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An explicit basis for the Grassmann T-space ☆

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Let S^3 denote the Grassmann T-space generated by the commutator $[x_1, x_2, x_3]$ in the free associative algebra $K\langle x_1, x_2, \ldots\rangle$ over a field K of characteristic zero. We construct an explicit linear basis for each $K\mathcal{S}_n$-module $S^3 \cap P_n$, where P_n is the space of all multilinear polynomials of degree n in indeterminates $x_1, \ldots, x_n$. This provides a solution to the problem of finding a linear basis for S^3, which was posed by Latyshev in circa 1990.

1. Introduction

Let K be a field of characteristic zero and fix a countably infinite set of indeterminates $X = \{x_1, x_2, \ldots\}$. Then by $K\langle X\rangle$ we shall denote the free associative algebra with unity over K generated by X. For each positive integer n, we shall use $\mathcal{S}_n$ to denote the group of permutations on the set $J_n = \{1, 2, \ldots, n\}$. By $P_n = P_n(x_1, \ldots, x_n)$ we shall mean the set of all multilinear polynomials of degree n in the indeterminates $x_1, x_2, \ldots, x_n$.

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We are now ready to recall the notion of a T-space of the free algebra $K\langle X\rangle$.

Definition 1.1. Let V be a linear subspace of $K\langle X\rangle$.

(1) If V is invariant under every endomorphism of $K\langle X\rangle$ then V is called a T-space of $K\langle X\rangle$. In other words, V is closed under "evaluation". If V is also an ideal of $K\langle X\rangle$ then V is called a T-ideal.
(2) We shall write S^2 for the commutator T-space in $K\langle X\rangle$ that is generated by the commutator $[x_1, x_2] = x_1x_2 - x_2x_1$. Similarly, the Grassmann T-space is the T-space S^3 in $K\langle X\rangle$ generated by the commutator $[x_1, x_2, x_3]$. We always use the left-normed convention; that is, $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$.
(3) The Grassmann T-ideal in $K\langle X\rangle$ generated by S^3 will be denoted by T^3.
(4) For each $n \geq 1$, we let $S^2_n = S^2 \cap P_n$, $S^3_n = S^3 \cap P_n$, and $T^3_n = T^3 \cap P_n$. More generally, if V is any T-space then we shall denote $V_n = V \cap P_n$. Observe that each V_n is naturally a $K\mathcal{S}_n$-module.

While it is theoretically possible to obtain a linear basis for S^3 using only the results proved by the present authors in [1] (see Theorem 1.11 below for a summary), it would be virtually impossible to calculate explicitly and hence would not be convenient for applications. The purpose of the present paper is to construct a different, but explicit, linear basis that should prove to be more useful in this regard. For its construction, we shall rely heavily on our results in [1], as well as classical results by Latyshev in [4] and Specht in [6].

The remainder of this section is a collection of various results from the literature that will be required in the sequel.

Our first lemma is straightforward and well-known.

Lemma 1.2. *(Lemma 1.6 in [1]) Let $t, u, v, w, x, y, z \in K\langle X\rangle$. Then the following identities hold.*

(1) $[xy, z] = x[y, z] + [x, z]y$.
(2) $[uxv, w] = [xv, wu] - [xvw, u]$.
(3) $[u, v, w] = [uv, w] + [uw, v] - [u, wv]$.
(4) $[x, y][z, t] + [x, t][z, y] \equiv 0 \,(\operatorname{mod} T^3)$.

Definition 1.3. Let $n \geq 1$.

(1) Let $\{i_1, \ldots, i_m\} \subseteq J_n$ be any subset. A monomial $x_{i_1}x_{i_2}\cdots x_{i_m}$ will be called regular if $i_1 < \cdots < i_m$. A multilinear commutator $[x_{i_1}, x_{i_2}, \ldots, x_{i_m}]$ (of degree ≥ 2) is called regular if i_1 is minimal in the set $\{i_1, \ldots, i_m\}$.
(2) A multilinear product $CY \in P_n$, where $C = C_1 \cdots C_s$ ($s \geq 0$) is a product of regular commutators and Y is a regular monomial, is also called regular whenever

(a) the degrees of the C_i do not increase from left to right; and
(b) the indices of the initial indeterminates in the commutators C_i of the same length increase from left to right.

(3) A regular product CY on a subset $\{i_1, \ldots, i_m\} \subseteq J_n$ will be called a regular subproduct in P_n. For convenience, we also include the possibility that CY is trivial, that is, $CY = 1$.

The next lemma is attributed to Krakowski and Regev ([3]) and Olsson and Regev ([5]) in the monograph [2]. It was also, inadvertently, reproved as Lemma 3.3 in [1], as pointed out by the referee of the present paper.

Lemma 1.4. *(Lemma 4.1.8 in [2]) Let $n \geq 1$ and consider the set of all Specht basis elements of the form*

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] x_{j_1} \cdots x_{j_{n-2k}}, \tag{1.1}$$

where $i_1 < \cdots < i_{2k}$, $j_1 < \cdots < j_{n-2k}$, and $0 \leq 2k \leq n$. Their images form a basis of the quotient vector space P_n/T_n^3. Furthermore,

$$\dim_K P_n/T_n^3 = 2^{n-1}.$$

Definition 1.5. For each $n \geq 2$, let Γ_n be the $K\mathcal{S}_n$-submodule of P_n spanned by all multilinear products of commutators of length at least 2.

(1) The elements of Γ_n are called proper multilinear forms (see [6]).
(2) Let G_n be the $K\mathcal{S}_n$-submodule of Γ_n generated by all multilinear products of commutators that involve at least one commutator of degree at least 3.

Notice that the Specht basis elements of the form CY with Y empty lie in Γ_n. According to a theorem of Specht ([6]), these elements form a basis of Γ_n, which is called the Specht basis for Γ_n.

Let k be any positive integer and set

$$u = [x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}] \in \Gamma_{2k}(x_1, \ldots, x_{2k}).$$

Fix an order on the set of all Specht basis elements $u_1 < u_2 < \cdots$ of the form $u_t = \sigma_t(u)$, for some $\sigma_t \in \mathcal{S}_{2k}$, such that $u_1 = u$. For each $t > 1$, define

$$z_t = u_1 - \epsilon_t u_t,$$

where $\epsilon_t = \mathrm{sgn}(\sigma_t)$. Observe that since the elements u_t are Specht basis elements, the elements z_t ($t > 1$) are linearly independent.

We shall require the following theorem due to Latyshev.

Theorem 1.6. *(Theorem 1 in [4]) Let k be any positive integer.*

(1) *The set of Specht basis elements of degree $2k+1$ forms a linear basis of $\Gamma_{2k+1} \cap T^3_{2k+1}$.*
(2) *The elements z_t $(t > 1)$ of degree $2k$ together with the Specht basis elements lying in G_{2k} form a linear basis of $\Gamma_{2k} \cap T^3_{2k}$.*

Definition 1.7. For each integer $n \geq 1$, we define two (possibly empty) subsets $\mathcal{A}_n$ and $\mathcal{B}_n$ of P_n as follows.

(1) Let $\mathcal{A}_n$ be the set of all Specht basis elements of the form $C_1 \cdots C_s Y$ with $\deg C_1 \geq 3$.
(2) Let $\mathcal{B}_n$ be the set of elements of the form $z_t W$, where $z_t = z_t(x_{i_1}, \ldots, x_{i_{2k}})$ lies in $\Gamma_{2k}(x_{i_1}, \ldots, x_{i_{2k}})$, $W = x_{j_1} \cdots x_{j_{n-2k}}$, $i_1 < \cdots < i_{2k}$, $j_1 < \cdots < j_{n-2k}$, $\{i_1, \ldots, i_{2k}, j_1, \ldots, j_{n-2k}\} = J_n$, and $4 \leq 2k \leq n$.

Definition 1.8. For each $n \geq 2$, we define $\mathcal{C}_n$ to be set of all elements of the form

$$v = [x_1 \cdots x_s, x_{s+1} x_{i_{s+2}} \cdots x_{i_{n-2k}} [x_{j_1}, x_{j_2}] \cdots [x_{j_{2k-1}}, x_{j_{2k}}]],$$

such that $0 \leq 2k < n$, $1 \leq s < n - 2k$,

$$\{i_{s+2}, \ldots, i_{n-2k}\} \cup \{j_1, \ldots, j_{2k}\} = \{s+2, \ldots, n\},$$

$i_{s+2} < \cdots < i_{n-2k}$, and $j_1 < \cdots < j_{2k}$.

Lemma 1.9. *(Lemma 5.1 in [1]) The following statements hold for each $n \geq 2$.*

(1) *The set of all multilinear elements of the form*

$$CYx_nC'Y' \in P_n,$$

where CY and $C'Y'$ are possibly trivial regular products, forms a linear basis for P_n.
(2) *The set of all multilinear elements of the form*

$$[CYx_n, C'Y'] \in P_n$$

where CY and $C'Y'$ are regular products with only CY possibly trivial, forms a linear basis of S^2_n.
(3) *We have a vector space decomposition*

$$P_n = S^2_n \oplus P_{n-1}x_n.$$

Definition 1.10. Let n be a positive integer.

(1) We shall denote, for each $a \in P_n$, the unique elements $a^{(1)} \in S_n^2$ and $a^{(2)} \in P_{n-1}x_n$ such that $a = a^{(1)} + a^{(2)}$.
(2) For each $n \geq 3$, we define the subset $\mathcal{D}_n$ of $S_n^2 \cap T_n^3$ by

$$\mathcal{D}_n = \{d^{(1)} \mid d \in (\mathcal{A}_n \cup \mathcal{B}_n) \setminus (\mathcal{A}_{n-1}x_n \cup \mathcal{B}_{n-1}x_n)\}.$$

Theorem 1.11. *(Theorem 6.6 in [1]) Then the following statements hold for each $n \geq 3$.*

(1) *The disjoint union $\mathcal{A}_n \cup \mathcal{B}_n$ forms a linear basis of T_n^3, so that* $\dim_K T_n^3 = n! - 2^{n-1}$.
(2) *The disjoint union $\mathcal{A}_n \cup \mathcal{B}_n \cup \mathcal{C}_n$ forms a linear basis of $S_n^2 + T_n^3$, so that* $\dim_K(S_n^2 + T_n^3) = n! - 2^{n-2}$.
(3) *The set $\mathcal{D}_n$ forms a linear basis of $S_n^3 = S_n^2 \cap T_n^3$, so that* $\dim_K S_n^3 = \dim_K S_n^2 \cap T_n^3 = (n-1)!(n-1) - 2^{n-2}$.
(4) *We have the following decomposition of linear spaces:*

$$T_n^3 = S_n^3 \oplus T_{n-1}^3 x_n = S_n^3 \oplus S_{n-1}^3 x_n \oplus \cdots \oplus S_3^3 x_4 x_5 \cdots x_n.$$

Consequently, the set $\bigcup_{m=3}^{n} \mathcal{D}_m x_{m+1} x_{m+2} \cdots x_n$ forms another basis of T_n^3.

2. A basis for P_n

In this section, we find a basis for each vector space P_n.

Definition 2.1. Let $n \geq 2$.

(1) We shall write E_n for the subspace of P_n spanned by all multilinear products of the form $e_{u,v} = u[v, x_n]$, where u and v are monomials with u possibly trivial.
(2) We define $\mathcal{E}_n$ to be the set of all multilinear elements in P_n of the form:

$$e_{CY,DZ} = CY[DZ, x_n], \tag{2.1}$$

where (CY, DZ) is any ordered pair of regular subproducts in P_n with DZ nontrivial.

Proposition 2.2. *For each $n \geq 2$, $\mathcal{E}_n$ forms a linear basis for E_n. Consequently,* $\dim_K E_n = (n-1)!(n-1)$ *and P_n has a linear basis of the form*

$$\mathcal{E}_n \cup \{EUx_n : EU \text{ is any regular product in } P_{n-1}\}.$$

Proof. To prove (1), notice first that

$$CY[DZ, x_n] = [DZ, CYx_n] - [DZ, CY]x_n. \tag{2.2}$$

We claim $e_{CY,DZ} = e_{C'Y',D'Z'}$ implies $(CY, DZ) = (C'Y', D'Z')$. Indeed, observe that if $e_{CY,DZ} = e_{C'Y',D'Z'}$ then (2.2) yields

$$[DZ, CYx_n] - [D'Z', C'Y'x_n] \in S_n^2 \cap P_{n-1}x_n = 0$$

by part 3 of Lemma 1.9. Thus, we must have $(CY, DZ) = (C'Y', D'Z')$ by part 2 of Lemma 1.9, proving the claim. A similar argument proves that $\mathcal{E}_n$ is a linearly independent set. To finish the proof of the first statement, it suffices to notice that $\mathcal{E}_n$ spans E_n. To prove the dimension argument, it suffices to notice that the elements of the form $e_{CY,DZ}$ are in one-to-one correspondence with the elements of the form $[DZ, CYx_n]$, which form a basis for S_n^2 by Lemma 1.9. To prove the last statement, we observe that $E_n \cap P_{n-1}x_n = 0$. It remains to remark that the total number of elements of the form $CY[DZ, x_n]$ and EUx_n is

$$(n-1)!(n-1) + (n-1)! = n!. \qquad \square$$

3. A basis for $E_n \cap T_n^3$

In this section, we find a linear basis for the vector space $E_n \cap T_n^3$.

Definition 3.1. For each integer $n \geq 3$, we define two (possibly empty) subsets $\mathcal{F}_n$ and $\mathcal{H}_n$ of $\mathcal{E}_n$ as follows.

(1) Let $\mathcal{F}_n$ be the subset of elements in $\mathcal{E}_n$ of the form $e_{CY,DZ}$, where $C = C_1 \cdots C_r$ and $D = D_1 \cdots D_s$ are such that $\deg C_1 \geq 3$, $\deg D_1 \geq 3$, or $\deg D_1 = 2$ and $Z = 1$.

(2) Let k be any integer such that $n \geq 2k \geq 2$, and consider any partition

$$\{i_1, \ldots, i_{2k}\} \cup \{j_1, \ldots j_{n-2k}\} = J_n$$

such that $i_1 < \cdots < i_{2k} = n$ and $j_1 < \cdots < j_{n-2k}$. Put $W = x_{j_1} \cdots x_{j_{n-2k}}$. We shall write $\mathcal{H}_n$ for the subset of $\mathcal{E}_n$ of elements of the form $h_W = e_{CW,x_{i_{2k-1}}}$, where $C = [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-3}}, x_{i_{2k-2}}]$ (for all possible W).

Clearly, $\mathcal{F}_n \cap \mathcal{H}_n = \varnothing$ and by identity (1) of Lemma 1.2, $\mathcal{F}_n \subseteq E_n \cap T_n^3$.

Definition 3.2. Let $n \geq 3$ and let $e_{CY,DZ} \in \mathcal{E}_n \setminus (\mathcal{F}_n \cup \mathcal{H}_n)$. Then C has the form

$$C = [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-3}}, x_{i_{2k-2}}]$$

for some $k \geq 1$, and $DZ = Dx_{l_1} \cdots x_{l_r}$ with $\deg D_1 \leq 2$, say.

(1) If $\deg DZ \geq 2$ then we define

$$g_{CY,DZ} = e_{CY,DZ} - CYD\sum_{s=1}^{r} x_{l_1}\cdots\hat{x}_{l_s}\cdots x_{l_r}[x_{l_s}, x_n]. \tag{3.1}$$

(2) If $\deg DZ < 2$ then $DZ = x_{l_1}$; in other words,

$$e_{CY,Z} = e_{CY,x_{l_1}} = [x_{i_1}, x_{i_2}]\cdots[x_{i_{2k-3}}, x_{i_{2k-2}}]Y[x_{l_1}, x_n].$$

Observe $k \geq 2$ since $e_{CY,x_{l_1}} \notin \mathcal{H}_n$. In this case, we define

$$g_{CY,x_{l_1}} = e_{CY,x_{l_1}} - \epsilon_{CY,x_{l_1}} h_Y, \tag{3.2}$$

where $\epsilon_{CY,x_{l_1}}$ is the sign of the permutation σ on $\{i_1 \ldots, i_{2k-2}, l_1\}$ such that $\sigma(i_1) < \cdots < \sigma(i_{2k-2}) < \sigma(l_1)$.

(3) Define $\mathcal{G}_n^1$ to be the set of all elements of type (3.1), $\mathcal{G}_n^2$ to be the set of all elements of type (3.2), and $\mathcal{G}_n = \mathcal{G}_n^1 \cup \mathcal{G}_n^2$.

Lemma 3.3. *Let $n \geq 3$. Then the following statements hold.*

(1) *The sets $\mathcal{F}_n, \mathcal{G}_n$, and $\mathcal{H}_n$ are mutually disjoint subsets of E_n.*

(2) *The set $\mathcal{F}_n \cup \mathcal{G}_n$ forms a linear basis of $E_n \cap T_n^3$ and*

$$\dim_K(E_n \cap T_n^3) = (n-1)!(n-1) - 2^{n-2}.$$

(3) *The set $\mathcal{H}_n + E_n \cap T_n^3$ forms a linear basis of $E_n/E_n \cap T_n^3$; thus,*

$$\dim_K E_n/E_n \cap T_n^3 = 2^{n-2}.$$

(4) *The set $\mathcal{F}_n \cup \mathcal{G}_n \cup \mathcal{H}_n$ forms a linear basis of E_n.*

Proof. First we prove $\mathcal{G}_n \subseteq E_n \cap T_n^3$. Suppose element $g_{CY,DZ} \in \mathcal{G}_n^1$. In this case, modulo T^3, we have

$$\begin{aligned} e_{CY,DZ} &\equiv CY[Dx_{l_1}\cdots x_{l_r}, x_n] \\ &\equiv CYD[x_{l_1}\cdots x_{l_r}, x_n] + CY[D, x_n]x_{l_1}\cdots x_{l_r} \\ &\equiv CYD\sum_{s=1}^{r} x_{l_1}\cdots\hat{x}_{l_s}\cdots x_{l_r}[x_{l_s}, x_n] \end{aligned}$$

by induction on the identity $[xy, z] = x[y, z] + [x, z]y$, as required. Now suppose element $g_{CY,x_{l_1}} \in \mathcal{G}_n^2$. Then, modulo T^3, we have

$$
\begin{aligned}
g_{CY,x_{l_1}} &\equiv e_{CY,x_{l_1}} - \epsilon_{CY,x_{l_1}} h_Y \\
&\equiv C[x_{l_1}, x_n]Y - \epsilon_{CY,x_{l_1}} [x_{\sigma(i_1)}, x_{\sigma(i_2)}] \cdots [x_{\sigma(i_{2k-3})}, x_{\sigma(i_{2k-2})}][x_{\sigma(l_1)}, x_n]Y \\
&\equiv 0,
\end{aligned}
$$

by Theorem 1.6, as required.

Next we claim that each element in $\mathcal{G}_n$ is nonzero. Indeed, suppose $g_{CY,DZ} \in \mathcal{G}_n^1$:

$$g_{CY,DZ} = CY[DZ, x_n] - CYD \sum_{s=1}^{r} x_{l_1} \cdots \hat{x}_{l_s} \cdots x_{l_r} [x_{l_s}, x_n].$$

Then using Proposition 2.2 and Specht's theorem, it follows that the element $e_{CY,DZ} = CY[DZ, x_n]$ is linearly independent to

$$CYD \sum_{s=1}^{r} x_{l_1} \cdots \hat{x}_{l_s} \cdots x_{l_r} [x_{l_s}, x_n]$$

since $\deg DZ \geq 2$. Thus, $g_{CY,DZ} \neq 0$, as claimed. If $g_{CY,x_{l_1}} \in \mathcal{G}_n^2$ then

$$g_{CY,x_{l_1}} = e_{CY,x_{l_1}} - \epsilon_{CY,x_{l_1}} h_Y \neq 0$$

since $e_{CY,x_{l_1}}$ and h_Y are distinct elements in the basis $\mathcal{E}_n$.

Similar arguments show that each element $g_{CY,DZ}$ in $\mathcal{G}_n$ are uniquely determined by the choice of (CY, DZ); consequently,

$$\dim_K E_n = |\mathcal{E}_n| = |\mathcal{F}_n| + |\mathcal{G}_n| + |\mathcal{H}_n|.$$

Furthermore, a similar degree argument to that made in the previous paragraph implies that $\mathcal{F}_n, \mathcal{G}_n$, and $\mathcal{H}_n$ are mutually disjoint and that their union $\mathcal{F}_n \cup \mathcal{G}_n \cup H_n$ is linearly independent set. This proves (1) and (4).

It follows from Lemma 1.4 that the elements $h_W + E_n \cap T_n^3$, $h_W \in \mathcal{H}_n$, are uniquely determined by the choice of W and are linearly independent. The fact that $h_W + E_n \cap T_n^3$ spans $E_n / E_n \cap T_n^3$ follows from the identity $[x, y][z, t] + [x, t][z, y] \equiv 0 \pmod{T^3}$ and the argument in the first paragraph. To complete the proof of (3), observe that $\dim_K(E_n / E_n \cap T_n^3) = 2^{n-2}$ since $|\mathcal{H}_n|$ concides with the number of partitions of the form

$$J_n = \{i_1, \ldots, i_{2k}\} \cup \{j_1, \ldots, j_{n-2k}\}$$

with $k \geq 1$ and $i_{2k} = n$.

Finally, notice that (2) follows from (1), (3), (4), and the fact that $\dim_K E_n = (n-1)!(n-1)$ as proved in Proposition 2.2. $\square$

4. A basis for S_n^3

We are now ready to describe an explicit linear basis for S_n^3.

Definition 4.1. For each integer $n \geq 3$, we define (possibly empty) subsets $\bar{\mathcal{F}}_n$, $\bar{\mathcal{G}}_n^1$, and $\bar{\mathcal{G}}_n^2$ of S_n^2 as follows.

(1) Let $\bar{\mathcal{F}}_n$ be the set of all elements of the form $\bar{e}_{CY,DZ} = [DZ, CYx_n]$, $e_{CY,DZ} \in \mathcal{F}_n$.
(2) Let $\bar{\mathcal{G}}_n^1$ be the set of all elements of the form

$$\bar{g}_{CY,DZ} = \bar{e}_{CY,DZ} - \sum_{s=1}^{r} [x_{l_s}, CYDx_{l_1} \cdots \hat{x}_{l_s} \cdots x_{l_r} x_n], \; g_{CY,DZ} \in \mathcal{G}_n^1;$$

let $\bar{\mathcal{G}}_n^2$ be the set of all elements of the form

$$\begin{aligned} &\bar{g}_{CY,x_{l_1}} = \bar{e}_{CY,x_{l_1}} - \epsilon_{CY,x_{l_1}} [x_{\sigma(l_1)}, [x_{\sigma(i_1)}, x_{\sigma(i_2)}] \cdots [x_{\sigma(i_{2k-3})}, x_{\sigma(i_{2k-2})}] Y x_n], \\ &g_{CY,x_{l_1}} \in \mathcal{G}_n^2; \end{aligned}$$

and, let $\bar{\mathcal{G}}_n = \bar{\mathcal{G}}_n^1 \cup \bar{\mathcal{G}}_n^2$.

Theorem 4.2. *For each $n \geq 3$, the sets $\bar{\mathcal{F}}_n$ and $\bar{\mathcal{G}}_n$ are disjoint and $\bar{\mathcal{F}}_n \cup \bar{\mathcal{G}}_n$ forms a linear basis for S_n^3.*

Proof. Obviously $\bar{\mathcal{F}}_n \cup \bar{\mathcal{G}}_n \subseteq S_n^2$. We claim that, in fact, $\bar{\mathcal{F}}_n \cup \bar{\mathcal{G}}_n \subseteq S_n^3$. Indeed, from the identity $[a, bc] = [a, b]c + b[a, c]$, we obtain

$$\begin{aligned} \bar{e}_{CY,DZ} &= e_{CY,DZ} + (\bar{e}_{CY,DZ}|_{x_n \to 1}) x_n, \\ \bar{g}_{CY,DZ} &= g_{CY,DZ} + (\bar{g}_{CY,DZ}|_{x_n \to 1}) x_n. \end{aligned}$$

But $e_{CY,DZ}$, $g_{CY,DZ} \in T_n^3$ by part (2) of Lemma 3.3; hence, $\bar{e}_{CY,DZ}$, $\bar{g}_{CY,DZ} \in S_n^3$ by Theorem 1.11. Since $\mathcal{F}_n \cap \mathcal{G}_n = \varnothing$ by part (1) of Lemma 3.3, the equations above also imply that $\bar{\mathcal{F}}_n \cap \bar{\mathcal{G}}_n = \varnothing$, and so

$$\begin{aligned} |\bar{\mathcal{F}}_n \cup \bar{\mathcal{G}}_n| = |\mathcal{F}_n \cup \mathcal{G}_n| &= (n-1)!(n-1) - 2^{n-2} \\ &= \dim_K S_n^3, \end{aligned}$$

by part (2) of Lemma 3.3 and Theorem 1.11. The linear independence of $\bar{\mathcal{F}}_n \cup \bar{\mathcal{G}}_n$ follows similarly from the linear independence of $\mathcal{F}_n \cup \mathcal{G}_n$ proved in Lemma 3.3. $\square$

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