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Research paper

Symmetry classification of time-fractional diffusion equation

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ABSTRACT

In this article, a new approach is proposed to construct the symmetry groups for a class of fractional differential equations which are expressed in the modified Riemann-Liouville fractional derivative. We perform a complete group classification of a nonlinear fractional diffusion equation which arises in fractals, acoustics, control theory, signal processing and many other applications. Introducing the suitable transformations, the fractional derivatives are converted to integer order derivatives and in consequence the nonlinear fractional diffusion equation transforms to a partial differential equation (PDE). Then the Lie symmetries are computed for resulting PDE and using inverse transformations, we derive the symmetries for fractional diffusion equation. All cases are discussed in detail and results for symmetry properties are compared for different values of α. This study provides a new way of computing symmetries for a class of fractional differential equations.

1. Introduction

The fractional differential equations arise in the study of fractals, acoustics, control theory, signal processing [1,2] and many other important studies such as in physical, chemical and biological processes [3–5]. One may encounter various processes in science and engineering where both spatial and temporal variations occur which result in a phenomena known as diffusion. Such processes include heat conduction in materials, transient flow in porous media, dispersion of chemicals and pollutants in its surrounding by gradual decrease in their concentration, transport of substances through cell membrane in cell biology, sedimentation and consolidation of geomaterials etc. In fact diffusion is a process in which molecules move around until they are evenly spread out in the area. Diffusion is often described by the power law $r^2(t) = Dt^\alpha$, where D is the diffusion coefficient, t is the elapsed time and $r^2(t)$ is the mean squared displacement. For $\alpha > 1$, the phenomenon is referred as a super diffusion and for $\alpha = 1$, it is called a normal diffusion whereas $\alpha < 1$ describes the subdiffusion. When $\alpha < 1$ or $\alpha > 1$, the diffusion process is anomalous due to nonlinear relationship with time. Often this irregularity is because of either nonhomogeneous medium or active cellular transport. There are number of frameworks to describe anomalous diffusion that are currently in fashion in statistical physics and biology which includes continuous time random walk (CRTW) and fractional Brownian motion. We consider the time-fractional diffusion equation [6]

$$^{R}D_t^\alpha u = (k(u)u_x)_x, \quad 0 < \alpha \leq 2, \tag{1}$$

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where ${}^{R}D_t^{\alpha}$ is the Riemann-Liouville fractional differential operator of order α defined as

$$ {}^{R}D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x f(\xi)(x-\xi)^{n-\alpha-1} d\xi, \qquad n-1 < \alpha \le n, \quad n \in \mathbb{N}. \tag{2} $$

In Eq. (2), $f \in \mathcal{C}_{\alpha}^{n}$, $x > 0$, $\alpha \in \mathcal{R}$ and there exists a real number $\beta > \alpha$ such that $f(x) = x^{\beta} g(x)$, where $g(x) \in \mathcal{C}[0, \infty)$. The fractional diffusion Eq. (1) investigates the mechanism of anomalous diffusion that arise in transport processes through complex and/or disordered systems including fractal media.

The definition of a fractional differential operator is not unique. The fractional differential equations can be expressed in terms of different differential operators defined by Riemann-Liouville [7], Caputo [8,9], Weyl [10] and many others (see e.g. [1,2,10] and references therein). Guy Jumaire [10] proposed some modifications in Riemann-Liouville fractional derivative and derived the fractional Taylor series of non differentiable functions. The new modified fractional derivative has some features similar with the classical derivative. The modified fractional derivatives are defined by

$$ {}^{J}D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (f(\xi) - f(0))(x-\xi)^{-\alpha-1} d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (f(\xi) - f(0))(x-\xi)^{-\alpha} d\xi, & 0 < \alpha \le 1, \\ (f^{(n-1)}(x))^{(\alpha-n+1)}, & n-1 < \alpha \le n, \quad n \ge 2. \end{cases} \tag{3} $$

The modified Riemann-Liouville fractional derivative bears some interesting properties:

$$ {}^{J}D_x^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}, \qquad \mu > 0, \tag{4} $$

$$ {}^{J}D_x^{\alpha} (f(x)g(x)) = g(x)\, {}^{J}D_x^{\alpha} f(x) + f(x)\, {}^{J}D_x^{\alpha} g(x), \tag{5} $$

$$ {}^{J}D_x^{\alpha} f[g(x)] = \frac{df[g(x)]}{dg(x)}\, {}^{J}D_x^{\alpha} g(x). \tag{6} $$

In Eq. (1), the fractional order $0 < \alpha < 1$ describes the sub-diffusion. Eq. (1) has been considered by various researchers. Numerous methods have been developed and used to find approximate and analytical solutions of these equations e.g. Laplace transform method, Green's function method [1], variational iteration method [11], Adomian decomposition method [12], Finite Sine transform method [13], homotopy perturbation method [14] and symmetries method [15]. The fundamental solution was obtained by Mainardi [16]. Wyss [17] obtained solutions to the Cauchy problem in terms of H-functions using the Mellin transform. Schneider and Wyss [18] converted the diffusion-wave equation with appropriate initial conditions into the integro-differential equation and found the corresponding Green functions in terms of Fox functions. Gazizov et. al. [6] considered the fractional diffusion equation with nonlinear diffusion coefficient and provided the complete Lie group classification ($0 < \alpha \le 2$) using Riemann- Liouville and Caputo fractional derivatives. The over-determined system of linear FDEs was obtained in the classification which was solved for infinitesimals. The invariance of a partial differential equation of fractional order under the Lie group of scaling transformations was studied by Buckwar and Luchko [19]. The point transformations of variables in fractional integrals and derivatives of different types were discussed by Gazizov et. al. [20]. They used the prolongation formulae for finding nonlocal symmetries of ordinary fractional differential equations. The algorithm for the computation of Lie point symmetries for fractional order differential equations using the method described by Buckwar and Luchko and Gazizov, Kasatkin and Lukashchuk is developed by Jefferson et. al. [21]. The method was generalized to calculate symmetries for FDEs with n independent variables and also for systems of partial FDEs. The systematic way to compute solutions for the linear time-fractional diffusion wave equation are presented in [22]. The integral transform technique is utilized which discusses the properties of the Mittag-Leffler, Wright and Mainardi functions that appear in the solution.

In this article, we provide the alternative way of computing symmetries which is more convenient and easier and provide more general results. These symmetries can be utilized to compute the invariant solutions for fractional differential equations. We first transform the Riemann-Liouville differential operator to the modified Riemann-Liouville fractional operator and then convert this operator in classical differential operator using suitable transformations. As a result an over-determined system of linear PDEs can be obtained after using invariance condition whose solution process is a well established procedure in the literature. Then solving these determining equations for unknown coefficients ξ^1, ξ^2, η and using inverse transformation we obtain the infinitesimal generators for fractional diffusion equation. This method is successfully applied to the fractional diffusion equation which is considered in Riemann-Liouville sense when $0 < \alpha \le 2$.

This article is organized in the following manner. In Section 2, we provide a complete classification of fractional diffusion equation to construct symmetries. All cases are discussed in detail. We compare our result with ones obtained by Gazizov et. al. [6]. The physical interpretation of symmetry properties of fractional diffusion equation is discussed in Section 3. Section 4 is devoted to the concluding remarks.

2. Lie symmetries of nonlinear fractional diffusion equation for $0 < \alpha \leq 1$

It is worthwhile to discuss the fractional diffusion equation for different cases namely sub-diffusion, normal diffusion, wave diffusion and for wave equation. We consider following two cases: *Case*1: $0 < \alpha \leq 1$, where $0 < \alpha < 1$ mentions the sub-diffusion and $\alpha = 1$ is a normal diffusion case. In *Case*2 ($1 < \alpha \leq 2$), $1 < \alpha < 2$ is a case of wave diffusion whereas $\alpha = 2$ describes the case for wave equation.

In this section, we will compute the Lie symmetries of fractional diffusion Eq. (1) for the case of sub diffusion $0 < \alpha < 1$ and normal diffusion $\alpha = 1$. The infinitesimal generator of Eq. (1) is

$$\Gamma = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{7}$$

Since the lower limit of the fractional differential operator (2) is fixed, hence the invariance condition of Eq. (1) provides (see e.g. [6])

$$\xi^1(t,x,u)\mid_{t=0} = 0. \tag{8}$$

Case 1: $0 < \alpha \leq 1$**.**

Eq. (1) can be written as

$${}^J D_t^\alpha u + {}^R D_t^\alpha u(0,x) = (k(u)u_x)_x, \quad 0 < \alpha < 1. \tag{9}$$

For $\alpha = 1$, Eq. (1) is a classical diffusion equation. Without loss of generality, we assume

$$u(0,x) = g(x). \tag{10}$$

With the aid of Eq. (10), Eq. (9) becomes

$${}^J D_t^\alpha u + {}^R D_t^\alpha g(x) = (k(u)u_x)_x. \tag{11}$$

When $\alpha = 1$, Eq. (11) reduces to the nonlinear heat equation which was considered by Ovsiannikov [24]. We introduce the transformations

$$T = \frac{pt^\alpha}{\Gamma(1+\alpha)}, \quad X = x, \quad U(T,X) = u(t,x). \tag{12}$$

Eq. (1) with the help of Eqs. (4)–(6) and Eq. (12) reduces to the following PDE:

$$pU_T + \frac{g(X)p}{\Gamma(1+\alpha)\Gamma(1-\alpha)T} = \big(k(U)U_X\big)_X. \tag{13}$$

For simplicity, we substitute

$$G(X) = \frac{g(X)p}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \tag{14}$$

in Eq. (13) to obtain

$$pU_T + \frac{G(X)}{T} = \big(k(U)U_X\big)_X. \tag{15}$$

To compute the Lie symmetries of Eq. (15), we use the invariance condition (see [23,24])

$$V^{[2]}\Big(pU_T + \frac{G(X)}{T} - \big(k(U)U_X\big)_X\Big)\mid_{(15)} = 0, \tag{16}$$

where $V^{[2]}$ is the second prolongation of infinitesimal generator

$$V = \xi^1(T,X,U)\frac{\partial}{\partial T} + \xi^2(T,X,U)\frac{\partial}{\partial X} + \eta(T,X,U)\frac{\partial}{\partial U}. \tag{17}$$

Taking Eq. (16) into an account and after expansion and then separation with respect to the derivatives of U, we obtain the following linear system of determining equations

$$\xi^1_X = 0, \quad \xi^1_U = 0, \tag{18a}$$

$$pk\xi^2_U + kk'\xi^1_X = 0, \tag{18b}$$

$$kk'\xi^2_U - k^2\xi^2_{UU} = 0, \tag{18c}$$

$$pk'\eta + pk\xi^1_T - k^2\xi^1_{XX} - 2pk\xi^2_X = 0, \tag{18d}$$

$$kk'\eta_U - (k')^2\eta - 2k^2\xi^2_{XU} + k^2\eta_{UU} + kk''\eta = 0, \tag{18e}$$

$$2kk'T\eta_X - k^2T\xi^2_{XX} - 3k\xi^2_U G(X) + 2k^2T\eta_{XU} + pkT\xi^2_T = 0, \tag{18f}$$

$$k'TG(X)\eta - 2kTG(X)\xi^2_X + kG(X)\xi^1 + kTG(X)\eta_U + k^2T^2\eta_{XX} - pkT^2\eta_T - kTG'(X)\xi^2 = 0. \tag{18g}$$

From Eq. (18a), we find $\xi^1 = A(T)$ whereas Eq. (18b) gives $\xi^2_U = 0$ which further implies $\xi^2 = B(T, X)$. With these values of ξ^1 and ξ^2, Eq. (18c) is identically satisfied. Thus

$$\xi^1 = A(T), \quad \xi^2 = B(T, X). \tag{19}$$

From Eq. (18d), we conclude that

$$\eta = \frac{k(2B_X - A')}{k'}, \quad k' \neq 0. \tag{20}$$

Substituting ξ^1, ξ^2 and η in Eq. (18e) results in

$$k^2\left(2B_X - A'\right)[(k')^2k'' - 2k(k'')^2 + kk'k'''] = 0. \tag{21}$$

In order to solve Eq. (21), the following cases need to be considered.

***Case 1.1* :** $2B_X - A' = 0$.

We find that

$$B = \frac{1}{2}XA' + C(T), \tag{22}$$

where $C(T)$ is an arbitrary function of integration. Eq. (18f) with the substitution of B from Eq. (22) reduces to

$$\frac{1}{2}XA'' + C' = 0. \tag{23}$$

Since A and C are functions of T hence we can make separation of Eq. (23) with respect to X which results in

$$A'' = 0, \quad C' = 0. \tag{24}$$

The solution of Eq. (24) is

$$A(T) = c_1T + c_2, \quad C(T) = c_3, \tag{25}$$

where c_1, c_2 and c_3 are arbitrary constants. Eq. (22) finally yields

$$B(T) = \frac{1}{2}c_1X + c_3. \tag{26}$$

Thus from Eqs. (19) and (20), we conclude that

$$\xi^1 = c_1T + c_2, \quad \xi^2 = \frac{1}{2}c_1X + c_3, \quad \eta = 0. \tag{27}$$

Using ξ^1, ξ^2 and η from Eq. (27) in Eq. (18g) yields

$$\left(\frac{1}{2}c_1X + c_3\right)G'(X) = 0, \quad c_2G(X) = 0. \tag{28}$$

At this point the following subcases of *Case*1.1 should be investigated.

***Case 1.1.1* :** $G(X) = 0$.

In this case, we obtain the following symmetries

$$V_0 = \frac{\partial}{\partial T}, \quad V_1 = 2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}, \quad V_2 = \frac{\partial}{\partial X}, \quad k(u) \text{ arbitrary}. \tag{29}$$

***Case 1.1.2*:** $G'(X) = 0,\ c_2 = 0$.

The straightforward but lengthy manipulations lead to

$$V_1 = 2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}, \quad V_2 = \frac{\partial}{\partial X}, \quad G(X) = \text{constant}, \quad k(u) \text{ arbitrary}. \tag{30}$$

***Case 1.2*:** $(k')^2k'' - 2k(k'')^2 + kk'k''' = 0$**.**

For this case the following subcases need to be looked at:

$$\textit{Case } 1.2.1: \quad k(U) = \text{constant}. \tag{31}$$

$$\textit{Case } 1.2.2: \quad k(U) = U^\beta, \quad \beta \neq 0. \tag{32}$$

$$\textit{Case } 1.2.3: \quad k(U) = \mathrm{e}^U. \tag{33}$$

***Case 1.2.1* :** $k(U)$ =constant.

For simplicity we assume $k(U) = 1$. Since $k' = 0$, therefore Eq. (18d) reduces to

$$2\xi^2_X = \xi^1_T, \tag{34}$$

which together with Eq. (18d) gives

$$B = \frac{1}{2}A'X + C(T). \tag{35}$$

From Eqs. (18e) and (18f), we obtain

$$\eta = \Big(-\frac{1}{8}pX^2A'' - \frac{1}{2}pXC' + F(T)\Big)U + E(T,X), \tag{36}$$

where $F(T)$ and $E(T, X)$ are arbitrary functions. Substituting ξ^1, ξ^2 and η in Eq. (18g) and then separating with respect to powers of U gives rise to

$$\frac{1}{8}p^2X^2A''' + \frac{1}{2}p^2XC'' - \frac{1}{4}A'' - pF' = 0, \tag{37}$$

$$G(X)A - TG(X)A' + TG\Big(-\frac{1}{8}pX^2A'' - \frac{1}{2}pXC' + F\Big) + T^2E_{XX} - pT^2E_T - TG'\Big(\frac{1}{2}XA' + C\Big) = 0. \tag{38}$$

Eq. (37) implies

$$A(T) = m_1T + m_2T + m_3, \quad C(T) = m_4T + m_5, \quad F(T) = -\frac{1}{4}m_iT + m_6. \tag{39}$$

The substitution from Eq. (39) in Eqs. (19) and (36) gives

$$\xi^1 = m_1T + m_2T + m_3, \quad \xi^2 = \frac{1}{2}(2m_1T + m_2)X + m_4T + m_5$$
$$\eta = \Big(-\frac{1}{4}pm_1X^2 - \frac{1}{2}pm_4X - \frac{1}{2}m_1T + m_6\Big)U + E(T,X), \tag{40}$$

and

$$\begin{aligned}&-\frac{5}{4}m_1T^2G(X) + m_3G(X) - \frac{1}{4}m_1X^2TG(X) - \frac{1}{2}m_4XTG(x) + m_6TG(X)\\ &+ T^2E_{XX} - pT^2E_T - m_1T^2XG'(X) - m_4T^2G'(X) - m_5TG'(X) = 0.\end{aligned} \tag{41}$$

The following Lie symmetries are computed by taking one by one constant equal to one and rest to zero

$$\begin{aligned}
V_0 &= \frac{\partial}{\partial T}, \quad G(X) = 0,\\
V_1 &= 2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}, \quad G(X) \text{ arbitrary},\\
V_2 &= \frac{\partial}{\partial X}, \quad G(X) = \text{constant},\\
V_3 &= U\frac{\partial}{\partial U}, \quad G(X) = 0,\\
V_4 &= 2T\frac{\partial}{\partial X} - pXU\frac{\partial}{\partial U}, \quad G(X) = 0,\\
V_5 &= 4T^2\frac{\partial}{\partial T} + 4TX\frac{\partial}{\partial X} - (pX^2 + 2T)\frac{\partial}{\partial U}, \quad G(X) = 0,\\
V_6 &= E(T,X)\frac{\partial}{\partial U}, \quad \text{where } E_{XX} - pE_T = 0, \quad G(X) = 0.
\end{aligned}$$

Case 1.2.2: $k(U) = U^{\beta}, \quad \beta \neq 0.$

Substituting $k(U)$ in Eq. (20) presents

$$\eta = \frac{U(2B_X - A')}{\beta}. \tag{42}$$

Employing ξ^1, ξ^2 and η in Eq. (18f) leads to

$$\big(3 + \frac{4}{\beta}\big)U^{\beta}B_{XX} + pB_T = 0. \tag{43}$$

Since B is a function of T and X hence separation of Eq. (43) for powers of U provides

$$\big(3 + \frac{4}{\beta}\big)B_{XX} = 0, \tag{44}$$

$$B_T = 0. \tag{45}$$

From Eqs. (44) and (45) either $3+\frac{4}{\beta}=0$ and $B_T=0$ or $B_{XX}=0$ and $B_T=0$. We discuss each case separately.

Case 1.2.2.1 **:** $3+\frac{4}{\beta}=0,\ B_T=0.$

In this case $k(U)=U^{-\frac{4}{3}}$ whereas $B=B(X)$. Using these values in Eq. (18g) and after some simplifications one can obtain the following symmetries.

When $G(X)=0$

$$V_0=\frac{\partial}{\partial T},\quad V_1=T\frac{\partial}{\partial T}+\frac{3}{4}U\frac{\partial}{\partial U},\quad V_2=X^2\frac{\partial}{\partial X}-3XU\frac{\partial}{\partial U},$$
$$V_3=2X\frac{\partial}{\partial X}-3U\frac{\partial}{\partial U},\quad V_5=\frac{\partial}{\partial X}. \tag{46}$$

When $G(X)\neq 0$

$$V_0=X^2\frac{\partial}{\partial X}-3XU\frac{\partial}{\partial U},\quad G(X)=\frac{c_1}{X^2},\quad c_1=\text{constant},$$
$$V_1=2X\frac{\partial}{\partial X}-3U\frac{\partial}{\partial U},\quad G(X)=-X,$$
$$V_2=\frac{\partial}{\partial X},\quad G(X)=\text{constant}. \tag{47}$$

Case 1.2.2.2**:** $B_{XX}=0,\ B_T=0.$

We compute $B=m_1X+m_2$, where m_1 and m_2 are arbitrary constants. By virtue of (18g), we find $A(T)=m_3T+m_4$ and $\eta=\frac{U(2m_1-m_3)}{\beta}$ which results in following symmetries

$$V_0=\frac{\partial}{\partial T},\quad G(X)\ \text{arbitrary},$$
$$V_1=\beta X\frac{\partial}{\partial X}+2U\frac{\partial}{\partial U},\quad G(X)=c_1X^{\frac{2}{\beta}},\quad c_1=\text{constant},$$
$$\beta T\frac{\partial}{\partial T}-U\frac{\partial}{\partial U},\quad G(X)=0,$$
$$V_4=\frac{\partial}{\partial X},\quad G(X)=\text{constant}. \tag{48}$$

Case 1.2.3**:** $k(U)=e^U$.

With the help of Eqs. (18e), (18f) and (18g), the following infinitesimal generators are computed.

When $G(X)=0$, we obtain

$$V_0=\frac{\partial}{\partial T},\quad V_1=T\frac{\partial}{\partial T}-\frac{\partial}{\partial U},\quad V_2=X\frac{\partial}{\partial X}+2\frac{\partial}{\partial U},\quad V_3=\frac{\partial}{\partial X}. \tag{49}$$

When $G(X)=$ constant, then

$$V_0=X\frac{\partial}{\partial X}+2\frac{\partial}{\partial U},\quad V_1=\frac{\partial}{\partial X}. \tag{50}$$

In Tables 1 and 3, all Lie symmetries of Eq. (13) and their corresponding counterparts of Eq. (1) are presented. Note that beside all the Lie symmetries obtained by Gazizov et. al. [6], we compute additional infinitesimal generators which were not reported before. These new infinitesimal generators possess interesting features which is the essence of fractional derivative and their physical interpretation is discussed in the next section. Moreover, we also find initial conditions under which these infinitesimal generators are appropriate. In Tables 2 and 4, we compare our results with those obtained by Gazizov et. al. [6].

It should be noted that new infinitesimal generators also satisfy the invariance condition (8).

3. Lie symmetries of nonlinear fractional diffusion equation for $1 < \alpha \le 2$

In this section, we will find the Lie symmetries of fractional diffusion equation for Case 2: $1<\alpha\le 2$, where $1<\alpha<2$ is a case of wave diffusion whereas $\alpha=2$ is a case of classical wave equation.

Eq. (1) can be read as

$$\frac{\partial}{\partial t}\left[JD_t^{(\alpha-1)}u(t,x)\right]+\frac{u(0,x)}{\Gamma(1-\alpha)t^{\alpha}}=(k(u)u_x)_x\,,\qquad 1<\alpha<2. \tag{51}$$

Table 1
Symmetries of Eq. (1) and Eq. (13).

$k(u)$	$u(0,x)$	$pU_T + \frac{g(X)p}{\Gamma(1+\alpha)\Gamma(1-\alpha)T} = \left(k(U)U_X\right)_X$	${}^{R}D_t^{\alpha}u = (k(u)u_x)_x$
Arbitrary	0	$\frac{\partial}{\partial T}$	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$
	Constant	$2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
Constant	0	$\frac{\partial}{\partial T}$	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$
	Arbitrary	$2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
	0	$U\frac{\partial}{\partial U}$	$u\frac{\partial}{\partial u}$
	0	$2T\frac{\partial}{\partial X} - pXU\frac{\partial}{\partial U}$	$\frac{2t^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$
	0	$4T^2\frac{\partial}{\partial T} + 4TX\frac{\partial}{\partial X} - (pX^2+2T)\frac{\partial}{\partial U}$	$\frac{4t^{\alpha+1}}{\alpha^2\Gamma(\alpha)}\frac{\partial}{\partial t} + \frac{4t^{\alpha}}{\Gamma(\alpha+1)}x\frac{\partial}{\partial x}$ $-(x^2 + \frac{2t^{\alpha}}{\Gamma(\alpha+1)})\frac{\partial}{\partial u}$
	0	$E(T,X)\frac{\partial}{\partial U},\quad E_{XX} - pE_T = 0$	$E(t,x)\frac{\partial}{\partial u},\quad E_{xx} - \frac{\Gamma(\alpha)}{t^{\alpha-1}}E_t = 0$
u^{β}	Arbitrary	$\frac{\partial}{\partial T}$	$\frac{\Gamma(\alpha)}{pt^{\alpha-1}}\frac{\partial}{\partial t}$
	$c_1x^{\frac{2}{\beta}}$	$\beta X\frac{\partial}{\partial X} + 2U\frac{\partial}{\partial U}$	$\beta x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$
	0	$2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
$u^{-\frac{4}{3}}$	0	$\frac{\partial}{\partial T}$	$\frac{\Gamma(\alpha)}{pt^{\alpha-1}}\frac{\partial}{\partial t}$
	0	$2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
	$\frac{c_1}{x^2}$	$X^2\frac{\partial}{\partial X} - 3XU\frac{\partial}{\partial U}$	$x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$
	$0,\ -x$	$2X\frac{\partial}{\partial X} - 3U\frac{\partial}{\partial U}$	$2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
e^{u}	0	$\frac{\partial}{\partial T}$	$\frac{\Gamma(\alpha)}{pt^{\alpha-1}}\frac{\partial}{\partial t}$
	0	$2T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
	Constant	$X\frac{\partial}{\partial X} + 2\frac{\partial}{\partial U}$	$x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$

$T = \frac{pt^{\alpha}}{\Gamma(\alpha+1)},\quad X = x,\quad U = u$

Introducing the change of variables

$$T = \frac{pt^{\alpha-1}}{\Gamma(\alpha)}, \qquad X = x, \qquad U(T,X) = u(t,x), \tag{52}$$

Eq. (51) is transformed to

$$AT^{\frac{\alpha-2}{\alpha-1}}U_{TT} + BU(0,X)T^{\frac{\alpha}{1-\alpha}} = (k(U)U_X)_X, \tag{53}$$

where

$$A = \left(\frac{\Gamma(\alpha)}{p}\right)^{\frac{\alpha-2}{\alpha-1}}\frac{p}{\Gamma(\alpha-1)}, \qquad B = \left(\frac{\Gamma(\alpha)}{p(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}. \tag{54}$$

The invariance condition

$$V^{[2]}\left(AT^{\frac{\alpha-2}{\alpha-1}}U_{TT} + BU(0,X)T^{\frac{\alpha}{1-\alpha}} = (k(U)U_X)_X\right)\big|_{(53)} = 0, \tag{55}$$

leads to an overdetermined system of linear PDEs which upon solving and classifying for ξ^1, ξ^2, η, $k(U)$ and $U(0,X)$ yields the Lie symmetries which are summarized in Tables 3 and 4.

4. Discussions

One cannot compare two time intervals simultaneously as they are available sequentially. We compare them by observing some processes which regularly repeated but we cannot verify whether the time intervals are the same. Clocks, including atomic clocks, repeat their “ticks”, and we simply count those ticks, calling them hours, minutes, seconds, milliseconds, etc. However there may exist some possible inhomogeneity between two ticks due to the loss of energy or for some other reason. Classical calculus and its uses demand time to flow equably. Geometrically time is depicted from the time axis by considering equal time intervals. Using the fractional differential operator this inhomogeneity can be described using transformed time axis which tells us how time is slowing down. Some cases are considered below to explain such phenomena.

Table 2
Comparison between results obtained using transformation and direct symmetry method.

$k(u)$	Using transformation	Direct symmetry method
Arbitrary	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$	–
	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
Constant	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$	–
	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
	$u\frac{\partial}{\partial u}$	$u\frac{\partial}{\partial u}$
	$\frac{2t^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial}{\partial x}-xu\frac{\partial}{\partial u}$	–
	$\frac{4t^{\alpha+1}}{\alpha^2\Gamma(\alpha)}\frac{\partial}{\partial t}+\frac{4t^{\alpha}}{\Gamma(\alpha+1)}x\frac{\partial}{\partial x}$ $-(x^2+\frac{2t^{\alpha}}{\Gamma(\alpha+1)})\frac{\partial}{\partial u}$	–
	$E(t,x)\frac{\partial}{\partial u},\quad E_{xx}-\frac{\Gamma(\alpha)}{t^{\alpha-1}}E_t=0$	$E(t,x)\frac{\partial}{\partial u},\quad {}^{R}D_t^{\alpha}E=E_{xx}$
u^{β}	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$	–
	$\beta x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}$	$\beta x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}$
	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
$u^{-\frac{4}{3}}$	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$	–
	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$	$x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$
	$2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u}$	$2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
e^u	$\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$	–
	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}$	$x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$

Table 3
Symmetries of Eq. (1) and Eq. (13).

$k(u)$	$u(0,x)$	$pU_T+\frac{g(X)p}{\Gamma(1+\alpha)\Gamma(1-\alpha)T}=\left(k(U)U_X\right)_X$	${}^{R}D_t^{\alpha}u=(k(u)u_x)_x$
Arbitrary	Constant	$2(\alpha-1)T\frac{\partial}{\partial T}+\alpha X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
Constant	Arbitrary	$2(\alpha-1)T\frac{\partial}{\partial T}+\alpha X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
	0	$U\frac{\partial}{\partial U}$	$u\frac{\partial}{\partial u}$
	0	$E(T,X)\frac{\partial}{\partial U},\quad E_{XX}-T^{\frac{\alpha-2}{\alpha-1}}E_{TT}=0$	$E(t,x)\frac{\partial}{\partial u},\quad E_{xx}-{}^{R}D_t^{\alpha}E=0$
u^{β}	$c_1x^{\frac{2}{\beta}}$	$\beta X\frac{\partial}{\partial X}+2U\frac{\partial}{\partial U}$	$\beta x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}$
	0	$2(\alpha-1)T\frac{\partial}{\partial T}+\alpha X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
$u^{-\frac{4}{3}}$	0	$2(\alpha-1)T\frac{\partial}{\partial T}+\alpha X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{c_1}{x^3}$	$X^2\frac{\partial}{\partial X}-3XU\frac{\partial}{\partial U}$	$x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$
	$\frac{c_1}{x^{\frac{3}{2}}}$	$2X\frac{\partial}{\partial X}-3U\frac{\partial}{\partial U}$	$2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$
e^u	0	$2(\alpha-1)T\frac{\partial}{\partial T}+\alpha X\frac{\partial}{\partial X}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	Constant	$X\frac{\partial}{\partial X}+2\frac{\partial}{\partial U}$	$x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}$
	Constant	$\frac{\partial}{\partial X}$	$\frac{\partial}{\partial x}$

$T=\frac{pt^{\alpha-1}}{\Gamma(\alpha)},\quad X=x,\quad U=u$

Table 4
Comparison between results obtained using transformation and direct symmetry method.

$k(u)$	Using transformation	Direct symmetry method
Arbitrary	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
Constant	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
	$u\frac{\partial}{\partial u}$	$u\frac{\partial}{\partial u}$
	$E(t,x)\frac{\partial}{\partial u},\quad E_{xx}-\frac{\Gamma(\alpha)}{t^{\alpha-1}}E_t=0$	$E(t,x)\frac{\partial}{\partial u},\quad {}^{R}D_t^{\alpha}E=E_{xx}$
u^{β}	$\beta x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}$	$\beta x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}$
	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
$u^{-\frac{4}{3}}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$	$x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}$
	$2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u}$	$2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$
e^{u}	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$\frac{2t}{\alpha}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$
	$x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}$	$x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}$
	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}$

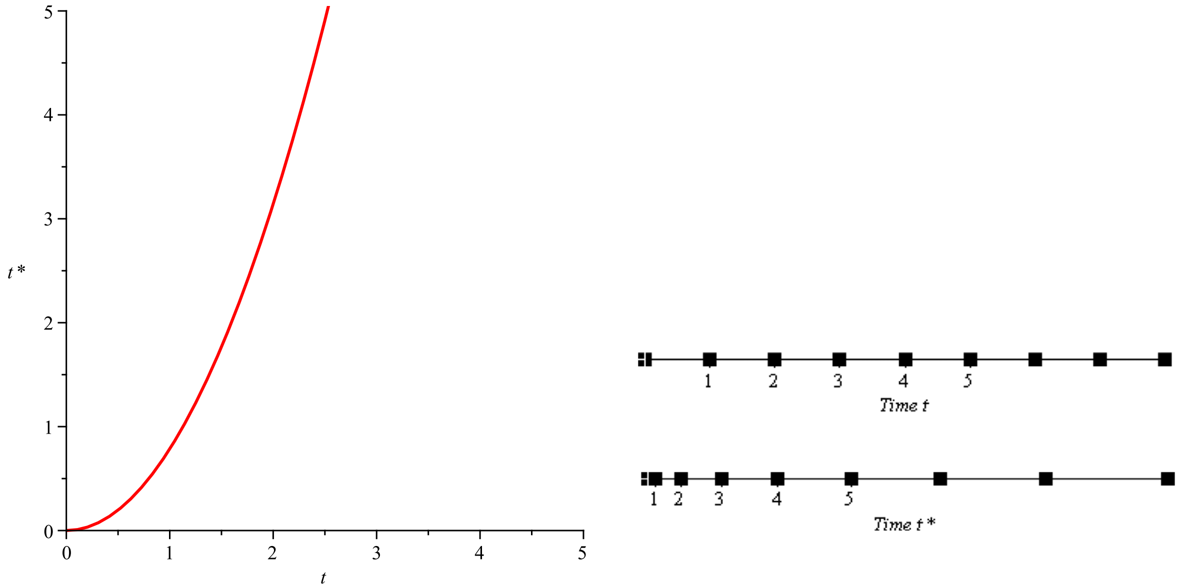

Fig. 1. Real and transformed time.

In case of infinitesimal generator $\frac{\Gamma(\alpha)}{t^{\alpha-1}}\frac{\partial}{\partial t}$, the corresponding group is

$$t^{*}=\left(\Gamma(\alpha+1)\epsilon+t^{\alpha}\right)^{\frac{1}{\alpha}},\quad x^{*}=x,\quad u^{*}=u. \tag{56}$$

The infinitesimal transformation method is a key technique to find invariant solutions of differential equations. An important property of these transformation is that the solution remain invariant under the action of these transformation that is one can derive new solutions from old ones. If $u(t,\ x)$ is a solution of fractional diffusion equation which is invariant under (56) then under the action of (56), $u(t^{*},\ x)$ is also a solution which describes that the process of diffusion at some point x takes too long with the passage of time because the transformed time decreases at increasing fashion. Geometrically it means that the time axis is inhomogeneous and as time increases the gap between any two consecutive time interval also increases. This situation is depicted in Fig. 1.

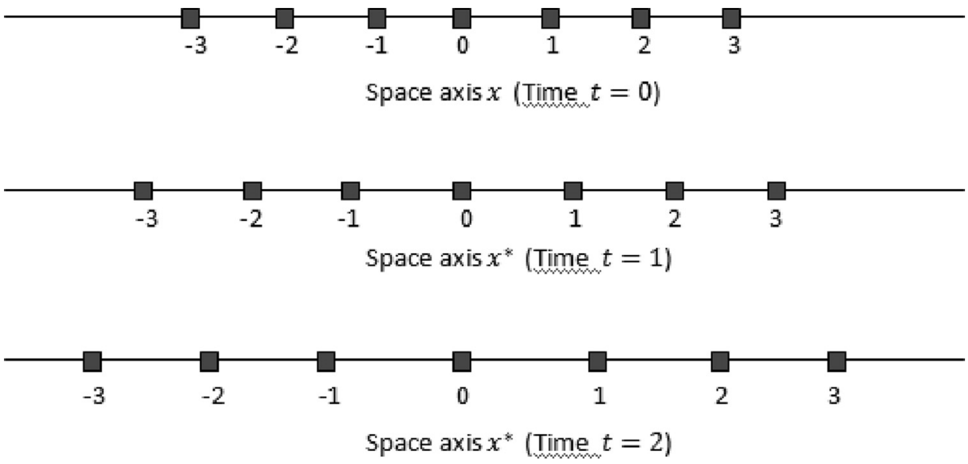

Fig. 2. Expansion of space axis.

For the infinitesimal generator $\frac{2t^\alpha}{\Gamma(1+\alpha)}\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$, the group representation is

$$t^* = t, \quad x^* = \frac{2t^\alpha}{\Gamma(\alpha+1)}\epsilon + x, \quad u^* = u\exp(-\frac{2t^\alpha}{\Gamma(\alpha+1)}\epsilon^2 - x\epsilon), \tag{57}$$

or, equivalently,

$$t^* = t, \quad x^* = \frac{2t^\alpha}{\Gamma(\alpha+1)}\epsilon + x, \quad u^* = ue^{-\epsilon\, x^*}, \tag{58}$$

From second equation of (58), it can be seen that the space axis is dependent on time and with the passage of time it expands as shown in Fig. 2, whereas first equation of (58) shows that the time flow equably i.e time axis is homogenous.

To illustrate this expansion, let us consider the following situation. Suppose a person x, using his clock observing that whether the distance between two milestones are the same while traveling in a train which is slowing down continuously but marginally and the person is not aware of it. The person x will observe that the first mile takes less time than the second mile and second mile takes less time than the third mile and so on. This leads him to the conclusion that distances between milestones are increasing and the milestones are not placed at the positions where they should have been. However, if there is an independent observer, knowing about the slowing-down train, then he would obtain a notably different conclusion.

To understand the diffusion process using (58), we consider a situation in which the molecule of a dye is diffusing in a container of water which is placed in an environment where the temperature is decreasing at an increasing rate. This continuous change in temperature will reduce the kinetic energy of molecule gradually. So the distance travel by a molecule in first unit of time will be greater than the second unit of time and is decreasing in the subsequent units of time. It gives an impression that space is expanding but actually it is the diffusion process which slows down because the space explored by the molecule does not grow linearly with respect to time as described by the third equation of (58).

5. Conclusion

In the study of fractional differential equations, one faces problems in finding exact solutions due to limited number of methods available in the literature. However the approximate/numerical techniques can be applied to compte approximate solutions but one has to face problem in convergence of the solution. The classical Lie methods can be implemented to derive symmetries for fractional differential equations, however the determining equations obtained in the procedure of finding symmetries are fractional differential equations and are of difficult to solve.

In this article, we proposed a new way of finding Lie symmetries for fractional differential equations. The appropriate transformations can be used to transform fractional differential equations to ordinary/partial differential equations and then the symmetries of reduced equations can be obtained using classical Lie's method. Using the inverse transformations one can obtain the Lie symmetries for fractional differential equations. We successfully applied this technique to derive the Lie symmetries for fractional diffusion equation with arbitrary diffusion coefficient which is defined in Riemann-Liouville sense. We transformed Riemann-Liouville fractional operator to modified Riemann-Liouville fractional operator (which has some characteristics similar with the classical calculus) and then introducing suitable transformation the fractional differential equation was reduced to PDE. We provided a complete classification of this resulting PDE and all cases were recovered as discussed by Gazizov et. al. [6]. Beside these we computed some extra symmetries and to the best of our knowledge these were not reported elsewhere. We also obtained the initial condition corresponding to each symmetry which would be helpful in solving certain initial value problems. In order to understand the phenomena of anomalous diffusion we discussed some important symmetries. This study points out a new way of finding Lie symmetries for fractional differential equations which can be used to construct exact solutions.

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