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Research paper

# Dynamic behaviors for a perturbed nonlinear Schrödinger equation with the power-law nonlinearity in a non-Kerr medium

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ABSTRACT

Effects of quantic nonlinearity on the propagation of the ultrashort optical pulses in a non-Kerr medium, like an optical fiber, can be described by a perturbed nonlinear Schrödinger equation with the power law nonlinearity, which is studied in this paper from a planar-dynamic-system view point. We obtain the equivalent two-dimensional planar dynamic system of such an equation, for which, according to the bifurcation theory and qualitative theory, phase portraits are given. Through the analysis of those phase portraits, we present the relations among the Hamiltonian, orbits of the dynamic system and types of the analytic solutions. Analytic expressions of the periodic-wave solutions, kink- and bell-shaped solitary-wave solutions are derived, and we find that the periodic-wave solutions can be reduced to the kink- and bell-shaped solitary-wave solutions.



## 1. Introduction

Nonlinear partial differential equations (NLPDEs) have been used to describe the nonlinear physical phenomena in such fields as optics, plasmas physics and condensed matter physics [1–4]. To study those nonlinear phenomena, people have paid their attention to seeking the solutions for the NLPDEs [5–7]. Nowadays, researchers' interest in searching for the analytic solutions has grown due to the availability of the computer symbolic softwares, like Maple, Mathematica and Matlab, which can help us to calculate the complicated and tedious algebraic calculations [7,8].

Methods have been proposed to obtain the analytic solutions for the NLPDEs, e.g., the Hirota method [9–11], $(G'/G)$-expansion method [12], first integral method [13,14], ansatz method [15,16], extended mapping method [17], Bäcklund transformation [18] and Darboux transformation [19,20]. Meanwhile, bifurcation method has also been developed to search for the analytic solutions [21–23]. Through the bifurcation method, one can display the phase portraits of the equivalent

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planar dynamic systems[1] for the NLPDEs, in which, different orbits correspond to different kinds of analytic solutions [26]. Certain NLPDEs have been investigated via the bifurcation method, including the analytic solutions for the Klein–Gordon equation [27], Klein–Gordon–Zakharov equations [28] and Zakharov–Kuznetsov–Burgers equation [29].

In this paper, we will employ the bifurcation method and qualitative theory of planar dynamic systems[2] to study the following perturbed nonlinear Schrödinger (NLS) equation with the power-law nonlinearity [31,32]:

$$iq_t + aq_{xx} + bq|q|^{2m} = icq_x - irq_{xxx} + i\lambda(q|q|^{2m})_x + i\nu(|q|^{2m})_x q, \tag{1}$$

which describes the effects of quantic nonlinearity on the propagation of the ultrashort optical pulses in a non-Kerr medium like an optical fiber, where $q$, a complex function of $x$ and $t$, refers to the wave profile, $x$ and $t$ respectively denote the space and time coordinates, the subscripts represent the partial derivatives, the real constants $a$, $b$, $c$, $r$, $\lambda$ and $\nu$ are respectively the coefficients of the group velocity dispersion, nonlinearity, inter-modal dispersion,[3] third-order dispersion, self-steepening term for the short pulses and higher-order dispersion, and the non-negative integer $m$ represents the power-law nonlinearity parameter [31,32]. Solitary-wave solutions for Eq. (1) have been obtained via the direct algebraic method [31]. With the solitary-wave ansatz, bright and dark solitary-wave solutions for Eq. (1) have been derived [32].

Bifurcation analysis and investigation for the analytic solutions will be the main focus of this paper. In Section 2, we will derive the equivalent two-dimensional planar dynamic system for Eq. (1). Then, the phase portraits of the dynamic system will be given, and the relations among the Hamiltonian, orbits of the dynamic system and types of the solutions for Eq. (1) will be investigated as well. In Section 3, we will derive the analytic expressions of the periodic-wave solutions, kink- and bell-shaped solitary-wave solutions for Eq. (1). Section 4 will be our conclusions.

## 2. Bifurcation analysis

Supposing that the solutions for Eq. (1) are in the following form:

$$q(x,t) = u(\xi)e^{i(p_1 x - p_2 t)}, \quad \xi = k_1 x - k_2 t, \tag{2}$$

where $u(\xi)$ is the real function, and $p_1$, $p_2$, $k_1$ and $k_2$ are the real constants, and substituting them into Eq. (1), we have

$$\begin{aligned}&(ak_1^2 - 3k_1^2 p_1 r)u_{\xi\xi} + (cp_1 - ap_1^2 + p_2 + p_1^3 r)u + (b + p_1\lambda)u^{2m+1}\\ &\quad + i\Big[k_1^3 r u_{\xi\xi\xi} - (ck_1 + k_2 - 2ak_1p_1 + 3k_1p_1^2 r)u_\xi - (2k_1 m\nu + k_1\lambda + 2k_1 m\lambda)u^{2m}u_\xi\Big] = 0.\end{aligned} \tag{3}$$

The real and imaginary parts of Expression (3), respectively, are

$$(ak_1^2 - 3k_1^2 p_1 r)u_{\xi\xi} + (cp_1 - ap_1^2 + p_2 + p_1^3 r)u + (b + p_1\lambda)u^{2m+1} = 0 \tag{4}$$

and

$$k_1^3 r u_{\xi\xi\xi} - (ck_1 + k_2 - 2ak_1p_1 + 3k_1p_1^2 r)u_\xi - (2k_1 m\nu + k_1\lambda + 2k_1 m\lambda)u^{2m}u_\xi = 0. \tag{5}$$

Integrating Expression (5) yields

$$k_1^3 r u_{\xi\xi} - (ck_1 + k_2 - 2ak_1p_1 + 3k_1p_1^2 r)u - \frac{(2k_1 m\nu + k_1\lambda + 2k_1 m\lambda)}{2m+1}u^{2m+1} = C, \tag{6}$$

where $C$ is the integral constant.

From Expressions (4) and (6), we get the same solutions, with $C = 0$ and

$$\frac{cp_1 - ap_1^2 + p_2 + p_1^3 r}{ck_1 + k_2 - 2ak_1p_1 + 3k_1p_1^2 r} = \frac{(b + p_1\lambda)(2m+1)}{2k_1 m\nu + k_1\lambda + 2k_1 m\lambda} = \frac{3p_1 r - a}{k_1 r}. \tag{7}$$

[1] According to Refs. [24,25], a planar dynamic system is a collection of $n$ interrelated differential equations of the form

$$\begin{aligned}x_1' &= g_1(\tau, x_1, x_2, \ldots, x_n),\\ x_2' &= g_2(\tau, x_1, x_2, \ldots, x_n),\\ &\vdots\\ x_n' &= g_n(\tau, x_1, x_2, \ldots, x_n),\end{aligned}$$

where $n$ represents the dimension of the planar dynamical system, $g_l$'s $(l = 1, 2, \ldots, n)$ are the real-valued functions of the $(n+1)$ variables $x_1$, $x_2$, ...,$x_n$ and $\tau$, while the superscript ′ represents the derivative with respect to $\tau$.

[2] That theory shows that, for the planar dynamic system, we can study the behaviors of its solutions by investigating the properties of the functions $g_l$'s without knowing the analytic expressions of the solutions [30]. For example, we can use the topological structure of the phase space to derive the qualitative information (such as the stability, periodicity and recurrence.) about the behaviors of the solutions.

[3] The inter-modal dispersion arises because the light passing along a fiber is distributed into a number of modes which vary in propagation velocities [1].

Solving Expressions (7) gives

$$p_1 = \frac{b(r+2mr)+a(\lambda+2m\nu+2m\lambda)}{2r(3m\nu+\lambda+2m\lambda)}, \tag{8a}$$

$$p_2 = \frac{2a^2k_1p_1 - a(ck_1+k_2+8k_1p_1^2r)+p_1r(2ck_1+3k_2+8k_1rp_1^2)}{k_1r}. \tag{8b}$$

Expression (6) can be rewritten as

$$u_{\xi\xi} - Au - \frac{B}{2m+1}u^{2m+1} = 0, \tag{9}$$

where

$$A = \frac{ck_1+k_2-2ak_1p_1+3k_1p_1^2r}{k_1^3r}, B = \frac{2k_1m\nu+k_1\lambda+2k_1m\lambda}{k_1^3r}.$$

With $X = u$ and $Y = u_\xi$, Expression (9) is equivalent to the two-dimensional planar dynamic system [17,33]

$$X_\xi = Y,$$
$$Y_\xi = AX + \frac{B}{2m+1}X^{2m+1} = f(X). \tag{10}$$

According to Ref. [34], we can see that System (10) is a Hamiltonian system[4] because we can obtain its Hamiltonian[5] as

$$H(X,Y) = \frac{1}{2}Y^2 - \frac{A}{2}X^2 - \frac{B}{(2m+1)(2m+2)}X^{2m+2}. \tag{11}$$

In addition, as $\frac{dH}{d\xi} = \frac{\partial H(X,Y)}{\partial X}X_\xi + \frac{\partial H(X,Y)}{\partial Y}Y_\xi = 0$, the Hamiltonian is a constant of motion, which means that the motion of System (10) is constrained to the curve

$$H(X,Y) = h = \text{constant}, \tag{12}$$

where $h$ is the value of the Hamiltonian $H(X, Y)$. The curve is just the orbit of System (10). For the different values of $h$, Expression (12) determines the different orbits of System (10), while the different orbits determine the different analytic solutions for Eq. (1).

On the $(X, Y)$ phase plane, all the singular points of System (10) are on the $X$ axis. Therefore, the number of the singular points of System (10) depends on the number of the real roots of $f(X)$. For System (10), we study two cases, as follows:

(1) If $AB > 0$, $f(X)$ has only one real root $X_0$, which suggests that System (10) has only one singular point $P_0 = (X_0, 0)$, where $f(X) = 0$ gives $X_0 = 0$.
(2) If $AB < 0$, $f(X)$ has three different real roots $X_0$, $X_-$ and $X_+$, which suggests that System (10) has three singular points $P_j = (X_j, 0)$ $(j = 0, -, +)$, where $f(X) = 0$ gives $X_0 = 0$, $X_+ = \sqrt[2m]{-\frac{(2m+1)A}{B}}$ and $X_- = -X_+$.

Then, we use

$$J = \begin{pmatrix} 0 & 1 \\ A+BX_j^{2m} & 0 \end{pmatrix}, \tag{13}$$

to denote the Jacobian matrix[6] of System (10) at $P_j$. The trace of $J$, $\text{Trace}(J) = 0$ and the determinant of $J$, $\text{Det}(J) = -(A + BX_j^{2m})$.

Via the bifurcation theory, when $\text{Trace}(J) = 0$, we know that if $\text{Det}(J) < 0$, $P_j$ is a saddle point; If $\text{Det}(J) > 0$, $P_j$ is a center point; If $\text{Det}(J) = 0$, $P_j$ is a degenerate saddle point [30]. Thus, we have

(1) if $A > 0$ and $B > 0$, System (10) has only one singular point $P_0$, and $P_0$ is a saddle point;
(2) if $A < 0$ and $B < 0$, System (10) has only one singular point $P_0$, and $P_0$ is a center point;
(3) if $A < 0$ and $B > 0$, System (10) has three singular points $P_0$, $P_-$ and $P_+$. $P_0$ is a center point, and $P_-$ and $P_+$ are the saddle points;

[4] A planar dynamic system, e.g., System (10), is a called a Hamiltonian system provided that there exists a function $H(X, Y)$ such that

$$X_\xi = \frac{\partial H}{\partial Y}, Y_\xi = -\frac{\partial H}{\partial X}.$$

Then $H$ is called a Hamiltonian of the system [34].

[5] Hereby, for System (10), as the Hamiltonian $H$ does not depend explicitly on $\xi$, i.e., $\frac{\partial H}{\partial \xi} = 0$ and $\frac{dH}{d\xi} = 0$, the Hamiltonian $H$ is the total energy of the system [34].

[6] The Jacobian matrix is the matrix of all the first-order partial derivatives of a vector-valued function [35].

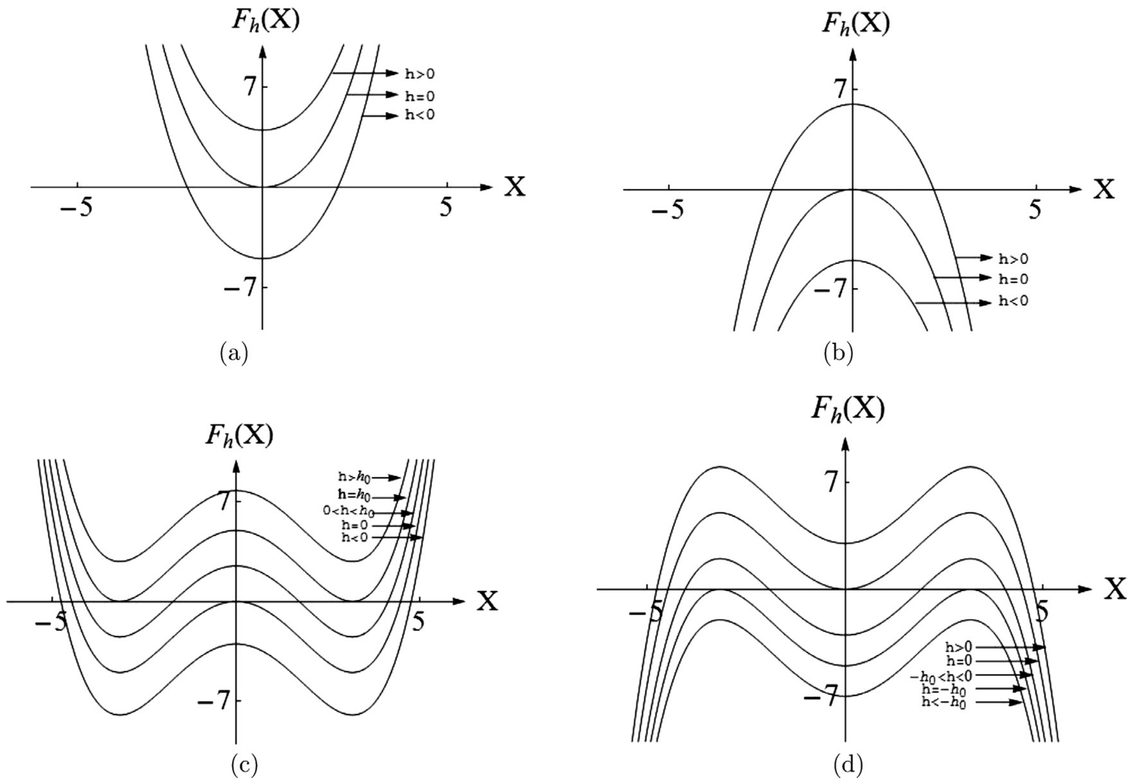


**Fig. 1.** $F_h(X)$ with (a) $A > 0$, $B > 0$; (b) $A < 0$, $B < 0$; (c) $A < 0$, $B > 0$; (d) $A > 0$, $B < 0$.

(4) if $A > 0$ and $B < 0$, System (10) has three singular points $P_0$, $P_-$ and $P_+$. $P_0$ is a saddle point, and $P_-$ and $P_+$ are the center points.

Next, we will consider the relations among $h$, orbits of System (10) and types of the analytic solutions for Eq. (1). Hereby, we denote

$$F_h(X) = h + \frac{A}{2}X^2 + \frac{B}{(2m+1)(2m+2)}X^{2m+2} \tag{14}$$

and

$$h_0 = |F_0(X_+)| = |F_0(X_-)|. \tag{15}$$

Through Expressions (11) and (12), we can have

$$Y = \pm\sqrt{2F_h(X)}, \tag{16}$$

where $F_h(X)$ needs to satisfy $F_h(X) \geq 0$. Via Expressions (14)–(16), graphs of $F_h(X)$ and phase portraits of System (10) are respectively displayed in Figs. 1 and 2, from which, we note that the different values of $h$ lead to the different orbits.

From the qualitative theory of planar dynamic systems, we know that a bell-shaped solitary-wave solution for Eq. (1) corresponds to a homoclinic orbit of System (10); a kink-shaped solitary-wave solution, to a heteroclinic orbit; a periodic-wave solution, to a periodic orbit [30]. Based on the above analysis, from Figs. 1 and 2, we have the following results:

**Case 1:** As shown in Figs. 1(a) and 2(a), when $A > 0$, $B > 0$, no closed curve can be found;
**Case 2:** As shown in Figs. 1(b) and 2(b), when $A < 0$, $B < 0$ and $h > 0$, a periodic orbit $\Gamma_1$ is observed, which suggests that there exist the periodic-wave solutions for Eq. (1);
**Case 3:** As shown in Figs. 1(b) and 2(b), when $A < 0$, $B < 0$ and $h \leq 0$, no closed curve can be found;
**Case 4**: As shown in Figs. 1(c) and 2(c), when $A < 0$, $B > 0$ and $h > h_0$, no closed curve can be found;
**Case 5**: As shown in Figs. 1(c) and 2(c), when $A < 0$, $B > 0$ and $h = h_0$, two heteroclinic orbits $\Gamma_2$ and $\Gamma_3$ are observed, which suggests that there exist the kink-shaped solitary-wave solutions for Eq. (1);
**Case 6**: As shown in Figs. 1(c) and 2(c), when $A < 0$, $B > 0$ and $0 < h < h_0$, a periodic orbit $\Gamma_4$ is observed, which suggests that there exist the periodic-wave solutions for Eq. (1);
**Case 7**: As shown in Figs. 1(c) and 2(c), when $A < 0$, $B > 0$ and $h \leq 0$, no closed curve can be seen;
**Case 8**: As shown in Figs. 1(d) and 2(d), when $A > 0$, $B < 0$ and $h \leq -h_0$, no closed curve can be seen;

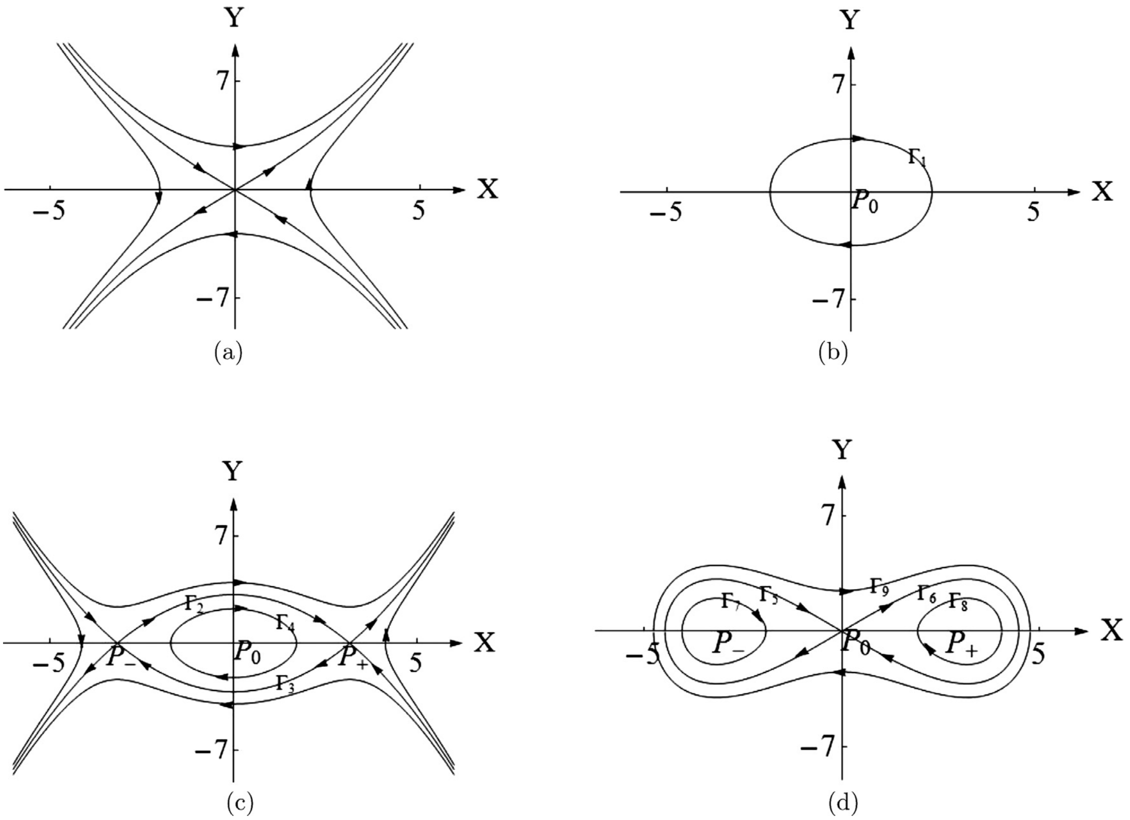


**Fig. 2.** Phase portraits of System (10) with (a) $A > 0$, $B > 0$; (b) $A < 0$, $B < 0$; (c) $A < 0$, $B > 0$; (d) $A > 0$, $B < 0$.

**Case 9**: As shown in Figs. 1(d) and 2(d), when $A > 0$, $B < 0$ and $h = 0$, two homoclinic orbits $\Gamma_5$ and $\Gamma_6$ are observed, which suggests that there exist the bell-shaped solitary-wave solutions for Eq. (1);

**Case 10**: As shown in Figs. 1(d) and 2(d), when $A > 0$, $B < 0$ and $-h_0 < h < 0$, two periodic orbits $\Gamma_7$ and $\Gamma_8$ are observed, which suggests that there exist the periodic-wave solutions for Eq. (1);

**Case 11**: As shown in Figs. 1(d) and 2(d), when $A > 0$, $B < 0$ and $h > 0$, a periodic orbit $\Gamma_9$ is seen, which suggests that there exist the periodic-wave solutions for Eq. (1).

## 3. Analytic solutions

In this part, we will derive the analytic expressions of the periodic-wave solutions, kink- and bell-shaped solitary-wave solutions for Eq. (1) with $m = 1$ and 2.

### 3.1. With $m = 1$

#### 3.1.1. Periodic-wave solutions

According to **Cases 2, 6, 10** and **11**, we know that the periodic-wave solutions correspond to the periodic orbits $\Gamma_1$, $\Gamma_4$, $\Gamma_7$, $\Gamma_8$ and $\Gamma_9$, respectively.

(1) Through Expression (16), $\Gamma_1$ can be given as

$$Y = \pm\sqrt{\frac{-B}{6}}\sqrt{(\sigma_1^2 - X^2)(X^2 + \sigma_2^2)}, \tag{17}$$

where $\sigma_1 = \sqrt{\frac{-3A-\sqrt{9A^2-12Bh}}{B}}$ and $\sigma_2 = \sqrt{\frac{-3A+\sqrt{9A^2-12Bh}}{-B}}$.

Substituting Expression (17) into $dX/d\xi = Y$ and then integrating the obtained expression along $\Gamma_1$, we have

$$\pm\int_X^{\sigma_1} \frac{1}{\sqrt{(\sigma_1^2 - s^2)(s^2 + \sigma_2^2)}} ds = \sqrt{\frac{-B}{6}}\int_0^{\xi} ds. \tag{18}$$

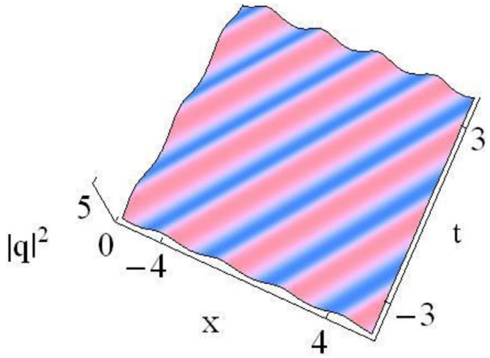


**Fig. 3.** Intensity profile of the periodic wave given by Solutions (20) with the parameters as $A=-1,\ B=-2,\ k_1=k_2=p_1=p_2=1,\ h=0.55$.

After completing the above integral with the integral constant vanishing, we obtain

$$X=\pm\sigma_1\mathrm{cn}\left[\sqrt{\frac{-B(\sigma_1^2+\sigma_2^2)}{6}}\xi,\ \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right]. \tag{19}$$

Then, from Expression (2), we can get the following periodic-wave solutions for Eq. (1):

$$q(x,t)=\pm\sigma_1\mathrm{cn}\left[\sqrt{\frac{-B(\sigma_1^2+\sigma_2^2)}{6}}(k_1x-k_2t),\ \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right]e^{i(p_1x-p_2t)}. \tag{20}$$

A periodic wave given by Solutions (20) is depicted in Fig. 3.

(2) Through Expression (16), $\Gamma_4$ is expressed as

$$Y=\pm\sqrt{\frac{B}{6}}\sqrt{(\sigma_3^2-X^2)(\sigma_4^2-X^2)}, \tag{21}$$

where $\sigma_3=\sqrt{-\frac{3A+\sqrt{9A^2-12Bh}}{B}}$ and $\sigma_4=\sqrt{-\frac{3A-\sqrt{9A^2-12Bh}}{B}}$.

Substituting Expression (21) into $dX/d\xi=Y$ and then integrating the obtained expression along $\Gamma_4$, we have

$$\pm\int_0^X\frac{1}{\sqrt{(\sigma_3^2-s^2)(\sigma_4^2-s^2)}}ds=\sqrt{\frac{B}{6}}\int_0^\xi ds. \tag{22}$$

After completing the above integral with the integral constant vanishing, we obtain

$$X=\pm\sigma_3\mathrm{sn}\left[\sqrt{\frac{B}{6}}\sigma_4\xi,\ \frac{\sigma_3}{\sigma_4}\right]. \tag{23}$$

Then, from Expression (2), we can get the following periodic-wave solutions for Eq. (1):

$$q(x,t)=\pm\sigma_3\mathrm{sn}\left[\sqrt{\frac{B}{6}}\sigma_4(k_1x-k_2t),\ \frac{\sigma_3}{\sigma_4}\right]e^{i(p_1x-p_2t)}. \tag{24}$$

A periodic wave given by Solutions (24) is depicted in Fig. 4.

(3) Through Expression (16), $\Gamma_7$ and $\Gamma_8$ can be given as

$$Y=\pm\sqrt{\frac{-B}{6}}\sqrt{(X^2-\sigma_5^2)(\sigma_6^2-X^2)}, \tag{25}$$

with $\sigma_5=\sqrt{\frac{-3A+\sqrt{9A^2-12Bh}}{B}}$ and $\sigma_6=\sqrt{\frac{-3A-\sqrt{9A^2-12Bh}}{B}}$.

Substituting Expression (25) into $dX/d\xi=Y$ and then integrating the obtained expression along $\Gamma_7$ and $\Gamma_8$, we have

$$\pm\int_X^{\sigma_6}\frac{1}{\sqrt{(s^2-\sigma_5^2)(\sigma_6^2-s^2)}}ds=\sqrt{\frac{-B}{6}}\int_0^\xi ds(X\geq 0) \tag{26}$$

and

$$\pm\int_{-\sigma_6}^X\frac{1}{\sqrt{(s^2-\sigma_5^2)(\sigma_6^2-s^2)}}ds=\sqrt{\frac{-B}{6}}\int_0^\xi ds(X<0). \tag{27}$$

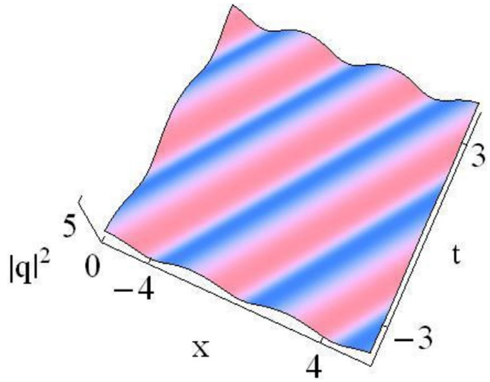


**Fig. 4.** Intensity profile of the periodic wave given by Solutions (24) with the parameters as $A = -1,\ B = 1,\ k_1 = k_2 = p_1 = p_2 = 1,\ h = 0.5$.

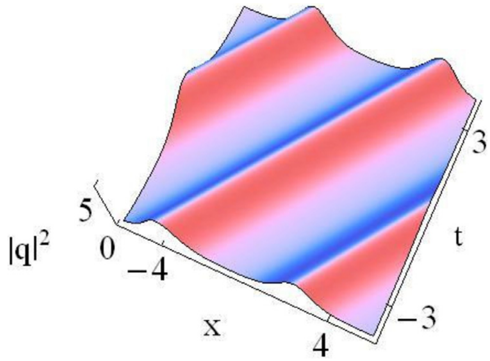


**Fig. 5.** Intensity profile of the periodic wave given by Solutions (29) with the parameters as $A = 1, B = -2, k_1 = k_2 = p_1 = p_2 = 1, h = -0.1$.

After completing the above integrals with the integral constants vanishing, we obtain

$$X = \pm\sigma_6 \mathrm{dn}\left[\sigma_6\sqrt{\frac{-B}{6}}\xi,\ \frac{\sqrt{\sigma_6^2-\sigma_5^2}}{\sigma_6}\right]. \tag{28}$$

Then, from Expression (2), we can get the following periodic-wave solutions for Eq. (1):

$$q(x,t) = \pm\sigma_6 \mathrm{dn}\left[\sigma_6\sqrt{\frac{-B}{6}}(k_1x-k_2t),\ \frac{\sqrt{\sigma_6^2-\sigma_5^2}}{\sigma_6}\right]e^{i(p_1x-p_2t)}. \tag{29}$$

A periodic wave given by Solutions (29) is depicted in Fig. 5.

(4) Through Expression (16), $\Gamma_9$ can be expressed as

$$Y = \pm\sqrt{\frac{-B}{6}}\sqrt{(\sigma_7^2-X^2)(\sigma_8^2+X^2)}, \tag{30}$$

where $\sigma_7 = \sqrt{\frac{-3A-\sqrt{9A^2-12Bh}}{B}}$ and $\sigma_8 = \sqrt{\frac{-3A+\sqrt{9A^2-12Bh}}{-B}}$.

Substituting Expression (30) into $dX/d\xi = Y$ and then integrating the obtained expression along $\Gamma_9$, we get

$$\pm\int_{-\sigma_7}^{X}\frac{1}{\sqrt{(\sigma_7^2-s^2)(\sigma_8^2+s^2)}}ds = \sqrt{\frac{-B}{6}}\int_0^{\xi}ds. \tag{31}$$

After completing the above integral with the integral constant vanishing, we obtain

$$X = \pm\sigma_7 \mathrm{cn}\left[\sqrt{\frac{-B(\sigma_7^2+\sigma_8^2)}{6}}\xi,\ \frac{\sigma_7}{\sqrt{\sigma_7^2+\sigma_8^2}}\right]. \tag{32}$$

Then, from Expression (2), the periodic-wave solutions for Eq. (1) can be given as

$$q(x,t) = \sigma_7 \mathrm{cn}\left[\sqrt{\frac{-B(\sigma_7^2+\sigma_8^2)}{6}}(k_1x-k_2t),\ \frac{\sigma_7}{\sqrt{\sigma_7^2+\sigma_8^2}}\right]e^{i(p_1x-p_2t)}. \tag{33}$$

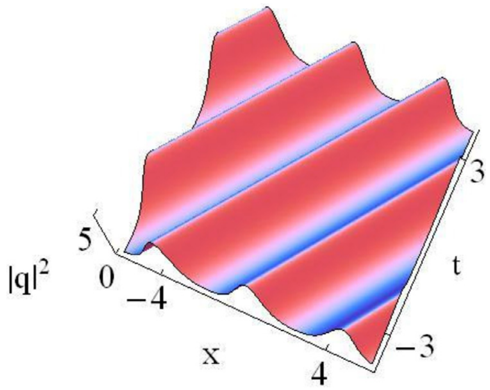


**Fig. 6.** Intensity profile of the periodic wave given by Solutions (33) with the parameters as $A=1,\ B=-2,\ k_1=k_2=p_1=p_2=1,\ h=0.5$.

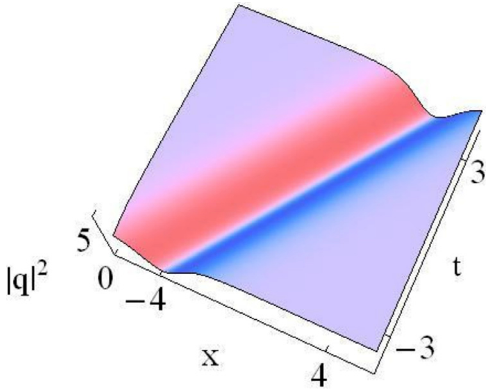


**Fig. 7.** Intensity profile of the kink-shaped solitary wave given by Solutions (37) with the parameters as $A=-1,\ B=1,\ k_1=k_2=p_1=p_2=1,\ h=0.75$.

A periodic wave given by Solutions (33) is depicted in Fig. 6.

*3.1.2. Kink-shaped solitary-wave solutions*

**Case 4** shows that the kink-shaped solitary-wave solutions for Eq. (1) correspond to the two heteroclinic orbits, $\Gamma_2$ and $\Gamma_3$, which can be expressed as

$$Y=\pm\sqrt{\frac{B}{6}}(\sigma_9^2-X^2), \tag{34}$$

with $\sigma_9=\sqrt{\frac{-3A}{B}}$.

Substituting Expression (34) into $dX/d\xi=Y$ and then integrating the obtained expression along $\Gamma_2$ and $\Gamma_3$, we have

$$\pm\int_0^X\frac{1}{\sigma_9^2-s^2}ds=\sqrt{\frac{B}{6}}\int_0^\xi ds. \tag{35}$$

After completing the above integral with the integral constant vanishing, we obtain

$$X=\pm\sigma_9\tanh\left(\sqrt{\frac{B}{6}}\sigma_9\xi\right). \tag{36}$$

Then, from Expression (2), the kink-shaped solitary-wave solutions for Eq. (1) are given as

$$q(x,t)=\pm\sigma_9\tanh\left[\sqrt{\frac{B}{6}}\sigma_9(k_1x-k_2t)\right]e^{i(p_1x-p_2t)}. \tag{37}$$

A kink-shaped solitary wave given by Solutions (37) is depicted in Fig. 7.

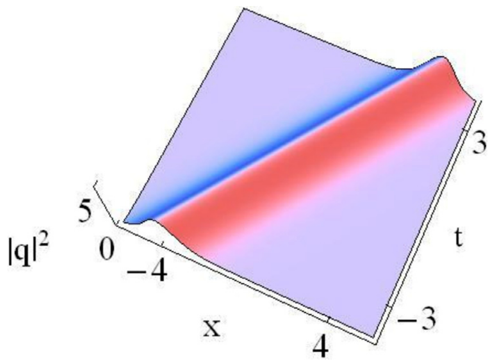


**Fig. 8.** Intensity profile of the bell-shaped solitary wave given by Solutions (43) with the parameters as $A = 1,\ B = -2,\ k_1 = k_2 = p_1 = p_2 = 1,\ h = 0$.

### 3.1.3. Bell-shaped solitary-wave solutions

**Case 8** shows that the bell-shaped solitary-wave solutions correspond to the two homoclinic orbits, $\Gamma_5$ and $\Gamma_6$, which can be expressed as

$$Y = \pm\sqrt{\frac{-B}{6}}X\sqrt{-X^2 - \frac{6A}{B}}. \tag{38}$$

Substituting Expression (38) into $dX/d\xi = Y$ and then integrating the obtained expression along $\Gamma_5$ and $\Gamma_6$, we have

$$\pm\int_X^{\sigma_{10}} \frac{1}{s\sqrt{-s^2 - \frac{6A}{B}}}ds = \sqrt{\frac{-B}{6}}\int_0^{\xi} ds (X \geq 0) \tag{39}$$

and

$$\pm\int_X^{-\sigma_{10}} \frac{1}{s\sqrt{-s^2 - \frac{6A}{B}}}ds = \sqrt{\frac{-B}{6}}\int_0^{\xi} ds (X < 0), \tag{40}$$

where

$$\sigma_{10} = \sqrt{-\frac{6A}{B}}. \tag{41}$$

After completing the above integrals with the integral constants vanishing, we obtain

$$X = \pm\sqrt{\frac{6A}{-B}}\operatorname{sech}\left(\sqrt{A}\xi\right). \tag{42}$$

From Expression (2), we can get the following bell-shaped solitary-wave solutions for Eq. (1):

$$q(x,t) = \pm\sqrt{\frac{6A}{-B}}\operatorname{sech}\left[\sqrt{A}(k_1 x - k_2 t)\right]e^{i(p_1 x - p_2 t)}. \tag{43}$$

A bell-shaped solitary wave given by Solutions (43) is depicted in Fig. 8.

Finally, we give the relations of the above solutions:

(1) $h \to (h_0)^-$ yields Solutions (24)→ Solutions (37);
(2) $h \to 0^-$ yields Solutions (29)→ Solutions (43);
(3) $h \to 0^+$ yields Solutions (33)→ Solutions (43).

## 3.2. With $m = 2$

### 3.2.1. Periodic-wave solutions

(1) $\Gamma_4$ can be expressed as

$$Y = \pm\sqrt{\frac{B}{15}}\sqrt{(\omega_1^2 - X^2)(\omega_2^2 - X^2)(\omega_3^2 + X^2)}, \tag{44}$$

with $\left(\frac{-5A}{B}\right)^{\frac{1}{4}} < \omega_1 < \left(\frac{-15A}{B}\right)^{\frac{1}{4}}$, $\omega_2 = \sqrt{\frac{-\omega_1^2 B + \sqrt{3}\sqrt{-B(20A + B\omega_1^4)}}{2B}}$ and $\omega_3 = \sqrt{\frac{\omega_1^2 B + \sqrt{3}\sqrt{-B(20A + B\omega_1^4)}}{2B}}$.

According to Expression (44), the corresponding periodic-wave solutions for Eq. (1) can be given as

$$q(x,t)=\pm\frac{\omega_1\omega_3\,\mathrm{sn}\left[\omega_2\sqrt{\frac{B}{15}}\sqrt{\omega_1^2+\omega_3^2}(k_1x-k_2t),\ \frac{\omega_1}{\omega_2}\sqrt{\frac{\omega_2^2+\omega_3^2}{\omega_1^2+\omega_3^2}}\right]}{\sqrt{\omega_3^2+\omega_1^2\mathrm{cn}^2\left[\omega_2\sqrt{\frac{B}{15}}\sqrt{\omega_1^2+\omega_3^2}(k_1x-k_2t),\ \frac{\omega_1}{\omega_2}\sqrt{\frac{\omega_2^2+\omega_3^2}{\omega_1^2+\omega_3^2}}\right]}}e^{i(p_1x-p_2t)}. \tag{45}$$

(2) $\Gamma_7$ and $\Gamma_8$ can be expressed as

$$Y=\pm\sqrt{\frac{-B}{15}}\sqrt{(X^2-\omega_4^2)(\omega_5^2-X^2)(\omega_6^2+X^2)}, \tag{46}$$

with $0<\omega_4<\left(\frac{-5A}{B}\right)^{\frac{1}{4}}$, $\omega_5=\sqrt{\frac{-\omega_4^2B-\sqrt{3}\sqrt{-B(20A+B\omega_4^4)}}{2B}}$ and $\omega_6=\sqrt{\frac{\omega_4^2B-\sqrt{3}\sqrt{-B(20A+B\omega_4^4)}}{2B}}$.

According to Expression (46), the corresponding periodic-wave solutions for Eq. (1) can be given as

$$q(x,t)=\pm\frac{\omega_4^2\omega_5^2e^{i(p_1x-p_2t)}}{\sqrt{\omega_5^2-(\omega_5^2-\omega_4^2)\mathrm{sn}^2\left[\omega_5\sqrt{\frac{-B}{15}}\sqrt{\omega_4^2+\omega_6^2}(k_1x-k_2t),\ \frac{\omega_6}{\omega_5}\sqrt{\frac{(\omega_5^2-\omega_4^2)}{(\omega_6^2+\omega_4^2)}}\right]}}. \tag{47}$$

(3) $\Gamma_9$ is expressed as

$$Y=\pm\sqrt{\frac{-B}{6}}\sqrt{(\omega_7^2-X^2)(\omega_8^2+X^2)(\omega_9^2+X^2)}, \tag{48}$$

where $\omega_7>\left(\frac{-15A}{B}\right)^{\frac{1}{4}}$, $\omega_8=\sqrt{\frac{\omega_7^2B-\sqrt{3}\sqrt{-B(20A+B\omega_7^4)}}{2B}}$ and $\omega_9=\sqrt{\frac{\omega_7^2B+\sqrt{3}\sqrt{-B(20A+B\omega_7^4)}}{2B}}$.

According to Expression (48), the corresponding periodic-wave solutions for Eq. (1) can be given as

$$q(x,t)=\pm\frac{\omega_7\omega_9\,\mathrm{sn}\left[\omega_8\sqrt{\frac{-B}{15}}\sqrt{\omega_7^2+\omega_9^2}(k_1x-k_2t),\ \frac{\omega_7}{\omega_8}\sqrt{\frac{\omega_8^2-\omega_9^2}{\omega_7^2+\omega_9^2}}\right]}{\sqrt{\omega_9^2+\omega_7^2\mathrm{cn}^2\left[\omega_8\sqrt{\frac{-B}{15}}\sqrt{\omega_7^2+\omega_9^2}(k_1x-k_2t),\ \frac{\omega_7}{\omega_8}\sqrt{\frac{\omega_8^2-\omega_9^2}{\omega_7^2+\omega_9^2}}\right]}}e^{i(p_1x-p_2t)}. \tag{49}$$

*3.2.2. Kink-shaped solitary-wave solutions*

$\Gamma_2$ and $\Gamma_3$ can be expressed as

$$Y=\pm\sqrt{\frac{B}{15}}(\omega_{10}^2-X^2)\sqrt{X^2+\omega_{11}^2}, \tag{50}$$

with $\omega_{10}=\left(\frac{-5A}{B}\right)^{\frac{1}{4}}$ and $\omega_{11}=\sqrt{2}\omega_{10}$.

According to Expression (50), the corresponding kink-shaped solitary-wave solutions for Eq. (1) are given as

$$q(x,t)=\pm\frac{\omega_{10}\omega_{11}\tanh\left[\omega_{10}\sqrt{\frac{B}{15}}\sqrt{\omega_{10}^2+\omega_{11}^2}(k_1x-k_2t)\right]}{\sqrt{\omega_{11}^2+\omega_{10}^2\mathrm{sech}^2\left[\omega_{10}\sqrt{\frac{B}{15}}\sqrt{\omega_{10}^2+\omega_{11}^2}(k_1x-k_2t)\right]}}e^{i(p_1x-p_2t)}. \tag{51}$$

*3.2.3. Bell-shaped solitary-wave solutions*

$\Gamma_5$ and $\Gamma_6$ can be expressed as

$$Y=\pm\sqrt{\frac{-B}{15}}X\sqrt{-X^4-\frac{15A}{B}}. \tag{52}$$

According to Expression (52), the corresponding bell-shaped solitary-wave solutions for Eq. (1) can be given as

$$q(x,t)=\pm\left\{\sqrt{\frac{15A}{-B}}\mathrm{sech}\left[2\sqrt{A}(k_1x-k_2t)\right]\right\}^{\frac{1}{2}}e^{i(p_1x-p_2t)}. \tag{53}$$

The relations of the above solutions are given as follows:

(1) $h\to(h_0)^-$ yields Solutions (45)→ Solutions (51);
(2) $h\to 0^-$ yields Solutions (47)→ Solutions (53);
(3) $h\to 0^+$ yields Solutions (49)→ Solutions (53).

## 4. Conclusions

This paper has been focused on a perturbed NLS equation with the power law nonlinearity, i.e., Eq. (1), which describes the effects of the quantic nonlinearity on the propagation of the ultrashort optical pulses in a non-Kerr medium like an optical fiber. We have derived the equivalent two-dimensional planar dynamic system for Eq. (1), i,e., System (10). According to the bifurcation theory and qualitative theory of planar dynamic systems, phase portraits of System (10) in different cases have been obtained, as seen in Figs. 1 and 2. Through the analysis of Figs. 1 and 2, relations among the Hamiltonian, i.e., Expression (11), orbits of System (10) and types of the analytic solutions for Eq. (1) have been discussed. For Eq. (1), we have given the analytic expressions of the periodic-wave solutions, i.e., Solutions (20), (24), (29), (33), (45), (47), and (49), kink-shaped solitary-wave solutions, i.e., Solutions (37) and (51), and bell-shaped solitary-wave solutions, i.e., Solutions (43) and (53). Figs. 3–7 have been shown to display Solutions (24), (29), (33), (37) and (43), respectively. Among those solutions, we have found that Solutions (24) can be reduced to Solutions (37), Solutions (29) to Solutions (43), Solutions (33) to Solutions (43), Solutions (45) to Solutions (51), Solutions (47) to Solutions (53), and Solutions (49) to Solutions (53).

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