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Solving system of linear Stratonovich Volterra integral equations via modification of hat functions

Farshid Mirzaee*, Elham Hadadiyan

Faculty of Mathematical Sciences and Statistics, Malayer University, P. O. Box 65719-95863, Malayer, Iran

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ABSTRACT

This paper proposes an efficient method for solving system of linear Stratonovich Volterra integral equations. Stochastic operational matrix of modification of hat functions (MHFs) is determined. By using MHFs and their stochastic operational matrix of integration, a system of linear Stratonovich Volterra integral equations can be reduced to a linear system of algebraic equations. Thus we can solve the problem by direct methods. Also, we prove that the rate of convergence is $O(h^3)$. Efficiency of this method and good degree of accuracy are confirmed by numerical examples.

1. Introduction

The Stratonovich integral equations developed simultaneously by Stratonovich [1] and Fisk [2]. In stochastic processes, the Stratonovich integral is a stochastic integral that is the most common alternative to the Itô integral. Although the Itô integral is the usual choice in applied mathematics, the Stratonovich integral is frequently used in physics. In physics, stochastic integrals occur as the solutions of Langevin equations. A Langevin equation is a coarse-grained version of a more microscopic model, depending on the problem in consideration, Stratonovich or Itô interpretation or even more exotic interpretations such as the isothermal interpretation.

The Wong–Zakai theorem [3] states the physical systems with non-white noise spectrum characterized by a finite noise correlation time t that can be approximated by a Langevin equations with white noise in Stratonovich interpretation in the limit where t tends to zero. Because the Stratonovich calculus satisfies the ordinary chain rule, stochastic differential equations (SDEs) in the Stratonovich sense can be meaningfully defined on arbitrary differentiable manifolds, rather than just on R^n. This is not possible in the Itô calculus, since here the choice of coordinate system would affect the SDEs solution.

In some circumstances, integrals in the Stratonovich definition are easier to manipulate. Unlike the Itô calculus, Stratonovich integrals are defined such that the chain rule of ordinary calculus holds. Perhaps the most common situation in which these are encountered is solving the Stratonovich SDEs. These are equivalent to Itô SDEs and it is possible to convert to each other.

The Stratonovich integral can be defined in a manner similar to the Riemann integral, that is as a limit of Riemann sums. Stochastic integrals can rarely be solved in analytic form that making stochastic numerical integration an important topic in all uses of stochastic integrals. Various numerical approximations converge to the Stratonovich integral, and are used to solve Stratonovich SDEs [4–6]. Numerical schemes to Stratonovich equations have been well developed [1–18]. However,

* Corresponding author. Tel.: +98 81 32355466; Fax: +98 81 32355466.
E-mail addresses: f.mirzaee@malayeru.ac.ir, fa_mirzaee@yahoo.com (F. Mirzaee).

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there are still very few papers discussing the numerical solutions for Stratonovich Volterra integral equations in comparison to many papers about deterministic integral equations [19–22].

In this paper, we use MHFs to solve the system of linear Stochastic Volterra integral equation as follows

$$\mathbf{f}(x) = \mathbf{g}(x) + \int_0^x \mathbf{k1}(x, y)\mathbf{f}(y)dy + \int_0^x \mathbf{k2}(x, y)\mathbf{f}(y) \circ dB_y, \quad x \in D = [0, T], \tag{1}$$

where

$$\mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T, \tag{2}$$

$$\mathbf{g}(x) = [g_1(x), g_2(x), \ldots, g_n(x)]^T, \tag{3}$$

$$\mathbf{k1}(x, y) = \left[k1_{ij}(x, y)\right], \quad i, j = 1, 2, \ldots, n, \tag{4}$$

$$\mathbf{k2}(x, y) = \left[k2_{ij}(x, y)\right], \quad i, j = 1, 2, \ldots, n, \tag{5}$$

$g_i(x) \in \mathcal{X} = C^3(D)$ and $k1_{ij}(x, y), k2_{i,j}(x, y) \in \mathcal{X} \times \mathcal{X}$ are known functions and $f_i(x) \in \mathcal{X}$ is unknown functions, for $i, j = 1, 2, \ldots, n$. Note that the symbol $\circ$ between integrand and the stochastic differential is used to indicate Stratonovich integrals.

2. MHFs and their properties

2.1. Definitions of MHFs

The family of first $(m+1)$ MHFs is defined as follows [23]

$$h_0(x) = \begin{cases} \dfrac{1}{2h^2}(x-h)(x-2h) & 0 \leqslant x \leqslant 2h, \\ 0 & \text{otherwise}, \end{cases}$$

if i is odd and $1 \leqslant i \leqslant m-1$,

$$h_i(x) = \begin{cases} \dfrac{-1}{h^2}(x-(i-1)h)(x-(i+1)h) & (i-1)h \leqslant x \leqslant (i+1)h, \\ 0 & \text{otherwise}, \end{cases}$$

if i is even and $2 \leqslant i \leqslant m-2$,

$$h_i(x) = \begin{cases} \dfrac{1}{2h^2}(x-(i-1)h)(x-(i-2)h) & (i-2)h \leqslant x \leqslant ih, \\ \dfrac{1}{2h^2}(x-(i+1)h)(x-(i+2)h) & ih \leqslant x \leqslant (i+2)h, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$h_m(x) = \begin{cases} \dfrac{1}{2h^2}(x-(T-h))(x-(T-2h)) & T-2h \leqslant x \leqslant T, \\ 0 & \text{otherwise}, \end{cases}$$

where $m \geq 2$ is an even integer and $h = \frac{T}{m}$. According to definition of MHFs, we have

$$h_i(jh) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

$$h_i(x)h_j(x) = \begin{cases} 0 & i \text{ is even and } |i-j| \geqslant 3, \\ 0 & i \text{ is odd and } |i-j| \geqslant 2, \end{cases}$$

and

$$\sum_{i=0}^{m} h_i(x) = 1.$$

Let us write the MHFs vector $H(x)$ as follows

$$H(x) = [h_0(x), h_1(x), \ldots, h_m(x)]^T; \; x \in D. \tag{6}$$

2.2. Function expansion with MHFs

An arbitrary function $f(x)$ on D can be expanded by the MHFs as

$$f(x) \simeq F^T H(x) = H^T(x)F,$$

where

$$F = [f_0, f_1, \ldots, f_m]^T,$$

and

$$f_i = f(ih), \qquad i = 0, 1, \ldots, m.$$

Similarly an arbitrary function of two variables, $k(x, y)$ on district $D \times D$ may be approximated with respect to MHFs such as

$$k(x, y) \simeq H^T(x) K H(y),$$

where $H(x)$ and $H(y)$ are MHFs vector of dimension $(m+1)$ and K is the $(m+1) \times (m+1)$ MHFs coefficients matrix.

2.3. Operational matrix

From [24], we have

$$H(x)H^T(x) \simeq diag(H(x)). \tag{7}$$

Integration of vector $H(x)$ defined in (6) can be expressed as

$$\int_0^x H(y)dy \simeq PH(x), \tag{8}$$

where P is the $(m+1) \times (m+1)$ matrix as follows

$$P = \frac{h}{12}\begin{pmatrix} 0 & 5 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 \\ 0 & 8 & 16 & 16 & 16 & \cdots & 16 & 16 & 16 & 16 \\ 0 & -1 & 4 & 9 & 8 & \cdots & 8 & 8 & 8 & 8 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & 9 & 8 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 4 \end{pmatrix}.$$

2.4. Stochastic operational matrix

In this section, we find stochastic operational matrix based on MHFs.

Theorem 1. *Let $H(x)$ be the vector defined in (6). The Stratonovich Volterra integral of $H(x)$ can be expressed as*

$$\int_0^x H(y) \circ dB_y \simeq P_s H(x), \tag{9}$$

where the $(m+1) \times (m+1)$ Stratonovich operational matrix of integration is given by

$$
Ps = \begin{pmatrix}
0 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 & \cdots & \alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
0 & B(h)+\beta_{1,1} & \beta_{2,1} & \beta_{2,1} & \beta_{2,1} & \cdots & \beta_{2,1} & \beta_{2,1} & \beta_{2,1} & \beta_{2,1} \\
0 & \gamma_{1,2} & B(2h)+\gamma_{2,2} & \gamma_{3,2} & \gamma_{4,2} & \cdots & \gamma_{4,2} & \gamma_{4,2} & \gamma_{4,2} & \gamma_{4,2} \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{1,m-2} & B(T-2h)+\gamma_{2,m-2} & \gamma_{3,m-2} & \gamma_{4,m-2} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & B(T-h)+\beta_{1,m-1} & \beta_{2,m-1} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{1,m} & B(T)+\gamma_{2,m}
\end{pmatrix},
$$

where

$$\alpha_1 = -\int_0^h \frac{1}{2h^2}(2y-3h)B(y)dy,$$

$$\alpha_2 = -\int_0^{2h} \frac{1}{2h^2}(2y-3h)B(y)dy,$$

$$\beta_{1,i} = \int_{(i-1)h}^{ih} \frac{1}{h^2}(2y-2ih)B(y)dy,$$

$$\beta_{2,i} = \int_{(i-1)h}^{(i+1)h} \frac{1}{h^2}(2y-2ih)B(y)dy,$$

$$\gamma_{1,i} = -\int_{(i-2)h}^{(i-1)h} \frac{1}{2h^2}(2y-(2i-3)h)B(y)dy,$$

$$\gamma_{2,i} = -\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2y-(2i-3)h)B(y)dy,$$

$$\gamma_{3,i} = -\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2y-(2i-3)h)B(y)dy - \int_h^{(i+1)h} \frac{1}{2h^2}(2y-(2i+3)h)B(y)dy,$$

and

$$\gamma_{4,i} = -\int_{(i-2)h}^{ih} \frac{1}{2h^2}(2y-(2i-3)h)B(y)dy - \int_h^{(i+2)h} \frac{1}{2h^2}(2y-(2i+3)h)B(y)dy.$$

Proof. For the proof, see [24]. □

3. Method of solution

In this section, MHFs are applied to present an effective method for approximating the solution of system of Stratonovich Volterra integral equations. For this purpose, we approximate $f_i(x)$, $g_i(x)$, $k1_{ij}(x, y)$ and $k2_{ij}(x, y)$, $i, j = 1, 2, \dots, n,$ by MHFs as follows

$$
\begin{aligned}
f_i(x) &\simeq H^T(x)F_i, \\
g_i(x) &\simeq H^T(x)G_i, \\
k1_{ij}(x,y) &\simeq H^T(x)K1_{ij}H(y), \\
k2_{ij}(x,y) &\simeq H^T(x)K2_{ij}H(y),
\end{aligned}
$$

where $i, j = 1, 2, \dots, n,$ $H(x)$ is defined by (6), F_i, G_i, $K1_{ij}$ and $K2_{ij}$ are MHFs coefficients of $f_i(x)$, $g_i(x)$, $k1_{i,j}(x, y)$ and $k2_{ij}(x, y)$, respectively. Let

$$\begin{aligned} \mathbf{F} &= [F_1, F_2, \ldots, F_n]^T, \\ \mathbf{G} &= [G_1, G_2, \ldots, G_n]^T, \\ \mathbf{K1} &= \left[K1_{ij}\right], \\ \mathbf{K2} &= \left[K2_{ij}\right], \end{aligned}$$

and

$$\mathbf{H}_n(x) = [\underbrace{H(x), H(x), \ldots, H(x)}_{n \ times}], \tag{10}$$

where $i, j = 1, 2, \ldots, n$.

Now by using above equations, we have

$$\mathbf{H}_n^T(x)\mathbf{F} \simeq \mathbf{H}_n^T(x)\mathbf{G} + \int_0^x \mathbf{H}_n^T(x)\mathbf{K1}\mathbf{H}_n(y)\mathbf{H}_n^T(y)\mathbf{F}dy + \int_0^x \mathbf{H}_n^T(x)\mathbf{K2}\mathbf{H}_n(y)\mathbf{H}_n^T(y)\mathbf{F} \circ dB_y.$$

From (7), we have

$$\mathbf{H}_n(x)\mathbf{H}_n^T(x) \simeq diag(\mathbf{H}_n(x)), \tag{11}$$

so

$$\mathbf{H}_n(x)\mathbf{H}_n^T(x)\mathbf{F} \simeq \widehat{\mathbf{F}}\mathbf{H}_n(x),$$

where $\widehat{\mathbf{F}}$ is an $n(m+1) \times n(m+1)$ diagonal matrix. Therefore, we have:

$$\mathbf{H}_n^T(x)\mathbf{F} \simeq \mathbf{H}_n^T(x)\mathbf{G} + \mathbf{H}_n^T(x)\mathbf{K1}\widehat{\mathbf{F}} \int_0^x \mathbf{H}_n(y)dy + \mathbf{H}_n^T(x)\mathbf{K2}\widehat{\mathbf{F}} \int_0^x \mathbf{H}_n(y) \circ dB_y.$$

By combination (8) and (10), we have

$$\int_0^x \mathbf{H}_n(y)dy \simeq \mathbf{P}_n\mathbf{H}_n(x),$$

where

$$\mathbf{P}_n = diag(\underbrace{P, P, \ldots, P}_{n \ times}).$$

From (9) and (10), we have

$$\int_0^x \mathbf{H}_n(y)dy \simeq \mathbf{Ps}_n\mathbf{H}_n(x),$$

where

$$\mathbf{Ps}_n = diag(\underbrace{Ps, Ps, \ldots, Ps}_{n \ times}),$$

so

$$\mathbf{H}_n^T(x)\mathbf{F} \simeq \mathbf{H}_n^T(x)\mathbf{G} + \mathbf{H}_n^T(x)\mathbf{K1}\widehat{\mathbf{F}}\mathbf{P}_n\mathbf{H}_n(x) + \mathbf{H}_n^T(x)\mathbf{K2}\widehat{\mathbf{F}}\mathbf{Ps}_n\mathbf{H}_n(x). \tag{12}$$

Let A be an $n(m+1) \times n(m+1)$-matrix, from (11) we have

$$\mathbf{H}_n^T(x)A\mathbf{H}_n(x) \simeq \mathbf{H}_n^T(x)\widetilde{A}, \tag{13}$$

where $\widetilde{A}$ is an $n(m+1)$-vector with elements equal to the diagonal entries of matrix A.

By using (13), we can rewrite (12) as follows

$$\mathbf{H}_n^T(x)\mathbf{F} \simeq \mathbf{H}_n^T(x)\mathbf{G} + \mathbf{H}_n^T(x)\widetilde{A} + \mathbf{H}_n^T(x)\widetilde{B}, \tag{14}$$

where $A = \mathbf{K1}\widehat{\mathbf{F}}\mathbf{P}_n$ and $B = \mathbf{K2}\widehat{\mathbf{F}}\mathbf{Ps}_n$. Therefore from (14) and replacing $\simeq$ by $=$, we have

$$\mathbf{F} = \mathbf{G} + \widetilde{A} + \widetilde{B}.$$

After solving the above linear system, we can find $\mathbf{F}$ and accordingly find $F_i, \ i = 1, 2, \ldots, n,$ so

$$f_i(x) \simeq H^T(x)F_i.$$

4. Convergence analysis

In this section, we show that the MHFs method, is convergent of order $O(h^3)$. Let $\mathcal{X}$ be a Banach space with sup-norm. Also we define

$$\|\mathbf{u}(x)\| = \max_{i=1,2,\dots,n} \|u_i(x)\|_\infty, \tag{15}$$

where

$$\mathbf{u}(x) = [u_1(x), u_2(x), \dots, u_n(x)], \tag{16}$$

and

$$\|\mathbf{u}(x,y)\| = \max_{i=1,2,\dots,n} \sum_{j=0}^{n} \|u_{ij}(x,y)\|_\infty, \tag{17}$$

where

$$\mathbf{u}(x,y) = \left[u_{ij}(x,y)\right], \quad i,j = 1,2,\dots,n. \tag{18}$$

Theorem 2. *Suppose that $x_j = jh$, $j = 0,1,\dots,m$, $u_i(x) \in \mathcal{X}$ and $u_{i,m}(x)$ is the MHFs expansions of $u_i(x)$ that defined as $u_{i,m}(x) = \sum_{j=0}^{m} u_i(x_j)h_j(x)$ where $i = 1,2,\dots,n$. Also, assume that $e_{i,m}(x) = u_i(x) - u_{i,m}(x)$ where $x \in D$, then*

$$\|e_{i,m}(x)\| = O(h^3).$$

Proof. For the proof, see [25]. □

Lemma 1. *Suppose $\mathbf{u}(x)$ was defined as (16), $\mathbf{u}_m(x)$ is the MHFs of $\mathbf{u}(x)$ and $e_{\mathbf{u}}(x) = \mathbf{u}(x) - \mathbf{u}_m(x)$. Then*

$$\|e_{\mathbf{u}}(x)\| = O(h^3).$$

Proof. From (15), we have

$$\|e_{\mathbf{u}}(x)\| = \max_{i=1,2,\dots,n} \|u_i(x) - u_{i,m}(x)\|_\infty,$$

and from Theorem 2, $\|u_i(x) - u_{i,m}(x)\|_\infty \leqslant C_i h^3$. Suppose C_r is the maximum of C_i, $i = 1,2,\dots,n$, then

$$\|e_{\mathbf{u}}(x)\| \leqslant C_r h^3. \tag{19}$$

□

Theorem 3. *Let $x_k = y_k = kh$, $k = 0,1,\dots,m$, $u_{ij}(x,y) \in \mathcal{X} \times \mathcal{X}$ and*

$$u_{ij,m}(x,y) = \sum_{k=0}^{m}\sum_{l=0}^{m} u_{ij}(x_k,y_l)h_k(x)h_l(y),$$

be the MHFs expansions of $u_{ij}(x,y)$. Then

$$\|e_{ij,m}(x,y)\| = O(h^3),$$

where $e_{ij,m}(x,y) = u_{ij}(x,y) - u_{ij,m}(x,y)$ and $(x,y) \in D \times D$.

Proof. For the proof, see [25]. □

Lemma 2. *Suppose $\mathbf{u}(x,y)$ was defined as (17), $\mathbf{u}_m(x,y)$ is the MHFs of $\mathbf{u}(x,y)$ and $e_{\mathbf{u}}(x,y) = \mathbf{u}(x,y) - \mathbf{u}_m(x,y)$. Then*

$$\|e_{\mathbf{u}}(x,y)\| = O(h^3).$$

Proof. From (16), we have

$$\|e_{\mathbf{u}}(x,y)\| = \max_{i=1,2,\dots,n} \sum_{j=0}^{n} \|u_{ij}(x,y) - u_{ij,m}(x,y)\|_\infty,$$

and from Theorem 3, we conclude that $\|u_{ij}(x,y) - u_{ij,m}(x,y)\|_\infty \leqslant C_{ij}h^3$. Suppose

$$\sum_{j=0}^{n} C_{rj} = \max_{i=1,2,\dots,n} \sum_{j=0}^{n} C_{ij},$$

therefore

$$\|e_{\mathbf{u}}(x,y)\| \leqslant \left(\sum_{j=0}^{n} C_{rj}\right)h^3. \tag{20}$$

From above equation, the lemma is established. □

Now, assume the following hypotheses

(M1) Suppose that the error of MHFs is denoted by:

$$E_m = \|\mathbf{f}(x) - \mathbf{f}_m(x)\|, \ \ x \in D,$$

where $\mathbf{f}(x)$ were defined in (2).

(M2) Let $x \in D$

$$\|\mathbf{f}(x)\| \leqslant N.$$

(M3) Let $(x, y) \in D \times D$,

$$\|\mathbf{ki}(x,y)\| \leqslant M_i, \quad i = 1, 2.$$

(M4) According to Lemmas 1 and 2, let

$$E_{\mathbf{g}} = \|e_{\mathbf{g}}(x)\| \leqslant Ch^3,$$

and

$$E_{\mathbf{ki}} = \|e_{\mathbf{ki}}(x,y)\| \leqslant C_i h^3,$$

where C and C_i, $i = 1, 2$ are constants that can be defined as coefficient in (19) and (20) and $\mathbf{g}(x)$, $\mathbf{k1}(x, y)$ and $\mathbf{k2}(x, y)$ are defined in (3)–(5), respectively.

(M5) Let $x \in D$

$$\|B(x)\| \leqslant L.$$

(M6) Let $M_1 + LM_2 + (C_1 + C_2 L)h^3 < 1$.

Theorem 4. *Suppose $\mathbf{f}(x)$ and $\mathbf{f}_m(x)$ be the exact and approximate solution of (1), respectively. Also above assumptions (M1)–(M6) are satisfied, then we have*

$$E_m \leqslant \frac{(C + C_1 TN + C_2 LN)h^3}{1 - TM_1 - LM_2 - (C_1 T + C_2 L)h^3}. \tag{21}$$

Proof. According to (1), we have

$$\mathbf{f}(x) - \mathbf{f}_m(x) = \mathbf{g}(x) - \mathbf{g}_m(x) + \int_0^x (\mathbf{k1}(x,y)\mathbf{f}(y) - \mathbf{k1}_m(x,y)\mathbf{f}_m(y))dy + \int_0^x (\mathbf{k2}(x,y)\mathbf{f}(y) - \mathbf{k2}_m(x,y)\mathbf{f}_m(y)) \circ dB_y,$$

therefore:

$$E_m \leqslant E_{\mathbf{g}} + \|x\| \ \|\mathbf{k1}(x,y)\mathbf{f}(y) - \mathbf{k1}_m(x,y)\mathbf{f}_m(y)\| + \|B(x)\| \ \|\mathbf{k2}(x,y)\mathbf{f}(y) - \mathbf{k2}_m(x,y)\mathbf{f}_m(y)\|.$$

It is clear that $\|x\| \le T$, so

$$E_m \leqslant E_{\mathbf{g}} + T\|\mathbf{k1}(x,y)\mathbf{f}(y) - \mathbf{k1}_m(x,y)\mathbf{f}_m(y)\| + \|B(x)\| \ \|\mathbf{k2}(x,y)\mathbf{f}(y) - \mathbf{k2}_m(x,y)\mathbf{f}_m(y)\|. \tag{22}$$

Now, according to assumptions (M2)–(M4), we have

$$\|\mathbf{ki}(x,y)\mathbf{f}(y) - \mathbf{ki}_m(x,y)\mathbf{f}_m(y)\| \leqslant \|\mathbf{ki}(x,y)\| \ E_m + E_{\mathbf{ki}}(E_m + \|\mathbf{f}(x)\|) \leqslant M_i E_m + C_i h^3 (E_m + N). \tag{23}$$

Also from assumptions (M4) and (M5), (22) and (23), we have

$$E_m \leqslant (C + C_1 TN + C_2 LN)h^3 + (TM_1 + LM_2 + (C_1 T + C_2 L)h^3)E_m.$$

Therefore according to (M6), (21) is satisfied and this completes the proof. Also we have $E_m = O(h^3)$. □

Lemma 3. *Suppose $\mathbf{f}(x)$ and $\mathbf{f}_m(x)$ are the exact and approximate solution of (1), respectively, where $\mathbf{f}(x)$ was defined in (2) and*

$$\mathbf{f}_m(x) = [f_{1,m}(x), f_{2,m}(x), \dots, f_{n,m}(x)]^T.$$

Then

$$e_{i,m} = \|f_i(x) - f_{i,m}(x)\| = O(h^3). \tag{24}$$

Proof. From Theorem 3, we have $E_m \le Ch^3$ and according to (15) we have:

$$e_{i,m} \leqslant E_m \leqslant Ch^3.$$

This completes the proof. □

Table 1
Numerical results of (25).

Nodes x	$f_1(x)$				$f_2(x)$			
	$m=32,\ k=25$		$m=64,\ k=75$		$m=32,\ k=25$		$m=64,\ k=75$	
	Exact solutions	Present method	Exact solutions	Present method	Exact solutions	Present method	Exact solutions	Present method
0.0	1.000000	1.000000	1.000000	1.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.607006	0.590342	0.908036	0.901431	−0.262667	−0.270182	−0.079773	−0.069891
0.2	0.582075	0.619869	0.676988	0.682518	−0.264879	−0.254586	−0.223220	−0.215227
0.3	0.682303	0.700610	0.550380	0.567854	−0.206256	−0.199568	−0.264620	−0.253712
0.4	0.893847	0.910406	0.463061	0.475646	−0.028360	−0.027065	−0.276158	−0.268886
0.5	1.257804	1.259793	0.486836	0.491483	0.636166	0.621312	−0.251382	−0.242507
0.6	0.985349	0.981187	0.305739	0.310194	0.181209	0.164100	−0.268081	−0.263160
0.7	0.831014	0.817254	0.429141	0.435997	0.051424	0.017227	−0.223793	−0.212277
0.8	0.983109	0.989433	0.467798	0.461071	0.386958	0.371419	−0.181305	−0.175181
0.9	0.864651	0.873423	0.359173	0.367520	0.274748	0.274438	−0.192712	−0.179055
1.0	0.641588	0.639991	0.282932	0.289212	0.037168	0.025985	−0.186340	−0.175791

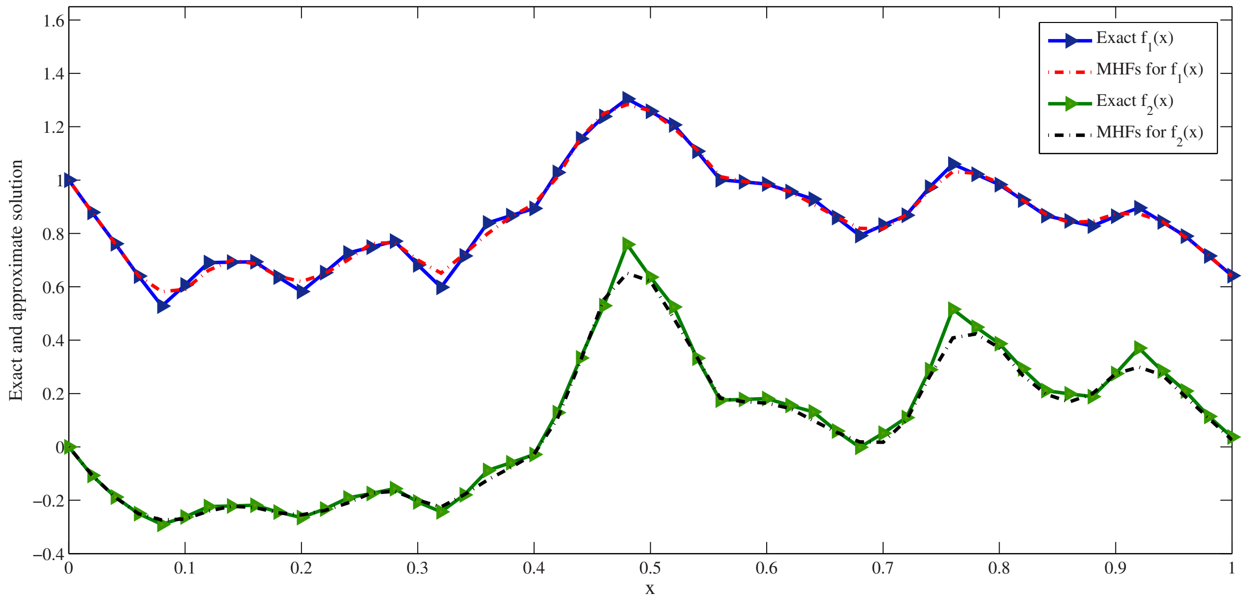

Fig. 1. Numerical results for Example 1, with $m = 32$ and $n = 25$.

5. Numerical examples

In this section, some examples are given to illustrate the application of the obtained method. All of them are performed on a computer using programs written in MATLAB. Also some interesting comparisons between proposed method and hat functions method are presented. Note that, m is the number of modification of hat functions and k is the number of iterations.

Example 1. Consider the following system of linear Stratonovich Volterra integral equations

$$\begin{cases} f_1(x) = 1 - \int_0^x y f_2(y)dy + \int_0^x f_1(y) \circ dB_y - \int_0^x f_2(y) \circ dB_y, \\ f_2(x) = \int_0^x y f_1(y)dy + \int_0^x f_1(y) \circ dB_y + \int_0^x f_2(y) \circ dB_y, \end{cases} \quad x \in [0, 1], \tag{25}$$

with the exact solution $\mathbf{f}(\mathbf{x}) = (f_1(x), f_2(x)) = (e^{B(x)} \cos(\frac{x^2}{2} + B(x)), e^{B(x)} \sin(\frac{x^2}{2} + B(x)))$.

Table 1 and Figs. 1 and 2 illustrate the comparison between the exact solutions and numerical solutions given by the proposed method (MHFs) for different values of m and k.

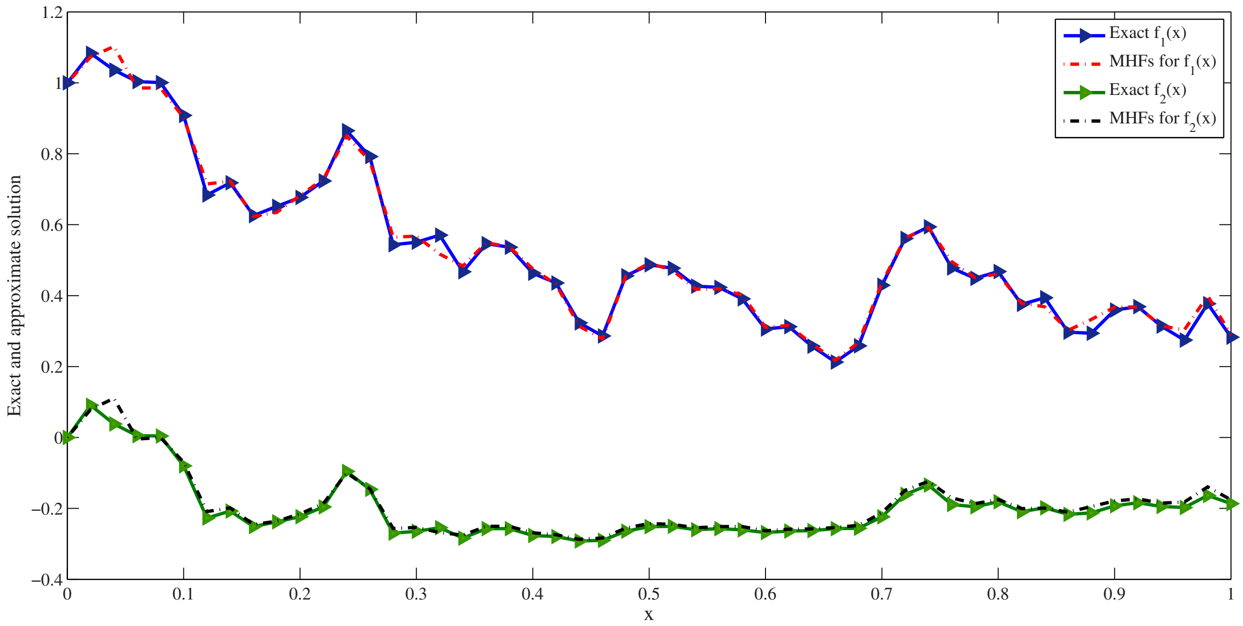

Fig. 2. Numerical results for Example 1, with $m = 64$ and $n = 75$.

Table 2
Numerical results of (26).

Nodes x	$f_1(x)$				$f_2(x)$			
	$m = 32,\ k = 25$		$m = 64,\ k = 75$		$m = 32,\ k = 25$		$m = 64,\ k = 75$	
	Exact solutions	Present method	Exact solutions	Present method	Exact solutions	Present method	Exact solutions	Present method
0.0	0.000000	0.000000	0.000000	0.000000	1.000000	1.000000	1.000000	1.000000
0.1	−0.009023	−0.011366	0.071837	0.080366	1.100037	1.100558	1.102343	1.100698
0.2	−0.311835	−0.350888	−0.599713	−0.595755	1.239855	1.255604	1.341512	1.339469
0.3	−0.878028	−0.883091	0.025574	0.030159	1.568736	1.577721	1.300252	1.291351
0.4	−0.832642	−0.841666	−0.017338	0.013856	1.628893	1.640803	1.400107	1.390652
0.5	−0.497654	−0.504040	0.237109	0.236089	1.580398	1.592002	1.518625	1.507259
0.6	−0.703721	−0.724039	−0.499547	−0.520894	1.747920	1.761594	1.676167	1.680933
0.7	−0.767917	−0.772216	0.337754	0.379468	1.865395	1.873326	1.733228	1.743265
0.8	−0.591026	−0.507240	−0.263310	−0.294521	1.894548	1.878153	1.819157	1.833962
0.9	−0.597885	−0.622634	0.401375	0.428392	1.991850	2.008083	1.941932	1.966238
1.0	−0.208308	−0.211948	0.690123	0.695588	2.010819	2.021696	2.115720	2.127036

Example 2. Consider the following system of linear Stratonovich Volterra integral equations

$$\begin{cases} f_1(x) = \int_0^x \frac{1}{y+1} f_1(y)dy + \int_0^x \frac{y^2}{2} f_2(y)dy - \int_0^x f_2(y) \circ dB_y, \\ f_2(x) = 1 + \int_0^x \frac{y^2}{2} f_1(y)dy + \int_0^x \frac{1}{y+1} f_2(y)dy - \int_0^x f_1(y) \circ dB_y, \end{cases} \quad x \in [0,1], \tag{26}$$

with the exact solution $\mathbf{f}(\mathbf{x}) = (f_1(x), f_2(x)) = \Big((x+1)\sin h(\frac{x^3}{6} - B(x)), (x+1)\cos h(\frac{x^3}{6} - B(x))\Big)$.

Table 2 and Figs. 3 and 4 illustrate the comparison between the exact solutions and numerical solutions given by the proposed method (MHFs) for different values of m and k.

6. Conclusion

Since the system of linear Stratonovich Volterra integral equations cannot be solved analytically, in this paper we present a new technique for solving system of linear Stratonovich Volterra integral equations numerically. Here, we consider MHFs and their applications. The convergence and error analysis of the proposed method were investigated. The main advantage of this method is its efficiency and simple applicability that were checked on some examples. The results of the numerical

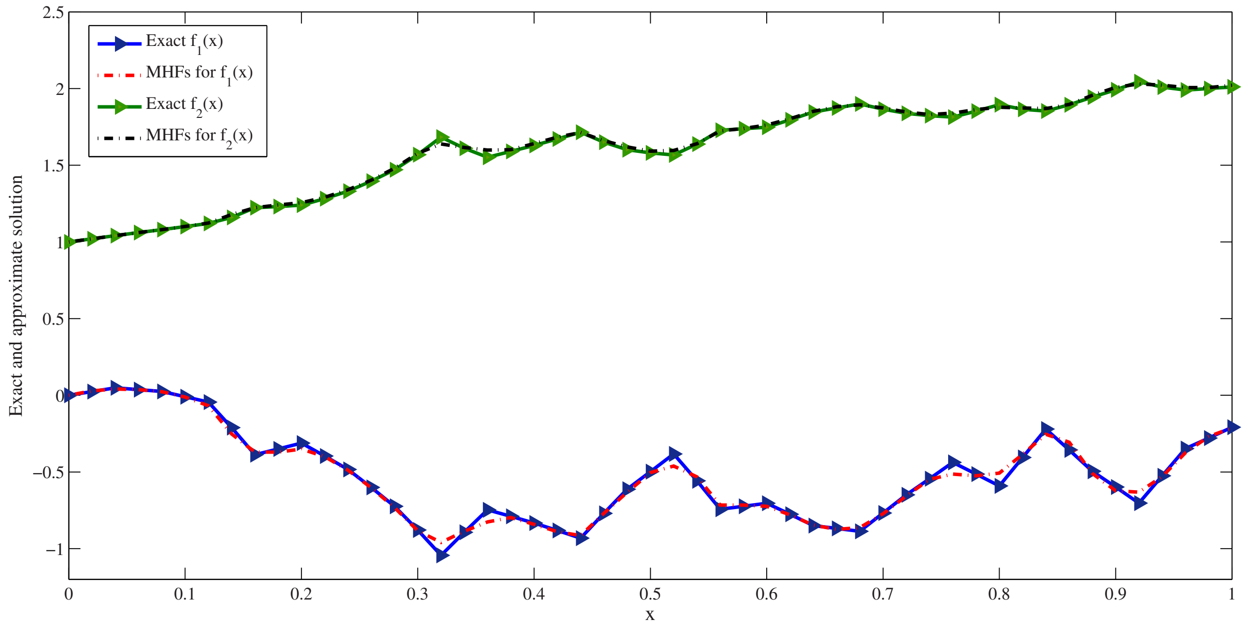

Fig. 3. Numerical results for Example 2, with $m = 32$ and $n = 25$.

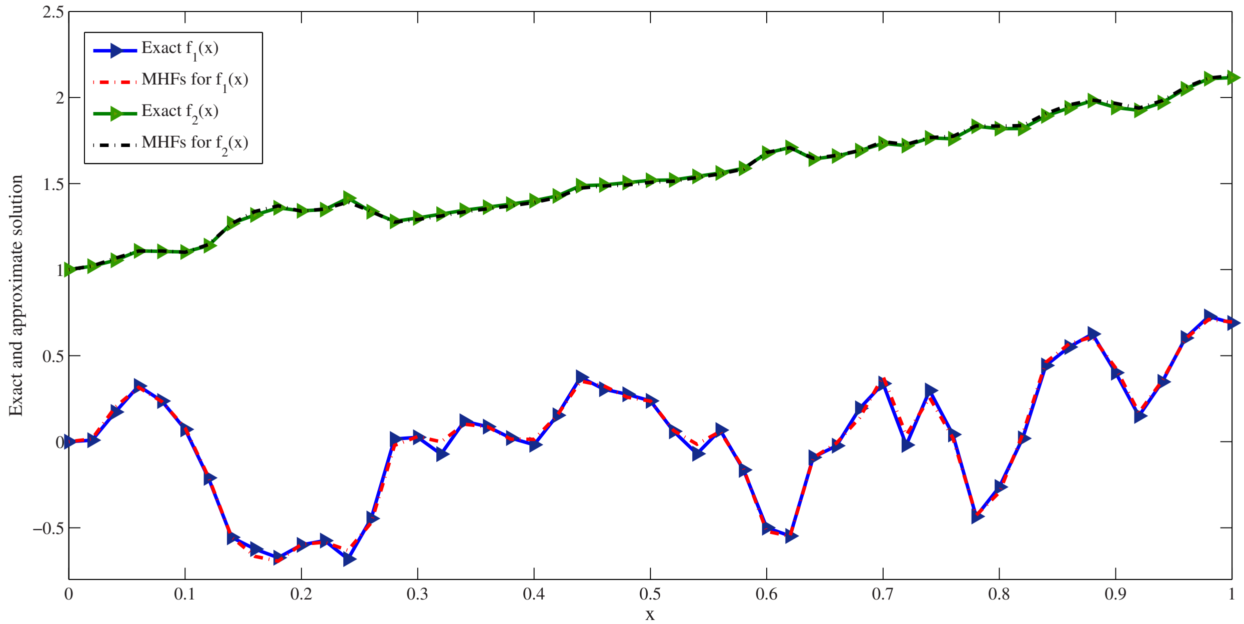

Fig. 4. Numerical results for Example 2, with $m = 64$ and $n = 75$.

solution were compared with the analytical solution. Applicability and accuracy of the proposed method were checked on some examples.

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