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A note on measure-geometric Laplacians

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Abstract We consider the measure-geometric Laplacians Δ^{μ} with respect to atomless compactly supported Borel probability measures μ as introduced by Freiberg and Zähle (Potential Anal. 16(1):265–277, 2002) and show that the harmonic calculus of Δ^{μ} can be deduced from the classical (weak) Laplacian. We explicitly calculate the eigenvalues and eigenfunctions of Δ^{μ}. Further, it is shown that there exists a measure-geometric Laplacian whose eigenfunctions are the Chebyshev polynomials and illustrate our results through specific examples of fractal measures, namely inhomogeneous self-similar Cantor measures and Salem measures.

Keywords Measure-geometric Laplacians · Spectral asymptotics · Singular measures · Chebyshev polynomials · Salem measures

Mathematics Subject Classification 35P20 · 42B35 · 47G30 · 45D05

1 Introduction

Kigami constructed a Laplacian for fractal sets, in particular post-critically finite attractors of iterated function systems, as the limit of (normalised) difference operators on a sequence of finite graphs, which approximate the attractor, see for instance [17,22].

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(Key examples of such attractors are the closed unit interval and the Sierpinski gasket.) This theory has been further developed and extensively studied by Kigami, Kusuoka, Rogers, Strichartz and Teplayev to name but a few. Another approach was developed by Goldstein [11], Kusuoka [18], Barlow and Perkins [3] and Lindstrøm [19] and has a probabilistic motivation. This approach defines a Laplacian as the infinitesimal generator of a Brownian motion, but certain assumptions on the class of attractors need to be made. Again this approach uses finite graph approximation methods. A different probabilistic approach has also been considered by Denker and Sato [5,6] and Ju et al. [14].

The previous approaches relied on graph approximation methods, with the following approach such approximations are not required. Given an atomless Borel probability measure μ supported on a compact subset K of $\mathbb{R}$, motivated by the fundamental theorem of calculus, Feller [8] and Kac and Kreĭn [15], introduced a derivative $\frac{\mathrm{d}}{\mathrm{d}\mu}$ with respect to μ, and subsequently the Laplacian $\Delta^{\mu,\nu} = \frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}\nu}$ has been extensively studied. For instance, when μ is self-similar measure and ν is the Lebesgue measure Bird, Ngai and Teplyaev [4] and Arzt [1] study the asymptotic behaviour of the eigenvalue counting function of $\Delta^{\mu,\nu}$. In [1], the author develops trigonometric identities for the eigenfunctions and in certain cases obtains information about the growth of the supremum of normalised eigenfunctions. In [4], under the weaker assumption that μ is an atomless Borel probability measures, existence, uniqueness, concavity, and properties of zeros of the eigenfunctions are proven. Further, when μ is a self-similar measure supported on the attractor of an iterated function system S and ν is S-homogenous, Freiberg [9] obtained the asymptotic behaviour of the eigenvalue counting function of $\Delta^{\mu,\nu}$. Moreover, Freiberg and Zähle [10] have studied the case when $\nu = \mu$ and developed a harmonic calculus of $\Delta^{\mu} := \Delta^{\mu,\mu}$. In the special case that μ is a self-similar measure supported on a Cantor-like set, it is shown that the eigenvalue counting function of Δ^{μ} is comparable to the square-root function.

In this article we extend the results of [10] in that we give a complete solution to the eigenvalue problem. Namely, for a given atomless Borel probability measure μ supported on K where $\{a, b\} \subset K \subset [a, b]$, for some real numbers $a < b$, with continuous distribution function F_{μ}, we prove the following Theorem. Indeed an indication that such a result should hold is given in [24].

Theorem 1.1 *Let the domain of the operator Δ^{μ} be denoted by $\mathscr{D}_2^{\mu}$ and set $\lambda_n := -(\pi n)^2$, for $n \in \mathbb{N}_0$.*

(i) *The eigenvalues of Δ^{μ} on $\mathscr{D}_2^{\mu}$ under homogeneous Dirichlet boundary conditions are λ_n, for $n \in \mathbb{N}$, with corresponding eigenfunctions*

$$f_n^{\mu}(x) := \sin(\pi n F_{\mu}(x)), \quad \text{for } x \in [a, b].$$

(ii) *The eigenvalues of Δ^{μ} on $\mathscr{D}_2^{\mu}$ under homogeneous von Neumann boundary conditions are λ_n, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions*

$$g_n^{\mu}(x) := \cos(\pi n F_{\mu}(x)), \quad \text{for } x \in [a, b].$$

In the proof of this result we will see that apart from the constant functions the μ-derivatives of von Neumann eigenfunctions are Dirichlet eigenfunctions and vice versa.

Remark 1.1 Theorem 1.1 demonstrates that the concept of fractal Fourier analysis (as defined by Bandt et al. [2]) has interesting connections to the measure-geometric Laplacian Δ^μ considered here.

Letting $N_D^\mu \colon \mathbb{R} \to \mathbb{N}$ denote the eigenvalue counting function of $-\Delta^\mu$ on $\mathscr{D}_2^\mu$ under homogeneous Dirichlet boundary conditions and letting $N_N^\mu \colon \mathbb{R} \to \mathbb{N}$ denote the eigenvalue counting function of $-\Delta^\mu$ on $\mathscr{D}_2^\mu$ under homogeneous von Neumann boundary conditions, the following can be obtained.

Corollary 1.2

$$\lim_{x\to\infty} \frac{N_D^\mu(x)}{(\pi x)^{1/2}} = \lim_{x\to\infty} \frac{N_N^\mu(x)}{(\pi x)^{1/2}} = 1$$

To illustrate our results, we show that there exists a measure-geometric Laplacian whose eigenfunctions are the Chebyshev polynomials and also consider fractal measures, namely inhomogeneous self-similar Cantor measures and Salem measures.

1.1 Outline

In Sect. 2 we present the necessary definitions and basic properties. In Sect. 3 we give the proof Theorem 1.1; first proving the result for when μ is the one-dimensional Lebesgue measure Λ restricted to the unit interval, and then using a transfer principle to prove the result for the general case. We conclude by considering specific examples, namely when $F_\mu(x) = \arccos(-x)/\pi$, when μ is an inhomogeneous self-similar Cantor measure (that is a measure singular to the Lebesgue measure) and where F_μ is a Salem function.

2 Preliminaries: derivatives and the Laplacian with respect to measures

Following conventions, the natural numbers will be denoted by $\mathbb{N}$, the non-negative integers by $\mathbb{N}_0$ and the real numbers by $\mathbb{R}$. Throughout let $a < b$ be two fixed real numbers. The one-dimensional Lebesgue measure will be denoted by Λ. Henceforth, we will be concerned with an atomless Borel probability measure μ supported on K where $\{a, b\} \subset K \subset [a, b]$. We denote the set of $\mathbb{R}$-valued functions with domain $[a, b]$ which are square integrable with respect to μ by $\mathfrak{L}^2(\mu)$. We set $L^2(\mu)$ to be the space of (measure) equivalence classes of $\mathfrak{L}^2(\mu)$-functions with respect to μ. For a measurable function f we write $f \in L^2(\mu)$ when there exists an equivalence class of $L^2(\mu)$ to which f belongs. For a given set $A \subseteq \mathbb{R}$, the characteristic function of A is denoted by $\mathbb{1}_A$.

We recall the definition of the operator Δ^μ as given in [10]. We define $\mathscr{D}_1^\mu \subseteq \mathfrak{L}^2(\mu)$ by

$$\mathscr{D}_1^\mu := \left\{ f \in \mathfrak{L}^2(\mu) \colon \exists\, f' \in L^2(\mu) \text{ s.t. } f(x) = f(a) + \int_a^x f'(y)\, \mathrm{d}\mu(y),\ x \in [a,b] \right\}. \tag{1}$$

For $f \in \mathscr{D}_1^\mu$ the function $f' \in L^2(\mu)$ given in (1) is unique. Moreover, every function in $\mathscr{D}_1^\mu$ is continuous on $[a,b]$. (For a proof of the first statement see [10, Proposition 2.1]; the second statement is a consequence of Lebesgue's dominated convergence theorem.) From these observations, we conclude that if $f, g \in \mathscr{D}_1^\mu$ with $f \neq g$, then $\mu(\{x \in [a,b] \colon f(x) \neq g(x)\}) > 0$. Hence, there exists a natural embedding $\pi \colon \mathscr{D}_1^\mu \to L^2(\mu)$. For both $\mathscr{D}_1^\mu$ and $\pi(\mathscr{D}_1^\mu)$ we write $\mathscr{D}_1^\mu$. Letting $f \in \mathscr{D}_1^\mu$ and letting f' be as in (1), the operator

$$\nabla^\mu : \mathscr{D}_1^\mu \to L^2(\mu),\ \ f \mapsto f',$$

is called the *μ-derivative*. In the literature this operator is also denoted by $\frac{\mathrm{d}}{\mathrm{d}\mu}$, however, here we follow the notation of Freiberg and Zähle [10] and use ∇^μ. We define $\mathscr{D}_2^\mu \subseteq \mathscr{D}_1^\mu$ by

$$\mathscr{D}_2^\mu := \left\{ f \in \mathscr{D}_1^\mu \colon \nabla^\mu f \in \mathscr{D}_1^\mu \right\}.$$

The *μ-Laplace operator* is defined by

$$\Delta^\mu \colon \mathscr{D}_2^\mu \to L_\mu^2,\ \ f \mapsto \nabla^\mu \circ \nabla^\mu f$$

and is a non-positive operator [10, Corollary 2.3]. (Note, this operator Δ^μ is different to the operator $\Delta_\mu := \nabla^\mu \nabla^\Lambda$ used in [4]; here Λ denotes the one-dimensional Lebesgue measure.)

For $f \in \mathscr{D}_2^\mu$ Fubini's theorem gives the representation

$$f(x) = f(a) + \nabla^\mu f(a) F_\mu(x) + \int_a^x (F_\mu(x) - F_\mu(y)) \Delta^\mu f(y)\, \mathrm{d}\mu(y). \tag{2}$$

A function $f \in \mathscr{D}_2^\mu$ is said to satisfy *homogenous Dirichlet boundary conditions* if and only if $f(a) = f(b) = 0$, whereas if $\nabla^\mu f(a) = \nabla^\mu f(b) = 0$, then we say that f satisfies *homogenous von Neumann boundary conditions*.

From this all *μ-harmonic functions* can be computed, namely functions for which $\Delta^\mu f \equiv 0$. Indeed the set of μ-harmonic functions is a two-dimensional space and equal to

$$\{x \mapsto A + BF_\mu(x) \colon A, B \in \mathbb{R}\}.$$

Notice that the operator Δ^Λ is the classical weak Laplacian.

It is shown in the proof of [10, Theorem 2.5] that if l_κ is a solution to the eigenvalue problem

$$\Delta^{\mu}(l_\kappa) = \kappa l_\kappa, \tag{3}$$

for some $\kappa \in \mathbb{R}$, under the boundary conditions $l_\kappa(a) = 0$ and $\nabla^{\mu} l_\kappa(a) = 1$, then it is unique and necessarily satisfies the Volterra type integral equation

$$l_\kappa(x) = F_\mu(x) + \kappa \int_a^x (F_\mu(x) - F_\mu(y)) l_\kappa(y) \, \mathrm{d}\mu(y), \;\; x \in [a, b]. \tag{4}$$

If l_κ is a solution to the eigenvalue problem (3) under the boundary conditions $l_\kappa(a) = 1$ and $\nabla^{\mu} l_\kappa(a) = 0$, then it is unique and necessarily satisfies the Volterra type integral equation

$$l_\kappa(x) = 1 + \kappa \int_a^x (F_\mu(x) - F_\mu(y)) l_\kappa(y) \, \mathrm{d}\mu(y), \;\; x \in [a, b].$$

3 Proof of Theorem 1.1: eigenfunctions and eigenvalues

Here we prove our main result Theorem 1.1. We divide the proof into two cases: the case when μ is the Lebesgue measure Λ supported on the closed unit interval, and the general case when the distribution function F_μ of μ is continuous. We will in fact deduce the general case from the specific case when $\mu = \Lambda$.

3.1 The Lebesgue case

Theorem 3.1 *Let Λ denote the Lebesgue measure restricted to the closed unit interval.*

(i) *Assuming homogeneous Dirichlet boundary conditions, the eigenvalues of Δ^{Λ} are $\lambda_n = -(\pi n)^2$, for $n \in \mathbb{N}$, with corresponding eigenfunctions $f_n^{\Lambda}(x) = \sin(\pi n x)$.*
(ii) *Assuming homogeneous von Neumann boundary conditions the eigenvalues of Δ^{Λ} are $\lambda_n = -(\pi n)^2$, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions $g_n^{\Lambda}(x) = \cos(\pi n x)$.*

Proof On the set of continuous functions supported on [0, 1] the Λ-Laplacian agrees with the (classical) weak Laplacian. Using this observation in tandem with the classical Sturm-Liouville theory (see for instance [23]) the result follows. □

3.2 The general case

Henceforth, we assume that μ is a Borel probability measure supported on K, where $\{a, b\} \subset K \subset [a, b]$, with continuous distribution function F_μ. Before we give the proof of our main theorem, we provide two essential Lemmas. The first lemma follows from elementary measure theory and is left to the reader. The second connects the initial value problem with the solution of an integral equation.

Lemma 3.2 *We have that*

(i) $\mu \circ F_\mu^{-1} = \Lambda$ *and*
(ii) $\Lambda \circ F_\mu = \mu$.

Lemma 3.3 *Let $\kappa \in \mathbb{R}$ be given. Under the boundary conditions $f(a) = A$ and $\nabla^\mu f(a) = B$, there exists a unique solution to the integral equation*

$$f(x) = A + BF_\mu(x) + \kappa \int_a^x (F_\mu(x) - F_\mu(y)) f(y) \, \mathrm{d}\mu(y),$$

for all $x \in [a, b]$.

Proof For $\alpha > 0$, the set of continuous functions supported on $[a, b]$ equipped with the norm $\|f\|_\alpha := \sup\{|f(x)|\mathrm{e}^{-\alpha F_\mu(x)} : x \in [a, b]\}$ is complete. For $\alpha > \kappa$ the operator

$$f(x) \mapsto A + BF_\mu(x) + \kappa \int_a^x (F_\mu(x) - F_\mu(y)) f(y) \, \mathrm{d}\mu(y)$$

is a contraction with respect to $\|\cdot\|_\alpha$. The result follows from Banach fixed point theorem. □

We define the *pseudoinverse* of the distribution function F_μ by

$$\check{F}_\mu^{-1} \colon [0, 1] \to [a, b], \ x \mapsto \inf\{y \in [a, b] \colon F_\mu(y) \geq x\}.$$

Notice that $F_\mu \circ \check{F}_\mu^{-1}(x) = x$, for all $x \in [0, 1]$, and that $\check{F}_\mu^{-1} \circ F_\mu(y) = y$, for μ-almost all $y \in [a, b]$. Also note that $\check{F}_\mu^{-1}(1) = b$ since b belongs to the support of μ.

We are now in a position to prove our main result, Theorem 1.1.

Proof of Theorem 1.1 As the proofs under Dirichlet and von Neumann boundary conditions follow in an identical manner, we only present the proof of the result under Dirichlet boundary conditions, namely part (i). In essence, to obtain (ii) from (i) one only needs to interchange the functions $x \mapsto \sin(x)$ and $x \mapsto \cos(x)$ in what follows.

First we show that the functions f_n^μ are eigenfunctions. Second we prove that if l_κ is an eigenfunction of Δ^μ with eigenvalue κ, then $l_\kappa \circ \check{F}_\mu^{-1}$ is an eigenfunction of Δ^Λ. Thus, the functions f_n^μ are all of the eigenfunctions of Δ^μ under Dirichlet boundary conditions.

By Lemma 3.2, we have that, for all $x \in [a, b]$,

$$\begin{aligned} \int_a^x \pi n \cos(\pi n F_\mu(y)) \, \mathrm{d}\mu(y) &= \int_0^{F_\mu(x)} \pi n \cos(\pi n y) \, \mathrm{d}\Lambda(y) \\ &= \sin(\pi n F_\mu(x)) = f_n^\mu(x). \end{aligned}$$

By the definition of the μ-derivative, for all $x \in [a, b]$, we have that

$$f_n^\mu(x) = f_n^\mu(a) + \int_a^x \nabla^\mu f_n^\mu(y)\, \mathrm{d}\mu(y) = \int_a^x \nabla^\mu f_n^\mu(y)\, \mathrm{d}\mu(y).$$

It therefore follows that $\nabla^\mu f_n^\mu = \pi n \cos(\pi n F_\mu)$. This together with Lemma 3.2 implies that, for all $x \in [a, b]$,

$$\begin{aligned} \nabla^\mu f_n^\mu(x) &= \pi n - \pi^2 n^2 \int_0^{F_\mu(x)} \sin(\pi n y)\, \mathrm{d}\Lambda(y) \\ &= \nabla^\mu f_n^\mu(a) + \int_a^x (-\pi^2 n^2) \sin(\pi n F_\mu(y))\, \mathrm{d}\mu(y), \end{aligned}$$

and hence that

$$\Delta^\mu f_n^\mu(x) = -\pi^2 n^2 \sin(\pi n F_\mu(x)) = -\pi^2 n^2 f_n^\mu(x).$$

Recall that Δ^μ is a non-positive operator. Suppose that l_κ is an eigenfunction of Δ^μ, with eigenvalue $\kappa \leq 0$ under Dirichlet boundary conditions. By (2) and Lemma 3.3, it follows that $\nabla^\mu l_\kappa(a) \neq 0$, and so without loss of generality, we may assume that $\nabla^\mu l_\kappa(a) = 1$. Using (4) we observe that, for $z \in [0, 1]$,

$$\begin{aligned} l_\kappa \circ \check{F}_\mu^{-1}(z) &= z + \kappa \int \mathbb{1}_{[a, \check{F}_\mu^{-1}(z)]}(\check{F}_\mu^{-1} \circ F_\mu(y))(z - F_\mu(y)) l_\kappa(y)\, \mathrm{d}\mu(y) \\ &= z + \kappa \int \mathbb{1}_{[0,z]}(F_\mu(y))(z - F_\mu(y)) l_\kappa \circ \check{F}_\mu^{-1} \circ F_\mu(y)\, \mathrm{d}\mu(y) \\ &= z + \kappa \int \mathbb{1}_{[0,z]}(y)(z - y) l_\kappa \circ \check{F}_\mu^{-1}(y)\, \mathrm{d}\mu \circ F_\mu^{-1}(y) \\ &= z + \kappa \int_0^z (z - y) l_\kappa \circ \check{F}_\mu^{-1}(y)\, \mathrm{d}\Lambda(y). \end{aligned}$$

For $\alpha > -\kappa$, on the set of continuous functions supported on $[0, 1]$ the operator

$$f(z) \mapsto z + \kappa \int_0^z (z - y) f(y)\, \mathrm{d}\Lambda(y)$$

is contractive with respect to the norm $\|\cdot\|_\alpha$, where the norm $\|\cdot\|_\alpha$ is as defined in the proof of Lemma 3.3. Banach's fixed point theorem implies that $l_\kappa \circ \check{F}_\mu^{-1}(z) = \sin(\sqrt{-\kappa} z)/\sqrt{-\kappa}$. By the fact that $l_\kappa(a) = l_\kappa(b) = 0$, it follows that $l_\kappa \circ \check{F}_\mu^{-1}(0) = l_\kappa \circ \check{F}_\mu^{-1}(1) = 0$. Therefore, $l_\kappa \circ \check{F}_\mu^{-1}$ is an eigenfunction of Δ^Λ, and hence of the form $z \mapsto \sin(n\pi z)$. □

Remark 3.1 An alternative proof for the fact that f_n^μ is an eigenfunction with eigenvalue $-\pi^2 n^2$ of Δ^μ may be obtained by adapting methods of [1], which we now outline. Assume homogeneous Dirichlet bounday condition and define $p_0(x) := 1$ and

$$p_n(x) := \int_a^x p_{n-1}(y)\,\mathrm{d}\mu(y),$$

for $n \in \mathbb{N}$ and $x \in [a,b]$. By an inductive argument it follows that $p_n(x) = (n!)^{-1}(F_\mu(x))^n$. Since, by definition, $\nabla^\mu p_{n+1} = p_n$ and noting that

$$\sin(\pi n F_\mu(x)) = \sum_{k=0}^{\infty}(-1)^k(\pi n)^{2k+1}\frac{F_\mu(x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k(\pi n)^{2k+1}p_{2k+1}(x),$$

we have $x \mapsto \sin(\pi n F_\mu(x))$ is an eigenfunction of Δ_μ under homogeneous Dirichlet boundary conditions. Similar arguments can be used when assuming von Neumann boundary conditions.

4 Examples: Chebyshev polynomials, Salem measures and inhomogeneous Cantor measures

4.1 Chebyshev polynomials as eigenfunctions

For this example we choose $a = -b = -1$ and let Λ denote the Lebesgue measure restricted to the interval $[-1,1]$. Consider the absolutely continuous probability measure μ supported on $[-1,1]$ given by

$$\frac{\mathrm{d}\mu}{\mathrm{d}\Lambda}(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

The distribution function F_μ can be determined explicitly:

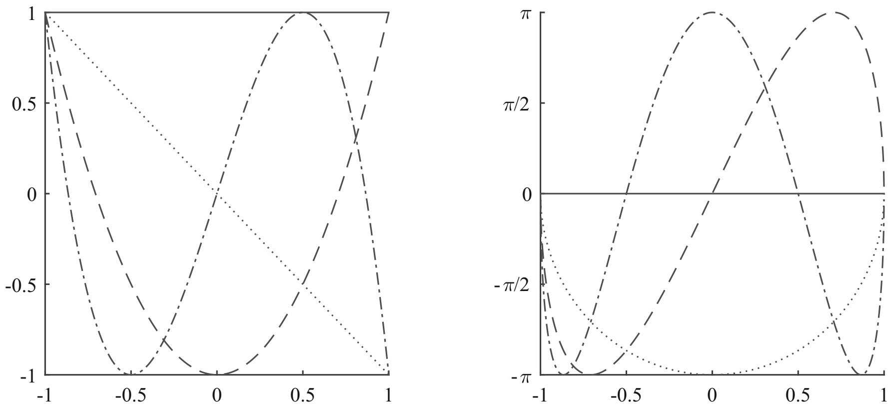
$$F_\mu(x) = \frac{1}{\pi}\int_{-1}^{x}\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}\Lambda(t) = \frac{1}{\pi}\left(\arcsin(x) + \frac{\pi}{2}\right) = \frac{1}{\pi}\arccos(-x).$$

We remind the reader that the Chebyshev polynomials (of the first kind) are given by

$$T_0(x) := 1, \quad T_1(x) := x, \quad \text{and} \quad T_{n+1}(x) := 2xT_n(x) - T_{n-1}(x);$$

hence T_{2k} is an even function and T_{2k+1} is an odd function, for $k \in \mathbb{N}_0$. Moreover it is well known that the Chebyshev polynomials can also be defined through the following formula:

$$T_n(x) = \cos(n\arccos(x)),\ \ x \in [-1,1].$$



(a) Plot of $g_0^\mu, g_1^\mu, g_2^\mu$ and g_3^μ **(b)** Plot of $\nabla^\mu g_0^\mu, \nabla^\mu g_1^\mu, \nabla^\mu g_2^\mu$ and $\nabla^\mu g_3^\mu$

Fig. 1 Eigenfunctions of Δ^μ, where $\mathrm{d}\mu/\mathrm{d}\Lambda = 1/(\pi\sqrt{1-x^2})$, under von Neumann boundary conditions, and their μ-derivatives

This together with Theorem 1.1 implies that the von Neumann eigenfunctions g_n^μ are closely related to the Chebyshev polynomials, that is, for all $n \in \mathbb{N}_0$ and $x \in [-1, 1]$,

$$g_n^\mu(x) = \cos(\pi n F_\mu(x)) = \cos(n \arccos(-x)) = T_n(-x) = (-1)^n T_n(x),$$

(see Fig. 1a). We also observe that, for $n \in \mathbb{N}_0$ and $x \in [-1, 1]$,

$$\nabla^\mu g_n^\mu(x) = (-1)^n \nabla^\mu T_n(x) = \pi\sqrt{1-x^2}\frac{\mathrm{d}T_n}{\mathrm{d}x}(x) = \frac{(-1)^n \pi n(xT_n(x) - T_{n+1}(x))}{\sqrt{1-x^2}},$$

(see Fig. 1b). Moreover, it can be shown that $f_n^\mu(x) = \nabla^\mu g_n^\mu(x)$, for all $x \in [-1, 1]$ and $n \in \mathbb{N}$. For further details and results concerning the Chebyshev polynomials see [20].

4.2 Inhomogeneous Cantor measures

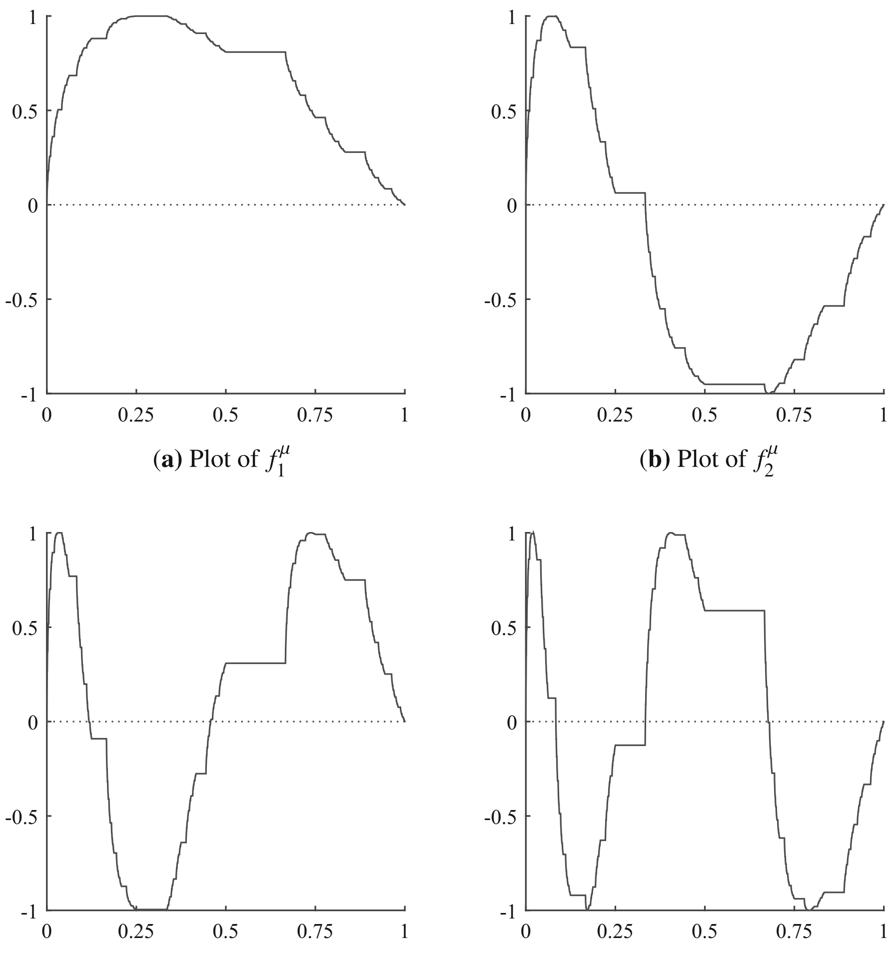
Given an interval $[a, b]$ and a finite set of contractions $S = \{s_i : [a, b] \circlearrowleft\}_{1 \leq i \leq N}$, such that $s_i([a, b]) \cap s_j([a, b]) = \emptyset$, there exists a unique non-empty set $E \subset [a, b]$ with

$$E = \bigcup_{i=1}^{N} s_i(E). \tag{5}$$

It is well known that such a set E is homeomorphic to the Cantor set, and, in particular, is totally disconnected. Further, if $\mathbf{p} = (p_1, \ldots, p_N)$ is a probability vector with $p_i \in (0, 1)$, for all $i \in \{1, 2, \ldots, N\}$, then there exists a unique atomless Borel probability measure μ supported on E satisfying

$$\mu(A) = \sum_{i=1}^{N} p_i \mu(s_i^{-1}(A)),$$

for all Borel measurable sets A. If all of the contractions in S are similarities then the set E is called a *self-similar set* and the measure μ is called a *self-similar measure*. Moreover, in this case, if all of the contraction ratios of the members of S are equal, then E is called a *homogeneous self-similar set*; otherwise E is called an *inhomogeneous self-similar set*. For further properties of the set E and the measure μ, and proof of the above results, we refer the reader to [7, 12].



(**a**) Plot of f_1^μ (**b**) Plot of f_2^μ

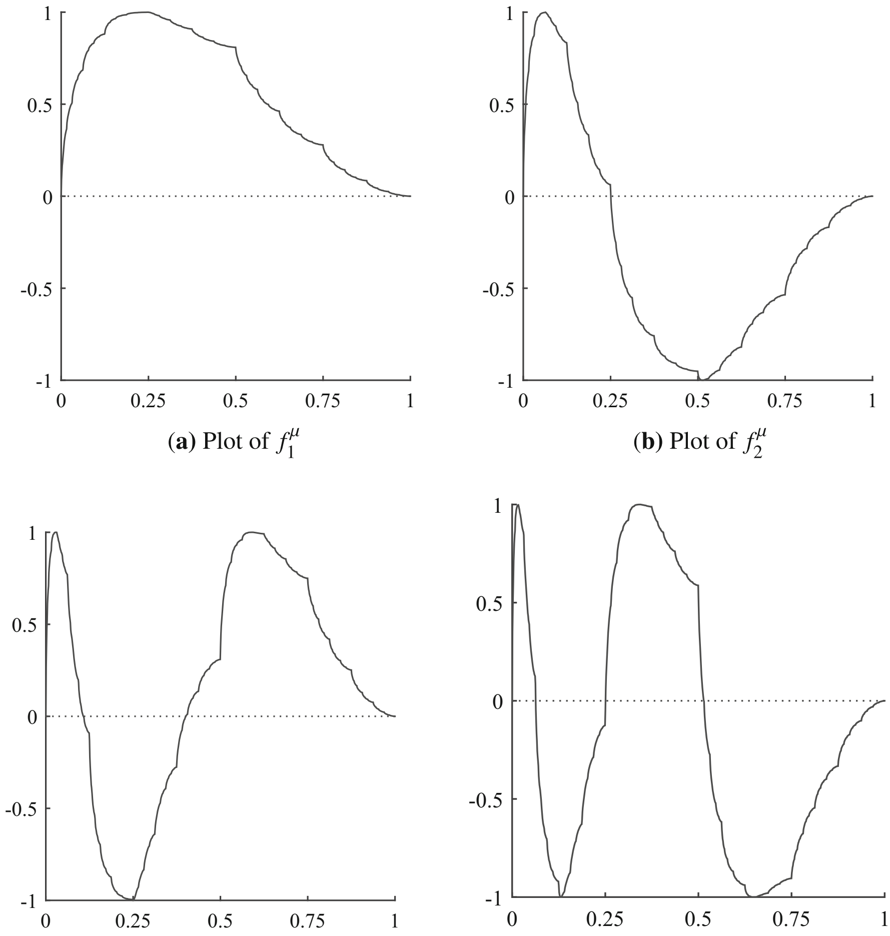
(**c**) Plot of f_3^μ (**d**) Plot of f_4^μ

Fig. 2 Eigenfunctions of Δ^μ, where $[a, b] = [0, 1]$ and μ is the self-similar measure associated to $S = \{x \mapsto x/2, x \mapsto x/3 + 2/3\}$ and $\mathbf{p} = (0.7, 0.3)$, under Dirichlet boundary conditions

Let us consider the specific case when $[a, b] = [0, 1]$, $S = \{x \mapsto x/2, x \mapsto x/3 + 2/3\}$ and $\mathbf{p} = (0.7, 0.3)$. Here the unique set E satisfying the equality given in (5) is an inhomogeneous self-similar set. Letting μ denote the associated self-similar measure, in Fig. 2, we give graphical representations of the eigenfunctions associated to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the μ-Laplacian, under Dirichlet boundary conditions.

4.3 Salem measures

Let us consider a class of examples of distribution functions which was studied by Salem [21]. Namely we consider the family of absolutely continuous measures



(a) Plot of f_1^μ

(b) Plot of f_2^μ

(c) Plot of f_3^μ

(d) Plot of f_4^μ

Fig. 3 Eigenfunctions of Δ^μ, where $\mu = \mu_{p,q}$ is the Salem measure on the unit interval with $p = 0.7$ and $q = 0.5$, under Dirichlet boundary conditions

$\{\mu_{p,q} : p, q \in (0, 1)\}$, whose distribution functions $\{F_{\mu_{p,q}} : p, q \in (0, 1)\}$ arise from the following endomorphisms of $[0, 1]$. For $r \in (0, 1)$, we define

$$S_r(x) := \begin{cases} x/r & \text{if } x \in [0, r], \\ (x - r)/(1 - r) & \text{if } x \in (r, 1]. \end{cases}$$

The maps $F_{\mu_{p,q}} : [0, 1] \circlearrowleft$ are then given by $S_p \circ F_{\mu_{p,q}} = F_{\mu_{p,q}} \circ S_q$. One can immediately verify that $F_{\mu_{p,q}}$ is strictly monotonically increasing and, for $p \neq q$, is differentiable Lebesgue almost everywhere with derivative equal zero, when it exists. For more details and further properties of these functions we refer the reader to [13, 16, 21]. In Fig. 3, we give graphical representations of the eigenfunctions associated to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the $\mu_{0.7,0.5}$-Laplacian, under Dirichlet boundary conditions.

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