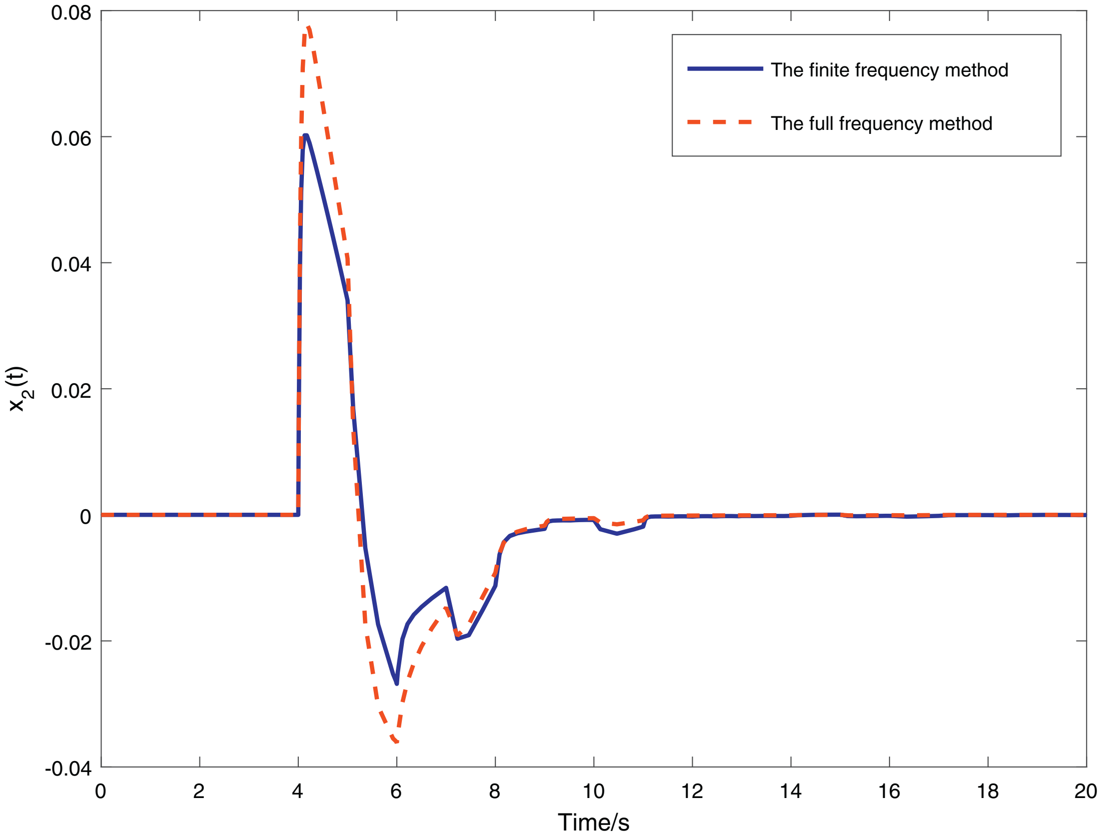

Fig. 2. The comparison of the state responses of $x_2(t)$ for $|\varpi| \le 1$(blue solid) and $\varpi \in (-\infty, +\infty)$ (red dot). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
The comparison of γ in different frequency intervals.

	$\lvert\varpi\rvert \le 1$	$\varpi \in (-\infty, \ +\infty)$
γ	0.7486	0.9459

where $\theta(t)$ is the arm's angle position, $u(t)$ is the control input, $w(t)$ is the external disturbance. M is the mass of the payload, J is the inertia moment, g is the acceleration gravity, L is the arm length, and $D(t)$ is the uncertain coefficient of viscous friction. The parameters of g, L and $D(t)$ are 9.81, 2 and 0.5, respectively. The parameters M and J have four different modes. In what follows, like [36], a linearized system model for (31) is given by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -gl & -\frac{2}{J(r(t))} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{J(r(t))} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \frac{1}{J(r(t))} \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{cases} \tag{32}$$

where $x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}$, $r(t) = \{1, 2, 3, 4\}$, $J(r(t))$ depends on the jump mode $r(t)$ and $J(1) = 0.1$, $J(2) = 0.25$, $J(3) = 0.5$, $J(4) = 0.8$.

The incomplete transition probability matrix is

$$\begin{bmatrix} -1.3 & 0.2 & ? & ? \\ ? & ? & 0.3 & 0.3 \\ 0.6 & ? & -1.5 & ? \\ ? & ? & ? & -1.2 \end{bmatrix} \tag{33}$$

Given $\tau_{1i} = \tau_{2i} = \tau_{3i} = \tau_{4i} = \tau_{1i}^s = \tau_{2i}^s = \tau_{3i}^s = 1$, set $\varpi_1 = -\varpi_2 = 1$ and $\tau = 1$. Namely, the finite frequency interval is $|\varpi| \le 1$. Additionally, the full frequency case ($\varpi \in (-\infty, +\infty)$) is also considered. By solving conditions proposed in Theorems 2 and 3, comparisons of γ for $|\varpi| \le 1$ and $\varpi \in (-\infty, +\infty)$ are given in Table 1.

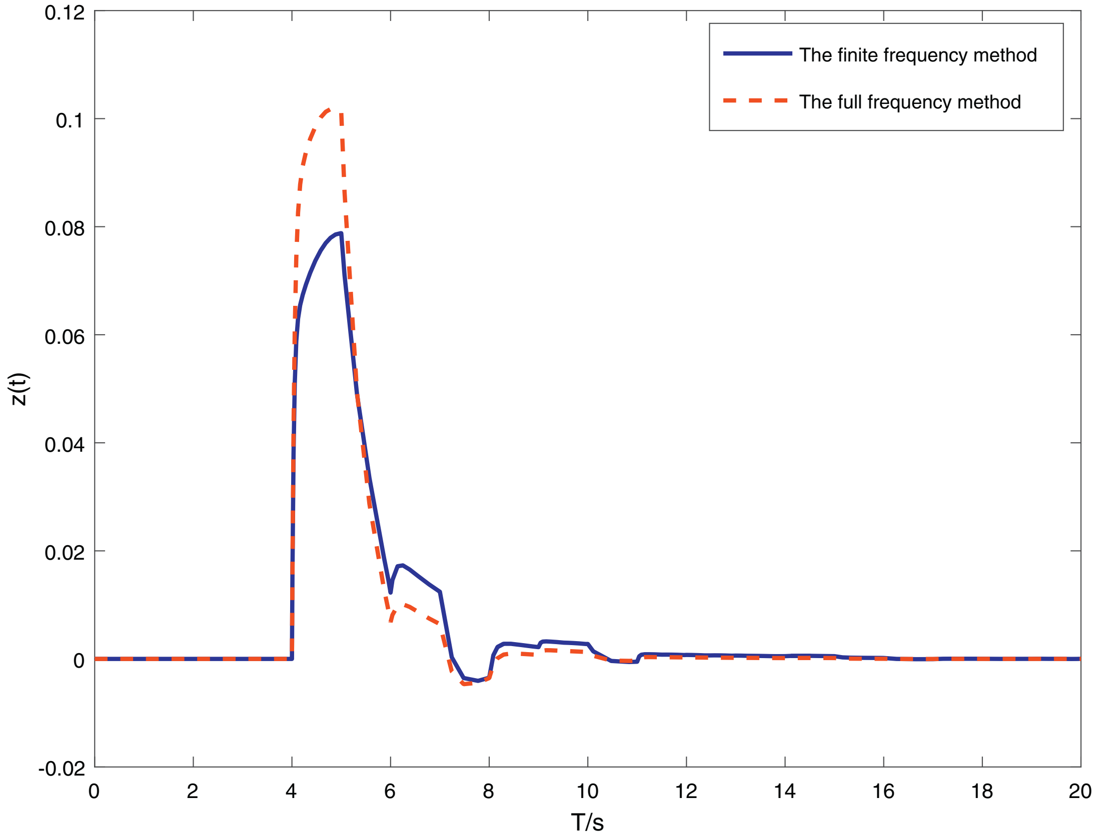

Fig. 3. The comparison of the measured output responses of $z(t)$ for $|\varpi| \le 1$(blue solid) and $\varpi \in (-\infty, +\infty)$ (red dot). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In addition, controller gains for $|\varpi| \le 1$ and $\varpi \in (-\infty, \ +\infty)$ are obtained respectively as

$$K_1 = [-0.0652 \quad -0.9748], K_2 = [-0.1227 \quad -0.9447],$$

$$K_3 = [-0.1459 \quad -0.4134], K_4 = [-0.1463 \quad -1.3424].$$

and

$$K_1 = [-0.1271 \quad -0.1474], K_2 = [-0.2430 \quad -0.2851],$$

$$K_3 = [-0.2650 \quad -0.3859], K_4 = [-0.3471 \quad -0.1302].$$

According to Table 1, it is demonstrated that the finite frequency performance index γ is smaller than its in full frequency. Consequently, a controller design method without the available frequency information could be conservative.

Under zero initial conditions and $w(t) = 0.2sin(0.5t)$ for $4 \le t \le 6$ (otherwise, $w(t) = 0$), simulation curves are obtained in Figs. 1, 2, and 3, respectively. In particular, Fig. 1 shows the comparison of the state responses of $x_1(t)$ for finite frequency and full frequency. The state response curves of $x_2(t)$ for two cases are compared in Fig. 2. The curves of the measured output $z(t)$ for two cases are drawn in Fig. 3.

As seen from these figures, simulation curves obtained in finite frequency interval are better than those in full frequency. Therefore, the frequency of disturbance should be integrated into the course of controller design.

5. Conclusions

A finite frequency approach to control of Makrov jump linear systems with incomplete transition probabilities is investigated in this paper. Based on the extended finite frequency performance definition, an effective controller design method is proposed by means of linear matrix inequalities. A robot arm system is given to show the effectiveness of the proposed method. How to reduce the possible conservativeness will be developed further in future.

Acknowledgments

The authors would like to thank the editor, the associate editor and the reviewers for their valuable comments and suggestions which help to significantly improve the quality and presentation of this paper. This work was supported in part by the National Natural Science Foundation of China under Grant 61403189, 61273155, 61322312, in part by the Natural Science Foundation of Jiangsu Province of China under Grant BK20130949, in part by the Doctoral Foundation of Ministry of Education of China under Grant 20133221120012, in part by the Jiangsu Postdoctoral Science Foundation under Grant1401015B, in part by the China Postdoctoral Science Foundation under Grant 2015M570397, in part by the peak of six talents in Jiangsu Province under Grant 2015XXRJ-011, in part by the Key Laboratory Open Foundation under grant 2015ACOCP01, in part by the Jiangsu Government Scholarship for Overseas Studies JS-2014-046, in part by the Fok Ying Tung Education Foundation under Grant 141060, the Fundamental Research Funds for the Central Universities under Grant N140405006.

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