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The quaternionic second weighted zeta function of a graph and the Study determinant

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ABSTRACT

We establish a generalization of the second weighted zeta function of a graph to the case of quaternions. For an arc-weighted graph whose weights are quaternions, we define the second weighted zeta function by using the Study determinant that is a quaternionic determinant for quaternionic matrices defined by Study. This definition is regarded as a quaternionic analogue of the determinant expression of Hashimoto type for the Ihara zeta function of a graph. We derive the Study determinant expression of Bass type and the Euler product for the quaternionic second weighted zeta function.

1. Introduction

The Ihara zeta function of a graph has achieved success in spectral theory of graphs. Zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara [8]. Ihara defined the p-adic Selberg zeta function $\mathbf{Z}_{\Gamma}(u)$ of a torsion-free discrete cocompact subgroup Γ of PGL_2 over a locally compact field under a discrete valuation, and showed

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that its reciprocal is a explicit polynomial. Ihara's motivation to define $\mathbf{Z}_\Gamma(u)$ was to count the number of primitive conjugacy classes of torsion-free discrete cocompact subgroups. Regarding Ihara's work, Serre [18] pointed out that the Ihara zeta function is the zeta function of the quotient $\Gamma \backslash T$ (a finite regular graph) of the one-dimensional Bruhat–Tits building T (an infinite regular tree) associated with $\mathrm{SL}_2(\mathbb{Q}_p)$. This observation led to rapid developments of zeta functions of graphs. Sunada [22,23] developed zeta functions of regular graphs equipped with unitary representations of fundamental groups of the graphs. Hashimoto [5] explored multivariable zeta functions of bipartite graphs and also gave a determinant expression for the Ihara zeta function of a general graph by using its edge matrix. Bass [2] generalized Ihara's result on zeta functions of regular graphs to irregular graphs, and showed that their reciprocals are again polynomials. Subsequently, various proofs of Bass' Theorem were given by Stark and Terras [19], Foata and Zeilberger [4], Kotani and Sunada [12]. Zeta functions of edge-weighted graphs were proposed by Hashimoto [6] and those of arc-weighted graphs by Stark and Terras [19] which are called edge zeta functions. Stark and Terras [19] gave their determinant expressions by using their edge matrices. Mizuno and Sato [14] focused on a special version of edge zeta functions, and defined the weighted zeta function by incorporating a variable t that measures the length of cycles into the edge zeta function. Subsequently, Sato [17] also defined a new class of zeta functions of graphs by modifying the determinant expression of the weighted zeta function defined in [14]. This new zeta function, which was named the second weighted zeta function by Sato, played essential roles in the concise proof of the spectral mapping theorem for the Grover walk on a graph in [11] and of the Smilansky's formula [21] for the characteristic polynomial of the bond scattering matrix of a graph in [15]. Thereby we expect that the second weighted zeta function brings about rich outcomes in quantum dynamics on graphs such as quantum walks on graphs or quantum graphs. In this paper, we aim to establish a quaternionic analogue of the second weighted zeta function of a graph and to derive its basic properties. Our results will play crucial roles in our future work on quaternionic quantum walks which was established by Konno [9] recently.

The quaternion was discovered by Hamilton in 1843. It can be considered as an extension of the complex number. However, quaternions do not commute mutually in general and the definition of determinant is invalid for quaternionic matrices. For many years, a number of researchers, for example Cayley, Study, Moore, Dieudonné, Dyson, Mehta, Xie, Chen, have given different definitions of determinants of quaternionic matrices. Detailed accounts on the determinants of quaternionic matrices can be found in, for example, [1,24]. In this paper, we extend the second weighted zeta function of a graph to the case of quaternions by using the quaternionic determinant defined by Study [20]. An advantage of this approach is that one can reduce a calculation of the Study determinant to that of the ordinary determinant, so is easier to handle and apply than other general and abstract approaches. The Study determinant enables us to derive the explicit determinant expression and the Euler product for the quaternionic second weighted zeta function analogous to the ordinary determinant.

The rest of the paper is organized as follows. In Section 2, we provide a summary of the Ihara zeta function and its variants. We give various zeta functions of graphs, including the second weighted zeta function, and present their determinant expressions. In the end of this section, we explain briefly that the second weighted zeta function can be viewed as a natural generalization of the Ihara zeta function of a tree lattice and is related to quantum systems on graphs. In Section 3, we explain the Study determinant of a quaternionic matrix and give some properties of it which are needed in later sections. In Section 4, we define the quaternionic second weighted zeta function of a graph by using the Study determinant which is considered as a quaternionic analogue of the determinant expression of Hashimoto type, and determine its determinant expression of Bass type (Theorem 4.1). In Section 5, we derive the Euler product for the quaternionic second weighted zeta function of a graph.

2. The Ihara zeta function of a graph

In this section, we provide a summary of the Ihara zeta function of a graph and its development which led the Ihara zeta function to the second weighted zeta function. Let $G = (V(G), E(G))$ be a finite connected graph with the set $V(G)$ of vertices and the set $E(G)$ of undirected edges uv joining two vertices u and v. We assume that G has neither loops nor multiple edges throughout. For $uv \in E(G)$, an arc (u, v) is the directed edge from u to v. Let $D(G) = \{ (u,v), (v,u) \mid uv \in E(G)\}$ and $|V(G)| = n$, $|E(G)| = m$, $|D(G)| = 2m$. For $e = (u, v) \in D(G)$, $o(e) = u$ denotes the *origin* and $t(e) = v$ the *terminal* of e respectively. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$. The *degree* $\deg v = \deg_G v$ of a vertex v of G is the number of edges incident to v. We denote by D_G the symmetric digraph whose vertex set is $V(G)$ and directed edge set is $D(G)$. A *path* P *of length* ℓ in G is a sequence $P = (e_1, \cdots, e_\ell)$ of ℓ arcs such that $e_r \in D(G)$ and $t(e_r) = o(e_{r+1})$ for $r \in \{1, \cdots, \ell-1\}$. We set $o(P) = o(e_1)$ and $t(P) = t(e_\ell)$. $|P|$ denotes the length of P. We say that a path $P = (e_1, \cdots, e_\ell)$ has a *backtracking* if $e_{r+1} = e_r^{-1}$ for some r $(1 \le r \le \ell-1)$, and that $P = (e_1, \cdots, e_\ell)$ has a *tail* if $e_\ell = e_1^{-1}$. A path P is said to be a *cycle* if $t(P) = o(P)$. The *inverse* of a path $P = (e_1, \cdots, e_\ell)$ is the path $(e_\ell^{-1}, \cdots, e_1^{-1})$ and is denoted by P^{-1}.

Two cycles $C_1 = (e_1, \cdots, e_\ell)$ and $C_2 = (f_1, \cdots, f_\ell)$ are said to be *equivalent* if there exists s such that $f_r = e_{r+s}$ for all r where indices are treated modulo ℓ. Let $[C]$ be the equivalence class which contains the cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is said to be *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is said to be *prime* if it is not a power of a strictly smaller cycle.

The *Ihara zeta function* of a graph G is a function of $t \in \mathbf{C}$ with $|t|$ sufficiently small, defined by

$$\mathbf{Z}(G, t) = \mathbf{Z}_G(t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G. Ihara's original definition was group theoretic and defined for a torsion-free discrete cocompact subgroup Γ of PGL_2 over a locally compact field under a discrete valuation. In the case of regular graphs, equivalence classes of prime, reduced cycles of G correspond to primitive conjugacy classes of Γ and $|C|$ the degree of the corresponding primitive conjugacy class. For the details, see [8]. Determinant expressions of Ihara zeta functions for finite graphs are obtained in the following way.

Let $\mathbf{B} = (\mathbf{B}_{ef})_{e,f\in D(G)}$ and $\mathbf{J}_0 = (\mathbf{J}_{ef})_{e,f\in D(G)}$ be $2m \times 2m$ matrices defined as follows:

$$\mathbf{B}_{ef} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{J}_{ef} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix $\mathbf{B} - \mathbf{J}_0$ is called the *edge matrix* of G.

Theorem 2.1 *(Hashimoto [5]; Bass [2]). Let G be a connected graph. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G,t)^{-1} = \det(\mathbf{I}_{2m} - t(\mathbf{B} - \mathbf{J}_0)) = (1-t^2)^{r-1}\det(\mathbf{I}_n - t\mathbf{A} + t^2(\mathbf{D} - \mathbf{I}_n)), \qquad (2.1)$$

where r and $\mathbf{A}$ are the Betti number and the adjacency matrix of G respectively, and $\mathbf{D} = (\mathbf{D}_{uv})_{u,v\in V(G)}$ is the diagonal matrix with $\mathbf{D}_{uu} = \deg u$ for all $u \in V(G)$.

We call the middle formula the determinant expression of *Hashimoto type* and the right hand side the determinant expression of *Bass type* in (2.1).

Now we shall give the definition of the second weighted zeta function. Consider an $n \times n$ complex matrix $\mathbf{W} = (\mathbf{W}_{uv})_{u,v\in V(G)}$ with (u,v)-entry equals 0 if $(u,v) \notin D(G)$. We call $\mathbf{W}$ a *weighted matrix* of G. Furthermore, let $w(u,v) = \mathbf{W}_{uv}$ for $u, v \in V(G)$ and $w(e) = w(u,v)$ if $e = (u,v) \in D(G)$. For a path $P = (e_1, \cdots, e_\ell)$ of G, the *norm* $w(P)$ of P is defined by $w(P) = w(e_1)w(e_2)\cdots w(e_\ell)$. For a weighted matrix $\mathbf{W}$ of G, let $\mathbf{B}_w = (\mathbf{B}^{(w)}_{ef})_{e,f\in D(G)}$ be the $2m \times 2m$ complex matrix as follows:

$$\mathbf{B}^{(w)}_{ef} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then the *second weighted zeta function* of G is defined by

$$\mathbf{Z}_1(G,w,t) = \det(\mathbf{I}_{2m} - t(\mathbf{B}_w - \mathbf{J}_0))^{-1}.$$

We call $\mathbf{B}_w - \mathbf{J}_0$ the $\mathbf{B}$-*weighted edge matrix* of G. If $w(e) = 1$ for any $e \in D(G)$, then the second weighted zeta function of G coincides with the Ihara zeta function of G.

Theorem 2.2 *(Sato [17]). Let G be a connected graph, and let $\mathbf{W}$ be a weighted matrix of G. Then the reciprocal of the second weighted zeta function of G is given by*

$$\mathbf{Z}_1(G,w,t)^{-1} = (1-t^2)^{m-n}\det(\mathbf{I}_n - t\mathbf{W} + t^2(\mathbf{D}_w - \mathbf{I}_n)),$$

where $n = |V(G)|$, $m = |E(G)|$ *and* $\mathbf{D}_w = (\mathbf{D}^{(w)}_{uv})_{u,v\in V(G)}$ *is the diagonal matrix with* $\mathbf{D}^{(w)}_{uu} = \sum_{e:o(e)=u} w(e)$ *for all* $u \in V(G)$.

In [15], Mizuno and Sato obtained the Euler product for $\mathbf{Z}_1(G,w,t)$. Let $\tilde{w}(e,f)$ be the (e,f)-entry of the matrix $\mathbf{B}_w - \mathbf{J}_0$. $\tilde{w}(e,f)$ is given by the following formula:

$$\tilde{w}(e,f) = \begin{cases} w(f) & \text{if } t(e)=o(f) \text{ and } f \neq e^{-1}, \\ w(f)-1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

We set $\tilde{w}(P) = \tilde{w}(e_1,e_2)\tilde{w}(e_2,e_3)\cdots\tilde{w}(e_{\ell-1},e_\ell)$ for a path $P = (e_1,\cdots,e_\ell)$.

Theorem 2.3 *(Mizuno and Sato [15]). Let G be a connected graph. Then*

$$\mathbf{Z}_1(G,w,t) = \prod_{[C]} (1-\tilde{w}(C)t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G.

We shall give some notable comments on the second weighted zeta function of a graph. Let T be a locally finite (possibly infinite) tree and Γ a group with an action on the symmetric digraph $X = D_T = (V(X), \overrightarrow{E}(X))$ where $V(X) = V(T)$ and $\overrightarrow{E}(X) = D(T)$. According to [2], we assume that Γ satisfies the following conditions:

(I) Γ acts without inversions, that is, $\Gamma e \neq \Gamma e^{-1}$ for all $e \in \overrightarrow{E}(X)$.
(D) Γ is discrete, that is, the stabilizer Γ_u of u is finite for every $u \in V(X)$.
(F) Γ is uniform (= cocompact), that is, $\Gamma\backslash X$ is finite.

Γ is called a *uniform tree lattice*. Let $Y = (V(Y), \overrightarrow{E}(Y)) = \Gamma\backslash X$ be the quotient digraph and $p : X \longrightarrow Y$ the projection. For $x \in X$ and $y = p(x) \in Y$, the *index* $i(e)$ of $e \in \overrightarrow{E}(Y)$ which satisfies $o(e) = y$ is defined by

$$i(e) = |\{\, e' \in \overrightarrow{E}(X) \mid o(e') = x,\ p(e') = e \,\}|.$$

The pair (Y, i) is called an *edge-indexed graph*. Now we assume Y has no loop and set $D_G = Y$ and $w(e) = i(e)$ for all $e \in \overrightarrow{E}(Y)$. Then $\mathbf{Z}_1(G,w,t)$ coincides with the zeta function of the edge-indexed graph (Y,i) which was defined by Bass [2]. Therefore we can consider $\mathbf{Z}_1(G,w,t)$ as a natural generalization of the zeta function of the edge-indexed graph.

The second weighted zeta function has remarkable connections with quantum dynamics on graphs. Replacing t with $1/\lambda$ in Theorem 2.2 we have

$$\det(\lambda \mathbf{I}_{2m} - (\mathbf{B}_w - \mathbf{J}_0)) = (\lambda^2 - 1)^{m-n} \det(\lambda^2 \mathbf{I}_n - \lambda \mathbf{W} + (\mathbf{D}_w - \mathbf{I}_n)). \tag{2.3}$$

Using this equation, several quantum systems on graphs have been investigated. Consequently, new proofs of spectral properties of them were given in [15,11] and spectra were newly determined by easy derivable parameters from $\mathbf{W}$ in [7,10]. Interestingly, time evolution operators of important quantum systems on graphs, for example, the Grover walk, the Szegedy walk, the bond scattering matrix of Smilansky, can be expressed by $\mathbf{B}$-weighted edge matrices. It enables us to apply (2.3) to derive spectra of time evolution operators. In this way, the generalization of Ihara zeta functions of tree lattices has significant applications in quantum mechanics. We expect that the second weighted zeta function brings about further outcomes in quantum dynamics on graphs such as quantum walks or quantum graphs.

3. The Study determinant of a quaternionic matrix

We shall establish the quaternionic second weighted zeta function of a graph from now on. In this section, we explain quaternions and quaternionic matrices that are needed in later sections. Let $\mathbb{H}$ be the set of quaternions. $\mathbb{H}$ is a noncommutative associative algebra over $\mathbb{R}$, whose underlying real vector space has dimension 4 with a basis $1, i, j, k$ which satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, x^* denotes the conjugate of x in $\mathbb{H}$ which is defined by $x^* = x_0 - x_1 i - x_2 j - x_3 k$, and $\operatorname{Re} x = x_0$ the real part of x. One can easily check $xx^* = x^* x$, $(x^*)^n = (x^n)^*$, and $x^{-1} = x^*/|x|^2$ for $x \neq 0$. Hence, $\mathbb{H}$ constitutes a skew field. We call $|x| = \sqrt{xx^*} = \sqrt{x^* x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ the norm of x. Indeed, $|\cdot|$ satisfies

(1) $|x| \geq 0$, and moreover $|x| = 0 \Leftrightarrow x = 0$,
(2) $|xy| = |x||y|$,
(3) $|x + y| \leq |x| + |y|$.

We notice that $\mathbb{H}$ is a complete space with respect to the metric derived from the norm of quaternions.

Any quaternion x can be presented by two complex numbers $x = a + jb$ uniquely. Explicitly, if $x = x_0 + x_1 i + x_2 j + x_3 k$ then $a = x_0 + x_1 i$ and $b = x_2 - x_3 i$. Such a presentation is called the *symplectic decomposition*. Two complex numbers a and b are called the *simplex part* and the *perplex part* of x respectively. We mean by a *quaternionic* *matrix* a matrix whose entries are quaternions. The symplectic decomposition is also valid for a quaternionic matrix. $\mathrm{Mat}(m\times n,\mathbb{H})$ denotes the set of $m\times n$ quaternionic matrices and $\mathrm{Mat}(m,\mathbb{H})$ the set of $m\times m$ quaternionic matrices. For $\mathbf{M}\in\mathrm{Mat}(m\times n,\mathbb{H})$, we can write $\mathbf{M}=\mathbf{M}^S+j\mathbf{M}^P$ uniquely where $\mathbf{M}^S,\mathbf{M}^P\in\mathrm{Mat}(m,n,\mathbb{C})$. $\mathbf{M}^S$ and $\mathbf{M}^P$ are called the *simplex part* and the *perplex part* of $\mathbf{M}$ respectively. We define ψ to be the map from $\mathrm{Mat}(m\times n,\mathbb{H})$ to $\mathrm{Mat}(2m\times 2n,\mathbb{C})$ as follows:

$$\psi:\mathrm{Mat}(m\times n,\mathbb{H})\longrightarrow\mathrm{Mat}(2m\times 2n,\mathbb{C})\quad \mathbf{M}\mapsto\begin{bmatrix}\mathbf{M}^S & -\overline{\mathbf{M}^P}\\ \mathbf{M}^P & \overline{\mathbf{M}^S}\end{bmatrix},$$

where $\overline{\mathbf{A}}$ is the complex conjugate of a complex matrix $\mathbf{A}$. Then ψ is an $\mathbb{R}$-linear map. We also have

Lemma 3.1. *Let* $\mathbf{M}\in\mathrm{Mat}(m\times n,\mathbb{H})$ *and* $\mathbf{N}\in\mathrm{Mat}(n\times m,\mathbb{H})$. *Then*

$$\psi(\mathbf{MN})=\psi(\mathbf{M})\psi(\mathbf{N}).$$

Proof. Let $\mathbf{M}=\mathbf{A}+j\mathbf{B}$ and $\mathbf{N}=\mathbf{C}+j\mathbf{D}$ be symplectic decompositions of $\mathbf{M}$ and $\mathbf{N}$. Then

$$\mathbf{MN}=(\mathbf{A}+j\mathbf{B})(\mathbf{C}+j\mathbf{D})=\mathbf{AC}+\mathbf{A}j\mathbf{D}+j\mathbf{BC}+j\mathbf{B}j\mathbf{D}.$$

Since $\mathbf{X}j=j\overline{\mathbf{X}}$ for every complex matrix $\mathbf{X}$, we obtain

$$\mathbf{MN}=\mathbf{AC}-\overline{\mathbf{B}}\mathbf{D}+j(\overline{\mathbf{A}}\mathbf{D}+\mathbf{BC}),$$

and therefore

$$\psi(\mathbf{MN})=\begin{bmatrix}\mathbf{AC}-\overline{\mathbf{B}}\mathbf{D} & -\mathbf{A}\overline{\mathbf{D}}-\overline{\mathbf{BC}}\\ \overline{\mathbf{A}}\mathbf{D}+\mathbf{BC} & \overline{\mathbf{AC}}-\mathbf{B}\overline{\mathbf{D}}\end{bmatrix}.$$

On the other hand,

$$\psi(\mathbf{M})\psi(\mathbf{N})=\begin{bmatrix}\mathbf{A} & -\overline{\mathbf{B}}\\ \mathbf{B} & \overline{\mathbf{A}}\end{bmatrix}\begin{bmatrix}\mathbf{C} & -\overline{\mathbf{D}}\\ \mathbf{D} & \overline{\mathbf{C}}\end{bmatrix}=\begin{bmatrix}\mathbf{AC}-\overline{\mathbf{B}}\mathbf{D} & -\mathbf{A}\overline{\mathbf{D}}-\overline{\mathbf{BC}}\\ \mathbf{BC}+\overline{\mathbf{A}}\mathbf{D} & -\mathbf{B}\overline{\mathbf{D}}+\overline{\mathbf{AC}}\end{bmatrix}.$$

Thus $\psi(\mathbf{MN})=\psi(\mathbf{M})\psi(\mathbf{N})$ holds. $\square$

From Lemma 3.1, it follows immediately that

Proposition 3.2. *If $m=n$, then ψ is an injective $\mathbb{R}$-algebra homomorphism.*

In [20], Study defined a determinant of an $n\times n$ quaternionic matrix which we denote by Sdet, by setting $\mathrm{Sdet}(\mathbf{M})=\det(\psi(\mathbf{M}))$, where det is the ordinary determinant. We call Sdet the *Study determinant*. The Study determinant is the unique, up to a real power factor, functional $d_{\mathbb{H}}$ which satisfies the following three axioms [1]:

(A1) $d_{\mathbb{H}}(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A}$ is singular.
(A2) $d_{\mathbb{H}}(\mathbf{AB}) = d_{\mathbb{H}}(\mathbf{A})d_{\mathbb{H}}(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathrm{Mat}(n, \mathbb{H})$.
(A3) If $\mathbf{A}'$ is obtained from $\mathbf{A}$ by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $d_{\mathbb{H}}(\mathbf{A}') = d_{\mathbb{H}}(\mathbf{A})$.

Therefore it is reasonable to adopt the Study determinant to investigate the quaternionization of determinant expressions for zeta functions of graphs. Before stating properties of Sdet, we mention a useful formula whose proof can be found in for example [25]:

Lemma 3.3. *If $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are complex square matrices with same size and $\mathbf{AC} = \mathbf{CA}$, then*

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{AD} - \mathbf{CB}).$$

Sdet has several basic properties as follows:

Proposition 3.4.

(i) $\mathrm{Sdet}(\mathbf{M}) \in \mathbb{R}_{\geq 0} = \{a \in \mathbb{R} \mid a \geq 0\}$ *for* $\mathbf{M} \in \mathrm{Mat}(n, \mathbb{H})$.
(ii) $\mathrm{Sdet}(\mathbf{M}) = 0 \Leftrightarrow \mathbf{M}$ *has no inverse.*
(iii) $\mathrm{Sdet}(\mathbf{MN}) = \mathrm{Sdet}(\mathbf{M})\,\mathrm{Sdet}(\mathbf{N})$ *for* $\mathbf{M}, \mathbf{N} \in \mathrm{Mat}(n, \mathbb{H})$.
(iv) *If* $\mathbf{N}$ *is obtained from* $\mathbf{M}$ *by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then* $\mathrm{Sdet}(\mathbf{N}) = \mathrm{Sdet}(\mathbf{M})$.
(v) $\mathrm{Sdet}(\alpha\mathbf{M}) = \mathrm{Sdet}(\mathbf{M}\alpha) = |\alpha|^{2n}\,\mathrm{Sdet}(\mathbf{M})$ *for* $\mathbf{M} \in \mathrm{Mat}(n, \mathbb{H})$, $\alpha \in \mathbb{H}$.
(vi) *If* $\mathbf{M} \in \mathrm{Mat}(n, \mathbb{H})$ *is of the form:*

$$\mathbf{M} = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ * & * & \cdots & \lambda_n \end{bmatrix},$$

then $\mathrm{Sdet}(\mathbf{M}) = \prod_{r=1}^{n} |\lambda_r|^2$.
(vii) *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{B}$ *an* $n \times m$ *matrix. Then*

$$\mathrm{Sdet}(\mathbf{I}_m - \mathbf{AB}) = \mathrm{Sdet}(\mathbf{I}_n - \mathbf{BA}).$$

Proof. The proofs of (i), (ii), (iii), (iv) can be found in [1]. We prove (v), (vi) and (vii).

(v) Let $\mathbf{M} \in \mathrm{Mat}(n, \mathbb{H})$ and $\alpha = \alpha_s + j\alpha_p \in \mathbb{H}(\alpha_s, \alpha_p \in \mathbb{C})$. Then using Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned}
\mathrm{Sdet}(\alpha\mathbf{M}) &= \det(\psi(\alpha\mathbf{M})) = \det(\psi(\alpha\mathbf{I}_n)\psi(\mathbf{M})) \\
&= \det\begin{bmatrix} \alpha_s\mathbf{I}_n & -\overline{\alpha}_p\mathbf{I}_n \\ \alpha_p\mathbf{I}_n & \overline{\alpha}_s\mathbf{I}_n \end{bmatrix}\det(\psi(\mathbf{M})) \\
&= \det((\alpha_s\mathbf{I}_n)(\overline{\alpha}_s\mathbf{I}_n) + (\alpha_p\mathbf{I}_n)(\overline{\alpha}_p\mathbf{I}_n))\mathrm{Sdet}(\mathbf{M}) \\
&= \det((|\alpha_s|^2 + |\alpha_p|^2)\mathbf{I}_n)\mathrm{Sdet}(\mathbf{M}) \\
&= |\alpha|^{2n}\mathrm{Sdet}(\mathbf{M}).
\end{aligned}$$

In the same way, we can deduce $\mathrm{Sdet}(\mathbf{M}\alpha) = |\alpha|^{2n}\,\mathrm{Sdet}(\mathbf{M})$.

(vi) For a $2n\times 2n$ matrix $\mathbf{N}$ and any two subsets $I = \{i_1, i_2, \cdots, i_r\}$, $J = \{j_1, j_2, \cdots, j_s\}$ of $[2n]$, $\mathbf{N}^{IJ}$ denotes the submatrix obtained from $\mathbf{N}$ by deleting $i_1, i_2, \cdots, i_r$ th rows and $j_1, j_2, \cdots, j_s$ th columns. Then by definitions of Sdet and ψ, we get the following:

$$\begin{aligned}
\mathrm{Sdet}(\mathbf{M}) &= \det(\psi(\mathbf{M})) = \det\begin{bmatrix} \mathbf{M}^S & -\overline{\mathbf{M}^P} \\ \mathbf{M}^P & \overline{\mathbf{M}^S} \end{bmatrix} \\
&= \det\begin{bmatrix}
\lambda_1^S & * & \cdots & * & -\overline{\lambda_1^P} & * & \cdots & * \\
0 & \lambda_2^S & & * & 0 & -\overline{\lambda_2^P} & & * \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n^S & 0 & 0 & \cdots & -\overline{\lambda_n^P} \\
\lambda_1^P & * & \cdots & * & \overline{\lambda_1^S} & * & \cdots & * \\
0 & \lambda_2^P & & * & 0 & \overline{\lambda_2^S} & & * \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n^P & 0 & 0 & \cdots & \overline{\lambda_n^S}
\end{bmatrix} \\
&= \lambda_1^S \det(\psi(\mathbf{M})^{\{1\}\{1\}}) + (-1)^{n+2}\lambda_1^P \det(\psi(\mathbf{M})^{\{n+1\}\{1\}}) \\
&= \lambda_1^S\overline{\lambda_1^S}\det(\psi(\mathbf{M})^{\{1,n+1\}\{1,n+1\}}) + \lambda_1^P\overline{\lambda_1^P}\det(\psi(\mathbf{M})^{\{1,n+1\}\{1,n+1\}}) \\
&= |\lambda_1|^2\det(\psi(\mathbf{M})^{\{1,n+1\}\{1,n+1\}}) \\
&= |\lambda_1|^2|\lambda_2|^2\det(\psi(\mathbf{M})^{\{1,2,n+1,n+2\}\{1,2,n+1,n+2\}}) = \cdots \\
&= \prod_{r=1}^{n}|\lambda_r|^2.
\end{aligned}$$

(vii) Let $\mathbf{A}$ be an $m\times n$ matrix and $\mathbf{B}$ an $n\times m$ matrix. Then by the definition of the Study determinant, we have

$$\begin{aligned}
\mathrm{Sdet}(\mathbf{I}_m - \mathbf{AB}) &= \mathrm{Sdet}(\psi(\mathbf{I}_m - \mathbf{AB})) = \det(\mathbf{I}_{2m} - \psi(\mathbf{A})\psi(\mathbf{B})) \\
&= \det(\mathbf{I}_{2n} - \psi(\mathbf{B})\psi(\mathbf{A})) = \det(\psi(\mathbf{I}_n - \mathbf{BA})) \\
&= \mathrm{Sdet}(\mathbf{I}_n - \mathbf{BA}).
\end{aligned}$$

Here the equation $\det(\mathbf{I}_{2m} - \psi(\mathbf{A})\psi(\mathbf{B})) = \det(\mathbf{I}_{2n} - \psi(\mathbf{B})\psi(\mathbf{A}))$ is based on the property of determinant that can be found in, for example, Problem 5 on page 47 in [25]. $\square$

Remark 3.5. Sdet is not multilinear as det is. Furthermore, $\mathrm{Sdet}({}^T\mathbf{M}) = \mathrm{Sdet}(\mathbf{M})$ does not hold in general where ${}^T\mathbf{M}$ is the transpose of $\mathbf{M}$.

4. The quaternionic second weighted zeta function of a graph

We follow symbols and notations in the previous section. We shall give the definition of a quaternionic analogue of the second weighted zeta function and derive its determinant expression of Bass type by the Study determinant. In the same way as the complex case, consider a quaternionic matrix $\mathbf{W} = (\mathbf{W}_{uv})_{u,v \in V(G)} \in \mathrm{Mat}(n, \mathbb{H})$ with (u,v)-entry equals 0 if $(u,v) \notin D(G)$. We call $\mathbf{W}$ a *quaternionic weighted matrix* of G. Furthermore, let $w(u,v) = \mathbf{W}_{uv}$ for $u, v \in V(G)$ and $w(e) = w(u,v)$ if $e = (u,v) \in D(G)$. For each path $P = (e_1, \cdots, e_\ell)$ of G, the *norm* $w(P)$ of P is defined by $w(P) = w(e_1)w(e_2)\cdots w(e_\ell)$.

For a quaternionic weighted matrix $\mathbf{W}$ of G, the $2m \times 2m$ quaternionic matrix $\mathbf{B}_w = (\mathbf{B}^{(w)}_{ef})_{e,f \in D(G)} \in \mathrm{Mat}(2m, \mathbb{H})$ is defined as follows:

$$\mathbf{B}^{(w)}_{ef} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We define the *quaternionic second weighted zeta function* of G to be as follows:

$$\mathbf{Z}^{\mathbb{H}}_1(G, w, t) = \mathrm{Sdet}(\mathbf{I}_{2m} - t(\mathbf{B}_w - \mathbf{J}_0))^{-1},$$

where t is a quaternionic variable.

The Study determinant expression of Bass type for the quaternionic second weighted zeta function of a graph is given as follows. The block diagonal sum $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ of square matrices $\mathbf{M}_1, \cdots, \mathbf{M}_s$ is defined as the square matrix:

$$\begin{bmatrix} \mathbf{M}_1 & & 0 \\ & \ddots & \\ 0 & & \mathbf{M}_s \end{bmatrix}.$$

If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{N}$, then we write $s \circ \mathbf{N} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The *Kronecker product* $\mathbf{A} \bigotimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{rs} replaced by the matrix $a_{rs}\mathbf{B}$.

Theorem 4.1. *Let G be a connected graph, and let $\mathbf{W}$ be a quaternionic weighted matrix of G. Then the reciprocal of the quaternionic second weighted zeta function of G is given by*

$$\mathbf{Z}_1^{\mathbb{H}}(G,w,t)^{-1} = |1-t^2|^{2m-2n}\mathrm{Sdet}(\mathbf{I}_n - \mathbf{W}t + (\mathbf{D}_w - \mathbf{I}_n)t^2),$$

where $n = |V(G)|$*,* $m = |E(G)|$ *and* $\mathbf{D}_w = (\mathbf{D}_{uv}^{(w)})_{u,v\in V(G)}$ *is the diagonal matrix with* $\mathbf{D}_{uu}^{(w)} = \sum_{e:o(e)=u} w(e)$*.*

Proof. Let $D(G) = \{f_1, \cdots, f_m, f_1^{-1}, \cdots, f_m^{-1}\}$. Arrange arcs of G as follows:

$$f_1, f_1^{-1}, \cdots, f_m, f_m^{-1}.$$

Note that $\mathbf{J}_0 = m \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by the arrangement of arcs. By the definition of the second weighted zeta function of G and Proposition 3.4, we have

$$\begin{aligned} \mathbf{Z}_1^{\mathbb{H}}(G,w,t)^{-1} &= \mathrm{Sdet}(\mathbf{I}_{2m} - t(\mathbf{B}_w - \mathbf{J}_0)) \\ &= \mathrm{Sdet}(\mathbf{I}_{2m} + t\mathbf{J}_0 - t\mathbf{B}_w) \\ &= \mathrm{Sdet}(\mathbf{I}_{2m} - t\mathbf{B}_w(\mathbf{I}_{2m} + t\mathbf{J}_0)^{-1})\mathrm{Sdet}(\mathbf{I}_{2m} + t\mathbf{J}_0). \end{aligned} \tag{4.2}$$

Let $t = t_s + jt_p \in \mathbb{H}$ be the symplectic decomposition. Then we have

$$\mathrm{Sdet}(\mathbf{I}_{2m} + t\mathbf{J}_0) = \det \begin{bmatrix} \mathbf{I}_{2m} + t_s\mathbf{J}_0 & -\overline{t_p}\mathbf{J}_0 \\ t_p\mathbf{J}_0 & \mathbf{I}_{2m} + \overline{t_s}\mathbf{J}_0 \end{bmatrix}, \tag{4.3}$$

where $\mathbf{I}_{2m} + t_s\mathbf{J}_0$ and $t_p\mathbf{J}_0$ are given by

$$\mathbf{I}_{2m} + t_s\mathbf{J}_0 = m \circ \begin{bmatrix} 1 & t_s \\ t_s & 1 \end{bmatrix}, \ t_p\mathbf{J}_0 = m \circ \begin{bmatrix} 0 & t_p \\ t_p & 0 \end{bmatrix}.$$

For any two complex numbers α and β,

$$\begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

holds. This implies any two blocks in the right-hand side of (4.3) commute. Thus by Lemma 3.3, we have

$$\begin{aligned} \mathrm{Sdet}(\mathbf{I}_{2m} + t\mathbf{J}_0) &= \det((\mathbf{I}_{2m} + t_s\mathbf{J}_0)(\mathbf{I}_{2m} + \overline{t_s}\mathbf{J}_0) + t_p\overline{t_p}\mathbf{J}_0^2) \\ &= \det(\mathbf{I}_{2m} + (t_s + \overline{t_s})\mathbf{J}_0 + (|t_s|^2 + |t_p|^2)\mathbf{I}_{2m}) \\ &= \det \mathbf{I}_m \bigotimes \begin{bmatrix} 1+|t|^2 & 2\,\mathrm{Re}\ t \\ 2\,\mathrm{Re}\ t & 1+|t|^2 \end{bmatrix} \\ &= \{(1+|t|^2)^2 - 4(\mathrm{Re}\ t)^2\}^m \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&= \{(1+tt^*)^2 - (t+t^*)^2\}^m \\
&= \{(1-t^2)(1-(t^*)^2)\}^m \\
&= \{(1-t^2)(1-t^2)^*\}^m \\
&= |1-t^2|^{2m}.
\end{aligned}
$$

On the other hand, since the following holds:

$$
t\frac{1}{1-t^2} = t\frac{(1-(t^2))^*}{|1-t^2|^2} = t\frac{1-t^*t^*}{|1-t^2|^2} = \frac{1-t^*t^*}{|1-t^2|^2}t = \frac{1}{1-t^2}t, \tag{4.5}
$$

and hence

$$
(\mathbf{I}_{2m} + t\mathbf{J}_0)(\mathbf{I}_{2m} - t\mathbf{J}_0)\frac{1}{1-t^2} = (\mathbf{I}_{2m} - t\mathbf{J}_0)\frac{1}{1-t^2}(\mathbf{I}_{2m} + t\mathbf{J}_0) = \mathbf{I}_{2m}. \tag{4.6}
$$

Thus, we have

$$
(\mathbf{I}_{2m} + t\mathbf{J}_0)^{-1} = (\mathbf{I}_{2m} - t\mathbf{J}_0)\frac{1}{1-t^2}.
$$

Now, let $\mathbf{K} = (\mathbf{K}_{ev})_{e\in D(G), v\in V(G)}$ and $\mathbf{L} = (\mathbf{L}_{ev})_{e\in D(G), v\in V(G)}$ be $2m \times n$ matrices defined as follows:

$$
\mathbf{K}_{ev} = \begin{cases} w(e) & \text{if } o(e) = v, \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{L}_{ev} = \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise.} \end{cases}
$$

Then

$$
\mathbf{L}^T\mathbf{K} = \mathbf{B}_w
$$

holds, where ${}^T\mathbf{K}$ is the transpose of $\mathbf{K}$. Thus, by Lemma 3.4, (4.5) and (4.6), we can show that

$$
\begin{aligned}
&\mathrm{Sdet}(\mathbf{I}_{2m} - t\mathbf{B}_w(\mathbf{I}_{2m} + t\mathbf{J}_0)^{-1}) = \mathrm{Sdet}(\mathbf{I}_{2m} - t\mathbf{L}^T\mathbf{K}(\mathbf{I}_{2m} + t\mathbf{J}_0)^{-1}) \\
&= \mathrm{Sdet}(\mathbf{I}_n - {}^T\mathbf{K}(\mathbf{I}_{2m} + t\mathbf{J}_0)^{-1}(t\mathbf{L})) = \mathrm{Sdet}(\mathbf{I}_n - {}^T\mathbf{K}(\mathbf{I}_{2m} + t\mathbf{J}_0)^{-1}\mathbf{L}t) \\
&= \mathrm{Sdet}(\mathbf{I}_n - {}^T\mathbf{K}(\mathbf{I}_{2m} - t\mathbf{J}_0)\frac{1}{1-t^2}\mathbf{L}t) = \mathrm{Sdet}(\mathbf{I}_n - {}^T\mathbf{K}(\mathbf{I}_{2m} - t\mathbf{J}_0)\mathbf{L}t\frac{1}{1-t^2}).
\end{aligned}
$$

For an arc $(u,v) \in D(G)$ with $u \neq v$,

$$
({}^T\mathbf{K}(\mathbf{I}_{2m} - t\mathbf{J}_0)\mathbf{L})_{uv} = w(u,v).
$$

In the case of $u = v$,

$$
({}^T\mathbf{K}(\mathbf{I}_{2m} - t\mathbf{J}_0)\mathbf{L})_{uu} = -\sum_{o(e)=u} w(e)t.
$$

Thus by Proposition 3.4, we have

$$\begin{aligned}\mathrm{Sdet}(\mathbf{I}_{2m}-t\mathbf{B}_w(\mathbf{I}_{2m}+t\mathbf{J}_0)^{-1}) &= \mathrm{Sdet}(\mathbf{I}_n-(\mathbf{W}-\mathbf{D}_w t)t\frac{1}{1-t^2})\\ &= \mathrm{Sdet}(((1-t^2)\mathbf{I}_n-\mathbf{W}t+\mathbf{D}_w t^2)\frac{1}{1-t^2})\\ &= \mathrm{Sdet}(\mathbf{I}_n-\mathbf{W}t+(\mathbf{D}_w-\mathbf{I}_n)t^2)\left|\frac{1}{1-t^2}\right|^{2n}.\end{aligned}$$

Finally, we conclude from (4.2) and (4.4) that

$$\begin{aligned}\mathbf{Z}_1^{\mathbb{H}}(G,w,t)^{-1} &= \mathrm{Sdet}(\mathbf{I}_{2m}-t\mathbf{B}_w(\mathbf{I}_{2m}+t\mathbf{J}_0)^{-1})\mathrm{Sdet}(\mathbf{I}_{2m}+t\mathbf{J}_0)\\ &= \mathrm{Sdet}(\mathbf{I}_n-\mathbf{W}t+(\mathbf{D}_w-\mathbf{I}_n)t^2)\left|\frac{1}{1-t^2}\right|^{2n}|1-t^2|^{2m}\\ &= |1-t^2|^{2m-2n}\mathrm{Sdet}(\mathbf{I}_n-\mathbf{W}t+(\mathbf{D}_w-\mathbf{I}_n)t^2). \quad \square\end{aligned}$$

5. The Euler product for the quaternionic second weighted zeta function

In this section, we derive the Euler product of the quaternionic second weighted zeta function. In order to obtain the Euler product, we make use of the notion of noncommutative formal power series. Our argument on this subject is based on the proofs of Amitsur's identity in [16] or [4] until (5.6). For the sake of argument, we will give a brief account of noncommutative formal power series at first. A detailed exposition of formal series can be found in [3]. Let $X = \{x_1, \cdots, x_N\}$ be a finite nonempty totally ordered set in which elements are arranged ascendingly. X^* denotes the free monoid generated by X. Let $<$ be the lexicographic order on X^* derived from the total order on X. For a word $w = x_{i_1}x_{i_2}\cdots x_{i_d} \in X^*$, d is called the *length* of w which is denoted by $|w|$. The length of the empty word is defined to be 0. A nonempty word w in X^* is called a *Lyndon word* if w is *prime*, namely, not a power $w'^{\,r}$ of any other word w' for any $r \geq 2$, and is minimal in the cyclic rearrangements of w. We denote by L_X the set of Lyndon words in X^*. It is well known that any nonempty word w can be formed uniquely as a nonincreasing sequence of Lyndon words.

Theorem 5.1. *For any nonempty word $w \in X^*$, there exists a unique nonincreasing sequence of Lyndon words $l_1, l_2, \cdots, l_d$ such that $w = l_1 l_2 \cdots l_d$.*

Proof. For the proof, see for example [13]. $\square$

Let us consider the ring of noncommutative formal power series $\mathbb{R}\langle\langle X^*\rangle\rangle$. Each element f of $\mathbb{R}\langle\langle X^*\rangle\rangle$ is displayed as

$$f = \sum_{w\in X^*} f_w w \quad (f_w \in \mathbb{R}).$$

$\mathbb{R}\langle\langle X^* \rangle\rangle$ can be equipped with the topology defined by the following manner. Let ω be the function defined as follows:

$$\omega : \mathbb{R}\langle\langle X^*\rangle\rangle \times \mathbb{R}\langle\langle X^*\rangle\rangle \longrightarrow \mathbb{N} \cup \{\infty\} = \{0, 1, 2, \cdots, \infty\}$$

$$\omega(\alpha, \beta) = \begin{cases} \infty & \text{if } \{w \in X^* \mid \alpha_w \neq \beta_w\} = \phi, \\ \inf\{n \in \mathbb{N} \mid \exists w \in X^*, \, |w| = n, \, \alpha_w \neq \beta_w\} & \text{otherwise.} \end{cases}$$

Then an ultrametric distance d_ω on $\mathbb{R}\langle\langle X^*\rangle\rangle$ is given by $d_\omega(\alpha, \beta) = 2^{-\omega(\alpha,\beta)}$ and a topology on $\mathbb{R}\langle\langle X^*\rangle\rangle$ is derived from d_ω. We notice that $\mathbb{R}\langle\langle X^*\rangle\rangle$ is complete for this topology. Since $(1 - l)^{-1} = 1 + l + l^2 + \cdots$ for every $l \in X^*$ in $\mathbb{R}\langle\langle X^*\rangle\rangle$, Theorem 5.1 implies

$$\prod_{l \in L_X}^{>} (1 - l)^{-1} = \sum_{w \in X^*} w, \tag{5.1}$$

in $\mathbb{R}\langle\langle X^*\rangle\rangle$, where $\prod\limits_{l \in L_X}^{>}$ means that the factors are multiplied in decreasing order. On the other hand, it follows that

$$\sum_{w \in X^*} w = \{1 - (x_1 + \cdots + x_N)\}^{-1}. \tag{5.2}$$

(5.1) and (5.2) imply the following equation:

$$\{1 - (x_1 + \cdots + x_N)\}^{-1} = \prod_{l \in L_X}^{>} (1 - l)^{-1}. \tag{5.3}$$

From (5.3), we obtain

Proposition 5.2.

$$1 - (x_1 + \cdots + x_N) = \prod_{l \in L_X}^{<} (1 - l), \tag{5.4}$$

where $\prod\limits_{l \in L_X}^{<}$ *means that the factors are multiplied in increasing order.*

Proof. In order to show that

$$\Big\{ \prod_{l \in L_X}^{>} (1 - l)^{-1} \Big\} \Big\{ \prod_{l \in L_X}^{<} (1 - l) \Big\} = 1, \tag{5.5}$$

we check that for an arbitrary nonnegative integer $r \geq 0$ the sum of words of length r equals 1 if $r = 0$ and 0 if $r > 0$. Since $\prod^{>}_{l\in L_X}(1-l)^{-1} = \prod^{>}_{l\in L_X}(1+l+l^2+\cdots)$, the sum of words of length at most d in the left hand side of (5.5) is the same as that of the product:

$$\Big\{\prod^{>}_{\substack{l\in L_X\\|l|\leq d}}(1+l+l^2+\cdots)\Big\}\Big\{\prod^{<}_{\substack{l\in L_X\\|l|\leq d}}(1-l)\Big\}.$$

This is a finite product since $|X| < \infty$ and therefore is equal to 1. Since d is arbitrary, (5.5) holds. $\square$

Let $[2m] = \{1, 2, \cdots, 2m\}$ and $[2m]\times[2m]$ the cartesian product with the lexicographic order derived from the natural order on $[2m]$. We say that a word $w = (i_1, j_1)(i_2, j_2)\cdots(i_d, j_d) \in ([2m]\times[2m])^*$ is *connected* if $j_r = i_{r+1}$ for $r = 1, 2, \cdots, d-1$. For a connected word $w = (i_1, i_2)(i_2, i_3)\cdots(i_d, j_d) \in ([2m]\times[2m])^*$, we set $o(w) = i_1$ and $t(w) = j_d$. Consider the finite nonempty set $X = \{x(r, s) \mid (r, s) \in [2m]\times[2m]\}$ equipped with the total order derived from $[2m]\times[2m]$. For each matrix $\mathbf{A} = (a_{rs}) \in \mathrm{Mat}(2m, \mathbb{H})$, we define $\rho^{\mathbf{A}}$ to be the $\mathbb{R}$-algebra homomorphism from the monoid ring $\mathbb{R}[X^*]$ to $\mathrm{Mat}(2m, \mathbb{H})$ defined by $\rho^{\mathbf{A}}(x(r, s)) = a_{rs}\mathbf{E}_{rs}$, where $\mathbf{E}_{rs}$ denotes the (r, s)-matrix unit. Let $\mathbf{A}(r, s) = a_{rs}\mathbf{E}_{rs}$. For a_{rs} $(1 \leq r, s \leq 2m)$ with $|a_{rs}|$ sufficiently small, we can apply $\rho^{\mathbf{A}}$ to (5.4) so that

$$\mathbf{I}_{2m} - \{\mathbf{A}(1,1) + \mathbf{A}(1,2) + \cdots + \mathbf{A}(2m, 2m)\} = \prod^{<}_{l\in L_{[2m]\times[2m]}}(\mathbf{I}_{2m} - \mathbf{A}_l), \tag{5.6}$$

where $\mathbf{A}_l = \mathbf{A}(i_1, j_1)\mathbf{A}(i_2, j_2)\cdots\mathbf{A}(i_d, j_d)$ for each $l = (i_1, j_1)(i_2, j_2)\cdots(i_d, j_d) \in L_{[2m]\times[2m]}$. Indeed, the following holds:

Proposition 5.3. *For a_{rs} $(1 \leq r, s \leq 2m)$ with $|a_{rs}|$ sufficiently small, all entries in the right hand side of (5.6) converge absolutely with respect to the norm of quaternions. Particularly all entries in the right hand side of (5.6) converge.*

Proof. We put $a_l = a_{i_1j_1}a_{i_2j_2}\cdots a_{i_dj_d}$ and $\mathbf{E}_l = \mathbf{E}_{i_1j_1}\mathbf{E}_{i_2j_2}\cdots\mathbf{E}_{i_dj_d}$ for a Lyndon word $l = (i_1, j_1)(i_2, j_2)\cdots(i_d, j_d) \in L_{[2m]\times[2m]}$. Since $\mathbf{A}_l = \mathbf{A}_{(i_1,j_1)\cdots(i_d,j_d)} = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{d-1}i_d}a_{i_dj_d}\mathbf{E}_l$, it follows that

$$\begin{aligned}\prod^{<}_{l\in L_{[2m]\times[2m]}}(\mathbf{I}_{2m} - \mathbf{A}_l) &= \prod^{<}_{l\in L_{[2m]\times[2m]}}(\mathbf{I}_{2m} - a_l\mathbf{E}_l)\\ &= \mathbf{I}_{2m} + \sum_{h=1}^{\infty}\sum_{\substack{w=l_1\cdots l_{n(w)},\, l_1<\cdots<l_{n(w)}\\ l_1,\cdots,l_{n(w)}\in L_{[2m]\times[2m]},\,|w|=h}}(-1)^{n(w)}a_w\mathbf{E}_w,\end{aligned} \tag{5.7}$$

where $a_w = a_{l_1} a_{l_2} \cdots a_{l_{n(w)}}$, $\mathbf{E}_w = \mathbf{E}_{l_1} \mathbf{E}_{l_2} \cdots \mathbf{E}_{l_{n(w)}}$ and $n(w)$ is the number of Lyndon words that are multiplied in w. If $|w| = h$ then a_w can be expressed by $a_w = a_{i_1 j_1} \cdots a_{i_h j_h}$. Then we notice that $|a_w| = |a_{i_1 j_1}| \cdots |a_{i_h j_h}|$. We can easily see that $\mathbf{E}_w = \mathbf{E}_{o(w)t(w)}$ if w is connected, and $\mathbf{E}_w = \mathbf{O}_{2m}$ otherwise. Therefore (r,s)-entry of (5.7) is expressed as follows:

$$\left(\prod_{l \in L_{[2m]\times[2m]}}^{<} (\mathbf{I}_{2m} - \mathbf{A}_l) \right)_{rs} = \delta_{rs} + \sum_{h=1}^{\infty} \sum_{\substack{w = l_1 \cdots l_{n(w)},\, l_1 < \cdots < l_{n(w)} \\ l_1, \cdots, l_{n(w)} \in L_{[2m]\times[2m]},\, |w| = h \\ w \text{ is connected},\, o(w)=r, t(w)=s}} (-1)^{n(w)} a_w. \tag{5.8}$$

In (5.8), the number of w of length h is at most the number of words in $([2m]\times[2m])^*$ of length h which equals $(2m)^{2h}$. Hence if $|a_{rs}| < 1/(8m^2)$ for all $r, s = 1, \cdots, 2m$, then

$$\begin{aligned} &|\delta_{rs}| + \sum_{h=1}^{\infty} \Bigg| \sum_{\substack{w = l_1 \cdots l_{n(w)},\, l_1 < \cdots < l_{n(w)} \\ l_1, \cdots, l_{n(w)} \in L_{[2m]\times[2m]},\, |w| = h \\ w \text{ is connected},\, o(w)=r, t(w)=s}} (-1)^{n(w)} a_w \Bigg| \\ &\le |\delta_{rs}| + \sum_{h=1}^{\infty} \sum_{\substack{w = l_1 \cdots l_{n(w)},\, l_1 < \cdots < l_{n(w)} \\ l_1, \cdots, l_{n(w)} \in L_{[2m]\times[2m]},\, |w| = h \\ w \text{ is connected},\, o(w)=r, t(w)=s}} |a_w| \\ &< 1 + \sum_{h=1}^{\infty} \frac{(2m)^{2h}}{(8m^2)^h} = 1 + \sum_{h=1}^{\infty} \frac{1}{2^h} = 2. \end{aligned}$$

Thus the right hand side of (5.8) converges absolutely with respect to the norm of quaternions. □

Since $\mathbf{A}(1,1) + \mathbf{A}(1,2) + \cdots + \mathbf{A}(2m,2m) = \mathbf{A}$, it follows from (5.6) that

Proposition 5.4. *Let $\mathbf{A} = (a_{rs})$ be a $2m\times 2m$ quaternionic matrix with $|a_{rs}|$ sufficiently small. Then*

$$\mathbf{I}_{2m} - \mathbf{A} = \prod_{\substack{(i_1,j_1)\cdots(i_d,j_d) \in L_{[2m]\times[2m]} \\ j_r = i_{r+1}\ (r=1,\cdots,d-1)}}^{<} (\mathbf{I}_{2m} - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{d-1} i_d} a_{i_d j_d} \mathbf{E}_{i_1 j_d}), \tag{5.9}$$

in $\mathrm{Mat}(2m, \mathbb{H})$.

Now we take Study determinants of both sides in (5.9).

$$\begin{aligned}
&\mathrm{Sdet}(\mathbf{I}_{2m}-\mathbf{A})\\
&=\mathrm{Sdet}\Bigg(\overset{<}{\prod_{\substack{(i_1,j_1)\cdots(i_d,j_d)\in L_{[2m]\times[2m]}\\ j_r=i_{r+1}\ (r=1,\cdots,d-1)}}}(\mathbf{I}_{2m}-a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{d-1}i_d}a_{i_dj_d}\mathbf{E}_{i_1j_d})\Bigg)\\
&=\prod_{\substack{(i_1,j_1)\cdots(i_d,j_d)\in L_{[2m]\times[2m]}\\ j_r=i_{r+1}\ (r=1,\cdots,d-1)}}\mathrm{Sdet}(\mathbf{I}_{2m}-a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{d-1}i_d}a_{i_dj_d}\mathbf{E}_{i_1j_d}).
\end{aligned} \tag{5.10}$$

We notice that the last formula does not depend on the order in which factors are multiplied since Study determinants take values in $\mathbb{R}$. It follows from Proposition 3.4 (vi) that if $j_d=i_1$, then

$$\mathrm{Sdet}(\mathbf{I}_{2m}-a_{i_1i_2}a_{i_2i_3}\cdots a_{i_di_1}\mathbf{E}_{i_1i_1})=|1-a_{i_1i_2}a_{i_2i_3}\cdots a_{i_di_1}|^2,$$

and otherwise,

$$\mathrm{Sdet}(\mathbf{I}_{2m}-a_{i_1i_2}a_{i_2i_3}\cdots a_{i_dj_d}\mathbf{E}_{i_1j_d})=1.$$

Let $\mathbf{W}=(\mathbf{W}_{uv})_{u,v\in V(G)}$ be an arbitrary quaternionic weighted matrix of G and t a quaternion with $|t|$ sufficiently small so that $|t\tilde{w}(e,f)|<1/(8m^2)$ for all $e,f\in D(G)$, where $\tilde{w}(e,f)$ is as in (2.2). Putting $\mathbf{A}=t(\mathbf{B}_w-\mathbf{J}_0)$ and indexing rows and columns with $e_1,e_2,\cdots,e_{2m}\in D(G)$, then we have $a_{rs}=a_{e_re_s}=t\tilde{w}(e_r,e_s)$. Therefore, (5.10) yields

$$\begin{aligned}
&\mathrm{Sdet}(\mathbf{I}_{2m}-t(\mathbf{B}_w-\mathbf{J}_0))\\
&=\prod_{(i_1,i_2)\cdots(i_d,i_1)\in L_{[2m]\times[2m]}}\mathrm{Sdet}(\mathbf{I}_{2m}-t\tilde{w}(e_{i_1},e_{i_2})t\tilde{w}(e_{i_2},e_{i_3})\cdots t\tilde{w}(e_{i_d},e_{i_1})\mathbf{E}_{i_1i_1})\\
&=\prod_{(i_1,i_2)\cdots(i_d,i_1)\in L_{[2m]\times[2m]}}|1-t\tilde{w}(e_{i_1},e_{i_2})t\tilde{w}(e_{i_2},e_{i_3})\cdots t\tilde{w}(e_{i_d},e_{i_1})|^2.
\end{aligned}$$

Each Lyndon word $(i_1,i_2)\cdots(i_d,i_1)$ in $L_{[2m]\times[2m]}$ corresponds to a Lyndon word $i_1i_2\cdots i_d$ in $L_{[2m]}$ bijectively. Hence we obtain

Theorem 5.5. *Let t be a quaternion with $|t|$ sufficiently small. Then*

$$\mathbf{Z}_1^{\mathbb{H}}(G,w,t)=\prod_{i_1i_2\cdots i_d\in L_{[2m]}}|1-t\tilde{w}(e_{i_1},e_{i_2})t\tilde{w}(e_{i_2},e_{i_3})\cdots t\tilde{w}(e_{i_d},e_{i_1})|^{-2}.$$

Since real numbers are central in $\mathbb{H}$, it follows that

Corollary 5.6. *Let t be a real number with $|t|$ sufficiently small. Then*

$$\mathbf{Z}_1^{\mathbb{H}}(G, w, t) = \prod_{i_1 i_2 \cdots i_d \in L_{[2m]}} |1 - \tilde{w}(e_{i_1}, e_{i_2})\tilde{w}(e_{i_2}, e_{i_3}) \cdots \tilde{w}(e_{i_d}, e_{i_1}) t^d|^{-2}.$$

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