Applied Mathematics Letters 64 (2017) 170–176

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Finite time stability of fractional delay differential equations[☆]

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ARTICLE INFO

Article history:
Received 15 July 2016
Received in revised form 2 September 2016
Accepted 3 September 2016
Available online 12 September 2016

Keywords:
Delayed Mittag-Leffler type matrix
Fractional delay differential equations
Finite time stability

ABSTRACT

In this paper, we firstly introduce a concept of delayed Mittag-Leffler type matrix function, an extension of Mittag-Leffler matrix function for linear fractional ODEs, which help us to seek explicit formula of solutions to fractional delay differential equations by using the variation of constants method. Secondly, we present the finite time stability results by virtue of delayed Mittag-Leffler type matrix. Finally, an example is given to illustrate our theoretical results.

1. Introduction

In recent decades, existence and finite time stability (FTS) problems of integer order and fractional order delay differential equations have been studied by using methods of linear matrix inequality, Lyapunov functions, and Gronwall's integral inequality, see for example [1–11].

Recently, Khusainov and Shuklin [12] give a new concept, delayed exponential matrix function, which is used as a representation of solution of a linear delay differential equation. Next, it is worth to mention that Diblík and Khusainov [13] transfer this idea to represent the solution of discrete delayed system by constructing discrete matrix delayed exponential. For more recent results about delay equations based on delayed exponential matrix, one can refer to [14–17] and the references therein.

In this paper, we apply a new method to study FTS of the following linear fractional delay differential equations:

$$\begin{cases} ({}^cD^{\alpha}_{0+}y)(x) = By(x-\tau),\ y(x) \in \mathbb{R}^n,\ x \in J := [0,T],\ \tau > 0, \\ y(x) = \varphi(x),\ \ -\tau \le x \le 0, \end{cases} \tag{1}$$

☆ This work is supported by National Natural Science Foundation of China (11661016), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), and Unite Foundation of Guizhou Province ([2015]7640), and Graduate ZDKC ([2015]003).

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http://dx.doi.org/10.1016/j.aml.2016.09.004

where ${}^cD^{\alpha}_{0+}y(\cdot)$ is the Caputo derivative of order $\alpha \in (0,1)$ and of lower limit zero, $B \in \mathbb{R}^{n\times n}$ denotes constant matrix, $T = k^*\tau$ for a fixed $k^* \in \Lambda := \{1, 2, \ldots\}$, τ is a fixed delay time, and $\varphi(\cdot)$ is an arbitrary continuously differentiable vector function, i.e., $\varphi \in C^1_\tau := C^1([-\tau, 0], \mathbb{R}^n)$.

We adopt similar idea from [12] and introduce delayed Mittag-Leffler type matrix function, which will be applied to seek a representation of solution of (1). Some new sufficient conditions to guarantee FTS of (1) are established by virtue of delayed Mittag-Leffler type matrix.

2. Preliminaries

Throughout the paper, we denote $\|y\| = \sum_{i=1}^n |y_i|$ and $\|A\| = \max_{1\le j\le n}\sum_{i=1}^n |a_{ij}|$, which are the Euclidean vector norm and matrix norm, respectively; y_i and a_{ij} are the elements of the vector y and the matrix A, respectively. Denote by $C(J, \mathbb{R}^n)$ the Banach space of vector-value continuous function from $J \to \mathbb{R}^n$ endowed with the norm $\|x\|_C = \max_{t\in J}\|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We introduce a space $C^1(J, \mathbb{R}^n) = \{x \in C(J, \mathbb{R}^n) : x' \in C(J, \mathbb{R}^n)\}$. In addition, we note $\|\varphi\|_C = \max_{\theta\in[-\tau,0]}\|\varphi(\theta)\|$.

We recall definitions of Caputo fractional derivative and finite time stability.

Definition 2.1 (*See [18]*)**.** The Caputo derivative of order $0 < \alpha < 1$ for a function $f : [0, \infty) \to \mathbb{R}$ can be written as $({}^cD^{\alpha}_{0+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{y'(t)}{(x-t)^\alpha}dt,\ x > 0.$

Definition 2.2 (*See [1]*)**.** System given by (1) is finite time stable with respect to $\{0, J, \tau, \delta, \beta\}$ if and only if $\|\varphi\|_C < \delta$ implies $\|y(x)\| < \beta,\ \forall\ x \in J$, where $\varphi(x),\ -\tau \le x \le 0$ is the initial time of observation, δ, β are real positive numbers and $\delta < \beta$.

Next, we introduce a concept of delayed Mittag-Leffler type matrix function, which is an analogy of delayed exponential matrix e^{Bt}_τ in [12].

Definition 2.3. Delayed Mittag-Leffler type matrix function $\mathbb{E}^{Bx^\alpha}_\tau : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by

$$\mathbb{E}^{Bx^\alpha}_\tau = \begin{cases} \Theta, & -\infty < x < -\tau, \\ I, & -\tau \le x \le 0, \\ I + B\dfrac{x^\alpha}{\Gamma(\alpha+1)} + B^2\dfrac{(x-\tau)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + B^k\dfrac{(x-(k-1)\tau)^{k\alpha}}{\Gamma(k\alpha+1)}, & \\ & (k-1)\tau \le x \le k\tau,\ k \in \Lambda, \end{cases} \tag{2}$$

where B is an $n\times n$ constant matrix, Θ and I are the zero and identity matrices, respectively.

Lemma 2.4. *For any $x \in [(k-1)\tau, k\tau]$ and $k \in \Lambda$, we have*

$$\left\|\mathbb{E}^{Bx^\alpha}_\tau\right\| \le \mathbb{E}_\alpha(\|B\|x^\alpha),$$

where $\mathbb{E}_\alpha(\cdot)$ is the Mittag-Leffler function defined by $\mathbb{E}_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+1)}, \alpha > 0, z \in \mathbb{R}$.

Proof. By the formula of (2), one has

$$\begin{aligned} \|\mathbb{E}^{Bx^\alpha}_\tau\| &\le 1 + \|B\|\frac{x^\alpha}{\Gamma(\alpha+1)} + \|B\|^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \|B\|^k\frac{x^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\le \sum_{k=0}^\infty \frac{(\|B\|x^\alpha)^k}{\Gamma(k\alpha+1)} = \mathbb{E}_\alpha(\|B\|x^\alpha). \end{aligned}$$

The proof is completed. $\square$

To end this section, we give the following integral inequality, which will be used in the sequel.

Lemma 2.5. *Let $(k-1)\tau \le x \le k\tau$ and $k \in \Lambda$, we have*

$$\int_{(k-1)\tau}^{x} (x-t)^{-\alpha}(t-(k-1)\tau)^{k\alpha-1}dt = (x-(k-1)\tau)^{(k-1)\alpha}\mathbb{B}[1-\alpha, k\alpha],$$

where $\mathbb{B}[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ is a Beta function.

Proof. Integration by parts, one has

$$\begin{aligned}\int_{(k-1)\tau}^{x} (x-t)^{-\alpha}(t-(k-1)\tau)^{k\alpha-1}dt &= \int_0^{x-(k-1)\tau} (x-(k-1)\tau)^{-\alpha}\left(1-\frac{z}{x-(k-1)\tau}\right)^{-\alpha} z^{k\alpha-1}dz \\ &= (x-(k-1)\tau)^{(k-1)\alpha}\mathbb{B}[1-\alpha, k\alpha].\end{aligned}$$

The proof is completed. □

3. Solutions of fractional delay system

In this section, we seek explicit formula of solutions to fractional delay system by adopting the classical ideas to find solution of linear fractional ODEs.

Theorem 3.1. *For delayed Mittag-Leffler type matrix $\mathbb{E}_\tau^{Bx^\alpha} : \mathbb{R} \to \mathbb{R}^{n\times n}$, one has*

$$({}^cD_{0+}^{\alpha}\mathbb{E}_\tau^{Bt^\alpha})(x) = B\mathbb{E}_\tau^{B(x-\tau)^\alpha}, \tag{3}$$

i.e., $\mathbb{E}_\tau^{Bx^\alpha}$ is a solution of $({}^cD_{0+}^{\alpha}y)(x) = By(x-\tau)$, that satisfy initial conditions $\mathbb{E}_\tau^{Bx^\alpha} = I, -\tau \le x \le 0$.

Proof. For arbitrary $x \in (-\infty, -\tau]$, $\mathbb{E}_\tau^{Bx^\alpha} = \mathbb{E}_\tau^{B(x-\tau)^\alpha} = \Theta$. Obviously, (3) holds. For arbitrary $x \in [-\tau, 0]$, $\mathbb{E}_\tau^{Bx^\alpha} = I$ and $\mathbb{E}_\tau^{B(x-\tau)^\alpha} = \Theta$. Note that ${}^cD_{0+}^{\alpha}I = \Theta = B\Theta$. Thus, (3) holds.

For arbitrary $x \in [(k-1)\tau, k\tau], k \in \Lambda$. We adopt mathematical induction to prove our result.

(i) For $k = 1, 0 \le x \le \tau$, we have

$$y(x) = \mathbb{E}_\tau^{Bx^\alpha} = I + \frac{Bx^\alpha}{\Gamma(\alpha+1)}, \quad y'(x) = \frac{\alpha Bx^{\alpha-1}}{\Gamma(\alpha+1)}. \tag{4}$$

Next, having Caputo fractional differentiated expression of $\mathbb{E}_\tau^{B\cdot^\alpha}$ via (4) and Lemma 2.5, we obtain

$$({}^cD_{0+}^{\alpha}\mathbb{E}_\tau^{Bt^\alpha})(x) = \frac{\alpha B}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}t^{\alpha-1}dt = \frac{\alpha B\Gamma(1-\alpha)\Gamma(\alpha)}{\Gamma(\alpha+1)\Gamma(1-\alpha)} = B. \tag{5}$$

(ii) For $k = 2, \tau \le x \le 2\tau$, we have

$$y(x) = \mathbb{E}_\tau^{Bx^\alpha} = I + \frac{Bx^\alpha}{\Gamma(\alpha+1)} + \frac{B^2(x-\tau)^{2\alpha}}{\Gamma(2\alpha+1)}, \quad y'(x) = \frac{\alpha Bx^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{2\alpha B^2(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha+1)}. \tag{6}$$

Next, having Caputo fractional differentiated expression of $\mathbb{E}_\tau^{B\cdot^\alpha}$ via (5), (6) and Lemma 2.5, we obtain

$$\begin{aligned}({}^cD_{0+}^{\alpha}\mathbb{E}_\tau^{Bt^\alpha})(x) &= B + \frac{2\alpha B^2}{\Gamma(1-\alpha)\Gamma(2\alpha+1)}\int_\tau^x (x-t)^{-\alpha}(t-\tau)^{2\alpha-1}dt \\ &= B + \frac{2\alpha B^2(x-\tau)^\alpha}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} = B + \frac{B^2(x-\tau)^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

(iii) Suppose $k = N, (N-1)\tau \le x \le N\tau$ and $N \in \Lambda$ the following relation holds:

$$({}^cD^\alpha_{0+}\mathbb{E}^{Bt^\alpha}_\tau)(x) = B + \frac{B^2(x-\tau)^\alpha}{\Gamma(\alpha+1)} + \frac{B^3(x-2\tau)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{B^N(x-(N-1)\tau)^{(N-1)\alpha}}{\Gamma((N-1)\alpha+1)}.$$

Next, for $k = N+1, N\tau \le x \le (N+1)\tau$, by elementary computation, one obtains

$$y'(x) = \frac{\alpha B x^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{2\alpha B^2(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha+1)} + \cdots + \frac{(N+1)\alpha B^{N+1}(x-N\tau)^{(N+1)\alpha-1}}{\Gamma((N+1)\alpha+1)}.$$

Having Caputo fractional differentiated expression of $\mathbb{E}^{B\cdot^\alpha}_\tau$ via (7) and Lemma 2.5, we obtain

$$\begin{aligned}({}^cD^\alpha_{0+}\mathbb{E}^{Bt^\alpha}_\tau)(x) &= \frac{\alpha B}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}dt + \frac{2\alpha B^2}{\Gamma(1-\alpha)}\int_\tau^x (x-t)^{-\alpha}\frac{(t-\tau)^{2\alpha-1}}{\Gamma(2\alpha+1)}dt \\ &\quad + \cdots + \frac{(N+1)\alpha B^{N+1}}{\Gamma(1-\alpha)}\int_{N\tau}^x (x-t)^{-\alpha}\frac{(t-N\tau)^{(N+1)\alpha-1}}{\Gamma((N+1)\alpha+1)}dt \\ &= B + \frac{B^2(x-\tau)^\alpha}{\Gamma(\alpha+1)} + \frac{B^3(x-2\tau)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{B^{N+1}(x-N\tau)^{N\alpha}}{\Gamma(N\alpha+1)},\end{aligned}$$

which implies that (3) holds for any $(k-1)\tau \le x \le k\tau$ and $k \in \Lambda$. The proof is completed. □

Theorem 3.2. *A solution $y \in C([-\tau, T], \mathbb{R}^n)$ of (1) can be expressed by the following formula*

$$y(x) = \mathbb{E}^{Bx^\alpha}_\tau \varphi(-\tau) + \int_{-\tau}^0 \mathbb{E}^{B(x-\tau-s)^\alpha}_\tau \varphi'(s)ds.$$

Proof. Let matrix $Y_0(x) = \mathbb{E}^{Bx^\alpha}_\tau$ satisfied Theorem 3.1 and any solution of (1) satisfy initial conditions $y(x) = \varphi(x), -\tau \le x \le 0$ should search in the form

$$y(x) = Y_0(x)c + \int_{-\tau}^0 Y_0(x-\tau-s)z(s)ds. \tag{7}$$

In this formula c is a vector of unknown constant, $z(\cdot)$ is an unknown continuously differentiable vector function. According to the matrix $Y_0(x)$ is a solution of (1), then, for arbitrary c and $z(\cdot)$ formula (7) is also a solution of (1). Therefore, we choose c and $z(\cdot)$ satisfy initial conditions, $y(x) = \varphi(x), -\tau \le x \le 0$.

Let us assume $x = -\tau$, as it following from (2), we obtain $Y_0(-\tau) = I$, $Y_0(-2\tau - s) = \Theta, -\tau < s \le 0$ and $Y_0(-2\tau - s) = I, s = -\tau$. Thus, $y(-\tau) = \varphi(-\tau) = c$, and the formula (7) takes a form

$$y(x) = \mathbb{E}^{Bx^\alpha}_\tau \varphi(-\tau) + \int_{-\tau}^0 \mathbb{E}^{B(x-\tau-s)^\alpha}_\tau z(s)ds.$$

Since $-\tau \le x \le 0$, one should divide interval into two subintervals, we have

(i) For the interval $-\tau \le s \le x$, so $-\tau \le x - \tau - s \le x$, the delayed Mittag-Leffler type matrix is equal to $\mathbb{E}^{B(x-\tau-s)^\alpha}_\tau = I, -\tau \le s \le x$.

(ii) For the interval $x \le s \le 0$, so $x - \tau \le x - \tau - s \le -\tau$, the delayed Mittag-Leffler type matrix is equal to

$$\mathbb{E}^{B(x-\tau-s)^\alpha}_\tau = \begin{cases}\Theta, & x < s \le 0,\\ I, & s = x.\end{cases}$$

Thus on the interval $-\tau \le x \le 0$, we have

$$\varphi(x) = \varphi(-\tau) + \int_{-\tau}^x z(s)ds. \tag{8}$$

Having differentiated on (8), we obtain $z(x) = \varphi'(x)$. The proof is completed. □

4. Finite time stability results

In this section, we show finite time stability results by using delayed Mittag-Leffler type matrix.

Theorem 4.1. *Suppose $M=\int_{-\tau}^{0}\|\varphi'(s)\|ds<\infty$. If*

$$\mathbb{E}_\alpha(\|B\|x^\alpha)<\frac{\beta}{\delta+M},\ \forall\ x\in J, \tag{9}$$

then (1) is finite time stable with respect to $\{0,J,\tau,\delta,\beta\}$.

Proof. According to Theorem 3.2, the solution of the system (1) can be expressed in the following form:

$$y(x)=\mathbb{E}_\tau^{Bx^\alpha}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{E}_\tau^{B(x-\tau-s)^\alpha}\varphi'(s)ds. \tag{10}$$

By Lemma 2.4 and the properties of norm $\|\cdot\|$ via (9), we have

$$\begin{aligned}\|y(x)\| &\le \|\varphi(-\tau)\|\|\mathbb{E}_\tau^{Bx^\alpha}\|+\int_{-\tau}^{0}\|\varphi'(s)\|\|\mathbb{E}_\tau^{B(x-\tau-s)^\alpha}\|ds\\ &\le \delta\mathbb{E}_\alpha(\|B\|x^\alpha)+\mathbb{E}_\alpha(\|B\|x^\alpha)\int_{-\tau}^{0}\|\varphi'(s)\|ds\\ &\le \mathbb{E}_\alpha(\|B\|x^\alpha)(\delta+M)<\beta,\ \forall\ x\in J,\end{aligned}$$

where we use $\|\mathbb{E}_\tau^{B(x-\tau-s)^\alpha}\|\le\|\mathbb{E}_\tau^{Bx^\alpha}\|$. The proof is completed. □

Theorem 4.2. *Suppose k,α are two constants and satisfying $\alpha\le\frac{1}{k}, k\in\Lambda$. If*

$$\mathbb{E}_\alpha(\|B\|x^\alpha)+\mathbb{E}_\alpha(\|B\|\tau^\alpha)<\frac{\beta}{\delta},\ \forall\ x\in J, \tag{11}$$

then (1) is finite time stable with respect to $\{0,J,\tau,\delta,\beta\}$.

Proof. Integration by parts, one can change the expression (10) to the form:

$$y(x)=\mathbb{E}_\tau^{B(x-\tau)^\alpha}\varphi(0)+\int_{-\tau}^{0}\sum_{i=1}^{k}\frac{i\alpha B^i(x-i\tau-s)^{i\alpha-1}}{\Gamma(i\alpha+1)}\varphi(s)ds, \tag{12}$$

where we use $\mathbb{E}_\tau^{B(x-\tau-s)^\alpha}=\sum_{i=0}^{k}B^i\frac{(x-i\tau-s)^{i\alpha}}{\Gamma(i\alpha+1)}$ and $\frac{d(\mathbb{E}_\tau^{B(x-\tau-s)^\alpha})}{ds}=-\sum_{i=1}^{k}\frac{i\alpha B^i(x-i\tau-s)^{i\alpha-1}}{\Gamma(i\alpha+1)}$.

By Lemma 2.4, and the properties of norm $\|\cdot\|$ via (11), we have

$$\begin{aligned}\|y(x)\| &\le \|\varphi(0)\|\|\mathbb{E}_\tau^{Bx^\alpha}\|+\sum_{i=1}^{k}\left\|\int_{-\tau}^{0}\frac{i\alpha B^i(x-i\tau-s)^{i\alpha-1}}{\Gamma(i\alpha+1)}ds\right\|\|\varphi\|_C\\ &\le \|\varphi(0)\|\|\mathbb{E}_\tau^{Bx^\alpha}\|+\sum_{i=1}^{k}\left\|\frac{B^i}{\Gamma(i\alpha+1)}((x-(i-1)\tau)^{i\alpha}-(x-i\tau)^{i\alpha})\right\|\|\varphi\|_C\\ &\le \|\varphi(0)\|\|\mathbb{E}_\tau^{Bx^\alpha}\|+\|\varphi\|_C\sum_{i=1}^{k}\left\|\frac{B^i}{\Gamma(i\alpha+1)}\tau^{i\alpha}\right\|\\ &\le \|\varphi(0)\|\|\mathbb{E}_\tau^{Bx^\alpha}\|+\|\varphi\|_C\sum_{i=1}^{\infty}\frac{\|B\|^i}{\Gamma(i\alpha+1)}\tau^{i\alpha}\\ &\le \delta\mathbb{E}_\alpha(\|B\|x^\alpha)+\delta\mathbb{E}_\alpha(\|B\|\tau^\alpha)<\beta,\ \forall\ x\in J,\end{aligned}$$

where we use inequality $a^\theta-b^\theta\le(a-b)^\theta, a>b>0, 0<\theta\le1$. The proof is completed. □

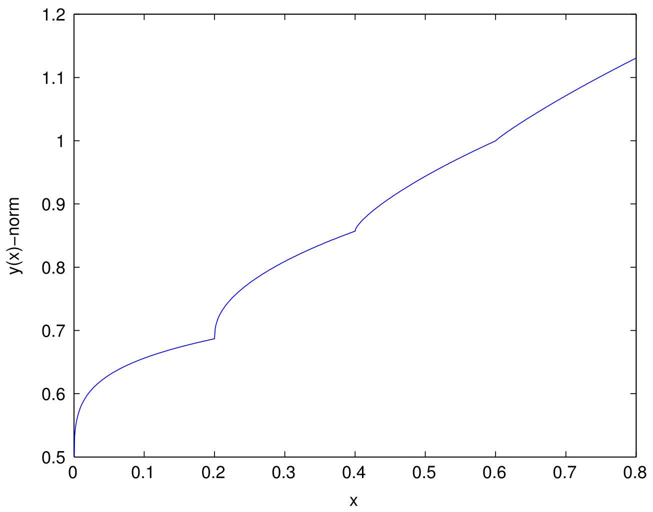

Fig. 1. The norm of the state vector of (13) when $T = 0.8$.

5. An example

In this section, we give an example to demonstrate the validity of our method and make some discussions. Let $\alpha = 0.2$, $\tau = 0.2$, $T = 0.8$, $k = 4$, $B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.8 \end{pmatrix}$. Consider

$$\begin{cases} {}^{c}D_{0+}^{0.2}y(x) = By(x-0.2), \quad y(x) = (y_1(x), y_2(x))^{\top}, \ x \in [0, 0.8], \\ \varphi(x) = (0.1, 0.2)^{\top}, \quad -0.2 \le x \le 0. \end{cases} \tag{13}$$

By Theorem 3.2, solution of (13) can be expressed by the following explicit form:

$$\begin{aligned} y(x) &= \mathbb{E}_{0.2}^{Bx^{0.2}}\varphi(-0.2) \\ &= \begin{cases} \left(I + B\dfrac{x^{0.2}}{\Gamma(1.2)}\right)\begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}, & 0 \le x \le 0.2, \\ \left(I + B\dfrac{x^{0.2}}{\Gamma(1.2)} + B^2\dfrac{(x-0.2)^{0.4}}{\Gamma(1.4)}\right)\begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}, & 0.2 < x \le 0.4, \\ \left(I + B\dfrac{x^{0.2}}{\Gamma(1.2)} + B^2\dfrac{(x-0.2)^{0.4}}{\Gamma(1.4)} + B^3\dfrac{(x-0.4)^{0.6}}{\Gamma(1.6)}\right)\begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}, & 0.4 < x \le 0.6, \\ \left(I + B\dfrac{x^{0.2}}{\Gamma(1.2)} + B^2\dfrac{(x-0.2)^{0.4}}{\Gamma(1.4)} + B^3\dfrac{(x-0.4)^{0.6}}{\Gamma(1.6)} + B^4\dfrac{(x-0.6)^{0.8}}{\Gamma(1.8)}\right)\begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}, & 0.6 < x \le 0.8. \end{cases} \end{aligned}$$

By calculation, $\|\varphi\| = 0.3$, $M = \int_{-0.2}^{0} \|\Theta\| ds = 0$, $\mathbb{E}_{0.2}(0.8^{1.2}) = 4.155$, $\mathbb{E}_{0.2}(\|B\|0.2^{0.2}) = \mathbb{E}_{0.2}(0.8 \times 0.2^{0.2})$ $= 2.4851$.

Discussions: Concerning on the definition of FTS (see Definition 2.2), we need to find a certain threshold β which make the state $\|y(x)\|$ of the system (1) does not exceed β during a fixed finite time interval J. On the one hand, we can use explicit formula of solution to (13) via numerical simulation to find a corresponding $\beta = 1.1307$ for a fixed $T = 0.8$ (see Fig. 1). On the other hand, by checking the conditions in Theorems 4.1, 4.2 and Theorem 4.2 in [1] for the finite time interval $[0, 0.8]$, we get the relative optimal threshold $\beta = 1.29$ by comparing the value of β in Table 1.

Table 1
FTS results of (13) and compare with Theorem 4.2 in [1] when $T = 0.8$.

Theorem	$\|\varphi\|_C$	τ	δ	$\|y(x)\|$	β	FTS
4.1	0.3	0.2	0.31	1.2882	1.29 (optimal)	Yes
4.2	0.3	0.2	0.31	2.0586	2.06	Yes
Theorem 4.2 in [1]	0.3	0.2	0.31	3.0168	3.1	Yes

Acknowledgments

The authors thank the referees for their careful reading of the manuscript and insightful comments, which help to improve the quality of the paper. The authors thank the help from the editor too.

References

[1] M.P. Lazarević, A.M. Spasić, Finite-time stability analysis of fractional order time-delay system: Grownwall's approach, Math. Comput. Model. 49 (2009) 475–481.
[2] S. Abbas, M. Benchohra, M. Rivero, J.J. Trujillo, Existence and stability results for nonlinear fractional order Riemann–Liouville Volterra-Stieltjes quadratic integral equations, Appl. Math. Comput. 247 (2014) 319–328.
[3] R. Rakkiyappan, G. Velmurugan, J. Cao, Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays, Nonlinear Dynam. 78 (2014) 2823–2836.
[4] Q. Wang, D.C. Lu, Y.Y. Fang, Stability analysis of impulsive fractional differential systems with delay, Appl. Math. Lett. 40 (2015) 1–6.
[5] R. Wu, Y. Lu, L. Chen, Finite-time stability of fractional delayed neural networks, Neurocomputing 149 (2015) 700–707.
[6] X. Hei, R. Wu, Finite-time stability of impulsive fractional-order systems with time-delay, Appl. Math. Model. 40 (2016) 4285–4290.
[7] S. Liu, W. Jiang, X. Li, X.F. Zhou, Lyapunov stability analysis of fractional nonlinear systems, Appl. Math. Lett. 51 (2016) 13–19.
[8] J. Wang, M. Fečkan, Y. Zhou, Ulam's type stability of impulsive ordinary differential equations, J. Math. Anal. Appl. 395 (2012) 258–264.
[9] J. Wang, Y. Zhou, M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, Comput. Math. Appl. 64 (2012) 3389–3405.
[10] J. Wang, Y. Zhang, On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives, Appl. Math. Lett. 39 (2015) 85–90.
[11] J. Wang, A.G. Ibrahim, M. Fečkan, Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on banach spaces, Appl. Math. Comput. 257 (2015) 103–118.
[12] D.Ya. Khusainov, G.V. Shuklin, Linear autonomous time-delay system with permutation matrices solving, Stud. Univ. Žilina 17 (2003) 101–108.
[13] J. Diblík, D.Ya. Khusainov, Representation of solutions of discrete delayed system $x(k+1) = Ax(k) + Bx(k-m) + f(k)$ with commutative matrices, J. Math. Anal. Appl. 318 (2006) 63–76.
[14] J. Diblík, D.Ya. Khusainov, Representation of solutions of linear discrete systems with constant coefficients and pure delay, Adv. Differential Equations 2006 (2006) 1–13. Art. ID80825.
[15] A. Boichuk, J. Diblík, D.Ya. Khusainov, M. Růžičková, Fredholm's boundary-value problems for differential systems with a single delay, Nonlinear Anal. 72 (2010) 2251–2258.
[16] M. Medveď, M. Pospišil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, Nonlinear Anal. 75 (2012) 3348–3363.
[17] J. Diblík, D.Ya. Khusainov, J. Baštinec, A.S. Sirenko, Exponential stability of linear discrete systems with constant coefficients and single delay, Appl. Math. Lett. 51 (2016) 68–73.
[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., 2006.