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Complex Analysis and Operator Theory

Approximation by (p, q)-Lorentz Polynomials on a Compact Disk

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Abstract The (p, q)-factors were introduced in order to generalize or unify several forms of q-oscillator algebras well known in the physics literature related to the representation theory of single parameter quantum algebras. This notion has been recently used in approximation by positive linear operators via (p, q)-calculus which has emerged a very active area of research. In this paper, we introduce a new analogue of Lorentz polynomials based on (p, q)-integers. We obtain quantitative estimate in the Voronovskaja's type theorem and exact orders in simultaneous approximation by the complex (p, q)-Lorentz polynomials of degree $n \in \mathbb{N}$ $(q > p > 1)$, attached to analytic functions on compact disks of the complex plane. In this way, we put in evidence the overconvergence phenomenon for the (p, q)-Lorentz polynomial, namely the extensions of approximation properties (with quantitative estimates) from real intervals to compact disks in the complex plane.

Keywords (p, q)-Integer · (p, q)-Lorentz polynomial · Voronovskaja's theorem · Iterates · Compact disk

Mathematics Subject Classification 41A10 · 41A25 · 41A36

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1 Introduction and Preliminaries

In 1986, Lorentz [12], introduced the following sequence of operators defined for any analytic function f in a domain containing the origin

$$L_n(f;z) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k f^{(k)}(0), \quad n \in \mathbb{N}. \tag{1.1}$$

In [7], Gal introduced and studied the q-analouge of the Lorentz operators for $q > 1$, for any analytic function f in a domain containing the origin as follows:

$$L_{n,q}(f;z) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{z}{[n]_q}\right)^k D_q^{(k)}(f)(0), \quad n \in \mathbb{N},\ z \in \mathbb{C} \tag{1.2}$$

Several authors have introduced and studied the approximation properties for different operators in compact disk. For instance, in [13–15] Mahmudov studied q-Stancu polynomials, q-Szász Mirakjan operators and generalised Kantorovich operators; In [5] Gal et al studied q-Szász–Kantorovich operators.

For the last two decades, there have been intensive researches on the approximation of functions by positive linear operators by using q-calculus. Recently, approximation by positive linear operators in post-quantum calculus, namely the (p,q)-calculus, has emerged a very active area of research. The (p,q)-calculus has many interesting applications in several areas of mathematics and mathematical sciences such as in field theory, differential equations, hypergeometric series, oscillator algebra, Lie group, physical sciences (see [3,10,11]). Most recently, Mursaleen et al. introduced the (p,q)-analogues of some well-known operators such as Bernstein operators [17], Bernstein–Stancu operators [18], Bleimann–Butzer–Hahn operators [19], Bernstein–Schurer operarors [20] and Kantorovich variant of (p,q)-Szász–Mirakjan operators [16]. They investigated the approximation properties of above mentioned operators using the techniques of (p,q)-calculus. In the sequel, some more articles on (p,q)-approximation have also been appeared, e.g. [1,2,4,9] and [21].

Let us recall certain notations and definitions of (p,q)-calculus. Let $0 < q < p \leq 1$.

The (p,q) integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots,$$
$$[k]_{p,q}! := \begin{cases} [k]_{p,q}\,[k-1]_{p,q} \ldots 1, & k \geq 1, \\ 1, & k = 0 \end{cases}$$

Now by some simple calculation and induction on n, we have (p,q) -binomial expansion as follows

$$(ax+by)_{p,q}^n := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x+y)^n_{p,q} = (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$
$$(1-x)^n_{p,q} = (1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x)$$

and the (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The (p, q)-derivative of the function f is defined as

$$D_{p,q}f(x) := \frac{f(px)-f(qx)}{(p-q)x},\ x \neq 0,\ \left(D_{p,q}f\right)(0) := f'(0)$$

and $(D_{p,q})f(0) = f'(0)$, provided that f is differentiable at 0. It can be easily seen that $D_{p,q}x^n = [n]_{p,q}x^{n-1}$. Details on (p, q)-calculus can be found in [8,22,23].

Motivated by the recent developments in approximation theory by using (p, q)-calculus, we introduce here (p, q)-Lorentz polynomials. We study the approximation properties based on Voronovskaj's type approximation theorem and also establish some approximation properties of the iterates of these polynomials. Our results generalize the results of Gal [7] (see also [6]) which can be obtained directly by taking $p = 1$ in our results. Note that it is not merely a generalization but working with an extra parameter p gives more flexibility to study a general class of positive linear operators.

Now, with the help of (p, q)-calculus and using above formula, we construct (p, q)-analogue of Lorentz operators (1.1) as follows:

$$L_{n,p,q}(f;z) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{z}{[n]_{p,q}}\right)^k D^{(k)}_{p,q}(f)(0), \quad n \in \mathbb{N},\ z \in \mathbb{C} \tag{1.3}$$

Note that for $p = 1$, (p, q)-Lorentz operators given by (1.3) turn out to be q-analogue of Lorentz operators (1.2).

2 Main Approximation Results

Firstly, we obtain an upper approximation estimate.

Theorem 2.1 *Let $R > q > p > 1$ and $D_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$.*

(i) *Let $1 \leq r < \frac{pr_1}{q} < \frac{pR}{q}$ be arbitrary fixed. Then, for all $|z| \leq r$ and $n \in \mathbb{N}$, we have the upper estimate as*

$$\left|L_{n,p,q}(f)(z) - f(z)\right| \leq \frac{p^n}{[n]_{p,q}} M_{r_1,p,q}(f),$$

where $M_{r_1,p,q}(f) = \frac{p(q-p+1)}{(q-p)^2} \sum_{k=0}^{\infty} |c_k|(k+1)r_1^k < \infty$.

(ii) *Let* $1 \leq r << r^* < \frac{pr_1}{q} < \frac{pR}{q}$ *be arbitrary fixed. Then, for all* $|z| \leq r$, $m, n \in \mathbb{N}$, *we have*

$$\left|L_{n,p,q}^{(m)}(f)(z) - f^{(m)}(z)\right| \leq \frac{p^n}{[n]_{p,q}} M_{r_1,p,q}(f) \frac{m!\, r^*}{(r^* - r)^{m+1}},$$

where $M_{r_1,p,q}(f)$ *is given as at the above point (i).*

Proof (i) For $e_j(z) = z^j$, it is to see that $L_{n,p,q}(e_0)(z) = 1$, $L_{n,p,q}(e_1)(z) = z$. Then we have

$$L_{n,p,q}(e_j)(z) = q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} [j]_{p,q}! \frac{z^j}{[n]_{p,q}^j}, \quad 2 \leq j \leq n, \text{ for all } j, n \in \mathbb{N},$$

by some simple calculation, we get

$$\begin{aligned} L_{n,p,q}(e_j)(z) &= z^j \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)\left(1 - p^{n-2}\frac{[2]_{p,q}}{[n]_{p,q}}\right) \\ &\times \ldots \left(1 - p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right). \end{aligned}$$

It is easy to see that for $j \geq n+1$, we get $L_{n,p,q}(e_j)(z) = 0$. Now it can be easily seen that

$$L_{n,p,q}(f)(z) = \sum_{j=0}^{\infty} c_j L_{n,p,q}(e_j)(z) \quad \text{for all} \quad |z| \leq r.$$

Hence

$$\begin{aligned} &\left|L_{n,p,q}(f)(z) - f(z)\right| \\ &\leq \sum_{j=0}^{n} |c_j| \left|L_{n,p,q}(e_j)(z) - e_j(z)\right| + \sum_{j=n+1}^{\infty} |c_j| \left|L_{n,p,q}(e_j)(z) - e_j(z)\right| \\ &\leq \sum_{j=2}^{n} |c_j| r^j \left| \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)\left(1 - p^{n-2}\frac{[2]_{p,q}}{[n]_{p,q}}\right) \right. \\ &\left. \times \ldots \left(1 - p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right) - 1 \right| \\ &\quad + \sum_{j=n+1}^{\infty} |c_j| r^j, \quad \text{for all} \quad |z| \leq r. \end{aligned}$$

By using the inequality proved in Gal ([7] p. 63), we get

$$\begin{aligned} &1 - \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)\left(1 - p^{n-2}\frac{[2]_{p,q}}{[n]_{p,q}}\right) \ldots \left(1 - p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right) \\ &\leq p^{n-(j-1)} \frac{(j-1)[j-1]_{p,q}}{[n]_{p,q}}. \end{aligned}$$

Thus, we obtain

$$\sum_{j=2}^{n}|c_j|r^j\ \frac{1}{p^{jn}}\ \Big|\left(1-p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)\left(1-p^{n-2}\frac{[2]_{p,q}}{[n]_{p,q}}\right)$$
$$\times\ldots\left(1-p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right)-1\Big|$$
$$\le\frac{p^n}{[n]_{p,q}}\sum_{j=2}^{n}|c_j|\ (j-1)\ [j-1]_{p,q}\ p^{-(j-1)}\ r^j,$$
$$\le\frac{p^{n+1}}{[n]_{p,q}(q-p)}\sum_{j=2}^{n}|c_j|\ j\left(\frac{q}{p}r\right)^j$$
$$\le\frac{p^{n+1}}{[n]_{p,q}(q-p)}\sum_{j=2}^{n}|c_j|\ (j+1)\ r_1^j\quad\text{as}\quad\frac{qr}{p}<r_1,$$

where by hypothesis on f we have $\sum_{j=2}^{n}|c_j|\ (j+1)\ r_1^j<\infty$.

On the other hand, the analyticity of f implies $c_j=\frac{f^{(k)}(0)}{j!}$, and by the Cauchy's estimates of the coefficients c_j in the disk $|z|\le r_1$, we have $|c_j|\le\frac{K_{r_1}}{r-1}$, for all $j\ge 0$, where

$$K_{r_1}=\max\{|f(x)|:|z|\le r_1\}\le\sum_{j=2}^{n}|c_j|r^j\le\sum_{j=2}^{n}|c_j|\ (j+1)\ r_1^j:=R_{r_1}(f)<\infty.$$

Therefore

$$\sum_{j=n+1}^{\infty}|c_j|r^j\le R_{r_1}(f)\left[\frac{r}{r_1}\right]^{n+1}\sum_{j=0}^{\infty}\left(\frac{r}{r_1}\right)^j=R_{r_1}(f)\left[\frac{r}{r_1}\right]^{n+1}\cdot\frac{r_1}{r_1-r}$$
$$=R_{r_1}(f)\frac{r}{r_1-r}\left[\frac{r}{r_1}\right]^n\le R_{r_1}(f)\frac{p^{n+1}}{[n]_{p,q}}\frac{1}{(q-p)^2},$$

and finally we get

$$\left|L_{n,p,q}(f)(z)-f(z)\right|\le\frac{p^{n+1}}{[n]_{p,q}}\frac{(q-p+1)}{(q-p)^2}R_{r_1}(f)$$

for all $n\in\mathbb{N}$ and $|z|\le r$.

(ii) Let γ be the circle of radius $r^*>r$ and center 0. Since for any $|z|\le r$ and $\upsilon\in\gamma$, we have $|\upsilon-z|\ge r^*-r$. By Cauchy's formula it follows that for all $n\in\mathbb{N}$

$$\left|L_{n,p,q}^{(m)}(f)(z)-f^{(m)}(z)\right|=\frac{m!}{2\pi}\left|\int_\gamma\frac{L_{n,p,q}(f)(\upsilon)-f(\upsilon)}{(\upsilon-z)^{m+1}}d\upsilon\right|$$

$$
\begin{aligned}
&\leq \frac{p^{n+1}}{[n]_{p,q}}\, M_{r_1,p,q}(f)\, \frac{m!}{2\pi}\, \frac{2\pi r^*}{(r^*-r)^{m+1}} \\
&= \frac{p^{n+1}}{[n]_{p,q}}\, M_{r_1,p,q}(f)\, \frac{m!\, r^*}{(r^*-r)^{m+1}}.
\end{aligned}
$$

This completes the proof.

Remark 2.1 Since $\frac{p^n}{[n]_{p,q}} = (q-p)\cdot\frac{p^n}{q^n-p^n}$, it follows that the order of approximation in Theorem 2.1 by the (p,q)-Lorentz polynomials is, in fact, $\left(\frac{p}{q}\right)^n$, which by $q > p$ implies that the order of approximation is still geometric. In the same time, the order of approximation by the simpler q -Lorentz polynomials in [4], is $\frac{1}{q^n}$, which clearly is better than $\left(\frac{p}{q}\right)^n$. However, the choice of $p > 1$ implies $\frac{pR}{q} > \frac{R}{q}$, which means that the upper estimates of geometric order in Theorem 2.1 hold in larger disks than those in the case when $p = 1$.

We have the following quantitative Voronovskaja-type results.

Theorem 2.2 *For $R > q^4 > p^4 > 1$, let $f : D_R \to \mathbb{C}$ be analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$ and let $1 \leq r < \frac{p^3 r_1}{q^3} < \frac{p^4 R}{q^4}$ be arbitrary fixed. Then, for all $n \in \mathbb{N}$, $|z| \leq r$, we have*

$$
\left| L_{n,p,q}(f)(z) - f(z) + \frac{S_{p,q}(f)(z)}{[n]_{p,q}} \right| \leq \frac{p^{2n}}{[n]_{p,q}^2}\, Q_{r_1,p,q}(f),
$$

where

$$
\begin{aligned}
S_{p,q}(f)(z) &= \sum_{k=2}^{\infty} p^{n-(k-1)} c_k \frac{[k]_{p,q}-[k]_q}{q-1} z^k \\
&= \sum_{k=2}^{\infty} p^{n-(k-1)} c_k \left([1]_{p,q} + \cdots + [k-1]_{p,q}\right)
\end{aligned}
$$

and $Q_{r_1,p,q} = \frac{pq-q+p}{(p-1)(q-p)} \sum |c_k|(k+1)(k+2)^2 \left(\frac{q}{p} r_1\right)^k < \infty$.

Proof We have

$$
\begin{aligned}
&\left| L_{n,p,q}(f)(z) - f(z) + \frac{S_{p,q}(f)(z)}{[n]_{p,q}} \right| \\
&= \left| \sum_{k=0}^{\infty} c_k \left[L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)}\, \frac{[k]_{p,q}-[k]_q}{p-1}\, e_k(z) \right] \right| \\
&\leq \left| \sum_{k=0}^{n} c_k \left[L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)}\, \frac{[k]_{p,q}-[k]_q}{p-1}\, e_k(z) \right] \right|
\end{aligned}
$$

$$+\left|\sum_{k=n+1}^{\infty} p^{n-(k-1)}\, c_k\, z^k \left(\frac{[k]_{p,q}-[k]_q}{p-1}-1\right)\right|$$
$$\leq \left|\sum_{k=0}^{n} c_k \left[L_{n,p,q}(e_k)(z)-e_k(z)+p^{n-(k-1)}\,\frac{[k]_{p,q}-[k]_q}{p-1}\, e_k(z)\right]\right|$$
$$+\sum_{k=n+1}^{\infty} |c_k|\, r^k \left(p^{n-(k-1)}\,\frac{[k]_{p,q}-[k]_q}{p-1}-1\right),$$

for all $|z| \leq r$ and $n \in \mathbb{N}$.

In what follows, firstly we will prove by mathematical induction with respect to k that

$$0 \leq E_{n,k,p,q}(z) \leq \frac{p^{2n}}{[n]_{p,q}^2}\frac{(k+1)(k-2)^2}{(q-p)}\left(\frac{qr_1}{p}\right)^k, \tag{2.1}$$

for all $2 \leq k \leq n$ (here $n \in \mathbb{N}$ is arbitrary fixed) and $|z| \leq r$, where

$$E_{n,k,p,q}(z) = L_{n,p,q}(e_k)(z)-e_k(z)+\frac{p^{n-(k-1)}}{[n]_{p,q}}\,\frac{[k]_{p,q}-[k]_q}{p-1}\, e_k(z)$$
$$= L_{n,p,q}(e_k)(z)-e_k(z)+\frac{p^{n-(k-1)}}{[n]_{p,q}}\left([1]_{p,q}+\cdots+[k-1]_{p,q}\right)\, e_k(z).$$

By mathematical induction, we easily

$$\frac{[k]_{p,q}-[k]_q}{p-1} = \left([1]_{p,q}+\cdots+[k-1]_{p,q}\right).$$

On the other hand, by the formula for $L_{n,p,q}(e_k)$ in the proof of Theorem 2.1 (i), simple calculation leads to $E_{n,2,p,q}(z) = 0$, for all $n \in \mathbb{N}$ and to the recurrence relation

$$E_{n,k,p,q}(z) = -\frac{z^2}{[n]_{p,q}}p^{n-(k-1)}\, D_{p,q}[L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)]$$
$$+\frac{p-1}{p}z[L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)]$$
$$+\frac{z}{p}\, E_{n,k-1,p,q}(z), \quad |z| \leq r.$$

Now, for $|z| \leq r$ and $3 \leq k \leq n$ and applying the mean value theorem in complex analysis, with notation $\|f\|_r = \max\{|f(z)| : |z| \leq r\}$, we get

$$\left|E_{n,k,p,q}(z)\right| = \frac{r^2}{[n]_{p,q}}\, p^{n-(k-1)}\, \|\left(L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)\right)'\|_{\frac{qr}{p}}$$
$$+\frac{(p-1)}{p}r\, \|L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)\|_{\frac{qr}{p}}$$

$$+\ \frac{r}{p}\ \left|E_{n,k-1,p,q}(z)\right|$$
$$=\frac{r^2}{[n]_{p,q}}\ p^{n-(k-1)}\ \frac{(k-1)}{\frac{q}{p}r}\ \|\left(L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)\right)\|_{\frac{qr}{p}}$$
$$+\frac{(p-1)}{p}r\ \|L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)\|_{\frac{qr}{p}}\ +\ \frac{r}{p}\ \left|E_{n,k-1,p,q}(z)\right|$$
$$=\frac{r^2}{[n]_{p,q}}\ p^{n-(k-1)}\ \frac{(k-1)}{\frac{qr}{p}}\ \left(\frac{qr}{p}\right)^{k-1}\ \frac{(k-2)[k-2]_{p,q}}{[n]_{p,q}}\ p^{n-(k-1)}$$
$$+\frac{(p-1)r}{p}\ \left(\frac{qr}{p}\right)^{k-1}\ \frac{(k-2)[k-2]_{p,q}}{[n]_{p,q}}\ p^{n-(k-1)}\ +\ \frac{r}{p}\ \left|E_{n,k-1,p,q}(z)\right|$$
$$=\left\{\frac{r^2}{[n]_{p,q}}p^{n-(k-1)}\frac{(k-1)}{qr/p}+\frac{(p-1)}{p}r\right\}\left(\frac{qr}{p}\right)^{k-1}\ \frac{(k-2)[k-2]_{p,q}}{[n]_{p,q}}\ p^{n-(k-1)}$$
$$+\frac{r}{p}\ \left|E_{n,k-1,p,q}(z)\right|$$
$$\leq\frac{p^{2n}}{[n]_{p,q}^2}(k+1)(k-2)[k-2]_{p,q}r_1^k\ +\ r_1\left|E_{n,k-1,p,q}(z)\right|$$

Now on taking $k=1,2,3,\ldots,$ step by step, we easily obtain the estimate

$$\left|E_{n,k,p,q}(z)\right|\leq\frac{p^{2n}}{[n]_{p,q}^2}r_1^k\sum_{j=3}^{k}(j-1)(j-2)[j-2]_{p,q}$$
$$\leq\frac{p^{2n}}{[n]_{p,q}^2}\frac{(k+1)(k-2)^2}{(q-p)}\left(\frac{qr_1}{p}\right)^k.$$

Now we calculate

$$\left|\sum_{k=0}^{n}|c_k|\left[L_{n,p,q}(e_k)(z)-e_k(z)+q^{n-(k-1)}\ \frac{[k]_{p,q}-[k]_q}{p-1}\ e_k(z)\right]\right|$$
$$\leq\sum_{k=0}^{n}\ |c_k|\ \left|E_{n,k,p,q}(z)\right|$$
$$\leq\frac{p^{2n}}{[n]_{p,q}^2}\frac{1}{(q-p)}\sum_{k=0}^{n}|c_k|(k+1)(k-2)^2\left(\frac{qr_1}{p}\right)^k$$
$$\leq\frac{p^{2n}}{[n]_{p,q}^2}\frac{1}{(q-p)}\sum_{k=0}^{n}|c_k|(k+1)(k+2)^2\left(\frac{qr_1}{p}\right)^k$$

On the other hand, since $(p^{n-(k-1)}\ \frac{[k]_{p,q}-[k]_q}{p-1}-1)\geq 0$ for all $k\geq n+1$, similar to proof of Theorem 2.1 (i), we get

$$\sum_{k=n+1}^{\infty}\ |c_k|r^k\left(p^{n-(k-1)}\frac{[k]_{p,q}-[k]_q}{p-1}-1\right)\leq\sum_{k=n+1}^{\infty}\ p^{n-(k-1)}\ |c_k|\ r^k\frac{[k]_{p,q}}{(p-1)[n]_{p,q}}$$

$$\leq \sum_{k=n+1}^{\infty} p^{n-(k-1)} \, |c_k| \, \frac{1}{(p-1)[n]_{p,q}} \frac{kq^k}{p^k(q-p)}$$
$$\leq \frac{R_{r_1}(f)\, p^{n+1}}{(p-1)[n]_{p,q}} \sum_{k=n+1}^{\infty} \frac{r^k}{r_1^k} \, q^k p^{-k}$$
$$\leq \frac{R_{r_1}(f)\, p^{n+1}}{(p-1)[n]_{p,q}} \sum_{k=n+1}^{\infty} \left[\left(\frac{r}{r_1}\right)^{1/3}\right]^k \left[\left(\frac{r}{r_1}\right)^{1/3}\right]^{2k} q^k p^{-k}$$
$$\leq \frac{R_{r_1}(f)\, p^{n+1}}{(p-1)[n]_{p,q}} \left(\frac{r}{r_1}\right)^{\frac{(n+1)}{3}} \sum_{k=0}^{\infty} \left[\left(\frac{r}{r_1}\right)^{1/3}\right]^k$$
$$= \frac{R_{r_1}(f)\, p^{n+1}}{(p-1)[n]_{p,q}} \left(\frac{r}{r_1}\right)^{\frac{n}{3}} \frac{r^{\frac{1}{3}}}{(r_1^{\frac{1}{3}} - r^{\frac{1}{3}})}$$
$$\leq \frac{p^{2n+2}}{[n]_{p,q}^2} \frac{R_{r_1}(f)}{(p-1)\,(q-p)^2}$$
$$\leq \frac{p^{2n+2}}{[n]_{p,q}^2\,(p-1)\,(q-p)^2} \sum_{k=0}^{n} |c_k| \, (k+1)(k+2)^2 \left(\frac{q}{p} r_1\right)^k,$$

where we used the inequalities, $[k]_{p,q} \leq \frac{kq^k}{p^k}$, $\frac{p^n}{q^n} \leq \frac{p^n}{(q-p)[n]_{p,q}}$ and $\frac{r^{1/3}}{(r_1^{1/3} - r^{1/3})} \leq \frac{p}{(q-p)}$. Hence, by combining all above estimates, we have

$$\left| L_{n,p,q}(f)(z) - f(z) + \frac{S_{p,q}(f)(z)}{[n]_{p,q}} \right| \leq \frac{(pq-q+p-1)}{(p-1)\,(q-p)^2} \, \frac{p^{2n}}{[n]_{p,q}^2} \times \sum_{k=0}^{n} |c_k| \, (k+1)(k+2)^2 \, \left(\frac{q}{p} r_1\right)^k.$$

This completes the proof.

The following result gives the lower approximation estimate.

Theorem 2.3 *Let $R > p^4/q^4 > 1$, $f : D_R \to \mathbb{C}$ be analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$ and let $1 \leq r < \frac{q^3 r_1}{p^3} < \frac{q^4 R}{p^4}$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then, we have*

$$\left\| L_{n,p,q}(f) - f \right\|_r \geq \frac{p^n}{[n]_{p,q}} C_{r,r_1,p,q}(f),$$

for all $n \in \mathbb{N}$ and $|z| \leq r$, where the constant $C_{r,r_1,p,q}(f)$ depends only on f, r and r_1. Here $\|f\|_r$ denotes $\max_{|z| \leq r}\{|f(z)|\}$.

Proof For $S_{n,p,q}(f)(z)$ as defined in Theorem 2.3, all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&L_{n,p,q}(f)(z) - f(z)\\
&= \frac{p^n}{[n]_{p,q}} \left\{ -S_{p,q}(f)(z) + \frac{p^n}{[n]_{p,q}} \left[\frac{[n]_{p,q}^2}{p^{2n}} \left(L_{n,p,q}(f)(z) \right.\right.\right.\\
&\left.\left.\left. -f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right) \right] \right\}.
\end{aligned}$$

Using the following inequality

$$\|F + G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r.$$

We have

$$\begin{aligned}
&\|L_{n,p,q}(f) - f\|_r\\
&\geq \frac{p^n}{[n]_{p,q}} \left\{ \|S_{p,q}(f)(z)\| \right.\\
&\left. -\frac{p^n}{[n]_{p,q}} \left[\frac{[n]_{p,q}^2}{p^{2n}} \left\| L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right\|_r \right] \right\}.
\end{aligned}$$

Since by hypothesis f is not a polynomial of degree ≤ 1 in D_R, we get $\|S_{p,q}(f)\|_r > 0$.

Indeed, supposing the contrary it follows that $S_{p,q}(f)(z) = 0$ for all $z \in \overline{D_R} = \{z \in \mathbb{C} : |z| \leq r\}$.

A simple calculation yields $S_{p,q}(f)(z) = z\, \frac{D_{p,q}(f)(z) - f'(z)}{p-q}$, $S_{p,q}(f)(z) = 0$ implies that $D_{p,q}(f)(z) = f'(z)$, for all $z \in \overline{D_r} \backslash \{0\}$. Taking into account the representation of f as $f(z) = \sum_{k=0}^{\infty} c_k z^k$, the last inequality immediately leads to $c_k = 0$, for all $k \geq 2$, which means that f is linear in $\overline{D_r}$, a contradiction with hypothesis.

Now, by Theorem 2.2 we have

$$\frac{[n]_{p,q}^2}{p^{2n}} \left\| L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right\|_r \leq Q_{r_1,p,q}(f),$$

where $Q_{r_1,p,q}(f)$ is a positive constant depending only on f, r_1, p and q.

Since $\frac{p^n}{[n_{p,q}]} \to 0$ as $n \to \infty$, there exists an index $n_\circ$ depending only on f, r, r_1, p and q such that for all $n > n_\circ$, we have

$$\begin{aligned}
&\|S_{p,q}(f)(z)\| - \frac{p^n}{[n]_{p,q}} \left[\frac{[n]_{p,q}^2}{p^{2n}} \left\| L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right\|_r \right]\\
&\geq \frac{1}{2} \left\| S_{p,q}(f) \right\|_r,
\end{aligned}$$

which immediately implies that

$$\|L_{n,p,q}(f) - f\|_r \geq \frac{p^n}{[n]_{p,q}} \frac{1}{2} \|S_{p,q}(f)(z)\|_r, \quad \text{for all } n > n_\circ.$$

For $n \in \{1, \ldots, n_\circ\}$, we have $\|L_{n,p,q}(f) - f\|_r \geq \frac{p^n}{[n]_{p,q}} M_{r,r_1,n,p,q}$ with $M_{r,r_1,n,p,q} = \frac{[n]_{p,q}}{p^n} \|L_{n,p,q}(f) - f\|_r > 0$ (if $\|L_{n,p,q}(f) - f\|_r$) would be equal to 0, this would imply that f is a linear function, a contradiction).

Therefore, finally we get $\|L_{n,p,q}(f) - f\|_r \geq \frac{p^n}{[n]_{p,q}} C_{r,r_1,p,q}(f)$ for all $n \in \mathbb{N}$, where

$$C_{r,r_1,p,q}(f) = \min \left\{ M_{r,r_1,1,p,q}(f), \ldots, M_{r,r_1,n_\circ,p,q}(f), \frac{1}{2}\|S_{p,q}(f)\| \right\}.$$

This completes the proof.

Combining Theorem 2.3, and Theorem 2.1 (i), we immediately get the following result.

Corollary 2.4 *Let $R > p^4/q^4 > 1$, $f : D_R \to \mathbb{C}$ be analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$ and let $1 \leq r < \frac{q^3 r_1}{p^3} < \frac{q^4 R}{p^4}$ be arbitrary fixed. If f is not a polynomial of degree ≤ 1, then for all $n \in \mathbb{N}$ and $|z| \leq r$, we have*

$$\|L_{n,p,q}(f) - f\|_r \sim \frac{p^n}{[n]_{p,q}},$$

where the constants in the equivalence depend only on f, r, r_1, p and q but are independent of n.

Remark 2.2 It can be easily verified that as $\lim_{n \longrightarrow \infty} \frac{p^n}{[n]_{p,q}} \longrightarrow 0$. Note that for $p = 1$, (p, q)-Lorentz operators given by (1.3) turn out to be q-Lorentz operators and also the rate of convergence is exactly same as given by Gal in [7].

Concerning the simultaneous approximation, we prove the following:

Theorem 2.5 *Let $R > p^4/q^4 > 1$, $f : D_R \to \mathbb{C}$ be analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$ and let $1 \leq r < r^* < \frac{p^3 r_1}{q^3} < \frac{p^4 R}{q^4}$ be arbitrary fixed. Also let $m \in \mathbb{N}$. If f is not a polynomial of degree $\leq \max\{1, m-1\}$, then for all $n \in \mathbb{N}$, we have*

$$\|L^{(m)}_{n,p,q}(f) - f^{(m)}\|_r \sim \frac{p^n}{[n]_{p,q}},$$

where the constants in the equivalence depend only on f, r, r_1, m, p and q but are independent of n.

Proof We already have the upper estimate for $\|L^{(m)}_{n,p,q}(f) - f^{(m)}\|_r$, by Theorem 2.1 (ii), so it remains to find the lower estimate.

Let us denote by Γ the circle of the radius r^* and center 0. We have that the inequality $|\upsilon - z| \geq r^* - r$ holds for all $|z| \leq r$ and $\upsilon \in \Gamma$. Cauchy's formula is expressed by

$$\left|L^{(m)}_{n,p,q}(f)(z) - f^{(m)}(z)\right| = \frac{m!}{2\pi}\left|\int_{\gamma} \frac{L_{n,p,q}(f)(\upsilon) - f(\upsilon)}{(\upsilon - z)^{m+1}} d\upsilon\right| \tag{2.2}$$

Now, as in the proof of Theorem 2.1 (ii), for all $\upsilon \in \Gamma$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} &L_{n,p,q}(f)(z) - f(z) \\ &= \frac{p^n}{[n]_{p,q}}\Bigg\{ - S_{p,q}(f)(z) \\ &+ \frac{p^n}{[n]_{p,q}}\left[\frac{[n]^2_{p,q}}{p^{2n}}\left(L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z)\right)\right]\Bigg\} \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we get

$$\begin{aligned} L^{(m)}_{n,p,q}(f) - f^{(m)}(f) &= \frac{p^n}{[n]_{p,q}}\Bigg\{ \frac{m!}{2\pi i}\int_{\Gamma} - \frac{S_{p,q}(f)(z)}{(\upsilon - z)^{m+1}} d\upsilon \\ &+ \frac{p^n}{[n]_{p,q}} \cdot \frac{m!}{2\pi i}\int_{\Gamma} \frac{[n]^2_{p,q}\left(L_{n,p,q}(f)(z) - f(z) + p^n \frac{S_{p,q}(f)(z)}{[n]_{p,q}}\right)}{p^{2n}(\upsilon - z)^{m+1}} d\upsilon\Bigg\} \\ &= \frac{p^n}{[n]_{p,q}}\Bigg\{[-S_{p,q}(f)(z)]^{(m)} + \frac{p^n}{[n]_{p,q}} \cdot \frac{m!}{2\pi i} \\ &\times \int_{\Gamma} \frac{[n]^2_{p,q}\left(L_{n,p,q}(f)(z) - f(z) + p^n \frac{S_{p,q}(f)(z)}{[n]_{p,q}}\right)}{p^{2n}(\upsilon - z)^{m+1}} d\upsilon\Bigg\}. \end{aligned}$$

Hence

$$\begin{aligned} \|L^{(m)}_{n,p,q} - f^{(m)}\|_r &\geq \frac{p^n}{[n]_{p,q}}\Bigg\{\left\|-\left[S_{p,q}(f)\right]^{(m)}\right\|_r \\ &- \frac{p^n}{[n]_{p,q}}\left\|\frac{m!}{2\pi}\int_{\Gamma} \frac{[n]^2_{p,q}\left(L_{n,p,q}(f)(z) - f(z) + p^n \frac{S_{p,q}(f)(z)}{[n]_{p,q}}\right)}{p^{2n}(\upsilon - z)^{m+1}} d\upsilon\right\|_r\Bigg\}. \end{aligned}$$

Now by using Theorem 2.2 for all $n \in \mathbb{N}$, we get

$$\left\|\frac{m!}{2\pi}\int_{\Gamma} \frac{[n]^2_{p,q}\left(L_{n,p,q}(f)(z) - f(z) + p^n \frac{S_{p,q}(f)(z)}{[n]_{p,q}}\right)}{p^{2n}(\upsilon - z)^{m+1}} d\upsilon\right\|_r$$

$$\leq \frac{m!}{2\pi} \frac{2\pi r^*[n]_{p,q}^2}{(r^*-r)^{m+1} p^{2n}} \left\| L_{n,p,q}(f) - f + p^n \frac{S_{p,q}(f)}{[n]_{p,q}} \right\|_{r^*}$$
$$\leq Q_{r_1,p,q}(f). \frac{m!\, r^*}{(r^*-r)^{m+1}}.$$

But by hypothesis on f, we have $\| - [S_{p,q}(f)]^{(m)} \|_{r^*} > 0$. Indeed, supposing the contrary, it would follow that $[S_{p,q}(f)]^{(m)}(z) = 0$, for all $|z| \leq r^*$, where by the statement of Theorem 2.2, we have

$$S_{p,q}(f)(z) = \sum_{k=2}^{\infty} q^{n-(k-1)} c_k \frac{[k]_{p,q} - [k]_q}{q-1} z^k$$
$$= \sum_{k=2}^{\infty} q^{n-(k-1)} c_k \left([1]_{p,q} + \cdots + [k-1]_{p,q}\right) z^k.$$

Firstly, supposing that $m = 1$, by $S'_{p,q}(f)(z) = \sum_{k=2}^{\infty} q^{n-(k-1)} c_k\, k\, ([1]_{p,q} + \cdots + [k-1]_{p,q}) z^{k-1} = 0$, for all $|z| \leq r^*$, which would imply that $c_k = 0$, for all $k \geq 2$, that is, f would be a polynomial of degree $1 = \max\{1, m-1\}$, a contradiction with the hypothesis.

Taking $m = 2$, we would get $S''_{p,q}(f)(z) = \sum_{k=2}^{\infty} q^{n-(k-1)}\, c_k\, k\, (k-1)\, ([1]_{p,q} + \cdots + [k-1]_{p,q}) z^{k-2} = 0$, for all $|z| \leq r^*$, which immediately would imply that $c_k = 0$, for all $k \geq 2$, that is, f would be a polynomial of degree $1 = \max\{1, m-1\}$, a contradiction with the hypothesis.

Now, taking $m > 2$, for all $|z| \leq r^*$, we would get

$$S_{p,q}^{(m)}(f)(z) = \sum_{k=m}^{\infty} q^{n-(k-1)}\, c_k\, k\, (k-1) \ldots (k-m+1)$$
$$\left([1]_{p,q} + \cdots + [k-1]_{p,q}\right) z^{k-m} = 0,$$

for all $|z| \leq r^*$, which would imply that $c_k = 0$, for all $k \geq m$, that is, f would be a polynomial of degree $m-1 = \max\{1, m-1\}$, a contradiction with the hypothesis.

This completes the proof.

Finally, we prove some approximation results for the iterates of (p, q) -Lorentz operators.

For f analytic in D_R that is of the form $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, let us define the iterates of complex Lorentz operators $L_{n,p,q}(f)(z)$, by $L_{n,p,q}^{[1]}(f)(z) = L_{n,p,q}(f)(z)$ and $L_{n,p,q}^{[m]}(f)(z) = L_{n,p,q}[L_{n,p,q}^{[m-1]}(f)](z)$, for any $m \in \mathbb{N},\ m \geq 2$.

Since we have $L_{n,p,q}(f)(z) = \sum_{k=0}^{\infty} c_k L_{n,p,q}(e_k)(z)$, by recurrence for all $m \geq 1$, we get that $L_{n,p,q}^{[m]}(f)(z) = \sum_{k=0}^{\infty} c_k L_{n,p,q}^{[m]}(e_k)(z)$, where $L_{n,p,q}^{[m]}(e_k)(z) = 1,$ if $k = 0,\ L_{n,p,q}^{[m]}(e_k)(z) = z$ if $k = 1,\ L_{n,p,q}^{[m]}(e_k)(z) = 0,$ if $k \geq n+1$ and

$$L_{n,p,q}^{[m]}(e_j)(z) = \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)^m \left(1 - p^{n-2}\frac{[2]_{p,q}}{[n]_{p,q}}\right)^m \\ \dots \left(1 - p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right)^m \; z^k,$$

for $2 \leq k \leq n$.

We present the following:

Theorem 2.6 *Let $R > p > q > 1$ and $1 \leq r < \frac{pr_1}{q} < \frac{pR}{q}$ be arbitrary fixed. Suppose that $f : D_R \to \mathbb{C}$ is analytic in $D_R := \{z \in \mathbb{C} : |z| < R\}$, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$.Then, we have*

$$\left\| L_{n,p,q}^{[m]}(f) - f \right\|_r \leq \frac{mp^n}{[n]_{p,q}} \frac{q-p+1}{(q-p)^2} \sum_{k=0}^{\infty} |c_k|(k+1)r_1^k.$$

Moreover, if $\lim_{n\to\infty} \frac{m_n p^n}{[n]_{p,q}} = 0$, *then*

$$\lim_{n\to\infty} \left\| L_{n,p,q}^{[m_n]}(f) - f \right\|_r = 0.$$

Proof For all $|z| \leq r$, we easily obtain

$$\left| f(z) - L_{n,p,q}^{[m]}(f)(z) \right| \\ \leq \sum_{k=2}^{n} |c_k| r^k \left[1 - \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)^m \dots \left(1 - p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right)^m \right] \\ + \sum_{k=n+1}^{\infty} |c_k| r^k.$$

Denoting $A_{k,n} = \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right) \dots \left(1 - p^{n-(j-1)}\frac{[k-1]_{p,q}}{[n]_{p,q}}\right)$, we get $1 - A_{k,n}^m = (1 - A_{k,n})(1 + A_{k,n} + A_{k,n}^2 + \dots + A_{k,n}^{m-1}) \leq m(1 - A_{k,n})$ and therefore since $1 - A_{k,n} \leq p^{n-(k-1)} \frac{(k-1)[k-1]_{p,q}}{[n]_{p,q}}$, for all $|z| \leq r$, we obtain

$$\sum_{k=2}^{n} |c_k| r^k \left[1 - \left(1 - p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}\right)^m \dots \left(1 - p^{n-(j-1)}\frac{[j-1]_{p,q}}{[n]_{p,q}}\right)^m \right] + \sum_{k=n+1}^{\infty} |c_k| r^k \\ \leq m \sum_{k=2}^{\infty} |c_k| r^k p^{1-(k-1)} \; [1 - A_{k,n}] \leq \frac{mp^{n+1}}{[n]_{p,q}} \sum_{k=2}^{\infty} |c_k| r^k (k-1)[k-1]_{p,q} r^k \\ \leq \frac{mp^{n+1}}{[n]_{p,q}} \sum_{k=2}^{\infty} |c_k| r^k \frac{kq^k/p^k}{q-p} \leq \frac{mp^{n+1}}{[n]_{p,q}} \frac{1}{q-p} \sum_{k=2}^{\infty} |c_k|(k+1)\left(\frac{q}{p}r\right)^k$$

$$\leq \frac{mp^{n+1}}{[n]_{p,q}} \frac{1}{q-p} \sum_{k=2}^{\infty} |c_k|(k+1)(r_1)^k.$$

On the other hand, following exactly the reasonings in the proof of the Theorem 2.1, we get the estimate

$$\sum_{k=n+1}^{\infty} |c_k| r^k \leq \frac{p^{n+1}}{[n]_{p,q}} \frac{\sum_{k=0}^{\infty} |c_k|(k+1)(r_1)^k}{(q-p)^2} \leq \frac{mp^{n+1}}{[n]_{p,q}} \cdot \frac{\sum_{k=0}^{\infty} |c_k|(k+1)(r_1)^k}{(q-p)^2}.$$

Collecting now all the estimates and taking into account that $\frac{1}{q-p} + \frac{1}{(q-p)^2} = \frac{q-p+1}{(q-p)^2}$, we arrive at the desired estimate.

Since $\frac{p^n}{[n]_{p,q}} \sim \frac{p6n}{q^n}$, it follows the conclusion that

$$\lim_{n\to\infty} \left\| L^{[m_n]}_{n,p,q}(f) - f \right\|_r = 0.$$

if $\lim_{n\to\infty} \frac{m_n p^n}{[n]_{p,q}} = 0$.

This completes the proof.

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