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Scale-free and small-world properties of Sierpinski networks[☆]

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HIGHLIGHTS

- We construct the evolving networks from Sierpinski carpet.
- We show that the evolving networks are scale-free and have the small-world effect.
- The exponent of power-law on cumulative degree distribution is $\log 8/\log 3$.
- We prove that the limit of the average clustering coefficient exists.

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ABSTRACT

In this paper, we construct the evolving networks from Sierpinski carpet, using the encoding approach in fractal geometry. We consider the small similar copies of unit square as nodes of network, where two nodes are neighbors if and only if their corresponding copies have common surface. For our networks, we check their scale-free and small-world effect by the self-similar structures, the exponent of power-law on degree distribution is $\log_3 8$ which is the Hausdorff dimension of the carpet.

1. Introduction

The complex network plays more and more important role in the scientific and social research. Readers are referred to Watts–Strogatz's [1] small-world network model and Barabási–Albert's [2] scale-free network model.

Song et al. [3] reveal that many real networks have self-similarity and fractality, also see Refs. [4–7]. On the other hand, we can generate complex network models from self-similar fractals. For example Andrade et al. [8] and Zhou et al. [9] discuss Apollonian networks generated from Apollonian fractal, Zhang et al. [10–12] construct evolving networks modeled from Sierpinski gasket by taking the line segments as nodes. Besides Zhang et al. [13] construct the networks produced from Vicsek fractals, Liu et al. [14] and Chen et al. [15] explore some Koch networks related to Koch curves, Song et al. [16] study complex networks modeled on Platonic solids, Chen et al. [17] investigate networks generated by Sierpinski tetrahedron.

The Sierpinski carpet, first introduced by Sierpinski in 1916, is a planar self-similar fractal: a generalization of the Cantor ternary set to two dimensions. The Hausdorff dimension of the Sierpinski carpet is $\log_3 8$.

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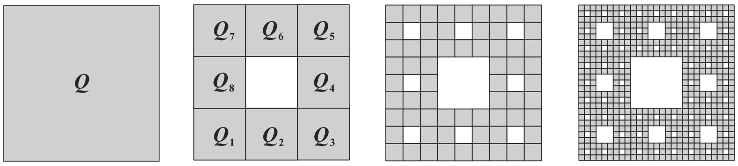

Fig. 1. The first 4 steps of construction of the Sierpinski carpet.

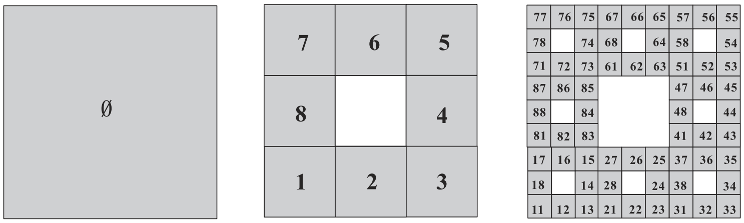

Fig. 2. Encoding 0-stage, 1-stage and 2-stage basic squares.

In topology, the Sierpinski carpet is a *universal* object like Cantor ternary set. Whyburn [18] gives an interesting characterization of the Sierpinski carpet. Suppose M is a nonempty connected compact subset in the plane, and its complement in the plane has countably many connected components such that their boundaries are simple closed curves pairwise disjoint. Then M is homeomorphic to the Sierpinski carpet.

On the other hand, the Sierpinski carpet is also a *special* object. For example, it is quite different from the Sierpinski gasket (triangle), Kigami [19] constructs Laplacian operator on the gasket. However his method cannot apply on the carpet, since the gasket is the post-critically finite self-similar set but carpet is not.

In this paper, we will construct the evolving networks modeled on the Sierpinski carpet, by using the encoding approach from IFS w.r.t. the carpet. We consider the small copies of unit square $[0, 1]^2$ as the nodes of network, and two nodes are neighbors if and only if their corresponding copies have common surface. This is similar to the case that liquid can be transferred from one copy to another through their common surface. For our networks, we check their scale-free and small-world effect. The exponent of power-law on degree distribution is $\log_3 8$ which is the Hausdorff dimension of the carpet.

1.1. Sierpinski carpet, IFS and encoding

We first recall the definition of Sierpinski carpet. Consider a solid square $Q = [0, 1]^2$ and the vectors $a_1 = (0, 0)$, $a_2 = (1/3, 0)$, $a_3 = (2/3, 0)$, $a_4 = (2/3, 1/3)$, $a_5 = (2/3, 2/3)$, $a_6 = (1/3, 2/3)$, $a_7 = (0, 2/3)$, $a_8 = (0, 1/3)$.

Let $\{S_i : \mathbb{R}^2 \to \mathbb{R}^2\}_{i=1}^8$ be similitudes defined by $S_i(x) = x/3 + a_i$. The self-similar set $E = \bigcup_{i=1}^8 S_i(E)$ with respect to the IFS (Iterated Function System) $\{S_i\}_{i=1}^8$ is called the Sierpinski carpet. Then the Hausdorff (or Box) dimension of the Sierpinski carpet is $\log_3 8$. Denote

$$S_\sigma = S_{i_1 i_2 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k} \quad \text{and} \quad Q_\sigma = S_\sigma(Q),$$

where the word $\sigma = i_1 i_2 \cdots i_k$ is composed of the letters in $\{1, \ldots, 8\}$. For $\sigma = i_1 i_2 \cdots i_k$, we let $|\sigma|(= k)$ denote the length of σ, we also call Q_σ a *k-stage basic square* with side length 3^{-k}. Note that $E = \bigcap_{k=1}^\infty \bigcup_{i_1 \cdots i_k \in \{1,\ldots,8\}^k} S_{i_1 i_2 \cdots i_k}(Q)$, from which we see that the kth stage of the construction of the Sierpinski gasket is given by $\bigcup_{i_1 \cdots i_k \in \{1,\ldots,8\}^k} S_{i_1 i_2 \cdots i_k}(Q)$. See Fig. 1.

Using the IFS as above, we obtain a natural way to encode the Sierpinski carpet. We can encode the solid square Q_σ by the word σ. We recall some notations on symbolic system. For any integer $k \geq 1$, we let $\Delta_k = \{1, \ldots, 8\}^k$, i.e., Δ_k is the collection of all words of length k. For $k = 0$, we have $\Delta_0 = \{\emptyset\}$, where $\emptyset$ is the empty word with $|\emptyset| = 0$. Given two words σ and τ, we say that σ is a prefix of τ, denoted by $\sigma \prec \tau$, if $\sigma = i_1 i_2 \cdots i_k$ and $\tau = i_1 i_2 \cdots i_k j_1 j_2 \cdots j_m$. In particular, we say that σ is the father of τ, if $\sigma \prec \tau$ and $|\sigma| + 1 = |\tau|$. Note that in this setting, anyone has 8 children while any child has a unique father. For any word $\sigma = i_1 i_2 \cdots i_k$, we denote by $\sigma^- = i_1 i_2 \cdots i_{k-1}$. See Fig. 2.

1.2. The construction of evolving networks G_t

Fix t, we will construct a network G_t with node set $V_t = \cup_{k=0}^t \Delta_k$. Given distinct words $\sigma, \tau \in V_t$, there is an edge between σ and τ, denoted by $\sigma \sim \tau$, if and only if the intersection of Q_σ and Q_τ is a line segment. See Fig. 3 for G_3. Let $\#A$ be the number of elements in the set A. Then we have

$$\#V_t = 1 + 8 + 8^2 + \cdots + 8^t = \frac{8^{t+1} - 1}{7}. \tag{1.1}$$

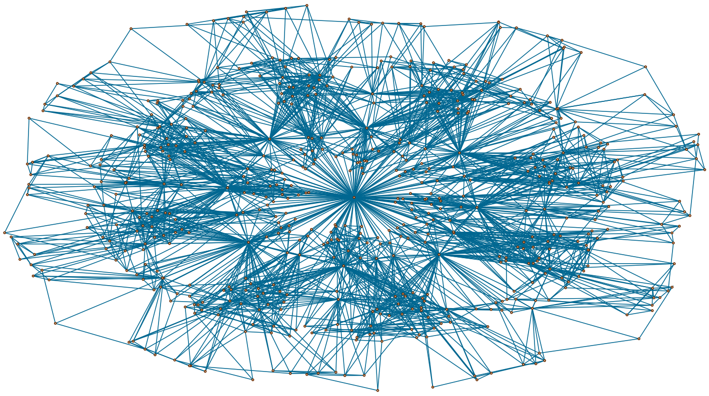

Fig. 3. G_3.

Since $Q_{\sigma\tau} = S_\sigma(Q_\tau)$, we notice that $\sigma\tau \sim \sigma\tau'$ if and only if $\tau \sim \tau'$.

If $\sigma = i_1 i_2 \cdots i_k$ is a word, whose every letter $i_1, i_2, \ldots, i_k \in \{1, 2, 3\}$, then Q_σ intersects Q with the bottom line segment. On the other hand, if there is a letter $i_t \notin \{1, 2, 3\}$ with $t < k$ and $i_p \in \{1, 2, 3\}$ for all $t < p \le k$, then $Q_\sigma \cap Q = \varnothing$. Replacing $\{1, 2, 3\}$ by $\{3, 4, 5\}$, $\{5, 6, 7\}$ or $\{7, 8, 1\}$, we have the similar result.

Remark 1. If $\tau = \sigma\beta$, then $\sigma \sim \tau$ if and only if all letters of β appear in one of the following four sets

$$\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}, \{7, 8, 1\}.$$

For example, the words 1596123 and 1596 are neighbors, and 1234567 and 1234 are also neighbors; but both the words 1596123 and 159 and the words 1234567 and 123 are not neighbors.

For any word σ, we can find a unique shortest word $f(\sigma)$ such that $f(\sigma) \prec \sigma$ and $f(\sigma) \sim \sigma$. For a word $\sigma = \tau\tau'$, where τ' is the maximal suffix such that all its letters lie in the one of the four sets

$$\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}, \{7, 8, 1\}$$

simultaneously. Using Remark 1 we have $f(\sigma) = \tau$. Iterating f repeatedly, we obtain a sequence from σ to $\emptyset$:

$$\sigma \sim f(\sigma) \sim \cdots f^{n-1}(\sigma) \sim f^n(\sigma) = \emptyset.$$

Write $\omega(\sigma) = n$.

Example 1. For example, for $\sigma = 4378182565 = (43)(7818)(2)(565)$, we have $f(\sigma) = (43)(7818)(2), f^2(\sigma) = (43)(7818), f^3(\sigma) = (43), f^4(\sigma) = \emptyset$.

2. Degree distribution

In this section, we consider the degree distribution of G_t. We use $\deg(\sigma)$ to denote respectively the numbers of friends of σ.

Theorem 1. *For sufficiently large t, the cumulative degree distribution of G_t*

$$P_{cum}(u) \propto u^{-\log 8/\log 3}.$$

See Fig. 4 for $t = 3, 4, 5, 6$. In order to prove Theorem 1, the following facts are useful.

Claim 1. (1) *For any given $k > 0$ and $\sigma \in \Delta_k$, σ has at most 4 and at least 2 friends in Δ_k, i.e.,*

$$2 \le \#\{\tau \in \Delta_k : \tau \sim \sigma\} \le 4. \tag{2.1}$$

Furthermore, we also have

$$\#\{\tau \in \Delta_k : \tau \sim \sigma \text{ and } \tau^- = \sigma^-\} \le 2, \tag{2.2}$$

$$\#\{\tau \in \Delta_k : \tau \sim \sigma \text{ and } \tau^- \ne \sigma^-\} \le 2. \tag{2.3}$$

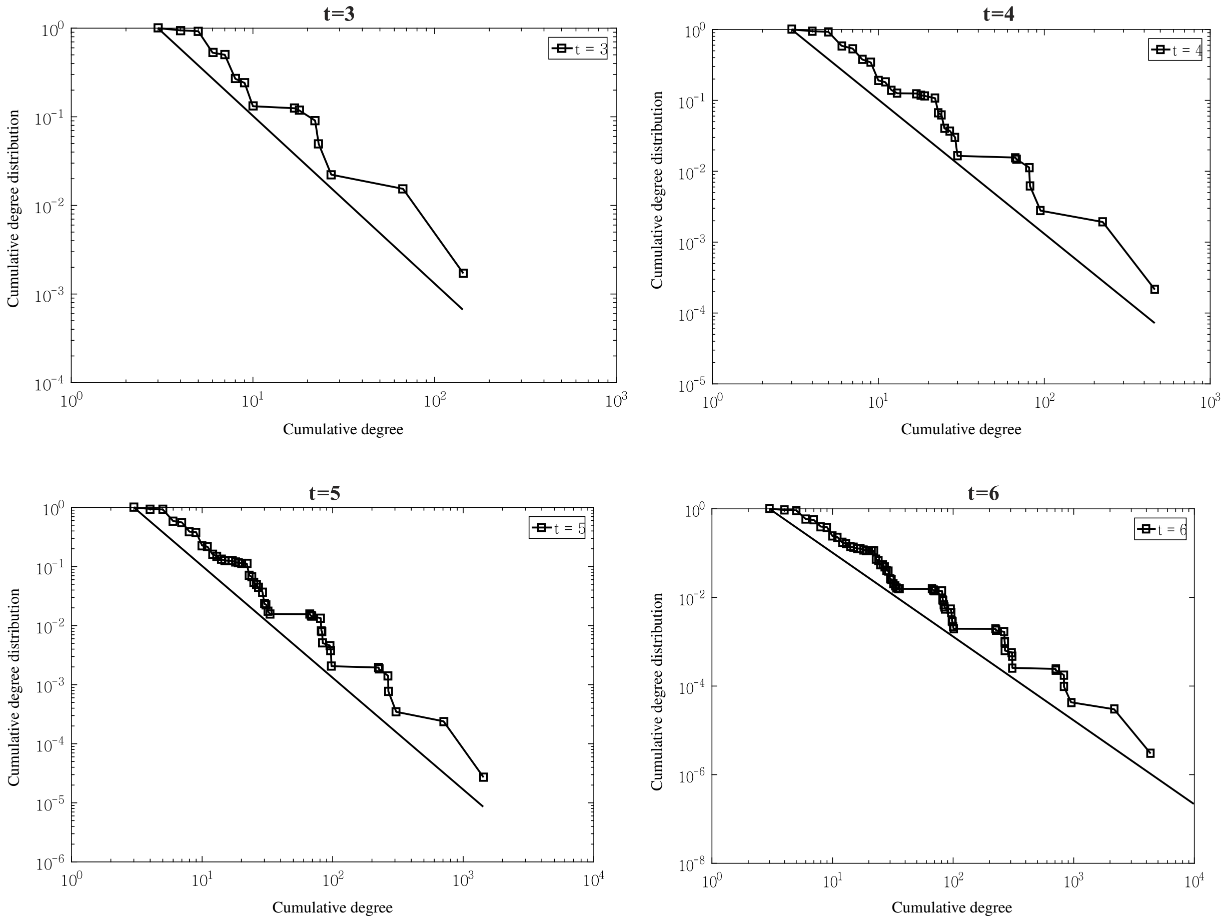

Fig. 4. The log–log graphs of cumulative degree distributions for $t = 3, 4, 5, 6$. The slope of lines is $-\log_3 8$.

(2) *Given $h > k$, the number of friends of σ in Δ_h is*

$$6 \times 3^{h-k} - 4 \le \#\{\tau \in \Delta_h : \tau \sim \sigma\} \le 8 \times 3^{h-k} - 4. \tag{2.4}$$

Furthermore, we have that

$$\#\{\tau \in \Delta_h : \tau \sim \sigma \text{ and } \sigma \prec \tau\} \le 4 \times 3^{h-k} - 4, \tag{2.5}$$

and

$$2 \times 3^{h-k} \le \#\{\tau \in \Delta_h : \tau \sim \sigma \text{ and } \sigma \not\prec \tau\} \le 4 \times 3^{h-k} \tag{2.6}$$

(3) *Given $h < k$, we have*

$$\#\{\tau \in \Delta_h : \tau \sim \sigma\} \le 3.$$

Lemma 1. *Given $t > 1$, and $k \le \frac{t}{2}$, we have*

$$\{\sigma : \deg(\sigma) \ge 18 \times 3^{t-k}\} = \{\sigma : |\sigma| < k\}. \tag{2.7}$$

Proof. In fact, we only need prove the following two statements:

(1) When $i < k$, $\deg(\sigma) \ge 18 \times 3^{t-k}$ for any $\sigma \in V_i$;
(2) When $i \ge k$, $\deg(\sigma) < 18 \times 3^{t-k}$ for any $\sigma \in V_i$.

Suppose $\sigma \in V_i$. Using Claim 1, we have

(1) If $i < k$, then

$$\begin{aligned}\deg(\sigma) &\ge (6 \times 3^{t-k+1} - 4) + (6 \times 3^{t-k} - 4) + \cdots + (6 \times 3 - 4) \\ &= 27 \times 3^{t-k} - 4(t-k) - 13 \ge 18 \times 3^{t-k}.\end{aligned}$$

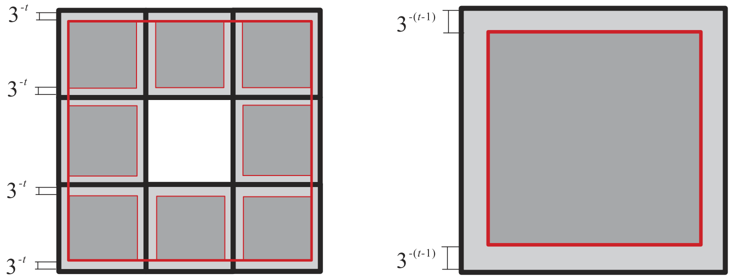

Fig. 5. The first 4 steps of construction of the Sierpinski carpet.

(2) If $i \geq k$, then

$$\begin{aligned}\deg(\sigma) &\leq (8\times 3^{t-k}-4)+\cdots+(8\times 3-4)+4+3\times k\\ &= 12\times 3^{t-k}-4(t-k)+3k-8 < 18\times 3^{t-k}. \quad \square\end{aligned}$$

The proof of Theorem 1. Using the definition of cumulative degree distribution, by Lemma 1, we have

$$P_{\text{cum}}(u) = \frac{\#\{\sigma : \deg(\sigma) \geq u\}}{\#V_t} = \frac{\#\{\sigma : |\sigma| < k\}}{\#V_t} = \frac{8^k-1}{8^{t+1}-1} \propto 8^{k-t},$$

where $u = 18\times 3^{t-k}$. Then using $k-t = -\log_3(u/18)$, we have

$$P_{\text{cum}}(u) \propto u^{-\log_3 8}. \quad \square$$

3. Clustering coefficient

For any t we consider the average clustering coefficient of G_t. Let

$$\bar{C}_t = \frac{\sum_{\sigma\in V_t} C_\sigma}{\#V_t},$$

where C_σ is the clustering coefficient of the node σ. We obtain the following list

$t =$	1	2	3	4	5	6
$\bar{C}(t)$	0.6243	0.64273	0.6458	0.6479	0.6491	0.6496

Theorem 2. *For any $t \geq 2$, the average clustering coefficient $\bar{C}_t$ of G_t always has a positive lower bound, i.e., for all $t \geq 2$,*

$$\bar{C}_t \geq 0.642.$$

At first, we need a weaker result.

Lemma 2. *For all $t \geq 2$, we have $\bar{C}_t \geq 0.2815$.*

Proof. Given t, we consider the following set

$$\mathcal{I}_t = \{\sigma : |\sigma| = t \text{ and } Q_\sigma \cap \partial Q = \varnothing\}, \tag{3.1}$$

where $\partial(A)$ is the boundary of the planar set A. We also can calculate that

$$\#\mathcal{I}_t = 8^t - 4(3^t-1).$$

For $t = 2$, we obtain

$$\sum_{\sigma\in\mathcal{I}_2} C_\sigma = \frac{144}{7}.$$

We shall verify that

$$\sum_{\sigma\in\mathcal{I}_t} C_\sigma \geq 8\sum_{\sigma\in\mathcal{I}_{t-1}} C_\sigma. \tag{3.2}$$

In fact, as shown in Fig. 5, we have

$$\sum_{\sigma\in\mathcal{I}_t} C_\sigma = \sum_{i=1}^{8}\sum_{i\tau\in\mathcal{I}_t} C_{i\tau} \geq \sum_{i=1}^{8}\sum_{\tau\in\mathcal{I}_{t-1}} C_{i\tau}.$$

Using Jordan curve theorem and self-similarity, we find out that

$$C_{i\tau} = C_\tau \quad \text{for all } \tau \in \mathcal{I}_{t-1}.$$

Then (3.2) follows.

Now, we have

$$\sum_{\sigma\in\mathcal{I}_t} C_\sigma \geq 8\sum_{\sigma\in\mathcal{I}_{t-1}} C_\sigma \geq \cdots \geq 8^{t-2}\sum_{\sigma\in\mathcal{I}_2} C_\sigma = 8^{t-2}\times\frac{144}{7}.$$

Since $\#V_t = \frac{8^{t+1}-1}{7} \leq \frac{8^{t+1}}{7}$, we can see that

$$\frac{\sum_{\sigma\in\mathcal{V}_t} C_\sigma}{\#\mathcal{V}_t} \geq \frac{\sum_{\sigma\in\mathcal{I}_t} C_\sigma}{\#\mathcal{V}_t} \geq \frac{7}{8^{t+1}}\frac{144}{7}8^{t-2} = 0.2815. \quad \square$$

Theorem 3. *The limit of the average clustering coefficient $\bar{C}_t$ of G_t exists and*

$$\lim_{t\to\infty}\bar{C}_t \in (0.642, 0.681).$$

Remark 2. For given $t \geq 2$, we consider the following set

$$\mathcal{J}_t = \{\sigma : |\sigma| \leq t \text{ and } Q_t \cap \partial Q = \varnothing\},$$

where $\#\mathcal{J}_2 = 32$, and

$$\#\mathcal{J}_t = \sum_{i=1}^{t}(8^i - 4\times 3^i + 4) = \frac{8\times(8^t-1)}{8-1} - 4\cdot\frac{3\times(3^t-1)}{3-1} + 4t. \tag{3.3}$$

Let $\alpha_t = \frac{\#\mathcal{J}_t}{8\times\#\mathcal{J}_{t-1}}$, and

$$e_t = \prod_{i=3}^{t} a_i = \frac{\#\mathcal{J}_t}{8\times\#\mathcal{J}_{t-1}}\cdot\frac{\#\mathcal{J}_{t-1}}{8\times\#\mathcal{J}_{t-2}}\cdots\frac{\#\mathcal{J}_3}{8\times\#\mathcal{J}_2} = \frac{1}{8^{t-2}}\frac{\#\mathcal{J}_t}{\#\mathcal{J}_2}.$$

Then we can find that $\lim_{t\to\infty} e_t = \lim_{t\to\infty}(\frac{1}{8^{t-2}}\frac{\#\mathcal{J}_t}{\#\mathcal{J}_2}) = \frac{16}{7}$.

Lemma 3. *For any t, let $\bar{\beta}_t = \frac{\sum_{\sigma\in\mathcal{J}_t} C_\sigma}{\#\mathcal{J}_t}$, then we have*

$$\lim_{t\to\infty}\bar{\beta}_t \textit{ exists and positive}.$$

Proof. For any $t \geq 3$,

$$\begin{aligned}
\bar{\beta}_t\cdot e_t &= \frac{\sum_{\sigma\in\mathcal{I}_t} C_\sigma}{\#\mathcal{I}_t}\cdot\prod_{i=3}^{t} a_i \\
&= \frac{\sum_{\sigma\in\mathcal{I}_t} C_\sigma}{\#\mathcal{I}_t}\cdot\frac{\#\mathcal{I}_t}{8\times\#\mathcal{I}_{t-1}}\cdot\frac{\#\mathcal{I}_{t-1}}{8\times\#\mathcal{I}_{t-2}}\cdots\frac{\#\mathcal{I}_4}{8\times\#\mathcal{I}_3} \\
&= \frac{\sum_{\sigma\in\mathcal{I}_t} C_\sigma}{8\times\#\mathcal{I}_{t-1}}\cdot\frac{\#\mathcal{I}_{t-1}}{8\times\#\mathcal{I}_{t-2}}\cdots\frac{\#\mathcal{I}_3}{8\times\#\mathcal{I}_2},
\end{aligned}$$

since $\sum_{\sigma\in\mathcal{J}_t} C_\sigma \geq 8\times\sum_{\sigma\in\mathcal{J}_{t-1}} C_\sigma$, and $e_{t-1}=\frac{\#\mathcal{J}_{t-1}}{8\times\#\mathcal{J}_{t-2}}\cdots\frac{\#\mathcal{J}_3}{8\times\#\mathcal{J}_2}$, we have

$$\bar{\beta}_t\cdot e_t \geq \frac{8\sum\limits_{\sigma\in\mathcal{J}_{t-1}} C_\sigma}{8\times\#\mathcal{J}_{t-1}}\cdot e_{t-1}=\bar{\beta}_{t-1}\cdot e_{t-1}.$$

That means $\left\{\bar{\beta}_t\cdot e_t\right\}_t$ is non-decreasing. Since $0\leq\bar{\beta}_t\leq 1$ and $\lim_{t\to\infty} e_t=\frac{16}{7}$, $\left\{\bar{\beta}_t\cdot e_t\right\}_t$ has the upper bound. Then we can see that the limit of $\left\{\bar{\beta}_t\cdot e_t\right\}_t$ exists. Since $\lim_{t\to\infty} e_t=\frac{16}{7}$, the limit $\lim_{t\to\infty}\bar{\beta}_t$ exists and

$$\lim_{t\to\infty}\bar{\beta}_t=\frac{\lim\limits_{t\to\infty}(\bar{\beta}_t\cdot e_t)}{\lim\limits_{t\to\infty} e_t}>0. \quad \square$$

Remark 3. By (1.1) and (3.3), we have

$$\lim_{t\to\infty}\frac{\#V_t-\#\mathcal{J}_t}{\#V_t}=0.$$

Lemma 4. $\lim_{t\to\infty}\frac{\bar{\beta}_t}{\bar{C}_t}=1$.

Proof. By Lemma 2,

$$\sum_{\sigma\in V_t} C_\sigma=\bar{C}_t\times\#V_t\geq 0.2815\#V_t,$$

then we obtain that

$$\sum_{\sigma\in V_t,\,\sigma\notin\mathcal{J}_t} C_\sigma\leq\#\{\sigma:\sigma\in V_t,\ \sigma\notin\mathcal{J}_t\}=\#V_t-\#\mathcal{J}_t,$$

by Remark 3, we also have

$$0\leq\frac{\sum\limits_{\sigma\in V_t,\,\sigma\notin\mathcal{J}_t} C_\sigma}{\sum\limits_{\sigma\in V_t} C_\sigma}\leq\frac{\#V_t-\#\mathcal{J}_t}{0.2815\#V_t}\to 0.$$

Then we find out that

$$\begin{aligned}\lim_{t\to\infty}\frac{\bar{\beta}_t}{\bar{C}_t}&=\lim_{t\to\infty}\frac{\left(\sum\limits_{\sigma\in\mathcal{J}_t} C_\sigma\right)/\#\mathcal{J}_t}{\left(\sum\limits_{\sigma\in V_t} C_\sigma\right)/\#V_t}\\&=\lim_{t\to\infty}\left(\frac{\sum\limits_{\sigma\in V_t} C_\sigma-\sum\limits_{\sigma\in V_t,\,\sigma\notin\mathcal{J}_t} C_\sigma}{\sum\limits_{\sigma\in V_t} C_\sigma}\cdot\frac{\#V_t}{\#\mathcal{J}_t}\right)\\&=\left(1-\lim_{t\to\infty}\frac{\sum\limits_{\sigma\in V_t,\sigma\notin\mathcal{J}_t} C_\sigma}{\sum\limits_{\sigma\in V_t} C_\sigma}\right)\cdot\lim_{t\to\infty}\frac{\#V_t}{\#\mathcal{J}_t}\\&=1. \quad \square\end{aligned}$$

It follows from Lemmas 3 and 4 that

$$\lim_{t\to\infty}\bar{C}_t=\lim_{t\to\infty}\left(\bar{\beta}_t\times\frac{\bar{\beta}_t}{\bar{C}_t}\right)=\lim_{t\to\infty}\bar{\beta}_t.$$

Therefore, the limit of clustering coefficient exists.

By calculating, we obtain that

$$\bar{\beta}_5=0.667673\ldots.$$

For every $t \geq 6$, using $\#V_t \leq \frac{8^{t+1}}{7}$, we have

$$\bar{C}_t \geq \frac{\sum\limits_{\sigma \in \mathcal{J}_t} C_\sigma}{\#V_t} \geq \frac{8^{t-5} \sum\limits_{\sigma \in \mathcal{J}_5} C_\sigma}{\frac{8^{t+1}}{7}} = \frac{8^{t-5}\bar{\beta}_5 \#\mathcal{J}_5}{\frac{8^{t+1}}{7}} > 0.642,$$

$$\bar{C}_t \leq \frac{8^{t-5} \sum\limits_{\sigma \in \mathcal{J}_5} C_\sigma + \#V_t - 8^{t-5}\#\mathcal{J}_5}{\#V_t} \leq 1 - \frac{8^{t-5}(1-\bar{\beta}_5)\#\mathcal{J}_5}{\frac{8^{t+1}}{7}} < 0.681.$$

Then Theorem 3 follows.

For every $t \geq 6$, $\bar{C}_t > 0.642$, then Theorem 2 is proved.

4. Average path length

For $\sigma \neq \emptyset$, let

$$L(\sigma) = \omega(\sigma) - 1. \tag{4.1}$$

Let $d_t(\sigma, \tau)$ be the shortest distance between σ and τ in G_t. Then, for any $\sigma \neq \emptyset$, $L(\sigma)$ is the least step to move for Q_σ to reach the boundary of K, i.e.

$$L(\sigma) = d_{|\sigma|}(\sigma, \emptyset) - 1.$$

Remark 4. An interesting fact is that

$$L(\sigma\tau) \geq L(\sigma) + L(\tau).$$

Given $\sigma \neq \tau$, we denote the geodesic distance between them by $\mathrm{d}(\sigma, \tau)$. Then we also denote the average path length at time t by

$$\overline{d}_t = \frac{\sum\limits_{\sigma \neq \tau \in V_t} \mathrm{d}(\sigma, \tau)}{\#V_t(\#V_t - 1)/2}.$$

In the section, we will prove the following

Theorem 4. *For all*

$$\frac{7}{64}(t-2) \leq \overline{d}_t \leq 2t.$$

In fact, for any $\sigma, \tau \in V_t$, we have

$$\mathrm{d}(\sigma, \tau) \leq \mathrm{d}(\sigma, \emptyset) + \mathrm{d}(\emptyset, \tau) \leq 2t. \tag{4.2}$$

To obtain the lower bound, we introduce the following technical index. Let

$$\overline{\alpha}_k = \frac{\sum\limits_{|\sigma|=k} L(\sigma)}{\#\{\sigma : |\sigma| = k\}}.$$

It follows immediately that $\overline{\alpha}_1 = 0$, and

$$\overline{\alpha}_2 = \frac{\#\{\sigma \nsim \emptyset : |\sigma| = 2\}}{\#\{\sigma : |\sigma| = k\}} = \frac{32}{64} = \frac{1}{2}.$$

Lemma 5. *For all* $k \geq 2$,

$$\overline{\alpha}_k \geq \frac{1}{4}(k-1). \tag{4.3}$$

Proof. For any $|\sigma| = k > 2$, suppose $\sigma = \tau\sigma'$ with $\left|\sigma'\right| = k-2$, we have

$$\frac{\sum\limits_{|\sigma|=k} L(\sigma)}{\#\{\sigma : |\sigma| = k\}} = \frac{\sum\limits_{|\tau|=2}\sum\limits_{|\sigma'|=k-2} L(\tau\sigma')}{\sum\limits_{|\tau|=2} \#\left\{\sigma' : |\sigma|' = k-2\right\}}.$$

If $\tau \nsim \emptyset$, $L(\sigma) \geq L(\sigma') + L(\tau) \geq L(\sigma') + 1$. In this case, we have

$$\frac{\sum\limits_{|\sigma'|=k-2} L(\tau\sigma')}{\#\{\sigma' : |\sigma'| = k-2\}} \geq \frac{\sum\limits_{|\sigma'|=k-2} (L(\sigma')+1)}{\#\left\{\sigma' : |\sigma|' = k-2\right\}} = \overline{\alpha}_{k-2} + 1. \tag{4.4}$$

If $\tau \sim \emptyset$, $L(\sigma) \geq L(\sigma')$. In this case, we have

$$\frac{\sum\limits_{|\sigma'|=k-2} L(\tau\sigma')}{\#\{\sigma' : |\sigma'| = k-2\}} \geq \frac{\sum\limits_{|\sigma'|=k-2} L(\sigma')}{\#\left\{\sigma' : |\sigma|' = k-2\right\}} = \overline{\alpha}_{k-2}. \tag{4.5}$$

It follows from (4.4)–(4.5) that

$$\begin{aligned}\bar{a}_k &\geq \frac{32}{64}(\overline{\alpha}_{k-2} + 1) + \frac{32}{64}\overline{\alpha}_{k-2} = \overline{\alpha}_{k-2} + \frac{1}{2}\\ &\geq \overline{\alpha}_{k-4} + \frac{1}{2} \times 2 \geq \overline{\alpha}_{k-6} + \frac{1}{2} \times 3 \geq \cdots .\end{aligned}$$

By induction, we have

$$\bar{a}_k \geq \frac{1}{4}(k-1). \quad \square$$

Suppose $\sigma = i_1\sigma'$ and $\tau = j_1\tau'$ with $i_1 \neq j_1$, using the Jordan curve theorem, we have

$$\mathrm{d}(\sigma, \tau) \geq L(\sigma') + L(\tau'). \tag{4.6}$$

Let $W_t = \{\sigma : |\sigma| = t\}$ with $\#W_t = 8^t$. Then by (4.3) and (4.6), we have

$$\begin{aligned}\sum_{\sigma\neq\tau\in W_t} \mathrm{d}(\sigma, \tau) &\geq \sum_{i_1\neq j_1\in\{1,\ldots,8\}} \sum_{i_1\prec\sigma, j_1\prec\tau,\quad \sigma,\tau\in W_t} \mathrm{d}(\sigma, \tau)\\ &\geq \binom{8}{2} \times 2 \times 8^{2(t-1)} \times \frac{\sum\limits_{\sigma'\in W_{t-1}} L(\sigma')}{\#W_{t-1}}\\ &= 56 \times \bar{\alpha}_{t-1} \times 8^{2(t-1)} \geq 56 \times \frac{1}{4}(t-2)8^{2(t-1)}\\ &\geq \frac{7}{32} \times 8^{2t}(t-2).\end{aligned}$$

Since $\sum_{\sigma\neq\tau\in V_t} \mathrm{d}(\sigma, \tau) \geq \sum_{\sigma\neq\tau\in W_t} \mathrm{d}(\sigma, \tau)$ and $\frac{\#V_t(\#V_t-1)}{2} \leq 4\frac{\#W_t(\#W_t-1)}{2}$, we have

$$\begin{aligned}\frac{\sum\limits_{\sigma\neq\tau} \mathrm{d}(\sigma, \tau)}{\#V_t(\#V_t-1)/2} &\geq \frac{1}{4}\frac{\sum\limits_{\sigma\neq\tau\in W_t} \mathrm{d}(\sigma, \tau)}{\#W_t(\#W_t-1)/2}\\ &\geq \frac{1}{4} \times \frac{7}{32} \times 8^{2t}(t-2) \times \frac{2}{8^t(8^t-1)}\\ &\geq \frac{7}{64}(t-2).\end{aligned} \tag{4.7}$$

Then Theorem 4 follows from (4.2) and (4.7). See Fig. 6.

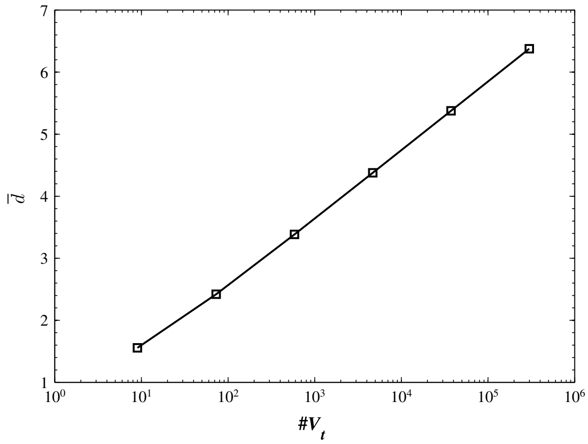

Fig. 6. Semilogarithmic plot of average path lengths $\bar{d}_t$ versus network orders $\#V_t$.

5. Conclusion

The IFS is a useful technique to understand fractals, especially for self-similar sets. The IFS provides a natural encoding approach for similar copies of small size.

In this paper, we use the Sierpinski carpet to construct evolving networks. When focusing on the Sierpinski carpet, the basic elements in the construction are solid squares of side length 3^{-k} with $k \geq 0$. Fix an integer t, take all the solid squares of side length greater than 3^{-t} as nodes of network G_t, in which two nodes are neighbors if and only if their corresponding squares have 1-dimensional intersection, i.e., a line segment. In fact, the neighbor relation has the geometric sense and we can use Jordan curve theorem to characterize the networks.

When our evolving networks $\{G_t\}_{t\geq 1}$ are constructed, we investigate their scale-free and small-world effects. In fact, we prove that the cumulative degree distribution of our networks G_t obey power-law $\log 8/\log 3$, which is the Hausdorff dimension of Sierpinski carpet. We also find the uniform positive lower bound of the average clustering coefficient of G_t by using the self-similarity,

$$\bar{C}_t \geq 0.642 \quad \text{for all } t, \quad \text{and the limit } \lim_{t\to\infty} \bar{C}_t \in (0.642, 0.681) \text{ exists},$$

where $\bar{C}_t$ is the average clustering coefficient of G_t. Finally, for the average path length, we obtain a lower bound and an upper bound being proportional to the logarithm of order of G_t, i.e.,

$$\frac{7}{64}(t-2) \leq \bar{d}_t \leq 2t \quad \text{for all } t,$$

where $\bar{d}_t$ is the average path length of G_t. An interesting conjecture is that the limit

$$\lim_{t\to\infty} \frac{\bar{d}_t}{t} \text{ exists.}$$

However, we notice that the eight small squares are not symmetric. More specifically four squares in corners and others have quite different properties. This is one great difference from Sierpinski gasket in the geometric point of view.

It is worth noting that the self-similar fractals can generate the complex networks satisfying the scale-free and small-world effects. An interesting question arises whether our networks are fractal networks in sense of Song et al. [3]. We will investigate their fractality in our next work.

References

[1] D.J. Watts, S.H. Strogatz, Collective dynamics of 'small-world' networks, Nature 393 (1998) 440–442.
[2] A.-L. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (1999) 509–512.
[3] C. Song, S. Havlin, H.A. Makse, Self-similarity of complex networks, Nature 433 (2005) 392–395.
[4] C. Song, S. Havlin, H.A. Makse, Origins of fractality in the growth of complex networks, Nat. Phys. 2 (2006) 275–281.
[5] J.S. Kim, K. Goh, B. Kahng, D. Kim, A box-covering algorithm for fractal scaling in scale-free networks, Chaos 17 (2007) 026116.
[6] C. Song, L.K. Gallos, S. Havlin, H.A. Makse, How to calculate the fractal dimension of a complex network: the box covering algorithm, J. Stat. Mech. 2007 (2007) P03006.
[7] L.K. Gallos, C. Song, H.A. Makse, Scaling of degree correlations and its influence on diffusion in scale-free networks, Phys. Rev. Lett. 100 (2008) 248701.
[8] J.S. Andrade Jr., H.J. Herrmann, R.F. Andrade, L.R. Da Silva, Apollonian networks: Simultaneously scale-free, small world, Euclidean, space filling, and with matching graphs, Phys. Rev. Lett. 94 (1) (2005) 018702.
[9] T. Zhou, G. Yan, B. Wang, Maximal planar networks with large clustering coefficient and power-law degree distribution, Phys. Rev. E 71 (4) (2005) 046141.

[10] Z. Zhang, S. Zhou, L. Fang, J. Guan, Y. Zhang, Maximal planar scale-free Sierpinski networks with small-world effect and power law strength-degree correlation, Europhys. Lett. 79 (2007) 38007.
[11] Z. Zhang, S. Zhou, Z. Su, T. Zou, J. Guan, Random Sierpinski network with scale-free small-world and modular structure, Eur. Phys. J. B 65 (2008) 141–147.
[12] J. Guan, Y. Wu, Z. Zhang, S. Zhou, Y. Wu, A unified model for Sierpinski networks with scale-free scaling and small-world effect, Physica A 388 (2009) 2571–2578.
[13] Z. Zhang, S. Zhou, L. Chen, M. Yin, J. Guan, The exact solution of the mean geodesic distance for Vicsek fractals, J. Phys. A 41 (2008) 485102.
[14] J. Liu, X. Kong, Establishment and structure properties of the scale-free Koch network, Acta Phys. Sin. 59 (2010) 2244–2249.
[15] R. Chen, X. Fu, Q. Wu, On topological properties of the octahedral Koch network, Physica A 391 (2012) 880–886.
[16] W.-M. Song, T. Di Matteo, T. Aste, Building complex networks with Platonic solids, Phys. Rev. E 85 (2012) 046115.
[17] J. Chen, F. Gao, A. Le, L. Xi, S. Yin, A small-world and scale-free network generated by Sierpinski tetrahedron, Fractals 24 (1) (2016) 1650001.
[18] G.T. Whyburn, Topological characterization of the Sierpinski curve, Fund. Math. 45 (1958) 320–324.
[19] J. Kigami, Analysis on Fractals, Cambridge University Press, Cambridge, 2001.