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Valuing American floating strike lookback option and Neumann problem for inhomogeneous Black–Scholes equation

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ABSTRACT

This paper presents our study of American floating strike lookback options written on dividend-paying assets. The valuation of these options can be mathematically formulated as a free boundary inhomogeneous Black–Scholes PDE with a Neumann boundary condition, which we, by using a Mellin transform, convert into a relatively simple ordinary differential equation with Dirichlet boundary conditions. We then use these results to derive an integral equation that can be used to calculate the price of American floating strike lookback options. In addition, we also used Mellin transforms to derive the closed-form of the perpetual case.

1. Introduction

Lookback options are path-dependent options with payoffs depending on the maximum or the minimum of the underlying asset price during the lifetime of the option. A popular form of lookback options in the insurance field is equity-indexed annuities (EIA), although other kinds of lookback options are also traded worldwide in the exchange market (refer to [1,2] for further details on the subject and related topics.) Various researchers have published results relating to the pricing of European lookback options; for example, Goldman et al. [3] and Conze and Viswanathan [4] derived the exact formula for the value function of European lookback options and Dai et al. [5] presented a formula for quanto lookback options regarding two underlying assets.

As American option holders can exercise their options at any instant before expiry, the early exercise policy should be considered when valuing American options. This is the reason why problems involving American options are usually referred to as optimal stopping problems or free boundary problems. Regarding American option theories, Kwok [6] gave an elaborate description, whereas Peskir and Shiryaev [7] established a number of theories related to optimal stopping problems. Furthermore, I. Kim [8] derived an integral equation satisfied by American options, because the closed-form solution of the American option did not yet exist at the time.

American lookback options can be thought of as a combination of American options and lookback options. Therefore, they have the properties of both of these types of options. Especially, valuing them requires a solution for the free boundary problems, an approach which is similar to the valuation of other American options. In addition, the presence of a lookback state variable results in a Neumann boundary condition. American lookback options can be divided into two categories: American fixed lookback options and American floating strike lookback options. Both of these types of options solve the same

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partial differential equation (PDE), but their payoff functions are different. There is also a relationship between American fixed strike lookback options and Russian options, where the latter could be considered a kind of perpetual version of the former. Anyone interested in Russian options can refer to [9–11]. The distinctive property of American floating strike lookback options is the homogeneity of their value functions. Such homogeneity makes it possible to reduce the dimension of the problem by one; thus, the structure of American floating strike lookback options is relatively simpler than that of the usual American fixed strike options. We focus on American floating strike options in this paper.

American floating strike lookback options have been studied previously. For example, Yu et al. studied the exercise boundary of American floating strike lookback options [12], and Dai and Kwok characterized the optimal stopping region of American lookback options [13,14]. Lai and Lim [15] proposed a way to calculate the value of American floating strike lookback options by using a numerical approach known as the Bernoulli walk approach. Kimura performed a premium decomposition for American floating lookback options employing Laplace transforms [16]. Finally, we remark that Dai succeeded in obtaining a closed-form solution of American options [17].

In this work, our approach was to mainly use the Mellin transform, which is a type of integral transform that can be considered a two-sided Laplace transform. Especially, a Mellin transform is widely used in solving option problems because it can be used to convert a Black–Scholes PDE into a simple ordinary differential equation (ODE). Remarkable results have been achieved by following the approach based on the Mellin transform; for example, Panini and Srivastav priced European, American, as well as perpetual American options using Mellin transforms [18,19], respectively. Frontczak [20] defined a modified Mellin transform and used it to derive an integral equation satisfied by an American call option. Yoon and Kim obtained a closed form of vulnerable options using double Mellin transforms [21]. Yoon also obtained a solution for European options with a stochastic interest model [22]. Buchen [23] analyzed the pricing of lookback type options and Jeon et al. [24] obtained integral equation representation of Russian option with finite time horizon by using Mellin transform techniques, respectively. In addition, Jeon et al. [25] derived a closed form solution of vulnerable geometric Asian options by utilizing double Mellin transforms.

This paper consists of five parts. In the first part (Section 2), we review the concepts of American floating lookback options and formulate the equivalent PDE problems. In the second part (Section 3), we derive the general solution of the inhomogeneous Black–Scholes equation with given Neumann boundary conditions with the aid of Mellin transform techniques. In the third part (Section 4), we apply the results of Section 3 to American floating lookback options to obtain the integral equation representation for American floating lookback options. In the fourth part (Section 5), we analyze the solution obtained in Section 4 to derive the closed form of the value function of perpetual American options. In the fifth part (Section 6), we summarize the work we have done. We added Appendix in which we summarize the basic definition, properties, and lemma regarding the Mellin transform for those who are not familiar with it.

2. Model formulation

Let S_t denote the underlying asset of the floating strike lookback option under a risk-neutral probability measure $\mathbb{P}$.

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t \quad (r > q)$$

where $r(> 0)$ is the riskless interest rate, σ and $q(> 0)$ are the volatility and dividend yield of X, respectively, and W_t is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0} \equiv \mathbb{F}$ is the natural filtration generated by $\mathbb{F}$.

For the process $(S_t)_{t\geq 0}$, define the minimum process as

$$m_t = \min_{0\leq \gamma\leq t} S_\gamma, \quad t \geq 0.$$

Consider an *American floating strike lookback call* option with a given finite time horizon $T > 0$. The payoff at maturity is given by $(S_T - m_T)$. In the absence of arbitrage opportunities, the value $C(t, S_t, m_t)$ is a solution of an *optimal stopping problem* (see [7])

$$C(t, s, m) = \sup_{\tau\in[t,T]} \mathbb{E}\left[e^{-r(\tau-t)}(S_\tau - m_\tau) \mid S_t = s, m_t = m\right] \tag{2.1}$$

where τ is the stopping time of the filtration $\mathbb{F}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$.

It is known that the *optimal stopping problem* (2.1) can be reduced to a *free boundary problem*. Define the differential operator $\mathcal{L}$ by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + (r-q)s\frac{\partial}{\partial s} - r.$$

Then, the free boundary problem can be written in a linear complementary form (see [6,26]) as

$$\begin{aligned} &\mathcal{L}\, C(t, s, m) \leq 0, \quad C(t, s, m) \geq S - m, \\ &(\mathcal{L}\, C(t, s, m))\,(C(t, s, m) - (s-m)) = 0, \quad s > m > 0,\ 0 \leq t < T, \end{aligned} \tag{2.2}$$

together with auxiliary conditions

$$C(T, s, m) = s - m$$
$$\lim_{x \downarrow m} \frac{\partial C}{\partial m} = 0. \tag{2.3}$$

The free boundary of problem (2.1) is given by the critical stock price $s^*(t, m)$ (this is termed the *early exercise boundary*). Arbitrage arguments show that the option price $C(t, s, m)$ must also satisfy the "smooth pasting conditions" at $s^*(t, m)$.

$$\lim_{s \uparrow s^*} C(t, s, m) = s^*(t, m) - m$$
$$\lim_{s \uparrow s^*} \frac{\partial C}{\partial s} = 1. \tag{2.4}$$

With the change of variables

$$x := \frac{m}{s} \tag{2.5}$$

and

$$\bar{C}(t, x) = \frac{C(t, s, m)}{s}. \tag{2.6}$$

Then, we can rewrite the linear complementary form (2.2) as

$$\begin{aligned} &\bar{\mathcal{L}}\bar{C}(t, x) \le 0, \quad \bar{C}(t, x) \ge 1 - x, \\ &\left(\bar{\mathcal{L}}\bar{C}(t, x)\right)(\bar{C}(t, x) - (1 - x)) = 0, \quad 0 < x < 1,\ 0 \le t < T, \end{aligned} \tag{2.7}$$

with auxiliary conditions:

$$\begin{aligned} &\bar{C}(T, x) = \phi(x) := 1 - x \\ &\lim_{x \downarrow 1} \frac{\partial \bar{C}}{\partial x} = 0 \\ &\lim_{x \uparrow x^*} \bar{C}(t, x) = 1 - x^*(t) \\ &\lim_{x \uparrow x^*} \frac{\partial \bar{C}}{\partial x} = -1 \end{aligned} \tag{2.8}$$

where the critical value is $x^* = \frac{m}{s^*}$ and the operator $\bar{\mathcal{L}}$ is given by

$$\bar{\mathcal{L}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (q - r)x\frac{\partial}{\partial x} - q.$$

Hence, solving the optimal stopping problem (2.1) is equivalent to finding the points $(t, x^*(t))$. Let

$$\mathcal{D} = \{(t, x) \mid 0 \le t \le T,\ 0 < x \le 1\}$$

and let $\mathcal{S}$ and $\mathcal{S}^C$ denote the *stopping region* and *continuation region*, respectively. In terms of the value function $\bar{C}(t, x)$, the stopping region $\mathcal{S}$ is defined by

$$\begin{aligned} \mathcal{S} &= \{(t, x) \in \mathcal{D} \mid \bar{C}(t, x) = 1 - x\} \\ &= \{(t, x) \mid 0 < x < x^*(t),\ 0 \le t \le T\}. \end{aligned}$$

The continuation region $\mathcal{S}^C$ is given by

$$\begin{aligned} \mathcal{S}^C &= \{(t, x) \in \mathcal{D} \mid \bar{C}(t, x) > 1 - x\} \\ &= \{(t, x) \mid x^*(t) < x < 1,\ 0 \le t \le T\}. \end{aligned}$$

Hence, the value function $\bar{C}(t, x)$ satisfies the following inhomogeneous Black–Scholes PDE:

$$\bar{\mathcal{L}}\bar{C}(t, x) = \frac{\partial \bar{C}}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \bar{C}}{\partial x^2} + (q - r)x\frac{\partial \bar{C}}{\partial x} - q\bar{C} = f(t, x) \tag{2.9}$$

where

$$f = f(t, x) = \begin{cases} rx - q & \text{for } 0 \le x \le x^*(t) \\ 0 & \text{for } x^*(t) < x < 1 \end{cases} \tag{2.10}$$

with Neumann boundary conditions in (2.8).

In the following section, we exhibit the closed-form representation of the solution of the general inhomogeneous Black–Scholes PDE with Dirichlet and Neumann boundary conditions. The application of such a closed-form formula to (2.9) leads to a representation of the value function $\bar{C}(t, x)$.

3. Inhomogeneous Black–Scholes equation with Neumann boundary condition

Although Buchen already analyzed real option problems with Neumann boundary conditions using a Mellin transform [1], he used a homogeneous Black–Scholes equation. In this section, we extend his idea to the inhomogeneous case, with the aim of converting an inhomogeneous Black–Scholes PDE with Neumann boundary conditions into an inhomogeneous Black–Scholes PDE with Dirichlet boundary conditions. We subsequently use these results to solve the reduced equation with the aid of Mellin transform techniques (the definition and properties of the Mellin transform are summarized in the Appendix of this paper).

Define the PDE operator $\mathcal{L}_{BS}$ as

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (q-r)x\frac{\partial}{\partial x} - q.$$

Consider the following *Neumann boundary condition* PDE problem:

$$\begin{aligned} &\mathcal{L}_{BS}\, P(t,x) = h(t,x), \qquad P(T,x) = g(x) \\ &\frac{\partial P}{\partial x}(t,1) = 0 \end{aligned} \tag{3.1}$$

on domain $\{(t,x) \mid 0 \le t < T,\ 0 \le x < 1\}$.

We assume that $h(t,x)$, $g(x)$ are smooth functions and Mellin transforms of $h, g, x\frac{dh}{dx}, x\frac{dg}{dx}$ are well-defined in proper domains, respectively.

Let $V(t,x) = x\frac{\partial P}{\partial x}(t,x)$, $\psi(t,x) = x\frac{\partial h}{\partial x}(t,x)$ and $\zeta(x) = x\frac{dg}{dx}(x)$, then PDE (3.1) is converted to

$$\begin{aligned} &\mathcal{L}_{BS}\, V(t,x) = \psi(t,x) \\ &V(t,1) = 0 \\ &V(T,x) = \zeta(x) \end{aligned} \tag{3.2}$$

on domain $\{(t,x) \mid 0 \le t < T,\ 0 \le x < 1\}$.

To solve PDE (3.2), we consider an unrestricted inhomogeneous PDE:

$$\begin{aligned} &\mathcal{L}_{BS}\, Q(t,x) = \psi(t,x)\mathbf{1}_{\{x<1\}} \\ &Q(T,x) = \zeta(x)\mathbf{1}_{\{x<1\}} \end{aligned} \tag{3.3}$$

on domain $\{(t,x) \mid 0 \le t < T,\ 0 \le x < \infty\}$.

Then, we can define $\hat{Q}(t,w)$ as the Mellin transform of $Q(t,x)$.

$$\hat{Q}(t,w) = \int_0^\infty Q(t,x)x^{w-1}dw.$$

From PDE (3.3), $\hat{Q}(t,w)$ satisfies the following ODE

$$\frac{d\hat{Q}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{Q} = \hat{\psi}(t,w) \tag{3.4}$$

where $\hat{Q}$ and $\hat{\psi}$ are the Mellin transforms of $Q(t,x)$ and $\psi(t,x)\mathbf{1}_{\{x<1\}}$, respectively.

Let $A(w) = w^2 + w(1-k_2) - k_1$ where $k_1 = \frac{2q}{\sigma^2}$, $k_2 = \frac{2(q-r)}{\sigma^2}$.

Then,

$$\hat{Q}(t,w) = e^{\frac{1}{2}\sigma^2 A(w)(T-t)}\hat{\zeta}(w) - \int_t^T e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}\hat{\psi}(\eta,w)d\eta \tag{3.5}$$

where $\hat{\zeta}(w)$ is the Mellin transform of $\zeta(x)\mathbf{1}_{\{x<1\}}$.

By the inverse Mellin transform,

$$Q(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 A(w)(T-t)}\hat{\zeta}(w)x^{-w}dw - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}\hat{\psi}(\eta,w)x^{-w}d\eta dw. \tag{3.6}$$

To compute (3.6), let

$$\mathcal{B}(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 A(w)t}x^{-w}dw. \tag{3.7}$$

Then,

$$\mathcal{B}(t,x)=e^{-\frac{\sigma^2}{2}\left\{\left(\frac{1-k_2}{2}\right)^2+k_1\right\}t}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\frac{\sigma^2}{2}\left(w+\frac{1-k_2}{2}\right)^2t}x^{-w}dw.$$

According to the property of the Mellin transform in Proposition A.3 of Appendix

$$\mathcal{B}(t,x)=e^{-\frac{\sigma^2}{2}\left\{\left(\frac{1-k_2}{2}\right)^2+k_1\right\}t}\frac{x^{\frac{1-k_2}{2}}}{\sigma\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{(\log x)^2}{\sigma^2 t}\right\}. \tag{3.8}$$

Because $e^{\frac{\sigma^2}{2}A(w)(T-t)}$, $\hat{\zeta}(w)$, and $\hat{\psi}(\eta,w)$ are the Mellin transforms of $\mathcal{B}(T-t,x)$, $\zeta(x)\mathbf{1}_{\{x<1\}}$, and $\psi(\eta,x)\mathbf{1}_{\{x<1\}}$, respectively, according to the Mellin convolution property in Proposition A.1 of Appendix,

$$Q(t,x)=\int_0^\infty\zeta(u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(T-t,\frac{x}{u}\right)\frac{1}{u}du-\int_t^T\int_0^\infty\psi(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}dud\eta. \tag{3.9}$$

The following lemma is essential in that it provides a tool for the domain extension of the PDE (3.2).

Lemma 3.1. *For $\mathcal{B}(t,x)$ defined in (3.7),*

$$\mathcal{B}(t,x)=x^{(1-k_2)}\mathcal{B}\left(t,\frac{1}{x}\right). \tag{3.10}$$

Proof. Because,

$$\begin{aligned}A(w)&=\frac{\sigma^2}{2}\{w^2+(1-k_2)w-k_1\}\\&=\frac{\sigma^2}{2}\left\{\left(w+\frac{1-k_2}{2}\right)^2-\left(\frac{1-k_2}{2}\right)^2-k_1\right\}.\end{aligned}$$

Hence, $A(w)=A(k_2-1-w)$ and

$$\begin{aligned}\mathcal{B}(t,x)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\frac{1}{2}\sigma^2A(w)t}x^{-w}dw\\&=\frac{1}{2\pi i}\int_{(k_2-1-c)+i\infty}^{(k_2-1-c)-i\infty}e^{\frac{1}{2}\sigma^2A(w)t}x^{w-(k_2-1)}d(-w)\\&=x^{1-k_2}\frac{1}{2\pi i}\int_{(c+1-k_2)-i\infty}^{(k_2-1-c)+i\infty}e^{\frac{1}{2}\sigma^2A(w)t}x^{w}dw\\&=x^{1-k_2}\mathcal{B}\left(t,\frac{1}{x}\right).\quad\square\end{aligned}$$

Using Lemma 3.1, we can prove the following theorem which extends the solution of the PDE (3.2) to $[0,T)\times[0,\infty)$.

Theorem 3.1 (***Inhomogeneous Black–Scholes Equation With Dirichlet Condition***). *In the domain $\{(t,x)\mid 0\le t<T, 0\le x<\infty\}$, $V(t,x)$ the solution of PDE (3.2) is given by*

$$V(t,x)=Q(t,x)-x^{(1-k_2)}Q\left(t,\frac{1}{x}\right) \tag{3.11}$$

where $Q(t,x)$ defined in (3.3) is expressed by

$$Q(t,x)=\int_0^\infty\zeta(u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(T-t,\frac{x}{u}\right)\frac{1}{u}du-\int_t^T\int_0^\infty\psi(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(\tau-t,\frac{x}{u}\right)\frac{1}{u}dud\eta.$$

Proof. By Lemma 3.1, $Q(t,x)$ of (3.7) is given by

$$Q(t,x)=\int_0^\infty\zeta(u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(T-t,\frac{x}{u}\right)\frac{1}{u}du-\int_t^T\int_0^\infty\psi(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}dud\eta \tag{3.12}$$

and it leads to

$$x^{(1-k_2)}Q\left(t,\frac{1}{x}\right)=\int_0^\infty \zeta\left(\frac{1}{u}\right)u^{(1-k_2)}\mathbf{1}_{\{u>1\}}\mathcal{B}\left(T-t,\frac{x}{u}\right)\frac{1}{u}du$$
$$-\int_t^T\int_0^\infty \psi\left(\eta,\frac{1}{u}\right)u^{(1-k_2)}\mathbf{1}_{\{u>1\}}\mathcal{B}\left(\tau-t,\frac{x}{u}\right)\frac{1}{u}dud\eta. \tag{3.13}$$

Let $Q^*(t,x) := x^{(1-k_2)}Q(t,\frac{1}{x})$.
Then, $Q^*(t,x)$ satisfies the following PDE:

$$\begin{aligned}\mathcal{L}_{BS}Q^*(t,x)&=\psi\left(t,\frac{1}{x}\right)x^{(1-k_2)}\mathbf{1}_{\{x>1\}}\\ Q^*(T,x)&=\zeta\left(\frac{1}{x}\right)x^{(1-k_2)}\mathbf{1}_{\{x>1\}}.\end{aligned} \tag{3.14}$$

Let $R(t,x)=Q(t,x)-Q^*(t,x)$, then by (3.2) and (3.14),

$$\mathcal{L}R(t,x)=\psi(t,x)\mathbf{1}_{\{x<1\}}-\psi\left(t,\frac{1}{x}\right)x^{(1-k_2)}\mathbf{1}_{\{x>1\}} \tag{3.15}$$

and $R(t,1)=Q(t,1)-Q^*(t,1)=0$.
In addition,

$$R(T,s)=\zeta(x)\mathbf{1}_{\{x<1\}}-\zeta\left(\frac{1}{x}\right)x^{(1-k_2)}\mathbf{1}_{\{x>1\}}. \tag{3.16}$$

Therefore, the solution $Q(t,s)$ of PDE (3.2), can be expressed by

$$V(t,x)=Q(t,x)-x^{(1-k_2)}Q\left(t,\frac{1}{x}\right) \tag{3.17}$$

where $Q(t,x)$ is the solution of PDE (3.3). □

Note that we extended the domain of the PDE (3.2) to non-negative real numbers using Theorem 3.1. Therefore, we can apply Mellin transforms to the extended PDE, which would enable us to obtain the representation for the solution of an inhomogeneous Black–Scholes PDE with Neumann boundary conditions.

Theorem 3.2 (***Inhomogeneous Black–Scholes Equation With Neumann Boundary Conditions***). *$P(t,x)$, the solution of PDE (3.1), satisfies the following PDE:*

$$\mathcal{L}_{BS}P(t,x)=h(t,x)\mathbf{1}_{\{x<1\}}+h\left(t,\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}}+(k_2-1)\left[\int_1^x h\left(t,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}$$
$$P(T,x)=g(x)\mathbf{1}_{\{x<1\}}+g\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}}+(k_2-1)\left[\int_1^x g\left(\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}$$

with the domain $\{(t,x)\mid 0\le t<T, 0<x<\infty\}$.
Further, $P(t,x)$ is expressed by

$$\begin{aligned}P(t,x)=&\int_0^\infty\left(g(u)\mathbf{1}_{\{u<1\}}+g\left(\frac{1}{u}\right)\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)}+(k_2-1)\left[\int_1^u g\left(\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\right)\\ &\times\mathcal{B}\left(T-t,\frac{x}{u}\right)\frac{1}{u}du-\int_t^T\int_0^\infty\left(h(\eta,u)\mathbf{1}_{\{u<1\}}+h\left(\eta,\frac{1}{u}\right)\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)}\right.\\ &\left.+(k_2-1)\left[\int_1^u h\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\right)\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}dud\eta.\end{aligned}$$

Proof. By Theorem 3.1, on domain $\{(t,x)|0\le t<T, 0<x<\infty\}$, $Q(t,x)$ satisfies the following PDE:

$$\begin{aligned}\mathcal{L}_{BS}V(t,x)&=\psi(t,x)\mathbf{1}_{\{x<1\}}-\psi\left(t,\frac{1}{x}\right)x^{(1-k_2)}\mathbf{1}_{\{x>1\}}\\ V(T,x)&=\zeta(x)\mathbf{1}_{\{x<1\}}-\zeta\left(\frac{1}{x}\right)x^{(1-k_2)}\mathbf{1}_{\{x>1\}}.\end{aligned} \tag{3.18}$$

Define

$$H(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{V}(t,w)\frac{1}{w}x^{-w}dw$$

i.e., $H(t,x)$ is the inverse Mellin transformation of $\frac{-\hat{V}(t,w)}{w}$.

Clearly, $-w\hat{H}(t,w) = \hat{V}(t,w)$. According to the property of the Mellin transform,

$$x\frac{\partial}{\partial x}H(t,x) = V(t,x).$$

Because $V(t,1) = 0$,

$$1\cdot\frac{\partial H}{\partial x}(t,1) = 0. \tag{3.19}$$

Similarly, by (3.4),

$$\frac{d\hat{V}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{V} = \hat{\psi}(t,w) - \hat{\psi}(t,k_2-1-w)$$

where $\hat{V}(t,w)$ is the Mellin transform of $V(t,x)$.

From $-w\hat{H}(t,w) = \hat{V}(t,w)$,

$$-w\left[\frac{d\hat{H}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{H}\right] = -w\hat{h}(t,w) + (k_2-1-w)\hat{h}(t,k_2-1-w)$$

where $\hat{h}$ is Mellin transform of $h(t,x)\mathbf{1}_{\{x<1\}}$.

Hence,

$$\frac{d\hat{H}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{H} = \hat{h}(t,w) + \hat{h}(t,k_2-1-w) - \frac{k_2-1}{w}\hat{h}(t,k_2-1-w). \tag{3.20}$$

By the inverse Mellin transform of both sides of (3.20),

$$\mathcal{L}_{BS}\,H(t,x) = h(t,x)\mathbf{1}_{\{x<1\}} + h\left(t,\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} + (k_2-1)\left[\int_1^x h\left(t,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}. \tag{3.21}$$

By the same procedure,

$$H(T,x) = g(x)\mathbf{1}_{\{x<1\}} + g\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} + (k_2-1)\left[\int_1^x g\left(\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}. \tag{3.22}$$

By (3.19), (3.21), and (3.22), $P(t,x)$, which is the solution of PDE (3.1), satisfies

$$\mathcal{L}_{BS}P(t,x) = h(t,x)\mathbf{1}_{\{x<1\}} + h\left(t,\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} + (k_2-1)\left[\int_1^x h\left(t,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}$$

$$P(T,x) = g(x)\mathbf{1}_{\{x<1\}} + g\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} + (k_2-1)\left[\int_1^x g\left(\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}$$

with the domain $\{(t,x) \mid 0 \le t < T, 0 < x < \infty\}$.

By Theorem 3.2 and (3.9), $P(t,x)$ is given by

$$\begin{aligned} P(t,x) = {}& \int_0^\infty \left(g(u)\mathbf{1}_{\{u<1\}} + g\left(\frac{1}{u}\right)\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)} + (k_2-1)\left[\int_1^u g\left(\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\right) \\ & \times \mathcal{B}\left(T-t,\frac{x}{u}\right)\frac{1}{u}du - \int_t^T\int_0^\infty \left(h(\eta,u)\mathbf{1}_{\{u<1\}} + h\left(\eta,\frac{1}{u}\right)\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)}\right. \\ & \left. + (k_2-1)\left[\int_1^u h\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\right)\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}dud\eta. \quad \square \end{aligned} \tag{3.23}$$

From Theorem 3.2, we define the absorbing Neumann boundary operator $\mathcal{T}$ as follows:

Definition 3.1. Let $U(t,x)$ be any function of t and x. Then, the image of the absorbing *Neumann boundary operator* of U with respect to $x=1$ is defined to be the function

$$\mathcal{T}[U(t,x)] = U(t,x)\mathbf{1}_{\{x<1\}} + U\left(t,\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} + (k_2-1)\left[\int_1^x U\left(t,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}. \tag{3.24}$$

4. Integral equation representation of American floating strike lookback option

The value of American type options is usually decomposed into the European option value and the early exercise value terms. Numerous approaches and considerable effort have been devoted to obtaining the value of American options. For example, I. Kim demonstrated the inclusion of early exercise premium terms in the integral equation [8], Lai and Lim derived the integral equation representation of American floating strike options using reflection Brownian motion [15], and Kimura performed a premium decomposition using Laplace transforms [16].

In this section, we derive an integral equation satisfied by American floating lookback options using our approach involving a PDE based on a Mellin transform, as described in Section 3.

By Theorem 3.2, the solution $\bar{C}(t,x)$ of the PDE (2.9) satisfies the following PDE when absorbing the Neumann boundary condition:

$$\tilde{\mathcal{L}}\bar{C}(t,x) = \frac{\partial\bar{C}}{\partial t} + \frac{1}{2}\sigma^2x^2\frac{\partial^2\bar{C}}{\partial x^2} + (q-r)x\frac{\partial\bar{C}}{\partial x} - q\bar{C} = \mathcal{T}[f(t,x)] \tag{4.1}$$

where $\mathcal{T}$ is the absorbing Neumann boundary operator in Definition 3.1 and

$$f = f(t,x) = \begin{cases} rx-q & \text{for } 0 \le x \le x^*(t) \\ 0 & \text{for } x^*(t) < x < 1 \end{cases}$$

with auxiliary condition

$$\begin{aligned} &\bar{C}(T,x) = \mathcal{T}[\phi(x)] \\ &\lim_{x\uparrow x^*}\bar{C}(t,x) = 1 - x^*(t) \\ &\lim_{x\uparrow x^*}\frac{\partial\bar{C}}{\partial x} = -1 \end{aligned} \tag{4.2}$$

in domain $\{(t,x) \mid 0 \le t < T, 0 < x < \infty\}$.

Let

$$\bar{C}(t,x) = \bar{C}_E(t,x) + \bar{C}_P(t,x)$$

where $\bar{C}_E(t,x)$ and $\bar{C}_P(t,x)$ satisfy following PDEs:

$$\begin{aligned} &\tilde{\mathcal{L}}\bar{C}_E(t,x) = 0 \\ &\bar{C}_E(T,x) = \mathcal{T}[\phi(x)] \end{aligned} \tag{4.3}$$

and

$$\tilde{\mathcal{L}}\bar{C}_P(t,x) = \mathcal{T}[f(t,x)], \quad \bar{C}_P(T,x) = 0. \tag{4.4}$$

Define $C_E(t,s,m) = s\tilde{C}_E(t,\frac{m}{s})$ and $C_P(t,s.m) = s\tilde{C}_P(t,\frac{m}{s})$.

Clearly, $C(t,s,m) = C_E(t,s,m) + C_P(t,s,m)$ and $C_E(t,s,m)$ is the value of the European lookback call option with terminal payoff $(S_t - m_t)$. We rewrite the value of $C_E(t,s,m)$, which was obtained by Kimura and can be found in [16], to obtain,

$$\begin{aligned} C_E(t,s,m) &= se^{-q(T-t)}\mathcal{N}\left(d_1^+\left(\frac{s}{m},T-t\right)\right) - me^{-r(T-t)}\mathcal{N}\left(d_1^-\left(\frac{s}{m},T-t\right)\right) \\ &\quad + \frac{\sigma^2s}{2(r-q)}\left\{e^{-r(T-t)}\left(\frac{m}{s}\right)^{\frac{2(r-q)}{\sigma^2}}\mathcal{N}\left(d_1^-\left(\frac{m}{s},T-t\right)\right) - e^{-q(T-t)}\mathcal{N}\left(-d_1^+\left(\frac{s}{m},T-t\right)\right)\right\} \end{aligned} \tag{4.5}$$

where N is the standard cumulative normal distribution and

$$d^{\pm}(\xi,T-t) := \frac{1}{\sigma\sqrt{T-t}}\left\{\log\xi + \left(r-q\pm\frac{1}{2}\sigma^2\right)(T-t)\right\}. \tag{4.6}$$

Therefore,

$$\bar{C}_E(t,x) = e^{-q(T-t)}\mathcal{N}\left(d^+\left(\frac{1}{x},T-t\right)\right) - xe^{-r(T-t)}\mathcal{N}\left(d^-\left(\frac{1}{x},T-t\right)\right) + \frac{\sigma^2}{2(r-q)}\left\{e^{-r(T-t)}x^{\frac{2(r-q)}{\sigma^2}}\mathcal{N}(d^-(x,T-t)) - e^{-q(T-t)}\mathcal{N}\left(-d^+\left(\frac{1}{x},T-t\right)\right)\right\}. \tag{4.7}$$

The following lemma is useful for the derivation of the integral equation representation of American lookback options.

Lemma 4.1.

$$\int_0^A u^{-\alpha}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}d\eta = x^{-\alpha}e^{-\frac{1}{2}\sigma^2(\eta-t)\left\{\left(\frac{1-k_2}{2}\right)^2+k_1-\left(\frac{1-k_2}{2}+\alpha\right)^2\right\}}\mathcal{N}\left(\frac{-\log\frac{x}{A}+\sigma^2(\eta-t)\left(\frac{1-k_2}{2}+\alpha\right)}{\sigma\sqrt{\eta-t}}\right)$$

$$\int_A^\infty u^{-\alpha}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}d\eta = x^{-\alpha}e^{-\frac{1}{2}\sigma^2(\eta-t)\left\{\left(\frac{1-k_2}{2}\right)^2+k_1-\left(\frac{1-k_2}{2}+\alpha\right)^2\right\}}\mathcal{N}\left(\frac{\log\frac{x}{A}-\sigma^2(\eta-t)\left(\frac{1-k_2}{2}+\alpha\right)}{\sigma\sqrt{\eta-t}}\right)$$

where $k_1 = \frac{2q}{\sigma^2}$, $k_2 = \frac{2(q-r)}{\sigma^2}$.

Now, we state the theorem regarding the price of an American lookback option as follows.

Theorem 4.1 (***Premium Decomposition of American Floating Strike Lookback Option***). *The price of an American floating strike lookback call option, as defined in (2.1), is*

$$C(t,s,m) = C_E(t,s,m) + C_P(t,s,m),$$

where

$$C_E(t,s,m) = se^{-q(T-t)}\mathcal{N}\left(d_1^+\left(\frac{s}{m}\right)\right) - me^{-r(T-t)}\mathcal{N}\left(d_1^-\left(\frac{s}{m}\right)\right) - \frac{1}{k_2}\left\{e^{-r(T-t)}\left(\frac{s}{m}\right)^{k_2}\mathcal{N}\left(d_1^-\left(\frac{m}{s}\right)\right) - e^{-q(T-t)}\mathcal{N}\left(-d_1^+\left(\frac{s}{m}\right)\right)\right\}$$

and

$$\begin{aligned} C_P(t,s,m) = &-r\cdot m\int_t^T e^{-r(\eta-t)}\mathcal{N}\left(d^-\left(\frac{s}{s^*(\eta,m)},\eta-t\right)\right)d\eta + q\cdot s\int_t^T \cdot e^{-q(\eta-t)} \\ &\times\mathcal{N}\left(d^+\left(\frac{s}{s^*(\eta,m)},\eta-t\right)\right)d\eta - \frac{r\cdot s}{k_2}\int_t^T\left(\frac{m}{s}\right)^{k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-\left(\frac{m^2}{s\cdot s^*(\eta,m)},\eta-t\right)\right)d\eta \\ &-r\left(1-\frac{1}{k_2}\right)\int_t^T e^{-q(\eta-t)}\left(\frac{m^2}{s^*(\eta,m)}\right)^{k_2}\mathcal{N}\left(-d^+\left(\frac{m^2}{s\cdot s^*(\eta,m)},\eta-t\right)\right)d\eta \\ &+q\cdot s\int_t^T e^{-q(\eta-t)}\left(\frac{m^2}{s^*(\eta,m)}\right)^{k_2-1}\mathcal{N}\left(-d^+\left(\frac{m^2}{s\cdot s^*(\eta,m)},\eta-t\right)\right)d\eta. \end{aligned}$$

For the free boundary of American floating lookback option $s^*(t,m)$, *we define* $x^*(t) = \frac{m}{s^*(t,m)}$. *Then* x^* *satisfies the following integral equation.*

$$\begin{aligned} 1-x^*(t) = &\bar{C}_E(t,x^*(t)) - r\int_t^T x^*(t)e^{-r(\eta-t)}\mathcal{N}\left(d^-\left(\frac{x^*(\eta)}{x^*(t)},\eta-t\right)\right)d\eta + q\int_t^T e^{-q(\eta-t)} \\ &\times\mathcal{N}\left(d^+\left(\frac{x^*(\eta)}{x^*(t)},\eta-t\right)\right)d\eta - \frac{r}{k_2}\int_t^T x^*(t)^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x^*(t),\eta-t)\right)d\eta \\ &-r\left(1-\frac{1}{k_2}\right)\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x^*(t)},\eta-t\right)\right)d\eta \\ &+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x^*(t)},\eta-t\right)\right)d\eta \end{aligned}$$

where $k_2 = \frac{2(q-r)}{\sigma^2}$, $d^\pm$ *is defined in (4.5) and* $\bar{C}_E(t,x)$ *is defined in (4.7).*

Proof. By (4.1),

$$f(t,x) = (rx-q)\mathbf{1}_{\{x<x^*(t)\}}.$$

In PDE (4.4), $\bar{C}_P(t,x)$ is expressed by

$$\begin{aligned}\bar{C}_P(t,x) &= -\int_t^T\int_0^\infty \left(f(\eta,u)\mathbf{1}_{\{u<1\}} + f\left(\eta,\frac{1}{u}\right)\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)}\right. \\ &\qquad \left. + (k_2-1)\left[\int_1^u f\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\right)\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta \\ &= I_1+I_2+I_3,\end{aligned}$$

where

$$\begin{aligned}I_1(t,x) &:= -\int_t^T\int_0^\infty f(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta \\ I_2(t,x) &:= -\int_t^T\int_0^\infty f\left(\eta,\frac{1}{u}\right)\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta \\ I_3(t,x) &:= -(k_2-1)\int_t^T\int_0^\infty\left[\int_1^u f\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta.\end{aligned}$$

By using Lemma 4.1,

$$\begin{aligned}I_1(t,x) &= -\int_t^T\int_0^\infty f(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta \\ &= -\int_t^T\int_0^{x^*(\eta)}(ru-q)\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta \\ &= -r\int_t^T\int_0^{x^*(\eta)} u\cdot\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta + q\int_t^T\int_0^{x^*(\eta)}1\cdot\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du d\eta \\ &= -r\int_t^T xe^{-r(\eta-t)}\mathcal{N}\left(d^-\left(\frac{x^*(\eta)}{x},\eta-t\right)\right)d\eta + q\int_t^T e^{-q(\eta-t)}\mathcal{N}\left(d^+\left(\frac{x^*(\eta)}{x},\eta-t\right)\right)d\eta.\end{aligned}$$

From $I_2(t,x) = \left(\frac{1}{x}\right)^{(k_2-1)} I_1(t,\frac{1}{x})$,

$$I_2(t,x) = -r\int_t^T\left(\frac{1}{x}\right)^{k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x,\eta-t)\right)d\eta + q\int_t^T e^{-q(\eta-t)}\left(\frac{1}{x}\right)^{(k_2-1)}\mathcal{N}\left(d^+(x^*(\eta)x,\eta-t)\right)d\eta.$$

Because $f(t,\frac{1}{y}) = (\frac{r}{y}-q)\mathbf{1}_{\{y>\frac{1}{x^*(\eta)}\}}$ and

$$\begin{aligned}&\int_0^\infty\left[\int_1^u f\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du \\ &= \int_0^\infty\left[\int_{\frac{1}{x^*(\eta)}}^u f\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\left\{u>\frac{1}{x^*(\eta)}\right\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du \\ &= \int_{\frac{1}{x^*(\eta)}}^\infty\left[\int_{\frac{1}{x^*(\eta)}}^u\left(\frac{r}{y}-q\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du \\ &= \int_{\frac{1}{x^*(\eta)}}^\infty\left[-r\frac{u^{-k_2}}{k_2}+q\frac{u^{-(k_2-1)}}{k_2-1}+r\frac{x^*(\eta)^{k_2}}{k_2}-q\frac{x^*(\eta)^{(k_2-1)}}{k_2-1}\right]\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}du.\end{aligned}$$

Hence,

$$\begin{aligned}
I_3 &= -(k_2-1)\int_t^T\int_0^\infty\left[\int_1^u f\left(\eta,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}dud\eta\\
&= \int_t^T\int_{\frac{1}{x^*(\eta)}}^\infty\left[r\frac{k_2-1}{k_2}u^{-k_2}-qu^{-(k_2-1)}-r\frac{k_2-1}{k_2}x^*(\eta)^{k_2}+qx^*(\eta)^{(k_2-1)}\right]\mathcal{B}\left(\eta-t,\frac{x}{u}\right)\frac{1}{u}dud\eta\\
&= r\left(1-\frac{1}{k_2}\right)\int_t^T x^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x,\eta-t)\right)d\eta-q\int_t^T x^{-(k_2-1)}e^{-q(\eta-t)}\\
&\quad\times\mathcal{N}\left(d^+(x^*(\eta)x,\eta-t)\right)d\eta-r\left(1-\frac{1}{k_2}\right)\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x},\eta-t\right)\right)d\eta\\
&\quad+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x},\eta-t\right)\right)d\eta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\bar{C}_P(t,x) &= -r\int_t^T xe^{-r(\eta-t)}\mathcal{N}\left(d^-\left(\frac{x^*(\eta)}{x},\eta-t\right)\right)d\eta+q\int_t^T e^{-q(\eta-t)}\mathcal{N}\left(d^+\left(\frac{x^*(\eta)}{x},\eta-t\right)\right)d\eta\\
&\quad-\frac{r}{k_2}\int_t^T x^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x,\eta-t)\right)d\eta-r\left(1-\frac{1}{k_2}\right)\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\\
&\quad\times\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x},\eta-t\right)\right)d\eta+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x},\eta-t\right)\right)d\eta. \qquad (4.8)
\end{aligned}$$

By (4.2), the critical value of $x^*(t)$ satisfies the following integral equation.

$$\begin{aligned}
1-x^*(t) &= \bar{C}_E(t,x^*(t))-r\int_t^T x^*(t)e^{-r(\eta-t)}\mathcal{N}\left(d^-\left(\frac{x^*(\eta)}{x^*(t)},\eta-t\right)\right)d\eta+q\int_t^T e^{-q(\eta-t)}\\
&\quad\times\mathcal{N}\left(d^+\left(\frac{x^*(\eta)}{x^*(t)},\eta-t\right)\right)d\eta-\frac{r}{k_2}\int_t^T x^*(t)^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x^*(t),\eta-t)\right)d\eta\\
&\quad-r\left(1-\frac{1}{k_2}\right)\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x^*(t)},\eta-t\right)\right)d\eta\\
&\quad+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+\left(\frac{1}{x^*(\eta)x^*(t)},\eta-t\right)\right)d\eta.
\end{aligned}$$

From $x^*(t)=\frac{m}{s^*(t,m)}$, we obtain the desired result. □

Remark 4.1. In fact, $f(t,x)$ is not differentiable at $x=x^*(t)$. But, by Appendix C.4 in [27], there exist sequences $\{f_n(t,x)\}\in C^\infty\left((0,T)\times(\frac{1}{n},1-\frac{1}{n})\right)$ such that $f_n\to f$ a.e. and $f_n\to f$ in $L^1((0,T)\times(0,1))$ as $n\to\infty$. Therefore, by applying Theorem 3.2 to a smooth sequence of functions $\{f_n(t,x)\}$ and letting $n\to\infty$, we obtain the same result for $f(t,x)$ as well.

5. Perpetual American floating strike lookback option

In this section, we derive the closed-form expressions of a perpetual American lookback call option using a Mellin transform and elementary complex analysis.

Theorem 5.1 (***Free Boundary of Perpetual American Floating Strike Lookback Call Option***). *If $T\to\infty$, we denote $s^*_\infty(m)$ as the free boundary of the perpetual American lookback call option. Then, $x^*_\infty=\frac{m}{s^*_\infty}$ satisfies*

$$x^{*\,(\lambda_2-\lambda_1)}_\infty=\frac{\lambda_1}{\lambda_2}\frac{(1+\lambda_2)x^*_\infty-\lambda_2}{(1+\lambda_1)x^*_\infty-\lambda_1},$$

where λ_1,λ_2 are the two roots of the equation $\frac{\sigma^2}{2}\lambda^2+(\frac{\sigma^2}{2}-(q-r))-q=0$.

Proof. In PDE (4.4), let $\bar{C}_P(t,x):=\bar{C}^1_P(t,x)+\bar{C}^2_P(t,x)+\bar{C}^3_P(t,x)$, where

$$\tilde{\mathcal{L}}\bar{C}^1_P(t,x)=f(t,x)\mathbf{1}_{\{x<1\}},\quad \bar{C}^1_P(T,x)=0 \qquad (5.1)$$

$$\tilde{\mathcal{L}}\bar{C}^2_P(t,x)=f\left(t,\frac{1}{x}\right)\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}},\quad \bar{C}^2_P(T,x)=0, \qquad (5.2)$$

and

$$\tilde{\mathcal{L}}\bar{C}_P^3(t,x) = (k_2-1)\left[\int_1^x f\left(t,\frac{1}{y}\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}, \quad \bar{C}_P^3(T,x)=0. \tag{5.3}$$

From PDE (5.1),

$$\frac{d\hat{C}_1}{dt}+\left(\frac{1}{2}\sigma^2 w(w+1)-(q-r)w-q\right)\hat{C}_1=\hat{f}(t,w),$$

where $\hat{C}_1(t,w)$ is the Mellin transform of $\bar{C}_P^1(t,x)$ and

$$\begin{aligned}\hat{f}(t,w) &= \int_0^\infty f(t,x)x^{w-1}dx\\ &= \int_0^{x^*(t)}(rx-q)x^{w-1}dx\\ &= \frac{rx^*(t)^{w+1}}{w+1}-\frac{qx^*(t)^w}{w}\end{aligned}$$

and by the inverse Mellin transform

$$\bar{C}_P^1(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-w}\int_t^T\left[\frac{rx^*(\eta)^{w+1}}{w+1}-\frac{qx^*(\eta)^w}{w}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw \tag{5.4}$$

and

$$\frac{\partial\bar{C}_P^1}{\partial x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-w-1}\int_t^T\left[\frac{rw}{w+1}x^*(\eta)^{w+1}-qx^*(\eta)^w\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw,$$

where $Re(w)>0$ and

$$\begin{aligned}A(w) &= w^2+w(1-k_2)-k_1\\ &= (w-\lambda_1)(w-\lambda_2), \quad \lambda_1<\lambda_2.\end{aligned}$$

Then, it is easy to verify that if $q>0$,

$$\lambda_1<-1, \qquad \lambda_2>0.$$

Letting $T\to\infty$,

$$\begin{aligned}\frac{\partial\bar{C}_P^1}{\partial x} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-w-1}\int_t^\infty\left[\frac{rw}{w+1}x_\infty^{*\ w+1}-qx_\infty^{*\ w}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2}{\sigma^2}\frac{1}{A(w)}\left[q\left(\frac{x_\infty^*}{x}\right)^{(w+1)}\frac{1}{x_\infty^*}-\frac{rw}{w+1}\left(\frac{x_\infty^*}{x}\right)^{(w+1)}\right]dw.\end{aligned} \tag{5.5}$$

Note that at any time t, there is infinite time to maturity, and therefore the free boundary of the perpetual American lookback call is constant, i.e., $S_\infty^*(t)=S_\infty^*$ for all t. Hence, $x_\infty^*=\frac{m}{s_\infty^*}$ is constant, too.

In addition, it is necessary that $Re(A(w))<0$ to ensure that (5.5) holds as $T\to\infty$. Hence, $0<Re(w)<\lambda_2$. Then,

$$\frac{\partial\bar{C}_P^1}{\partial x}(x_\infty^*) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{k_1}{(w-\lambda_1)(w-\lambda_2)}\frac{1}{x_\infty^*}-(k_1-k_2)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{w}{(w-\lambda_1)(w-\lambda_2)(w+1)}d\eta dw.$$

Because $0<Re(w)<\lambda_2$, by application of the residue theorem,

$$\frac{\partial\bar{C}_P^1}{\partial x}(x_\infty^*) = -\frac{k_1}{x_\infty^*}\frac{1}{(\lambda_2-\lambda_1)}+(k_1-k_2)\frac{\lambda_2}{(\lambda_2-\lambda_1)(1+\lambda_2)}. \tag{5.6}$$

In case of $\bar{C}_P^2$, the Mellin transform of $f(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}$ is $\hat{f}(t,k_2-1-w)$ and

$$\hat{f}(t,k_2-1-w) = \frac{rx^*(t)^{(k_2-w)}}{k_2-w}-\frac{qx^*(t)^{(k_2-1-w)}}{k_2-1-w}$$

with $Re(w)<k_2-1$.

Similarly, as $T \to \infty$,

$$\bar{C}_P^2(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{rx^*(\eta)^{(k_2-w)}}{k_2-w} - \frac{qx^*(\eta)^{(k_2-1-w)}}{k_2-1-w}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \tag{5.7}$$

and

$$\begin{aligned}\frac{\partial \bar{C}_P^2}{\partial x} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w-1}\int_t^\infty \left[\frac{rw}{k_2-w}{x_\infty^*}^{(k_2-w)} - \frac{qw}{(k_2-1-w)}{x_\infty^*}^{(k_2-1-w)}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2}{\sigma^2}\frac{1}{A(w)} x^{-(w+1)}\left[q\frac{w}{k_2-w-1}{x_\infty^*}^{(k_2-1-w)} - r\frac{w}{k_2-w}{x_\infty^*}^{(k_2-w)}\right] dw\end{aligned}$$

with $\lambda_1 < Re(w) < k_2 - 1$.

By application of the residue theorem,

$$\begin{aligned}\frac{\partial \bar{C}_P^2}{\partial x}(x_\infty^*) &= k_1\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{w{x_\infty^*}^{(k_2-2-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-1-w)} \\ &\quad - (k_1-k_2)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{w{x_\infty^*}^{(k_2-1-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-w)} d\eta dw \\ &= k_1\frac{\lambda_1 {x_\infty^*}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} - (k_1-k_2)\frac{\lambda_1 {x_\infty^*}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)}.\end{aligned} \tag{5.8}$$

In case of $\bar{C}_P^3$, the Mellin transform of $(k_2-1)\left[\int_1^x f(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}$ is $-\frac{(k_2-1)}{w}\hat{f}(t,k_2-1-w)$ and

$$-\frac{(k_2-1)}{w}\hat{f}(t,k_2-1-w) = r\frac{(1-k_2)x^*(t)^{(k_2-w)}}{w(k_2-w)} - q\frac{(1-k_2)x^*(t)^{(k_2-1-w)}}{w(k_2-1-w)}$$

with $Re(w) < 0$.

Similarly,

$$\bar{C}_P^3(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{r(1-k_2)x^*(\eta)^{(k_2-w)}}{w(k_2-w)} - \frac{q(1-k_2)x^*(\eta)^{(k_2-1-w)}}{w(k_2-1-w)}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \tag{5.9}$$

and as $T \to \infty$,

$$\begin{aligned}\frac{\partial \bar{C}_P^3}{\partial x} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w-1}\int_t^\infty \left[r\frac{(1-k_2)}{k_2-w}{x_\infty^*}^{(k_2-w)} - q\frac{(1-k_2)}{(k_2-1-w)}{x_\infty^*}^{(k_2-1-w)}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2}{\sigma^2}\frac{1}{A(w)} x^{-(w+1)}\left[q\frac{(1-k_2)}{(k_2-1-w)}{x_\infty^*}^{(k_2-1-w)} - \frac{(1-k_2)}{k_2-w}{x_\infty^*}^{(k_2-w)}\right] dw\end{aligned}$$

with $\lambda_1 < Re(w) < \min\{0, k_2-1\}$.

By application of the residue theorem,

$$\begin{aligned}\frac{\partial \bar{C}_P^3}{\partial x}(x_\infty^*) &= \frac{k_1(1-k_2)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{{x_\infty^*}^{(k_2-2-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-1-w)} \\ &\quad - \frac{(k_1-k_2)(1-k_2)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{{x_\infty^*}^{(k_2-1-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-w)} dw \\ &= k_1(1-k_2)\frac{{x_\infty^*}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} - (1-k_2)(k_1-k_2)\frac{{x_\infty^*}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)}.\end{aligned} \tag{5.10}$$

By (4.5) and $x = \frac{m}{s}$,

$$\begin{aligned}\bar{C}_E(t,x) &= e^{-q(T-t)}\mathcal{N}\left(d_1^+\left(\frac{1}{x}, T-t\right)\right) - xe^{-r(T-t)}\mathcal{N}\left(d_1^-\left(\frac{1}{x}, T-t\right)\right) \\ &\quad + \frac{\sigma^2}{2(r-q)}\left\{e^{-r(T-t)}x^{\frac{2(r-q)}{\sigma^2}}\mathcal{N}(d_1^-(x,T-t)) - e^{-q(T-t)}\mathcal{N}\left(-d_1^+\left(\frac{1}{x}, T-t\right)\right)\right\}.\end{aligned} \tag{5.11}$$

By performing a computation, it is not hard to show that

$$\frac{\partial \bar{C}_E}{\partial x} \to 0 \quad \text{as } T \to \infty. \tag{5.12}$$

By applying "smooth pasting conditions",

$$\lim_{x \to x^*(t)} \frac{\partial \bar{C}}{\partial x} = -1 \tag{5.13}$$

and (5.6), (5.8), (5.10), and (5.12),

$$\begin{aligned} -1 &= -\frac{k_1}{x^*_\infty}\frac{1}{(\lambda_2-\lambda_1)} + (k_1-k_2)\frac{\lambda_2}{(\lambda_2-\lambda_1)(1+\lambda_2)} + k_1\frac{\lambda_1 {x^*_\infty}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} - (k_1-k_2)\frac{\lambda_1 {x^*_\infty}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)} \\ &\quad + k_1(1-k_2)\frac{{x^*_\infty}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} - (1-k_2)(k_1-k_2)\frac{{x^*_\infty}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)}. \end{aligned}$$

Using $\lambda_1+\lambda_2 = k_2 - 1$ and $\lambda_1\lambda_2 = -k_1$, we obtain

$${x^*_\infty}^{(\lambda_2-\lambda_1)} = \frac{\lambda_1}{\lambda_2}\frac{(1+\lambda_2)x^*_\infty - \lambda_2}{(1+\lambda_1)x^*_\infty - \lambda_1}, \tag{5.14}$$

which is the critical value of an American floating strike lookback call option derived by Dai [17]. □

We can also prove the following theorem using Theorem 5.1.

Theorem 5.2 (***Price of Perpetual American Floating Strike Call Option***). *The closed-form price formula of the perpetual American lookback call option is given by*

$$C(s,m) = s\Lambda_1\left(\frac{s}{m}\right)^{\lambda_2} + s\Lambda_2\left(\frac{s}{m}\right)^{\lambda_1}, \quad s > s^*_\infty$$

where

$$\Lambda_1 = \frac{(1+\lambda_1)x^*_\infty - \lambda_1}{(\lambda_2-\lambda_1){x^*_\infty}^{-\lambda_2}}, \qquad \Lambda_2 = \frac{(1+\lambda_2)x^*_\infty - \lambda_2}{(\lambda_1-\lambda_2){x^*_\infty}^{-\lambda_1}}$$

and λ_1, λ_2, and x^*_∞ are defined in Theorem 5.1.

Proof. By (5.4),

$$\bar{C}^1_P(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{rx^*(\eta)^{w+1}}{w+1} - \frac{qx^*(\eta)^w}{w}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw$$

with $0 < Re(w) < \lambda_2$.

Letting $T \to \infty$, then

$$\begin{aligned} \bar{C}^1_P(x) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^\infty \left[\frac{r{x^*_\infty}^{w+1}}{w+1} - \frac{q{x^*_\infty}^{w}}{w}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\ &= \frac{1}{2\pi i}\frac{2r}{\sigma^2}\int_{c-i\infty}^{c+i\infty} \frac{x^*_\infty}{(w+1)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x^*_\infty}{x}\right)^w dw \\ &\quad - \frac{1}{2\pi i}\frac{2q}{\sigma^2}\int_{c-i\infty}^{c+i\infty} \frac{1}{w(w-\lambda_1)(w-\lambda_2)}\left(\frac{x^*_\infty}{x}\right)^w dw. \end{aligned}$$

By application of the residue theorem,

$$\bar{C}^1_P(x) = \frac{(1+\lambda_1)x^*_\infty - \lambda_1}{(\lambda_2-\lambda_1)}\left(\frac{x^*_\infty}{x}\right)^{\lambda_2}. \tag{5.15}$$

By (5.7),

$$\bar{C}^2_P(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{rx^*(\eta)^{(k_2-w)}}{k_2-w} - \frac{qx^*(\eta)^{(k_2-1-w)}}{k_2-1-w}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw$$

with $\lambda - 1 < Re(w) < k_2 - 1$.

Letting $T \to \infty$, then

$$\begin{aligned}\bar{C}_P^2(x) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^{\infty}\left[\frac{rx_\infty^{*\ (k_2-w)}}{k_2-w} - \frac{qx_\infty^{*\ (k_2-1-w)}}{k_2-1-w}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw \\ &= \frac{1}{2\pi i}\frac{2r}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{x_\infty^{*\ (k_2-2w)}}{(k_2-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw \\ &\quad - \frac{1}{2\pi i}\frac{2q}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{x_\infty^{*\ (k_2-1-2w)}}{(k_2-1-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw.\end{aligned}$$

By application of the residue theorem,

$$\bar{C}_P^2(x) = \frac{-(1+\lambda_1)x_\infty^* + \lambda_1}{(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1} x_\infty^{*\ (\lambda_2-\lambda_1)}. \tag{5.16}$$

By (5.9),

$$\bar{C}_P^3(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^{T}\left[\frac{r(1-k_2)x^*(\eta)^{(k_2-w)}}{w(k_2-w)} - \frac{q(1-k_2)x^*(\eta)^{(k_2-1-w)}}{w(k_2-1-w)}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw$$

with $Re(w) < \min\{0, k_2-1\}$.

Letting $T \to \infty$, then

$$\begin{aligned}\bar{C}_P^3(x) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^{\infty}\left[\frac{r(1-k_2)x_\infty^{*\ (k_2-w)}}{w(k_2-w)} - \frac{q(1-k_2)x_\infty^{*\ (k_2-1-w)}}{w(k_2-1-w)}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw \\ &= \frac{1}{2\pi i}\frac{2r}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{(1-k_2)x_\infty^{*\ (k_2-2w)}}{w(k_2-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw \\ &\quad - \frac{1}{2\pi i}\frac{2q}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{(1-k_2)x_\infty^{*\ (k_2-1-2w)}}{w(k_2-1-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw.\end{aligned}$$

By application of the residue theorem,

$$\bar{C}_P^3(x) = \frac{(\lambda_1+\lambda_2)\{(1+\lambda_1)x_\infty^* - \lambda_1\}}{\lambda_1(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1} x_\infty^{*\ (\lambda_2-\lambda_1)}. \tag{5.17}$$

By Theorem 5.1 and (5.15), (5.16), and (5.17)

$$\bar{C}_P(x) = \frac{(1+\lambda_1)x_\infty^* - \lambda_1}{(\lambda_2-\lambda_1)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_2} + \frac{(1+\lambda_2)x_\infty^* - \lambda_2}{(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1}.$$

From (4.5),

$$\bar{C}_E(t,x) \to 0 \quad \text{as } T \to \infty; \tag{5.18}$$

therefore,

$$\bar{C}(x) = \bar{C}_P(x) = \frac{(1+\lambda_1)x_\infty^* - \lambda_1}{(\lambda_2-\lambda_1)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_2} + \frac{(1+\lambda_2)x_\infty^* - \lambda_2}{(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1}. \tag{5.19}$$

Because $x = \frac{m}{s}$, we obtained the desired result. □

6. Conclusion

This paper describes our examination of the valuation of American floating strike lookback options written on dividend-paying assets. Usually, the valuation of American options is equivalent to solving a free boundary problem of the inhomogeneous Black–Scholes PDE. However, in the case of American floating strike lookback options, the Neumann boundary condition should be considered due to the existence of the lookback state variable. Therefore, our examination started with an analysis of the inhomogeneous Black–Scholes PDE with a Neumann boundary condition. The analysis consisted of two steps.

We first converted the PDE with a Neumann boundary condition into the same PDE with a Dirichlet boundary condition by changing the variables such that they were particularly apt to accommodate the properties of the Mellin transform.

The second step consisted of extending the domain of the given PDE. Whereas the Mellin transform is defined on the domain, the given PDE involving the value function is defined only on the domain, and we used a scaling and reflection method to introduce the absorbing Neumann boundary operator. We then used this operator to extend the PDE onto the domain. Using these two steps, we derived the integral equation satisfied by American floating strike lookback options, after which we also derived the closed form of the value of perpetual American options.

In conclusion, we proposed an approach based on the Mellin transform for solving inhomogeneous Black–Scholes PDEs with a Neumann boundary condition. As a result of the general applicability of our methodology, we expect this approach to be useful for solving a variety of option problems.

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Appendix. Review of Mellin transforms

This appendix summarizes the definition and basic properties of Mellin transforms (refer to [28,29], and [30]), which are useful for the transformation of PDEs.

Definition A.1 (*Definition of Mellin Transform and Inverse Mellin Transform*). Let $f(x)$ be a *locally integrable function* on $(0, \infty)$. Then, the *Mellin transform* $\mathcal{M}(f(x), w)$ of $f(x)$ is defined by

$$\mathcal{M}(f(x); w) := \hat{f}(w) = \int_0^\infty f(x) \quad x^{w-1} dw, \quad w \in \mathbb{C} \tag{A.1}$$

and if this integral converges for $a < Re(w) < b$ and $a < c < b$, then the inverse of the Mellin transform is given by

$$f(x) = \mathcal{M}^{-1}\left(\hat{f}(w)\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(w) x^{-w} dw. \tag{A.2}$$

Proposition A.1 (*Convolution Property of Mellin Transform*). *Let $g(x)$ and $h(x)$ be locally integrable functions on $(0, \infty)$. For $a < w < b$, let the Mellin transforms $\hat{g}(w)$ and $\hat{h}(w)$ exist. Then, the Mellin convolution of $g(x)$ and $h(x)$ is given by the inverse Mellin transform of $\hat{g}(w)\hat{h}(w)$ as follows:*

$$\begin{aligned} g(x) \vee h(x) &:= \int_{c-i\infty}^{c+i\infty} \hat{g}(w)\hat{h}(w) x^{-w} dw, \quad a < c < b \\ &= \int_0^\infty g\left(\frac{x}{u}\right) h(u) \frac{du}{u}. \end{aligned} \tag{A.3}$$

Proposition A.2 (*Basic Properties of Mellin Transform*).

(1) *For constant α, $\mathcal{M}(x^\alpha f(x); w) = \hat{f}(w+\alpha)$;*
(2) *For constant α,*

$$\mathcal{M}(f(x^\alpha); w) = \begin{cases} \dfrac{1}{\alpha}\hat{f}\left(\dfrac{w}{\alpha}\right) & \text{for } \alpha > 0 \\ -\dfrac{1}{\alpha}\hat{f}\left(\dfrac{w}{\alpha}\right) & \text{for } \alpha < 0. \end{cases}$$

(3) *For positive integer n,*

$$\mathcal{M}\left(\left(x\frac{\partial}{\partial x}\right)^n f(x); w\right) = (-w)^n \hat{f}(w)$$

and

$$\mathcal{M}\left(\left(\frac{\partial}{\partial x}x\right)^n f(x); w\right) = (1-w)^n \hat{f}(w).$$

(4) *For constant* α, β, *define*

$$f(x) = \begin{cases} x^{\beta} & \text{for } x < \alpha \\ 0 & \text{for } x > \alpha. \end{cases}$$

Then,

$$\hat{f}(w) = \alpha^{(w+\beta)} \frac{1}{w+\beta},$$

where $Re(w) > -Re(\beta)$.

Proposition A.3 (*Inverse Mellin Transform of Exponential Function*)**.** *For* α *with* $Re(\alpha) > 0$, *let* $\hat{f}(w) = e^{\alpha(w+\beta)^2}$. *Then, the inverse Mellin transform of* $f(x)$ *is given by*

$$\mathcal{M}^{-1}\left(\hat{f}(w); x\right) = \frac{1}{2}(\pi\alpha)^{-\frac{1}{2}} x^{\beta} e^{-\frac{1}{4\alpha}(\ln x)^2}.$$

References

[1] H.U. Gerber, E.S.W. Shiu, Pricing lookback options and dynamic guarantees, N. Am. Actuar. J. 7 (2003) 48–67.
[2] H. Lee, Pricing equity-indexed annuities with path-dependent options, Insurance Math. Econom. 33 (2003) 677–690.
[3] M.B. Goldman, H.B. Sosin, M.A. Gatto, Path dependent options: buy at the low, sell at the high, J. Finance 34 (1979) 1111–1127.
[4] A. Conze, R. Viswanathan, Path dependent options: The case of lookback options, J. Finance 46 (1991) 1893–1907.
[5] M. Dai, H.Y. Wong, Y.K. Kwok, Quanto lookback options, Math. Finance 14 (2004) 445–467.
[6] Y.K. Kwok, Mathematical Models of Financial Derivatives, second ed., Springer-Verlag, Berlin, 2008.
[7] G. Peskir, A. Shiryaev, Optimal Stopping and Free-Boundary Problems, Birkhauser, Basel, Switzerland, 2006.
[8] I. Kim, The analytic valuation of American options, Rev. Financ. Stud. 3 (4) (1990) 547–572.
[9] G. Peskir, The Russian option: Finite horizon, Finance Stoch. 9 (2005) 251–267.
[10] L.A. Shepp, A.N. Shiryaev, The Russian option: Reduced regret, Ann. Appl. Probab. 3 (1993) 631–640.
[11] E. Ekstrom, Russian options with a finite time horizon, J. Appl. Probab. 41 (2004) 313–326.
[12] H. Yu, Y.K. Kwok, L. Wu, Early exercise policies of American floating and fixed strike lookback options, Nonlinear Anal. 47 (2001) 4591–4602.
[13] M. Dai, Y.K. Kwok, Characterization of optimal stopping regions of American Asian and lookback options, Math. Finance 16 (2006) 63–82.
[14] M. Dai, Y.K. Kwok, American options with lookback payoff, SIAM J. Appl. Math. 66 (2005) 206–227.
[15] T.L. Lai, T.W. Lim, Efficient Valuation of American Floating Strike Lookback Options, Technical Report, Department of Statistics, Stanford University, Stanford, CA, 2004.
[16] T. Kimura, American fractional lookback options: Valuation and premium decomposition, SIAM J. Appl. Math. 71 (2011) 517–539.
[17] M. Dai, A closed-form solution for perpetual American floating strike lookback options, J. Comput. Finance 4 (2000) 63–68.
[18] R. Panini, R.P. Srivastav, Option pricing with Mellin transforms, Math. Comput. Modelling 40 (2004) 43–56.
[19] R. Panini, R.P. Srivastav, Pricing perpetual options using Mellin transforms, Appl. Math. Lett. 18 (4) (2005) 471–474.
[20] R. Frontczak, R. Schobel, On modified Mellin transforms, Gauss–Laguerre quadrature, and the valuation of American call options, J. Comput. Appl. Math. 234 (5) (2010) 1559–1571.
[21] J.H. Yoon, J.H. Kim, The pricing of vulnerable options with double Mellin transforms, J. Math. Anal. Appl. 422 (2) (2015) 838–857.
[22] J.H. Yoon, Mellin transform method for European option pricing with Hull–White stochastic interest rate, J. Appl. Math. (2014) 7. Article ID 759562.
[23] P. Buchen, An Introduction to Exotic Option Pricing, CRC Press, Boca Raton, FL, 2012.
[24] J. Jeon, H. Han, H. Kim, M. Kang, An integral equation representation approach for Russian options with finite time horizon, Commun. Nonlinear Sci. Numer. Simul. 36 (2016) 496–516.
[25] J. Jeon, J.H. Yoon, M. Kang, Valuing vulnerable geometric Asian options, Comput. Math. Appl. 71 (2) (2016) 676–691.
[26] P. Wilmott, J. Dewynne, J. Howison, Option Pricing: Mathematical Models and Computation, Oxford Financial Press, Oxford, 1993.
[27] L.C. Evans, Partial Differential Equations, in: Graduate Studies in Mathematics, vol. 60, American Mathematical Society, 1998.
[28] A. Erd lyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Tables of Integral Transforms, Vols. 1–2, McGraw-Hill, New York, 1954.
[29] I.N. Sneddon, The Use of Integral Transforms, McGraw-Hill, New York, 1972.
[30] J. Bertrand, P. Bertrand, J. Ovarlez, The Mellin transform, in: D. Alexander (Ed.), The Transforms and Applications Handbook, second ed., CRC Press, Boca Raton, FL, 2000.