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Stability of a fourth order bi-parametric family of iterative methods[☆]

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ABSTRACT

In this paper we present a dynamical study of the Ostrowski–Chun family of iterative methods on quadratic polynomials. We will use dynamical tools such as the analysis of fixed and critical points, and the calculation of parameter planes, to find the most stable members of the family. These results have been checked on the unidimensional Bratu's problem.

1. Introduction

The dynamical analysis of a method is becoming a trend in recent publications on iterative methods because it allows us to classify the various iterative formulas, not only from the point of view of its order of convergence, but also analyzing how these formulas behave as a function of the initial estimate that is taken. It also provides valuable information about the stability and reliability of the iterative method. In this sense, Varona in [1] and Amat in [2], among others, described the dynamical behavior of several well-known families of iterative methods. More recently, in [3–7], the authors analyze, under the point of view of complex dynamics, the quantitative behavior of different known iterative methods and families, such as King's or Chebyshev–Halley's. When this kind of study is made, different pathological numerical behaviors appear, such as periodic orbits, attracting fixed points different from the solutions of the problem, etc. A very useful tool to understand the behavior of the different members of a family is the parameter plane, that helps to select the most stable members of the class.

In this paper we study the dynamical behavior of the Ostrowski–Chun (OC) family of methods introduced in [8], a bi-parametric class of iterative predictor–corrector schemes with order of convergence 4 for solving nonlinear equations or

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systems. We will study the dynamics of OC family of methods on a generic quadratic polynomial, analyzing the fixed and critical points of the associated rational operator. With this analysis we will select the members of the family with good numerical properties. Finally we will perform numerical tests to confirm the qualitative results.

Given a rational function $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0\in\hat{\mathbb{C}}$ is defined as the sequence of points:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

A z^* is called a fixed point of an operator R if $z_0 = R(z_0)$. The roots of an equation $f(x) = 0$ are fixed points of the operator associated with the iterative method, but fixed points not corresponding to any root of $f(x) = 0$ may appear. These points are called *strange fixed points*. We can classify fixed points according to the absolute value of the derivated operator on them, so, a fixed point z^* can be:

- Attractor, if $|R'(z^*)| < 1$;
- Superattractor, if $|R'(z^*)| = 0$;
- Repulsor, if $|R'(z^*)| > 1$; and
- Parabolic or indifferent, if $|R'(z^*)| = 1$.

The basin of attraction of an attractor z is the set of preimages of any order: $A(z) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z, n \to \infty\}$. The Fatou set of the rational function R, $F(R)$, is the set of points $z\in\hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the Julia set, $J(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The presence of attractive strange fixed points can show a dangerous behaviour of the method, as in this case the scheme can converge to a point that is not a solution of the problem to be solved.

The solutions of $R'(z) = 0$ are called critical points of R. A classical result establishes that there is at least one critical point associated with each immediate invariant Fatou component. A detailed reference of dynamical concepts is given by Devaney in [9].

Two known optimal fourth-order methods are Ostrowski' [10] and Chun's [11] schemes, with iterative expressions

$$x_{k+1} = y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)}\frac{f(y_k)}{f'(x_k)},$$
$$x_{k+1} = y_k - \frac{f(x_k) + 2f(y_k)}{f(x_k)}\frac{f(y_k)}{f'(x_k)},$$

respectively, where y_k is Newton's step.

We can combine these methods to design a new family of iterative methods. In [8] using Newton's method as predictor we constructed a combination of Ostrowski' and Chun's schemes in the form

$$\begin{aligned} y_k &= x_k - \alpha\frac{f(x_k)}{f'(x_k)},\\ x_{k+1} &= y_k - \left[\frac{f(x_k)}{a_1 f(x_k) + a_2 f(y_k)} + \frac{b_1 f(x_k) + b_2 f(y_k)}{f(x_k)}\right]\frac{f(y_k)}{f'(x_k)}, \end{aligned} \tag{1}$$

where α, a_1, a_2, b_1 and $b_2\in\mathbb{R}$. We found their values so that the order of convergence is at least 4.

Theorem 1 ([8]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function at each point of the open interval I such that $\xi \in I$ is a simple root of the nonlinear equation $f(x) = 0$. Then, the sequence $\{x_k\}_{k\geq 0}$ obtained using expression (1) converges to ξ and the local order of convergence is at least four if $\alpha = 1$, $a_2 = a_1^2(b_2 - 2)$, $b_1 = 1 - \frac{1}{a_1}$ and for all a_1 and $b_2 \in \mathbb{R}$, with $a_1 \neq 0$.*

Then, the iterative formula obtained from Ostrowski–Chun's method (OC) is

$$z_{k+1} = y_k - \frac{1}{a_1}\left[\frac{f(z_k)}{f(z_k) + a_1(b_2-2)f(y_k)} + \frac{(a_1-1)f(z_k) + a_1 b_2 f(y_k)}{f(z_k)}\right]\frac{f(y_k)}{f'(z_k)}, \tag{2}$$

that defines a two-parameter family of optimal methods with order of convergence four. We can define the operator $\bar{K_f}$ corresponding to OC class.

$$\bar{K_f} = y - \frac{1}{a_1}\left[\frac{f(z)}{f(z) + a_1(b_2-2)f(y)} + \frac{(a_1-1)f(z) + a_1 b_2 f(y)}{f(z)}\right]\frac{f(y)}{f'(z)} \tag{3}$$

being $y = z - \frac{f(z)}{f'(z)}$ where $z \in \hat{\mathbb{C}}$.

2. Scaling Theorem and analysis of the fixed and critical points

The *Scaling Theorem* allows to match the dynamical behavior of an operator with another operator, related by an affine transformation.

Theorem 2 (*Scaling Theorem*). *Let $f(z)$ be an analytic function in the Riemann sphere and $T(z) = \alpha z + \beta$ an affine map. If $g(z) = (f \circ T)(z)$, then $(T \circ \bar{K}_g \circ T^{-1}) = \bar{K}_f(z)$ for all z, i.e., $\bar{K}_f$ is analytically conjugate to $\bar{K}_g$ through T.*

Proof. We need to prove that

$$(T \circ \bar{K}_g \circ T^{-1})(z) = \bar{K}_f(z),$$

that is

$$(T \circ \bar{K}_g)(z) = (\bar{K}_f \circ T)(z). \tag{4}$$

The left side of Eq. (4) gives

$$(T \circ \bar{K}_g)(z) = \alpha z - \alpha \frac{g(z)}{g'(z)} - \frac{\alpha}{a_1}\left(\frac{g(z)}{g(z) + a_1(b_2 - 1)g(y)} + \frac{(a_1 - 1)g(z) + a_1 b_2 g(y)}{g(z)}\right)\frac{g(y)}{g'(z)} + \beta.$$

To develop the right side we will use that from the definition of g we have $g(z) = (f \circ T)(z) = f(\alpha z + \beta)$. By applying the chain rule we can write $g'(z) = f'(T(z))T'(z) = f'(T(z))\alpha$. Besides $y = N_f(z) = z - \frac{f(z)}{f'(z)}$ is Newton's iteration. In [2] it can be seen that Newton's method meets the Scaling Theorem, so

$$f(N_f \circ T)(z) = f(T \circ N_g)(z) = g(y)$$

thus, the right side of (4) is

$$(\bar{K}_f \circ T)(z) = \alpha z - \alpha \frac{g(z)}{g'(z)} + \beta - \frac{1}{a_1}\left(\frac{g(z)}{g(z) + a_1(b_2 - 1)g(y)} + \frac{(a_1 - 1)g(z) + a_1 b_2 g(y)}{g(z)}\right)\alpha \frac{g(y)}{g'(z)}$$

that coincides with the left side of (4), so the theorem is proved. □

On the other hand, any quadratic polynomial with simple roots is conjugate to $z^2 + c$, as we establish in the following result:

Theorem 3. *Let be $p(z) = a_1 z^2 + a_2 z + a_3$, with $a_1 \neq 0$ a generic quadratic polynomial with simple roots, then through a coordinate affine map, $p(z)$ reduces to $q(z) = z^2 + c$, where $c = 4a_1 a_3 - a_2^2$. This affine transformation provides a conjugation between $\bar{K}_p$ and $\bar{K}_q$.*

Theorems 2 and 3 state that a quadratic polynomial can be transformed using an affine map with no qualitative changes on the dynamical behavior of the associated operator, so to study its dynamical behavior we can use the quadratic polynomial $p(z) = z^2 + c$ without loss of generality.

Applying operator (3) on $p(z)$ we get the rational function:

$$\begin{aligned} K_{a_1,b_2}(c,z) = {} & \frac{1}{32z^5\left(4z^2 - a_1(b_2 - 2)\left(c - z^2\right)\right)}\Big[8z^2\left(c^3 - 5c^2z^2 + 15cz^4 + 5z^6\right) \\ & - a_1\left(c - z^2\right)\left(b_2^2\left(c - z^2\right)^3 - 2b_2\left(c^3 - c^2z^2 - 9cz^4 - 7z^6\right) + 8z^2\left(c^2 - 6cz^2 - 3z^4\right)\right)\Big], \end{aligned} \tag{5}$$

which depends on c. By considering the conjugacy map:

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}, \tag{6}$$

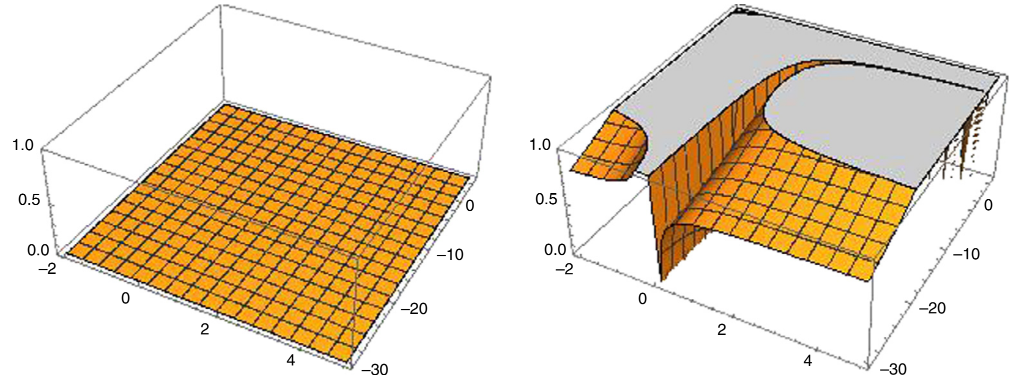
known as *Möbius transformation*, with properties:

$$\text{(i) } h(i\sqrt{c}) = 0, \qquad \text{(ii) } h(-i\sqrt{c}) = \infty, \qquad \text{(iii) } h(\infty) = 1$$

and applying (6) to the operator (5) we get $O_{a_1,b_2}(z) = \left(h \circ K_{a_1,b_2} \circ h^{-1}\right)$, where

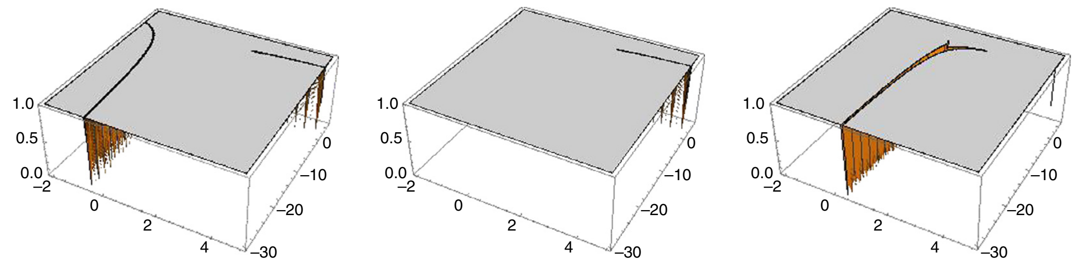
$$O_{a_1,b_2}(z) = \frac{-z^4\left((z+1)^2\left(z^2 + 4z + 5\right) - a_1\left(b_2^2 - b_2\left(z^3 + 4z^2 + 5z + 4\right) + 2(z+1)^2(z+2)\right)\right)}{z^4\left(a_1(b_2 - 2)^2 - 5\right) + z^3(-5a_1(b_2 - 2) - 14) - 2z^2(2a_1(b_2 - 2) + 7) - z(a_1(b_2 - 2) + 6) - 1}$$

which no longer depends on c.

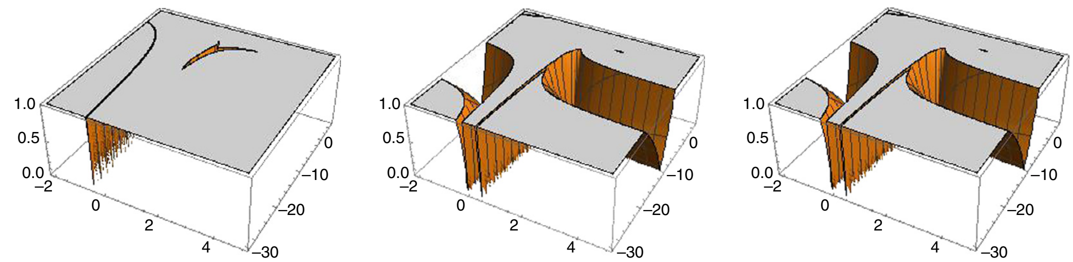


(a) Stability of $z = 0$. (b) Stability of $z = 1$.

Fig. 1. Stability diagrams.



(a) $z = s_1$. (b) $z = s_2$. (c) $z = s_3$.



(d) $z = s_4$. (e) $z = s_5$. (f) $z = s_6$.

Fig. 2. Stability diagrams.

The fixed points of $O_{a_1,b_2}(z)$ are the roots of the equation $O_{a_1,b_2}(z) = z$, that is: $z = 0, z = \infty$, that are associated to the roots of $p(z)$, and the strange fixed points are $z = 1, s_1, s_2, s_3, s_4, s_5, s_6$, where s_i, depending on a_1 and b_2, are the roots of the polynomial (7)

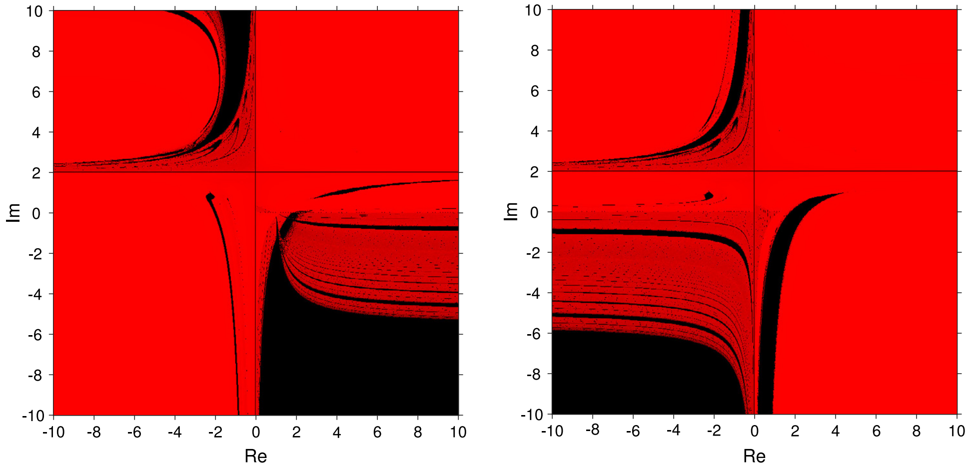
$$1 + (7 - 2a_1 + a_1b_2)z + (21 - 10a_1 + 5a_1b_2)z^2 + (30 - 16a_1 + 6a_1b_2 + a1b_2^2)z^3 + (21 - 10a_1 + 5a_1b_2)z^4 + (7 - 2a_1 + a_1b_2)z^5 + z^6. \tag{7}$$

The two roots of $p(z)$ after Möbius map, $z = 0$ and $z = \infty$, are always superattractive fixed points, but the stability of the strange fixed points depends on the values of a_1 and b_2.

To check the stability of the fixed points we can use stability diagrams, Figs. 1 and 2, which are built as a 3D representation of $|O'_{a_1,b_2}(z)|$ in terms of a_1 and b_2.

Fig. 1(a) shows that $z = 0$ is always superattractor, this is a consequence of the (higher than 2) order of convergence of the method, but this behavior should not be desirable for strange fixed points, because if they are attractors OC method could converge to them instead of to an actual root of the equation. For $z = 1$, in Fig. 1(b), and for the other strange fixed points in Fig. 2, we see combinations of parameters that could make them simultaneously repulsive. We expect these combinations to define stable iterative methods due to the lack of attractive strange fixed points.

It is clear that the roots of the polynomial after Möbius transformation, $z = 0$ and $z = \infty$, are critical points and give rise to their respective Fatou components, but there exist in the family some critical points different from $z = 0$ and $z = \infty$, called free critical points, some of them depending on the value of the parameters. The free critical points of $O_{a_1,b_2}(z)$ are



(a) $z = cr_1$.

(b) $z = cr_3$.

Fig. 3. Parameter planes. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$z = -1$ and:

$$cr_{1,2} = \frac{1}{16}\left(\frac{E}{F} - \sqrt{\frac{G}{F^2}} \mp \sqrt{2\left(\frac{E^2}{F^2} + \frac{I\sqrt{G/F^2}}{F} + \frac{H}{F} - 64\right)}\right),$$

$$cr_{3,4} = \frac{1}{16}\left(\frac{E}{F} + \sqrt{\frac{G}{F^2}} \mp \sqrt{2\left(\frac{E^2}{F^2} + \frac{E\sqrt{G/F^2}}{F} + \frac{H}{F} - 64\right)}\right)$$

where $cr_1 = 1/cr_2$ and $cr_3 = 1/cr_4$ and

$$E = -3a_1^2(b_2-2)^3 - 2a_1\left(b_2^2 - 24b_2 + 44\right) + 80,$$

$$F = a_1(b_2-2)^2 - 5,$$

$$G = a_1^2(b_2-2)^3\left(9a_1^2(b_2-2)^3 + 4a_1\left(19b_2^2 - 56b_2 + 36\right) + 196b_2 - 72\right),$$

$$H = 32\left(a_1^2(b_2+3)(b_2-2)^2 + a1\left(b_2^2 + 16b_2 - 36\right) + 30\right),$$

$$I = 3a_1^2(b_2-2)^3 + 2a_1\left(b_2^2 - 24b_2 + 44\right) - 80.$$

The asymptotic behavior of the critical points plays an important role in the dynamics of the operator because a classical result establishes that there is at least one critical point associated with each invariant Fatou component. In the next section we are going to construct the parameter planes associated to the independent critical points.

3. Parameter planes

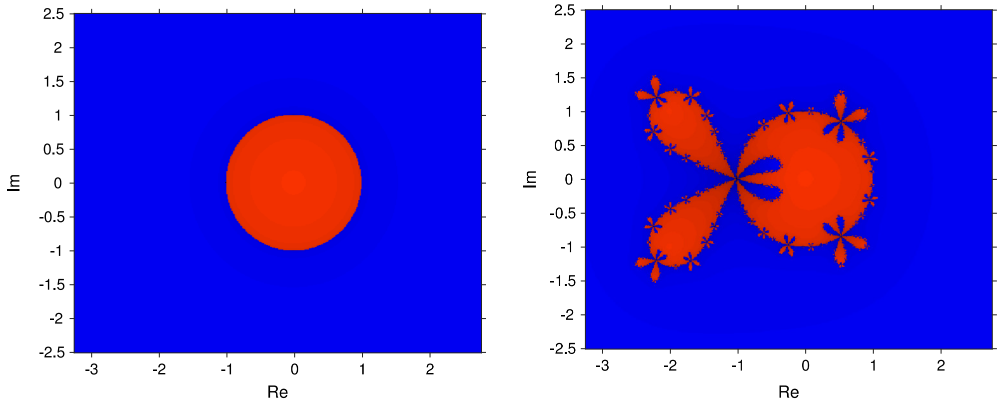
As the dynamics of critical points could lead to a Fatou component, we will draw parameter planes to see the behavior of the method when the initial estimate is a critical point. The parameter space of a free critical point is obtained by associating each point of the parameter plane with real values of a_1 and b_2, so every point on the plane represents a different member of the iterative family. Parameter planes in Fig. 3 have been created using a vectorized version of the MATLAB programs used in [12]. We have used 800×800 different combinations of a_1 and b_2. The points of the plane shown in black correspond to the parameter values for which the associated iterative method does not converge to zero or infinity with a tolerance of 10^{-3} after 500 iterations, taking as the initial estimate the same free independent critical point. Points shown in red converge to zero or infinity, i.e., correspond to stable methods.

As $cr_1 = \frac{1}{cr_2}$ and $cr_3 = \frac{1}{cr_4}$, and the critical point $z = -1$ is a preimage of the fixed point $z = 1$, we have two independent free critical points. In Fig. 3 we can see the parameter planes for the independent critical points.

Combinations of parameters corresponding to red regions for all parameter planes are stable methods, leading to one of the roots, whereas combinations corresponding to a black region could lead to a strange fixed point or periodic orbits.

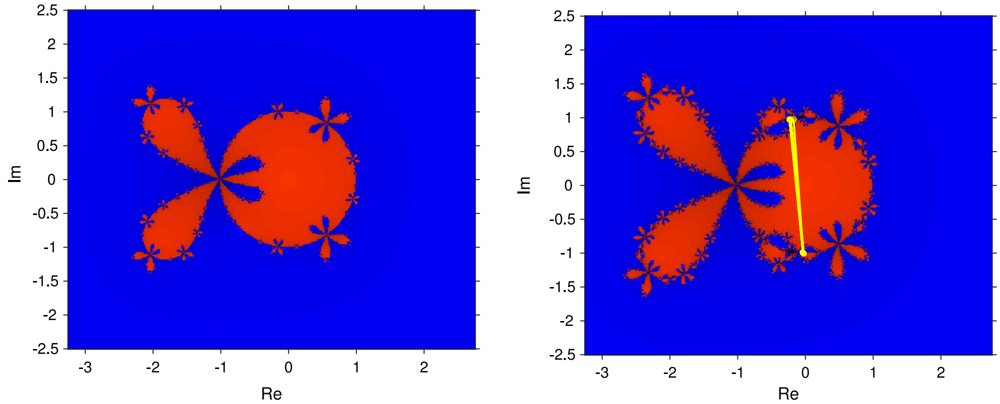
4. Dynamical planes

Dynamical planes are built by selecting particular values of the parameters and applying the method with different initial estimates distributed in a mesh. The dynamical planes of Figs. 4–6 show the basins of attraction corresponding to different



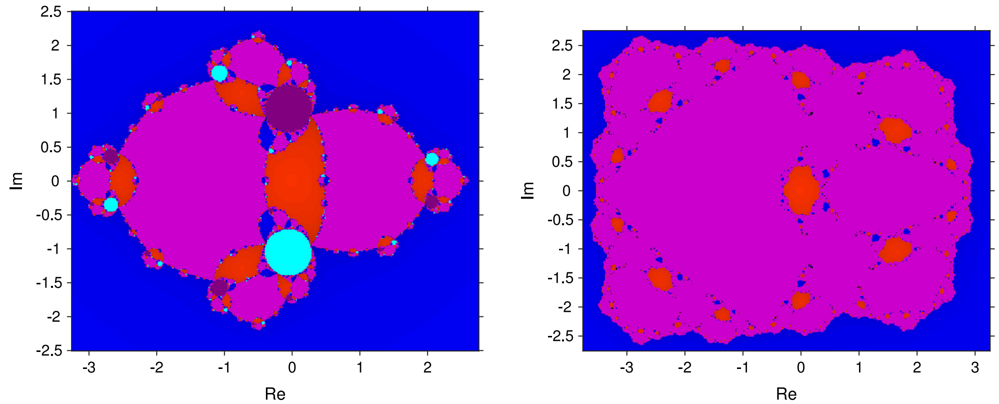
(a) $a_1 = 1, b_2 = 0.$ (b) $a_1 = 1, b_2 = 2.$

Fig. 4. Dynamical planes of Ostrowski' and Chun's methods. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) $a_1 = 1, b_2 = 1.$ (b) $a_1 = -1, b_2 = 3.$

Fig. 5. Dynamical planes of different elements of the family. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) $a_1 = 1, b_2 = -2.$ (b) $a_1 = -0.75, b_2 = 8.$

Fig. 6. Dynamical planes of different elements of the family. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

choices of the parameters. We have chosen combinations of parameters in the red and in the black regions of Fig. 3 to study their dynamical planes.

These dynamical planes have been generated by using 400×400 points of the complex plane as initial estimation. We paint in blue points whose orbits converge to infinity and in orange points converging to zero with a tolerance of 10^{-3}.

Colors different from blue and orange indicate convergence to strange fixed points, while black indicates non convergence to any fixed point after 40 iterations.

After Möbius map, the two roots of $p(z)$ are $z = 0$, in the orange basin, and $z = \infty$, in the blue basin. Fig. 4(a) and (b) represent dynamical planes corresponding to $a_1 = 1$ and $b_2 = 0$ (Ostrowski's method) and $a_1 = 1$ and $b_2 = 2$ (Chun's method) respectively, which are in the red region of the parameter plane, there are only two basins of attraction associated with the roots of $p(z)$.

In Fig. 5(a) we see a member of the family with a very stable behavior, similar to Chun. But if we select methods in the black regions of any parameter plane, for instance $a_1 = -1$, $b_2 = 3$ (Fig. 5(b)), we observe black areas of the dynamical plane that indicate that the method converges to a periodic orbit.

By choosing $a_1 = 1$ and $b_2 = -2$ and $a_1 = -0.75$ and $b_2 = 8$ in the parameter space, both in black regions, we obtain the dynamical planes shown in Fig. 6. In Fig. 6(a) we observe five basins of attraction, three of them corresponding to strange fixed points. While in Fig. 6(b) the basin of attraction of the strange fixed point $z = 1$ is bigger than the basins of attraction of the roots of the polynomial.

5. Numerical solution of Bratu equation

As it was proved in [8], OC family can be easily extended to solve a system of equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$. This can be done by expressing

$$\frac{f(y_k)}{f(x_k)} = 1 - \frac{f[x_k, y_k]}{f'(x_k)},$$

to avoid the function evaluation in the denominator. The resulting iterative expression is

$$\begin{aligned} y^{(k+1)} &= x^{(k)} - \alpha[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - \left(G_1(x^{(k)}, y^{(k)}) + G_2(x^{(k)}, y^{(k)})\right)[F'(x^{(k)})]^{-1}F(y^{(k)}), \end{aligned}$$

where

$$\begin{aligned} G_1(x^{(k)}, y^{(k)}) &= \frac{1}{a_1}\left[(1 + a_1b_2 - 2a_1)I - a_1(b_2 - 2)[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right]^{-1}, \\ G_2(x^{(k)}, y^{(k)}) &= \frac{1}{a_1}\left[(a_1 + a_1b_2 - 1)I - a_1b_2[F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]\right]. \end{aligned}$$

We are going to obtain an approximated solution of *Bratu problem* [13] by applying this class of schemes. Bratu equation appears in a variety of applications, such as ignition model of fuel thermal combustion, chemical reactor theory, or Chandrasekhar model of the expansion of the universe, among others. In the unidimensional case, Bratu problem reduces to

$$\frac{d^2u}{dx^2} + Ce^u = 0, \quad 0 < x < 1 \tag{8}$$

with the boundary conditions

$$u(0) = u(1) = 0,$$

where $C > 0$. The exact solution of (8) is given by

$$u(x) = 2\ln\left[\frac{\cosh\alpha}{\cosh(\alpha(1 - 2x))}\right], \tag{9}$$

where α satisfies the transcendental equation

$$\cosh\alpha = \frac{4}{\sqrt{2C}}\alpha. \tag{10}$$

The solution of (10) can be found using an iterative method, for instance Newton's method.

Bratu's problem has 2 solutions for $C < C_c$, so we have 2 branches, called upper branch and lower branch. If $C = C_c$ (where $C_c \approx 3.513830719$) there is only one solution and if $C > C_c$ there is no solution. To solve Bratu's equation numerically, first we will use the finite difference method to discretize the problem, and then the OC family to solve the resulting system of nonlinear equations.

By using a symmetric standard finite-difference scheme, the discretized version of Bratu problem is:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + Ce^{u_i} = 0 \quad i = 1, 2, \dots, n \tag{11}$$

where $h = \frac{1}{n+1}$ is the discretization step and $u_0 = u_{n+1} = 0$. The unknowns of this nonlinear system are u_i, $i = 1, \dots, n$, that denote approximated values of function u at the points $x_i = 0 + ih$, $i = 1, \dots, n$. The discretization (11) leads to

Table 1
Solution of Bratu equation for $C = 2.65$.

Method	$\hat{\rho}$	Iter	$\|u^{(k+1)} - u^{(k)}\|$	$\|F(u^{(k+1)})\|$
Newton	1.8501	6	1.70×10^{-39}	3.29×10^{-40}
Jarratt	3.8089	4	1.67×10^{-39}	4.37×10^{-40}
Ostrowski	n.c			
Chun	3.7244	4	1.47×10^{-39}	3.25×10^{-40}
$a_1 = 1, b_2 = 1$	3.7350	4	1.14×10^{-39}	2.57×10^{-40}
$a_1 = 1, b_2 = -2$	n.c			
$a_1 = -1, b_2 = 3$	3.7155	4	3.64×10^{-40}	4.73×10^{-41}
$a_1 = -0.75, b_2 = 8$	3.6212	4	2.67×10^{-39}	4.73×10^{-41}

Table 2
Solution of Bratu equation for $C = 3$.

Method	$\hat{\rho}$	Iter	$\|u^{(k+1)} - u^{(k)}\|$	$\|F(u^{(k+1)})\|$
Newton	1.9997	6	2.76×10^{-36}	2.37×10^{-40}
Jarratt	3.7798	4	5.97×10^{-40}	2.50×10^{-40}
Ostrowski	3.7758	4	3.22×10^{-39}	2.39×10^{-40}
Chun	n.c			
$a_1 = 1, b_2 = 1$	3.6596	4	1.71×10^{-39}	6.58×10^{-40}
$a_1 = 1, b_2 = -2$	3.4314	4	2.44×10^{-39}	7.13×10^{-40}
$a_1 = -1, b_2 = 3$	n.c			
$a_1 = -0.75, b_2 = 8$	3.4683	4	1.35×10^{-38}	2.32×10^{-39}

Table 3
Solution of Bratu equation for $C = 3.35$.

Method	$\hat{\rho}$	Iter	$\|u^{(k+1)} - u^{(k)}\|$	$\|F(u^{(k+1)})\|$
Newton	1.9991	7	1.33×10^{-37}	3.69×10^{-40}
Jarratt	3.6356	4	2.27×10^{-35}	3.31×10^{-40}
Ostrowski	3.6349	4	1.87×10^{-35}	2.60×10^{-40}
Chun	3.3686	4	4.12×10^{-28}	1.97×10^{-40}
$a_1 = 1, b_2 = 1$	3.4106	4	2.93×10^{-29}	3.27×10^{-40}
$a_1 = 1, b_2 = -2$	4.0211	5	1.90×10^{-39}	3.40×10^{-40}
$a_1 = -1, b_2 = 3$	3.3336	4	4.15×10^{-27}	2.63×10^{-40}
$a_1 = -0.75, b_2 = 8$	3.8996	5	9.80×10^{-39}	6.77×10^{-40}

a system of n nonlinear equations we will solve using different methods to compare their behavior. Numerical results for three different values of C with $n = 10$ and taking as an initial estimate $u^{(0)} = (0.5, \dots, 0.5)$ are shown in Tables 1–3. We will compare the *approximated computational order of convergence (ACOC)* of each method, a numerical estimation of the order of convergence, defined in [14] as

$$p \approx \hat{\rho} = \frac{\ln(\|u^{(k+1)} - u^{(k)}\| / \|u^{(k)} - u^{(k-1)}\|)}{\ln(\|u^{(k)} - u^{(k-1)}\| / \|u^{(k-1)} - u^{(k-2)}\|)},$$

where $u^{(k)}$ denotes the kth iterate.

In Tables 1–3 we solve the Bratu system of equations for different values of C with Newton', Jarratt's, and different elements of OC family. We have used variable precision arithmetics with 100 digits of mantissa. The stopping criterion is $\|u^{(k+1)} - u^{(k)}\| < 10^{-25}$ or $\|F(u^{(k+1)})\| < 10^{-25}$.

For $C = 2.65$, Ostrowski's method and $a_1 = 1, b_2 = -2$ fail to converge. Also for $C = 3$, Chun's method and $a_1 = -1, b_2 = 3$ fail to converge. As we approach $C = C_c$ convergence takes more iterations. For $C_c = 3.35$ all methods converge, but methods shown to be dynamically stable converge with less iterations.

Method corresponding to $a_1 = 1$ and $b_2 = 1$ seems to be very stable, as it converges for the three values of C and gets a precise value of the root because the value of $\|F(u^{(k+1)})\|$ is among the best methods.

6. Conclusions

In this paper, we have studied the dynamics associated to Ostrowski–Chun family of iterative methods acting on a generic quadratic polynomial, and we have used dynamical tools to select the values of the parameters that yield specially stable and unstable methods of the family. We have solved Bratu's equation with OC methods and compared the results with other known iterative schemes. We can conclude that methods with a good dynamical behavior with a quadratic polynomial are stable with Bratu's equation.

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