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Synchronization of complex dynamical networks with hybrid coupling delays on time scales by handling multitude Kronecker product terms

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ABSTRACT

The problem of synchronization is studied for complex dynamical networks with hybrid coupling delays on time scales. The hybrid coupling delays consist of both discrete and distributed coupling delays. Some novel and useful synchronization criteria of complex dynamical networks are derived based on stability theory of error dynamical system. By employing the standard Lyapunov–Krasovskii functional, matrix expansion method to handle multitude Kronecker product terms and the modified Jensen's inequalities on time scale, new sufficient conditions guaranteeing the global exponential stability of the origin of complex dynamical networks are established interms of linear matrix inequality (LMI). Numerical examples are included to demonstrate the effectiveness of the proposed method.

1. Introduction

Complex network is a large number of highly interconnected fundamental units and therefore exhibit very complicated dynamics [1]. For example, if the neural networks has many interconnected nodes, it can even be considered as complex networks [49]. Many large scale systems can be modeled by complex dynamical networks (CDNs) with the nodes representing individuals in the system and the edges representing the special connection among them [2]. Many of these networks exhibit complexity in the overall topological and dynamical properties of the network nodes and the coupled units. Many natural and technological systems can be modeled as complex networks, which is very common in biological and physical systems such as gene regulatory networks, wireless communication networks, World Wide Web, epidemiological models, electrical power grid, internet, metabolic networks and so on (For example [3–8], and references therein). One of the most significant and interesting phenomena in complex networks is the synchronization of all dynamical nodes.

As collective behavior of complex networks, synchronization has been studied extensively [9]. The synchronization phenomena are common and important in real-world networks, such as synchronization on the Internet, synchronization transfer of digital or analog signals in communication networks and synchronization related to biological neural networks [10–14]. Hence, synchronization analysis of complex networks is important both in theory and practice [15]. In recent years, in order to make a deep understanding of the synchronization phenomenon and to make a good use of the synchronization behavior, researchers have paid increasing attentions to the synchronization problem of complex dynamical networks (For example

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[44–48], and references therein). In the past few years, the synchronization problems in coupled dynamic networks have been widely investigated due to its applications in secure communication and signal generators design [16,17].

Time delays are ubiquitous in many dynamical systems and may modify drastically dynamic behavior of the system. Time delay can be found in many systems, such as nuclear reactors, population dynamic models, aircraft stabilization, biological systems, chemical engineering systems, ship stabilization and so on [18–22]. It has been shown that such kind of time delay has a tendency to destabilize a system. In recent decades, considerable attention has been devoted to the time delay systems due to their extensive applications in practical systems including circuit theory, chemical processing, bio engineering, complex dynamical networks, automatic control and so on. In the implementation of complex dynamical networks, time delay is unavoidably encountered due to the finite speed of signal transmission over the link and the network traffic congestions [23–26].

In real-world systems, the interaction among agents can happen at any time, may be some continuous time intervals accompanying some discrete moments. So it is necessary and meaningful to consider both continuous-time and discrete-time cases at the same time in networked systems. Empirical results show that the theory of time scales is not only a pure theoretical field of mathematics but also a useful tool to deal with many practical problems. The field of dynamic equations on time scales contains links and extends the classical theory of differential and difference equations. Recently, the theory of time scale calculus has been applied in neural networks and complex networks [27,28,42].

However, in reality, the networks might always be expected to achieve synchronization as quickly as possible, particularly in engineering fields. To achieve faster convergence rate in time-delay complex networks, an effective method is to use synchronization control techniques on time scales [29]. The theory of dynamic equations on time scales is undergoing a rapid development as it provides a powerful tool to generalize the discussion of these systems on time scales. The dynamic systems on time scales have tremendous applications in some mathematical models, such as neural networks, population dynamics, physics technology, and so on [30,31].

In [32–34], the authors studied the continuous-time synchronization in arrays of complex networks. In [35,36], the authors discussed the discrete-time synchronization in arrays of complex networks. The above mentioned complex dynamical networks are either continuous-time CDNs or discrete-time CDNs, it is troublesome to study synchronization in two kinds of models. Therefore, it is necessary to unify the study of continuous-time and discrete-time CDNs under the same framework. In [29], based on the theory of time scales, authors considered a CDNs on time scales and established a main criterion guaranteeing the global exponential synchronization of system. In [37], authors considered a complex-valued neural networks with both leakage time delay and discrete constant delay on time scales. By constructing appropriate Lyapunov–Krasovskii functionals with multitude Kronecker products, and employing the free weighting matrix method, several delay-dependent criteria for global exponential synchronization of the addressed CDNs were established.

By constructing appropriate Lyapunov–Krasovskii functionals with multitude Kronecker products and using matrix inequality technique, a new delay-dependent criterion for checking the global exponential synchronization of the addressed CDNs was established in terms of real linear matrix inequalities (LMIs). However, the previous criteria [29] for checking the synchronization criterion of the addressed CDNs are somewhat conservative due to the construction of constructed Lyapunov functionals and technicality of used mathematical method. Hence, it is our intention in this paper to reduce the possible conservatism. In fact, the problem of synchronization of complex dynamical networks on time scales remains to be important and challenging. However, to the best of authors knowledge, delay-dependent synchronization analysis of complex dynamical networks with hybrid coupling delays on time scale has not been investigated yet. Motivated by this work, we investigate synchronization of complex dynamical networks with hybrid coupling delays and multitude Kronecker products terms on time scales.

The main contributions of this paper are complex dynamical networks on time scales is equivalently expressed as the error dynamical networks on time scales. In this paper, we will combine continuous-time and discrete-time cases together and design the consensus/synchronization protocols under a unified framework. By constructing a set of Lyapunov–Krasovskii functional, new delay-dependent synchronization criteria for complex dynamical networks with hybrid coupling delays on time scales is established in terms of LMIs, which allow simultaneous computation that characterize the synchronization rate of the solution and can be easily determined by utilizing Matlab LMI Control Toolbox.

Notation: The following notations are used in this paper. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denotes the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The superscript "T" denotes the transpose of a matrix or vector. N and Z denote the positive integer collection and integer collection, respectively. For square matrices M and N, the notation M $>$ ($\geq$, $<$, $\leq$) N denotes M - N is a positive-definite (positive-semi-definite, negative, negative-semi-definite) matrix. $\mathbb{T}$ is a time scale. Set $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T}, a \leq t \leq b\}$. $\mathbb{T}^+ = \{t \in \mathbb{T}, t \geq 0\}$. I and 0 denote the identity matrix and the zero matrix with compatible dimensions, respectively; and diag{...} stands for a block-diagonal matrix. The Kronecker product of matrices $U \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{p\times q}$ is a matrix in $\mathbb{R}^{np\times mq}$ and denoted as $U \otimes R$. Let $\omega \geq 0$ and $C([-\omega, 0]_{\mathbb{T}}; \mathbb{R}^n)$ denote the family of continuous functions ϕ from $[-\omega, 0]_{\mathbb{T}}$ to $\mathbb{R}^n$ with the norm $\|\phi\| = sup_{-\omega\leq\theta\leq 0} \parallel \phi(\theta) \parallel$, where $\|.\|$ is the Euclidean norm in $\mathbb{R}^n$.

2. Model description and preliminaries

Let $\mathbb{T}$ be an arbitrary nonempty closed subset (time scale) of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ and $\mu(t) := \sigma(t) - t$. Assume that time scale $\mathbb{T}$ has a bounded graininess $\mu(t) \le \mu < \infty$.

For a point $t \in \mathbb{T}$, t is called left-dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, right-scattered if $\sigma(t) > t$. we call $g^{\Delta}(t)$ the delta derivative of g at t. It is easy to see that

$$g^{\Delta}(t) = \begin{cases} \lim_{s\to t, s\in\mathbb{T}} \frac{g(t)-g(s)}{t-s} & \text{if } \mu(t) = 0 \\ \frac{g(\sigma(t))-g(t)}{\sigma(t)-t} & \text{if } \mu(t) > 0. \end{cases}$$

Definition 1 [38]**.** For a function $g : \mathbb{T} \to \mathbb{R},\ t \in \mathbb{T}$, the delta derivative of $g(t)$, $g^{\Delta}(t)$, is the number (if it exists) with the property that, for a given $\epsilon > 0$, there exists a neighborhood U of t such that

$$| [g(\sigma(t)) - g(s)] - g^{\Delta}(t)[\sigma(t) - s] | < \epsilon \mid \sigma(t) - s \mid,$$

for all $s \in U$.

A function $g : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous provided it is continuous at right-dense point of $\mathbb{T}$ and the left side limit exists (finite) at left-dense point of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is defined by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. A function $g : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)g(t) \ne 0$ for all $t \in \mathbb{T}$. The set of all regressive and right-dense continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

We denote that $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{g \in \mathcal{R} : 1 + \mu(t)g(t) > 0, \text{ for all } t \in \mathbb{T}\}$. Obviously, $\mathcal{R}^+$ is the set of all positively regressive elements of $\mathcal{R}$.

Lemma 1 [38]**.** *Suppose φ and ψ are differentiable$n \times n$-matrix-valued functions. Then*

(a) $(\varphi + \psi)^{\Delta} = \varphi^{\Delta} + \psi^{\Delta}$;
(b) $(\alpha\varphi)^{\Delta} = \alpha\varphi^{\Delta}$ *if α is a constant;*
(c) $(\varphi\psi)^{\Delta} = \varphi^{\Delta}\psi^{\sigma} + \varphi\psi^{\Delta}$.

Definition 2 [29]**.** If $p \in \mathcal{R}$, then the generalized exponential function $e_p(t, s)$ is defined by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \quad \text{for } s, t \in \mathbb{T}$$

with the cylinder transformation

$$\xi_k(z) = \begin{cases} \frac{\log(1+kz)}{k} & \text{if } k \ne 0 \\ z & \text{if } k = 0. \end{cases}$$

If g and h are two differentiable functions, then the product rule for the derivative of product gh is that

$$(gh)^{\Delta} = g^{\Delta}h + (g + \mu g^{\Delta})h\Delta.$$

Definition 3 [29]**.** A function $G : \mathbb{T} \to \mathbb{R}$ is called a delta-antiderivative of $g : \mathbb{T} \to \mathbb{R}$ provided $G^{\Delta} = g$ holds for all $t \in \mathbb{T}$. In this case, the integral of f is defined by

$$\int_a^t g(s)\Delta s = G(t) - G(a) \quad \textit{for } t \in \mathbb{T}$$

Then we have

$$\int_t^{\sigma(t)} g(s)\Delta s = \mu(t)g(t).$$

If $p \in \mathcal{R}$, then the exponential function $e_p(t, t_0)$ is the only solution of the initial value problem

$$y^{\Delta} = py(t), \ \ y(t_0) = 1,$$

on a time scale $\mathbb{T}$.

Remark 1. There are two special cases:

(i) If $\mathbb{T} = \mathbb{R}$, then $g : \mathbb{R} \to \mathbb{R}$ is delta differential at $t \in \mathbb{R}$ iff

$$g'(t) = lim_{s\to t}\frac{g(t) - g(s)}{t - s}$$

exists, i.e., iff g is differentiable (in the ordinary sense) at t. In this case, we have $g^{\Delta}(t) = g'(t)$.

(ii) If $\mathbb{T} = \mathbb{Z}$, then $g : \mathbb{Z} \to \mathbb{R}$ is delta differential at $t \in \mathbb{Z}$ with

$$g^{\Delta}(t) = \frac{g^{\sigma}(t) - g(t)}{\mu(t)} = g(t+1) - g(t) = \Delta g(t),$$

where Δ is the usual forward difference operator defined by the last equation above.

Remark 2. This delta differential or derivatives is present in theory of time scales. Recently, the theory of time scale calculus has been applied in neural networks [27], hopfield neural networks [28], BAM systems [30], multi-agent systems [31], complex-valued neural networks [37] and so on. Since the delta derivatives are used in many network systems [27,28,30,31,37] so it is clear that this delta derivatives are applicable to complex dynamical network also. In [29], authors considered the synchronization of complex dynamical networks with discrete time delays on time scales in which to derive the Lyapunov–Krasovskii functionals the delata derivative is used.

The exponential function on time scales has some properties as follows;

If $p, q \in \mathcal{R},\ t,\ s,\ r \in \mathbb{T}$, then

(1) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(2) $e_p(\sigma(t), s) = (1 + p\mu(t))e_p(t, s)$;
(3) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;
(4) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(5) $e_p^{\Delta}(t, t_0) = (e_p(t, t_0))\Delta = pe_p(t, t_0)$;
(6) $e_p(t, s) > 0$ for $p \in \mathcal{R}^+$.

Consider the model in [50,51]. We extend this model as complex dynamical networks with hybrid coupling delays on time scales consisting of N coupled nodes with each node being an n-dimensional dynamical system

$$u_i^{\Delta}(t) = -Du_i(t) + B_1g(u_i(t)) + B_2g(u_i(t - d(t))) + J(t) + \sum_{j=1}^{N} E_{ij}^{(1)} L_1 u_j(t) + \sum_{j=1}^{N} E_{ij}^{(2)} L_2 u_j(t - d(t)) + \sum_{j=1}^{N} E_{ij}^{(3)} L_3 \int_{t-d(t)}^{t} u_j(s)\Delta s, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $t \in \mathbb{T},\ u_i(t) = (u_{i1}(t),\ u_{i2}(t), \ldots, u_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the ith network at time t. $g(u_i(t))$ and $g(u_i(t - d(t)))$ are nonlinear vector functions satisfying certain conditions given later. D denotes a known connection matrix; B_1 and B_2 denote the connection weight matrices; $L_1,\ L_2,\ L_3 \in \mathbb{R}^n$ are the matrices describing the inner-coupling between the subsystems at time t; $E = (E_{ij}^{(a)})_{N\times N}(a = 1, 2, 3)$ is the outer-coupling configuration matrices representing the coupling strength and the topological structure of the complex networks. $J(t)$ is the constant external input vector. The term $d(t)$ stands for the coupling time-varying delay which satisfy that $0 \le d(t) \le d$ and $t - d(t) \in \mathbb{T}$ for all $t \in \mathbb{T}$. $g(u_i(t)) = (g_1(u_{i1}(t)), g_2(u_{i2}(t)), \ldots, g_n(u_{in}(t)))^T$ is an unknown but sector-bounded nonlinear function. There are coupling terms between the state variables, delayed state variables and distributed state variables, in which the delayed coupling could describe the decentralized nature of real-world coupled systems.

The initial conditions associated with system (1) are given by

$$u_i(s) = \phi_i(s) \in C([-d, 0]_{\mathbb{T}}, \mathbb{R}^n), \quad i = 1, 2, \ldots, N, \tag{2}$$

where $\phi_i(s)$ is rd-continuous of the dynamical system and the corresponding state trajectory is denoted as $u_i(t, \phi_i)$.

Assumption 1. The outer-coupling configuration matrix of the complex networks (1) satisfies

$$E_{ij}^{(a)} = E_{ji}^{(a)} \ge 0\ (i \ne j), \quad E_{ii}^{(a)} = -\sum_{j=1, j\ne 1}^{N} E_{ij}^{(a)} \quad (a = 1, 2, 3)\ (i, j = 1, 2, \ldots, N). \tag{3}$$

Assumption 2 [39]**.** For any constants $\gamma_r^-,\ \gamma_r^+$, the active function satisfy:

$$\gamma_r^- \le \frac{g_r(u_i) - g_r(u_j)}{u_i - u_j} \le \gamma_r^+, \quad r = 1, 2, \ldots, n$$

We denote $\Upsilon_1 = diag(\gamma_1^+\gamma_1^-, \ldots, \gamma_n^+\gamma_n^-)$ and $\Upsilon_2 = diag(\frac{\gamma_1^+ + \gamma_1^-}{2}, \ldots, \frac{\gamma_n^+ \gamma_n^-}{2})$.

Lemma 2 [38]**.** *Let $y, g \in C_{rd}$, and $p \in \mathcal{R}^+$, If y is differentiable on $[t_0, +\infty) \cap \mathbb{T}$ such that*

$$y^{\Delta}(t) \le p(t)y(t) + g(t)$$

for all $[t_0, +\infty) \cap \mathbb{T}$, then

$$y(t) \le y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))g(s)\Delta s$$

for all $[t_0, +\infty) \cap \mathbb{T}$.

Definition 4 [29]**.** System (1) is said to be globally exponentially synchronized, if there exist constants $\epsilon > 0$, $M > 0$, such that for any $\phi_i(s) \in C([-\omega, 0]_{\mathbb{T}}, \mathbb{R}^n)$, there exists a large constant $\mathbb{T} > 0$ such that

$$\| u_i(t) - u_j(t) \| \le Me_{\ominus\epsilon}(t, 0) \quad t \in \mathbb{T}, \quad i, j = 1, 2, \ldots, N. \tag{4}$$

Remark 3. According to Definition 4, if $\epsilon \in \mathcal{R}^+$, we can get $e_{\ominus\epsilon}(t,0) \le e^{\ominus\epsilon t}$ by using mathematical induction. Thus, we have $\lim_{t\to\infty} e_{\ominus\epsilon}(t,0) = 0$. The addition '$\oplus$' is defined by $p \oplus q := p + q + \mu pq$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition '$\oplus$'. The additive inverse in this group is denoted by $\ominus p := -p/(1+\mu p)$. Then the subtraction $\ominus$ on the set of regressive functions is defined by $p\ominus q := p\oplus(\ominus q)$. It can be shown easily that $p \ominus q = -(p-q)/(1+\mu p)$. One can easily verify that if $g \in \mathcal{R}^+$, then $\ominus g \in \mathcal{R}^+$ where $\mathcal{R}^+$ is the set of all positively regressive elements of $\mathcal{R}$

Lemma 3 [40]**.** *The Kronecker product has the following properties:*

(1) $(\alpha A)\otimes B = A\otimes(\alpha B),\ \alpha$ *is a constant;*
(2) $(A+B)\otimes P = A\otimes P + B\otimes P$;
(3) $(A\otimes B)(P\otimes Q) = (AP)\otimes(BQ)$;
(4) $(A\otimes B)^T = A^T\otimes B^T$.

Lemma 4 [41]**.** *Let* $W = (\alpha_{ij})_{N\times N},\ \ M\in\mathbb{R}^{n\times n},\ \ u = (u_1^T, u_2^T, \ldots, u_N^T)^T$ *where* $u_i = (u_{i1}, u_{i2}, \ldots, u_{in})^T \in \mathbb{R}^n$ *and* $v = (v_1^T, v_2^T, \ldots, v_N^T)^T$ *where* $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})^T \in \mathbb{R}^n (i = 1, 2, \ldots., N)$*. If* $W = W^T$ *and each row sum of W is zero, then*

$$u^T(W\otimes M)v = -\sum_{1\le i<j\le N}\alpha_{ij}(u_i-u_j)^T M(v_i - v_j).$$

Lemma 5 [37]**.** *If for any constant matrix* $R \in \mathbb{R}^{n\times n},\ \ R = R^T > 0$*, scalar* $a < b$ *and a scalar function* $\phi : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ *such that the integrations concerned are well defined, the following inequality holds:*

$$-(b-a)\int_a^b \phi^T(s)R\phi(s)\Delta s \le -\left[\int_a^b \phi(s)\Delta s\right]^T R\left[\int_a^b \phi(s)\Delta s\right].$$

Lemma 6. *According to Lemma 5, for constant matrices*

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ & * & S_{33}\end{bmatrix} > 0$$

$S_{kq} \in \mathbb{R}^{n\times n},\ 1\le q\le 3,\ 1\le k\le 3,\ d>0,$ *and vector function* $u^{\Delta}(t) : [-d, 0]_{\mathbb{T}} \to \mathbb{R}^{n\times N}$ *such that the following integration is well-defined, it holds that*

$$-d\int_{t-d}^{t}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}^T \begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}\Delta s$$
$$\le -\psi^T(t)\begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} & -W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} & -W\otimes S_{23} \\ * & * & W\otimes S_{33} & -W\otimes S_{33} \\ * & * & * & W\otimes S_{33}\end{bmatrix}\psi(t)$$

where $\psi^T(t) = [(\int_{t-d}^t u(s)\Delta s)^T,\ (\int_{t-d}^t G(u(s))\Delta s)^T,\ u^T(t),\ u^T(t-d)]$.

Proof.

$$-d\int_{t-d}^{t}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}^T \begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}\Delta s$$
$$\le -\int_{t-d}^{t}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}^T\Delta s \begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\int_{t-d}^{t}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}\Delta s$$
$$= -\begin{bmatrix} \int_{t-d}^t u(s)\Delta s \\ \int_{t-d}^t G(u(s))\Delta s \\ u(t)-u(t-d)\end{bmatrix}^T \begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix} \int_{t-d}^t u(s)\Delta s \\ \int_{t-d}^t G(u(s))\Delta s \\ u(t)-u(t-d)\end{bmatrix}$$

$$= -\psi^T(t)\begin{bmatrix} I_N\otimes I_n & 0 & 0 \\ 0 & I_N\otimes I_n & 0 \\ 0 & 0 & I_N\otimes I_n \\ 0 & 0 & -I_N\otimes I_n \end{bmatrix} \times \begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33} \end{bmatrix}$$

$$\times \begin{bmatrix} I_N\otimes I_n & 0 & 0 \\ 0 & I_N\otimes I_n & 0 \\ 0 & 0 & I_N\otimes I_n \\ 0 & 0 & -I_N\otimes I_n \end{bmatrix}^T \psi(t)$$

$$= -\psi^T(t)\begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} & -W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} & -W\otimes S_{23} \\ * & * & W\otimes S_{33} & -W\otimes S_{33} \\ * & * & * & W\otimes S_{33} \end{bmatrix}\psi(t).$$

□

3. Main results

In this section, we derive globally exponential synchronization criteria of the complex dynamical networks on time scale.

For simplicity, let $u(t) = (u_1^T(t), u_2^T(t), \ldots, u_N^T(t))^T$, $J(t) = (J^T(t), J^T(t), \ldots, J^T(t))^T$, $G(u(t)) = (g^T(u_1(t)), g^T(u_2(t)), \ldots, g^T(u_N(t)))^T$. The error dynamical network (1) in virtue of the Kronecker product as

$$u^{\Delta}(t) = -(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) \\ + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\int_{t-d(t)}^{t} u(s)\Delta s. \tag{5}$$

Theorem 3.1. *For given constant $\mu > 0$, the dynamical network system (5) is globally exponentially synchronized on time scale $\mathbb{T}$ if there exist positive definite matrices P, Q_{kk}, $(k = 1, 2)$, R_{qq}, S_{qq}, $(q = 1, 2, 3)$, any real matrices $T_1, Q_{kq}, R_{kq}, S_{kq}, (1 \le k < q \le 3)$, and positive diagonal matrices H_1, H_2 with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all $(1 \le i < j \le N)$*

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ & Q_{22} \end{bmatrix} > 0, \quad R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & * & R_{33} \end{bmatrix} > 0, \quad S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ & * & S_{33} \end{bmatrix} > 0, \tag{6}$$

$$\Psi_{ij} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & \Psi_{16} & \Psi_{17} & \Psi_{18} \\ * & \Psi_{22} & \Psi_{23} & \Psi_{24} & \Psi_{25} & \Psi_{26} & \Psi_{27} & \Psi_{28} \\ * & * & \Psi_{33} & \Psi_{34} & \Psi_{35} & \Psi_{36} & \Psi_{37} & \Psi_{38} \\ * & * & * & \Psi_{44} & \Psi_{45} & \Psi_{46} & \Psi_{47} & \Psi_{48} \\ * & * & * & * & \Psi_{55} & \Psi_{56} & \Psi_{57} & \Psi_{58} \\ * & * & * & * & * & \Psi_{66} & \Psi_{67} & \Psi_{68} \\ * & * & * & * & * & * & \Psi_{77} & \Psi_{78} \\ * & * & * & * & * & * & * & \Psi_{88} \end{bmatrix} < 0, \tag{7}$$

where

$$\Psi_{11} = \epsilon P - (1+\mu\epsilon)PD - (1+\mu\epsilon)D^TP^T - (1+\mu\epsilon)NE_{ij}^{(1)}PL_1 - (1+\mu\epsilon)NE_{ij}^{(1)}L_1^TP^T + (1+\mu\epsilon)\mu \\ \times D^TPD + (1+\mu\epsilon)\mu NE_{ij}^{(1)}D^TPL_1 + (1+\mu\epsilon)\mu NE_{ij}^{(1)}L_1^TPD - (1+\mu\epsilon)\mu N(E_{ij}^{(1)})^2L_1^TPL_1 \\ + e_{\epsilon}(t+d,t)R_{11} + e_{\epsilon}(t+d,t)dS_{11} - \frac{1}{d}S_{33} - H_1\Upsilon_1,$$

$$\Psi_{12} = (1+\mu\epsilon)PB_1 - (1+\mu\epsilon)\mu D^TPB_1 - (1+\mu\epsilon)\mu NE_{ij}^{(1)}L_1^TPB_1 + e_{\epsilon}(t+d,t)R_{12} + e_{\epsilon}(t+d,t)dS_{12} \\ + H_1\Upsilon_2,\ \ \Psi_{13} = -(1+\mu\epsilon)NE_{ij}^{(2)}PL_2 + (1+\mu\epsilon)\mu NE_{ij}^{(2)}D^TPL_2 - (1+\mu\epsilon)\mu NE_{ij}^{(1)}E_{ij}^{(2)}L_1^TPL_2 \\ + \frac{1}{d}S_{33},\ \ \Psi_{14} = (1+\mu\epsilon)PB_2 - (1+\mu\epsilon)\mu D^TPB_1 - (1+\mu\epsilon)\mu NE_{ij}^{(1)}L_1^TPB_2,$$

$$\Psi_{15}=e_\epsilon(t+d,t)R_{13}+e_\epsilon(t+d,t)dS_{13}-D^TT_1^T-NE_{ij}^{(1)}L_1^TT_1^T,\ \ \Psi_{16}=-(1+\mu\epsilon)NE_{ij}^{(3)}PL_3$$
$$+(1+\mu\epsilon)\mu NE_{ij}^{(3)}D^TPL_3-(1+\mu\epsilon)\mu NE_{ij}^{(1)}E_{ij}^{(3)}L_1^TPL_3+(1+\mu\epsilon)Q_{11}^T-\frac{1}{d}S_{13}^T,$$
$$\Psi_{17}=(1+\mu\epsilon)Q_{12}-\frac{1}{d}S_{23}^T,\ \ \Psi_{22}=(1+\mu\epsilon)\mu B_1^TPB_1+e_\epsilon(t+d,t)R_{22}+e_\epsilon(t+d,t)dS_{22}-H_1,$$
$$\Psi_{23}=-(1+\mu\epsilon)\mu NE_{ij}^{(2)}B_1^TPL_2,\ \ \Psi_{24}=(1+\mu\epsilon)\mu B_1^TPB_2,\ \ \Psi_{25}=e_\epsilon(t+d,t)R_{23}+e_\epsilon(t+d,t)dS_{23}$$
$$+B_1^TT_1^T,\ \ \Psi_{26}=-(1+\mu\epsilon)\mu NE_{ij}^{(3)}B_1^TPL_3+(1+\mu\epsilon)Q_{12}^T,\ \ \Psi_{27}=(1+\mu\epsilon)Q_{22}^T,$$
$$\Psi_{33}=-(1+\mu\epsilon)\mu N(E_{ij}^{(2)})^2L_2^TPL_2-R_{11}-\frac{1}{d}S_{33}-H_2\Upsilon_1,\ \ \Psi_{34}=-(1+\mu\epsilon)\mu NE_{ij}^{(2)}L_2^TPB_2-R_{12}$$
$$+H_2\Upsilon_2,\ \Psi_{35}=-NE_{ij}^{(2)}L_2^TT_1^T,\ \ \Psi_{36}=-(1+\mu\epsilon)\mu NE_{ij}^{(2)}E_{ij}^{(3)}L_2^TPL_3-(1+\mu\epsilon)Q_{11}^T+\frac{1}{d}S_{13}^T,$$
$$\Psi_{37}=-(1+\mu\epsilon)Q_{12}+\frac{1}{d}S_{23}^T,\ \ \Psi_{38}=-R_{13},\ \ \Psi_{44}=(1+\mu\epsilon)\mu B_2^TPB_2-R_{22}-H_2,\ \ \Psi_{45}=B_2^TT_1^T,$$
$$\Psi_{46}=-(1+\mu\epsilon)\mu NE_{ij}^{(3)}B_2^TPL_3-(1+\mu\epsilon)Q_{12}^T,\ \ \Psi_{47}=-(1+\mu\epsilon)Q_{22}^T,\ \ \Psi_{48}=-R_{23},\ \ \Psi_{55}=e_\epsilon(t+d,t)$$
$$\times R_{33}+e_\epsilon(t+d,t)dS_{33}-T_1-T_1^T,\ \ \Psi_{56}=-NE_{ij}^{(3)}T_1L_3,\ \ \Psi_{66}=-(1+\mu\epsilon)\mu N(E_{ij}^{(3)})^2L_3^TPL_3$$
$$+\epsilon Q_{11}-\frac{1}{d}S_{11},\ \ \Psi_{67}=\epsilon Q_{12}-\frac{1}{d}S_{12},\ \ \Psi_{77}=\epsilon Q_{22}-\frac{1}{d}S_{22},\ \ \Psi_{88}=-R_{33},$$
$$\Psi_{18}=\Psi_{28}=\Psi_{57}=\Psi_{58}=\Psi_{68}=\Psi_{78}=0.$$

Proof. Let W be defined in Lemma 4, consider the following Lyapunov–Krasovskii functional candidate for model (5) as

$$V(t)=\sum_{r=1}^{4}V_r(t), \tag{8}$$

where

$$V_1(t)=e_\epsilon(t,0)u^T(t)(W\otimes P)u(t),$$
$$V_2(t)=e_\epsilon(t,0)\begin{bmatrix}\int_{t-d(t)}^{t}u(s)\Delta s\\ \int_{t-d(t)}^{t}G(u(s))\Delta s\end{bmatrix}^T\begin{bmatrix}W\otimes Q_{11} & W\otimes Q_{12}\\ * & W\otimes Q_{22}\end{bmatrix}\begin{bmatrix}\int_{t-d(t)}^{t}u(s)\Delta s\\ \int_{t-d(t)}^{t}G(u(s))\Delta s\end{bmatrix},$$
$$V_3(t)=\int_{t-d(t)}^{t}e_\epsilon(s+d,0)\begin{bmatrix}u(s)\\ G(u(s))\\ u^\Delta(s)\end{bmatrix}^T\begin{bmatrix}W\otimes R_{11} & W\otimes R_{12} & W\otimes R_{13}\\ * & W\otimes R_{22} & W\otimes R_{23}\\ * & * & W\otimes R_{33}\end{bmatrix}\begin{bmatrix}u(s)\\ G(u(s))\\ u^\Delta(s)\end{bmatrix}\Delta s,$$
$$V_4(t)=\int_{-d(t)}^{0}\int_{t+\theta}^{t}e_\epsilon(s+d,0)\begin{bmatrix}u(s)\\ G(u(s))\\ u^\Delta(s)\end{bmatrix}^T\begin{bmatrix}W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13}\\ * & W\otimes S_{22} & W\otimes S_{23}\\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix}u(s)\\ G(u(s))\\ u^\Delta(s)\end{bmatrix}$$
$$\times\Delta s\Delta\theta,$$

and

$$W=\begin{bmatrix}N-1 & -1 & \cdots & -1\\ -1 & N-1 & \cdots & -1\\ \cdots & \cdots & \cdots & \cdots\\ -1 & -1 & \cdots & N-1\end{bmatrix}_{N\times N}.$$

Noting that $WE=EW=NE$, for any matrix M with appropriate dimension, we obtain

$$\begin{aligned}(E\otimes L)^T(W\otimes M)(E\otimes L)&=(E^T\otimes L^T)(W\otimes M)(E\otimes L)\\ &=(E^TWE)\otimes(L^TML)\\ &=NE^2\otimes(LML).\\ (W\otimes M)(E\otimes L)&=(WE)\otimes(ML)\\ &=(NE)\otimes(ML).\end{aligned}$$

Calculating the delta derivative of $V(t)$, we obtain that

$$V_1^\Delta(t)=e_\epsilon^\Delta(t,0)u^T(t)(W\otimes P)u(t)+e_\epsilon(\sigma(t),0)(u^T(t)(W\otimes P)u(t))\Delta$$

$$
\begin{aligned}
&= \epsilon e_\epsilon(t,0)u^T(t)(W\otimes P)u(t) + (1+\mu(t)\epsilon)e_\epsilon(t,0)[u^{\Delta T}(t)(W\otimes P)u(t)\\
&\quad + u^T(t)(W\otimes P)u^\Delta(t) + \mu(t)u^{\Delta T}(t)(W\otimes P)u^\Delta(t)]\\
&\le e_\epsilon(t,0)\Bigg\{\epsilon u^T(t)(W\otimes P)u(t) + (1+\mu\epsilon)\Bigg[(-(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)\\
&\quad\times G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\\
&\quad\times\int_{t-d(t)}^{t} u(s)\Delta s)^T(W\otimes P)u(t) + u^T(t)(W\otimes P)(-(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t))\\
&\quad + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\\
&\quad\times\int_{t-d(t)}^{t} u(s)\Delta s) + \mu(-(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t)))\\
&\quad + J(t) + (E^{(1)}\otimes L_1)u(t) + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\int_{t-d(t)}^{t} u(s)\Delta s)^T(W\otimes P)\\
&\quad\times(-(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t)\\
&\quad + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\int_{t-d(t)}^{t} u(s)\Delta s)\Bigg]\Bigg\},
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
V_2^\Delta(t) &\le \epsilon e_\epsilon(t,0)\begin{bmatrix}\int_{t-d(t)}^{t} u(s)\Delta s\\ \int_{t-d(t)}^{t} G(u(s))\Delta s\end{bmatrix}^T\begin{bmatrix}W\otimes Q_{11} & W\otimes Q_{12}\\ * & W\otimes Q_{22}\end{bmatrix}\begin{bmatrix}\int_{t-d(t)}^{t} u(s)\Delta s\\ \int_{t-d(t)}^{t} G(u(s))\Delta s\end{bmatrix}\\
&\quad + 2(1+\mu\epsilon)e_\epsilon(t,0)\begin{bmatrix}\int_{t-d(t)}^{t} u(s)\Delta s\\ \int_{t-d(t)}^{t} G(u(s))\Delta s\end{bmatrix}^T\begin{bmatrix}W\otimes Q_{11} & W\otimes Q_{12}\\ * & W\otimes Q_{22}\end{bmatrix}\times\begin{bmatrix}u(t)-u(t-d(t))\\ G(u(t))-G(u(t-d(t)))\end{bmatrix}\\
&= e_\epsilon(t,0)\Bigg\{\epsilon\begin{bmatrix}\int_{t-d(t)}^{t} u(s)\Delta s\\ \int_{t-d(t)}^{t} G(u(s))\Delta s\end{bmatrix}^T\begin{bmatrix}W\otimes Q_{11} & W\otimes Q_{12}\\ * & W\otimes Q_{22}\end{bmatrix}\begin{bmatrix}\int_{t-d(t)}^{t} u(s)\Delta s\\ \int_{t-d(t)}^{t} G(u(s))\Delta s\end{bmatrix}\\
&\quad + (1+\mu\epsilon)\Bigg[2\Bigg\{\left(\int_{t-d(t)}^{t} u(s)\Delta s\right)^T(W\otimes Q_{11}) + \left(\int_{t-d(t)}^{t} G(u(s))\Delta s\right)^T(W\otimes Q_{12}^T)\Bigg\}[u(t)\\
&\quad - u(t-d(t))] + 2\Bigg\{\left(\int_{t-d(t)}^{t} u(s)\Delta s\right)^T(W\otimes Q_{12}) + \left(\int_{t-d(t)}^{t} G(u(s))\Delta s\right)^T(W\otimes Q_{22})\Bigg\}\\
&\quad\times[G(u(t)) - G(u(t-d(t)))]\Bigg]\Bigg\},
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
V_3^\Delta(t) &= e_\epsilon(t+d,0)\begin{bmatrix}u(t)\\ G(u(t))\\ u^\Delta(t)\end{bmatrix}^T\begin{bmatrix}W\otimes R_{11} & W\otimes R_{12} & W\otimes R_{13}\\ * & W\otimes R_{22} & W\otimes R_{23}\\ * & * & W\otimes R_{33}\end{bmatrix}\begin{bmatrix}u(t)\\ G(u(t))\\ u^\Delta(t)\end{bmatrix}\\
&\quad - e_\epsilon(t,0)\begin{bmatrix}u(t-d(t))\\ G(u(t-d(t)))\\ u^\Delta(t-d(t))\end{bmatrix}^T\begin{bmatrix}W\otimes R_{11} & W\otimes R_{12} & W\otimes R_{13}\\ * & W\otimes R_{22} & W\otimes R_{23}\\ * & * & W\otimes R_{33}\end{bmatrix}\begin{bmatrix}u(t-d(t))\\ G(u(t-d(t)))\\ u^\Delta(t-d(t))\end{bmatrix}\\
&= e_\epsilon(t,0)\Bigg\{e_\epsilon(t+d,t)\begin{bmatrix}u(t)\\ G(u(t))\\ u^\Delta(t)\end{bmatrix}^T\begin{bmatrix}W\otimes R_{11} & W\otimes R_{12} & W\otimes R_{13}\\ * & W\otimes R_{22} & W\otimes R_{23}\\ * & * & W\otimes R_{33}\end{bmatrix}\begin{bmatrix}u(t)\\ G(u(t))\\ u^\Delta(t)\end{bmatrix}\\
&\quad - \begin{bmatrix}u(t-d(t))\\ G(u(t-d(t)))\\ u^\Delta(t-d(t))\end{bmatrix}^T\begin{bmatrix}W\otimes R_{11} & W\otimes R_{12} & W\otimes R_{13}\\ * & W\otimes R_{22} & W\otimes R_{23}\\ * & * & W\otimes R_{33}\end{bmatrix}\begin{bmatrix}u(t-d(t))\\ G(u(t-d(t)))\\ u^\Delta(t-d(t))\end{bmatrix}\Bigg\},
\end{aligned} \tag{11}
$$

$$V_4^{\Delta}(t) \le e_\epsilon(t+d,0)d\begin{bmatrix} u(t) \\ G(u(t)) \\ u^{\Delta}(t)\end{bmatrix}^T\begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix} u(t) \\ G(u(t)) \\ u^{\Delta}(t)\end{bmatrix}$$

$$- e_\epsilon(t,0)\int_{t-d(t)}^{t}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}^T\begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix} u(s) \\ G(u(s)) \\ u^{\Delta}(s)\end{bmatrix}\Delta s$$

$$\le e_\epsilon(t,0)\Bigg\{e_\epsilon(t+d,t)d\begin{bmatrix} u(t) \\ G(u(t)) \\ u^{\Delta}(t)\end{bmatrix}^T\begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} \\ * & * & W\otimes S_{33}\end{bmatrix}\begin{bmatrix} u(t) \\ G(u(t)) \\ u^{\Delta}(t)\end{bmatrix}$$

$$- \frac{1}{d}\psi^T(t)\begin{bmatrix} W\otimes S_{11} & W\otimes S_{12} & W\otimes S_{13} & -W\otimes S_{13} \\ * & W\otimes S_{22} & W\otimes S_{23} & -W\otimes S_{23} \\ * & * & W\otimes S_{33} & -W\otimes S_{33} \\ * & * & * & W\otimes S_{33}\end{bmatrix}\psi(t)\Bigg\}, \tag{12}$$

where $\psi^T(t) = [(\int_{t-d(t)}^t u(s)\Delta s)^T,\ (\int_{t-d(t)}^t G(u(s))\Delta s)^T,\ u^T(t),\ u^T(t-d(t))]$.

Noting that $W\otimes$ (any arbitrary constant matrix) $J(t) = 0$, from Lemma 4, we can obtain

$$V_1^{\Delta}(t)e_{\Theta\epsilon}(t,0) \le \sum_{1\le i<j\le N}\Bigg\{(u_i(t)-u_j(t))^T\Bigg[\{\epsilon P - (1+\mu\epsilon)PD - (1+\mu\epsilon)D^TP^T - (1+\mu\epsilon)NE_{ij}^{(1)}$$
$$\times PL_1 - (1+\mu\epsilon)NE_{ij}^{(1)}L_1^TP^T + (1+\mu\epsilon)\mu D^TPD + (1+\mu\epsilon)\mu NE_{ij}^{(1)}D^TPL_1$$
$$+ (1+\mu\epsilon)\mu NE_{ij}^{(1)}L_1^TPD - (1+\mu\epsilon)\mu N(E_{ij}^{(1)})^2L_1^TPL_1\}(u_i(t)-u_j(t)) + \{(1+\mu\epsilon)$$
$$\times PB_1 - (1+\mu\epsilon)\mu D^TPB_1 - (1+\mu\epsilon)\mu NE_{ij}^{(1)}L_1^TPB_1\}(g(u_i(t)) - g(u_j(t)))$$
$$\times +\{(1+\mu\epsilon)PB_2 - (1+\mu\epsilon)\mu D^TPB_1 - (1+\mu\epsilon)\mu NE_{ij}^{(1)}L_1^TPB_2\}(g(u_i(t-d(t)))$$
$$- g(u_j(t-d(t)))) + \{-(1+\mu\epsilon)NE_{ij}^{(2)}PL_2 + (1+\mu\epsilon)\mu NE_{ij}^{(2)}D^TPL_2 - (1+\mu\epsilon)\mu$$
$$\times NE_{ij}^{(1)}E_{ij}^{(2)}L_1^TPL_2\}(u_i(t-d(t)) - u_j(t-d(t))) + \{-(1+\mu\epsilon)NE_{ij}^{(3)}PL_3$$
$$+ (1+\mu\epsilon)\mu NE_{ij}^{(3)}D^TPL_3 - (1+\mu\epsilon)\mu NE_{ij}^{(1)}E_{ij}^{(3)}L_1^TPL_3\}\left(\int_{t-d(t)}^t (u_i(s)-u_j(s))\Delta s\right)\Bigg]$$
$$+ (g(u_i(t)) - g(u_j(t)))^T\Bigg[\{(1+\mu\epsilon)B_1^TP^T - (1+\mu\epsilon)\mu B_1^TPD - (1+\mu\epsilon)\mu NE_{ij}^{(1)}B_1^T$$
$$\times PL_1\}(u_i(t)-u_j(t)) + \{(1+\mu\epsilon)\mu B_1^TPB_1\}(g(u_i(t)) - g(u_j(t))) + \{(1+\mu\epsilon)\mu B_1^T$$
$$\times PB_2\}(g(u_i(t-d(t))) - g(u_j(t-d(t)))) + \{-(1+\mu\epsilon)\mu NE_{ij}^{(2)}B_1^TPL_2\}(u_i(t-d(t))$$
$$- u_j(t-d(t))) + \{-(1+\mu\epsilon)\mu NE_{ij}^{(3)}B_1^TPL_3\}\left(\int_{t-d(t)}^t (u_i(s)-u_j(s))\Delta s\right)\Bigg]$$
$$+ (g(u_i(t-d(t))) - g(u_j(t-d(t))))^T\Bigg[(1+\mu\epsilon)B_2^TP^T - (1+\mu\epsilon)\mu B_2^TPD - (1+\mu\epsilon)$$
$$\times \mu NE_{ij}^{(1)}B_2^TPL_1\Bigg\}(u_i(t)-u_j(t)) + \{(1+\mu\epsilon)\mu B_2^TPB_1\}(g(u_i(t)) - g(u_j(t)))$$
$$+ \{(1+\mu\epsilon)\mu B_2^TPB_2\}(g(u_i(t-d(t))) - g(u_j(t-d(t)))) + \{-(1+\mu\epsilon)\mu NE_{ij}^{(2)}B_2^T$$
$$\times PL_2\}(u_i(t-d(t)) - u_j(t-d(t))) + \{-(1+\mu\epsilon)\mu NE_{ij}^{(3)}B_2^TPL_3\}\left(\int_{t-d(t)}^t (u_i(s)-u_j(s))\Delta s\right)\Bigg]$$
$$+ (u_i(t-d(t)) - u_j(t-d(t)))^T\Bigg[\{-(1+\mu\epsilon)NE_{ij}^{(2)}L_2^TP^T + (1+\mu\epsilon)\mu$$
$$\times NE_{ij}^{(2)}L_2^TPD - (1+\mu\epsilon)\mu NE_{ij}^{(2)}E_{ij}^{(1)}L_2^TPL_1\}(u_i(t)-u_j(t)) + \{-(1+\mu\epsilon)\mu NE_{ij}^{(2)}$$
$$\times L_2^TPB_1\}(g(u_i(t)) - g(u_j(t))) + \{-(1+\mu\epsilon)\mu NE_{ij}^{(2)}L_2^TPB_2\}(g(u_i(t-d(t)))$$

$$
\begin{aligned}
&- g(u_j(t-d(t)))) + \{-(1+\mu\epsilon)\mu N (E_{ij}^{(2)})^2 L_2^T P L_2\}(u_i(t-d(t)) - u_j(t-d(t)))\\
&+ \{-(1+\mu\epsilon)\mu N E_{ij}^{(2)} E_{ij}^{(3)} L_2^T P L_3\}\left(\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\right)\Bigg]\\
&+ \left(\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\right)^T\Bigg[\{-(1+\mu\epsilon)N E_{ij}^{(3)} L_3^T P^T + (1+\mu\epsilon)\mu N E_{ij}^{(3)} L_3^T P D - (1+\mu\epsilon)\mu N E_{ij}^{(2)}\\
&\times E_{ij}^{(1)} L_3^T P L_1\}(u_i(t)-u_j(t)) + \{-(1+\mu\epsilon)\mu N E_{ij}^{(3)} L_3^T P B_1\}(g(u_i(t)) - g(u_j(t)))\\
&+ \{-(1+\mu\epsilon)\mu N E_{ij}^{(3)} L_3^T P B_2\}(g(u_i(t-d(t))) - g(u_j(t-d(t)))) + \{-(1+\mu\epsilon)\mu N\\
&\times E_{ij}^{(3)} E_{ij}^{(2)} L_3^T P L_2\}(u_i(t-d(t)) - u_j(t-d(t))) + \{-(1+\mu\epsilon)\mu N (E_{ij}^{(3)})^2 L_3^T P L_3\}\\
&\times \left(\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\right)\Bigg] \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
V_2^{\Delta}(t)e_{\Theta\epsilon}(t,0) \le \sum_{1\le i<j\le N}&\left\{\epsilon\begin{bmatrix}\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\\ \int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\end{bmatrix}^T\begin{bmatrix}Q_{11} & Q_{12}\\ * & Q_{22}\end{bmatrix}\right.\\
&\times\begin{bmatrix}\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\\ \int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\end{bmatrix} + (1+\mu\epsilon)\Bigg[2\Bigg\{\left(\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\right)^T\\
&\times Q_{11} + \left(\int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\right)^T Q_{12}^T\Bigg\}[(u_i(t)-u_j(t)) - (u_i(t-d(t)) - u_j(t-d(t)))]\\
&+ 2\Bigg\{\left(\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\right)^T Q_{12} + \left(\int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\right)^T Q_{22}\Bigg\}\\
&\times\left.[(g(u_i(t)) - g(u_j(t))) - (g(u_i(t-d(t))) - g(u_j(t-d(t))))]\Bigg]\right\}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
V_3^{\Delta}(t)e_{\Theta\epsilon}(t,0) = \sum_{1\le i<j\le N}&\left\{e_\epsilon(t+d,t)\begin{bmatrix}(u_i(t)-u_j(t))\\ (g(u_i(t))-g(u_j(t)))\\ (u_i^{\Delta}(t)-u_j^{\Delta}(t))\end{bmatrix}^T\begin{bmatrix}R_{11} & R_{12} & R_{13}\\ * & R_{22} & R_{23}\\ * & * & R_{33}\end{bmatrix}\right.\\
&\times\begin{bmatrix}(u_i(t)-u_j(t))\\ (g(u_i(t))-g(u_j(t)))\\ (u_i^{\Delta}(t)-u_j^{\Delta}(t))\end{bmatrix} - \begin{bmatrix}(u_i(t-d(t))-u_j(t-d(t)))\\ (g(u_i(t-d(t)))-g(u_j(t-d(t))))\\ (u_i^{\Delta}(t-d(t))-u_j^{\Delta}(t-d(t)))\end{bmatrix}^T\\
&\times\left.\begin{bmatrix}R_{11} & R_{12} & R_{13}\\ * & R_{22} & R_{23}\\ * & * & R_{33}\end{bmatrix}\begin{bmatrix}(u_i(t-d(t))-u_j(t-d(t)))\\ (g(u_i(t-d(t)))-g(u_j(t-d(t))))\\ (u_i^{\Delta}(t-d(t))-u_j^{\Delta}(t-d(t)))\end{bmatrix}\right\}, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
V_4^{\Delta}(t)e_{\Theta\epsilon}(t,0) \le \sum_{1\le i<j\le N}&\left\{e_\epsilon(t+d,t)d\begin{bmatrix}(u_i(t)-u_j(t))\\ (g(u_i(t))-g(u_j(t)))\\ (u_i^{\Delta}(t)-u_j^{\Delta}(t))\end{bmatrix}^T\begin{bmatrix}S_{11} & S_{12} & S_{13}\\ * & S_{22} & S_{23}\\ * & * & S_{33}\end{bmatrix}\right.\\
&\times\begin{bmatrix}(u_i(t)-u_j(t))\\ (g(u_i(t))-g(u_j(t)))\\ (u_i^{\Delta}(t)-u_j^{\Delta}(t))\end{bmatrix} - \frac{1}{d}\begin{bmatrix}\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\\ \int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\\ (u_i(t)-u_j(t))\\ (u_i(t-d(t))-u_j(t-d(t)))\end{bmatrix}\\
&\times\left.\begin{bmatrix}S_{11} & S_{12} & S_{13} & -S_{13}\\ * & S_{22} & S_{23} & -S_{23}\\ * & * & S_{33} & -S_{33}\\ * & * & * & S_{33}\end{bmatrix}\begin{bmatrix}\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\\ \int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\\ (u_i(t)-u_j(t))\\ (u_i(t-d(t))-u_j(t-d(t)))\end{bmatrix}\right\}. \qquad (16)
\end{aligned}
$$

From (5), the following equation holds for any matrices $T_1 \in R^{n\times n}$:

$$0 = 2u^{\Delta T}(t)(W\otimes T_1)\bigg(-u^{\Delta}(t) - (I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\int_{t-d(t)}^{t} u(s)\Delta s.\bigg). \quad (17)$$

According to [26] and Assumption 2, for any diagonal matrix $H_1 > 0, H_2 > 0$ with appropriate dimensions, it follows that

$$0 \le \begin{bmatrix} u_i(t)-u_j(t) \\ g(u_i(t))-g(u_j(t)) \end{bmatrix}^T \begin{bmatrix} -H_1\Upsilon_1 & H_1\Upsilon_2 \\ * & -H_1 \end{bmatrix} \begin{bmatrix} u_i(t)-u_j(t) \\ g(u_i(t))-g(u_j(t)) \end{bmatrix} + \begin{bmatrix} u_i(t-d(t))-u_j(t-d(t)) \\ g(u_i(t-d(t)))-g(u_j(t-d(t))) \end{bmatrix}^T \begin{bmatrix} -H_2\Upsilon_1 & H_2\Upsilon_2 \\ * & -H_2 \end{bmatrix} \begin{bmatrix} u_i(t-d(t))-u_j(t-d(t)) \\ g(u_i(t-d(t)))-g(u_j(t-d(t))) \end{bmatrix}. \quad (18)$$

From (13)–(18), we get

$$V^{\Delta}(t)e_{\Theta\epsilon}(t,0) \le \sum_{1\le i<j\le N} \zeta_{ij}^T(t)\Psi_{ij}\zeta_{ij}(t), \quad (19)$$

where

$$\zeta_{ij}^T(t) = \bigg[(u_i(t)-u_j(t))^T \ (g(u_i(t))-g(u_j(t)))^T \ (u_i(t-d(t))-u_j(t-d(t)))^T \ (g(u_i(t-d(t))) - g(u_j(t-d(t))))^T \ (u^{\Delta}(t)-u^{\Delta}(t))^T \left(\int_{t-d(t)}^{t}(u_i(s)-u_j(s))\Delta s\right)^T \left(\int_{t-d(t)}^{t}(g(u_i(s))-g(u_j(s)))\Delta s\right)^T \ (u_i^{\Delta}(t-d(t))-u_j^{\Delta}(t-d(t)))^T\bigg]$$

and Ψ_{ij} is as defined in (7).

From the LMIs (7), we have

$$V^{\Delta}(t)e_{\Theta\epsilon}(t,0) \le \sum_{1\le i<j\le N} \zeta_{ij}^T(t)\Psi_{ij}\zeta_{ij}(t) < 0.$$

This implies that $V(t) < V(0)$, $V(t)$ is a bounded function. Hence, $e_\epsilon(t,0)u^T(t)(W\otimes P)u(t)$ is also bounded, and it yields that

$$\lambda_{min}(P) \parallel u_i(t)-u_j(t) \parallel^2 \le \sum_{1\le i<j\le N} (u_i(t)-u_j(t))^T P(u_i(t)-u_j(t)) = O(e_{\Theta\epsilon}(t,0)).$$

According to Definition 4, we can conclude that dynamical system (1) is globally exponentially synchronized. The proof is completed. □

Remark 4. In this paper, we investigate time-varying delay on time scales by referring the paper [52] and we study the two cases synchronization problems on time scales.

1. We have dealt with the continuous-time synchronization in the complex dynamical network as $\mathbb{T} = [t_0, \infty)$, then $\sigma(t) = t$, $\mu(t) = 0$, for all $t \in \mathbb{T}$.
2. For the discrete-time synchronization in the complex dynamical network, we considered $\mathbb{T} = \mathbb{N}$, then $\sigma(k) = k+1$, $\mu(k) = 1$, for all $t = k \in \mathbb{T}$. We obtain that $e_\epsilon(t,0) = (1+\epsilon)^k$, $e_\epsilon(t+d,t) = (1+\epsilon)^d$.

Let us derive the following much less conservative synchronization criteria in a similar fashion as in the proof of Theorem 3.1.

Remark 5. When $L_3 = 0$, the dynamical network system (5) correspondingly reduces to

$$u^{\Delta}(t) = -(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) + (E^{(2)}\otimes L_2)u(t-d(t)). \quad (20)$$

Corollary 3.2. *For given constant $\mu > 0$, the dynamical network system (20) is globally exponentially synchronized on time scale $\mathbb{T}$ if there exist positive definite matrices P, Q_{kk}, $(k=1,2)$, R_{qq}, S_{qq}, $(q=1,2,3)$, any real matrices $T_1, Q_{kq}, R_{kq}, S_{kq}, (1\le k<q \le 3)$, and positive diagonal matrices H_1, H_2 with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all $(1\le i<j\le N)$*

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ & Q_{22} \end{bmatrix} > 0, \quad R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & * & R_{33} \end{bmatrix} > 0, \quad S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ & * & S_{33} \end{bmatrix} > 0, \quad (21)$$

$$\Psi = \Psi_{(r\times s)} < 0 \quad (r,s = 1,2,\dots,8), \tag{22}$$

where

$$\Psi_{16} = (1+\mu\epsilon)Q_{11}^T - \frac{1}{d}S_{13}^T,\ \ \Psi_{26} = (1+\mu\epsilon)Q_{12}^T,\ \ \Psi_{36} = -(1+\mu\epsilon)Q_{11}^T + \frac{1}{d}S_{13}^T,\ \ \Psi_{46} = -(1+\mu\epsilon)Q_{12}^T,$$

$\Psi_{56} = 0,\ \Psi_{66} = \epsilon Q_{11} - \frac{1}{d}S_{11}$, *and* $\Psi_{(r\times s)}$ $(r,s=1,2,\dots,8)$ *are same as defined in Theorem 3.1.*

Remark 6. When $L_2 = 0$, the dynamical network system (5) correspondingly reduces to

$$\begin{aligned} u^{\Delta}(t) &= -(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) \\ &\quad + (E^{(3)}\otimes L_3)\int_{t-d(t)}^{t} u(s)\Delta s. \end{aligned} \tag{23}$$

Corollary 3.3. *For given constant* $\mu > 0$, *the dynamical network system (23) is globally exponentially synchronized on time scale* $\mathbb{T}$ *if there exist positive definite matrices* $P,\ Q_{kk},\ (k=1,2),\ R_{qq},\ S_{qq},\ (q=1,2,3)$, *any real matrices* $T_1, Q_{kq}, R_{kq}, S_{kq}, (1\le k<q \le 3)$, *and positive diagonal matrices* H_1, H_2 *with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all* $(1\le i<j\le N)$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ & Q_{22} \end{bmatrix} > 0,\quad R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & * & R_{33} \end{bmatrix} > 0,\quad S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ & * & S_{33} \end{bmatrix} > 0, \tag{24}$$

$$\Psi = \Psi_{(r\times s)} < 0 \quad (r,s = 1,2,\dots,8), \tag{25}$$

where

$$\Psi_{13} = \frac{1}{d}S_{33},\ \ \Psi_{23} = 0,\ \ \Psi_{33} = -R_{11} - \frac{1}{d}S_{33} - H_2\Upsilon_1,\ \ \Psi_{34} = -R_{12} + H_2\Upsilon_2,\ \ \Psi_{35} = 0,$$

$\Psi_{36} = -(1+\mu\epsilon)Q_{11}^T + \frac{1}{d}S_{13}^T$, *and* $\Psi_{(r\times s)}$ $(r,s=1,2,\dots,8)$ *are same as defined in Theorem 3.1.*

Remark 7. When $L_2 = L_3 = 0$, the dynamical network system (5) correspondingly reduces to

$$u^{\Delta}(t) = -(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t). \tag{26}$$

Corollary 3.4. *For given constant* $\mu > 0$, *the dynamical network system (26) is globally exponentially synchronized on time scale* $\mathbb{T}$ *if there exist positive definite matrices* $P,\ Q_{kk},\ (k=1,2),\ R_{qq},\ S_{qq},\ (q=1,2,3)$, *any real matrices* $T_1, Q_{kq}, R_{kq}, S_{kq}, (1\le k<q \le 3)$, *and positive diagonal matrices* H_1, H_2 *with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all* $(1\le i<j\le N)$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ & Q_{22} \end{bmatrix} > 0,\quad R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & * & R_{33} \end{bmatrix} > 0,\quad S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ & * & S_{33} \end{bmatrix} > 0, \tag{27}$$

$$\Psi = \Psi_{(r\times s)} < 0 \quad (r,s = 1,2,\dots,8), \tag{28}$$

where

$$\begin{aligned} &\Psi_{13} = \frac{1}{d}S_{33},\quad \Psi_{16} = (1+\mu\epsilon)Q_{11}^T - \frac{1}{d}S_{13}^T,\quad \Psi_{23} = 0,\quad \Psi_{26} = (1+\mu\epsilon)Q_{12}^T,\quad \Psi_{33} = -R_{11} - \frac{1}{d}S_{33} - H_2\Upsilon_1, \\ &\Psi_{34} = -R_{12} + H_2\Upsilon_2,\quad \Psi_{35} = 0,\quad \Psi_{36} = -(1+\mu\epsilon)Q_{11}^T + \frac{1}{d}S_{13}^T,\quad \Psi_{46} = -(1+\mu\epsilon)Q_{12}^T,\quad \Psi_{56} = 0, \end{aligned}$$

$\Psi_{66} = \epsilon Q_{11} - \frac{1}{d}S_{11}$, *and* $\Psi_{(r\times s)}$ $(r,s=1,2,\dots,8)$ *are same as defined in Theorem 3.1.*

4. Illustrative examples

In this section, a numerical examples are presented to demonstrate the effectiveness of the results derived above.

Example 1. Consider the following complex dynamical networks with hybrid coupling delays on time scales with 3-nodes:

$$\begin{aligned} u^{\Delta}(t) &= -(I_N\otimes D)u(t) + (I_N\otimes B_1)G(u(t)) + (I_N\otimes B_2)G(u(t-d(t))) + J(t) + (E^{(1)}\otimes L_1)u(t) \\ &\quad + (E^{(2)}\otimes L_2)u(t-d(t)) + (E^{(3)}\otimes L_3)\int_{t-d(t)}^{t} u(s)\Delta s. \end{aligned}$$

where the relevant parameters are given as follows:

$$D = diag(0.17, 0.15), \quad \Upsilon_1 = diag(1.5, 1.5), \quad \Upsilon_2 = diag(0.5, 0.5),$$

$$B_1 = \begin{bmatrix} 0.2 & -1.5 \\ 1.4 & -1.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.12 & -0.13 \\ 0.12 & -0.11 \end{bmatrix}, \quad L_1 = L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

The time scale is chosen as $\mathbb{T} = \cup_{i=1}^{\infty}[t_i, t_i + k_i]$, where $t_0 = 0, t_{i+1} = t_i + k_i + i$ and k_i are some real numbers randomly selected from (0, 1]. We choose the coupling matrices as

$$E^{(1)} = \begin{bmatrix} -4 & 3 & 0 \\ 1 & -5 & 2 \\ 0 & 1 & -2 \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 1 & -2 \end{bmatrix}, \quad E^{(3)} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 1 & -2 \end{bmatrix}.$$

Let $d = 0.5, \ \mu = 0.5$. Then by using the Matlab LMI Toolbox, we solve the LMIs in Theorem 3.1, we obtain the feasible solutions as follows

$$P = \begin{bmatrix} 0.8777 & -0.2271 \\ -0.2271 & 0.3723 \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} 15.2928 & 0.7980 \\ 0.7980 & 18.1563 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} -21.9802 & 1.1115 \\ 1.1115 & -18.9973 \end{bmatrix},$$

$$Q_{22} = \begin{bmatrix} 22.8734 & -0.0425 \\ -0.0425 & 23.0065 \end{bmatrix}, \quad R_{11} = \begin{bmatrix} 490.0173 & 10.0873 \\ 10.0873 & 513.6800 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} -135.9287 & 3.9602 \\ 3.9602 & -182.9125 \end{bmatrix},$$

$$R_{13} = \begin{bmatrix} -28.5908 & 5.7256 \\ 5.7256 & -14.2968 \end{bmatrix}, \quad R_{22} = \begin{bmatrix} 293.6943 & -4.4863 \\ -4.4863 & 284.2086 \end{bmatrix}, \quad R_{23} = \begin{bmatrix} 18.4266 & -2.9720 \\ -2.9720 & 11.9164 \end{bmatrix},$$

$$R_{33} = \begin{bmatrix} 6.7405 & -1.1847 \\ -1.1847 & 3.5474 \end{bmatrix}, \quad S_{11} = \begin{bmatrix} 791.8948 & 14.6112 \\ 14.6112 & 692.9432 \end{bmatrix}, \quad S_{12} = \begin{bmatrix} -351.4432 & 18.5923 \\ 18.5923 & -325.1264 \end{bmatrix},$$

$$S_{13} = \begin{bmatrix} -246.6694 & 5.4999 \\ 5.4999 & -179.2267 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} 546.4348 & -2.7393 \\ -2.7393 & 550.2723 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} -37.9290 & 4.7776 \\ 4.7776 & -6.6002 \end{bmatrix},$$

$$S_{33} = \begin{bmatrix} 5.9131 & -0.9373 \\ -0.9373 & 3.3085 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 21.7770 & -3.7119 \\ -3.7119 & 12.0209 \end{bmatrix}, \quad H_1 = 10^3 \begin{bmatrix} 1.5156 & 0 \\ 0 & 1.5156 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 655.0934 & 0 \\ 0 & 655.0934 \end{bmatrix},$$

and $\epsilon = 526.6788$. Therefore, by Theorem 3.1, the complex dynamical networks with hybrid coupling delays on time scale (5) achieve synchronization with the above mentioned parameters.

Example 2. Consider the complex dynamical networks (5) with hybrid coupling delays on time scales with 5-nodes and the following parameters:

$$D = diag(0.17, 0.15, 0.18), \quad \Upsilon_1 = diag(1.5, 1.5, 1.5), \quad \Upsilon_2 = diag(0.5, 0.5, 0.5),$$

$$B_1 = \begin{bmatrix} 0.2 & -1.5 & 0.5 \\ 1.4 & -1.3 & 0.2 \\ 0 & 0.2 & 1.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.12 & -0.13 & 0 \\ 0.12 & 0.4 & -0.11 \\ 0.6 & 0.5 & 1.3 \end{bmatrix}.$$

The time scale is chosen as $\mathbb{T} = \cup_{i=1}^{\infty}[t_i, t_i + k_i]$, where $t_0 = 0, t_{i+1} = t_i + k_i + i$ and k_i are some real numbers randomly selected from (0, 1]. We consider the coupling matrices [43] as

$$E^{(1)} = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 2 & 0 & 0 & 0 & -2 \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 \\ 1 & 0 & 0 & -2 & 1 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix},$$

$$E^{(3)} = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

Take $L_1 = L_2 = I_3, \ L_3 = 0.5I_3$. Let $d = 0.5, \ \mu = 0.5$. Then by using the Matlab LMI Toolbox, we solve the LMIs in Theorem 3.1, we obtain the feasible solutions as follows

$$P = \begin{bmatrix} 0.0417 & -0.0010 & -0.0046 \\ -0.0010 & 0.0410 & -0.0043 \\ -0.0046 & -0.0043 & 0.0194 \end{bmatrix}, \quad Q_{11} = \begin{bmatrix} 3.7575 & -0.0090 & 0.2214 \\ -0.0090 & 3.7920 & 0.1827 \\ 0.2214 & 0.1827 & 4.2967 \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} -1.7551 & 0.0593 & 0.1474 \\ 0.0593 & -1.5655 & 0.0716 \\ 0.1474 & 0.0716 & -1.7255 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 2.8914 & 0.0135 & -0.0229 \\ 0.0135 & 2.8959 & 0.0221 \\ -0.0229 & 0.0221 & 2.9023 \end{bmatrix},$$

$$R_{11} = \begin{bmatrix} 57.7566 & 0.2511 & -2.1938 \\ 0.2511 & 58.3811 & -2.5970 \\ -2.1938 & -2.5970 & 53.7168 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} -24.0925 & -1.2423 & -1.2607 \\ -1.2423 & -23.3783 & -1.9365 \\ -1.2607 & -1.9365 & -22.4895 \end{bmatrix},$$

$$R_{13} = \begin{bmatrix} -0.4084 & -0.0007 & 0.1119 \\ -0.0007 & -0.4584 & 0.0496 \\ 0.1119 & 0.0496 & -0.2795 \end{bmatrix}, \quad R_{22} = \begin{bmatrix} 38.8627 & -0.1327 & 0.0722 \\ -0.1327 & 39.4885 & 0.4469 \\ 0.0722 & 0.4469 & 41.4897 \end{bmatrix},$$

$$R_{23} = \begin{bmatrix} 0.4255 & -0.0293 & -0.1757 \\ -0.0293 & 0.4756 & -0.1123 \\ -0.1757 & -0.1123 & 0.1295 \end{bmatrix}, \quad R_{33} = \begin{bmatrix} 0.2456 & -0.0055 & -0.0209 \\ -0.0055 & 0.2370 & -0.0208 \\ -0.0209 & -0.0208 & 0.2189 \end{bmatrix},$$

$$S_{11} = \begin{bmatrix} 125.2307 & -0.0511 & 6.2574 \\ -0.0511 & 124.8284 & 5.9177 \\ 6.2574 & 5.9177 & 133.1551 \end{bmatrix}, \quad S_{12} = \begin{bmatrix} -29.2985 & 1.0257 & 1.7778 \\ 1.0257 & -27.6126 & 1.2501 \\ 1.7778 & 1.2501 & -31.3725 \end{bmatrix},$$

$$S_{13} = \begin{bmatrix} -16.0555 & 1.0019 & 1.2196 \\ 1.0019 & -15.2932 & 1.1497 \\ 1.2196 & 1.1497 & -13.7369 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} 69.0065 & 0.4144 & -0.3596 \\ 0.4144 & 69.1801 & 0.5358 \\ -0.3596 & 0.5358 & 68.8026 \end{bmatrix},$$

$$S_{23} = \begin{bmatrix} -0.5149 & 0.1268 & 0.2364 \\ 0.1268 & 1.0305 & 0.1563 \\ 0.2364 & 0.1563 & -0.8508 \end{bmatrix}, \quad S_{33} = \begin{bmatrix} 0.2462 & -0.0051 & -0.0193 \\ -0.0051 & 0.2371 & -0.0195 \\ -0.0193 & -0.0195 & 0.2207 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 1.0725 & -0.0657 & -0.1174 \\ -0.0657 & 1.0271 & -0.1038 \\ -0.1174 & -0.1038 & 0.9032 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 92.2971 & 0 & 0 \\ 0 & 192.2971 & 0 \\ 0 & 0 & 192.2971 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 76.3730 & 0 & 0 \\ 0 & 76.3730 & 0 \\ 0 & 0 & 76.3730 \end{bmatrix},$$

and $\epsilon = 66.9639$. Therefore, by Theorem 3.1, the complex dynamical networks with hybrid coupling delays on time scale (5) achieve synchronization with the above mentioned parameters.

5. Conclusion

In this paper, we have developed the exponential synchronization criteria for a class of complex dynamical networks with hybrid coupling delays on time scales. Both continuous-time and discrete-time synchronization problems have been discussed under a unified framework. Based on the theory of calculus on time scales, some synchronization have been derived by using a suitable Lyapunov functional with Kronecker product term. All the developed results are formulated as LMIs. Numerical results can demonstrate the effectiveness of the obtained result. The idea and approach developed in this paper can be further generalized to deal with some other problems on impulse control and stability of complex dynamical networks with hybrid coupling on time scale.

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