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A fast trigonometric collocation method for some elliptic pseudodifferential equations

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ABSTRACT

In this paper, we propose and analyze a fast trigonometric collocation method for a class of periodic elliptic pseudodifferential equations, whose pseudodifferential operators can always be represented as the sum of a principal part and a smoothing operator. We show that the whole matrix representation for the principal part in our discrete linear system can be generated by only computing $\mathcal{O}(n)$ entries rather than computing all entries of the matrix, where $2n$ or $2n+1$ is the size of the matrix. The dense matrix for the smoothing operator can be compressed into a sparse matrix with only $\mathcal{O}(n\log n)$ nonzero entries. We also prove that our proposed method preserves the optimal convergent order the same as without compression. Some numerical experiments for solving three cases of boundary integral equations are presented to demonstrate its approximate accuracy and computational efficiency, verifying the theoretical estimates.

1. Introduction

We consider in this paper solving a class of singular integral equations or system of elliptic pseudodifferential equations

$$\mathcal{T}u = f, \tag{1.1}$$

where u and f are 2π-periodic, complex-valued functions. Those periodic pseudodifferential operators $\mathcal{T}$ have always the form $\mathcal{A}+\mathcal{B}$, where $\mathcal{B}$ is a smoothing operator and the principal part $\mathcal{A}$ has a homogeneous symbol $\sigma(x, l)$, that is,

$$\mathcal{A}u(x) = \sum_{l\in\mathbb{Z}} \sigma(x, l)\hat{u}(l)e^{ilx}, \tag{1.2}$$

where $\hat{u}(l), l \in \mathbb{Z}$ are the Fourier coefficients of u and $\sigma(x, l)$ are the global symbol. This class of pseudodifferential equations includes boundary integral equations of various types, such as Cauchy singular integral equations, hypersingular integral equations and elliptic integro-differential equations. All of these equations are used to solve many different problems in applications (see for example [1,31,35,36,39–42]).

Some traditional methods, such as Nyström, Galerkin, Petrov–Galerkin, collocation methods and spectral methods, are developed for numerically solving those problems (see, for example, [2–14,30,32–34,37,38]). These methods enjoy nice convergence properties but at the same time suffer from having large computational costs. This is due to the coefficient matrix of the corresponding discrete linear system is usually a dense matrix which leads to the computation on generating the matrix and as well solving the resulting linear system is huge.

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To overcome this drawback, developing a fast method for generating those dense matrix is highly desirable. Some fast methods, such as fast wavelet methods, fast multipole methods and H-matrix methods, are developed for solving some boundary integral equations (see, for example, [15,2,16–19]). Fast wavelet Galerkin methods and fast wavelet collocation methods were developed for solving Fredholm integral equations of the second kind by compressing the dense coefficient matrix into a sparse matrix without loss of convergence (see, for example, [20,21,2,17]). Fast Fourier Galerkin methods was developed in [16] for solving a class of singular boundary integral equation which is a special case of pseudodifferential equation (1.1). The boundary integral operator considered in [16] can also be written as the form $\mathcal{A}+\mathcal{B}$, and the principal part $\mathcal{A}$ has the Fourier basis functions as its eigenfunctions. This property leads to the matrix representation of the operator $\mathcal{A}$ under the Fourier basis is a diagonal matrix. A truncation strategy was developed for compressing the dense matrix generated from the compact operator $\mathcal{B}$ into a sparse matrix without loss of critical information encoded in the matrix. This leads to this Fourier Galerkin method fast.

In comparison, collocation methods receive the most favorable attention due to the lower computational cost in evaluating the integrals and generating the coefficient matrix of the corresponding discrete equations (see, for example, [22–24,3,25,8,9]). A trigonometric collocation method and its fully discrete version were considered for singular integral equations and periodic pseudodifferential equations in [6–8]. Some spectral collocation methods and their modified versions were also considered for the first kind integral equations (see, for example, [11,13,18]). Multiscale collocation methods were studied thoroughly for solving Fredholm integral equations of the second kind and applied to ill-posed integral equations (see, for example, [20,26,21,11]).

In this paper, we aim to develop a fast trigonometric collocation scheme for numerically solving this class of elliptic pseudodifferential equations (1.1), where the pseudodifferential operator $\mathcal{T}$ can be written as $\mathcal{A}+\mathcal{B}$, and the principal part $\mathcal{A}$ has a homogeneous symbol $\sigma(x,l)$ and $\mathcal{B}$ is a smoothing operator. We choose suitable trial function spaces and collocation functionals to construct a trigonometric collocation scheme, which leads to the whole matrix for the operator $\mathcal{A}$ in our discrete linear system can be generated by only computing $\mathcal{O}(n)$ entries rather than computing all entries of the matrix, where $2n$ or $2n+1$ is the size of matrix. The truncation technique is also used thoroughly to compress the dense matrix for the operator $\mathcal{B}$ into a sparse matrix without loss of critical information encoded in the matrix. Our fast trigonometric collocation method still preserves the convergence property of the trigonometric collocation method but reduce the number of nonzero entries of the matrix from $\mathcal{O}(n^2)$ to $\mathcal{O}(n\log n)$. This leads to the fast method.

This paper is organized as follows. In Section 2, we introduce the trigonometric collocation method for solving a class of singular integral equations or pseudodifferential equations by choosing suitable collocation functionals and trial function spaces. In Section 3, we describe theoretically the fast trigonometric collocation method and present the convergence analysis for the approximation solution. In the last section, we apply our fast theory to some boundary integral equations and illustrate the theoretical estimates by some numerical examples.

2. Trigonometric collocation methods for PPDEs

Let us first describe the periodic pseudodifferential equations. To this end, we introduce periodic pseudodifferential operator (PPDO) first (see [8]). We denote the complex Fourier coefficient of u by $\hat{u}(l) := \int_I e^{-i2\pi lx}u(x)dx$, for $l\in\mathbb{Z}$. An operator $\mathcal{A}$ is called the pseudodifferential operator with order $\beta\in\mathbb{R}$, if it admits a Fourier series representation

$$\mathcal{A}u(x)=\sum_{l\in\mathbb{Z}}\sigma(x,l)\hat{u}(l)e^{ilx},\quad \text{for } x\in I \tag{2.3}$$

where the global symbol $\sigma:\mathbb{R}\times\mathbb{Z}\to\mathbb{C}$ is $\mathcal{C}^\infty(\mathbb{R})$ in its first argument and is homogeneous of degree β in its second argument, that is,

$$\sigma(x,l)=[l]^\beta\sigma(x,\operatorname{sgn}(l)),\qquad x\in I,\ l\in\mathbb{Z},\ \beta\in\mathbb{R}. \tag{2.4}$$

Here we have used the notation

$$[l]=\begin{cases}|l|, & l\neq 0,\\ 1, & l=0,\end{cases}\quad \text{and}\quad \operatorname{sgn}(l)=\begin{cases}1, & l\geq 0,\\ -1, & l<0.\end{cases}$$

Our problem is to solve a periodic pseudodifferential equation

$$(\mathcal{A}+\mathcal{B})u=f, \tag{2.5}$$

where u and f are 2π-periodic, complex-valued functions. The operator $\mathcal{A}$ is a pseudodifferential operator with order β defined by (2.3) and the operator $\mathcal{B}$ is always a smoothing operator denoted by

$$\mathcal{B}u(x)=\int_I b(x,y)u(y)dy, \tag{2.6}$$

with a $\mathcal{C}^\infty$ kernel $b(x,y):I\times I\to\mathbb{C}$.

A large class periodic pseudodifferential equation has the form (2.5) including boundary integral equations of various types, such as Cauchy singular integral equations, hypersingular integral equation and elliptic integro-differential equations. Specially, when the global symbol $\sigma(x,l)$ is independent on the first argument x, such as $\sigma(x,l) = \delta(l)$, then the pseudodifferential operator $\mathcal{A}$ can admit a Fourier series representation

$$\mathcal{A}u(x) = \sum_{l\in\mathbb{Z}} \delta(l)\hat{u}(l)e^{ilx}, \quad \text{for } x \in I. \tag{2.7}$$

The operator $\mathcal{A}$ has obviously the Fourier function e^{ilx} as its eigenfunctions. The periodic pseudodifferential equation $(\mathcal{A}+\mathcal{B})u = f$ includes many boundary integral equations (see, for example, [16]).

Now we consider the trigonometric collocation method for Eq. (2.5). To this end, we denote $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_n^+ := \{1, 2, \ldots, n-1\}$, $\mathbb{Z}_n := \mathbb{Z}_n^+ \bigcup \{0\}$ for $n \in \mathbb{N}$. We use $e_l(x)$ to represent trigonometric monomials $e_l(x) = \frac{1}{\sqrt{2\pi}}e^{ilx}$, $l \in \mathbb{Z}$. We let $\mathbb{X} := C(I)$ be the set of all 2π-periodic, continuous functions and construct finite dimensional subspace $\mathbb{X}_n \subset \mathbb{X}$ as

$$\mathbb{X}_n = \operatorname{span}\{e_l(x) : |l| \le n\}, \quad n \in \mathbb{N}_0.$$

To construct the collocation method, we choose $2n + 1$ points to set up the collocation point set

$$\mathbb{W}_{2n+1} := \left\{ x_m : x_m = \frac{2m\pi}{2n+1}, m \in \mathbb{Z}_{2n+1} \right\}.$$

To construct the collocation scheme, we also need a linear functional l_j given by

$$l_j = \frac{2\pi}{2n+1} \sum_{m\in\mathbb{Z}_{2n+1}} e_{-j}(x_m)\delta_{x_m}, \quad |j| \in \mathbb{Z}_{n+1}, \tag{2.8}$$

where we use δ_s to denote the linear functional in $\mathbb{X}^*$ defined by the equation $\langle \delta_s, v\rangle = v(s)$ for $v \in \mathbb{X}$ and for any $s \in I$. The functionals $\{l_j, |j| \in Z_{n+1}\}$ and the Fourier basis $\{e_l, |l| \in \mathbb{Z}_{n+1}\}$ satisfy the orthogonality

$$\langle l_j, e_l \rangle = \begin{cases} 1, & j = l, \\ 0, & j \ne l, \end{cases} \quad |j| \in \mathbb{Z}_{n+1}, \ |l| \in \mathbb{Z}_{n+1}.$$

We also introduce a trigonometric interpolation operator $\mathcal{Q}_n : \mathbb{X} \to \mathbb{X}_n$ defined by (see [27])

$$\mathcal{Q}_n u(x) = \frac{1}{2n+1} \sum_{m\in\mathbb{Z}_{2n+1}} D_n(x - x_m)u(x_m), \quad \text{for any } x_m \in \mathbb{W}_{2n+1}, \tag{2.9}$$

where $D_n(x)$ is the Dirichlet kernel defined as $D_n(x) := \sum_{|l|\in\mathbb{Z}_{n+1}} e^{ilx}$, $x \in I$. Obviously, the trigonometric interpolation operator $\mathcal{Q}_n$ satisfy the interpolation conditions

$$(\mathcal{Q}_n u)(x_m) = u(x_m), \quad x_m \in \mathbb{W}_{2n+1},$$

and has the property that

$$\mathcal{Q}_n u(x) = \sum_{|j|\in\mathbb{Z}_{n+1}} \langle l_j, u \rangle e_j(x). \tag{2.10}$$

It is known from [22] that $\|\mathcal{Q}_n\| = \mathcal{O}(\log n)$.

The Hölder–Zygmund space $\mathcal{H}^s$ ($s \in \mathbb{R}$) of order s is defined in this paper by

$$\mathcal{H}^s := \left\{ f : f \in C^m \text{ and } \left[\mathcal{D}^m f\right]^\alpha \le \infty \right\}$$

for $s = m + \alpha$ with $m \in \mathbb{N}_0$, $0 < \alpha \le 1$ with the finite norm

$$\|f\|_{\mathcal{H}^s} := \|f\|_{C^m} + \left[\mathcal{D}^m f\right]^\alpha,$$

where $\|f\|_{C^m} = \sum_{j=0}^m \|\mathcal{D}^j f\|_\infty$, and $[\cdot]^\alpha$ is the seminorm

$$[f]^\alpha := \begin{cases} \sup\limits_{h>0} \dfrac{\|\triangle_h f\|_\infty}{h^\alpha}, & 0 < \alpha < 1 \\ \sup\limits_{h>0} \dfrac{\|\triangle_h^2 f\|_\infty}{h^\alpha}, & \alpha = 1 \end{cases}$$

with $(\triangle_h f)(t) := f(t+h) - f(t)$ and $(\triangle_h^2 f)(t) := f(t+2h) - 2f(t+h) + f(t)$.

Next we recall a result from [6] about the approximation property of trigonometric interpolation operator $\mathcal{Q}_n$.

Lemma 2.1. *If* $0 < s < r < \infty$ *and* $f \in \mathcal{H}^r$, *then*

$$\|\mathcal{Q}_n f - f\|_{\mathcal{H}^s} \leq c n^{s-r} \log n \|f\|_{\mathcal{H}^r},$$

where the constant c *depends only on the integer part of* r.

The trigonometric collocation method for solving (2.5) is to seek an approximation $u_n \in \mathbb{X}_n$ such that

$$\mathcal{Q}_n (\mathcal{A} + \mathcal{B}) u_n = \mathcal{Q}_n f. \tag{2.11}$$

We define the operator $\mathcal{A}_n = \mathcal{Q}_n \mathcal{A}|_{\mathbb{X}_n}$, $\mathcal{B}_n = \mathcal{Q}_n \mathcal{B}|_{\mathbb{X}_n}$ and $f_n = \mathcal{Q}_n f$, then the operator Eq. (2.11) can be reformed as

$$(\mathcal{A}_n + \mathcal{B}_n) u_n = f_n. \tag{2.12}$$

The matrix form is to seek a vector $\mathbf{u}_n := \left[\alpha_{-n}, \alpha_{-(n-1)}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n\right]^T$ such that the function

$$u_n(x) := \sum_{l=-n}^{n} \alpha_l [l]^{-\beta} e_l(x), \quad x \in I, \tag{2.13}$$

satisfy the property that

$$\left\langle l_j, (\mathcal{A} + \mathcal{B}) u_n \right\rangle = \left\langle l_j, f \right\rangle, \quad |j| \in \mathbb{Z}_{n+1}. \tag{2.14}$$

Equivalently, we obtain the linear system of equations

$$(\mathbf{A}_n + \mathbf{B}_n) \mathbf{u}_n = \mathbf{f}_n, \tag{2.15}$$

where $\mathbf{f}_n := \left[f_j : |j| \in \mathbb{Z}_{n+1}\right]^T$ with $f_j := \left\langle l_j, f \right\rangle$, and

$$\mathbf{A}_n = \left[a_{j,l} : |j|, |l| \in \mathbb{Z}_{n+1}\right], \qquad \mathbf{B}_n = \left[b_{j,l} : |j|, |l| \in \mathbb{Z}_{n+1}\right],$$

where $a_{j,l}, b_{j,l}, |j|, |l| \in \mathbb{Z}_{n+1}$ represent the entries defined respectively by

$$a_{j,l} := \left\langle l_j, \sigma(x, \operatorname{sgn}(l)) e_l \right\rangle, \qquad b_{j,l} := \left\langle l_j, [l]^{-\beta} \mathcal{B} e_l \right\rangle. \tag{2.16}$$

Next we provide some approximation properties of u_n. The proof can be found in [6,7].

Lemma 2.2. *If* $\mathcal{T} : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ $(s \in \mathbb{N}_0,\ \beta \in \mathbb{R}$ *and* $\beta \leq s)$ *is invertible, then for all n sufficiently large, the solutions of* (2.5) *and* (2.11) *satisfy the following:*

(i)

$$\|u_n\|_{C^s} \leq c \log^2 n \|u\|_{C^s}; \tag{2.17}$$

(ii) *if* $\beta \in \mathbb{Z}$, *then*

$$\|u_n\|_{C^s} \leq c \log n \left\{ \left\|\mathcal{P}^+ u\right\|_{C^s} + \left\|\mathcal{P}^- u\right\|_{C^s} \right\}, \tag{2.18}$$

with $\mathcal{P}^+ u(t) := \sum_{l \geq 0} \hat{u}(l) e^{ilt}$, $\mathcal{P}^- u(t) := \sum_{l \leq -1} \hat{u}(l) e^{ilt}$.

Theorem 2.3. *If* $\mathcal{T} : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ $(s \in \mathbb{N}_0,\ \beta \in \mathbb{R}$ *and* $\beta \leq s)$ *is invertible and* $u \in \mathcal{H}^t$ $(s < t)$ *is the exact solution satisfying* (2.5), *then for all n sufficiently large, there exists an approximation solution* u_n *of* (2.11), *and*

(i)

$$\|u_n - u\|_{C^s} \leq c n^{s-t} \log^2 n \|u\|_{\mathcal{H}^t}; \tag{2.19}$$

(ii) *when* $\beta \in \mathbb{Z}$,

$$\|u_n - u\|_{C^s} \leq c n^{s-t} \log n \|u\|_{\mathcal{H}^t}. \tag{2.20}$$

3. Fast trigonometric collocation methods

In this section, we develop a fast trigonometric collocation method for solving Eq. (2.12). It is known that the coefficient matrix $\mathbf{A}_n$ and $\mathbf{B}_n$ are dense due to the global property of integral operators $\mathcal{A}$ and $\mathcal{B}$. Then when the order of the full coefficient matrix is large, the computational cost for generating the matrix and then solving the corresponding linear system is huge. So we need to choose a fast method to generate two dense matrix $\mathbf{A}_n$ and $\mathbf{B}_n$. To this aim, we develop an algorithm to generate the whole dense matrix $\mathbf{A}_n$ by only computing at most $\mathcal{O}(n)$ entries of $\mathbf{A}_n$ rather than all of the entries. We also need to choose a truncation strategy to compress the dense matrix $\mathbf{B}_n$ into a sparse one which need to compute only $\mathcal{O}(n \log n)$

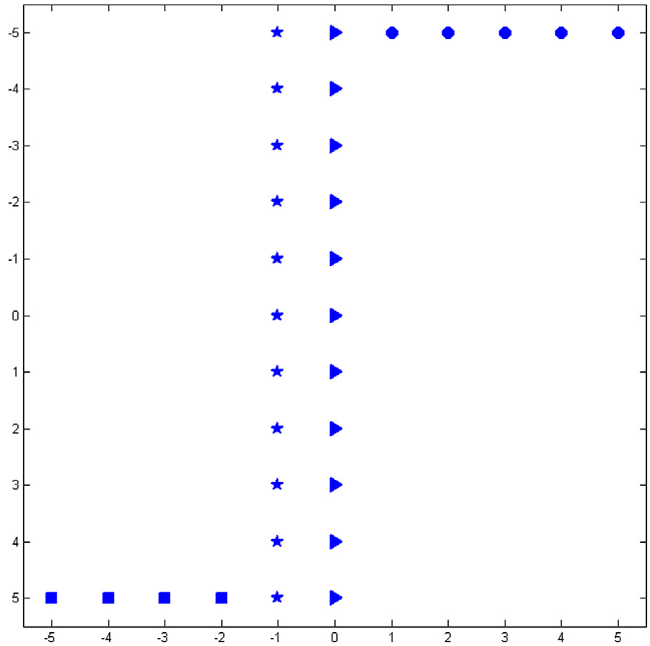

Fig. 1. $6n+1$ entries of the set $\mathcal{C}(\mathbf{A}_n)$ $(n=5)$.

nonzeros entries of $\mathbf{B}_n$ while preserving the convergence property of the method. This leads the method fast to generate both dense matrix $\mathbf{A}_n$ and $\mathbf{B}_n$.

For convenience, we use $\mathcal{C}(\mathbf{G})$ in this paper to denote a set of its entries which can generate the whole matrix $\mathbf{G}$ and we use $|\mathcal{C}(\mathbf{G})|$ to denote the number of $\mathcal{C}(\mathbf{G})$. We will show that the computation of the dense matrix $\mathbf{A_n}$ can be reduced largely from $\mathcal{O}(n^2)$ to $\mathcal{O}(n)$ under our collocation scheme.

Proposition 3.1. *Under our collocation scheme (2.11), there exists a set $\mathcal{C}(\mathbf{A}_n)$ for generating the matrix $\mathbf{A}_n$ as following*

$$\mathcal{C}(\mathbf{A}_n)=\left\{a_{j,l}|(j,l)\in\triangle,\ or\ (j,l)\in\heartsuit,\ or\ (j,l)\in\diamondsuit,\ or\ (j,l)\in\nabla\right\},\tag{3.21}$$

where $\triangle=\{(j,l)|j=n \text{ and } -l\in\mathbb{Z}^+_{n+1}\backslash\{1\}\}$, $\heartsuit=\{(j,l)|\,|j|\in\mathbb{Z}_{n+1} \text{ and } l=-1\}$, $\diamondsuit=\{(j,l)|\,|j|\in\mathbb{Z}_{n+1} \text{ and } l=0\}$, $\nabla=\{(j,l)|j=-n \text{ and } l\in\mathbb{Z}^+_{n+1}\}$.

Proof. It follows from (2.16) that

$$a_{j,l}=\frac{2\pi}{2n+1}\sum_{m=0}^{2n}e_{l-j}(x_m)\sigma(x_m,\operatorname{sgn}(l)),\quad x_m=\frac{2m\pi}{2n+1},\ m\in\mathbb{Z}_{2n+1},\ |j|,|l|\in\mathbb{Z}_{n+1}.$$

For any (l,j) and (l_0,j_0), $|j_0|\in\mathbb{Z}_{n+1}$, $|l_0|\in\mathbb{Z}_{n+1}$, it is clear that when $\operatorname{sgn}(l)=\operatorname{sgn}(l_0)$ and $l-j=l_0-j_0$, we have $a_{j,l}=a_{j_0,l_0}$. With this property and the entries of $\triangle$, $\heartsuit$, $\diamondsuit$ and ∇, it is straightforward to generate the whole matrix $\mathbf{A}_n$. In fact, after having the entries of the set $\triangle$, we can obtain all the entries $a_{j,l}=a_{n,l_0}$ satisfy the condition $\{(j,l):2\le j\le n,\ -n\le l\le -2\}$ and $l-j=l_0-n$ where $-l_0\in\mathbb{Z}^+_{n+1}\backslash\{1\}$. Similarly, having the entries of the set $\heartsuit$ in hand, we can obtain entries $a_{j,l}=a_{j_0,-1}$ satisfy the condition $\{(j,l):-n\le j\le n,\ -n\le l\le -2\}$ and $l-j=-1-j_0$ where $|j_0|\in\mathbb{Z}_{n+1}$. We can obtain the entries $a_{j,l}$, $1\le j\le n$, $-n\le l\le n$ after having the set $\diamondsuit$ similarly and the entries $a_{j,l}$, $1\le j\le n$, $-1\le l\le n$ by the set ∇. $\square$

Theorem 3.2. *For all $n\in\mathbb{N}$, there exists a set $\mathcal{C}(\mathbf{A}_n)$ such that*

$$|\mathcal{C}(\mathbf{A}_n)|\le 6n+1.$$

Proof. It is easy to obtain the result by the set $\mathcal{C}(\mathbf{A}_n)$ in the Proposition 3.1. $\square$

The following algorithm will describe the detail of generating the matrix $\mathbf{A}_n$.

Algorithm 3.3 (*Generate the Matrix $\mathbf{A}_n$*)**.** We first compute the entries in the set $\mathcal{C}(\mathbf{A}_n)$ from Step 1 to Step 4, then generate the whole matrix by Step 5 and Step 6 in the following.

Step 1. Set $j=n$ and compute $a_{j,l}$ for $-l\in\mathbb{Z}^+_{n+1}\setminus\{1\}$. We use ■ in Fig. 1 to denote the entries from Step 1.
Step 2. Set $l=-1$ and compute $a_{j,l}$ for $|j|\in\mathbb{Z}_{n+1}$. We use ★ in Fig. 1 to denote the entries from Step 2.

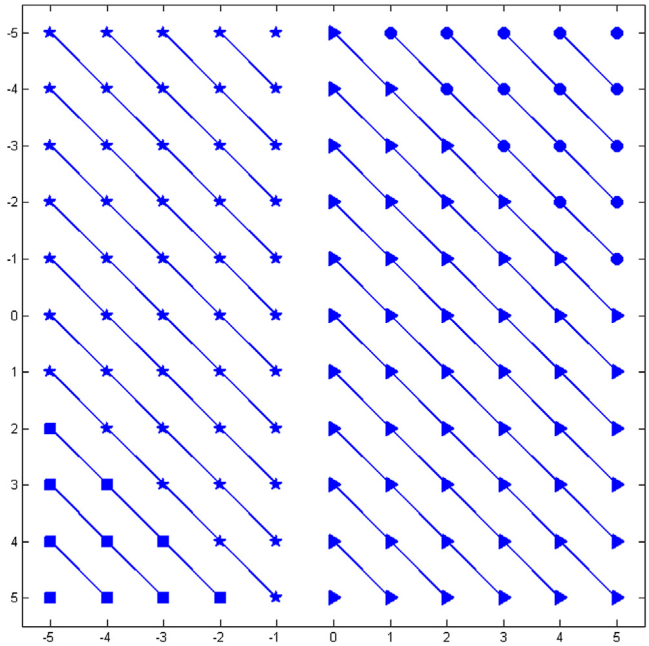

Fig. 2. The dense matrix $\mathbf{A_n}$ ($n = 5$).

Step 3. Set $l = 0$ and compute $a_{j,l}$ for $|j| \in \mathbb{Z}_{n+1}$. We use ▶ in Fig. 1 to denote the entries from Step 3.
Step 4. Set $j = -n$ and compute $a_{j,l}$ for $l \in \mathbb{Z}^+_{n+1}$. We use • in Fig. 1 to denote the entries from Step 4.
Step 5. For $-n \le j \le n-1$ and $-n \le l \le -2$, we set $a_{j,l} := a_{j+1,l+1}$. We use the line connecting the points of equal in Fig. 2.
Step 6. For $-n+1 \le j \le n$ and $1 \le l \le n$, we set $a_{j,l} := a_{j-1,l-1}$. We use the line connecting the points of equal in Fig. 2.

Remark. We remark here that when the global symbol $\sigma(x, l)$ is independent on the first argument x, such as, $\sigma(x, l) = \lambda(l)$, $\lambda(l) : \mathbb{Z} \to \mathbb{C}$ is homogeneous of degree β, the set

$$\mathcal{C}(\mathbf{A}_n) = \{a_{-1,-1}, a_{0,0}\}.$$

The operator $\mathcal{A}$ in this special case can have a Fourier series representation

$$\mathcal{A}u(x) = \sum_{l\in\mathbb{Z}} \lambda(l)\hat{u}(l)e^{ilx}, \tag{3.22}$$

and we can conclude that the operator $\mathcal{A}$ has the Fourier basis functions e^{ilx} as its eigenfunctions, that is, $\mathcal{A}e^{ilx} = \lambda(l)e^{ilx}$. The coefficient matrix $\mathbf{A}_n$ is diagonal, that is,

$$\mathbf{A}_n = \operatorname{diag}(a_{-1,-1}, \dots, a_{-1,-1}, a_{0,0}, \dots, a_{0,0}). \tag{3.23}$$

Next, we consider the second dense matrix $\mathbf{B}_n$. Since the operator $\mathcal{B}$ has a smooth kernel $b(x, y)$, then the coefficient matrix $\mathbf{B}_n$ has decay property, that is, the entries $b_{j,l}$ become smaller when j or l go larger. So we can choose a truncation strategy to compress the dense matrix $\mathbf{B}_n$ into a sparse matrix. Inspiring by the hyperbolic curve truncation strategy in [16] for fast Fourier Galerkin methods, we use the similar strategy to compress the matrix $\mathbf{B}_n$ and develop a fast trigonometric collocation method for solving Eq. (2.12).

To do this, we choose the truncation strategy

$$\tilde{b}_{j,l} := \begin{cases} b_{j,l}, & |jl| \le n, \\ 0, & \text{otherwise} \end{cases} \tag{3.24}$$

and let $\tilde{\mathbf{B}}_n = \left[\tilde{b}_{j,l} : |j|, |l| \in \mathbb{Z}_{n+1}\right]$. The matrix $\widetilde{\mathbf{B}}_n$ is illustrated in Fig. 3.

In next theorem we show the number of nonzero entries in matrix $\widetilde{\mathbf{B}}_n$ should be $\mathcal{O}(n)$, which leads the method fast. To this end, let $\mathcal{N}(\mathbf{G})$ be the number of nonzero entries in matrix $\mathbf{G}$. The proof can be found in [15,16].

Theorem 3.4. *For the truncation strategy (3.24), for all n we have that*

$$\mathcal{N}\left(\widetilde{\mathbf{B}}_n\right) \le cn\log n.$$

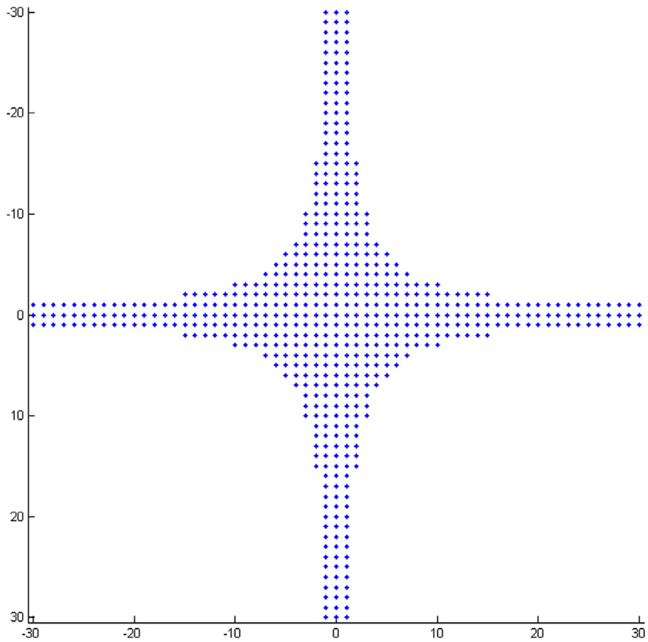

Fig. 3. The matrix $\widetilde{\mathbf{B}}_n$ ($n = 30$).

Using the sparse matrix $\widetilde{\mathbf{B}}_n$ to replace the dense matrix $\mathbf{B}_n$ in (2.15), we can obtain the truncated linear system

$$\left(\mathbf{A}_n + \widetilde{\mathbf{B}}_n\right) \tilde{\mathbf{u}}_n = \mathbf{f}_n, \tag{3.25}$$

where $\tilde{\mathbf{u}}_n := \left[\tilde{\alpha}_{-l}, \tilde{\alpha}_l : l \in \mathbb{Z}_{n+1}\right]^T$ is the corresponding solution to be determined. The truncated linear system (3.25) leads to a fast method for solving the periodic pseudodifferential equation (2.5).

In the rest of this section, we will show the approximation solution of the truncated linear system (3.25) still preserve the convergence property of the collocation method (2.15).

To this purpose, we let $\tilde{\mathcal{B}}_n$ be the linear operator such that its matrix representation under the basis $\{e_l : |l| \in \mathbb{Z}_{n+1}\}$ and the collocation functionals $\left\{l_j : \lfloor j \rfloor \in \mathbb{Z}_{n+1}\right\}$ having the matrix representation $\mathbf{A}_n^{-1}\widetilde{\mathbf{B}}_n$. Solving the linear system (3.25) is equivalent to seeking $\widetilde{u}_n := \sum_{l=-n}^{n} \tilde{\alpha}_l [l]^{-\beta} e_l(x) \in \mathbb{X}_n$ such that

$$\left(\mathcal{A}_n + \tilde{\mathcal{B}}_n\right) \tilde{u}_n = f_n. \tag{3.26}$$

By the definition of $\mathcal{B}_n$ and $\tilde{\mathcal{B}}_n$, we have $\mathcal{Q}_n\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right) = \mathcal{B}_n - \tilde{\mathcal{B}}_n$. Therefore, (3.26) is equivalent to

$$\mathcal{Q}_n\left(\mathcal{T} + \tilde{\mathcal{B}}_n - \mathcal{B}_n\right) \tilde{u}_n = \mathcal{Q}_n f. \tag{3.27}$$

The next result concerns the difference between $\mathcal{B}_n$ and $\tilde{\mathcal{B}}_n$. For this purpose, we denote by $\lceil x \rceil$ the smallest integer larger than x and set $\Pi_j := \{\lceil \frac{n}{j} \rceil, \lceil \frac{n}{j} \rceil + 1, \ldots, n\}$ for $j \in \mathbb{Z}_n^+$.

Lemma 3.5. *For any $w \in \mathbb{X}_n$ and $s, r > 0$, if $k \geq \lceil r \rceil + \lceil s \rceil + 2$ and $b(x, y) \in C^{k,k}(\mathbb{R}^2)$, then*

$$\left\|(\mathcal{B}_n - \tilde{\mathcal{B}}_n)w\right\|_{\mathcal{H}^s} \leq cn^{-r} \|w\|_\infty. \tag{3.28}$$

Proof. For any $w \in \mathbb{X}_n$, it can be written as the following form

$$w(x) := \sum_{\lfloor j \rfloor \in \mathbb{Z}_{n+1}} w_j e_j(x), \tag{3.29}$$

where the coefficients $w_j = \langle l_j, w \rangle$, $\lfloor j \rfloor \in \mathbb{Z}_{n+1}$. We denote the vectors in (3.29) by $\mathbf{w} = \left[w_{-j}, w_j : j \in \mathbb{Z}_{n+1}\right]$ for any $\lfloor j \rfloor \in \mathbb{Z}_{n+1}$.

Recall the error in $\mathcal{H}^s$-norm [6], there holds

$$\|f\|_{\mathcal{H}^s} \leq cn^s \|f\|_\infty, \quad \text{for all } f \in \mathbb{X}_n. \tag{3.30}$$

Since $(\mathcal{B}_n - \tilde{\mathcal{B}}_n)w \in \mathbb{X}_n$, by (3.30), we obtain

$$\left\|(\mathcal{B}_n - \tilde{\mathcal{B}}_n)w\right\|_{\mathcal{H}^s} \leq cn^s \left\|(\mathcal{B}_n - \tilde{\mathcal{B}}_n)w\right\|_\infty. \tag{3.31}$$

Let $b_{j,l} = \langle l_j, \mathcal{B}e_l(x)\rangle$, by the definition of operator $\mathcal{B}_n$ and operator $\tilde{\mathcal{B}}_n$, we have

$$\mathcal{B}_n w(x) = \sum_{|j|\in\mathbb{Z}_{n+1}} \sum_{|l|\in\mathbb{Z}_{n+1}} b_{j,l} w_l e_j(x), \qquad \tilde{\mathcal{B}}_n w(x) = \sum_{|j|\in\mathbb{Z}_{n+1}} \sum_{|l|\in\mathbb{Z}_{n+1}\setminus\Pi_j} b_{j,l} w_l e_j(x),$$

and

$$(\mathcal{B}_n - \tilde{\mathcal{B}}_n) w(x) = \sum_{|j|\in\mathbb{Z}_{n+1}} \sum_{|l|\in\Pi_j} b_{j,l} w_l e_j(x).$$

Therefore,

$$\left\|(\mathcal{B}_n - \tilde{\mathcal{B}}_n)w\right\|_\infty \le \frac{1}{\sqrt{2\pi}} \left\|(\mathbf{B}_n - \widetilde{\mathbf{B}}_n)\mathbf{w}^T\right\|_1. \tag{3.32}$$

For convenience, we let $g(x) := \int_I b(x,y)e_l(y)dy,\ x \in I$. It follows from (2.10) that $\left\langle l_j, g(x)\right\rangle = \int_I e_{-j}(x)\mathcal{Q}_n g(x)dx$, then we have

$$\left|\left\langle l_j, g(x)\right\rangle - \int_I g(x)e_{-j}(x)dx\right| \le \int_I |\mathcal{Q}_n g(x) - g(x)|\, e_{-j}(x)dx \le cn^{-k}(\log n)\left\|g^{(k)}(x)\right\|_\infty,$$

and

$$\left|\int_I g(x)e_{-j}(x)dx\right| = \left|\left(-\frac{i}{j}\right)^k \int_I g^{(k)}(x)e_{-j}(x)\,dx\right| \le \sqrt{2\pi}\,|j|^{-k}\left\|g^{(k)}(x)\right\|_\infty.$$

Therefore, for $|j| = 2, 3, \ldots, n$ and $|l| \in \Pi_j$, there exists a constant c independent of n such that

$$\begin{aligned}\left|b_{j,l}\right| &\le \left|\left\langle l_j, g(x)\right\rangle - \int_I g(x)\,e_{-j}(x)\,dx\right| + \left|\int_I g(x)\,e_{-j}(x)dx\right| \\ &\le c\left(n^{-k}\log n + |j|^{-k}\right)\left\|g^{(k)}(x)\right\|_\infty \\ &\le c\,n^{-k},\end{aligned} \tag{3.33}$$

as $\left\|g^{(k)}(x)\right\|_\infty = \left\|\int_I b^{(k,0)}(x,y)e_l(y)dy\right\|_\infty \le \sqrt{2\pi}\left\|b^{(k,k)}(x,y)\right\|_\infty |l|^{-k}$.

Since $w \in \mathbb{X}_n$, we have

$$\left|w_j\right| \le c\|w\|_\infty \quad \text{for } |j| \in \mathbb{Z}_{n+1}. \tag{3.34}$$

Using (3.32)–(3.34), there exists a constant $c > 0$ such that $|j| = 2, 3, \ldots, n$,

$$\left|b_{j,l}w_l + b_{j,-l}w_{-l}\right| \le c\,n^{-k}\|w\|_\infty.$$

For $j \in \mathbb{Z}_{n+1}$, let $h_j = \sum_{l\in\Pi_j}(b_{j,l}w_l + b_{j,-l}w_{-l})$, and denote $\mathbf{h} := [h_{-j}, h_j],\ j \in \mathbb{Z}_{n+1}$. Then it is clear that the vector $\mathbf{h} = (\mathbf{B}_n - \widetilde{\mathbf{B}}_n)\mathbf{w}^T$, and there holds

$$\|\mathbf{h}\|_\infty \le \max_{j\in\mathbb{Z}_{n+1}}\left\{\sum_{l\in\Pi_j}\left|b_{j,l}w_l + b_{j,-l}w_{-l}\right|\right\} \le cn^{-(k-1)}\|w\|_\infty,$$

which implies

$$\|\mathbf{h}\|_1 \le cn^{-(k-2)}\|w\|_\infty. \tag{3.35}$$

Hence, by (3.31) and (3.35), we have

$$\left\|(\mathcal{B}_n - \tilde{\mathcal{B}}_n)w\right\|_{\mathcal{H}^s} \le cn^{-(k-2-s)}\|w\|_\infty \le cn^{-r}\|w\|_\infty. \quad \square$$

Lemma 3.6. *If $\mathcal{T} : \mathcal{H}^s \longrightarrow \mathcal{H}^{s-\beta}$ ($s \in \mathbb{N}_0,\ \beta \in \mathbb{R}$ and $\beta \le s$) is invertible, then for all n sufficiently large, the solutions of (2.5) and (3.26) satisfy the following*

(i)

$$\left\|\tilde{u}_n\right\|_{C^s} \le c\log^2 n\,\|u\|_{C^s}; \tag{3.36}$$

(ii) *If $\beta \in \mathbb{Z}$, then*

$$\left\|\tilde{u}_n\right\|_{C^s} \le c\log n\left\{\left\|\mathcal{P}^+u\right\|_{C^s} + \left\|\mathcal{P}^-u\right\|_{C^s}\right\}. \tag{3.37}$$

Proof. Eliminating f between (2.5) and (3.27) yields the equation

$$\mathcal{Q}_n\mathcal{T}\tilde{u}_n = \mathcal{Q}_n\mathcal{T}\left(u + \mathcal{T}^{-1}(\mathcal{B}_n - \tilde{\mathcal{B}}_n)\tilde{u}_n\right),$$

so we can look upon $\tilde{u}_n$ as the collocation solution of the equation whose exact solution is not u but $u + \mathcal{T}^{-1}(\mathcal{B}_n - \tilde{\mathcal{B}}_n)\tilde{u}_n$. Hence, the stability estimate (2.17) implies

$$\left\|\tilde{u}_n\right\|_{C^s} \le c\log^2 n\left\|u + \mathcal{T}^{-1}(\mathcal{B}_n - \tilde{\mathcal{B}}_n)\tilde{u}_n\right\|_{C^s},$$

for n sufficiently large. Since $\mathcal{T}:\mathcal{H}^s \longrightarrow \mathcal{H}^{s-\beta}$ is invertible, $\mathcal{T}^{-1}:\mathcal{H}^{s-\beta} \longrightarrow \mathcal{H}^s$ exists and is bounded. By Lemma 3.5, for any $r > 0$,

$$\left\|\mathcal{T}^{-1}(\mathcal{B}_n - \tilde{\mathcal{B}}_n)\tilde{u}_n\right\|_{C^s} \le c\left\|\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right)\tilde{u}_n\right\|_{\mathcal{H}^{s-\beta}} \le cn^{-r}\left\|\tilde{u}_n\right\|_{\infty}.$$

Since $\left\|\tilde{u}_n\right\|_{\infty} \le \left\|\tilde{u}_n\right\|_{C^s}$, if n large enough so that $cn^{-r}\log^2 n \le \frac{1}{2}$, then

$$\left\|\tilde{u}_n\right\|_{C^s} \le c\log^2 n\left\|u\right\|_{C^s} + \frac{1}{2}\left\|\tilde{u}_n\right\|_{C^s}.$$

Thus, Eq. (3.36) is proved.

The estimate (3.37) can be proved in the same manner. Using (2.18), we have

$$\left\|\tilde{u}_n\right\|_{C^s} \le c\log n\left\{\left\|\mathcal{P}^+\left(u + \mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right)\tilde{u}_n\right)\right\|_{C^s} + \left\|\mathcal{P}^-\left(u + \mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right)\tilde{u}_n\right)\right\|_{C^s}\right\}.$$

Using Lemma 3.5, for any $r > 0$,

$$\begin{aligned}\left\|\mathcal{P}^+\left[\mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right)\tilde{u}_n\right]\right\|_{C^s} &\le \left\|\mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right)\tilde{u}_n\right\|_{C^s}\\ &\le c\|(\mathcal{B}_n - \tilde{\mathcal{B}}_n)\tilde{u}_n\|_{\mathcal{H}^{s-\beta}}\\ &\le cn^{-r}\|\tilde{u}_n\|_{\infty},\end{aligned}$$

and

$$\|\mathcal{P}^-(\mathcal{T}^{-1}(\mathcal{B}_n - \tilde{\mathcal{B}}_n)\tilde{u}_n)\|_{C^s} \le cn^{-r}\|\tilde{u}_n\|_{\infty}.$$

Since $\left\|\tilde{u}_n\right\|_{\infty} \le \left\|\tilde{u}_n\right\|_{C^s}$, if n large enough so that $cn^{-r}\log n \le \frac{1}{4}$, then

$$\left\|\tilde{u}_n\right\|_{C^s} \le c\log n\left\{\left\|\mathcal{P}^+u\right\|_{C^s} + \left\|\mathcal{P}^-u\right\|_{C^s}\right\} + \frac{1}{2}\left\|\tilde{u}_n\right\|_{C^s}. \quad \Box$$

Next we establish the convergence order of the approximate solution $\tilde{u}_n$. To this end, we first recall an approximation result from Jackson's theorem (see [23]). Suppose $t > 0$ and a function $f \in \mathcal{H}^t$, for any $s \in \mathbb{N}_0$ and $s < t < \infty$, there exits a unique element $v \in \mathbb{X}_n$ satisfies

$$\|f - v\|_{C^s} = \inf_{z\in\mathbb{X}_n}\|f - z\|_{C^s} \le cn^{s-t}\|f\|_{\mathcal{H}^t}.$$

We now return to the convergence order of the approximate solution $\tilde{u}_n$.

Theorem 3.7. *Suppose that $\mathcal{T}:\mathcal{H}^s \longrightarrow \mathcal{H}^{s-\beta}$ ($s \in \mathbb{N}_0$, $\beta \in \mathbb{R}$ and $\beta \le s$) is invertible and $u \in \mathcal{H}^t$ ($s < t$) is the exact solution satisfying (2.5), then for all sufficiently large n and for each $f \in \mathcal{H}^{t-\beta}$, there exists a unique fast collocation solution $\tilde{u}_n \in \mathbb{X}_n$, satisfying (3.26). Furthermore,*

(i)

$$\left\|\tilde{u}_n - u\right\|_{C^s} \le cn^{s-t}\log^2 n\|u\|_{\mathcal{H}^t}; \tag{3.38}$$

(ii) *when $\beta \in \mathbb{Z}$,*

$$\left\|\tilde{u}_n - u\right\|_{C^s} \le cn^{s-t}\log n\|u\|_{\mathcal{H}^t}. \tag{3.39}$$

Proof. It follows by Jackson's theorem that there exists a unique $v \in \mathbb{X}_n$ satisfies

$$\|u - v\|_{C^s} \le cn^{s-t}\|u\|_{\mathcal{H}^t}. \tag{3.40}$$

Using (3.27), we have

$$\mathcal{Q}_n\left(\mathcal{T} + \tilde{\mathcal{B}}_n - \mathcal{B}_n\right)\left(\tilde{u}_n - v\right) = \mathcal{Q}_n\mathcal{T}\left((u - v) + \mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right)v\right),$$

so we can look upon $\tilde{u}_n - v$ as the fast collocation solution of the equation whose exact solution is not u but $(u-v) + \mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right) v$. Hence, Lemma 3.6 implies that

$$\begin{aligned}\left\|\tilde{u}_n - v\right\|_{C^s} &\le c\log^2 n\left\|(u-v) + \mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right) v\right\|_{C^s},\\ &\le c\log^2 n\left(\|u-v\|_{C^s} + \left\|\mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right) v\right\|_{C^s}\right).\end{aligned} \tag{3.41}$$

Choosing $r = t - s > 0$ in Lemma 3.5, we have

$$\left\|\mathcal{T}^{-1}\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right) v\right\|_{C^s} \le c\left\|\left(\mathcal{B}_n - \tilde{\mathcal{B}}_n\right) v\right\|_{\mathcal{H}^{s-\beta}} \le cn^{-(t-s)}\|v\|_{C^s},$$

as $\|v\|_\infty \le \|v\|_{C^s}$, and then since $\|v\|_{C^s} \le \|u\|_{C^s} + \|u - v\|_{C^s}$, by (3.41), we have

$$\left\|\tilde{u}_n - v\right\|_{C^s} \le c\log^2 n\left(\|u-v\|_{C^s} + n^{-(t-s)}\|u\|_{\mathcal{C}^s}\right). \tag{3.42}$$

Therefore, it follows from (3.40), (3.42) and $\|\tilde{u}_n - u\|_{C^s} \le \|\tilde{u}_n - v\|_{C^s} + \|u - v\|_{C^s}$ that

$$\|\tilde{u}_n - u\|_{C^s} \le cn^{s-t}\log^2 n\|u\|_{\mathcal{H}^t},$$

which reach (3.38).

For the case $\beta \in \mathbb{Z}$, we obtain

$$\left\|\tilde{u}_n - u\right\|_{C^s} \le c\log n\left\{\left\|\mathcal{P}^+(u-v)\right\|_{C^s} + \left\|\mathcal{P}^-(u-v)\right\|_{C^s} + n^{s-t}\|u\|_{\mathcal{H}^t}\right\}. \tag{3.43}$$

As shown in [28,10], there exists an operator $\mathcal{L}_n$ satisfies

$$\|(I - \mathcal{L}_n)f\|_{C^s} \le cn^{s-t}\|f\|_{\mathcal{H}^t}, \tag{3.44}$$

and commutes with $\mathcal{P}^+$ and $\mathcal{P}^-$, by putting $v := \mathcal{L}_n u$ in (3.43) we obtain

$$\begin{aligned}\left\|\tilde{u}_n - u\right\|_{C^s} &\le c\log n\left\{\left\|(I - \mathcal{L}_n)\mathcal{P}^+ u\right\|_{C^s} + \left\|(I - \mathcal{L}_n)\mathcal{P}^- u\right\|_{C^s} + n^{s-t}\|u\|_{\mathcal{H}^t}\right\}\\ &\le cn^{s-t}\log n\|u\|_{\mathcal{H}^t}.\end{aligned}$$

This completes the proof. □

4. Applications to boundary integral equations and numerical results

We will now consider the applications of our theory on boundary integral equations. The pseudodifferential equation (2.5) can represent many boundary integral equations, such as weakly singular integral equations, strong singular integral equations and hypersingular integral equations, formulated from boundary value problems of Laplace equations (see [23,29,1]). To this end, we let D be a bounded open simply connected region in $\mathbb{R}^2$ and Γ be its boundary having parametrization γ defined by $\Gamma : \gamma(x) = (\gamma_1(x), \gamma_2(x)),\ x \in [0, 2\pi]$.

In the first case, we consider the weakly singular integral equations

$$(\mathcal{T}u)(x) = -\frac{1}{\pi}\int_0^{2\pi}\log|\gamma(x) - \gamma(y)|\, u(y)dy = f(x), \tag{4.45}$$

which can be reformulated by the interior Dirichlet problem of Laplace equation. The operator $\mathcal{T}$ can be decomposed as the form $\mathcal{A} + \mathcal{B}$ with

$$\mathcal{A}u(x) = \int_0^{2\pi} -\frac{1}{\pi}\log\left|2e^{-\frac{1}{2}}\sin\frac{y-x}{2}\right| u(y)dy, \quad \text{and} \quad \mathcal{B}u(x) = \int_0^{2\pi} b(x, y)dy, \tag{4.46}$$

with

$$b(x, y) = \begin{cases} -\dfrac{1}{\pi}\log\left|\dfrac{e^{\frac{1}{2}}|\gamma(x) - \gamma(y)|}{\left|2\sin\left(\frac{x-y}{2}\right)\right|}\right| u(y), & x - y \ne 2k\pi,\, k \in \mathbb{Z} \\ -\dfrac{1}{\pi}\log|e^{\frac{1}{2}}\gamma'(y)|, & \text{otherwise}, \end{cases} \quad x, y \in I. \tag{4.47}$$

It is easy to check that $b \in C^{\infty,\infty}(\mathbb{R}^2)$. The operator $\mathcal{A}$ is a classical pseudodifferential operator of order -1 admitting a Fourier series representation $\mathcal{A}u(x) = \sum_{|l|\in\mathbb{Z}}\sigma(x, l)\hat{u}(l)e^{ilx}$ with $\sigma(x, l) = \frac{1}{[l]}$. Thus, $\mathcal{T} : \mathcal{H}^s \to \mathcal{H}^{s+1}$ is a isomorphism for any $s \in \mathbb{R}$. This contribute the coefficient matrix $\mathbf{A}_n$ in (3.25) is an identity matrix with the form $\mathbf{A}_n = \mathrm{diag}(1, 1, 1, \ldots, 1, 1)$.

We present two numerical examples to illustrate the efficiency and accuracy of the methods proposed in the previous sections. In this section, the notations $C.R.$ and $C.O.$ stand for the compression rate and convergence order defined, respectively, by

$$C.R. = \frac{\mathcal{N}(\tilde{\mathbf{B}}_n)}{\mathcal{N}(\mathbf{B}_n)}, \qquad C.O. = \log\frac{\|u - u_n\|}{\|u - u_{2n}\|}/\log(2).$$

Table 1

n	$\|u-u_n\|_\infty$	C.O.	$\|u-\tilde{u}_n\|_\infty$	C.O.	C.R.
16	1.11517694e−1		1.312719696e−1		0.1809
32	6.05971510e−2	0.879951	6.608180930e−2	0.9902	0.1020
64	3.16642460e−2	0.936395	3.310906084e−2	0.9970	0.0568
128	1.61950505e−2	0.967302	1.695648252e−2	0.9654	0.0312
256	8.09957532e−3	0.999635	8.479955806e−3	0.9997	0.0170
512	4.05021371e−3	0.999848	4.264944817e−3	0.9915	0.0092
1024	2.03086285e−3	0.995905	2.133000826e−3	0.9996	0.0040

Example 4.1. In this example, we consider solving the boundary integral equation (4.45), and the kernel $b(x,y)$ defined by

$$b(x,y) = e^{\sin x \cos 2y} \log|\cos y + 5|.$$

Choose the righthand side function f such that $u(x) = x$ is the exact solution of the equation. Here $t = 1$ and $s = 0$ as shown in Theorem 3.7, and, thus, the theoretical convergence order is close to 1. The numerical results of our algorithms (3.25) are given in Table 1. Note that, the fast collocation solution $\tilde{u}_n(x)$ is nearly can be seen as the Fourier expansion of $u(x)$, it will be shock when x close to 0 or 2π. Therefore, we use $\|u-\tilde{u}_n\|_\infty$ denote the max norm of $u(x)-\tilde{u}_n(x)$ in $[1, 2\pi-1]$.

In the second case, we consider the interior Neumann problem

$$\begin{cases} \triangle U(P) = 0, & P \in D, \\ \dfrac{\partial U(P)}{\partial \mathbf{n}_P} = g(P), & P \in \Gamma \end{cases} \tag{4.48}$$

where $\triangle$ denotes the Laplace operator defined on $\mathbb{R}^2$, and $\mathbf{n}$ denotes the outer unit normal vector to the boundary Γ. The solution U can be expressed by a double layer potential

$$U(P) = \int_\Gamma \rho(Q) \frac{\partial}{\partial \mathbf{n}_Q} \log|P-Q|\, ds_Q, \quad P \in D. \tag{4.49}$$

The problem can be transferred for solving the equation

$$\int_0^{2\pi} \frac{\sin(y-x)}{2(1-\cos(y-x))} u(y)dy + \int_0^{2\pi} \frac{\gamma'(x)\cdot(\gamma(y)-\gamma(x))}{|\gamma(y)-\gamma(x)|^2} - \frac{\sin(y-x)}{2(1-\cos(y-x))} u(y)dy = f(x), \tag{4.50}$$

where $f(x) = g(\gamma(x))\gamma'(x)$ and $u(x) = \frac{d\rho(\gamma(x))}{dx}$. The strong singularity operator

$$(\mathcal{A}u)(x) = \int_0^{2\pi} \frac{\sin(y-x)}{2(1-\cos(y-x))} u(y)dy, \quad \text{and} \quad \mathcal{B}u(x) = \int_0^{2\pi} b(x,y)dy, \tag{4.51}$$

with

$$b(x,y) = \frac{\gamma'(x)\cdot(\gamma(y)-\gamma(x))}{|\gamma(y)-\gamma(x)|^2} - \frac{\sin(y-x)}{2(1-\cos(y-x))}. \tag{4.52}$$

The operator $\mathcal{A}$ is a classical pseudodifferential operator of order 0, admitting a Fourier series representation $\mathcal{A}u(x) = \sum_{l\neq 0} \sigma(x,l)\hat{u}(l)e^{ilx}$ with $\sigma(x,l) = -i\pi\,\mathrm{sgn}(l)$. Note that the kernel of $\mathcal{A}$ has a strong singularity along the diagonal, it means that, $\mathcal{A}e_l(x)$ does not exist when $l = 0$ and u is the exact solution of (4.50) if and only if $\hat{u}(0) = 0$. We denote $\mathcal{H}_0^s := \{u \in \mathcal{H}^s : \hat{u}(0) = 0\}$. Thus, $\mathcal{T}$ is an isomorphism from $\mathcal{H}_0^s$ to $\mathcal{H}_0^s$ for any $s \in \mathbb{R}$. In order to use the fast trigonometric collocation method, we let $\mathcal{A}e_0 = -\pi i$. This contributes the coefficient matrix $\mathbf{A}_n$ in (3.25) to be a diagonal matrix with the form

$$\mathbf{A}_n = \mathrm{diag}(-\pi i, \pi i, -\pi i, \pi i, -\pi i, \ldots, \pi i, -\pi i).$$

Example 4.2. In this example, we consider the integral equation (4.50) with the smooth kernel

$$b(x,y) = e^{\cos x \sin y} \sin x \sin 2y \log|\sin y + 5|.$$

The right-hand side function f is chosen so that the exact solution $u(x) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$. Here $t = 2$ and $s = 0$ as shown in Theorem 3.7, and, thus, the theoretical convergence order is close to 2. In Table 2, we list the numerical results for this example and use $\|u-\tilde{u}_n\|_\infty$ denote the max norm of $u(x)-\tilde{u}_n(x)$ in $[1, 2\pi-1]$.

Table 2

n	$\|u-u_n\|_\infty$	C.O.	$\|u-\tilde{u}_n\|_\infty$	C.O.	C.R.
16	3.51767967e−3		3.61130842e−3		0.1809
32	9.69733436e−4	1.89744	9.69345932e−4	1.8590	0.1020
64	2.49843363e−4	1.95599	2.49843331e−4	1.9566	0.0568
128	6.23092490e−5	2.00351	6.23092490e−5	2.0035	0.0312
256	1.57003754e−5	1.98865	1.57003754e−5	1.9886	0.0170
512	3.96241877e−6	1.98013	3.97953366e−6	1.9863	0.0092
1024	9.92540038e−7	1.92804	1.04576843e−6	1.9972	0.0040

Table 3

n	$\|u-u_n\|_{C^1}$	C.O.	$\|u-\tilde{u}_n\|_{C^1}$	C.O.	C.R.
16	5.93701557e−2		6.92472930e−2		0.1809
32	3.12679214e−2	0.92505	3.41027032e−2	1.0219	0.1020
64	1.60819663e−2	0.95924	1.76323840e−2	0.9517	0.0568
128	8.15983452e−2	0.97883	8.97056467e−3	0.9750	0.0312
256	4.06548803e−2	1.00511	4.46940450e−3	1.0051	0.0170
512	2.02906927e−3	1.00261	2.24474349e−3	0.9935	0.0092
1024	1.02490739e−3	0.98532	1.12036927e−3	1.0026	0.0040

In the third case, we consider the hypersingular integral equation reformulation of interior Neumann problem (4.48). The hypersingular integral equation

$$(\mathcal{T}u)(x) = \int_0^{2\pi} \frac{\gamma'(x)\cdot(\gamma(y)-\gamma(x))}{|\gamma(y)-\gamma(x)|^2} u(y)dy = f(x), \tag{4.53}$$

with $f(x) = g(\gamma(x))\,\gamma'(x)$ and $u(x) = \frac{d\rho(\gamma(x))}{dx}$ can transfer into the form

$$\int_0^{2\pi} -\frac{1}{4\pi}\csc^2\frac{y-x}{2}u(y)dy + \int_0^{2\pi} \frac{1}{\pi}\frac{\partial}{\partial y}\left[\frac{\gamma(x)\cdot(\gamma(y)-\gamma(x))}{|\gamma(y)-\gamma(x)|^2} - \frac{\sin(y-x)}{2(1-\cos(y-x))}\right]u(y)dy = f(x). \tag{4.54}$$

The hypersingular operator

$$(\mathcal{A}u)(x) = \int_0^{2\pi} -\frac{1}{4\pi}\csc^2\frac{y-x}{2}u(y)dy = \sum_{l\neq 0} -l\cdot\hat{u}(l)e^{ilx},$$

and $\mathcal{B}u(x) = \int_0^{2\pi} b(x,y)dy$ with the smooth kernel

$$b(x,y) = \frac{1}{\pi}\frac{\partial}{\partial y}\left[\frac{\gamma(x)\cdot(\gamma(y)-\gamma(x))}{|\gamma(y)-\gamma(x)|^2} - \frac{\sin(y-x)}{2(1-\cos(y-x))}\right] \in C^{\infty,\infty}(\mathbb{R}^2).$$

The operator $\mathcal{A}$ is a pseudodifferential operator of order 1. Thus, $\mathcal{T}$ is a isomorphism from $\mathcal{H}_0^s$ to $\mathcal{H}_0^{s-1}$ for any $s \in \mathbb{R}$. We let $\mathcal{A}e_0 = -1$, then the coefficient matrix $\mathbf{A}_n$ in (3.25) is a diagonal matrix with the form $\mathbf{A}_n = \mathrm{diag}(-1, 1, -1, 1, -1, \dots, 1, -1)$.

Example 4.3. We consider solving the integral equation (4.54) with the smooth kernel

$$b(x,y) = \log(\cos x + \sin y + 5)\sin x.$$

The right-hand side function f is chosen so that the exact solution $u(x) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$. Here $t = 2$ and $s = 1$ as shown in Theorem 3.7, and, thus, the theoretical convergence order is close to 1. We list the numerical results of this example in Table 3 and use $\left\|u-\tilde{u}_n\right\|_{C^1}$ denote the C^1-norm of $u(x)-\tilde{u}_n(x)$ in $[1, 2\pi-1]$.

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References

[1] J. Saranen, G. Vainikko, Periodic Integral and Pseudodifferential Equations with Numerical Approximation, Springer-Verlag, Berlin, Heidelberg, New York, 2001.
[2] Z. Chen, C.A. Micchelli, Y. Xu, The Petrov–Galerkin method for second kind integral equationg II: multiwavelet schemes, Adv. Comput. Math. 7 (1997) 199–233.
[3] R. Celorrio, F.-J. Sayas, Extrapolation techniques and the collocation method for a class of boundary integral equations, ANZIAM J. 42 (2001) 413–437.
[4] V. Domínguez, F.-J. Sayas, Full asymptotics of spline Petrov–Galerkin methods for some periodic pseudodifferential equations, Adv. Comput. Math. 14 (2001) 75–101.
[5] R. Kress, A collocation method for a hypersingular boundary integral equation via trigonometric differentiation, J. Integral Equations Appl. 26 (2014) 197–213.
[6] W. Mclean, S.B. Pröbdorf, W.L. Wendland, Pointwise error estimates for the trigonometric collocation method applied to singular integral equations and periodic pseudodifferential equations, J. Integral Equations Appl. 2 (1989) 125–146.
[7] W. Mclean, S.B. Pröbdorf, W.L. Wendland, A Fully-discrete trigonometric collocation method, J. Integral Equations Appl. 5 (1993) 103–129.
[8] W. Mclean, Fully-discrete collocation methods for an integral equation of the first kind, J. Integral Equations Appl. 6 (1994) 537–571.
[9] T.M. Morton, M. Neamtu, Error bounds for solving pseudodifferential equations on spheres by collocation with zonal kernels, J. Approx. Theory 114 (2002) 242–268.
[10] W. McLean, W.L. Wendland, Trigonometric approximation of solutions of periodic pseudodifferential equations, Oper. Theory Adv. Appl. 41 (1989) 359–383. The Gohberg Anniversary Collection.
[11] S. Prössdorf, R. Schneider, A spline collocation methods for multidimensional strongly elliptic pseudodifferential operations of order zero, Integral Equations Operator Theory 1 (1991) 399–435.
[12] S. Prössdorf, R. Schneider, Spline approximation methods for multidimensional periodic pseudodifferential equations, Integral Equations Operator Theory 15 (1992) 626–672.
[13] J. Saranen, A modified discrete spectral collocation method for first kind integral equations with logarithmic kernel, J. Integral Equations Appl. 5 (1993) 547–551.
[14] J. Saranen, G. Vainikko, Trigonometric collocation methods with product integration for boundary integral equations on closed curves, SIAM J. Numer. Anal. 33 (1996) 1577–1596.
[15] H. Cai, A fast Fourier collocation method for second boundary integral equations, J. Comput. Appl. Math. 234 (2010) 165–173.
[16] H. Cai, Y. Xu, A fast Fourier-Galerkin method for solving singular boundary integral equations, SIAM J. Numer. Anal. 46 (2008) 1965–1984.
[17] W. Dahmen, S. Prössdorf, R. Schneider, Wavelet approximation methods for pseudodifferential equations: I Stability and convergence, Math. Z. 215 (1994) 583–620.
[18] J. Saranen, G. Vainikko, Two-grid solutions of Symm's integral equation, Math. Nachr. 177 (1996) 265–279.
[19] J. Saranen, G. Vainikko, Fast solvers of integral and pseudodifferential equations on closed curves, Math. Comp. 67 (1998) 1473–1491.
[20] Z. Chen, C.A. Micchelli, Y. Xu, Fast collocation methods for second kind integral equations, SIAM J. Numer. Anal. 40 (2002) 344–375.
[21] Z. Chen, G. Long, G. Nelakanti, Y. Zhang, Iterated fast collocation methods for integral equations of the second kind, J. Sci. Comput. (2013) 502–517.
[22] K.E. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, Springer-Verlag, Berlin, Heidelberg, New York, 2005.
[23] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, Cambridge, UK, 1997.
[24] R. Celorrio, F.-J. Sayas, Full collocation methods for some boundary integral equations, Numer. Algorithms 22 (1999) 327–351.
[25] J. Elschner, E.P. Stephan, A discrete collocation method for Symm's integral equation on curves with corners, J. Comput. Appl. Math. 75 (1996) 131–146.
[26] Z. Chen, Y. Xu, H. Yang, Fast collocation method for solving ill-posed integral equations of the first kind, Inverse Problems 24 (2008) 1–21.
[27] J. Prestin, Trigonometric interpolation in Hölder spaces, J. Approx. Theory 53 (1988) 145–154.
[28] M. Nikol'skiĭ, Approximation of Functions of Several Variables and Imbedding Theorems, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
[29] R. Kress, Linear Integral Equations, Springer-Verlag, New York, 1989.
[30] Z. Chen, S. Ding, H. Yang, Multilevel augmentation algorithms based on fast collocation methods for solving ill-posed integral equations, Comput. Math. Appl. 62 (2011) 2071–2082.
[31] B. Booß-Bavnbek, G. Chen, M. Lesch, C. Zhu, Perturbation of sectorial projections of elliptic pseudo-differential operators, J. Pseudo-Diff. Oper. Appl. 3 (2012) 49–79.
[32] Y. Jeon, A Nyström method for boundary integral equations on domains with a piecewise smooth boundary, J. Integral Equations Appl. 5 (1993) 221–242.
[33] Y. Jeon, New boundary element formulas for the biharmonic equation, Adv. Comput. Math. 9 (1998) 97–115.
[34] R. Kress, A Nyström method for boundary integral equations in domains with corners, Numer. Math. 61 (1990) 145–161.
[35] V.D. Kryakvin, Characterization of pseudodifferential operators in Hölder-Zygmund Spaces, Differential Equations 49 (2013) 306–312.
[36] R. Kress, I. Sloan, On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation, Numer. Math. 66 (1993) 119–214.
[37] W. Mclean, A spectral Galerkin method for boundary integral equation, Math. Comp. 47 (1986) 597–607.
[38] W. McLean, I.H. Sloan, A fully discrete and symmetric boundary element method, IMA J. Numer. Anal. 14 (1994) 311–345.
[39] T.D. Pham, T. Tran, Strongly elliptic pseudodifferential equations on the sphere with radial basis functions, Numer. Math. 128 (2014) 589–614.
[40] R. Plato, G. Vainikko, The fast solution of periodic integral and pseudodifferential equations by GMRES, Comput. Methods Appl. Math. 1 (2001) 383–397.
[41] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, New York, Oxford, 1978.
[42] T. Tran, Q.T. Le Gia, I.H. Sloan, E.P. Stephan, Preconditioners for pseudodifferential equations on the sphere with radial basis functions, Numer. Math. 115 (2010) 114–163.