Applied Mathematics Letters 65 (2017) 7–13

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A generalization of the Bernfeld–Haddock conjecture

Bingwen Liu[1]

College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing, Zhejiang 314001, PR China

ARTICLE INFO

Article history:
Received 12 July 2016
Received in revised form 23 September 2016
Accepted 23 September 2016
Available online 2 October 2016

Keywords:
Bernfeld–Haddock conjecture
Non-autonomous differential equation
Time-varying delay
Convergence

ABSTRACT

In this paper, we investigate the convergence of a non-autonomous differential equation with a time-varying delay. The equation has important practical applications. It is shown that every solution of the equation is bounded and tends to a constant as $t \to +\infty$, which generalizes the Bernfeld–Haddock conjecture. Our result improves and extends some existing ones.

1. Introduction

In the 1976, Bernfeld and Haddock [1] proposed the following conjecture.

Conjecture ([1]). *Every solution of the delay differential equation*

$$x'(t) = -x^{\frac{1}{3}}(t) + x^{\frac{1}{3}}(t-r), \tag{1.1}$$

where $r > 0$, tends to a constant as $t \to +\infty$.

To confirm the above conjecture, variants of the above equation, which have been used as models for some population growth and the spread of epidemics, have received considerable attention (see, for example, [2–13] and the references therein). In the discussion of [2–6], Proposition 4 and Proposition 5 of [2] played a crucial role. In [2], Ding considered the following differential equation

$$x'(t) = -F(x(t)) + F(c) \tag{1.2}$$

with the initial condition

$$x(t_0) = x_0 \quad \text{for all } t_0, x_0 \in \mathbf{R}, \tag{1.3}$$

E-mail address: liubw007@aliyun.com.
[1] Fax: +86 057383643075.

http://dx.doi.org/10.1016/j.aml.2016.09.018

where $F : \mathbf{R} \to \mathbf{R}$ is continuous and strictly increasing. Recently, Yi and Huang [14] gave a counterexample to demonstrate that the two propositions are not true because the uniqueness of the left-hand solution of the initial value problem (1.2) and (1.3) was employed in their proofs. It follows from the Peano existence theorem that for any given constant c, the right-hand solution of the above initial value problem exists and is unique. But the counterexample in [14] implies that the uniqueness of the left-hand solution does not necessarily hold. Then Ding improved these propositions, which are included as Appendix of [14]. Precisely, under the assumption,

(H) If $c \neq 0$, then the left-hand solution of Eq. (1.2) with the initial value $x(t_0) = x_0$ is unique.

Proposition 4 and Proposition 5 of [2] are corrected as follows.

Proposition 4*. *Let* (H) *hold. Consider the differential equation*

$$u' = -F(u) + F(c + \varepsilon), \tag{1.4}$$

where $c \neq 0$ is a given constant, ε is a parameter satisfying $0 \leq \varepsilon \leq \frac{|c|}{2}$. Moreover, assume the initial condition

$$u(t_0) = u_0 \qquad (u_0 < c). \tag{1.5}$$

Let $u = u(t; t_0, u_0)$ be the solution of the initial value problem (1.4) *and* (1.5), *and $\alpha > 0$ be a given constant. Then there exists a positive real number μ independent of t_0 and ε such that*

$$(c + \varepsilon) - u(t; t_0, u_0) \geq \mu > 0 \quad \textit{for} \quad t \in [t_0, t_0 + \alpha].$$

Proposition 5*. *Let* (H) *hold. Consider the differential equation*

$$u' = -F(u) + F(c - \varepsilon), \tag{1.6}$$

where $c \neq 0$ is a given constant, ε is a parameter satisfying $0 \leq \varepsilon \leq \frac{|c|}{2}$. Moreover, assume the initial condition

$$u(t_0) = u_0 \qquad (u_0 > c). \tag{1.7}$$

Let $u = u(t; t_0, u_0)$ be the solution of the initial value problem (1.6) *and* (1.7), *and $\alpha > 0$ be a given constant. Then there exists a positive real number ν independent of t_0 and ε such that*

$$u(t; t_0, u_0) - (c - \varepsilon) \geq \nu > 0 \quad \textit{for } t \in [t_0, t_0 + \alpha].$$

It should be mentioned that there is a typo in (H) of [14]. The initial value should be $x(t_0) = x_0$ instead of $x(t_0) = c$, which can be found by examining the proof of Proposition 4* in [14]. This has also been cited wrongly as a key technical assumption, (A1), in the most recent paper by Zhou [15].

It is difficult to provide some sufficient conditions on guaranteeing the uniqueness of the left-hand solution of the initial value problem (1.2). So far it has not been shown which kind of technical conditions should be added on the function F that satisfies (H). Therefore, the proof in the appendix of [14] needs further improvement. Moreover, delays in population and ecology models are usually time-varying and hence (1.1) usually can be generalized as the following non-autonomous equation,

$$x'(t) = -F(x(t)) + F(x(t - \tau(t))), \tag{1.8}$$

where $\tau : \mathbf{R} \to (0, \infty)$ is a bounded and continuous function. Throughout this paper, we assume that $F : \mathbf{R} \to \mathbf{R}$ is continuous and strictly increasing, and

$$F(0) = 0,\ F(x) \text{ is continuous differentiable on } \mathbf{R} \setminus \{0\},\ F'(x) > 0 \text{ for all } x \in \mathbf{R} \setminus \{0\}. \tag{1.9}$$

It is well known that a non-autonomous delay differential equation generally does not generate a semiflow and hence the methods for differential equations with constant delays [7–9,11–14] are not suitable for (1.8).

Motivated by the above discussion, we aim to employ a novel argument to show that every solution of (1.8) tends to a constant as $t \to +\infty$. For convenience, F^{-1} denotes the inverse function of F. We let $r = \sup_{t\in\mathbf{R}}\tau(t) \geq \inf_{t\in\mathbf{R}}\tau(t) > 0$ and $C = C([-r, 0], \mathbf{R})$. If $\sigma \geq 0$, $t_0 \in \mathbf{R}$, and $x \in C([t_0 - r, t_0 + \sigma], \mathbf{R})$, then, for any $t \in [t_0, t_0 + \sigma]$, $x_t \in C$ is defined by $x_t(t_0, \theta) = x(t_0, t + \theta)$, $-r \leq \theta \leq 0$. Moreover, for $\varphi \in C$, we use $x_t(t_0, \varphi)$ $(x(t; t_0, \varphi))$ to denote the solution of (1.8) with the initial data $x_{t_0}(t_0, \varphi) = \varphi$. For $V(t) \in C([a, \infty),\ \mathbf{R})$, let

$$D^+V(t) = \limsup_{h\to 0^+} \frac{V(t+h) - V(t)}{h} \quad \text{and} \quad D^-V(t) = \liminf_{h\to 0^+} \frac{V(t+h) - V(t)}{h}.$$

The remaining part of this paper is organized as follows. In Section 2, we show the boundedness and global existence of every solution for (1.8) with the initial data $x_{t_0} = \varphi \in C$. Based on the preparation in Section 2, we state and prove our main result in Section 3.

2. Preliminaries

Lemma 2.1. *For any constants $c \neq 0$, t_0 and x_0, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + F(c), \\ x(t_0) = x_0 \end{cases} \tag{2.1}$$

has a unique left-hand solution $x(t; t_0, x_0)$.

Proof. By way of contradiction, we assume that $x(t) = x(t; t_0, x_0)$ and $\bar{x}(t) = \bar{x}(t; t_0, x_0)$ are two different left-hand solutions of (2.1) with the maximal existence interval $(\eta, t_0]$ and $(\bar{\eta}, t_0]$, respectively. Then, there exists $t_1 \in (\max\{\eta, \bar{\eta}\}, t_0)$ such that $x(t_1) \neq \bar{x}(t_1)$. Let

$$t_2 = \sup\{t | t_1 < t \leq t_0, x(s) \neq \bar{x}(s) \text{ for all } t_1 < s < t\}.$$

It follows that either $x(s) > \bar{x}(s)$ or $x(s) < \bar{x}(s)$ must hold for all $t_1 < s < t_2$. It is easy to show that $x(t_2) = \bar{x}(t_2) = 0$ since $-F(u) + F(c)$ is continuous differentiable in the neighborhood of $x_0 \neq 0$, which guarantees the uniqueness of the left-hand solution of (2.1) with the initial value $x_0 \neq 0$.

Without loss of generality, we assume that $x(s) > \bar{x}(s)$ for all $t_1 < s < t_2$ and also assume $c > 0$ since the case $c < 0$ can be dealt with similarity. Then $x'(t_2) = \bar{x}'(t_2) = F(c) > 0$, which ensures the existence of a constant $\delta > 0$ such that $(t_2 - \delta, t_2) \subset (\max\{\eta, \bar{\eta}\}, t_0)$ and

$$\bar{x}(t) < x(t) < 0 \quad \text{for all} \quad t \in (t_2 - \delta, t_2).$$

Let $y_1(t) = -F(x(-t))$ and $y_2(t) = -F(\bar{x}(-t))$ for all $t \in (-t_2, \delta - t_2)$. Consequently, $y_1(t)$ and $y_2(t)$ are two different solutions of the following differential equation problem:

$$\begin{cases} \dfrac{1}{F'(F^{-1}(-y(t)))} y'(t) = y(t) + a, \quad a = F(c), t \in (-t_2, \delta - t_2), \\ \lim\limits_{t\to -t_2} y(t) = 0. \end{cases} \tag{2.2}$$

Let $G(y) = \int_0^y \frac{1}{(y+a)F'(F^{-1}(-y))} dy$. It follows from the first equation of (2.2) that

$$G(y(t)) = t + \tilde{C} \text{ for all } t \in (-t_2, \delta - t_2),$$

which, together with the second equation in (2.2), implies that

$$\tilde{C} = t_2. \tag{2.3}$$

From the fact that $G'(y) = \frac{1}{(y+a)F'(F^{-1}(-y))} > 0$ for all $y \in [0, +\infty)$, we obtain that G is strictly increasing on $[0, +\infty)$. However,

$$G(y_1(t)) = t + t_2 = G(y_2(t)),$$

which gives a contradiction since $0 < y_1(t) < y_2(t)$ are two different functions satisfying (2.2). This completes the proof.

Remark 2.1. From Lemma 2.1, one can easily see that $F(x) = x^{\frac{1}{3}}$ satisfies (1.9) and hence Proposition 4* and Proposition 5* hold in this case.

With the application of Lemma 3.2 in [11], we can easily obtain the following result.

Lemma 2.2. *Let $k \in (0, +\infty)$ be given and $d \in C([t_0, t_0 + k], \mathbf{R})$. Then, for any constant x_0, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + d(t), \\ x(t_0) = x_0 \end{cases}$$

has a unique solution $x(t)$ on $[t_0, t_0 + k]$.

Lemma 2.3. *Let $\varphi \in C$. Then $x_t(t_0, \varphi)$ exists and is unique on $[t_0, +\infty)$. Moreover, $x_t(t_0, \varphi)$ is bounded on $[t_0, +\infty)$.*

Proof. Let $\kappa = \inf_{t \in \mathbf{R}} \tau(t)$ and $d(t) = F(\varphi(t - \tau(t) - t_0)), \quad t \in [t_0, t_0 + \kappa]$. Consider the solution $x(t)$ of the following initial value problem,

$$\begin{cases} x'(t) = -F(x(t)) + d(t), \\ x(t_0) = \varphi(0). \end{cases}$$

By Lemma 2.2, $x(t)$ exists and is unique on $[t_0, t_0 + \kappa]$, that is, $x_t(\varphi)$ exists and is unique on $[t_0, t_0 + \kappa]$. Then $x_t(\varphi)$ exists and is unique on $[t_0, +\infty)$ by induction. Furthermore, we claim that

$$\alpha < x(t; t_0, \varphi) < \beta \quad \text{for all} \quad t \in [t_0, +\infty),$$

where α and β are two constants such that $\alpha < \varphi(s) < \beta$ for all $s \in [-r, 0]$. Suppose that the claim is not true. Then one of the following two cases must occur.

Case I. There exists $\theta_1 > t_0$ such that

$$x(\theta_1; t_0, \varphi) = \beta \text{ and } x(t; t_0, \varphi) < \beta \text{ for all } t \in [t_0 - r, \theta_1). \tag{2.4}$$

Case II. There exists $\theta_2 > t_0$ such that

$$x(\theta_2; t_0, \varphi) = \alpha \text{ and } \alpha < x(t; t_0, \varphi) \text{ for all } t \in [t_0 - r, \theta_2). \tag{2.5}$$

If Case I holds, then in view of (1.8) and (2.4), we have

$$0 \le x'(\theta_1) = -F(x(\theta_1)) + F(x(\theta_1 - \tau(\theta_1))) < -F(\beta) + F(\beta) = 0,$$

which is a contradiction. If Case II holds, similarly we will have

$$0 \ge x'(\theta_2) = -F(x(\theta_2)) + F(x(\theta_2 - \tau(\theta_2))) > -F(\alpha) + F(\alpha) = 0,$$

which is also a contradiction. Thus we have proved the claim and the proof is complete.

3. The main result

Theorem 3.1. *Let $\varphi \in C$. Then $x(t; t_0, \varphi)$ tends to a constant as $t \to +\infty$.*

Proof. Let $x(t) = x(t; t_0, \varphi)$, $y(t) = \max_{t-r \le s \le t} x(s)$, $u(t) = \min_{t-r \le s \le t} x(s)$ for all $t \ge t_0$ and $S = \{t | t \in [t_0, \infty), y(t) = x(t)\}$. Firstly, we show $D^+y(t) \le 0$ for all $t \ge t_0$. We distinguish two cases to finish the proof.

Case 1. $t \in [t_0, \infty) \setminus S$. Then there exists $t^* \in [t - r, t)$ such that

$$y(t) = \max_{t-r \le s \le t} x(s) = x(t^*) > x(t).$$

From the continuity of $x(\cdot)$ at t, we can choose a positive constant $\delta < r$ such that

$$x(s) < x(t^*) \quad \text{for all } s \in [t, t + \delta],$$

which yields

$$x(s) \le x(t^*) = \max_{t-r \le s \le t} x(s) = y(t) \quad \text{for all } s \in [t - r, t + \delta].$$

It follows that

$$y(t + h) = \max_{t+h-r \le s \le t+h} x(s) \le \max_{t-r \le s \le t+\delta} x(s) = x(t^*) = y(t) \quad \text{for all } h \in (0, \delta)$$

and hence

$$D^+y(t) = \limsup_{h \to 0^+} \frac{y(t + h) - y(t)}{h} \le \limsup_{h \to 0^+} \frac{y(t) - y(t)}{h} = 0.$$

Case 2. $t \in S$. Then (1.8) implies that

$$0 \le x'(t) = -F(x(t)) + F(x(t - \tau(t))) \le -F(x(t)) + F(x(t)) = 0,$$

which gives $x'(t) = 0$. Let $\rho = \frac{1}{2}\inf_{t \le s \le t+r} \tau(s)$. Obviously, $\rho > 0$. First assume that $y(s) = x(s)$ for all $s \in (t, t + \rho]$. Then we have

$$D^+y(t) = \limsup_{h \to 0^+} \frac{y(t + h) - y(t)}{h} = \limsup_{h \to 0^+} \frac{x(t + h) - x(t)}{h} = x'(t) = 0.$$

Now assume that there exists $s_1 \in (t, t + \rho]$ such that $y(s_1) > x(s_1)$. Then we can choose a constant $\tilde{t} \in [s_1 - r, s_1)$ such that

$$y(s_1) = x(\tilde{t}) = \max_{s_1 - r \le s \le s_1} x(s).$$

This, together with the fact that $t-r<s_1-r\le t+\rho-r<t<s_1$, implies

$$x(\tilde{t})\ge x(t)=y(t)=\max_{t-r\le s\le t}x(s).$$

We claim that $x(\tilde{t})=x(t)=y(t)$. Otherwise, $x(\tilde{t})>x(t)$. Then $t<\tilde{t}<s_1$ and

$$0\le x'(\tilde{t})=-F(x(\tilde{t}))+F(x(\tilde{t}-\tau(\tilde{t}))),$$

which follows that

$$x(\tilde{t}-\tau(\tilde{t}))\ge x(\tilde{t})>x(t). \tag{3.1}$$

Noting that $t-r\le t-\tau(\tilde{t})<\tilde{t}-\tau(\tilde{t})<\tilde{t}-\rho<t<s_1$, we have

$$x(\tilde{t})\le x(\tilde{t}-\tau(\tilde{t}))\le \max_{t-r\le s\le t}x(s)=x(t),$$

which contradicts with (3.1). Thus we have proved the claim. It follows that $\max_{t-r\le s\le s_1}x(s)=x(t)$ and hence

$$D^+y(t)=\limsup_{h\to0^+}\frac{y(t+h)-y(t)}{h}=\limsup_{h\to0^+}\frac{y(t+h)-x(t)}{h}\le\limsup_{h\to0^+}\frac{x(t)-x(t)}{h}=0.$$

Secondly, using similar arguments as those in the proof of $D^+y(t)\le0$ for all $t\ge t_0$, we can get

$$D^-u(t)\ge0\quad\text{for all}\quad t\ge t_0.$$

From the above results, we see that y is non-increasing and u is non-decreasing on $[t_0,+\infty)$. In view of the boundedness of x, we obtain

$$\limsup_{t\to+\infty}x(t)=\lim_{t\to+\infty}y(t)=A,\quad\text{and}\quad\liminf_{t\to+\infty}x(t)=\lim_{t\to+\infty}u(t)=B.$$

It suffices to show that $A=B$. Suppose that, on the contrary, $B<A$. Then A and B are not zero simultaneously. Without loss of generality, we assume that $A\ne0$ since the proof for the case of $B\ne0$ is quite similar. For $H\in(B,A)$, we can choose $t_0^*>t_0$ and $\{\tau_m\}_{m=1}^{\infty}\subset[t_0^*+r,+\infty)$ such that

$$x(\tau_m)=H,\quad\lim_{m\to+\infty}\tau_m=+\infty,\quad\text{and}\quad x(t)\le A+\frac{|A|}{2}\text{ for all }t\in[t_0^*,+\infty).$$

Then, for arbitrary positive integer m,

$$\begin{aligned}&F(A)\le F(y(\tau_m))=F(A+\varepsilon_m),0\le\varepsilon_m\le\frac{|A|}{2},\ \varepsilon_m=y(\tau_m)-A\to0\text{ (as }m\to+\infty),\\&x'(t)\le-F(x(t))+F(y(\tau_m))=-F(x(t))+F(A+\varepsilon_m)\text{ for all }t\in[\tau_m,\tau_m+2r].\end{aligned}\tag{3.2}$$

Denote $v(t)=v(t;\tau_m,\varepsilon_m)$ the solutions of the initial-value problem

$$v'(t)=-F(v(t))+F(A+\varepsilon_m),\ v(\tau_m)=H. \tag{3.3}$$

Note that $H<A$. Proposition 4* implies that

$$A+\varepsilon_m-v(t;\tau_m,\varepsilon_m)\ge\mu>0,\ t\in[\tau_m,\tau_m+2r],$$

where the positive constant μ is independent of τ_m and ε_m. Furthermore, from (3.2) and (3.3), we have

$$x(t)\le v(t)<A+\varepsilon_m-\mu,\ t\in[\tau_m,\tau_m+2r]\text{ and }\ y(\tau_m+r)<A+\varepsilon_m-\mu,$$

which contradicts with the fact that $\lim_{m\to+\infty}y(\tau_m+r)=\lim_{t\to+\infty}y(t)=A$.

Remark 3.1. Note that the Bernfeld–Haddock conjecture only deals with the special case where $F(x) = x^{\frac{1}{3}}$ and $\tau(t) = r$ is a constant, and the results on the variants of Eq. (1.1) in [7–9,11–14] give no information about functional differential equations with time-varying delays. Here we have given a detailed proof on the uniqueness of the left-hand solution of the initial value problem (1.2) and (1.3) with $F(x)$ satisfying (1.9). This has not been touched in [2–6,14,15]. In particular, in view of Lemma 2.2. in [10], the method of this paper can be readily modified and made applicable to an equation such as $x'(t) = \beta(t)[-F(x(t)) + F(x(t-\tau(t)))](\beta \in C(\mathbf{R}, (0, +\infty)))$. Hence, our results improve and extend the corresponding ones in the above references.

Acknowledgments

I wish to thank the anonymous reviewers whose valuable comments and suggestions led to significant improvement of the manuscript. This work was supported by the Natural Scientific Research Fund of Zhejiang Province of China (Grant No. LY16A010018), and the Natural Scientific Research Fund of Hunan Province of China (Grant Nos. 2016JJ6103, 2016JJ6104).

References

[1] S.R. Bernfeld, J.R.A. Haddock, Variation of Razumikhin's method for retarded functional equations, in: Variation of Razumikhin's method for retarded functional equations, Academic Press, New York, 1977, pp. 561–566.
[2] T. Ding, Asymptotic behavior of solutions of some retarded differential equations, Sci. China, Ser. A. 8 (1981) 939–945 (in Chinese).
[3] T. Ding, Asymptotic behavior of solutions of some retarded differential equations, Sci. China, Ser. A. 25 (1982) 263–371.
[4] T. Ding, Applications of the Qualitative Methods in Ordinary Differential Equations, China Higher Education Press, Peking, 2004, pp. 155–163 (in Chinese).
[5] B.S. Chen, Asymptotic behavior of a class of nonautonomous retarded differential equations, Chinese Sci. Bull. 6 (1988) 413–415 (in Chinese).
[6] B.S. Chen, Asymptotic behavior of solutions of some infinite retarded differential equations, Acta Math. Sinica 3 (1990) 353–358 (in Chinese).
[7] O. Arino, P. Seguier, About the behavior at infinity of solutions of: $x'(t) = f(t-1, x(t-1)) - f(t, x(t))$, J. Math. Anal. Appl. 96 (1983) 420–436.
[8] K.L. Cooke, J.L. Kaplan, A periodicity threshold theorem for epidemic and population growth, Math. Biosci. 31 (1976) 87–104.
[9] K.L. Cooke, J. Yorke, Some equations modelling growth process and gonorrhea epidemics, Math. Biosci. 16 (1973) 75–107.
[10] B. Liu, Asymptotic behavior of solutions to a class of non-autonomous delay differential equations, J. Math. Anal. Appl. 446 (2017) 580–590.
[11] T. Yi, L. Huang, Convergence for pseudo monotone semi-flows on product ordered topological spaces, J. Differential Equations 214 (2005) 429–456.
[12] T. Yi, L. Huang, Convergence of solution to a class of systems of delay differential equations, Nonlinear Dyn. Syst. Theory 5 (2005) 189–200.
[13] M. Xu, W. Chen, X. Yi, New generalization of the two-dimensional Bernfeld-Haddock conjecture and its proof, Nonlinear Anal. Real World Appl. 11 (2010) 3413–3420.
[14] T. Yi, L. Huang, A generalization of the Bernfeld-Haddock conjecture and its proof, Acta Math. Sinica (Chin. Ser.) 50 (2) (2007) 261–270.
[15] Q. Zhou, Asymptotic behavior of solutions to a class of non-autonomous delay differential equations, Electron J. Differential Equations 103 (2011) 1–8.