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The Cauchy problem for a Bardina–Oldroyd model to the incompressible viscoelastic flow

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ABSTRACT

In this paper, we study the Cauchy problem for a regularized viscoelastic fluid model in space dimension two, the Bardina–Oldroyd model, which is inspired by the simplified Bardina model for the turbulent flows of fluids, introduced by Cao et al. (2006). In particular, we obtain the local existence of smooth solutions to this model via the contraction mapping principle. Furthermore, we prove the global existence of smooth solutions to this system.

1. Introduction

In this article, our study is concerned with the following incompressible Oldroyd model describing the incompressible non-Newtonian fluid:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu \Delta u + \nabla P = \nabla \cdot (FF^T), \\ \partial_t F + (u \cdot \nabla)F = \nabla u F, \\ \nabla \cdot u = 0, \quad x \in \mathbb{R}^n, \ n = 2, 3, \\ u(0, x) = u_0, \end{cases} \tag{1.1}$$

where $u(t, x)$ denotes the fluid velocity vector field, $P = P(t, x)$ is the scalar pressure, $F = F(t, x) \in \mathbb{R}^n \times \mathbb{R}^n$ the deformation tensor, $\mu > 0$ is the constant kinematic viscosity, while u_0 is the given initial velocity with $\nabla \cdot u_0 = 0$.

There has been a lot of work on the existence theory of Oldroyd model (1.1). In particular, Lin–Liu–Zhang [1] proved global existence in the two-dimensional case by introducing an auxiliary vector field to replace the transport variable F, while Lei–Zhou [2] proved the same results via the incompressible limit working directly on the deformation tensor F. It is worthy mentioning that Lei–Liu–Zhou [3] proved

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the existence of both local and global smooth solutions to the Cauchy problem in the whole space and the periodic problem in the n-dimensional torus ($n = 2, 3$) in the case of near equilibrium initial data, while Lin–Zhang [4] established the global well-posedness of the initial–boundary value problem of the viscoelastic fluid system of the Oldroyd model with Dirichlet conditions. More discussions can be found in [5–12].

Viscoelastic materials include a wide range of fluids with elastic properties, as well as solids with fluid properties. To introduce our model, we need to give the standard description of general mechanical evolutions to introduce some notations and definitions and simplify the system (1.1). In the context of hydrodynamics, the basic variable is the particle trajectory $x(t, X)$, where X is the original labeling (Lagrangian coordinate) of the particle and referred to as the material coordinate, while x is the current (Eulerian) coordinate and referred to as the reference coordinate. For a given velocity field $u(t, x)$, the flow map $x(t, X)$ is defined by the following ordinary differential equation:

$$\frac{\partial x(t, X)}{\partial t} = u(t, x(t, X)), \quad x(0, X) = X.$$

The deformation tensor is then defined by $\tilde{F}(t, X) = \frac{\partial x(t,X)}{\partial X}$. In the Eulerian coordinate, the corresponding deformation tensor $F(t, x)$ is defined as $F(t, x(t, X)) = \tilde{F}(t, X)$. Using the chain rule, one can see that $F(t, x)$ satisfies the following transport equation, i.e. the second equation of (1.1):

$$\partial_t F + (u \cdot \nabla)F = \nabla u F.$$

If $\nabla \cdot F^T(0, x) = 0$, then we get from the second equations of (1.1):

$$\partial_t(\nabla \cdot F^T) + (u \cdot \nabla)(\nabla \cdot F^T) = 0.$$

Therefore, if $\nabla \cdot F^T(0, x) = 0$, it will remain so for later times, namely, $\nabla \cdot F^T = 0$ for any time $t > 0$. In what follows, we will make this assumption. Denote $F_k = Fe_k$ as the columns of F, then $\nabla \cdot (FF^T) = \sum_{k=1}^{n}(F_k \cdot \nabla)F_k$ by the fact $\nabla \cdot F_k = 0$. So the system (1.1) can be rewritten equivalently as

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu\Delta u + \nabla P = \displaystyle\sum_{k=1}^{n}(F_k \cdot \nabla)F_k, \\ \partial_t F_k + (u \cdot \nabla)F_k = (F_k \cdot \nabla)u, \\ \nabla \cdot u = \nabla \cdot F_k = 0, \end{cases} \tag{1.2}$$

with $k = 1, \ldots, n$, and $u(0, x) = u_0, F_k(0, x) = F_{k,0}$.

It is known that the global regularity problem for the 3D Navier–Stokes equations (NSE) is one of the most challenging outstanding problems in nonlinear analysis. The main difficulty lies in controlling certain norms of vorticity. More specifically, the vorticity stretching term in the 3D vorticity equation forms the main obstacle to achieving this control. Numerical solution of the Navier–Stokes equations for problems of engineering and geophysical relevance is not possible at present even on the most powerful computers [13,14]. In turbulent fluid flows, current scientific methods and tools are unable to compute the turbulent behavior of three-dimensional fluids analytically or via direct numerical simulation due to the large range of scales of motion that need to be resolved when the Reynolds number is high. To overcome this difficulty, much effort is being made to produce reliable turbulence models which parameterize the effect of the active small scales in terms of the large scales. The simplified Bardina model, introduced by Cao et al. [15], enjoys the most important property that it compares successfully with empirical data from turbulent channel and pipe flows, for a wide range of Reynolds numbers. So it is proved that it is a good sub-grid scale large-eddy simulation model of turbulence, just as the viscous Camassa–Holm equations (also known as the Lagrangian-averaged Navier–Stokes-α (LANS-α) model) (more discussions can be found in [16–21]). Based on the success of the simplified Bardina model in producing solutions in excellent agreement with empirical data, it is natural to lead us to consider such a kind of regularization also for the Oldroyd model.

In this paper, we consider the following Bardina type model for Oldroyd system in space dimension two, where we call it the Bardina–Oldroyd model in brevity:

$$\begin{cases} \partial_t v + (u\cdot\nabla)u - \mu\Delta v + \nabla P = \sum_{k=1}^{2}(F_k\cdot\nabla)F_k, \\ \partial_t F_k + (u\cdot\nabla)F_k = (F_k\cdot\nabla)u, \\ v = (1-\alpha^2\Delta)u, \\ \nabla\cdot v = \nabla\cdot u = \nabla\cdot F_k = 0, \end{cases} \tag{1.3}$$

with $v(0,x) = v_0, u(0,x) = u_0, F_k(0,x) = F_{k,0}, x \in \mathbb{R}^2$.

The main results in this paper are stated as follows, which are the local existence of the smooth solutions to the system (1.3) for Theorem 1.1 and the global existence of the smooth solution to the system (1.3) for Theorem 1.2.

Theorem 1.1 (*Local Existence*). *Let $v_0 \in H^m(\mathbb{R}^2)$ and $F_{k,0} \in H^{m+2}(\mathbb{R}^2)$ with $\nabla\cdot u_0 = \nabla\cdot v_0 = \nabla\cdot F_{k,0} = 0$ for $k = 1,2$ in $\mathbb{R}^2$, where $m \geq 2$ is an integer number. Then there exists a positive time T_0 such that the Bardina–Oldroyd model (1.3) admits a unique smooth solution (v, F_k) satisfying*

$$v \in L^\infty(0,T_0;H^m(\mathbb{R}^2)) \cap L^1(0,T_0;H^{m+1}(\mathbb{R}^2)) \cap L^2(0,T_0;H^2(\mathbb{R}^2)),$$
$$F_k \in L^\infty(0,T_0;H^{m+2}(\mathbb{R}^2)).$$

Theorem 1.2 (*Global Existence*). *Let $v_0 \in H^3(\mathbb{R}^2)$ and $F_{k,0} \in H^3(\mathbb{R}^2)$ with $\nabla\cdot u_0 = \nabla\cdot v_0 = \nabla\cdot F_{k,0} = 0$ for $k = 1,2$ in $\mathbb{R}^2$. Then the Bardina–Oldroyd model (1.3) has a unique global smooth solution (v, F_k) satisfying*

$$v \in L^\infty(0,T;H^3(\mathbb{R}^2)) \cap L^2(0,T;H^2(\mathbb{R}^2)),$$
$$F_k \in L^\infty(0,T;H^3(\mathbb{R}^2)),$$

for any time $T > 0$.

In the rest of this section, we give some lemmas, which will be used in the proof of our main results.

Firstly, we give the well-known commutator's estimates proved by Kato–Ponce [22].

Lemma 1.3. *Let $s > 0, 1 < p < \infty$, and suppose that $f, g \in \mathcal{S}(\mathbb{R}^2)$. Then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}),$$
$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}),$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, \Lambda = (-\Delta)^{\frac{1}{2}}$, $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz class of rapidly decreasing functions.

The following logarithmic Sobolev inequality is obtained by Brezis–Gallouet [23] and Brezis–Wainger [24].

Lemma 1.4. *Let $u \in H^2(\mathbb{R}^2)$. Then*

$$\|u\|_{L^\infty} \leq C\|u\|_{H^1}\ln(e + \|u\|_{H^2}),$$

where C is a constant.

Finally, the following well-known Gagliardo–Nirenberg inequalities in two space dimensions play an important role throughout our proofs.

Lemma 1.5. *In two space dimensions, the following inequalities*

$$\begin{cases} \|u\|_{L^4} \le C\|u\|_{L^2}^{\frac{3}{4}}\|\nabla^2 u\|_{L^2}^{\frac{1}{4}}, \\ \|u\|_{L^\infty} \le C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}, \\ \|\nabla u\|_{L^2} \le C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}, \\ \|\nabla u\|_{L^4} \le C\|u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 u\|_{L^2}^{\frac{3}{4}} \end{cases}$$

hold, where C is a generic constant.

This paper is organized as follows. Section 2 provides the proof of Theorem 1.1 using the fixed-point method. Section 3 details the proof of Theorem 1.2.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In this section, we will prove that there exists a positive time T_0 and a unique smooth solution (v, F_k) to the Bardina–Oldroyd model (1.3) in suitable spaces for $(t, x) \in [0, T_0] \times \mathbb{R}^2$.

Firstly, we give some notations which will be used in our proof. Without loss of generality, we assume $\mu = 1$ in what follows. Furthermore, for ease of notation, we will write $\sum_k$ for $\sum_{k=1}^2$, and denote $\partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_1}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. For any integer number $m \ge 0$, denote the functional spaces by

$$\mathbb{X}_m = L^\infty(0, T_0; H^m_\sigma(\mathbb{R}^2)), \qquad \mathbb{Y}_m = L^1(0, T_0; H^m_\sigma(\mathbb{R}^2)),$$

equipped with the following norms

$$\|u\|_{\mathbb{X}_m} = ess \sup_{0\le t\le T_0} \|u(t)\|_{H^m(\mathbb{R}^2)} = ess \sup_{0\le t\le T_0} \sum_{|\alpha|\le m} \|\partial^\alpha u(t)\|_{L^2(\mathbb{R}^2)},$$

$$\|u\|_{\mathbb{Y}_m} = \int_0^{T_0} \|u(t)\|_{H^m(\mathbb{R}^2)} dt = \int_0^{T_0} \sum_{|\alpha|\le m} \|\partial^\alpha u(t)\|_{L^2(\mathbb{R}^2)} dt,$$

respectively. Here the subscript σ represents divergence free function spaces. Now, let $\mathbb{Z}_m = \mathbb{X}_m \cap \mathbb{Y}_m$, equipped with the norm

$$\|u\|_{\mathbb{Z}_m} = \max\{K^{-1}\|u\|_{\mathbb{X}_m}, L^{-1}\|u\|_{\mathbb{Y}_{m+1}}\},$$

where the positive constants K and L will be chosen appropriately later on. And, denote by P_σ the projection on the subspace of the divergence-free functions. It is obvious that P_σ is continuous from $H^m(\mathbb{R}^2)$ into $H^m(\mathbb{R}^2)$ for each integer number $m \ge 0$ and $P_\sigma \nabla \equiv 0$.

We will prove our first main result via the contraction mapping principle by virtue of constructing a contraction map $f : S_{m+1} \longrightarrow S_{m+1}$ for $u = (1 - \alpha^2 \Delta)^{-1} v$ motivated by [25], which is used in [26] later. Here S_k represents the unitary ball, centered at the origin of $\mathbb{Z}_{k+1}$. We will split the proof into several steps.

Step 1. Energy estimates

Multiplying the first equation of (1.3) by u and the second equation of (1.3) by F_k, and integrating on $\mathbb{R}^2$, then

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \alpha^2\|\nabla u\|_{L^2}^2) + \int_{\mathbb{R}^2} (u \cdot \nabla)u \cdot u dx + \|\nabla u\|_{L^2}^2 + \alpha^2\|\Delta u\|_{L^2}^2 = \sum_k \int_{\mathbb{R}^2} (F_k \cdot \nabla)F_k \cdot u dx, \quad (2.1)$$

and

$$\frac{1}{2}\frac{d}{dt}\|F_k\|_{L^2}^2 = \int_{\mathbb{R}^2} (F_k \cdot \nabla)u \cdot F_k dx - \int_{\mathbb{R}^2} (u \cdot \nabla)F_k \cdot F_k dx. \quad (2.2)$$

Notice the facts that

$$\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot udx=\int_{\mathbb{R}^2}(u\cdot\nabla)F_k\cdot F_kdx=0,$$
$$\int_{\mathbb{R}^2}(F_k\cdot\nabla)F_k\cdot udx+\int_{\mathbb{R}^2}(F_k\cdot\nabla)u\cdot F_kdx=0,$$

by the conditions $\nabla\cdot u=\nabla\cdot F_k=0$. Then, summing over k for (2.2) and combining with (2.1), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2+\alpha^2\|\nabla u\|_{L^2}^2+\sum_k\|F_k\|_{L^2}^2\right)+\|\nabla u\|_{L^2}^2+\alpha^2\|\Delta u\|_{L^2}^2=0,$$

from which it implies that by integrating on time t,

$$\|u(T)\|_{L^2}^2+\alpha^2\|\nabla u(T)\|_{L^2}^2+\sum_k\|F_k(T)\|_{L^2}^2+2\int_0^T(\|\nabla u(\tau)\|_{L^2}^2+\alpha^2\|\Delta u(\tau)\|_{L^2}^2)d\tau=C_0, \tag{2.3}$$

where

$$C_0=\|u_0\|_{L^2}^2+\alpha^2\|\nabla u_0\|_{L^2}^2+\sum_k\|F_{k,0}\|_{L^2}^2. \tag{2.4}$$

Furthermore, from the above energy equality, we have

$$\int_0^T\|v(t)\|_{L^2}^2dt\le C_0\left(T+\frac{\alpha^2}{2}\right). \tag{2.5}$$

Step 2. Formulation of the contraction map f

Now fix $\bar u\in\mathbb{Z}_{m+2}$ and let F_k be the solution to the following equation

$$\begin{cases}\partial_tF_k+(\bar u\cdot\nabla)F_k=(F_k\cdot\nabla)\bar u,\\ F_k(0,x)=F_{k,0}.\end{cases} \tag{2.6}$$

It is easy to obtain this solution using the characteristic method, where similar arguments can be found in [27]. Here we omit it. Multiplying (2.6) by F_k and integrating on $\mathbb{R}^2$, from (2.4) and $\nabla\cdot F_k=0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|F_k(t)\|_{L^2}^2&=\int_{\mathbb{R}^2}(F_k\cdot\nabla)\bar u\cdot F_kdx\\&\le\|\nabla\bar u\|_{L^\infty}\|F_k\|_{L^2}^2\\&\le C\|\bar u\|_{H^3}\|F_k\|_{L^2}^2,\end{aligned}$$

where we have used the Hölder inequality and $\|f\|_{L^\infty}\le C\|f\|_{H^2}$. Hence,

$$\frac{d}{dt}\|F_k(t)\|_{L^2}\le C\|\bar u\|_{H^3}\|F_k\|_{L^2},$$

which implies that by the Gronwall inequality

$$\begin{aligned}\|F_k\|_{L^2}&\le\|F_{k,0}\|_{L^2}e^{C\|\bar u\|_{\mathbb{Y}_3}}\\&\le C\|F_{k,0}\|_{L^2}e^{CL\|\bar u\|_{\mathbb{Z}_2}}.\end{aligned}$$

Furthermore, the following property for F_k holds, namely, $F_k\in\mathbb{X}_{m+2}$ for any positive time $T_0>0$ and

$$\|F_k\|_{H^n}\le\|F_{k,0}\|_{H^n}e^{CL\|\bar u\|_{\mathbb{Z}_3}},\quad\text{as } n=1,2, \tag{2.7}$$
$$\|F_k\|_{H^n}\le\|F_{k,0}\|_{H^n}e^{CL\|\bar u\|_{\mathbb{Z}_n}},\quad\text{as } n=3,4,\ldots,m+2. \tag{2.8}$$

In fact, applying $\partial^n F_k$ on both sides of (2.6) and multiplying the resulting equation by $\partial^n F_k$, integrating on $\mathbb{R}^2$ and from the Leibniz rule, then

$$\int_{\mathbb{R}^2} \partial^n \partial_t F_k \cdot \partial^n F_k dx = \sum_{0\le i\le n} C_n^i \int_{\mathbb{R}^2} [\partial^{n-i} F_k \cdot \nabla \partial^i \bar{u} - \partial^{n-i}\bar{u} \cdot \nabla \partial^i F_k] \cdot \partial^n F_k dx, \tag{2.9}$$

where i is a multi-index. Since the term $\partial^n F_k \cdot \partial^n F_k$ on the right side of the above equality (2.9) disappears as $i = n$ according to $\nabla \cdot F_k = 0$, then the right side of (2.9) has the form $\sum_{0\le i\le n} \int_{\mathbb{R}^2} \partial^{n-i} F_k \cdot \partial^{i+1}\bar{u} \cdot \partial^n F_k dx$, which of them can be estimated by, as $n \ge 3$:

$$\begin{aligned}
&\sum_{i=0}^{n-2} \int_{\mathbb{R}^2} \partial^{n-i} F_k \cdot \partial^{i+1}\bar{u} \cdot \partial^n F_k dx + \sum_{i=n-1}^{n} \int_{\mathbb{R}^2} \partial^{n-i} F_k \cdot \partial^{i+1}\bar{u} \cdot \partial^n F_k dx \\
&\quad \le \sum_{i=0}^{n-2} \|\partial^{n-i} F_k\|_{L^2} \|\partial^{i+1}\bar{u}\|_{L^\infty} \|\partial^n F_k\|_{L^2} + \sum_{i=n-1}^{n} \|\partial^{n-i} F_k\|_{L^\infty} \|\partial^{i+1}\bar{u}\|_{L^2} \|\partial^n F_k\|_{L^2} \\
&\quad \le C\|\bar{u}\|_{H^{n+1}} \|F_k\|_{H^n}^2.
\end{aligned}$$

Inserting the above inequality into (2.9), then

$$\frac{d}{dt}\|\partial^n F_k(t)\|_{L^2}^2 \le C\|\bar{u}\|_{H^{n+1}} \|F_k\|_{H^n}^2.$$

Moreover,

$$\frac{d}{dt}\|F_k(t)\|_{H^n} \le C\|\bar{u}\|_{H^{n+1}} \|F_k\|_{H^n},$$

which infers that

$$\begin{aligned}
\|F_k\|_{H^n} &\le \|F_{k,0}\|_{H^n} e^{C\|\bar{u}\|_{\mathbb{Y}_{n+1}}} \\
&\le \|F_{k,0}\|_{H^n} e^{C\|\bar{u}\|_{\mathbb{Z}_n}}
\end{aligned}$$

in the case $n = 3, 4, \dots, m+2$ by the Gronwall inequality and the fact $F_{k,0} \in H^{m+2}(\mathbb{R}^2)$. Then we get that $F_k \in \mathbb{X}_{m+2}$ and

$$\|F_k\|_{\mathbb{X}_n} \le C_n = C_n(\|F_{k,0}\|_{H^n}, \|\bar{u}\|_{\mathbb{Z}_n}), \quad n = 3, 4, \dots, m+2.$$

As we choose $L \in (0, \frac{1}{C})$, the above inequality will be independent of C_n. And we also observe that $F_k \in \mathbb{Z}_n$, and

$$\begin{aligned}
\|F_k\|_{\mathbb{Z}_n} &\le \max\{K^{-1}\|F_k\|_{\mathbb{X}_n}, L^{-1}\|F_k\|_{\mathbb{Y}_{n+1}}\} \\
&\le \max\{K^{-1}\|F_k\|_{\mathbb{X}_{n+1}}, L^{-1}T_0\|F_k\|_{\mathbb{X}_{n+1}}\} \\
&\le \max\{K^{-1}, L^{-1}T_0\}C_{n+1},
\end{aligned}$$

which gives that

$$\|F_k\|_{\mathbb{Z}_n} \le \frac{L + KT_0}{KL} C_{n+1}, \quad n = 3, 4, \dots, m+1. \tag{2.10}$$

From above, there exists a map g such that $F_k = g(\bar{u}) \in \mathbb{X}_{m+2}$ solves Eq. (2.6) associated with each function $\bar{u} \in \mathbb{Z}_{m+2}$.

Since $\bar{u} \in \mathbb{Z}_{m+2}$ is fixed and $F_k = g(\bar{u})$, then we can denote v as the solution of the following linear system:

$$\begin{cases}
\partial_t v - \Delta v + \nabla p = \sum_k (F_k \cdot \nabla) F_k - (\bar{u} \cdot \nabla)\bar{u}, \\
\nabla \cdot v = 0, \\
v(0, x) = v_0 \in H_\sigma^m(\mathbb{R}^2).
\end{cases}$$

Projecting the above system by P_σ gives that

$$\begin{cases} \partial_t v - P_\sigma \Delta v = P_\sigma \left[\sum_k (F_k \cdot \nabla) F_k - (\bar{u} \cdot \nabla)\bar{u} \right], \\ v(0,x) = v_0 \in H^m_\sigma(\mathbb{R}^2). \end{cases}$$

This implies from continuous semigroup method that the solution $v = P_\sigma v$ can be represented by

$$v(t) = e^{-tA} v_0 + \int_0^t e^{-(t-s)A} P_\sigma h(s) ds, \quad \forall x \in \mathbb{R}^2. \tag{2.11}$$

Here $A = -P_\sigma \varDelta$ is a self-adjoint positive operator and $h = \sum_k (F_k \cdot \nabla) F_k - (\bar{u} \cdot \nabla)\bar{u}$.

According to $v = (1 - \alpha^2 \varDelta) u$, we know that $u = (1 - \alpha^2 \varDelta)^{-1} v$. Combining with (2.11), $\bar{u}$ and $F_k = g(\bar{u})$, we claim that there exists a continuous map f satisfying $u = f(\bar{u}) \in S_{m+1}$ for each function $\bar{u} \in S_{m+1}$. For convenience, we will detail the proof of this assertion (namely, f is well-defined) in next subsection.

Step 3. Well-definedness of the contraction map f

In this subsection, our aim is to give the well-definedness of the contraction map f. Before doing this, we will provide the estimates for f in $\mathbb{X}_{m+2}, \mathbb{Y}_{m+3}, \mathbb{Z}_{m+2}$ and the estimate of $f(\mathbf{0})$.

Firstly, we give the estimate of f in $\mathbb{X}_{m+2}$. Take two functions $\bar{u}_1, \bar{u}_2 \in S_{m+1} \subset \mathbb{Z}_{m+2}$ and denote the corresponding F_k by F_k^1 and F_k^2 respectively, from (2.11), $v = (1 - \alpha^2 \varDelta) u$ and the continuity of P_σ, then

$$\begin{aligned} \|f(\bar{u}_1) - f(\bar{u}_2)\|_{H^{m+2}} &\le C \|v_1 - v_2\|_{H^m} \\ &\le C \int_0^t \|h_1(s) - h_2(s)\|_{H^m} ds \\ &\le \|\bar{u}_1 \cdot \nabla \bar{u}_1 - \bar{u}_2 \cdot \nabla \bar{u}_2\|_{H^m} + \sum_k \|F_k^1 \cdot \nabla F_k^1 - F_k^2 \cdot \nabla F_k^2\|_{H^m} \\ &\doteq J_1 + J_2, \end{aligned} \tag{2.12}$$

where we have used the boundedness of e^{-tA} as an operator from H^m_σ into H^m_σ, and $v_1 = (1 - \alpha^2 \varDelta)\bar{u}_1, v_2 = (1 - \alpha^2 \varDelta)\bar{u}_2$.

For $m \ge 2$, as $m > \frac{d}{2}$ (d is space dimension), we have

$$\begin{aligned} \int_0^t J_1(s) ds &= \int_0^t \|\bar{u}_1 \cdot \nabla \bar{u}_1 - \bar{u}_2 \cdot \nabla \bar{u}_2\|_{H^m} ds \\ &= \int_0^t \|(\bar{u}_1 - \bar{u}_2) \cdot \nabla \bar{u}_1 - \bar{u}_2 \cdot \nabla(\bar{u}_1 - \bar{u}_2)\|_{H^m} ds \\ &\le C \int_0^t \|\bar{u}_1 - \bar{u}_2\|_{H^m} \|\bar{u}_1\|_{H^{m+1}} ds + C \int_0^t \|\bar{u}_1 - \bar{u}_2\|_{H^{m+1}} \|\bar{u}_2\|_{H^m} ds \\ &\le C \|\bar{u}_1 - \bar{u}_2\|_{\mathbb{X}_m} \|\bar{u}_1\|_{\mathbb{Y}_{m+1}} + C \|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Y}_{m+1}} \|\bar{u}_2\|_{\mathbb{X}_m}, \end{aligned}$$

which infers that

$$\int_0^t J_1(s) ds \le CKL(\|\bar{u}_1\|_{\mathbb{Z}_m} + \|\bar{u}_2\|_{\mathbb{Z}_m}) \|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Z}_m} \tag{2.13}$$

by the definition of $\mathbb{Z}_m$. Similarly, we also have

$$\int_0^t J_2(s) ds \le CKL \sum_k (\|F_k^1\|_{\mathbb{Z}_m} + \|F_k^2\|_{\mathbb{Z}_m}) \|F_k^1 - F_k^2\|_{\mathbb{Z}_m}. \tag{2.14}$$

Now we show that for each $n = 3, 4, \ldots, m+2$

$$\|F_k^1 - F_k^2\|_{H^m} \le C_{n+1}(L + KT_0) \|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Z}_m}. \tag{2.15}$$

In fact, substituting $(\bar{u}_1, F_k^1)$ and $(\bar{u}_2, F_k^2)$ into (2.6) and subtracting these two resulting equations, we get

$$\begin{aligned}\partial_t(F_k^1 - F_k^2) &= F_k^1 \cdot \nabla\bar{u}_1 - \bar{u}_1 \cdot \nabla F_k^1 + \bar{u}_2 \cdot \nabla F_k^2 - F_k^2 \cdot \nabla\bar{u}_2 \\ &= (F_k^1 - F_k^2) \cdot \nabla\bar{u}_1 + F_k^2 \cdot \nabla(\bar{u}_1 - \bar{u}_2) - (\bar{u}_1 - \bar{u}_2) \cdot \nabla F_k^2 - \bar{u}_1 \cdot \nabla(F_k^1 - F_k^2). \end{aligned}\tag{2.16}$$

Multiplying (2.16) by $(F_k^1 - F_k^2)$ and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|F_k^1 - F_k^2\|_{L^2}^2 &= \int_{\mathbb{R}^2}(F_k^1 - F_k^2) \cdot \nabla\bar{u}_1 \cdot (F_k^1 - F_k^2)dx + \int_{\mathbb{R}^2} F_k^2 \cdot \nabla(\bar{u}_1 - \bar{u}_2) \cdot (F_k^1 - F_k^2)dx \\ &\quad - \int_{\mathbb{R}^2}(\bar{u}_1 - \bar{u}_2) \cdot \nabla F_k^2 \cdot (F_k^1 - F_k^2)dx \\ &\le C\|\nabla\bar{u}_1\|_{L^\infty}\|F_k^1 - F_k^2\|_{L^2}^2 + C\|F_k^2\|_{L^2}\|\nabla(\bar{u}_1 - \bar{u}_2)\|_{L^\infty}\|F_k^1 - F_k^2\|_{L^2} \\ &\quad + C\|\nabla F_k^2\|_{L^2}\|\bar{u}_1 - \bar{u}_2\|_{L^\infty}\|F_k^1 - F_k^2\|_{L^2} \\ &\le C\|\bar{u}_1\|_{H^3}\|F_k^1 - F_k^2\|_{L^2}^2 + C\|F_k^2\|_{L^2}\|\bar{u}_1 - \bar{u}_2\|_{H^3}\|F_k^1 - F_k^2\|_{L^2} \\ &\quad + C\|F_k^2\|_{H^1}\|\bar{u}_1 - \bar{u}_2\|_{H^2}\|F_k^1 - F_k^2\|_{L^2}.\end{aligned}$$

Here we have used $\nabla \cdot F_k = 0$, the Hölder inequality and $\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{H^2(\mathbb{R}^2)}$. Then

$$\frac{d}{dt}\|F_k^1 - F_k^2\|_{L^2} \le C\|\bar{u}_1\|_{H^3}\|F_k^1 - F_k^2\|_{L^2} + C\|F_k^2\|_{L^2}\|\bar{u}_1 - \bar{u}_2\|_{H^3} + C\|F_k^2\|_{H^1}\|\bar{u}_1 - \bar{u}_2\|_{H^2}.$$

Integrating in time on $[0, t]$ for the above inequality, we have

$$\|F_k^1 - F_k^2\|_{L^2} \le C\int_0^t \|\bar{u}_1\|_{H^3}\|F_k^1 - F_k^2\|_{L^2}ds + C\|F_k^2\|_{\mathbb{X}_0}\|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Y}_3} + C\|F_k^2\|_{\mathbb{Y}_1}\|\bar{u}_1 - \bar{u}_2\|_{\mathbb{X}_2},$$

which implies that

$$\|F_k^1 - F_k^2\|_{L^2} \le CKL\|F_k^2\|_{\mathbb{Z}_0}e^{CL\|\bar{u}_1\|_{\mathbb{Z}_2}}\|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Z}_2}$$

by the Gronwall inequality according to the definition of $\mathbb{Z}_m$.

As $n > 1$, applying ∂^n on both sides of (2.6) and multiplying the resulting equation by $\partial^n(F_k^1 - F_k^2)$, integrating on $\mathbb{R}^2$ gives that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^n(F_k^1 - F_k^2)\|_{L^2}^2 &= \int_{\mathbb{R}^2}\sum_{i=0}^n C_n^i[\partial^i(F_k^1 - F_k^2) \cdot \nabla\partial^{n-i}\bar{u}_1 + \partial^i F_k^2 \cdot \nabla\partial^{n-i}(\bar{u}_1 - \bar{u}_2) \\ &\quad - \partial^{n-i}(\bar{u}_1 - \bar{u}_2) \cdot \nabla\partial^i F_k^2 - \partial^{n-i}\bar{u}_1 \cdot \nabla\partial^i(F_k^1 - F_k^2)] \cdot \partial^n(F_k^1 - F_k^2)dx \\ &\le C\int_{\mathbb{R}^2}\sum_{i=0}^n |\partial^i(F_k^1 - F_k^2)| \cdot |\nabla\partial^{n-i}\bar{u}_1| \cdot |\partial^n(F_k^1 - F_k^2)|dx \\ &\quad + C\int_{\mathbb{R}^2}\sum_{i=0}^n |\partial^i F_k^2| \cdot |\nabla\partial^{n-i}(\bar{u}_1 - \bar{u}_2)| \cdot |\partial^n(F_k^1 - F_k^2)|dx \\ &\quad + C\int_{\mathbb{R}^2}|\nabla\partial^n F_k^2| \cdot |\bar{u}_1 - \bar{u}_2| \cdot |\partial^n(F_k^1 - F_k^2)|dx \\ &\le C\|\bar{u}_1\|_{H^{n+1}}\|F_k^1 - F_k^2\|_{H^n}^2 + C\|F_k^2\|_{H^n}\|\bar{u}_1 - \bar{u}_2\|_{H^{n+1}}\|F_k^1 - F_k^2\|_{H^n} \\ &\quad + C\|F_k^2\|_{H^{n+1}}\|\bar{u}_1 - \bar{u}_2\|_{H^n}\|F_k^1 - F_k^2\|_{H^n},\end{aligned}$$

where we have used the Leibniz rule, the condition $\nabla \cdot F_k = 0$, the Hölder inequality, and the embedding relation $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Integrating the above inequality in time on $[0, t]$ and by the definition of $\mathbb{X}_m, \mathbb{Y}_m$

and $\mathbb{Z}_m$, we have

$$\begin{aligned}\|F_k^1-F_k^2\|_{H^n} &\le C\int_0^t \|\bar u_1\|_{H^{n+1}}\|F_k^1-F_k^2\|_{H^n}ds\\ &\quad +C\|F_k^2\|_{\mathbb{X}_n}\|\bar u_1-\bar u_2\|_{\mathbb{Y}_{n+1}}+C\|F_k^2\|_{\mathbb{Y}_{n+1}}\|\bar u_1-\bar u_2\|_{\mathbb{X}_n}\\ &\le C\int_0^t \|\bar u_1\|_{H^{n+1}}\|F_k^1-F_k^2\|_{H^n}ds+CKL\|F_k^2\|_{\mathbb{Z}_n}\|\bar u_1-\bar u_2\|_{\mathbb{Z}_n},\end{aligned}$$

which deduces that by the Gronwall inequality

$$\|F_k^1-F_k^2\|_{H^n}\le CKL\|F_k^2\|_{\mathbb{Z}_n}e^{CL\|\bar u_1\|_{\mathbb{Z}_n}}\|\bar u_1-\bar u_2\|_{\mathbb{Z}_n},$$

that is to obtain (2.15). Notice that, as before mentioned, the constant C_{n+1} will be independent of L by choosing $L\in(0,\frac{1}{C})$. According to (2.17),

$$\begin{aligned}\|F_k^1-F_k^2\|_{\mathbb{X}_m} &\le C_{m+1}(L+KT_0)\|\bar u_1-\bar u_2\|_{\mathbb{Z}_m},\\ \|F_k^1-F_k^2\|_{\mathbb{Y}_{m+1}} &\le C_{m+2}(L+KT_0)\|\bar u_1-\bar u_2\|_{\mathbb{Z}_{m+1}},\end{aligned}$$

then,

$$\|F_k^1-F_k^2\|_{\mathbb{Z}_m}\le C_{m+2}\frac{(L+KT_0)^2}{KL}\|\bar u_1-\bar u_2\|_{\mathbb{Z}_{m+1}}. \tag{2.17}$$

Hence, inserting (2.17) into (2.14), we have

$$\int_0^t J_2(s)ds\le C_{m+2}\frac{(L+KT_0)^3}{KL}\|\bar u_1-\bar u_2\|_{\mathbb{Z}_{m+1}}. \tag{2.18}$$

Then, from (2.12), (2.13) and (2.18), we get

$$\|f(\bar u_1)-f(\bar u_2)\|_{\mathbb{X}_{m+2}}\le C^*\|\bar u_1-\bar u_2\|_{\mathbb{Z}_{m+1}}, \tag{2.19}$$

where $C^*=C_{m+2}KL[KL+\frac{(L+KT_0)^3}{KL}]$.

Secondly, we present the estimate for f in $\mathbb{Y}_{m+3}$. Notice that the operator e^{-tA} is from $H_\sigma^m(\mathbb{R}^2)$ into $H_\sigma^{m+1}(\mathbb{R}^2)$ with the boundedness of $t^{-\frac12}e^t$ [28], and

$$\begin{aligned}\|f(\bar u_1)-f(\bar u_2)\|_{\mathbb{Y}_{m+3}} &\le C\|v_1-v_2\|_{\mathbb{Y}_{m+1}}\\ &= C\int_0^{T_0}\|v_1(t)-v_2(t)\|_{H^{m+1}}dt.\end{aligned} \tag{2.20}$$

From (2.11), we have

$$\begin{aligned}\|v_1(t)-v_2(t)\|_{H^{m+1}} &\le \int_0^t \|e^{-(t-s)A}P_\sigma[h_1(s)-h_2(s)]\|_{H^{m+1}}ds\\ &\le C\int_0^t (t-s)^{-\frac12}e^{t-s}\|h_1(s)-h_2(s)\|_{H^m}ds,\end{aligned}$$

where $h_1=\sum_k(F_k^1\cdot\nabla)F_k^1-(\bar u_1\cdot\nabla)\bar u_1, h_2=\sum_k(F_k^2\cdot\nabla)F_k^2-(\bar u_2\cdot\nabla)\bar u_2$. By changing the order of integration for the last term above, from (2.12), (2.13) and (2.18), we have

$$\begin{aligned}\|v_1-v_2\|_{\mathbb{Y}_{m+1}} &\le C\int_0^{T_0}dt\int_0^t (t-s)^{-\frac12}e^{t-s}\|h_1(s)-h_2(s)\|_{H^m}ds\\ &= C\int_0^{T_0}ds\int_s^{T_0}(t-s)^{-\frac12}e^{t-s}\|h_1(s)-h_2(s)\|_{H^m}ds\\ &= C\int_0^{T_0}\|h_1(s)-h_2(s)\|_{H^m}ds\int_s^{T_0}(t-s)^{-\frac12}e^{t-s}dt\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^{T_0}\|h_1(s)-h_2(s)\|_{H^m}dse^{T_0}\int_s^{T_0}(t-s)^{-\frac{1}{2}}dt\\
&\leq 2CT_0^{\frac{1}{2}}e^{T_0}\int_0^{T_0}\|h_1(s)-h_2(s)\|_{H^m}ds\\
&\leq C^*\varepsilon_0\|\bar{u}_1-\bar{u}_2\|_{\mathbb{Z}_{m+1}},
\end{aligned}
\tag{2.21}
$$

where $\varepsilon_0 = 2T_0^{\frac{1}{2}}e^{T_0}$.
Inserting (2.12) into (2.20), we have

$$
\|f(\bar{u}_1)-f(\bar{u}_2)\|_{\mathbb{Y}_{m+3}} \leq C^*\varepsilon_0\|\bar{u}_1-\bar{u}_2\|_{\mathbb{Z}_{m+1}}. \tag{2.22}
$$

Then, from (2.19) and (2.22), it gives that

$$
\|f(\bar{u}_1)-f(\bar{u}_2)\|_{\mathbb{Z}_{m+2}} \leq C^*(K^{-1}+\varepsilon_0L^{-1})\|\bar{u}_1-\bar{u}_2\|_{\mathbb{Z}_{m+1}}. \tag{2.23}
$$

Specially, as $\bar{u}\equiv\mathbf{0}$, then we have $F_k(t,x)=F_{k,0}(x)$ and $h=\sum_k F_{k,0}\cdot\nabla F_{k,0}$. Then, from (2.11), we get

$$
v(t) = e^{-tA}v_0 + \int_0^t e^{-(t-s)A}P_\sigma\left(\sum_k F_{k,0}\cdot\nabla F_{k,0}\right)ds. \tag{2.24}
$$

Using the boundedness of e^{-tA} as an operator from $H^m_\sigma(\mathbb{R}^2)$ into $H^m_\sigma(\mathbb{R}^2)$ for the above equality (2.24), we obtain

$$
\begin{aligned}
\|v\|_{H^m} &\leq \|v_0\|_{H^m} + C\int_0^t\sum_k\|F_{k,0}\|_{H^m}\|F_{k,0}\|_{H^{m+1}}ds\\
&\leq \|v_0\|_{H^m} + CT_0\sum_k\|F_{k,0}\|_{H^m}\|F_{k,0}\|_{H^{m+1}}.
\end{aligned}
$$

So,

$$
\|v\|_{\mathbb{X}_m} \leq \|v_0\|_{H^m} + CT_0\sum_k\|F_{k,0}\|_{H^m}\|F_{k,0}\|_{H^{m+1}}. \tag{2.25}
$$

Furthermore, since the operator e^{-tA} is from $H^m_\sigma(\mathbb{R}^2)$ into $H^{m+1}_\sigma(\mathbb{R}^2)$ with the boundedness of $t^{-\frac{1}{2}}e^t$, then, for (2.24), we have

$$
\begin{aligned}
\|v\|_{H^{m+1}} &\leq \|e^{-tA}\|_{H^{m+1}} + \int_0^t\left\|e^{-(t-s)A}P_\sigma\left(\sum_k F_{k,0}\cdot\nabla F_{k,0}\right)\right\|_{H^{m+1}}ds\\
&\leq t^{-\frac{1}{2}}e^t\|v_0\|_{H^m} + C\int_0^t(t-s)^{-\frac{1}{2}}e^{t-s}\|h(s)\|_{H^m}ds.
\end{aligned}
$$

Then, by changing the order of integration, we have

$$
\begin{aligned}
\|v\|_{\mathbb{Y}_{m+1}} &\leq \int_0^{T_0}t^{-\frac{1}{2}}e^t\|v_0\|_{H^m}ds + C\int_0^{T_0}\int_0^t(t-s)^{-\frac{1}{2}}e^{t-s}\|h(s)\|_{H^m}dsdt\\
&\leq \varepsilon_0\|v_0\|_{H^m} + C\int_0^{T_0}\|h(s)\|_{H^m}ds\int_s^{T_0}(t-s)^{-\frac{1}{2}}e^{t-s}dt\\
&\leq \varepsilon_0\left(\|v_0\|_{H^m} + C\int_0^{T_0}\|h(s)\|_{H^m}ds\right)\\
&\leq \varepsilon_0\left(\|v_0\|_{H^m} + CT_0\sum_k\|F_{k,0}\|_{H^m}\|F_{k,0}\|_{H^{m+1}}\right).
\end{aligned}
\tag{2.26}
$$

Therefore, as we take $K > 0$ such that $L = \varepsilon_0 K$, we have

$$\|f(\mathbf{0})\|_{\mathbb{Z}_{m+2}} \le C\|v\|_{\mathbb{Z}_m} \le \frac{M_0\varepsilon_0}{L}, \tag{2.27}$$

where $M_0 = \|v_0\|_{H^m} + CT_0 \sum_k \|F_{k,0}\|_{H^m}\|F_{k,0}\|_{H^{m+1}}$.

From (2.23) and (2.27) with $L = \varepsilon_0 K$, we have

$$\begin{aligned} \|f(\bar{u})\|_{\mathbb{Z}_{m+2}} &\le \|f(\bar{u}) - f(\mathbf{0})\|_{\mathbb{Z}_{m+2}} + \|f(\mathbf{0})\|_{\mathbb{Z}_{m+2}} \\ &\le C^*(K^{-1} + \varepsilon_0 L^{-1})\|\bar{u}\|_{\mathbb{Z}_{m+1}} + M_0\varepsilon_0 L^{-1} \\ &= \frac{\varepsilon_0}{L}(2C^*\|\bar{u}\|_{\mathbb{Z}_{m+1}} + M_0), \end{aligned} \tag{2.28}$$

which deduces that f is a map from $\mathbb{Z}_{m+2}$ into $\mathbb{Z}_{m+2}$ by C^* depending on C_{m+2} according to (2.19).

Now, it only need to show that $f(S_{m+1}) \subset S_{m+1}$ (S_{m+1} denotes the unitary ball centered at the origin in $\mathbb{Z}_{m+2}$). Without loss of generality, we can assume $T_0 \le 1$ in order to complete the proof of local existence in time. As we take $T_0 = 1$ in M_0, we can obtain a positive constant $M_1(\ge M_0)$ depending only on $m, \|v_0\|_{H^m}, \|F_{k,0}\|_{H^{m+1}}$. Similarly, as we take T_0 and their norms of $\bar{u}_1$ and $\bar{u}_2$ less than or equal to 1, the positive constant C_{m+2} will be less than or equal to a positive constant C depending only on m and $\|F_{k,0}\|_{H^{m+2}}$. Then, as we take $\varepsilon_0^{-1}T_0 \le \frac{1}{2}$ and $K = \varepsilon_0^{-1}L$, we have

$$\begin{aligned} C^* &= C_{m+2}\left[KL + \frac{(L+KT_0)^3}{KL}\right] \\ &\le CKL\left[1 + \frac{(L+KT_0)^3}{(KL)^2}\right] \\ &= C\varepsilon_0^{-1}L^2 + C\cdot\frac{(L+KT_0)^3}{KL} \\ &= C\varepsilon_0^{-1}L^2 + C\cdot\frac{(L+L\varepsilon_0^{-1}\cdot T_0)^3}{\varepsilon_0^{-1}L\cdot L} \\ &\le C\varepsilon_0^{-1}L^2 + C\cdot\left(\frac{3}{2}\right)^3\varepsilon_0 L \\ &\le C\varepsilon_0^{-1}L^2 + C\varepsilon_0 L \\ &\doteq C_1^*. \end{aligned} \tag{2.29}$$

Combining (2.28) and (2.29), then

$$\|f(\bar{u})\|_{\mathbb{Z}_{m+2}} \le \frac{\varepsilon_0}{L}(2C_1^* + M_0).$$

Now, we show that there exists $L \in (0, \frac{1}{C})$ such that

$$\frac{\varepsilon_0}{L}(2C_1^* + M_0) \le \frac{1}{2}, \tag{2.30}$$

which is equivalent to

$$4CL^2 + (4C\varepsilon_0^2 - 1)L + 2M_0\varepsilon_0 \le 0,$$

according to (2.29).

Noticing that we can choose ε_0 sufficiently small such that

$$\varDelta_L = (4C\varepsilon_0^2 - 1)^2 - 32CM_0\varepsilon_0 \ge 0,$$

and

$$L = \frac{1 - 4C\varepsilon_0^2 - \sqrt{\varDelta_L}}{8C} < \frac{1}{8C},$$

which asserts that (2.30) holds. Furthermore, from (2.23) and (2.30) with $K = \varepsilon_0^{-1}L$, we have

$$\begin{aligned}\|f(\bar{u}_1) - f(\bar{u}_2)\|_{\mathbb{Z}_{m+2}} &\le C^*(K^{-1} + \varepsilon_0 L^{-1})\|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Z}_{m+1}} \\ &\le 2C^*\varepsilon_0 L^{-1}\|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Z}_{m+1}} \\ &\le \frac{1}{2}\|\bar{u}_1 - \bar{u}_2\|_{\mathbb{Z}_{m+1}},\end{aligned}$$

which concludes that f is a well-defined contraction map from S_{m+1} into S_{m+1}. So we can let $u = \bar{u}$ be the unique fixed point of f. According to $u = \bar{u} \in S_{m+1}$, we know that $v \in \mathbb{Z}_m$ for each integer number $m \ge 2$. This completes the proof of Theorem 1.1 for local existence of the smooth solution to the system (1.3).

3. Proof of Theorem 1.2

In this section, we will prove the global existence result for Theorem 1.2 to the system (1.3). Denote $\omega = \mathrm{curl}v$. According to the conditions $\nabla \cdot u = \nabla \cdot F_k = 0$ and through proper computations, we know that

$$\begin{aligned}&\mathrm{curl}(u\cdot\nabla)u = (u\cdot\nabla)\mathrm{curl}u, \\ &\mathrm{curl}(F_k\cdot\nabla)F_k = (F_k\cdot\nabla)\mathrm{curl}F_k,\end{aligned}$$

then, applying the operator *curl* to the first equation of (1.3), we have

$$\partial_t\omega + (u\cdot\nabla)\mathrm{curl}u - \Delta\omega = \sum_k (F_k\cdot\nabla)\mathrm{curl}F_k. \tag{3.1}$$

Multiplying both sides of (3.1) by ω, integrating on $\mathbb{R}^2$, and using integration by parts, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &= \sum_k \int_{\mathbb{R}^2} (F_k\cdot\nabla)\mathrm{curl}F_k\omega dx - \int_{\mathbb{R}^2}(u\cdot\nabla)\mathrm{curl}u\cdot\omega dx \\ &\le \sum_k \|F_k\cdot\nabla\mathrm{curl}F_k\|_{\mathcal{H}^1}\|\omega\|_{BMO} + \|u\cdot\nabla\mathrm{curl}u\|_{\mathcal{H}^1}\|\omega\|_{BMO} \\ &\le C\sum_k \|F_k\|_{L^2}\|\nabla\mathrm{curl}F_k\|_{L^2}\|\nabla\omega\|_{L^2} + C\|u\|_{L^2}\|\nabla\mathrm{curl}u\|_{L^2}\|\nabla\omega\|_{L^2} \\ &\le C\sum_k \|\Delta F_k\|_{L^2}\|\nabla\omega\|_{L^2} + C\|\Delta u\|_{L^2}\|\nabla\omega\|_{L^2} \\ &\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\left(\sum_k \|\Delta F_k\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right),\end{aligned} \tag{3.2}$$

where we have used the conditions $\nabla\cdot u = \nabla\cdot F_k = 0, \mathrm{curl}\nabla \equiv 0$ and the theory of Hardy spaces from Coifman–Lions–Meyer–Semmes [29], the Young inequality and (2.3).

Then, applying the operator Δ on both sides of the second equations of (1.3), multiplying the resulting equation by ΔF_k, integrating on $\mathbb{R}^2$ and using the conditions $\nabla\cdot u = \nabla\cdot F_k = 0$ and Lemma 1.3, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\sum_k \|\Delta F_k\|_{L^2}^2\right) &= \sum_k \int_{\mathbb{R}^2} \left(\Delta[(F_k\cdot\nabla)u]\cdot\Delta F_k - \Delta[(u\cdot\nabla)F_k]\cdot\Delta F_k\right) dx \\ &= \sum_k \int_{\mathbb{R}^2} \Delta[(F_k\cdot\nabla)u]\cdot\Delta F_k dx - \sum_k \int_{\mathbb{R}^2} \left(\Delta[(u\cdot\nabla)F_k] - (u\cdot\nabla)\Delta F_k\right)\cdot\Delta F_k dx \\ &\le \sum_k (C\|\nabla u\|_{L^\infty}\|\Delta F_k\|_{L^2}^2 + C\|\Delta u\|_{L^4}\|\nabla F_k\|_{L^4}\|\Delta F_k\|_{L^2} \\ &\quad + C\|F_k\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\|\Delta F_k\|_{L^2}) \\ &\doteq K_1 + K_2 + K_3.\end{aligned} \tag{3.3}$$

For the term II_1, using Lemma 1.4, then

$$\begin{aligned}
K_1 &= \sum_k C\|\nabla u\|_{L^\infty}\|\Delta F_k\|_{L^2}^2 \\
&\le \sum_k C\|u\|_{H^2}\log(e+\|u\|_{H^3})\|\Delta F_k\|_{L^2}^2 \\
&\le \sum_k C\|v\|_{L^2}\ln(e+\|\omega\|_{L^2}+\|\Delta F_k\|_{L^2})\|\Delta F_k\|_{L^2}^2.
\end{aligned}$$

For the terms K_2 and K_3, using Lemma 1.5, (2.3) and the Young inequality, then

$$\begin{aligned}
K_2 &= \sum_k C\|\Delta u\|_{L^4}\|\nabla F_k\|_{L^4}\|\Delta F_k\|_{L^2} \\
&\le \sum_k C\|v\|_{L^4}\|\nabla F_k\|_{L^4}\|\Delta F_k\|_{L^2} \\
&\le \sum_k C\|v\|_{L^2}^{\frac{3}{4}}\|\Delta v\|_{L^2}^{\frac{1}{4}}\|F_k\|_{L^2}^{\frac{1}{4}}\|\Delta F_k\|_{L^2}^{\frac{3}{4}}\|\Delta F_k\|_{L^2} \\
&\le \sum_k C\|v\|_{L^2}^{\frac{3}{4}}\|\nabla\omega\|_{L^2}^{\frac{1}{4}}\|\Delta F_k\|_{L^2}^{\frac{7}{4}} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|v\|_{L^2}^{\frac{6}{7}}\sum_k\|\Delta F_k\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
K_3 &= \sum_k C\|F_k\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\|\Delta F_k\|_{L^2} \\
&\le \sum_k C\|F_k\|_{L^\infty}\|\nabla v\|_{L^2}\|\Delta F_k\|_{L^2} \\
&\le \sum_k C\|F_k\|_{L^2}^{\frac{1}{2}}\|\Delta F_k\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{1}{2}}\|\Delta F_k\|_{L^2} \\
&\le \sum_k C\|\Delta F_k\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|v\|_{L^2}^{\frac{2}{3}}\sum_k\|\Delta F_k\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates for K_1–K_3 into (3.3) in combination with (3.2), one follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2+\sum_k\|\Delta F_k\|_{L^2}^2\right)+\frac{1}{4}\|\nabla\omega\|_{L^2}^2 \\
&\quad\le C\left(1+\|v\|_{L^2}^{\frac{6}{7}}+\|v\|_{L^2}^{\frac{2}{3}}+\|v\|_{L^2}\ln\left(e+\|\omega\|_{L^2}+\sum_k\|\Delta F_k\|_{L^2}\right)\right)\sum_k\|\Delta F_k\|_{L^2}^2+C\|\omega\|_{L^2}^2.
\end{aligned}$$

Then we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(e+\|\omega\|_{L^2}^2+\sum_k\|\Delta F_k\|_{L^2}^2\right)+\frac{1}{4}\|\nabla\omega\|_{L^2}^2 \le C\left(1+\|v\|_{L^2}^{\frac{6}{7}}+\|v\|_{L^2}\right)\ln\left(e+\|\omega\|_{L^2}+\sum_k\|\Delta F_k\|_{L^2}\right) \\
\left(e+\|\omega\|_{L^2}^2+\sum_k\|\Delta F_k\|_{L^2}^2\right), \qquad (3.4)
\end{aligned}$$

which infers by (2.3) that

$$\|\omega\|_{L^\infty(0,T;L^2)}+\|\omega\|_{L^2(0,T;H^1)}\le C, \qquad (3.5)$$

$$\|F_k\|_{L^\infty(0,T;H^2)}\le C. \qquad (3.6)$$

Therefore,

$$\|v\|_{L^\infty(0,T;H^1)} + \|v\|_{L^2(0,T;H^2)} \le C, \tag{3.7}$$

$$\|u\|_{L^\infty(0,T;H^3)} + \|u\|_{L^2(0,T;H^4)} \le C. \tag{3.8}$$

For completeness, the proof of (3.5)–(3.8) will be sketched here, whose similar arguments can be found in [26]. Noticing that the inequality (3.4) has the form

$$y'(t) + \|\nabla\omega\|_{L^2}^2 \le C(1 + \|v\|_{L^2}^{\frac{6}{7}} + \|v\|_{L^2})y(t)\ln y(t), \tag{3.9}$$

where $y(t) = e + \|\omega\|_{L^2}^2 + \sum_k \|\Delta F_k\|_{L^2}^2$, then

$$\int_{y_0}^{y} \frac{d\bar{y}}{\bar{y}\ln\bar{y}} \le C\int_0^t (1 + \|v(\tau)\|_{L^2}^{\frac{6}{7}} + \|v(\tau)\|_{L^2})d\tau.$$

From (2.3), one can get

$$\ln\ln y - \ln\ln y_0 \le C(t+1),$$

which gives that

$$y(t) \le e^{Ce^{C(T+1)}}$$

for each $t \in [0,T]$. Then we have

$$\|\omega\|_{L^\infty(0,T;L^2)} + \sum_k \|\Delta F_k\|_{L^\infty(0,T;L^2)} \le C. \tag{3.10}$$

Noticing that $\|\nabla F_k\|_{L^2} \le C\|F_k\|_{L^2}^{\frac{1}{2}}\|\nabla^2 F_k\|_{L^2}^{\frac{1}{2}}$ and by Lemma 1.5, it infers that (3.6) holds by (2.3). Moreover, from (3.9), one can get $\|\nabla\omega\|_{L^2(0,T:L^2)} \le C$, whose combination with (3.10) concludes that (3.5), (3.7) and (3.8) hold.

Now, applying ∂^3 on both sides of the second equation of (1.3), multiplying the resulting equation by $\partial^3 F_k$, using Lemma 1.3, (3.8) and the Sobolev inequality, then we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial^3 F_k(t)\|_{L^2} &= \int_{\mathbb{R}^2} \partial^3[(F_k\cdot\nabla)u]\cdot\partial^3 F_k dx - \int_{\mathbb{R}^2} \partial^3[(u\cdot\nabla)F_k]\cdot\partial^3 F_k dx \\
&= \int_{\mathbb{R}^2} \partial^3[(F_k\cdot\nabla)u]\cdot\partial^3 F_k dx - \int_{\mathbb{R}^2} (\partial^3[(u\cdot\nabla)F_k] - (u\cdot\nabla)\partial^3 F_k)\cdot\partial^3 F_k dx \\
&\le C\|\nabla u\|_{L^\infty}\|\partial^3 F_k\|_{L^2}^2 + C\|\nabla F_k\|_{L^\infty}\|\partial^3 u\|_{L^2}\|\partial^3 F_k\|_{L^2} + C\|F_k\|_{L^\infty}\|\partial^4 u\|_{L^2}\|\partial^3 F_k\|_{L^2} \\
&\le C\|\partial^3 F_k\|_{L^2}^2 + C\|\nabla F_k\|_{L^\infty}\|\partial^3 F_k\|_{L^2} + C\|\partial^4 u\|_{L^2}\|\partial^3 F_k\|_{L^2} \\
&\le C\|\partial^3 F_k\|_{L^2}^2 + C(1 + \|\partial^3 F_k\|_{L^2})\|\partial^3 F_k\|_{L^2} + C\|\partial^4 u\|_{L^2}\|\partial^3 F_k\|_{L^2},
\end{aligned}$$

which implies that by using the Gronwall inequality

$$\|F_k\|_{L^\infty(0,T;H^3)} \le C. \tag{3.11}$$

Finally, applying the operator Δ on both sides of (3.1), multiplying the resulting equation by $\Delta\omega$ and integrating on $\mathbb{R}^2$, then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_{L^2}^2 + \|\nabla\Delta\omega\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \Delta[(u\cdot\nabla)\mathrm{cur}u]\Delta\omega dx + \sum_k \int_{\mathbb{R}^2} \Delta[(F_k\cdot\nabla)\mathrm{curl}F_k]\Delta\omega dx \\
&\doteq L_1 + L_2.
\end{aligned} \tag{3.12}$$

For the terms L_1 and L_2, by integration by parts, using Lemma 1.3, the Young inequality, (3.5)–(3.8) and (3.11), then

$$\begin{aligned} L_1 &= -\int_{\mathbb{R}^2} \Delta[(u\cdot\nabla)\mathrm{curl}u]\Delta\omega dx \\ &= \int_{\mathbb{R}^2} \nabla[(u\cdot\nabla)\mathrm{curl}u]\cdot\nabla\Delta\omega dx \\ &\le C\|u\|_{L^\infty}\|\Delta\mathrm{curl}u\|_{L^2}\|\nabla\Delta\omega\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\nabla\mathrm{curl}u\|_{L^2}\|\nabla\Delta\omega\|_{L^2} \\ &\le \frac{1}{4}\|\nabla\Delta\omega\|_{L^2}^2 + C, \end{aligned}$$

and

$$\begin{aligned} L_2 &= \sum_k \int_{\mathbb{R}^2} \Delta[(F_k\cdot\nabla)\mathrm{curl}F_k]\Delta\omega dx \\ &= -\sum_k \int_{\mathbb{R}^2} \Delta\partial_i(F_k^{(i)}F_k)\cdot\mathrm{curl}\Delta\omega dx \\ &\le C\|F_k\|_{L^\infty}\|\partial^3 F_k\|_{L^2}\|\nabla\Delta\omega\|_{L^2} \\ &\le C\|\nabla\Delta\omega\|_{L^2} \\ &\le \frac{1}{4}\|\nabla\Delta\omega\|_{L^2}^2 + C, \end{aligned}$$

where we have used the fact that $(F_k\cdot\nabla)\mathrm{curl}F_k = \mathrm{curl}[(F_k\cdot\nabla)F_k] = \mathrm{curl}(\partial_i(F_k^{(i)}F_k))$ by $\nabla\cdot F_k = 0$, and $F_k^{(i)}$ denotes the (i,k)-element of F in the second equality for L_2.

Inserting the above estimates for $L_1 - L_3$ into (3.12) gives that

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_{L^2}^2 + \|\nabla\Delta\omega\|_{L^2}^2 \le \frac{1}{2}\|\nabla\Delta\omega\|_{L^2}^2 + C,$$

which implies that

$$\|\omega\|_{L^\infty(0,T;H^2)} \le C.$$

Therefore,

$$\|v\|_{L^\infty(0,T;H^3)} \le C.$$

This completes the proof of Theorem 1.2.

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