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A note on fractional Duhamel's principle and its application to a class of fractional partial differential equations

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ABSTRACT

The aim of this study is to develop a fractional version of Duhamel's principle for a class of fractional partial differential equations. Also we establish the existence of unique solution.

1. Introduction

In recent years, fractional differential equations have been investigated and applied in many research fields, such as fluid mechanics, chemistry, mathematical biology and finance. There has been considerable development of the qualitative theory of fractional differential equations (see, for instance, [1–13] and references therein). Fractional partial differential equations are becoming increasingly popular due to their practical applications in various fields of science and engineering.

Since the analytic solutions of nonlinear fractional order partial differential equations are rarely available, one needs to exploit some efficient and reliable criteria for the existence and uniqueness of solutions for nonlinear fractional partial differential equations. In literature, few works are found for obtaining the existence criteria for solutions of nonlinear fractional partial differential equations as far as we know [14,15].

In this work, we will introduce a fractional Duhamel principle and use it to establish an existence result for nonlinear fractional order diffusion-wave equation of the form

$$\mathcal{D}_t^{\alpha} u(x,t) - c u_{xx}(x,t) = q(x,t,u), \qquad u(x,0) = f(x), \qquad u_t(x,0) = 0, \quad x \in \mathbb{R},\ t > 0 \tag{1.1}$$

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where $\mathcal{D}_t^\alpha$ is the standard Caputo fractional derivative with $1 < \alpha \leq 2$. We work in an arbitrary Banach space of bounded continuous functions $u(x,t)$ on $\mathbb{R} \times \mathbb{R}^+$ and the nonlinear function q is assumed to satisfy certain conditions to be specified later.

The rest of the paper is organized as follows. In Section 2, we shall present basic definitions and preliminary results that are needed in the sequel. In Section 3, we develop a fractional version of Duhamel's principle and conditions for existence of unique solution are established in Section 4.

2. Fractional derivative and integral

For convenience, this section summarizes some concepts, definitions and basic results from fractional calculus, which are useful for the further developments in this paper.

Definition 2.1. Let $\alpha > 0$, $n = \lceil \alpha \rceil$ and $u(x,t) \in C^n(\mathbb{R} \times \mathbb{R}_+)$. Then the Caputo fractional derivative of $u(x,t)$ with respect to t is defined as $\mathcal{D}_t^\alpha u(x,t) = \mathcal{I}_t^{n-\alpha} \frac{\partial^n}{\partial t^n} u(x,t)$, where $\mathcal{I}_t^\alpha$ is the Riemann–Liouville fractional integral, given as

$$\mathcal{I}_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x,s) ds. \tag{2.1}$$

We review few basic properties of fractional differential and integral operators that are needed for this work. For details, see [16] and references therein.

(P1) $\mathcal{I}_t^\alpha \mathcal{I}_t^\beta u(x,t) = \mathcal{I}_t^{\alpha+\beta} u(x,t) = \mathcal{I}_t^\beta \mathcal{I}_t^\alpha u(x,t)$, (P2) $\mathcal{D}_t^\alpha \mathcal{I}_t^\alpha u(x,t) = \mathcal{I}_t^{\alpha-\beta} u(x,t)$.
(P3) The Caputo fractional derivative of order $\alpha > 0$ for $g(t) := t^\beta$, is given as

$$\mathcal{D}_t^\alpha g(t) = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{if } \beta \in \mathbb{N}, \text{ and } \beta \geq m \text{ or } \beta \notin \mathbb{N}, \text{ and } \beta > m-1, \\ 0, & \text{if } \beta \in \{0,1,2,\ldots,m-1\}. \end{cases} \tag{2.2}$$

Lemma 2.2. *Let $m-1 < \alpha \leq m$, and $u(x,.) \in AC^m([0,T])$, then $\mathcal{I}_t^\alpha \mathcal{D}_t^\alpha u(x,t) = u(x,t) - \sum_{j=0}^{n-1} \mu_j(x) t^j$, where, $\mu_j(x) = \frac{1}{j!} \frac{\partial^j u(x,t)|_{t=0}}{\partial t^j}$, $j = 0,1,2,\ldots$.*

Definition 2.3. The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(t)$ is defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)} \quad t \in \mathbb{R},\ \alpha,\ \beta > 0. \tag{2.3}$$

On the basis of numerical evidences, F. Mainardi [17] conjectured that for every $\alpha \in (0,1)$ the uniform estimate

$$\frac{1}{1+\Gamma(1-\alpha)x} \leq E_{\alpha,1}(-x) \leq \frac{1}{1+\Gamma(1-\alpha)^{-1}x}$$

holds for $x > 0$. T. Simon [18] proved the Mainardi's [17] conjecture. His proof is based on probabilistic arguments. R. Spigler [19] proved, in part, the Mainardi conjecture for $x := t^\alpha$ in some right neighborhood of $t = 0$. The Mittag-Leffler function $E_{\alpha,1}(-t^\alpha)$ for $t > 0$, $\alpha \in (0,1]$ solve the Caputo differential equation $\mathcal{D}_t^\alpha u + u = 0$, $u(0) = 1$. So it plays essential role in the theory of fractional differential equations. As we shall see later in Section 3, that the Mittag-Leffler functions $E_{\alpha,1}(-t^\alpha)$ and $E_{\alpha,\alpha}(-t^\alpha)$ for $t > 0$ and $\alpha \in [1,2]$

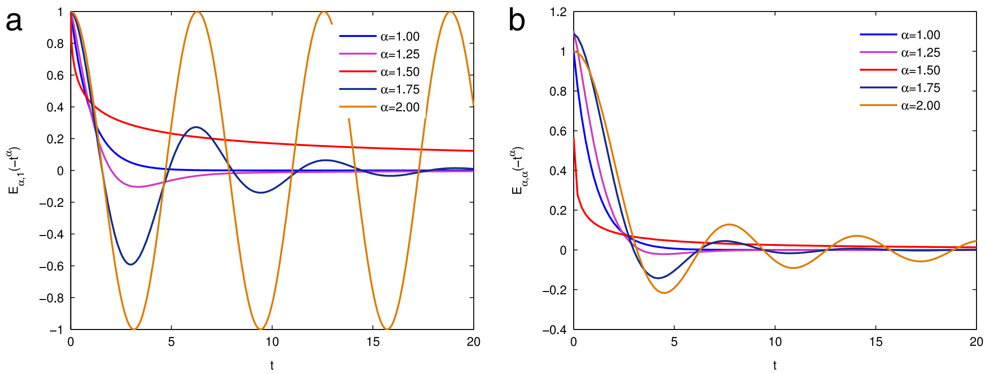

Fig. 1. Graphs of $E_{\alpha,1}(-t^{\alpha})$ and $E_{\alpha,\alpha}(-t^{\alpha})$ for $\alpha = 1.00, 1.25, 1.50, 1.75, 2.00$.

appear in the solutions of fractional partial differential equations. From numerical evidences, as depicted in Fig. 1 we formulate following conjecture.

Let $t > 0$, $\alpha \in [1, 2]$ and σ is unique solution of the equation $\tan\sigma - \sigma = 0$, for $4 < \sigma < 5$. Then following inequalities hold:

$$-1 \leq E_{\alpha,1}(-t^{\alpha}) \leq 1, \qquad \frac{\sin\sigma}{\sigma} \leq E_{\alpha,\alpha}(-t^{\alpha}) \leq \frac{1}{\Gamma(\alpha)}.$$

To proceed further, we state the following auxiliary lemma.

Lemma 2.4. *If $n - 1 < \alpha \leq n$ and $\eta(t) := t^{n-1}E_{\alpha,n}(-at^{\alpha})$ then $\mathcal{D}_t^{\alpha}\eta(t) - a\eta(t) = 0$.*

3. Duhamel's principle

The classical Duhamel principle reduces the Cauchy problem for an inhomogeneous partial differential equation to Cauchy problem for corresponding homogeneous equation. In [20,21], authors have established fractional analog of Duhamel principle. Here we introduce a simple version of fractional Duhamel principle that helps us to reduce the problem (1.1) to an equivalent integral equation.

Theorem 3.1 (*Duhamel's Principle*)**.** *If $n - 1 < \alpha \leq n$, $\phi \in L_1[a, b]$ and $f(t) = \mathcal{I}_t^{n-\alpha}\phi(t)$. Then solution of the problem*

$$\mathcal{D}_t^{\alpha}u(t) + au(t) = f(t), \qquad u(0) = 0, \qquad u'(0) = 0, \ldots, u^{(n-1)}(0) = 0 \tag{3.1}$$

is

$$u(t) = \int_0^t \nu(t - \tau; \tau)d\tau \tag{3.2}$$

where $\nu(t; \tau)$ is a solution of

$$\mathcal{D}_t^{\alpha}\nu(t; \tau) + a\nu(t; \tau) = 0, \qquad \nu(0; \tau) = 0, \qquad \nu'(0; \tau) = 0, \ldots, \nu^{(n-1)}(0; \tau) = \phi(\tau). \tag{3.3}$$

Proof. By Lemma 2.4, $\nu(t; \tau) = t^{n-1}E_{\alpha,n}(-at^{\alpha})\phi(\tau)$ satisfies (3.3). Also, by (2.2) we have $\nu^{(k)}(t; \tau) = \sum_{i=0}^{\infty} \frac{(-a)^i t^{\alpha i-k+n-1}}{\Gamma(\alpha i-k+n)}\phi(\tau)$. Now, it is easy to see that $\nu(0; \tau) = 0$, $\nu'(0; \tau) = 0, \ldots,$ $\nu^{(n-1)}(0; \tau) = \phi(\tau)$. Finally it remains to prove that u defined by (3.2) satisfies (3.1).

Eq. (3.2) can be written as

$$\begin{aligned} u(t) &= \int_0^t \nu(t-\tau;\tau)d\tau = \int_0^t (t-\tau)^{n-1} E_{\alpha,n}(-a(t-\tau)^\alpha)\phi(\tau)d\tau \\ &= \sum_{k=0}^{\infty}(-a)^k \int_0^t \frac{(t-\tau)^{\alpha k+n-1}}{\Gamma(\alpha k+n)}\phi(\tau)d\tau. \end{aligned}$$

Thus

$$u(t) = \sum_{k=0}^{\infty}(-a)^k \mathcal{I}_t^{\alpha k+n}\phi(t). \tag{3.4}$$

Applying $\mathcal{D}_t^\alpha$ on both sides of (3.4) and using (P2) we get

$$\begin{aligned} \mathcal{D}_t^\alpha u(t) &= \sum_{k=0}^{\infty}(-a)^k \mathcal{D}_t^\alpha \mathcal{I}_t^{\alpha k+n}\phi(t) = \sum_{k=0}^{\infty}(-a)^k \mathcal{I}_t^{\alpha(k-1)+n}\phi(t) \\ &= \mathcal{I}_t^{n-\alpha}\phi(t) + \sum_{k=1}^{\infty}(-a)^k \mathcal{I}_t^{\alpha(k-1)+n}\phi(t) \\ &= f(t) - a\sum_{k=0}^{\infty}(-a)^k \mathcal{I}_t^{\alpha k+n}\phi(t) = f(t) - au(t). \end{aligned}$$

Also, from (3.4), we have $u^{(k)}(0) = 0$ for $k = 1, 2, \dots, n-1$. □

Remark 3.2. (i) In the statement of Theorem 3.1, we do not need to know the function ϕ explicitly. It is sufficient to know that such a function exists. (ii) If $\alpha = n$, then fractional Duhamel's principle reduces to classical Duhamel's principle for ordinary differential equations.

Lemma 3.3. *The functions G_1, G_2 defined by $G_1(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}E_{\alpha,1}(-ck^2t^\alpha)dk$, and $G_2(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}E_{\alpha,2}(-ck^2t^\alpha)dk$, satisfy following properties:*

(i) $\mathcal{D}_t^\alpha G_1(x,t) - cG_{1xx}(x,t) = 0$.
(ii) $G_1(x,t) \le \delta(x)$, $G_2(x,t) \le \delta(x)$, *where* $1 \le \alpha \le 2$ *and* $t > 0$.

Proof. (i) Since $\mathcal{D}_t^\alpha E_{\alpha,1}(-ck^2t^\alpha) = -ck^2E_{\alpha,1}(-ck^2t^\alpha)$ for $1 \le \alpha \le 2$, therefore

$$\mathcal{D}_t^\alpha G_1(x,t) = \frac{-ck^2}{2\pi}\int_{\mathbb{R}} e^{ikx}E_{\alpha,1}(-ck^2t^\alpha)dk, \quad \text{and} \quad G_{1xx}(x,t) = \frac{-k^2}{2\pi}\int_{\mathbb{R}} e^{ikx}E_{\alpha,1}(-ck^2t^\alpha)dk.$$

Hence $\mathcal{D}_t^\alpha G_1(x,t) - cG_{1xx}(x,t) = 0$.

(ii) The Fourier transform of Dirac delta function is $\mathcal{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}\delta(x)dx = \frac{1}{\sqrt{2\pi}}$.

Therefore $\delta(x) = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}dk$, which is integral representation of Dirac delta function commonly used in quantum mechanics. Hence, we have

$$\begin{aligned} G_1(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}E_{\alpha,1}(-ck^2t^\alpha)dk \le \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}dk = \delta(x), \quad \text{and} \\ G_2(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}E_{\alpha,2}(-ck^2t^\alpha)dk \le \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ikx}}{\Gamma(2)}dk = \delta(x). \quad \square \end{aligned}$$

Consider the initial value problem for linear inhomogeneous diffusion-wave equation

$$\mathcal{D}_t^\alpha u(x,t) - cu_{xx}(x,t) = h(x,t), \qquad u(x,0) = f(x), \qquad u_t(x,0) = 0, \quad 1 \le \alpha \le 2,\ x \in \mathbb{R},\ t > 0. \tag{3.5}$$

By linear properties of the operator $Lu := \mathcal{D}_t^\alpha u(x,t) - cu_{xx}(x,t)$ the solution $u(x,t)$ of (3.5) is the sum of solutions $\nu(x,t)$ and $w(x,t)$ of problems

$$\mathcal{D}_t^\alpha v(x,t) - cv_{xx}(x,t) = 0, \qquad v(x,0) = f(x), \qquad v_t(x,0) = 0, \quad x \in \mathbb{R},\ t > 0, \tag{3.6}$$

$$\mathcal{D}_t^\alpha w(x,t) - cw_{xx}(x,t) = h(x,t), \qquad w(x,0) = 0, \qquad w_t(x,0) = 0, \quad x \in \mathbb{R},\ t > 0, \tag{3.7}$$

respectively. By Lemma 2.4 the function

$$v(x,t) = \int_{\mathbb{R}} G_1(x-\xi,t) f(\xi) d\xi \tag{3.8}$$

satisfies (3.6) and also $v(x,0) = \int_{\mathbb{R}} \delta(x-\xi) f(\xi) d\xi = f(x)$ and will show that, for $\psi(x,.) \in L_1[a,b]$ and $h(x,.) = \mathcal{I}_t^{n-\alpha}\psi(x,.)$,

$$w(x,t) = \int_0^t \int_{\mathbb{R}} (t-\tau) G_2(x-\xi,t-\tau)\psi(\xi,\tau) d\xi d\tau \tag{3.9}$$

is the solution of (3.7). From (3.8) and (3.9) it follows that solution of (3.5) is

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t) f(\xi) d\xi + \int_0^t \int_{\mathbb{R}} (t-\tau) G_2(x-\xi,t-\tau)\psi(\xi,\tau) d\xi d\tau. \tag{3.10}$$

Theorem 3.4 (*Duhamel's Principle for Diffusion-Wave Equation*)**.** *Let $f \in C(\mathbb{R}\times\mathbb{R}^+)$, $1 < \alpha \le 2$, $\psi(x,.) \in L_1[a,b]$ and $h(x,t) = \mathcal{I}_t^{2-\alpha}\psi(x,.)$. Then solution of the problem (3.7) is*

$$w(x,t) = \int_0^t \theta(x-\xi,t-\tau;\tau) d\tau \tag{3.11}$$

where $\theta(x,t;\tau) := \int_{\mathbb{R}} tG_2(x-\xi,t)\psi(\xi,\tau)d\xi$ is a solution of

$$\mathcal{D}_t^\alpha \theta(x,t;\tau) + a\theta_{xx}(x,t;\tau) = 0, \qquad \theta(x,0;\tau) = 0, \qquad \theta_t(x,0;\tau) = \psi(x,\tau). \tag{3.12}$$

Proof. By Lemma 2.4, $\theta(x,t;\tau)$ satisfies Eq. (3.12) and also it is easy to see that initial conditions in (3.12) are satisfied. Using the method of proof of Theorem 3.1 it follows that $w(x,t)$ defined in (3.11) is solution of the problem (3.7). □

At this point we are in position to consider the question of existence of solution to the nonlinear initial value problem (1.1). Assume there exists $\chi(x,.,u,v) \in L_1[a,b]$ and $q(x,t,u) = \mathcal{I}_t^{2-\alpha}\chi(x,t,u)$. If the function $u(x,t)$ is solution of differential equation in (1.1) then $\mathcal{D}_t^\alpha u(x,t) + cu_{xx}(x,t) = q(x,t,u) = h(x,t)$. Thus from (3.10) we expect that

$$u(x,t) = \int_{\mathbb{R}} G_1(x-\xi,t) f(\xi) d\xi + \int_0^t \int_{\mathbb{R}} (t-\tau) G_2(x-\xi,t-\tau)\chi(\xi,\tau,u) d\xi d\tau, \tag{3.13}$$

which is a nonlinear integral equation. A function $u(x,t)$ is solution of (1.1) if and only if $u(x,t)$ is solution of (3.13). Eq. (3.13) can be written as $u = \mathcal{A}(u)$ where $\mathcal{A}$ is mapping defined as

$$\mathcal{A}(u)(x,t) = \int_{\mathbb{R}} G_1(x-\xi,t) f(\xi) d\xi + \int_0^t \int_{\mathbb{R}} (t-\tau) G_2(x-\xi,t-\tau)\chi(\xi,\tau,u) d\xi d\tau. \tag{3.14}$$

4. Existence of solution

Using tools from functional analysis we now formulate existence theorems for the nonlinear problem (1.1). For fixed time t, $u(x,t)$ can be regarded as function of $x \in \mathbb{R}$. Let B be the Banach space of all bounded continuous function $u(x,t)$ on $\mathbb{R}$ (with fixed t) equipped with the norm $\|u(.,t)\|_B = \sup_{x\in\mathbb{R}} |u(x,t)|$, for fixed t. For $T > 0$ consider the Banach space $\tilde{B} = C([0,t];B)$ of all continuous functions defined on $[0,T]$ that have values in Banach space B equipped with norm $\|u\|_{\tilde{B}} = \sup_{t\in[0,T]} |u(.,t)|_B$. Following theorem is a generalization existence theorem [22] from integer order to fractional settings.

Theorem 4.1. *Assume there exists $\chi(x,.,u,v) \in L_1[a,b]$ and $q(x,t,u) = I_t^{2-\alpha}\chi(x,t,u)$ with $q(x,t,0) = 0$ is jointly continuous on $\mathbb{R}\times[0,T]\times\mathbb{R}$. Furthermore assume for any $M > 0$, there exists constant K such that*

$$\|q(.,t,u(.,t)) - q(.,t,v(.,t))\|_B \le Kt^{\alpha-2}\|u(.,t) - v(.,t)\|_B, \tag{4.1}$$

for all $t \in [0,T]$ and all $u, v \in B$ with $\|u(.,t)\|_B \le M$ and $\|v(.,t)\|_B \le M$. Then there exists $t_0 > 0$ such that the initial value problem (1.1) has unique solution in $\Gamma = C([0,t_0];B)$ and $\|u\|_{\tilde{B}} \le 2\|f(.)\|_B$.

Proof. Define the set $\Omega = \left\{u \in \tilde{B} : \|u(.,t) - (G_1 * f)(.,t)\|_B \le \|f(.)\|_B, \text{ for } t \in [0,t_0]\right\}$ where $t_0 = \frac{1}{\sqrt{2\Gamma(\alpha-1)K}}$ and $(G_1 * f)(x,t) = \int_{\mathbb{R}} G_1(x-\xi,t)f(\xi)d\xi$ is convolution of G_1 and f. Note that $\|(G_1 * u)(.,t)\|_B \le \int_{\mathbb{R}} \|\delta(x-\xi)u(\xi,t)\|_B d\xi = \|u(x,t)\|_B$. Similarly $\|(G_2 * u)(.,t)\|_B \le \|u(x,t)\|_B$. By defining property of Ω, triangular inequality and the last inequality, we have $\|u(.,t)\|_B \le 2\|f(.)\|_B$. Taking supremum with respect to t, we have $\|u\|_{\tilde{B}} \le 2\|f(.)\|_B$.

Now we prove that $\mathcal{A}$ maps Ω into Ω.

$$\begin{aligned}
\|\mathcal{A}(u)(.,t) - (G_1 * f)(.,t)\|_B &= \left\|\int_0^t (t-\tau)(G_2 * \chi)(.,t-\tau)d\tau\right\|_B \\
&= \int_0^t (t-\tau)\|(G_2 * \chi)(.,t-\tau)\|_B d\tau \\
&\le \int_0^t (t-\tau)\|\chi(.,\tau,u(.,\tau))\|_B d\tau = \int_0^t (t-\tau)\mathcal{D}_\tau^{2-\alpha}\|q(.,\tau,u(.,\tau))\|_B d\tau.
\end{aligned}$$

Using (4.1) in above inequality (for $v = 0$) we have

$$\begin{aligned}
\|\mathcal{A}(u)(.,t) - (G_1 * f)(.,t)\|_B &\le K\int_0^t (t-\tau)\mathcal{D}_\tau^{2-\alpha}\tau^{\alpha-2}\|u(.,\tau)\|_B d\tau \\
&\le K\Gamma(\alpha-1)\|u\|_{\tilde{B}}\int_0^t (t-\tau)d\tau \le K\Gamma(\alpha-1)t^2\|f(.)\|_B < \|f(.)\|_B.
\end{aligned}$$

Now we prove that $\mathcal{A}$ is contraction.

$$\begin{aligned}
\|\mathcal{A}(u)(.,t) - \mathcal{A}(v)(.,t)\|_B &\le \sup_{x\in\mathbb{R}}\int_0^t\int_{\mathbb{R}} (t-\tau)G_2(x-\xi,t-\tau)|\chi(\xi,\tau,u) - \chi(\xi,\tau,u)|d\xi d\tau, \\
&\le \int_0^t (t-\tau)\|\chi(.,\tau,u) - \chi(.,\tau,u)\|_B d\xi d\tau, \\
&= \int_0^t (t-\tau)\mathcal{D}_\tau^{2-\alpha}\tau^{\alpha-1}\|q(.,\tau,u) - q(.,\tau,u)\|_B d\xi d\tau \\
&\le K\Gamma(\alpha-1)t_0^2\|u-v\|_{\tilde{B}} = \frac{1}{2}\|u-v\|_{\tilde{B}}.
\end{aligned}$$

Thus

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{\tilde{B}} = \sup_{t\in[0,T]}\|\mathcal{A}(u)(.,t) - \mathcal{A}(v)(.,t)\|_B \le \frac{1}{2}\|u-v\|_{\tilde{B}}.$$

Therefore $\mathcal{A}$ is contraction on closed set Ω of the Banach space $\tilde{B}$. Hence, by contraction mapping principle, $\mathcal{A}$ has unique fixed point in Ω which is solution of (1.1). □

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