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Restricted one-dimensional central extensions of restricted simple Lie algebras

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ABSTRACT

We study the restricted one-dimensional central extensions of an arbitrary finite dimensional restricted simple Lie algebra for $p \geq 5$. For $H^2(\mathfrak{g}) = 0$, we explicitly describe the cocycles spanning $H^2_*(\mathfrak{g})$, and in the case $H^2(\mathfrak{g}) \neq 0$, we give a procedure to describe a basis for $H^2_*(\mathfrak{g})$.

1. Introduction

In [3], the authors give an explicit description of the cocycles parameterizing the space of restricted one-dimensional central extensions of the Witt algebra $W(1) = W(1, \mathbf{1})$ defined over fields of characteristic $p \geq 5$. The Witt algebra is a finite dimensional

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restricted simple Lie algebra, and such algebras have been completely classified for primes $p \geq 5$ [10]. In this paper, we study the restricted one-dimensional central extensions of an arbitrary finite dimensional restricted simple Lie algebra for $p \geq 5$.

The one-dimensional central extensions of a Lie algebra $\mathfrak{g}$ defined over a field $\mathbb{F}$ are classified by the Lie algebra cohomology group $H^2(\mathfrak{g}) = H^2(\mathfrak{g}, \mathbb{F})$ where $\mathbb{F}$ is taken as a trivial $\mathfrak{g}$-module. The restricted one-dimensional central extensions of a restricted Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ are likewise classified by the restricted Lie algebra cohomology group $H^2_*(\mathfrak{g}) = H^2_*(\mathfrak{g}, \mathbb{F})$. We refer the reader to [3] and [4] for descriptions of the complexes used to compute the ordinary and restricted Lie algebra cohomology groups as well as a review of the correspondence between one-dimensional central extensions (restricted central extensions) and the second cohomology (restricted cohomology) group. In particular, we adopt the notation and terminology in [3].

The dimensions of the ordinary cohomology groups $H^2(\mathfrak{g})$ for finite dimensional simple restricted Lie algebras are known [1,2,5,6,9]. Following the technique used in [12], we use these results along with Hochschild's six term exact sequence relating the first two ordinary and restricted cohomology groups to analyze the restricted cohomology group $H^2_*(\mathfrak{g})$. Our theorem states that if $\mathfrak{g}$ is a restricted simple Lie algebra, this sequence reduces to a short exact sequence relating $H^2(\mathfrak{g})$ and $H^2_*(\mathfrak{g})$. In the case that $H^2(\mathfrak{g}) = 0$, we explicitly describe the cocycles spanning $H^2_*(\mathfrak{g})$. If $H^2(\mathfrak{g}) \neq 0$, we give a procedure for describing a basis for $H^2_*(\mathfrak{g})$.

The paper is organized as follows. Section 2 gives an overview of the classification of finite dimensional simple restricted Lie algebras $\mathfrak{g}$ defined over fields of characteristic $p \geq 5$. Section 3 contains the statement and proof of the theorem relating $H^2(\mathfrak{g})$ and $H^2_*(\mathfrak{g})$ as well as an explicit description of the cocycles spanning $H^2_*(\mathfrak{g})$ when $H^2(\mathfrak{g}) = 0$. Section 4 outlines a procedure for describing a basis for $H^2_*(\mathfrak{g})$ in the case where $H^2(\mathfrak{g}) \neq 0$.

2. Restricted simple Lie algebras

Finite dimensional simple Lie algebras over fields of characteristic zero were classified more than a century ago in the work of Killing and Cartan. The well known classification theorem states that in characteristic zero, any simple Lie algebra is isomorphic to one of the linear Lie algebras A_l, B_l, C_l or D_l ($l \geq 1$), or one of the exceptional Lie algebras E_6, E_7, E_8, F_4 or G_2. Levi's theorem implies that in characteristic 0, all extensions of a semi-simple Lie algebra split, and hence all one-dimensional central extensions of such an algebra are trivial.

The classification of modular simple Lie algebras over fields of characteristic $p \geq 5$ was completed more recently in the work by Block, Wilson, Premet and Strade in multiple papers spanning decades of research. We refer the reader to [10] for detailed account of this work. The classification states that simple Lie algebras of characteristic $p \geq 5$ fall into one of three types: *classical* Lie algebras, algebras of *Cartan type* and *Melikian algebras* (Melikian algebras are defined only for $p = 5$).

The classical type simple Lie algebras are constructed by using a Chevalley basis to construct a Lie algebra L and tensoring over $\mathbb{F}$ to yield a Lie algebra $L_{\mathbb{F}}$ over $\mathbb{F}$. The algebras $L_{\mathbb{F}}$ are all restrictable, and they are all simple unless $L_{\mathbb{F}} \simeq A_l$ where $p|(l+1)$. In this case $L_{\mathbb{F}}$ has a one dimensional center C and the quotient $\mathfrak{psl}(l+1) = L_{\mathbb{F}}/C$ is simple [1,10]. Thus the simple restricted algebras of classical type are:

$$A_l(p \nmid (l+1)), \mathfrak{psl}(l+1)(p|(l+1)), B_l, C_l, D_l, G_2, F_4, E_6, E_7, E_8.$$

Block shows in [1] (Theorem 3.1) that if $\mathfrak{g}$ is a simple modular Lie algebra of classical type and $\mathfrak{g} \not\simeq \mathfrak{psl}(l+1)$ where $p|(l+1)$, then $\mathfrak{g}$ has no non-trivial central extensions at all so that in particular $H^2(\mathfrak{g}) = 0$. Moreover, the same theorem implies that if $p|(l+1)$, any one-dimensional central extension of $\mathfrak{psl}(l+1)$ is equivalent to the trivial one-dimensional central extension so that $H^2(\mathfrak{psl}(l+1)) = 0$ as well.

Algebras of Cartan type were constructed by Kostrikin, Shafarevich and Wilson and are divided into four families, called *Witt–Jacobson* ($W(n, \mathbf{m})$), *Special* ($S(n, \mathbf{m})$), *Hamiltonian* ($H(n, \mathbf{m})$) and *Contact* ($K(n, \mathbf{m})$) *algebras* [10]. Unlike the classical type algebras, not all algebras of Cartan type are restrictable. The restrictable simple Lie algebras of Cartan type are $W(n) = W(n, \mathbf{1}), S(n) = S(n, \mathbf{1}), H(n) = H(n, \mathbf{1})$ and $K(n) = K(n, \mathbf{1})$ [10]. We refer the reader to [12] for explicit descriptions of these algebras. Some of these algebras have non-trivial one-dimensional central extensions as Lie algebras [2,5,6].

If $p = 5$, there is one more family of simple modular Lie algebras called *Melikian algebras* that were first introduced in [8]. The only restrictable algebra in this family is the algebra $M = M(1, 1)$, and M has no non-trivial one-dimensional central extensions as a Lie algebra [9]. We refer the reader to [11] for an explicit description of this algebra.

3. Restricted one-dimensional central extensions

If $\mathfrak{g}$ is a restricted Lie algebra, there is a six-term exact sequence in [7] that relates the ordinary and restricted cohomology groups in degrees one and two.

$$\begin{aligned} 0 \longrightarrow H^1_*(\mathfrak{g}, M) \to H^1(\mathfrak{g}, M) \longrightarrow \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, M^{\mathfrak{g}}) \longrightarrow \\ \longrightarrow H^2_*(\mathfrak{g}, M) \to H^2(\mathfrak{g}, M) \to \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, H^1(\mathfrak{g}, M)) \end{aligned} \tag{1}$$

Here we use the same notation as in [12] and write $\mathrm{Hom}_{\mathrm{fr}}(V, W)$ for the set of *Frobenius homomorphisms* from the $\mathbb{F}$-vector space V to the $\mathbb{F}$-vector space W. That is

$$\mathrm{Hom}_{\mathrm{fr}}(V, W) = \{f : V \to W \mid f(\alpha x + \beta y) = \alpha^p f(x) + \beta^p f(y)\}$$

for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$. We remark that $f \in \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, \mathbb{F})$ if and only if f is p-semilinear as defined in [3].

Theorem 3.1. *If $\mathfrak{g}$ is a restricted simple Lie algebra defined over a field $\mathbb{F}$ of characteristic $p > 0$, then as vector spaces over $\mathbb{F}$,*

$$H^2_*(\mathfrak{g}) = H^2(\mathfrak{g}) \oplus \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, \mathbb{F}),$$

so that

$$\dim H^2_*(\mathfrak{g}) = \dim H^2(\mathfrak{g}) + \dim \mathfrak{g}.$$

Proof. With trivial coefficients, the ordinary coboundary map $\delta^1 : C^1(\mathfrak{g}) \to C^2(\mathfrak{g})$ reduces to

$$\delta^1\varphi(g_1, g_2) = \varphi([g_1, g_2]).$$

Since $\mathfrak{g}$ is simple, this shows $H^1(\mathfrak{g}) = 0$, and it follows from (1) that $H^1_*(\mathfrak{g}) = 0$ as well. Moreover $\mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, H^1(\mathfrak{g}, M)) = \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, 0) = 0$ and $\mathbb{F}^{\mathfrak{g}} = \mathbb{F}$. Therefore the sequence (1) reduces to a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, \mathbb{F}) \longrightarrow H^2_*(\mathfrak{g}) \longrightarrow H^2(\mathfrak{g}) \longrightarrow 0 \qquad \square$$

Corollary 3.2. *If $\dim \mathfrak{g} = n$, and $\{x_1, \dots, x_n\}$ is a basis for $\mathfrak{g}$, then there is a n-dimensional subspace of $H^2_*(\mathfrak{g})$ spanned by restricted cohomology classes represented by restricted cocycles $(0, \omega_i)$ $(i = 1, \dots, n)$ where $\omega_i : \mathfrak{g} \to \mathbb{F}$ is defined by*

$$\omega_i\left(\sum_{j=1}^{n} \alpha_j x_j\right) = \alpha_j^p.$$

Moreover, if E_i denotes the one-dimensional restricted central extension of $\mathfrak{g}$ determined by the cohomology class of the cocycle $(0, \omega_i)$, then $E_i = \mathfrak{g} \oplus \mathbb{F}c$ as a $\mathbb{F}$-vector space, and we have for all $1 \le j, k \le n$,

$$\begin{aligned} [x_j, x_k] &= [x_j, x_k]_{\mathfrak{g}}; \\ [x_j, c] &= 0; \\ e_j^{[p]} &= x_j^{[p]_{\mathfrak{g}}} + \delta_{i,j} c; \\ c^{[p]} &= 0, \end{aligned} \tag{2}$$

where $[x_j, x_k]_{\mathfrak{g}}$ and $x_j^{[p]_{\mathfrak{g}}}$ denote the Lie bracket and $[p]$-operation in $\mathfrak{g}$ respectively, and δ denotes the Kronecker delta-function.

Proof. Easy computations show the maps ω_i are a basis for $\mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, \mathbb{F})$, $(0, \omega_i)$ is a cocycle for all i, and the corresponding cohomology classes are linearly independent in $H^2_*(\mathfrak{g})$. The Lie bracket and $[p]$-operation in E_i are given by the equations (4) in [3]. $\square$

4. Case by case study

In this section, all restricted Lie algebras are defined over fields of characteristic $p \geq 5$. Corollary 3.2 implies if $H^2(\mathfrak{g}) = 0$, then the extensions in (2) are the only restricted one-dimensional central extensions of a restricted simple Lie algebra $\mathfrak{g}$. Table 1 lists the restricted simple Lie algebras $\mathfrak{g}$ for which $H^2(\mathfrak{g}) = 0$ along with the dimensions of these algebras (i.e. the dimensions of the space of restricted one-dimensional central extensions). This table contains all the simple restricted Lie algebras of classical type [1], the Witt algebras $W(n)$ for $n > 1$ [6], the Contact algebras $K(n)$ for $n+3 \not\equiv 0 \pmod p$ [6] and the Melikian algebra M [9]. For each of these algebras, we have explicit descriptions (2) of all restricted one-dimensional central extensions.

Table 2 lists the restricted simple Lie algebras $\mathfrak{g}$ for which $H^2(\mathfrak{g}) \neq 0$ along with the dimensions of these algebras, the dimensions of the ordinary cohomology groups and the dimensions of the space of restricted one-dimensional central extensions. This table includes the Witt algebra $W(1)$ [3,6], the Special algebras $S(n)$ [2], Hamiltonian algebras $H(n)$ [2], and the Contact algebras $K(n)$ for $n + 3 \equiv 0 \pmod p$ [6].

Table 1
$\dim H^2_*(\mathfrak{g})$ for $\mathfrak{g}$ for which $H^2(\mathfrak{g}) = 0$.

$\mathfrak{g}$	$\dim \mathfrak{g} = \dim H^2_*(\mathfrak{g})$
A_l $(l \geq 1,\ p \nmid (l+1))$	$(l+1)^2 - 1$
$\mathfrak{psl}(l+1)$ $(l \geq 1,\ p \mid (l+1))$	$(l+1)^2 - 2$
B_l $(l \geq 2)$	$2l^2 + l$
C_l $(l \geq 3)$	$2l^2 + l$
D_l $(l \geq 4)$	$2l^2 - l$
G_2	14
F_4	52
E_6	78
E_7	133
E_8	248
$W(n)$ $(n > 1)$	np^n
$K(n)$ $(n+3 \not\equiv 0 \pmod p)$	p^n
M	125

Table 2
$\dim H^2_*(\mathfrak{g})$ for $\mathfrak{g}$ for which $H^2(\mathfrak{g}) \neq 0$.

$\mathfrak{g}$	$\dim \mathfrak{g}$	$\dim H^2(\mathfrak{g})$	$\dim H^2_*(\mathfrak{g})$
$W(1)$	p	1	$p+1$
$S(3)$	$2p^3 - 2$	3	$2p^3 + 1$
$S(n)$ $(n > 3)$	$(n-1)(p^n - 1)$	$n(n-1)/2$	$(n-1)(2p^n + n - 2)/2$
$H(n)$ $(n+4 \equiv 0 \pmod p)$	$p^n - 2$	$n+2$	$p^n + n$
$H(n)$ $(n+4 \not\equiv 0 \pmod p)$	$p^n - 2$	$n+1$	$p^n + n - 1$
$K(n)$ $(n+3 \equiv 0 \pmod p)$	$p^n - 1$	$n+1$	$p^n + n$

For $W(1)$, the non-trivial ordinary cohomology class determines a restricted one-dimensional central extension $E = W(1) \oplus \mathbb{K}c$ where we have for all $-1 \leq j, k \leq p-2$,

$$\begin{aligned} [e_j, e_k] &= (k-j)e_{j+k} + \frac{j(j^2-4)}{3}\delta_{0,j+k}c; \\ [e_j, c] &= 0; \\ e_j^{[p]} &= \delta_{0,j}e_0; \\ c^{[p]} &= 0 \end{aligned}$$

and $\{e_{-1}, \dots, e_{p-2}\}$ is a basis for $W(1)$ [3].

Finally, we remark that in [12], the map

$$H^2(\mathfrak{g}, M) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{fr}}(\mathfrak{g}, H^1(\mathfrak{g}, M))$$

from the exact sequence (1) is given explicitly. In the case of trivial coefficients, this map Δ is given by

$$\Delta_\varphi(g) \cdot h = \varphi(g, h^{[p]}) - \varphi([g, \underbrace{h, \cdots, h}_{p-1}], h)$$

where $\varphi \in C^2(\mathfrak{g})$ and $g, h \in \mathfrak{g}$. Since $\mathfrak{g}$ is simple, $H^1(\mathfrak{g}) = 0$ and hence Δ is the zero map so that

$$\varphi(g, h^{[p]}) - \varphi([g, \underbrace{h, \cdots, h}_{p-1}], h) = 0$$

for all $\varphi \in C^2(\mathfrak{g})$ and $g, h \in \mathfrak{g}$. Therefore if $\varphi \in C^2(\mathfrak{g})$ is a cocycle and $\omega : \mathfrak{g} \to \mathbb{F}$ has the $*$-property with respect to φ (as defined in [3]), then $(\varphi, \omega) \in C^2_*(\mathfrak{g})$ is a restricted cocycle. Moreover, given $\varphi \in C^2(\mathfrak{g})$, we can set the value of ω on every basis vector for $\mathfrak{g}$ to be 0 and use equation (1) in [3] to define ω to have the $*$-property with respect to φ. If φ_1 and φ_2 are linearly independent cocycles in $C^2(\mathfrak{g})$, the resulting restricted cocycles (φ_1, ω_1) and (φ_2, ω_2) are linearly independent in $C^2_*(\mathfrak{g})$. This gives us a canonical process of producing a basis for $H^2_*(\mathfrak{g})$ from a basis for $H^2(\mathfrak{g})$ and the cocycles $(0, \omega_i)$ in Corollary 3.2. This process was used to produce the basis for $H^2_*(W(1))$ in [3].

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