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Nonlinear Hartree equation in high energy-mass

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ABSTRACT

This paper is concerned with the Cauchy problem of the nonlinear Hartree equation. By constructing a constrained variational problem, we get a refined Gagliardo–Nirenberg inequality and the best constant for this inequality. We thus derive two conclusions. Firstly, by establishing and analyzing the invariant manifolds, we obtain a new criteria for global existence and blowup of the solutions. Secondly, we get other sufficient condition for global existence with the discussing of the Bootstrap argument. And based on these two conclusions, we also deduce so-called energy-mass control maps, which expose the relationship between the initial data and the solutions.

1. Introduction

Bose–Einstein condensation (BEC) in gases with weak attractive two-body interactions can be found in system of ^{7}Li atoms as long as the gas in the trap has a sufficiently low density [1,2]. By the heuristic discussion, we have that the dynamical evolution of the bosonic system in its mean-field regime is described by nonlinear Hartree equations. Thus, the aim of this paper is to study the following Hartree equation:

$$\begin{cases} i\varphi_t + \Delta\varphi + (V(x) * |\varphi|^2)\varphi = 0, \\ \varphi(0,x) = \varphi_0, \quad x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

where $\varphi(t,x) : \mathbb{R}^+ \times \mathbb{R}^N \longrightarrow \mathbb{C}$ is a complex function, Δ is Laplacian operator on $\mathbb{R}^N, V(x) = \frac{1}{|x|^\alpha}$ $(0 < \alpha < \min\{N, 4\})$ and $*$ is the standard convolution in $\mathbb{R}^N$. The initial data $\varphi_0 \in H^1(\mathbb{R}^N)$, and $H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N)$ is the standard Sobolev space with the norm $\|\varphi\|^2_{H^1} = \int_{\mathbb{R}^N}(|\nabla\varphi|^2 + |\varphi|^2)dx$.

The success of experiment in atomic BEC has stimulated great interest in the properties of Cauchy problem (1.1). For instance, Cazenave [3] stated the local well-posedness of Cauchy problem (1.1) in $H^1(\mathbb{R}^N)$

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when $0 < \alpha < \min\{N, 4\}$ Lions [4] studied the existence of the standing wave of Cauchy problem (1.1) for $\alpha = 1$ (mass subcritical case). Miao [5] showed the existence of the standing waves and the blow up criterion of Cauchy problem (1.1) for L^2 critical (i.e., $\alpha = 2$). And for L^2 supercritical (i.e., $2 < \alpha < \min\{N, 4\}$), Wang [6] and Chen and Guo [7] discussed strong instability of the standing waves of Cauchy problem (1.1) in supercritical case with and without harmonic potential respectively.

In this paper, we plan to discuss the Cauchy problem (1.1) for L^2 supercritical. As we mentioned, Wang [6] and Chen and Guo [7] had discussed this issue before, however. We notice that in [6,7], they obtained the condition for blowup of Cauchy problem (1.1) by constructing a type of cross constrained variational problem and some of these functionals haven't clear physical meanings. Can we use the energy and mass to character the criterion for global existence and blowup? Cazenave also mentioned this topic in their monographs [3]. At the meantime, a collapse processing is observed in the experiments if the number of ^{7}Li atoms in the trap exceeds some threshold value. So this problem is also pursued strongly in Physics (see [8] and the references therein). This is one of the motivations for discussing this problem.

However, in spite of quite a number of contributions dealing with this problem (see [9–12] as well as the other relevant references), almost all of them still need to restrict the energy-mass functional(or some other functionals)to be less than its minimum in one subset. What will happen when the energy-mass is less than the minmax value of the energy-mass functional in the whole space? Obviously, this minmax value is larger than its minimum in one subset. Consequently, we also can answer the question: what will happen when the energy-mass functional is less than a number which is larger than the minimum in one subset? To fill this gap, we do more detailed discussion for the properties of the equation and energy-mass functional and extend the argument to the case that we mentioned before. Actually, we get a sharp condition for the global existence of the solution in higher energy-mass. In addition, these conditions are precisely computed.

Finally, we expect to find some details of the energy-mass criteria, which is called energy-mass control map. This control map shows the relationships between the initial data and some behaviors of the solutions corresponding to these initial data. Furthermore, we find a set which indicate both global existence and blowup. To get more information of this set, we divide it into two parts, and then strip a global existence subset from one of the parts. In fact, to our knowledge no results have been already known in this direction.

The plan of this paper is as follows. In Section 2, we give some concerned preliminaries and obtain a refined Gagliardo–Nirenberg inequality. And in Section 3, we give a new criterion of global existence and blowup for Cauchy problem (1.1). And the other sufficient condition for global existence is also obtained by Bootstrap argument. In the last section, we derive the so-called energy-mass control map by discussing the relations of the results which is obtained in the Section 3.

2. Preliminaries

2.1. Notations and some known lemmas

In this subsection, we will give some concerned preliminaries. Throughout this paper, C denotes various positive constants. For simplicity, here and hereafter, we denote $\|\cdot\|_p$ to denote the norm of $L^p(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \cdot dx$ by $\int \cdot dx$ unless stated otherwise. Moreover, we denote $\Sigma := \{u \in H^1(\mathbb{R}^N) : |x|u \in L^2(\mathbb{R}^N)\}$ and

$$\|\varphi\|_V = \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|^\alpha} dxdy \right)^{\frac{1}{4}}. \tag{2.1}$$

Now, we begin by recalling the Hardy–Littlewood–Sobolev inequality.

Lemma 2.1 *([13,14]). Let $0 < \beta < N$ and suppose that $f \in L^s(\mathbb{R}^N), h \in L^r(\mathbb{R}^N)$ with $\frac{1}{s} + \frac{1}{r} + \frac{\beta}{N} = 2$ and $1 < s, r < \infty$, then*

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^\beta} dxdy \le C \left(\int |f(x)|^s dx \right)^{\frac{1}{s}} \left(\int |h(x)|^r dx \right)^{\frac{1}{r}}$$

for some constant $C > 0$.

Proposition 2.1 *([3]). Suppose $N \ge 3$ and $2 < \alpha < \min\{N, 4\}$. Then for any $\varphi_0 \in H^1(\mathbb{R}^N)$, there is a unique solution $\varphi(t,x)$ of Cauchy problem (1.1) in $C([0,T); H^1(\mathbb{R}^N))$ for time $T \in (0,1]$ (maximal existence time) with the following property: either $T = \infty$ or else $T < \infty$ and $\lim_{t \to T^-} \| \, |\nabla \varphi(t,x)|^2 \| = \infty$, where $t \to T^-$ means that $t \to T$ and $t < T$.*

Furthermore, for all $t \in [0,T)$, the solution φ satisfies the following conservative laws:

(1) *Conservation of mass:*

$$M(\varphi) = \int_{\mathbb{R}^N} |\varphi(t,x)|^2 dx = \int_{\mathbb{R}^N} |\varphi_0|^2 dx \equiv M(\varphi_0). \tag{2.2}$$

(2) *Conservation of energy:*

$$E(\varphi) = \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|^\alpha} dxdy \equiv E(\varphi_0). \tag{2.3}$$

Lemma 2.2 *([15] Gagliardo–Nirenberg's inequality). Let $1 < p < \frac{n+2}{(n-2)^+}$ and Θ be the positive and spherically symmetric solution of the nonlinear elliptic equation*

$$-\Delta u + u - |u|^{p-1} u = 0, \quad u \in H^1(\mathbb{R}^N).$$

Then the best constant $C_* > 0$ of the Gagliardo–Nirenberg's inequality,

$$\int |u|^{p+1} dx \le C_* \left(\int |u|^2 dx \right)^{\frac{p+1}{2} - \frac{N(p-1)}{4}} \left(\int |\nabla u|^2 dx \right)^{\frac{N(p-1)}{4}} \tag{2.4}$$

is given by

$$C_* = \frac{2(p+1)}{N(p-1)} \left(\frac{4-(p-1)(N-2)}{N(p-1)} \right)^{\frac{N(p-1)-4}{4}} \|\Theta\|_2^{-(p-1)}.$$

Next, by direct computations, we get the following variance identity.

Lemma 2.3. *Let $\varphi_0 \in \Sigma$ and $\varphi(t,x)$ be a solution of the Cauchy problem (1.1). Then*

$$\frac{d^2}{dt^2} \int |x|^2 |\varphi|^2 dx = 8 \int |\nabla \varphi|^2 dx - 2\alpha \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|^\alpha} dxdy. \tag{2.5}$$

By a similar way as in [7], we can prove this lemma. Here we omit the proof.

Also, we need the following concentration-compactness principle which was given in [16,17].

Lemma 2.4. *For a bounded sequence $\{u_n\}_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^N)$, there is a subsequence of $\{u_n\}_{n \in \mathbb{N}}$ (still denoted by $\{u_n\}$) and a sequence $\{U^{(j)}\}_{j \ge 1}$ in $H^1(\mathbb{R}^N)$ and for any $j \ge 1$, a family $(x_n^j) \subset \mathbb{R}^N$ such that*

(i) *If* $j \neq k, |x_n^j - x_n^k| \to \infty$, *as* $n \to \infty$.
(ii) *For every* $l \geq 1$,

$$u_n(x) = \sum_{j=1}^{l} U^{(j)}(x - x_n^j) + r_n^l(x). \tag{2.6}$$

Moreover, for any $p \in (2, \frac{2N}{N-2})$,

$$\limsup_{n\to\infty} \|r_n^l\|_p \to 0 \quad \text{as } l \to \infty. \tag{2.7}$$

(iii)

$$\|u_n\|_2^2 = \sum_{j=1}^{l} \|U^{(j)}\|_2^2 + \|r_n^l\|_2^2 + o(1), \tag{2.8}$$

$$\|\nabla u_n\|_2^2 = \sum_{j=1}^{l} \|\nabla U^{(j)}\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1). \tag{2.9}$$

Lemma 2.5 (*[15]*)**.** *Let* $\varphi \in \Sigma$. *One has*

$$\int |\varphi|^2 dx \leq \frac{2}{N} \left(\int |x\varphi|^2 dx \right)^{\frac{1}{2}} \left(\int |\nabla \varphi|^2 dx \right)^{\frac{1}{2}}.$$

2.2. A refined Gagliardo–Nirenberg inequality

In this subsection, we obtained a refined Gagliardo–Nirenberg inequality of convolution type which is a general case of the result in [5]. Moreover, we determine the best constant of the refined Gagliardo–Nirenberg inequality. First, we begin with the following lemma:

By Lemmas 2.1 and 2.2, we have

$$\|u\|_V^4 \leq C_\alpha \|\nabla u\|_2^\alpha \|u\|_2^{4-\alpha}. \tag{2.10}$$

To compute C_α, it will suffice to minimize the functional:

$$J(u) := \frac{\|\nabla u\|_2^\alpha \|u\|_2^{4-\alpha}}{\|u\|_V^4}$$

and

$$J_\alpha = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} J(u). \tag{2.11}$$

Lemma 2.6. *If* U *is the minimizer of* $J(u)$, *then* U *satisfies*

$$-\Delta U + AU - B(V(x) * |U|^2)U = 0, \tag{2.12}$$

where

$$A = \frac{4-\alpha}{\alpha} \frac{\|\nabla U\|_2^2}{\|U\|_2^2}, \qquad B = \frac{4J_\alpha}{\alpha \|U\|_2^{4-\alpha} \|\nabla U\|_2^{\alpha-2}}.$$

Proposition 2.2. *For* $0 \le \alpha < \min\{N, 4\}$, *then*

$$J_\alpha = \min_{u \in H^1(\mathbb{R}^N)} J(u).$$

Proof. Choose a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ with $J(u_n) \to J_\alpha$. Assume, without loss of generality, that $\|u_n\|_2 = 1$ and $\|\nabla u_n\|_2 = 1$, then

$$J(u_n) = \frac{1}{\|u_n\|_V^4} \to J_\alpha.$$

Obviously, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Then by Lemma 2.4, we have (2.6)–(2.9) hold. It follows from (2.8) and (2.9) that

$$\sum_{j=1}^{l} \|U^{(j)}\|_2^2 \le 1, \qquad \sum_{j=1}^{l} \|\nabla U^{(j)}\|_2^2 \le 1. \tag{2.13}$$

Then, by Lemma 2.1, we have $\|r_n^l\|_V^4 \le C\|r_n^l\|^4_{\frac{4N}{2N-\alpha}}$. It follows from (2.7) that

$$\limsup_{n\to\infty} \|r_n^l\|_V \to 0.$$

Hence, by (2.6), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left|\sum_{j=1}^{l} U^{(j)}(x-x_n^j)\right|^2 \left|\sum_{j=1}^{l} U^{(j)}(y-x_n^j)\right|^2}{|x-y|^\alpha} dxdy \\
&= \sum_{j=1}^{l} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|U^{(j)}(x-x_n^j)|^2 |U^{(j)}(y-x_n^j)|^2}{|x-y|^\alpha} dxdy \\
&\quad + 2\sum_{j=1}^{l-1}\sum_{k=j+1}^{l} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|U^{(j)}(x-x_n^j)U^{(k)}(x-x_n^k)| \left|\sum_{i=1}^{l} U^{(i)}(y-x_n^i)\right|^2}{|x-y|^\alpha} dxdy \\
&\quad + 2\sum_{j=1}^{l-1}\sum_{k=j+1}^{l} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|U^{(j)}(x-x_n^j)|^2 |U^{(k)}(y-x_n^k)|^2}{|x-y|^\alpha} dxdy \\
&= I + 2II + 2III.
\end{aligned} \tag{2.14}$$

By (i) of Lemmas 2.4 and 2.1, we have $II \to 0$ and $III \to 0$ as $n \to \infty$. And without loss of generality, we assume that all $U^{(j)}$ are continuous and compactly supported. Thus,

$$I = \sum_{j=1}^{l} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^\alpha} dxdy.$$

It follows that

$$\left\|\sum_{j=1}^{l} U^{(j)}(x-x_n^j)\right\|_V^4 \to \sum_{j=1}^{l} \|U^{(j)}\|_V^4 \quad \text{as } n \to \infty.$$

Hence, we have

$$\sum_{j=1}^{l} \|U^{(j)}\|_V^4 = \frac{1}{J_\alpha}.$$

Thus, by (2.11), we have

$$J_\alpha \|U^{(j)}\|_V^4 \le \|U^{(j)}\|_2^{4-\alpha} \|\nabla U^{(j)}\|_2^\alpha \le \sup_{j\ge 1}\{\|U^j\|_2^{4-\alpha}\} \|\nabla U^{(j)}\|_2^\alpha. \tag{2.15}$$

Next, we discuss the problem in two cases, $0 < \alpha < 2$ and $2 \le \alpha < \min\{N, 4\}$. And it is easy to have $\|U^{(j_0)}\|_2 = 1$.

Above all, we have that there exists a unique j_0 such that

$$\|U^{(j_0)}\|_2 = 1, \qquad \|\nabla U^{(j_0)}\|_2 = 1 \quad \text{and} \quad \|U^{(j_0)}\|_V^4 = \frac{1}{J_\alpha}.$$

It follows that

$$-\Delta U^{(j_0)} + \frac{4-\alpha}{\alpha} U^{(j_0)} - \frac{4}{\alpha} J_\alpha (V(x) * |U^{(j_0)}|^2) U^{(j_0)} = 0. \tag{2.16}$$

The proof is complete. $\square$

Remark 2.1. If U is the minimizer of $J(u)$, then $|U|$ is also a minimizer. Moreover, $|U| \in H^1(\mathbb{R}^N)$ and $J(|U|) \le J(U)$. Hence we can assume that $U > 0$.

Remark 2.2. The uniqueness of U can be obtained by means of the method in Kwong [18].

Hence, we obtain a refined Gagliardo–Nirenberg inequality:

Theorem 2.1. *Let $0 < \alpha < \min\{N, 4\} (N \ge 3)$ and Q be the ground state solution of the nonlinear elliptic equation:*

$$-\Delta u + u - (V(x) * |u|^2) u = 0, \quad u \in H^1(\mathbb{R}^N). \tag{2.17}$$

Then the best constant $C_\alpha > 0$ of the Gagliardo–Nirenberg's inequality,

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)|^2|u(y)|^2}{|x-y|^\alpha} dxdy \le C_\alpha \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{4-\alpha}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{\alpha}{2}}, \tag{2.18}$$

is given by

$$C_\alpha = \frac{4}{4-\alpha}\left(\frac{\alpha}{4-\alpha}\right)^{\frac{\alpha}{2}} \|Q\|_2^{-2}. \tag{2.19}$$

Proposition 2.3. *If Q satisfies (2.17), then I can be obtained by $u(x) = \mu Q(\lambda x + \eta)$ for some $\mu \in \mathbb{C}, \lambda > 0$ and $\eta \in \mathbb{R}^N$. Moreover,*

$$J_\alpha = \frac{4-\alpha}{4}\left(\frac{4-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} \|Q\|_2^2.$$

Proof. On the one hand, multiplying (2.17) with $x\nabla Q$ and integrating on $\mathbb{R}^N$, we get

$$\frac{N-2}{2}\|\nabla Q\|_2^2 + \frac{N}{2}\|Q\|_2^2 = \frac{2N-\alpha}{4}\|Q\|_V^4.$$

On the other hand, multiplying (2.17) with Q and integrating on $\mathbb{R}^N$, we have

$$\|\nabla Q\|_2^2 + \|Q\|_2^2 - \|Q\|_V^4 = 0.$$

Thus, we have

$$\|Q\|_V^4 = \frac{4}{4-\alpha}\|Q\|_2^2.$$

Taking $\mu = \left(\frac{\alpha}{4J_\alpha}\right)^{\frac{1}{2}} \left(\frac{4-\alpha}{\alpha}\right)^{\frac{N+2-\alpha}{4}}$ and $\lambda = \left(\frac{4-\alpha}{\alpha}\right)^{\frac{1}{2}}$. It follows that

$$J_\alpha = \frac{1}{\|U^{(j_0)}\|_V^4} = \frac{4-\alpha}{4}\left(\frac{4-\alpha}{\alpha}\right)^{\frac{\alpha}{2}} \|Q\|_2^2.$$

The proof is complete. □

3. Global existence and blowup results

3.1. A new criteria for global existence and blowup

In this subsection, we shall give some invariant evolution flows generated by the Cauchy problem (1.1). For convenience, if $2 < \alpha < \min\{N, 4\}$, we define

$$\mathcal{G} := \left\{u \in H^1(\mathbb{R}^N) : F(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{8\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2,\ \|u\|_{H^1}^2 < \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{4\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2\right\}$$

and

$$\mathcal{B} := \left\{u \in H^1(\mathbb{R}^N) : F(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2,\ \|u\|_{H^1}^2 > \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}}\|Q\|_2^2\right\},$$

where $F(S) := S - \frac{C_\alpha}{2}S^2$ for any $S > 0$. Then, denote $\bar{\mathcal{G}} := \mathcal{G} \cup \{0\}$.

Proposition 3.1. *Let $2 < \alpha < \min\{N, 4\}$ and $N \geq 3$. Then $\bar{\mathcal{G}}$ and $\mathcal{B}$ are invariant under the flows generated by the Cauchy problem (1.1). More precisely, if the initial data $\varphi_0 \in \bar{\mathcal{G}}$ (resp. $\mathcal{B}$), then the solution $\varphi(t,x)$ of the Cauchy problem (1.1) still satisfies $\varphi(t,x) \in \bar{\mathcal{G}}$ (resp. $\mathcal{B}$) for all $t \in [0,T)$.*

Proof. Suppose $\varphi_0 \in \bar{\mathcal{G}}$ and $\varphi(t,x)$ is the unique solution of Cauchy problem (1.1) with the initial data φ_0. By local well-posedness of Cauchy problem (1.1), one has $E(\varphi(t)) = E(\varphi_0)$ and $M(\varphi(t)) = M(\varphi_0) t \in [0,T)$. Thus from $E(\varphi_0) + M(\varphi_0) < \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{8\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2 = \frac{1}{2C_\alpha}$, it follows that

$$E(\varphi(t)) + M(\varphi(t)) < \frac{1}{2C_\alpha}, \quad t \in [0,T). \tag{3.1}$$

To check that $\varphi(t) \in \bar{\mathcal{G}}$, we only need to prove

$$\|\varphi(t)\|_{H^1}^2 < \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{4\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2 = \frac{1}{C_\alpha}, \quad t \in [0,T). \tag{3.2}$$

If otherwise, we have $\|\varphi(t)\|_{H^1}^2 \geq \frac{1}{C_\alpha}$. Then there would exist, by continuity, a $t_1 \in [0,T)$ such that

$$\|\varphi(t)\|_{H^1}^2 = \frac{1}{C_\alpha}. \tag{3.3}$$

However, it follows from Lemma 2.2 that

$$\begin{aligned} E(\varphi(t_1)) + M(\varphi(t_1)) &= \int (|\nabla\varphi(t_1)|^2 + |\varphi(t_1)|^2)dx - \frac{1}{2}\|\varphi(t_1)\|_V^4 \\ &\geq \int (|\nabla\varphi(t_1)|^2 + |\varphi(t_1)|^2)dx - \frac{C_\alpha}{2}\|\varphi(t_1)\|_2^{4-\alpha}\|\nabla\varphi(t_1)\|_2^\alpha \\ &\geq \int (|\nabla\varphi(t_1)|^2 + |\varphi(t_1)|^2)dx - \frac{C_\alpha}{2}\left[\int (|\nabla\varphi(t_1)|^2 + |\varphi(t_1)|^2)dx\right]^2. \end{aligned} \tag{3.4}$$

Recall the real value function

$$F(S) := S - \frac{C_\alpha}{2} S^2 \quad \text{for any } S > 0. \tag{3.5}$$

Obviously, $F(S)$ reaches the maximum

$$F_{\max} = F\left(\frac{1}{C_\alpha}\right) = \frac{1}{2C_\alpha}. \tag{3.6}$$

Therefore it follows from (3.4) and (3.6) that

$$E(\varphi(t_1)) \geq \frac{1}{2C_\alpha}, \tag{3.7}$$

which violates $E(\varphi(t_1)) + M(\varphi(t_1)) = E(\varphi_0) + M(\varphi(t_0)) < \frac{1}{2C_\alpha}$. Then, combining with (3.4), we obtain that (3.2) is true. Hence, $\bar{\mathcal{G}}$ is invariant under the flow generated by the Cauchy problem (1.1).

By the similar argument as above, we can show that $\mathcal{B}$ are also invariant under the flows generated by the Cauchy problem (1.1). □

We can now state the main result of this subsection.

Theorem 3.1. *Let $2 < \alpha < \min\{N, 4\}(N \geq 3)$ and Q be the ground state solution of the nonlinear elliptic equation (2.15). Denote $\varphi(t, x)$ the solution of the Cauchy problem (1.1) corresponding to the initial datum φ_0. Then one has that*

(1) *if $\varphi_0 \in \bar{\mathcal{G}}$, then $\varphi(t, x)$ globally exists in $H^1(\mathbb{R}^N)$;*
(2) *if $\varphi_0 \in \mathcal{B} \cap \Sigma$, then $\varphi(t, x)$ blows up in a finite time.*

Proof. It is obvious that for any $\varphi \in H^1(\mathbb{R}^N) \setminus \{0\}$, we get $E(\varphi) + M(\varphi) > F(\|\varphi\|_{H^1}^2)$.

(1) Let $\varphi(t, x)$ be the solution of the Cauchy problem (1.1) with the initial datum φ_0 in $t \in [0, T)$. If $\varphi_0 \in \bar{\mathcal{G}}$, it follows from Proposition 3.1 that $\varphi \in \bar{\mathcal{G}}$. Hence

$$\int [| \bigtriangledown \varphi|^2 + |\varphi|^2] dx < \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{4\alpha^{\frac{\alpha}{2}}} \|Q\|_2^2 \tag{3.8}$$

and we easily obtain φ is bounded in $H^1(\mathbb{R}^N)$. Therefore, the solution φ of the Cauchy problem (1.1) globally exists in $H^1(\mathbb{R}^N)$.

(2) Suppose $\varphi_0 \in \mathcal{B}$, it follows from Proposition 3.1 that $\varphi \in \mathcal{B}$, which reads

$$E(\varphi) + M(\varphi) < \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}} \|Q\|_2^2 \quad \text{and} \quad \|\varphi\|_{H^1}^2 > \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}} \|Q\|_2^2.$$

It follows that

$$\begin{aligned}
\frac{d^2}{dt^2} \int |x|^2 |\varphi|^2 dx &= 8 \int |\nabla\varphi|^2 dx - 2\alpha \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|^\alpha} dxdy \\
&= 4\alpha(E(\varphi) + M(\varphi)) - (4\alpha - 8) \int |\nabla\varphi|^2 dx - 4\alpha |\varphi|^2 dx \\
&< 4\alpha(E(\varphi) + M(\varphi)) - (4\alpha - 8) \int (|\nabla\varphi|^2 + |\varphi|^2) dx \\
&< 4\alpha \left(E(\varphi_0) + M(\varphi_0) - \frac{\alpha-2}{\alpha} \|\varphi\|_{H^1}^2 \right) \\
&< 4\alpha \left[(E(\varphi_0) + M(\varphi_0)) - \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}} \|Q\|_2^2 \right].
\end{aligned} \tag{3.9}$$

Therefore, $\frac{d^2}{dt^2}\int |x|^2|\varphi|^2dx \le -C < 0$ is got. Then using the method of Glassey [19] and Lemma 2.5, we get that the solution φ of the Cauchy problem (1.1) blows up in a finite time. The proof is complete. $\square$

Remark 3.1. It is clear that $\frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{8\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2 > \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2$ for $2<\alpha<\min\{N,4\}(N\ge 3)$. If lowering the energy-mass to $\frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2$ in case (1), Theorem 3.1 becomes a sharp condition of global existence and blowup for Cauchy problem (1.1).

In fact, the set $\{\varphi\in H^1(\mathbb{R}^N) : E(\varphi)+M(\varphi) < \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2,\ \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}}\|Q\|_2^2 < \|\varphi_0\|_{H^1}^2 < \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{4\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2\}$ equals $\emptyset$. (We refer to Proposition 4.2 in Section 4 for more details.) So we can use any number between $\frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}}\|Q\|_2^2$ and $\frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{4\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2$ to divide the subspace $\{u\in H^1(\mathbb{R}^N) : E(\varphi_0)+M(\varphi_0) < \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2\}$ by norm into two parts, in which one corresponds to global existence and the other corresponds to blowup.

3.2. Bootstrap argument

Now we are at the point to do the bootstrap argument of Cauchy problem (1.1). Let us begin with the following proposition.

Proposition 3.2. *Let $2<\alpha<\min\{N,4\}(N\ge 3)$ and Q be the ground state solution of the nonlinear elliptic equation (2.15). If the initial data of Cauchy problem (1.1) φ_0 satisfy $\int|\nabla\varphi_0|^2dx < \frac{(4-\alpha)^{\frac{\alpha+2}{\alpha-2}}}{2^{\frac{2}{\alpha-2}}\alpha^{\frac{\alpha}{\alpha-2}}(\int|\varphi_0|^2dx)^{\frac{4-\alpha}{\alpha-2}}}\|Q\|_2^{\frac{4}{\alpha-2}}$, then $E(\varphi(t,x))>0$.*

Proof. Let $\varphi\in H^1(\mathbb{R}^N)\setminus\{0\}$, we have

$$\begin{aligned} E(\varphi(t)) &= \int|\nabla\varphi(t)|^2dx - \frac{1}{2}\|\varphi(t)\|_V^4 \ge \int|\nabla\varphi(t)|^2dx - \frac{C_\alpha}{2}\|\varphi(t)\|_2^{4-\alpha}\|\nabla\varphi(t)\|_2^\alpha \\ &= \int|\nabla\varphi(t)|^2dx\left[1-\frac{C_\alpha}{2}\left(\int|\varphi(t)|^2dx\right)^{2-\alpha/2}\left(\int|\nabla\varphi(t)|^2dx\right)^{\alpha/2-1}\right]. \end{aligned} \tag{3.10}$$

Then, by Proposition 2.1, the result is proved. $\square$

Now, by Theorem 2.1 and Proposition 2.1, we have

$$\begin{aligned} \int|\nabla\varphi(t)|^2dx &= E(\varphi(t)) + \frac{1}{2}\|\varphi(t)\|_V^4 \le E(\varphi(t)) + \frac{C_\alpha}{2}\|\varphi(t)\|_2^{4-\alpha}\|\nabla\varphi(t)\|_2^\alpha \\ &= E(\varphi_0) + \frac{C_\alpha}{2}M(\varphi_0)^{2-\alpha/2}\|\nabla\varphi(t)\|_2^\alpha. \end{aligned} \tag{3.11}$$

With this estimate in hand, the result follows from the following bootstrap argument:

Lemma 3.1 ([20] *Bootstrap argument*)**.** *Let $f=f(t)$ be a nonnegative continuous function on $[0,T]$ such that, for every $t\in[0,T]$,*

$$f(t)\le\varepsilon_1+\varepsilon_2 f(t)^\theta,$$

where $\varepsilon_1,\varepsilon_2>0$ and $\theta>1$ are constant such that

$$\varepsilon_1 < \left(1-\frac{1}{\theta}\right)\frac{1}{(\theta\varepsilon_2)^{1/(\theta-1)}}, \qquad f(0)\le\frac{1}{(\theta\varepsilon_2)^{1/(\theta-1)}}.$$

Then, for every $t \in [0, T]$, we have

$$f(t) \le \frac{\theta}{\theta - 1}\varepsilon_1.$$

From this lemma, Propositions 2.1 and 3.2, taking $f(t) = \int |\nabla\varphi(t)|^2 dx, \theta = \frac{\alpha}{2}$, $\varepsilon_1 = E(\varphi(t))$ and then $\varepsilon_2 = \frac{C_\alpha}{2} M(\varphi(t))^{4-\alpha}$, we can infer the following discussion of global existence.

Now, denote

$$\mathcal{K} := \left\{u \in H^1(\mathbb{R}^N) : F(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{2(4-\alpha)^{\alpha-1}}{\alpha^\alpha}\|Q\|_2^2, \|u\|_{H^1}^2 \le \frac{2(4-\alpha)^{\alpha-1}}{\alpha^{\frac{\alpha}{2}+1}(\alpha-2)^{\frac{\alpha}{2}-1}}\|Q\|_2^2\right\}.$$

Then we give the main results of this subsection.

Theorem 3.2. *Let $2 < \alpha < \min\{N, 4\}(N \ge 3)$ and Q be the ground state solution of the nonlinear elliptic equation (2.15). Denote $\varphi(t, x)$ the solution of the Cauchy problem (1.1) corresponding to the initial datum φ_0. If $\varphi_0 \in \mathcal{K}$, then $\varphi(t, x)$ globally exists in $H^1(\mathbb{R}^N)$.*

4. Energy-mass control map

From Section 3, we have known some sufficient conditions for global existence and blowup. But, there is no results about the relation of them. So in this section we will try to explore more details of these results in the above section, and show the so called energy-mass control map of Cauchy problem (1.1).

To give a concise and clear representation both in expressions and graphs, we denote

$$H(\alpha) := \frac{2(4-\alpha)^{\alpha-1}}{\alpha^\alpha}\|Q\|_2^2, \qquad h(\alpha) := \frac{2(4-\alpha)^{\alpha-1}}{\alpha^{\frac{\alpha}{2}+1}(\alpha-2)^{\frac{\alpha}{2}-1}}\|Q\|_2^2,$$

$$G(\alpha) := \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{8\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2, \qquad g(\alpha) := \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{4\alpha^{\frac{\alpha}{2}}}\|Q\|_2^2,$$

and

$$B(\alpha) := \frac{(\alpha-2)(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2, \qquad b(\alpha) := \frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}}\|Q\|_2^2.$$

Hence, in order to show the comparison of $H(\alpha), G(\alpha)$ and $B(\alpha)$, we give Fig. 1 for details. Furthermore, we also give the relationship among $h(\alpha), g(\alpha)$ and $b(\alpha)$ in Fig. 2.Thus, we obtain the following propositions.

Proposition 4.1. *If $2 < \alpha < \min\{N, 4\}(N \ge 3)$, we have $\bar{\mathcal{K}} := \mathcal{K} \cup \{0\} \supset \bar{\mathcal{G}}$.*

Now, consider $\mathcal{E} := \mathcal{K} \cap \mathcal{B}$. Thus,

$$\mathcal{E} = \left\{u \in H^1(\mathbb{R}^N) : F(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{2(4-\alpha)^{\alpha-1}}{\alpha^\alpha}\|Q\|_2^2,\right.$$
$$\left.\frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}}\|Q\|_2^2 < \|u\|_{H^1}^2 \le \frac{2(4-\alpha)^{\alpha-1}}{\alpha^{\frac{\alpha}{2}+1}(\alpha-2)^{\frac{\alpha}{2}-1}}\|Q\|_2^2\right\}.$$

But from Theorems 3.1 and 3.2, we find if the initial data $\varphi_0 \in \mathcal{E} \cap \Sigma$, the solutions of Cauchy problem (1.1) are global existence and also blow up in a finite time, which is a contradiction. Hence, we conjecture that $\mathcal{E} \cap \Sigma = \emptyset$. But there still exists a problem: does $\mathcal{E}$ equal to empty set? To solve this problem, we first need the exact position relation of $\mathcal{E}$ and $\mathcal{B}$, we give Fig. 3 for details.

Here $k(\alpha) := \frac{2(4-\alpha)^{\frac{3\alpha}{2}-3}[\alpha(4-\alpha)^{2-\frac{\alpha}{2}}(\alpha-2)^{\frac{\alpha}{2}-1}-4]}{(\alpha-2)^{\alpha-2}\alpha^{\frac{\alpha}{2}+2}}\|Q\|_2^2$. Hence, we divide $\mathcal{E}$ into two subsets: $\mathcal{E}_+ := \mathcal{E} \cap \{u \in H^1(\mathbb{R}^N) : E(u) + M(u) \ge 0\}$ and $\mathcal{E}_- := \mathcal{E} \cap \{u \in H^1(\mathbb{R}^N) : E(u) + M(u) < 0\}$. Thus, by Proposition 3.2, we obtain the following proposition.

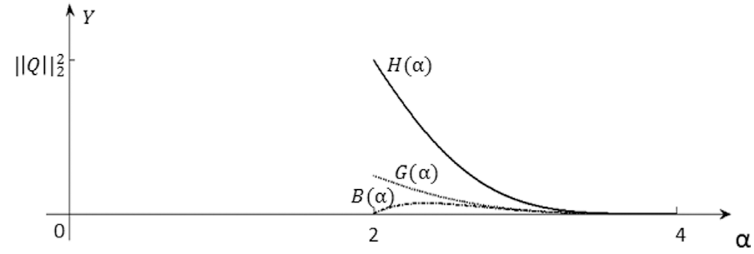

Fig. 1. Comparison of energy-mass.

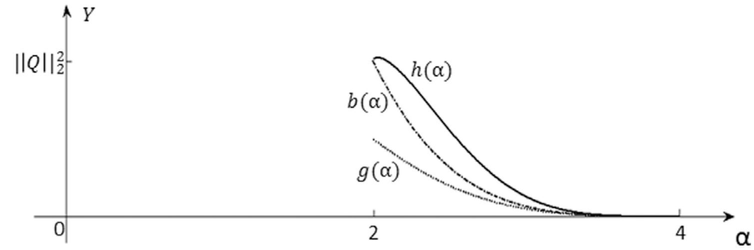

Fig. 2. Comparison of norm.

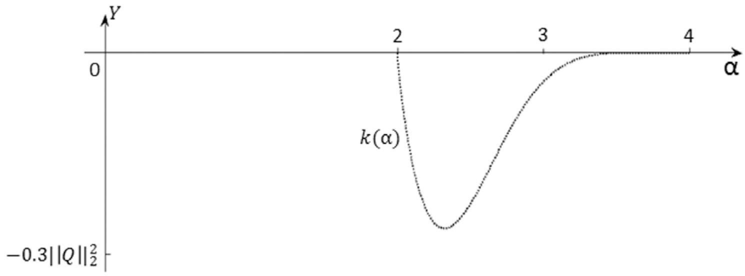

Fig. 3. Second coordinate of the lowest point in $\mathcal{E}$.

Proposition 4.2. *If* $2 < \alpha < \min\{N, 4\}, N \geq 3$, *then*

$$\mathcal{E}_- \subset \mathcal{D} := \left\{u \in H^1(\mathbb{R}^N) : E(u) + M(u) < 0,\ \|u\|_{H^1}^2 < \frac{(4-\alpha)^{\alpha-1}}{(\alpha-2)^{\alpha/2-1}\alpha^{\alpha/2}}\|Q\|_2^2\right\} = \emptyset.$$

Proposition 4.3. *Let* $2 < \alpha < \min\{N, 4\}$ *and* $N \geq 3$. *There exists a* $\alpha_* \in (2, 4)$ *such that if* $2 < \alpha < \alpha_*$, *then* $\mathcal{E}_+ \setminus \Sigma \neq \emptyset$.

Proof. For any $u \in H^1(\mathbb{R}^N)$, put $\hat{u} = \lambda\frac{\|Q\|_2}{\|u\|_{H^1}}u$ $(\lambda > 0)$. Then, on the one hand, if $\lambda \in (\frac{(4-\alpha)^{\frac{\alpha+2}{4}}}{\alpha^{\frac{\alpha+2}{4}}}, \frac{\sqrt{2}(4-\alpha)^{\frac{\alpha-1}{2}}}{\alpha^{\frac{\alpha+2}{4}}(\alpha-2)^{\frac{\alpha-2}{4}}})$, it follows $\frac{(4-\alpha)^{\frac{\alpha}{2}+1}}{\alpha^{\frac{\alpha}{2}+1}}\|Q\|_2^2 < \|\hat{u}\|_{H^1}^2 \leq \frac{2(4-\alpha)^{\alpha-1}}{\alpha^{\frac{\alpha}{2}+1}(\alpha-2)^{\frac{\alpha}{2}-1}}\|Q\|_2^2$. On the other hand, if $\lambda \in (0, \frac{\sqrt{2}(4-\alpha)^{\frac{\alpha-1}{2}}}{\alpha^{\frac{\alpha}{2}}})$, the energy-mass satisfies $0 < E(\hat{u}) + M(\hat{u}) < \frac{2(4-\alpha)^{\alpha-1}}{\alpha^{\alpha}}\|Q\|_2^2$. Thus, $\mathscr{E}_+ \neq \emptyset$ for $\lambda \in (\frac{(4-\alpha)^{\frac{\alpha+2}{4}}}{\alpha^{\frac{\alpha+2}{4}}}, \frac{\sqrt{2}(4-\alpha)^{\frac{\alpha-1}{2}}}{\alpha^{\frac{\alpha}{2}}})$. From calculation by computer, we find there exists a number 3.385 such that if $\alpha \in (2, 3.385) \cap (2, \min\{N, 4\})$, then $(\frac{(4-\alpha)^{\frac{\alpha+2}{4}}}{\alpha^{\frac{\alpha+2}{4}}}, \frac{\sqrt{2}(4-\alpha)^{\frac{\alpha-1}{2}}}{\alpha^{\frac{\alpha}{2}}}) \neq \emptyset$. This completes the proof. $\square$

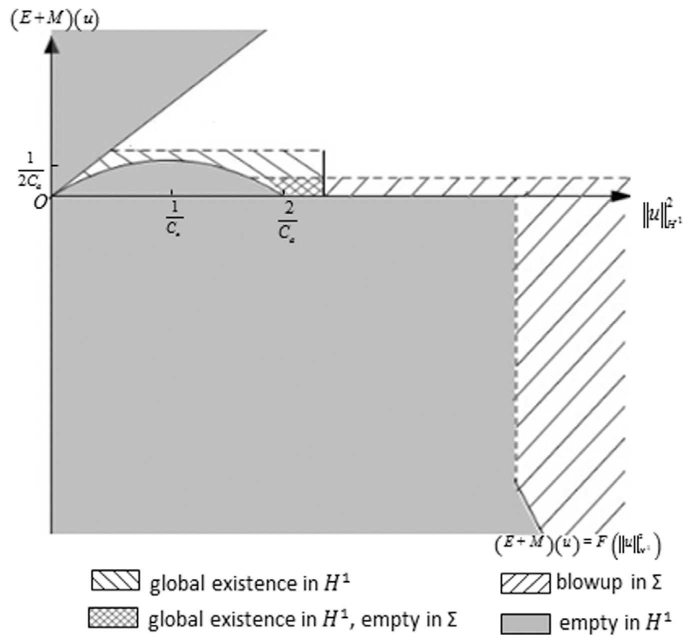

Fig. 4. Energy-mass control map.

Remark 4.1. Actually, from the proof of Proposition 4.3, we have 3.385 is not the best number of this case and the best number should be less than 3.385. And by a direct computation, we also find the best number is larger than 3.

Hence, we have the main results of this section, which can be treated as the improvement of Theorems 3.1 and 3.2.

Theorem 4.1. *Let $2 < \alpha < \min\{N, 4\}, N \geq 3$ and Q be the ground state solution of the nonlinear elliptic equation* (2.15)*. Denote $\varphi(t,x)$ the solution of the Cauchy problem* (1.1) *corresponding to the initial datum φ_0.*

(1) *If the initial datum $\varphi_0 \in \bar{\mathcal{K}} \setminus (\mathcal{E} \cap \Sigma)$, then $\varphi(t,x)$ globally exists in $H^1(\mathbb{R}^N)$;*
(2) *If the initial datum $\varphi_0 \in (\mathcal{B} \setminus \mathcal{D}) \cap \Sigma$, then $\varphi(t,x)$ blows up in a finite time.*

To give an intuitive and visual feeling of Theorem 4.1, we draw Fig. 4. It is worth to point out that we do not know what are the blank parts of the first quadrants in the energy-mass control maps correspond to. And this is the next problem we will pursue.

Remark 4.2. From [21], we know that the condition $\varphi_0 \in \Sigma$ can be changed into $\varphi_0 \in H^1_{radial}$. That means $\mathcal{E} \cap H^1_{radial} = \emptyset$ and there are also have some corresponding changes in Theorem 4.1 and Fig. 4.

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References

[1] J. Froehlich, E. Lenzmann, Mean-field limit of quantum bose gases and nonlinear Hartree equation, in: Seminaire EDP, Ecole Polytechnique, 2003–2004.

[2] E.P. Gross, in: E. Meeron (Ed.), Physics of Many-Particle Systems, Vol. 1, Gordon Breach, New York, 1966, pp. 231–406.

[3] T. Cazenave, Semilinear Schrödinger Equations, in: Courant Lecture Notes, vol. 10, American Mathematical Society, Providence, RI, 2003.

[4] P.L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact case Part 1, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145.

[5] C. Miao, G. Xu, L. Zhao, On the blow up phenomenon for the L^2-critical focusing Hartree equation in $\mathbb{R}^3$, Colloq. Math. 119 (2010) 23–50.

[6] Y. Wang, Strong instability of standing waves for Hartree equation with harmonic potential, Physica D 237 (2008) 998–1005.

[7] J. Chen, B. Guo, Strong instability of standing waves for a nonlocal Schrödinger equation, Physica D 227 (2007) 142–148.

[8] E.A. Kuznetsov, J.J. Rasmussen, K. Rypdal, S.K. Turitsyn, Sharper criteria for the wave collapse, Physica D 87 (1995) 273–284.

[9] G. Chen, J. Zhang, Energy criterion of global existence for supercritical nonlinear Schrödinger equation with harmonic potential, J. Math. Phys. 48 (2007) 073513.

[10] J. Huang, J. Zhang, X. Li, Stability of standing waves for the L^2-critical Hartree equations with harmonic potential, Appl. Anal. 92 (10) (2013) 2076–2083.

[11] J. Huang, J. Zhang, Sharp conditions of global existence and scattering for a focusing energy-critical Schrödinger equation, J. Math. Anal. Appl. 410 (2) (2014) 561–576.

[12] J. Huang, J. Zhang, The Cauchy problem of a focusing energy -critical nonlinear Schrödinger equation, Nonlinear Anal. Ser. A: Theory Methods Appl. 80 (2013) 122–134.

[13] M. Reed, B. Simon, Fourier analysis, self-adjointness, in: Methods of Modern Mathematical Physics, Vol. II, Elsevier (Singapore) Pvt. Ltd., 2003.

[14] E.M. Stein, Singular Integral and Differentiability Properties of Functions, in: Princeton Math. Series, vol. 30, Princeton Univ. Press, 1970.

[15] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolations estimates, Comm. Math. Phys. 87 (1983) 567–576.

[16] T. Hmidi, S. Keraani, Blowup theory for the critical nonlinear Schrödinger equations revisited, Int. Math. Res. Not. (46) (2005) 2815–2828.

[17] S. Zhu, J. Zhang, H. Yang, Limiting profile of the blow-up solutions for the fourth-order nonlinear Schrödinger equation, Dyn. Partial Differ. Equ. 7 (2) (2010) 187–205.

[18] M.K. Kwong, Uniqueness of positive solutions of $\triangle u - u + u^p = 0$ in $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 105 (3) (1989) 243–266.

[19] R.T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. 18 (1977) 1794–1797.

[20] R. Carles, P.A. Markowich, C. Sparber, On the Gross-Pitaevskii equation for trapped dipolar quantum gases, Nonlinearity 21 (2008) 2569–2590.

[21] T. Ogawa, Y. Tsutsumi, Blow-up of H^1 solutions of the nonlinear Schrödinger equation, J. Differential Equations 92 (1991) 317–330.