Commun Nonlinear Sci Numer Simulat 44 (2017) 506–512

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Line graphs for fractals

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ARTICLE INFO

Article history:
Received 15 July 2016
Accepted 14 September 2016
Available online 20 September 2016

Keywords:
Fractals
Lindenmayes system
Classes of nodes

ABSTRACT

We found the Lindenmayer systems for line graphs built on selected fractals. We show that the fractal dimension of such obtained graphs in all analysed cases is the same as for their original graphs. Both for the original graphs and for their line graphs we identified classes of nodes which reflect symmetry of the graph.

1. Introduction

Fractals are very interesting and ubiquitous objects. They are very common in nature, from blood vessels [1], through plants [2], coastlines [3] and lightnings [4] to the structure of the Universe [5]. But, as it turns out, fractals are also observed in a street network [6] or literary works [7].

One of methods which allow for a fractal construction is the Lindenmayer system [8]. It defines rules of transformation of a fractal structure from a given generation to the subsequent one. Some rules are presented in [8–10], from among which we chose three examples of three different fractal categories: Cyclical, linear and higher orders. The last category means that when we present fractal as a graph there are nodes with degree k higher than 2. For each obtained graph we then construct its line graph [11]. Nodes in a line graph constructed for a given graph replace its edges, and two nodes in a new graph are connected if they have a common node in the original graph. The question we ask here is if the obtained line graphs show fractal properties, as do graphs on which they are constructed. In this case we should be able to find the Lindenmayer system which allows for the line graph construction. We are also interested in checking the fractal dimension of the obtained line graphs.

In our earlier paper [12] some symmetric fractals were analysed in terms of the number of classes of nodes. The concept of classes was proposed by us in [13] to express symmetry of the network, as class is formed by a set of nodes with the same structure of connection with other nodes in the network. Now, the method is applied for graphs and their line graphs constructed on fractals to compare their symmetry.

The paper is organised as follows: In the next section we present line graphs obtained for some fractals and the Lindenmayer systems found for them. The following section is devoted to the class identification method applied for analysed graph and line graphs. The last section concludes the obtained results.

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http://dx.doi.org/10.1016/j.cnsns.2016.09.007

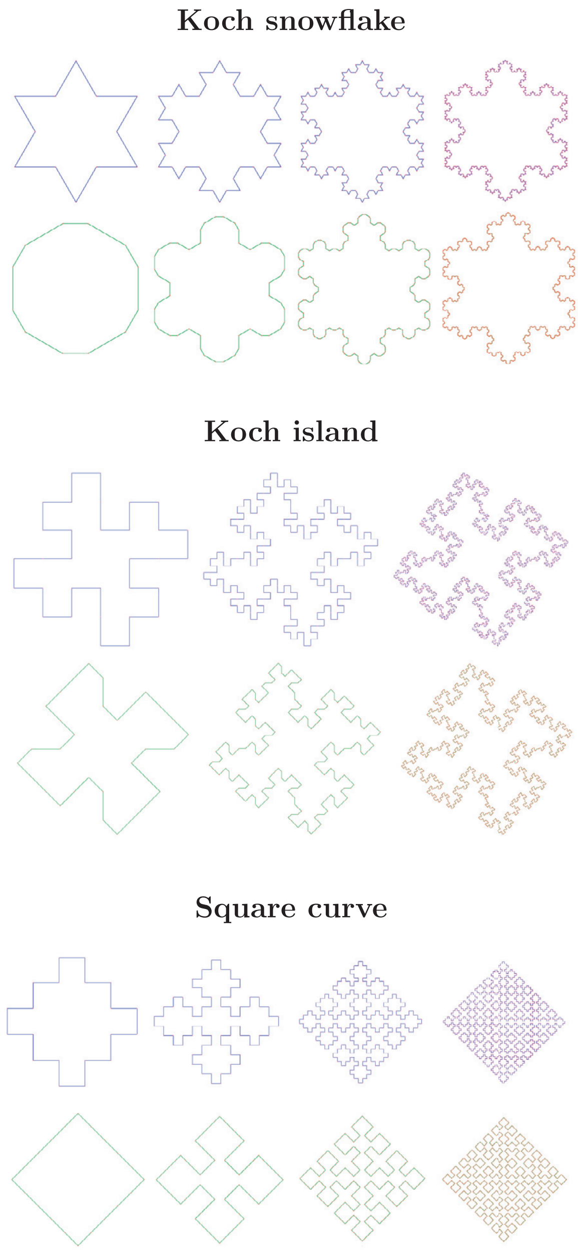

Fig. 1. Graphs (top) and it's corresponding line graphs (bottom) for cyclic fractals.

2. Line graphs and lindenmayer systems

As it was mentioned in the Introduction we focused on three types of fractals: Cyclic, linear and higher orders. From each category, three examples are chosen: Koch snowflake [9], Koch island [8], and Square curve [10] which are cyclical; Koch curve [10], Minkowski sausage [9], and Sierpinski arrowhead curve [9] which are linear, and for higher orders - Sierpinski triangle [10], Koch anti-snowflake [9], and Modified Koch curve [10].

Constructing line graphs, positions of nodes in line graphs are taken at the geometrical center of edges in original graphs. Original graphs and their line graphs analysed in the paper are presented in Figs. 1–3. While in original graphs lengths of all edges are equal, in their line graphs their are not. For all analysed here fractals, an angle between subsequent edges in the line graph is equal to half of the angle in the original graph. For such constructed line graphs we found their Lindenmayer systems. The rules obtained by us for line graphs, together with the rules for original fractals [8–10] are presented in Tables 1–3. Besides Lindenmayer rules for all analysed graphs we also calculated fractal dimension, which is defined

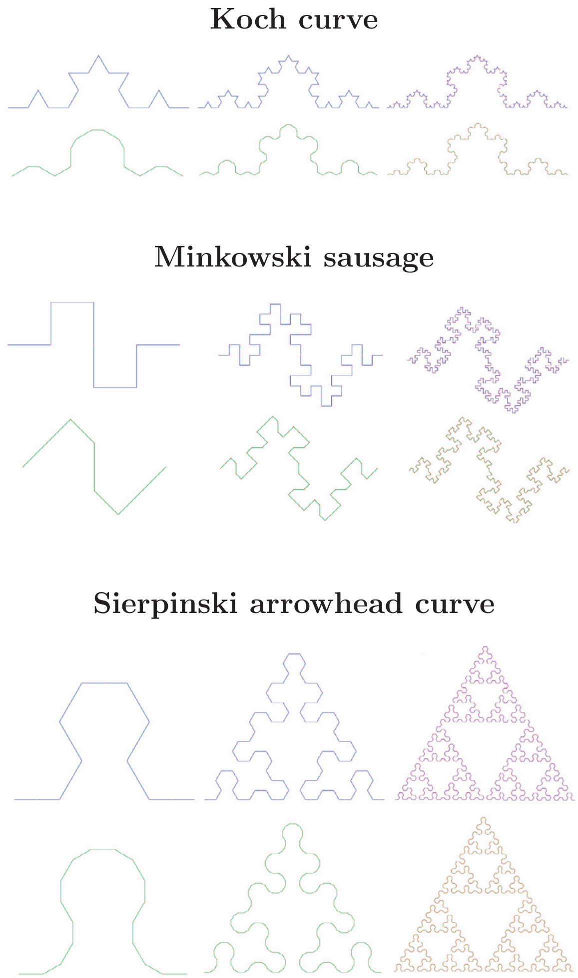

Fig. 2. Graphs (top) and it's corresponding line graphs (bottom) for linear fractals.

as [14]:

$$D_S = \lim_{n\to\infty} \frac{\ln(\frac{M_n}{M_{n-1}})}{\ln(\frac{1}{s})}, \tag{1}$$

where: M_n - weight (expressed as the number of graph nodes) in nth generation, s - scale.

The results are presented in Table 4. For all analysed cases the fractal dimension for the original graph and the one of its line graph are the same.

3. Classes analysis

The concept of classes of nodes has been proposed by us in [13]. The idea is based on a fact that a graph symmetry allows us to group nodes which have the same structure of connections with other elements of this graph. It means that nodes belong to the same class if they have the same number of neighbours which belong to the same classes [12,15–19]. The first step in the procedure of specifying classes is to determine the degrees of all nodes of the network. Then one should check whether the neighbours of all nodes that have the same degree have also consistent degrees. If it is the case, the classes identification procedure is finished, otherwise one should introduce distinction of the nodes with the same degrees but different types of neighbours. This procedure is repeated until all nodes belonging to a given class have neighbours belonging to the same classes.

The procedure of classes identification is now applied for graphs and line graphs presented in paragraph 2. In the case of linear and cyclic fractals their line graphs are very similar, so the classes structure will be the same. In the case of cyclic graphs (line graphs) each node has degree $k = 2$, so all nodes belong to the same class. In the case of linear graphs (line graphs) nodes at the ends of the chain have degree $k = 1$, and all remaining nodes have degree $k = 2$. The number of classes

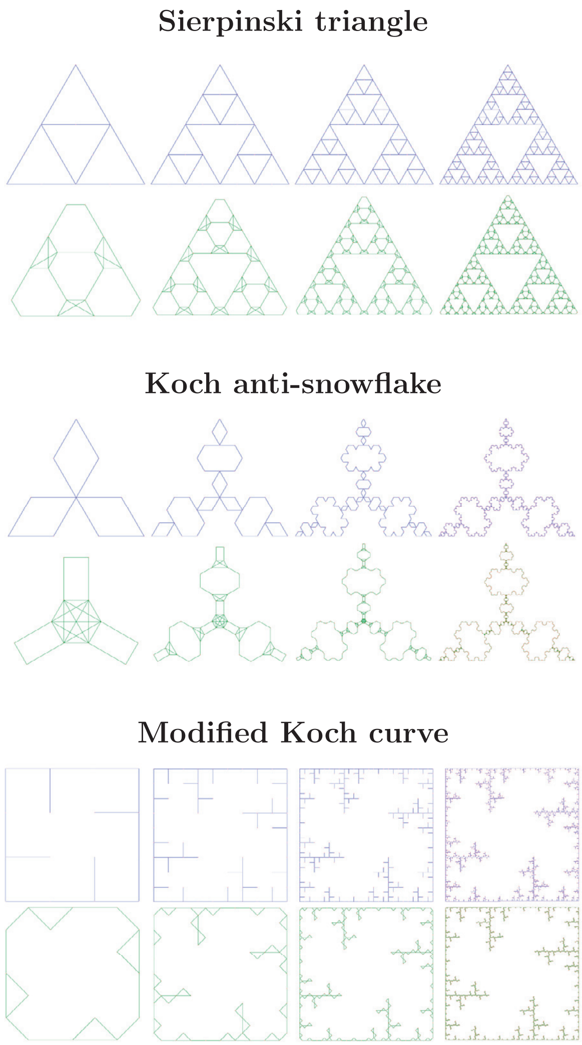

Fig. 3. Graphs (top) and it's corresponding line graphs (bottom) for higher orders fractals.

for those graphs is equal to $N/2$, as their positions are symmetric in respect of the chain center. For those two families of fractals the symmetry of the original graphs and their line graphs is preserved.

In the case of higher orders fractals the situation is more complex. The results obtained for the *Koch anti-snowflake* indicate that the number of classes for graph and line graph differ by one. The number of classes C changes with the generation n as $C_{n+1} = 4 * C_n - 3$ for the original graph and as $C_{n+1} = 4 * C_n$ for its line graph.

In the case of the *modified Koch curve* also the number of classes of the original graph and its line graph is the same, and it increases exponentially with the generation number ($C = 0.17\exp(1.48n)$). In this case however, we were not able to find simple relation between number of classes in the subsequent generations.

For the last analysed fractal, i.e. the *Sierpinski triangle*, the number of classes for the original graph and the line graph constructed on this graph is different. The number of classes for the original graph changes as: $C_{n+1} = 3(C_n - 1) - A_n$, where $A_{n+1} = A_n + n - 2$. This relation is true from $n = 3$ with $C_2 = 4$ and $A_2 = 2$. The number of classes in the line graph for the *Sierpinski triangle* changes as $C_{n+1} = 3(C_n - 1)$. Interestingly, the number of classes for the line graph is equal to the number of nodes in the original graph two generations earlier. In Fig. 4 graphs of classes for the original and the line graph for $n = 3$ are presented.

4. Conclusions

In the paper we have shown that line graphs constructed on fractal graphs demonstrate fractal properties. We found the related Lindenmayer systems and calculated the fractal dimension for a set of line graphs, which for all analysed examples is equal to the fractal dimension of original graphs.

Table 1
Lindenmayer systems for graphs [8–10] and line graphs (found by us) for circular fractals, where: d - length of edge and δ - angle between subsequent edges. In *turtle graphics* a letter in the Lindenmayer system is interpreted as a forward movement of a given length, and + $(-)$ means turn left(right) of a specified angle.

	initialiser	generator			d	δ
Koch snowflake						
graph	$F--F--F$	F	$\rightarrow$	$F+F--F+F$	1	60°
line graph	$+F-G-F-G$	F	$\rightarrow$	$F-G-F++F$	$\sqrt{3}/2$	30°
		G	$\rightarrow$	$+++F-G-F-G$	$1/2$	
Koch island						
graph	$F+F+F+F$	F	$\rightarrow$	$F+F-F-FF+F+F-F$	1	90°
line graph	$+G++G++G++G$	F	$\rightarrow$	$+GH--H-F+G++GH-F$	1	45°
		G	$\rightarrow$	$GH--H-F+G++GHG$	$\sqrt{2}/2$	
		H	$\rightarrow$	$++GH--H-F+G++GH--H$	$\sqrt{2}/2$	
Square curve						
graph	$F+XF+F+XF$	F	$\rightarrow$	F	1	90°
		X	$\rightarrow$	$XF-F+F-XF+F+XF-F+F-X$	0	
line graph	$+F++F++F++F$	F	$\rightarrow$	$FLGRF$	1	45°
		G	$\rightarrow$	LGR	$\sqrt{2}/2$	
		L	$\rightarrow$	$++FLGR--$	$\sqrt{2}/2$	
		R	$\rightarrow$	$--LGRF++$	$\sqrt{2}/2$	

Table 2
Lindenmayer systems for graphs [9,10] and line graphs (found by us) for linear fractals, where: d - length of edge and δ - angle between subsequent edges. In *turtle graphics* a letter in the Lindenmayer system is interpreted as a forward movement of a given length, and + $(-)$ means turn left(right) of a specified angle.

	initialiser	generator			d	δ
Koch curve						
graph	F	F	$\rightarrow$	$F+F--F+F$	1	60°
line graph	F	F	$\rightarrow$	$F-G-F++F$	$\sqrt{3}/2$	30°
		G	$\rightarrow$	$+++F-G-F-G$	$1/2$	
Minkowski sausage						
graph	F	F	$\rightarrow$	$F+F-F-FF+F+F-F$	1	90°
line graph	G	F	$\rightarrow$	$+GH--H-F+G++GH-F$	1	45°
		G	$\rightarrow$	$GH--H-F+G++GHG$	$\sqrt{2}/2$	
		H	$\rightarrow$	$++GH--H-F+G++GH--H$	$\sqrt{2}/2$	
Sierpinski arrowhead curve						
graph	HF	F	$\rightarrow$	F	1	60°
		G	$\rightarrow$	$HF+GF+H$		
		H	$\rightarrow$	$GF-HF-G$		
line graph	F	F	$\rightarrow$	$H-G-F-G-H$	1	30°
		G	$\rightarrow$	G	$\sqrt{3}/2$	
		H	$\rightarrow$	$F+G+H+G+F$	1	

As the structure of cyclic and linear graphs and their line graphs is very similar, the related number of classes is also the same. This is no more true for higher order fractals. Also in this case pattern of a change of the number of classes with the graph size is not trivial. For the fractals analysed in this paper the number of classes is much lower than the number of nodes; this means that the ratio of the reduction of the system size when described by classes is high. This is due to the high symmetry of the considered examples.

Table 3
Lindenmayer systems for graphs [8–10] and line graphs (found by us) for higher orders fractals, where: d - length of edge and δ - angle between subsequent edges. In *turtle graphics* a letter in the Lindenmayer system is interpreted as a forward movement of a given length, and + $(-)$ means turn left(right) of a specified angle.

	initialiser	generator			d	δ
Sierpinski triangle						
graph	$FXF++FF++FF$	F	$\rightarrow$	FF	1	60°
		X	$\rightarrow$	$++FXF--FXF--FXF++$	0	
line graph	$A++H++B++H++B++H$	F	$\rightarrow$	F	1	30°
		G	$\rightarrow$	G	$\sqrt{3}/2$	
		H	$\rightarrow$	H	$1/2$	
		A	$\rightarrow$	$BF+++++G+++H+++++G$ $+++++H--A-G-----H--$ $H--H-----G-A-G-----H$ $--H--H-----G-A++H++B$	1	
		B	$\rightarrow$	BFB	1	
Koch anti-snowflake						
graph	$F++F++F$	F	$\rightarrow$	$F+F--F+F$	1	60°
line graph	$X------H------X---$ $---H------X$	F	$\rightarrow$	F	1	30°
		G	$\rightarrow$	G	$\sqrt{3}/2$	
		H	$\rightarrow$	H	$1/2$	
		A	$\rightarrow$	$A+H+H--B--H+H--H---$ $--G++++++F++++++G--A$	$\sqrt{3}/2$	
		B	$\rightarrow$	$+++H+H--B--H+H+H--B$ $--H+H+H--B--H+H+++$	$1/2$	
		X	$\rightarrow$	$H--H-----G++++++F+++$ $++G--A---H---A+H$	0	
Modified Koch curve						
graph	$F+F+F+F$	F	$\rightarrow$	$FF+F++F+F$	1	90°
line graph	$F+G-X-G++G+F+G-X-G+$ $+G+F+G-X-G++G+F+G-X-$ $G++G+AAY+G+AAY+G+AAY+$ $G+AAY$	F	$\rightarrow$	F	1	45°
		G	$\rightarrow$	G	$\sqrt{2}/2$	
		A	$\rightarrow$	AAA	1	
		X	$\rightarrow$	$X-G+FF+G-X-G++G+F+G-X$ $-G---F+G-X-G++G+F+G-X$	0	
		Y	$\rightarrow$	AAY	0	

Table 4
Masses of analysed fractals in $n+1$st generation (where: M_n - mass in nth generation) and fractal dimension D_s. For cases marked by $^{(*)}$ the mass in the next generation is expressed by masses in two previous generations.

fractal	graph: M_{n+1}	line graph: M_{n+1}	D_s
Koch snowflake	$4M_n$	$4M_n$	1.26
Koch island	$8M_n$	$8M_n$	1.50
Square curve	$4(M_n+1)$	$4(M_n+1)$	1.26
Koch curve	$4M_n$	$4M_n$	1.26
Minkowski sausage	$8M_n$	$8M_n$	1.50
Sierpinski arrowhead curve	$3M_n-2$	$3M_n$	1.00
Sierpinski triangle	$3(M_n-1)$	$3M_n$	1.58
Koch anti-snowflake	$4+3(4^n-2^n)^{(*)}$	$4M_n$	1.26
Modified Koch curve	$2(5^n+3^n)^{(*)}$	$2(5^n+3^n)^{(*)}$	1.46

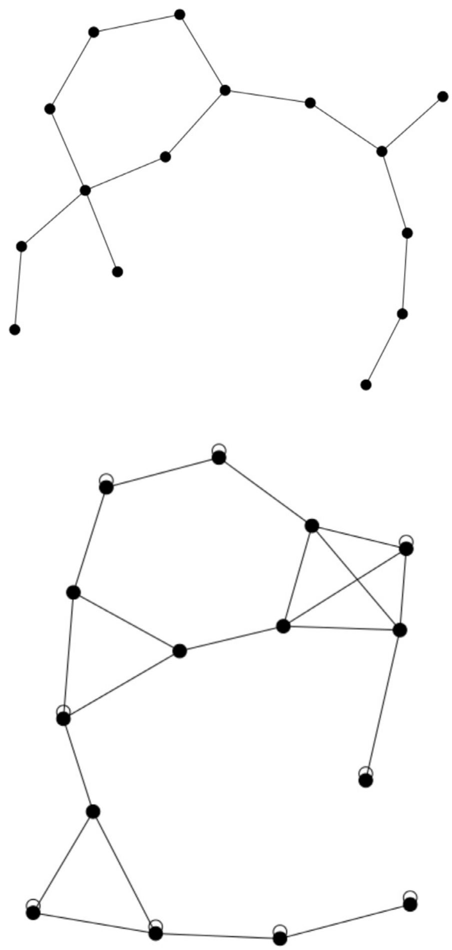

Fig. 4. Graphs of classes for the Sierpinski triangle (top) and it's corresponding line graph (bottom) for $n = 3$.

Acknowledgments

The authors are grateful to Krzysztof Kułakowski for critical reading of the manuscript and helpful discussions.

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