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# A new double integral inequality and application to stability test for time-delay systems☆

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A B S T R A C T

This paper is concerned with stability analysis for linear systems with time delays. Firstly, a new double integral inequality is proposed. Then, it is used to derive a new delay-dependent stability criterion in terms of linear matrix inequalities (LMIs). Two numerical examples are given to demonstrate the effectiveness and merits of the present result.



## 1. Introduction

Consider the following system with state and distributed delays:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h) + A_D \int_{t-h}^{t} x(s)ds, \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A$, $A_d$, $A_D \in \mathbb{R}^{n\times n}$ are constant matrices, $h$ is a constant time delay satisfying $h > 0$, and $\phi(t)$ is a continuous vector-valued initial function.

The stability of system (1) keeps attracting researchers for many years. In order to reduce conservatism of stability criteria, a number of techniques are presented, including for instance, the free-weighting matrix method [1–4], reciprocally convex approach [5,6] and various integral inequality methods [7–15]. The well-known Jensen's inequality is commonly adopted as it could lead to a stability test with fewer matrix variables.

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Recently, a so-called Wirtinger-based integral inequality developed in [8] is shown more powerful than Jensen's inequality. Later, some other types of integral inequalities have been reported in [3,4,10–14] to further reduce the conservatism of the stability test.

This paper will present a new double integral inequality which includes those in [10,11] as special cases. A new stability criteria is established by applying the newly proposed inequality. Two numerical examples are given to illustrate the present results.

## 2. Main result

The following two lemmas are useful in the sequel.

**Lemma 2.1** (*[10,11]*). *For a positive definite matrix $R > 0$, and any continuously differentiable function $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{b-a}{\Omega_1}^T R\Omega_1 + \frac{3}{b-a}{\Omega_2}^T R\Omega_2 + \frac{5}{b-a}{\Omega_3}^T R\Omega_3 \tag{2}$$

*where*

$$\begin{aligned}
\Omega_1 &= x(b) - x(a), \\
\Omega_2 &= x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds, \\
\Omega_3 &= x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu.
\end{aligned}$$

**Lemma 2.2** (*[10,11]*). *For a positive definite matrix $R > 0$, and any continuously differentiable function $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b\int_u^b \dot{x}^T(s)R\dot{x}(s)dsdu \geq 2{\Omega_4}^T R\Omega_4 + 4{\Omega_5}^T R\Omega_5 \tag{3}$$

*where*

$$\begin{aligned}
\Omega_4 &= x(b) - \frac{1}{b-a}\int_a^b x(s)ds, \\
\Omega_5 &= x(b) + \frac{2}{b-a}\int_a^b x(s)ds - \frac{6}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu.
\end{aligned}$$

In order to reduce the conservatism of inequality (3), we propose the following result which will be used in the development.

**Lemma 2.3.** *For a positive definite matrix $R > 0$, and any continuously differentiable function $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b\int_u^b \dot{x}^T(s)R\dot{x}(s)dsdu \geq 2{\Omega_4}^T R\Omega_4 + 4{\Omega_5}^T R\Omega_5 + 6{\Omega_6}^T R\Omega_6 \tag{4}$$

*where $\Omega_4$ and $\Omega_5$ are the same as in (3), and*

$$\Omega_6 = x(b) - \frac{3}{b-a}\int_a^b x(s)ds + \frac{24}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu - \frac{60}{(b-a)^3}\int_a^b\int_u^b\int_s^b x(r)drdsdu.$$

**Proof.** For $\varphi(s) \in \mathbb{R}$ and an integrable function $w(s)$ in $[a, b] \to \mathbb{R}^n$, we define

$$p_i = \int_a^b \int_u^b \varphi_i^2(s)dsdu, \quad \Omega_i(w) = \int_a^b \int_u^b \varphi_i(s)w(s)dsdu$$

and let $V = \int_a^b \int_u^b w^T(s)Rw(s)dsdu$, $z(s) = \sum_{i=1}^{\infty} \frac{1}{p_i}\varphi_i(s)\Omega_i(w)$. If

$$\int_a^b \int_u^b \varphi_i(s)\varphi_j(s)dsdu = 0 \tag{5}$$

for $i = 1, 2, 3 \dots$ and $i \neq j$, it is obvious that

$$\begin{aligned}
0 &\leq \int_a^b \int_u^b [w(s) - z(s)]^T R[w(s) - z(s)]dsdu \\
&= \int_a^b \int_u^b \{w^T(s)Rw(s) - 2z^T(s)Rw(s) + z^T(s)Rz(s)\}dsdu \\
&= \int_a^b \int_u^b w^T(s)Rw(s)dsdu - 2\int_a^b \int_u^b \left(\sum_{i=1}^{\infty} \frac{1}{p_i}\varphi_i(s)\Omega_i(w)\right)^T Rw(s)dsdu \\
&\quad + \int_a^b \int_u^b \left(\sum_{i=1}^{\infty} \frac{1}{p_i}\varphi_i(s)\Omega_i(w)\right)^T R\left(\sum_{i=1}^{\infty} \frac{1}{p_i}\varphi_i(s)\Omega_i(w)\right) dsdu \\
&= V - 2\sum_{i=1}^{\infty} \frac{1}{p_i}\Omega_i^T(w)R\int_a^b \int_u^b \varphi_i(s)w(s)dsdu + \int_a^b \int_u^b \left\{\sum_{i=1}^{\infty} \frac{1}{p_i^2}\varphi_i^2(s)\Omega_i^T(w)R\Omega_i^T(w)\right\} dsdu \\
&= V - \sum_{i=1}^{\infty} \frac{1}{p_i}\Omega_i^T(w)R\Omega_i(w)
\end{aligned}$$

which gives

$$\int_a^b \int_u^b w^T(s)Rw(s)dsdu \geq \sum_{i=1}^{\infty} \frac{1}{p_i}\Omega_i^T(w)R\Omega_i(w). \tag{6}$$

Let

$$\varphi_1 = 1, \qquad \varphi_2 = s - \frac{2b + a}{3}, \qquad \varphi_3 = \left(s - \frac{3b + 2a}{5}\right)^2 - \frac{3(b - a)^2}{50}. \tag{7}$$

It is easy to see that $\varphi_i$ $(i = 1, 2, 3)$ in (7) satisfy (5) , and hence we can easily derive

$$p_1 = \frac{(b - a)^2}{2}, \qquad p_2 = \frac{(b - a)^4}{36}, \qquad p_3 = \frac{(b - a)^6}{600}.$$

Then, we have from (6) that

$$V \geq \frac{2}{(b - a)^2}\Omega_1{}^T R\Omega_1 + \frac{36}{(b - a)^4}\Omega_2{}^T R\Omega_2 + \frac{600}{(b - a)^6}\Omega_3{}^T R\Omega_3. \tag{8}$$

Letting $w(s) = \dot{x}(s)$, we obtain

$$\begin{aligned}
\Omega_1(\dot{x}) &= (b - a)\left\{x(b) - \frac{1}{b - a}\int_a^b x(s)ds\right\}, \\
\Omega_2(\dot{x}) &= \frac{(b - a)^2}{3}\left\{x(b) + \frac{2}{b - a}\int_a^b x(s)ds - \frac{6}{(b - a)^2}\int_a^b \int_u^b x(s)dsdu\right\}, \\
\Omega_3(\dot{x}) &= \frac{(b - a)^3}{10}\left\{x(b) - \frac{3}{b - a}\int_a^b x(s)ds + \frac{24}{(b - a)^2}\int_a^b \int_u^b x(s)dsdu\right. \\
&\qquad \left. - \frac{60}{(b - a)^3}\int_a^b \int_u^b \int_s^b x(r)drdsdu\right\}.
\end{aligned}$$

Substituting the above into (8) yields (4). This completes the proof.

**Remark 2.1.** Lemma 2.3 is based on inequality (6) with special choices of $w(s) = \dot{x}(s)$ and $i = 1, 2, 3$. It is easy to see that the result of Lemma 2.3 improves that of Lemma 2.2 and thus it could provide improvement over those using double integral inequalities in [9–11].

We now present our main result using integral inequality (4).

**Theorem 2.1.** *Given $h > 0$, system (1) is asymptotically stable if there exist positive definite matrices $P \in \mathbb{R}^{4n\times 4n}$, $Q, S, R \in \mathbb{R}^{n\times n}$, such that the following LMI holds:*

$$\begin{aligned} \Psi &= sym(\Pi_1^T P \Pi_2) + e_1^T Q e_1 - e_2^T Q e_2 + h^2 e_0^T S e_0 + \frac{h^2}{2} e_0^T R e_0 - \Pi_3^T S \Pi_3 \\ &\quad - 3\Pi_4^T S \Pi_4 - 5\Pi_5^T S \Pi_5 - 2\Pi_6^T R \Pi_6 - 4\Pi_7^T R \Pi_7 - 6\Pi_8^T R \Pi_8 < 0 \end{aligned} \tag{9}$$

*where*

$$\Pi_1 = \begin{bmatrix} e_1^T, & e_3^T, & e_4^T, & e_5^T \end{bmatrix}^T, \quad \Pi_2 = \begin{bmatrix} e_0^T, & e_1^T - e_2^T, & he_1^T - e_3^T, & \frac{h^2}{2} e_1^T - e_4^T \end{bmatrix}^T,$$

$$\Pi_3 = e_1 - e_2, \quad \Pi_4 = e_1 + e_2 - \frac{2}{h} e_3,$$

$$\Pi_5 = e_1 - e_2 + \frac{6}{h} e_3 - \frac{12}{h^2} e_4, \quad \Pi_6 = e_1 - \frac{1}{h} e_3,$$

$$\Pi_7 = e_1 + \frac{2}{h} e_3 - \frac{6}{h^2} e_4, \quad \Pi_8 = e_1 - \frac{3}{h} e_3 + \frac{24}{h^2} e_4 - \frac{60}{h^3} e_5,$$

$$e_0 = Ae_1 + A_d e_2 + A_D e_3,$$

*and $e_i \in \mathbb{R}^{n\times 5n}$ is defined as $e_i = \begin{bmatrix} 0_{n\times(i-1)n}, & I_n, & 0_{n\times(5-i)n} \end{bmatrix}$ for $i = 1, 2, \ldots, 5$.*

**Proof.** Define

$$\eta(t) = \left[ x^T(t), \quad \int_{t-h}^t x^T(s)ds, \quad \int_{t-h}^t \int_u^t x^T(s)dsdu, \quad \int_{t-h}^t \int_u^t \int_s^t x^T(r)drdsdu \right]^T,$$

$$\xi(t) = \left[ x^T(t), \quad x^T(t-h), \quad \int_{t-h}^t x^T(s)ds, \quad \int_{t-h}^t \int_u^t x^T(s)dsdu, \quad \int_{t-h}^t \int_u^t \int_s^t x^T(r)drdsdu \right]^T.$$

Consider a Lyapunov–Krasovskii functional candidate

$$V(t) = \sum_{i=1}^4 V_i(t) \tag{10}$$

where

$$V_1(t) = \eta^T(t) P \eta(t), \qquad V_2(t) = \int_{t-h}^t x^T(s) Q x(s) ds,$$

$$V_3(t) = h \int_{t-h}^t \int_u^t \dot{x}^T(s) S \dot{x}(s) dsdu, \qquad V_4(t) = \int_{t-h}^t \int_u^t \int_s^t \dot{x}^T(r) R \dot{x}(r) drdsdu.$$

The time derivative of $V(t)$ can be computed as follows:

$$\begin{aligned} \dot{V}_1(t) &= 2\dot{\eta}^T(t) P \eta(t), \\ \dot{V}_2(t) &= x(t)^T Q x(t) - x(t-h)^T Q x(t-h), \\ \dot{V}_3(t) &= h^2 \dot{x}^T(t) S \dot{x}(t) - h \int_{t-h}^t \dot{x}^T(s) S \dot{x}(s) ds, \\ \dot{V}_4(t) &= \frac{h^2}{2} \dot{x}^T(t) R \dot{x}(t) - \int_{t-h}^t \int_u^t \dot{x}^T(s) R \dot{x}(s) dsdu, \end{aligned}$$

**Table 1**
Upper bounds on $h$ obtained for Example 3.1.

| Methods | Maximum $h$ allowed | NoDv |
|---|---|---|
| [2] | 1.6339 | 85 |
| Theorem 6 [8] | 1.877 | 16 |
| Theorem 1 [9] | 1.9504 | 59 |
| Theorem 3 [4] | 2.0395 | 75 |
| Theorem 1[11] | 2.0395 | 27 |
| Theorem 1 | 2.0402 | 45 |
| Analytical bound | 2.0412 | – |

**Table 2**
Upper bounds on $h$ obtained for Example 3.2.

| Methods | Maximum $h$ allowed | NoDv |
|---|---|---|
| Theorem 6 [8] | 0.126 | 16 |
| Theorem 1 [9] | 0.126 | 59 |
| Theorem 3 [4] | 0.577 | 75 |
| Theorem 1 [10] | 0.577 | 96 |
| Theorem 1 [11] | 0.577 | 27 |
| Theorem 1 | 0.675 | 45 |

and it can be rewritten as

$$\begin{aligned}\dot{V}(t) = \xi^T(t)\left\{sym(\Pi_1^T P\Pi_2) + e_1^T Qe_1 - e_2^T Qe_2 + h^2 e_0^T Se_0 + \frac{h^2}{2}e_0^T Re_0\right\}\xi(t)\\ - h\int_{t-h}^{t}\dot{x}^T(s)S\dot{x}(s)ds - \int_{t-h}^{t}\int_{u}^{t}\dot{x}^T(s)R\dot{x}(s)dsdu.\end{aligned} \tag{11}$$

Applying Lemmas 2.1 and 2.3 to the above leads to

$$-h\int_{t-h}^{t}\dot{x}^T(s)S\dot{x}(s)ds \le \xi^T(t)(-\Pi_3^T S\Pi_3 - 3\Pi_4^T S\Pi_4 - 5\Pi_5^T S\Pi_5)\xi(t), \tag{12}$$

$$-\int_{t-h}^{t}\int_{u}^{t}\dot{x}^T(s)R\dot{x}(s)dsdu \le \xi^T(t)[-2\Pi_6^T R\Pi_6 - 4\Pi_7^T R\Pi_7 - 6\Pi_8^T R\Pi_8]\xi(t). \tag{13}$$

Hence, we have $\dot{V}(t) \le \xi^T(t)\Psi\xi(t)$. This completes the proof.

## 3. Numerical examples

In this section, two examples are used to illustrate the effectiveness of the proposed method.

**Example 3.1.** Consider system (1) with:

$$A = \begin{bmatrix}0.2 & 0\\ 0.2 & 0.1\end{bmatrix}, \qquad A_d = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad A_D = \begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}.$$

Table 1 lists the computed upper bounds which keep the system stability by different methods. It is obvious that our proposed approach produces better results, which demonstrate the advantage of our method. It should be stressed that our method returns a satisfactory result which is quite close to the analytical limit.

**Example 3.2.** Consider system (1) with:

$$A = \begin{bmatrix}0 & 1\\ -100 & -1\end{bmatrix}, \qquad A_d = \begin{bmatrix}0 & 0.1\\ 0.1 & 0.2\end{bmatrix}, \qquad A_D = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Table 2 lists the computed upper bounds by different methods and it shows that our method produces better results.

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