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A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-differential equations with weakly singular kernels

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ABSTRACT

A spectral method based on operational matrices of the second kind Chebyshev polynomials (SKCPs) is employed for solving fractional integro-differential equations with weakly singular kernels. Firstly, properties of shifted SKCPs, operational matrix of fractional integration and product operational matrix are introduced and then utilized to reduce the given equation to the solution of a system of linear algebraic equations. This new approach provides a significant computational advantage by converting the given original problem to an equivalent linear Volterra integral equation of the second kind with the same initial conditions. Approximate solution is achieved by expanding the functions in terms of SKCPs and employing operational matrices. Unknown coefficients are determined by solving final system of linear equations. An estimation of the error is given. Finally, illustrative examples are included to demonstrate the high precision, fast computation and good performance of the new scheme.

1. Introduction

Fractional order dynamics arise from many real-world physical problems such as dynamics of interfaces between nanoparticles and substrates [1], viscoelasticity [2,3], bioengineering [4] and electromagnetic waves [5]. Hence, a considerable attention has been given to the exact and the numerical solutions of such equations. Among these methods, different numerical techniques have been proposed for solving fractional integro-differential equations. For example, authors of [6] have considered fractional differential transform method to solve linear and nonlinear fractional integro-differential equations of Volterra type. Second kind Chebyshev wavelet method has been proposed to solve nonlinear fractional Fredholm integro-differential equations and nonlinear fractional Volterra integro-differential equations in [7,8], respectively. In [9], the authors have used a collocation method to obtain approximate solution for nonlinear fractional integro-differential equations.

Fractional integro-differential equations with a weakly singular kernel are used in modeling different physical problems. These equations appear in the radiative equilibrium [10], heat conduction problem [11], elasticity and fracture mechanics [12], etc.

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In this paper, we consider the following fractional integro-differential equation with weakly singular kernel

$$ {}_0^C D_t^{\alpha} y(t) = g(t) + p(t)y(t) + \int_0^t (t-s)^{-\beta} y(s)ds, \quad \alpha > 0,\ 0 \le \beta < 1,\ t \in I(T), \tag{1} $$

$$ y^{(i)}(0) = y_0^{(i)}, \quad i = 0, 1, \ldots, n-1, \tag{2} $$

where $y(t)$ is the unknown function, $g(t)$ and $p(t)$ are known continuous functions on $I(T) := [0, T]$, $y_0^{(i)}$ $(i = 0, 1, \ldots, n-1)$ are given real numbers, $n = \lceil \alpha \rceil$ is the ceiling function of α and ${}_0^C D_t^{\alpha}$ is the Caputo fractional differential operator of order α.

The existence and uniqueness of solution for differential equations with fractional order have been studied in [13,14]. The local and global existence and uniqueness results for the solution of fractional integro-differential equations have been obtained in [15,16], respectively.

Zhao et al. have applied collocation methods for problem (1)–(2) and analyzed the convergence of these methods [17]. In this work, we present a numerical solution for problem (1)–(2) in terms of the shifted SKCPs series. Among all of the well-known set of polynomials such as Legendre polynomials, first kind Chebyshev polynomials, and Taylor polynomials, the motivation of choosing SKCPs is that the product operational matrix of the SKCPs basis (as explained in Section 3) has a special form, such that the elements of this matrix are given by summation of the elements of the corresponding considered vector without any multiplication factor. This leads to decrease of the number of required computations and therefore the suggested algorithm gives a fast result. On the other hand, the computations can be handled in a simple way making use of the operational matrix technique. In Section 2, we give some preliminaries in fractional calculus. Section 3 is devoted to introducing the operational matrices of SKCPs basis. In Section 4, we suggest a numerical method to solve problem (1)–(2) using the results obtained in [17]. An estimation of the error is given in Section 5. Numerical examples are given in Section 6 to illustrate the applicability and accuracy of our method. Finally, concluding remarks are given in Section 7.

2. Preliminaries of the fractional calculus

In this section, we give some necessary definitions and mathematical preliminaries on fractional calculus which will be used further in this paper.

Definition 2.1. The Riemann–Liouville fractional integral operator I_t^{α} of order α is given by

$$ I_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, & \alpha > 0, \\ y(t), & \alpha = 0, \end{cases} \tag{3} $$

where

$$ \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt. $$

Definition 2.2. The Caputo definition of fractional differential operator ${}_0^C D_t^{\alpha}$ of order α is given by

$$ {}_0^C D_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} y(\tau) d\tau, & n-1 < \alpha < n, \\ y^{(n)}(t), & \alpha = n. \end{cases} $$

Some properties of the Riemann–Liouville fractional integral operator I_t^{α} and the Caputo fractional differential operator ${}_0^C D_t^{\alpha}$ are as follows:

$$ I_t^{\alpha} t^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1+\alpha)} t^{\upsilon+\alpha}, \quad \alpha \ge 0,\ \upsilon > -1, \tag{4} $$

$$ {}_0^C D_t^{\alpha} I_t^{\alpha} y(t) = y(t), $$

$$ I_t^{\alpha}\, {}_0^C D_t^{\alpha} y(t) = y(t) - \sum_{i=0}^{n-1} y^{(i)}(0) \frac{t^i}{i!}, \quad n-1 < \alpha \le n,\ t > 0. $$

3. Properties of shifted SKCPs

3.1. Definition and function approximation

Definition 3.1. Shifted SKCP of order i is defined on $I(T)$ as

$$ \psi_i(t) = U_i\left(\frac{2}{T} t - 1\right), \quad i = 0, 1, 2, \ldots, $$

where $U_i(t)$ is the well-known SKCP of order i. We note that the SKCPs are orthogonal functions on the interval $[-1, 1]$ and can be determined with the aid of the following recursive formula:

$$U_i(t) = 2tU_{i-1}(t) - U_{i-2}(t), \quad i \geq 2,$$

with $U_0(t) = 1$ and $U_1(t) = 2t$.

Some properties of the shifted SKCPs are as follows:

$$\psi_i(t) = \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k+1)!2^{2k}}{(i-k)!(2k+1)!T^k} t^k. \tag{5}$$

$$\psi_i(t)\psi_j(t) = \sum_{k=0}^{\frac{i+j-|i-j|}{2}} \psi_{i+j-2k}(t). \tag{6}$$

$$\int_0^T w(t)\psi_i(t)\psi_j(t)dt = \frac{T\pi}{4}\delta_{ij},$$

where $w(t) = \sqrt{1 - (\frac{2}{T}t - 1)^2}$.

A function $y(t)$, square integrable on $I(T)$, may be expanded in terms of the shifted SKCPs as follows:

$$y(t) = \sum_{i=0}^{\infty} a_i \psi_i(t), \tag{7}$$

where the coefficient a_i is given by

$$a_i = \frac{4}{T\pi} \int_0^T w(t)y(t)\psi_i(t)dt, \quad i = 0, 1, 2, \ldots. \tag{8}$$

Hence, $y(t)$ can be approximated by truncating the infinite series in (7) as

$$y(t) \simeq \sum_{i=0}^{N} a_i \psi_i(t) = A^T \psi(t) = \psi^T(t)A,$$

where

$$A = [a_0, a_1, \ldots, a_N]^T, \tag{9}$$

$$\psi(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_N(t)]^T. \tag{10}$$

The interested readers can refer to [18, Chapter 5], to see the truncation error.

3.2. Product operational matrix

The following property of the product of two vectors $\psi(t)$ and $\psi^T(t)$ will be used,

$$\psi(t)\psi^T(t)A \simeq \tilde{A}\psi(t), \tag{11}$$

where A is defined by (9) and $\tilde{A}$ is the product operational matrix of dimension $(N + 1) \times (N + 1)$. In order to obtain the elements of matrix $\tilde{A}$, we use Eq. (6) and get

$$\psi(t)\psi^T(t)A = \begin{bmatrix} \sum_{j=0}^{N} a_j\psi_j(t)\psi_0(t) \\ \sum_{j=0}^{N} a_j\psi_j(t)\psi_1(t) \\ \vdots \\ \sum_{j=0}^{N} a_j\psi_j(t)\psi_i(t) \\ \vdots \\ \sum_{j=0}^{N} a_j\psi_j(t)\psi_N(t) \end{bmatrix}$$

Table 1
Numerical results for Example 6.1.

N	ξ_N	Computing time
2	1.34264×10^{-2}	0.344
4	5.13797×10^{-4}	0.406
6	1.55784×10^{-4}	0.484
8	6.46120×10^{-5}	0.531

$$= \begin{bmatrix} \sum_{j=0}^{N} a_j\psi_j(t) \\ \sum_{j=0}^{N} \sum_{k=0}^{\frac{1+j-|1-j|}{2}} a_j\psi_{1+j-2k}(t) \\ \vdots \\ \sum_{j=0}^{N} \sum_{k=0}^{\frac{i+j-|i-j|}{2}} a_j\psi_{i+j-2k}(t) \\ \vdots \\ \sum_{j=0}^{N} \sum_{k=0}^{\frac{N+j-|N-j|}{2}} a_j\psi_{N+j-2k}(t) \end{bmatrix}. \tag{12}$$

To retain only the elements of $\psi(t)$ in Eq. (12), we need to have

$$i + j - 2k \le N, \quad \text{for } i, j = 0, 1, 2, \dots, N,$$

hence, we obtain

$$k \ge \max\left\{0, \left\lceil \frac{i+j-N}{2} \right\rceil\right\}. \tag{13}$$

Considering condition (13) in (12) yields

$$\psi(t)\psi^T(t)A \simeq \begin{bmatrix} \sum_{j=0}^{N} a_j\psi_j(t) \\ \sum_{j=0}^{N} \sum_{k=\max\left\{0,\left\lceil\frac{1+j-N}{2}\right\rceil\right\}}^{\frac{1+j-|1-j|}{2}} a_j\psi_{1+j-2k}(t) \\ \vdots \\ \sum_{j=0}^{N} \sum_{k=\max\left\{0,\left\lceil\frac{i+j-N}{2}\right\rceil\right\}}^{\frac{i+j-|i-j|}{2}} a_j\psi_{i+j-2k}(t) \\ \vdots \\ \sum_{j=0}^{N} \sum_{k=\max\left\{0,\left\lceil\frac{N+j-N}{2}\right\rceil\right\}}^{\frac{N+j-|N-j|}{2}} a_j\psi_{N+j-2k}(t) \end{bmatrix}$$

$$
= \begin{bmatrix}
\sum_{j=0}^{N} a_j \psi_j(t) \\
\sum_{j=0}^{N} \sum_{k=\max\left\{0,\left\lceil \frac{1+j-N}{2} \right\rceil\right\}}^{\frac{1+j-|1-j|}{2}} a_{1+j-2k}\psi_j(t) \\
\vdots \\
\sum_{j=0}^{N} \sum_{k=\max\left\{0,\left\lceil \frac{i+j-N}{2} \right\rceil\right\}}^{\frac{i+j-|i-j|}{2}} a_{i+j-2k}\psi_j(t) \\
\vdots \\
\sum_{j=0}^{N} \sum_{k=\max\left\{0,\left\lceil \frac{N+j-N}{2} \right\rceil\right\}}^{\frac{N+j-|N-j|}{2}} a_{N+j-2k}\psi_j(t)
\end{bmatrix}. \tag{14}
$$

From (14), we have (11) with

$$\tilde{A} = [\tilde{a}_{ij}], \quad i, j = 0, 1, 2, \dots, N,$$

in which

$$\tilde{a}_{ij} = \sum_{k=\max\left\{0,\left\lceil \frac{i+j-N}{2} \right\rceil\right\}}^{\frac{i+j-|i-j|}{2}} a_{i+j-2k}.$$

3.3. Operational matrix of fractional integration

In order to obtain the operational matrix of fractional integration, we apply the fractional operator I_α^t defined by (3) to the function $\psi_i(t)$. To do this, we use (5) and then (4) and gain

$$
\begin{aligned}
I_t^\alpha \psi_i(t) &= \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k+1)!2^{2k}}{(i-k)!(2k+1)!T^k} I_t^\alpha t^k \\
&= \sum_{k=0}^{i} (-1)^{i-k} \frac{2^{2k}(i+k+1)!k!}{(i-k)!(2k+1)!T^k \Gamma(k+1+\alpha)} t^{k+\alpha}, \quad i = 0, 1, \dots, N.
\end{aligned} \tag{15}
$$

Now, approximating $t^{k+\alpha}$ using the shifted SKCPs yields

$$t^{k+\alpha} \simeq \sum_{j=0}^{N} c_{kj} \psi_j(t), \tag{16}$$

where c_{kj} are obtained using Eq. (8) with $y(t) = t^{k+\alpha}$ as

$$c_{kj} = \frac{4(j+1)T^{k+\alpha}\Gamma\left(k+\alpha+\frac{3}{2}\right)\Gamma(k+\alpha+1)}{\sqrt{\pi}\,\Gamma(j+k+\alpha+3)\Gamma(k+\alpha-j+1)}, \quad j = 0, 1, 2, \dots, N.$$

Substituting (16) into (15) and after some lengthy manipulation, we have

$$I_t^\alpha \psi_i(t) \simeq \sum_{j=0}^{N} \Theta_{ij}^\alpha \psi_j(t),$$

where

$$\Theta_{ij}^\alpha = \frac{4(j+1)T^\alpha}{\sqrt{\pi}} \sum_{k=0}^{i} (-1)^{i-k} \frac{2^{2k}(i+k+1)!k!\Gamma\left(k+\alpha+\frac{3}{2}\right)}{(i-k)!(2k+1)!\Gamma(j+k+\alpha+3)\Gamma(k+\alpha-j+1)}.$$

Therefore, for the vector $\psi(t)$ defined by (10) we get

$$I_t^{\alpha}\psi(t) \simeq P^{(\alpha)}\psi(t), \tag{17}$$

where $P^{(\alpha)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional integration as

$$P^{(\alpha)} = \left[\Theta_{ij}^{\alpha}\right], \quad i, j = 0, 1, \dots, N.$$

4. Numerical method

In this section, we use the results obtained in [17] in order to introduce a numerical method to solve problem (1)–(2).

Theorem 4.1 *([17, Theorem 1]). Let $\alpha > 0$ and $\alpha \notin \mathbb{N}$, if the right hand side of (1) is continuous, then the initial problem (1)–(2) is equivalent to the linear Volterra integral equation of the second kind with the same initial condition (2),*

$$y(t) = f(t) + \int_0^t (t-w)^{(\alpha-1)}k(t, w)y(w)dw, \quad t \in I(T), \tag{18}$$

where

$$f(t) = f_1(t) + I_t^{\alpha}g(t),$$

with

$$f_1(t) = \sum_{i=0}^{\lceil\alpha\rceil-1} y^{(i)}(0)\frac{t^i}{i!},$$

and

$$k(t, w) = \frac{1}{\Gamma(\alpha)}\left[p(w) + (t-w)^{1-\beta}B(\alpha, 1-\beta)\right], \tag{19}$$

in which $B(a, b)$ is Beta function defined by

$$B(a, b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds.$$

We rewrite the second part of the right hand side in Eq. (18) using Eq. (19) as

$$\begin{aligned}
\int_0^t (t-w)^{(\alpha-1)}k(t, w)y(w)dw &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-w)^{(\alpha-1)}p(w)y(w)dw \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1+1-\beta}B(\alpha, 1-\beta)y(w)dw \\
&= I_t^{\alpha}(p(t)y(t)) + \frac{B(\alpha, 1-\beta)}{\Gamma(\alpha)}\int_0^t (t-w)^{(1+\alpha-\beta)-1}y(w)dw \\
&= I_t^{\alpha}(p(t)y(t)) + \frac{B(\alpha, 1-\beta)\Gamma(1+\alpha-\beta)}{\Gamma(\alpha)}I_t^{1+\alpha-\beta}y(t).
\end{aligned}$$

Let us introduce

$$\chi_{\alpha,\beta} = \frac{B(\alpha, 1-\beta)\Gamma(1+\alpha-\beta)}{\Gamma(\alpha)},$$

then, Eq. (18) can be written as follows:

$$y(t) = f_1(t) + I_t^{\alpha}g(t) + I_t^{\alpha}(p(t)y(t)) + \chi_{\alpha,\beta}I_t^{1+\alpha-\beta}y(t). \tag{20}$$

The functions in Eq. (20) can be approximated by the shifted second kind Chebyshev series using the way mentioned in Section 3 as

$$y(t) \simeq A^T\psi(t), \tag{21}$$

$$f_1(t) \simeq F^T\psi(t), \tag{22}$$

$$g(t) \simeq G^T\psi(t), \tag{23}$$

$$p(t) \simeq C^T \psi(t), \tag{24}$$

where the vector A is unknown. Substituting Eqs. (21)–(24) into Eq. (20) yields

$$A^T \psi(t) = F^T \psi(t) + I_t^\alpha G^T \psi(t) + I_t^\alpha C^T \psi(t)\psi^T(t)A + \chi_{\alpha,\beta} I_t^{1+\alpha-\beta} A^T \psi(t).$$

Using Eqs. (11) and (17), we have

$$A^T \psi(t) = F^T \psi(t) + G^T P^{(\alpha)} \psi(t) + C^T \tilde{A} P^{(\alpha)} \psi(t) + \chi_{\alpha,\beta} A^T P^{(1+\alpha-\beta)} \psi(t). \tag{25}$$

Finally, we get

$$A^T - F^T - G^T P^{(\alpha)} - C^T \tilde{A} P^{(\alpha)} - \chi_{\alpha,\beta} A^T P^{(1+\alpha-\beta)} = 0, \tag{26}$$

which is a system of linear algebraic equations in terms of the unknown elements of the vector A. After solving system (26), we obtain the numerical solution of problem (1)–(2) as (21). In our implementation, we have solved these systems using the Mathematica function "Solve".

5. Error analysis

The purpose of this section is to obtain an estimate of the error of the numerical solution obtained by the presented method in the previous section. To this aim, we assume that the known functions in Eq. (1) provide the conditions that the exact solution $y(t)$ belongs to the Sobolev space $H^{m+1}(I(T))$ [19], where m is a positive even integer number. With the help of the properties of Sobolev spaces we obtain the following results.

Theorem 5.1. *Assume that $f(t) \in H^{m+1}(I(T))$, and $P_N(f(t)) = \sum_{i=0}^{N} a_i \psi_i(t)$ is the truncated second kind Chebyshev series of $f(t)$. Then, the truncation error $f(t) - P_N(f(t))$ can be estimated as follows*

$$|f(t) - P_N(f(t))| = O(N^{-m+1}),$$

as $N \to \infty$ for all $t \in I(T)$.

Proof. Multiplying both sides of the equation $f(t) = \sum_{i=0}^{N} a_i \psi_i(t)$ by $\frac{1}{w(t)} T_N(t)$ and integrating the result from 0 to T, yield

$$\int_0^T \frac{1}{w(t)} f(t) T_N(t) dt = \sum_{i=0}^{N} \int_0^T \frac{1}{w(t)} a_i \psi_i(t) T_N(t) dt, \tag{27}$$

where $T_N(t) = \cos(N\theta)$ with $\theta = \cos^{-1}(\frac{2}{T}t - 1)$ is the well-known shifted first kind Chebyshev polynomial of order N.

From [20], we have

$$T_N(t) = \psi_N(t) - (2t-1)\psi_{N-1}(t).$$

On the other hand, we have

$$\psi_N(t) = \begin{cases} 2\left(\dfrac{T_0}{2} + T_2 + T_4 + \cdots + T_N\right), & N \text{ is even}, \\ 2(T_1 + T_3 + T_5 + \cdots + T_N), & N \text{ is odd}. \end{cases} \tag{28}$$

By using (28), we obtain

$$\psi_N(t) = 2T_N(t) + \sum_{k<N} \alpha_k T_k(t). \tag{29}$$

Then from (27) and (29), a_N is given by an integral which can be regarded as Fourier coefficients as

$$a_N = \frac{2}{T\pi} \int_0^T w(t) f(t) T_N(t) dt = \frac{1}{\pi} \int_0^\pi F(\theta) \cos(N\theta) d\theta,$$

where $\frac{2}{T}t - 1 = \cos\theta$. That is, if $F(\theta) = f(\frac{T(\cos\theta+1)}{2})$ then the Fourier coefficients of F are the Chebyshev coefficients of f. Integrating by parts m (even) times respect to the variable θ gives

$$a_N = \frac{1}{\pi} \cdot \frac{1}{N^m} \int_0^\pi F^{(m)}(\theta) \cos(N\theta) d\theta.$$

Since $F(\theta) \in H^{m+1}(I(T))$ then $F^{(m)}(\theta)$ is continuous [19] and therefore, we know (by the Riemann–Lebesgue lemma [21], specifically) that $a_N \to 0$ faster than the rate of N^{-m}, that is

$$a_N = O(N^{-m}). \tag{30}$$

Also we have

$$|\psi_N(t)| \le N+1. \tag{31}$$

On the other hand

$$f(t) - \sum_{i=0}^{N} a_i\psi_i(t) = \sum_{i=N+1}^{\infty} a_i\psi_i(t).$$

Provided that the coefficients a_i decrease in magnitude sufficiently rapidly, the error made by truncating the second kind Chebyshev expansion will be given approximately by

$$\left| f(t) - \sum_{i=0}^{N} a_i\psi_i(t) \right| \simeq |a_{N+1}\psi_{N+1}(t)|.$$

Hence, using (30) and (31) we have

$$|f(t) - P_N(f(t))| = O(N^{-m+1}),$$

as $N \to \infty$ for all $t \in I(T)$. □

In our numerical method, we have used the operational matrix of fractional integration and the product operational matrix. By neglecting the error of the operational matrices we will have the following result.

Theorem 5.2. *Let $y(t) \in H^{m+1}(I(T))$ be the exact solution of problem (1)–(2) and $y_N(t) = A^T\psi(t)$ be its approximation obtained by the presented method in Section 4, then*

$$|y(t) - y_N(t)| = O(N^{-m+1}),$$

as $N \to \infty$ for all $t \in I(T)$.

Proof. As it was obtained in Section 4, problem (1)–(2) and Eq. (20) are equivalent. Subtracting (25) from (20) yields

$$\begin{aligned} |y(t) - y_N(t)| \le\ & |f_1(t) - F^T\psi(t)| + |I_t^{\alpha}g(t) - G^TP^{(\alpha)}\psi(t)| \\ & + |I_t^{\alpha}(p(t)y(t)) - C^T\tilde{A}P^{(\alpha)}\psi(t)| + \chi_{\alpha,\beta}|I_t^{1+\alpha-\beta}y(t) - A^TP^{(1+\alpha-\beta)}\psi(t)|. \end{aligned} \tag{32}$$

By employing Theorem 5.1, we obtain the following estimates

$$|f_1(t) - F^T\psi(t)| = O(N^{-m+1}), \tag{33}$$

$$|I_t^{\alpha}g(t) - G^TP^{(\alpha)}\psi(t)| = O(N^{-m+1}), \tag{34}$$

$$|I_t^{\alpha}(p(t)y(t)) - C^T\tilde{A}P^{(\alpha)}\psi(t)| = O(N^{-m+1}), \tag{35}$$

$$|I_t^{1+\alpha-\beta}y(t) - A^TP^{(1+\alpha-\beta)}\psi(t)| = O(N^{-m+1}). \tag{36}$$

Therefore, using (32)–(36), we get

$$|y(t) - y_N(t)| = O(N^{-m+1}),$$

provided that N is sufficiently large. □

6. Illustrative examples

In this section, two examples are given to show the accuracy and applicability of the proposed method. In order to demonstrate the error of the method, we introduce the notations:

$$e_N(t) = |y(t) - y_N(t)|,$$

$$\xi_N = \left(\int_0^T w(t)e_N^2(t)dt \right)^{\frac{1}{2}},$$

where $y(t)$ is the exact solution and $y_N(t)$ is the computed solution by the presented method. The computations were carried out in a personal computer with Intel(R) Pentium(R) CPU G620 @ 2.60 GHz with 4.00 GB random access memory and the codes were written in Mathematica 10. The computing times (seconds) to obtain the numerical solution are given for the examples.

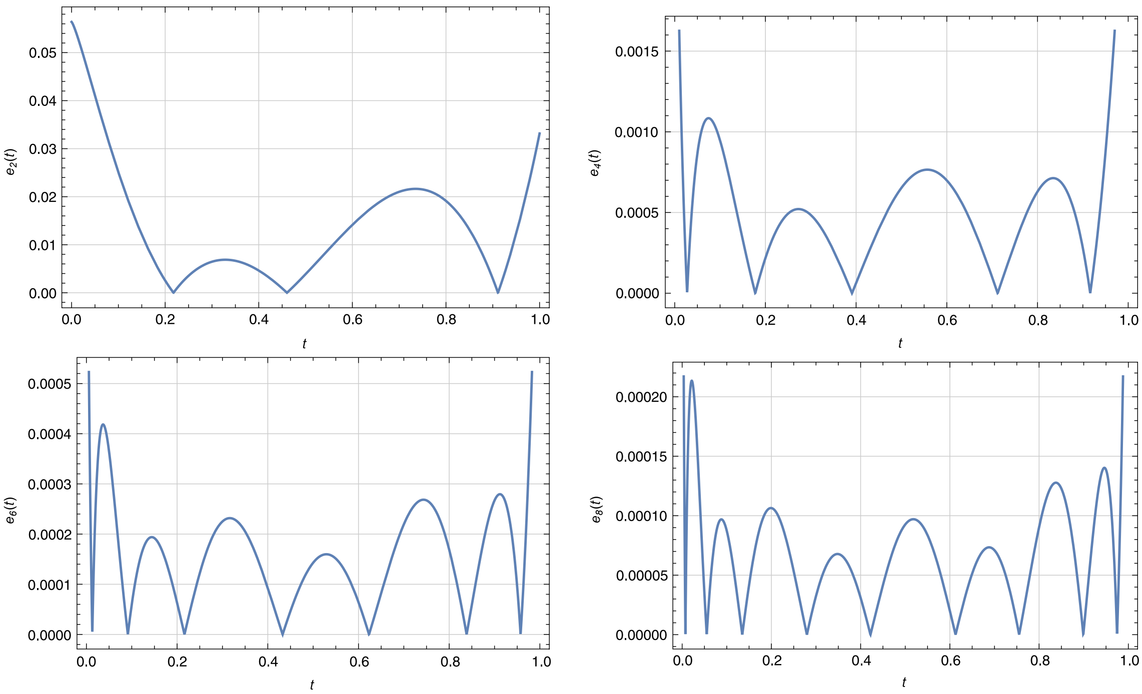

Fig. 1. Plot of the absolute error function with $N = 2, 4, 6, 8$ for Example 6.1.

Example 6.1. Consider the following fractional order integro-differential equation with weakly singular kernel [17]:

$$ {}_0^C D_t^{\frac{1}{3}} y(t) = g(t) + p(t)y(t) + \int_0^t (t-s)^{-\frac{1}{2}} y(s)ds, \quad t \in [0, 1], \tag{37} $$

where

$$ g(t) = \frac{6t^{8/3}}{\Gamma\left(\frac{11}{3}\right)} + \left(\frac{32}{35} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)}\right) t^{11/6} + \Gamma\left(\frac{7}{3}\right) t, $$

$$ p(t) = -\frac{32}{35} t^{\frac{1}{2}}, $$

with initial value $y(0) = 0$.

The exact solution of this problem is $y(t) = t^3 + t^{\frac{4}{3}}$. We have applied the method presented in this paper to this example with $N = 2, 4, 6, 8$. The operational matrices for $N = 2$ are given in the following forms:

$$ P^{(\alpha)} = P^{\left(\frac{1}{3}\right)} = \begin{bmatrix} 0.855692 & 0.171138 & -0.0394935 \\ -0.299492 & 0.526579 & 0.160442 \\ 0.0846288 & -0.192531 & 0.430271 \end{bmatrix}, $$

$$ P^{(1+\alpha-\beta)} = P^{\left(\frac{5}{6}\right)} = \begin{bmatrix} 0.584608 & 0.254177 & -0.0131471 \\ -0.39282 & 0.0806356 & 0.152506 \\ 0.160085 & -0.124778 & 0.0467524 \end{bmatrix}, $$

$$ \tilde{A} = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_0 + a_2 & a_1 \\ a_2 & a_1 & a_0 + a_2 \end{bmatrix}. $$

The final system of linear algebraic equations is obtained as follows:

$$ \begin{cases} 0.439447a_0 + 0.68596a_1 - 0.312118a_2 - 0.574115 = 0 \\ -0.245169a_0 + 1.16465a_1 + 0.195667a_2 - 0.42135 = 0 \\ 0.0145253a_0 - 0.106124a_1 + 1.20118a_2 - 0.104628 = 0. \end{cases} $$

By solving this system, we get

$$ a_0 = 0.647191, \qquad a_1 = 0.477613, \qquad a_2 = 0.121475. $$

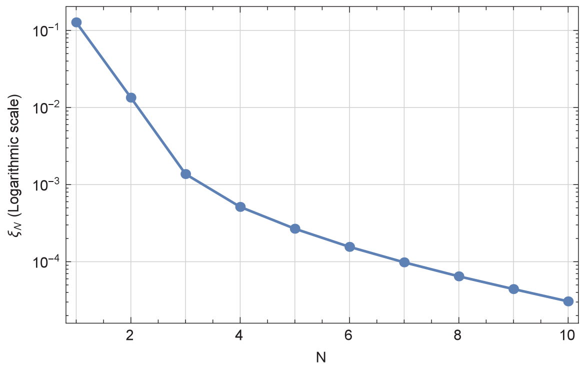

Fig. 2. ξ_N on logarithmic scale for Example 6.1.

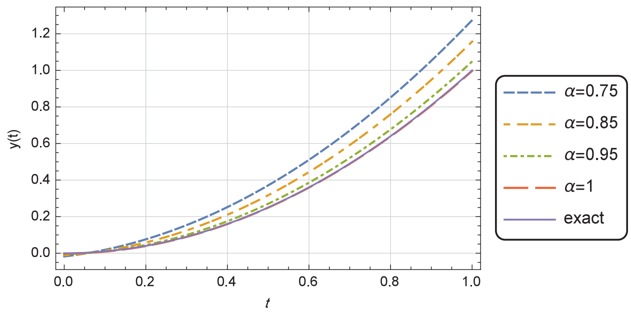

Fig. 3. Plot of the function $y_2(t)$ with $\alpha = 0.75, 0.85, 0.95, 1$ for Example 6.2.

Table 2
Numerical results for Example 6.2 with $\alpha = 1$.

N	ξ_N	Computing time
2	1.06802×10^{-3}	0.296
4	5.14924×10^{-5}	0.313
6	1.70223×10^{-5}	0.375
8	6.56125×10^{-6}	0.452

Finally, substituting these values into Eq. (21), we obtain an approximate solution for Eq. (37) as

$$y_2(t) = 1.9436t^2 - 0.0331502t + 0.0563899,$$

which leads to

$$e_2(t) \leq 0.0563899.$$

Numerical results for different values of N are displayed in Table 1, Figs. 1 and 2.

Example 6.2. Consider the following equation

$${}_0^C D_t^\alpha y(t) = g(t) + p(t)y(t) + \int_0^t (t-s)^{-\frac{1}{2}} y(s)ds, \quad t \in [0, 1], \tag{38}$$

where $g(t) = 2t$ and $p(t) = -\frac{16}{15}t^{\frac{1}{2}}$, with this initial condition $y(0) = 0$.

The exact solution, when $\alpha = 1$, is $y(t) = t^2$.

By setting $N = 2$, we obtain numerical solutions for various α. In Fig. 3, we show the numerical solutions obtained by the presented method and the exact solution for various values of α. It is obvious from Fig. 3 that, as α is close to 1, the numerical solutions converge to the exact solution. Also, the proposed method has been applied to Eq. (38) with different N and $\alpha = 1$. Table 2, Figs. 4 and 5 display the numerical results for different values for N.

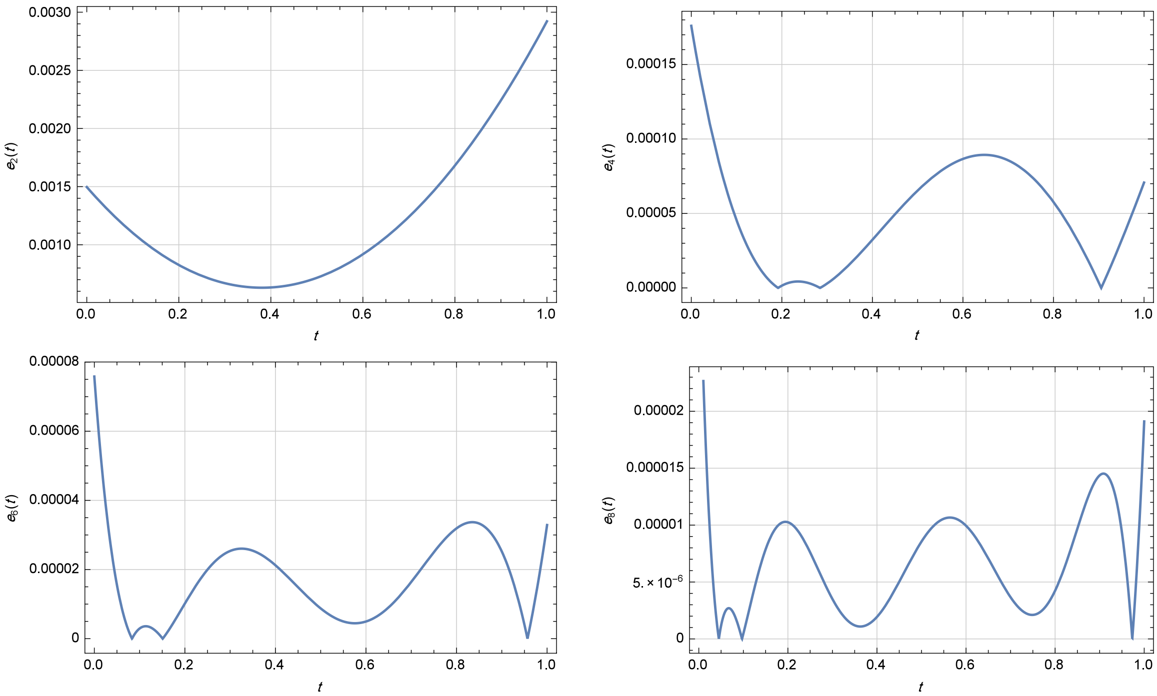

Fig. 4. Plot of the absolute error function with $N = 2, 4, 6, 8$ for Example 6.2.

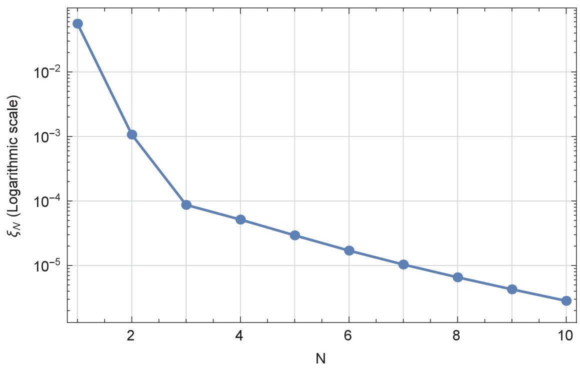

Fig. 5. ξ_N on logarithmic scale for Example 6.2.

7. Conclusion

In this paper, a numerical method has been proposed for numerical solution of the fractional order integro-differential equations with a weakly singular kernel. By using the definition of Riemann–Liouville fractional integral operator and properties of the shifted SKCPs, the operational matrix of fractional integration and the product operational matrix have been introduced. With the help of the results obtained in [17] and some manipulation techniques, we transformed the initial problem into Eq. (20). The operational matrix method has been employed to obtain a system of linear algebraic equations. By solving the linear system, a numerical solution is obtained. An estimation of the error was gained. The method was tested on two examples and the obtained results were compared with exact solutions. The numerical results approved that the proposed method has very high accuracy and also, the computing times reported in Tables 1 and 2 confirmed that the suggested algorithm is very fast.

References

[1] T.S. Chow, Fractional dynamics of interfaces between soft-nanoparticles and rough substrates, Phys. Lett. A 342 (2005) 148–155.
[2] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol. 27 (1983) 201–210.
[3] R.L. Bagley, P.J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, AIAA J. 23 (1985) 918–925.
[4] R.L. Magin, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng. 32 (2004) 1–104.

[5] M. Ichise, Y. Nagayanagi, T. Kojima, An analog simulation of noninteger order transfer functions for analysis of electrode process, J. Electroanal. Chem. 33 (1971) 253–265.

[6] A. Arikoglu, I. Ozkol, Solution of fractional integro-differential equations by using fractional differential transform method, Chaos Solitons Fractals 40 (2009) 521–529.

[7] Li Zhu, Qibin Fa, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 2333–2341.

[8] Li Zhu, Qibin Fa, Numerical solution of nonlinear fractional-order Volterra integro-differential equations by SCW, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 1203–1213.

[9] M.R. Eslahchi, M. Dehghan, M. Parvizi, Application of collocation method for solving nonlinear fractional integro-differential equations, J. Comput. Appl. Math. 257 (2014) 105–128.

[10] P.K. Kythe, P. Puri, Computational Method for Linear Integral Equations, Birkhauser, Boston, 2002.

[11] B.Q. Tang, X.F. Li, Solution of a class of Volterra integral equations with singular and weakly singular kernels, Appl. Math. Comput. 199 (2008) 406–413.

[12] V.V. Zozulya, P.I. Gonzalez-Chi, Weakly singular, singular and hypersingular integrals in 3-D elasticity and fracture mechanics, J. Chin. Inst. Eng. 22 (1999) 763–775.

[13] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002) 229–248.

[14] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[15] S. Momani, Local and global existence theorems on fractional integro-differential equations, J. Frac. Calc. 18 (2000) 81–86.

[16] S. Momani, A. Jameel, S. Al-Azawi, Local and global uniqueness theorems on fractional integro-differential equations via Biharis and Gronwalls inequalities, Soochow J. Math. 33 (2007) 619–627.

[17] J. Zhao, J. Xiao, N.J. Ford, Collocation methods for fractional integro-differential equations with weakly singular kernels, Numer. Algorithms 65 (2014) 723–743.

[18] J.C. Mason, D.C. Handscomb, Chebyshev Polynomials, CRC Press LLC, 2003.

[19] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods Fundamentals in Single Domains, Springer-Verlag, 2006.

[20] A. Jeffrey, H.H. Dai, Handbook of Mathematical Formulas and Integrals, Academic Press, USA, 2008.

[21] D.S. Kurtz, C.W. Swartz, Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock–Kurzweil, and Mcshane, World Scientific, USA, 2004.