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Fractal nil graded Lie superalgebras

Victor Petrogradsky [1]

Department of Mathematics, University of Brasilia, 70910-900 Brasilia, DF, Brazil

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A B S T R A C T

The Grigorchuk and Gupta–Sidki groups play fundamental role in modern group theory. They are natural examples of self-similar finitely generated periodic groups. The author constructed their analogue in case of restricted Lie algebras of characteristic 2 [31], Shestakov and Zelmanov extended this construction to an arbitrary positive characteristic [41]. Thus, we have examples of (self-similar) finitely generated restricted Lie algebras with a nil p-mapping. In characteristic zero, similar examples of Lie algebras do not exist by a result of Martinez and Zelmanov [27].

The goal of the present paper is to construct analogues of the Grigorchuk and Gupta–Sidki groups in the world of Lie superalgebras of an arbitrary characteristic. We construct two examples $\mathbf{R}$, $\mathbf{Q}$ of finitely generated self-similar Lie superalgebras over a field K of an arbitrary characteristic and study their properties and properties of their associative hulls. In case $\operatorname{char} K = 2$, these examples turn into restricted Lie algebras.

The virtue of the present construction is that the Lie superalgebras have clear monomial bases. These Lie superalgebras have slow polynomial growth and are multigraded by multidegree in the generators. The Lie superalgebra $\mathbf{R}$ is $\mathbb{Z}^2$-graded, while $\mathbf{Q}$ has a multidegree $\mathbb{Z}^3$-gradation and a $\mathbb{Z}^2$-gradation. Both algebras $\mathbf{R}$ and $\mathbf{Q}$ have similar constructions, computations for $\mathbf{R}$ are simpler, but $\mathbf{Q}$ enjoys some more specific interesting properties. The $\mathbb{Z}^3$-components of $\mathbf{Q}$ lie inside an elliptic paraboloid in space, they are at most one-dimensional, thus, the $\mathbb{Z}^3$-grading of $\mathbf{Q}$ is fine. In the $\mathbb{Z}^2$-gradation of $\mathbf{Q}$, all components $\mathbf{Q}_{nm}$, $n, m \in \mathbb{Z}$, are infinite dimensional except

E-mail address: petrogradsky@rambler.ru.

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for $\mathbf{Q}_{00} = \{0\}$. The $\mathbb{Z}^2$-gradation also yields a continuum of different decompositions into a direct sum of two locally nilpotent subalgebras $\mathbf{Q} = \mathbf{Q}_+ \oplus \mathbf{Q}_-$.
In both examples, $\operatorname{ad} a$ is nilpotent, a being even or odd with respect to the $\mathbb{Z}_2$-grading as Lie superalgebras. This property is an analogue of the periodicity of the Grigorchuk and Gupta–Sidki groups. In particular, $\mathbf{Q}$ is a nil finely-graded Lie superalgebra, which shows that an extension of a theorem due to Martinez and Zelmanov [27] for the Lie superalgebras of characteristic zero is not valid.
Both Lie superalgebras are self-similar, contain infinitely many copies of themselves, let us also call them fractal due to this property.

1. Introduction: self-similar groups and algebras

In this section we provide a survey of related results and give motivations for construction of our examples, in Section 2 we supply basic definitions. Main results are formulated in Section 3.

1.1. Golod–Shafarevich algebras and groups

The General Burnside Problem puts the question whether a finitely generated periodic group is finite. The first negative answer was given by Golod and Shafarevich, who proved that, for each prime p, there exists a finitely generated infinite p-group [13,14]. The construction is based on a famous construction of a family of finitely generated infinite dimensional associative nil-algebras [13]. This construction also yields examples of infinite dimensional finitely generated Lie algebras L such that $(\operatorname{ad} x)^{n(x,y)}(y) = 0$, for all $x, y \in L$, the field being arbitrary [15]. The field being of positive characteristic p, one obtains an infinite dimensional finitely generated restricted Lie algebra L such that the p-mapping is nil, namely, $x^{[p^{n(x)}]} = 0$, for all $x \in L$. This gives a negative answer to a question of Jacobson whether a finitely generated restricted Lie algebra L is finite dimensional provided that each element $x \in L$ is algebraic, i.e. satisfies some p-polynomial $f_{p,x}(x) = 0$ [20, Ch. 5, ex. 17]. It is known that the construction of Golod yields associative nil-algebras of exponential growth. Using specially chosen relations, Lenagan and Smoktunowicz constructed associative nil-algebras of polynomial growth [26]. On further developments concerning Golod–Shafarevich algebras and groups see [47,11].

A close by spirit but different construction was motivated by respective group-theoretic results. A restricted Lie algebra G is called *large* if there is a subalgebra $H \subset G$ of finite codimension such that H admits a surjective homomorphism on a nonabelian free restricted Lie algebra. Let K be a perfect at most countable field of positive characteristic. Then there exist infinite-dimensional finitely generated nil restricted Lie algebras over K that are residually finite dimensional and direct limits of large restricted Lie algebras [3].

1.2. Grigorchuk and Gupta–Sidki groups

The construction of Golod is rather undirect, Grigorchuk gave a direct and elegant construction of an infinite 2-group generated by three elements of order 2 [16]. This group was defined as a group of transformations of the interval $[0,1]$ from which rational points of the form $\{k/2^n \mid 0 \le k \le 2^n,\ n \ge 0\}$ are removed. For each prime $p \ge 3$, Gupta and Sidki gave a direct construction of an infinite p-group on two generators, each of order p [19]. This group was constructed as a subgroup of an automorphism group of an infinite regular tree of degree p.

The Grigorchuk and Gupta–Sidki groups are counterexamples to the General Burnside Problem. Moreover, they gave answers to important problems in group theory. So, the Grigorchuk group and its further generalizations are first examples of groups of intermediate growth [17], thus answering in negative to a conjecture of Milnor that groups of intermediate growth do not exist. The construction of Gupta–Sidki also yields groups of subexponential growth [12]. The Grigorchuk and Gupta–Sidki groups are *self-similar*, in particular, they contain infinitely many copies of themselves. Now self-similar, in particular so called *branch groups*, form a well-established area in group theory, see [18] for further developments. Below we discuss existence of analogues of the Grigorchuk and Gupta–Sidki groups for other algebraic structures.

1.3. Self-similar nil graded associative algebras

The study of these groups leads to investigation of group rings and other related associative algebras [43]. In particular, there appeared self-similar associative algebras defined by matrices in a recurrent way [5]. The recurrent matrices were also applied to number theoretical problems [1]. Sidki suggested two examples of self-similar associative matrix algebras [42]. A more general family of self-similar associative algebras was introduced in [34], this family generalizes the second example of Sidki [42], also it yields a realization of a Fibonacci restricted Lie algebras (see below) in terms of self-similar matrices [34]. Another important feature of some associative self-similar algebras A constructed in [34] is that they are sums of two locally nilpotent subalgebras $A = A_+ \oplus A_-$ (see similar decompositions (1) below). Recall that an algebra is said *locally nilpotent* if every finitely generated subalgebra is nilpotent. But the desired analogues of the Grigorchuk and Gupta–Sidki groups should be associative self-similar nil-algebras, in a standard way yielding new examples of finitely generated periodic groups. But such examples are not known yet. On close open problems in theory of infinite dimensional algebras see review [48].

1.4. Nil restricted Lie algebras

Unlike associative algebras, for restricted Lie algebras, there are natural analogues of the Grigorchuk and Gupta–Sidki groups. Namely, over a field of characteristic 2, the author constructed an example of an infinite dimensional restricted Lie algebra $\mathbf{L}$ generated by two elements, called a *Fibonacci restricted Lie algebra* [31]. Let char $K = 2$ and $R = K[t_i | i \geq 0]/(t_i^p | i \geq 0)$ a truncated polynomial ring. Put $\partial_i = \frac{\partial}{\partial t_i}$, $i \geq 0$. Define the following two derivations of R:

$$\begin{aligned} v_1 &= \partial_1 + t_0(\partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots))))); \\ v_2 &= \qquad\quad \partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots)))). \end{aligned}$$

These two derivations generate a restricted Lie algebra $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2) \subset \mathrm{Der}\, R$ and an associative algebra $\mathbf{A} = \mathrm{Alg}(v_1, v_2) \subset \mathrm{End}\, R$. By Bergman's theorem, the Gelfand–Kirillov dimension of an associative algebra cannot belong to the interval $(1, 2)$ [22]. Such a gap for Lie algebras does not exist, the Gelfand–Kirillov dimension of a finitely generated Lie algebra can be an arbitrary number $\{0\} \cup [1, +\infty]$ [30]. The Fibonacci Lie algebra has slow polynomial growth with Gelfand–Kirillov dimension $\mathrm{GKdim}\, \mathbf{L} = \log_{\frac{\sqrt{5}+1}{2}} 2 \approx 1.44$ [31]. The restricted Lie algebra $\mathbf{L}$ is self-similar. Further properties of the Fibonacci restricted Lie algebra and its generalizations are studied in [33,35].

Probably, the most interesting property of $\mathbf{L}$ is that it has a nil p-mapping [31], which is an analog of the periodicity of the Grigorchuk and Gupta–Sidki groups. We do not know whether the associative hull $\mathbf{A}$ is a nil-algebra. We have a weaker statement. The algebras $\mathbf{L}$, $\mathbf{A}$, and the augmentation ideal of the restricted enveloping algebra $\mathbf{u} = \omega u(L)$ are direct sums of two locally nilpotent subalgebras [33]:

$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_-. \tag{1}$$

There are examples of infinite dimensional associative algebras which are direct sums of two locally nilpotent subalgebras [24,9]. Infinite dimensional restricted Lie algebras can have finitely many different decompositions into a direct sum of two locally nilpotent subalgebras [36].

In case of an arbitrary prime characteristic, Shestakov and Zelmanov suggested an example of a finitely generated restricted Lie algebra with a nil p-mapping [41]. That example yields the same decompositions (1) for some primes [25,34].

An example of a p-generated nil restricted Lie algebra L, characteristic p being arbitrary, was studied in [36]. The virtue of that example is that for all primes p we have the same decompositions (1) into direct sums of two locally nilpotent subalgebras. But computations for that example are rather complicated.

Observe that only the original example has a clear monomial basis [31,33]. In other examples, elements of a Lie algebra are linear combinations of monomials, working with such linear combinations is sometimes an essential technical difficulty, see e.g. [41,36]. A continuum family of nil restricted Lie algebras of slow growth having good monomial basis is constructed in [37], these algebras are close relatives of the Lie superalgebra $\mathbf{R}$ of the present paper.

Let G be a group and $G = G_1 \supseteq G_2 \supseteq \cdots$ its lower central series. One constructs the related $\mathbb{Z}$-graded Lie algebra $L_K(G) = \oplus_{i\geq 1} L_i$, where $L_i = G_i/G_{i+1} \otimes_{\mathbb{Z}} K$, $i \geq 1$, and $\operatorname{char} K = p$. A product is given by $[a_i G_{i+1}, b_j G_{j+1}] = (a_i, b_j) G_{i+j+1}$, where $(a_i, b_j) = a_i^{-1} b_j^{-1} a_i b_j$ stands for the group commutator. A residually p-group G is said of *finite width* if all factors G_i/G_{i+1} are finite groups with uniformly bounded orders.

The Grigorchuk group G is of finite width, namely, it is proved that $\dim_{\mathbb{F}_2} G_i/G_{i+1} \in \{1, 2\}$ for $i \geq 2$ [40,7]. In particular, the respective Lie algebra $L = L_K(G) = \oplus_{i\geq 1} L_i$ has a linear growth. Bartholdi recently proved that the restricted Lie algebra $L_{\mathbb{F}_2}(G)$ is nil while $L_{\mathbb{F}_4}(G)$ is not nil [6].

1.5. Lie algebras over a field of characteristic zero

In case of characteristic zero a similar example does not exist by the next result. Informally speaking, there are no natural analogues of the Grigorchuk and Gupta–Sidki groups in the world of Lie algebras of zero characteristic.

Theorem 1.1 *(Martinez and Zelmanov [27]). Let $L = \oplus_{\alpha \in \Gamma} L_\alpha$ be a Lie algebra over a field K of zero characteristic graded by an abelian group Γ. Suppose that*

(1) *there exists $d > 0$ such that $\dim_K L_\alpha \leq d$ for all $\alpha \in \Gamma$,*
(2) *every homogeneous element $a \in L_\alpha$, $\alpha \in \Gamma$, is ad-nilpotent.*

Then the Lie algebra L is locally nilpotent.

1.6. Fractal nil graded Lie superalgebras $\mathbf{R}$ and $\mathbf{Q}$

The goal of the present paper is to construct analogues of the Grigorchuk and Gupta–Sidki groups in the world of *Lie superalgebras* of an *arbitrary characteristic*. We construct two Lie superalgebras $\mathbf{R}$, $\mathbf{Q}$, which are also analogues of the Fibonacci restricted Lie algebra and other self-similar Lie algebras mentioned above. In both examples, $\operatorname{ad} a$ is nilpotent, a being an even or odd element with respect to $\mathbb{Z}_2$-grading as Lie superalgebras. This property is an analogue of the periodicity of the Grigorchuk and Gupta–Sidki groups. The Lie superalgebra $\mathbf{R}$ is $\mathbb{Z}^2$-graded, while $\mathbf{Q}$ has a natural fine $\mathbb{Z}^3$-gradation and $\mathbb{Z}^2$-gradation. Constructions of both algebras $\mathbf{R}$ and $\mathbf{Q}$ look similar, computations for $\mathbf{R}$ are simpler, but $\mathbf{Q}$ enjoys some specific interesting properties.

In particular, the example $\mathbf{Q}$ shows that an extension of Theorem 1.1 for Lie superalgebras of zero characteristic is not valid. Both Lie superalgebras are self-similar, contain infinitely many copies of themselves, let us also call them *fractal* due to this property. A detailed exposition of properties of the Lie superalgebras $\mathbf{R}$ and $\mathbf{Q}$ is given in Section 3.

It is interesting that there are many examples of $\mathbb{Z}^2$-graded simple Lie algebras in characteristic zero with one-dimensional components [21].

2. Basic definitions: (restricted) Lie superalgebras, growth

In this section we supply basic definitions. Superalgebras appear naturally in physics and mathematics [23,44,2]. Let K denote the ground field, $\langle S\rangle_K$ a linear span of a subset S in a K-vector space. A *superalgebra* A is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar 0}\oplus A_{\bar 1}$. The elements $a \in A_\alpha$ are called *homogeneous of degree* $\deg a = \alpha$, $\alpha \in \mathbb{Z}_2$. The elements of $A_{\bar 0}$ are called *even*, those of $A_{\bar 1}$ *odd*. Throughout what follows, if $\deg a$ enters an expression, then it is assumed that a is homogeneous of degree $\deg a \in \mathbb{Z}_2$, and the expression extends to the other elements by linearity. Let A, B be superalgebras, a *tensor product* $A \otimes B$ is the superalgebra whose space is the tensor product of the spaces A and B with the induced $\mathbb{Z}_2$-grading and the product defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\deg b_1\cdot \deg a_2} a_1a_2 \otimes b_1b_2, \quad a_i \in A,\ b_i \in B.$$

An *associative superalgebra* A is just a $\mathbb{Z}_2$-graded associative algebra $A = A_{\bar 0} \oplus A_{\bar 1}$. A *Lie superalgebra* is a $\mathbb{Z}_2$-graded algebra $L = L_{\bar 0} \oplus L_{\bar 1}$ with an operation $[\ ,\ ]$ satisfying the following axioms:

- $[x, y] = -(-1)^{\deg x\cdot \deg y}[y, x]$, (superanticommutativity);
- $[x, [y, z]] = [[x, y], z] + (-1)^{\deg x\cdot \deg y}[y, [x, z]]$, (Jacobi identity).

In particular, these conditions imply that $L_{\bar 0}$ is a Lie algebra and $L_{\bar 1}$ a $L_{\bar 0}$-module. All commutators in the present paper are supercommutators. Long commutators are *right-normed*: $[x, y, z] = [x, [y, z]]$. We use standard notation $\operatorname{ad} x(y) = [x, y]$, where $x, y \in L$.

Assume that $A = A_{\bar 0} \oplus A_{\bar 1}$ is an associative superalgebra. Then one defines on the same vector space A a *supercommutator*

$$[x, y] = xy - (-1)^{\deg x\cdot \deg y} yx, \quad x, y \in A.$$

This product turns A into a Lie superalgebra denoted by $A^{(-)}$. Assume that L is a Lie superalgebra, one defines the *universal enveloping algebra* $U(L) = T(L)/(x \otimes y - (-1)^{\deg x\cdot \deg y} y \otimes x - [x, y] \mid x, y \in L)$, where $T(L)$ is the tensor algebra of the vector space L. Now, the product in L coincides with the supercommutator in $U(L)^{(-)}$. A basis of $U(L)$ is given by PBW-theorem [2,44].

In case of small characteristics $\operatorname{char} K = 2, 3$ the axioms of the Lie superalgebra have to be modified, see e.g. [2, section 1.10], [8]. Substituting $x = y = z \in L_{\bar 1}$ in the Jacobi identity and using anticommutativity, we get $3[z, [z, z]] = 0$, $z \in L_{\bar 1}$. Thus, in case $\operatorname{char} K \neq 3$, we have an identity: $[z, [z, z]] = 0$ for all $z \in L_{\bar 1}$. Otherwise we add one more axiom for the Lie superalgebra.

- $(\operatorname{char} K = 3)$ $[z, [z, z]] = 0$, $z \in L_{\bar 1}$.

Substituting $x = y \in L_{\bar{1}}$ in the Jacobi identity, we get $2(\operatorname{ad} x)^2 z = [[x, x], z]$. In case $\operatorname{char} K \neq 2$ we get an identity

$$(\operatorname{ad} x)^2 z = \frac{1}{2}[[x, x], z], \qquad x \in L_{\bar{1}},\ z \in L.$$

The present paper deals with superalgebras of the form $A^{(-)}$, they have squares "for free" $[x, x] = 2x^2$, $x \in A^{(-)}_{\bar{1}}$. Thus, the identity above takes the following form, which we shall use below without mention (one directly verifies that it is also valid for $\operatorname{char} K = 2$):

$$(\operatorname{ad} x)^2 z = [x^2, z], \qquad x \in A^{(-)}_{\bar{1}},\ z \in A^{(-)}. \tag{2}$$

In case $\operatorname{char} K = 2$, we add more axioms for Lie superalgebras:

- $[x, x] = 0$ for all $x \in L_{\bar{0}}$;
- there exists a *quadratic mapping (formal square)*: $(*)^{[2]} : L_{\bar{1}} \to L_{\bar{0}}$, $x \mapsto x^{[2]}$, $x \in L_{\bar{1}}$, that satisfies:

$$(\lambda x)^{[2]} = \lambda^2 x^{[2]}, \quad (x + y)^{[2]} = x^{[2]} + [x, y] + y^{[2]}; \quad x, y \in L_{\bar{1}}, \lambda \in K;$$

  (By putting $y = x$, we get $[y, y] = 0$, $y \in L_{\bar{1}}$.)
- a formal substitute of (2) is fulfilled:

$$(\operatorname{ad} x)^2 z = [x^{[2]}, z], \quad x \in L_{\bar{1}},\ z \in L.$$

In case $p = 2$, to get the universal enveloping algebra, we additionally factor out $\{y \otimes y - y^{[2]} \mid y \in L_{\bar{1}}\}$. Also, one has to modify the definition of the *derived series*:

$$L^{(0)} = L, \quad L^{(i+1)} = [L^{(i)}, L^{(i)}] + \langle y^2 \mid y \in (L^{(i)})_{\bar{1}} \rangle_K,\ i \geq 0.$$

Roughly speaking, a Lie superalgebra in case $\operatorname{char} K = 2$ is the same as a $\mathbb{Z}_2$-graded Lie algebra supplied with a quadratic mapping $L_{\bar{1}} \to L_{\bar{0}}$, which is similar to the p-mapping (see below). In all cases, the quadratic mapping will be denoted by x^2, $x \in L_{\bar{1}}$, it also coincides with the ordinary square in $U(L)$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector space, we say that it is $\mathbb{Z}_2$-graded. The associative algebra of all endomorphisms of the space $\operatorname{End} V$ is an associative superalgebra: $\operatorname{End} V = \operatorname{End}_{\bar{0}} V \oplus \operatorname{End}_{\bar{1}} V$, where $\operatorname{End}_\alpha V = \{\phi \in \operatorname{End} V \mid \phi(V_\beta) \subset V_{\alpha+\beta}, \beta \in \mathbb{Z}_2\}$, $\alpha \in \mathbb{Z}_2$. Thus, $\operatorname{End}^{(-)} V$ is a Lie superalgebra, called the *general linear superalgebra* $gl(V)$.

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a $\mathbb{Z}_2$-graded algebra of arbitrary signature. A linear mapping $\phi \in \operatorname{End}_\beta A$, $\beta \in \mathbb{Z}_2$, is a *superderivative* of degree β if it satisfies

$$\phi(a \cdot b) = \phi(a) \cdot b + (-1)^{\beta \cdot \deg(a)} a \cdot \phi(b), \quad a, b \in A.$$

Denote by $\operatorname{Der}_\alpha A \subset \operatorname{End}_\alpha A$ the space of all superderivatives of degree $\alpha \in \mathbb{Z}_2$. An easy check shows that $\operatorname{Der} A = \operatorname{Der}_{\bar 0} A \oplus \operatorname{Der}_{\bar 1} A$ is a subalgebra of the Lie superalgebra $\operatorname{End}^{(-)} A$. In this paper by a derivation we always mean a superderivation.

Let L be a Lie algebra over a field K of characteristic $p > 0$. The Lie algebra L is called a *restricted Lie algebra* (or *Lie p-algebra*), if it is additionally supplied with a unary operation $x \mapsto x^{[p]}$, $x \in L$, that satisfies the following axioms [20,45,46]:

- $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, for $\lambda \in K$, $x \in L$;
- $\operatorname{ad}(x^{[p]}) = (\operatorname{ad} x)^p$, $x \in L$;
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$, $x, y \in L$, where $is_i(x,y)$ is the coefficient of t^{i-1} in the polynomial $\operatorname{ad}(tx+y)^{p-1}(x) \in L[t]$.

This notion is motivated by the following observation. Let A be an associative algebra over a field K, $\operatorname{char} K = p > 0$. Then the mapping $x \mapsto x^p$, $x \in A^{(-)}$, satisfies these conditions considered in the Lie algebra $A^{(-)}$.

A *restricted Lie superalgebra* $L = L_{\bar 0} \oplus L_{\bar 1}$ is a Lie superalgebra such that the even component $L_{\bar 0}$ is a restricted Lie algebra and $L_{\bar 0}$-module $L_{\bar 1}$ is restricted, i.e. $\operatorname{ad}(x^{[p]})y = (\operatorname{ad} x)^p y$, for all $x \in L_{\bar 0}$, $y \in L_{\bar 1}$ (see e.g. [28,2]). Remark that in case $\operatorname{char} K = 2$, the restricted Lie superalgebras and $\mathbb{Z}_2$-graded restricted Lie algebras are the same objects. (Let $L = L_{\bar 0} \oplus L_{\bar 1}$ be a restricted Lie superalgebra, we extend the p-mapping on the whole of algebra by $(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x,y]$, $x \in L_{\bar 0}$, $y \in L_{\bar 1}$.)

Suppose that L is a restricted Lie algebra. Let J be an ideal of the universal enveloping algebra $U(L)$ generated by all elements $x^{[p]} - x^p$, $x \in L$. Then $u(L) = U(L)/J$ is called the *restricted enveloping algebra*. In this algebra, the formal operation $x^{[p]}$ coincides with the pth power x^p for any $x \in L$. One has an analogue of Poincare–Birkhoff–Witt's theorem describing the basis of the restricted enveloping algebra [20, p. 213].

We recall the notion of *growth*. Let A be an associative (or Lie) algebra generated by a finite set X. Denote by $A^{(X,n)}$ a subspace of A spanned by all monomials in X of length not exceeding n, $n \ge 0$. In case of a Lie superalgebra of $\operatorname{char} K = 2$ we also consider formal squares of odd monomials of length at most $n/2$. If A is a restricted Lie algebra, put $A^{(X,n)} = \langle [x_1, \dots, x_s]^{p^k} \mid x_i \in X,\ sp^k \le n \rangle_K$ [29]. Similarly, one defines the growth for restricted Lie superalgebras. In either situation, one obtains a *growth function*:

$$\gamma_A(n) = \gamma_A(X,n) = \dim_K A^{(X,n)}, \quad n \ge 0.$$

Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two eventually monotone increasing and positive valued functions. Write $f(n) \preccurlyeq g(n)$ if and only if there exist two positive constants N, C such that $f(n) \le g(Cn)$ for all $n \ge N$. Introduce equivalence $f(n) \sim g(n)$ if and only if $f(n) \preccurlyeq g(n)$ and $g(n) \preccurlyeq f(n)$. Different generating sets of an algebra yield equivalent growth functions [22].

It is well known that the exponential growth is the highest possible growth for finitely generated Lie and associative algebras. A growth function $\gamma_A(n)$ is compared with polynomial functions n^k, $k \in \mathbb{R}^+$, by computing *upper and lower Gelfand–Kirillov dimensions* [22]:

$$\overline{\mathrm{GKdim}}\, A = \overline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n} = \inf\{\alpha > 0 \mid \gamma_A(n) \preccurlyeq n^\alpha\};$$
$$\underline{\mathrm{GKdim}}\, A = \underline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n} = \sup\{\alpha > 0 \mid \gamma_A(n) \succcurlyeq n^\alpha\}.$$

Assume that generators $X = \{x_1, \dots, x_k\}$ are assigned positive weights $\mathrm{wt}(x_i) = \lambda_i$, $i = 1, \dots, k$. Define a *weight growth function*:

$$\tilde{\gamma}_A(n) = \dim_K \langle x_{i_1} \cdots x_{i_m} \mid \mathrm{wt}(x_{i_1}) + \cdots + \mathrm{wt}(x_{i_m}) \le n,\ x_{i_j} \in X \rangle_K, \quad n \ge 0.$$

Set $C_1 = \min\{\lambda_i \mid i = 1, \dots, k\}$, $C_2 = \max\{\lambda_i \mid i = 1, \dots, k\}$, then $\tilde{\gamma}_A(C_1 n) \le \gamma_A(n) \le \tilde{\gamma}_A(C_2 n)$ for $n \ge 1$. Thus, we obtain an equivalent growth function $\tilde{\gamma}_A(n) \sim \gamma_A(n)$. Therefore, we can use the weight growth function $\tilde{\gamma}_A(n)$ in order to compute the Gelfand–Kirillov dimensions.

Suppose that L is a Lie (super)algebra and $X \subset L$. By $\mathrm{Lie}(X)$ denote the subalgebra of L generated by X, (including application of the quadratic mapping in case $\mathrm{char}\, K = 2$). Let L be a restricted Lie (super)algebra, by $\mathrm{Lie}_p(X)$ denote a restricted subalgebra of L generated by X. Similarly, assume that X is a subset of an associative algebra A. Write $\mathrm{Alg}(X) \subset A$ to denote an associative subalgebra (without unit) generated by X.

3. Main results: two examples of Lie superalgebras R, Q and their properties

In this section we give two examples of Lie superalgebras and formulate their main properties.

Let $\Lambda_n = \Lambda(x_1, \dots, x_n)$ be the Grassmann algebra in n variables. By setting $\deg x_i = 1$, $i = 1, \dots, n$, one has the associative superalgebra $\Lambda_n = \Lambda_{\bar 0} \oplus \Lambda_{\bar 1}$, called the *Grassmann superalgebra*. Since the superbracket in $\Lambda_n^{(-)}$ is trivial, it is called *supercommutative*. The relations $\partial_{x_i}(x_j) = \delta_{ij}$, $i, j \in \{1, \dots, n\}$, define superderivations $\partial_{x_i} \in \mathrm{Der}\, \Lambda_n$. Any superderivation $D \in \mathrm{Der}\, \Lambda_n$ can be uniquely written as [23]:

$$D = \sum_{i=1}^{n} P_i \partial_{x_i}, \quad P_i \in \Lambda_n.$$

If $\mathrm{char}\, K = 0$ one obtains a simple finite dimensional *Lie superalgebra of Cartan type* $\mathbf{W}_n$ [23]. For more information on theory of Lie superalgebras we refer the reader to [23,44,2].

Assume that I is a well-ordered set of arbitrary cardinality. Put $\mathbb{Z}_2 = \{0, 1\}$. Let $\mathbb{Z}_2^I = \{\alpha : I \to \mathbb{Z}_2\}$ be a set of functions with finitely many nonzero values. Suppose that $\alpha \in \mathbb{Z}_2^I$ has nonzero values at $\{i_1, \dots, i_t\} \subset I$, where $i_1 < \cdots < i_t$, put $\mathbf{x}^\alpha = x_{i_1} x_{i_2} \cdots x_{i_t}$ and $|\alpha| = t$. Now $\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{Z}_2^I\}$ is a basis of the Grassmann algebra $\Lambda_I = \Lambda(x_i \mid i \in I)$, which is an associative superalgebra $\Lambda_I = \Lambda_{\bar 0} \oplus \Lambda_{\bar 1}$, all x_i, $i \in I$, being odd. Let ∂_{x_i}, $i \in I$, denote the superderivatives of Λ_I, which are determined by the values $\partial_{x_i}(x_j) = \delta_{ij}$, $i, j \in I$. Consider a space of all formal sums

$$\mathbf{W}(\Lambda_I) = \left\{ \sum_{\alpha \in \mathbb{Z}_2^I} \mathbf{x}^\alpha \sum_{j=1}^{m(\alpha)} \lambda_{\alpha,i_j}\, \partial_{x_{i_j}} \;\middle|\; \lambda_{\alpha,i_j} \in K,\ i_j \in I \right\}.$$

It is essential that the sum at each $\mathbf{x}^\alpha$, $\alpha \in \mathbb{Z}_2^I$, is finite. This construction is similar to so called Lie algebra of *special derivations*, see [38,39,32]. It is similarly verified that the product of elements of $\mathbf{W}(\Lambda_I)$ is well defined and $\mathbf{W}(\Lambda_I)$ acts on Λ_I by superderivations. Our superalgebras will be constructed as subalgebras in $\mathbf{W}(\Lambda_I) \subset \operatorname{Der} \Lambda_I$.

Our goal is to study the following finitely generated (restricted) Lie superalgebras. We consider Grassmann superalgebras Λ in infinitely many variables. We write our elements as infinite sums, they belong to the Lie superalgebra $\mathbf{W}(\Lambda) \subset \operatorname{Der} \Lambda$. We consider that the algebra Λ is naturally embedded into $\operatorname{End} \Lambda$ by sending each element to the left multiplication. We also shall consider respective finitely generated associative subalgebras of $\operatorname{End} \Lambda$.

Example 1. Let $\Lambda = \Lambda[x_i, y_i \mid i \geq 0]$ and consider the following odd elements in $\mathbf{W}(\Lambda)$:

$$\begin{aligned} a_i &= \partial_{x_i} + y_i x_i(\partial_{x_{i+1}} + y_{i+1}x_{i+1}(\partial_{x_{i+2}} + y_{i+2}x_{i+2}(\partial_{x_{i+3}} + \cdots))), \\ b_i &= \partial_{y_i} + x_i y_i(\partial_{y_{i+1}} + x_{i+1}y_{i+1}(\partial_{y_{i+2}} + x_{i+2}y_{i+2}(\partial_{y_{i+3}} + \cdots))), \end{aligned} \qquad i \geq 0. \tag{3}$$

We define a Lie superalgebra $\mathbf{R} = \operatorname{Lie}(a_0, b_0) \subset \mathbf{W}(\Lambda)$ and its associative hull $\mathbf{A} = \operatorname{Alg}(a_0, b_0) \subset \operatorname{End} \Lambda$. In case $\operatorname{char} K = 2$, assume that the quadratic mapping on odd elements is just the square of the respective operator in $\operatorname{End} \Lambda$.

We call $\{a_i, b_i \mid i \geq 0\}$ the *pivot elements*. We establish the following main properties of $\mathbf{R} = \operatorname{Lie}(a_0, b_0)$ and its associative hull $\mathbf{A} = \operatorname{Alg}(a_0, b_0)$.

(i) Section 4 yields basic relations of $\mathbf{R}$ and shows that $\mathbf{R}$ is self-similar.
(ii) $\mathbf{R}$ has a monomials' basis consisting of standard monomials of three types ($\operatorname{char} K \neq 2$, Theorem 5.1). In case $\operatorname{char} K = 2$, a basis of $\mathbf{R}$ is given by monomials of the first and second type and squares of the pivot elements (Corollary 5.2).
(iii) In case $\operatorname{char} K = p > 0$, we also consider a restricted Lie superalgebra $\operatorname{Lie}_p(a_0, b_0)$, which in case $p = 2$ coincides with the restricted Lie algebra also denoted by $\operatorname{Lie}_p(a_0, b_0)$, the bases coincide with the base of $\mathbf{R}$.
(iv) We introduce two weight functions wt, swt, which are additive on products of monomials (Section 6).
(v) $\mathbf{R}$ is $\mathbb{Z}^2$-graded by multidegree in the generators (Lemma 7.1). We introduce two coordinate systems on plane: multidegree coordinates (X_1, X_2), and weight coordinates (Z_1, Z_2) (Section 7).

(vi) We find bounds on weights of monomials of $\mathbf{R}$ and $\mathbf{A}$ (Section 8). Monomials of $\mathbf{R}$ are in a region of plane bounded by two logarithmic curves (Theorem 8.6, Fig. 1). A similar bound holds for $\mathbf{A}$.

(vii) $\operatorname{GKdim} \mathbf{R} = \underline{\operatorname{GKdim}}\, \mathbf{R} = \log_3 4 \approx 1.26$ (Theorem 8.5).

(viii) $\operatorname{GKdim} \mathbf{A} = \underline{\operatorname{GKdim}}\, \mathbf{A} = 2\log_3 4 \approx 2.52$ (Theorem 8.8).

(ix) We describe generating functions of $\mathbf{R}$ (Section 9) and observe a fractal structure of its homogeneous components (Fig. 1).

(x) $\mathbf{R} = \mathbf{R}_{\bar{0}} \oplus \mathbf{R}_{\bar{1}}$ is a nil graded Lie superalgebra (Theorem 10.1). We have a triangular decomposition into a direct sum of three locally nilpotent subalgebras $\mathbf{R} = \mathbf{R}_{+} \oplus \mathbf{R}_0 \oplus \mathbf{R}_{-}$ (Corollary 10.2).

Example 2. Let $\Lambda = \Lambda[x_i, y_i, z_i \mid i \geq 0]$ and consider the following odd elements in $\mathbf{W}(\Lambda)$:

$$
\begin{aligned}
a_i &= \partial_{x_i} + y_i x_i(\partial_{x_{i+1}} + y_{i+1}x_{i+1}(\partial_{x_{i+2}} + y_{i+2}x_{i+2}(\partial_{x_{i+3}} + \cdots))),\\
b_i &= \partial_{y_i} + z_i y_i(\partial_{y_{i+1}} + z_{i+1}y_{i+1}(\partial_{y_{i+2}} + z_{i+2}y_{i+2}(\partial_{y_{i+3}} + \cdots))), \qquad i \geq 0. \qquad (4)\\
c_i &= \partial_{z_i} + x_i z_i(\partial_{z_{i+1}} + x_{i+1}z_{i+1}(\partial_{z_{i+2}} + x_{i+2}z_{i+2}(\partial_{z_{i+3}} + \cdots))),
\end{aligned}
$$

We define a Lie superalgebra $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0) \subset \mathbf{W}(\Lambda)$ and its associative hull $\mathbf{A} = \operatorname{Alg}(a_0, b_0, c_0) \subset \operatorname{End}\Lambda$. In case $\operatorname{char} K = 2$, assume that the quadratic mapping on odd elements is just the square of the respective operator in $\operatorname{End}\Lambda$.

We also call $\{a_i, b_i, c_i \mid i \geq 0\}$ the *pivot elements*. We establish the following main properties of $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$ and $\mathbf{A} = \operatorname{Alg}(a_0, b_0, c_0)$:

(i) Section 11 yields basic relations of $\mathbf{Q}$ and shows that $\mathbf{Q}$ is self-similar.

(ii) $\mathbf{Q}$ has a monomials' basis consisting of standard monomials of three types ($\operatorname{char} K \neq 2$, Theorem 12.1). In case $\operatorname{char} K = 2$, a basis is given by monomials of the first and second type and squares of the pivot elements (Corollary 12.2).

(iii) In case $\operatorname{char} K = p > 0$, we also consider a restricted Lie superalgebra $\operatorname{Lie}_p(a_0, b_0, c_0)$, which in case $p = 2$ coincides with the restricted Lie algebra also denoted by $\operatorname{Lie}_p(a_0, b_0, c_0)$, the bases coincide with the base of $\mathbf{Q}$.

(iv) We introduce three weight functions wt, swt, $\overline{\operatorname{swt}}$, additive on products of monomials (Section 13).

(v) $\mathbf{Q}$ is $\mathbb{Z}^3$-graded by multidegree in the generators (Lemma 14.1). We introduce three coordinate systems in space: multidegree coordinates (X_1, X_2, X_3), weight coordinates (Z_1, Z_2, Z_3) and orthogonal coordinates (Y_1, Y_2, Y_3) (Section 14).

(vi) We establish bounds on weights of monomials of $\mathbf{Q}$ and $\mathbf{A}$ (Section 15). Monomials of $\mathbf{Q}$ are in a region of space bounded by an elliptic paraboloid (Theorem 15.6 and Fig. 2). A similar bound holds for $\mathbf{A}$. A situation when homogeneous components of a graded restricted Lie algebra are in a region of space bounded by a paraboloid-like surface was observed before [36].

(vii) $\operatorname{GKdim} \mathbf{Q} = \underline{\operatorname{GKdim}}\, \mathbf{Q} = \log_3 8 \approx 1.89$ (Theorem 15.5).

(viii) $\operatorname{GKdim} \mathbf{A} = \underline{\operatorname{GKdim}}\, \mathbf{A} = 2 \log_3 8$ (Theorem 15.9).

(ix) We describe generating functions of $\mathbf{Q}$ (Section 16). The initial coefficients show an interesting pattern: the coefficients are at most one and the diagonal coefficients are zero. These observations are true in general, see below.

(x) $\mathbf{Q} = \mathbf{Q}_{\bar 0} \oplus \mathbf{Q}_{\bar 1}$ is also a nil graded Lie superalgebra (Theorem 16.2).

The following properties are specific to $\mathbf{Q}$:

(xi) The components of the $\mathbb{Z}^3$-grading of $\mathbf{Q}$ are at most one-dimensional (Theorem 17.1), so the $\mathbb{Z}^3$-grading of $\mathbf{Q}$ is fine. The diagonal components $\mathbf{Q}_{nnn}$, $n \geq 0$, are empty (Lemma 17.2). Asymptotically, a density of the standard monomials of $\mathbf{Q}$ inside the elliptic paraboloid is zero (Lemma 15.7).

(xii) $\mathbf{Q}$ has a $\mathbb{Z}^2$-grading such that all components $\mathbf{Q}_{nm}$, $n, m \in \mathbb{Z}$, are infinite dimensional except for $\mathbf{Q}_{00} = \{0\}$ (Theorem 18.1), a part of this grading is shown on Fig. 3.

(xiii) There exists a continuum of different decompositions into a direct sum of two locally nilpotent subalgebras: $\mathbf{Q} = \mathbf{Q}_+ \oplus \mathbf{Q}_-$ (Theorem 18.2).

(xiv) $\mathbf{Q}$ shows that an extension of Theorem 1.1 (Martinez and Zelmanov [27]) for Lie superalgebras of characteristic zero is not valid.

4. Example 1: Lie superalgebra R, relations

In this section we start to study the algebras of Example 1. Let $\Lambda = \Lambda[x_i, y_i | i \geq 0]$ be the Grassmann algebra in infinitely many variables. The Grassmann letters and superderivatives $\{x_i, y_i, \partial_{x_i}, \partial_{y_i} | i \geq 0\}$ are odd elements of $\operatorname{End}\Lambda$. As a rule, they anticommute, e.g.:

$$\begin{aligned}
&x_i x_j = -x_j x_i, \quad \partial_{x_i}\partial_{x_j} = -\partial_{x_j}\partial_{x_i}, \quad x_i y_j = -y_j x_i;\\
&\partial_{x_i} x_j = -x_j \partial_{x_i}, \ i \neq j;\\
&x_i^2 = y_i^2 = (\partial_{x_i})^2 = (\partial_{y_i})^2 = 0, \quad i \geq 0.
\end{aligned}$$

The nontrivial relations are:

$$\partial_{x_i} x_i + x_i \partial_{x_i} = 1, \quad \partial_{y_i} y_i + y_i \partial_{y_i} = 1, \quad i \geq 0.$$

For convenience, we present the pivot elements (3) recursively

$$\begin{aligned}
a_i &= \partial_{x_i} + y_i x_i a_{i+1},\\
b_i &= \partial_{y_i} + x_i y_i b_{i+1};
\end{aligned} \qquad i \geq 0. \tag{5}$$

The pivot elements act on Grassmann variables as follows:

$$
a_n(y_k) = 0, \qquad n, k \geq 0;
$$

$$
a_n(x_k) = \begin{cases} 0, & n > k; \\ 1, & n = k; \\ y_n x_n y_{n+1} x_{n+1} \cdots y_{k-1} x_{k-1}, & n < k. \end{cases} \tag{6}
$$

The action of b_n is similar. Recall that we consider the Lie superalgebra $\mathbf{R} = \mathrm{Lie}(a_0, b_0) = \mathbf{R}_{\bar{0}} \oplus \mathbf{R}_{\bar{1}} \subset \mathbf{W}(\Lambda) \subset \operatorname{Der} \Lambda$ and its associative hull, the associative superalgebra $\mathbf{A} = \mathrm{Alg}(a_0, b_0) \subset \operatorname{End} \Lambda$.

Define the *shift* mappings $\tau : \Lambda \to \Lambda$ and $\tau : \mathbf{W}(\Lambda) \to \mathbf{W}(\Lambda)$ by

$$
\tau(x_i) = x_{i+1},\ \tau(y_i) = y_{i+1}, \qquad \tau(\partial_{x_i}) = \partial_{x_{i+1}},\ \tau(\partial_{y_i}) = \partial_{y_{i+1}}, \quad i \geq 0.
$$

Consider also the *permuting* mappings $\theta : \Lambda \to \Lambda$ and $\theta : \mathbf{W}(\Lambda) \to \mathbf{W}(\Lambda)$:

$$
\theta(x_i) = y_i,\ \theta(y_i) = x_i, \qquad \theta(\partial_{x_i}) = \partial_{y_i},\ \theta(\partial_{y_i}) = \partial_{x_i}, \quad i \geq 0.
$$

Clearly, τ is an endomorphism and θ an isomorphism, $\theta^2 = 1$.

Lemma 4.1. *We have a sequence of even elements:*

$$
c_{i+1} = [a_i, b_i] = [b_i, a_i] = x_i a_{i+1} + y_i b_{i+1} \in \mathbf{R}_{\bar{0}}, \quad i \geq 0. \tag{7}
$$

Remark that by our construction we have:

$$
\tau(a_i) = a_{i+1},\ \tau(b_i) = b_{i+1},\ \tau(c_i) = c_{i+1}, \qquad \theta(a_i) = b_i,\ \theta(b_i) = a_i,\ \theta(c_i) = c_i, \qquad i \geq 0.
$$

Below by degree of relations we mean the degree in the set $\{a_i, b_i\}$, i being fixed. Thus, relation (7) has degree 2. We group relations with those obtained by the action of θ.

Lemma 4.2. *We have the following relations of degree 2 and 3, $i \geq 0$:*

(1) $[a_i^2, b_i] = [a_i, a_i, b_i] = [a_i, c_{i+1}] = a_{i+1}$;
(2) $[b_i^2, a_i] = [b_i, b_i, a_i] = [b_i, c_{i+1}] = b_{i+1}$;
(3) $a_i^2 = -y_i a_{i+1}$;
(4) $b_i^2 = -x_i b_{i+1}$;
(5) $[a_i, a_i] = 2a_i^2 = -2y_i a_{i+1}$;
(6) $[b_i, b_i] = 2b_i^2 = -2x_i b_{i+1}$;
(7) $[a_i, a_i, a_i] = [b_i, b_i, b_i] = 0$.

Proof. We use (2), (7), (5) and that $x_i^2 = y_i^2 = 0$:

$$
[a_i, a_i, b_i] = [a_i, c_{i+1}] = [\partial_{x_i} + y_i x_i a_{i+1}, x_i a_{i+1} + y_i b_{i+1}] = [\partial_{x_i}, x_i] a_{i+1} = a_{i+1}.
$$

One has the identity $(a+b)^2 = a^2 + [a,b] + b^2$, where a, b are odd elements of an associative superalgebra. We get

$$a_i^2 = (\partial_{x_i} + y_i x_i a_{i+1})^2 = [\partial_{x_i}, y_i x_i a_{i+1}] = -y_i[\partial_{x_i}, x_i]a_{i+1} = -y_i a_{i+1}. \qquad \square$$

The notion of self-similarity plays important role in group theory [18]. The Fibonacci Lie algebra is self-similar [33]. A general definition was given by Bartholdi [6]. A Lie algebra L is called *self-similar* if it affords a homomorphism

$$\psi : L \to \operatorname{Der} R \leftthreetimes R \otimes L,$$

where R is a commutative algebra and $\operatorname{Der} R$ its Lie algebra of derivations. This definition easily extends to superalgebras, we consider R to be a supercommutative associative superalgebra.

Consider Lie superalgebras generated by two elements: $L_i = \mathrm{Lie}(a_i, b_i)$, $i \geq 0$, so $L_0 = \mathbf{R}$.

Corollary 4.3. *We have*

1. (1) $a_i, b_i \in \mathbf{R}$, $i \geq 0$;
2. (2) $\tau^i : \mathbf{R} \to L_i$ *is an isomorphism for any* $i \geq 0$;
3. (3) $\mathbf{R}$ *is infinite dimensional;*
4. (4) $\theta : \mathbf{R} \to \mathbf{R}$ *is an automorphism;*
5. (5) *there exists a self-similarity embedding:*

$$\mathbf{R} \hookrightarrow \langle \partial_{x_0}, \partial_{y_0} \rangle_K \leftthreetimes \Lambda[x_0, y_0] \otimes \tau(\mathbf{R}).$$

Lemma 4.4. *We have the following relations of degree 4,* $i \geq 0$*:*

(1) $[b_i, a_i, a_i, b_i] = [b_i, a_{i+1}] = x_i y_i c_{i+2}$;
(2) $[a_i, b_i, b_i, a_i] = [a_i, b_{i+1}] = -x_i y_i c_{i+2}$;
(3) $[a_i, a_i, a_i, b_i] = [a_i, a_{i+1}] = 2x_i y_i\, y_{i+1} a_{i+2}$;
(4) $[b_i, b_i, b_i, a_i] = [b_i, b_{i+1}] = -2x_i y_i\, x_{i+1} b_{i+2}$;
(5) $[[a_i, b_i], [a_i, b_i]] = [c_i, c_i] = 0$.

Proof. Recall that by Lemma 4.2, $[a_i, a_i, b_i] = a_{i+1}$, $[b_i, b_i, a_i] = b_{i+1}$. Let us check claims (1), (3).

$$[b_i, a_{i+1}] = [\partial_{y_i} + x_i y_i b_{i+1}, a_{i+1}] = x_i y_i [b_{i+1}, a_{i+1}] = x_i y_i c_{i+2};$$
$$[a_i, a_{i+1}] = [\partial_{x_i} + y_i x_i a_{i+1}, a_{i+1}] = y_i x_i [a_{i+1}, a_{i+1}] = 2x_i y_i y_{i+1} a_{i+2}.$$

Relations (2), (4) follow by application of θ. For the last relation recall that $c_i \in \mathbf{R}_{\bar{0}}$. $\square$

5. Monomial bases of R and $\tilde{A}$

Let us introduce a notation widely used below. By r_n denote a *tail* monomial:

$$r_n = x_0^{\xi_0} y_0^{\eta_0} \cdots x_n^{\xi_n} y_n^{\eta_n} \in \Lambda, \quad \xi_i, \eta_i \in \{0,1\};\ n \geq 0. \tag{8}$$

For $n < 0$ we assume that $r_n = 1$. If needed, other monomials of type (8) will be denoted by r'_n, $\tilde{r}_n$, etc.

Theorem 5.1. *Let* $\operatorname{char} K \neq 2$. *A basis of the Lie superalgebra* $\mathbf{R} = \operatorname{Lie}(a_0, b_0)$ *is given by the following standard monomials of three types (where* r_n *denote all tail monomials* (8)*)*

(1) *monomials of the first type:*

$$\{r_{n-2}a_n,\ r_{n-2}b_n \mid n \geq 0\};$$

(2) *monomials of the second type:*

$$\{r_{n-2}c_n \mid n \geq 1\};$$

(3) *monomials of the third type:*

$$\{r_{n-2}y_{n-1}a_n, r_{n-2}x_{n-1}b_n \mid\ n \geq 1\}$$

(in case $n = 1$ *we have* $\{y_0a_1, x_0b_1\}$*).*

Proof. Let us call n the *length*, a_n, b_n, c_n the *heads*, r_{n-2} the *tails*, and the remaining products the *necks* of the respective monomials.

A) We check by induction on length n that all these monomials belong to $\mathbf{R}$. By Lemma 4.1 and Lemma 4.2, we have $\{a_0, b_0, c_1, a_1, b_1\} \subset \mathbf{R}$. In case $\operatorname{char} K \neq 2$ by Lemma 4.2, we get $\{y_0a_1, x_0b_1\} \subset \mathbf{R}$. So, we get the desired monomials of all three types of lengths $n = 0, 1$. This is the base of induction.

An integer $n \geq 1$ being fixed, assume that all monomials of the first type of lengths $i = 0, \dots, n$ belong to $\mathbf{R}$. Take a monomial $w = r_{n-2}a_n \in \mathbf{R}$. Using Lemma 4.4 and Lemma 4.2, $\mathbf{R}$ contains the following elements

$$w' = [b_{n-1}, r_{n-2}a_n] = \pm r_{n-2}[b_{n-1}, a_n] = \pm r_{n-2}x_{n-1}y_{n-1}c_{n+1},$$

$$[a_n, w'] = \pm r_{n-2}x_{n-1}y_{n-1}[a_n, c_{n+1}] = \pm r_{n-2}x_{n-1}y_{n-1}a_{n+1},$$

$$[b_n, w'] = \pm r_{n-2}x_{n-1}y_{n-1}[b_n, c_{n+1}] = \pm r_{n-2}x_{n-1}y_{n-1}b_{n+1}.$$

Now we multiply all the elements above by a_{n-1} or b_{n-1} and delete the letters x_{n-1} or y_{n-1}, or both. Thus, we can obtain all desired monomials of the first and second type of length $n+1$.

Assume that char $K \neq 2$. Take an arbitrary $w = r_{n-2}a_n \in \mathbf{R}$, using Lemma 4.4, we get

$$w'' = [a_{n-1}, r_{n-2}a_n] = \pm 2r_{n-2}x_{n-1}y_{n-1}\, y_n a_{n+1}.$$

We multiply this element by a_{n-1} or b_{n-1} and delete letters x_{n-1} or y_{n-1}, or both. So, we obtain all monomials of the third type of length $n+1$.

B) Now let us prove that a product of the standard monomials is expressed via the standard monomials.

1) Consider products of monomials of the same length. Until the end of the proof, we use a somewhat different notation, we write monomials of the first and third type in the same way: $r_{n-1}a_n$, $r_{n-1}b_n$, and present monomials of the second type as $c_n = r_{n-1}a_n + r'_{n-1}b_n$. Thus, we get products of the respective heads $[a_n, b_n]$, $[a_n, a_n]$, $[b_n, b_n]$, given by Lemma 4.1 and Lemma 4.2. We obtain standard monomials:

$$\begin{aligned} &[r_{n-1}a_n, r'_{n-1}b_n] = \pm r''_{n-1}[a_n, b_n] = \pm r''_{n-1}c_{n+1},\\ &[r_{n-1}a_n, r'_{n-1}a_n] = \pm r''_{n-1}[a_n, a_n] = \pm 2r''_{n-1}y_n a_{n+1},\\ &[r_{n-1}b_n, r'_{n-1}b_n] = \pm r''_{n-1}[b_n, b_n] = \pm 2r''_{n-1}x_n b_{n+1}. \end{aligned} \tag{9}$$

2) Consider a product of two standard monomials of different lengths $n < m$. We present a monomial of shorter length n as follows:

$$\begin{aligned} r_{n-1}a_n &= r_{n-1}\Big(\partial_{x_n} + y_n x_n(\partial_{x_{n+1}}\\ &\qquad + y_{n+1}x_{n+1}(\cdots + y_{m-2}x_{m-2}(\partial_{x_{m-1}} + y_{m-1}x_{m-1}a_m)\cdots))\Big)\\ &= \sum_{j=n}^{m-1} \bar r_{j-1}\partial_{x_j} + \bar r_{m-2}y_{m-1}x_{m-1}a_m. \end{aligned} \tag{10}$$

We present $r_{n-1}b_n$ similarly. Using $c_n = x_{n-1}a_n + y_{n-1}b_n$, we present monomials of the second type as a respective linear combination. We multiply a monomial (10) by a standard monomial of a bigger length m of the first or third type, and proceed similar to (9):

$$\begin{aligned} [r_{n-1}a_n, r'_{m-1}b_m] &= \sum_{j=n}^{m-1} \bar r_{j-1}\partial_{x_j}(r'_{m-1})b_m \pm \bar r_{m-2}y_{m-1}x_{m-1}r'_{m-1}[a_m, b_m]\\ &= \sum_{j=n}^{m-2} \bar r_{j-1}\partial_{x_j}(r'_{m-1})b_m + \bar r_{m-2}\partial_{x_{m-1}}(r'_{m-1})b_m \pm r''_{m-1}c_{m+1}. \end{aligned} \tag{11}$$

The derivatives ∂_{x_j}, $j = n, \dots, m-2$ do not change a possible neck in position $m-1$ of r'_{m-1}, the next derivative $\partial_{x_{m-1}}$ can only delete this neck. Thus, (11) yields standard monomials. Consider the case that the longest monomial has the same head letter: $r'_{m-1}a_m$. The arguments remain valid, where the last term in (11) has the factor $[a_m, a_m] = -2y_m a_{m+1}$.

Consider the case that a standard monomial of a bigger length m is of the second type. We use presentation (10):

$$\begin{aligned}[r_{n-1}a_n, r'_{m-2}c_m] &= \sum_{j=n}^{m-2} \bar r_{j-1}\partial_{x_j}(r'_{m-2})c_m \pm \bar r_{m-2} r'_{m-2}\partial_{x_{m-1}}(c_m) \\ &\quad \pm \bar r_{m-2} r'_{m-2}[y_{m-1}x_{m-1}a_m, x_{m-1}a_m + y_{m-1}b_m] \\ &= \Big(\sum_{j=n}^{m-2} \pm r^{(j)}_{m-2}\Big) c_m \pm r''_{m-2}a_m. \qquad \square\end{aligned}$$

Corollary 5.2. *Let* char $K = 2$. *A basis of the Lie superalgebra* $\mathbf{R} = \mathrm{Lie}(a_0, b_0)$ *as well as a basis of the restricted Lie algebra* $\mathrm{Lie}_p(a_0, b_0)$ *is given by the standard monomials of the first and second type and squares of the pivot elements:*

$$\{y_{n-1}a_n, x_{n-1}b_n \mid n \ge 1\}.$$

Proof. Let $L = L_{\bar 0} \oplus L_{\bar 1} \subset \mathbf{W}(\Lambda)$ be the subalgebra generated by a_0, b_0 using only the commutator bracket. By proof of Theorem, its basis $\{v_j \mid j \in J\}$ consists of all monomials of the first and second types. In order to obtain the restricted Lie algebra $\mathrm{Lie}_p(a_0, b_0) = \mathrm{Lie}_p(L)$ it is sufficient to add all powers: $\{v_j^{[p^n]} \mid n \ge 1, j \in J\}$. These powers are trivial except squares of the pivot elements (see Lemma 4.2) and one more case:

$$c_m^2 = (x_{m-1}a_m + y_{m-1}b_m)^2 = x_{m-1}y_{m-1}[a_m, b_m] = x_{m-1}y_{m-1}c_{m+1}, \quad m \ge 1,$$

these are monomials of the second type.

Consider the case of the Lie superalgebra $\mathrm{Lie}(a_0, b_0)$. We need to add the squares of a basis of the odd component $L_{\bar 1}$. Again, we add the same squares of the pivot elements. $\square$

Corollary 5.3. *Let* char $K = p \ge 3$. *The restricted Lie superalgebra* $\mathrm{Lie}_p(a_0, b_0)$ *coincides with the Lie superalgebra* $\mathbf{R} = \mathrm{Lie}(a_0, b_0)$, *both have the same bases.*

Proof. We need to add p^k-powers of a basis of the even component of the Lie superalgebra, a check shows that we add nothing. $\square$

We consider a little bigger Lie superalgebra $\tilde R \supset \mathbf{R}$ spanned by all monomials that contain at most one variable x_{n-1} or y_{n-1}, the head letter being a_n or b_n. Namely, consider

$$\{r_{n-2}x_{n-1}^{\xi_{n-1}}y_{n-1}^{\eta_{n-1}}a_n,\ r_{n-2}x_{n-1}^{\xi_{n-1}}y_{n-1}^{\eta_{n-1}}b_n \mid \xi_{n-1},\eta_{n-1}\in\{0,1\},\ \xi_{n-1}+\eta_{n-1}\le 1,\ n\ge 0\}, \tag{12}$$

where r_n are tail monomials. We refer to these elements as *quasi-standard monomials of length n.*

Corollary 5.4. *The Lie superalgebra $\mathbf{R}=\mathrm{Lie}(a_0,b_0)$ is contained in a bigger (restricted) Lie superalgebra $\tilde{R}\subset \mathbf{W}(\Lambda)$ which basis is given by quasi-standard monomials (12).*

Proof. Computations of Theorem show that a product of quasi-standard monomials is expressed via quasi-standard monomials. $\square$

We consider an associative subalgebra generated by quasi-standard monomials.

Theorem 5.5. *Let $\tilde{A}=\mathrm{Alg}(\tilde{R})\subset\mathrm{End}(\Lambda)$. Then*

(1) *a basis of $\tilde{A}$ is given by monomials*

$$\{r_{n-2}x_{n-1}^{\xi_{n-1}}y_{n-1}^{\eta_{n-1}}a_n^{\alpha_n}b_n^{\beta_n}\cdots a_0^{\alpha_0}b_0^{\beta_0} \mid \xi_{n-1},\eta_{n-1},\alpha_i,\beta_i\in\{0,1\},$$
$$\xi_{n-1}+\eta_{n-1}\le\alpha_n+\beta_n>0,\ n\ge 0\},$$

where r_{n-2} are tail monomials (8).

(2) *$\tilde{A}$ contains $\mathbf{A}=\mathrm{Alg}(a_0,b_0)$;*

(3) *monomials with $\xi_{n-1}=\eta_{n-1}=0$ are linearly independent and belong to $\mathbf{A}$.*

Proof. The desired products having at most one Grassmann variable in position $n-1$ are easily obtained as products of quasi-standard monomials. Consider the described product having both Grassmann variables in position $n-1$. Then $\alpha_n=\beta_n=1$. We have $r_{n-2}x_{n-1}a_n\cdot y_{n-1}b_n=-r_{n-2}x_{n-1}y_{n-1}a_nb_n$. Thus, the described monomials belong to $\tilde{A}$ as well. The linear independence follows by the same arguments as [33, proof of Theorem 4.1].

Let us check that $\tilde{A}$ is spanned by these monomials. First, we work with products of quasi-standard monomials, we reorder such products using PBW-like arguments, where a total order is fixed obeying to the length n of quasi-standard monomials. In this process, not only powers of odd quasi-standard monomials but also products of quasi-standard monomials of the same length with the same head letter disappear:

$$(r_{n-1}a_n)\cdot(r'_{n-1}a_n)=\pm r''_{n-1}a_n^2=\pm r''_{n-1}y_na_{n+1}.$$

As a result, we get products of quasi-standard monomials, each given head a_j, b_j appearing at most once, written in the length-decreasing order. For example, we obtain:

$$r_{n-1}a_n\cdot r'_{n-1}b_n\cdot r_{n-2}a_{n-1}\cdot r'_{n-2}b_{n-1}\cdots a_0\cdot b_0, \tag{13}$$

where there is at least one quasi-standard monomial with the head letter a_n or b_n, while the remaining quasi-standard monomials of lengths $n-1, \dots, 0$ are optional. We observe the following *senior part property* of the product (13) implied by construction of quasi-standard monomials. The product containing both senior head letters a_n, b_n can contain both smaller variables x_{n-1}, y_{n-1}. But the product with exactly one of the senior head letters a_n, b_n can contain at most one of $\{x_{n-1}, y_{n-1}\}$.

Second, we move all Grassmann letters in (13) to the left. Let x_i (or y_i) be a Grassmann variable in a quasi-standard monomial $w = r_{j-1}a_j$, then $i < j$. The quasi-standard monomials before w in (13) have heads a_k, b_k with $k \ge j > i$. Hence, all Grassmann variables x_i, y_i, $i = 0, \dots, n-1$ appearing in (13) super-commute with the preceding heads (see (6)), and the migration of Grassmann variables is a supercommutation only without additional commutators. In particular, the process does not change the senior letters a_n, b_n and existing variables x_{n-1}, y_{n-1}. Therefore, the senior part property remains valid, yielding the required monomials of the theorem.

The second claim is trivial. By Theorem 5.1, we have $r_{n-2}a_n$, $r_{n-2}b_n \in \mathbf{R} \subset \mathbf{A}$. It follows that monomials specified in the third claim belong to $\mathbf{A}$ as well. $\square$

6. Two weight functions on R

Assume that the superderivatives and variables have the weights as follows:

$$\begin{aligned} \mathsf{wt}(\partial_{x_i}) &= -\mathsf{wt}(x_i) = \alpha_i \in \mathbb{C}, \\ \mathsf{wt}(\partial_{y_i}) &= -\mathsf{wt}(y_i) = \beta_i \in \mathbb{C}, \end{aligned} \qquad i \ge 0.$$

We want all terms in (5) be homogeneous. Thus, we get equalities:

$$\begin{aligned} \alpha_i &= -\alpha_i - \beta_i + \alpha_{i+1}, \\ \beta_i &= -\alpha_i - \beta_i + \beta_{i+1}, \end{aligned} \qquad i \ge 0.$$

Thus, we get a recurrence relation:

$$\begin{pmatrix} \alpha_{i+1} \\ \beta_{i+1} \end{pmatrix} = A \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \quad i \ge 0, \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{14}$$

Lemma 6.1. *The matrix A has the following properties*

(1) *the eigenvalues are*

$$\lambda = 3, \quad \mu = 1;$$

(2) *the respective eigenvectors are*

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

(3)

$$A^n = \frac{1}{2}\begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}, \qquad n \ge 0.$$

Consider a set of sequences $\mathcal{W} = \{(\alpha_i, \beta_i \mid \alpha_i, \beta_i \in \mathbb{C}, i \ge 0, \text{ satisfy (14)})\}$. Clearly, this space is two-dimensional. Let us choose a convenient basis in this space. Namely, according to the eigenvalues and eigenvectors we introduce two *weight functions*.

Corollary 6.2. *We have the following weight functions:*

(1) $\mathrm{wt}(\partial_{x_n}) = \mathrm{wt}(\partial_{y_n}) = \mathrm{wt}(a_n) = \mathrm{wt}(b_n) = 3^n$, $n \ge 0$ *(the* weight *function);*
(2) $\mathrm{swt}(\partial_{x_n}) = \mathrm{swt}(a_n) = 1$, $\mathrm{swt}(\partial_{y_n}) = \mathrm{swt}(b_n) = -1$, $n \ge 0$ *(the* superweight *function);*
(3) $\mathrm{Wt}(a) = (\mathrm{wt}(a), \mathrm{swt}(a))$ *(*vector weight *function, here a is one of a_n, b_n).*

Proof. For example, we start with $\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and obtain

$$\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = A^n v_1 = 3^n v_1 = \begin{pmatrix} 3^n \\ 3^n \end{pmatrix}, \qquad n \ge 0. \qquad \square$$

By *monomials* we call any monomials in letters $\{x_i, y_i,\ \partial_{x_i}, \partial_{y_i},\ a_i, b_i, c_i \mid i \ge 0\}$, belonging to the associative algebra $\operatorname{End}\Lambda$.

Lemma 6.3. *Let a, b be any monomials. Then*

(1) *the weight functions are well defined on monomials;*
(2) *the weight functions are additive on products of monomials, e.g.:*

$$\mathrm{Wt}(a \cdot b) = \mathrm{Wt}(a) + \mathrm{Wt}(b).$$

Proof. It follows by our construction of weights, see also [33] and [36]. $\square$

Corollary 6.4. *The vector weight function has the values:*

$$\begin{aligned} \mathrm{Wt}(a_n) &= (3^n, 1), \qquad n \ge 0; \\ \mathrm{Wt}(b_n) &= (3^n, -1), \qquad n \ge 0; \\ \mathrm{Wt}(c_n) &= (2 \cdot 3^{n-1}, 0), \qquad n \ge 1. \end{aligned} \tag{15}$$

Proof. Follows by definition (Corollary 6.2) and observation that $\mathrm{Wt}(c_n) = \mathrm{Wt}([a_{n-1}, b_{n-1}]) = \mathrm{Wt}(a_{n-1}) + \mathrm{Wt}(b_{n-1})$. $\square$

7. $\mathbb{Z}^2$-gradation of R and two coordinate systems

Lemma 7.1. *The algebras* $\mathbf{R} = \mathrm{Lie}(a_0, b_0)$ *and* $\mathbf{A} = \mathrm{Alg}(a_0, b_0)$ *are* $\mathbb{Z}^2$*-graded by multidegree in* $\{a_0, b_0\}$*:*

$$\mathbf{A} = \bigoplus_{n_1,n_2\ge 0} \mathbf{A}_{n_1n_2}, \qquad \mathbf{R} = \bigoplus_{n_1,n_2\ge 0} \mathbf{R}_{n_1n_2}.$$

Proof. By (15), we have two vectors

$$\mathrm{Wt}(a_0) = (1,1),\ \ \mathrm{Wt}(b_0) = (1,-1) \in \mathbb{R}^2. \tag{16}$$

They are linearly independent and form a *weight lattice*:

$$\Gamma = \mathbb{Z}\,\mathrm{Wt}(a_0) \oplus \mathbb{Z}\,\mathrm{Wt}(b_0) \subset \mathbb{R}^2.$$

Let $\mathbf{A}_{n_1n_2} \subset \mathbf{A}$ be a span of all monomials of degrees n_1, n_2 in the generators a_0, b_0, respectively, where $n_1, n_2 \ge 0$. By Lemma 6.3, for any $v \in \mathbf{A}_{n_1n_2}$ we have

$$\mathrm{Wt}(v) = n_1\,\mathrm{Wt}(a_0) + n_2\,\mathrm{Wt}(b_0). \tag{17}$$

Since vectors (16) are linearly independent we conclude that elements belonging to different components $\mathbf{A}_{n_1n_2}$ are linearly independent. $\square$

Let $v \in \mathbf{A}_{n,m}$, then $\mathrm{swt}(v) = n - m = 0$ iff $n = m$. We collect such monomials and call $\oplus_{n\ge 0}\mathbf{A}_{nn}$ a *diagonal* of the $\mathbb{Z}^2$-gradation of $\mathbf{A}$.

Corollary 7.2.

(1) *The algebras* $\mathbf{R}$, $\mathbf{A}$, *and* $\mathbf{u} = u(\mathbf{R})$ *allow triangular decomposition into direct sums of three subalgebras:*

$$\mathbf{R} = \mathbf{R}_+ \oplus \mathbf{R}_0 \oplus \mathbf{R}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_0 \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_0 \oplus \mathbf{u}_-,$$

where the respective components are spanned by monomials of positive, zero, and negative superweight.

(2) *The zero component coincides with the diagonal:* $\mathbf{R}_0 = \oplus_{n\ge 0}\mathbf{R}_{nn}$.

Proof. Follows by additivity of the superweight function $\mathrm{swt}(v)$. $\square$

Consider a monomial $0 \ne v \in \mathbf{A}_{n_1n_2}$, $n_1, n_2 \ge 0$, we introduce a *multidegree* vector and put it on plane identifying with standard coordinates $(X_1, X_2) \in \mathbb{R}^2$. Denote

$$\mathrm{Gr}(v) = (n_1, n_2) \in \mathbb{Z}^2 \subset \mathbb{R}^2.$$

From (16), (17) we get the following relations with the weight functions:

$$\begin{aligned} \mathrm{wt}(v) &= n_1 + n_2; \\ \mathrm{swt}(v) &= n_1 - n_2. \end{aligned} \tag{18}$$

Now let (X_1, X_2) be the *standard coordinates* of an arbitrary point in $\mathbb{R}^2$, which we also refer to as the *multidegree coordinates*. We introduce *weight coordinates* (Z_1, Z_2) in $\mathbb{R}^2$:

$$\begin{cases} Z_1 = X_1 + X_2; \\ Z_2 = X_1 - X_2; \end{cases} \tag{19}$$

$$\begin{cases} X_1 = (Z_1 + Z_2)/2; \\ X_2 = (Z_1 - Z_2)/2. \end{cases} \tag{20}$$

Lemma 7.3. *Let $v \in \mathbf{A}$ be a monomial,* $\mathrm{Gr}(v) = (X_1, X_2)$ *its multidegree, and* (Z_1, Z_2) *its weight coordinates introduced above. The latter coincide with the vector weight function:*

$$(Z_1, Z_2) = \mathrm{Wt}(v) = (\mathrm{wt}(v), \mathrm{swt}(v)).$$

Proof. Follows from (18) and (19). □

Lemma 7.4. *The multidegrees of the pivot elements a_n, b_n and c_n are as follows:*

(1)

$$\begin{aligned} \mathrm{Gr}(a_n) &= \frac{1}{2}(3^n + 1, 3^n - 1), \quad n \geq 0; \\ \mathrm{Gr}(b_n) &= \frac{1}{2}(3^n - 1, 3^n + 1), \quad n \geq 0; \\ \mathrm{Gr}(c_n) &= (3^{n-1}, 3^{n-1}), \quad n \geq 1. \end{aligned}$$

(2) *Sets of points for pivot elements $\{a_n | n \geq 0\}$, $\{b_n | n \geq 0\}$, $\{c_n | n \geq 0\}$ belong to three lines of plane. The equations in the standard and the weight coordinates are as follows, respectively:*

$$\begin{array}{lll} a_n: & X_1 - X_2 = 1; & Z_2 = 1; \\ b_n: & X_1 - X_2 = -1; & Z_2 = -1; \\ c_n: & X_1 - X_2 = 0; & Z_2 = 0. \end{array}$$

Proof. The weights of a_n, b_n, c_n are known (15), the standard (multidegree) coordinates are obtained by (20). □

Throughout what follows, by using $\mathrm{Wt}(v)$ or $\mathrm{Gr}(v)$, where $v \in \mathbf{A}$, we assume that v is homogeneous with respect to the $\mathbb{Z}^2$-gradation of $\mathbf{A}$.

8. Bounds on weights and growth of R

The goal of this section is to compute the growth and describe position of points on plane that correspond to homogeneous elements of $\mathbf{R}$ and $\mathbf{A}$. Let us find bounds on weights of standard monomials of $\mathbf{R}$.

Lemma 8.1. *We have the following bounds on weights of the standard monomials of* $\mathbf{R}$ *of three types (applying θ to a_n we get the head b_n) of length $n \geq 1$:*

$$\begin{aligned} 2\cdot 3^{n-1}+1 &\leq \mathrm{wt}(r_{n-2}a_n) \leq 3^n; \\ 3^{n-1}+1 &\leq \mathrm{wt}(r_{n-2}c_n) \leq 2\cdot 3^{n-1}; \\ 3^{n-1}+1 &\leq \mathrm{wt}(r_{n-2}y_{n-1}a_n) \leq 2\cdot 3^{n-1}. \end{aligned}$$

Proof. Consider a tail $r_m = x_0^{\xi_0} y_0^{\eta_0} \cdots x_m^{\xi_m} y_m^{\eta_m}$, we get bounds on its weight:

$$0 \geq \mathrm{wt}(r_m) \geq -2(1+\cdots+3^{m-1}+3^m) = -2\frac{3^{m+1}-1}{3-1} = -3^{m+1}+1, \quad m \geq 0. \tag{21}$$

Formally, these inequalities are valid also for $m = -1$. Using this bound, we evaluate weights of monomials of the three different types as follows

$$\begin{aligned} 3^n \geq \mathrm{wt}(r_{n-2}a_n) &\geq -3^{n-1}+1+3^n = 2\cdot 3^{n-1}+1; \\ 2\cdot 3^{n-1} \geq \mathrm{wt}(r_{n-2}c_n) &\geq -3^{n-1}+1+2\cdot 3^{n-1} = 3^{n-1}+1; \\ 2\cdot 3^{n-1} = -3^{n-1}+3^n \geq \mathrm{wt}(r_{n-2}y_{n-1}a_n) &\geq -3^{n-1}+1-3^{n-1}+3^n = 3^{n-1}+1. \qquad \square \end{aligned}$$

Corollary 8.2. *Let w be a quasi-standard monomial. Then* $\mathrm{wt}(w) \geq 1$.

Proof. Let w be of length $n \geq 1$, the estimate is the same as that for the standard monomials of the third type. For monomials of length $n = 0$, i.e. a_0, b_0, the bound is trivial. $\square$

Lemma 8.3. *We have estimates on superweights of the standard monomials of* $\mathbf{R}$ *of length $n \geq 1$:*

$$\begin{aligned} -(n-2) &\leq \mathrm{swt}(r_{n-2}a_n) \leq n, \\ -n &\leq \mathrm{swt}(r_{n-2}b_n) \leq n-2, \\ -(n-1) &\leq \mathrm{swt}(r_{n-2}c_n) \leq n-1, \\ -(n-3) &\leq \mathrm{swt}(r_{n-2}y_{n-1}a_n) \leq n+1, \\ -n-1 &\leq \mathrm{swt}(r_{n-2}x_{n-1}b_n) \leq n-3. \end{aligned}$$

Proof. Recall that $\operatorname{swt}(a_n) = 1$, $\operatorname{swt}(b_n) = -1$, $\operatorname{swt}(c_n) = 0$, $n \geq 1$. Consider a tail $r_m = x_0^{\xi_0} y_0^{\eta_0} \cdots x_m^{\xi_m} y_m^{\eta_m}$, $m \geq 0$. Observe that $\operatorname{swt}(x_j^{\xi_j} y_j^{\eta_j}) \in \{-1, 0, 1\}$, $j = 0, \dots, m$. We get

$$-m - 1 \leq \operatorname{swt}(r_m) \leq m + 1, \quad m \geq -1. \qquad \square$$

Now we give rough general estimates.

Lemma 8.4. *Let $w \in \mathbf{R}$ be a standard monomial of length $n \geq 0$. Then*

$$3^{n-1} < \operatorname{wt}(w) \leq 3^n, \qquad |\operatorname{swt}(w)| \leq n + 1.$$

Proof. The previous lemmas exclude only a_0, b_0, for which the claim is clear. $\square$

Theorem 8.5. *Let $\mathbf{R} = \operatorname{Lie}(a_0, b_0)$. Then* $\operatorname{GKdim} \mathbf{R} = \underline{\operatorname{GKdim}} \mathbf{R} = \log_3 4 \approx 1.26$.

Proof. Let a natural number m be fixed and set $n = [\log_3 m]$. Consider standard monomials of the first type with the head a_n and of length exactly n: $w = x_0^{\xi_0} y_0^{\eta_0} \cdots x_{n-2}^{\xi_{n-2}} y_{n-2}^{\eta_{n-2}} a_n$, where $\xi_i, \eta_i \in \{0, 1\}$. By Lemma 8.4, $\operatorname{wt}(w) \leq 3^n \leq m$. The number of such monomials yields a lower bound on the growth:

$$\tilde{\gamma}_{\mathbf{R}}(m) \geq 2^{2(n-1)} \geq 4^{\log_3 m - 2} = \frac{1}{16} m^{\log_3 4}.$$

Let a natural number m be fixed, set $n = [\log_3(m)] + 1$, so $m < 3^n$. By Lemma 8.4, every standard monomial of length at least $n + 1$ has the weight greater than $3^n > m$. Thus, the number of standard monomials w with $\operatorname{wt}(w) \leq m$ is evaluated by the number of all standard monomials of length at most n. Let $j = 1, \dots, n$ be a length of a standard monomial, there are $2^{2(j-1)}$ different tails r_{j-2} and 5 different types of monomials (see a list in Lemma 8.3). We get an upper bound on the growth:

$$\tilde{\gamma}_{\mathbf{R}}(m) \leq 2 + 5 \sum_{j=1}^{n} 2^{2(j-1)} < 2 + 5 \frac{4^n}{4 - 1} \leq 2 + \frac{20}{3} 4^{\log_3 m} = 2 + \frac{20}{3} m^{\log_3 4}. \qquad \square$$

Theorem 8.6. *Let $w \in \mathbf{R} = \operatorname{Lie}(a_0, b_0)$ be a standard monomial. The respective point of plane is bounded by two logarithmic curves in terms of the weight coordinates $\operatorname{Wt}(w) = (Z_1, Z_2)$:*

$$|Z_2| < \log_3 Z_1 + 2.$$

Proof. Assume that w has length $n \geq 0$. By Lemma 8.4, we have bounds on weights

$$Z_1 = \operatorname{wt}(w) > 3^{n-1}, \quad |Z_2| = |\operatorname{swt}(w)| \leq n + 1.$$

It follows that $|Z_2| \leq n + 1 < \log_3 Z_1 + 2$. $\square$

Now we skip to study the associative algebras $\mathbf{A} \subset \tilde{A}$. Recall that $\tilde{A}$ is spanned by monomials:

$$w = r_{n-2} x_{n-1}^{\xi_{n-1}} y_{n-1}^{\eta_{n-1}} a_n^{\alpha_n} b_n^{\beta_n} \cdots a_0^{\alpha_0} b_0^{\beta_0}, \tag{22}$$

under the restrictions of Theorem 5.5. Such a monomial will be said of length n.

Lemma 8.7. *Let w be a monomial (22) of $\tilde{A}$ described in Theorem 5.5, $n \geq 0$. Then*

$$3^{n-1} < \mathrm{wt}(w) < 3^{n+1}, \qquad |\,\mathrm{swt}(w)| \leq 2n + 1.$$

Proof. We evaluate weights of different parts of a monomial (22) applying estimate (21) and using additional restrictions specified in Theorem 5.5:

$$-3^{n-1} < \mathrm{wt}(r_{n-2}) \leq 0;$$
$$2 \cdot 3^{n-1} \leq \mathrm{wt}(x_{n-1}^{\xi_{n-1}} y_{n-1}^{\eta_{n-1}} a_n^{\alpha_n} b_n^{\beta_n}) \leq 2 \cdot 3^n;$$
$$0 \leq \mathrm{wt}(a_{n-1}^{\alpha_{n-1}} b_{n-1}^{\beta_{n-1}} \cdots a_0^{\alpha_0} b_0^{\beta_0}) \leq 2(3^{n-1} + \cdots + 3^0) < 3^n.$$

Summing these inequalities, we get the first estimates. Recall that $\mathrm{swt}(a_i^{\alpha_i} b_i^{\beta_i})$ and $\mathrm{swt}(x_i^{\xi_i} y_i^{\eta_i})$, $i \geq 0$, take values in the set $\{-1, 0, 1\}$. This observation yields the second estimate. □

Theorem 8.8. *Let $\mathbf{A} = \mathrm{Alg}(a_0, b_0)$. Then* $\mathrm{GKdim}\, \mathbf{A} = \underline{\mathrm{GKdim}}\, \mathbf{A} = 2\log_3 4 \approx 2.52$.

Proof. Let a natural number m be fixed and set $n = [\log_3 m] - 1$. Consider monomials w of type (22) such that $\xi_{n-1} = \eta_{n-1} = 0$, having the senior part one of three types: $a_n b_n$, a_n, or b_n. By Theorem 5.5, they are linearly independent and belong to $\mathbf{A}$. By Lemma 8.7, $\mathrm{wt}(w) < 3^{n+1} \leq m$. The number of such monomials gives a lower bound on the growth:

$$\tilde{\gamma}_{\mathbf{A}}(m) \geq 3 \cdot 2^{2(n-1)+2n} \geq 3 \cdot 2^{4\log_3 m - 10} = 3 \cdot 2^{-10} m^{\log_3 16}.$$

Let $m \geq 1$ be fixed, set $n = [\log_3 m] + 1$, then $m < 3^n$. By the lower bound of Lemma 8.7, every monomial (22) of length at least $n + 1$ has the weight greater than $3^n > m$. Thus, the number of monomials $w \in \tilde{A}$ of type (22) with $\mathrm{wt}(w) \leq m$ is evaluated by the number of all such monomials of length at most n. These monomials have the senior part $a_j b_j$, a_j, or b_j, where $0 \leq j \leq n$, and their number is evaluated as:

$$\tilde{\gamma}_{\mathbf{A}}(m) \leq \tilde{\gamma}_{\tilde{A}}(m) \leq 3 \sum_{j=0}^{n} 2^{4j} < 3 \frac{2^{4(n+1)}}{16 - 1} \leq \frac{2^{4\log_3 m + 8}}{5} = \frac{2^8}{5} m^{\log_3 16}. \qquad \square$$

Theorem 8.9. *Let w be a monomial of $\mathbf{A} = \mathrm{Alg}(a_0, b_0)$. The respective point of plane is bounded by two logarithmic curves in terms of the weight coordinates* $\mathrm{Wt}(w) = (Z_1, Z_2)$*:*

$$|Z_2| < 2\log_3 Z_1 + 3.$$

Proof. Let w be a monomial (22) of length n. We have bounds on weights by Lemma 8.7:

$$Z_1 = \mathrm{wt}(w) > 3^{n-1}, \quad |Z_2| = |\,\mathrm{swt}(w)| \le 2n + 1.$$

It follows that $|Z_2| \le 2n + 1 < 2\log_3 Z_1 + 3$. □

9. Generating functions of R

Let $A = \mathop{\oplus}\limits_{n,m\in\mathbb{Z}} A_{nm}$ be a $\mathbb{Z}^2$-graded algebra, one has an induced $\mathbb{Z}$-gradation: $A = \oplus_n A_n$, where $A_n = \mathop{\oplus}\limits_{m+k=n} A_{mk}$. Define respective *generating functions*:

$$\mathcal{H}(A, t_1, t_2) = \sum_{n,m} \dim A_{nm} t_1^n t_2^m;$$
$$\mathcal{H}(A, t) = \sum_{n} \dim A_n t^n = \mathcal{H}(A, t, t).$$

Similarly, define generating functions for graded sets and spaces.

Lemma 9.1. *Let $\mathbf{R} = \mathrm{Lie}(a_0, b_0)$, where $\mathrm{char}\, K \neq 2$. The generating function $\mathcal{H}(\mathbf{R}, t_1, t_2)$ satisfies a functional relation:*

$$\mathcal{H}(\mathbf{R}, t_1, t_2) = (1 + t_1^{-1})(1 + t_2^{-1})\mathcal{H}(\mathbf{R}, t_1^2 t_2, t_1 t_2^2) - t_1 t_2.$$

Proof. Denote by T the set of all standard monomials, by T_n the standard monomials of length n, $n \ge 0$. Recall that $T_0 = \{a_0, b_0\}$, $T_1 = \{a_1, b_1, c_1, y_0 a_1, x_0 b_1\}$. Using Lemma 4.2, we have

$$\mathcal{H}(T_0, t_1, t_2) = t_1 + t_2;$$
$$\mathcal{H}(T_1, t_1, t_2) = t_1 t_2 + t_1^2 + t_2^2 + t_1^2 t_2 + t_1 t_2^2.$$

We have $\mathcal{H}(\mathbf{R}, t_1, t_2) = \mathcal{H}(T, t_1, t_2)$. Due to the structure of the standard monomials of all three types (see Theorem 5.1) we obtain bijections between sets of monomials:

$$T_{n+1} = \{1, x_0, y_0, x_0 y_0\} \cdot \tau(T_n), \qquad n \ge 1; \tag{23}$$
$$T \setminus (T_1 \cup T_0) = \{1, x_0, y_0, x_0 y_0\} \cdot \tau(T \setminus T_0). \tag{24}$$

By Lemma 7.4,

$$
\begin{aligned}
&\mathrm{Gr}(x_0) = -\mathrm{Gr}(a_0) = (-1,0), \qquad \mathcal{H}(\{x_0\},t_1,t_2) = t_1^{-1};\\
&\mathrm{Gr}(y_0) = -\mathrm{Gr}(b_0) = (0,-1), \qquad \mathcal{H}(\{y_0\},t_1,t_2) = t_2^{-1};\\
&\mathcal{H}(\{1,x_0,y_0,x_0y_0\},t_1,t_2) = 1+t_1^{-1}+t_2^{-1}+t_1^{-1}t_2^{-1} = (1+t_1^{-1})(1+t_2^{-1}).
\end{aligned}
$$

Using (24), we compute the generating function

$$
\begin{aligned}
\mathcal{H}(\mathbf{R},t_1,t_2) &= \mathcal{H}(T_0\cup T_1,t_1,t_2)\\
&\quad + (1+t_1^{-1})(1+t_2^{-1})\cdot\Big(\mathcal{H}(\mathbf{R},t_1,t_2)-\mathcal{H}(T_0,t_1,t_2)\Big)\Big|_{t_1:=t_1^2t_2,\ t_2:=t_1t_2^2}\\
&= t_1+t_2+t_1t_2+t_1^2+t_2^2+t_1^2t_2+t_1t_2^2\\
&\quad + (1+t_1^{-1})(1+t_2^{-1})(\mathcal{H}(\mathbf{R},t_1^2t_2,t_1t_2^2)-t_1^2t_2-t_1t_2^2)\\
&= (1+t_1^{-1})(1+t_2^{-1})\mathcal{H}(\mathbf{R},t_1^2t_2,t_1t_2^2)-t_1t_2. \qquad \square
\end{aligned}
$$

Corollary 9.2. *Let* $\mathrm{char}\, K \neq 2$, $\mathbf{R} = \mathrm{Lie}(a_0,b_0)$, *and* T_n *the set of standard monomials of length* n. *Then*

$$
\mathcal{H}(T_{n+1},t_1,t_2) = (1+t_1^{-1})(1+t_2^{-1})\mathcal{H}(T_n,t_1^2t_2,t_1t_2^2), \quad n\geq 1.
$$

Proof. Follows by the proof and (23). $\square$

Now, using computer calculations, we have the following.

Corollary 9.3. *Let* $\mathrm{char}\, K \neq 2$, $\mathbf{R} = \mathrm{Lie}(a_0,b_0)$.

(1) *The generating functions are:*

$$
\begin{aligned}
\mathcal{H}(\mathbf{R},t_1,t_2) &= t_1+t_2+t_1^2+t_1t_2+t_2^2+t_1^2t_2+t_1t_2^2\\
&\quad + t_1^3t_2+t_1^2t_2^2+t_1t_2^3+t_1^4t_2+2t_1^3t_2^2+2t_1^2t_2^3+t_1t_2^4+t_1^4t_2^2+t_1^3t_2^3+t_1^2t_2^4\\
&\quad + t_1^4t_2^3+t_1^3t_2^4+t_1^5t_2^3+2t_1^4t_2^4+t_1^3t_2^5+t_1^5t_2^4+t_1^4t_2^5+\dots;\\
\mathcal{H}(\mathbf{R},t) &= 2t+3t^2+2t^3+3t^4+6t^5+3t^6+2t^7+4t^8+2t^9+3t^{10}+6t^{11}\\
&\quad + 3t^{12}+6t^{13}+12t^{14}\dots;
\end{aligned}
$$

(2) *The diagonal* $\mathbf{R}_0$ *(the zero component of the triangular decomposition in Corollary 7.2) is nontrivial:*

$$
\begin{aligned}
\mathcal{H}(\mathbf{R}_0,t_1,t_2) &= t_1t_2+t_1^2t_2^2+t_1^3t_2^3+2t_1^4t_2^4+t_1^5t_2^5+t_1^6t_2^6+4t_1^7t_2^7+t_1^8t_2^8+t_1^9t_2^9+2t_1^{10}t_2^{10}\\
&\quad +2t_1^{11}t_2^{11}+2t_1^{12}t_2^{12}+2t_1^{13}t_2^{13}+t_1^{14}t_2^{14}+t_1^{15}t_2^{15}+4t_1^{16}t_2^{16}+t_1^{17}t_2^{17}+\dots
\end{aligned}
$$

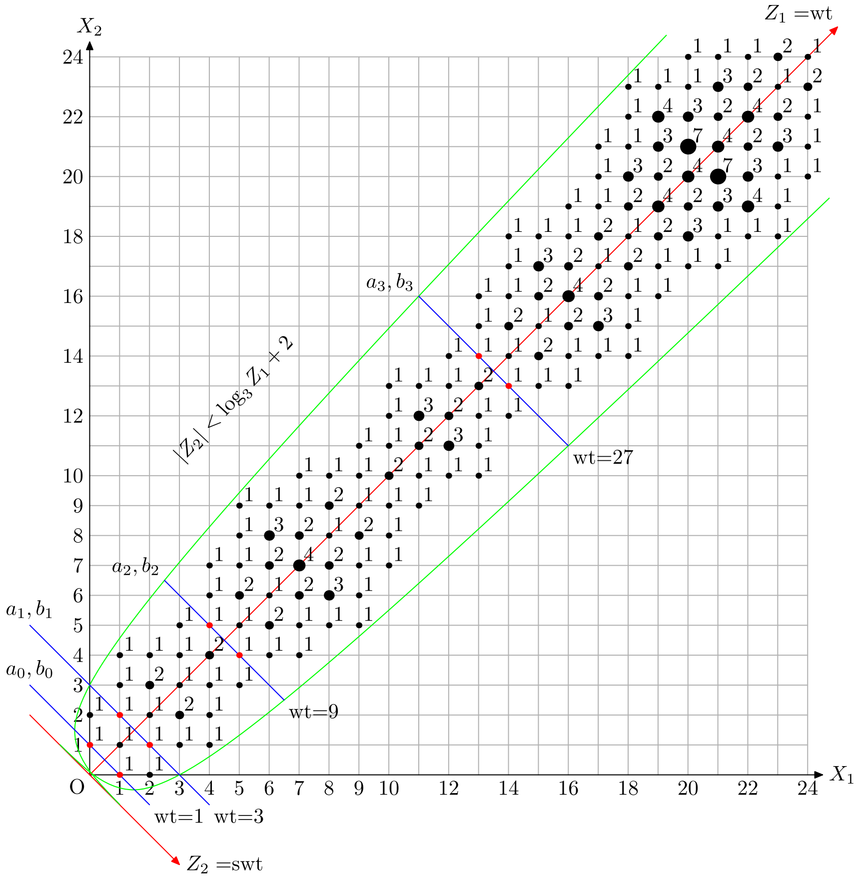

Fig. 1. $\mathbb{Z}^2$-components of the Lie superalgebra $\mathbf{R}$, char $K \neq 2$.

10. Graded nillity of R

The following result shows that the Lie superalgebra $\mathbf{R}$ is an analogue of the Grigorchuk and Gupta–Sidki groups. Namely, we prove that $\mathbf{R}$ is nil $\mathbb{Z}_2$-graded. The question whether $\operatorname{ad} a$ is nil for non-homogeneous elements $a \in \mathbf{R}$ is open.

Theorem 10.1. *Let* $\mathbf{R} = \operatorname{Lie}(a_0, b_0) = \mathbf{R}_{\bar 0} \oplus \mathbf{R}_{\bar 1}$. *For any* $a \in \mathbf{R}_{\bar n}$, $\bar n \in \{\bar 0, \bar 1\}$, *the operator* $\operatorname{ad}(a)$ *is nilpotent.*

Proof. Let $a \in \mathbf{R}_{\bar 1}$, then $(\operatorname{ad} a)^2 = \operatorname{ad}(a^2)$, where $a^2 = \frac{1}{2}[a, a] \in \mathbf{R}_{\bar 0}$ (see (2), in case char $K = 2$ we use the square mapping a^2, which is added to the axioms of the Lie superalgebra). Thus, the proof is reduced to the case of an even element $a \in \mathbf{R}_{\bar 0}$. Observe that the even standard monomials contain at least one Grassmann variable (Theorem 5.1).

We prove a more general statement. Fix a number $N > 0$. Let V be a set of all quasi-standard monomials (see (12)) of length at most N, each containing at least one Grassmann letter. We are going to prove that the associative algebra $\operatorname{Alg}(V) \subset \operatorname{End} \Lambda$ is nilpotent.

Let w be an M-fold non-zero product of v_i's, $v_i \in V$. Since by Corollary 8.2 $\mathrm{wt}(v_i) \ge 1$, we get $\mathrm{wt}(w) \ge M$. We transform the product as follows. We move to the left and order Grassmann variables, while not changing an order of the heads and doing nothing with their mutual products. We only tackle the following interaction. A Grassmann variable x_j (or y_j) can disappear by commuting with an appropriate head a_k (or b_k) with $k \le j$. There are two options illustrated by the following examples (see (6)):

$$\begin{aligned} x_3 b_5 \cdot y_5 a_7 &= -x_3 y_5 \cdot b_5 a_7 + x_3 \cdot a_7;\\ x_3 b_4 \cdot y_5 a_7 &= -x_3 y_5 \cdot b_4 a_7 + x_3 x_4 y_4 \cdot a_7. \end{aligned} \tag{25}$$

As a result, the product is written as a linear combination of monomials of the form:

$$u = \underbrace{x_{i_1} \cdots x_{i_{n_1}} y_{j_1} \cdots y_{j_{n_2}}}_{n \text{ different Grassmann letters}} \cdot \underbrace{\cdots a_{k_l} \cdots b_{s_l} \cdots}_{m \text{ heads with original order}}, \qquad i_l, j_l < N, \quad k_l, s_l \le N. \tag{26}$$

Since $x_{i_l}, y_{j_l} \in \{x_0, y_0, \ldots, x_{N-1}, y_{N-1}\}$ and the product is non-zero, the number of different Grassmann letters above satisfies $n \le 2N$. The number of Grassmann letters in the original product w is greater or equal to the number of heads. This property is kept by transformations (25), hence $n \ge m$. We evaluate weight of the resulting monomial (26):

$$M \le \mathrm{wt}(w) = \mathrm{wt}(u) \le m \,\mathrm{wt}\, a_N \le m3^N \le n3^N \le 2N3^N.$$

Therefore, $\mathrm{Alg}(V)^{N_1} = 0$, where $N_1 = 2N3^N + 1$.

Now let $a \in \mathbf{R}_{\bar 0}$ be a linear combination of standard monomials of length at most N. By statement above, $a^{N_1} = 0$. Now standard arguments show that $(\mathrm{ad}\, a)^{2N_1-1} = 0$. Theorem is proved. □

Corollary 10.2. *All three components of the triangular decomposition (Corollary 7.2) of the Lie superalgebra* $\mathbf{R} = \mathbf{R}_+ \oplus \mathbf{R}_0 \oplus \mathbf{R}_-$ *are locally nilpotent subalgebras.*

Proof. Recall that the superweights of the pivot elements $\{a_i, b_i \mid i \ge 0\}$, are nonzero (Corollary 6.2). So, basis monomials spanning $\mathbf{R}_0$ contain at least one variable. By observation made in the proof of Theorem, $\mathbf{R}_0$ is locally nilpotent.

The homogeneous components of $\mathbf{R}$ are bounded by two logarithmic curves (Theorem 8.6, see Fig. 1), and $\mathbf{R}_+$, $\mathbf{R}_-$ are locally nilpotent by the same geometric arguments as in [33]. □

11. Example 2: Lie superalgebra Q, relations

From now on we skip to Example 2, arguments are similar to those of Example 1. Let $\Lambda = \Lambda[x_i, y_i, z_i \mid i \ge 0]$ be the Grassmann algebra in infinitely many variables. The Grassmann variables and respective superderivatives $\{x_i, y_i, z_i, \partial_{x_i}, \partial_{y_i}, \partial_{z_i} \mid i \ge 0\}$ are odd elements of $\mathrm{End}\,\Lambda$. They anticommute except for nontrivial relations:

$$\partial_{x_i} x_i + x_i \partial_{x_i} = 1, \quad \partial_{y_i} y_i + y_i \partial_{y_i} = 1, \quad \partial_{z_i} z_i + z_i \partial_{z_i} = 1;$$
$$x_i^2 = y_i^2 = z_i^2 = (\partial_{x_i})^2 = (\partial_{y_i})^2 = (\partial_{z_i})^2 = 0, \quad i \geq 0.$$

We present the pivot elements of Example 2 (4) recursively:

$$\begin{aligned} a_i &= \partial_{x_i} + y_i x_i a_{i+1}, \\ b_i &= \partial_{y_i} + z_i y_i b_{i+1}, \qquad\qquad i \geq 0. \\ c_i &= \partial_{z_i} + x_i z_i c_{i+1}, \end{aligned} \tag{27}$$

Recall that we consider the Lie superalgebra $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0) \subset \mathbf{W}(\Lambda) \subset \mathrm{Der}\,\Lambda$ and the associative algebra $\mathbf{A} = \mathrm{Alg}(a_0, b_0, c_0) \subset \mathrm{End}\,\Lambda$.

Define *shift* mappings $\tau : \Lambda \to \Lambda$ and $\tau : \mathbf{W}(\Lambda) \to \mathbf{W}(\Lambda)$ by

$$\tau(x_i) = x_{i+1},\ \tau(y_i) = y_{i+1},\ \tau(z_i) = z_{i+1},$$
$$\tau(\partial_{x_i}) = \partial_{x_{i+1}},\ \tau(\partial_{y_i}) = \partial_{y_{i+1}},\ \tau(\partial_{z_i}) = \partial_{z_{i+1}}, \quad i \geq 0.$$

Also, consider *permuting* mappings $\theta : \Lambda \to \Lambda$ and $\theta : \mathbf{W}(\Lambda) \to \mathbf{W}(\Lambda)$:

$$\theta(x_i) = y_i,\ \theta(y_i) = z_i,\ \theta(z_i) = x_i, \quad \theta(\partial_{x_i}) = \partial_{y_i},\ \theta(\partial_{y_i}) = \partial_{z_i},\ \theta(\partial_{z_i}) = \partial_{x_i}, \quad i \geq 0.$$

Remark that

$$\tau(a_i) = a_{i+1},\ \tau(b_i) = b_{i+1},\ \tau(c_i) = c_{i+1}, \quad \theta(a_i) = b_i,\ \theta(b_i) = c_i,\ \theta(c_i) = a_i, \quad i \geq 0.$$

Clearly, τ is an endomorphism and θ an isomorphism of order 3.

Below by degree of relations we mean the degree in the set $\{a_i, b_i, c_i\}$, i being fixed. We group relations with their conjugates under θ, θ^2.

Lemma 11.1. *We have the following relations of degree 2, $i \geq 0$:*

(1) $[a_i, b_i] = x_i a_{i+1}$,
(2) $[b_i, c_i] = y_i b_{i+1}$,
(3) $[c_i, a_i] = z_i c_{i+1}$,
(4) $a_i^2 = -y_i a_{i+1}$,
(5) $b_i^2 = -z_i b_{i+1}$,
(6) $c_i^2 = -x_i c_{i+1}$,
(7) $[a_i, a_i] = 2a_i^2 = -2y_i a_{i+1}$;
(8) $[b_i, b_i] = 2b_i^2 = -2z_i b_{i+1}$;
(9) $[c_i, c_i] = 2c_i^2 = -2x_i c_{i+1}$.

Proof. We proceed as in Lemma 4.1 and Lemma 4.2. $\square$

Lemma 11.2. *All relations of degree 3, $i \geq 0$:*

(1) $[a_i, a_i, b_i] = [a_i^2, b_i] = a_{i+1}$,
(2) $[b_i, b_i, c_i] = [b_i^2, c_i] = b_{i+1}$,
(3) $[c_i, c_i, a_i] = [c_i^2, a_i] = c_{i+1}$,
(4) $[a_i, a_i, c_i] = [a_i^2, c_i] = x_i y_i z_i\, z_{i+1} c_{i+2}$,
(5) $[b_i, b_i, a_i] = [b_i^2, a_i] = x_i y_i z_i\, x_{i+1} a_{i+2}$,
(6) $[c_i, c_i, b_i] = [c_i^2, b_i] = x_i y_i z_i\, y_{i+1} b_{i+2}$,
(7) $[a_i, b_i, c_i] = [b_i, c_i, a_i] = [c_i, a_i, b_i] = 0$,
(8) $[a_i, a_i, a_i] = [b_i, b_i, b_i] = [c_i, c_i, c_i] = 0$.

Proof. Let us check claims 1 and 4:

$$[a_i, a_i, b_i] = [a_i, x_i a_{i+1}] = [\partial_{x_i} + y_i x_i a_{i+1}, x_{i+1} a_{i+1}] = a_{i+1};$$
$$[a_i, a_i, c_i] = [a_i, z_i c_{i+1}] = [\partial_{x_i} + y_i x_i a_{i+1}, z_i c_{i+1}] = -y_i x_i z_i [a_{i+1}, c_{i+1}]$$
$$= x_i y_i z_i\, z_{i+1} c_{i+2}. \qquad \square$$

Consider Lie superalgebras $L_i = \mathrm{Lie}(a_i, b_i, c_i)$, $i \geq 0$, so $L_0 = \mathbf{Q}$.

Corollary 11.3. *We have*

(1) $a_i, b_i, c_i \in \mathbf{Q}$ *for all* $i \geq 0$;
(2) $\tau^i : \mathbf{Q} \to L_i$ *is an isomorphism for any* $i \geq 1$;
(3) $\mathbf{Q}$ *is infinite dimensional;*
(4) $\theta : \mathbf{Q} \to \mathbf{Q}$ *is an automorphism of order 3;*
(5) *there exists a natural self-similarity embedding:*

$$\mathbf{Q} \hookrightarrow \langle \partial_{x_0}, \partial_{y_0}, \partial_{z_0} \rangle_K \leftthreetimes \Lambda[x_0, y_0, z_0] \otimes \tau(\mathbf{Q}).$$

All nonzero relations of degree 3 given by Lemma 11.2 are θ-conjugates of those of items 1) and 4). To get relations of degree 4 we multiply those formulas by a_i, b_i, c_i and consider θ-conjugates.

Lemma 11.4. *Relations of degree 4 are:*

(1) $[a_i, a_i, a_i, b_i] = [a_i, a_{i+1}] = 2x_i y_i\, y_{i+1} a_{i+2}$,
(2) $[b_i, a_i, a_i, b_i] = [b_i, a_{i+1}] = -y_i z_i\, x_{i+1} a_{i+2}$,
(3) $[c_i, a_i, a_i, b_i] = [c_i, a_{i+1}] = x_i z_i\, z_{i+1} c_{i+2}$,
(4) $[a_i, a_i, a_i, c_i] = y_i z_i\, z_{i+1} c_{i+2}$,
(5) $[b_i, a_i, a_i, c_i] = -x_i z_i\, z_{i+1} c_{i+2}$,
(6) $[c_i, a_i, a_i, c_i] = x_i y_i\, z_{i+1} c_{i+2}$,

and their θ-conjugates as well, where $i \geq 0$.

Proof. Using Claim 1) of Lemma 11.2, we get:

$$[a_i, a_{i+1}] = [\partial_{x_i} + y_i x_i a_{i+1}, a_{i+1}] = y_i x_i [a_{i+1}, a_{i+1}] = 2x_i y_i\, y_{i+1} a_{i+2};$$

$$[b_i, a_{i+1}] = [\partial_{y_i} + z_i y_i b_{i+1}, a_{i+1}] = z_i y_i [b_{i+1}, a_{i+1}] = -y_i z_i\, x_{i+1} a_{i+2};$$

$$[c_i, a_{i+1}] = [\partial_{z_i} + x_i z_i c_{i+1}, a_{i+1}] = x_i z_i [c_{i+1}, a_{i+1}] = x_i z_i\, z_{i+1} c_{i+2}.$$

Using Claim 4) of Lemma 11.2, we get:

$$[a_i, a_i, a_i, c_i] = [\partial_{x_i} + y_i x_i a_{i+1}, x_i y_i z_i\, z_{i+1} c_{i+2}] = y_i z_i\, z_{i+1} c_{i+2};$$

$$[b_i, a_i, a_i, c_i] = [\partial_{y_i} + z_i y_i b_{i+1}, x_i y_i z_i\, z_{i+1} c_{i+2}] = -x_i z_i\, z_{i+1} c_{i+2};$$

$$[c_i, a_i, a_i, c_i] = [\partial_{z_i} + x_i z_i c_{i+1}, x_i y_i z_i\, z_{i+1} c_{i+2}] = x_i y_i\, z_{i+1} c_{i+2}. \qquad \square$$

12. Monomial bases of Q and $\tilde{A}$

As above, by r_n denote a *tail* monomial:

$$r_n = x_0^{\xi_0} y_0^{\eta_0} z_0^{\zeta_0} \cdots x_n^{\xi_n} y_n^{\eta_n} z_n^{\zeta_n} \in \Lambda, \quad \xi_i, \eta_i, \zeta_i \in \{0,1\};\ n \geq 0. \tag{28}$$

If $n < 0$, we consider that $r_n = 1$. Another monomials of type (28) will be denoted by r'_n, $\tilde{r}_n$, etc.

Theorem 12.1. *Let* $\operatorname{char} K \neq 2$. *A basis of the Lie superalgebra* $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$ *is given by the following standard monomials of three types (where* r_n *denote tail monomials* (28)*)*

(1) *monomials of the first type:*

$$\{r_{n-2} a_n,\ r_{n-2} b_n,\ r_{n-2} c_n \mid n \geq 0\};$$

(2) *monomials of the second type:*

$$\{r_{n-2} x_{n-1} a_n,\ r_{n-2} y_{n-1} b_n,\ r_{n-2} z_{n-1} c_n \mid n \geq 1\};$$

(3) *monomials of the third type:*

$$\begin{aligned} &\{r_{n-3} x_{n-2}^{\alpha} y_{n-2}^{\beta} y_{n-1} a_n, \\ &\ r_{n-3} y_{n-2}^{\alpha} z_{n-2}^{\beta} z_{n-1} b_n, \\ &\ r_{n-3} z_{n-2}^{\alpha} x_{n-2}^{\beta} x_{n-1} c_n \mid \alpha, \beta \in \{0,1\},\ n \geq 1\} \end{aligned}$$

(in case $n = 1$ *we have* $\{y_0 a_1, z_0 b_1, x_0 c_1\}$*).*

Proof. Let us call n the *length*, a_n, b_n, c_n the *heads*, r_{n-2} (or r_{n-3}) the *tails*, and the remaining products the *necks* of the respective monomials.

A) We check by induction on the length n that all these monomials belong to $\mathbf{Q}$. By Lemma 11.2 and Lemma 11.1, we get $\{a_0, b_0, c_0, a_1, b_1, c_1\} \subset \mathbf{Q}$, $\{x_0a_1, y_0b_1, z_0c_1\} \subset \mathbf{Q}$, and $\{y_0a_1, z_0b_1, x_0c_1\} \subset \mathbf{Q}$. Thus, we have monomials of all three types for $n = 0, 1$. This is the base of induction.

Let a number $n \geq 1$ be fixed and assume that all monomials of the first type of length $i = 0, \dots, n$ belong to $\mathbf{Q}$. Then we take $w = r_{n-3}a_{n-1} \in \mathbf{Q}$ and use Lemma 11.2:

$$w' = [b_{n-1}, b_{n-1}, w] = r_{n-3}[b_{n-1}, b_{n-1}, a_{n-1}] = r_{n-3}x_{n-1}y_{n-1}z_{n-1}\, x_n a_{n+1}. \qquad (29)$$

Observe that we can delete the letters $\{x_{n-1}, y_{n-1}, z_{n-1}, x_n\}$ in (29) in any combination, for example:

$$\begin{aligned}
[a_{n-1}, w'] &= \pm r_{n-3}[\partial_{x_{n-1}} + y_{n-1}x_{n-1}a_n, x_{n-1}y_{n-1}z_{n-1}\, x_n a_{n+1}] \\
&= \pm r_{n-3}y_{n-1}z_{n-1}\, x_n a_{n+1}; \\
[a_n, w'] &= \pm r_{n-3}[\partial_{x_n} + y_n x_n a_{n+1}, x_{n-1}y_{n-1}z_{n-1}\, x_n a_{n+1}] \\
&= \pm r_{n-3}x_{n-1}y_{n-1}z_{n-1}\, a_{n+1}.
\end{aligned}$$

This argument yields all desired monomials of the first and second type of length $n+1$.

We start with a monomial of the first type $w = r_{n-2}a_n \in \mathbf{Q}$ and use the first relation of Lemma 11.4:

$$[a_{n-1}, w] = \pm r_{n-2}[a_{n-1}, a_n] = \pm 2r_{n-2}x_{n-1}y_{n-1}\, y_n a_{n+1}.$$

As above, we can delete any of the letters x_{n-1}, y_{n-1}, thus getting all the desired monomials of the third type of length $n+1$.

B) Now let us prove that a product of the standard monomials is expressed via the standard monomials.

1) Consider products of monomials of the same length. Assume that the heads are different letters, e.g. "a" and "c". We write monomials of all three types uniformly: $v = r_{m-1}a_m$, and $w = r'_{m-1}c_m$. Then

$$[v, w] = \pm r_{m-1}r'_{m-1}[a_m, c_m] = \pm r''_{m-1}z_m c_{m+1},$$

we get a monomial of the second type.

Assume that the heads are the same letters, e.g. both are "a". Take two monomials of the first type $v = r_{m-2}a_m$, and $w = r'_{m-2}a_m$. Then

$$[v, w] = \pm r_{m-2}r'_{m-2}[a_m, a_m] = \pm 2r''_{m-2}y_m a_{m+1}. \qquad (30)$$

Consider a product of monomials of the same length of the second and third type:

$$
\begin{aligned}
[r_{m-2}x_{m-1}a_m, r'_{m-2}y_{m-1}a_m] &= \pm r''_{m-2}x_{m-1}y_{m-1}[a_m, a_m] \\
&= \pm 2r''_{m-2}x_{m-1}y_{m-1}y_m a_{m+1}.
\end{aligned}
\tag{31}
$$

Similarly, all products of monomials of the same length with the same head yield monomials of the third type.

2) Consider monomials of different lengths. Assume that the heads are different letters, e.g. "a" and "c". We present monomials of all three types uniformly: $v = r_{n-1}a_n$, and $w = r'_{m-1}c_m$. Assume that $n < m$, we have a presentation

$$
\begin{aligned}
r_{n-1}a_n = r_{n-1}\Big(&\partial_{x_n} + y_n x_n(\partial_{x_{n+1}} \\
&+ y_{n+1}x_{n+1}(\cdots + y_{m-2}x_{m-2}(\partial_{x_{m-1}} + y_{m-1}x_{m-1}a_m)\cdots))\Big) \qquad (32)\\
= &\sum_{j=n}^{m-1} \bar r_{j-1}\partial_{x_j} + \bar r_{m-2}y_{m-1}x_{m-1}a_m.
\end{aligned}
$$

We use this presentation

$$
\begin{aligned}
[r_{n-1}a_n, r'_{m-1}c_m] &= \sum_{j=n}^{m-1} \bar r_{j-1}\partial_{x_j}(r'_{m-1})c_m \pm \bar r_{m-2}y_{m-1}x_{m-1}r'_{m-1}[a_m, c_m] \\
&= \sum_{j=n}^{m-2} \bar r_{j-1}\partial_{x_j}(r'_{m-1})c_m + \bar r_{m-2}\partial_{x_{m-1}}(r'_{m-1})c_m \pm r''_{m-1}z_m c_{m+1}.
\end{aligned}
\tag{33}
$$

The last term is of the second type. The factors $\partial_{x_i}(r'_{m-1})$, $i = n, \dots, m-2$, keep the letter x_{m-1} or z_{m-1} of the neck (r'_{m-1} can have at most one of those), and do not add new Grassmann letters with index $m-2$. Thus, the summation in (33) yields standard monomials of the same type as $r'_{m-1}c_m$. The middle monomial in (33) is non-zero only in the case $\partial_{x_{m-1}}(r'_{m-1}) \neq 0$ which implies that the only letter x_{m-1} of the neck disappears, we get a monomial of the first type.

Now consider monomials with the same head letter of different lengths. Take two monomials $v = r_{n-1}a_n$, and $w = r'_{m-1}a_m$ such that $n < m$. We use presentation (32)

$$
[v, w] = \sum_{j=n}^{m-1} \bar r_{j-1}\partial_{x_j}(r'_{m-1})a_m + \bar r_{m-2}[y_{m-1}x_{m-1}a_m, r'_{m-1}a_m].
$$

We treat the summation as above. Consider the commutator of the last term, it is nonzero only in the case $r'_{m-1}a_m$ is of the first type, similar to (31), we get a monomial of the third type. □

The following corollaries are proved as above.

Corollary 12.2. *Let* $\operatorname{char} K = 2$. *A basis of the Lie superalgebra* $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$ *as well as a basis of the restricted Lie algebra* $\operatorname{Lie}_p(a_0, b_0, c_0)$ *is given by standard monomials of the first and second type and squares of the pivot elements:*

$$\{y_{n-1}a_n, z_{n-1}b_n, x_{n-1}c_n \mid \ n \geq 1\}.$$

Corollary 12.3. *Let* $\operatorname{char} K = p \geq 3$. *The restricted Lie superalgebra* $\operatorname{Lie}_p(a_0, b_0, c_0)$ *coincides with the Lie superalgebra* $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$, *both have the same bases.*

Define monomials

$$\{r_{n-2}x_{n-1}^{\xi_{n-1}}y_{n-1}^{\eta_{n-1}}z_{n-1}^{\zeta_{n-1}}a_n \mid \xi_{n-1}, \eta_{n-1}, \zeta_{n-1} \in \{0,1\},\ \xi_{n-1} + \eta_{n-1} + \zeta_{n-1} \leq 1,\ n \geq 0\}, \tag{34}$$

and their θ-conjugates. As above, we refer to them as *quasi-standard monomials*.

Corollary 12.4. *The algebra* $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$ *is contained in a bigger (restricted) Lie superalgebra* $\tilde{Q} \subset \mathbf{W}(\Lambda)$, *which basis is given by quasi-standard monomials (34).*

We consider an associative subalgebra generated by quasi-standard monomials.

Theorem 12.5. *Let* $\tilde{A} = \operatorname{Alg}(\tilde{Q}) \subset \operatorname{End}(\Lambda)$. *Then*

(1) *a basis of* $\tilde{A}$ *is given by monomials*

$$\begin{aligned}\{r_{n-2}x_{n-1}^{\xi_{n-1}}y_{n-1}^{\eta_{n-1}}z_{n-1}^{\zeta_{n-1}}a_n^{\alpha_n}b_n^{\beta_n}c_n^{\gamma_n}\cdots a_0^{\alpha_0}b_0^{\beta_0}c_0^{\gamma_0} \mid \xi_{n-1}, \eta_{n-1}, \zeta_{n-1}, \alpha_i, \beta_i, \gamma_i \in \{0,1\};\\ \xi_{n-1} + \eta_{n-1} + \zeta_{n-1}\\ \leq \alpha_n + \beta_n + \gamma_n > 0;\ n \geq 0\};\end{aligned}$$

(the inequality above we call the senior part property*);*

(2) $\tilde{A}$ *contains* $\mathbf{A} = \operatorname{Alg}(a_0, b_0, c_0)$*;*

(3) *monomials with* $\xi_{n-1} = \eta_{n-1} = \zeta_{n-1} = 0$ *are linearly independent and belong to* $\mathbf{A}$.

Proof. We proceed as in Theorem 5.5. □

13. Three weight functions on Q

Assume that the superderivatives and Grassmann letters have the *weights* as follows:

$$\begin{aligned}\mathsf{wt}(\partial_{x_i}) &= -\mathsf{wt}(x_i) = \alpha_i \in \mathbb{C},\\ \mathsf{wt}(\partial_{y_i}) &= -\mathsf{wt}(y_i) = \beta_i \in \mathbb{C},\\ \mathsf{wt}(\partial_{z_i}) &= -\mathsf{wt}(z_i) = \gamma_i \in \mathbb{C}, \quad i \geq 0.\end{aligned}$$

In order to define weight for the elements a_n, b_n, c_n, $n \ge 0$, we want all terms in (27) be homogeneous. Thus, we get equalities:

$$\alpha_i = -\alpha_i - \beta_i + \alpha_{i+1},$$
$$\beta_i = -\beta_i - \gamma_i + \beta_{i+1},$$
$$\gamma_i = -\alpha_i - \gamma_i + \gamma_{i+1}, \quad i \ge 0.$$

Thus, we get a recurrence relation

$$\begin{pmatrix}\alpha_{i+1}\\ \beta_{i+1}\\ \gamma_{i+1}\end{pmatrix} = A\begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix}, \quad i \ge 0; \quad \text{where} \qquad A = \begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}. \tag{35}$$

Below $i \in \mathbb{C}$, $i^2 = -1$, $\bar{z}$ denotes the complex conjugation of $z \in \mathbb{C}$. We shall use the following constants:

$$\epsilon = \frac{-1+\sqrt{3}i}{2} = e^{2\pi i/3}, \qquad \bar{\epsilon} = \frac{-1-\sqrt{3}i}{2} = e^{-2\pi i/3};$$
$$\mu = 2+\epsilon = \sqrt{3}e^{\pi i/6}, \qquad \bar{\mu} = 2+\bar{\epsilon} = \sqrt{3}e^{-\pi i/6}, \qquad \lambda = 3.$$

In the next lemma we use sequences of numbers:

$$\Theta_1^n := 3^n + \mu^n + \bar{\mu}^n = 3^n + 2\sqrt{3^n}\cos\left(\frac{\pi n}{6}\right),$$
$$\Theta_2^n := 3^n + \epsilon\mu^n + \bar{\epsilon}\,\bar{\mu}^n = 3^n + 2\sqrt{3^n}\cos\left(\frac{\pi n}{6} + \frac{2\pi}{3}\right),$$
$$\Theta_3^n := 3^n + \bar{\epsilon}\mu^n + \epsilon\bar{\mu}^n = 3^n + 2\sqrt{3^n}\cos\left(\frac{\pi n}{6} - \frac{2\pi}{3}\right); \qquad n \ge 0.$$

Lemma 13.1. *The matrix A (35) has the following properties:*

(1) *$\lambda = 3$, μ, $\bar{\mu}$ are the eigenvalues, where the respective eigenvectors are as follows:*

$$v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\ \epsilon\\ \bar{\epsilon}\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\ \bar{\epsilon}\\ \epsilon\end{pmatrix};$$

(2) *the transition matrix C to the basis $\{v_1, v_2, v_3\}$ and its inverse are:*

$$C = \begin{pmatrix}1 & 1 & 1\\ 1 & \epsilon & \bar{\epsilon}\\ 1 & \bar{\epsilon} & \epsilon\end{pmatrix}, \qquad C^{-1} = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & \bar{\epsilon} & \epsilon\\ 1 & \epsilon & \bar{\epsilon}\end{pmatrix};$$

(3)

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 4 & -2 & 1 \\ 1 & 4 & -2 \\ -2 & 1 & 4 \end{pmatrix};$$

(4)

$$A^n = \frac{1}{3}\begin{pmatrix} \Theta_1^n & \Theta_3^n & \Theta_2^n \\ \Theta_2^n & \Theta_1^n & \Theta_3^n \\ \Theta_3^n & \Theta_2^n & \Theta_1^n \end{pmatrix}, \qquad n \geq 0.$$

Proof. The first three claims are checked by multiplication of matrices. The last claim is checked by induction. □

According to three eigenvalues and eigenvectors we introduce three *weight functions*.

Corollary 13.2. *For all $n \geq 0$ let*

(1) $\mathrm{wt}(a_n) = \mathrm{wt}(b_n) = \mathrm{wt}(c_n) = 3^n$ *(*a weight function*);*
(2) $\mathrm{swt}(a_n) = \mu^n$, $\mathrm{swt}(b_n) = \epsilon\mu^n$, $\mathrm{swt}(c_n) = \bar{\epsilon}\mu^n$ *(a* superweight *function);*
(3) $\overline{\mathrm{swt}}(a_n) = \bar{\mu}^n$, $\overline{\mathrm{swt}}(b_n) = \bar{\epsilon}\,\bar{\mu}^n$, $\overline{\mathrm{swt}}(c_n) = \epsilon\bar{\mu}^n$ *(a* conjugated superweight *function);*
(4) $\mathrm{Wt}(a) = (\mathrm{wt}(a), \mathrm{swt}(a), \overline{\mathrm{swt}}(a))$ *(a* vector weight function*, were a is one of a_n, b_n, c_n);*
(5) *the space of weight functions is three-dimensional and* $\{\mathrm{wt}, \mathrm{swt}, \overline{\mathrm{swt}}\}$ *is its basis.*

Proof. We proceed as in the proof of Corollary 6.2. □

By *monomials* we call any monomials in letters $\{x_i, y_i, z_i, \partial_{x_i}, \partial_{y_i}, \partial_{z_i}, a_i, b_i, c_i \mid i \geq 0\}$, belonging to $\mathrm{End}\,\Lambda$.

Lemma 13.3. *Let a, b be any monomials. Then*

(1) *the weight functions are well defined on monomials;*
(2) *the weight functions are additive on products of monomials, e.g.:*

$$\mathrm{Wt}(a \cdot b) = \mathrm{Wt}(a) + \mathrm{Wt}(b).$$

(3) $\overline{\mathrm{swt}}(a) = \overline{\mathrm{swt}(a)}$ *is the complex conjugation.*

Proof. It follows by our construction of weights, see also [33] and [36]. □

Corollary 13.4. *The vector weight function has the following values on the pivot elements:*

$$\begin{aligned}\mathrm{Wt}(a_n) &= (3^n, \mu^n, \bar\mu^n),\\ \mathrm{Wt}(b_n) &= (3^n, \epsilon\mu^n, \bar\epsilon\,\bar\mu^n),\\ \mathrm{Wt}(c_n) &= (3^n, \bar\epsilon\mu^n, \epsilon\bar\mu^n), \qquad n \ge 0.\end{aligned}$$

14. $\mathbb{Z}^3$-gradation of Q and three coordinate systems

Lemma 14.1. *The Lie superalgebra* $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0)$ *and its associative hull* $\mathbf{A} = \mathrm{Alg}(a_0, b_0, c_0)$ *are* $\mathbb{Z}^3$*-graded by multidegree in the generators* $\{a_0, b_0, c_0\}$*:*

$$\mathbf{A} = \mathop{\oplus}_{n_1,n_2,n_3\ge 0} \mathbf{A}_{n_1,n_2,n_3}, \qquad \mathbf{Q} = \mathop{\oplus}_{n_1,n_2,n_3\ge 0} \mathbf{Q}_{n_1,n_2,n_3}.$$

Proof. By Corollary 13.4, we get three vectors

$$\mathrm{Wt}(a_0) = (1,1,1),\ \mathrm{Wt}(b_0) = (1,\epsilon,\bar\epsilon),\ \mathrm{Wt}(c_0) = (1,\bar\epsilon,\epsilon) \in \mathbb{C}^3. \tag{36}$$

Let $\mathbf{A}_{n_1,n_2,n_3}$ be spanned by all monomials of $\mathbf{A}$ of degrees n_1, n_2, n_3 in the generators a_0, b_0, c_0, respectively, where $n_1, n_2, n_3 \ge 0$. For any $v \in \mathbf{A}_{n_1,n_2,n_3}$ we apply Lemma 13.3

$$\mathrm{Wt}(v) = n_1\,\mathrm{Wt}(a_0) + n_2\,\mathrm{Wt}(b_0) + n_3\,\mathrm{Wt}(c_0). \tag{37}$$

Since the vectors (36) are linearly independent we conclude that elements belonging to different components $\mathbf{A}_{n_1,n_2,n_3}$ are linearly independent. □

We put relations (36) and (37) in the matrix form:

$$\mathrm{Wt}^T(v) = \begin{pmatrix}\mathrm{wt}(v)\\ \mathrm{swt}(v)\\ \overline{\mathrm{swt}}(v)\end{pmatrix} = \begin{pmatrix}1&1&1\\1&\epsilon&\bar\epsilon\\1&\bar\epsilon&\epsilon\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}. \tag{38}$$

Consider a monomial $0 \ne v \in \mathbf{A}_{n_1,n_2,n_3}$, $n_1, n_2, n_3 \ge 0$, we introduce a *multidegree* vector and put it in space identifying with the standard coordinates $(X_1, X_2, X_3) \in \mathbb{R}^3$.

$$\mathrm{Gr}(v) := (n_1, n_2, n_3) \in \mathbb{Z}^3 \subset \mathbb{R}^3 \subset \mathbb{C}^3$$

(we warn that in [33] the function $\mathrm{Gr}(*)$ was denoted as $\mathrm{Wt}(*)$).

Now let (X_1, X_2, X_3) be the *standard coordinates* of an arbitrary point in $\mathbb{C}^3$, which we also refer to as the *multidegree coordinates*. We introduce the *weight coordinates* (Z_1, Z_2, Z_3) in $\mathbb{C}^3$ by the relations:

$$\begin{pmatrix}Z_1\\Z_2\\Z_3\end{pmatrix} = \begin{pmatrix}1&1&1\\1&\epsilon&\bar\epsilon\\1&\bar\epsilon&\epsilon\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}. \tag{39}$$

We use the inverse matrix for C given in Lemma 13.1:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\epsilon} & \epsilon \\ 1 & \epsilon & \overline{\epsilon} \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}. \tag{40}$$

Lemma 14.2. *Let $v \in \mathbf{A}$ be a monomial, $\mathrm{Gr}(v) = (X_1, X_2, X_3)$ its multidegree, and (Z_1, Z_2, Z_3) its weight coordinates introduced above. They coincide with the vector weight function:*

$$(Z_1, Z_2, Z_3) = \mathrm{Wt}(v) = (\mathrm{wt}(v), \mathrm{swt}(v), \overline{\mathrm{swt}}(v)).$$

Proof. Follows from (38) and (39). □

Let (X_1, X_2, X_3) be the standard coordinates in $\mathbb{R}^3 \subset \mathbb{C}^3$. Using (39), we introduce real *orthogonal coordinates* as follows:

$$\begin{aligned} Y_1 &= \frac{Z_1}{\sqrt{3}} = \frac{X_1 + X_2 + X_3}{\sqrt{3}}; \\ Y_2 &= \frac{\sqrt{2}}{\sqrt{3}}\,\mathrm{Re}(Z_2) = \frac{2X_1 - X_2 - X_3}{\sqrt{6}}; \\ Y_3 &= \frac{\sqrt{2}}{\sqrt{3}}\,\mathrm{Im}(Z_2) = \frac{X_2 - X_3}{\sqrt{2}}. \end{aligned} \tag{41}$$

The transition matrix (41) is orthogonal, thus the inverse transition is given by the transpose matrix:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & \sqrt{2/3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}. \tag{42}$$

For any monomial $v \in \mathbf{A}$, we refer to these coordinates as the *orthogonal coordinates* $\mathrm{Ort}(v) = (Y_1, Y_2, Y_3)$. Two last equations (41) imply that

$$Y_2^2 + Y_3^2 = \frac{2}{3}\Big(X_1^2 + X_2^2 + X_3^2 - X_1X_2 - X_2X_3 - X_1X_3\Big). \tag{43}$$

Thus, given a point of space $P = (X_1, X_2, X_3) \in \mathbb{R}^3$, we have three different coordinates related by (39), (40), (41), and (42), which we write $\mathrm{Gr}(P) = (X_1, X_2, X_3)$, $\mathrm{Ort}(P) = (Y_1, Y_2, Y_3)$, and $\mathrm{Wt}(P) = (Z_1, Z_2, Z_3) \in \mathbb{C}^3$.

Lemma 14.3. *Using notations of Lemma 13.1, the multidegrees of the pivot elements a_n, b_n, c_n are as follows:*

$$\begin{aligned} \mathrm{Gr}(a_n) &= \frac{1}{3}(\Theta_1^n, \Theta_3^n, \Theta_2^n), \\ \mathrm{Gr}(b_n) &= \frac{1}{3}(\Theta_2^n, \Theta_1^n, \Theta_3^n), \\ \mathrm{Gr}(c_n) &= \frac{1}{3}(\Theta_3^n, \Theta_2^n, \Theta_1^n), \quad n \geq 0. \end{aligned}$$

Proof. Any initial data $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{C}^3$ yields a weight function given by the recurrence relation (35). In particular, by setting $\alpha_0 = 1$, $\beta_0 = \gamma_0 = 0$, we get a weight function

$$\begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad n \geq 0. \tag{44}$$

By additivity, this tuple yields the degrees of a_n, b_n, c_n in the letter a_0. On the other hand, this is the first column of A^n. By similar arguments, we conclude that the rows of A^n yield the multidegrees of the elements a_n, b_n, c_n in $\{a_0, b_0, c_0\}$. □

Lemma 14.4.

(1) *The orthogonal coordinates of the pivot elements a_n, b_n, c_n, $n \geq 0$, are:*

$$\begin{aligned}
\operatorname{Ort}(a_n) &= \frac{1}{\sqrt{3}}\left(3^n, \sqrt{2{\cdot}3^n}\cos\left(\frac{\pi n}{6}\right), \sqrt{2{\cdot}3^n}\sin\left(\frac{\pi n}{6}\right)\right);\\
\operatorname{Ort}(b_n) &= \frac{1}{\sqrt{3}}\left(3^n, \sqrt{2{\cdot}3^n}\cos\left(\frac{\pi n}{6}+\frac{2\pi}{3}\right), \sqrt{2{\cdot}3^n}\sin\left(\frac{\pi n}{6}+\frac{2\pi}{3}\right)\right);\\
\operatorname{Ort}(c_n) &= \frac{1}{\sqrt{3}}\left(3^n, \sqrt{2{\cdot}3^n}\cos\left(\frac{\pi n}{6}-\frac{2\pi}{3}\right), \sqrt{2{\cdot}3^n}\sin\left(\frac{\pi n}{6}-\frac{2\pi}{3}\right)\right).
\end{aligned}$$

(2) *The respective points of space for a_n, b_n, c_n, $n \geq 0$, belong to an elliptic paraboloid (a smaller one on Fig. 2), which equation in orthogonal and standard coordinates is as follows:*

$$\begin{aligned}
Y_1 &= \frac{\sqrt{3}}{2}\left(Y_2^2 + Y_3^2\right);\\
X_1 + X_2 + X_3 &= X_1^2 + X_2^2 + X_3^2 - X_1X_2 - X_2X_3 - X_1X_3.
\end{aligned}$$

Proof. The formulas for the orthogonal coordinates follow from the expression of Y's via Z's given by (41) and values of the weight coordinates (Corollary 13.4). Now the equation of the elliptic paraboloid in orthogonal coordinates is directly checked. The equation in standard coordinates is obtained using (41) and (43). □

Denote by Π the plane $X_1 + X_2 + X_3 = 0$ in the standard coordinates of $\mathbb{R}^3$ (equivalently, $Y_1 = 0$). Consider a point in space given in orthogonal coordinates $\bar{Y} = (Y_1, Y_2, Y_3)$, let $\operatorname{pr}_\Pi(\bar{Y}) := (Y_2, Y_3)$ be its *orthogonal projection* on Π (this projection is parallel to line $Y_2 = Y_3 = 0$, or $X_1 = X_2 = X_3$ using the standard coordinates, see (41)). For convenience, let us identify $(Y_2, Y_3) \in \Pi$ with $Y_2 + iY_3 \in \mathbb{C}^2$.

Now we can describe a geometrical meaning of the superweight function.

Lemma 14.5. *Let $v \in \mathbf{A}$, $\operatorname{Ort}(v) = (Y_1, Y_2, Y_3) \in \mathbb{R}^3$. The orthogonal projection on Π coincides with the superweight function up to scalar:*

$$\mathrm{pr}_\Pi(v) = (Y_2, Y_3) = Y_2 + iY_3 = \sqrt{2/3}\,\mathrm{swt}(v).$$

Proof. Recall that $\mathrm{swt}(v) = Z_2$. The result follows by relations between Y's and Z's given by two last formulas (41). $\square$

We call $\mathbf{A}_0 = \sum_n \mathbf{A}_{nnn}$ a *diagonal* of the $\mathbb{Z}^3$-gradation, it is also described as follows.

Lemma 14.6. *Let $v \in \mathbf{A}_{X_1X_2X_3}$ and $\mathrm{Ort}(v) = (Y_1, Y_2, Y_3)$. The following conditions are equivalent:*

(1) $X_1 = X_2 = X_3$;
(2) $\mathrm{swt}(v) = 0$;
(3) $\mathrm{pr}_\Pi(v) = 0$;
(4) $Y_2 = Y_3 = 0$.

Proof. By Corollary 13.2, we have three vectors on plane

$$\mathrm{swt}(a_0) = 1, \quad \mathrm{swt}(b_0) = \epsilon = e^{2\pi i/3}, \quad \mathrm{swt}(c_0) = \bar\epsilon = e^{-2\pi i/3},$$

that yield their linear combination:

$$\mathrm{swt}(v) = X_1\,\mathrm{swt}(a_0) + X_2\,\mathrm{swt}(b_0) + X_3\,\mathrm{swt}(c_0) = X_1 + X_2e^{2\pi i/3} + X_3e^{-2\pi i/3}.$$

Put $X = \min\{X_1, X_2, X_3\}$, and $\bar X_i = X_i - X$, $i = 1, 2, 3$, assume that $X = X_3$. Then the condition $\mathrm{swt}(v) = 0$ implies that

$$\mathrm{swt}(v) = \bar X_1 + \bar X_2e^{2\pi i/3} + \bar X_3e^{-2\pi i/3} + X(1 + e^{2\pi i/3} + e^{-2\pi i/3}) = \bar X_1 + \bar X_2e^{2\pi i/3} = 0,$$

hence $\bar X_1 = \bar X_2 = 0$, and $X_1 = X_2 = X_3$. We showed that the first two conditions are equivalent.

Since $\mathrm{swt}(v) = \sqrt{3/2}(Y_2 + iY_3)$, these conditions are also equivalent to $Y_2 = Y_3 = 0$. $\square$

We shall need a technical fact describing an action of the endomorphism τ on homogeneous elements.

Lemma 14.7. *Consider $0 \neq v \in \mathbf{A}_{n_1n_2n_3}$, having standard coordinates $\mathrm{Gr}(v) = (n_1, n_2, n_3)$, $n_1, n_2, n_3 \geq 0$. Let its image under endomorphism τ have coordinates $\mathrm{Gr}(\tau(v)) = (m_1, m_2, m_3)$. Then*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 2n_1 + n_3 \\ n_1 + 2n_2 \\ n_2 + 2n_3 \end{pmatrix} = A^T \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$$

Proof. By assumption, v is a linear combination of products with n_1, n_2, n_3 factors a_0, b_0, c_0, respectively. By the first three relations of Lemma 11.2, $\mathrm{Gr}(a_1) = (2, 1, 0)$, $\mathrm{Gr}(b_1) = (0, 2, 1,)$, $\mathrm{Gr}(c_1) = (1, 0, 2)$. Since $\tau(a_0) = a_1$, $\tau(b_0) = b_1$, $\tau(c_0) = c_1$, we get

$$\begin{aligned}\mathrm{Gr}(\tau(v)) &= n_1 \,\mathrm{Gr}(a_1) + n_2 \,\mathrm{Gr}(b_1) + n_3 \,\mathrm{Gr}(c_1) = (2n_1 + n_3, n_1 + 2n_2, n_2 + 2n_3)\\ &= (n_1, n_2, n_3) \cdot A. \qquad \square\end{aligned}$$

Now we describe the action of the endomorphism τ on homogeneous elements in terms of orthogonal coordinates and the projection on Π.

Lemma 14.8. *Let $0 \neq v \in \mathbf{A}$, $\mathrm{Ort}(v) = (Y_1, Y_2, Y_3)$, and $\mathrm{Ort}(\tau(v)) = (Y_1', Y_2', Y_3')$. Then $Y_1' = 3Y_1$ and τ acts on projections onto the plane Π as a dilation $\sqrt{3}$ times and rotation by $\pi/6$ around the center:*

$$\mathrm{pr}_\Pi(\tau(v)) = \sqrt{3}e^{\pi i/6}\,\mathrm{pr}_\Pi(v), \quad \mathrm{swt}(\tau(v)) = \sqrt{3}e^{\pi i/6}\,\mathrm{swt}(v).$$

Proof. By additivity of the weight functions, it is sufficient to check the action of the endomorphism τ on the generators. Consider e.g. $v = a_0$, we use Corollary 13.2:

$$\begin{aligned}Y_1' &= \mathrm{wt}(\tau(a_0)) = \mathrm{wt}(a_1) = 3 = 3\,\mathrm{wt}(a_0) = 3Y_1;\\ \mathrm{swt}(\tau(a_0)) &= \mathrm{swt}(a_1) = \mu = \sqrt{3}e^{\pi i/6}\,\mathrm{swt}(a_0). \qquad \square\end{aligned}$$

15. Bounds on weights, growth, and elliptic paraboloid for Q

First, let us find bounds on weights of standard monomials.

Lemma 15.1. *We have bounds on weights of standard monomials of $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0)$ (bounds for the heads b_n, c_n are the same):*

$$\begin{aligned}3^n/2 &< \mathrm{wt}(r_{n-2}a_n) \le 3^n, \quad n \ge 0;\\ 3^{n-1}/2 &< \mathrm{wt}(r_{n-2}x_{n-1}a_n) \le 2 \cdot 3^{n-1}, \quad n \ge 1;\\ 3^{n-1} \cdot 5/6 &< \mathrm{wt}(r_{n-3}x_{n-2}^{\alpha}y_{n-2}^{\beta}y_{n-1}a_n) \le 2 \cdot 3^{n-1}, \quad n \ge 1.\end{aligned}$$

Proof. The upper bound of the first inequality is evident. The upper bounds of the remaining inequalities follow from $\mathrm{wt}(x_{n-1}a_n) = 3^n - 3^{n-1} = 2 \cdot 3^{n-1}$. To prove the lower bounds we use an estimate for the tail (28):

$$0 \ge \mathrm{wt}(r_{n-2}) \ge -\sum_{j=0}^{n-2} 3 \cdot 3^j > -\frac{3^{n-1}}{1 - 1/3} = -\frac{3^n}{2}. \qquad \square \tag{45}$$

Corollary 15.2. *Let w be a standard monomial of* $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0)$ *of length* $n \geq 0$. *Then*

$$3^n/6 < \mathrm{wt}(w) \leq 3^n.$$

Corollary 15.3. *Let w be a quasi-standard monomial of length $n \geq 1$. Then* $\mathrm{wt}(w) \geq 2$.

Proof. The lower estimate of Corollary 15.2 applies to quasi-standard monomials as well. In case $n \geq 3$ the result follows. The cases $n = 1, 2$ are checked directly. The lowest values are given by the following examples: $\mathrm{wt}(x_0 a_1) = 3 - 1 = 2$, and $\mathrm{wt}(x_0 y_0 z_0 x_1 a_2) = 9 - 3 - 3 = 3$. $\square$

Lemma 15.4. *Let w be a standard monomial of* $\mathbf{Q}$ *of length n. We have bounds on the superweights:*

$$|\,\mathrm{swt}(w)| = |\,\overline{\mathrm{swt}}(w)| < \theta\sqrt{3^n}, \quad n \geq 0,$$

where θ has values 1.8, 1.4, *and* 2, *respectively, for three types of monomials.*

Proof. Let $\xi_j, \eta_j, \zeta_j \in \{0, 1\}$, where $j \geq 0$, then by definition of the superweight function (claim 2 of Corollary 13.2):

$$\mathrm{swt}(x_j^{\xi_j} y_j^{\eta_j} z_j^{\zeta_j}) = -\mu^j(\xi_j + \epsilon\eta_j + \bar{\epsilon}\zeta_j).$$

By observing respective regular hexagonal, we get the following possible values for the factor above:

$$\{\xi_j + \eta_j e^{2\pi i/3} + \zeta_j e^{-2\pi i/3} \mid \xi_j, \eta_j, \zeta_j \in \{0,1\}\} = \{e^{k\pi i/3} \mid k = 0, \dots, 5\} \cup \{0\}. \tag{46}$$

Thus,

$$\left|\,\mathrm{swt}(x_j^{\xi_j} y_j^{\eta_j} z_j^{\zeta_j})\right| \leq |\mu|^j = (\sqrt{3})^j, \quad j = 0, \dots, n-2. \tag{47}$$

Consider standard monomials listed in Lemma 15.1. Using (47), we evaluate superweights of the tails:

$$|\,\mathrm{swt}(r_{n-2})| \leq \sum_{j=0}^{n-2} (\sqrt{3})^j < \frac{(\sqrt{3})^{n-1}}{\sqrt{3} - 1} = \frac{3 + \sqrt{3}}{6}\sqrt{3^n}.$$

Next, we evaluate superweights of the necks and heads for three types of monomials using Corollary 13.2:

$$\begin{aligned}
|\,\mathrm{swt}(a_n)| &= |\mu^n| = (\sqrt{3})^n;\\
|\,\mathrm{swt}(x_{n-1} a_n)| &= |-\mu^{n-1} + \mu^n| = |\mu^{n-1}(\mu - 1)| = |\mu^{n-1} e^{\pi i/3}| = (\sqrt{3})^{n-1};\\
|\,\mathrm{swt}(y_{n-1} a_n)| &= |-\epsilon\mu^{n-1} + \mu^n| = |\mu^{n-1}(\mu - \epsilon)| = |2\mu^{n-1}| = 2(\sqrt{3})^{n-1}.
\end{aligned}$$

Finally, we get total estimates for our three types of monomials:

$$
\begin{aligned}
|\operatorname{swt}(r_{n-2}a_n)| &\le \frac{3+\sqrt{3}}{6}\sqrt{3^n}+\sqrt{3^n}=\frac{9+\sqrt{3}}{6}\sqrt{3^n}<1.8\sqrt{3^n};\\
|\operatorname{swt}(r_{n-2}x_{n-1}a_n)| &\le \frac{3+\sqrt{3}}{6}\sqrt{3^n}+\frac{\sqrt{3}}{3}\sqrt{3^n}=\frac{3+3\sqrt{3}}{6}\sqrt{3^n}<1.4\sqrt{3^n};\\
|\operatorname{swt}(r_{n-2}y_{n-1}a_n)| &\le \frac{3+\sqrt{3}}{6}\sqrt{3^n}+\frac{2\sqrt{3}}{3}\sqrt{3^n}=\frac{3+5\sqrt{3}}{6}\sqrt{3^n}<2\sqrt{3^n}. \qquad \square
\end{aligned}
$$

Theorem 15.5. *Let* $\mathbf{Q}=\operatorname{Lie}(a_0,b_0,c_0)$. *Then* $\operatorname{GKdim}\mathbf{Q}=\underline{\operatorname{GKdim}}\,\mathbf{Q}=\log_3 8\approx 1.89$.

Proof. Let a natural number m be fixed, set $n=[\log_3 m]$. Consider standard monomials of the first type having the head a_n, of length exactly n: $w=x_0^{\xi_0}y_0^{\eta_0}z_0^{\zeta_0}\cdots x_{n-2}^{\xi_{n-2}}y_{n-2}^{\eta_{n-2}}z_{n-2}^{\zeta_{n-2}}a_n$. By Corollary 15.2, $\operatorname{wt}(w)\le 3^n\le m$. Then number of such monomials gives a lower bound on the growth:

$$\tilde{\gamma}_{\mathbf{Q}}(m)\ge 2^{3(n-1)}\ge 8^{\log_3 m-2}=\frac{1}{64}m^{\log_3 8}.$$

Let a natural number m be fixed, set $n=[\log_3(6m)]$, then $m<3^{n+1}/6$. By Corollary 15.2, all standard monomials of length at least $n+1$ have weights greater than $3^{n+1}/6>m$. Thus, a number of the standard monomials w with $\operatorname{wt}(w)\le m$ is evaluated by a number of all standard monomials of length at most n. We count the number of the standard monomials of the first, second, and third type, of length $j=0,1,\dots,n$. We get an upper bound on the growth:

$$
\begin{aligned}
\tilde{\gamma}_{\mathbf{Q}}(m) &\le 3\Bigg(1+\sum_{j=1}^{n}2^{3(j-1)}+\sum_{j=1}^{n}2^{3(j-1)}+1+\sum_{j=2}^{n}2^{2+3(j-2)}\Bigg)\le C_0 2^{3n}\\
&\le C_0 8^{\log_3(6m)}\le C_1 m^{\log_3 8}. \qquad \square
\end{aligned}
$$

Theorem 15.6. *Let standard monomials* $w\in\mathbf{Q}=\operatorname{Lie}(a_0,b_0,c_0)$ *be drawn in* $\mathbb{R}^3$ *using the multidegree coordinates* $\operatorname{Gr}(w)=(X_1,X_2,X_3)\in\mathbb{Z}^3$. *The respective points are inside an elliptic paraboloid which equations in standard and orthogonal coordinates* $\operatorname{Ort}(w)=(Y_1,Y_2,Y_3)$ *are as follows:*

$$
\begin{aligned}
X_1+X_2+X_3 &> \frac{1}{15}\big(X_1^2+X_2^2+X_3^2-X_1X_2-X_2X_3-X_1X_3\big);\\
Y_1 &> \frac{1}{10\sqrt{3}}\big(Y_2^2+Y_3^2\big).
\end{aligned}
$$

Proof. Let w be a standard monomial of length n. We have bounds on weights by Corollary 15.2 and Lemma 15.4

$$Z_1=\operatorname{wt}(w)>\sigma 3^n,\qquad |Z_2|=|\operatorname{swt}(w)|<\theta\sqrt{3^n},$$

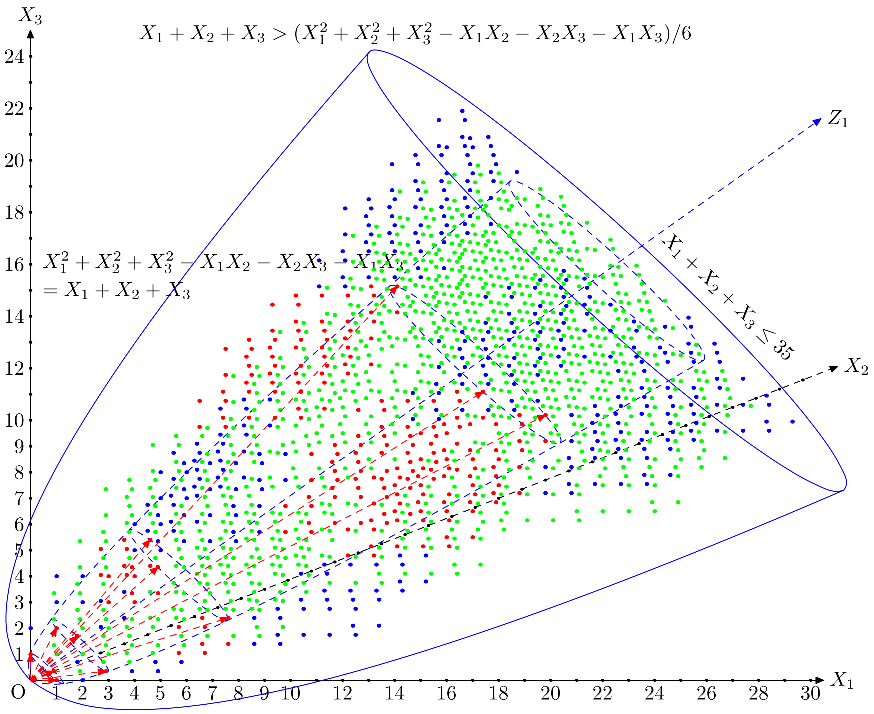

Fig. 2. Standard monomials of **Q** (first type – red, second type – green, third type – blue, pivot elements are marked by arrows) and two elliptic paraboloids cut by plane. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where the constants σ and θ depend on the type. We apply these bounds and transition formulas (41)

$$Y_1 = \frac{Z_1}{\sqrt{3}} > \frac{\sigma 3^n}{\sqrt{3}} > \frac{\sigma}{\sqrt{3}}\left(\frac{|Z_2|}{\theta}\right)^2 = \frac{\sigma}{\sqrt{3}\theta^2}\Big(\mathrm{Re}^2(Z_2) + \mathrm{Im}^2(Z_2)\Big)$$
$$= \frac{3\sigma}{2\sqrt{3}\theta^2}(Y_2^2 + Y_3^2) = \frac{1}{C\sqrt{3}}(Y_2^2 + Y_3^2), \qquad C = \frac{2\theta^2}{3\sigma}.$$

For the first type $\sigma = 1/2$, $\theta = 1.8$ we get $C = 4.32$; for the second type $\sigma = 1/6$, $\theta = 1.4$ and $C = 7.84$; for the third type, $\sigma = 5/18$, $\theta = 2$ and $C = 9.6 < 10$. Thus, the formula in the orthogonal coordinates is established. Now the formula in the standard coordinates follows by (41) and (43). $\square$

It seems that the standard monomials are contained inside a smaller paraboloid, such as one shown on Fig. 2. But a proof of this fact requires more heavy computations.

The next result shows that, asymptotically, a probability that an integer point inside the elliptic paraboloid corresponds to a standard monomial is zero.

Lemma 15.7. *Fix $m > 0$ and let $g(m)$ be a number of all points $u \in \mathbb{Z}^3$ inside the elliptic paraboloid with* $\operatorname{wt}(u) \le m$. *Then*

$$\lim_{m\to+\infty} \frac{\tilde\gamma_{\mathbf{Q}}(m)}{g(m)} = 0.$$

Proof. By Theorem 15.6, the standard monomials $w \in \mathbf{Q}$ with $\operatorname{wt}(w) \le m$ are represented by integer points (in the standard coordinates) inside a cut of an elliptic paraboloid:

$$Y_1 = C\big(Y_2^2 + Y_3^2\big), \quad Y_1 \le m,$$

where C is a constant. We choose a parallelepiped that fits properly inside the elliptic paraboloid:

$$P = \{(Y_1, Y_2, Y_3) \mid m/2 + 2 \le Y_1 \le m - 2, \\ |Y_2| < \sqrt{m/(4C)} - 2,\ |Y_3| < \sqrt{m/(4C)} - 2\}.$$

Consider a lattice point $(X_1, X_2, X_3) \in \mathbb{Z}^3$, it is a center of a unitary cube with vertices $(X_1 \pm 1/2, X_2 \pm 1/2, X_3 \pm 1/2)$ and volume 1. Consider the cubes that intersect P, then they cover P. Since we stepped by 2 which is less than their diagonals $\sqrt{3}$, the cubes lie inside our cut of the elliptic paraboloid. Therefore,

$$g(m) \ge \text{Volume(cubes intersecting } P) \ge \text{Volume}(P) \approx C_1 m^2, \quad m \to \infty.$$

On the other hand, by proof of Theorem 15.5, we have an estimate for the nominator: $\tilde\gamma_{\mathbf{Q}}(m) \le C_2 m^{\log_3 8}$. The result follows. $\square$

Now let us study associative algebras $\mathbf{A} \subset \tilde A$. Recall that $\tilde A$ is spanned by monomials:

$$w = r_{n-2} x_{n-1}^{\xi_{n-1}} y_{n-1}^{\eta_{n-1}} z_{n-1}^{\zeta_{n-1}} a_n^{\alpha_n} b_n^{\beta_n} c_n^{\gamma_n} \cdots a_0^{\alpha_0} b_0^{\beta_0} c_0^{\gamma_0}, \tag{48}$$

which satisfy restrictions of Theorem 12.5. Such a monomial is said of *length n*.

Lemma 15.8. *Let w be a monomial (48) of $\tilde A$ described in Theorem 12.5, $n \ge 0$. We have bounds*

$$\frac{3^{n-1}}{2} < \operatorname{wt}(w) < \frac{3^{n+2}}{2}, \qquad |\operatorname{swt}(w)| \le 4\sqrt{3^n}.$$

Proof. We have an upper bound $\operatorname{wt}(w) \le 3\sum_{j=0}^{n} 3^j < 3^{n+2}/2$. By the senor part property of Theorem 12.5,

$$\operatorname{wt}(x_{n-1}^{\xi_{n-1}} y_{n-1}^{\eta_{n-1}} z_{n-1}^{\zeta_{n-1}} a_n^{\alpha_n} b_n^{\beta_n} c_n^{\gamma_n}) \ge \operatorname{wt}(x_{n-1} a_n) = 2 \cdot 3^{n-1}.$$

We use (45) and get the lower bound on the weight.

Similar to (47) we also have

$$\left|\operatorname{swt}(a_j^{\alpha_j} b_j^{\beta_j} c_j^{\gamma_j})\right| \le |\mu|^j = (\sqrt{3})^j, \quad j = 0, \dots, n.$$

Thus,

$$|\operatorname{swt}(w)| \le 2\sum_{j=0}^{n-1} \sqrt{3^j} + \sqrt{3^n} < \sqrt{3^n}\left(\frac{2}{\sqrt{3}-1} + 1\right) = \sqrt{3^n}(2+\sqrt{3}) < 4\sqrt{3^n}. \qquad \square$$

Theorem 15.9. *Let* $\mathbf{A} = \operatorname{Alg}(a_0, b_0, c_0)$*. Then* $\operatorname{GKdim} \mathbf{A} = \underline{\operatorname{GKdim}}\, \mathbf{A} = 2\log_3 8$.

Proof. Let a natural number m be fixed and set $n = [\log_3(2m)] - 2$. Consider monomials w of type (48) such that $\xi_{n-1} = \eta_{n-1} = \zeta_{n-1} = 0$. By Theorem 12.5, they are linearly independent and belong to $\mathbf{A}$. By Lemma 15.8, $\operatorname{wt}(w) < 3^{n+2}/2 \le m$. There are 7 types of the senior part $a_n^{\alpha_n} b_n^{\beta_n} c_n^{\gamma_n}$, the remaining factors are arbitrary and give a lower bound:

$$\tilde{\gamma}_{\mathbf{A}}(m) \ge 7 \cdot 2^{3(n-1)+3n} \ge 7 \cdot 2^{6\log_3(2m)-21} = C_0 m^{\log_3 64}.$$

Let $m \ge 1$ be fixed, set $n = [\log_3(2m)] + 1$, then $m < 3^n/2$. By the lower bound of Lemma 15.8, every monomial (48) of length at least $n+1$ has weight greater than $3^n/2 > m$. Thus, the number of monomials $w \in \tilde{A}$ of type (48) with $\operatorname{wt}(w) \le m$ is evaluated by the number of all such monomials of length at most n. There are 7 types of the senior part, the remaining factors for a monomial of length j are arbitrary and give an upper bound:

$$\tilde{\gamma}_{\mathbf{A}}(m) \le 7\sum_{j=0}^{n} 2^{6j} < 7\frac{2^{6(n+1)}}{64-1} \le \frac{7}{63} 2^{6\log_3(2m)+12} = C_1 m^{\log_3 64}. \qquad \square$$

Theorem 15.10. *Lattice points in* $\mathbb{R}^3$ *for monomials of* $\mathbf{A} = \operatorname{Alg}(a_0, b_0, c_0)$ *are inside an elliptic paraboloid which equations in standard and orthogonal coordinates are as follows:*

$$X_1 + X_2 + X_3 > \frac{1}{96}\left(X_1^2 + X_2^2 + X_3^2 - X_1X_2 - X_2X_3 - X_1X_3\right);$$
$$Y_1 > \frac{1}{64\sqrt{3}}\left(Y_2^2 + Y_3^2\right).$$

Proof. Let w be a monomial (48) of length n. By Lemma 15.8, we have bounds

$$Z_1 = \operatorname{wt}(w) > 3^n/6, \quad |Z_2| = |\operatorname{swt}(w)| < 4\sqrt{3^n}.$$

We apply these bounds and transition formulas (41)

$$Y_1 = \frac{Z_1}{\sqrt{3}} > \frac{3^n}{6\sqrt{3}} > \frac{1}{6\sqrt{3}}\left(\frac{|Z_2|}{4}\right)^2 = \frac{1}{96\sqrt{3}}\Big(\operatorname{Re}^2(Z_2) + \operatorname{Im}^2(Z_2)\Big) = \frac{1}{64\sqrt{3}}(Y_2^2 + Y_3^2).$$

The formula in the standard coordinates follows by (41) and (43). $\square$

16. Generating functions, nillity of Q

In this section we describe the generating function of $\mathbf{Q}$. The computation of its initial coefficients showed an interesting pattern: the coefficients are at most one and the diagonal coefficients are zero. These observations are true in general, we show below that the components of $\mathbb{Z}^3$-grading of $\mathbf{Q}$ are at most one-dimensional (Theorem 17.1), so $\mathbb{Z}^3$-grading of $\mathbf{Q}$ is fine. Also, the diagonal components $\mathbf{Q}_{nnn}$, $n \geq 0$, are empty (Lemma 17.2).

Let $A = \oplus_{n,m,k} A_{nmk}$ be a $\mathbb{Z}^3$-graded algebra, one has a induced $\mathbb{Z}$-gradation: $A = \oplus_n A_n$, where $A_l = \mathop{\oplus}\limits_{n+m+k=l} A_{nmk}$. Define respective *generating functions*:

$$\mathcal{H}(A, t_1, t_2, t_3) = \sum_{n,m,k} \dim A_{nmk} t_1^n t_2^m t_3^k;$$
$$\mathcal{H}(A, t) = \sum_n \dim A_n t^n = \mathcal{H}(A, t, t, t).$$

By T_n^k denote the set of all standard monomials of length n, $n \geq 0$, and type $k \in \{1, 2, 3\}$, described in Theorem 12.1.

Lemma 16.1. *Let* $\operatorname{char} K \neq 2$, $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$, *and* T_n^k *the set of standard monomials of length* $n \geq 1$ *and type* $k \in \{1, 2, 3\}$, *except* T_1^3. *Then*

(1) *we have a bijection:*

$$T_{n+1}^k = \{x_0^{\alpha_0} y_0^{\beta_0} z_0^{\gamma_0} \mid \alpha_0, \beta_0, \gamma_0 \in \{0, 1\}\} \cdot \tau(T_n^k);$$

(2) *we have a recurrence relation between generating functions:*

$$\mathcal{H}(T_{n+1}^k, t_1, t_2, t_3) = (1 + t_1^{-1})(1 + t_2^{-1})(1 + t_3^{-1}) \mathcal{H}(T_n^k, t_1^2 t_2, t_2^2 t_3, t_3^2 t_1).$$

Proof. The first claim follows by structure of the standard monomials. To prove the second claim we proceed as in proof of Theorem 9.1 or its Corollary 9.2. □

Computer calculations yield the following series.

$$\begin{aligned}\mathcal{H}(\mathbf{Q}, t_1, t_2, t_3) = {} & t_1 + t_2 + t_3 + t_1^2 + t_1 t_2 + t_1 t_3 + t_2^2 + t_2 t_3 + t_3^2 + t_1^2 t_2 + t_1^2 t_3 + t_1 t_2^2 \\ & + t_1 t_3^2 + t_2^2 t_3 + t_2 t_3^2 + t_1^3 t_2 + t_1^3 t_3 + t_1^2 t_2^2 + t_1^2 t_2 t_3 + t_1^2 t_3^2 + t_1 t_2^3 + t_1 t_2^2 t_3 \\ & + t_1 t_2 t_3^2 + t_1 t_3^3 + t_2^3 t_3 + t_2^2 t_3^2 + t_2 t_3^3 + t_1^4 t_2 + t_1^3 t_2^2 + t_1^3 t_2 t_3 + t_1^3 t_3^2 + t_1^2 t_2^3 \\ & + t_1^2 t_2^2 t_3 + t_1^2 t_2 t_3^2 + t_1^2 t_3^3 + t_1 t_2^3 t_3 + t_1 t_2^2 t_3^2 + t_1 t_2 t_3^3 + t_1 t_3^4 + t_2^4 t_3 + t_2^3 t_3^2 \\ & + t_2^2 t_3^3 + t_1^4 t_2^2 + t_1^4 t_2 t_3 + t_1^3 t_2^3 + t_1^3 t_2 t_3^2 + t_1^3 t_3^3 + t_1^2 t_2^3 t_3 + t_1^2 t_3^4 + t_1 t_2^4 t_3 \\ & + t_1 t_2^2 t_3^3 + t_1 t_2 t_3^4 + t_2^4 t_3^2 + t_2^3 t_3^3 + \ldots;\end{aligned}$$

$$\begin{aligned}\mathcal{H}(\mathbf{Q},t) = 3(&t + 2t^2 + 2t^3 + 4t^4 + 5t^5 + 4t^6 + 6t^7 + 6t^8 + 6t^9 + 12t^{10} + 12t^{11}\\ &+ 9t^{12} + 15t^{13} + 15t^{14} + 9t^{15} + 12t^{16} + 12t^{17} + 10t^{18} + 18t^{19} + 18t^{20} + 12t^{21}\\ &+ 18t^{22} + 18t^{23} + 12t^{24} + 18t^{25} + 18t^{26} + 18t^{27} + 36t^{28} + 36t^{29} + 24t^{30}\\ &+ 36t^{31} + \cdots).\end{aligned}$$

Theorem 16.2. *Let* $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0) = \mathbf{Q}_{\bar 0} \oplus \mathbf{Q}_{\bar 1}$. *For any* $a \in \mathbf{Q}_{\bar n}$, $\bar n \in \{\bar 0, \bar 1\}$, *the operator* $\mathrm{ad}(a)$ *is nilpotent.*

Proof. The same as that of Theorem 10.1. □

17. Properties of $\mathbb{Z}^3$-grading of Q

We start with an observation on the $\mathbb{Z}^3$-grading of $\mathbf{Q}$. The computed above coefficients of $\mathcal{H}(\mathbf{Q}, t_1, t_2, t_3)$ are 0 and 1. Below, we establish this fact for the whole series. In particular, $\mathbb{Z}^3$-grading of $\mathbf{Q}$ is *fine*, i.e. it cannot be split by taking a bigger grading group (see a definition in [4,10]).

Theorem 17.1. *The components of the multidegree* $\mathbb{Z}^3$*-grading of* $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0)$ *are at most one-dimensional.*

Proof. By way of contradiction assume that u, v are different standard monomials and $\mathrm{Gr}(u) = \mathrm{Gr}(v) = (X_1, X_2, X_3)$. Assume that u has length 0. If v has also length zero, then $u, v \in \{a_0, b_0, c_0\}$ and $u = v$. Let v have a nonzero length, by Corollary 15.3, $\mathrm{wt}(v) \geq 2$, a contradiction with $\mathrm{wt}(u) = 1$.

Thus, we can assume that u, v have nonzero length and our example is minimal, namely, the minimum of lengths of u, v is minimal among such examples. The first claim of Lemma 16.1 yields bijections, we need a weaker observation, by structure of the standard monomials (Theorem 12.1) we have

$$u = x_0^{\alpha_0} y_0^{\beta_0} z_0^{\gamma_0} \tau(\tilde u), \qquad v = x_0^{\alpha_0'} y_0^{\beta_0'} z_0^{\gamma_0'} \tau(\tilde v), \qquad \alpha_0, \beta_0, \gamma_0, \alpha_0', \beta_0', \gamma_0' \in \{0, 1\},$$

where $\tilde u$, $\tilde v$ are standard monomials, which lengths are equal to lengths of u, v minus one, respectively.

Let $\mathrm{Gr}(\tilde u) = (n_1, n_2, n_3)$ and $\mathrm{Gr}(\tilde v) = (m_1, m_2, m_3)$. Using Lemma 14.7, we get a relation for the multidegree coordinates:

$$A^T \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} - \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} = A^T \cdot \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} - \begin{pmatrix} \alpha_0' \\ \beta_0' \\ \gamma_0' \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

We use Lemma 13.1

$$\begin{pmatrix} n_1 - m_1 \\ n_2 - m_2 \\ n_3 - m_3 \end{pmatrix} = (A^T)^{-1} \cdot \begin{pmatrix} \alpha_0 - \alpha_0' \\ \beta_0 - \beta_0' \\ \gamma_0 - \gamma_0' \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 4 & 1 & -2 \\ -2 & 4 & 1 \\ 1 & -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix},$$

where $\delta_i \in \{-1, 0, 1\}$ for all $i = 1, 2, 3$. One directly checks that such nonzero vector $(\delta_1, \delta_2, \delta_3)$ cannot yield a nonzero integer vector on the left hand side.

Therefore, $\delta_1 = \delta_2 = \delta_3 = 0$, and by the formula above, we have $(\alpha_0, \beta_0, \gamma_0) = (\alpha_0', \beta_0', \gamma_0')$ and $(n_1, n_2, n_3) = (m_1, m_2, m_3)$, i.e., $\mathrm{Gr}(\tilde{u}) = \mathrm{Gr}(\tilde{v})$. By minimality of the example, we have $\tilde{u} = \tilde{v}$. Hence, $u = v$, this contradiction proves the result. □

The series above has trivial diagonal coefficients, this is true in general. We show that the diagonal (see Lemma 14.6) $\mathbf{Q}_0 = \oplus_{n \geq 0} \mathbf{Q}_{nnn}$ of the $\mathbb{Z}^3$-grading is trivial.

Lemma 17.2. *Let* $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0)$. *Then* $\dim \mathbf{Q}_{nnn} = 0$, $n \geq 0$.

Proof. Easy check shows that the standard monomials of lengths 0, 1 do not belong to the diagonal (the multidegrees of the monomials of length 1 follow from relations of Lemma 11.1, where $i = 0$).

Let w be a standard monomial with $w \in \mathbf{Q}_{NNN}$, we assume that w has a minimal length among monomials belonging to the diagonal. Then w is of length at least two. By structure of the standard monomials (Theorem 12.1) we have

$$w = x_0^{\alpha_0} y_0^{\beta_0} z_0^{\gamma_0} \tau(\tilde{w}), \quad \alpha_0, \beta_0, \gamma_0 \in \{0, 1\},$$

where $\tilde{w}$ is a standard monomial of smaller length. Let $\mathrm{Gr}(\tilde{w}) = (n_1, n_2, n_3) \in \mathbb{Z}^3$. Applying Lemma 14.7,

$$A^T \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} - \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} = \mathrm{Gr}^T(w) = \begin{pmatrix} N \\ N \\ N \end{pmatrix}, \quad \alpha_0, \beta_0, \gamma_0 \in \{0, 1\}.$$

We are going to prove that this is impossible provided that some n_1, n_2, n_3 are different.

Due to the cyclic symmetry assume that $n_3 = \min\{n_1, n_2, n_3\}$. Put $m_1 = n_1 - n_3$, $m_2 = n_2 - n_3$, $m_3 = 0$. Then $m_1, m_2 \geq 0$, using (35) we get:

$$A^T \cdot \begin{pmatrix} m_1 \\ m_2 \\ 0 \end{pmatrix} - \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} N' \\ N' \\ N' \end{pmatrix},$$

where $N' = N - 3n_3$. Now use the formula for A (35):

$$\begin{aligned} 2m_1 - \alpha_0 &= N', \\ m_1 + 2m_2 - \beta_0 &= N', \\ m_2 - \gamma_0 &= N'. \end{aligned}$$

Subtracting the third equality from the second one and using that $\alpha_0, \beta_0, \gamma_0 \in \{0,1\}$, we get:

$$0 = |(m_1 + 2m_2 - \beta_0) - (m_2 - \gamma_0)| \geq m_1 + m_2 - |\beta_0 - \gamma_0| \geq m_1 + m_2 - 1.$$

Therefore, $m_1 + m_2 \leq 1$. Consider the case: $m_1 = 1$, $m_2 = 0$. A difference of the first and third equalities yields a contradiction:

$$0 = |(2m_1 - \alpha_0) - (m_2 - \gamma_0)| = |2 - (\alpha_0 - \gamma_0)| \geq 2 - |\alpha_0 - \gamma_0| \geq 1.$$

Consider the case: $m_1 = 0$, $m_2 = 1$. A difference of the second and first equalities yields a contradiction:

$$0 = |(m_1 + 2m_2 - \beta_0) - (2m_1 - \alpha_0)| = |2 - (\beta_0 - \alpha_0)| \geq 2 - |\beta_0 - \alpha_0| \geq 1.$$

Consider the remaining case: $m_1 = m_2 = 0$. Hence, $n_1 = n_2 = n_3$, and $\tilde{w}$ belongs to the diagonal, a contradiction with minimality of w. $\square$

Observe that, in contrast to the result above, the associative hull $\mathbf{A} = \mathrm{Alg}(a_0, b_0, c_0)$ has a nontrivial diagonal, because it contains the following elements

$$\mathbf{A}_0 \supset \langle x_0^{\kappa_0} y_0^{\kappa_0} z_0^{\kappa_0} \cdots x_{n-2}^{\kappa_{n-2}} y_{n-2}^{\kappa_{n-2}} z_{n-2}^{\kappa_{n-2}} a_n b_n c_n a_{n-1}^{\delta_{n-1}} b_{n-1}^{\delta_{n-1}} c_{n-1}^{\delta_{n-1}} \\ \cdots a_0^{\delta_0} b_0^{\delta_0} c_0^{\delta_0} \mid \kappa_j, \delta_j \in \{0,1\}\rangle_K.$$

18. $\mathbb{Z}^2$-grading of Q

We have the multidegree vectors of the generators: $\mathrm{Gr}(a_0) = (1,0,0)$, $\mathrm{Gr}(b_0) = (0,1,0)$, $\mathrm{Gr}(c_0) = (0,0,1)$, they generate a lattice:

$$\Gamma = \langle \mathrm{Gr}(a_0), \mathrm{Gr}(b_0), \mathrm{Gr}(c_0)\rangle_{\mathbb{Z}} = \{(n_1, n_2, n_3) \mid n_i \in \mathbb{Z}\} = \mathbb{Z}^3 \subset \mathbb{R}^3.$$

Consider a projection of Γ on plane Π (see its definition before Lemma 14.5, where we also identify Π with $\mathbb{C}$). For convenience, instead of the projection, we shall use $\mathrm{swt}(P) = \sqrt{3/2}\,\mathrm{pr}_{\Pi}(P)$, $P \in \mathbb{R}^3$ (see Lemma 14.5). We get a lattice generated by images of the generators:

$$\mathrm{swt}(a_0) = 1, \quad \mathrm{swt}(b_0) = \epsilon = e^{2\pi i/3}, \quad \mathrm{swt}(c_0) = \bar{\epsilon} = e^{-2\pi i/3}.$$

We obtain a lattice $\bar{\Gamma} = \mathrm{swt}(\Gamma) = \langle 1, \epsilon, \bar{\epsilon}\rangle_{\mathbb{Z}} \subset \mathbb{C} = \Pi$, isomorphic to $\mathbb{Z}^2$ (a basis is given by any pair of these vectors). Points of $\bar{\Gamma}$ are naturally represented by points of a triangular grid (see Fig. 3). For example, $3 \in \bar{\Gamma}$ is marked there by z_2c_3, this means that z_2c_3 is projected on that point, namely, $\mathrm{swt}(z_2c_3) = 3$.

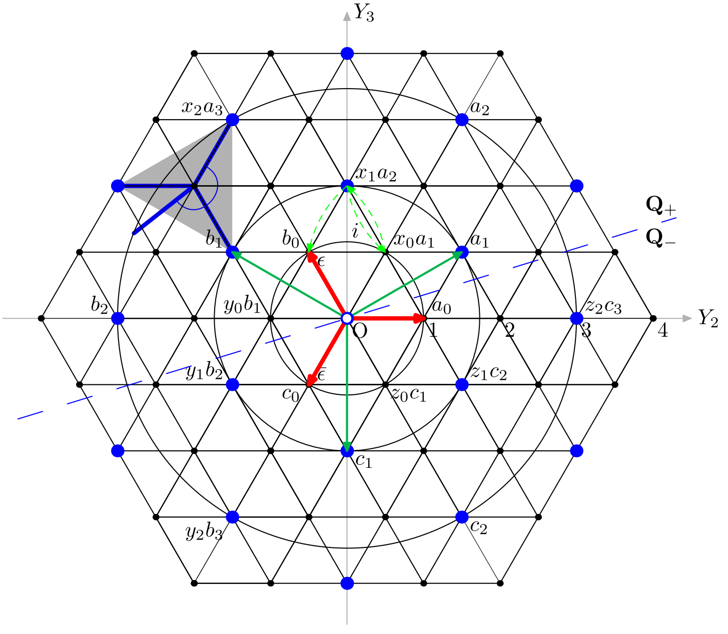

Fig. 3. Components of projection of $\mathbf{Q}$ on plane Π: $Y_1 = 0$; $\mathbb{Z}^2$-grading of $\mathbf{Q}$.

By additivity of the function $\operatorname{swt}(*)$, the projection yields a $\mathbb{Z}^2$-gradation of $\mathbf{Q}$ and $\mathbf{A}$. We have the following interesting property.

Theorem 18.1. *The Lie superalgebra* $\mathbf{Q} = \operatorname{Lie}(a_0, b_0, c_0)$ *is* $\mathbb{Z}^2$*-graded so that*

$$\dim \mathbf{Q}_{nm} = \begin{cases} 0, & (n,m) = (0,0); \\ \infty, & (n,m) \neq (0,0). \end{cases}$$

Proof. Triviality of the zero component follows from Lemma 17.2 and Lemma 14.6.

We shall prove that any nonzero component contains infinitely many standard monomials of the second type. Let v be a standard monomial of the second type. (Below we are dealing with standard monomials of the second type only.) By structure of such monomials (Theorem 12.1), all elements

$$\{x_0^{\alpha_0} y_0^{\beta_0} z_0^{\gamma_0} \tau(v) \mid \alpha_0, \beta_0, \gamma_0 \in \{0,1\}\} \tag{49}$$

are again standard monomials of the second type. Consider the induced action of τ on lattice $\bar{\Gamma}$, its image is spanned by images of the generators:

$$\tau(\bar{\Gamma}) = \langle \operatorname{swt}(\tau(a_0)), \operatorname{swt}(\tau(b_0)), \operatorname{swt}(\tau(c_0)) \rangle_{\mathbb{Z}} = \langle \operatorname{swt}(a_1), \operatorname{swt}(b_1), \operatorname{swt}(c_1) \rangle_{\mathbb{Z}}.$$

On Fig. 3, points of $\tau(\bar{\Gamma})$ are marked by thick points. By Lemma 14.8, τ acts on projections on plane Π as a multiplication by $\lambda = \sqrt{3} e^{\pi i/6}$.

As we remarked above (see (46)),

$$\begin{aligned}
&\operatorname{swt}(\{x_0^{\alpha_0} y_0^{\beta_0} z_0^{\gamma_0} \mid \alpha_0, \beta_0, \gamma_0 \in \{0,1\}\}) \\
&\quad = -\{\alpha_0 + \beta_0 e^{2\pi i/3} + \gamma_0 e^{-2\pi i/3} \mid \alpha_0, \beta_0, \gamma_0 \in \{0,1\}\} \\
&\quad = \{e^{k\pi i/3} \mid k = 0, \dots, 5\} \cup \{0\}.
\end{aligned}$$

Thus, the first factor in (49) yields a step of length one (or none) on the triangular grid (see Fig. 3).

As a base of induction, let us prove that all six points on a circle of radius one have infinite dimensional components. Put $w_0 = x_0 a_1$, $w_{n+1} = y_0 \tau(w_n)$, $n \ge 0$, they belong to the same component. Indeed, $\tau(x_0 a_1) = x_1 a_2$ and the factor $\operatorname{swt}(y_0) = -\operatorname{swt}(b_0)$ returns back to $\operatorname{swt}(x_0 a_1)$. These movements are shown by dashed arrows on the picture. We obtain an infinite dimensional component: $\operatorname{swt}(w_n) = \operatorname{swt}(x_0 a_1)$, $n \ge 0$. Put $w'_{n+1} = x_0 y_0 \tau(w_n)$, $n \ge 0$, yielding another infinite dimensional component: $\operatorname{swt}(w'_{n+1}) = \operatorname{swt}(\tau(w_n)) + \operatorname{swt}(x_0) + \operatorname{swt}(y_0) = \operatorname{swt}(x_1 a_2) - \operatorname{swt}(a_0) - \operatorname{swt}(b_0) = \operatorname{swt}(b_0)$ for all $n \ge 0$. Thus, the components of $x_0 a_1$ and b_0 are infinite dimensional. Applying the automorphisms θ, θ^2, we conclude that the components corresponding to all six lattice points on the unitary circle are infinite dimensional.

Let us make a geometric observation. Points of the sublattice $\tau(\bar\Gamma) \subset \bar\Gamma$ are drawn thick on the picture. Observe the following property: any non-thick point $P \in \bar\Gamma \setminus \tau(\bar\Gamma)$ has three adjacent thick points $S_1, S_2, S_3 \in \bar\tau(\Gamma)$ with $|P - S_i| = 1$, $i = 1, 2, 3$, situated in vertices of an equilateral triangle, P being the center of the triangle (see a shaded triangle on the picture as an example). Moreover, let a radius $R \ge 1$ be fixed and assume that $|P| \le R$, then at least one vertex satisfies the same condition $|S_i| \le R$. Indeed, let $\{h \in \mathbb{C} \mid |h| = R\} \cap \{h \in \mathbb{C} \mid |h - P| = 1\} = K_1 \cup K_2$, where $K_1 = K_2$ is possible. (If the intersection is empty then all S_1, S_2, S_3 will do.) Consider an angle $\measuredangle K_1 P K_2$ containing O, geometric observations show that $\measuredangle K_1 P K_2 \ge 2\pi/3$. (This angle is shown on the picture, where we also have $R = 3$.) (The extreme case is as follows: $R = 1$, $|P| = 1$, then we get the angle $2\pi/3$, one can find this situation on the picture. Otherwise the angle is bigger. For example, if we keep the point $|P| = 1$ and increase R then the angle increases.) Therefore, the angle contains at least one desired point S_i (see the shaded triangle on the picture).

Now we are ready to prove an inductive step. We assume that all lattice points $P \in \bar\Gamma$ with $|P| \in [1, R]$, where $R \ge 1$, have infinite dimensional components and prove that all lattice points with $|P'| \in [1, \sqrt{3}R]$ also have infinite dimensional components. Indeed, the endomorphism τ yields that all thick points $S \in \tau(\bar\Gamma)$, $|S| \le \sqrt{3}R$, have infinite dimensional components. Take a non-thick point: $P \in \bar\Gamma \setminus \tau(\bar\Gamma)$, $|P| \le \sqrt{3}R$. By the geometric observation above, there exists a thick point: $S \in \tau(\bar\Gamma)$, $|S| \le \sqrt{3}R$, $|P - S| = 1$. Now we choose an appropriate first factor in (49) that makes the step on the triangular grid, in order to move the infinite dimensional component of S into the point P. The inductive step is proved. $\square$

Theorem 18.2. *Let* $\mathbf{Q} = \mathrm{Lie}(a_0, b_0, c_0)$.

(1) *There exists a decomposition into a direct sum of two locally nilpotent subalgebras:* $\mathbf{Q} = \mathbf{Q}_+ \oplus \mathbf{Q}_-$*;*
(2) *There is a continuum of such different decompositions.*

Proof. Pick any line on plane Π passing through zero and not intersecting any point of the triangular grid except zero. Let $\mathbf{Q}_+$, $\mathbf{Q}_-$ be sums of homogeneous components that lie on different sides of the line. Since, $\mathbf{Q}_{00} = \{0\}$, we get a direct decomposition $\mathbf{Q} = \mathbf{Q}_+ \oplus \mathbf{Q}_-$. A position of a homogeneous monomial v on Π is given by the superweight function $\mathrm{swt}(v)$, additivity of which implies that $\mathbf{Q}_+$ and $\mathbf{Q}_-$ are subalgebras. The line also yields a plane in space $\mathbb{R}^3$ passing through the diagonal line $X_1 = X_2 = X_3$. This plane splits the elliptic hyperboloid (Theorem 15.6) into two halves. Now the same geometric arguments as in [36] prove that $\mathbf{Q}_+$, $\mathbf{Q}_-$ are locally nilpotent.

Points of the triangular grid yield a countable set of directions that the line should avoid. There is a continuum of remaining directions. Different lines yield different decompositions because directions to the grid points are dense among all directions. □

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