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A Method for Identification of Critical States of Open Stochastic Dynamical Systems Based on the Analysis of Acceleration

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Abstract A new method of fractal analysis of nonstationary time series for recognition of critical (precatastrophic) and noncritical (quiet) modes of behaviour of open stochastic dynamical systems is developed. The method is a modification of the conventional detrended fluctuation analysis (DFA) technique. Unlike the classical DFA that originates in the R/S analysis and implies investigation of the time series by studying the properties of a mixture of velocities and accelerations, the new method focuses on the study of accelerations only. Because at the most basic level the equations of motion of stochastic dynamical systems are expressions for the acceleration, the suggested method results to be more suitable for recognition of critical and noncritical states. Using both model and real data, we demonstrate superiority of the newly developed method over the conventional DFA and provide a detailed discussion on the topic.

Keywords Open stochastic dynamical systems · Critical states · Precursors of catastrophes · Fractal analysis · Measure of chaoticity

1 Introduction

A large number of practically important evolutionary problems can be reduced to the analysis of behaviour of open stochastic dynamical systems. Examples include the tasks of diagnostics of heart diseases [9,20], early detection of disastrous seismic events [15,16], prediction of price crashes in financial markets [23,24], just to name a few.

During the evolution stochastic dynamical systems pass through a sequence of states characterised by different types of behaviour. The low chaotic behaviour implies gradual and continuous changes of states, whereas the highly chaotic behaviour consists of fast and abrupt alterations. Empirically it has been found out that when the system is in a normal (noncritical)

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state, it is evolving in a highly chaotic regime; however, when it is in a pathological (critical) state, either a low chaotic behaviour is observed (see, e.g., [1,20]) or, on the contrary, extreme levels of chaoticity are reached [6,14]. From the practical point of view entering of the system into a critical mode implies that after some time a catastrophe may occur unless appropriate measures are duly taken to get the system back into a highly chaotic regime, and therefore the recognition of pathological regimes is of crucial importance [7,11,12]. The point is that for open stochastic dynamical systems adequate equations of motion are often unknown, which leads to the necessity of developing smart methods of analysis of historical records of the system's evolution in order to foresee a catastrophe before it happens.

For the last a few decades fractal analysis of time series has proved to be an efficient method for detection of precatastrophic states of stochastic dynamical systems. These methods imply quantitative analysis of a certain characteristic that serves as a measure of chaoticity of the system's behaviour. If the chaoticity is out of an experimentally found range of noncritical values, then it is very likely that the system is evolving in a critical—precatastrophic—mode.

Among various methods of fractal analysis detrended fluctuation analysis (DFA) is of particular interest [10]. The original first-order DFA is a modification of the empirically invented R/S analysis suggested by H. Hurst in the 1950s when solving the conundrum of the Nile's great floods [8]. Since the appearance in the middle of the 1990s the DFA has been successfully used in many research studies aimed to identify critical states of open dynamical systems of diverse nature [1,2,5,6,9,14,18–21]. It was also extended to high-order methods [10]. Nevertheless, the empirical foundations of the DFA methods originating in the R/S analysis make it useful to investigate their physical meaning and suggest possible modifications intended to increase the accuracy of discrimination of critical and noncritical states. That would allow to identify the modes of systems' evolution more precisely, and hence make the advance detection of catastrophes more reliable.

The rest of the paper is organised as follows. In Sect. 2 we start from providing a general formula of unifractal analysis methods and then demonstrate how the well-known Hurst and conventional DFA methods follow from it as particular cases. After that we introduce a modification of the conventional DFA and reveal the difference of the newly developed method by analysing the physical meaning of the conventional and modified methods. In Sect. 3, using an analytical model that allows simulating both critical and noncritical regimes of evolution of an open dynamical system, we prove superiority of the modification over the original DFA. Finally in Sect. 4 we apply the modified DFA to study the behaviour of a real dynamical system: the human heart. There we show how the quantitative measure of chaoticity based on the modified DFA can be used to distinguish critical and quiet states, providing better discrimination accuracy compared to the conventional method. In Sect. 5 we give a conclusion.

2 Fractal Analysis: The Conventional and Modified Methods

Fractal analysis of a finite sample $\{\Delta x_k\}_{k=1}^{N}$ of a time series of increments defined on the interval $[t_0, t_N]$ with the step $\Delta t = t_{k+1} - t_k$ implies introducing a quantity $j \geq 1$ (called 'scale') and partitioning the time interval into several adjacent segments of the size $2j\Delta t$, so that the number of the segments is $n = \frac{N\Delta t}{2j\Delta t}$. Then at each scale j the measure of chaoticity of the system's evolution on the interval $[t_0, t_N]$ can be written as

$$D = \frac{\ln \mu(j + \Delta j) - \ln \mu(j)}{\ln(j + \Delta j) - \ln j}, \tag{1}$$

where the quantity $\mu(j)$ is called 'data measure' at the scale j, while Δj is the scale increment. For such a setup, conventional methods of fractal analysis imply

$$\mu(j) = \left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{R_k^{(j)}}{S_k^{(j)}}\right)^q\right]^{1/q}, \quad q \geq 1, \tag{2}$$

where

$$R_k^{(j)} = \left[\frac{1}{N/n}||\mathbf{r}_k^{(j)}||_p^p\right]^{1/p}, \quad p \geq 1, \tag{3}$$

$$\mathbf{r}_k^{(j)} = \{r_{k,m}^{(j)}\}, \quad r_{k,m}^{(j)} = \sum_{i=1}^{m}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right), \quad m = 1, \ldots, \frac{N}{n}, \tag{4}$$

$$\overline{\Delta x^{(k)}} = \frac{1}{N/n}\sum_{i=1}^{N/n}\Delta x_i^{(k)}, \qquad \Delta x_i^{(k)} = \Delta x_l, \quad l = (k-1)\frac{N}{n} + i. \tag{5}$$

Formulas (2)–(5) are the general form of the conventional fractal analysis methods. For example, if one defines $S_k^{(j)} = \left[\frac{1}{N/n}\sum_{i=1}^{N/n}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right)^2\right]^{1/2}$ then it will lead to the R/S analysis; in particular, under $p \to +\infty$ and $q = 1$ it will hold

$$\mu(j) = \frac{1}{n}\sum_{k=1}^{n}\frac{R_k^{(j)}}{S_k^{(j)}} \tag{6}$$

and

$$R_k^{(j)} = \max_m \left|\sum_{i=1}^{m}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right)\right|, \tag{7}$$

which are the well-known Hurst formulas [4,8]. (To be rigorous, according to Hurst $R_k^{(j)} = \max_m |r_{k,m}^{(j)}| - \min_m |r_{k,m}^{(j)}|$. However, this is solely a nonstrict extension of the norm $||\cdot||_{+\infty}$.) If one defines $S_k^{(j)} \equiv 1$ then (2)–(5) will become the first-order DFA; in the particular case of $p = q = 2$ we obtain the most frequently used formula (cf. [10, p. 17])

$$\mu(j) = \left[\frac{1}{N}\sum_{k=1}^{n}\sum_{m=1}^{N/n}\left(\sum_{i=1}^{m}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right)\right)^2\right]^{1/2}. \tag{8}$$

From (2)–(5) it is seen that the difference between various conventional methods is determined exclusively by the quantity $S_k^{(j)}$. As for the parameters p and q, these can be chosen in an arbitrary manner, as they merely specify a concrete pair of norms in which the quantities $R_k^{(j)}$ and $\mu(j)$ will be calculated. Following the traditional notation, hereinafter for the R/S analysis we shall denote D as H, while for the DFA methods D will be denoted as H_{DFA}.

Remark The parameters p and q, being once chosen, are considered to be fixed afterwards. Therefore, in this paper we address the so-called '(uni)fractal' methods [17, pp. 45, 60, 159] (also [4,10]). Otherwise, if p and q are supposed to vary then that would lead to the 'multifractal' methods [4,10], which are out of our scope. □

When identifying critical states of real dynamical systems, the DFA methods are experienced to provide more reliable results than the R/S analysis methods do. The reason for that is a wider range of values which the quantity H_{DFA} can take compared to the quantity H: $H_{DFA} \in [0, 2]$ vs $H \in [0, 1]$. The point is that for many dynamical systems, when the system is in a noncritical regime, the power spectrum at low frequencies is close to the pink noise ($|\widehat{\Delta x}(\omega)|^2 \sim \omega^{-1}$), for which at major scales ($j \gg 1$) $H_{DFA} \approx 1$, while $H \approx 0.92$; however, when the system transits to a critical regime, the power spectrum at low frequencies usually becomes warmer, up to brown ($|\widehat{\Delta x}(\omega)|^2 \sim \omega^{-2}$), for which at major scales $H_{DFA} \approx 1.5$, whereas $H \approx 1$. It is seen that the measure of chaoticity of the R/S analysis is less sensitive than the DFA's, and hence when using H critical and noncritical regimes are harder to distinguish.

In [21] it was shown that the conventional DFA methods provide a rather good discrimination between different modes of behaviour of a dynamical system. Spefically, having been applied to the human heart interbeat interval time series, the quantity H_{DFA} permitted to differentiate healthy subjects and subjects with congestive heart failure: a scatter plot of H_{DFA} at major scales versus H_{DFA} at minor scales demonstrated two apparent clusters. However, there was also a plain overlap meaning that some healthy subjects are still erroneously treated as ill, whereas some ill ones are wrongly treated as healthy. Clearly that the misidentification of critical and noncritical regimes may have unpleasant consequences from the practical standpoint.

The reason of the misidentification is apparently the numerical inaccuracy of the method. In order to reduce it and make the distinction between critical and noncritical states more reliable, we suggest to modify the conventional fractal analysis methods. For this, let us first notice that for the R/S analysis the quantity $S_k^{(j)}$ can be rewritten as

$$\begin{aligned}\left(S_k^{(j)}\right)^2 &= \frac{1}{N/n}\sum_{m=1}^{N/n}\left(\overline{\Delta x^{(k)}} - \Delta x_m^{(k)}\right)^2\\ &= \frac{1}{N/n}\sum_{m=1}^{N/n}\left(\sum_{i=1}^{m}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right) - \sum_{i=1}^{m-1}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right)\right)^2\\ &= \frac{1}{N/n}\sum_{m=1}^{N/n}\left(r_{k,m}^{(j)} - r_{k,m-1}^{(j)}\right)^2. \end{aligned} \quad (9)$$

From formulas (2)–(5) and (9) it is seen that the quantity $\mu(j)$ (and hence, the measures of chaoticity H and H_{DFA}) is (are) fully determined by the way of computing the components of the vector $\mathbf{r}_k^{(j)} = \left\{r_{k,m}^{(j)}\right\}$. Therefore, instead of the way given in (4) we suggest

$$r_{k,m}^{(j)} = \frac{1}{2}\left(\sum_{i=m+\frac{N}{2}}^{m+\frac{N}{2}+j-1}\Delta x_i^{(k)} - \sum_{i=m+\frac{N}{2}-j}^{m+\frac{N}{2}-1}\Delta x_i^{(k)}\right), \quad m = 1, \ldots, \frac{N}{n} + 1. \quad (10)$$

Formulas (2)–(3), (5) and (10) are the general form of the modified fractal analysis methods. For the resulting measures of chaoticity we shall use the notation $\widetilde{H}$ (for the modified R/S analysis) and $\widetilde{H}_{DFA}$ (for the modified first-order DFA). In order not to go outside the bounds of the data $\{\Delta x_k\}$ when using (10), the parameter N has to be reduced accordingly (e.g., two times if N is a power of two, see Fig. 1, on the right).

In order to reveal the difference between the conventional and modified methods, let us denote $\Delta x_{\text{left},m}^{(k)} = \sum_{i=1}^{m}\Delta x_i^{(k)}$, $\Delta x_{\text{right},N/n-m}^{(k)} = \sum_{i=m+1}^{N/n}\Delta x_i^{(k)}$ and $\xi = \frac{m}{N/n}$. Because $\Delta x_{\text{left},m}^{(k)} + \Delta x_{\text{right},N/n-m}^{(k)} = \frac{N}{n}\overline{\Delta x^{(k)}}$ for all $m = 1, \ldots, N/n$, for $r_{k,m}^{(j)}$ defined by (4) it holds

$$\begin{aligned} r_{k,m}^{(j)} &= \sum_{i=1}^{m}\left(\overline{\Delta x^{(k)}} - \Delta x_i^{(k)}\right) \\ &= \xi\left(\Delta x_{\text{left},m}^{(k)} + \Delta x_{\text{right},N/n-m}^{(k)}\right) - \Delta x_{\text{left},m}^{(k)} \\ &= \xi\,\Delta x_{\text{right},N/n-m}^{(k)} - (1-\xi)\Delta x_{\text{left},m}^{(k)} \\ &= \xi\left(\Delta x_{\text{right},N/n-m}^{(k)} - \Delta x_{\text{left},m}^{(k)}\right) - (1-2\xi)\Delta x_{\text{left},m}^{(k)}. \end{aligned} \tag{11}$$

Up to the factor $1/\Delta t$, for a fixed m the right-hand side of (11) is, qualitatively, a linear combination of an acceleration and a velocity. For example, if $N/n = 8$ and $m = 1$ then we have $r_{k,1}^{(j)} = \frac{1}{8}\left(\Delta x_{\text{right},7}^{(k)} - \Delta x_{\text{left},1}^{(k)}\right) - \frac{6}{8}\Delta x_{\text{left},1}^{(k)}$, where the first summand is proportionate to a difference of velocities defined on time intervals of the lengths $7\Delta t$ and Δt respectively, which is, qualitatively, a function of acceleration, while the second summand is proportional to velocity; if $m = 4$ ($\xi = 0.5$) then the quantity $r_{k,4}^{(j)}$ is simply proportionate to a difference of velocities, i.e. is purely acceleration; the case of $m = 7$ is the mirror image of $m = 1$, since we have $r_{k,7}^{(j)} = \frac{1}{8}\left(\Delta x_{\text{right},1}^{(k)} - \Delta x_{\text{left},7}^{(k)}\right) + \frac{6}{8}\Delta x_{\text{right},1}^{(k)}$. Because in the median case of $\xi = 0.5$ we deal with the acceleration only, *at the average* the quantities $r_{k,m}^{(j)}$ defined via (4) can be thought of as accelerations, keeping in mind that additional velocity-like terms appear when $m \neq \frac{N/n}{2}$. Consequently, since at the most basic level the equation of motion of any stochastic dynamical system is an expression for the acceleration defined by the Langevin equation with a specific (though unknown) forcing on the right-hand side [7, 12], the measures of chaoticity H and H_{DFA} based on $r_{k,m}^{(j)}$ defined via (4) characterise the system's evolution with an introduced inaccuracy due to the presence of the non-accelerational terms in (4). Unlike that, if $r_{k,m}^{(j)}$'s are defined via (10) then under any m both sums contain the same number (namely, j) of summands, and hence we *always* deal with accelerations only—no additional terms appear. Therefore, the resulting measures of chaoticity $\widetilde{H}$ and $\widetilde{H}_{DFA}$ describe the system's evolution without possibly non-physical artefacts that may perturb the result. A geometrical illustration of both methods is given in Fig. 1.

Remark To give more insight into the motivation to modify the conventional DFA, we observe that the Langevin equation that governs evolution of an open stochastic dynamical system can be written in the form

$$m\frac{d^2x}{dt^2} = F, \tag{12}$$

where F is the sum of deterministic and stochastic forces acting on the particle of the mass m (here the particle may be thought of as either real or abstract, depending on the dynamical system being studied), whereas $\frac{d^2x}{dt^2}$ is the particle's acceleration. Although the term F may contain a summand that depends on the velocity $\frac{dx}{dt}$ (and even be nonlinear, e.g. $\alpha\frac{dx}{dt} - \beta\left(\frac{dx}{dt}\right)^3$ [7]), for the sake of generality of subsequent analysis it makes sense to treat the right-hand side of (12) as the total forcing regardless of its internal structure, and thus deal with the acceleration only on the left-hand side.

At the same time, one may suggest that a method based on the analysis of pure velocities rather than pure accelerations can also be used. This would follow if one integrates both sides of (12), which would yield

$$m\frac{dx}{dt} = \int F dt. \tag{13}$$

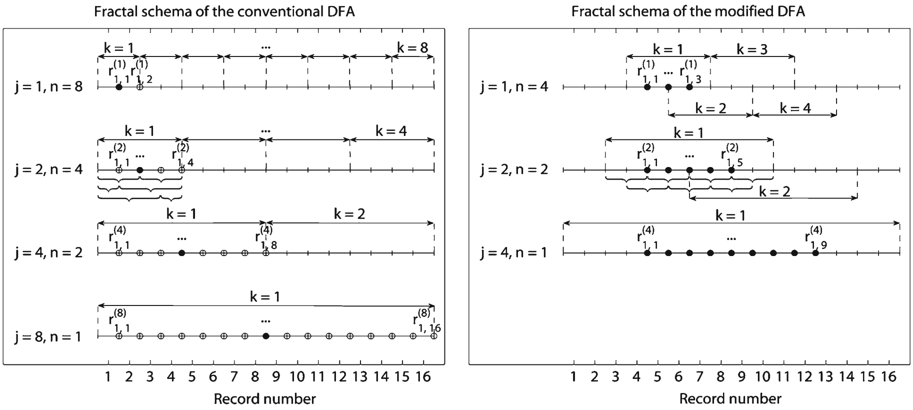

Fig. 1 Geometrical illustration of the use of records of the sample $\{\Delta x_k\}$ by the conventional and modified DFAs at $N = 16$ and the scale j varying exponentially with the factor 2. The common place of both methods is the scale, i.e. the number of records (shown by the *curly brackets*) involved in the computation of the central $r^{(j)}_{k,m}$'s. In what the methods differ is that the conventional DFA involves nonequal numbers of records from the *left* and *right sides* when computing the noncentral $r^{(j)}_{k,m}$'s (shown by the *hollow circles*). Unlike that, the modified DFA always uses equal numbers of records for all m, which guarantees that all $r^{(j)}_{k,m}$'s are pure accelerations (shown by the *filled circles*). This is achieved by the double size of the window at every k in comparison with the conventional DFA. The double window size implies to overlap the windows at different k's a half of the size in order not to have gaps when computing $r^{(j)}_{k,m}$'s. This also requires to treat the number of records as if it is reduced two times as $N \to N/2$ (that is why at a fixed j the parameter n in the modified DFA is two times smaller than that in the conventional DFA) and hence to decrease the number of available scales in one—otherwise at the most major scale we would need records outside the bounds of the data $\{\Delta x_k\}$

Either way, analysis of purely observable quantities rather than a combination thereof seems to be more appropriate for accurate discrimination of critical and noncritical states, unless specific information about the governing forcing F is available a priori. In favour of the acceleration-based method it should still be observed that normally a real time series of velocities $\{\Delta x_k\}$ is nonstationary, and hence a reduction of the nonstationarity will naturally lead to accelerations. □

Comparison of both methods reveals that at the most major scale the modified DFA involves all available records of the sample $\{\Delta x_k\}_{k=1}^{N}$, as it actually should do (see the case $j = 4$ in Fig. 1, on the right, as well as formula (10)). However, at the rest minor scales ($j = 1, 2$ in the right Fig. 1) some lateral records fall out of the calculations when using the modified DFA; unlike it, the conventional DFA always uses all available records, i.e. no skipping of information takes place. The skipping of information in the modified DFA is the payment for the ability to base the measures of chaoticity $\widetilde{H}$ and $\widetilde{H}_{DFA}$ on the analysis of pure accelerations. However, in the subsequent sections we shall demonstrate that the skipping of information does not affect the accuracy of the modified DFA—its performance is superior than that of the conventional method.

3 Identification of Critical States of a Model Dynamical System

In order to show the difference between the two methods numerically, compare them on a model test. In [21] a stochastic model for the simulation of critical states was suggested which was subsequently proved to fit the real data with a good accuracy. However, the suggested model did not account for quiet (noncritical) modes of behaviour. Therefore, consider the model

$$\begin{aligned}&\Delta(\Delta x)_k = Aa_k + Bb_k + Cc_k,\\&|\hat{a}(\omega)|^2 \sim \omega^{\beta-\Delta\beta_{\text{low}}}, \quad |\hat{b}(\omega)|^2 \sim \omega^{\beta}, \quad |\hat{c}(\omega)|^2 \sim \omega^{\beta+\Delta\beta_{\text{high}}},\end{aligned} \tag{14}$$

where $A, B, C \in \mathbb{R}$, $\beta > 0$, $\Delta\beta_{\text{low}} \geq 0$, $\Delta\beta_{\text{high}} \geq 0$. The time series $\{\Delta x_k\}$ is made by cumulative summation of the accelerations $\{\Delta(\Delta x)_k\}$. It is remarkable that model (14) allows simulating both regimes of system's behaviour, critical and noncritical. Speficially, under $\Delta\beta_{\text{low}} \approx \Delta\beta_{\text{high}} \approx 0$ the power spectrum of the time series $\{\Delta(\Delta x)_k\}$ is close to single-coloured with the colour exponent β (and hence, $\left|\widehat{\Delta x}(\omega)\right|^2 \sim \omega^{\beta-2}$), which corresponds to a noncritical, highly chaotic behaviour; otherwise, under $\Delta\beta_{\text{low}} > 0$ the influence of low frequencies increases and the system transits to a low chaotic, critical regime (Fig. 2). These properties are consistent with the observations about the behaviour of real dynamical systems (see, e.g., [20,21,24]). The point of using model rather than real data for comparison of the methods is that for each of the simulated states we thereby have objective information about whether it is critical or noncritical. This makes the criterion of comparison straightforward: for the two clusters of states appearing on the plane 'the measure of chaoticity at major scales versus it at minor scales' (1) the smaller the radii of the clusters, as well as (2) the farther the clusters one from another, the smaller the clusters' overlap and hence the better the method's accuracy. In the ideal case the two clusters should have an empty overlap—this would mean that all the states are identified properly and the corresponding method has no numerical errors. To estimate the radius of a cluster, one can use the standard deviation from the cluster's centre, while the distance between the clusters can be computed as the distance between the clusters' centres.

In Fig. 3 we show scatter plots of the measures of chaoticity based on the conventional and modified DFAs using 75 critical and 75 noncritical states simulated with model (14) under $A = 0.4$, $B = C = 1$, $\beta = 1$. For the critical states we had $\Delta\beta_{\text{low}} = 1$, $\Delta\beta_{\text{high}} = 2.5$, while for the noncritical ones $\Delta\beta_{\text{low}} = \Delta\beta_{\text{high}} = 0$. For the simulation of each state we set $N = 8192$, the scale varied from 1 to 4096 as $j = N2^{-i}, i = 1, \ldots, 13$ (for the modified DFA it was from 1 to 2048 as $j = N2^{-(i+1)}, i = 1, \ldots, 12$,—in order not to go outside the bounds of the data in (10), see Fig. 1), the boundary scale j_B that separates the major and minor scales was computed by finding the best least square piecewise linear approximation of the dependence $\log \mu$ vs $\log j$ (see Fig. 2, at the bottom). In Table 1 we summarise the statistics over 100 repetitions of the simulation: for each method and for each regime of behaviour we present the mean standard deviation of the measure of chaoticity from the cluster's centre, as well as the mean distance between the centres of the two clusters corresponding to the different regimes of behaviour. As one can see, the modified DFA methods demonstrate a superior performance over the conventional ones. Specifically, the noncritical states are properly identified about 20 % more frequently, whereas for the critical states the accuracy is even higher, roughly 33–50 %. As for the distance between the clusters of states, both methods seem to provide comparable results—the absolute value of the relative variation $\delta V = \frac{V_{\text{mod}}-V_{\text{conv}}}{V_{\text{conv}}} \times 100\ \%$, where V_{conv} and V_{mod} are the mean values produced by the conventional and modified methods respectively, is about 5 % pro and contra the modified DFA. Altogether, these result in a smaller overlap between the clusters and thus in more reliable identifications of the regimes of the system's behaviour—critical states are up to twice less likely to be identified as noncritical and vice versa.

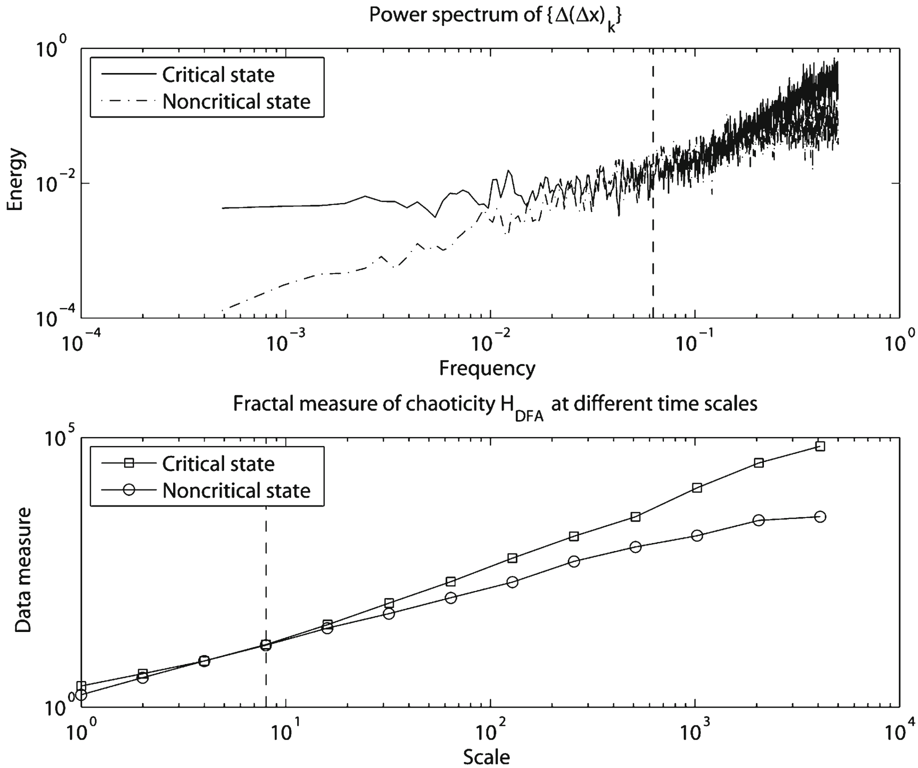

Fig. 2 Power spectra of a sample of the time series $\{\Delta(\Delta x)_k\}$ (at the *top*) and graphs of the dependence $\log \mu$ versus $\log j$ (at the *bottom*) at two different regimes of the system's behaviour simulated by model (14) with $A = 0.4$, $B = C = 1$, $\beta = 1$; $\Delta\beta_{\text{low}} = 1$, $\Delta\beta_{\text{high}} = 2.5$ at the critical state, $\Delta\beta_{\text{low}} = \Delta\beta_{\text{high}} = 0$ at the noncritical (quiet) state. The increasing influence of low frequencies results in transition of the system from a highly chaotic (noncritical) to a low chaotic (critical) regime. This is accompanied by the appearance of two crossovers—a boundary (or critical) frequency ω_B and the related boundary (critical) time scale $j_B = \frac{1}{2\omega_B}$ marked by the *vertical dashed lines*, so that the power spectrum exponent and the measure of chaoticity (here we used H_{DFA}) have essentially different values at both sides of the borders

4 Identification of Critical States of a Real Dynamical System

Now we are going to apply the developed method for recognition of critical states of a real stochastic dynamical system. For our tests we shall take cardiac interbeat interval time series from the PhysioNet archive freely available on the Internet [22].

Cardiac interbeat interval time series exhibit specific behaviour depending on the state of the heart of a subject. A noncritical state corresponds to the normal sinus rhythm, while a critical state corresponds to the congestive heart failure. Following [21], for our tests we have taken time series of the length $N = 8192$ records. Each record represents a number of samples (measured at the rate 250 samples per second) between successive heart beats. We analysed time series corresponding to 18 healthy and 15 ill subjects. In Table 2 there are statistical outcomes obtained with the conventional and modified DFAs. In Fig. 4 we depict scatter plots of the measures of chaoticity computed for the critical and noncritical states. It is seen that the modified method again shows a better performance: the noncritical states are

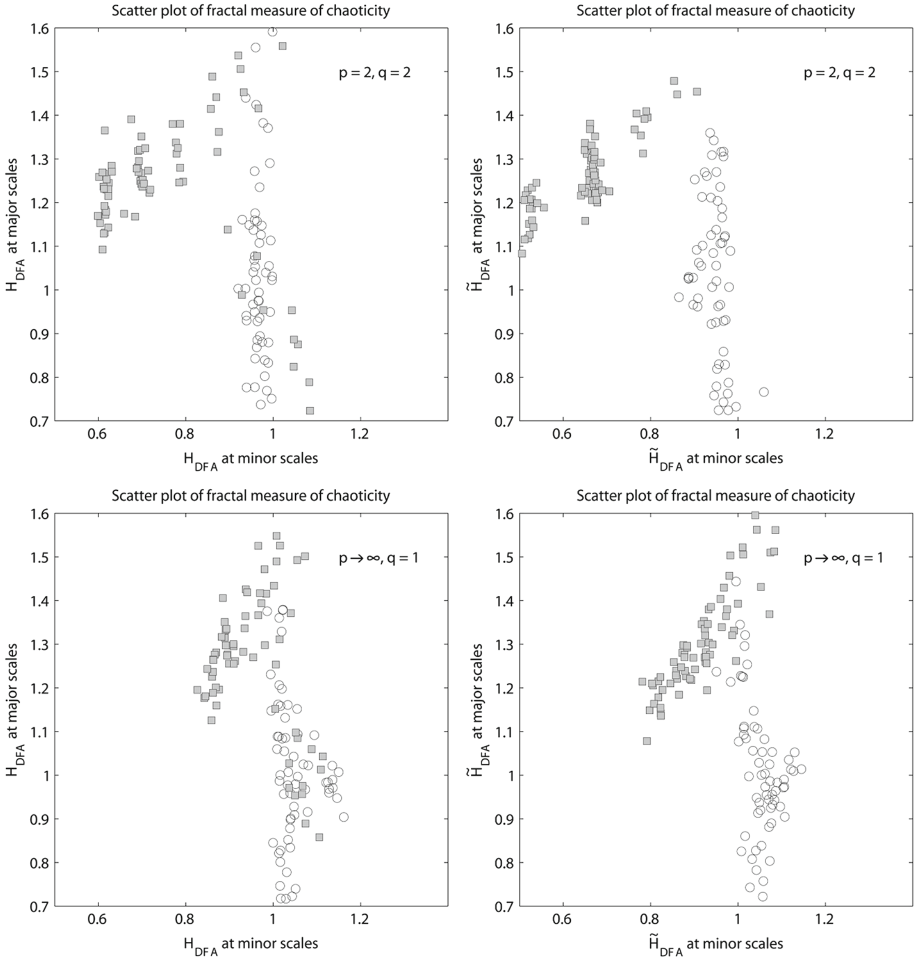

Fig. 3 Scatter plots of the measures of chaoticity H_{DFA} (on the *left*) and $\widetilde{H}_{DFA}$ (on the *right*) for 75 critical (*grey squares*) and 75 noncritical (*hollow circles*) states simulated by model (14). Each marker is a state of the system during its evolution characterised by a pair of values of a measure of chaoticity at the major and minor scales. It is seen that independently of the parameters p and q the modified DFA yields a better distinction of the clusters than the conventional DFA does: the number of critical states falling near the noncritical states' cluster (and vice versa) is much smaller when the modified DFA is used. Yet, no qualitative difference between the outcomes related to the different pairs p, q is observed, which confirms that the difference between the R/S analysis and the DFA is solely due to the way of computing $S_k^{(j)}$ in (2) and not affected by the choice of norms

correctly recognised 16–24 % more often, for the critical states this value is a bit smaller but still significant—about 13 %. Besides, the distance between the clusters of states increases essentially, up to 20–25 %.

Compared to the test with the model data presented in the previous section, this one demonstrates a slight decay of accuracy. We explain this by the fact that when analysing real data there is no truly objective information about criticality or noncriticality of the system's states. In other words, for a real dynamical system currently seeming to evolve in a critical mode, among a sequence of mostly critical states there may occasionally appear

Table 1 Mean radii of the clusters of states and mean distances between the clusters' centres over 100 repetitions of the simulation via model (14)

	Conventional DFA	Modified DFA	Rel. variation (%)
$p = q = 2$			
Critical mode	0.29 ± 0.05	0.15 ± 0.03	**−48.28**
Noncritical mode	0.36 ± 0.03	0.29 ± 0.02	**−19.44**
Intercluster distance	0.44 ± 0.04	0.46 ± 0.03	**4.55**
$p \to +\infty, q = 1$			
Critical mode	0.28 ± 0.03	0.18 ± 0.02	**−35.71**
Noncritical mode	0.30 ± 0.02	0.24 ± 0.02	**−20.00**
Intercluster distance	0.39 ± 0.04	0.37 ± 0.03	−5.13

The results are given in the format 'the mean $\pm$ the standard deviation' of the corresponding quantity; the relative variation's values in support of the modified DFA are shown in bold

Table 2 Mean radii of the clusters of states and mean distances between the clusters' centres for the cardiac interbeat interval time series

	Conventional DFA	Modified DFA	Rel. variation (%)
$p = q = 2$			
Critical mode	0.32 ± 0.05	0.28 ± 0.05	**−12.50**
Noncritical mode	0.41 ± 0.02	0.31 ± 0.03	**−24.39**
Intercluster distance	0.62 ± 0.12	0.74 ± 0.16	**19.35**
$p \to +\infty, q = 1$			
Critical mode	0.31 ± 0.04	0.27 ± 0.02	**−12.90**
Noncritical mode	0.31 ± 0.03	0.26 ± 0.04	**−16.13**
Intercluster distance	0.41 ± 0.14	0.52 ± 0.17	**26.83**

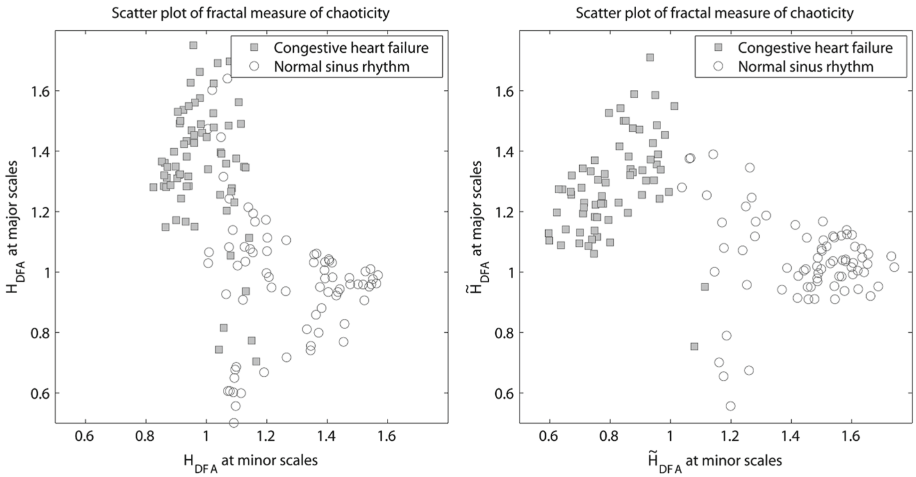

Fig. 4 Scatter plots of the measures of chaoticity H_{DFA} (on the *left*) and $\widetilde{H}_{DFA}$ (on the *right*) for noncritical (normal sinus rhythm) and critical (congestive heart failure) states of the human heart. Here we used $p \to +\infty$, $q = 1$. The newly developed method provides superior results over the conventional DFA: the critical and noncritical clusters have smaller deviations and are located farther one from another (see also Table 2). Yet, it is seen that these plots nicely repeat the model plots given in Fig. 3, which indicates that (14) is an adequate model for describing both critical and noncritical regimes of dynamics of the human heart

a small number of quiet states, and vice versa. Such inclusions of the opposite states result in an increase of the sizes of the critical and noncritical clusters of states and hence in an increase of the overlap between the clusters. Thereby, the accuracy of the states discrimination becomes lower compared to the model data which are always compounded of either critical or noncritical states of behaviour (unless one suggests a sophisticated model that would allow to generate time series with inclusions of the opposite states), and this is what is reflected in Table 2.

From the plots of Fig. 4 one can also see that these closely repeat the model plots given in Fig. 3. This suggests that (14) is an adequate model for describing both critical and noncritical regimes of dynamics of the human heart, and the relation of the model parameters to specific neuroautonomic control mechanisms [3,13] could be a subject of further studies.

5 Conclusion

We have developed a new method for recognition of critical and noncritical states alternating during the evolution of open stochastic dynamical systems. We began from analysing the physical meaning of the conventional first-order DFA method and found out that it implies the study of time series via investigating statistical properties of a combination of velocities and accelerations. Then, taking into account the general form of equations of motion of dynamical systems, we suggested a modification of the conventional DFA which possesses more physical fidelity, as it focuses on the analysis of pure accelerations being the key point of equations of motion. In the subsequent numerical tests we showed that the newly developed method provides a higher accuracy on the discrimination of states of both model and real dynamical systems. In doing so we also suggested a stochastic model that allows simulating both critical and quiet modes of behaviour of a real dynamical system—the human heart. The suggested model was demonstrated to generate time series with the properties consistent with those of real data.

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