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Research paper

Complexity dynamics and Hopf bifurcation analysis based on the first Lyapunov coefficient about 3D IS-LM macroeconomics system

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ABSTRACT

Based on the study of scholars at home and abroad, this paper improves the three-dimensional IS-LM model in macroeconomics, analyzes the equilibrium point of the system and stability conditions, focuses on the parameters and complex dynamic characteristics when Hopf bifurcation occurs in the three-dimensional IS-LM macroeconomics system. In order to analyze the stability of limit cycles when Hopf bifurcation occurs, this paper further introduces the first Lyapunov coefficient to judge the limit cycles, i.e. from a practical view of the business cycle. Numerical simulation results show that within the range of most of the parameters, the limit cycle of 3D IS-LM macroeconomics is stable, that is, the business cycle is stable; with the increase of the parameters, limit cycles becomes unstable, and the value range of the parameters in this situation is small. The research results of this paper have good guide significance for the analysis of macroeconomics system.

1. Introduction

In recent years, nonlinear dynamics and chaos theory are more and more frequent combined with economic theory, and construct some nonlinear economic model, which have well reflection on the variation characteristics of economic variables, reveal the internal evolution trend and the complex dynamical characteristics of the system. The most common models among them are Van der Pol [1] model, Kaldorian model [2] and IS-LM model [3] et al, which have achieved good results.

In chaos theory, due to limit cycle of Hopf bifurcation can effectively simulate the "business cycle" phenomenon in macroeconomics, i.e. “recovery, prosperity, recession, depression” business cycle. Therefore, Hopf bifurcation is studied by scholars within a long time. Q. Gao et al [4] had an analysis on a finance system, and described its complex dynamics. Y. Chen [5] paid attention on food chain system, and analyzed the stability and Hopf bifurcation of a delayed three-level food chain system, which obtained good result. J. Ma et al [6] studied a kind of economic system, focused on Hopf bifurcation and its complexity, and obtained conclusions of practical significance.

But it is difficult to judge whether the limit cycle of Hopf bifurcation is stable, that is, the stability of the business cycle of macroeconomics. In order to analyze the stability of the limit cycle, we need to discuss it by means of first Lyapunov coefficient. At present, the research of the first Lyapunov coefficient is mainly concentrated in the field of mathematics, less involved in the field of economic & management, most of these studies focused on discrete system. Junhai Ma et al [7] had complexity research of a three parameters’ economic system, and described the complex dynamic characteristics of

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the system in detail. G.A. Leonov [8,9] had a detailed analysis on time-varying linearization and the perron effects, which obtained good results.

Based on the research results of scholars, this paper introduces 3D IS-LM macroeconomics model, solve the equilibrium of product market and money market, analyze the Hopf bifurcation of the system, and describe the business cycle formed by the market. The emphasis is on the first Lyapunov coefficient of the system to analyze the limit cycle of the Hopf bifurcation, the stability of the business cycle.

2. The establishment of model

Based on the previous work, we construct the corresponding 3D IS-LM macroeconomics model [10-12]:

$$\begin{cases} \dot{y}(t) = \alpha[I(y(t), r(t)) + g - S(y^D(t)) - T(t)] \\ \dot{r}(t) = \beta[L(y(t), r(t)) - m(t)] \\ \dot{m}(t) = g - T(t) \end{cases} \tag{2.1}$$

The information contained three variables is the key structure of the feedback loop: $\dot{y}(t) = dy/dt, \dot{r}(t) = dr/dt$ and $\dot{m}(t) = dm/dt$ respectively. Among them, $y(t)$ is the national income; $I(Y(t), R(t))$ is investment; g is government porcument; $S(y^D(t))$ is savings, related to disposable income $y^D(t)$; $T(t)$ is tax revenue; $r(t)$ is interest rate; $L(Y(t), R(t))$ is actual demand for money; $m(t)$ is nominal supply of money, α, β are positive parameters, indicate the change rate of the national income and interest rate.

For relevant economic variables in system (2.1):

Investment:

$$I(Y(t), R(t)) = a\frac{y(t)}{r(t)}$$

Actual demand for money:

$$L(Y(t), R(t)) = mY(t) + \frac{l_1}{r(t) - l2}$$

Where, l_1, $l_2 > 0$, l_2 is a fixed value, when l_2 is too large, the macroeconomics is likely to have a liquidity trap, at this time, the market is not sensitive to interest rate, and will lead to macroeconomics policy failure.

Savings:

$$S(y^D(t)) = s(1-\varepsilon)y(t), 0 < s < 1.$$

Tax revenue:

$$T(t) = \varepsilon y(t).$$

Through investment function, actual demand for money function, savings function and tax revenue functions above, the system (2.1) is improved, and the IS-LM macroeconomics can be expressed as follows:

$$\begin{cases} \dot{y}(t) = \alpha\left[a\frac{y(t)}{r(t)} + g - s(1-\varepsilon)y(t) - \varepsilon y(t)\right] \\ \dot{r}(t) = \beta\left[my(t) + \frac{l_1}{r(t) - l2} - m(t)\right] \\ \dot{m}(t) = g - \varepsilon y(t) \end{cases} \tag{2.2}$$

3. System model analysis

3.1. Stability analysis of equilibrium point

When 3D IS-LM macroeconomics evolves to equilibrium state, variables should in $\dot{y}(t) = 0, \dot{r}(t) = 0, \dot{m}(t) = 0$, that is, the national income, interest rate and money supply will reach a stable state, and have no change, at this time we can get the equation as follows:

$$\begin{cases} \alpha\left[a\frac{y(t)}{r(t)} + g - s(1-\varepsilon)y(t) - \varepsilon y(t)\right] = 0 \\ \beta\left[my(t) + \frac{l_1}{r(t) - l2} - m(t)\right] = 0 \\ g - \varepsilon y(t) = 0 \end{cases} \tag{3.1}$$

Solve the equilibrium point of the Eq. (3.1), we know that the system have only one equilibrium solution. $E_0=(y_0,\ r_0,\ m_0)$:

$$y_0=\frac{g}{\varepsilon},\ r_0=\frac{a}{s(1-\varepsilon)},$$
$$m_0=\frac{\varepsilon gmsl_2-\varepsilon^2 sl_1-gmsl_2+agm+\varepsilon sl_1}{\varepsilon(\varepsilon sl_2-sl_2+a)}$$

We can easily know that, $E_0=(y_0,\ r_0,\ m_0)$ is the Nash equilibrium solution of the system.

Since the analysis object is the actual macroeconomics, we only consider the positive equilibrium point of the system; at this time, the equilibrium point should meet the conditions:

$(C_1)\ a>sl_2(1-\varepsilon)$ and $\varepsilon s(gml_2+l_1)+agm>s(\varepsilon^2 l_1+gml_2)$, or

$(C_2)\ a<sl_2(1-\varepsilon)$ and $\varepsilon s(gml_2+l_1)+agm<s(\varepsilon^2 l_1+gml_2)$

Have the linear translation to the system (3.1); transfer the equilibrium point of the system to the origin. As shown below:

$$Y(t)=y(t)-\frac{g}{\varepsilon};\ R(t)=r(t)-\frac{a}{s(1-\varepsilon)};$$
$$M(t)=m(t)-\frac{\varepsilon gmsl_2-\varepsilon^2 sl_1-gmsl_2+agm+\varepsilon sl_1}{\varepsilon(\varepsilon sl_2-sl_2+a)}$$

At this time, the system (3.1) has transformed into:

$$\begin{cases}\dot{Y}(t)=\dfrac{\alpha}{\varepsilon(R\varepsilon s-Rs-a)}(RY\varepsilon^3 s^2-RY\varepsilon^3 s-2RY\varepsilon^2 s^2+\\ \quad R\varepsilon^2 gs^2+RY\varepsilon^2 s+RY\varepsilon s^2-2R\varepsilon gs^2+Rgs^2+Ya\varepsilon^2)\\ \dot{R}(t)=\dfrac{\beta}{(R\varepsilon s-\varepsilon sl_2-Rs+sl_2-a)(\varepsilon sl_2-sl_2+a)}\cdot\\ \quad(RY\varepsilon^2 ms^2 l_2-Y\varepsilon^2 ms^2 {l_2}^2-2RY\varepsilon ms^2 l_2-RM\varepsilon^2 s^2 l_2\\ \quad+2Y\varepsilon ms^2 {l_2}^2+M\varepsilon^2 s^2 {l_2}^2+RYa\varepsilon ms+RYms^2 l_2\\ \quad+2RM\varepsilon s^2 l_2+R\varepsilon^2 s^2 l_1-2Ya\varepsilon msl_2-Yms^2 {l_2}^2\\ \quad-2M\varepsilon s^2 {l_2}^2-RYams-RMa\varepsilon s-RMs^2 l_2-2R\varepsilon s^2 l_1\\ \quad+2Yamsl_2+2Ma\varepsilon sl_2+Ms^2 {l_2}^2+RMas+Rs^2 l_1\\ \quad-Ya^2 m-2Masl_2+Ma^2)\\ \dot{M}(t)=-Y\varepsilon\end{cases} \tag{3.2}$$

The stability analysis of the system at the equilibrium point becomes the stability of the system (3.2) at the origin.

Compute the Jacobi matrix J of the system (3.2) at the origin

$$J=\begin{bmatrix}-\alpha\varepsilon & -\dfrac{\alpha gs^2(-1+\varepsilon)^2}{a\varepsilon} & 0\\ \beta m & -\dfrac{\beta l_1 s^2(-1+\varepsilon)^2}{(\varepsilon sl_2-sl_2+a)^2} & -\beta\\ -\varepsilon & 0 & 0\end{bmatrix} \tag{3.3}$$

The characteristic equation of system (3.2) is:

$$P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3$$

Where:

$$a_1=\alpha\varepsilon+\frac{\beta l_1 s^2(\varepsilon-1)^2}{(\varepsilon sl_2-sl_2+a)^2}$$
$$a_2=\frac{\alpha\beta l_1\varepsilon s^2(\varepsilon-1)^2}{(\varepsilon sl_2-sl_2+a)^2}+\frac{\alpha\beta gms^2(\varepsilon-1)^2}{a\varepsilon}$$
$$a_3=\frac{\alpha\beta gs^2(\varepsilon-1)^2}{a}$$

Therefore, we can judge the state of the parameters when the Hopf bifurcation occurs according to the conditions of Hopf bifurcation. The characteristic equation of system (3.2) $P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3$ exist a pair of pure imaginary eigenvalues, the rest of the eigenvalues is less than 0, and the cross section condition is not equal to 0.

3.2. Numerical simulation of Hopf bifurcation

In order to reflect the dynamic evolution behavior and the complex dynamics characteristics of the system (3.2), we make a numerical simulation to the system.

According to the relevant research and the actual data, we set the parameters for:

$$\begin{aligned} &a = 0.37, \alpha = 0.95, l_1 = 1, l_2 = 0.0075, \\ &m = 0.0035, g = 30, s = 0.25, \varepsilon = 0.01 \end{aligned} \tag{3.4}$$

And the initial point of the system can be obtained:

$$E_0 = (y_0 = 3000, r_0 = 1.494949495, m_0 = 11.17229173)$$

Analyze the condition of the parameter β when the Hopf bifurcation occurs, and the characteristic equation of the system (3.2) is:

$$P(\lambda) = \lambda^3 + (0.452\beta + 0.0095)\lambda^2 + (0.0043\beta + 1.6514)\lambda + 4.7181\beta$$

At this time, we get $\beta = 6.385$ when the Hopf bifurcation occurs, and the system (3.2) produces a Hopf bifurcation state, as shown in Fig. 1.

From Fig. 1, we can visually verify the correctness of the theory, when $\beta = 6.385$, Hopf bifurcation occurs, the limit cycle of Hopf bifurcation is generated, as shown in Fig 1(b). At this time, the macroeconomics form a periodic solution under the corresponding economic parameters, macroeconomics operates in the mode of "business cycle".

Analyze the situation of macroeconomics when it passes through Hopf bifurcation point, when $\beta = 5.8 < 6.385$, the system is in divergence state, as shown in Fig 1(a). The trajectories of system gradually spread, the waveform of the system losing stability, and the range expand. The macroeconomics is unstable, national income, interest rates, currency nominal supply quantity will be shock; macroeconomics is in disorder. At this time, macroeconomics regulation and control should be carried out by the government or the central bank, to ensure the stability of the system. When $\beta = 7 > 6.385$, the system is in a convergence state, as shown in Fig 1(c), the trajectories of system gradually shrink, the waveform display that the system gradually enter into stable state. At this time, macroeconomics stability, the corresponding economic variables are stable and controllable, the national economic stability and develop healthy.

When β gradually passes through $\beta = 6.385$, Hopf bifurcation occurs, and the stability of the system (3.2) changes, the system gradually shifted from instability to the convergence state.

4. First Lyapunov coefficient analysis of IS-LM system

According to the chaos and bifurcation theory, we know that when a Hopf bifurcation occurs, the stability of bifurcation point is uncertain, the limit cycle may be stable, may be unstable. If the limit cycle of Hopf bifurcation is unstable, it shows that the same way in business cycle and the operating state of the economic system is not controllable [13-15]. Therefore, in the process of the Hopf bifurcation analysis of IS-LM macroeconomics, it is necessary to study the stability of Hopf bifurcation [16].

In this paper, we judge the stability of Hopf bifurcation by calculating the first Lyapunov coefficient at the bifurcation point [17-19]:

4.1. First Lyapunov coefficient and its solution

For the linear translation system (3.2), we can express it as:

$$\dot{x} = Ax + F(x), \ x \in \mathbb{R}^n$$

At the origin, the Taylor expansion of this differential system is:

$$F(x, 0) = \frac{1}{2}B(x, y) + \frac{1}{6}C(x, y, z) + O(||x||^4)$$

$F(x) = O(\|x\|^2)$ is a smooth function, $B(x, y)$ and $C(x, y, z)$ are multiple linear function, expressed as follows:

$$\begin{cases} B_i(x, y) = \sum\limits_{j,k=1}^{n} \left. \dfrac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k} \right|_{\xi=0} x_j y_k \\ C_i(x, y, z) = \sum\limits_{j,k,l=1}^{n} \left. \dfrac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\xi=0} x_j y_k z_l \end{cases}, i = 1, 2, \cdots, n$$

Suppose: The matrix A, A^T has a pair of pure imaginary eigenvalues $\pm i\omega$, we can know that ω meet the following conditions:

$$trace(A) = i\omega - i\omega + \frac{|A|}{\omega^2} = -\alpha\varepsilon - \frac{\beta l_1 s^2(-1+\varepsilon)^2}{(\varepsilon s l_2 - s l_2 + a)^2}$$

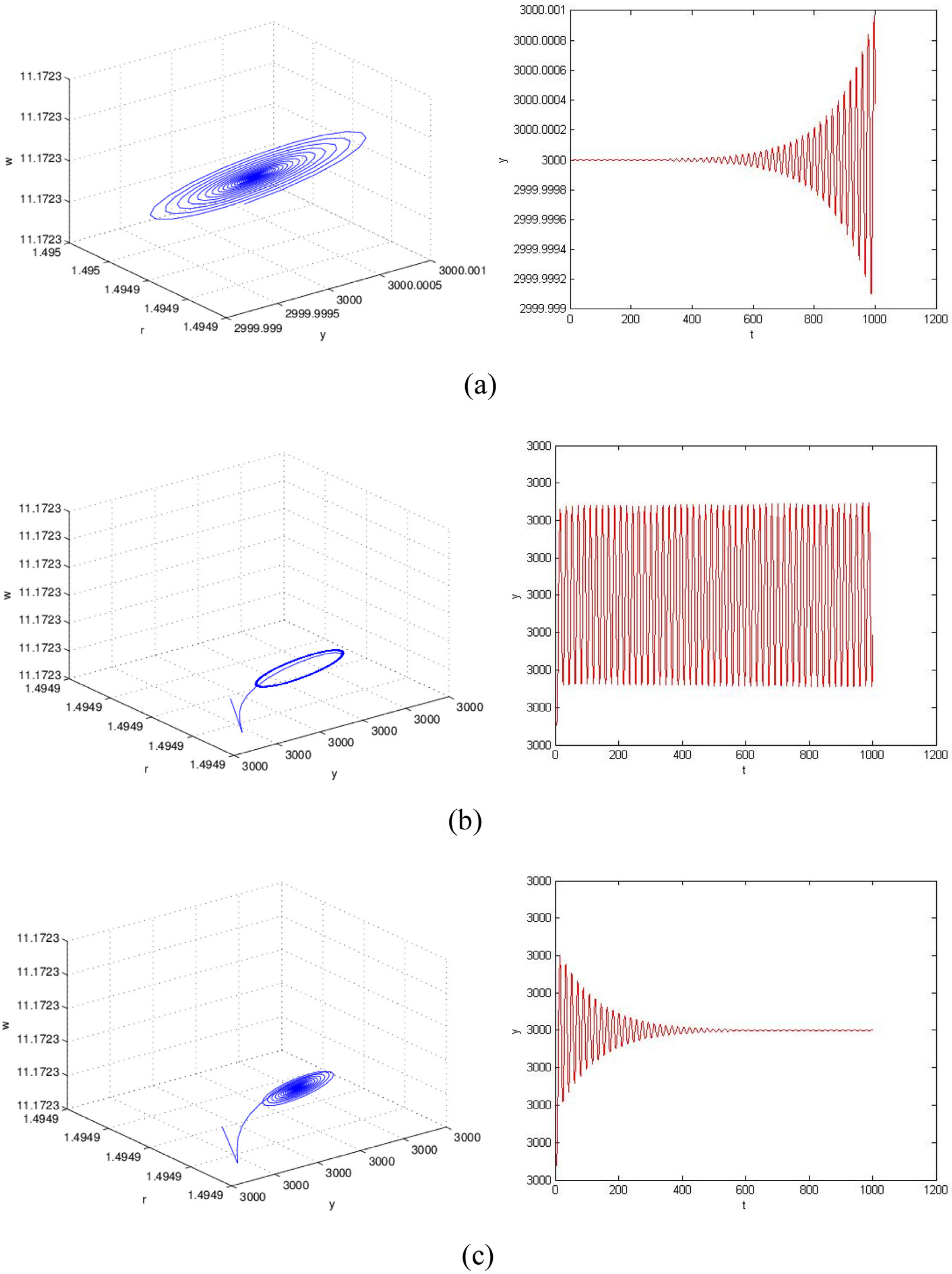

Fig. 1. The running state of system (3.2) near the Hopf bifurcation. (a) Phase diagram and waveform when $\beta = 5.8$. (b) Phase diagram and waveform when $\beta = 6.385$ (c) Phase diagram and waveform when $\beta = 7$.

Where $|A| = -\frac{\alpha g s^2(-1+\varepsilon)^2\beta}{a}$
Calculation to obtain:

$$\omega = \sqrt{\frac{\alpha g s^2(-1+\varepsilon)^2\beta}{a\left(-\alpha\varepsilon - \frac{\beta l_1 s^2(-1+\varepsilon)^2}{(\varepsilon s l_2 - s l_2 + a)^2}\right)}} \tag{4.1}$$

Through the equations:

$$Aq = iwq, A\bar{q} = -iw\bar{q}, A^T p = -iwp, A^T\bar{p} = iw\bar{p}$$

We can obtain the characteristic vector of the system q and its adjoint vector p.
For q, when $q=(q_1, q_2, q_3)^T \neq 0$, the basic solutions of linear equations:

$$\begin{bmatrix} -\alpha\varepsilon - i\omega & -\frac{\alpha g s^2(-1+\varepsilon)^2}{a\varepsilon} & 0 \\ \beta m & -\frac{\beta l_1 s^2(-1+\varepsilon)^2}{(\varepsilon s l_2 - s l_2 + a)^2} - i\omega & -\beta \\ -\varepsilon & 0 & -i\omega \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exist non-zero solutions, i.e.$|A|=0$, we can get the following formula:

$$\begin{aligned} |A| = & \frac{1}{(\varepsilon s l_2 - s l_2 + a)^2 a\varepsilon}(-ia\alpha\beta\varepsilon^4\omega s^2 l_1 - i\alpha\beta g m\omega s^4 {l_2}^2 + 2ia\alpha\beta\varepsilon^3\omega s^2 l_1 \\ & - ia^2\alpha\beta g m\omega s^2 - ia\alpha\beta\varepsilon^2\omega s^2 l_1 - \alpha\beta\varepsilon^5 g s^4 {l_2}^2 + 4\alpha\beta\varepsilon^4 g s^4 {l_2}^2 + a\alpha\varepsilon^4\omega^2 s^2 {l_2}^2 \\ & - 6\alpha\beta\varepsilon^3 g s^4 {l_2}^2 + 4\alpha\beta\varepsilon^2 g s^4 {l_2}^2 - a^2\alpha\beta\varepsilon^3 g s^2 - \alpha\beta\varepsilon g s^4 {l_2}^2 + 2a^2\alpha\beta\varepsilon^2 g s^2 \\ & - a^2\alpha\beta\varepsilon g s^2 - 2a\alpha\varepsilon^3\omega^2 s^2 {l_2}^2 + 2a^2\alpha\varepsilon^3\omega^2 s l_2 + a\alpha\varepsilon^2\omega^2 s^2 {l_2}^2 + a\beta\varepsilon^3\omega^2 s^2 l_1 \\ & - 2a^2\alpha\varepsilon^2\omega^2 s l_2 - 2a\beta\varepsilon^2\omega^2 s^2 l_1 + a\beta\varepsilon\omega^2 s^2 l_1 + ia\varepsilon^3\omega^3 s^2 {l_2}^2 + ia\varepsilon\omega^3 s^2 {l_2}^2 \\ & - 2ia\varepsilon^2\omega^3 s^2 {l_2}^2 + 2ia^2\varepsilon^2\omega^3 s l_2 - 2ia^2\varepsilon\omega^3 s l_2 + a^3\alpha\varepsilon^2\omega^2 + ia^3\varepsilon\omega^3 \\ & + 4i\alpha\beta\varepsilon g m\omega s^4 {l_2}^2 - ia^2\alpha\beta\varepsilon^2 g m\omega s^2 + 2ia^2\alpha\beta\varepsilon g m\omega s^2 + 2ia\alpha\beta g m\omega s^3 l_2 \\ & - i\alpha\beta\varepsilon^4 g m\omega s^4 {l_2}^2 + 4i\alpha\beta\varepsilon^3 g m\omega s^4 {l_2}^2 - 6i\alpha\beta\varepsilon^2 g m\omega s^4 {l_2}^2 - 2ia\alpha\beta\varepsilon^3 g m\omega s^3 l_2 \\ & + 6ia\alpha\beta\varepsilon^2 g m\omega s^3 l_2 - 6ia\alpha\beta\varepsilon g m\omega s^3 l_2 - 2a\alpha\beta\varepsilon^4 g s^3 l_2 + 6a\alpha\beta\varepsilon^3 g s^3 l_2 \\ & - 6a\alpha\beta\varepsilon^2 g s^3 l_2 + 2a\alpha\beta\varepsilon g s^3 l_2) = 0 \end{aligned}$$

$i\omega$ is the corresponding eigenvalues of A, and its eigenvectors is q.

$$q = \begin{bmatrix} -\frac{i\omega}{\varepsilon} & \frac{a\omega(i\alpha - \omega)}{\alpha g s^2(\varepsilon^2 - 2\varepsilon + 1)} & 1 \end{bmatrix}^T$$

$-i\omega$ is the corresponding eigenvalues of A^T, and its eigenvectors is p.

$$p = \begin{bmatrix} 1 & -\frac{i\omega(-\alpha\omega + i\omega)}{\beta(im\omega - \varepsilon)} & -\frac{-\alpha\omega + i\omega}{im\omega - \varepsilon} \end{bmatrix}^T$$

Have the normalized calculation to the vector q and p, we can obtain the *mol*.

$$\begin{aligned} mol = \langle q, p\rangle = & \frac{1}{\varepsilon\beta\left(m^2\omega^2 + \varepsilon^2\right)\alpha g s^2\left(\varepsilon^2 - 2\varepsilon + 1\right)}(i\alpha^2\beta\varepsilon^4 g m\omega s^2 \\ & + 2i\alpha\beta\varepsilon g m^2\omega^3 s^2 - i\alpha\beta\varepsilon^2 g m^2\omega^3 s^2 - 2i\alpha\beta\varepsilon^2 g\omega s^2 - 2i\alpha\beta\varepsilon^4 g\omega s^2 \\ & - 2i\alpha^2\beta\varepsilon^3 g m\omega s^2 - 2ia\alpha\varepsilon^3\omega^3 - ia\varepsilon m\omega^5 - \alpha^2\beta\varepsilon^5 g s^2 - \alpha\beta\varepsilon^3 g m\omega^2 s^2 \\ & + ia\alpha^2\varepsilon^3 m\omega^3 + 2\alpha^2\beta\varepsilon^4 g s^2 + 2\alpha\beta\varepsilon^2 g m\omega^2 s^2 + 4i\alpha\beta\varepsilon^3 g\omega s^2 \\ & + i\alpha^2\beta\varepsilon^2 g m\omega s^2 + a\varepsilon^2\omega^4 - i\alpha\beta g m^2\omega^3 s^2 - a\alpha^2\varepsilon^4\omega^2 - 2a\alpha\varepsilon^2 m\omega^4 \\ & - \varepsilon^3\alpha^2\beta g s^2 - m\omega^2\varepsilon\beta\alpha g s^2) \end{aligned}$$

Then the normalized vector p is:

$$p = \frac{1}{mol}\begin{bmatrix} 1 & -\frac{i\omega(-\alpha\omega + i\omega)}{\beta(im\omega - \varepsilon)} & -\frac{-\alpha\omega + i\omega}{im\omega - \varepsilon} \end{bmatrix}^T$$

Calculate $B_i(\xi, \eta)$ and $C_i(\xi, \eta, \zeta)$:

$$B_i(\xi,\eta)=\left[-\frac{\alpha s^2(-1+\varepsilon)^2\xi_1\eta_2}{a},\quad -\frac{2\beta l_1\xi_2\eta_2 s^3(-1+\varepsilon)^3}{(a+l_2 s(-1+\varepsilon))^3},\quad 0\right]$$

$$C_i(\xi,\eta,\zeta)=\left[-\frac{2\alpha s^3(-1+\varepsilon)^3\xi_1\eta_2\zeta_2}{a^2},\quad -\frac{6\beta l_1\xi_2\eta_2\zeta_2 s^4(-1+\varepsilon)^4}{(a+l_2 s(-1+\varepsilon))^4},\quad 0\right]$$

Will the A, A^T, p, q obtained above into the following equation, we can calculate the first Lyapunov coefficient of the system (3-2) at the Hopf bifurcation point.

$$l_1(0)=\frac{1}{2\omega_0}\mathrm{Re}\left[\langle p, C(q,q,\bar{q})\rangle-2\left\langle p, B(q, A^{-1}B(q,\bar{q}))\right\rangle+\left\langle p, B(\bar{q}, (2i\omega_0 Y_n-A)^{-1}B(q,q))\right\rangle\right] \tag{4.2}$$

Generally speaking, the first Lyapunov coefficients of high dimensional nonlinear systems are usually very large, and are not listed in this paper.

For the first Lyapunov coefficient $l_1(0)$, when $l_1(0) < 0$, the Hopf bifurcation of limit cycles is stable; when $l_1(0) > 0$, the limit cycle is not stable; when $l_1(0)=0$, the first Lyapunov coefficient lose efficacy, we need to judge the second Lyapunov coefficient.

4.2. Numerical simulation of the first Lyapunov coefficient of the system

Will the assignment (3.4) into A, A^T, p, q and $l_1(0)$, and judge the symbol of $l_1(0)$. In this paper, we simulate the first Lyapunov coefficient by using Maple 17 software. The results are as follows:

When the eigenvalues of the system satisfy:

$$\omega_0=\sqrt{\frac{\alpha g s^2(-1+\varepsilon)^2\beta}{a\left(-\alpha\varepsilon-\frac{\beta l_1 s^2(-1+\varepsilon)^2}{(\varepsilon s l_2-s l_2+a)^2}\right)}}$$

Namely, the system (3.2) has a pair of pure imaginary eigenvalues $\pm i\omega_0$. We have the calculation on the first Lyapunov coefficients of the system in three aspects, the results are as follows:

(1) When $\varepsilon=0.15$, $\alpha \in (0, 4.5)$, $\beta \in (0, 10)$, the first Lyapunov coefficients of system (3.2) are shown in Fig. 2.

Fig 2(a) is the first Lyapunov coefficients when the system (3.2) is in the situation of $\varepsilon=0.15$, $\alpha \in (0, 4.5)$, $\beta \in (0, 10)$, Fig 2(b) $\sim$ (d) are the projection on $l_1(0)-\alpha$ plane. From the Fig. 2(a) we can clearly see, although most of $l_1(0)$ are near 0, but we can clearly know that, the system is in $l_1(0)=0$ only at certain points, most of the state is in $l_1(0) < 0$, the limit cycles of Hopf bifurcation of in most cases is stable.

When the system is in the situation of $\varepsilon=0.15$, $\beta=0.9$, $\alpha \in (0, 4.5)$, $l_1(0) < 0$ when $\alpha \in (0, 3.437)$, and limit cycle is stable; $l_1(0) > 0$ when $\alpha \in (3.437, 4, 5)$, and limit cycle is unstable; $l_1(0)=0$ when $\alpha=3.437$, at this time, first Lyapunov coefficients failure, the stability of the Hopf bifurcation of the system should be studied by means of second Lyapunov coefficients.

Similarly, under the situation of $\beta=5$, $l_1(0) < 0$ when $\alpha \in (0.01, 4.5)$, and limit cycle is stable; $l_1(0) > 0$ when $\alpha \in (0, 0.01)$, and limit cycle is unstable. The same situation happens when $\beta=8$, the point to distinguish between stable and unstable regions is $\alpha=0.2588$.

We can see that, with the gradual increase of the parameters α, the first Lyapunov coefficient of the system (3–2) is gradually becoming more complex, which is gradually developing towards instability. And when the parameters α is small, the first Lyapunov coefficient of the system $l_1(0) < 0$

(2) When $\beta=0.9$, $\alpha \in (0, 10)$, $\varepsilon \in (0, 10)$, the first Lyapunov coefficients of system (3.2) are shown in Fig. 3.

Fig 3(a) is the first Lyapunov coefficients when the system (3.2) is in the situation of $\beta=0.9$, $\varepsilon=0.3$, $\alpha \in (0, 10)$, Fig 3(b) $\sim$ (e) are the projection on $l_1(0)-\alpha$ plane. From the Fig. 3(a) we can clearly see that, although most of $l_1(0)$ are near 0, but we can clearly know that, the system is in $l_1(0)=0$ only at certain points, most of the state is $l_1(0) < 0$, the limit cycles of Hopf bifurcation of in most cases is stable.

When the system is in the situation of $\beta=0.9$, $\varepsilon=0.3$, $\alpha \in (0, 10)$, $l_1(0) < 0$ when $\alpha \in (0, 1.304)\cup(8.74, 10)$, and limit cycle is stable; $l_1(0) > 0$ when $\alpha \in (1.304, 8.74)$, and limit cycle is unstable; $l_1(0)=0$ when $\alpha=1.304$ and 8.74.

Similarly, under the situation of $\varepsilon=0.68$, $l_1(0) < 0$ when $\alpha \in (0, 0.149) \bigcup (1.258, 1.875) \bigcup (1.875, 7.598)$; under the situation of $\varepsilon=1.2$, $l_1(0) < 0$ when $\alpha \in (0, 8.14)$; under the situation of $\varepsilon=7.1$, $l_1(0) < 0$ when $\alpha \in (0, 0.798) \bigcup (0.895, 1.7335)$, and limit cycle is stable.

We can see that, the effect of the parameter ε to the system is similar with the parameter α, with the gradual increase of the parameters ε, the first Lyapunov coefficient of the system (3.2) is gradually becoming more complex, which is gradually developing towards instability. And when the parameters ε is small, the first Lyapunov coefficient of the system $l_1(0) < 0$

(3) When $\alpha=0.95$, $\beta \in (0, 10)$, $\varepsilon \in (0, 10)$, the first Lyapunov coefficients of system (3.2) are shown in Fig. 4.

Fig 4(a) is the first Lyapunov coefficients when the system (3.2) is in the situation of $\alpha=0.95$, $\varepsilon=0.7$, Fig 4(b) $\sim$ (e) are the projection on $l_1(0)-\beta$ plane. From the Fig. 4(a) we can clearly see, although most of $l_1(0)$ are near 0, but we can clearly know that, the system is in $l_1(0)=0$ only at certain points, most of the state is $l_1(0) < 0$.

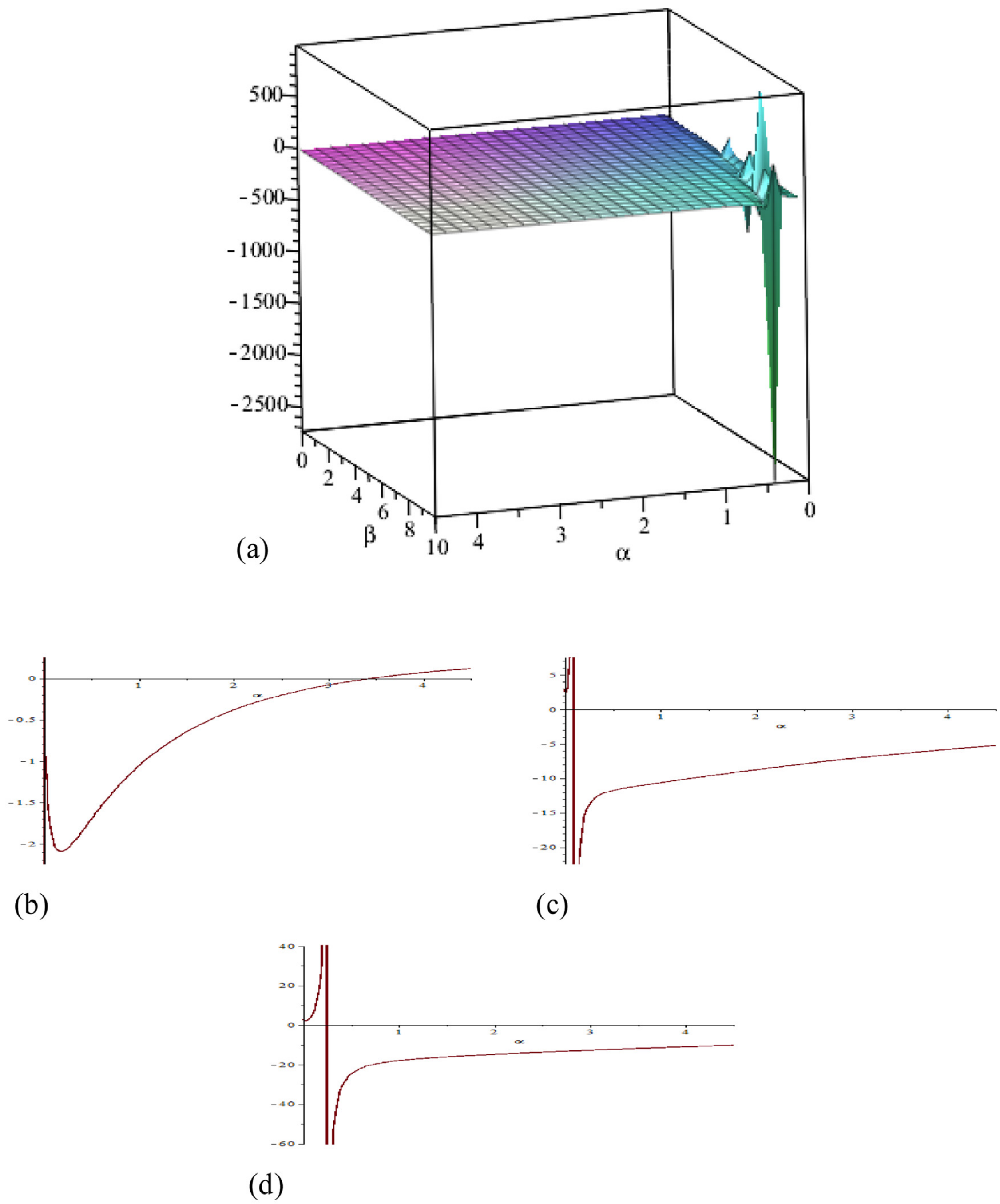

Fig. 2. First Lyapunov coefficients when $\varepsilon=0.15$, $\alpha \in (0, 10)$, $\beta \in (0, 10)$. (a) $l_1(0)$ when $\varepsilon=0.15$, $\alpha \in (0, 4.5)$, $\beta \in (0, 10)$. (b) $l_1(0)$ when $\beta=0.9$, $\alpha \in (0, 4.5)$. (c) $l_1(0)$ when $\beta=5$, $\alpha \in (0, 4.5)$. (d) $l_1(0)$ when $\beta=8$, $\alpha \in (0, 4.5)$.

When the system is in the situation of $\alpha=0.95$, $\varepsilon=0.7$, $\beta \in (0, 10)$, $l_1(0) < 0$ when $\beta \in (0, 0.7058)\cup(4.648, 10)$, and limit cycle is stable; $l_1(0) < 0$ when $\beta \in (0.7058, 4, 648)$, and limit cycle is unstable; $l_1(0)=0$ when $\alpha=0.7058$ and 4.648.

Similarly, under the situation of $\varepsilon=0.68$, $l_1(0) < 0$ when $\beta \in (0, 0.7788)\cup(8.7426, 10)$, but this example is special, its stable and unstable regions are not continuous; $l_1(0) > 0$ when $\beta \in (0.7788, 5.238)\cup(5.337, 8.7246)$, and limit cycle is unstable; $l_1(0) < 0$ when $\varepsilon=6$ and $\beta \in (5.238, 10)$; $l_1(0) < 0$ when $\varepsilon=9$ and $\beta \in (0.148, 0.6514)$, at this time, the limit cycle of Hopf bifurcation is stable.

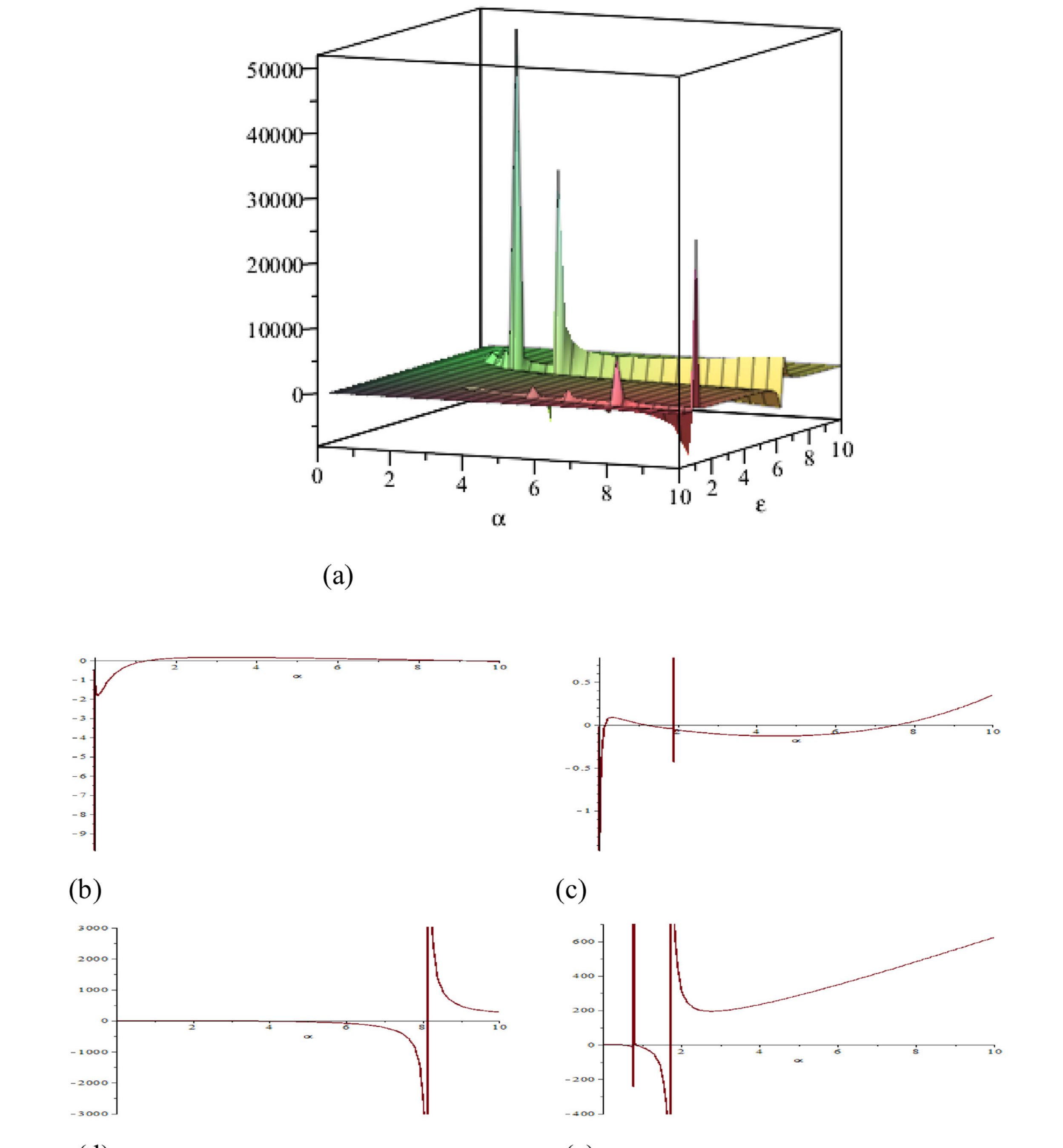

Fig. 3. First Lyapunov coefficients when $\beta = 0.9$, $\alpha \in (0, 10)$, $\varepsilon \in (0, 10)$. (a) $l_1(0)$ when $\beta = 0.9$, $\alpha \in (0, 10)$, $\varepsilon \in (0, 10)$. (b) $l_1(0)$ when $\varepsilon = 0.3$, $\alpha \in (0, 10)$. (c) $l_1(0)$ when $\varepsilon = 0.68$, $\alpha \in (0, 10)$. (d) $l_1(0)$ when $\varepsilon = 1.2$, $\alpha \in (0, 10)$. (e) $l_1(0)$ when $\varepsilon = 7.1$, $\alpha \in (0, 10)$.

We can see that, similarly with the two situations above, with the gradual increase of the parameters ε, the first Lyapunov coefficient of the system (3.2) is gradually becoming more complex. And when the parameters ε is small, the first Lyapunov coefficient of the system $l_1(0) < 0$.

In the process of Hopf bifurcation analysis of the system (3.2), the parameters of this paper are: $\alpha = 0.95$, $\beta = 0.9$, $\varepsilon = 0.15$ we can verify that the first Lyapunov coefficient of this system is less than 0, the limit cycle is stable, and this is consistent with the analysis result. At this time, IS-LM macroeconomics formed periodic solutions, economic system will form

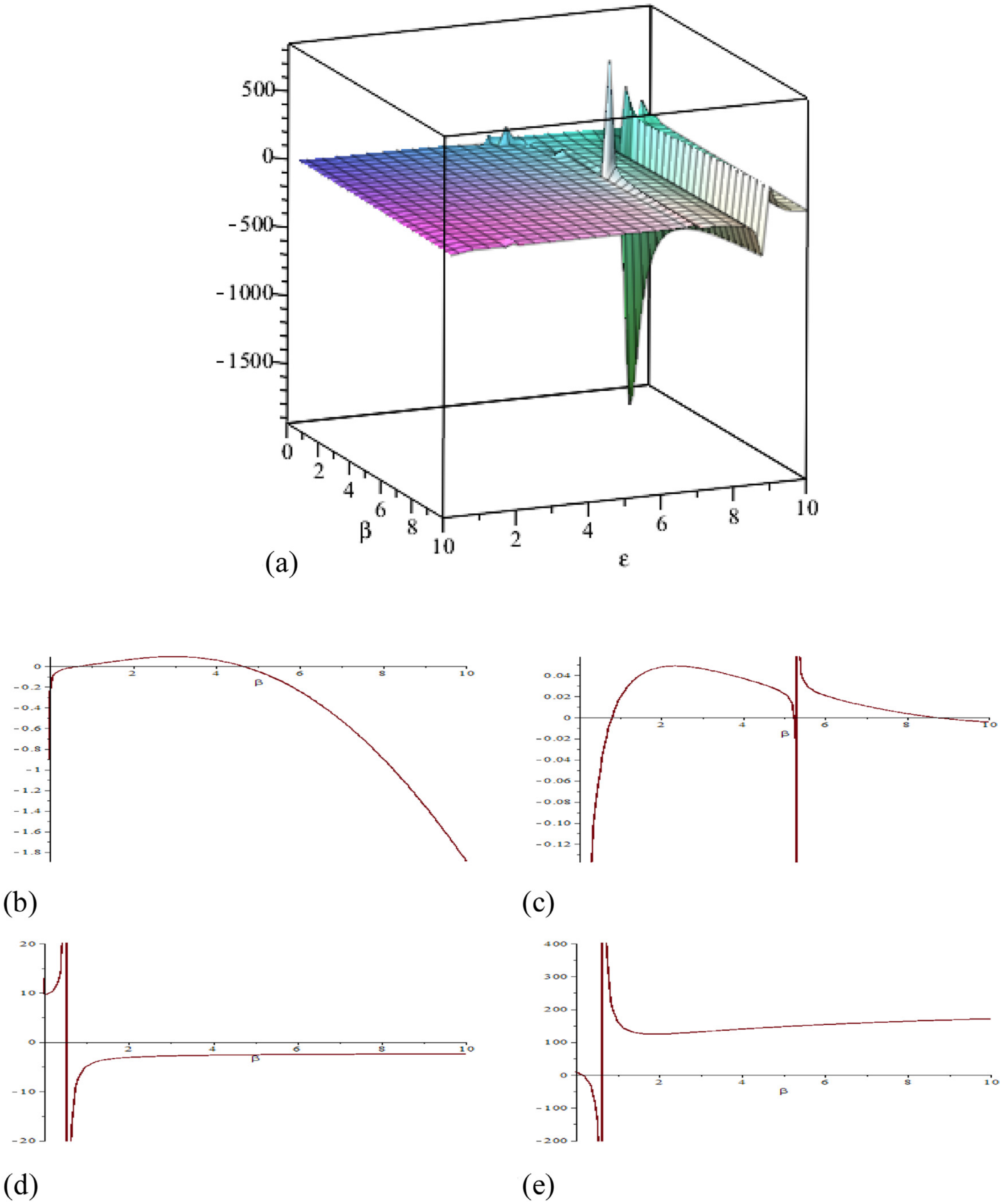

Fig. 4. First Lyapunov coefficients when $\alpha = 0.95$, $\beta \in (0, 10)$, $\varepsilon \in (0, 10)$. (a) $l_1(0)$ when $\alpha = 0.95$, $\beta \in (0, 10)$, $\varepsilon \in (0, 10)$. (b) $l_1(0)$ when $\varepsilon = 0.7$, $\beta \in (0, 10)$. (c) $l_1(0)$ when $\varepsilon = 0.9$, $\beta \in (0, 10)$. (d) $l_1(0)$ when $\varepsilon = 6$, $\beta \in (0, 10)$. (e) $l_1(0)$ when $\varepsilon = 9$, $\beta \in (0, 10)$.

prosperity and recession in period, and this trend is stable. Therefore, the government and the central bank can adjust the macroeconomics by changing the system parameters to maintain the sustainable prosperity of the economy and society.

5. Conclusion

Based on the previous studies, this paper improves nonlinear three-dimensional IS-LM macroeconomics, and focuses on the parameters and complex dynamic characteristics of the system when Hopf bifurcation occurs. In this paper, the stability of the Hopf bifurcation limit cycle is calculated and analyzed by the means of first Lyapunov coefficient, and the stability of the business cycle is judged. Through numerical simulation in three aspects, we found that in most range of parameters, the Hopf bifurcation limit cycles of 3D IS-LM macroeconomics is stable and unstable situation appears gradually with the increase of parameters.

For macroeconomics, if Hopf bifurcation exists in nonlinear economic model under certain parameter's situation, we can effectively have the analysis on the periodic behavior of the macroeconomics, inhibit rapid growth of some important parameters, and ensure the stability of business cycle. We can obtain the parameter situation when macroeconomics start to recovery, prosperity, recession and depression, governments and central banks can formulate economic policies according to the analysis results, and give guidance to the operation of macroeconomics. The results of this study have a good guiding significance for the macroeconomics regulation and control.

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