Applied Mathematics and Computation 295 (2017) 16–23

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Sweep method for solving the Roesser type equation describing the motion in the pipeline

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ARTICLE INFO

MSC:
35L20
65N22

Keywords:
Gas lift
System of hyperbolic equations
Balancing parameter
Discretization
Roesser model

ABSTRACT

The initial problem for the system of hyperbolic equations describing the motion in oil production with gas lift method is considered. Introducing a new variable which is the difference of gas pressure and volume (or gas–liquid mixture (GLM)) multiplied by a constant number (balancing unit of measurements), the original system of equations is reduced to the such form of equations which after appropriate discretization becomes a Roesser type discrete equation. Searching the new variable as a linear function of the volume of gas (or GLM), it is shown that the coefficients satisfy the two difference equations of the first order, one of which corresponds to the quadratic equation and the second is a linear difference equation of the first order whose coefficients depend on the solution of the first one. In the case when the volume of the assessment gas and the motion (initial conditions) are constant at the mouth, it is shown that the results obtained by the Roesser model coincide with the known results, where the concrete analytical expression for the parameter of the balancing unit of measurements is provided.

1. Introduction

As is known, in different stages of oil production there exist numerous methods one of which is the gas-lift method [4–6,18,20]. There exist different ways for modeling [4,12,18] and then solving the corresponding equations arising in the gas-lift [3,8], where at first partial differential equations at time or at coordinates, are averaged [5,9]. Then the solution of corresponding ordinary differential equations is studied. The identification problems [17] of definition of the hydraulic resistance coefficient [6], formation of GLM [4–6] are investigated. Note that these problems concerning averaging are approximate and the obtained results are not in general adequate for the initial problem. Therefore, it is natural to study the initial problem where the motion is already described by the first order hyperbolic equation. On the other hand, in the gas-lift method, in the ring space and lift the motions are described by partial differential equations, but on the layer by the finite-difference equation. Therefore, it makes sense to discretize these partial equations and then to describe the general system of partial equations for the gas-lift process by a finite-difference equations.

It is shown that the general systems of equations can be reduced [11] to the useful form by using the Roesser model [2,13,19]. For these Roesser difference equations [2] the sweep method is suggested that makes easy the finding of approxi-

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http://dx.doi.org/10.1016/j.amc.2016.09.022

mate solutions of initial hyperbolic equations coinciding with the known solutions by the more degree of accuracy. Note that for the periodic solutions of Roesser equation can be used the results [7], which can increase the debit in oil production.

2. Problem statement. Roesser type equation in continuous case

The Cauchy problem for a system of the first order hyperbolic type differential equations is considered [10,14]

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial x} = -\dfrac{c}{F}\cdot\dfrac{\partial Q(x,t)}{\partial t}, \\ \dfrac{\partial Q(x,t)}{\partial x} = -F\dfrac{\partial P(x,t)}{\partial t} - 2aQ(x,t), \quad x\in R,\ \ t>0, \end{cases} \tag{1}$$

$$\begin{aligned} &P(x,0) = P_o(x), \\ &Q(x,0) = Q_0(x),\ \ x\in R. \end{aligned} \tag{2}$$

where c, F, Q are constants and defined as in [4,5,7,12,18,20], $P_0(x)$ and $Q_0(x)$ known continuous functions in $x \in R$.

Note that immediately after discretization the problems (1) and (2) it is impossible to obtain the Rosser model [19] because the derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial t}$, also $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial t}$ are involved into the same equation of the considering system. Therefore for applicability of the Roesser scheme [2] to Eq. (1), we use the following substitution into (2)

$$P(x,t) = R(x,t) + \alpha\, Q(x,t). \tag{3}$$

In (3) α is a parameter that balances the unit of measurements between P and Q.

Then we can write system (1) in the form:

$$\begin{cases} \dfrac{\partial R(x,t)}{\partial x} = -\alpha\dfrac{\partial Q(x,t)}{\partial x} - \dfrac{c}{F}\cdot\dfrac{\partial Q(x,t)}{\partial t}, \\ \dfrac{\partial Q(x,t)}{\partial x} = -F\left[\dfrac{\partial R(x,t)}{\partial t} + \alpha\dfrac{\partial Q(x,t)}{\partial t}\right] - 2aQ(x,t), \end{cases}$$

or

$$\begin{cases} \dfrac{\partial R(x,t)}{\partial x} + \alpha\dfrac{\partial Q(x,t)}{\partial x} = -\dfrac{c}{F}\cdot\dfrac{\partial Q(x,t)}{\partial t}, \\ \dfrac{\partial Q(x,t)}{\partial t} = -\dfrac{1}{F\alpha}\dfrac{\partial Q(x,t)}{\partial x} - \dfrac{1}{\alpha}\dfrac{\partial R(x,t)}{\partial t} - \dfrac{2a}{F\alpha}Q(x,t). \end{cases} \tag{4}$$

Now, taking into account the second equation of system (4) in the first one, we have

$$\begin{cases} \dfrac{\partial R(x,t)}{\partial x} = -\alpha\dfrac{\partial Q(x,t)}{\partial x} - \dfrac{c}{F}\left[-\dfrac{1}{F\alpha}\dfrac{\partial Q(x,t)}{\partial x} - \dfrac{1}{\alpha}\dfrac{\partial R(x,t)}{\partial t} - \dfrac{2a}{F\alpha}Q(x,t)\right], \\ \dfrac{\partial Q(x,t)}{\partial t} = -\dfrac{1}{F\alpha}\dfrac{\partial Q(x,t)}{\partial x} - \dfrac{1}{\alpha}\dfrac{\partial R(x,t)}{\partial t} - \dfrac{2a}{F\alpha}Q(x,t), \end{cases}$$

After grouping we obtain

$$\begin{cases} \dfrac{\partial R(x,t)}{\partial x} = \left(\dfrac{c}{F^2\alpha} - \alpha\right)\dfrac{\partial Q(x,t)}{\partial x} + \dfrac{c}{F\alpha}\dfrac{\partial R(x,t)}{\partial t} + \dfrac{2ac}{F^2\alpha}Q(x,t), \\ \dfrac{\partial Q(x,t)}{\partial t} = -\dfrac{1}{F\alpha}\dfrac{\partial Q(x,t)}{\partial x} - \dfrac{1}{\alpha}\dfrac{\partial R(x,t)}{\partial t} - \dfrac{2a}{F\alpha}Q(x,t). \end{cases} \tag{5}$$

Let accept the notations

$$\begin{cases} \dfrac{\partial Q(x,t)}{\partial x} = W(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = \chi(x,t),\ \ x\in R,\ \ t>0. \end{cases} \tag{6}$$

From systems (5) and (6) we have:

$$\begin{cases} \dfrac{\partial R(x,t)}{\partial x} = \left(\dfrac{c}{F^2\alpha} - \alpha\right)W(x,t) + \dfrac{c}{F\alpha}\chi(x,t) + \dfrac{2ac}{F^3\alpha}Q(x,t) \\ \dfrac{\partial Q(x,t)}{\partial t} = -\dfrac{1}{F\alpha}W(x,t) - \dfrac{1}{\alpha}\chi(x,t) - \dfrac{2a}{F\alpha}Q(x,t),\ \ x\in R,\ \ t>0. \end{cases} \tag{7}$$

Thus, we obtain two Roesser type systems, (6) and (7).

Lemma. Let the data c, F, Q, $P_0(x)$, $Q_0(x)$ be from an initial problem (1) and (2) such that its solution exists and is unique. Then using transformation (3) the systems of Eq. (1) may be reduced to two systems (6) and (7), which are continuous analog of Rosser model.

Note. As one can see from the systems (6) and (7) using the immediate discretization there is the discrete Rosser model.

3. Discretization method and discrete Roesser equation

For hyperbolic systems (6) and (7) we give the following discretization [1,15]

$$Q(x_i, t_i) = Q_i^j, R(x_i, t_i) = R_i^j,\ x_i = ih,\ \ t_j = j\tau.$$

$$\begin{cases} \dfrac{Q_{i+1}^j - Q_i^j}{h} = W_i^j, \\ \dfrac{R_i^{j+1} - R_i^j}{\tau} = \chi_i^j, \end{cases} \tag{8}$$

$$\begin{cases} \dfrac{R_{i+1}^j - R_i^j}{h} = \left(\dfrac{c}{F^2\alpha} - \alpha\right)W_i^j + \dfrac{c}{F\alpha}\chi_i^j + \dfrac{2ac}{F^2\alpha}Q_i^j, \\ \dfrac{Q_i^{j+1} - Q_i^j}{\tau} = \dfrac{-1}{F\alpha}W_i^j - \dfrac{1}{\alpha}\chi_i^j - \dfrac{2a}{F\alpha}Q_i^j, \end{cases} \tag{9}$$

or

$$\begin{cases} Q_{i+1}^j = Q_i^j + hW_i^j, \\ R_i^{j+1} = R_i^j + \tau\chi_i^j, \end{cases} \tag{10}$$

$$\begin{cases} R_{i+1}^j = R_i^j + \left(\dfrac{ch}{F^2\alpha} - \alpha h\right)W_i^j + \dfrac{ch}{F\alpha}\chi_i^j + \dfrac{2ach}{F^2\alpha}Q_i^j, \\ Q_i^{j+1} = Q_i^j - \dfrac{\tau}{F\alpha}W_i^j - \dfrac{\tau}{\alpha}\chi_i^j - \dfrac{2a\tau}{F\alpha}Q_i^j. \end{cases} \tag{11}$$

where the difference equations (10) and (11) describe the discrete Rosser model [19].

Finally, in (9)–(11) we make the following substitution [3]

$$R_i^j = S_i^j Q_i^j + K_i^j, \tag{12}$$

then the systems (10) and (11) take the form (Roesser type discrete equation):

$$\begin{cases} Q_{i+1}^j = Q_i^j + hW_i^j, \\ S_i^{j+1}Q_i^{j+1} + K_i^{j+1} = S_i^jQ_i^j + K_i^j + \tau\chi_i^j, \end{cases} \tag{13}$$

$$\begin{cases} S_{i+1}^jQ_{i+1}^j + K_{i+1}^j = S_i^jQ_i^j + K_i^j + \left(\dfrac{ch}{F^2\alpha} - \alpha h\right)W_i^j + \dfrac{ch}{F\alpha}\chi_i^j + \dfrac{2ach}{F^2\alpha}Q_i^j, \\ Q_i^{j+1} = Q_i^j - \dfrac{\tau}{F\alpha}W_i^j - \dfrac{\tau}{\alpha}\chi_i^j - \dfrac{2a\tau}{F\alpha}Q_i^j. \end{cases} \tag{14}$$

where new unknown values S_i^j, K_i^j, appropriate to the definition.

Note that W_i^j are determined from the first equation of system (8), and χ_i^j is obtained from the second equation of (13) in the form

$$\chi_i^j = \frac{S_i^{j+1}Q_i^{j+1} + K_i^{j+1} - S_i^jQ_i^j - K_i^j}{\tau},$$

and substituting them to the system of Eq. (14), we get:

$$\begin{cases} S_{i+1}^jQ_{i+1}^j + K_{i+1}^j = S_i^jQ_i^j + K_i^j + \left(\dfrac{ch}{F^2\alpha} - \alpha h\right)\dfrac{Q_{i+1}^j - Q_i^j}{h} + \dfrac{ch}{F\alpha}\dfrac{S_i^{j+1}Q_i^{j+1} + K_i^{j+1} - S_i^jQ_i^j - K_i^j}{\tau} + \dfrac{2ach}{F^2\alpha}Q_i^j, \\ Q_i^{j+1} = Q_i^j - \dfrac{\tau}{F\alpha}\dfrac{Q_{i+1}^j - Q_i^j}{h} - \dfrac{\tau}{\alpha}\dfrac{S_i^{j+1}Q_i^{j+1} + K_i^{j+1} - S_i^jQ_i^j - K_i^j}{\tau} - \dfrac{2a\tau}{F\alpha}Q_i^j, \end{cases}$$

that after grouping on Q has the form:

$$\begin{cases} \left(S_{i+1}^j - \dfrac{c}{F^2\alpha} + \alpha\right)Q_{i+1}^j - \dfrac{ch}{F\alpha\tau}S_i^{j+1}Q_i^{j+1} = \left(S_i^j - \dfrac{c}{F^2\alpha} + \alpha - \dfrac{ch}{F\alpha\tau}S_i^j + \dfrac{2ach}{F^2\alpha}\right)Q_i^j - K_{i+1}^j + K_i^j + \dfrac{ch}{F\alpha\tau}K_i^{j+1} - \dfrac{ch}{F\alpha\tau}K_i^j, \\ \left(1 + \dfrac{1}{\alpha}S_i^{j+1}\right)Q_i^{j+1} + \dfrac{\tau}{F\alpha h}Q_{i+1}^j = \left(1 + \dfrac{\tau}{F\alpha h} + \dfrac{1}{\alpha}S_i^j - \dfrac{2a\tau}{F\alpha}\right)Q_i^j - \dfrac{1}{\alpha}K_i^{j+1} + \dfrac{1}{\alpha}K_i^j. \end{cases}$$

(15)

So we have the following statement:

Theorem. By the condition of above-stated lemma using the “sweep” transformations (12) the Rosser type discrete systems (10) and (11) relatively Q_i^j, R_i^j are reduced to the Rosser type system (15) relatively S_i^j and Q_i^j.

4. Determination of $S_i^j, K_i^j, Q_i^j, P_i^j$ in order $O(\tau)$ for $\tau = h$

Now taking into account the relation

$$Q_{i+1}^j + Q_i^{j+1} = 2Q_i^j \tag{16}$$

that is valid within h and τ of first degree, i.e. $O(\tau)$ and $O(h)$, from (15) we get:

$$\begin{cases} \left(S_{i+1}^j - \frac{c}{F^2\alpha} + \alpha\right)2Q_i^j - \left(S_{i+1}^j - \frac{c}{F^2\alpha} + \alpha\right)Q_i^{j+1} - \frac{c}{F\alpha}S_i^{j+1}Q_i^{j+1} \\ \qquad = \left(S_i^j - \frac{c}{F^2\alpha} + \alpha - \frac{c}{F\alpha}S_i^j + \frac{2ach}{F^2\alpha}\right)Q_i^j - K_{i+1}^j + K_i^j + \frac{c}{F\alpha}K_i^{j+1} - \frac{c}{F\alpha}K_i^j, \\ \left(1 + \frac{1}{\alpha}S_i^{j+1}\right)Q_i^{j+1} + \frac{1}{F\alpha}2Q_i^j - \frac{1}{F\alpha}Q_i^{j+1} = \left(1 + \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j - \frac{2a\tau}{F\alpha}\right)Q_i^j - \frac{1}{\alpha}K_i^{j+1} + \frac{1}{\alpha}K_i^j, \end{cases}$$

After appropriate grouping, the last system has the form:

$$\begin{cases} \left(\frac{c}{F^2\alpha} - S_{i+1}^j - \alpha - \frac{c}{F\alpha}S_i^{j+1}\right)Q_i^{j+1} \\ \qquad = \left(S_i^j - \frac{c}{F^2\alpha} + \alpha - \frac{c}{F\alpha}S_i^j + \frac{2ach}{F^2\alpha} - 2S_{i+1}^j + \frac{2c}{F^2\alpha} - 2\alpha\right)Q_i^j - K_{i+1}^j + K_i^j + \frac{c}{F\alpha}K_i^{j+1} - \frac{c}{F\alpha}K_i^j, \\ \left(1 + \frac{1}{\alpha}S_i^{j+1} - \frac{1}{F\alpha}\right)Q_i^{j+1} = \left(1 + \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j - \frac{2a\tau}{F\alpha} - \frac{2}{F\alpha}\right)Q_i^j - \frac{1}{\alpha}K_i^{j+1} + \frac{1}{\alpha}K_i^j. \end{cases} \tag{17}$$

Finally we determine Q_i^{j+1} from the second equation of system (17) and substituting it into the first expression from (17) we get:

$$\begin{cases} \left[\left(\frac{c}{F^2\alpha} - S_{i+1}^j - \alpha - \frac{c}{F\alpha}S_i^{j+1}\right) \cdot \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j - \frac{2\tau a}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^{j+1}} \right. \\ \left. - \left(S_i^j - \frac{c}{F\alpha}S_i^j + \frac{2ach}{F^2\alpha} - 2S_{i+1}^j + \frac{c}{F^2\alpha} - \alpha\right)\right]Q_i^j \\ = -K_{i+1}^j + K_i^j + \frac{c}{F\alpha}K_i^{j+1} - \frac{c}{F\alpha}K_i^j \\ -\left(\frac{c}{F^2\alpha} - S_{i+1}^j - \alpha - \frac{c}{F\alpha}S_i^{j+1}\right) \cdot \frac{K_i^j - K_i^{j+1}}{\alpha - \frac{1}{F} + S_i^{j+1}}. \end{cases} \tag{18}$$

Taking into account that (18) is fulfilled regardless of Q_i^j, the coefficient Q_i^j converges to zero, i.e.

$$\begin{aligned} &S_{i+1}^j - \frac{1}{F\alpha}S_{i+1}^j - \frac{1}{\alpha}S_i^jS_{i+1}^j + \frac{2a\tau}{F\alpha}S_{i+1}^j - 2S_i^j + \frac{2a\tau}{F} - \frac{c}{F\alpha}S_i^{j+1} \\ &\quad + \frac{1}{F\alpha}S_i^j - \frac{1}{\alpha}S_i^jS_i^{j+1} + \frac{c}{F\alpha}S_i^j - \frac{2ach}{F^2\alpha} + \frac{2}{\alpha}S_i^{j+1}S_{i+1}^j + S_i^{j+1} = 0 \end{aligned} \tag{19}$$

and

$$\begin{aligned} &-\alpha K_{i+1}^j + 2\alpha K_i^j + \frac{c}{F}K_i^{j+1} - \frac{c}{F}K_i^j + \frac{1}{F}K_{i+1}^j - \frac{1}{F}K_i^j \\ &\quad - S_i^{j+1}K_{i+1}^j + S_i^{j+1}K_i^j + S_{i+1}^jK_i^j - S_{i+1}^jK_i^{j+1} - \alpha K_i^{j+1} = 0. \end{aligned} \tag{20}$$

So after some transformations (19) takes the form:

$$\begin{aligned} &-\frac{c}{F\alpha}S_i^{j+1} - \frac{1}{F\alpha}S_{i+1}^j - \frac{1}{\alpha}S_i^j(S_i^{j+1} + S_{i+1}^j) - 2S_i^j + (S_i^{j+1} + S_{i+1}^j) \\ &\quad + \frac{2a\tau}{F\alpha}S_{i+1}^j + \frac{2a\tau}{F} + \frac{c}{F\alpha}S_i^j - \frac{2ach}{F^2\alpha} + \frac{1}{F\alpha}S_i^j + \frac{2}{\alpha}S_i^{j+1}S_{i+1}^j = 0. \end{aligned}$$

Taking into account the equalities similar to (16) for S_i^j

$$S_{i+1}^j + S_i^{j+1} = 2S_i^j \tag{21}$$

from the last one we get:

$$-\frac{c}{F\alpha}(S_i^{j+1}-S_i^j)-\frac{1}{F\alpha}S_i^j+\frac{1}{F\alpha}S_i^{j+1}-\frac{2}{\alpha}S_i^{j2}+\frac{4a\tau}{F\alpha}S_i^j-\frac{2a\tau}{F\alpha}S_i^{j+1}+\frac{2a\tau}{F}-\frac{2ach}{F^2\alpha}+\frac{4}{\alpha}S_i^{j+1}S_i^j-\frac{2}{\alpha}\left(S_i^{j+1}\right)^2=0$$

or

$$-\frac{2}{\alpha}(S_i^{j+1}-S_i^j)^2-\frac{c}{F\alpha}(S_i^{j+1}-S_i^j)+\frac{1}{F\alpha}(S_i^{j+1}-S_i^j)+\frac{4a\tau}{F\alpha}S_i^j-\frac{2a\tau}{F\alpha}S_i^{j+1}+\frac{2a\tau}{F}-\frac{2ach}{F^2\alpha}=0.$$

Thus, for determining S_i^j we arrive to the following quadratic equation

$$-\frac{2}{\alpha}(S_i^{j+1}-S_i^j)^2-\left(\frac{c}{F\alpha}-\frac{1}{F\alpha}+\frac{2a\tau}{F\alpha}\right)\left(S_i^{j+1}-S_i^j\right)+\left(\frac{2a\tau}{F}-\frac{2ach}{F^2\alpha}+\frac{2a\tau}{F\alpha}S_i^j\right)=0, \tag{22}$$

the solution of which is represented in the form:

$$S_i^{j+1}-S_i^j=\frac{\frac{c}{F\alpha}-\frac{1}{F\alpha}+\frac{2a\tau}{F\alpha}\pm\sqrt{\left(\frac{c}{F\alpha}-\frac{1}{F\alpha}+\frac{2a\tau}{F\alpha}\right)^2+\frac{8}{\alpha}\left(\frac{2a\tau}{F}-\frac{2ach}{F^2\alpha}+\frac{2a\tau}{F\alpha}S_i^j\right)}}{-\frac{4}{\alpha}}. \tag{23}$$

In order (23) have the sense, in $\pm$ we should take the sign "–". Then from (23) we have:

$$S_i^{j+1}=S_i^j+\frac{1}{4}\sqrt{\left(\frac{c}{F}-\frac{1}{F}+\frac{2a\tau}{F}\right)^2+\frac{16a}{F}\left(\alpha\tau-\frac{ch}{F}+\tau S_i^j\right)}-\frac{1}{4F}(c+2a\tau-1). \tag{24}$$

Similar to (21), considering

$$K_{i+1}^j+K_i^{j+1}=2K_i^j, \tag{25}$$

from (20) we have:

$$\frac{c}{F}\left(K_i^{j+1}-K_i^j\right)+\frac{1}{F}K_i^j-\frac{1}{F}K_i^{j+1}-2S_i^{j+1}K_i^j+2S_i^{j+1}K_i^{j+1}+2K_i^jS_i^j-2K_i^{j+1}S_i^j=0.$$

or

$$\frac{c}{F}\left(K_i^{j+1}-K_i^j\right)-\frac{1}{F}\left(K_i^{j+1}-K_i^j\right)+2S_i^{j+1}\left(K_i^{j+1}-K_i^j\right)-2S_i^j\left(K_i^{j+1}-K_i^j\right)=0$$

or

$$\left(\frac{c}{F}-\frac{1}{F}+2S_i^{j+1}-2S_i^j\right)\left(K_i^{j+1}-K_i^j\right)=0. \tag{26}$$

As is seen from (23)

$$2\left(S_i^{j+1}-S_i^j\right)+\frac{c}{F}-\frac{1}{F}\neq 0. \tag{27}$$

Then from (26) and (27) we have:

$$K_i^{j+1}=K_i^j. \tag{28}$$

From (28) should that K_i^j is independent from j i.e. $K(x, t)$ is independent from t.

$$K(x,t)\equiv K_0(x). \tag{29}$$

Coming back to (25) we can easily see that if S_i^0 is known, then we can determine all S_i^j.

As is seen from (3)

$$R(x,0)=P(x,0)-\alpha Q(x,0)=P_0(x)-\alpha Q_0(x) \tag{30}$$

i.e. from (30) we have:

$$R_i^0=P_i^0-\alpha Q_i^0, \tag{31}$$

and from (29) we obtain

$$K_i^j=K_i^0. \tag{32}$$

Then from (12) we find:

$$S_i^0=\frac{R_i^0-K_i^0}{Q_i^0}=\frac{P_i^0-\alpha Q_i^0-K_i^0}{Q_i^0}. \tag{33}$$

If to take

$$K_i^0 = 0,$$

then from (31)–(33) we have

$$K_i^j \equiv 0, \tag{34}$$

and

$$S_i^0 = \frac{P_i^0}{Q_i^0} - \alpha. \tag{35}$$

Taking into account (34) from the second equation of system (17) we have:

$$Q_i^{j+1} = \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j - \frac{2a\tau}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^{j+1}} Q_i^j, \tag{36}$$

and from (12) we find

$$R_i^j = S_i^j \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^{j-1} - \frac{2a\tau}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j} Q_i^{j-1}.$$

Finally P_i^j is determined from discretization (3)

$$P_i^j = R_i^j + \alpha Q_i^j = S_i^j \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^{j-1} - \frac{2a\tau}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j} Q_i^{j-1} + \alpha \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^{j-1} - \frac{2a\tau}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^j} Q_i^{j-1}, \tag{37}$$

$$P(x,t) = S(x,t)Q(x,t) + \alpha Q(x,t) = (\alpha + S)Q.$$

Now, we calculate the approximate square root from (24), i.e.

$$\sqrt{\left(\frac{c}{F} - \frac{1}{F} + \frac{2a\tau}{F}\right)^2 + \frac{16a}{F}\left(\alpha\tau - \frac{ch}{F} + \tau S_i^j\right)} \approx \left(\frac{c}{F} - \frac{1}{F} + \frac{2a\tau}{F}\right) + \frac{8a}{F}\frac{\alpha\tau - \frac{ch}{F} + \tau S_i^j}{\frac{c}{F} - \frac{1}{F} + \frac{2a\tau}{F}}.$$

Then from (24) we get:

$$S_i^{j+1} = S_i^j + \frac{2a}{F}\frac{\alpha\tau - \frac{ch}{F} + \tau S_i^j}{\frac{c}{F} - \frac{1}{F} + \frac{2a\tau}{F}} = \left(1 + \frac{2\tau a}{c + 2a\tau - 1}\right)S_i^j + \frac{2a}{F}\frac{\alpha\tau - \frac{ch}{F}}{-\frac{1}{F} + \frac{2a\tau}{F} + \frac{c}{F}}$$

or

$$S_i^{j+1} = S_i^j + 2a\frac{\alpha\tau - \frac{ch}{F} + \tau S_i^j}{c + 2a\tau - 1} = \frac{cS_i^j - S_i^j + 2a\tau S_i^j + 2a\tau\alpha - \frac{2ach}{F} + 2a\tau S_i^j}{c + 2a\tau - 1},$$

i.e.

$$S_i^{j+1} = \frac{cS_i^j - S_i^j + 4ahS_i^j + 2a\alpha h - \frac{2ac\tau}{F}}{c - 1 + 2a\tau}. \tag{38}$$

Thus, from (35) and (38) step by step are uniquely determined S_i^j. In its turn, it allows one to find K_i^j, P_i^j, Q_i^j respectively. So, for this case we have the following computational algorithm for finding P_i^j, Q_i^j .
Algorithm.

1. The parameters F, a, c are given from (1)
2. All S_i^j are determined from (35) and (38) and K_i^j are determined from (34)
3. Q_i^j are defined from (36) and (2) ($Q_0(x) \approx Q_i^0$)
4. P_i^j are defined from (37) and (2) ($P_0(x) \approx P_i^0$)
5. The residual of the system (1) is calculated. If it is sufficient, then the process stops, otherwise reduce the steps h and τ, and go to 1.

5. Example

Let us consider the following example with initial condition when S is constant and $\tau = h$, i.e. and S_i^j from (4) is of the form

$$S_i^0 = const = S,$$

$$S_i^1 = S + 2a\frac{\left(\alpha - \frac{c}{F} + S\right)\tau}{c - 1 + 2a\tau}.$$

In the same way

$$Q_i^1 = \frac{\left(1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S\right)Q_i^0}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^1} - \frac{\frac{2a}{F\alpha}Q_i^0}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^1}\tau.$$

Similar to Q_i^1 for P_i^1 we have:

$$P_i^1 = S_i^1 \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^0 - \frac{2a\tau}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^1} Q_i^0 + \alpha \frac{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^0 - \frac{2a\tau}{F\alpha}}{1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S_i^1} Q_i^0.$$

Thus, restricting the first term of Taylor's formula, we have:

$$\begin{aligned} S_i^1 &= S + 2a \left.\frac{\left(\alpha - \frac{c}{F} + S\right)(c - 1 + 2a\tau) - \left(\alpha - \frac{c}{F} + S\right)\tau 2a}{(c - 1 + 2a\tau)^2}\right|_{\tau=0} \tau \\ &= S + 2a \frac{\left(\alpha - \frac{c}{F} + S\right)(c-1)}{(c-1)^2}\tau. \end{aligned}$$

With regard to the expression S_i^1 forP_i^1 we get:

$$P_i^1 = P_0 + \alpha Q_0 + \left[\frac{2a\left(\alpha - \frac{c}{F} + S\right)\left(1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S\right) - S(c-1)\left(\frac{2a}{F\alpha} + \frac{S}{\alpha}\right)}{(c-1)\left(1 - \frac{1}{F\alpha} + \frac{S}{\alpha}\right)} - (c-1)\frac{2a}{F\alpha}\right] Q\tau.$$

Taking into account that the new scheme of $S + \alpha$ coincides with S from the old scheme in expression P_0^1, then we select α by the following way [4]:

$$\frac{2a\left(\alpha - \frac{c}{F} + S\right)\left(1 - \frac{1}{F\alpha} + \frac{1}{\alpha}S\right) - S(c-1)\left(\frac{2a}{F\alpha} + \frac{S}{\alpha}\right) - \frac{2a}{F\alpha}(c-1)}{(c-1)\left(1 - \frac{1}{F\alpha} + \frac{S}{\alpha}\right)} = -\frac{2a}{F\alpha}.$$

This expression allows us to determine $\alpha$from the following equations

$$\frac{2a(F\alpha - c + SF)(F\alpha - 1 + SF) - SF(c-1)(2a + FS) - 2aF(c-1)}{F(c-1)(F\alpha - 1 + SF)} = -\frac{2a}{F\alpha},$$

or

$$\begin{aligned} &2aF^3\alpha^3 - \left(2aF^2 - 4aF^3S + 2aF^2c\right)\alpha^2 \\ &+ \left(2acF - 4aF^2Sc + 2aF^3S^2 - F^3Sc - F^3S^2 + 4aF^2c\right)\alpha \\ &- \left(2aFc - 2aF^2Sc - 2aF + 2aF^2S\right) = 0. \end{aligned}$$

The obtained third order equation with real coefficients has always of least one real root and for finding it we can use the Kardano formula [16].

6. Conclusion

We present here Roesser's discrete model for the system of hyperbolic equation describing the motion in gas lift process. For the first time, the sweep method is proposed for Roesser discrete equation. Efficiency of the suggested algorithm is given on the example.

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