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On the analytic assessment of the impact of traffic correlation on queues in continuous time domain

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ABSTRACT

Given only the traffic correlations of counts and intervals, a Batch Renewal Arrival Process (BRAP) is completely determined, as the least biased choice and thus, it provides the analytic means to construct suitable traffic models for the study of queueing systems independently of any other traffic characteristics. In this context, the BRAP and the Batch Markovian Arrival Process (BMAP) are employed in the continuous time domain towards the analysis of the stable BRAP/GE/1 and BMAP/GE/1 queues with infinite capacity, single servers and generalized exponential (GE) service times. Novel closed form expressions for the steady state probabilities of these queues are obtained, based on the embedded Markov chains (EMCs) technique and the matrix-geometric (M-G) method, respectively. Moreover, the stable $GE^{sGGeo}/GE/1$ queue with GE-type service times and a GE^{sGGeo} BRAP consisting of bursty GE-type batch interarrival times and a shifted generalized geometric (sGGeo) batch size distribution is adopted to assess analytically the combined adverse effects of varying degrees of correlation of intervals between individual arrivals and the burstiness of service times upon the typical quality of service (QoS) measure of the mean queue length (MQL). Moreover, a comprehensive experimental study is carried out to investigate numerically the relative impact of count and interval traffic correlations as well as other traffic characteristics upon the performance of stable BRAP/GE/1 and BMAP/GE/1 queues. It is suggested via a conjecture that the BRAP/GE/1 queue is likely to yield pessimistic performance metrics in comparison to those of the stable BMAP/GE/1 queues under the worst case scenario (i.e., a worst case scenario) of the same positive count and interval traffic correlations arising from long sojourn in each phase.

1. Introduction

Correlated traffic modelling has received considerable attention in the literature, where a variety of processes have been applied to model correlated traffic flows. Markov modulated processes, semi-Markov and Markov renewal processes are the most frequently employed analytic models [1,2] and appropriate correlation metrics, such as indices of dispersion for counts and intervals, were established [3–5].

In the discrete time domain a Batch Renewal Arrival Process (BRAP) has been found to exhibit both interarrival correlations between individual arrivals and count correlation between successive epochs [6]. In the continuous time domain, a BRAP has continuous inter-batch arrival times and discrete counts.

As the measures of correlation do not characterize the traffic completely, it is possible to have more than one type of stochastic models (with appropriate parameterization) generating traffic with the same measures of correlation. However, given the count and interval traffic correlations, a BRAP can be completely determined as the least-biased choice and thus, it provides the analytic means to investigate the effect of traffic correlation independently of any other characteristics.

The discovery of the BRAP and its applicability into the analysis of the stable single server BRAP/D/1 and BRAP/D/1/N queues with BRAP arrivals and deterministic (D) service times in the discrete time domain can be traced in [6,7]. Moreover, a rigorous steady-state analysis of a multiserver BRAP/Geo/c queue with a BRAP and geometric (Geo) service times in the discrete time domain was carried out in [8] based on the probability generating functions (PGFs) approach and contour integration.

In a similar context, the Batch Markovian Arrival Process (BMAP) was introduced earlier by Lucantoni [1] to keep some aspects of the tractability of the Poisson Arrival Process (PAP) but significantly generalize it in ways that allow the inclusion of non-exponential and/or dependent interarrival times with correlated arriving batch sizes.

The BRAP and BMAP with the same interval and count traffic

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correlations are employed in this study to analyse in the continuous time domain the stable single server BRAP/GE/1 and BMAP/GE/1 queues with generalized exponential (GE) (cf. [9]) service times. Moreover, a comparative study is undertaken to assess the impact of traffic correlation on the performance of these queues.

Note that the GE distribution is of the form [9]

$$F(t) = 1 - \frac{2}{C^2+1}\exp\left[-\frac{2\nu t}{C^2+1}\right], \quad t \geq 0 \tag{1}$$

and may be used to represent bursty interarrival and service times with known mean, $1/\nu$ and squared coefficient of variation (SCV) C^2, respectively. For $C^2 > 1$, the GE is a proper mixed probability distribution and it can be interpreted as either

- an extremal case of the family of two-phase exponential distributions (e.g. Hyper-exponential-2 (H_2)) having the same ν and C^2, where one of the two phases has zero service time [10,11] and
- a bulk type distribution with an underlying counting process equivalent to a compound Poisson process (CPP) with parameter $2\nu/(1+C^2)$ and geometrically distributed batch sizes with mean $(C^2+1)/2$ and SCV $(C^2-1)/(C^2+1)$.

The GE-type distribution exhibits neither correlation of counts nor correlation of intervals. However, it is versatile, possessing pseudo-memoryless properties, which makes the solution of many GE-type queueing systems and networks analytically tractable [11].

The organization of the rest of the paper is as follows: Sections 2 and 3 focus on the definitions of BRAP and BMAP and the PGFs of their count and interval correlations, the full proofs of which are derived in Appendices A and B, respectively. A novel closed form solution of the queue length distribution (QLD) (or random observer's steady state probabilities) of a stable BRAP/GE/1 queue with GE-type service times is obtained in Section 4 by adopting the technique of two embedded Markov chains (EMCs). Moreover, Section 5 investigates, for illustration purposes, the adverse impact of interval correlation between individual arrivals and the burstiness of service times upon the typical quality of service (QoS) measure of the mean queue length (MQL) for the stable $\mathrm{GE}^{\mathrm{sGGeo}}$/GE/1 queue with GE-type batch inter-arrival times and service times and a shifted generalized geometric (sGGeo) batch size distribution. The proof of the full derivation of the QLD of the stable $\mathrm{GE}^{\mathrm{sGGeo}}$/GE/1 queue is presented in Appendix C. An explicit new solution for the QLD of the stable BMAP/GE/1 queue, based on the application of the matrix-geometric (M-G) method, is derived in Section 6. A comprehensive comparative study on the impact of traffic correlation on performance of the stable BRAP/GE/1 and BMAP/GE/1 queues with the same count and interval correlations is presented in Section 7. Concluding remarks and some future directions are drawn in Section 8.

2. Batch Renewal Arrival Process (BRAP)

This section is concerned with the PGFs of count and interval distributions of the BRAP and in particular, how they are related to traffic correlation and fitted to measured correlation.

The BRAP allows simultaneous events such that

- The number of individual customers in different batches are independent and identically distributed (i.i.d.).
- The intervals between batches are i.i.d.
- The batch sizes (number of individual customers in one batch) are independent of intervals between batches.

In the discrete time domain the BRAP has been found to exhibit both interarrival correlation between individual arrivals and count correlation between successive epochs [6]. In the continuous time domain, a BRAP has continuous inter-batch arrival times and discrete counts. In the next two subsections the corresponding correlations are investigated in the context of a continuous time BRAP.

A continuous time BRAP is specified by the batch size random variable b_k and the inter-batch time random variable a_s.

2.1. The PGF of count correlation of BRAP

In order to compute the count correlation, consider an instant s_0 selected at random and start counting the number of arriving batches from s_0. The probability that k ($k \geq 1$) batch arrivals in an interval of duration t, beginning at s_0, has a probability density function (PDF) that is the convolution of three probability densities, namely

- The time s from s_0 to the instant that the first batch arrives (residual time) (s could be zero).
- The time $u - s$ of the $k-1$ intervals between k successive batches ($u - s = 0$ if k=1).
- The remaining time $t - u$ after the arrival of the kth batch, which might be zero and must be no longer than the interarrival time between the kth and $(k+1)$th batches

as illustrated by graph in Fig. 1.

Considering the two PDFs that there be (i) k ($k \geq 1$) batch arrivals and (ii) no arriving batches during t, respectively, and combining their Laplace transforms of the PDF with the PGFs that there will be n customers in k batch arrivals, the PGF of count correlation, $K(\theta)$ is determined by

$$K(\theta) = b\left(C_b^2 + \frac{1+A(\theta)}{1-A(\theta)} - \frac{2}{a}\cdot\frac{1}{\theta}\right) \tag{2}$$

where $A(\theta)$ is the Laplace transform of interval between batches, b is the mean batch size, C_b^2 is the squared coefficient of variation (SCV) of batch sizes and a is the mean inter-batch interval, i.e., $\lambda = b/a$ is the mean arrival rate of individual customers.

2.2. The PGF of interval correlation of BRAP

In order to determine the interval correlation, consider the probability that there will be j customers following a randomly selected customer in the same arriving batch, i.e., $\sum_{v=j+1}^{\infty} b(v)/b$, where $b(v)$ is the probability mass function (PMF) of batch size v, $v = 1, 2, \ldots$ (cf. [12]). Moreover, by naming an arriving customer selected at random as customer 0 (denoted c_0) and the batch it belongs to as the 0th batch, by counting the number of customers following c_0, the probability that the nth customer (denoted c_n) will be in the sth batch has mass function that is clearly the convolution of two PMFs that

- c_n is in the 0th batch (i.e., c_n is in the same batch as c_0)
- n customers (from c_1 through c_n) span from the 0th to the sth batches, consisting of
  1. j ($j \geq 0$) customers following c_0 in the 0th batch.
  2. $k - j$ ($0 \leq k - j < n$) customers in the 1st to the $(s-1)$ th batches.
  3. $n - k - 1$ ($n - k - 1 \geq 0$) customers in the sth batch before c_n.

as illustrated by graph in Fig. 2.

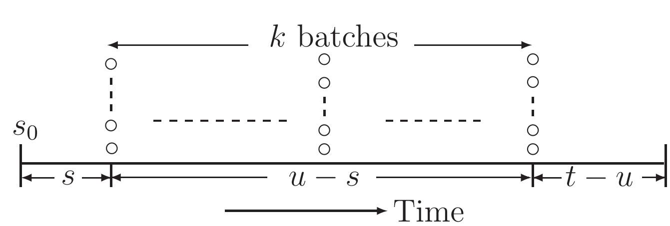

Fig. 1. IDC structure.

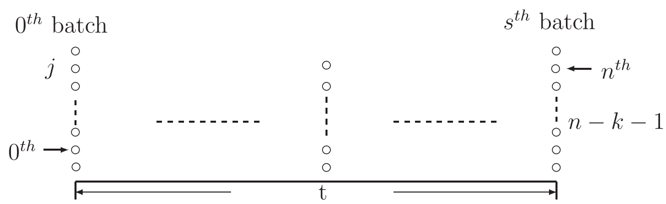

Fig. 2. IDI structure.

Focusing on the convolution of two PMFs that n ($n \geq 1$) customers that i) are in the same arriving tagged batch of a randomly selected customer and ii) span in $s + 1 (s \geq 1)$ arriving batches following the tagged customer and taking into account the 0th batch and that n customers span in $s + 1$ arriving batches and combining them with the PGF of the probability of n intervals between $n + 1$ successive individual arrivals during a time interval t and integrating over t, the PGF of interval correlation, $L(z)$ is expressed by

$$L(z) = b\left(C_a^2 + \frac{1 + B(z)}{1 - B(z)} - \frac{1}{b}\cdot\frac{1 + z}{1 - z}\right) \tag{3}$$

where $B(z)$ is the PGF of batch size and C_a^2 is the SCV of intervals between batches.

Note that the detailed proofs of both expressions of the PGFs of a BRAP, namely $K(\theta)$ and $L(z)$ (cf. (2) and (3), respectively) can be seen in Appendix A (which also displays a complete notation of all relevant to the Sections 2.1 and 2.2 mathematical symbols).

2.3. Theorem 1: determine B(z) and $A(\theta)$ in terms of count and interval correlations

From the PGFs of count and interval correlations, i.e., Eqs. (2) and (3), $B(z)$ and $A(\theta)$ can be expressed in terms of $L(z)$ and $K(\theta)$ and are given, respectively, by

$$1 - B(z) = \frac{2b}{L(z) - b(C_a^2 - 1) + \dfrac{1 + z}{1 - z}} \tag{4}$$

$$1 - A(\theta) = \frac{2b}{K(\theta) - b(C_b^2 - 1) + 2\lambda\cdot\dfrac{1}{\theta}} \tag{5}$$

It has been shown [13] that Eqs. (4) and (5) define $B(z)$ and $A(\theta)$ uniquely given $L(z)$ and $K(\theta)$.

2.4. Construction of a BRAP from measures of correlations

It is not easy to compute $A(\theta)$ and $B(z)$ from Eqs. (4) and (5) as they contain SCVs. However, the following relations can be clearly obtained:

$$L(1) = L(z \to 1) = b(C_a^2 + C_b^2) \tag{6}$$

$$K(0) = K(\theta \to 0) = b(C_a^2 + C_b^2) \tag{7}$$

$$L(0) = L(z \to 0) = b\left(C_a^2 + 1 - \frac{1}{b}\right) \tag{8}$$

$$K(\infty) = K(\theta \to \infty) = b(C_b^2 + 1) \tag{9}$$

Therefore, it follows that

$$b(C_a^2 - 1) = L(1) - K(\infty) \tag{10}$$

$$b(C_b^2 - 1) = L(1) - L(0) - 1 \tag{11}$$

Substituting Eqs. (6)–(11) to (4) and (5)**?**, $A(\theta)$ and $B(z)$ can be expressed by

$$A(\theta) = 1 - \frac{K(\infty) + L(0) - L(1) + 1}{K(\theta) + L(0) - L(1) + 1 + 2\lambda\cdot\dfrac{1}{\theta}} \tag{12}$$

$$B(z) = 1 - \frac{K(\infty) + L(0) - L(1) + 1}{K(\infty) + L(z) - L(1) + \dfrac{1 + z}{1 - z}} \tag{13}$$

For computation purposes, provided λ is known either analytically or numerically from measurements, expressions (12) and (13) are most suitable. It is theoretically feasible to derive simple recurrence relations which can be used to describe the $a(t)$ and $b(n)$ in terms of λ, I_t and J_n, where I_t and J_n are the indices of dispersion of counts (IDCs) and intervals (IDIs), respectively. The main objection to direct calculation of the component distributions $a(t)$ and $b(n)$ is that, for fixed precision arithmetic, rounding errors accumulate and are likely to become significant when dealing with covariances at the longer lags. This calculation requires the difference between numbers of similar magnitude and so the effect of rounding errors might accumulate rapidly.

To overcome such computational issues, the adopted approach in this study (cf. [12]) is based on the conversion of the actual traffic measurements to an algebraic representation and then apply the algebraic method to obtain the constitute distributions.

3. Batch Markovian Arrival Process (BMAP)

This section presents the analytic expressions for the PGFs of count and interval correlations relating to traffic generated by a BMAP. For exposition purposes, it is assumed that the first two moments of batch size exist for each phase. Note that the BMAP is determined from not only the traffic correlation of counts and intervals (as it is the case of the BRAP) but also other traffic characteristics such as cross-correlations between counts and intervals etc.

3.1. The PGF of count correlation of BMAP

By (i) deriving the z and Laplace transforms of $\mathbf{p}(n, t)$, the probability density of n arrivals in the interval $(0, t]$, (ii) combining these transforms with the stationary probability vector $\boldsymbol{\pi} = \{\pi_i\}$, of the Markov process with the infinitesimal generator $\mathbf{D}$, $\mathbf{D} = \{\mathbf{D}_{ij}\}$ of the underlying Markov process, where $\mathbf{D}_{ij}$ is the infinitesimal rate at which a transition from phase i to phase j takes place (i.e., $\pi\mathbf{D} = 0$, $\pi\mathbf{e} = 1$, where $\mathbf{e} \approx 2.71828$ is Euler's number), and (iii) following a similar procedure as the one used in deriving the PGF $K(\theta)$ of BRAP (cf. (2)) in Section 2, the PGF for count correlation, $K(\theta)$ of BMAP can be determined by

$$K(\theta) = \frac{1}{\lambda}\boldsymbol{\pi}(2\mathbf{D}'(1)(\theta\mathbf{I} - \mathbf{D})^{-1}\mathbf{D}'(1) + \mathbf{D}''(1))\mathbf{e} + 1 - \frac{2\lambda}{\theta} \tag{14}$$

where $\mathbf{D}'(1)$ and $\mathbf{D}''(1)$ is the first and second derivatives of $\mathbf{D}(z)$, the PGF of the generator of the rate of arrivals of size k, D_k, with the appropriate phase change with elements $(\mathbf{D}_k)_{ij} = \lambda_i p_i(k, j)$, $k \geq 1$, $1 \leq i, j$, i.e., $\sum_{k=0}^{\infty}\mathbf{D}_k = \mathbf{D}$ and $p_i(k, j)$, $k \geq 1$ is the probability that there be a transition from i to j with a batch arrival of size k.

3.2. The PGF of interval correlation of BMAP

The PGF of interval correlations of BMAP, $L(z)$ can be devised by following similar procedure as the ones used in deriving the PGF $L(z)$ of interval correlation of BRAP in Section 2.2, and is obtained by

$$L(z) = 2\lambda\boldsymbol{\pi}\mathbf{D}^{-1}(z)\mathbf{e} + \lambda - \frac{1 + z}{1 - z} \tag{15}$$

Note that the detailed proofs of both expressions of the PGFs of counts and intervals of BMAP, namely $K(\theta)$ and $L(z)$ of (14) and (15), respectively, can be seen in Appendix B.

3.3. BRAP with the same correlations as BMAP

Following the same pattern as the one presented in Section 2.4, $L(0)$, $K(\infty)$, $L(1)$ and $K(0)$ can be calculated by using Eqs. (14) and (15), namely

$$L(0) = 2\lambda\boldsymbol{\pi}\mathbf{D}_0^{-1}\mathbf{e} + \lambda - 1 \tag{16}$$

$$K(\infty) = \frac{1}{\lambda}\boldsymbol{\pi}\mathbf{D}''(1)\mathbf{e} + 1 \tag{17}$$

$$L(1) = \lim_{z\to 1}\left(2\lambda\boldsymbol{\pi}\mathbf{D}^{-1}(z)\mathbf{e} - \frac{2}{1-z}\right) + \lambda + 1 = 2\lim_{z\to 1}\frac{\lambda(1-z)\boldsymbol{\pi}\mathbf{D}^{-1}(z)\mathbf{e} - 1}{1-z} + \lambda + 1K(0) = L(1) \tag{18}$$

Substituting $L(0)$, $K(\infty)$, $L(1)$, $K(\theta)$ and $L(z)$ into Eqs. (12) and (13), a BRAP can be uniquely determined, sharing the same correlations with BMAP.

4. On the analysis of the stable BRAP/GE/1 queue

Consider a stable single server BRAP/GE/1 queue with a GE-type service time having a mean service rate, μ and squared coefficient of variation, $C_s{}^2$. Let $a(t)$ for $t > 0$ be the distribution of interarrival times between batches and $b(n)$ for $n = 1, 2, \ldots$. be the distribution of batch sizes.

Focusing first on the departures from the BRAP/GE/1 queue, let the state of system be the number of individual customers in the queue (including both those waiting for service and those being served, if any).

Let two processes be embedded at epochs immediately before and after each batch arrival. Each process may be described independently by a Markov chain but the processes are mutually dependent. To this end,

- For the first chain (chain 'A'), the state is the number of customers in the queue immediately before the batch arrival. Let $p_\infty^A(n)$ be the steady state probability at equilibrium that the state be $n = 0, 1, \ldots, \infty$.
- For the second chain (chain 'D'), the state is the number of customers in the queue immediately after the arrival of new batch. Let $p_\infty^D(n)$ be the steady state probability at equilibrium that state be $n = 1, 2, \ldots, \infty$.
- Let at any time $p-\infty(n)$ be the QLD at equilibrium, i.e., the steady state probability that there are n ($n = 0, 1, \ldots, \infty$) customers in the system (including queueing and receiving service, if any) at any time.

To derive the relation between the two EMCs, firstly consider the state of each chain immediately before and immediately after one arrival instant as shown by Fig. 3. Chain 'D' may be in state $n = 1, 2, \ldots, \infty$ when chain 'A' is in state $k = 0, 1, \ldots, n-1$ and there are $n-k$ arrivals in the batch. Thus, the following relationship can be clearly established:

$$p_\infty^D(n) = \sum_{k=0}^{n-1} p_\infty^A(k)b(n-k) \quad \text{where } n = 1, 2, \ldots, \infty \tag{19}$$

Secondly, consider the states of each chain at two successive arrival instants as illustrated in Fig. 4. At the later instant, the chain 'A' may be in state $n = 1, 2, \ldots, \infty$ when the chain 'D' may be in state $k = n+1, \ldots, \infty$ at earlier arrival instant and there are $k-n$ departures in the interval between two arrivals of batches. In other words, k customers are served and depart from the system given there are $n+k$ customers in system immediately after the latest batch arrival and new arriving batch sees a non-empty system, i.e., $n = 1, 2, \ldots, \infty$.

Let $g(k, t)$ denoting the probability that exactly k customers depart during interval of duration t given sufficient number of customers in the queue. Considering a GE-type service time with parameter $\tau\mu$, where $\tau = 2/(1 + C_s^2)$ is the selection probability of a non-zero GE-type service time [9,11] it follows that

$$g(k, t) = \begin{cases} e^{-\tau\mu t} & k = 0 \\ \sum_{\ell=1}^{k}\binom{k-1}{\ell-1}(1-\tau)^{k-\ell}\tau^{\ell}\dfrac{(\tau\mu t)^{\ell}}{\ell!}e^{-\tau\mu t} & k = 1, 2, \ldots \end{cases} \tag{20}$$

Note that the expression (20) for $g(k, t)$ counting process can be devised by

(i) Considering the probability density function of (exactly) k customers departing from the stable BRAP/GE/1 queue in n batches ($n \geq k$) during an interval of time [0, t], $t \geq 0$, i.e.,

$$\frac{d\mathsf{P}[N(t) = k]}{dt} = \sum_{n=0}^{k}(s^{n*}(t)b^{n*}(k) - s^{(n+1)*}(t)b^{n*}(k)) \tag{21}$$

where $\mathsf{P}[N(t)]$ is the random variable of k customers departing during [0, t], $t \geq 0$ and $s^{n*}(t)$, $b^{n*}(k)$ are the n-fold convolutions of the PDF of inter-departure times between batches and PMF of batch sizes, respectively, and

(ii) Formulating the PGF of $d\mathsf{P}[N(t) = k]/dt$ (cf. Eq (21)) denoted by $G(z, \theta)$, i.e.,

$$G(z, \theta) = \int_0^\infty \sum_{k=0}^{\infty}\frac{d\mathsf{P}[N(t) = k]}{dt}z^k e^{-\theta t} = G^n(z, \theta) - S(\theta)G^n(z, \theta) \tag{22}$$

where $S(\theta)$ is Laplace transform of GE service time distribution and $G^*(z, \theta)$ is the n-fold convolution of $G(z, \theta)$.

By inverting $G^*(z, \theta)$ and integrating with respect to t, the expression (20) for $g(k, t)$ can be devised and may be interpreted as follows: l batches of customers are served during the time interval [0, t], $t \geq 0$ with probability $\frac{(\tau\mu t)^\ell}{\ell!}e^{-\tau\mu t}$, where the batch sizes are geometrically distributed according to a negative binomial distribution $\binom{k-1}{l-1}(1-\tau)^{k-\ell}\tau^{\ell}$ is convolution of geometric distribution (cf. [12,13]).

Moreover, let us consider the states of each chain at two successive batch arrival instants as illustrated in Fig. 4. At the later instant, the chain 'A' is in state $n = 1, 2, \ldots, \infty$ when the chain 'D' is in state $k = n+1, \ldots, \infty$ at the earlier arrival instant and there are $k-n$ departures in the interval between the two batch arrivals. In other words, k customers are served and depart from the system given that there are $n+k$ customers in system immediately after the later batch arrival and the new arriving batch meets a non-empty system with $n = 1, 2, \ldots, \infty$.

Alternatively, at the later instant the chain 'A' may be in state 0 when chain 'D' is in state $k = 1, \ldots, \infty$ at the earlier arrival instance and all the k customers depart before the next batch arrives. All the customers previously in the system must be 'serviced' and depart if the new arriving batch 'sees' empty system. $\sum_{r=k}^{\infty} g(r, t)$ denotes the service capacity, i.e., indicates the service capacity of the stable BRAP/GE/1 queue, i.e., the server is able to serve at least k customers in the interval of duration t. The server becomes idle after serving all the k customers and system keeps empty until the next batch arrives.

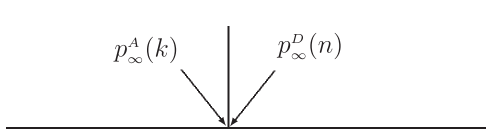

Fig. 3. Relationship between two EMCs at two successive arrival instants.

Fig. 4. Relationship between two EMCs at two successive arrival instants.

Thus, it clearly follows that the relationship between p_∞^A and p_∞^D of the two EMCs at two successive arrival instants satisfies the expressions

$$p_\infty^A(n) = \begin{cases} \sum_{k=1}^{\infty} p_\infty^D(k) \int_0^\infty \sum_{r=k}^{\infty} g(r, t) a(t) dt & n = 0 \\ \sum_{k=0}^{\infty} p_\infty^D(n+k) \int_0^\infty g(k, t) a(t) dt & n = 1, 2, \ldots, \infty \end{cases} \quad (23)$$

4.1. The queue length distribution (QLD) of the BRAP/GE/1 queue

Under the condition that the residual service time of customers in service (if any) is memoryless, the stationary QLD can be derived by considering the case of having $n + k$ customers in the system at point D (immediately after a batch of arrivals) when the interval to the next arrival is at least t, of k customers completing service in time s, $0 < s < t$, whereupon there are just n customers in the system. The probability that inter-batch arrival time be of length t is $t{\cdot}a(t)/a$ and the probability that a small interval δt occupies any particular position within t is $\delta t/t$. Therefore, $a(t)/a{\cdot}\delta t$ is the probability that a small interval δt is at any particular position within t given the inter-batch time is t. Consequently, the steady state QLD $p_\infty(n)$ is given by (cf. [13])

$$p_\infty(n) = \begin{cases} \sum_{k=1}^{\infty} p_\infty^D(k) \int_0^\infty \int_0^t \sum_{r=k}^{\infty} g(r, s) ds \cdot \dfrac{a(t)}{a} dt & n = 0 \\ \sum_{k=0}^{\infty} p_\infty^D(n+k) \int_0^\infty \int_0^t g(k, s) ds \cdot \dfrac{a(t)}{a} dt & n = 1, 2, \ldots, \infty \end{cases} \quad (24)$$

The actual computation of QLDs $p_\infty^D(n)$, $p_\infty^A(n)$ and $p_\infty(n)$, $n = 0, 1, \ldots, \infty$ is based on the analytic expressions (20), (24) and (25), respectively, and clearly requires the full characterization of the selected BRAP and GE-type service time distribution of the stable BRAP/GE/1 queue. Specifically, this computation can be achieved by (i) establishing a recursive relationship of $p_\infty^A(n)$ in terms of $p_\infty^A(n+1)$ and $p_\infty^D(n+1)$, $n = 0, 1, \ldots, \infty$, (ii) determining the probability of exactly k customers depart from the stable BRAP/GE/1 queue during an interval of duration t, $g(k, t)$, $k = 1, \ldots, \infty$ and (iii) making use of the particular parameters of the BRAP and GE-type service time distribution, as appropriate. Moreover, this computation requires the adoption of the expression for the QLD $p_\infty^D(n)$ as a function of $p_\infty^A(k)$, $k = 0, 1, \ldots, n-1$ and the PMF of batch size, $b(n-k)$ and also defines and inverts the z-transform $p_\infty^A(z)$ of QLD $p_\infty A(n)$ towards the determination of the QLD $p_\infty^A(n)$ and subsequently, the QLDs $p\infty^D(n)$ and $p_\infty(n)$. Note that the QLD $p_\infty(n)$ can be used to devise analytic expressions for other queueing performance measures of the stable BRAP/GE/1 queue, such as MQL and waiting time distribution (cf. [13]).

For illustration purposes, the adverse impact of traffic correlation of intervals on the performance of a stable GE$^{\text{sGGeo}}$/GE/1 queue with GE-type interarrival times between consecutive batch arrivals and individual service times in conjunction with sGGeo batch size is demonstrated in Section 5.

5. On the impact of interval correlation and the GE$^{\text{sGGeo}}$/GE/1 Queue

This section derives the MQL and mean waiting time for the GE$^{\text{sGGeo}}$/GE/1 queue and demonstrates the adverse effect of the correlation of intervals between individual arrivals when arrival batch size is sGGeo distributed (cf. [14]), namely

$$b(n) = \begin{cases} 1 - \eta & n = 1 \\ \eta\nu(1-\nu)^{n-2} & n = 2, 3, \ldots, \infty \end{cases} \quad (25)$$

where

$$\eta = \frac{\lambda^2 e^{-C}}{1 + \lambda^2 e^{-C}}(1 - e^{-m})\nu = \frac{1}{1 + \lambda^2 e^{-C}}(1 - e^{-m})$$

in which C and m are constants that are used to plot [14] the logarithm of covariances of intervals between successive arrivals $X(t)$ against lags l and fit a straight line to the plot, i.e.,

$$logCov[X(t), X(t+l)] \simeq -C - ml$$

The consideration of a batch size distribution in the study of a BRAP is concerned, in a similar fashion to the discrete time analogue (cf. [18]), with geometrically bounded distributions implying, for all practical purposes, weighted sums of geometric distributions. As it was pointed out by Fretwell and Kouvatsos [18], just two geometric components (also known as phases) may have significance contribution to the overall shape of the distribution and thus, on the overall performance impact on the stable BRAP/GE/1 queue. Moreover, the analytic forms of count and interval correlation functions of BMAP (cf. Appendix B) indicate that only the first two moments of the batch size distribution have the greatest effect and therefore, a 2-phase count distribution should be sufficient to reveal the dominant performance impact on the stable BMAP/GE/1 queue. Finally, the selection in this context of a BRAP with an sGGeo distribution of batch sizes is further motivated by the fact that, given its first two moments, it is completely defined and thus, in information theoretic terms, it is least biased (cf. [18]).

Moreover, the interval between arriving batches is GE distributed [11], i.e., with a PDF given by

$$a(t) = \begin{cases} 1 - \sigma & t = 0 \\ \sigma^2\lambda e^{-\sigma\lambda t} & t > 0 \end{cases} \quad (26)$$

where $\sigma = 2/(C_a^2 + 1)$ and $C_a{}^2$ is the SCV of intervals between individual batch arrivals whilst the service times are also GE distributed with a PDF

$$s(t) = \begin{cases} 1 - \tau & t = 0 \\ \tau^2\mu e^{-\tau\mu t} & t > 0 \end{cases} \quad (27)$$

where $\tau = 2/(C_s^2 + 1)$ and $C_s{}^2$ is the SCV of individual service times. Clearly, the ordinary GE-type batch interarrival time distribution of the stable GE$^{\text{sGGeo}}$/GE/1 queue does not induce any count traffic correlations.

Substituting Eqs. (21), (26)–(28) into (24), the QLD of GE$^{\text{sGGeo}}$/GE/1 queue is expressed explicitly as

$$p_\infty(n) = \begin{cases} \dfrac{1}{\nu}\dfrac{\sigma\lambda}{\sigma\lambda + \tau\mu}(\sigma\nu - \tau\eta) & n = 0 \\ \dfrac{\eta}{\nu}\dfrac{\sigma\lambda}{\sigma\lambda + \tau\mu}\dfrac{\tau\mu}{\sigma}(1 - x)x^n & n = 1, 2, \ldots, \infty \end{cases} \quad (28)$$

where

$$x = \frac{\sigma(1 - \eta - \nu) + \eta + \tau}{\sigma + (1 - \sigma)\eta\tau\mu} \quad (29)$$

The complete proofs of the analytic derivations of the QLDs of $p_\infty^D(n)$, $p_\infty^A(n)$ and $p_\infty(n)$, based on the expressions (20), (24) and (25), respectively, can be seen in Appendix C.

Finally, the MQL and mean waiting time (W) can be determined, after some manipulations and the application of Little's law [12], by

$$MQL = \frac{\lambda}{\sigma + \lambda}\frac{\eta}{\nu}\left((\eta + \nu) + \frac{1 - (1 - \sigma)(1 - \eta - \nu)\tau\mu}{\sigma\nu - \lambda\eta}\right) \quad (30)$$

$$W = 1 + \frac{\eta}{\eta + \nu}\frac{1 - (1 - \sigma)(1 - \eta - \nu)\tau\mu}{\sigma\nu - \lambda\eta} \quad (31)$$

To view the significance of the correlation, define symbol $\beta_b \triangleq (1 - \eta - \nu)$ as the geometric factor in the correlation function for intervals between individual arrivals, as in [13–15]. Further, it is convenient to investigation of queue behaviour when β_b be close to 1, therefore define an additional symbol $\kappa_b \triangleq (1 - \beta_b)^{-1}$.

5.1. Choice of reference system

As it was first observed by Lucantoni David [15], the factor β_b is a good indicator of the type of correlation in GE^{sGGeo} batch renewal process. Specifically,

- A β_b value of 0 implies free of correlation, i.e., if $\beta_b = 0$, the process is renewal and there is no correlation between intervals.
- A positive (negative) value for the β_b implies positive (negative) correlation. A greater magnitude of the β_b value implies stronger (positive or negative) correlation.

In order to determine effects on queueing behaviour of interval correlation arising from a GE^{sGGeo} BRAP, the performance distributions and statistics for the queue must be compared by varying the degree of correlation but invariant in other significant characteristics. The GE^{sGGeo} process is determined by 4 parameters and, since two degrees of freedom are determined by the choice of the β_b. An obvious requirement is that the arrival intensity λ be invariant.

5.2. Numerical results

Figs. 5 and 6 illustrate, respectively, the effects of increasing correlations of intervals between individual arrivals on MQL of the GE^{sGGeo}/GE/1 infinite buffer queue at two typical but different values of the service time SCV, $C_s^2 = \{5.66, 99.00\}$ (or $\tau = \{0.3, 0.02\}$).

The adverse impact of correlation on the MQL of the stable GE^{sGGeo}/GE/1 is shown in Figs. 5 and 6 with $\kappa_b = 1, 5, 10, 20, 100$. It is observed that as κ_b increases from 1 to 100, i.e., the geometric factor (in the correlation function of the intervals between individual arrivals) increases, the corresponding values of the MQLs also increase (cf. Figs. 5 and 6). This MQL pattern due to the increase in correlation deteriorates further as it combines with increased values of $C_s{}^2$ from 5.66 (Fig. 5) to 99.00 (cf. Fig. 6).

6. On the analysis of the stable BMAP/GE/1 queue

The solution of the QLD for a stable BMAP/GE/1 queue, is based on the comprehensive analysis of the BMAP/G/1 queue, which was solved by Lucantoni David (cf. [15]) using the Matrix-Geometric (M-G) method introduced earlier by Neuts (cf. [16,17]).

Consider a single-server queue with a BMAP specified by the sequence of matrices $\{\mathbf{D}_k, \mathrm{k} \geq 0\}$. Let the service times be GE distributed and independent of the arrival process. The PDF of service times is specified as

$$S(t) = 1 - \sigma e^{-\sigma\mu t}, \quad t \geq 0 \tag{32}$$

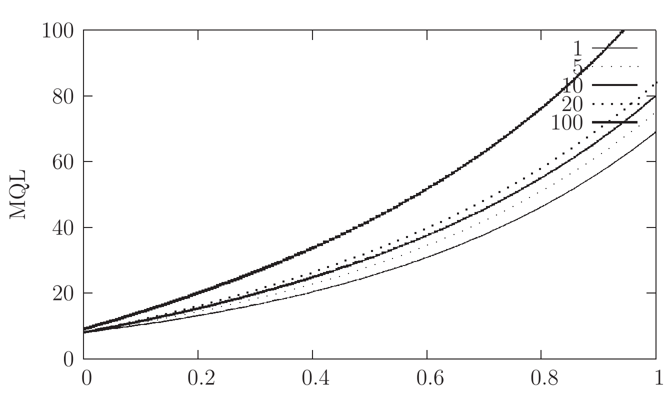

Fig. 5. MQL of GE^{sGGeo}/GE/1 versus GE-type parameter $\sigma = 2/(1 + C_a^2)$, $\sigma = 0, 0.2, 0.4, 0.6, 0.8, 1$ (or $C_a^2 = +\infty, 9.0, 4.0, 2.33, 1.5, 1.0$) with mean arrival rate $\lambda = 0.2$, mean service rate $\mu = 2$ and $\tau = 0.3$ (or $C_s^2 = 5.66$), 99.00 and various values of $\kappa_b = 1, 5, 10, 20, 100$.

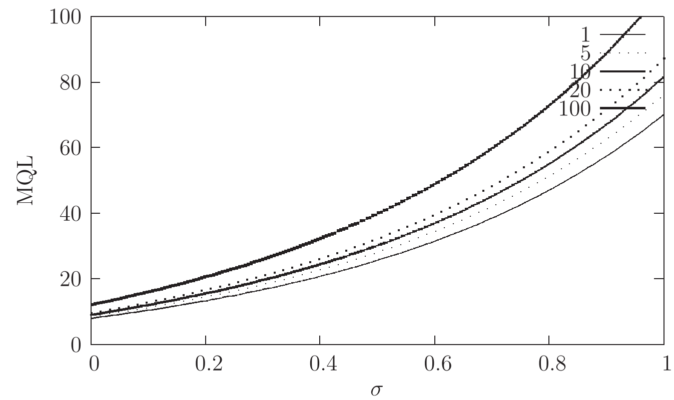

Fig. 6. MQL of GE^{sGGeo}/GE/1 versus GE-type parameter $\sigma = 2/(1 + C_a^2)$, $\sigma = 0, 0.2, 0.4, 0.6, 0.8, 1$ (or $C_a^2 = +\infty, 9.0, 4.0, 2.33, 1.5, 1.0$) for mean arrival rate $\lambda = 0.2$, mean service rate $\mu = 2$ and $\tau = 0.02$ (or $C_s^2 = 99.00$) and various values of $\kappa_b = 1, 5, 10, 20, 100$.

Use σ to replace $\frac{2}{C^2+1}$ in Eq. (1) for short expression. Let us assume that the service rate μ is finite and define the traffic intensity by $\rho = \lambda/\mu$.

6.1. Embedded Markov renewal process at departures

Let us define the sequence of matrices $\mathbf{A}_n(t)$ by the elements

$\mathbf{A}_n(t)_{ij} = \mathsf{P}$ [Given a departure at time 0 which left at least one customer in the system and the arrival process in phase i, the next departure occurs no later than time t with the arrival process in phase j, and during that service time there were n arrivals]

and introduce the transform matrix

$$\mathbf{A}(z, \theta) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\theta t} d\mathbf{A}_n(t) z^n = \int_0^{\infty} e^{\mathbf{D}(z)t} e^{-\theta t} dS(t) \tag{33}$$

and

$$\mathbf{A}(z) = \mathbf{A}(z, 0) = (1 - \sigma)\mathbf{I} + \sigma^2\mu(\sigma\mu\mathbf{I} - \mathbf{D}(z))^{-1} \tag{34}$$

6.2. Stationary QLD at arbitrary time

In the context of the BMAP/G/1 queue, $\mathbf{G}\ (z, \theta)$ is the two-dimensional transform of the number served during, and the duration of the busy period (with appropriate phase change information). It can be shown that the joint transform matrix governing the number served during and the duration of a busy period starting with r customers, is given by $\mathbf{G}\ (z, \theta)^r$

$$\mathbf{G}(z, \theta) = z \int_0^{\infty} e^{-\theta t} e^{\mathbf{D}[G(z,\theta)]t} dS(t) \tag{35}$$

and

$$\mathbf{G} = \mathbf{G}(1, 0) = \int_0^{\infty} e^{\mathbf{D}[\mathbf{G}]t} dS(t) \tag{36}$$

The matrix $\mathbf{G}$ is stochastic when $\rho \leq 1$ (cf. [15]) and is the key ingredient in the solution of the stationary version of this system. For $\rho \leq 1$, the invariant probability vector $\mathbf{g}$, of the positive stochastic matrix $\mathbf{G}$, satisfies

$$\mathbf{gG} = \mathbf{g}, \quad \mathbf{ge} = 1 \tag{37}$$

The vector generating function of QLD at arbitrary time is clearly given by

$$\mathbf{Y}(z) = (1 - \rho)\mathbf{g}(z - 1)\mathbf{A}(z)[z\mathbf{I} - \mathbf{A}(z)]^{-1} \tag{38}$$

Thus, the generating function of stationary probability of QLD at arbitrary time is given by the vector generating function of QLD multiplied by the transpose of the stationary probability of the underlying Markov process, i.e.,

$$Y(z) = (1-\rho)\mathbf{g}(z-1)\mathbf{A}(z)(z\mathbf{I}-\mathbf{A}(z))^{-1}\boldsymbol{\pi}^{\mathrm{T}} \tag{39}$$

7. A comparative study on the effects of BRAP and BMAP on queues

This section presents a comparative numerical study on the effects of traffic correlation and other traffic characteristics on the performance of the stable BRAP/GE/1 and BMAP/GE/1 queues with GE-type service time distributions and BRAP and BMAP arrivals with the same interval correlations (of individual arrivals).

7.1. General considerations

The essence of the numerical experiments is to compare the effects of the correlated traffic flows with those due to the traffic correlation alone. In this context, the relevant performance measures include the cell loss rate, waiting time, jitter and buffer occupancy, etc. All these statistics are related to the QLDs of the stable BRAP/GE/1 and BMAP/GE/1 queues. In particularly, the queueing model $\mathrm{GE}^{\mathrm{sGGeo}}$/GE/1 is chosen as the specific BRAP/GE/1 and its QLD is derived as expressed by Eq. (28).

In the experimental model adopted in this section the BRAP/GE/1 and BMAP/GE/1 queues have infinite buffer capacity because, if the buffer is "too small", buffer congestion and other measures of interests are largely determined by the buffer size (cf. [18]) in order to illustrate the effect of "small" buffer size upon the traffic characteristics. To this end, the experimental study focuses on the comparisons of the QLDs of the stable BRAP/GE/1 and BMAP/GE/1 queues with identical GE-type servers and the same count and interval correlations at all epochs. The remaining design issues of the numerical experiments are concerned with the appropriate selection of representative members of the class of the traffic processes of interest such that the experiment should be based on a credible parameterization.

7.2. The batch size distribution

In a similar fashion to the discrete time analogue in [18], the choice of the batch size distribution is restricted to geometrically bounded distributions which, for all practical purposes, means weighted sums of geometric distributions. As it was pointed out in [18], "Just two geometric components (also known as phases) have a significance contribution to the overall distribution". Moreover, as the form of the BMAP correlation functions (14) and (15) indicates, only the first two moments of the batch size distribution are of great influence. So a 2-phase count distribution should be of sufficient robustness to reveal the dominant effects of traffic correlation.

7.3. Modulating two phases

By arguments essentially similar to those for the batch size distributions, it is clearly implied that the modulating process (i.e., the underlying Markov process) should consist of just two phases.

7.4. The issue of intensity

It appears to be generally true that high traffic intensity leads to an

Table 1
Parameters of the BMAP.

Curves of graphs 1–27	λ_1	λ_2	λ	μ	η_1	ν_1	β_{b1}	η_2	ν_2	β_{b2}	$\sigma = 0.8$, $\rho = 0.85$
(01)	1	1	44.51	52.36	0.02	0.025	0.955				
(02)	1	1	26.01	30.60	0.2	0.25	0.55	0.022	0.02	0.958	
(03)	1	1	24.99	29.40	0.4	0.5	0.1				$p = 0.0070$
*(04)	1	1	22.95	27.00	0.02	0.025	0.955				
(05)	1	1	4.46	5.25	0.2	0.25	0.55	0.22	0.2	0.58	$q = 0.0075$
*(06)	1	1	3.43	4.04	0.4	0.5	0.1				
(07)	1	1	21.74	25.58	0.02	0.025	0.955				$\beta = 0.9855$
*(08)	1	1	3.25	3.82	0.2	0.25	0.55	0.44	0.4	0.16	
*(09)	1	1	2.23	2.62	0.4	0.5	0.1				
(10)	1	1	36.99	43.52	0.02	0.025	0.955				
(11)	0.62	15	36.85	43.35	0.02	0.025	0.955				
(12)	1	1	18.01	21.19	0.2	0.25	0.55				
(13)	0.622	16	18.03	21.19	0.2	0.25	0.55	0.022	0.02	0.958	
(14)	1	1	16.96	19.95	0.4	0.5	0.1				$p = 0.13$
(15)	10	0.42	16.98	19.98	0.4	0.5	0.1				
(16)	1	1	22.82	26.85	0.02	0.025	0.955				$q = 0.20$
(17)	1	1	3.83	4.51	0.2	0.25	0.55	0.22	0.2	0.58	
(18)	1	1	2.78	3.27	0.4	0.5	0.1				$\beta = 0.67$
(19)	1	1	22.03	25.92	0.02	0.025	0.955				
(10)	1	1	3.05	3.59	0.2	0.25	0.55	0.44	0.4	0.16	
(21)	1	1	1.99	2.34	0.4	0.5	0.1				
(22)	1	1	23.78	27.98	0.02	0.025	0.955	0.022	0.02	0.958	
(23)	1	1	2.48	2.92	0.2	0.25	0.55	0.22	0.2	0.58	
(24)	1	1	18.77	22.08	0.02	0.025	0.955	0.22	0.2	0.58	$p = 0.333$
(25)	1	1	1.59	1.87	0.4	0.5	0.1				$q = 0.666$
(26)	1	1	18.49	21.95	0.02	0.025	0.955	0.44	0.4	0.16	$\beta = 0.001$
(27)	1	1	2.49	2.93	0.2	0.25	0.55				

extremal behaviour at the queue. So the experimental model should be chosen to be compatible at high intensity.

Moreover, as the form of BMAP-related correlation function indicates, the difference in mean intensity between phases is significant factor. Thus the experimental model should adopt a large difference in intensity between phases. The experiments should also illustrate the effect of the variation of traffic intensity at each phase under the same overall traffic intensity.

7.5. Typical numerical experiments

In this section, the generalized geometric (GGeo) distribution (cf. [19]), a generalization of the geometric distribution, is used to model the transition probabilities $p_i(0, j)$, $j \neq i$ and $p_i(k, j)$, $k \geq 1$ from phase i to phase j without and with, respectively, an arriving batch of size k (cf. Section 3). In this context, for each phase i, the GGeo is generalization of geometric distribution. For each phase i, the GGeo distribution is given by the probability $b_i(n)$ of n arrivals being generated in one batch:

$$b_i(n) = \begin{cases} 1 - \eta_i & n = 0 \\ \eta_i \nu_i (1 - \nu_i)^{n-1} & n = 1, 2, \ldots \end{cases} \tag{40}$$

Let ϕ_{ij} be the conditional probability that the phase changes to j given that the process phase was i:

$$\phi_{12} = p, \quad \phi_{11} = 1 - p, \quad \phi_{21} = q, \quad \phi_{22} = 1 - q \tag{41}$$

The parameters $\beta_{bi} = 1 - \eta_i - \nu_i$ and $\beta = 1 - p - q$ are useful indicators of correlation. When equal to zero, they indicate independence. When β be close to 1, the sojourn of the BMAP is long in each phase. Count correlation increases with β for any given stationary

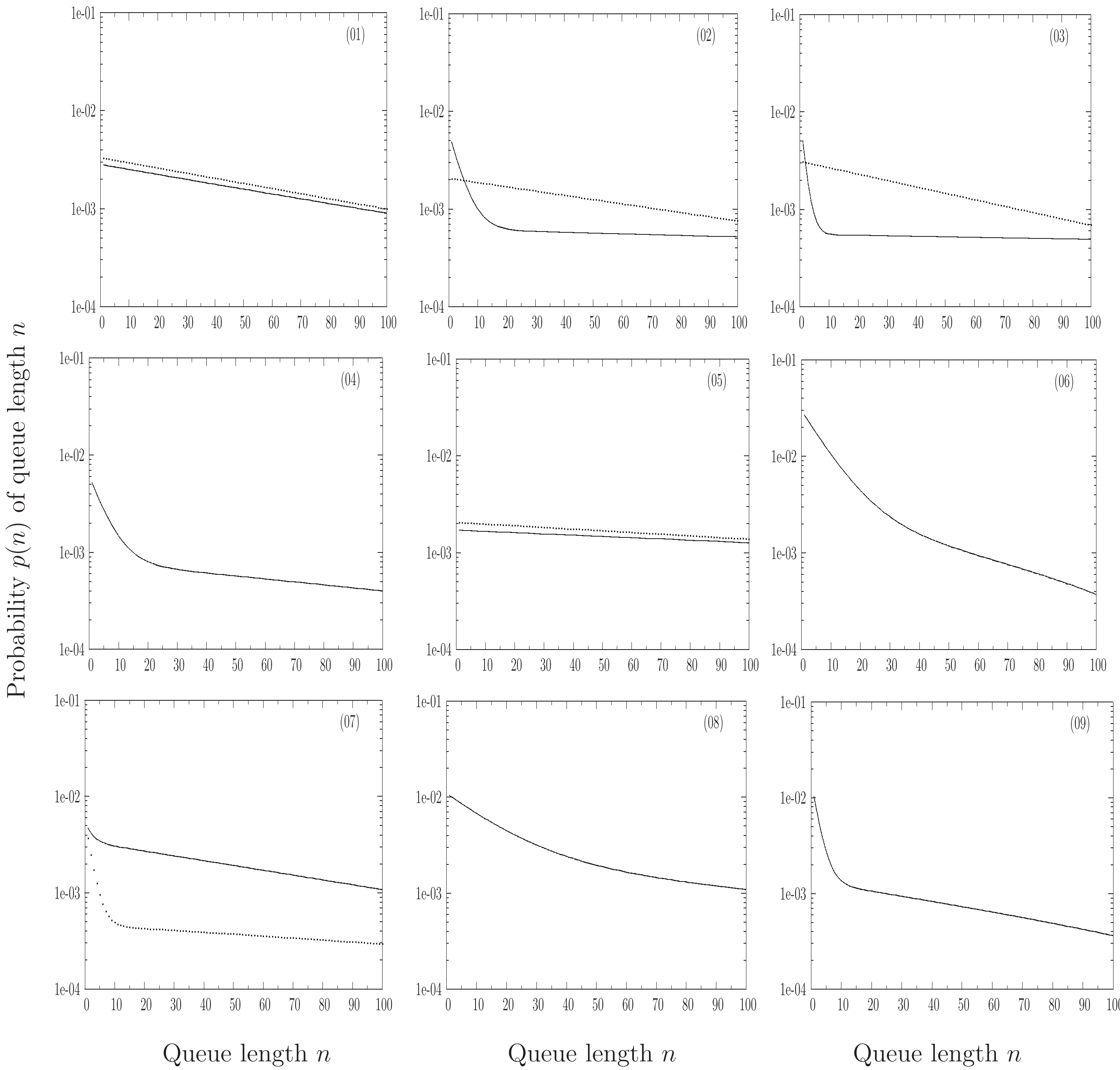

Fig. 7. QLDs for BMAP and for the correlation indicator of the BMAP (part 1 with $\beta = 0.9855$ – graphs 01–09), the solid line shows the BMAP/GE/1 QLD and the dotted line shows the BRAP/GE/1 QLD. See Table 1 for the relevant parameter values.

phase distribution. When β_{bi} be close to 1 the variability of counts is high during phase i whilst interval correlation increases with β_{bi}.

The generating function of the BMAP (infinitesimal generator) $\mathbf{D}(z)$ is expressed explicitly as

$$\mathbf{D}(z) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} -1 + (1-p)\frac{\nu_1 z}{1-(1-\nu_1)z} & p\left(1 - \eta_1 + \frac{\eta_1 \nu_1 z}{1-(1-\nu_1)z}\right) \\ q\left(1 - \eta_2 + \frac{\eta_2 \nu_2 z}{1-(1-\nu_2)z}\right) & -1 + (1-q)\frac{\nu_2 z}{1-(1-\nu_2)z} \end{pmatrix} \tag{42}$$

All the related quantities in BMAP can be calculated from $\mathbf{D}(z)$, namely

$$\mathbf{D}(1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} -p & p \\ q & -q \end{pmatrix} \tag{43}$$

$$\boldsymbol{\pi} = \frac{(\lambda_2 q, \lambda_1 p)}{\lambda_2 q + \lambda_1 p} \tag{44}$$

$$\lambda = \boldsymbol{\pi}\mathbf{D}'(1)\mathbf{e} = \frac{\lambda_1 \lambda_2}{\nu_1 \nu_2} \cdot \frac{q\nu_2(1 - p + p\eta_1) + p\nu_1(1 - q + q\eta_2)}{\lambda_2 q + \lambda_1 p} \tag{45}$$

Table 1 shows the parameter sets, arranged in decreasing order of magnitude of β, β_{b1}, β_{b2}, and the corresponding graphs of QLD are shown in Figs. 7–9. The experimental results are typical of those obtained from the sets of parameters selected in accordance with the aforementioned design scheme. In each graph of Fig. 7, the solid and dotted lines indicate the behaviours of the QLDs of the BMAP/GE/1 and the BRAP/GE/1 queues. However

- For some of the graphs (viz. 14–17, 19, and 20) the initial portion has been expanded to show more clearly the difference between the

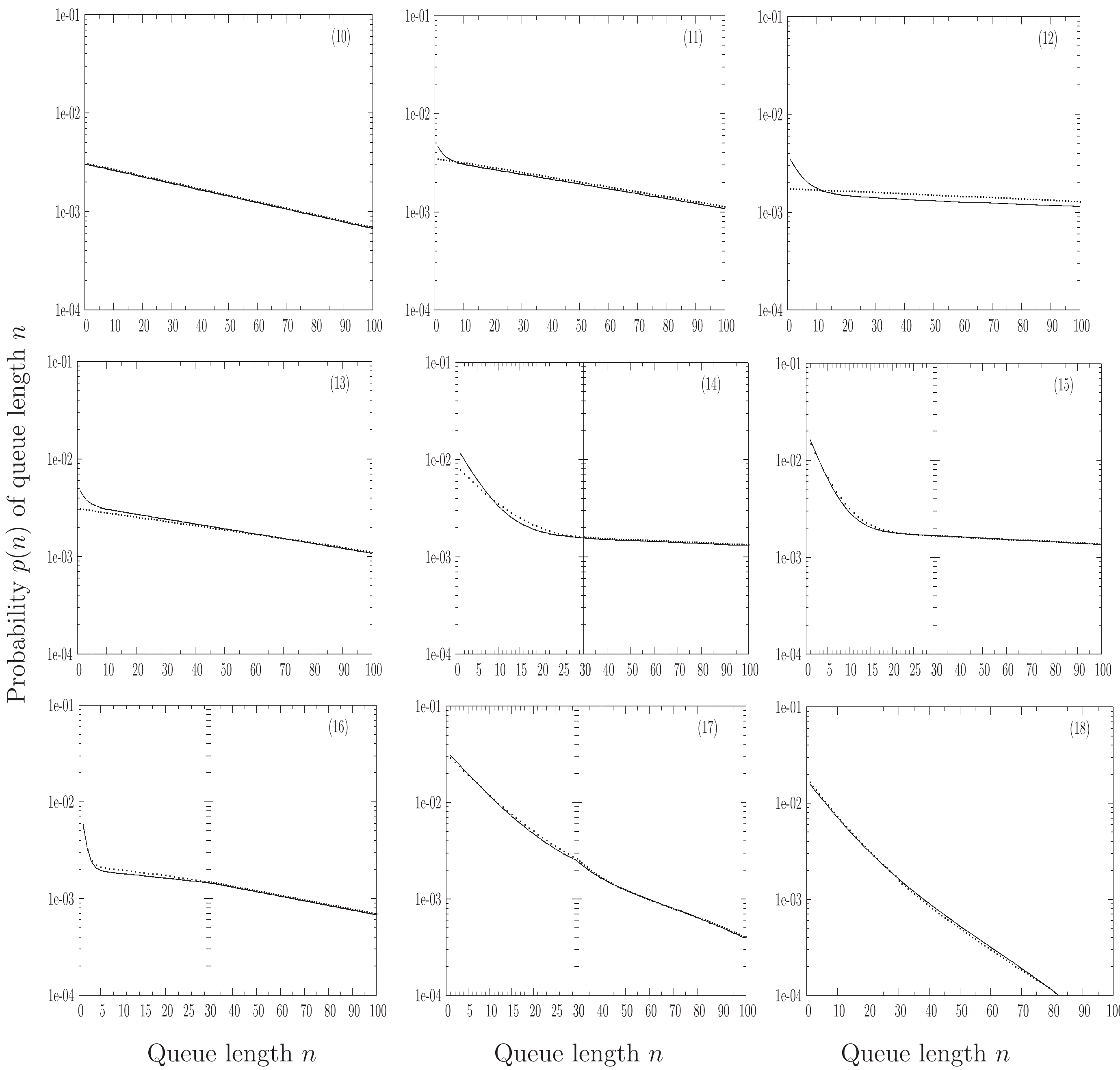

Fig. 8. QLDs for BMAP and for the correlation indicator of the BMAP (part 2 with $\beta = 0.67$ – graphs 10–21), the solid line shows the BMAP/GE/1 QLD and the dotted line shows the BRAP/GE/1 QLD. See Table 1 for the relevant parameter values.

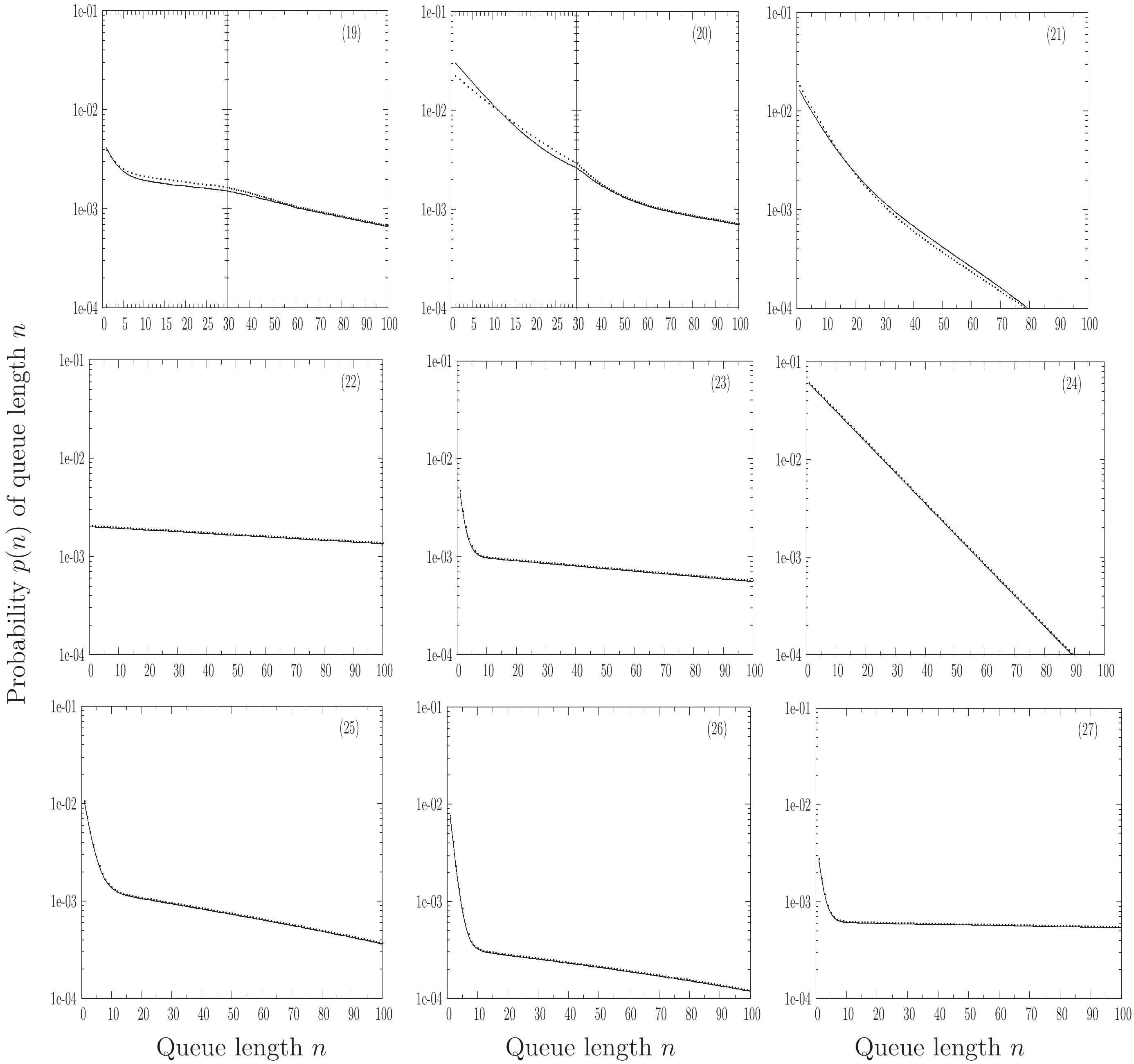

Fig. 9. QLDs for BMAP and for the correlation indicator of the BMAP (part 3 with β=0.001 – graphs 22–27), the solid line shows the BMAP/GE/1 QLD and the dotted line shows the BRAP/GE/1 QLD. See Table 1 for the relevant parameter values.

two curves.

- No dotted line is shown on graphs 04, 06, 08 and 09 because the corresponding BRAP is not proper and the practical usefulness of improper BRAPs has not yet been investigated.

7.6. Discussion

The form of the QLDs of the stable BMAP/GE/1 and BRAP/GE/1 queues is of a modified weighted sum of two geometrics with parameters x_1 and x_2, respectively, and it is given by

$$p(n) = \begin{cases} 1 - \rho & n = 0 \\ \rho A(1 - x_1)x_1^n + \rho(1 - A)(1 - x_2)x_2^n & n = 1, 2, \cdots \end{cases} \tag{46}$$

where the BRAP has the same indices of dispersion for counts and intervals as the BMAP, i.e., the QLD of the BRAP/GE/1 queue is only attributable to the correlation parameters of the BMAP.

Most of the graphs in Figs. 7–9 have the characteristic "knee" form (cf. graph 14 of Fig. 8) with a steep first segment (corresponding to the smaller geometric rate x_1) followed by a flatter second segment (corresponding to the larger geometric rate x_2). In some graphs (cf. graphs 17 and 18 of Fig. 8) the rates x_1 x_2 are of similar magnitude and the "knee" is not very apparent. In some other graphs (cf. graphs 01, 05, 10 and 22 of Figs. 7–9) the first segment is not obvious because the corresponding weight A is small.

In the last group of graphs (cf. graphs 22–27 of Fig. 9) the two QLDs are so close in magnitude (agreement to several significant digits) that the difference between solid line and the dotted line is indistinguishable.

However, in every case, the smaller geometric rate x_1 of the BMAP/GE/1 QLD is smaller than the QLD of the stable BRAP/GE/1 queue. In the typical result (e.g. graph 14 of Fig. 8), the smaller rate (with a steeper first segment) of the QLD of the BMAP/GE/1 queue clearly implies lower waiting time, less jitter and, extrapolating to the finite buffer case, lower blocking probability as compared with the distribution attributable to the correlation alone. In other words, the BMAP yields optimistic results. On the other hand, the cases (cf. graphs 01, 05, 10 and 22 of Figs. 7–9) in which the QLD of the BMAP/GE/1 queue is pessimistic do not correspond to acceptable operating conditions: jitter would be high and, extrapolating to the finite buffer case, the blocking probability would be high.

It follows from the outcomes of the numerical experiment of Figs. 7–9 and Table 1 that BMAP-based traffic is likely to yield optimistic results under the condition of positive traffic correlation of counts and intervals arising from long sojourn in each phase (cf. graphs 02, 03 and 07 of Fig. 7). The BMAP and BRAP-based traffic flows are prone to give almost same results when the count correlation is very low (i.e., the case β=0.001 in Table 1).

It is, therefore suggested that the stable BRAP/GE/1 queue is likely to yield pessimistic performance metrics in comparison to those of the family of stable BMAP/GE/1 queues under the worst case scenario of the same positive count and interval traffic correlations arising from long sojourn in each phase. To this end, the following conjecture is proposed.

Conjecture 7.1. Given a stable BRAP/GE/1 queue and a family of stable BMAP/GE/1 queues with the same positive count and interval traffic correlations arising from long sojourn in each phase, it is implied under the same parameterization and acceptable operational conditions that the BRAP induces a 'worst case' (pessimistic) performance scenario on the stable BRAP/GE/1 queue in comparison to the corresponding 'better case' (optimistic) scenarios imposed by the family of BMAPs on the stable BMAP/GE/1 queues.

8. Conclusions

The BRAP and BMAP with the same positive count and interval traffic correlations arising from long sojourn in each phase were employed in continuous time domain to investigate the adverse impact of traffic correlation and other traffic features on the performance of the stable BMAP/GE/1 and BRAP/GE/1 queues. The count and interval traffic correlations were determined by the means of the PGFs of the corresponding counting and timing processes, respectively, and their first two moments. Based on the one-to-one correspondence between the BRAP and its indices of correlation, a theoretical procedure was devised to construct as a least biased traffic process, from the statistics of covariances alone, a unique BRAP in continuous time domain. The EMCs technique and the M-G approach were applied, respectively, to derive novel closed form expressions or the QLDs of the stable BRAP/GE/1 and BMAP/GE/1 queues. Moreover, the stable GE^{sGGeo}/GE/1 queue with a GE^{sGGeo} BRAP, consisting of GE-type batch interarrival times and a sGGeo batch size distribution, was adopted to assess the combined adverse effects of varying degrees of the correlation of intervals between individual arrivals and the burstiness of service times upon the typical QoS measure of the MQL. Furthermore, given the same measures of count and interval traffic correlations at various lags, it became evident from numerical experiments plotting the QLD against queue length n that the stable BRAP/GE/1 queue is likely to give pessimistic values for typical performance measures in comparison to those obtained from the family of stable BMAP/GE/1 queues. Thus, the BRAP may be employed, as a suitable arrival traffic process, for the performance prediction of worst case scenarios in queueing systems and networks with correlated traffic flows.

Further research is required for the modelling and characterization of the departure process of the of the stable BMAP/GE/1 and BRAP/GE/1 queues in order to investigate the impact of correlated traffic flows on the behaviour of buffered queueing network models (QNMs) with blocking and particularly, into the propagation of correlation across high speed computer networks. These are subjects of current study.

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Appendix A. Probability generating functions of count and interval correlations of BRAP

This Appendix presents the derivation of analytic expressions for the PGFs of correlation of counts $c(\cdot)$ and correlation of intervals $x(\cdot)$ in traffic generated by a BRAP.

The relevant notation of this section is described below.

$b(n) = \mathsf{P}[b_k = n]$	$n = 1, 2, \ldots$ the probability mass function (PMF) of batch size
b	the mean batch size
C_b^2	the squared coefficient of variation (SCV) of batch sizes
$B(z) = \sum_{n=1}^{\infty} b(n) z^n$	the probability generating function (PGF) of batch size
$a(t)$, $t > 0$	the probability density function (PDF) of intervals between consecutive batch arrivals (batch interarrival times)
$a = \int_0^{\infty} v\, a(v) dv$	the mean inter-batch interval
C_a^2	the SCV of intervals between batches
$A(\theta) = \int_0^{\infty} a(t) e^{-\theta t} dt$	the Laplace transform of interval between batches
$\lambda = \frac{b}{a}$	the mean arrival rate of individual customer
$c(t)$	the number of individual customers that arrive during a randomly chosen interval of duration t, it does not depends on the initial time of the chosen interval
$I_t = \frac{\mathsf{Var}[c(t)[}{\mathsf{E}[c(t)]}$	the index of dispersion for counts (IDC), which is defined to be the ratio of the variance of the number of individual arrivals during an interval of duration t to the average number of arrivals in t;

$K(\theta)$	the generating function of count correlation
$x(n)$	the n consecutive intervals between $n+1$ individual arrivals
$J_n = \frac{n \cdot \mathsf{Var}[x(n)]}{\mathsf{E}[x(n)]^2}$	the index of dispersion for intervals (IDI), which is defined to be the ratio of the variance of the n consecutive intervals between $n+1$ individual arrivals, $x(n)$ multiplied by n to the square of the average $x(n)$, $\mathsf{E}[x(n)]^2$
$L(z)$	the generating function of interval correlation

A.1. The PGF of count correlation of BRAP

A continuous time batch renewal process is specified by the batch size random variable b_k and the inter-batch time random variable a_s such that

- the distribution of batch size is given by the PMF $b(n)$, i.e., $\mathsf{P}[b_k = n] = b(n)$, $n = 1, 2, \ldots$, with mean b, SCV C_b^2 and PGF $B(z) = \sum_{n=1}^{\infty} b(n) z^n$,
- the distribution of interval between batches is given by the PDF $a(t)$, i.e., $\mathsf{P}[a_s = t] = a(t)$, $t > 0$, with mean a, SCV C_a^2 and Laplace transform $A(\theta) = \int_0^{\infty} a(t) e^{-\theta t} dt$.

Consider an instant s_0 selected at random and start counting number of arriving batches from s_0. The probability that there be k batches of arrivals in an interval of duration t, beginning at s_0, has a density function that is the convolution of three probability densities:

- that of the time s from s_0 to the instant that the first batch arrives (residual time) (s could be zero)
- that of the $k-1$ intervals $u - s$ between the k batches ($u - s = 0$ if $k = 1$)
- that of the remaining time $t - u$ after the arrival of the kth batch, which might be zero and must be no longer than the inter-arrival time between kth and $(k+1)$th batches

as illustrated by graph in Fig. A1.

The probability that a randomly selected instant be s before the next arriving batch has density $\int_s^{\infty} \frac{a(v)}{a} dv$, where $a = \int_0^{\infty} v\, a(v) dv$ is the mean interval between batches.

The Laplace transform of the density (over time) that there be k batches in the randomly selected interval of duration t is

$$\int_{t=0}^{\infty} e^{-\theta t} \int_{u=0}^{t} \int_{s=0}^{u} \int_{v=s}^{\infty} \frac{a(v)}{a} dv \cdot a^{(k-1)*}(u-s) \cdot \int_{w=t-u}^{\infty} a(w) dw\, ds\, du\, dt = \frac{1}{a} \left(\frac{1 - A(\theta)}{\theta} \right)^2 A(\theta)^{k-1} \tag{A.1}$$

where $a^{k*}(\cdot)$ is the k-fold convolution of $a(\cdot)$.

The probability that there be no batch in the interval of duration t following a randomly selected instant (i.e., $s > t$) has density $\int_t^{\infty} \int_s^{\infty} \frac{a(v)}{a} dv\, ds$, for which the Laplace transform is

$$\int_0^{\infty} e^{-\theta t} \left(\int_t^{\infty} \int_s^{\infty} \frac{a(v)}{a} dv\, ds \right) dt = \frac{1}{\theta} - \frac{1}{a} \cdot \frac{1 - A(\theta)}{\theta^2} \tag{A.2}$$

Let $\nu(n, t)$ be the probability that exactly n individual customers arrive during the interval of duration t, $n = 0, 1, \ldots, t > 0$. By combining Eq. (A.1) with the PGF for there being n customers in k batches and summing over all $k \in \mathbb{N}_0$ ($\mathbb{N}_0$ denotes the set of nonnegative integers), the probability of n individual customers arrive in duration t has the generating function

$$N(\theta, z) = \int_{t=0}^{\infty} \sum_{n=0}^{\infty} \nu(n, t) z^n e^{-\theta t} dt = \frac{1}{\theta} - \frac{1}{a} \cdot \frac{1 - A(\theta)}{\theta^2} + \frac{1}{a} \left(\frac{1 - A(\theta)}{\theta} \right)^2 \frac{B(z)}{1 - A(\theta) B(z)} = \frac{1}{\theta} - \frac{1}{a} \cdot \frac{1}{\theta^2} \cdot \frac{(1 - A(\theta))(1 - B(z))}{1 - A(\theta) B(z)} \tag{A.3}$$

The mean and variance of $c(t)$ can be obtained from (A.3)

$$\int_0^{\infty} \mathsf{E}[c(t)] e^{-\theta t} = \left. \frac{\partial N(\theta, z)}{\partial z} \right|_{z=1} = - \frac{1}{a\theta^2} \cdot \left. \frac{(A(\theta) - 1) B'(z)(1 - A(\theta) B(z) - A(\theta) + A(\theta) B(z))}{(1 - A(\theta) B(z))^2} \right|_{z=1} = \frac{1}{a\theta^2} \cdot \left. \frac{b(1 - A(\theta))^2}{(1 - A(\theta))^2} \right|_{z=1} = \frac{\lambda}{\theta^2} \tag{A.4}$$

Recognize that $\frac{\lambda}{\theta^2}$ is the Laplace transform of λt, i.e., the average number of customers arrive during t is λt. The Laplace transform of $(\lambda t)^2$ is $\frac{2\lambda^2}{\theta^3}$, which will be used in the derivation of $\mathsf{Var}[c(t)]$

$$\left. \frac{\partial^2 N(\theta, z)}{\partial z^2} \right|_{z=1} = \frac{\partial}{\partial z} \frac{1}{a\theta^2} \left. \left(\frac{(1 - A(\theta))^2 B'(z)}{(1 - A(\theta) B(z))^2} \right) \right|_{z=1} = \frac{1}{a\theta^2} \left(B''(1) + \frac{2b^2 A(\theta)}{1 - A(\theta)} \right) \tag{A.5}$$

Clearly,

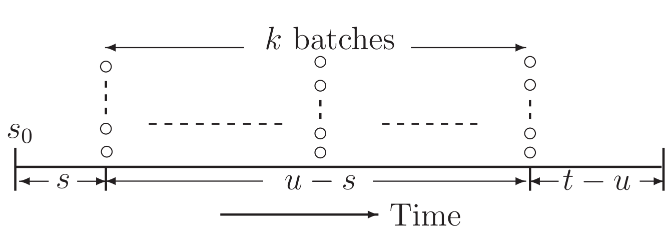

Fig. A1. IDC structure.

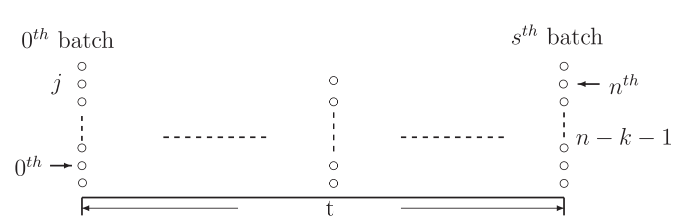

Fig. A2. IDI structure.

$$\left.\frac{\partial^2 N(\theta, z)}{\partial z^2}\right|_{z=1} = \int_0^\infty (\mathsf{E}[c^2(t)] - \mathsf{E}[c(t)])e^{-\theta t}dt \tag{A.6}$$

By the definition of variance,

$$\int_0^\infty \mathsf{Var}[c(t)]e^{-\theta t}dt = \int_0^\infty (\mathsf{E}[c^2(t)] - \mathsf{E}[c(t)]^2)e^{-\theta t}dt \tag{A.7}$$

$$\int_0^\infty \mathsf{Var}[c(t)]e^{-\theta t}dt = \left.\frac{\partial^2 N(\theta, z)}{\partial z^2}\right|_{z=1} + \int_0^\infty \mathsf{E}[c(t)]e^{-\theta t}dt - \int_0^\infty \mathsf{E}[c(t)]^2 e^{-\theta t}dt = \left.\frac{\partial^2 N(\theta, z)}{\partial z^2}\right|_{z=1} + \left.\frac{\partial N(\theta, z)}{\partial z}\right|_{z=1} - \left(\left.\frac{\partial N(\theta, z)}{\partial z}\right|_{z=1}\right)^2$$
$$= \frac{1}{a\theta^2}\left(B''(1) + \frac{2b^2 A(\theta)}{1 - A(\theta)}\right) + \frac{\lambda}{\theta^2} - \frac{2\lambda^2}{\theta^3} \tag{A.8}$$

The IDC, I_t, is defined to be the ratio of the variance of the number of individual arrivals during an interval of duration t to the average number of arrivals in t:

$$I_t = \frac{\mathsf{Var}[c(t)]}{\mathsf{E}[c(t)]} \tag{A.9}$$

So,

$$\lambda t \cdot I_t = \mathsf{Var}[c(t)] \tag{A.10}$$

And the Laplace transform of both sides of (A.10) is

$$\int_0^\infty \lambda t \cdot I_t e^{-\theta t}dt = \int_0^\infty \mathsf{Var}[c(t)]e^{-\theta t}dt \tag{A.11}$$

Then, define $I(\theta)$ to be the generating function of I_t, and

$$\lambda \cdot I'(\theta) = \frac{1}{a\theta^2}\left(B''(1) + \frac{2b^2 A(\theta)}{1 - A(\theta)}\right) + \frac{\lambda}{\theta^2} - \frac{2\lambda^2}{\theta^3} \tag{A.12}$$

By the well-known relationship between count correlation generating function $K(\theta)$ and IDC, we have

$$K(\theta) = \theta^2 I'(\theta) = \frac{1}{b}\left(B''(1) + \frac{2b^2 A(\theta)}{1 - A(\theta)}\right) + 1 - \frac{2\lambda}{\theta} \tag{A.13}$$

Note that

$$B''(z)|_{z=1} = \mathsf{E}[b_k^2] - b \tag{A.14}$$

$$C_b^2 = \frac{\mathsf{E}[b_k^2]}{b^2} - 1 \tag{A.15}$$

Therefore

$$B''(1) = (C_b^2 + 1)b^2 - b \tag{A.16}$$

Replacing $B''(1)$ in (A.13) by (A.16), $K(\theta)$ is obtained as

$$K(\theta) = \frac{1}{b}\left((C_b^2 + 1)b^2 - b + \frac{2b^2 A(\theta)}{1 - A(\theta)}\right) + 1 - \frac{2\lambda}{\theta} = bC_b^2 + b\cdot\frac{1 + A(\theta)}{1 - A(\theta)} - \frac{2\lambda}{\theta} = b\left(C_b^2 + \frac{1 + A(\theta)}{1 - A(\theta)} - \frac{2}{a}\cdot\frac{1}{\theta}\right) \tag{A.17}$$

Note that the expression (A.17) is referred to as expression (2) in the main body of the paper.

A.2. The PGFs of interval correlation of BRAP

Consider a customer selected at random, the probability that there be a future j customers following the selected customer in the same batch is $\sum_{v=j+1}^{\infty}\frac{b(v)}{b}$. Counting the randomly selected customer as customer 0 (denote c_0) and the batch it is in as the 0th batch, the probability that nth customer be in the sth batch has mass function that is the convolution of two probability mass functions

- that the nth customer be in the 0th batch (i.e., it is in the same batch as c_0) is

$$\sum_{j=n}^{\infty}\sum_{v=j+1}^{\infty}\frac{1}{b}b(v) \tag{A.18}$$

and the generating function of this mass function is

$$\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}\sum_{v=j+1}^{\infty}\frac{1}{b}b(v)z^n = \frac{1}{1 - z} - \frac{z}{b}\cdot\frac{1 - B(z)}{(1 - z)^2} \tag{A.19}$$

- that j ($j \ge 0$) customers following c_0 in the 0th batch, $k - j$ ($0 \le k - j < n$) customers in the batches between the 1st and the $(s - 1)$ th $n - k - 1$ ($n - k - 1 \ge 0$) customers before the nth customer in the sth batch and the generating function of this mass function is

$$\sum_{n=s}^{\infty}\sum_{k=s-1}^{n-1}\sum_{j=0}^{k-s-1}\left(\sum_{v=j+1}^{\infty}\frac{b(v)}{b}\right)b^{s-1*}(k-j)\left(\sum_{w=n-k}^{\infty}b(w)\right)z^n=\frac{z}{b}\left(\frac{1-B(z)}{1-z}\right)^2B(z)^{s-1} \tag{A.20}$$

Let $\tau(n, t)$ be the probability that there are n intervals between $n+1$ successive individual arrivals during time of duration t. Combining (A.20) with PGF of the probability of n intervals summing over t, then the transform function of $\tau(n, t)$ is defined as (Fig. A2).

$$T(z,\theta)=\int_{t=0}^{\infty}\sum_{n=0}^{\infty}\tau(n,t)z^ne^{-\theta t}dt=\frac{1}{1-z}-\frac{z}{b}\cdot\frac{1-B(z)}{(1-z)^2}+\frac{z}{b}\left(\frac{1-B(z)}{1-z}\right)^2\frac{A(\theta)}{1-A(\theta)B(z)}=\frac{1}{1-z}-\frac{z}{b\cdot(1-z)^2}\cdot\frac{(1-A(\theta))(1-B(z))}{1-A(\theta)B(z)} \tag{A.21}$$

The mean and variance of $x(n)$ can be obtained from (A.21).

$$\sum_{n=1}^{\infty}\mathsf{E}[x(n)]z^n=\left.\frac{\partial T(z,\theta)}{\partial\theta}\right|_{\theta=0}=\frac{1}{\lambda}\frac{z}{(1-z)^2} \tag{A.22}$$

recognize that $\frac{1}{\lambda}\frac{z}{(1-z)^2}$ is the z-transform of $\frac{n}{\lambda}$, i.e., the average intervals between $n+1$ individual arrivals is $\frac{n}{\lambda}$. z-Transform of $(\frac{n}{\lambda})^2$ is $\frac{z(1+z)}{\lambda^2(1-z)^3}$, which will be used in derivation of $\mathsf{Var}[x(n)]$.

$$\left.\frac{\partial^2T(z,\theta)}{\partial\theta^2}\right|_{\theta=0}=\frac{1}{b}\cdot\frac{z}{(1-z)^2}\left(A''(\theta)+\frac{2a^2B(z)}{1-B(z)}\right) \tag{A.23}$$

$$\left.\frac{\partial^2T(z,\theta)}{\partial\theta^2}\right|_{\theta=0}=\sum_{n=1}^{\infty}(\mathsf{E}[x^2(n)]-\mathsf{E}[x(n)])z^n \tag{A.24}$$

$$\sum_{n=1}^{\infty}\mathsf{Var}[x(n)]z^n=\sum_{n=1}^{\infty}\mathsf{E}[x^2(n)]z^n-\sum_{n=1}^{\infty}\mathsf{E}[x(n)]^2z^n \tag{A.25}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\mathsf{Var}[x(n)]z^n=\left.\frac{\partial^2T(z,\theta)}{\theta^2}\right|_{\theta=0}+\sum_{n=1}^{\infty}\mathsf{E}[x(n)]z^n-\sum_{n=1}^{\infty}\mathsf{E}[x(n)]^2z^n=\sum_{n=1}^{\infty}\mathsf{E}[x^2(n)]z^n\\-\sum_{n=1}^{\infty}\mathsf{E}[x(n)]^2z^n=\left.\frac{\partial^2T(z,\theta)}{\partial\theta^2}\right|_{\theta=0}+\left.\frac{\partial T(z,\theta)}{\partial\theta}\right|_{\theta=0}-\left(\left.\frac{\partial T(z,\theta)}{\partial\theta}\right|_{\theta=0}\right)^2\\=\frac{1}{b}\frac{z}{(1-z)^2}\left(A''(0)+\frac{2a^2B(z)}{1-B(z)}\right)+\frac{z}{\lambda(1-z)^2}-\frac{z(1+z)}{\lambda^2(1-z)^3}\end{aligned} \tag{A.26}$$

The IDI, J_n is defined to be

$$J_n=\frac{n\cdot\mathsf{Var}[x(n)]}{\mathsf{E}[x(n)]^2} \tag{A.27}$$

Thus define $J(z)$ to be the z-transform of J_n and introducing the related interval correlation function $L(z)$

$$L(z)=(1-z)^2J'(z)=\frac{A''(0)}{a}+\frac{2aB(z)}{1-B(z)}+1-\frac{1+z}{\lambda(1-z)} \tag{A.28}$$

As

$$A''(0)=(C_a^2+1)a^2-a \tag{A.29}$$

Replacing $A''(0)$ in (A.28) by (A.29), $L(z)$ becomes

$$L(z)=b\left(C_a^2+\frac{1+B(z)}{1-B(z)}-\frac{1}{b}\cdot\frac{1+z}{1-z}\right) \tag{A.30}$$

Note that the expression (A.30) is referred to as expression (3) in the main body of the manuscript.

Appendix B. Probability generating functions of count and interval correlations of BMAP

This Appendix presents the derivation of analytic expressions for the PGFs of correlation of counts and correlation of intervals in traffic generated by a BMAP. For convenience of exposition, it is assumed that the first two moments of batch size exist for each phase.

The relevant notation of this section is described below.

E	the countable phase space of the BMAP;
$\mathbf{D}=\{D_{ij}\}$	the infinitesimal generator of the underlying Markov process, i.e., D_{ij} is the infinitesimal rate at which a transition from phase i to phase j takes place;
$\mathbf{e}=\{1\}^T$	the column vector with 1 in every location, $\mathbf{De}=0$;

$\boldsymbol{\pi} = \{\pi_i\}$ — the stationary probability vector of the Markov process with generator $\mathbf{D}$, i.e., $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi}\mathbf{D}$=0, $\boldsymbol{\pi}\mathbf{e} = 1$;

$p_i(0, j),\ j \neq i$ — the probability that there be a transition from i to j without an arrival;

$p_i(k, j),\ k \geq 1$ — the probability that there be a transition from i to j with an arrival of size k;

$\lambda_i,\ i = 1, 2, \ldots$ — the exponential distribution parameter for the sojourn in phase i;

$\mathbf{D}_0$ — the matrix governs transitions in the phase process without generating arrivals, $(\mathbf{D}_0)_{ii} = -\lambda_i,\ 1 \leq i$, $(\mathbf{D}_0)_{ij} = \lambda_i p_i(0, j),\ 1 \leq i, j,\ i \neq j$

The matrix $\mathbf{D}_0$ is stable matrix which implies that it is nonsingular and the sojourn in each state is finite with probability 1. This implies that the arrival process does not terminate;

$\mathbf{D}_k$ — the rate of arrivals of size k with the appropriate phase change, $(\mathbf{D}_k)_{ij} = \lambda_i p_i(k, j),\ k \geq 1,\ 1 \leq i, j$, this definition implies that $\sum_{k=0}^{\infty}\mathbf{D}_k$=$\mathbf{D}$;

$\lambda = \boldsymbol{\pi}\sum_{k=1}^{\infty} k\mathbf{D}_k\mathbf{e} = \boldsymbol{\pi}\boldsymbol{d}$ — the stationary arrival rate of the process, i.e., the overall arrival rate, where $\boldsymbol{d} = \sum_{k=1}^{\infty} k\mathbf{D}_k\mathbf{e}$;

$\mathbf{D}(z) = \sum_{k=0}^{\infty}\mathbf{D}_k z^k$ — the generating function of $\mathbf{D}$;

$N(t)$ — the counting function of number of arrivals in $(0, t]$;

$J(t)$ — the auxiliary phase at time $t+$;

$\mathbf{p}(n, t)$ — the probability density of n arrivals in $(0, t]$;

$\mathbf{P}(z, t) = \sum_{n=0}^{\infty}\mathbf{p}(n, t)z^n$ — the z-transform of $\mathbf{p}(n,t)$;

$\mathbf{P}^*(z, \theta) = \sum_{n=0}^{\infty}\int_{t=0}^{\infty}\mathbf{p}(n, t)z^n e^{-\theta t}dt$ — the 2-dimension transform of $\mathbf{p}(n, t)$;

$p_{ij}(n, t) = \mathsf{P}[N(t) = n, J(t) = j | N(0) = 0, J(0) = i]$ — the (i,j) element of $\mathbf{p}(n, t)$;

$N(z, \theta) = \sum_{n=0}^{\infty}\int_{t=0}^{\infty}\mathsf{P}[\int_{s=0}^{t} c(s)ds = n]z^n e^{-\theta t}dt$ — the generating function for the random observer's probabilities that n arrivals be generated in time of length t, where $c(s)$ is the number of arrivals in time of length s and time 0 is chosen at random;

$T(z, \theta) = \sum_{n=1}^{\infty}\int_{t=0}^{\infty}\mathsf{P}[\sum_{r=1}^{n} x(r) = t]z^n e^{-\theta t}dt$ — the generating function for the random observer's probabilities that n arrivals between $n+1$ successive arrivals span time of length t, where $x(n)$ is the n^{th} interval and point 0 of the interarrival process is chosen at random.

The function $N(z, \theta)$ is closely related to the generating function of the IDC and $T(z, \theta)$ to that of the IDI. Both $N(z, \theta)$ and $T(z, \theta)$ may be derived in terms of the probability $p_{ij}(n, t)$.

B.1. The PGF of count correlation of BMAP

From the definition, the (i, j)th entry of $\mathbf{p}(n,t)$ satisfies the forward Chapman–Kolmogorov equations

$$\frac{\partial\mathbf{p}(n, t)}{\partial t} = \sum_{k=0}^{n}\mathbf{p}(n-k, t)\mathbf{D}_k \quad n \geq 1, \quad t \geq 0 \tag{B.1}$$

$$\mathbf{p}(0, 0) = \mathbf{I} \tag{B.2}$$

The z-transform $\mathbf{P}(z,t)$ satisfies

$$\frac{\partial\mathbf{P}(z, t)}{\partial t} = \mathbf{P}(z, t)\mathbf{D}(z) \tag{B.3}$$

$$\mathbf{P}(z, 0) = \mathbf{I} \tag{B.4}$$

so that $\mathbf{P}(z,t)$ is explicitly given by

$$\mathbf{P}(z, t) = e^{\mathbf{D}(z)t} \tag{B.5}$$

and Laplace transform of (B.5) with respect to t is

$$\mathbf{P}^*(z, \theta) = (\mathbf{D}(z) - \theta\mathbf{I})^{-1} \tag{B.6}$$

Then

$$N(z, \theta) = \sum_{i,j\in E}\sum_{n=0}^{\infty}\int_{t=0}^{\infty}\pi_i p_{ij}(n, t)z^n e^{-\theta t}dt = \sum_{i,j\in E}\pi_i p_{ij}(z, \theta) = \boldsymbol{\pi}(\mathbf{D}(z) - \theta\mathbf{I})^{-1}\mathbf{e} \tag{B.7}$$

From Eq. (B.7) we have the first two moments

$$\left.\frac{\partial N(z, \theta)}{\partial z}\right|_{z=1} = \boldsymbol{\pi}(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{D}'(1)(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{e} \tag{B.8}$$

$$\left.\frac{\partial N^2(z, \theta)}{\partial z^2}\right|_{z=1} = 2\boldsymbol{\pi}(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{D}'(1)(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{D}'(1)(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{e} + \boldsymbol{\pi}(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{D}''(1)(\mathbf{D} - \theta\mathbf{I})^{-1}\mathbf{e} \tag{B.9}$$

We have the identities

$$\boldsymbol{\pi}(\mathbf{D}-\theta\mathbf{I})^{-1}=\frac{1}{1-\theta}\boldsymbol{\pi}\quad(\mathbf{D}-\theta\mathbf{I})^{-1}\mathbf{e}=\frac{1}{1-\theta}\mathbf{e} \tag{B.10}$$

Follow the same procedure used in deriving count correlation generating function of BRAP equations (A.7)–(A.13) and use (B.10), the generating function for count correlation of BMAP is obtained as

$$K(\theta)=\frac{1}{\lambda}\boldsymbol{\pi}(2\mathbf{D}'(1)(\mathbf{D}-\theta\mathbf{I})^{-1}\mathbf{D}'(1)+\mathbf{D}''(1))\mathbf{e}+1-\frac{2\lambda}{\theta} \tag{B.11}$$

Note that the expression (B.11) is referred to as expression (14) in the main body of the paper.

B.2. The PGF of interval correlation of BMAP

The interval correlation is derived by argument essentially similar to that used in section A.2.

- the probability that an arbitrarily selected arrival be generated by the phase i is $\frac{\lambda_i\pi_i}{\lambda}$,
- the probability that the arrival be in a batch of size u and phase changes to k from i is $\frac{u\lambda_i p_i(u,k)}{\lambda_i}$ and
- the probability that the arrival be at any particular position in the batch of size u is $\frac{1}{u}$.

The probability that n arrivals be contained in the same batch (i.e., t=0) is

$$\mathsf{P}[\sum_{r=1}^{n}x(r)=0]=\sum_{i,k\in E}\frac{\lambda_i\pi_i}{\lambda}\sum_{u=n+1}^{\infty}\frac{u\lambda_i p_i(u,k)}{\lambda_i}\cdot\frac{u-n}{u}=\frac{1}{\lambda}\sum_{i,k\in E}\sum_{u=n+1}^{\infty}\pi_i\lambda_i p_i(u,k)(u-n) \tag{B.12}$$

which is generated by

$$\sum_{n=1}^{\infty}\mathsf{P}[\sum_{r=1}^{n}x(r)=0]z^n=\frac{1}{\lambda}\sum_{i,k\in E}\sum_{n=1}^{\infty}\sum_{u=n+1}^{\infty}\pi_i\lambda_i p_i(u,k)(u-n)z^n=\frac{1}{\lambda}\cdot\frac{z}{1-z}\sum_{i,k\in E}\pi_i\lambda_i\sum_{u=1}^{\infty}up_i(u,k)-\frac{1}{\lambda}\cdot\frac{z}{(1-z)^2}\sum_{i,k\in E}\pi_i\lambda_i\sum_{u=1}^{\infty}p_i(u,$$
$$k)+\frac{1}{\lambda}\cdot\frac{z}{(1-z)^2}\sum_{i,k\in E}\pi_i\lambda_i\sum_{u=1}^{\infty}p_i(u,k)z^u=\frac{z}{1-z}-\frac{1}{\lambda}\cdot\frac{z}{(1-z)^2}\boldsymbol{\pi}(\mathbf{D}(z)-\mathbf{D})\mathbf{e} \tag{B.13}$$

The probability that n intervals span an interval t ($t>0$) is

$$\mathsf{P}[\sum_{r=1}^{n}x(r)=t|t>0]=\sum_{i,j,k,l\in E}\sum_{r=1}^{n}\sum_{s=1}^{r}\frac{\lambda_i\pi_i}{\lambda}\sum_{u=n+1-r}^{\infty}\frac{u\lambda_i p_i(u,j)}{\lambda_i}\cdot\frac{1}{u}\cdot p_{jk}(r-s,t)\sum_{v=s}^{\infty}\lambda_k p_k(v,l)=\frac{1}{\lambda}\sum_{i,j,k,l\in E}\sum_{r=1}^{n}\sum_{s=1}^{r}\sum_{u=n+1-r}^{\infty}\pi_i\lambda_i p_i(u,j)p_{jk}(r-s,t)\sum_{v=s}^{\infty}\lambda_k p_k(v,l) \tag{B.14}$$

which is generated by

$$\sum_{n=1}^{\infty}\mathsf{P}[\sum_{r=1}^{n}x(r)=t|t>0]z^n=\sum_{i,j,k,l\in E}\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{s=1}^{r}\frac{\lambda_i\pi_i}{\lambda}\sum_{u=n+1-r}^{\infty}p_i(u,j)p_{jk}(r-s,t)\sum_{v=s}^{\infty}\lambda_k p_k(v,l)z^n=\sum_{i,j,k,l\in E}\sum_{r=1}^{\infty}\sum_{n=r}^{\infty}\sum_{s=1}^{r}\frac{\lambda_i\pi_i}{\lambda}\sum_{u=n+1-r}^{\infty}p_i(u,j)p_{jk}(r-s,$$
$$t)\sum_{v=s}^{\infty}\lambda_k p_k(v,l)z^n=\sum_{i,j,k,l\in E}\frac{\lambda_i\pi_i}{\lambda}\sum_{n=0}^{\infty}\sum_{u=n+1}^{\infty}p_i(u,j)\sum_{s=1}^{\infty}\sum_{r=s}^{\infty}p_{jk}(r-s,t)z^{r-s}\sum_{v=s}^{\infty}\lambda_k p_k(v,l)z^{n+s}=\sum_{i,j,k,l\in E}\frac{\lambda_i\pi_i}{\lambda(1-z)}\left(\sum_{u=0}^{\infty}p_i(u,j)-\sum_{u=0}^{\infty}p_i(u,j)z^u\right)p_{jk}(z,$$
$$t)\times\left(\sum_{s=1}^{\infty}\sum_{v=0}^{\infty}\lambda_k p_k(v,l)z^s-\sum_{s=1}^{\infty}\sum_{v=0}^{s-1}\lambda_k p_k(v,l)z^s\right) \tag{B.15}$$

therefore

$$\int_0^{\infty}\sum_{n=1}^{\infty}\mathsf{P}[\sum_{r=1}^{n}x(r)=t|t>0]z^n e^{-\theta t}dt=\frac{1}{\lambda}\cdot\frac{z}{(1-z)^2}\boldsymbol{\pi}(\mathbf{D}(z)-\mathbf{D})(\mathbf{D}(z)-\theta\mathbf{I})^{-1}(\mathbf{D}(z)-\mathbf{D})\mathbf{e} \tag{B.16}$$

By combining (B.13) and (B.16), $T(z,\theta)$ is obtained as

$$T(z,\theta)=\int_0^{\infty}\sum_{n=1}^{\infty}\mathsf{P}[\sum_{r=1}^{n}x(r)=t|t\geq 0]z^n e^{-\theta t}dt=\frac{z}{1-z}-\frac{1}{\lambda(1-z)^2}\boldsymbol{\pi}\mathbf{D}(z)\mathbf{e}+\frac{z}{\lambda(1-z)^2}\boldsymbol{\pi}\mathbf{D}(z)(\mathbf{D}(z)-\theta\mathbf{I})^{-1}\mathbf{D}(z)\mathbf{e} \tag{B.17}$$

From (B.17) we have the first two moments

$$\left.\frac{\partial T(z,\theta)}{\partial\theta}\right|_{\theta=0}=\frac{1}{\lambda}\cdot\frac{z}{(1-z)^2} \tag{B.17}$$

$$\left.\frac{\partial T^2(z,\theta)}{\partial\theta^2}\right|_{\theta=0}=\frac{2}{\lambda}\cdot\frac{z}{(1-z)^2}\boldsymbol{\pi}\mathbf{D}^{-1}(z)\mathbf{e} \tag{B.18}$$

Follow the same procedure used in deriving interval correlation generating function of BRAP equations (A.25)–(A.28), the generating function for interval correlation of BMAP is obtained as

$$L(z)=2\lambda\boldsymbol{\pi}\mathbf{D}^{-1}(z)\mathbf{e}+\lambda-\frac{1+z}{1-z} \tag{B.19}$$

Note that the expression (B.19) is referred to as expression (15) in the main body of the paper.

Appendix C. The derivation of the QLD of the stable GE^{sGGeo}/GE/1 queue

This Appendix presents the complete proof of the derivation of the expressions for the QLDs $p_\infty^A(n)$, $p_\infty^D(n)$ and $p_\infty(n)$ of the stable GE^{sGGeo}/GE/1 queue.

The relevant notation of this section is described below.

$a(t),\ t>0$	the PDF of interval between consecutive batch arrivals, i.e., GE distribution in GE^{sGGeo}/GE/1 queue
$a=\int_0^\infty v\,a(v)dv$	the mean inter-batch interval
C_a^2]	the SCV of intervals between consecutive batch arrivals
$\sigma=2/(C_a^2+1)$; $b(n)\mathsf{P}[b_k=n]$, $n=1,2,\ldots$	the PMF of batch size, i.e., sGGeo distribution in GE^{sGGeo}/GE/1 queue
b	the mean batch size
C_b^2	the SCV of batch sizes
$\eta=2/(C_b^2+1)$; $s(t),\ t>0$	the PDF of service time, i.e., GE distribution in GE^{sGGeo}/GE/1 queue
C_s^2	the SCV of individual service times
$\tau=2/(C_s^2+1)$; $\lambda=\frac{b}{a}$	the mean arrival rate of individual customer
μ	the mean service rate of GE service time distribution
$p_\infty^A(n),\ n=0,1,\ldots,\infty$	the steady state probability that there are n customers in the queue immediately before a batch arrival
$p_\infty^D(n),\ n=1,2,\ldots,\infty$	the steady state probability that there are n customers in the queue immediately after a batch arrival
$p_\infty(n),\ n(n=0,1,\ldots,\infty)$	the steady state probability that there are n customers in the system (including queueing and receiving service, if any) at any time
$g(k,t)$	the probability that exactly k customers depart during interval of duration t given sufficient number of customers in the queue

Inter-arrival time between batches and service time follow GE distributions, respectively, and specialized as

$$a(t)=\begin{cases}1-\sigma & t=0\\ \sigma^2\lambda e^{-\sigma\lambda t} & t>0\end{cases} \tag{C.1 (i.e., (27) in paper)}$$

$$s(t)=\begin{cases}1-\tau & t=0\\ \tau^2\mu e^{-\tau\mu t} & t>0\end{cases} \tag{C.2 (i.e., (28) in paper)}$$

Arrival batch size is sGGeo distributed as

$$b(n)=\begin{cases}1-\eta & n=1\\ \eta\nu(1-\nu)^{n-2} & n=2,3,\ldots,\infty\end{cases} \tag{C.3 (i.e., (26) in paper)}$$

We also have relationships between $p_\infty^A(\cdot)$ and $p_\infty^D(\cdot)$, $p_\infty(\cdot)$ and $p_\infty^D(\cdot)$ as well as $p_\infty^D(\cdot)$ and $p_\infty^A(\cdot)$

$$p_\infty^A(n)=\begin{cases}\sum_{k=1}^\infty p_\infty^D(k)\int_0^\infty \sum_{r=k}^\infty g(r,t)a(t)dt & n=0\\ \sum_{k=0}^\infty p_\infty^D(n+k)\int_0^\infty g(k,t)a(t)dt & n=1,2,\ldots,\infty\end{cases} \tag{C.4 (i.e., (24) in paper)}$$

$$p_\infty(n)=\begin{cases}\sum_{k=1}^\infty p_\infty^D(k)\int_0^\infty \int_0^t \sum_{r=k}^\infty g(r,s)ds\cdot\frac{a(t)}{a}dt & n=0\\ \sum_{k=0}^\infty p_\infty^D(n+k)\int_0^\infty \int_0^t g(k,s)ds\cdot\frac{a(t)}{a}dt & n=1,2,\ldots,\infty\end{cases} \tag{C.5 (i.e., (25) in paper)}$$

$$p_\infty^D(n)=\sum_{k=0}^{n-1} p_\infty^A(k)b(n-k)\quad \text{where } n=1,2,\ldots,\infty \tag{C.6}$$

The counting function of GE process is

$$g(k,t)=\begin{cases}e^{-\tau\mu t} & k=0\\ \sum_{\ell=1}^k \binom{k-1}{\ell-1}(1-\tau)^{k-\ell}\tau^\ell\frac{(\tau\mu t)^\ell}{\ell!}e^{-\tau\mu t} & k=1,2,\cdots\end{cases} \tag{C.7 (i.e., (21) in paper}$$

Substitute (C.7) and (C.1) into (C.4), one will obtain

$$p_\infty^A(n)=\begin{cases}\sum_{k=1}^\infty p_\infty^D(k)\frac{\tau\mu}{\sigma\lambda+\tau\mu}\left(1-\frac{\tau\sigma^2\lambda}{\sigma\lambda+\tau\mu}\right)^{k-1} & n=0\\ \frac{\tau\mu}{\sigma\lambda+\tau\mu}p_\infty^D(n)+\frac{\tau^2\sigma^2\lambda\mu}{(\sigma\lambda+\tau\mu)^2}\sum_{k=1}^\infty p_\infty^D(k)\left(1-\frac{\tau\sigma^2\lambda}{\sigma\lambda+\tau\mu}\right)^{k-1} & n=1,2,\ldots,\infty\end{cases} \tag{C.8}$$

Write the expressions of $p_\infty^A(0)$ and $p_\infty^A(1)$ explicitly and consider the relationship between $p_\infty^A(0)$ and $p_\infty^A(1)$, we have

$$p_\infty^A(1)=\frac{\sigma\lambda}{\sigma\lambda+\tau\mu}p_\infty^D(1)+\frac{\tau\sigma\lambda}{\sigma\lambda+\tau\mu}p_\infty^A(0) \tag{C.9}$$

Substituting $p_\infty^D(1)$ in (C.9) by (C.6)

$$p_\infty^A(1) = \frac{\sigma\lambda(1-\eta) + \tau\sigma\lambda}{\sigma\lambda + \tau\mu} p_\infty^A(0) \tag{C.10}$$

and, moreover, the relationship between $p_\infty^A(n)$ and $p_\infty^A(n+1)$, where $n = 1, 2, \ldots, \infty$ is determined by

$$p_\infty^A(n) = (1 - \frac{\tau\sigma\lambda}{\sigma\lambda + \tau\mu}) p_\infty^A(n+1) + \frac{\sigma\lambda}{\sigma\lambda + \tau\mu} p_\infty^D(n) - \frac{(1-\tau)\sigma\lambda}{\sigma\lambda + \tau\mu} p_\infty^D(n+1) \tag{C.11}$$

It follows that the z-transform $p_\infty^A(z)$ of $p_\infty^A(n)$ is given by

$$p_\infty^A(z) = \sum_{n=0}^{\infty} p_\infty^A(n) z^n = \frac{(\tau\sigma\lambda - \sigma\lambda - \tau\mu) p_\infty^A(0)}{\sigma\lambda B(z) - \sigma\lambda - \tau\mu + \dfrac{1 - B(z)}{1 - z}\tau\sigma\lambda} \tag{C.12}$$

Let z=1 in (C.12), we get

$$p_\infty^A(0) = \frac{\tau\mu - \sigma\lambda}{\tau\mu - \tau\sigma\lambda + \sigma\lambda} \tag{C.13}$$

Substituting (C.13) into (C.12)

$$p_\infty^A(z) = \frac{\tau\mu - \sigma\lambda}{\tau\mu - \tau\sigma\lambda + \sigma\lambda} \cdot \left(\frac{1 - (1-\sigma)z}{1 - xz}\right) \tag{C.14}$$

where

$$x = \frac{\sigma(1 - \eta - \nu) + \eta + \tau}{\sigma + (1-\sigma)\eta\tau\mu} \tag{C.15}$$

Thus, the derivation of $p_\infty^A(n)$ is given by

$$p_\infty^A(n) = \begin{cases} \frac{\tau\mu - \sigma\lambda}{\tau\mu - \tau\sigma\lambda + \sigma\lambda} & n = 0 \\ \frac{\tau\mu - \sigma\lambda}{\tau\mu - \tau\sigma\lambda + \sigma\lambda} \cdot \frac{\sigma\lambda(1-\eta) + \tau\sigma\lambda}{\sigma\lambda + \tau\mu} \cdot \left(\frac{\sigma\lambda + \tau\mu(1-\sigma)}{\tau\mu + \sigma\lambda(1-\tau)}\right)^{n-1} & n = 1, 2, \ldots, \infty \end{cases} \tag{C.16}$$

By (C.6)

$$p_\infty^D(n) = \begin{cases} \dfrac{\tau\mu - \sigma\lambda}{\tau\mu - \tau\sigma\lambda + \sigma\lambda} \cdot (1-\eta) & n = 1 \\ \dfrac{\tau\mu - \sigma\lambda}{\tau\mu + \sigma\lambda} \cdot \dfrac{(1-\nu)^{n-1} - (\sigma\lambda + \tau\mu(1-\sigma))^{n-1}}{1-\nu} & n = 2, 3, \ldots, \infty \end{cases} \tag{C.17}$$

By (C.5), one has

$$p_\infty(n) = \begin{cases} \dfrac{1}{\nu} \dfrac{\sigma\lambda}{\sigma\lambda + \tau\mu} (\sigma\nu - \tau\eta) & n = 0 \\ \dfrac{\eta}{\nu} \dfrac{\sigma\lambda}{\sigma\lambda + \tau\mu} \dfrac{\tau\mu}{\sigma} (1-x) x^n & n = 1, 2, \ldots, \infty \end{cases} \tag{C.18}$$

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