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# Delay-dependent $H_\infty$ filtering for singular Markovian jump systems with general uncomplete transition probabilities

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ABSTRACT

This paper is devoted to the investigation of the delay-dependent $H_\infty$ filtering problem for a kind of singular Markovian jump time-delay systems with general unknown transition probabilities. In this model, the transition rates of the jumping process are assumed to be partly available, that is, some elements have been exactly known, some ones have been merely known with lower and upper bounds, others may have no information to use. Using the Lyapunov functional theory, a stochastically stable filter is designed to guarantee both the mean-square exponential admissibility and a prescribed level of $H_\infty$ performance for the singular Markovian jump time-delay systems with general unknown transition probabilities. A sufficient condition is derived for the existence of such a desired filter in terms of linear matrix inequalities (LMIs). A numerical example is provided to demonstrate the effectiveness of the proposed theory.



## 1. Introduction

Singular systems, also referred to as generalized systems, descriptor systems, implicit systems, differential-algebraic systems or semi-state systems, have widespread applications in electrical circuits, power systems, economics and other areas [1]. Singular system is more complicated than nonsingular systems because not only asymptotic stability but also the system regularity and impulse elimination are needed to be tackled [2–4]. Very recently, more and more attention has been paid to the problem of stochastic stability and stochastic admissibility for singular Markovian jump systems (SMJSs), due to their popularity in modelling many physical systems which suffer from abrupt variations in their structure, see [5–16] and the references therein.

On the other hand, state estimation has been extensively studied for various applications. $H_\infty$ control has attracted a lot of interests, due to its capability of dealing with the problems of robustness, optimality and disturbance rejection [9,13,14, 16–19]. It should be pointed out that compared with traditional Kalman filtering, the $H_\infty$ filtering approach is more useful in many applications because it does not require knowledge of the statistical properties of the external noises [20]. Therefore, the $H_\infty$ filtering problem for SMJSs with or without time-delay has been widely discussed via linear matrix inequality (LMI) approach, see [21–27] and the references therein. The authors in [22] considered $H_\infty$ filtering for time-delayed singular Markovian jump systems with time-varying switching by a quantized method. Ma and Boukas [23] investigated the robust

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$H_\infty$ filtering problem for mode-dependent time-delay discrete Markov jump singular systems, and designed a Markov jump filter guaranteeing the filtering error system to be regular, causal, stochastically stable and satisfy $H_\infty$ performance. Besides, it is well known that delay-dependent conditions are generally less conservative than delay-independent ones, especially when the size of the time delay is small. Therefore, delay-dependent $H_\infty$ filtering for singular Markovian jump time-delay systems was discussed in [25–27]. However, the results in the above mentioned literatures are under the assumption of completely known transition probabilities.

In practice, owing to all kinds of complex factors, the exact values of transition probabilities cannot be completely available. For example, as a result of the existence of the packet dropout and channel delays in networked control systems, it will be more expensive to acquire complete transition rates (TRs). Therefore, the study of the stabilization of Markovian jump systems with unknown transition rates becomes interesting [28–40]. At present, there are three types of descriptions about uncertain TRs. One is Bounded Uncertain TRs (BUTRs) [28], where the precise value of each TR may be unknown but its bounds (upper bounds and lower bounds) are known. The other is Partly Unknown TRs (PUTRs) [29,30,36–39], where each TR is either exactly known or completely unknown. However, this two types may be too restrictive in many practical situations, because it is hard to precisely estimate every TR in practice. Therefore, another type named General Unknown Transition Rates (GUTRs) was put up in [31], which is more applicable and has been studied in recent years [32–34,40]. In this model, each transition rate can be completely unknown or only its estimate is known. Both BUTR models and PUTR models can be seen as special cases of GUTR models. Recently, there are some pioneer works on $H_\infty$ filtering for singular Markovian jump systems with partially unknown transition probabilities, see [35,36] and the references therein. Wang and Xu [35] probed robust $H_\infty$ filtering for singular time-delay systems with uncertain Markovian switching probabilities. Lin et al. [36] considered delay-dependent $H_\infty$ filtering for discrete-time singular Markovian jump systems with time-varying delay and partially unknown transition probabilities. There also appears some other important results about Markovian jump systems [37–44], however the systems discussed do not belong to singular Markovian jump systems, which are more complicated than nonsingular systems. To the best of our knowledge, the problem of delay-dependent $H_\infty$ filter design for continuous singular Markovian jump time-delay systems with general unknown transition rates (GUTRs) has not been fully investigated.

Motivated by the aforementioned discussions, this paper deals with the problem of delay-dependent $H_\infty$ filtering for singular Markovian jump systems with both time varying delay and general unknown transition probabilities. In Section 2, problem statement and preliminaries are formulated. Our new uncertain model is more general than the existing ones and can be applicable to more practical situations due to each transition rate can be completely unknown or only its estimate value is known. In Section 3, using a bounded real lemma and an LMI-based approach, the desired filters are designed to guarantee the considered system to be delay-dependent exponentially admissible in mean-square and satisfy a prescribed $H_\infty$ performance level. In Section 4, an illustrative example is provided to demonstrate the effectiveness of the proposed methods. Section 5 is conclusions.

*Notation.* $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, and $\mathbb{R}^n$ the $n$-dimensional Euclidean space. $\|\cdot\|$ represents for the Euclidean norm for a vector and $C_{n,d} = C([-d, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d, 0]$ into $\mathbb{R}^n$ with norm $\|\phi(t)\|_d = \sup_{-d\le s\le 0} \|\phi(s)\|$. $\mathcal{L}_2[0, \infty)$ stands for the space of square integrable functions on $[0, \infty)$. $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, $\Omega$ is the sample space, $\mathcal{F}$ is the $\sigma$-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $\varepsilon\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathcal{P}$. The superscript "$T$" and + represent the transpose and the Moore–Penrose inverse, respectively, and "$*$" denotes the term that is induced by symmetry.

## 2. Preliminaries and problem formulation

Consider the singular time-delay systems with Markovian jump parameters on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ as follows:

$$\begin{aligned} E\dot{x}(t) &= A(r_t)x(t) + A_d(r_t)x(t-d(t)) + B_\omega(r_t)\omega(t),\\ y(t) &= C(r_t)x(t) + C_d(r_t)x(t-d(t)) + D_\omega(r_t)\omega(t),\\ z(t) &= L(r_t)x(t),\\ x(t) &= \phi(t), \quad t\in[-d_2, 0], \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $y(t)\in\mathbb{R}^s$ is the measurement, $z(t)\in\mathbb{R}^q$ is the signal to be estimated and $\omega(t)\in\mathbb{R}^p$ is the disturbance input that belongs to $\mathcal{L}_2[0,\infty)$. The matrix $E\in\mathbb{R}^{n\times n}$ may be singular and it is assumed that rank $E = r \le n$. $A(r_t)$, $A_d(r_t)$, $B_\omega(r_t)$, $C(r_t)$, $C_d(r_t)$, $D_\omega(r_t)$ and $L(r_t)$ are known real constant matrices with appropriate dimensions for each $r_t\in\mathcal{S}$. $d(t)$ is a time-varying continuous function that satisfies

$$0 \le d_1 \le d(t) \le d_2, \quad \dot{d}(t) \le \mu, \tag{2}$$

where $d_1$ and $d_2$ are the time delay lower and upper bounds, respectively, and $0 \le \mu < 1$ is the time delay variation rate. $\phi(t)\in C_{n,d_2}$ is a compatible vector valued initial function.

**Remark 1.** In Assumption (2), the delays are assumed to be differentiable and bounded. The constraint of the delay derivative $\dot{d}(t) \le \mu < 1$ is strong. The condition can be relaxed. In fact, now, many papers have no longer required the delay derivative less than 1 [46]. However, in real world, time delays in some of the systems change slowly, therefore, the constraint of the delay derivative $\dot{d}(t) \le \mu < 1$ is still of significance. Of course, we also can use the similar method in [46] to deal with the relaxed condition.

Let $\{r_t, t \ge 0\}$ be a continuous-time Markovian process with right continuous trajectories and take values in a finite set $\mathcal{S} = \{1, 2, \ldots, s\}$ with transition probability matrix $\prod \triangleq \{\pi_{ij}\}$ given by

$$Pr = \{r_{t+h} = j | r_t = i\} = \begin{cases} \pi_{ij} h + o(h), & j \ne i, \\ 1 + \pi_{ii} h + o(h), & j = i, \end{cases} \tag{3}$$

where $h > 0$, $\lim_{h \to 0} \frac{o(h)}{h} = 0$, and $\pi_{ij} \ge 0$, for $j \ne i$, is the transition rate from mode $i$ at time $t$ to mode $j$ at time $t+h$ and $\pi_{ii} = -\sum_{j=1, j \ne i}^{s} \pi_{ij}$. The mode transition probability matrix $\Pi \triangleq (\pi_{ij})$ is considered to be generally uncertain. For instance, the transition probability matrix for system (1) with GUTRs with $s$ operation modes may be expressed as

$$\begin{bmatrix} \hat{\pi}_{11} + \Delta_{11} & ? & \hat{\pi}_{13} + \Delta_{13} & \cdots & ? \\ ? & ? & \hat{\pi}_{23} + \Delta_{23} & \cdots & \hat{\pi}_{2s} + \Delta_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & \hat{\pi}_{s2} + \Delta_{s2} & ? & \cdots & \hat{\pi}_{ss} + \Delta_{ss} \end{bmatrix} \tag{4}$$

where $\hat{\pi}_{ij}$ and $\Delta_{ij} \in [-\delta_{ij}, \delta_{ij}] (\delta_{ij} \ge 0)$ represent the estimate value and estimate error of the uncertain transition probability $\pi_{ij}$ respectively, where $\hat{\pi}_{ij}$ and $\delta_{ij}$ are known. "?" represents the complete unknown transition probability, which means its estimate value $\hat{\pi}_{ij}$ and estimate error bound are unknown. For notational clarity, for all $i \in \mathcal{S}$, the set $U^i$ denotes $U^i = U^i_K \cup U^i_{UK}$, with $U^i_K \triangleq \{j : \text{The estimate value of } \pi_{ij} \text{ is known for } j \in \mathcal{S}\}$, and $U^i_{UK} \triangleq \{j : \text{The estimate value of } \pi_{ij} \text{ is unknown for } j \in \mathcal{S}\}$. Moreover, if $U^i_K \ne \emptyset$, it is described as $U^i_K = \{K^i_1, K^i_2, \ldots, K^i_m\}$, where $K^i_m \in \mathbb{N}^+$ represent the $m$th bound-known element with the index $K^i_m$ in the $i$th row of matrix $\Pi$.

We assume that the known estimate values of the transition probabilities are well defined. That is

**Assumption 1.** If $U^i_K = \mathcal{S}$, then $\hat{\pi}_{ij} - \delta_{ij} \ge 0, (\forall j \in \mathcal{S}, j \ne i), \hat{\pi}_{ii} = -\sum_{j=1, j \ne i}^{s} \hat{\pi}_{ij} \le 0$, and $\delta_{ii} = \sum_{j=1, j \ne i}^{s} \delta_{ij} > 0$;

**Assumption 2.** If $U^i_K \ne \mathcal{S}$, *and* $i \in U^i_K$, then $\hat{\pi}_{ij} - \delta_{ij} \ge 0, (\forall j \in U^i_K, j \ne i), \hat{\pi}_{ii} + \delta_{ii} \le 0$, and $\sum_{j \in U^i_K} \hat{\pi}_{ij} \le 0$;

**Assumption 3.** If $U^i_K \ne \mathcal{S}$ and $i \notin U^i_K$, then $\hat{\pi}_{ij} - \delta_{ij} \ge 0, (\forall j \in U^i_K)$.

**Definition 1** (Wu et al. [27]). (i). The singular Markovian jump time-delay system

$$\begin{aligned} E\dot{x}(t) &= A(r_t)x(t) + A_d(r_t)x(t - d(t)), \\ x(t) &= \phi(t), \quad t \in [-d_2, 0], \end{aligned} \tag{5}$$

is said to be regular and impulse free for any time delay $d(t)$ satisfying (2), if the pairs $(E, A(r_t))$ and $(E, A(r_t) + A_d(r_t))$ are regular and impulse free for every $r_t \in \mathcal{S}$.

(ii). The singular Markovian jump time-delay system (5) is said to be mean-square exponentially stable, if there exist scalars $\alpha > 0$ and $\beta > 0$ such that $\varepsilon\{\|x(t)\|^2\} \le \alpha e^{-\beta t} \|\phi(t)\|^2_{d_2}, t > 0$.

(iii). The singular Markovian jump time-delay system (5) is said to be mean-square exponentially admissible, if it is regular, impulse free and mean-square exponentially stable.

**Definition 2** (Wu et al. [27]). Given a scalar $\gamma > 0$, the singular Markovian jump time-delay system (1) is said to be mean-square exponentially admissible with $H_\infty$ performance $\gamma$, if the system with $\omega(t) \equiv 0$ is mean-square exponentially admissible, and under zero initial condition, it satisfies $\|z(t)\|_{E_2} < \gamma \|\omega(t)\|_2$ for any non-zero $\omega(t) \in \mathcal{L}_2[0, \infty)$, where $\|z(t)\|_{E_2} = \sqrt{\varepsilon\{\int_0^\infty z(t)^T z(t) dt\}}$.

In this paper, in order to estimate $z(t)$, we aim to design a filter of the following structure:

$$\begin{aligned} E_f \dot{\hat{x}}(t) &= A_f(r_t)\hat{x}(t) + B_f(r_t)y(t), \\ \hat{z}(t) &= C_f(r_t)\hat{x}(t), \end{aligned} \tag{6}$$

where $\hat{x} \in \mathbb{R}^n$, $\hat{z}(t) \in \mathbb{R}^q$, and the constant matrices $E_f$, $A_f(r_t)$, $B_f(r_t)$ and $C_f(r_t)$ are the filter matrices with appropriate dimensions.

Define

$$\begin{aligned} \bar{z}(t) &= z(t) - \hat{z}(t), \\ \bar{x}(t) &= [x^T(t) \quad \hat{x}^T(t)]^T, \end{aligned}$$

and combining (1) and (6), we obtain the filtering error dynamics as follows:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}(r_t)\bar{x}(t) + \bar{A}_d(r_t)\bar{x}(t - d(t)) + \bar{B}_\omega(r_t)\omega(t), \\ \bar{z}(t) &= \bar{L}(r_t)\bar{x}(t), \end{aligned} \tag{7}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & E_f \end{bmatrix}, \ \bar{A}(r_t) = \begin{bmatrix} A(r_t) & 0 \\ B_f(r_t)C(r_t) & A_f(r_t) \end{bmatrix}, \bar{A}_d(r_t) = \begin{bmatrix} A_d(r_t) & 0 \\ B_f(r_t)C_d(r_t) & 0 \end{bmatrix},$$
$$\bar{B}_\omega(r_t) = \begin{bmatrix} B_\omega(r_t) \\ B_f(r_t)D_\omega(r_t) \end{bmatrix}, \ \bar{L}(r_t) = \begin{bmatrix} L(r_t) & -C_f(r_t) \end{bmatrix}.$$

For simplicity, in the sequel, for each possible $r_t = i, \ i \in \mathcal{S}$, a matrix $M(r_t)$ will be denoted by $M_i$, $A(r_t)$ by $A_i$, $A_d(r_t)$ by $A_{di}$, and so on.

**Lemma 1** (Xu and Lam [5]). *The matrix inequality*

$$\begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix} \geq 0$$

*holds if and only if*

$$Z_3 \geq 0, \ Z_1 - Z_2 Z_3^{+} Z_2^T \geq 0, \ Z_2(I - Z_3 Z_3^{+}) = 0.$$

**Lemma 2** (Xu and Lam [5]). *Given any real number a and any matrix Q, the matrix inequality*

$$a(Q + Q^T) \leq a^2 T + Q T^{-1} Q^T.$$

*holds for any matrix $T > 0$.*

**Lemma 3** (Wu et al. [27]). *For given scalars $0 \leq d_1 < d_2$ and $\mu$, the singular Markovian jump time-delay system (1) is mean-square exponentially admissible with $H_\infty$ performance $\gamma$ for any time delay $d(t)$ satisfying (2), if there exist symmetric positive-definite matrices $Q_j$, $j = 1, 2, 3$, $Z_l$, $l = 1, 2$ and matrices $P_i$ such that for every $i \in \mathcal{S}$,*

$$E^T P_i = P_i^T E \geq 0, \tag{8a}$$

$$\Xi_i = \begin{bmatrix} \Xi_{11i} & \Xi_{12i} & 0 & 0 & P_i^T B_{\omega i} & A_i^T W \\ * & \Xi_{22i} & E^T Z_2 E & \Xi_{24i} & 0 & A_{di}^T W \\ * & * & \Xi_{33i} & 0 & 0 & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & B_{\omega i}^T W \\ * & * & * & * & * & -W \end{bmatrix} < 0, \tag{8b}$$

*where $d_{12} = d_2 - d_1, \ W = d_{12} d_2^2 Z_1 + d_{12}^2 Z_2$ and*

$$\begin{aligned}
\Xi_{11i} &= P_i^T A_i + A_i^T P_i + \sum_{k=1}^{3} Q_k - d_{12} E^T Z_1 E + L_i^T L_i + \sum_{j=1}^{s} \pi_{ij} E^T P_j, \\
\Xi_{12i} &= P_i^T A_{di} + d_{12} E^T Z_1 E, \\
\Xi_{22i} &= -(1-\mu) Q_3 - E^T((d_{12} + d_2) Z_1 + 2 Z_2) E, \\
\Xi_{33i} &= -Q_1 - E^T Z_2 E, \\
\Xi_{24i} &= E^T(d_2 Z_1 + Z_2) E, \\
\Xi_{44i} &= -Q_2 - E^T(d_2 Z_1 + Z_2) E.
\end{aligned}$$

## 3. Main results

In this section, we aim to design an $H_\infty$ filter of the form (6) for the system (1) with GUTRs such that the filtering error system (7) is mean-square exponentially admissible with $H_\infty$ performance $\gamma$, and the criteria are delay-dependent. And the following theorems are based on Lemma 3.

**Theorem 1.** *For given scalars $0 \leq d_1 < d_2$ and $\mu$, the singular Markovian jump time-delay system (1) with GUTRs is mean-square exponentially admissible with $H_\infty$ performance $\gamma$ for any time delay $d(t)$ satisfying (2), if there exist symmetric positive-definite matrices $Q_j, \ j = 1, 2, 3, \ Z_l, \ l = 1, 2$ and matrices $P_i$ such that for every $i \in \mathcal{S}$,*

$$E^T P_i = P_i^T E \geq 0, \tag{9}$$

*Case I. If* $i \notin U_K^i$ *and* $U_K^i = \{K_1^i, \ldots, K_m^i\}$, *there exist a set of positive definite matrices* $T_{ij} \in R^{n\times n}(i \notin U_K^i, j \in U_K^i)$ *such that*

$$\begin{bmatrix} \tilde{\Xi}_{11i} & \Xi_{12i} & 0 & 0 & P_i^T B_{\omega i} & A_i^T W & E^T(P_{K_1^i} - P_i) & \cdots & E^T(P_{K_m^i} - P_i) \\ * & \Xi_{22i} & E^T Z_2 E & \Xi_{24i} & 0 & A_{di}^T W & 0 & \cdots & 0 \\ * & * & \Xi_{33i} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & B_{\omega i}^T W & 0 & \cdots & 0 \\ * & * & * & * & * & -W & 0 & \cdots & 0 \\ * & * & * & * & * & * & -T_{iK_1^i} & \cdots & 0 \\ * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & \cdots & -T_{iK_m^i} \end{bmatrix} < 0, \tag{10}$$

*with* $E^T(P_j - P_i) \le 0,\ j \in U_{UK}^i, j \ne i.$

*Case II. If* $i \in U_K^i, U_{UK}^i \ne \emptyset$ *and* $U_K^i = \{K_1^i, \ldots, K_m^i\}$, *there exist a set of positive definite matrices* $V_{ijl} \in R^{n\times n}(i, j \in U_K^i, l \in U_{UK}^i)$ *such that*

$$\begin{bmatrix} \bar{\Xi}_{11i} & \Xi_{12i} & 0 & 0 & P_i^T B_{\omega i} & A_i^T W & E^T(P_{K_1^i} - P_l) & \cdots & E^T(P_{K_m^i} - P_l) \\ * & \Xi_{22i} & E^T Z_2 E & \Xi_{24i} & 0 & A_{di}^T W & 0 & \cdots & 0 \\ * & * & \Xi_{33i} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & B_{\omega i}^T W & 0 & \cdots & 0 \\ * & * & * & * & * & -W & 0 & \cdots & 0 \\ * & * & * & * & * & * & -V_{iK_1^i l} & \cdots & 0 \\ * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & \cdots & -V_{iK_m^i l} \end{bmatrix} < 0, \tag{11}$$

*Case III. If* $i \in U_K^i, U_{UK}^i = \emptyset$, *there exist a set of positive definite matrices* $R_{ij} \in R^{n\times n}(i, j \in U_K^i)$ *such that*

$$\begin{bmatrix} \hat{\Xi}_{11i} & \Xi_{12i} & 0 & 0 & P_i^T B_{\omega i} & A_i^T W & E^T(P_1 - P_i) & \cdots & E^T(P_{i-1} - P_i) \\ * & \Xi_{22i} & E^T Z_2 E & \Xi_{24i} & 0 & A_{di}^T W & 0 & \cdots & 0 \\ * & * & \Xi_{33i} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \Xi_{44i} & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & B_{\omega i}^T W & 0 & \cdots & 0 \\ * & * & * & * & * & -W & 0 & \cdots & 0 \\ * & * & * & * & * & * & -R_{i1} & \cdots & 0 \\ * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & \cdots & -R_{i(i-1)} \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{bmatrix}$$

$$\begin{array}{ccc} E^T(P_{i+1}-P_i) & \cdots & E^T(P_s-P_i) \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ -R_{i(i+1)} & \cdots & 0 \\ * & \ddots & \vdots \\ * & \cdots & -R_{is} \end{array}\Bigg] < 0, \tag{12}$$

*where*

$$\tilde{\Xi}_{11i} = P_i^T A_i + A_i^T P_i + \sum_{k=1}^{3} Q_k - d_{12} E^T Z_1 E + L_i^T L_i + \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T (P_j - P_i) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} T_{ij},$$

$$\bar{\Xi}_{11i} = P_i^T A_i + A_i^T P_i + \sum_{k=1}^{3} Q_k - d_{12} E^T Z_1 E + L_i^T L_i + \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T (P_j - P_l) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} V_{ijl},$$

$$\hat{\Xi}_{11i} = P_i^T A_i + A_i^T P_i + \sum_{k=1}^{3} Q_k - d_{12} E^T Z_1 E + L_i^T L_i + \sum_{j\in \mathcal{S}, j\neq i} \left( \hat{\pi}_{ij} E^T (P_j - P_i) + \frac{\delta_{ij}^2}{4} R_{ij} \right).$$

*And the other notations are defined as in Lemma 3.*

**Proof.** Now we prove that (8b) is guaranteed by inequalities (10)–(12) in three different cases respectively. And then based on Lemma 3, we can derive that system (1) with GUTRs is mean-square exponentially admissible. Firstly, we prove the regularity and impulse free of the system (1) with GUTRs. Since rank $E = r \le n$, there exist non-singular matrices $G$ and $H$ such that

$$GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{13}$$

Denote

$$GA_iH = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, \quad G^{-T} P_i H = \begin{bmatrix} \bar{P}_{i1} & \bar{P}_{i2} \\ \bar{P}_{i3} & \bar{P}_{i4} \end{bmatrix}. \tag{14}$$

From (9) and using the expressions in (13) and (14), it is easy to obtain that $\bar{P}_{i2} = 0$ for every $i \in \mathcal{S}$. Pre-multiplying and post-multiplying $\tilde{\Xi}_{11i} < 0$, $\bar{\Xi}_{11i} < 0$ and $\hat{\Xi}_{11i} < 0$ by $H^T$ and $H$, respectively. In all the three cases, we have

$$A_{i4}^T \bar{P}_{i4} + \bar{P}_{i4}^T A_{i4} < 0,$$

which implies $A_{i4}$ are non-singular for every $i \in \mathcal{S}$ and thus the pairs $(E, A_i)$ are regular and impulse free for every $i \in \mathcal{S}$. On the other hand, it can be seen from (10)–(12) that

*Case I.* If $i \notin U_K^i$ and $U_K^i = \{K_1^i, \ldots, K_m^i\}$, from (10), we know there exists

$$\begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix}^T \begin{bmatrix} \tilde{\Xi}_{11i} & \Xi_{12i} & 0 & 0 \\ * & \Xi_{22i} & E^T Z_2 E & \Xi_{24i} \\ * & * & \Xi_{33i} & 0 \\ * & * & * & \Xi_{44i} \end{bmatrix} \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} < 0,$$

which implies

$$\sum_{j\in U_K^i} \hat{\pi}_{ij} E^T (P_j - P_i) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} T_{ij} + P_i^T (A_i + A_{di}) + (A_i + A_{di})^T P_i < 0. \tag{15}$$

*Case II.* If $i \in U_K^i, U_{UK}^i \neq \emptyset$ and $U_K^i = \{K_1^i, \ldots, K_m^i\}$, from (11), we know there exists

$$\begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix}^T \begin{bmatrix} \bar{\Xi}_{11i} & \Xi_{12i} & 0 & 0 \\ * & \Xi_{22i} & E^T Z_2 E & \Xi_{24i} \\ * & * & \Xi_{33i} & 0 \\ * & * & * & \Xi_{44i} \end{bmatrix} \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} < 0,$$

which implies

$$\sum_{j\in U_K^i}\hat{\pi}_{ij}E^T(P_j-P_l)+\sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}V_{ijl}+P_i^T(A_i+A_{di})+(A_i+A_{di})^TP_i<0. \tag{16}$$

*Case III.* If $i\in U_K^i, U_{UK}^i=\emptyset$, from (12), we know there exists

$$\begin{bmatrix} I\\ I\\ I\\ I\end{bmatrix}^T\begin{bmatrix}\hat{\Xi}_{11i} & \Xi_{12i} & 0 & 0\\ * & \Xi_{22i} & E^TZ_2E & \Xi_{24i}\\ * & * & \Xi_{33i} & 0\\ * & * & * & \Xi_{44i}\end{bmatrix}\begin{bmatrix} I\\ I\\ I\\ I\end{bmatrix}<0,$$

which implies

$$\sum_{j\in\mathcal{S},j\neq i}\left(\hat{\pi}_{ij}E^T(P_j-P_i)+\frac{\delta_{ij}^2}{4}R_{ij}\right)+P_i^T(A_i+A_{di})+(A_i+A_{di})^TP_i<0. \tag{17}$$

According to Theorem 10.1 of [5], we have from (9) and (15)–(17) that the pairs $(E, A_i+A_{di})$ are regular, impulse free and stochastically stable for every $i\in\mathcal{S}$. Thus, by Definition 1, the system (1) with GUTRs is regular and impulse free for any time delay $d(t)$ satisfying (2).

Next, we will show that LMI (8b) is guaranteed by inequalities (10)–(12). Now, we denote

$$\Xi_i=\begin{bmatrix}\Xi'_{11i} & \Xi_{12i} & 0 & 0 & P_i^TB_{\omega i} & A_i^TW\\ * & \Xi_{22i} & E^TZ_2E & \Xi_{24i} & 0 & A_{di}^TW\\ * & * & \Xi_{33i} & 0 & 0 & 0\\ * & * & * & \Xi_{44i} & 0 & 0\\ * & * & * & * & -\gamma^2I & B_{\omega i}^TW\\ * & * & * & * & * & -W\end{bmatrix}+diag\left\{\sum_{j=1}^s\pi_{ij}E^TP,0,0,0,0,0\right\}.$$

where

$\Xi'_{11i}=P_i^TA_i+A_i^TP_i+\sum_{k=1}^3Q_k-d_{12}E^TZ_1E+L_i^TL_i.$

*Case I.* If $i\notin U_K^i$ and $U_K^i=\{K_1^i,\ldots,K_m^i\}$. Note that in this case, $\sum_{j\in U_{UK}^i,j\neq i}\pi_{ij}=-\pi_{ii}-\sum_{j\in U_K^i}\pi_{ij}$ and $\pi_{ij}\geq 0$, we have

$$\begin{aligned}\sum_{j=1}^s\pi_{ij}E^TP_j &= \sum_{j\in U_K^i}\pi_{ij}E^TP_j+\pi_{ii}E^TP_i+\sum_{j\in U_{UK}^i,j\neq i}\pi_{ij}E^TP_j\\ &\leq \sum_{j\in U_K^i}\pi_{ij}E^TP_j+\pi_{ii}E^TP_i+\left(-\pi_{ii}-\sum_{j\in U_K^i}\pi_{ij}\right)E^TP_i\\ &= \sum_{j\in U_K^i}\pi_{ij}E^T(P_j-P_i)\\ &= \sum_{j\in U_K^i}(\hat{\pi}_{ij}+\Delta_{ij})E^T(P_j-P_i)\quad (noticing\ \ \pi_{ij}=\hat{\pi}_{ij}+\Delta_{ij}, for\ \ j\in U_K^i)\\ &= \sum_{j\in U_K^i}\hat{\pi}_{ij}E^T(P_j-P_i)+\sum_{j\in U_K^i}\Delta_{ij}E^T(P_j-P_i)\end{aligned} \tag{18}$$

On the other hand, in view of Lemma 2, we have

$$\begin{aligned}\sum_{j\in U_K^i}\Delta_{ij}E^T(P_j-P_i) &= \sum_{j\in U_K^i}\left[\frac{1}{2}\Delta_{ij}E^T(P_j-P_i)+\frac{1}{2}\Delta_{ij}E^T(P_j-P_i)\right]\\ &\leq \sum_{j\in U_K^i}\left[\frac{\delta_{ij}^2}{4}T_{ij}+E^T(P_j-P_i)T_{ij}^{-1}(P_j-P_i)^TE\right]\end{aligned} \tag{19}$$

From Eqs. (18) and (19), we have

$$\sum_{j=1}^s\pi_{ij}E^TP_j\leq\sum_{j\in U_K^i}\hat{\pi}_{ij}E^T(P_j-P_i)+\sum_{j\in U_K^i}\left[\frac{\delta_{ij}^2}{4}T_{ij}+E^T(P_j-P_i)T_{ij}^{-1}(P_j-P_i)^TE\right] \tag{20}$$

Hence, according to Schur complement and inequality (10), we have $\Xi_i<0$, therefore, (8b) holds.

*Case II*. If $i \in U_K^i, U_{UK}^i \neq \emptyset$ and $U_K^i = \{K_1^i, \dots, K_m^i\}$. There must be an $l \in U_{UK}^i$, for $\forall j \in U_{UK}^i$, it holds that

$$\begin{aligned}\sum_{j=1}^{s} \pi_{ij} E^T P_j &\leq \sum_{j\in U_K^i} \pi_{ij} E^T P_j - \sum_{j\in U_K^i} \pi_{ij} E^T P_l \\ &= \sum_{j\in U_K^i} (\hat{\pi}_{ij} + \Delta_{ij}) E^T (P_j - P_l) \\ &= \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T (P_j - P_l) + \sum_{j\in U_K^i} \Delta_{ij} E^T (P_j - P_l)\end{aligned} \tag{21}$$

On the other hand, in view of Lemma 2, we have

$$\begin{aligned}\sum_{j\in U_K^i} \Delta_{ij} E^T (P_j - P_l) &= \sum_{j\in U_K^i} \left[ \frac{1}{2} \Delta_{ij} E^T (P_j - P_l) + \frac{1}{2} \Delta_{ij} E^T (P_j - P_l) \right] \\ &\leq \sum_{j\in U_K^i} \left[ \frac{\delta_{ij}^2}{4} V_{ijl} + E^T (P_j - P_l) V_{ijl}^{-1} (P_j - P_l)^T E \right]\end{aligned} \tag{22}$$

From Eqs. (21) and (22), we have

$$\sum_{j=1}^{s} \pi_{ij} E^T P_j \leq \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T (P_j - P_l) + \sum_{j\in U_K^i} \left[ \frac{\delta_{ij}^2}{4} V_{ijl} + E^T (P_j - P_l) V_{ijl}^{-1} (P_j - P_l)^T E \right], \quad \forall l \in U_{UK}^i \tag{23}$$

Finally, according to Schur complement and inequality (11), we have $\Xi_i < 0$, which means (8b) holds.

*Case III*. If $i \in U_K^i, U_{UK}^i = \emptyset$. Similarly, it holds that

$$\begin{aligned}\sum_{j=1}^{s} \pi_{ij} E^T P_j &= \sum_{j=1, j\neq i}^{s} \pi_{ij} E^T (P_j - P_i) \\ &= \sum_{j=1, j\neq i}^{s} \hat{\pi}_{ij} E^T (P_j - P_i) + \sum_{j=1, j\neq i}^{s} \Delta_{ij} E^T (P_j - P_i)\end{aligned} \tag{24}$$

On the other hand, in view of Lemma 2, we have

$$\sum_{j=1, j\neq i}^{s} \Delta_{ij} E^T (P_j - P_i) \leq \sum_{j=1, j\neq i}^{s} \left[ \frac{\delta_{ij}^2}{4} R_{ij} + E^T (P_j - P_i) R_{ij}^{-1} (P_j - P_i)^T E \right] \tag{25}$$

From Eqs. (24) and (25), we have

$$\sum_{j=1}^{s} \pi_{ij} E^T P_j \leq \sum_{j=1, j\neq i}^{s} \hat{\pi}_{ij} E^T (P_j - P_i) + \sum_{j=1, j\neq i}^{s} \left[ \frac{\delta_{ij}^2}{4} R_{ij} + E^T (P_j - P_i) R_{ij}^{-1} (P_j - P_i)^T E \right] \tag{26}$$

Therefore, According to Schur complement and inequality (12), we have $\Xi_i < 0$, which means (8b) holds.

So, if inequalities (10)–(12) hold, we conclude that system (1) with GUTRs is mean-square exponentially admissible based on Lemma 3. This completes the proof. □

Next, we will design an $H_\infty$ filtering of the form of (6). The following theorem presents a sufficient condition of the existence of the filtering for system (1) with GUTRs.

**Theorem 2.** *For given scalars $0 \leq d_1 < d_2$ and $\mu$, the filtering error system (7) is mean-square exponentially admissible with $H_\infty$ performance $\gamma$ for any time delay $d(t)$ satisfying (2), if there exist symmetric positive-definite matrices $S_j$, $j = 1, 2, 3$, $Z_l$, $l = 1, 2$ and matrices $X_i$, $U_i$, $\bar{A}_{fi}$, $\bar{B}_{fi}$, $\bar{C}_{fi}$, and small enough scalar $\sigma > 0$ such that for every $i \in \mathcal{S}$,*

$$E^T X_i = X_i^T E \geq 0, \tag{27}$$

$$E^T U_i = U_i^T E \geq 0, \tag{28}$$

$$E^T (X_i - U_i) = (X_i - U_i)^T E \geq 0, \tag{29}$$

*Case I. If $i \notin U_K^i$ and $U_K^i = \{K_1^i, \dots, K_m^i\}$, there exist a set of positive definite matrices $T_{ij} \in R^{n\times n} (i \notin U_K^i, j \in U_K^i)$ such that*

$$\begin{bmatrix} E^T ((X_j - X_i) - (U_j - U_i)) & E^T ((X_j - X_i) - (U_j - U_i)) \\ & E^T (X_j - X_i) \end{bmatrix} \leq 0, \quad j \in U_{UK}^i, j \neq i, \tag{30a}$$

*and*

$$
\begin{bmatrix}
\tilde{\Delta}_{1i} & \tilde{\Delta}_{2i} & \Delta_{4i} & 0 & 0 & (X_i-U_i)^T B_{\omega i} & A_i^T U & L_i^T-\bar{C}_{fi}^T & \tilde{\Pi}_{K_1^i} & 0 & \cdots & \tilde{\Pi}_{K_m^i} & 0 \\
* & \tilde{\Delta}_{3i} & \Delta_{5i} & 0 & 0 & X_i^T B_{\omega i}-\bar{B}_{fi} D_{\omega i} & A_i^T U & L_i^T & E^T(X_{K_1^i}-X_i) & E^T(U_i-U_{K_1^i}) & \cdots & E^T(X_{K_m^i}-X_i) & E^T(U_i-U_{K_m^i}) \\
* & * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & A_{di}^T U & 0 & 0 & 0 & \cdots & 0 & 0 \\
* & * & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
* & * & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I & B_{\omega i}^T U & 0 & 0 & 0 & \cdots & 0 & 0 \\
* & * & * & * & * & * & -U & 0 & 0 & 0 & \cdots & 0 & 0 \\
* & * & * & * & * & * & * & -I & 0 & 0 & \cdots & 0 & 0 \\
* & * & * & * & * & * & * & * & -T_{iK_1^i} & 0 & \cdots & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\sigma I & \cdots & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \ddots & \vdots & 0 \\
 & & & & & & & & * & * & \cdots & -T_{iK_m^i} & 0 \\
 & & & & & & & & * & * & \cdots & * & -\sigma I
\end{bmatrix} < 0, \tag{30b}
$$

*where* $d_{12}=d_2-d_1$, $U=d_{12}d_2^2 Z_1+d_{12}^2 Z_2$ *and*

$$
\begin{aligned}
\tilde{\Delta}_{1i} &= A_i^T(X_i-U_i)+(X_i-U_i)^T A_i+\sum_{j\in U_K^i}\hat{\pi}_{ij}E^T((X_j-X_i)-(U_j-U_i))+\sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}T_{ij}-d_{12}E^T Z_1 E+\sum_{k=1}^{3}S_k,\\
\tilde{\Delta}_{2i} &= A_i^T X_i-C_i^T\bar{B}_{fi}^T-\bar{A}_{fi}^T+(X_i-U_i)^T A_i+\sum_{j\in U_K^i}\hat{\pi}_{ij}E^T((X_j-X_i)-(U_j-U_i))+\sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}T_{ij}-d_{12}E^T Z_1 E+\sum_{k=1}^{3}S_k,\\
\tilde{\Delta}_{3i} &= A_i^T X_i-C_i^T\bar{B}_{fi}^T+X_i^T A_i-\bar{B}_{fi}C_i+\sum_{j\in U_K^i}\hat{\pi}_{ij}E^T(X_j-X_i)+\sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}T_{ij}-d_{12}E^T Z_1 E+\sum_{k=1}^{3}S_k,\\
\Delta_{4i} &= (X_i-U_i)^T A_{di}+d_{12}E^T Z_1 E, \quad \Delta_{5i}=X_i^T A_{di}-\bar{B}_{fi}C_{di}+d_{12}E^T Z_1 E,\\
\Delta_{6i} &= -(1-\mu)S_3-E^T((d_{12}+d_2)Z_1+2Z_2)E, \quad \Delta_{7i}=-S_1-E^T Z_2 E,\\
\Delta_{8i} &= E^T(d_2 Z_1+Z_2)E, \quad \Delta_{9i}=-S_2-E^T(d_2 Z_1+Z_2)E,\\
\tilde{\Pi}_j &= E^T((X_j-X_i)-(U_j-U_i)) \quad j\in U_K^i.
\end{aligned}
$$

*Case II. If* $i \in U_K^i, U_{UK}^i \neq \emptyset$ *and* $U_K^i = \left\{K_1^i, \dots, K_s^i\right\}$, *there exist a set of positive definite matrices* $V_{ijl} \in R^{n\times n}(i, j \in U_K^i, l \in U_{UK}^i)$ *such that*

$$
\left[\begin{array}{cccccccc}
\bar{\Delta}_{1i} & \bar{\Delta}_{2i} & \Delta_{4i} & 0 & 0 & (X_i - U_i)^T B_{\omega i} & A_i^T U & L_i^T - \bar{C}_{fi}^T \\
* & \bar{\Delta}_{3i} & \Delta_{5i} & 0 & 0 & X_i^T B_{\omega i} - \bar{B}_{fi} D_{\omega i} & A_i^T U & L_i^T \\
* & * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & A_{di}^T U & 0 \\
* & * & * & \Delta_{7i} & 0 & 0 & 0 & 0 \\
* & * & * & * & \Delta_{9i} & 0 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I & B_{\omega i}^T U & 0 \\
* & * & * & * & * & * & -U & 0 \\
* & * & * & * & * & * & * & -I \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{array}\right.
$$

$$
\left.\begin{array}{ccccc}
\bar{\Pi}_{K_1^i} & 0 & \cdots & \bar{\Pi}_{K_s^i} & 0 \\
E^T(X_{K_1^i} - X_l) & E^T(U_l - U_{K_1^i}) & \cdots & E^T(X_{K_s^i} - X_l) & E^T(U_l - U_{K_s^i}) \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
-V_{iK_1^i l} & 0 & \cdots & 0 & 0 \\
* & -\sigma I & \cdots & 0 & 0 \\
* & * & \ddots & \vdots & 0 \\
* & * & \cdots & -V_{iK_s^i l} & 0 \\
* & * & \cdots & * & -\sigma I
\end{array}\right] < 0, \tag{31}
$$

*where*

$$
\bar{\Delta}_{1i} = A_i^T(X_i - U_i) + (X_i - U_i)^T A_i + \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T((X_j - X_l) - (U_j - U_l)) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} V_{ijl} - d_{12} E^T Z_1 E + \sum_{k=1}^{3} S_k,
$$

$$
\bar{\Delta}_{2i} = A_i^T X_i - C_i^T \bar{B}_{fi}^T - \bar{A}_{fi}^T + (X_i - U_i)^T A_i + \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T((X_j - X_l) - (U_j - U_l)) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} V_{ijl} - d_{12} E^T Z_1 E + \sum_{k=1}^{3} S_k,
$$

$$
\bar{\Delta}_{3i} = A_i^T X_i - C_i^T \bar{B}_{fi}^T + X_i^T A_i - \bar{B}_{fi} C_i + \sum_{j\in U_K^i} \hat{\pi}_{ij} E^T(X_j - X_l) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} V_{ijl} - d_{12} E^T Z_1 E + \sum_{k=1}^{3} S_k,
$$

$$
\bar{\Pi}_j = E^T((X_j - X_l) - (U_j - U_l)) \quad j \in U_K^i.
$$

*Case III. If $i \in U_K^i, U_{UK}^i = \emptyset$, there exist a set of positive definite matrices $R_{ij} \in R^{n\times n}(i, j \in U_K^i)$ such that*

$$
\left[\begin{array}{cccccccc}
\hat{\Delta}_{1i} & \hat{\Delta}_{2i} & \Delta_{4i} & 0 & 0 & (X_i-U_i)^T B_{\omega i} & A_i^T U & L_i^T - \bar{C}_{fi}^T \\
* & \hat{\Delta}_{3i} & \Delta_{5i} & 0 & 0 & X_i^T B_{\omega i} - \bar{B}_{fi} D_{\omega i} & A_i^T U & L_i^T \\
* & * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & A_{di}^T U & 0 \\
* & * & * & \Delta_{7i} & 0 & 0 & 0 & 0 \\
* & * & * & * & \Delta_{9i} & 0 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I & B_{\omega i}^T U & 0 \\
* & * & * & * & * & * & -U & 0 \\
* & * & * & * & * & * & * & -I \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{array}\right.
$$

$$
\begin{array}{ccccc}
\hat{\Pi}_1 & 0 & \cdots & \hat{\Pi}_{i-1} & 0 \\
E^T(X_1 - X_i) & E^T(U_i - U_1) & \cdots & E^T(X_{i-1} - X_i) & E^T(U_i - U_{i-1}) \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
-R_{i1} & 0 & \cdots & 0 & 0 \\
* & -\sigma I & \cdots & 0 & 0 \\
* & * & \ddots & \vdots & 0 \\
* & * & \cdots & -R_{i(i-1)} & 0 \\
* & * & \cdots & * & -\sigma I
\end{array}
$$

$$
\left.\begin{array}{ccccc}
\hat{\Pi}_{i+1} & 0 & \cdots & \hat{\Pi}_s & 0 \\
E^T(X_{i+1} - X_i) & E^T(U_i - U_{i+1}) & \cdots & E^T(X_s - X_i) & E^T(U_i - U_s) \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
-R_{i(i+1)} & 0 & \cdots & 0 & 0 \\
* & -\sigma I & \cdots & 0 & 0 \\
* & * & \ddots & \vdots & 0 \\
* & * & \cdots & -R_{is} & 0 \\
* & * & \cdots & * & -\sigma I
\end{array}\right] < 0, \tag{32}
$$

*where*

$$\hat{\Delta}_{1i} = A_i^T(X_i - U_i) + (X_i - U_i)^T A_i + \sum_{j\in\mathcal{S}, j\neq i}\left(\hat{\pi}_{ij}E^T((X_j - X_i) - (U_j - U_i)) + \frac{\delta_{ij}^2}{4}R_{ij}\right) - d_{12}E^T Z_1 E + \sum_{k=1}^{3} S_k,$$

$$\hat{\Delta}_{2i} = A_i^T X_i - C_i^T \bar{B}_{fi}^T - \bar{A}_{fi}^T + (X_i - U_i)^T A_i + \sum_{j\in\mathcal{S}, j\neq i}\left(\hat{\pi}_{ij}E^T((X_j - X_i) - (U_j - U_i)) + \frac{\delta_{ij}^2}{4}R_{ij}\right) - d_{12}E^T Z_1 E + \sum_{k=1}^{3} S_k,$$

$$\hat{\Delta}_{3i} = A_i^T X_i - C_i^T \bar{B}_{fi}^T + X_i^T A_i - \bar{B}_{fi}C_i + \sum_{j\in\mathcal{S}, j\neq i}(\hat{\pi}_{ij}E^T(X_j - X_i) + \frac{\delta_{ij}^2}{4}R_{ij}) - d_{12}E^T Z_1 E + \sum_{k=1}^{3} S_k,$$

$$\hat{\Pi}_j = E^T((X_j - X_i) - (U_j - U_i)) \quad j \in \mathcal{S}, j \neq i.$$

*Moreover, the parameters of the desired filter can be chosen by*

$$A_{fi} = U_i^{-T}\bar{A}_{fi},\ B_{fi} = U_i^{-T}\bar{B}_{fi},\ C_{fi} = \bar{C}_{fi},\ E_f = E. \tag{33}$$

**Proof.** Considering (30b)–(32), we have that

*Case I.* If $i \notin U_K^i$ and $U_K^i = \{K_1^i, \ldots, K_m^i\}$, from (30b), we know there exists

$$\begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}_{1i} & \Delta_{4i} & 0 & 0 \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} \\ * & * & \Delta_{7i} & 0 \\ * & * & * & \Delta_{9i} \end{bmatrix} \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} < 0,$$

which implies

$$(A_i + A_{di})^T(X_i - U_i) + (X_i - U_i)^T(A_i + A_{di}) + \sum_{j\in U_K^i}\hat{\pi}_{ij}E^T((X_j - X_i) - (U_j - U_i)) + \sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}T_{ij} < 0. \tag{34}$$

*Case II.* If $i \in U_K^i, U_{UK}^i \neq \emptyset$ and $U_K^i = \{K_1^i, \ldots, K_s^i\}$, from (31), we know there exists

$$\begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix}^T \begin{bmatrix} \bar{\Delta}_{1i} & \Delta_{4i} & 0 & 0 \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} \\ * & * & \Delta_{7i} & 0 \\ * & * & * & \Delta_{9i} \end{bmatrix} \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} < 0,$$

which implies

$$(A_i + A_{di})^T(X_i - U_i) + (X_i - U_i)^T(A_i + A_{di}) + \sum_{j\in U_K^i}\hat{\pi}_{ij}E^T((X_j - X_l) - (U_j - U_l)) + \sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}V_{ijl} < 0. \tag{35}$$

*Case III.* If $i \in U_K^i, U_{UK}^i = \emptyset$, from (32), we know there exists

$$\begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix}^T \begin{bmatrix} \hat{\Delta}_{1i} & \Delta_{4i} & 0 & 0 \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} \\ * & * & \Delta_{7i} & 0 \\ * & * & * & \Delta_{9i} \end{bmatrix} \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} < 0,$$

which implies

$$(A_i + A_{di})^T(X_i - U_i) + (X_i - U_i)^T(A_i + A_{di}) + \sum_{j\in\mathcal{S}, j\neq i}(\hat{\pi}_{ij}E^T((X_j - X_l) - (U_j - U_l)) + \frac{\delta_{ij}^2}{4}R_{ij}) < 0. \tag{36}$$

Using the same approach in Theorem 1, we can find from (29) and (34)–(36) that $X_j - U_j$ are non-singular for every $i \in \mathcal{S}$. Define

$$P_i = \begin{bmatrix} X_i & -U_i \\ -U_i & U_i \end{bmatrix},\ J_i = \begin{bmatrix} (X_i - U_i)^{-1} & I \\ (X_i - U_i)^{-1} & 0 \end{bmatrix},\ \hat{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix},\ H = \begin{bmatrix} I \\ 0 \end{bmatrix}^T.$$

$$\bar{T}_{ij} = \begin{bmatrix} T_{ij} & 0 \\ 0 & \sigma I \end{bmatrix}, \bar{V}_{ijl} = \begin{bmatrix} V_{ijl} & 0 \\ 0 & \sigma I \end{bmatrix}, \bar{R}_{ij} = \begin{bmatrix} R_{ij} & 0 \\ 0 & \sigma I \end{bmatrix},$$

Then noting (27)–(29), we have

$$\hat{E}^T P_i = \begin{bmatrix} E^T X_i & -E^T U_i \\ -E^T U_i & E^T U_i \end{bmatrix} = P_i^T \hat{E}.$$

It can be deduced from (29) that

$$E^T X_i - (-E^T U_i)(E^T U_i)^+(-E^T U_i) = E^T(X_i - U_i) \geq 0, \tag{37}$$

and

$$\begin{aligned} -E^T U_i(I - (-E^T U_i)(E^T U_i)^+) &= -E^T U_i + (E^T U_i)[(E^T U_i)]^T[(E^T U_i)^+]^T \\ &= -E^T U_i + (E^T U_i)[(E^T U_i)^+(E^T U_i)]^T \\ &= -E^T U_i + E^T U_i = 0. \end{aligned} \tag{38}$$

Considering (28), (37) and (38), and using Lemma 1, we have

$$\hat{E}^T P_i = P_i^T \hat{E} \geq 0. \tag{39}$$

Now, pre-multiplying and post-multiplying (30b)–(32) by $diag\{(X_i - U_i)^{-T}, I, I, I, I, I, \ldots, I\}$ and its transpose, respectively, we obtain

*Case I.* If $i \notin U_K^i$ and $U_K^i = \{K_1^i, \ldots, K_m^i\}$, there exists

$$\begin{bmatrix} \tilde{\Sigma}_{1i} & \Sigma_{2i} & 0 & 0 & J_i^T P_i^T \bar{B}_{\omega i} & J_i^T \bar{A}_i^T H^T U & J_i^T \bar{L}_i^T & J_i^T \hat{E}^T(P_{K_1^i} - P_i) & \cdots & J_i^T \hat{E}^T(P_{K_m^i} - P_i) \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & H\bar{A}_{di}^T H^T U & 0 & 0 & \cdots & 0 \\ * & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T H^T U & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & -U & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & -I & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & -\bar{T}_{iK_1^i} & \cdots & 0 \\ * & * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & * & \cdots & -\bar{T}_{iK_m^i} \end{bmatrix} < 0, \tag{40}$$

where

$$\tilde{\Sigma}_{1i} = J_i^T \left( \sum_{j \in U_K^i} \hat{\pi}_{ij} \hat{E}^T(P_j - P_i) + \sum_{j \in U_K^i} \frac{\delta_{ij}^2}{4} H^T T_{ij} H + P_i^T \bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^{3} H^T S_k H - d_{12} \hat{E}^T H^T Z_1 H \hat{E} \right) J_i,$$

$$\Sigma_{2i} = J_i^T(P_i^T \bar{A}_{di} H^T + d_{12} \hat{E}^T H^T Z_1 \hat{E}),$$

*Case II.* If $i \in U_K^i, U_{UK}^i \neq \emptyset$ and $U_K^i = \{K_1^i, \ldots, K_s^i\}$, there exists

$$\begin{bmatrix} \bar{\Sigma}_{1i} & \Sigma_{2i} & 0 & 0 & J_i^T P_i^T \bar{B}_{\omega i} & J_i^T \bar{A}_i^T H^T U & J_i^T \bar{L}_i^T & J_i^T \hat{E}^T(P_{K_1^i} - P_l) & \cdots & J_i^T \hat{E}^T(P_{K_s^i} - P_l) \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & H\bar{A}_{di}^T H^T U & 0 & 0 & \cdots & 0 \\ * & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T H^T U & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & -U & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & -I & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & -\bar{V}_{iK_1^i l} & \cdots & 0 \\ * & * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & * & \cdots & -\bar{V}_{iK_s^i l} \end{bmatrix} < 0, \tag{41}$$

where

$$\bar{\Sigma}_{1i} = J_i^T \left( \sum_{j \in U_K^i} \hat{\pi}_{ij} \hat{E}^T(P_j - P_l) + \sum_{j \in U_K^i} \frac{\delta_{ij}^2}{4} H^T V_{ijl} H + P_i^T \bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^{3} H^T S_k H - d_{12} \hat{E}^T H^T Z_1 H \hat{E} \right) J_i,$$

*Case III*. If $i \in U_K^i, U_{UK}^i = \emptyset,$ there exists

$$\begin{bmatrix} \hat{\Sigma}_{1i} & \Sigma_{2i} & 0 & 0 & J_i^T P_i^T \bar{B}_{\omega i} & J_i^T \bar{A}_i^T H^T U & J_i^T \bar{L}_i^T & J_i^T \hat{E}^T(P_1 - P_i) & \cdots & J_i^T \hat{E}^T(P_{i-1} - P_i) & J_i^T \hat{E}^T(P_{i+1} - P_i) & \cdots & J_i^T \hat{E}^T(P_s - P_i) \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & H\bar{A}_{di}^T H^T U & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T H^T U & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & -U & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & -I & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & -\bar{R}_{i1} & \cdots & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & * & \ddots & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & * & \cdots & -\bar{R}_{i(i-1)} & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & * & \cdots & * & -\bar{R}_{i(i+1)} & \cdots & 0 \\ * & * & * & * & * & * & * & * & \cdots & * & * & \ddots & \vdots \\ * & * & * & * & * & * & * & * & \cdots & * & * & \cdots & -\bar{R}_{is} \end{bmatrix} < 0, \tag{42}$$

where $\hat{\Sigma}_{1i} = J_i^T(\sum_{j\in\mathcal{S}, j\neq i}(\hat{\pi}_{ij}\hat{E}^T(P_j - P_i) + \frac{\delta_{ij}^2}{4}H^T R_{ij} H) + P_i^T \bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^3 H^T S_k H - d_{12}\hat{E}^T H^T Z_1 H\hat{E})J_i$, and the matrices $\bar{A}_i$, $\bar{A}_{di}$, $\bar{B}_{\omega i}$ and $\bar{L}_i$ are given in (7) with the parameters $E_f$, $A_{fi}$, $B_{fi}$ and $C_{fi}$ are given in (33), respectively. Then pre-multiplying and post-multiplying (40)–(42) by $diag\{J_i^{-T}, I, I, I, I, \ldots, I\}$ and its transpose, respectively, we obtain

*Case I*. If $i \notin U_K^i$ and $U_K^i = \{K_1^i, \ldots, K_m^i\}$, there exists

$$\begin{bmatrix} \tilde{\Upsilon}_{1i} & \Upsilon_{2i} & 0 & 0 & P_i^T \bar{B}_{\omega i} & \bar{A}_i^T H^T U & \bar{L}_i^T & \hat{E}^T(P_{K_1^i} - P_i) & \cdots & \hat{E}^T(P_{K_m^i} - P_i) \\ * & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & H\bar{A}_{di}^T H^T U & 0 & 0 & \cdots & 0 \\ * & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T H^T U & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & -U & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & -I & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & -\bar{T}_{iK_1^i} & \cdots & 0 \\ * & * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & * & \cdots & -\bar{T}_{iK_m^i} \end{bmatrix} < 0, \tag{43}$$

where

$$
\begin{aligned}
\tilde{\Upsilon}_{1i} &= \sum_{j\in U_K^i} \hat{\pi}_{ij}\hat{E}^T(P_j - P_i) + \sum_{j\in U_K^i} \frac{\delta_{ij}^2}{4} H^T T_{ij} H + P_i^T \bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^{3} H^T S_k H - d_{12}\hat{E}^T H^T Z_1 H \hat{E},\\
\Upsilon_{2i} &= P_i^T \bar{A}_{di} H^T + d_{12}\hat{E}^T H^T Z_1 E.
\end{aligned}
$$

*Case II.* If $i \in U_K^i, U_{UK}^i \neq \emptyset$ and $U_K^i = \left\{K_1^i, \ldots, K_s^i\right\}$, there exists

$$
\begin{bmatrix}
\bar{\Upsilon}_{1i} & \Upsilon_{2i} & 0 & 0 & P_i^T\bar{B}_{\omega i} & \bar{A}_i^T H^T U & \bar{L}_i^T & \hat{E}^T(P_{K_1^i} - P_l) & \cdots & \hat{E}^T(P_{K_m^i} - P_l)\\
* & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & H\bar{A}_{di}^T H^T U & 0 & 0 & \cdots & 0\\
* & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
* & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & \cdots & 0\\
* & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T H^T U & 0 & 0 & \cdots & 0\\
* & * & * & * & * & -U & 0 & 0 & \cdots & 0\\
* & * & * & * & * & * & -I & 0 & \cdots & 0\\
* & * & * & * & * & * & * & -\bar{V}_{iK_1^i l} & \cdots & 0\\
* & * & * & * & * & * & * & * & \ddots & 0\\
* & * & * & * & * & * & * & * & \cdots & -\bar{V}_{iK_s^i l}
\end{bmatrix} < 0, \tag{44}
$$

where

$$
\bar{\Upsilon}_{1i} = \sum_{j\in U_k^i} \hat{\pi}_{ij}\hat{E}^T(P_j - P_l) + \sum_{j\in U_k^i} \frac{\delta_{ij}^2}{4} H^T V_{ijl} H + P_i^T \bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^{3} H^T S_k H - d_{12}\hat{E}^T H^T Z_1 H \hat{E},
$$

*Case III.* If $i \in U_K^i, U_{UK}^i = \emptyset$, there exists

$$
\begin{bmatrix}
\hat{\Upsilon}_{1i} & \Upsilon_{2i} & 0 & 0 & P_i^T\bar{B}_{\omega i} & \bar{A}_i^T H^T U & \bar{L}_i^T & \hat{E}^T(P_1 - P_i) & \cdots & \hat{E}^T(P_{i-1} - P_i)\\
* & \Delta_{6i} & E^T Z_2 E & \Delta_{8i} & 0 & H\bar{A}_{di}^T H^T U & 0 & 0 & \cdots & 0\\
* & * & \Delta_{7i} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
* & * & * & \Delta_{9i} & 0 & 0 & 0 & 0 & \cdots & 0\\
* & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T H^T U & 0 & 0 & \cdots & 0\\
* & * & * & * & * & -U & 0 & 0 & \cdots & 0\\
* & * & * & * & * & * & -I & 0 & \cdots & 0\\
* & * & * & * & * & * & * & -\bar{R}_{i1} & \cdots & 0\\
* & * & * & * & * & * & * & * & \ddots & 0\\
* & * & * & * & * & * & * & * & \cdots & -\bar{R}_{i(i-1)}\\
* & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & *
\end{bmatrix}
$$

$$\left.\begin{matrix} \hat{E}^T(P_{i+1}-P_i) & \cdots & \hat{E}^T(P_s-P_i) \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ -\bar{R}_{i(i+1)} & \cdots & 0 \\ * & \ddots & \vdots \\ * & \cdots & -\bar{R}_{is} \end{matrix}\right] < 0, \tag{45}$$

where

$$\hat{\Upsilon}_{1i} = \sum_{j\in\mathcal{S},j\neq i}\left(\hat{\pi}_{ij}\hat{E}^T(P_j-P_i)+\frac{\delta_{ij}^2}{4}H^TR_{ij}H\right)+P_i^T\bar{A}_i+\bar{A}_i^TP_i+\sum_{k=1}^{3}H^TS_kH-d_{12}\hat{E}^TH^TZ_1H\hat{E},$$

Since $\sigma > 0$ is small enough, so we have that

*Case I*. If $i \notin U_K^i$ and $U_K^i = \left\{K_1^i, \ldots, K_m^i\right\}$, there exists

$$\begin{bmatrix} \tilde{\Lambda}_{1i} & \Lambda_{2i} & 0 & 0 & P_i^T\bar{B}_{\omega i} & \bar{A}_i^T\bar{U} & \bar{L}_i^T & \hat{E}^T(P_{K_1^i}-P_i) & \cdots & \hat{E}^T(P_{K_m^i}-P_i) \\ * & \Lambda_{3i} & \hat{E}^T\bar{Z}_2\hat{E} & \hat{E}^T(d_2\bar{Z}_1+\bar{Z}_2)\hat{E} & 0 & \bar{A}_{di}^T\bar{U} & 0 & 0 & \cdots & 0 \\ * & * & -\bar{S}_1-\hat{E}^T\bar{Z}_2\hat{E} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & -\bar{S}_2-\hat{E}^T(d_2\bar{Z}_1+\bar{Z}_2)\hat{E} & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma^2I & \bar{B}_{\omega i}^T\bar{U} & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & -\bar{U} & 0 & 0 & \cdots & 0 \\ * & * & * & * & * & * & -I & 0 & \cdots & 0 \\ * & * & * & * & * & * & * & -\bar{T}_{iK_1^i} & \cdots & 0 \\ * & * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & * & \cdots & -\bar{T}_{iK_m^i} \end{bmatrix} < 0, \tag{46}$$

where

$$\tilde{\Lambda}_{1i} = \sum_{j\in U_K^i}\hat{\pi}_{ij}\hat{E}^T(P_j-P_i)+\sum_{j\in U_K^i}\frac{\delta_{ij}^2}{4}\bar{T}_{ij}+P_i^T\bar{A}_i+\bar{A}_i^TP_i+\sum_{k=1}^{3}\bar{S}_k-d_{12}\hat{E}^T\bar{Z}_1\hat{E},$$

$$\Lambda_{2i} = P_i^T\bar{A}_{di}+d_{12}\hat{E}^T\bar{Z}_1\hat{E},$$

$$\begin{aligned}\Lambda_{3i} &= -(1-\mu)\bar{S}_3 - \hat{E}^T((d_{12}+d_2)\bar{Z}_1 + 2\bar{Z}_2)\hat{E},\\ \bar{S}_l &= \begin{bmatrix} S_l & 0\\ 0 & \sigma I\end{bmatrix},\quad l=1,2,3,\\ \bar{Z}_k &= \begin{bmatrix} Z_k & 0\\ 0 & \sigma I\end{bmatrix},\quad k=1,2,\\ \bar{U} &= d_{12}d_2^2\bar{Z}_1^2 + d_{12}^2\bar{Z}_2.\end{aligned}$$

*Case II.* If $i \in U_K^i, U_{UK}^i \neq \emptyset$ and $U_K^i = \{K_1^i, \ldots, K_s^i\}$, there exists

$$\begin{bmatrix}
\bar{\Lambda}_{1i} & \Lambda_{2i} & 0 & 0 & P_i^T\bar{B}_{\omega i} & \bar{A}_i^T\bar{U} & \bar{L}_i^T & \hat{E}^T(P_{K_1^i}-P_l) & \cdots & \hat{E}^T(P_{K_s^i}-P_l)\\
* & \Lambda_{3i} & \hat{E}^T\bar{Z}_2\hat{E} & \hat{E}^T(d_2\bar{Z}_1+\bar{Z}_2)\hat{E} & 0 & \bar{A}_{di}^T\bar{U} & 0 & 0 & \cdots & 0\\
* & * & -\bar{S}_1-\hat{E}^T\bar{Z}_2\hat{E} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
* & * & * & -\bar{S}_2-\hat{E}^T(d_2\bar{Z}_1+\bar{Z}_2)\hat{E} & 0 & 0 & 0 & 0 & \cdots & 0\\
* & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T\bar{U} & 0 & 0 & \cdots & 0\\
* & * & * & * & * & -\bar{U} & 0 & 0 & \cdots & 0\\
* & * & * & * & * & * & -I & 0 & \cdots & 0\\
* & * & * & * & * & * & * & -\bar{V}_{iK_1^i l} & \cdots & 0\\
* & * & * & * & * & * & * & * & \ddots & 0\\
* & * & * & * & * & * & * & * & \cdots & -\bar{V}_{iK_s^i l}
\end{bmatrix} < 0, \tag{47}$$

where

$$\bar{\Lambda}_{1i} = \sum_{j\in U_k^i}\hat{\pi}_{ij}\hat{E}^T(P_j-P_l) + \sum_{j\in U_k^i}\frac{\delta_{ij}^2}{4}\bar{V}_{ijl} + P_i^T\bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^{3}\bar{S}_k - d_{12}\hat{E}^T\bar{Z}_1\hat{E},$$

*Case III.* If $i \in U_K^i, U_{UK}^i = \emptyset$, there exists

$$
\left[\begin{matrix}
\hat{\Lambda}_{1i} & \Lambda_{2i} & 0 & 0 & P_i^T \bar{B}_{\omega i} & \bar{A}_i^T \bar{U} & \bar{L}_i^T \\
* & \Lambda_{3i} & \hat{E}^T \bar{Z}_2 \hat{E} & \hat{E}^T (d_2 \bar{Z}_1 + \bar{Z}_2) \hat{E} & 0 & \bar{A}_{di}^T \bar{U} & 0 \\
* & * & -\bar{S}_1 - \hat{E}^T \bar{Z}_2 \hat{E} & 0 & 0 & 0 & 0 \\
* & * & * & -\bar{S}_2 - \hat{E}^T (d_2 \bar{Z}_1 + \bar{Z}_2) \hat{E} & 0 & 0 & 0 \\
* & * & * & * & -\gamma^2 I & \bar{B}_{\omega i}^T \bar{U} & 0 \\
* & * & * & * & * & -\bar{U} & 0 \\
* & * & * & * & * & * & -I \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
\hat{E}^T (P_1 - P_i) & \cdots & \hat{E}^T (P_{i-1} - P_i) & \hat{E}^T (P_{i+1} - P_i) & \cdots & \hat{E}^T (P_s - P_i) \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
-\bar{R}_{i1} & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
* & \cdots & -\bar{R}_{i(i-1)} & 0 & \cdots & 0 \\
* & \cdots & * & -\bar{R}_{i(i+1)} & \cdots & 0 \\
* & \cdots & * & * & \ddots & \vdots \\
* & \cdots & * & * & \cdots & -\bar{R}_{is}
\end{matrix}\right] < 0, \tag{48}
$$

where

$$
\hat{\Lambda}_{1i} = \sum_{j \in \mathcal{S}, j \neq i} \left( \hat{\pi}_{ij} \hat{E}^T (P_j - P_i) + \frac{\delta_{ij}^2}{4} \bar{R}_{ij} \right) + P_i^T \bar{A}_i + \bar{A}_i^T P_i + \sum_{k=1}^{3} \bar{S}_k - d_{12} \hat{E}^T \bar{Z}_1 \hat{E},
$$

Therefore, by Schur complement and Theorem 1, the filtering error system (7) is mean square exponentially admissible with $H_\infty$ performance $\gamma$ for any time delay $d(t)$ satisfying (2). This completes the proof. □

## 4. Numerical example

In this section, we shall give a numerical example to demonstrate the applicability of the proposed approaches. Consider the singular Markovian jump time-delay system (1) with three modes, and suppose the transition probability matrix is given by

$$
\begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} = \begin{bmatrix} ? & 1.4 + \Delta_{12} & ? \\ ? & -2.5 + \Delta_{22} & ? \\ 0.8 + \Delta_{31} & 1.9 + \Delta_{32} & -2.7 + \Delta_{33} \end{bmatrix}.
$$

where $\Delta_{12} \in [-0.1, 0, 1]$, $\Delta_{22} \in [-0.15, 0.15]$, $\Delta_{31}$, $\Delta_{32} \in [-0.13, 0.13]$, $\Delta_{33} \in [-0.26, 0.26]$.

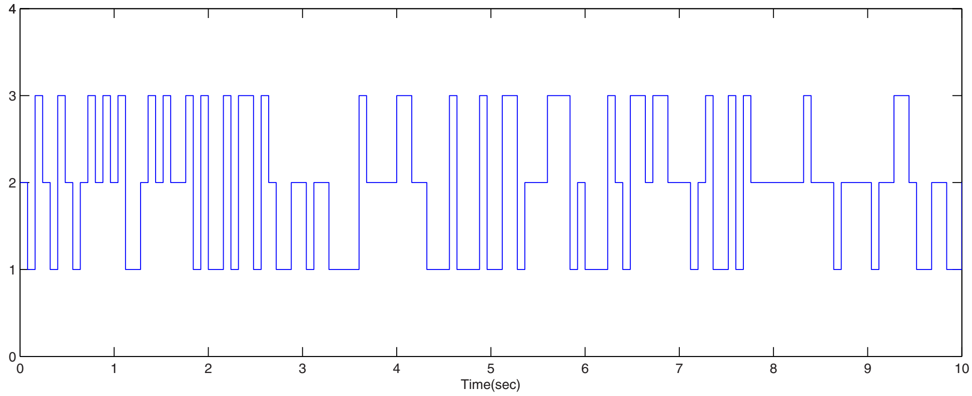


**Fig. 1.** System jumping modes.

The system parameters are given as follows: for mode 1, the dynamics of the system are described as

$$A_1 = \begin{bmatrix} -8.1523 & -1.4521 \\ 2.2013 & -3.2098 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.3125 & -0.5122 \\ 0.2 & -0.4 \end{bmatrix},\ B_{\omega 1} = \begin{bmatrix} -1.3159 \\ -0.1921 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -0.5215 & 1.4327 \end{bmatrix},\ C_{d1} = \begin{bmatrix} -0.5215 & 1.4327 \end{bmatrix},\ D_{\omega 1} = 2.2121,\ L_1 = \begin{bmatrix} -0.9800 & -1.1210 \end{bmatrix}.$$

For mode 2, the dynamics of the system are described as

$$A_2 = \begin{bmatrix} -4.2111 & -1.4321 \\ -1 & -2.8649 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.8001 & 1.5421 \\ -0.35 & -0.3 \end{bmatrix},\ B_{\omega 2} = \begin{bmatrix} 1.500 \\ -0.1200 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} -0.2197 & 1.5421 \end{bmatrix},\ C_{d2} = \begin{bmatrix} -0.2197 & 1.5421 \end{bmatrix},\ D_{\omega 2} = 2.5490,\ L_2 = \begin{bmatrix} -0.9721 & -1.5412 \end{bmatrix}.$$

For mode 3, the dynamics of the system are described as

$$A_3 = \begin{bmatrix} -5.5082 & -1.9563 \\ 0.8825 & -2.5005 \end{bmatrix},\ A_{d3} = \begin{bmatrix} -0.8049 & -1.0731 \\ 0.9856 & -1.5 \end{bmatrix},\ B_{\omega 3} = \begin{bmatrix} -1.0001 \\ 0.7812 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} -0.3657 & 1.0301 \end{bmatrix},\ C_{d3} = \begin{bmatrix} 0.5056 & -1.4432 \end{bmatrix},\ D_{\omega 2} = 2.3908,\ L_3 = \begin{bmatrix} 0.8532 & 1.3306 \end{bmatrix}.$$

In this example, we assume

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$d_1 = 0.1,\ d_2 = 0.5$ and $\mu = 0.3$. The purpose is the design of a delay-dependent $H_\infty$ filter in the form of (6) such that the filtering error system (7) achieves mean-square exponentially admissible with $H_\infty$ performance $\gamma = 0.5$. Considering transition probability matrix, we know that $U_K^1 = \{2\},\ U_K^2 = \{2\},\ U_K^3 = \{1, 2, 3\}$, so we need to solve inequalities (30b)–(32) simultaneously. For equality constraints (27)–(29), we use the same method in [34], and for simplicity, the $H_\infty$ filter is given as follows:

$$A_{f1} = \begin{bmatrix} -20.8088 & -9.2563 \\ 7.2596 & -9.1019 \end{bmatrix},\ B_{f1} = \begin{bmatrix} -0.3876 \\ -1.2289 \end{bmatrix},\ C_{f1} = \begin{bmatrix} 0.5000 & 0.7006 \end{bmatrix},$$
$$A_{f2} = \begin{bmatrix} -15.8109 & 0.5712 \\ 1.9908 & -7.9458 \end{bmatrix},\ B_{f2} = \begin{bmatrix} -0.5941 \\ 1.0863 \end{bmatrix},\ C_{f2} = \begin{bmatrix} 1.5533 & 0.0451 \end{bmatrix},$$
$$A_{f3} = \begin{bmatrix} -30.4708 & -11.5712 \\ 7.6857 & -7.6810 \end{bmatrix},\ B_{f3} = \begin{bmatrix} -1.5941 \\ -2.2069 \end{bmatrix},\ C_{f3} = \begin{bmatrix} -0.1343 & 0.9476 \end{bmatrix},$$
$$E_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let the time-varying $d(t) = 0.5 + 0.1 \sin t$, the simulation results of the real states response $x(t)$ and their estimations $\hat{x}(t)$ are displayed in Figs. 2 and 3. Fig. 1 depicts the system random switching modes under given transition rates. Both of the real states and their estimations' initial conditions are the same, $x_1(t) = x_2(t) = 5$ and $\hat{x}_1(t) = \hat{x}_2(t) = 5$. For the simulation results, it demonstrate that the filtering error system (7) is mean-square exponentially stable, so the obtained $H_\infty$ filter satisfies the prescribed performance requirements.

**Remark 2.** The above example shows that the proposed theories are feasible for the design of $H_\infty$ filter for singular system (1) when some of the system transition probabilities are completely unknown. This is the main element that [27] do not take into consideration. The theorems in [27,33–45] cannot solve the $H_\infty$ filter design problem with above parameters and transition probabilities, thus our results are new.

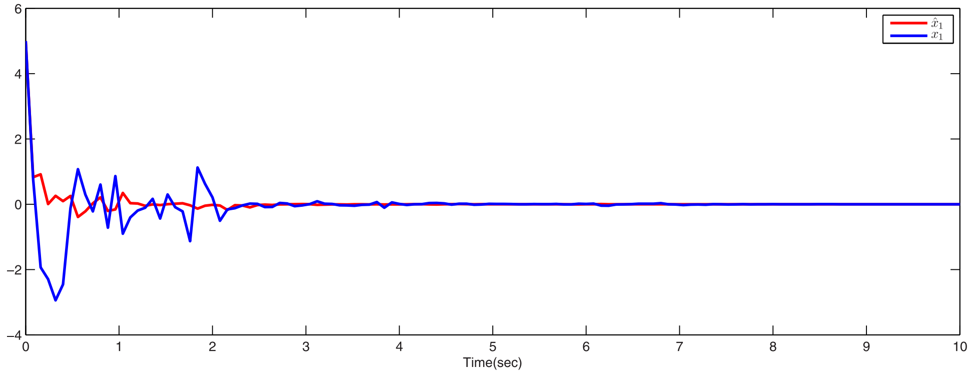


**Fig. 2.** State response of $\hat{x}_1(t)$ and $x_1(t)$.

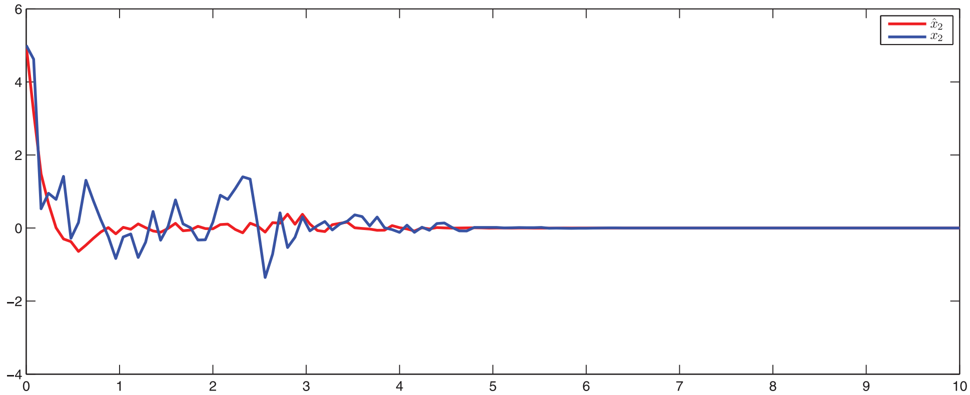


**Fig. 3.** State response of $\hat{x}_2(t)$ and $x_2(t)$.

## 5. Conclusions

In this paper the $H_\infty$ filtering problem for a class of singular Markovian jump time-delay systems with GUTRs has been investigated. Applying the Lyapunov functional theory and the LMI approach, delay-dependent sufficient conditions have been established for the solvability of the filtering problem. And the obtained results guaranteed the corresponding filtering error system is delay-dependent mean-square exponentially admissible with $H_\infty$ performance $\gamma$. Our future work will try to find some new results for Markovian jump time-delay systems based on the results in the references [37–45].


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