Applied Mathematics Letters 64 (2017) 34–41

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Generalized Prüfer angle and oscillation of half-linear differential equations

O. Došlý[a], J. Jaroš[b], M. Veselý[a,*]

[a] Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
[b] Department of Mathematical Analysis and Numerical Mathematics, Comenius University, 842 48 Bratislava, Slovakia

ARTICLE INFO

Article history:
Received 9 August 2016
Accepted 9 August 2016
Available online 22 August 2016

Keywords:
Prüfer angle
Half-linear equations
Oscillation theory
Conditional oscillation
Oscillation constant

ABSTRACT

In this paper, we introduce a new modification of the half-linear Prüfer angle. Applying this modification, we investigate the conditional oscillation of the half-linear second order differential equation

$$\left[t^{\alpha-1}r(t)\,\Phi(x')\right]' + t^{\alpha-1-p}s(t)\,\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \tag{$*$}$$

where $p > 1$, $\alpha \neq p$, and r, s are continuous functions such that $r(t) > 0$ for large t. We present conditions on the functions r, s which guarantee that Eq. $(*)$ behaves like the Euler type equation $[t^{\alpha-1}\Phi(x')]' + \lambda t^{\alpha-1-p}\Phi(x) = 0$, which is conditionally oscillatory with the oscillation constant $\lambda_0 = |p-\alpha|^p/p^p$.

1. Introduction

The half-linear differential equation is an equation of the form

$$[R(t)\,\Phi(x')]' + S(t)\,\Phi(x) = 0 \tag{1.1}$$

with the odd power function $\Phi(x) = |x|^{p-1}\operatorname{sgn} x$ for some $p > 1$ and with continuous coefficients R, S, where R is positive. The oscillation theory of Eq. (1.1) has attracted considerable attention in the recent years. We refer to papers below and books [1,2] for references up to the publication years of these books. One of the reasons for this interest is that the qualitative theory of Eq. (1.1) is very similar to that of the Sturm–Liouville differential equation $[R(t)x']' + S(t)x = 0$. This linear equation is the particular case of Eq. (1.1) for $p = 2$ and its qualitative theory is very deeply developed. In particular, the classical Sturmian oscillation theory for linear equations is extended to Eq. (1.1). This means that Eq. (1.1) can be classified as oscillatory or non-oscillatory (analogously as in the linear case).

* Corresponding author.
E-mail address: xvesely@math.muni.cz (M. Veselý).

http://dx.doi.org/10.1016/j.aml.2016.08.004

One of the half-linear differential equations which can be solved explicitly (at least partially) is the Euler half-linear differential equation

$$\left[t^{\alpha-1}\Phi(x')\right]' + \gamma t^{\alpha-p-1}\Phi(x) = 0, \quad \alpha \neq p. \tag{1.2}$$

Especially, it is known (see [3,4] or Theorem A) that Eq. (1.2) is oscillatory if $\gamma p^p > |p-\alpha|^p$, and non-oscillatory in the opposite case. Eq. (1.2) is a typical example of the so-called *conditionally oscillatory* half-linear differential equation. Let us consider Eq. (1.1), where $S(t) = \lambda c(t)$ for $\lambda \in \mathbb{R}$ and a continuous function c. We recall that Eq. (1.1) is said to be conditionally oscillatory if there exists the so-called *oscillation constant* $\lambda_0 > 0$ such that Eq. (1.1) with $S(t) = \lambda c(t)$ is oscillatory for $\lambda > \lambda_0$ and non-oscillatory for $\lambda < \lambda_0$.

Concerning the conditional oscillation, a natural question is what happens when one considers the more general equation in the form

$$\left[t^{\alpha-1}r(t)\Phi(x')\right]' + t^{\alpha-p-1}s(t)\Phi(x) = 0. \tag{1.3}$$

We remark that Eq. (1.2) is the special case of Eq. (1.3) with $r(t) \equiv 1$, $s(t) \equiv \gamma$. This problem is partially studied, e.g., in [5–10] (and, e.g., in [11–16] for the linear case). In this paper, we introduce a new very general modification of the half-linear Prüfer angle. Using this concept of the Prüfer angle, we obtain conditions on the functions r, s which guarantee that Eq. (1.3) remains conditionally oscillatory; i.e., we show that Eq. (1.3) behaves essentially in the same way as Eq. (1.2).

This paper is organized as follows. In the next section, we present the modified Prüfer transformation which is the main tool in our paper. Section 3 presents our main result about the conditional oscillation of Eq. (1.3).

2. Generalized Prüfer angle

Throughout this paper, we will consider all equations for large t, say in an interval $[b, \infty)$. Intervals of this form will be denoted by $\mathbb{R}_b$. Let $p > 1$ and $\alpha \in \mathbb{R} \setminus \{p\}$ be arbitrarily given. The symbol q will denote the number conjugated with p which means that q is given by the equality $p + q = pq$. We put

$$\gamma_{p,\alpha} := \left(\frac{|p-\alpha|}{p}\right)^p. \tag{2.1}$$

Let us consider the half-linear differential equation of the form

$$\left[R(t)\Phi(x')\right]' + S(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \tag{2.2}$$

where $R, S : \mathbb{R}_b \to \mathbb{R}$ are continuous functions and R is positive. Let x be a non-trivial solution of (2.2). For $x(t) \neq 0$, the so-called Riccati transformation $w(t) = R(t)\Phi(x'(t)/x(t))$ leads to the Riccati equation

$$w'(t) + S(t) + (p-1)R^{1-q}(t)|w(t)|^q = 0. \tag{2.3}$$

Concerning the correctness of the used transformations, we can refer to [2, Sections 1.1.3–1.1.6].

Let an arbitrary positive function $f \in C^1(\mathbb{R}_b)$ be given. The substitution $v(t) = f^p(t)w(t)$ and Eq. (2.3) give

$$v'(t) = pf^{p-1}(t)f'(t)w(t) + f^p(t)w'(t) = p\frac{f'(t)}{f(t)}v(t) - f^p(t)S(t) - (p-1)R^{1-q}(t)f^{-q}(t)|v(t)|^q. \tag{2.4}$$

Using the above function f, we introduce the modified Prüfer transformation

$$x(t) = \rho(t)\sin_p\varphi(t), \quad R^{q-1}(t)x'(t) = \frac{\rho(t)}{f^q(t)}\cos_p\varphi(t) \tag{2.5}$$

for a non-trivial solution of (2.2), where $\sin_p$ is the so-called *generalized sine function* and $\cos_p$ is the *generalized cosine function*. We recall that $\sin_p$ is defined as the odd extension of the solution of the initial problem

$$[\Phi(x')]' + (p-1)\Phi(x) = 0, \quad x(0) = 0, \quad x'(0) = 1 \tag{2.6}$$

with the period $4\pi/(p\sin(\pi/p))$ and that $\cos_p$ is introduced as the derivative of $\sin_p$. See also [2, Section 1.1.2].

We obtain (see (2.5))

$$v(t) = f^p(t)w(t) = f^p(t)R(t)\Phi\left(\frac{x'(t)}{x(t)}\right) = f^p(t)R(t)\Phi\left(\frac{\rho(t)R^{1-q}(t)\cos_p\varphi(t)}{f^q(t)\rho(t)\sin_p\varphi(t)}\right) = \Phi\left(\frac{\cos_p\varphi(t)}{\sin_p\varphi(t)}\right). \tag{2.7}$$

In [2, Remark 1.1.3], there is shown

$$v'(t) = (1-p)\left[1 + \left|\frac{\cos_p\varphi(t)}{\sin_p\varphi(t)}\right|^p\right]\varphi'(t) = \frac{1-p}{|\sin_p\varphi(t)|^p}\,\varphi'(t), \tag{2.8}$$

where the generalized Pythagorean identity

$$|\sin_p y|^p + |\cos_p y|^p = 1, \quad y \in \mathbb{R}, \tag{2.9}$$

is applied. Note that (2.8) can be shown using (2.7) and the fact that $\sin_p$ is a solution of the equation in (2.6). The identity (2.9) can be found in [2, Section 1.1.2] as well. Combining (2.4) and (2.8), we have

$$\frac{1-p}{|\sin_p\varphi(t)|^p}\,\varphi'(t) = p\frac{f'(t)}{f(t)}v(t) - f^p(t)S(t) - (p-1)R^{1-q}(t)f^{-q}(t)|v(t)|^q. \tag{2.10}$$

Thus (see (2.7) and (2.10)), it holds

$$(1-p)\,\varphi'(t) = p\frac{f'(t)}{f(t)}\,\Phi\left(\cos_p\varphi(t)\right)\sin_p\varphi(t) - f^p(t)S(t)|\sin_p\varphi(t)|^p - (p-1)R^{1-q}(t)f^{-q}(t)|\cos_p\varphi(t)|^p,$$

i.e.,

$$\varphi'(t) = \frac{1}{R^{q-1}(t)f^q(t)}|\cos_p\varphi(t)|^p - \frac{p}{p-1}\,\frac{f'(t)}{f(t)}\,\Phi\left(\cos_p\varphi(t)\right)\sin_p\varphi(t) + \frac{f^p(t)S(t)}{p-1}|\sin_p\varphi(t)|^p. \tag{2.11}$$

In this paper, we assume that the functions R and S have the forms $R(t) = t^{\alpha-1}r(t)$, $S(t) = t^{\alpha-1-p}s(t)$, $t \in \mathbb{R}_b$, i.e., we consider the equation

$$\left[t^{\alpha-1}r(t)\,\Phi(x')\right]' + t^{\alpha-1-p}s(t)\,\Phi(x) = 0, \tag{2.12}$$

where $r, s : \mathbb{R}_b \to \mathbb{R}$ are continuous functions with $r(t) > 0$ for $t \in \mathbb{R}_b$ satisfying

$$\lim_{t\to\infty} \frac{\int_t^{t+a} r^{1-q}(\tau)\,\mathrm{d}\tau}{\sqrt{t}} = \lim_{t\to\infty} \frac{\int_t^{t+a} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = 0 \tag{2.13}$$

for some $a > 0$. Now, for the choice $f(t) = t^{\frac{p-\alpha}{p}}$, Eq. (2.11) takes the form

$$\varphi'(t) = \frac{1}{t}\left[r^{1-q}(t)|\cos_p\varphi(t)|^p - \frac{p-\alpha}{p-1}\,\Phi(\cos_p\varphi(t))\sin_p\varphi(t) + s(t)\frac{|\sin_p\varphi(t)|^p}{p-1}\right]. \tag{2.14}$$

We will use Eq. (2.14) to study Eq. (2.12). We will also use the auxiliary “average type” function $\psi_a : \mathbb{R}_b \to \mathbb{R}$ defined by the formula

$$\psi_a(t) := \frac{1}{a}\int_t^{t+a} \varphi(\tau)\,\mathrm{d}\tau, \quad t \geq b, \tag{2.15}$$

where $a > 0$ is the constant given in (2.13) and φ is an arbitrarily given solution of (2.14) on $\mathbb{R}_b$.

3. Conditional oscillation of half-linear equations

First, we mention one known result and a series of lemmas which will be applied in the sequel.

Theorem A. *Let $C > 0$ and $D \in \mathbb{R}$ be arbitrary. The equation $[Ct^{\alpha-1}\Phi(x')]' + Dt^{\alpha-1-p}\Phi(x) = 0$ is oscillatory for $C^{-1}D > \gamma_{p,\alpha}$ and non-oscillatory for $C^{-1}D < \gamma_{p,\alpha}$.*

Proof. See, e.g., [2, Theorem 1.4.4] (or directly [3] and [4]). □

It is well-known that the non-oscillation of Eq. (2.2) is equivalent to the boundedness of the Prüfer angle φ given by Eq. (2.11). See, e.g., [5,12,15,17] for similar cases or directly consider the transformation in (2.5), positivity of the functions R and f, and Eq. (2.11) when $\sin_p\varphi(t) = 0$. Considering the periodicity of $\sin_p$ together with the right-hand side of Eq. (2.11), we get that the set of all values of φ is unbounded if and only if $\lim_{t\to\infty}\varphi(t) = \infty$. Thus, the above theorem implies the first lemma.

Lemma 3.1. *Let $C > 0$ and $D \in \mathbb{R}$ be arbitrary and let $\eta : \mathbb{R}_b \to \mathbb{R}$ be a solution of the equation*

$$\eta'(t) = \frac{1}{t}\left[C^{1-q}|\cos_p\eta(t)|^p - \frac{p-\alpha}{p-1}\,\Phi(\cos_p\eta(t))\sin_p\eta(t) + D\frac{|\sin_p\eta(t)|^p}{p-1}\right].$$

If $C^{-1}D > \gamma_{p,\alpha}$, then $\lim_{t\to\infty}\eta(t) = \infty$. If $C^{-1}D < \gamma_{p,\alpha}$, then $\limsup_{t\to\infty}\eta(t) < \infty$.

Lemma 3.2. *Let $\varphi : \mathbb{R}_b \to \mathbb{R}$ be a solution of (2.14). It holds uniformly with respect to $\tau \in [t, t+a]$*

$$\lim_{t\to\infty}\sqrt{t}\,|\varphi(\tau) - \psi_a(t)| = 0. \tag{3.1}$$

Proof. The generalized Pythagorean identity (2.9) gives

$$|\sin_p y|^p \le 1, \quad |\cos_p y|^p \le 1, \quad |\Phi(\cos_p y)\sin_p y| \le 1, \quad y \in \mathbb{R}. \tag{3.2}$$

Hence, we obtain (consider the mean value theorem and (2.13))

$$\begin{aligned}
0 &\le \limsup_{t\to\infty}\sqrt{t}\,|\varphi(\tau) - \psi_a(t)| \le \limsup_{t\to\infty}\sqrt{t}\max_{s_1,s_2\in[0,a]}|\varphi(t+s_1) - \varphi(t+s_2)| \le \limsup_{t\to\infty}\sqrt{t}\int_t^{t+a}|\varphi'(\tau)|\,\mathrm{d}\tau \\
&= \limsup_{t\to\infty}\sqrt{t}\int_t^{t+a}\left|\frac{1}{\tau}\left[r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p - \frac{p-\alpha}{p-1}\,\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau) + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\right]\right|\mathrm{d}\tau \\
&\le \limsup_{t\to\infty}\frac{1}{\sqrt{t}}\int_t^{t+a}\left(r^{1-q}(\tau) + \frac{|p-\alpha|}{p-1} + \frac{1}{p-1}\,|s(\tau)|\right)\mathrm{d}\tau = 0
\end{aligned}$$

uniformly with respect to $\tau \in [t, t+a]$. □

Lemma 3.3. *Let $\varphi : \mathbb{R}_b \to \mathbb{R}$ be a solution of (2.14). There exists a continuous function $g_a : \mathbb{R}_b \to \mathbb{R}$ such that $\lim_{t\to\infty}g_a(t) = 0$ and that*

$$\begin{aligned}
\psi_a'(t) = \frac{1}{t}\Bigg[&\frac{|\cos_p\psi_a(t)|^p}{a}\int_t^{t+a}r^{1-q}(\tau)\,\mathrm{d}\tau - \frac{p-\alpha}{p-1}\,\Phi\left(\cos_p\psi_a(t)\right)\sin_p\psi_a(t) \\
&+ \frac{|\sin_p\psi_a(t)|^p}{(p-1)a}\int_t^{t+a}s(\tau)\,\mathrm{d}\tau + g_a(t)\Bigg], \quad t > b.
\end{aligned}$$

Proof. Obviously, for all $t > b$, we can express

$$\begin{aligned}\psi_a'(t) &= \frac{1}{a}\int_t^{t+a} \varphi'(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{a}\int_t^{t+a} \frac{1}{\tau}\left[r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p - \frac{p-\alpha}{p-1}\,\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau) + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\right]\mathrm{d}\tau.\end{aligned}$$

By (2.13) and (3.2), we have

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t}}\int_t^{t+a}\left(r^{1-q}(\tau) + \frac{|p-\alpha|}{p-1} + |s(\tau)|\,\frac{1}{p-1}\right)\mathrm{d}\tau = 0$$

and

$$\begin{aligned}&\frac{1}{a}\left|\int_t^{t+a}\frac{1}{\tau}\left[r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p - \frac{p-\alpha}{p-1}\,\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau) + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\right]\mathrm{d}\tau\right. \\ &\qquad \left. - \int_t^{t+a}\frac{1}{t}\left[r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p - \frac{p-\alpha}{p-1}\,\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau) + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\right]\mathrm{d}\tau\right| \\ &\quad \le \frac{1}{a}\int_t^{t+a}\left(\frac{1}{t}-\frac{1}{\tau}\right)\left[r^{1-q}(\tau) + \frac{|p-\alpha|}{p-1} + |s(\tau)|\,\frac{1}{p-1}\right]\mathrm{d}\tau \\ &\quad \le \frac{1}{t^2}\int_t^{t+a}\left(r^{1-q}(\tau) + \frac{|p-\alpha|}{p-1} + |s(\tau)|\,\frac{1}{p-1}\right)\mathrm{d}\tau.\end{aligned}$$

The above estimations guarantee that it suffices to prove that

$$\lim_{t\to\infty}\left(\frac{|\cos_p\psi_a(t)|^p}{a}\int_t^{t+a} r^{1-q}(\tau)\,\mathrm{d}\tau - \frac{1}{a}\int_t^{t+a} r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p\,\mathrm{d}\tau\right) = 0, \tag{3.3}$$

$$\lim_{t\to\infty}\left(\frac{p-\alpha}{p-1}\,\Phi\left(\cos_p\psi_a(t)\right)\sin_p\psi_a(t) - \frac{p-\alpha}{(p-1)a}\int_t^{t+a}\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\,\mathrm{d}\tau\right) = 0, \tag{3.4}$$

$$\lim_{t\to\infty}\left(\frac{|\sin_p\psi_a(t)|^p}{(p-1)a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau - \frac{1}{a}\int_t^{t+a} s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\,\mathrm{d}\tau\right) = 0. \tag{3.5}$$

We begin with (3.4) which follows from the uniform continuity of $G(y) = \Phi\left(\cos_p y\right)\sin_p y$, $y \in \mathbb{R}$, and from (3.1) in Lemma 3.2. To prove (3.3) and (3.5), we use the fact that $\Phi(\sin_p)$ and $\Phi(\cos_p)$ are continuously differentiable and periodic functions (see, e.g., [2, Section 1.1.2]). Hence, there exist $L_1, L_2 > 0$ such that $||\sin_p y|^p - |\sin_p z|^p| \le L_1|y-z|$ and $||\cos_p y|^p - |\cos_p z|^p| \le L_2|y-z|$ for all $y, z \in \mathbb{R}$.

It holds (see (2.13) and (3.1))

$$\begin{aligned}&\limsup_{t\to\infty}\int_t^{t+a} r^{1-q}(\tau)\cdot\left||\cos_p\psi_a(t)|^p - |\cos_p\varphi(\tau)|^p\right|\mathrm{d}\tau \\ &\quad \le \limsup_{t\to\infty}\frac{\sqrt{t}}{\sqrt{t}}\int_t^{t+a} r^{1-q}(\tau)L_2\left|\psi_a(t)-\varphi(\tau)\right|\,\mathrm{d}\tau = 0\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_t^{t+a}|s(\tau)|\cdot\left||\sin_p\psi_a(t)|^p - |\sin_p\varphi(\tau)|^p\right|\mathrm{d}\tau \\ &\quad \le \limsup_{t\to\infty}\frac{\sqrt{t}}{\sqrt{t}}\int_t^{t+a}|s(\tau)|\,L_1\left|\psi_a(t)-\varphi(\tau)\right|\,\mathrm{d}\tau = 0.\end{aligned} \tag{3.7}$$

Of course, (3.6) and (3.7) imply (3.3) and (3.5), respectively. □

Using the above lemmas, we can analyse the conditional oscillation of Eq. (2.12) with coefficients satisfying (2.13) for some $a > 0$.

Theorem 3.4. *Let $A > 0$ and $B \in \mathbb{R}$.*

(i) *If $A^{p-1}B > \gamma_{p,\alpha}$ and if the inequalities*
$$\overline{r}(t) := \frac{1}{a}\int_t^{t+a} r^{1-q}(\tau)\,\mathrm{d}\tau \geq A, \quad \overline{s}(t) := \frac{1}{a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau \geq B$$
hold for all sufficiently large t, then Eq. (2.12) is oscillatory.

(ii) *If $A^{p-1}B < \gamma_{p,\alpha}$ and if the inequalities $\overline{r}(t) \leq A$, $\overline{s}(t) \leq B$ hold for all sufficiently large t, then Eq. (2.12) is non-oscillatory.*

Proof. In both parts of the proof, we use Eq. (2.14) for the Prüfer angle and the generalized Pythagorean identity (2.9) which gives
$$|\mathrm{cos}_p y|^p + \frac{|\mathrm{sin}_p y|^p}{p-1} \geq \min\left\{\frac{1}{p-1}, 1\right\} \tag{3.8}$$
for all $y \in \mathbb{R}$. We recall that it suffices to show that a solution of (2.14) is unbounded in the case (i) and bounded from above in the case (ii). Let us consider a solution $\varphi : \mathbb{R}_b \to \mathbb{R}$ of (2.14) and the corresponding function ψ_a given in (2.15). We also know that it suffices to check the boundedness of ψ_a due to Lemma 3.2.

Part (i). Considering Lemma 3.3, we have
$$\begin{aligned}\psi_a'(t) &= \frac{1}{t}\left[\frac{|\mathrm{cos}_p\psi_a(t)|^p}{a}\int_t^{t+a} r^{1-q}(\tau)\,\mathrm{d}\tau - \frac{p-\alpha}{p-1}\varPhi\left(\mathrm{cos}_p\psi_a(t)\right)\mathrm{sin}_p\psi_a(t)\right.\\ &\quad\left. + \frac{|\mathrm{sin}_p\psi_a(t)|^p}{(p-1)a}\int_t^{t+a} s(\tau)\,\mathrm{d}\tau + g_a(t)\right]\\ &\geq \frac{1}{t}\left[|\mathrm{cos}_p\psi_a(t)|^p A - \frac{p-\alpha}{p-1}\varPhi\left(\mathrm{cos}_p\psi_a(t)\right)\mathrm{sin}_p\psi_a(t) + \frac{|\mathrm{sin}_p\psi_a(t)|^p}{p-1}B + g_a(t)\right]\end{aligned}$$
for all sufficiently large t. Since $\lim_{t\to\infty} g_a(t) = 0$ and (3.8) is valid on $\mathbb{R}$, for any $\varepsilon > 0$, we have
$$\psi_a'(t) > \frac{1}{t}\left[|\mathrm{cos}_p\psi_a(t)|^p(A-\varepsilon) - \frac{p-\alpha}{p-1}\varPhi\left(\mathrm{cos}_p\psi_a(t)\right)\mathrm{sin}_p\psi_a(t) + \frac{|\mathrm{sin}_p\psi_a(t)|^p}{p-1}(B-\varepsilon)\right] \tag{3.9}$$
for all sufficiently large t. Let us choose $\varepsilon > 0$ so small that $(A-\varepsilon)^{p-1}(B-\varepsilon) > \gamma_{p,\alpha}$. Now we apply Lemma 3.1. If we put $C^{1-q} = A - \varepsilon$ and $D = B - \varepsilon$, then any solution $\zeta : \mathbb{R}_b \to \mathbb{R}$ of the equation
$$\zeta'(t) = \frac{1}{t}\left[(A-\varepsilon)\,|\mathrm{cos}_p\zeta(t)|^p - \frac{p-\alpha}{p-1}\varPhi\left(\mathrm{cos}_p\zeta(t)\right)\mathrm{sin}_p\zeta(t) + (B-\varepsilon)\,\frac{|\mathrm{sin}_p\zeta(t)|^p}{p-1}\right] \tag{3.10}$$
has the property that $\lim_{t\to\infty}\zeta(t) = \infty$. Indeed, $C^{-1}D = (A-\varepsilon)^{\frac{1}{q-1}}(B-\varepsilon) = (A-\varepsilon)^{p-1}(B-\varepsilon) > \gamma_{p,\alpha}$. Comparing Eq. (3.10) with the right-hand side of (3.9), we obtain $\lim_{t\to\infty}\psi_a(t) = \lim_{t\to\infty}\varphi(t) = \infty$ which proves the first part of the theorem.

Part (ii). We can proceed analogously as in the first part. Let $\varepsilon > 0$ be arbitrary. Applying Lemma 3.3 and (3.8), we obtain
$$\psi_a'(t) < \frac{1}{t}\left[|\mathrm{cos}_p\psi_a(t)|^p(A+\varepsilon) - \frac{p-\alpha}{p-1}\varPhi\left(\mathrm{cos}_p\psi_a(t)\right)\mathrm{sin}_p\psi_a(t) + \frac{|\mathrm{sin}_p\psi_a(t)|^p}{p-1}(B+\varepsilon)\right] \tag{3.11}$$
for all sufficiently large t. Now we choose the value of ε so that $(A+\varepsilon)^{p-1}(B+\varepsilon) < \gamma_{p,\alpha}$ and we put $C^{1-q} = A + \varepsilon$, $D = B + \varepsilon$. Considering $C^{-1}D = (A+\varepsilon)^{p-1}(B+\varepsilon) < \gamma_{p,\alpha}$ in Lemma 3.1, any solution $\xi : \mathbb{R}_b \to \mathbb{R}$ of the equation
$$\xi'(t) = \frac{1}{t}\left[(A+\varepsilon)\,|\mathrm{cos}_p\xi(t)|^p - \frac{p-\alpha}{p-1}\varPhi\left(\mathrm{cos}_p\xi(t)\right)\mathrm{sin}_p\xi(t) + (B+\varepsilon)\,\frac{|\mathrm{sin}_p\xi(t)|^p}{p-1}\right] \tag{3.12}$$
satisfies $\limsup_{t\to\infty}\xi(t) < \infty$. From (3.11) and (3.12), we see that $\limsup_{t\to\infty}\psi_a(t) = \limsup_{t\to\infty}\varphi(t) < \infty$. □

A frequently studied case of the considered problem is given by $\alpha = 1$. In this case, we also get a new result. We point out that the strongest results concerning Eq. (2.12) for $\alpha = 1$ are proved in [8,9,18] and that any result in those papers does not cover our result. For periodic coefficients with the same period, we get a new result as well (cf. results from [7,11,16] and also [19–21] concerning discrete and time scales counterparts). The case $\alpha = p$, which is not treated in this paper, is substantially different from the considered situation, when $\alpha \neq p$. In particular, the same transformations cannot be used and the corresponding results are not valid. For more details, we refer to [22,23].

Acknowledgements

The first author was supported by Grants P201/11/0768, GA16-00611S of the Czech Science Foundation. The second author was supported by the Grant No. 1/0071/14 of the Slovak Grant Agency VEGA. The third author was supported by the project "Employment of Best Young Scientists for International Cooperation Empowerment" (CZ.1.07/2.3.00/30.0037) co-financed by European Social Fund and the state budget of the Czech Republic. The authors would like to thank the anonymous referee for her/his valuable comments and suggestions.

References

[1] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory of Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations, Kluwer Academic Publishers, Dordrecht, 2002.

[2] O. Došlý, P. Řehák, Half-linear Differential Equations, Elsevier, Amsterdam, 2005.

[3] Á. Elbert, Asymptotic behaviour of autonomous half-linear differential systems on the plane, Studia Sci. Math. Hungar. 19 (2–4) (1984) 447–464.

[4] Á. Elbert, Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations, in: Ordinary and Partial Differential Equations (Dundee, 1982), in: Lecture Notes in Math., vol. 964, Springer, Berlin, 1982, pp. 187–212.

[5] O. Došlý, P. Hasil, Critical oscillation constant for half-linear differential equations with periodic coefficients, Ann. Mat. Pura Appl. 190 (3) (2011) 395–408.

[6] O. Došlý, N. Yamaoka, Oscillation constants for second-order ordinary differential equations related to elliptic equations with p-Laplacian, Nonlinear Anal. 113 (2015) 115–136.

[7] P. Hasil, Conditional oscillation of half-linear differential equations with periodic coefficients, Arch. Math. 44 (2) (2008) 119–131.

[8] P. Hasil, R. Mařík, M. Veselý, Conditional oscillation of half-linear differential equations with coefficients having mean values, Abstr. Appl. Anal. 2014 (2014) 1–14. Art. ID 258159.

[9] J. Jaroš, M. Veselý, Conditional oscillation of Euler type half-linear differential equations with unbounded coefficients, Studia Sci. Math. Hungar. 53 (1) (2016) 22–41.

[10] M. Veselý, P. Hasil, Conditional oscillation of Riemann–Weber half-linear differential equations with asymptotically almost periodic coefficients, Studia Sci. Math. Hungar. 51 (3) (2014) 303–321.

[11] F. Gesztesy, M. Ünal, Perturbative oscillation criteria and hardy-type inequalities, Math. Nachr. 189 (1998) 121–144.

[12] H. Krüger, G. Teschl, Effective Prüfer angles and relative oscillation criteria, J. Differential Equations 245 (12) (2008) 3823–3848.

[13] H. Krüger, G. Teschl, Relative oscillation theory for Sturm–Liouville operators extended, J. Funct. Anal. 254 (6) (2008) 1702–1720.

[14] H. Krüger, G. Teschl, Relative oscillation theory, weighted zeros of the Wronskian, and the spectral shift function, Comm. Math. Phys. 287 (2) (2009) 613–640.

[15] K.M. Schmidt, Critical coupling constant and eigenvalue asymptotics of perturbed periodic Sturm–Liouville operators, Comm. Math. Phys. 211 (2000) 465–485.

[16] K.M. Schmidt, Oscillation of perturbed Hill equation and lower spectrum of radially periodic Schrödinger operators in the plane, Proc. Amer. Math. Soc. 127 (1999) 2367–2374.

[17] O. Došlý, M. Veselý, Oscillation and non-oscillation of Euler type half-linear differential equations, J. Math. Anal. Appl. 429 (2015) 602–621.

[18] P. Hasil, M. Veselý, Oscillation of half-linear differential equations with asymptotically almost periodic coefficients, Adv. Differential Equations 2013 (122) (2013) 1–15.

[19] P. Hasil, M. Veselý, Critical oscillation constant for difference equations with almost periodic coefficients, Abstr. Appl. Anal. 2012 (2012) 1–19. Art. ID 471435.

[20] P. Hasil, J. Vítovec, Conditional oscillation of half-linear Euler-type dynamic equations on time scales, Electron. J. Qual. Theory Differ. Equ. 2015 (6) (2015) 1–24.

[21] M. Veselý, P. Hasil, Oscillation and non-oscillation of asymptotically almost periodic half-linear difference equations, Abstr. Appl. Anal. 2013 (2013) 1–12. Art. ID 432936.
[22] P. Hasil, M. Veselý, Non-oscillation of periodic half-linear equations in the critical case, Electron. J. Differential Equations 2016 (120) (2016) 1–12.
[23] P. Hasil, M. Veselý, Oscillation constant for modified Euler type half-linear equations, Electron. J. Differential Equations 2015 (220) (2015) 1–14.