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Optimal convergence rates for the strong solutions to the compressible Navier–Stokes equations with potential force

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ABSTRACT

In this paper, we consider the effect of external force on the large-time behavior of solutions to the Cauchy problem for the three-dimensional full compressible Navier–Stokes equations. We construct the global unique solution near the stationary profile to the system for the small $H^2(\mathbb{R}^3)$ initial data. Moreover, the optimal L^p–L^2 $(1 \le p \le 2)$ time decay rates of the solution to the system are established via a low frequency and high frequency decomposition.

1. Introduction

This paper is concerned with the Cauchy problem of the full compressible Navier–Stokes equations affected by the external potential force in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho[u_t + (u \cdot \nabla)u] + \nabla P(\rho, \theta) = \mu \Delta u + (\mu + \mu')\nabla(\nabla \cdot u) + \rho F, \\ \rho c_V [\theta_t + (u \cdot \nabla)\theta] + \theta P_\theta(\rho, \theta)\nabla \cdot u = \kappa \Delta \theta + \Psi[u], \end{cases} \tag{1.1}$$

and the initial data

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x) \to (\rho_\infty, 0, \theta_\infty), \quad \text{as } |x| \to \infty. \tag{1.2}$$

Here the unknown functions $\rho > 0$, $u = (u_1, u_2, u_3)$, and θ denote the density, the velocity and the temperature; $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, $t > 0$ is the time variable; $P = P(\rho, \theta)$, μ, μ', $\kappa > 0$, and c_V are the pressure, the first and second viscosity coefficients, the coefficient of heat conduction,

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and the specific heat at constant volume, respectively. In addition, $F = F(x)$ is an external force and $\Psi = \Psi[u]$ is the dissipation function:

$$\Psi[u] = \frac{\mu}{2}\sum_{i,j=1}^{3}(\partial_i u_j + \partial_j u_i)^2 + \mu' \sum_{j=1}^{3}(\partial_j u_j)^2. \tag{1.3}$$

Throughout this paper, we assume that the above physical parameters satisfy $\mu > 0$ and $2\mu + 3\mu' \geq 0$ which deduce $\mu + \mu' > 0$. ρ_∞ and θ_∞ are positive constants, and $P(\rho, \theta)$ is smooth in a neighborhood of $(\rho_\infty, \theta_\infty)$ with $P_\rho(\rho_\infty, \theta_\infty) > 0$ and $P_\theta(\rho_\infty, \theta_\infty) > 0$.

In this work, we only consider the potential force, that is, $F = -\nabla\Phi(x)$. Under aforementioned assumptions, the existence of the stationary solution to the problem (1.1) and (1.2) has been established in [1]. The solution (ρ_*, u_*, θ_*) in a neighborhood of $(\rho_\infty, 0, \theta_\infty)$ is given by

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P_\rho(\eta, \theta_\infty)}{\eta} \mathrm{d}\eta + \Phi(x) = 0, \qquad u_*(x) = 0, \qquad \theta_*(x) = \theta_\infty, \tag{1.4}$$

and satisfies

$$\|\rho_* - \rho_\infty\|_{H^k(\mathbb{R}^3)} \leq C\|\Phi\|_{H^k(\mathbb{R}^3)}, \quad 0 \leq k \leq 4, \tag{1.5}$$

$$\sum_{k=1}^{4}\|(1+|x|)\nabla^k(\rho_* - \rho_\infty)\|_{L^2(\mathbb{R}^3)} \leq C\sum_{k=1}^{4}\|(1+|x|)\nabla^k\Phi\|_{L^2(\mathbb{R}^3)}. \tag{1.6}$$

We will construct the global unique solution to (1.1) near the steady state $(\rho_*, 0, \theta_\infty)$ when the initial perturbation belongs to the Sobolev space $H^2(\mathbb{R}^3)$. Our main results are stated as the following theorem.

Theorem 1.1. *Let $(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty) \in H^2(\mathbb{R}^3)$, there exists some small constant $\varepsilon > 0$ such that if*

$$\|(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty)\|_{H^2(\mathbb{R}^3)} + \|\Phi\|_{H^4(\mathbb{R}^3)} + \sum_{k=1}^{4}\|(1+|x|)\nabla^k\Phi\|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \tag{1.7}$$

then the initial value problem (1.1) and (1.2) admits a unique solution (ρ, u, θ) globally in time which satisfies

$$\rho - \rho_* \in C^0([0,\infty); H^2(\mathbb{R}^3)) \cap C^1([0,\infty); H^1(\mathbb{R}^3)),$$
$$u, \theta - \theta_\infty \in C^0([0,\infty); H^2(\mathbb{R}^3)) \cap C^1([0,\infty); L^2(\mathbb{R}^3)).$$

Moreover, if the initial data $(\rho_0 - \rho_\infty, u_0, \theta_0 - \theta_\infty)$ is bounded in $L^p(\mathbb{R}^3)$ for any given $1 \leq p \leq 2$, the solution (ρ, u, θ) enjoys the following decay-in-time estimates:

$$\|\nabla(\rho - \rho_*, u, \theta - \theta_\infty)\|_{H^1(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \quad \text{for all } t \geq 0, \tag{1.8}$$

$$\|(\rho - \rho_*, u, \theta - \theta_\infty)\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } t \geq 0,\ 2 \leq q \leq 6, \tag{1.9}$$

$$\|\partial_t(\rho - \rho_*, u, \theta - \theta_\infty)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \quad \text{for all } t \geq 0, \tag{1.10}$$

for some positive constant C.

Remark 1.1. In Theorem 1.1, using the Sobolev imbedding inequalities in Lemma 2.1, (1.7) together with (1.5) and (1.6) yields

$$\|\rho_* - \rho_\infty\|_{H^4(\mathbb{R}^3)} + \sum_{k=1}^{3}\|(1+|x|)\nabla^k(\rho_* - \rho_\infty)\|_{L^2(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)} \leq C\varepsilon. \tag{1.11}$$

Remark 1.2. From Theorem 3.1 in [2], we know the time-decay rates of solutions to the linearized system of (1.1). Compared to the linearized system, the decay rates of the perturbation $(\rho-\rho_*, u, \theta-\theta_\infty)$ given by (1.8)–(1.9) in Theorem 1.1 are optimal.

There are a lot of research work which were devoted to proving the global existence, unique and time decay rates of solutions to the compressible Navier–Stokes equations with or without external forces, cf. [3–10,2, 11–13,1,14–16] and references therein. We only refer to time decay rates of solutions to the compressible Navier–Stokes equations with external forces. When there is an external potential force $F=-\nabla\Phi(x)$, the first work to give explicit estimates for the decay rates for solutions to the problem (1.1) and (1.2) was represented by Deckelnick [3]. In [3], when the initial perturbation only belongs to the Sobolev space $H^3(\Omega)$, the following decay estimates were established in an unbounded domain $\Omega\subset\mathbb{R}^3$ (the half space $\{\mathbb{R}^3_+=x\in\mathbb{R}^3|x_3>0\}$ or the exterior of a bounded domain with smooth boundary):

$$\|\nabla(\rho-\rho_*, u, \theta-\theta_\infty)\|_{L^2(\Omega)}\le C(1+t)^{-\frac14},$$

and

$$\|\partial_t(\rho-\rho_*, u, \theta-\theta_\infty)\|_{L^2(\Omega)}\le C(1+t)^{-\frac12}.$$

It is worth mentioning that the convergence rate is an important topic in the study of the fluid dynamics for the purpose of the computation. In particular, the background profile here is non-trivial due to the effect of the external force.

For the isentropic case, Ukai,Yang and Zhao [15] obtained an almost optimal convergence rate $(1+t)^{-\frac{n}{4}+\kappa'}$ for any $\kappa'>0$ if the initial disturbance belongs to $H^s(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ with $s\ge[\frac{n}{2}]+2$ and $n\ge3$. Moreover, based on the energy method and the spectral analysis on the linearized system, Duan et al. established the optimal L^p–L^q type of the time decay estimates, for any $1\le p<\frac65$,

$$\|\nabla^k(\rho-\rho_*, u)\|_{L^q(\mathbb{R}^3)}\le\begin{cases}C(1+t)^{-\frac32(\frac1p-\frac1q)}, & \text{for } k=0,\ 2\le q\le6,\\ C(1+t)^{-\frac32(\frac1p-\frac1q)-\frac12}, & \text{for } 1\le k\le3,\ q=2,\end{cases}$$

when the initial perturbation belongs to $H^3(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ with only H^3-norm small, see [4]. For the non-isentropic case, the same optimal decay rates were obtained in [5] for $p=1$. When a general (non-potential) external force $F=F(x)$ is involved, the velocity of the stationary solution (ρ_*, u_*, θ_*) may not be zero. The global existence and the almost optimal convergence rate were obtained by Shibata and Tanaka [17,18] for the isentropic viscous fluid and by Qian and Yin [19] for non-isentropic case if the initial perturbation belongs to $H^3(\mathbb{R}^3)\cap L^{\frac65}(\mathbb{R}^3)$.

However, all known decay results were established in H^3-framework. In H^2-framework, to the best of our knowledge, so far there is no result on the large-time behaviors of the problem (1.1) and (1.2). Therefore, the main purpose of this paper is to investigate the optimal convergence rates in time to the stationary solution in H^2-framework. As we know, the key point is to establish the following two types of estimates by using the usual method, cf. [4,5,15,16]. One is a Lyapunov-type energy inequality as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t)+C\mathcal{H}(t)\le C\|\nabla U(t)\|^2_{L^2(\mathbb{R}^3)},$$

where $U:=(\rho-\rho_*, u, \theta-\theta_\infty)$ and $\mathcal{H}(t)$ is an energy functional including all the derivatives of U at least one order. The other is a decay estimate on the first order derivative $\nabla U(t)$ as follows

$$\|\nabla U(t)\|_{L^2(\mathbb{R}^3)}\le C(1+t)^{-\frac32(\frac1p-\frac12)-\frac12}\|U_0\|_{L^p(\mathbb{R}^3)}+C\varepsilon_0\int_0^t(1+t-\tau)^{-\frac32(\frac1p-\frac12)-\frac12}\|\nabla U(\tau)\|_{H^2(\mathbb{R}^3)}\mathrm{d}\tau,$$

where $\varepsilon_0>0$ be a small constant and $-\frac32(\frac1p-\frac12)-\frac12<-1$ with $1\le p<\frac65$. Unfortunately, this method is not available for the case in Theorem 1.1. The main difficulties in the analysis come from the low regularity of the solution in H^2-framework and $1 \le p \le 2$. To overcome the difficulties, a new idea is to consider the low-frequency and high-frequency decomposition of the solution to (1.1) and to use the decay properties of the low-frequency part. More specifically, we are going to prove an energy inequality in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) + C\mathcal{H}(t) \le C\|\nabla U_L(t)\|^2_{L^2(\mathbb{R}^3)},$$

where U_L is the low-frequency part of U (see (4.7)). Based on the decay estimates on the low-frequency part of the semigroup $E(t)$, we show that the decay estimates on $\nabla U_L(t)$ satisfies, for any $p \in [1, 2]$,

$$\|\nabla U_L(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}\|U_0\|_{L^p(\mathbb{R}^3)} + C\varepsilon_0 \int_0^t (1+t-\tau)^{-\frac{5}{4}}\|\nabla U(\tau)\|_{H^1(\mathbb{R}^3)}\mathrm{d}\tau.$$

Combining these two types of estimates, the optimal convergence rates can be obtained by the Gronwall inequality. It should be pointed out that the methods used in this paper only work if the spatial domain is the entire space.

Notations. Let C be denoted as a generic positive constant. For multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, $C_\alpha^\beta = \frac{\alpha!}{(\alpha-\beta)!\beta!}$ with $\beta \le \alpha$ which means $\beta_i \le \alpha_i$ for all $1 \le i \le n$. $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\nabla = (\partial_1, \partial_2, \partial_3)$, $\partial_i = \partial_{x_i}$, $i = 1, 2, 3$ and $|\alpha| = \sum_{i=1}^3 \alpha_i$. We denote a set composed of all mth partial derivatives with respect to the variable x by ∇^m. $H^m(\mathbb{R}^3)$, $m \in \mathbb{Z}_+$, denotes the usual Sobolev space with its norm

$$\|f\|_{H^m(\mathbb{R}^3)} := \sum_{k=0}^m \|\nabla^k f\|_{L^2(\mathbb{R}^3)}.$$

In particular, we use $\|\cdot\|_m = \|\cdot\|_{H^m(\mathbb{R}^3)}$ and $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^3)}$. As usual, $\langle\cdot,\cdot\rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. In addition, we apply the Fourier transform to the variable $x \in \mathbb{R}^3$ by $\hat{f}(\xi, t) = \int_{\mathbb{R}^n} f(x,t)\mathrm{e}^{-\sqrt{-1}x\cdot\xi}\mathrm{d}x$ and the inverse Fourier transform to the variable $\xi \in \mathbb{R}^3$ by $(\mathcal{F}^{-1}\hat{f})(x,t) = (2\pi)^{-n}\int_{\mathbb{R}^3}\hat{f}(\xi,t)\mathrm{e}^{\sqrt{-1}x\cdot\xi}\mathrm{d}\xi$.

The rest of paper is organized as follows. In Section 2, we reformulate the Cauchy problem (1.1)–(1.2) into a more suitable form. In Section 3, we establish the global existence of solutions to the problem in H^2-framework. In Section 4, we give the L^p–L^q type of the time decay rates of the solution and complete the proof of Theorem 1.1.

2. Preliminary

In this section, we are going to reformulate the Cauchy problem of the compressible Navier–Stokes system (1.1) with the initial condition (1.2) and give some basic lemmas for later use. Define

$$\tilde{\rho}(x,t) = \rho(x,t) - \rho_*(x), \qquad \tilde{u}(x,t) = u(x,t), \qquad \tilde{\theta}(x,t) = \theta(x,t) - \theta_\infty,$$

and

$$\bar{\rho}(x) = \rho_*(x) - \rho_\infty.$$

Then, by using (1.4), the Navier–Stokes equations (1.1) are transformed into the following system

$$\begin{cases} \tilde{\rho}_t + \rho_\infty \nabla\cdot\tilde{u} = \tilde{S}_1, \\ \tilde{u}_t - \dfrac{\mu}{\rho_\infty}\Delta\tilde{u} - \dfrac{\mu+\mu'}{\rho_\infty}\nabla\nabla\cdot\tilde{u} + \dfrac{P_\rho(\rho_\infty,\theta_\infty)}{\rho_\infty}\nabla\tilde{\rho} + \dfrac{P_\theta(\rho_\infty,\theta_\infty)}{\rho_\infty}\nabla\tilde{\theta} = \tilde{S}_2, \\ \tilde{\theta}_t - \dfrac{\kappa}{c_V\rho_\infty}\Delta\tilde{\theta} + \dfrac{\theta_\infty P_\theta(\rho_\infty,\theta_\infty)}{c_V\rho_\infty}\nabla\cdot\tilde{u} = \tilde{S}_3, \end{cases} \tag{2.1}$$

and the initial condition (1.2) becomes

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x,t)|_{t=0} = (\rho_0 - \rho_*, u_0, \theta_0)(x) \to (0,0,0), \quad \text{as } |x| \to \infty, \tag{2.2}$$

where $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$ are the source terms with

$$\tilde{S}_1 = -\nabla\cdot(\tilde{\rho}\tilde{u}) - \nabla\cdot(\bar{\rho}\tilde{u}), \tag{2.3}$$

$$\begin{aligned}\tilde{S}_2 &= -(\tilde{u}\cdot\nabla)\tilde{u} + \left[\frac{\mu}{\tilde{\rho}+\rho_*} - \frac{\mu}{\rho_\infty}\right]\Delta\tilde{u} + \left[\frac{\mu+\mu'}{\tilde{\rho}+\rho_*} - \frac{\mu+\mu'}{\rho_\infty}\right]\nabla\nabla\cdot\tilde{u} \\ &\quad - \left[\frac{P_\rho(\tilde{\rho}+\rho_*,\tilde{\theta}+\theta_\infty)}{\tilde{\rho}+\rho_*} - \frac{P_\rho(\rho_\infty,\theta_\infty)}{\rho_\infty}\right]\nabla\tilde{\rho} - \left[\frac{P_\theta(\tilde{\rho}+\rho_*,\tilde{\theta}+\theta_\infty)}{\tilde{\rho}+\rho_*} - \frac{P_\theta(\rho_\infty,\theta_\infty)}{\rho_\infty}\right]\nabla\tilde{\theta} \\ &\quad - \left[\frac{P_\rho(\tilde{\rho}+\rho_*,\tilde{\theta}+\theta_\infty)}{\tilde{\rho}+\rho_*} - \frac{P_\rho(\rho_*,\theta_\infty)}{\rho_*}\right]\nabla\bar{\rho},\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\tilde{S}_3 &= -(\tilde{u}\cdot\nabla)\tilde{\theta} - \left[\frac{(\tilde{\theta}+\theta_\infty)P_\theta(\tilde{\rho}+\rho_*,\tilde{\theta}+\theta_\infty)}{c_V(\tilde{\rho}+\rho_*)} - \frac{\theta_\infty P_\infty(\rho_\infty,\theta_\infty)}{c_V\rho_\infty}\right]\nabla\cdot\tilde{u} \\ &\quad + \left[\frac{\kappa}{c_V(\tilde{\rho}+\rho_*)} - \frac{\kappa}{c_V\rho_\infty}\right]\Delta\tilde{\theta} + \left[\frac{1}{c_V(\tilde{\rho}+\rho_*)} - \frac{1}{c_V\rho_\infty}\right]\Psi[\tilde{u}] + \frac{1}{c_V\rho_\infty}\Psi[\tilde{u}].\end{aligned} \tag{2.5}$$

To obtain a symmetric system, we denote

$$\sigma(x,t) = \tilde{\rho}(x,t), \qquad \omega(x,t) = \sqrt{\frac{P_1}{\rho_\infty}}\tilde{u}(x,t), \qquad z(x,t) = \sqrt{\frac{P_2\rho_\infty}{P_1P_3}}\tilde{\theta}(x,t),$$

and

$$\mu_1 = \frac{\mu}{\rho_\infty}, \qquad \mu_2 = \frac{\mu+\mu'}{\rho_\infty}, \qquad \gamma_1 = \sqrt{P_1\rho_\infty}, \qquad \gamma_2 = \sqrt{P_2P_3}, \qquad \kappa_1 = \frac{\kappa}{c_V}\sqrt{\frac{P_2}{P_1P_3\rho_\infty}},$$

with

$$P_1 = \frac{P_\rho(\rho_\infty,\theta_\infty)}{\rho_\infty}, \qquad P_2 = \frac{P_\theta(\rho_\infty,\theta_\infty)}{\rho_\infty}, \qquad P_3 = \frac{\theta_\infty P_\theta(\rho_\infty,\theta_\infty)}{c_V\rho_\infty}.$$

Then the initial value problem (2.1) and (2.2) can be rewritten as

$$\begin{cases}\sigma_t + \gamma_1\nabla\cdot\omega = S_1, \\ \omega_t - \mu_1\Delta\omega - \mu_2\nabla\nabla\cdot\omega + \gamma_1\nabla\sigma + \gamma_2\nabla z = S_2, \\ z_t - \kappa_1\Delta z + \gamma_2\nabla\cdot\omega = S_3, \\ (\sigma,\omega,z)(x,t)|_{t=0} = (\sigma_0,\omega_0,z_0)(x),\end{cases} \tag{2.6}$$

where $S_1 = \tilde{S}_1$, $S_2 = \sqrt{\frac{P_1}{\rho_\infty}}\tilde{S}_2$ and $S_3 = \sqrt{\frac{P_2\rho_\infty}{P_1P_3}}\tilde{S}_3$, and

$$(\sigma_0,\omega_0,z_0)(x) = \left(\rho_0-\rho_*, \sqrt{\frac{\rho_\infty}{P_1}}u_0, \sqrt{\frac{P_2\rho_\infty}{P_1P_3}}(\theta_0-\theta_\infty)\right)(x) \to (0,0,0), \quad \text{as } |x|\to\infty.$$

In what follows, we will consider the global existence and time decay rates of the solution (ρ,u,θ) to the steady state $(\rho_*,0,\theta_\infty)$, that is, the existence and decay rates of the perturbed solution (σ,ω,z) to the problem (2.6). For later use, some inequalities are listed as follows. One can found them in [20,16].

Lemma 2.1. *Let $f\in H^2(\mathbb{R}^3)$. Then*

(i) $\|f\|_{L^\infty} \le C\|\nabla f\|^{1/2}\|\nabla f\|_{H^1}^{1/2} \le C\|\nabla f\|_{H^1}$;
(ii) $\|f\|_{L^6} \le C\|\nabla f\|$;
(iii) $\|f\|_{L^q} \le C\|f\|_{H^1}$, $2\le q\le 6$.

Lemma 2.2. *Let r_1, $r_2 > 0$, then it holds that*

$$\int_0^t (1+t-\tau)^{-r_1}(1+\tau)^{-r_2}\mathrm{d}\tau \le C(r_1,r_2)(1+t)^{-\min\{r_1,r_2,r_1+r_2-1-\eta\}},$$

for an arbitrarily small $\eta > 0$.

3. Global existence

In this section, we will establish the global existence of solutions to the problem (2.6) in H^2-framework by using the energy method. First of all, let us define the solution space and the solution norm of the initial value problem (2.6) by

$$\begin{aligned}X(0,T) = \{(\sigma,\omega,z)|\sigma \in\ & C^0(0,T;H^2(\mathbb{R}^3))\cap C^1(0,T;H^1(\mathbb{R}^3)),\\ & \omega, z\in C^0(0,T;H^2(\mathbb{R}^3))\cap C^1(0,T;L^2(\mathbb{R}^3)),\\ & \nabla\sigma \in L^2(0,T;H^1(\mathbb{R}^3)), \nabla(\omega,z)\in L^2(0,T;H^2(\mathbb{R}^3))\},\end{aligned}$$

and

$$N(0,T)^2 = \sup_{0\le t\le T}\|(\sigma,\omega,z)(t)\|_2^2 + \int_0^T (\|\nabla\sigma(t)\|_1^2 + \|\nabla(\omega,z)(t)\|_2^2)\mathrm{d}t,$$

for any $0 \le T \le +\infty$. By the standard continuity argument, the global existence of solutions to (2.6) will be obtained from the combination of the local existence result with some *a priori* estimates.

Proposition 3.1 (*Local Existence*). *Suppose that the initial data satisfy $(\sigma_0,\omega_0,z_0) \in H^2(\mathbb{R}^3)$ and (1.7). Then there exists a positive constant $T_1 > 0$ depending on $\|(\sigma_0,\omega_0,z_0)\|_2$, such that the initial value problem (2.6) has a unique solution $(\sigma,\omega,z)\in X(0,T_1)$ which satisfies $N(0,T_1)\le 2N(0,0)$.*

Proof. The proof can be done by using the standard iteration arguments. Refer, for instance, to [21,1]. □

Proposition 3.2 (*A Priori Estimates*). *Suppose that the initial value problem (2.6) has a solution $(\sigma,\omega,z)\in X(0,T)$, where T is a positive constant. Under the assumptions of Theorem 1.1, there exist a positive constant C_0 and a small positive constant δ which are independent of T, such that if*

$$\sup_{0\le t\le T}\|(\sigma,\omega,z)(t)\|_2 \le \delta, \tag{3.1}$$

where $\delta > \max\{\varepsilon, \sqrt{C_0}\varepsilon\}$ and ε is given by (1.7), then it holds that for any $t\in[0,T]$

$$\|(\sigma,\omega,z)(t)\|_2^2 + \int_0^t (\|\nabla\sigma(\tau)\|_1^2 + \|\nabla(\omega,z)(\tau)\|_2^2)\mathrm{d}\tau \le C_0\|(\sigma_0,\omega_0,z_0)\|_2^2 < \delta^2, \tag{3.2}$$

where C_0 is independent of δ.

Remark 3.1. The global existence and uniqueness of the solutions stated in Theorem 1.1 follow from Propositions 3.1 and 3.2.

By (1.11), (3.1) and the Sobolev inequality, we have

$$\frac{1}{2}\rho_\infty \le \sigma + \rho_* \le 2\rho_\infty.$$

This will be often used in the rest of paper. Before proving Proposition 3.2, we need Lemmas 3.1 and 3.2.

Lemma 3.1. *For $0 \le k \le 2$, it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k(\sigma,\omega,z)(t)\|^2+\frac{\mu_1}{2}\|\nabla^k\nabla\omega(t)\|^2+\frac{\mu_2}{2}\|\nabla^k\nabla\cdot\omega(t)\|^2+\frac{k_1}{2}\|\nabla^{k+1}z(t)\|^2\\&\quad\le C\delta\left\{\|\nabla\sigma(t)\|_1^2+\|\nabla(\omega,z)(t)\|_2^2\right\},\end{aligned} \tag{3.3}$$

for any $0 \le t \le T$.

Proof. For each multi-index α with $|\alpha| = k$, multiplying ∂_x^α (2.6) $_1$, ∂_x^α (2.6) $_2$ and ∂_x^α (2.6) $_3$ by $\partial_x^\alpha\sigma$, $\partial_x^\alpha\omega$ and $\partial_x^\alpha z$, respectively, summing up and then integrating the result over $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha(\sigma,\omega,z)(t)\|^2+\mu_1\|\nabla\partial_x^\alpha\omega(t)\|^2+\mu_2\|\nabla\cdot\partial_x^\alpha\omega(t)\|^2+\kappa_1\|\nabla\partial_x^\alpha z(t)\|^2\\&\quad=\langle\partial_x^\alpha\sigma(t),\partial_x^\alpha S_1(t)\rangle+\langle\partial_x^\alpha\omega(t),\partial_x^\alpha S_2(t)\rangle+\langle\partial_x^\alpha z(t),\partial_x^\alpha S_3(t)\rangle\\&\quad:=I_1^k(t)+I_2^k(t)+I_3^k(t),\end{aligned} \tag{3.4}$$

where the nonlinear terms given in (2.3)–(2.5) have the following equivalence properties:

$$\begin{aligned}S_1 &\sim \partial_i\sigma\omega_i+\sigma\partial_i\omega_i+\partial_i\bar\rho\omega_i+\bar\rho\partial_i\omega_i,\\ S_2 &\sim \omega_i\partial_i\omega_j+\sigma\partial_i\partial_i\omega_j+\sigma\partial_j\partial_i\omega_i+\sigma\partial_j\sigma+z\partial_j\sigma+\sigma\partial_j z+z\partial_j z\\&\quad+\bar\rho\partial_i\partial_i\omega_j+\bar\rho\partial_j\partial_i\omega_i+\bar\rho\partial_j\sigma+\sigma\partial_j\bar\rho+z\partial_j\bar\rho+\bar\rho\partial_j z,\\ S_3 &\sim \omega_i\partial_i z+\sigma\partial_i\partial_i z+\sigma\partial_i\omega_i+z\partial_i\omega_i+\sigma\,\Psi[\omega]+\Psi[\omega]+\bar\rho\partial_i\partial_i z+\bar\rho\partial_i\omega_i+\bar\rho\,\Psi[\omega].\end{aligned} \tag{3.5}$$

When $k = 0$, by using the Hölder inequality, Lemma 2.1, (1.11), the *a priori* assumption (3.1) and the Young inequality, we obtain

$$\begin{aligned}I_1^0(t)&\le C\{|\langle\sigma(t),(\partial_i\sigma\omega_i)(t)\rangle|+|\langle\sigma(t),(\sigma\partial_i\omega_i)(t)\rangle|+|\langle\sigma(t),(\partial_i\bar\rho\omega_i)(t)\rangle|+|\langle\sigma(t),(\bar\rho\partial_i\omega_i)(t)\rangle|\}\\&\le C\left\{\|\sigma\|_{L^6}\|\partial_i\sigma\|\,\|\omega_i\|_{L^3}+\|\sigma\|_{L^6}\|\sigma\|_{L^3}\|\partial_i\omega\|\right.\\&\quad\left.+\left\|\frac{\sigma}{1+|x|}\right\|\,\|(1+|x|)\partial_i\bar\rho\|_{L^3}\|\omega_i\|_{L^6}+\|\sigma\|_{L^6}\|\bar\rho\|_{L^3}\|\partial_i\omega_i\|\right\}\\&\le C\{\|\nabla\sigma\|^2\|\omega\|_1+\|\nabla\sigma\|\,\|\sigma\|_1\|\nabla\omega\|+\|\nabla\sigma\|\,\|(1+|x|)\partial_i\bar\rho\|_1\|\nabla\omega\|+\|\nabla\sigma\|\,\|\bar\rho\|_1\|\nabla\omega\|\}\\&\le C\delta\left\{\|\nabla\sigma\|^2+\|\nabla\omega\|^2\right\},\end{aligned} \tag{3.6}$$

where we have used the Hardy inequality as follows

$$\left\|\frac{\sigma}{1+|x|}\right\|\le\|\nabla\sigma\|.$$

In a similar way, we get

$$I_2^0(t)\le C\delta\left\{\|\nabla\sigma\|^2+\|\nabla\omega\|_1^2+\|\nabla z\|^2\right\}, \tag{3.7}$$

and

$$I_3^0(t)\le C\delta\left\{\|\nabla z\|_1^2+\|\nabla\omega\|^2\right\}. \tag{3.8}$$

Putting (3.6)–(3.8) into (3.4) and taking δ sufficiently small, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\sigma,\omega,z)(t)\|^2+\frac{\mu_1}{2}\|\nabla\omega(t)\|^2+\frac{\mu_2}{2}\|\nabla\cdot\omega(t)\|^2+\frac{\kappa_1}{2}\|\nabla z(t)\|^2\\&\quad\le C\delta\left\{\|\nabla\sigma(t)\|^2+\|\nabla^2\omega(t)\|^2+\|\nabla^2 z(t)\|^2\right\}.\end{aligned} \tag{3.9}$$

When $1 \le k \le 2$, the terms on the right hand side of (3.4) can be estimated as follows. For $I_1^k(t)$, from (3.5), we get

$$\begin{aligned}I_1^k(t)\le C\{&|\langle\partial_x^\alpha\sigma(t),\partial_x^\alpha(\partial_i\sigma\omega_i)(t)\rangle|+|\langle\partial_x^\alpha\sigma(t),\partial_x^\alpha(\sigma\partial_i\omega_i)(t)\rangle|\\&+|\langle\partial_x^\alpha\sigma(t),\partial_x^\alpha(\partial_i\bar\rho\omega_i)(t)\rangle|+|\langle\partial_x^\alpha\sigma(t),\partial_x^\alpha(\bar\rho\partial_i\omega_i)(t)\rangle|\}.\end{aligned} \tag{3.10}$$

For the first term on the r.h.s. of (3.10), by using integration by parts, the Hölder inequality, the Young inequality, Lemma 2.1 and the *a priori* assumption (3.1), we have

$$\begin{aligned}
|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha(\partial_i\sigma\omega_i)(t)\rangle| &\le C|\langle \partial_x^\alpha\sigma, \partial_x^\alpha\partial_i\sigma\omega_i\rangle| + C\sum_{\beta\le\alpha,|\beta|\le|\alpha|-1} C_\alpha^\beta|\langle\partial_x^\alpha\sigma, \partial_x^\beta\partial_i\sigma\partial_x^{\alpha-\beta}\omega_i\rangle| \\
&= \frac{C}{2}|\langle(\partial_x^\alpha\sigma)^2, \partial_i\omega_i\rangle| + C\left\{\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|-1}\right\}|\langle\partial_x^\alpha\sigma, \partial_x^\beta\partial_i\sigma\partial_x^{\alpha-\beta}\omega_i\rangle| \\
&\le C\|\partial_i\omega_i\|_{L^\infty}\|\partial_x^\alpha\sigma\|^2 + C\|\partial_x^\alpha\sigma\|\|\partial_i\sigma\|_{L^3}\|\partial_x^\alpha\omega_i\|_{L^6} + \delta\|\partial_x^\alpha\sigma\|^2 \\
&\quad + \frac{C}{\delta}\sum_{1\le|\beta|\le|\alpha|-1}\|\nabla\partial_x^{\alpha-\beta}\omega_i\|^2\|\partial_x^\beta\partial_i\sigma\|^2 \\
&\le C\delta\|\partial_x^\alpha\sigma\|\,\|\nabla\omega\|_2 + C\delta\|\partial_x^\alpha\sigma\|\|\nabla\partial_x^\alpha\omega\| + \delta\|\partial_x^\alpha\sigma\|^2 + C\delta\|\nabla\omega\|_1^2 \\
&\le C\delta\left\{\|\partial_x^\alpha\sigma\|^2 + \|\nabla\omega\|_2^2\right\},
\end{aligned}\tag{3.11}$$

where we note that the terms including the sum of β with $1 \le |\beta| \le |\alpha| - 1$ will be vanished if $|\alpha| = 1$. Similarly, we have

$$\begin{aligned}
|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\sigma\partial_i\omega_i)(t)\rangle| &= \left\{\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|-1} + \sum_{|\beta|=|\alpha|}\right\} C_\alpha^\beta|\langle\partial_x^\alpha\sigma, \partial_x^{\alpha-\beta}\sigma\partial_x^\beta\partial_i\omega_i\rangle| \\
&\le C\|\partial_i\omega_i\|_{L^\infty}\|\partial_x^\alpha\sigma\|^2 + C\sum_{1\le|\beta|\le|\alpha|}\|\partial_x^\alpha\sigma\|\,\|\partial_x^{\alpha-\beta}\sigma\|_{L^6}\|\partial_x^\beta\partial_i\omega_i\|_{L^3} \\
&\quad + C\|\partial_x^\alpha\sigma\|\,\|\sigma\|_{L^\infty}\|\partial_x^\alpha\partial_i\omega\| \\
&\le C\delta\|\partial_x^\alpha\sigma\|\,\|\partial_i\omega_i\|_2 + \delta\|\partial_x^\alpha\sigma\|^2 + \frac{C}{\delta}\sum_{1\le|\beta|\le|\alpha|-1}\|\partial_x^{\alpha-\beta}\nabla\sigma\|^2\|\partial_x^\beta\partial_i\omega_i\|_1^2 \\
&\quad + C\delta\|\partial_x^\alpha\sigma\|\,\|\partial_x^\alpha\partial_i\omega_i\| \\
&\le C\delta\left\{\|\partial_x^\alpha\sigma\|^2 + \|\nabla\omega\|_2^2\right\}.
\end{aligned}\tag{3.12}$$

By using the Hölder inequality, the Young inequality, Lemma 2.1 and (1.11), the third term on the r.h.s. of (3.10) can be estimated as follows

$$\begin{aligned}
|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\partial_i\bar\rho\omega_i)(t)\rangle| &\le \sum_{|\beta|\le|\alpha|}|\langle\partial_x^\alpha\sigma, \partial_x^\beta\partial_i\bar\rho\partial_x^{\alpha-\beta}\omega_i\rangle| \\
&\le C\sum_{|\beta|\le|\alpha|}\|\partial_x^\alpha\sigma\|\,\|\partial_x^\beta\partial_i\bar\rho\|_{L^3}\|\partial_x^{\alpha-\beta}\omega_i\|_{L^6} \\
&\le \delta\|\partial_x^\alpha\sigma\|^2 + \frac{C}{\delta}\sum_{|\beta|\le|\alpha|}\|\partial_x^\beta\partial_i\bar\rho\|_1^2\|\partial_x^{\alpha-\beta}\nabla\omega_i\|^2 \\
&\le C\delta\left\{\|\partial_x^\alpha\sigma\|^2 + \|\nabla\omega\|_2^2\right\}.
\end{aligned}\tag{3.13}$$

By a similar argument, we have

$$\begin{aligned}
|\langle\partial_x^\alpha\sigma(t), \partial_x^\alpha(\bar\rho\partial_i\omega_i)(t)\rangle| &= \left\{\sum_{|\beta|=0} + \sum_{1\le|\beta|\le|\alpha|}\right\}|\langle\partial_x^\alpha\sigma, \partial_x^\beta\bar\rho\partial_x^{\alpha-\beta}\partial_i\omega_i\rangle| \\
&\le C\|\partial_x^\alpha\sigma\|\,\|\bar\rho\|_{L^\infty}\|\partial_x^\alpha\partial_i\omega_i\| + C\sum_{1\le|\beta|\le|\alpha|}\|\partial_x^\alpha\sigma\|\,\|\partial_x^\beta\bar\rho\|_{L^3}\|\partial_x^{\alpha-\beta}\partial_i\omega_i\|_{L^6} \\
&\le C\delta\left\{\|\partial_x^\alpha\sigma\|^2 + \|\nabla\omega\|_2^2\right\}.
\end{aligned}\tag{3.14}$$

Therefore, putting (3.11)–(3.14) into (3.10) yields

$$I_1^k(t) \le C\delta \left\{ \|\partial_x^\alpha \sigma\|^2 + \|\nabla \omega\|_2^2 \right\}. \tag{3.15}$$

For $I_2^k(t)$, from (3.5), we have

$$\begin{aligned} I_2^k(t) \le\ & |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\omega_i \partial_i \omega_j)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\sigma \partial_i \partial_i \omega_j)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\sigma \partial_j \partial_i \omega_i)\rangle| \\ & + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\sigma \partial_j \sigma)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(z \partial_j \sigma)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\sigma \partial_j z)\rangle| \\ & + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(z \partial_j z)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\bar\rho \partial_i \partial_i \omega_j)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\bar\rho \partial_j \partial_i \omega_i)\rangle| \\ & + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\bar\rho \partial_j \sigma)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\sigma \partial_j \bar\rho)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(z \partial_j \bar\rho)\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha(\bar\rho \partial_j z)\rangle|. \end{aligned} \tag{3.16}$$

For the second term on the r.h.s. of (3.16), by using integration by parts, the Hölder inequality, Lemma 2.1 and (3.1), we have

$$\begin{aligned} |\langle \partial_x^\alpha \omega(t), \partial_x^\alpha(\sigma \partial_i \partial_i \omega_j)\rangle| =\ & \left\{ \sum_{|\beta|=0} + \sum_{1 \le |\beta| \le |\alpha|} \right\} |\langle \partial_x^\alpha \omega, \partial_x^\beta \sigma \partial_x^{\alpha-\beta} \partial_i \partial_i \omega_j\rangle| \\ \le\ & |\langle (\partial_x^\alpha \partial_i \omega)^2, \sigma\rangle| + |\langle \partial_x^\alpha \omega, \partial_i \sigma \partial_x^\alpha \partial_i \omega_j\rangle| \\ & + \sum_{1 \le |\beta| \le |\alpha|-1} |\langle \partial_x^\alpha \omega, \partial_x^\beta \sigma \partial_x^{\alpha-\beta} \partial_i \partial_i \omega_j\rangle| + |\langle \partial_x^\alpha \omega, \partial_x^\alpha \sigma \partial_i \partial_i \omega_j\rangle| \\ \le\ & C\|\sigma\|_{L^\infty} \|\partial_x^\alpha \partial_i \omega\|^2 + \|\partial_x^\alpha \omega\|_{L^6} \|\partial_i \sigma\|_{L^3} \|\partial_x^\alpha \partial_i \omega_j\| \\ & + \sum_{1 \le |\beta| \le |\alpha|-1} \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\beta \sigma\|_{L^3} \|\partial_x^{\alpha-\beta} \partial_i \partial_i \omega_j\| + \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\alpha \sigma\| \, \|\partial_i \partial_i \omega_j\|_{L^3} \\ \le\ & C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 + \|\nabla^2 \omega\|_1^2 \right\}. \end{aligned}$$

For the 11th term on the r.h.s. of (3.16), by using the Hölder inequality, the Hardy inequality, Lemma 2.1 and (1.11), we obtain

$$\begin{aligned} |\langle \partial_x^\alpha \omega(t), \partial_x^\alpha(\sigma \partial_j \bar\rho)\rangle| =\ & \left\{ \sum_{|\beta|=0} + \sum_{1 \le |\beta| \le |\alpha|} \right\} |\langle \partial_x^\alpha \omega(t), \partial_x^\beta \sigma \partial_x^{\alpha-\beta} \partial_j \bar\rho\rangle| \\ \le\ & C\|\partial_x^\alpha \omega\|_{L^6} \|(1+|x|) \partial_x^\alpha \partial_j \bar\rho\|_{L^3} \left\| \frac{\sigma}{1+|x|} \right\| + \sum_{1 \le |\beta| \le |\alpha|} \|\partial_x^\alpha \omega\|_{L^6} \|\partial_x^\beta \sigma\| \, \|\partial_x^{\alpha-\beta} \partial_j \bar\rho\|_{L^3} \\ \le\ & C\delta \|\partial_x^\alpha \nabla \omega\| \, \|\nabla \sigma\| + C\delta \sum_{1 \le |\beta| \le |\alpha|} \|\partial_x^\alpha \nabla \omega\| \, \|\partial_x^\beta \sigma\| \\ \le\ & C\delta \left\{ \|\partial_x^\alpha \nabla \omega\|^2 + \|\nabla \sigma\|_1^2 \right\}. \end{aligned}$$

The other terms on the r.h.s. of (3.16) can be estimated similarly. Thus,

$$I_2^k(t) \le C\delta \left\{ \|\partial_x^\alpha \omega\|^2 + \|\partial_x^\alpha \nabla \omega\|^2 + \|\nabla \sigma\|_1^2 + \|\nabla \omega\|_2^2 + \|\nabla z\|_1^2 \right\}. \tag{3.17}$$

Now it remains to estimate $I_3^k(t)$. Similar to the estimate of $I_2^k(t)$, it holds that

$$I_3^k(t) \le C\delta \left\{ \|\partial_x^\alpha z\|^2 + \|\partial_x^\alpha \nabla z\|^2 + \|\nabla \omega\|_2^2 + \|\nabla z\|_2^2 \right\}. \tag{3.18}$$

Thus (3.4) together with (3.15), (3.17) and (3.18) yields

$$\begin{aligned} & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x^\alpha (\sigma, \omega, z)(t)\|^2 + \frac{\mu_1}{2} \|\nabla \partial_x^\alpha \omega(t)\|^2 + \frac{\mu_2}{2} \|\nabla \cdot \partial_x^\alpha \omega(t)\|^2 + \frac{\kappa_1}{2} \|\nabla \partial_x^\alpha z(t)\|^2 \\ & \quad \le C\delta \left\{ \|\partial_x^\alpha (\sigma, \omega, z)\|^2 + \|\partial_x^\alpha \nabla (\omega, z)\|^2 + \|\nabla \sigma\|_1^2 + \|\nabla (\omega, z)\|_2^2 \right\}, \end{aligned} \tag{3.19}$$

where we have taken δ sufficiently small. Therefore, (3.9) and (3.19) give (3.3). $\square$

Now, the dissipative estimates for $\nabla\sigma$ are obtained in the following lemma.

Lemma 3.2. *For $0 \le k \le 1$, it holds that*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla^k\omega(t), \nabla^k\nabla\sigma(t)\rangle + \frac{\gamma_1}{2}\|\nabla^k\nabla\sigma(t)\|^2 \\ &\quad \le C\left\{\|\nabla^k\nabla\omega(t)\|_1^2 + \|\nabla^k\nabla z(t)\|^2\right\} + C\delta\left\{\|\nabla\sigma(t)\|_1^2 + \|\nabla(\omega,z)(t)\|_2^2\right\}, \end{aligned} \tag{3.20}$$

for any $0 \le t \le T$.

Proof. From $(2.6)_2$, we get

$$\gamma_1\nabla\sigma = -\omega_t + \mu_1\Delta\omega + \mu_2\nabla\nabla\cdot\omega - \gamma_2\nabla z + S_2. \tag{3.21}$$

By applying ∂_x^α ($|\alpha| = k$) to (3.21), multiplying it by $\nabla\partial_x^\alpha\sigma$, and then integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} \gamma_1\|\nabla\partial_x^\alpha\sigma\|^2 &= -\langle\partial_x^\alpha\omega_t, \nabla\partial_x^\alpha\sigma\rangle + \mu_1\langle\partial_x^\alpha\Delta\omega, \nabla\partial_x^\alpha\sigma\rangle + \mu_2\langle\partial_x^\alpha\nabla\nabla\cdot\omega, \nabla\partial_x^\alpha\sigma\rangle \\ &\quad - \gamma_2\langle\partial_x^\alpha\nabla z, \nabla\partial_x^\alpha\sigma\rangle + \langle\partial_x^\alpha S_2, \nabla\partial_x^\alpha\sigma\rangle. \end{aligned} \tag{3.22}$$

Using integration by parts and $(2.6)_1$, we obtain

$$\begin{aligned} -\langle\partial_x^\alpha\omega_t, \nabla\partial_x^\alpha\sigma\rangle &= -\frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\sigma\rangle + \langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\sigma_t\rangle \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\sigma\rangle - \gamma_1\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\nabla\cdot\omega\rangle + \langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha S_1\rangle. \end{aligned} \tag{3.23}$$

Adding (3.22) and (3.23) gives

$$\begin{aligned} \gamma_1\|\nabla\partial_x^\alpha\sigma\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\sigma\rangle &= \mu_1\langle\partial_x^\alpha\Delta\omega, \nabla\partial_x^\alpha\sigma\rangle + \mu_2\langle\partial_x^\alpha\nabla\nabla\cdot\omega, \nabla\partial_x^\alpha\sigma\rangle \\ &\quad - \gamma_2\langle\partial_x^\alpha\nabla z, \nabla\partial_x^\alpha\sigma\rangle + \langle\partial_x^\alpha S_2, \nabla\partial_x^\alpha\sigma\rangle \\ &\quad - \gamma_1\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\nabla\cdot\omega\rangle + \langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha S_1\rangle. \end{aligned}$$

Then by using integration by parts and the Young inequality, it follows from the above equality that

$$\begin{aligned} &\frac{3\gamma_1}{4}\|\nabla\partial_x^\alpha\sigma\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\sigma\rangle \\ &\quad \le C\left\{\|\partial_x^\alpha\nabla^2\omega\|^2 + \|\partial_x^\alpha\nabla\cdot\omega\|^2 + \|\partial_x^\alpha\nabla z\|^2\right\} + C|\langle\nabla\partial_x^\alpha\omega, \partial_x^\alpha S_1\rangle| + C|\langle\nabla\partial_x^\alpha\sigma, \partial_x^\alpha S_2\rangle|. \end{aligned} \tag{3.24}$$

Similar to the estimates of $I_1^k(t)$ and $I_2^k(t)$ in the proof of Lemma 3.1, we have

$$|\langle\nabla\partial_x^\alpha\omega, \partial_x^\alpha S_1\rangle| \le C\|\nabla\partial_x^\alpha\omega\|^2 + C\delta\left\{\|\nabla\sigma\|^2 + \|\nabla\omega\|_2^2\right\}, \tag{3.25}$$

and

$$|\langle\nabla\partial_x^\alpha\sigma, \partial_x^\alpha S_2\rangle| \le C\delta\left\{\|\nabla\partial_x^\alpha\sigma\|^2 + \|\nabla\sigma\|_1^2 + \|\nabla\omega\|_2^2 + \|\nabla z\|_1^2\right\}. \tag{3.26}$$

Putting (3.25) and (3.26) into (3.24) and taking δ sufficiently small yield

$$\frac{\gamma_1}{2}\|\nabla\partial_x^\alpha\sigma\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x^\alpha\omega, \nabla\partial_x^\alpha\sigma\rangle \le C\left\{\|\partial_x^\alpha\nabla\omega\|_1^2 + \|\partial_x^\alpha\nabla z\|^2\right\} + C\delta\left\{\|\nabla\sigma\|_1^2 + \|\nabla\omega\|_2^2 + \|\nabla z\|_1^2\right\}.$$

Thus, we complete the proof of Lemma 3.2. $\square$

Let $0 \le l \le 1$. By summing up (3.3) for from $k = l$ to 2, since δ is small, there exists a constant $C_1 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\le k\le 2}\|\nabla^k(\sigma,\omega,z)(t)\|^2 + \sum_{l\le k\le 2}\|\nabla^{k+1}(\omega,z)(t)\|^2 \le C_1\delta\|\nabla\sigma(t)\|_1^2. \tag{3.27}$$

Summing up (3.20) for from $k = l$ to 1, since δ is small, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\le k\le 1}\langle\nabla^k\omega(t),\nabla^k\nabla\sigma(t)\rangle + C_2\sum_{l\le k\le 1}\|\nabla^k\nabla\sigma(t)\|^2\\ &\le C\left\{\sum_{l\le k\le 2}\|\nabla^{k+1}\omega(t)\|^2+\sum_{l\le k\le 1}\|\nabla^{k+1}z(t)\|^2+\delta\|\nabla(\sigma,\omega,z)(t)\|^2\right\},\end{aligned}\tag{3.28}$$

where $C_2 > 0$ is a constant.

Multiplying (3.28) by $2C_1\delta/C_2 < \frac{1}{2}$, adding it with (3.27), taking $l = 0$ and using the smallness of δ, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{0\le k\le 2}\|\nabla^k(\sigma,\omega,z)(t)\|^2+\frac{2C_1\delta}{C_2}\sum_{0\le k\le 1}\langle\nabla^k\omega(t),\nabla^k\nabla\sigma(t)\rangle\right\}\\ &+C\left\{\sum_{0\le k\le 1}\|\nabla^{k+1}\sigma(t)\|^2+\sum_{0\le k\le 2}\|\nabla^{k+1}(\omega,z)(t)\|^2\right\}\le 0.\end{aligned}\tag{3.29}$$

Integrating (3.29) with respect to t and using the smallness of δ, we get (3.2). The proof of Proposition 3.2 is complete.

4. Decay rates

In this section, we get decay rates of solutions to the problem (2.6). Let $\mathbb{A}$ be the following 5×5 matrix of the differential operators of the form

$$\mathbb{A} = \begin{pmatrix} 0 & \gamma_1\mathrm{div} & 0\\ \gamma_1\nabla & -\mu_1\Delta-\mu_2\nabla\mathrm{div} & \gamma_2\nabla\\ 0 & \gamma_2\mathrm{div} & -\kappa_1\Delta\end{pmatrix}.$$

If we set

$$\tilde{U} = (\tilde{\sigma},\tilde{\omega},\tilde{z})^t,\qquad U(0) = (\sigma_0,\omega_0,z_0)^t,$$

the linearized problem of (2.6) can be written as

$$\begin{cases}\tilde{U}_t + \mathbb{A}\tilde{U} = 0, & \text{for } t>0,\\ \tilde{U}|_{t=0} = U(0).\end{cases}\tag{4.1}$$

By taking the Fourier transform of $(4.4)_1$ with respect to the x-variable and solving the ordinary differential equation with respect to t, we have

$$\tilde{U}(t) = E(t)U(0),\quad t\ge 0,\tag{4.2}$$

where $E(t) = \mathrm{e}^{-t\mathbb{A}}$ $(t\ge 0)$ is the semigroup generated by the linear operator $-\mathbb{A}$. For a function $f(x,t)$, we define $E(t)f = \mathcal{F}^{-1}(\mathrm{e}^{-t\hat{\mathbb{A}}(\xi)}\hat{f}(\xi,t))$ with

$$\hat{\mathbb{A}}(\xi) = \begin{pmatrix} 0 & \sqrt{-1}\gamma_1\xi^t & 0\\ \sqrt{-1}\gamma_1\xi & \mu_1|\xi|^2+\mu_2\xi\xi^t & \sqrt{-1}\gamma_2\xi\\ 0 & \sqrt{-1}\gamma_2\xi^t & \bar{\kappa}|\xi|^2\end{pmatrix},\quad \text{for } \xi = \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}.$$

In what following, we denote the solutions to the nonlinear problem (2.6) by

$$U(t) = (\sigma(t),\omega(t),z(t))^t.\tag{4.3}$$

Then (2.6) can be rewritten as follows:

$$\begin{cases} U_t + \mathbb{A}U = S(U), & \text{for } t > 0, \\ U|_{t=0} = U(0), \end{cases} \tag{4.4}$$

where $S(U) := (S_1, S_2, S_3)^t$. By using the Duhamel principle, we have

$$U(t) = E(t)U_0 + \int_0^t E(t-\tau)S(U)(\tau)\mathrm{d}\tau, \quad t \geq 0. \tag{4.5}$$

Let $\varphi_0(\xi)$ be a function in $C_0^\infty(\mathbb{R}^3)$ such that

$$\varphi_0(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \dfrac{R}{2}, \\ 0, & \text{for } |\xi| \geq R, \end{cases} \tag{4.6}$$

where $R > 0$ be some fixed constant. Now, based on the Fourier transform and (4.6), we can define the low frequency and high frequency decomposition $(f_L(x), f_H(x))$ for a function $f(x)$ as follows

$$f_L = \mathcal{F}^{-1}(\varphi_0(\xi)\hat{f}(\xi)), \quad \text{and} \quad f_H = f - f_L. \tag{4.7}$$

By noticing the definitions (4.6) and (4.7) and using the Plancherel theorem, we can obtain directly the following estimates

$$\|\nabla f\| \leq C(\|\nabla f_L\| + \|\nabla f_H\|), \tag{4.8}$$

and

$$C\|\nabla f_H\| \leq \|\nabla^k f_H\|, \quad \text{and} \quad C\|\nabla^k f_H\| \leq \|\nabla^k f\|, \quad \text{for } \ k \geq 1. \tag{4.9}$$

In terms of the definition (4.7), we denote $U_L(t) := \mathcal{F}^{-1}(\varphi_0(\xi)\hat{U}(\xi,t))$ and $E_L(t)f := \mathcal{F}^{-1}(\varphi_0(\xi)\mathrm{e}^{-t\hat{\mathbb{A}}(\xi)}$ $\hat{f}(\xi,t))$. Then from (4.5), we have

$$U_L(t) = E_L(t)U_0 + \int_0^t E_L(t-\tau)S(U)(\tau)\mathrm{d}\tau. \tag{4.10}$$

To estimate $U_L(t)$, we need the L^p–L^q type of the time decay estimates on the low-frequency part of the semigroup $E(t)$. From Theorem 3.1 in [2], one has the following decay estimates.

Lemma 4.1 ([2]). *Let $k \geq 0$ be integers and $1 \leq p \leq 2 \leq q \leq \infty$, then for any $t > 0$, it holds that*

$$\|\nabla^k E_L(t)f\|_{L^q} \leq C(k,p,q)(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{k}{2}}\|f\|_{L^p}. \tag{4.11}$$

4.1. Decay estimates of the low-frequency part

In this subsection, based on the L^p–L^q type estimate (4.11), we show the decay estimates of $U_L(t)$ as follows.

Lemma 4.2. *For $k \geq 0$, it holds that*

$$\|\nabla^k U_L(t)\| \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{k}{2}}K_0 + C(\delta+\varepsilon)\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|\nabla U(\tau)\|_1\mathrm{d}\tau, \tag{4.12}$$

where $K_0 = \|U_0\|_{L^p}$ is finite.

Proof. From (4.10) and Lemma 4.1, we have

$$\|\nabla^k U_L(t)\| \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|U_0\|_{L^p} + C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|S(U)(\tau)\|_{L^1}\mathrm{d}\tau. \tag{4.13}$$

For the terms including $\bar\rho$ on the r.h.s. of (4.13), by using the Hölder inequality, (1.11) and the Hardy inequality, we get

$$\begin{aligned}
&\|\nabla\bar\rho\cdot\omega\|_{L^1} \leq \|(1+|x|)\nabla\bar\rho\| \left\|\frac{\omega}{1+|x|}\right\| \leq C\varepsilon\|\nabla\omega\|,\\
&\|\bar\rho\nabla\cdot\omega\|_{L^1} \leq \|\bar\rho\|\,\|\nabla\cdot\omega\| \leq C\varepsilon\|\nabla\omega\|.
\end{aligned}$$

Thus, the above inequalities give that

$$\|\nabla\cdot(\bar\rho\omega)\|_{L^1} \leq C\varepsilon\|\nabla\omega\|.$$

Similarly, it holds that

$$\begin{aligned}
&\|\nabla(\bar\rho\sigma)\|_{L^1} \leq C\varepsilon\|\nabla\sigma\|_1, \qquad \|\nabla(\bar\rho z)\|_{L^1} \leq C\varepsilon\|\nabla z\|_1,\\
&\|\bar\rho\nabla\nabla\cdot\omega\|_{L^1} \leq C\varepsilon\|\nabla^2\omega\|, \qquad \|\bar\rho\Delta z\|_{L^1} \leq C\varepsilon\|\nabla^2 z\|.
\end{aligned}$$

The other terms can be estimated easily so that we omit for brevity. Then, from (3.5) and the *a priori* assumption (3.1), we have

$$\begin{aligned}
\|S(U)(t)\|_{L^1} &\leq C(\|(\sigma,\omega)\|_1\|\nabla(\sigma,\omega,z)\|_1 + \|\nabla\cdot(\bar\rho\omega)\|_{L^1} + \|\nabla(\bar\rho\sigma)\|_{L^1} + \|\nabla(\bar\rho z)\|_{L^1} + \|\bar\rho\nabla^2(\omega,z)\|_{L^1})\\
&\leq C(\delta+\varepsilon)\|\nabla(\sigma,\omega,z)\|_1.
\end{aligned} \tag{4.14}$$

Thus, putting (4.14) into (4.13), we get (4.12). □

4.2. Decay rates for the nonlinear problem

In this subsection, from the *a priori* estimates obtained in Section 3 and the low frequency and high frequency decomposition (4.7), we first derive a new Lyapunov-type inequality. From (3.27) and (3.28), by taking $l=1$, similar to the estimate (3.29), we get

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} + \|\nabla^2(\sigma,\omega,z)(t)\|^2 + \|\nabla^2(\omega,z)(t)\|_1^2 \leq C\delta\|\nabla(\sigma,\omega,z)(t)\|^2, \tag{4.15}$$

where $\mathcal{H}(t)$ is an energy functional which is equivalent to $\|\nabla(\sigma,\omega,z)(t)\|_1^2$, that is, there exists a positive constant C_3 such that

$$\frac{1}{C_3}\|\nabla(\sigma,\omega,z)(t)\|_1^2 \leq \mathcal{H}(t) \leq C_3\|\nabla(\sigma,\omega,z)(t)\|_1^2.$$

By using (4.8) and (4.9), from (4.15), there exists a constant $C_4>0$ such that

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} + C_4\|\nabla(\sigma_H,\omega_H,z_H)(t)\|^2 + \frac{1}{2}\|\nabla^2(\sigma,\omega,z)(t)\|^2 + \frac{1}{2}\|\nabla^2(\omega,z)(t)\|_1^2\\
&\quad\leq C\delta\left\{\|\nabla(\sigma_L,\omega_L,z_L)(t)\|^2 + \|\nabla(\sigma_H,\omega_H,z_H)(t)\|^2\right\}.
\end{aligned} \tag{4.16}$$

It follows from (4.16) and the smallness of δ that

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} + C_5\|\nabla(\sigma_H,\omega_H,z_H)(t)\|^2 + \frac{1}{2}\|\nabla^2(\sigma,\omega,z)(t)\|^2 + \frac{1}{2}\|\nabla^2(\omega,z)(t)\|_1^2\\
&\quad\leq C\delta\|\nabla(\sigma_L,\omega_L,z_L)(t)\|^2,
\end{aligned} \tag{4.17}$$

where $C_5 > 0$ is a constant. Adding the both sides of (4.17) by $C_5\|\nabla(\sigma_L, \omega_L, z_L)(t)\|^2$ and noticing the definition of $\mathcal{H}(t)$, there exists a constant $D > 0$ such that

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} + D\mathcal{H}(t) \le C\|\nabla U_L(t)\|^2. \tag{4.18}$$

To prove the decay estimates stated in Theorem 1.1, we first define

$$M(t) = \sup_{0\le\tau\le t}(1+\tau)^{3\left(\frac{1}{p}-\frac{1}{2}\right)+1}\mathcal{H}(\tau).$$

Notice that $M(t)$ is non-decreasing and

$$\|\nabla U(\tau)\|_1 \le C\sqrt{\mathcal{H}(\tau)} \le C(1+\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}\sqrt{M(t)}, \quad 0\le\tau\le t. \tag{4.19}$$

Then, from (4.12), by using Lemma 2.2 and (4.19), we obtain

$$\begin{aligned}\|\nabla U_L(t)\| &\le CK_0(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} + C(\delta+\varepsilon)\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}\mathrm{d}\tau\sqrt{M(t)}\\ &\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}\big(K_0 + (\delta+\varepsilon)\sqrt{M(t)}\big).\end{aligned} \tag{4.20}$$

By the Gronwall inequality and (4.20), from (4.18), we have

$$\begin{aligned}\mathcal{H}(t) &\le \mathcal{H}(0)\mathrm{e}^{-Dt} + C_2\int_0^t \mathrm{e}^{-D(t-\tau)}\|\nabla U_L(\tau)\|^2\mathrm{d}\tau\\ &\le \mathcal{H}(0)\mathrm{e}^{-Dt} + C\int_0^t \mathrm{e}^{-D(t-\tau)}(1+\tau)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)-1}\mathrm{d}\tau(K_0^2 + (\delta+\varepsilon)^2M(t))\\ &\le (1+t)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)-1}\big(\mathcal{H}(0) + K_0^2 + (\delta+\varepsilon)^2M(t)\big),\end{aligned} \tag{4.21}$$

where $\mathcal{H}(0)$ is equivalent to $\|\nabla(\sigma_0, \omega_0, z_0)\|_1^2$.

In terms of the definition of $M(t)$, it follows from (4.21) that

$$M(t) \le C(\mathcal{H}(0) + K_0^2) + C(\delta+\varepsilon)^2M(t),$$

which implies that if δ and ε are sufficiently small then

$$M(t) \le C(\mathcal{H}(0) + K_0^2).$$

By noticing the definition of $\mathcal{H}(t)$ and $M(t)$, we have

$$\|\nabla(\sigma, \omega, z)(t)\|_1 \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}. \tag{4.22}$$

Then, we get (1.8).

Next, we define the temporal energy functional

$$\mathcal{H}_1(t) = \|(\sigma, \omega, z)(t)\|_2^2 + \frac{2C_1\delta}{C_2}\sum_{0\le k\le 2}\langle\nabla^k\sigma(t), \nabla^k\nabla\omega(t)\rangle,$$

where it is noticed that $\mathcal{H}_1(t)$ is equivalent to $\|(\sigma, \omega, z)(t)\|_2^2$ since the positive constants δ can be sufficiently small. Then, from (3.29), we have

$$\frac{\mathrm{d}\mathcal{H}_1(t)}{\mathrm{d}t} + \|\nabla(\sigma, \omega, z)(t)\|^2 + \|\nabla(\omega, z)(t)\|_1^2 \le 0. \tag{4.23}$$

Similar to the estimate (4.18), from (4.23), there exists a positive constant D_1 such that

$$\frac{\mathrm{d}\mathcal{H}_1(t)}{\mathrm{d}t} + D_1\mathcal{H}_1(t) \le C\|U_L(t)\|^2. \tag{4.24}$$

Taking $k = 0$ in (4.12) and using (4.22), for $1 \le p \le 2$, we obtain

$$\begin{aligned}\|U_L(t)\| &\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}K_0 + C(\delta+\varepsilon)\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}\mathrm{d}\tau \\ &\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}(K_0+C),\end{aligned} \tag{4.25}$$

where we have used Lemma 2.2 with $r_1 = \frac{3}{4}$, $r_2 = \frac{3}{2}(\frac{1}{p}-\frac{1}{2}) + \frac{1}{2}$ and $\eta = \frac{1}{4}$.

From (4.24), by using the Gronwall inequality and (4.25), we get

$$\begin{aligned}\mathcal{H}_1(t) &\le \mathcal{H}_1(0)\mathrm{e}^{-D_1 t} + C\int_0^t \mathrm{e}^{-D_1(t-\tau)}\|U_L(\tau)\|^2\mathrm{d}\tau \\ &\le \mathcal{H}(0)\mathrm{e}^{-D_1 t} + (C + CK_0^2)\int_0^t \mathrm{e}^{-D_1(t-\tau)}(1+\tau)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)}\mathrm{d}\tau \\ &\le C(1+t)^{-3\left(\frac{1}{p}-\frac{1}{2}\right)}(\mathcal{H}_1(0) + C + K_0^2),\end{aligned} \tag{4.26}$$

where $\mathcal{H}_1(0)$ is equivalent to $\|(\sigma_0, \omega_0, z_0)\|_2^2$. Then, it follows from the definition of $\mathcal{H}_1(t)$ that

$$\|(\sigma, \omega, z)(t)\| \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}. \tag{4.27}$$

Finally, by the interpolation, for any $2 \le q \le 6$, combining (4.22) and (4.27) yields

$$\|U(t)\|_{L^q} \le \|U(t)\|_{L^6}^{\eta_0}\|U(t)\|^{1-\eta_0} \le C\|\nabla U(t)\|^{\eta_0}\|U(t)\|^{1-\eta_0} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)},$$

where $\eta_0 = \frac{3(q-2)}{2q}$. Then, we get (1.9).

On the other hand, by using (2.6), Lemma 2.1, (1.8) and (1.11), we have

$$\begin{aligned}\|\partial_t(\sigma,\omega,z)(t)\|_{L^2} &\le C\|\nabla\cdot\omega(t)\|_{L^2} + \|S_1(t)\|_{L^2} + C\|\nabla\sigma(t)\|_{L^2} + C\|\nabla z(t)\|_{L^2} + C\|\Delta\omega(t)\|_{L^2} \\ &\quad + C\|\nabla\nabla\cdot\omega(t)\|_{L^2} + \|S_2(t)\|_{L^2} + C\|\Delta z(t)\|_{L^2} + \|S_3(t)\|_{L^2} \\ &\le C\left\{\|\nabla\omega(t)\|_{H^1} + \|\nabla\sigma(t)\|_{L^2} + \|\nabla z(t)\|_{H^1}\right\} \\ &\le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}},\end{aligned}$$

for any $0 \le t \le T$. Thus, we complete the proof of Theorem 1.1.

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