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Construction of a full row-rank matrix system for multiple scanning directions in discrete tomography

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ABSTRACT

A full row-rank system matrix generated by scans along two directions in discrete tomography was recently studied. In this paper, we generalize the result to multiple directions. Let $A\mathbf{x} = \mathbf{h}$ be a reduced binary linear system generated by scans along three directions. Using geometry, it is shown in this paper that the linearly dependent rows of the system matrix A can be explicitly identified and a full row-rank matrix can be obtained after the removal of those rows. The results could be extended to any number of multiple directions. Therefore, certain software packages requiring a full row-rank system matrix can be adopted to reconstruct an image. Meanwhile, the cost of computation is reduced by using a full row-rank matrix.

1. Introduction

The algebraic reconstruction of an image in discrete tomography from the projections in a few directions involves solving an underdetermined linear system of projection equations

$$M\mathbf{x} = \mathbf{b}, \tag{1}$$

where M is an $m \times n^2$ matrix, $\mathbf{x} \in \mathbb{R}^{n^2}$ a reconstructed image vector for an $n \times n$ 2-dimensional image and $\mathbf{b} \in \mathbb{R}^m$ the projection vector. The projection equations are formulated from projection data based on different models. The strip-based projection model [1–4] formulates projection equations according to the fractional areas where each strip-shaped beam intersects with the rectangular lattice of the image to be reconstructed. Thus, it is more realistic than the line-based projection model used in some applications. The two matrices of systems (1) generated by the two projection models along the same direction set are proven to be row equivalent [4].

The l_1-minimization algorithm has been applied to reconstruct an image from the underdetermined system (1) [5,6]. A full row-rank system reduces the cost for solving the system and it is required for the usage of some current l_1-minimization software packages, such as the l_1-magic and the sparselab software packages [7,8]. However, the matrix M is often row-rank deficient so it is desired to convert the matrix M into a full row-rank matrix.

Consider an $n \times n$ 2-dimensional image f defined on a lattice set

$$\Omega = \{(u, v) | 0 \le u, v \le n - 1, u, v \in \mathbb{Z}\}$$

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of n^2 lattice points. Suppose that the n^2 lattice points of Ω are arranged in such an order from bottom to top column-wise and the corresponding vector $\mathbf{x}$ representing the unknown image values is given by

$$\mathbf{x} = [f(0,0)\ \cdots f(0,n-1)\, f(1,0)\cdots f(1,n-1)\ \cdots f(n-1,0)\ \cdots f(n-1,n-1)]^T.$$

In general, a direction for parallel beam scanning can be represented by (q,p) for rays with slope $\frac{p}{q}$, $q>0$ and $\gcd(|p|,q)=1$, in addition to the horizontal and vertical directions.

Projection data obtained by each ray or beam provides one equation in system (1). In the case of one direction $(q,-|p|)$ with a negative slope $-\frac{|p|}{q}$, it is known in [4,9] that M is a $(|p|+q)n$ by n^2 binary matrix with $\mathrm{rank}(M)=(|p|+q)n-|p|\,q$, in the form of

$$M = [M_1\ M_2\ \cdots\ M_n],$$
$$M_i = [\mathbf{m}_1^{(i)}\ \mathbf{m}_2^{(i)}\ \cdots\ \mathbf{m}_n^{(i)}] \in \mathbb{R}^{(p+q)n\times n}, \quad 1\le i\le n,$$

where

$$\mathbf{m}_j^{(i)} = \begin{bmatrix} \mathbf{o}_{(i-1)|p|+(j-1)q} \\ \mathbf{e}_{|p|+q} \\ \mathbf{o}_{(n-i)|p|+(n-j)q} \end{bmatrix}, \quad 1\le i,j\le n.$$

Here $\mathbf{o}_s$ represents the zero vector of dimension s and $\mathbf{e}_{|p|+q}$ represents the first column of the identity matrix of order $|p|+q$. The linear dependence of the rows of M has been studied and can be described with the following result.

Lemma 1 (*[10]*). *The linearly dependent rows of the matrix M can be precisely located so that the removal of these rows will result in a full row-rank matrix. The rows of M with row indices $i=|p|\,u+qv+1$ for nonnegative integers $u,v\le n-1$ are maximal linearly independent rows of M.*

The terminology of a reduced binary system matrix is naturally introduced for discussing scans along multiple directions.

Definition 1 (*[10]*). If the rows of M with row indices $i\ne |p|\,u+qv+1$, for any nonnegative integers $u,v\le n-1$, are replaced by zero rows and the corresponding components of $\mathbf{b}$ by zeros, the resultant system,

$$A\mathbf{x}=\mathbf{h}, \tag{2}$$

is called the reduced binary system (RBS) generated along a scanning direction (q,p). The reduced matrix A is called the reduced binary system matrix (RBSM) generated along a scanning direction (q,p).

It is clear that the last $|p|+q-1$ rows of A are zero rows. More properties of the RBSM generated along a single direction are summarized in the following Lemma 2.

Lemma 2 (*[10]*). *The RBSM A generated along a scanning direction (q,p), where $q>0$, is a binary matrix of dimension $(|p|+q)n\times n^2$ with rank $(|p|+q)n-|p|\,q$, having $|p|\,q$ zero rows. The tth row of A is a zero row if and only if $t\ne |p|\,u+qv+1$ for any nonnegative integers $u,v\le n-1$. Each column of A has exactly one entry $a_{ij}=1$ if and only if $i=|p|\,u+qv+1$ and $j=un+v+1$ if $p<0$, or $j=(n-1-u)n+v+1$ if $p>0$, for some nonnegative integers $u,v\le n-1$.*

For scans along two distinct directions (q_1,p_1) and (q_2,p_2) with $\gcd(|p_s|,q_s)=1$, $q_s>0$, $s=1,2$, let $A_1\mathbf{x}=\mathbf{h}_1$ and $A_2\mathbf{x}=\mathbf{h}_2$ be RBSs generated along the two directions, respectively. System (2) is considered with the formulation of

$$A=\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad \text{and} \quad \mathbf{h}=\begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}, \tag{3}$$

where A is a $(|p_1|+q_1+|p_2|+q_2)n$ by n^2 binary matrix with rank $(|p_1|+q_1+|p_2|+q_2)n-(|p_1|+|p_2|)(q_1+q_2)$, having $|p_1|\,q_1+|p_2|\,q_2$ zero rows. For convenience we define a *minimal linearly dependent set* in this context.

Definition 2. A linearly dependent set E is said to be minimal if every proper subset of E is linearly independent.

A full row-rank matrix can be constructed explicitly as summarized in the following lemma.

Lemma 3 (*[10]*). *For scans along two distinct directions (q_1,p_1) and (q_2,p_2), the nonzero rows of the matrix A are partitioned into $|p_1|\,q_2+|p_2|\,q_1$ minimal linearly dependent sets. The removal of all zero rows and one row from each set will result in a full row-rank matrix.*

It is observed that every nonzero row of the matrix A lies in one and only one minimal linearly dependent set of rows.

In general, for scans along multiple directions $\{(q_s,p_s)\}_{s=1}^k$, the system matrix M consists of k submatrices M_s, $s=1,2,\ldots,k$, generated by the k scanning directions, respectively. The corresponding matrix A is formed by k RBSMs, A_s's,

$s = 1, 2, \ldots, k$, where A_s is a RBSM generated along the sth scanning direction (q_s, p_s). It is known [11,4] that $A \in \mathbb{R}^{m\times n^2}$ with $\sum_{s=1}^{k}(|p_s|\, q_s)$ zero rows, where $m = n\sum_{s=1}^{k}(|p_s| + q_s)$ and

$$\mathrm{rank}(A) = n\sum_{s=1}^{k}(|p_s| + q_s) - \sum_{s=1}^{k}|p_s|\sum_{s=1}^{k}q_s.$$

It is in our interest to explicitly identify $\sum_{s=1}^{k}|p_s|\sum_{s=1}^{k}q_s - \sum_{s=1}^{k}(|p_s|\, q_s)$ nonzero rows of A such that the removal of these rows results in a full row-rank matrix.

In the following sections, we study how to construct a full row-rank system matrix for scans along three directions. The main contribution of this paper is to identify explicitly and efficiently the dependent rows of A without performing any row operations to obtain a full row-rank matrix. The paper is organized as follows: Section 2 develops geometric techniques and applies them to provide an alternate proof of a known result for the system generated along two scanning directions, thus establishing the equivalence of geometric and analytical proofs. The main result is then presented in Section 3. A procedure for explicitly determining a minimally dependent set of rows of A is developed using a row of A_3 and then applied to obtain a full row-rank matrix. Numerical examples are presented in Section 4. Finally, concluding remarks are given in Section 5.

2. Equivalent results

The goal of this section is to develop geometric techniques to better understand established results and prove future results. We begin by introducing terminology.

We employ the notion of a *multiset* [12,13] which is a generalization of a set allowing repetition of elements. The *multiplicity* of an element is the number of times that element is listed in the set. We make use of the union, intersection, and direct union as defined for multisets (see [12]). For example, if $S = \{a, a, a, b\}$ and $T = \{a, a, b, b, c\}$ then $S \cup T = \{a, a, a, b, b, c\}$, $S \cap T = \{a, a, b\}$, and $S \uplus T = \{a, a, a, a, a, b, b, b, c\}$. Two multisets are equal if they contain the same elements and equivalent elements have the same multiplicity.

Definition 3. Given a RBSM A, a rowline is a line intersecting the lattice points indicated by a nonzero row in A. The set of all lattice points intersected by a rowline l is denoted $\mathrm{lp}(l)$. The lattice points of a set of rowlines R is the direct union of the set of lattice points intersected by each rowline in R, denoted $\mathrm{lp}(R) = \uplus_{l\in R}\,\mathrm{lp}(l)$.

As a note to readers, by Lemma 1 the lattice points along a rowline l are evenly spaced.

Definition 4. Let R be a subset of all rowlines from the RBSMs A_s, $1 \le s \le k$. If $R = R_1 \cup R_2$, where $R_1 \cap R_2 = \emptyset$, for which $\exists\widetilde{R}_1$ that is the direct union of some subsets of R_1 and $\exists\widetilde{R}_2$ that is the direct union of some subsets of R_2 such that $\mathrm{lp}(\widetilde{R}_1) = \mathrm{lp}(\widetilde{R}_2)$ then R is said to be balanced. Otherwise, R is said to be unbalanced. If R is balanced and every proper subset of R is unbalanced we say that R is minimally balanced.

Lemma 4. *Let V_{A_s} be a subset of row vectors from the RBSMs A_s, $1 \le s \le k$, and let R_{A_s} be the set of corresponding rowlines from A_s. Define $V = \cup_{s=1}^{k} V_{A_s}$ and $R = \cup_{s=1}^{k} R_{A_s}$. Then V is a minimal linearly dependent set if and only if R is minimally balanced.*

Proof. We first show a set of row vectors being linearly dependent is equivalent to the set of corresponding rowlines being balanced. Let R be a balanced set of rowlines from the RBSMs A_s, $1 \le s \le k$. By definition, $R = R_1 \cup R_2$, where $R_1 \cap R_2 = \emptyset$, for which $\exists\widetilde{R}_1$ that is the direct union of subsets of R_1 and $\exists\widetilde{R}_2$ that is the direct union of subsets of R_2 such that $\mathrm{lp}(\widetilde{R}_1) = \mathrm{lp}(\widetilde{R}_2)$. Using the one-to-one correspondence between the rowlines and nonzero row vectors of A_s, there exist disjoint sets of row vectors $\widetilde{V}_1$ and $\widetilde{V}_2$ corresponding to the sets $\widetilde{R}_1$ and $\widetilde{R}_2$, including multiplicity, such that

$$\sum_{v\in\widetilde{V}_1} v = \sum_{w\in\widetilde{V}_2} w. \tag{4}$$

Letting V_1 and V_2 be the sets $\widetilde{V}_1$ and $\widetilde{V}_2$ without multiplicity of elements, Eq. (4) is equivalent to saying $V = V_1 \cup V_2$ is linearly dependent.

We now show that a set of row vectors being minimally linearly dependent is equivalent to the corresponding set of rowlines being minimally balanced. Assume that R is now minimally balanced. Using the one-to-one correspondence, any $V' \subsetneq V$ corresponds to some $R' \subsetneq R$. By the minimality of R, R' is unbalanced which is equivalent to V' being linearly independent. Thus, V is a minimal linearly dependent set. □

Given two RBSMs, A_1 and A_2, with scanning directions (q_1, p_1) and (q_2, p_2), respectively, we first introduce a procedure for generating a minimally balanced set of rowlines. For convenience we let $(q_0, p_0) = (q_2, p_2)$. The notation $d(l, l')$ is used to represent the perpendicular distance between two parallel rowlines l and l'.

Procedure 1 (*Minimal Balanced Set Generated with a Rowline from A_2*).

1. Input: two RBSMs A_1 and A_2; two directions (q_1, p_1) and (q_2, p_2); a rowline l_2 from A_2.
2. Choose a rowline l_1 from A_1 that intersects l_2 at a lattice point.

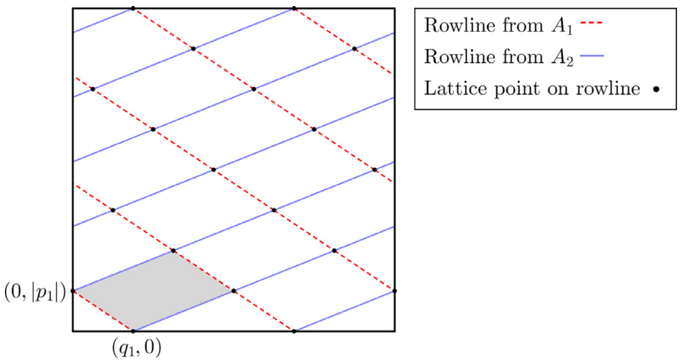

Fig. 1. Minimally balanced set of rowlines with opposite slope signs on the lattice Ω.

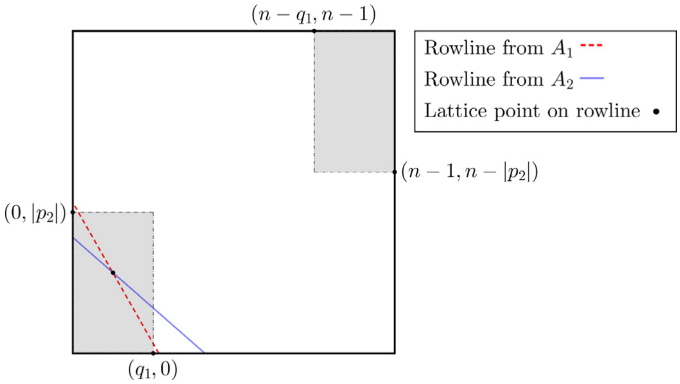

Fig. 2. Minimally balanced set of rowlines with the same slope sign on the lattice Ω.

3. for $i = 0$ to 1

   generate $R_{i+1} = \{l: l \parallel l_{i+1}, d(l, l_{i+1}) = \left| k \frac{q_i p_{i+1} - p_i q_{i+1}}{\sqrt{p_i^2 + q_i^2}} \right|$, for any $k \in \mathbb{Z}$ s.t. l intersects $\Omega\}$.

   end
4. Output: $R = R_1 \cup R_2$.

Procedure 1 will be generalized into Procedure 2 for the case of multiple directions in the next section. It is pointed out that the understanding of Procedure 1 is helpful for establishing Procedure 2.

We now provide an alternate proof for Lemma 3 using the notion of rowlines and Procedure 1.

Proof of Lemma 3. We then consider two possible cases for the problem.

Case 1: Scanning directions have opposite slope signs.

Consider scanning directions (q_1, p_1) and (q_2, p_2) for the RBSMs A_1 and A_2, respectively, where $p_1/q_1 < 0 < p_2/q_2$. Following Procedure 1, we generate a minimally balanced set, $R = R_1 \cup R_2$, using the rowline from A_2 passing through the point $(0, |p_1|)$ on the lattice Ω, as displayed in Fig. 1.

For each lattice point P inside of the shaded region in Fig. 1, we can use the rowline from A_2 that intersects P to create a minimally balanced set of rowlines disjoint from the previous set using Procedure 1. Thus, the number of disjoint minimally balanced sets is $|p_1|q_2 + |p_2|q_1$. In addition to the zero rows, the rows from A_2 that need to be removed correspond to the rowlines from A_2 that pass through the shaded region and one of the rowlines from A_2 along the edge of the shaded region.

Case 2: Scanning directions have same slope sign.

Consider scanning directions (q_1, p_1) and (q_2, p_2) for the RBSMs A_1 and A_2, respectively, where $p_1/q_1 < p_2/q_2$. Using Procedure 1, any rowline selected from A_2 that intersects the point (u, v) in the region where $u < q_1$, $v < |p_2|$ or the region where $u > n - q_1$, $v > n - |p_2|$ will generate a minimally balanced set containing one rowline from A_1 and one rowline from A_2, as illustrated in Fig. 2. This results in $2|p_2|q_1$ rowlines of A_2 that intersect a lattice point in the shaded regions from Fig. 2.

If we use a rowline from A_2 that does not intersect one of the shaded regions in Fig. 2 then Procedure 1 will generate disjoint minimally balanced sets similar in form to the sets from Case 1. In this instance, we use the same idea from Case 1 yielding an additional $|p_1|q_2 - |p_2|q_1$ disjoint minimally balanced sets of rowlines. Finally, we remove one row from A_2 corresponding to a rowline from A_2 in each disjoint minimally balanced set. This results in removing a total of $|p_1|q_2 + |p_2|q_1$ rows from A_2. $\square$

It should be noted that in the case where the scanning directions have opposite slope signs we can remove any $|p_1|q_2 + q_1|p_2|$ consecutive nonzero rows from A_2 having an index in the set W together with the zero rows of A_1 and A_2, where W is given by

$$W = \{i = u|p_2| + vq_2 + 1 : n - q_1 \le u \le n - 1, n - |p_2| \le v \le n - 1\}.$$

The ideas from the geometric proofs of Lemmas 3 and 4 will be used in the following section to prove the main result in three scanning directions.

3. Main results

Let

$$A = [A_1; A_2; A_3] \in R^{m\times n^2} \tag{5}$$

be a system matrix generated by the line projection model for reconstruction of an $n \times n$ 2-dimensional image, where A_s's are RBSMs generated along three scanning directions (q_s, p_s), $q_s > 0$ with gcd $(q_s, |p_s|) = 1,\ s = 1, 2, 3$, respectively. Without loss of generality, we assume that the slopes are in ascending order, i.e.,

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \frac{p_3}{q_3}, \quad q_s > 0,\ s = 1, 2, 3.$$

Then

$$m = \sum_{s=1}^{3}(q_s + |p_s|)n \quad \text{and rank } (A) = \sum_{s=1}^{3}(q_s + |p_s|)n - \left(\sum_{s=1}^{3} q_s\right)\left(\sum_{s=1}^{3} |p_s|\right) = m_1.$$

We explicitly construct a full rank matrix $F \in \mathbb{R}^{m_1\times n^2}$ by the removal of certain $(\sum_{s=1}^{3} q_s)(\sum_{s=1}^{3} |p_s|)$ rows from A in this section.

With an application of Lemma 3, we have the following remark.

Remark 1. If $T = \{v_1, v_2, \ldots, v_t\}$ is a subset of rows of A_3, where $t = |p_3|(q_1 + q_2) + (|p_1| + |p_2|)q_3$, such that all rows v_i's in T are linear combinations of other rows of A then the removal of all rows of T from A_3, $|p_1|q_2 + |p_2|q_1$ rows of A_2 in Lemma 3, and all zero rows of A will result in a desired full rank matrix.

Furthermore, we have the following equivalent lemma in terms of minimally balanced sets:

Lemma 5. *Assume that a set of t rowlines from A_3 is denoted by $D = \{r_1, r_2, \ldots, r_t\}$, where $t = |p_3|(q_1+q_2)+(|p_1|+|p_2|)q_3$, and all the other rowlines from A are denoted by $\{r_{t+1}, r_{t+2}, \ldots, r_m\}$. Let S_j be a minimally balanced set of A containing the rowline $r_j \in D$, respectively, $j = 1, 2, \ldots, t$, such that*

$$\{r_{j+1}, r_{j+2}, \ldots, r_{j+t-1}\} \cap S_j = \varnothing, \quad \text{for } j = 1, 2, \ldots, t. \tag{6}$$

Then the removal of the rows of A_3 corresponding to the rowlines in D, $|p_1|q_2 + |p_2|q_1$ rows of A_2 in Lemma 3, and all zero rows of A will result in a full row-rank matrix.

Proof. We label the row vectors of A corresponding to the rowline r_j of A as $v_j, j = 1, 2, \ldots, m$. It follows from (6) that v_j is a linear combination of $v_1, \ldots, v_{j-1}, v_{j+t}, \ldots, v_m$, for $1 \le j \le t$. By induction it can be shown that $\{v_1, \ldots, v_t\} \subset \text{Span}\{v_{t+1}, \ldots, v_m\}$. We apply Remark 1 to complete the proof. □

For Lemma 5, we will focus on constructing a set of rowlines, D, from A_3 and the corresponding minimally balanced sets S_j's satisfying (6) in this section. For convenience, we denote

$$A_0 = A_3, \quad \text{and} \quad A_k = A_{\text{mod}(k,3)} \text{ for } k \ge 3.$$

Let H be a hexagon $P_1 \ldots P_6$ such that the vertices, P_i, are lattice points of Ω in counterclockwise order satisfying

$$-\overrightarrow{P_sP_{s+1}} = \overrightarrow{P_{s+3}P_{s+4}} = q_s\mathbf{i} + p_s\mathbf{j}, \quad s = 1, 2, 3,$$

where $P_7 = P_1$ and $\mathbf{i}, \mathbf{j}$ are the unit vectors of the axes in the uv-plane. See Fig. 3 for an illustration. Before describing a procedure to generate a minimally balanced set using a rowline from A_3, we introduce some necessary notations, for $i = 1, 2, \ldots, 6$:

l_i: a rowline passing through the side P_iP_{i+1}.
H_i: an open half plane bounded by l_i excluding the hexagon H.
$\overline{H_i}$: the closure of H_i.
$d_{i,+}$: the perpendicular distance from P_{i+2} to l_i.
$d_{i,-}$: the perpendicular distance from P_{i-1} to l_i.
D_i: a set of distances, $\{kd_{i,+} + hd_{i,-} : k, h \text{ nonnegative integers}\}$.
$d(l, l_i)$: the distance between two parallel rowlines l and l_i.
O_i: The open region in Ω located between l_i and l_{i+3}.

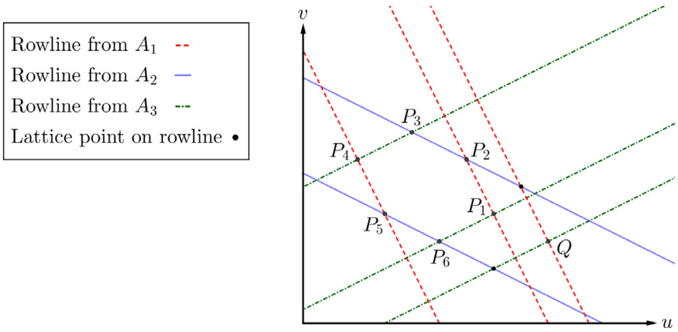

Fig. 3. Hexagon and a lattice point Q.

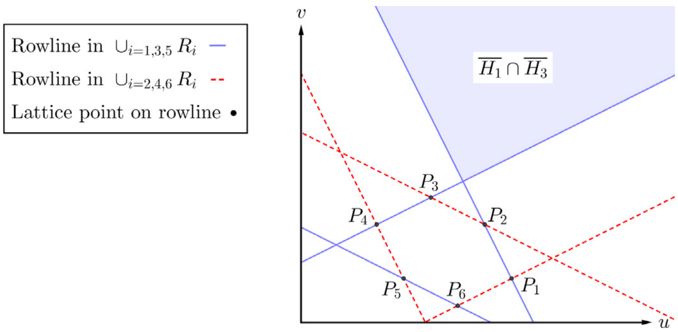

Fig. 4. Some rowlines in $\cup_{i=1,3,5} R_i$ or $\cup_{i=2,4,6} R_i$.

Procedure 2 (*Minimally Balanced Set Generated with a Rowline from* A_3).

1. Input: RMSMs A_i, directions (q_i, p_i), $i = 1, 2, 3$; a rowline l from A_3.
2. Construct a hexagon $H = P_1P_2 \dots P_6$ such that P_6P_1 lies on l.
3. for $i = 1$ to 6
   generate $R_i = \{l: l \parallel l_i$, located in H_i, intersecting Ω, and $d(l, l_i) \in D_i\}$.
   end
4. Output: $R = \cup_{i=1}^{6} R_i$.

Procedure 2 is a geometric interpretation of a minimally balanced set generated from a nonzero row of A_3. It should be noted that there is no rowline in the set R that intersects a lattice point in the region $\cap_{s=1}^{3} O_s$, which is equivalent to the interior of hexagon H. We now prove that the output of Procedure 2 is a minimally balanced set.

Theorem 6. *The set of rowlines, $R = \cup_{i=1}^{6} R_i$, yielded by Procedure 2, is minimally balanced.*

Proof. We first show that

$$R \subset S, \tag{7}$$

where S is a minimally balanced set containing the rowline l_i, $1 \le i \le 6$. For each i, $1 \le i \le 6$, we define two sets of rowlines:

$$\begin{aligned} R_{i,+} &= \{l : l \parallel l_i, \text{ located in } \overline{H_i}, \text{ intersecting } \Omega, \text{ and } d(l, l_i) = kd_{i,+}, k \ge 0\}; \\ R_{i,-} &= \{l : l \parallel l_i, \text{ located in } \overline{H_i}, \text{ intersecting } \Omega, \text{ and } d(l, l_i) = kd_{i,-}, k \ge 0\}. \end{aligned} \tag{8}$$

The rowlines in $R_{i,+}$ and $R_{i+1,-}$ intersect at lattice points of Ω. Since S is a minimally balanced set we have

$$R_{i,+} \cup R_{i+1,-} \subset S, \quad i = 1, \dots, 6.$$

Next, for a fixed i, we denote $l_{i,+}^1$ as the rowline in $R_{i,+}$ with $k = 1$ (8). There exists a lattice point Q on the rowline $l_{i,+}^1$ located in the open region O_{i-1}. See Fig. 4. By a similar argument, a rowline parallel to l_{i-1} in H_{i-1} must be an element of S. Thus, the rowline in R_i with distance $d_{i,+} + d_{i,-}$ must be an element of S. By induction, it can be shown that R_i is a subset of S which implies that (7) holds.

We now show that R is a balanced set. In other words, we prove that

$$\mathrm{lp}(\cup_{i=1,3,5} \widetilde{R}_i) = \mathrm{lp}(\cup_{i=2,4,6} \widetilde{R}_i). \tag{9}$$

Recall that $d_{i,+}$ and $d_{i,-}$ are the perpendicular distances from P_{i+2} and P_{i-1} to l_i, respectively. We claim that there exist smallest positive integers k_i and h_i such that

$$d_i = k_i d_{i,+} = h_i d_{i,-}. \tag{10}$$

For simplicity, we consider the case of $\frac{p_1}{q_1} < \frac{p_2}{q_2} < 0 < \frac{p_3}{q_3}$, $q_s > 0$, $s = 1, 2, 3$. Using geometry, it can be shown that

$$d_{1,+} = \frac{-p_1q_2 + p_2q_1}{\sqrt{p_1^2 + q_1^2}} \quad \text{and} \quad d_{1,-} = \frac{-p_1q_3 + p_3q_1}{\sqrt{p_1^2 + q_1^2}}.$$

So $\frac{d_{1,+}}{d_{1,-}} = \frac{|p_1q_2 - p_2q_1|}{|p_1|q_3 + p_3q_1}$. Let

$$k_1 = \frac{-p_1q_3 + p_3q_1}{\gcd(-p_1q_2 + p_2q_1, -p_1q_3 + p_3q_1)} \quad \text{and} \quad h_1 = \frac{-p_1q_2 + p_2q_1}{\gcd(-p_1q_2 + p_2q_1, -p_1q_3 + p_3q_1)}.$$

Then k_1 and h_1 are smallest positive integers such that

$$d_1 = k_1 d_{1,+} = h_1 d_{1,-}.$$

The claim (10) can be proved for each $i = 1, 2, \dots, 6$, in an analogous way.

The rowline $l^* \in R_{i,+} \cap R_{i,-}$ with $d(l^*, l_i) = d_i$ provided in the claim intersects rowlines in both $\overline{H_{i-1}}$ and $\overline{H_{i+1}}$. Thus, it needs to be included twice in the set $\widetilde{R}_i$, requiring the use of a direct union. Additionally, the rowline $l^* \in R_{i,+} \cap R_{i,-}$ with $d_i \le d(l^*, l_i) < 2d_i$ needs to be included in the set $\widetilde{R}_i$ twice. In general, the rowline $l^* \in R_{i,+} \cap R_{i,-}$ with $(k-1)d_i \le d(l^*, l_i) < kd_i$ needs to be included in the set $\widetilde{R}_i$ k times, requiring the use of a direct union.

In Fig. 4, all rowlines in R_i, for odd i, are indicated by solid rowlines while all rowlines in R_i, for even i, are represented by dashed rowlines. In the intersection of two closed sets $\overline{H_1} \cap \overline{H_3}$, we have

$$\mathrm{lp}(\cup_{i=1,3,5} \widetilde{R}_i) = (\mathrm{lp}(\widetilde{R}_1) \cap \mathrm{lp}(\widetilde{R}_2)) \uplus (\mathrm{lp}(\widetilde{R}_2) \cap \mathrm{lp}(\widetilde{R}_3)) = \mathrm{lp}(\cup_{i=2,4,6} \widetilde{R}_i),$$

since $H_4 \cup H_5 \cup H_6$ and $\overline{H_1} \cap \overline{H_3}$ are disjoint. In general, we have

$$\mathrm{lp}(\cup_{i=1,3,5} \widetilde{R}_i) = \mathrm{lp}(\cup_{i=2,4,6} \widetilde{R}_i), \quad \text{on } \cup_{i=1}^{6} (\overline{H_i} \cap \overline{H_{i+2}}).$$

Furthermore, by the previous argument, every lattice point from the set $\mathrm{lp}(R)$ in the open region $\cup_{s=1}^{3} O_s$ is the intersection of rowlines from disjoint sets $\cup_{i=1,3,5} \widetilde{R}_i$ and $\cup_{i=2,4,6} \widetilde{R}_i$. Note that $\cup_{s=1}^{3} O_s$ is the complement of $\cup_{i=1}^{6} (\overline{H_i} \cap \overline{H_{i+1}})$. Therefore, (9) is proved.

Finally, the removal of any rowline of R results in the loss of the balance of R yielding that R is minimally balanced. □

The rowline l used in Procedure 2 to generate a minimal balanced set is selected to be l_6. A special case may occur when there is no hexagon located in Ω for a given rowline l from A_3. (For example, see the proof of Case 2 in Theorem 7.) In such an instance, using l, we first construct part of a hexagon consisting of two or three consecutive rowlines, where the first and last edges have one vertex of the hexagon outside of Ω. Using these rowlines, a minimally balanced set is formed.

We now show the main result.

Theorem 7. *Let A be given in (5). One can explicitly determine $|p_3|(q_1 + q_2) + (|p_1| + |p_2|)q_3$ rows of A_3 and $|p_1|q_2 + |p_2|q_1$ rows of A_2 such that the removal of those rows and all zero rows of A will result in a full rank matrix.*

Proof. We consider the following two cases. The remaining cases can be proved in an analogous way.

Case 1. Scanning directions have two negative slopes and one positive slope.

Assume that

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < 0 < \frac{p_3}{q_3}, \quad q_s > 0, s = 1, 2, 3. \tag{11}$$

We define a hexagon $H = P_1P_2 \dots P_6$ using the rowline $l = l_6$ from A_3 passing through the points $P_6(q_1 + q_2, 0)$ and $P_1(q_1 + q_2 + q_3, |p_3|)$ on the lattice Ω. See Fig. 5. Let D be the union of l_6 and R_{A_3} in the open region O_3. We claim

$$|D| = (q_1 + q_2)|p_3| + q_3(|p_1| + |p_2|). \tag{12}$$

In fact, there are $q_1|p_3| + q_3|p_1| - 1$ and $q_2|p_3| + q_3|p_2| - 1$ rowlines in R_{A_3} that intersect the open parallelogram regions $P_6P_1P_2Q$ and $P_2P_3P_4Q$ in the hexagon H, respectively. Using each rowline, we generate a minimally balanced set. Distinct rowlines in these regions generate different minimally balanced sets. Including the rowline l_6 and the rowline intersecting lattice point P_2 concludes the proof of (12).

We construct a set of minimally balanced sets of A, $\{S_j\}_{j=1}^{t}$, using all rowlines in D by Procedure 2. Moreover, (6) holds and we complete the proof with an application of Lemma 5.

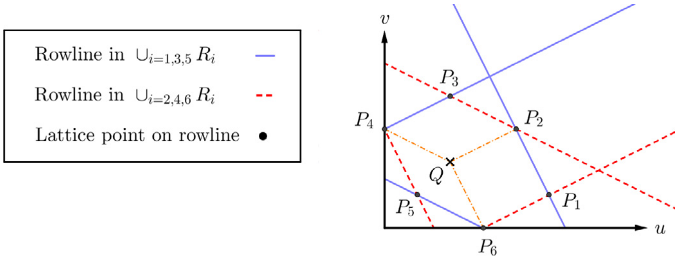

Fig. 5. Hexagon and a point Q in Case 1 of Theorem 7.

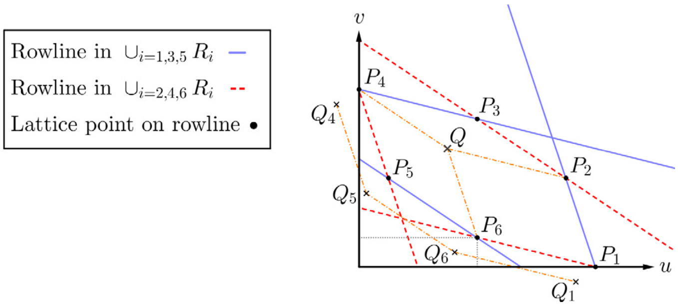

Fig. 6. Hexagon and a point Q in Case 2 of Theorem 7.

Case 2. Scanning directions have all negative slopes.
Assume that

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \frac{p_3}{q_3} < 0, \quad q_s > 0, \; s = 1, 2, 3. \tag{13}$$

We construct a hexagon $H = P_1P_2 \dots P_6$ using the rowline $l = l_6$ from A_3 passing through the points $P_6(q_1 + q_2, |p_3|)$ and $P_1(q_1 + q_2 + q_3, 0)$ on Ω. See Fig. 6. Then $P_4 = (0, |p_1| + |p_2| + |p_3|)$. Let D_1 be the union of l_6 and rowlines from A_3 in the open region O_3. We claim

$$|D_1| = q_3(|p_1| + |p_2|) - (q_1 + q_2)|p_3|. \tag{14}$$

We find that there are $q_3|p_1| - q_1|p_3| - 1$ and $q_3|p_2| - q_2|p_3| - 1$ rowlines from A_3 that intersect the open parallelogram regions $P_6P_1P_2Q$ and $P_2P_3P_4Q$ in the hexagon H, respectively. Including the rowline l_6 and the rowline intersecting lattice point P_2 proves (14).

Any rowline from A_3 that passes through a lattice point (u, v) in the region where $u < q_1$, $v < |p_3|$ or the region where $u \geq n - q_1$, $v \geq n - |p_3|$ will generate a minimally balanced set containing one rowline from A_1 and one rowline from A_3. Let D_2 be the set of such rowlines from A_3. Then

$$|D_2| = q_1|p_3|.$$

For each lattice point Q_6 in the region where $q_1 \leq u < q_1 + q_2$, $v < |p_3|$, we set segments $Q_4Q_5Q_6Q_1$ such that

$$Q_iQ_{i+1} // P_iP_{i+1} \quad \text{and} \quad Q_iQ_{i+1} = |P_iP_{i+1}|, \quad i = 4, 5, 6.$$

Then both Q_4 and Q_1 are out of Ω. So the minimal balanced set generated from the rowline from C passing through Q_6 contains only one rowline from A_3 and other rowlines from A_2 and A_3. Let D_3 be the set of those rowlines from A_3. Then

$$|D_3| = q_2|p_3|.$$

There are $q_2|p_3|$ such minimal balanced sets. They are disjoint from the minimal balanced sets generated by rowlines in $D_1 \cup D_2$.

We can make a similar argument for the region where $u \geq n - (q_1 + q_2)$, $v \geq n - |p_3|$. Let D_4 be the set of those rowlines from A_3. Then

$$|D_4| = (q_1 + q_2)|p_3|.$$

Letting $D = D_1 \cup D_2 \cup D_3 \cup D_4$ yields that

$$|D| = (q_1 + q_2)|p_3| + q_3(|p_1| + |p_2|).$$

Applying Lemma 5 completes the proof. $\square$

Remark 2. In Case 1, particularly, any $(|p_1| + |p_2|)q_3 + |p_3|(q_1 + q_2)$ consecutive nonzero rows of A_3 with index in the set W_1 can be selected as rows of A_3 to be removed, where

$$W_1 = \{|p_3|q_3 + 1, |p_3|q_3 + 2, \ldots, (|p_3| + q_3)n - (|p_3| + q_3 - 1) - |p_3|q_3\}.$$

On the other hand, in Case 2, particularly, any $(|p_1| + |p_2|)q_3 - |p_3|(q_1 + q_2)$ consecutive nonzero rows of A_3 with index in the set W_2 can be removed with all nonzero rows of A_3 with index in $W_3 \cup W_4$ where

$$\begin{aligned} W_2 &= \{|p_3|(q_1 + q_2 + q_3) + 1, \ldots, (|p_3| + q_3)n - (|p_3| + q_3 - 1) - |p_3|(q_1 + q_2 + q_3)\}, \\ W_3 &= \{i = u|p_3| + vq_3 + 1 : 0 \le u \le q_1 + q_2 - 1, 0 \le v \le |p_3| - 1\}, \\ W_4 &= \{i = u|p_3| + vq_3 + 1 : n - (q_1 + q_2) \le u \le n - 1, n - |p_3| \le v \le n - 1\}. \end{aligned} \tag{15}$$

4. Examples

In this section, we demonstrate the main results in Section 3 by considering two numerical examples as representatives of same slope signs and different slope signs using three slopes. Consider the scanning of an image with size 24×24 along three directions (q_i, p_i), $i = 1, 2, 3$, where the slopes of projections in real cases are approximated by rational numbers. For simplicity, numerates and denominates of rational numbers are chosen small integers in absolute value. The system matrix in (2) is defined as in (5).

Example 1. The first example is selected as an experiment for the case of different slope signs. Suppose that the three directions are $(q_1, p_1) = (4, -3)$, $(q_2, p_2) = (3, -2)$, and $(q_3, p_3) = (2, 3)$. The matrix A is of size 408×576 with $\text{rank}(A) = 336$.

The dimensions of submatrices A_1, A_2, and A_3 are 168×576, 120×576, and 120×576, respectively [4]. The submatrices A_1, A_2, and A_3 have 6, 12, and 6 zero rows, respectively. In addition to the 24 zero rows, the matrix A has 48 nonzero rows that should be removed to result in a full row-rank matrix F:

(i) $|p_1|\,q_2 + |p_2|\,q_1 = 17$ nonzero dependent rows from A_2 by Lemma 3: 1, 3-8, 10, 15, 107, 109-114, 116.
(ii) $|p_3|\,(q_1 + q_2) + (|p_1| + |p_2|)q_3 = 31$ consecutive nonzero dependent rows from A_3 by Remark 2, for example, rows $7 - 37$.

The numerical experiment verifies that the removal of the 24 zero rows and above nonzero rows in A_2 and A_3 results in a full row-rank matrix F with $\text{rank}(F) = 336$.

Example 2. Suppose the three directions with same slope sign are $(q_1, p_1) = (2, -3)$, $(q_2, p_2) = (4, -3)$, and $(q_3, p_3) = (3, -2)$. The matrix A is of size 408×576 with $\text{rank}(A) = 336$.

In this case, A_1, A_2, and A_3 are 120×576, 168×576, and 120×576 submatrices having 6, 12, and 6 zero rows, respectively. The matrix A is converted to a full row-rank matrix F after the removal of the following rows:

(i) 24 zero rows from A.
(ii) $|p_1|\,q_2 + |p_2|\,q_1 = 18$ nonzero dependent rows from A_2 by Lemma 3: 1, 4, 5, 8, 9, 12, 19-24, 151, 154, 155, 158, 159, 162.
(iii) $|p_3|\,(q_1 + q_2) + (|p_1| + |p_2|)q_3 = 30$ nonzero dependent rows from A_3 with indices in $W_2 \cup W_3 \cup W_4$, by Theorem 7 and Remark 2:

$$\begin{aligned} W_2 &= \{19, 20, 21, 22, 23, 24\}, \\ W_3 &= \{i = 2u + 3v + 1 : 0 \le u \le 5, 0 \le v \le 1\} = \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14\}, \\ W_4 &= \{i = 2u + 3v + 1 : 18 \le u \le 23, 22 \le v \le 23\} \\ &= \{103, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116\}. \end{aligned}$$

The numerical experiment verifies that the removal of above rows results in a full row-rank matrix F with $\text{rank}(F) = 336$.

5. Conclusion

In the case of three scanning directions, it is observed that every nonzero dependent row of the system matrix A may lie in more than one minimal linearly dependent set of rows. The structure of a minimal linearly independent set and a partition for those minimal linearly independent sets are more complicated than the case of two scanning directions.

For scans along more than two directions, we first show the equivalence of a minimal linearly dependent set of rows of A and the corresponding minimally balanced set of rowlines. In the case of three scanning directions, our work in this paper provides the exact locations of the linearly dependent rows of the system matrix A.

It is remarked that the result for three directions can be naturally extended to arbitrarily many scanning directions. Thus, theoretically, the linearly dependent rows of the system matrix A generated by scans along multiple directions can be explicitly and efficiently identified explicitly and a full row-rank matrix can be obtained after the removal of these rows. Therefore, software packages requiring a full row-rank system matrix can be adopted to reconstruct an image. Meanwhile, the cost of computation is reduced by using a full row-rank matrix.

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References

[1] A.C. Kak, Malcolm Slaney, Principles of Computerized Tomographic Imaging, Society of Industrial and Applied Mathematics, Philadelphia, 2001.
[2] X. Li, J. Zhu, A note of reconstruction algorithm of the strip-based projection model for discrete tomography, J. X-Ray Sci. Technol. 16 (2008) 253–260.
[3] Y. Ye, J. Zhu, G. Wang, Linear diophantine equations for discrete tomography, J. X-Ray Sci. Technol. 10 (2001) 59–66.
[4] J. Zhu, X. Li, Y. Ye, G. Wang, Analysis on the strip-based projection for discrete tomography, Discrete Appl. Math. 156 (2008) 2359–2367.
[5] E. Candes, T. Tao, Decoding by linear programming, IEEE Trans. Inform. Theory 51 (2005) 4203–4215.
[6] E. Candes, T. Tao, Near-optimal signal recovery from random projections: universal encoding strategies, IEEE Trans. Inform. Theory 52 (2006) 5406–5425.
[7] E. Candes, J. Romberg, l_1-magic, http://statweb.stanford.edu/~candes/l1magic/.
[8] V. Stodden, L. Carlin, D. Donoho, I. Drori, D. Dunson, M. Elad, S. Ji, J. Starck, J. Tanner, V. Temlyakov, Y. Tsaig, Y. Xue, Sparselab, https://sparselab.stanford.edu/.
[9] J. Zhu, X. Li, A full row-rank system matrix generated by the strip-based projection model in discrete tomography, Appl. Math. Comput. 216 (2010) 3536–3540.
[10] X. Li, H. Wang, Y. Wu, J. Zhu, A full row-rank system matrix along two directions in discrete tomography, Appl. Math. Comput. 218 (2011) 107–114.
[11] L. Hajdu, R. Tjideman, Algebraic aspects of discrete tomography, J. Reine Angew. Math. 534 (2001) 119–128.
[12] D. Singh, A. Ibrahim, T. Yohanna, J. Singh, An overriew of the applications of multiset, Novi Sad J. Math. 37 (2007) 73–92.
[13] J. Hein, Discrete Mathematics, second ed., Jones & Bartlett Publishers, Burlington, MA, 2003.