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Bifurcation of limit cycles by perturbing piecewise smooth integrable non-Hamiltonian systems

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ABSTRACT

This paper deals with bifurcation of limit cycles for piecewise smooth integrable non-Hamiltonian systems. We derive the first order Melnikov function, which plays an important role in the study of the number of limit cycles bifurcated from the periodic annulus of a center. As an application, we consider a class of cubic isochronous centers, which has a non-rational first integral. Using the first order Melnikov function, we obtain the sharp upper bound of the number of limit cycles which bifurcate from the periodic annulus of the center under piecewise smooth polynomial perturbations.

1. Introduction and statement of main results

One of the main topics in the qualitative theory of planar differential systems is to determine the numbers and distributions of limit cycles. The restriction of this problem to polynomial differential systems is the well-known Hilbert's 16th problem, see [1]. Since Hilbert's 16th problem turns out to be a strongly difficult one, many mathematicians began to study some special kinds of polynomial systems, such as the weak Hilbert's 16th problem [2], Liénard system [3–5], quadratic system [6,7] and cubic system [8].

Consider the perturbed planar smooth Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = H_y(x,y) + \varepsilon f(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -H_x(x,y) + \varepsilon g(x,y), \end{cases} \tag{1}$$

where $f(x,y)$ and $g(x,y)$ are analytic functions with respect to x, y. We suppose that the unperturbed system (1)$|_{\varepsilon=0}$ has a family of periodic orbits $L_h \subset \{(x,y) : H(x,y) = h \in J = (\alpha,\beta)\}$.

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A classic way to generate limit cycles is by perturbing a system which has a center via Poincaré bifurcation, in such a way that limit cycles bifurcate in the perturbed system from the periodic annulus of the center for the unperturbed system. In order to study the maximum number of limit cycles of system (1) which bifurcate from the periodic annulus of the center, it is necessary to study the number of zeros of the first order Melnikov function, also known as Abelian integral, i.e.,

$$M_1(h) = \oint_{L_h} g(x,y)\mathrm{d}x - f(x,y)\mathrm{d}y. \tag{2}$$

In the case that (1) is a polynomial system, the study of the upper bound of the number of zeros of (2) is the so called weak Hilbert's 16th problem, see [2]. Many works have been done on limit cycles by perturbing Hamiltonian system with a family of periodic orbits, see the survey paper [2].

For the perturbed planar smooth integrable non-Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{H_y(x,y)}{R(x,y)} + \varepsilon f(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{H_x(x,y)}{R(x,y)} + \varepsilon g(x,y), \end{cases} \tag{3}$$

with integrating factor $R(x,y)$, the first order Melnikov function can be expressed as

$$\overline{M}_1(h) = \oint_{L_h} R(x,y)g(x,y)\mathrm{d}x - R(x,y)f(x,y)\mathrm{d}y. \tag{4}$$

There are some works on limit cycles by perturbing integrable non-Hamiltonian system, see for instance [9,10].

In recent years, stimulated by the discontinuous phenomena in the real world, there has been considerable interest in studying the bifurcation of piecewise smooth differential system, see for instance [11] and the references therein. There are many authors generalizing the Hilbert's 16 problem to the piecewise smooth case, that is to say, they consider the limit cycles for the piecewise smooth Hamiltonian system, see for instance [12–17].

The authors in the paper [18] extend the averaging method to piecewise smooth system. In order to apply the averaging method, a transformation of the original system into the standard form must be done. Generally speaking, if the system is integrable but no-Hamiltonian, then the translation is difficult.

In the paper [15], the authors consider the perturbed planar piecewise smooth Hamiltonian systems

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{cases} \begin{pmatrix} H_y^+(x,y) + \varepsilon f^+(x,y) \\ -H_x^+(x,y) + \varepsilon g^+(x,y) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} H_y^-(x,y) + \varepsilon f^-(x,y) \\ -H_x^-(x,y) + \varepsilon g^-(x,y) \end{pmatrix}, & x \leqslant 0, \end{cases} \tag{5}$$

where $f^\pm(x,y), g^\pm(x,y)$ and $H^\pm(x,y)$ are analytic functions with respect to x and y. The switching manifold $x = 0$ divides $\mathbb{R}^2$ into two regions, where the systems are smooth in each region. We call system (5) with $x > 0$ the right subsystem and $x < 0$ the left subsystem respectively.

Suppose that the unperturbed system $(5)|_{\varepsilon=0}$ has a family of periodic orbits around the origin and satisfy the following two assumptions:

Assumption (I). There exist an open interval $J = (\alpha, \beta)$, and two points $A(h) = (0, a(h))$, $A_1(h) = (0, a_1(h))$, where $a(h) \neq a_1(h)$. For $h \in J$, we have

$$H^+(A(h)) = H^+(A_1(h)) = h, \qquad H^-(A(h)) = H^-(A_1(h)). \tag{6}$$

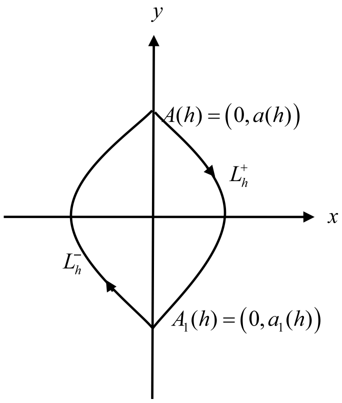

Fig. 1. The closed orbits of system $(1.5)|_{\varepsilon=0}$.

Assumption (II). The unperturbed right subsystem of $(5)|_{\varepsilon=0}$ has an orbital arc L_h^+ starting from $A(h)$ and ending at $A_1(h)$ which is defined by $H^+(x,y)=h, x>0$. The unperturbed left subsystem of $(5)|_{\varepsilon=0}$ has an orbital arc L_h^- starting from $A_1(h)$ and ending at $A(h)$ which is defined by $H^-(x,y)=H^-(A(h)), x\leqslant 0$.

Under the above two Assumptions (I) and (II), the unperturbed system $(5)|_{\varepsilon=0}$ has a family of periodic orbits $L_h=L_h^+\cup L_h^-$ for $h\in J$. Each of the closed curves L_h is piecewise smooth in general. Further, we suppose that L_h has a clockwise orientation without loss of generality, see Fig. 1.

From Theorem 1.1 of the paper [15], the first order Melnikov function of system (5) can be written as:

$$\widehat{M}_1(h)=\frac{H_y^+(A(h))}{H_y^-(A(h))}\left(\frac{H_y^-(A_1(h))}{H_y^+(A_1(h))}\int_{L_h^+} g^+(x,y)\mathrm{d}x-f^+(x,y)\mathrm{d}y+\int_{L_h^-} g^-(x,y)\mathrm{d}x-f^-(x,y)\mathrm{d}y\right). \quad (7)$$

Stimulated by the paper [15], we consider the following planar piecewise smooth integrable non-Hamiltonian systems

$$\begin{pmatrix}\dfrac{\mathrm{d}x}{\mathrm{d}t}\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}\end{pmatrix}=\begin{cases}\begin{pmatrix}\dfrac{H_y^+(x,y)}{R^+(x,y)}+\varepsilon f^+(x,y)\\ -\dfrac{H_x^+(x,y)}{R^+(x,y)}+\varepsilon g^+(x,y)\end{pmatrix}, & x>0,\\ \begin{pmatrix}\dfrac{H_y^-(x,y)}{R^-(x,y)}+\varepsilon f^-(x,y)\\ -\dfrac{H_x^-(x,y)}{R^-(x,y)}+\varepsilon g^-(x,y)\end{pmatrix}, & x\leqslant 0.\end{cases} \quad (8)$$

Note that the right subsystem and left subsystem of (8) are integrable non-Hamiltonian systems with integrating factors $R^+(x,y)$ and $R^-(x,y)$ respectively. We suppose that the unperturbed system $(8)|_{\varepsilon=0}$ satisfies Assumptions (I) and (II), that is, the unperturbed system $(8)|_{\varepsilon=0}$ has a family of periodic orbits around the origin.

Consider the right subsystem of (8) starting from the point $A(h)$. Let $A_\varepsilon(h)=(0,a_\varepsilon(h))$ denote the first intersection point of the right subsystem of (8) with the negative y-axis. Let $B_\varepsilon(h)=(0,b_\varepsilon(h))$ denote the first intersection point starting from $A_\varepsilon(h)$ of the left subsystem of (8) with the positive y-axis, see Fig. 2.

Similar to the smooth system, we define the Poincaré map of piecewise smooth system (8) as follows

$$H^+(B_\varepsilon(h))-H^+(A(h))=\varepsilon\widetilde{M}_1(h)+O(\varepsilon^2), \quad (9)$$

where $\widetilde{M}_1(h)$ is called the first order Melnikov function of the piecewise smooth system (8). Since the first order Melnikov function $\widetilde{M}_1(h)$ is the first order approximation in ε of the Poincaré map, the number of isolated zeros of $\widetilde{M}_1(h)$ (taking into account the multiplicities) gives an upper bound of the number of limit

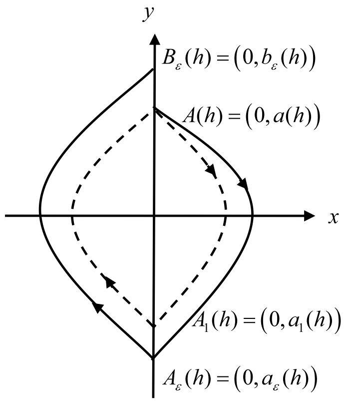

Fig. 2. The Poincaré map related to $x = 0$.

cycles of system (8) (taking into account the multiplicities) with ε sufficiently small. We have the following theorem:

Theorem 1. *Consider system (8) with ε sufficiently small. Under Assumptions (I) and (II), the first order Melnikov function can be expressed as:*

$$\widetilde{M}_1(h) = \frac{H_y^+(A(h))}{H_y^-(A(h))}\left(\frac{H_y^-(A_1(h))}{H_y^+(A_1(h))}\int_{L_h^+} R^+(x,y)g^+(x,y)\mathrm{d}x - R^+(x,y)f^+(x,y)\mathrm{d}y \right.$$
$$\left. + \int_{L_h^-} R^-(x,y)g^-(x,y)\mathrm{d}x - R^-(x,y)f^-(x,y)\mathrm{d}y\right). \tag{10}$$

Further, if $\widetilde{M}_1(h^*) = 0$ and $\widetilde{M}_1'(h^*) \neq 0$ for some $h^* \in J$, then for $|\varepsilon| > 0$ sufficiently small, system (8) has a unique limit cycle near L_{h^*}.

Remark 2. (i) If $R^+(x,y) \equiv R^-(x,y) \equiv 1$, then piecewise smooth integrable non-Hamiltonian system (8) becomes piecewise smooth Hamiltonian system (5), and the first order Melnikov function $\widehat{M}_1(h)$ of piecewise smooth Hamiltonian system (5) has been studied in [15].

(ii) If $R^+(x,y) \equiv R^-(x,y), H^+(x,y) \equiv H^-(x,y), f^+(x,y) \equiv f^-(x,y), g^+(x,y) \equiv g^-(x,y)$, then piecewise integrable non-Hamiltonian system (8) becomes smooth integrable non-Hamiltonian system (3). The first order Melnikov function $\overline{M}_1(h)$ of smooth integrable non-Hamiltonian system (3) is a classical result, see for example [1].

As an application, we consider the piecewise polynomial perturbation of an integrable non-Hamiltonian system

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y\left(3x^2+y^2\right) + \varepsilon P^+(x,y) \\ x\left(x^2-y^2\right) + \varepsilon Q^+(x,y) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} -y\left(3x^2+y^2\right) + \varepsilon P^-(x,y) \\ x\left(x^2-y^2\right) + \varepsilon Q^-(x,y) \end{pmatrix}, & x \leqslant 0, \end{cases} \tag{11}$$

where

$$\begin{aligned} P^+(x,y) &= \sum_{i+j=0}^{n} p_{i,j}^+ x^i y^j, \qquad P^-(x,y) = \sum_{i+j=0}^{n} p_{i,j}^- x^i y^j, \\ Q^+(x,y) &= \sum_{i+j=0}^{n} q_{i,j}^+ x^i y^j, \qquad Q^-(x,y) = \sum_{i+j=0}^{n} q_{i,j}^- x^i y^j. \end{aligned} \tag{12}$$

Note that the unperturbed system (11)$|_{\varepsilon=0}$ has a non-rational first integral $H(x,y) = \left(x^2+y^2\right) e^{-\frac{2x^2}{x^2+y^2}}$ with integrating factor $R(x,y) = \frac{2}{x^2+y^2} e^{-\frac{2x^2}{x^2+y^2}}$. The origin $(0,0)$ is a center. Apply the above first order Melnikov function of (10), we obtain the number of limit cycles which bifurcate from the periodic annulus of the origin of system (11).

Theorem 3. *Consider system (11) with ε sufficiently small. Using the first order Melnikov function of (10), the upper bound of the number of limit cycles which bifurcate from the periodic annulus of the origin of system (11)$|_{\varepsilon=0}$ is n. Moreover, this bound is sharp.*

Remark 4. If $p^+_{i,j} = p^-_{i,j}$ and $q^+_{i,j} = q^-_{i,j}$ for $0 \leqslant i+j \leqslant n$, then system (11) becomes a smooth system and has been studied in the papers [19–21]. In the paper [20], applying the first order averaging method, the authors prove that the sharp upper bound of the number of limit cycles which bifurcate from the periodic annulus of the origin of the smooth system (11)$|_{\varepsilon=0}$ is $[(n-1)/2]$, where $[\eta]$ denotes the integer part of real number η. Our result shows that piecewise smooth differential system (11) can bifurcate at least twice the number of limit cycles than the smooth one. In the papers [19,21], the authors bound the number of limit cycles which bifurcate from the periodic annulus of the origin of the smooth system (11)$|_{\varepsilon=0}$ via second order averaging method and averaging method at any order respectively.

2. Proof of Theorem 1

In this section, we will prove Theorem 1. According to (9), in order to derive the first order Melnikov function $\widetilde{M}_1(h)$, we need to deduce the Poincaré map

$$\begin{aligned} H^+(B_\varepsilon(h)) - H^+(A(h)) &= \left(H^+(B_\varepsilon(h)) - H^-(B_\varepsilon(h))\right) + \left(H^-(B_\varepsilon(h)) - H^-(A_\varepsilon(h))\right) \\ &\quad + \left(H^-(A_\varepsilon(h)) - H^+(A_\varepsilon(h))\right) + \left(H^+(A_\varepsilon(h)) - H^+(A(h))\right) \\ &\overset{\text{def}}{=} l_1 + l_2 + l_3 + l_4. \end{aligned} \tag{13}$$

Firstly, we have

$$\begin{aligned} l_4 &= H^+(A_\varepsilon(h)) - H^+(A(h)) \\ &= \int_{\widehat{AA_\varepsilon}} \mathrm{d}H^+(x,y) \\ &= \int_{\widehat{AA_\varepsilon}} H^+_x(x,y)\mathrm{d}x + H^+_y(x,y)\mathrm{d}y. \end{aligned} \tag{14}$$

From the right subsystem of (8), we can substitute $\mathrm{d}x$ with $\left(\frac{H^+_y(x,y)}{R^+(x,y)} + \varepsilon f^+(x,y)\right)\mathrm{d}t$, and $\mathrm{d}y$ with $\left(-\frac{H^+_x(x,y)}{R^+(x,y)} + \varepsilon g^+(x,y)\right)\mathrm{d}t$ respectively, then

$$\begin{aligned} l_4 &= \int_{\widehat{AA_\varepsilon}} H^+_x(x,y)\left(\frac{H^+_y(x,y)}{R^+(x,y)} + \varepsilon f^+(x,y)\right)\mathrm{d}t + \int_{\widehat{AA_\varepsilon}} H^+_y(x,y)\left(-\frac{H^+_x(x,y)}{R^+(x,y)} + \varepsilon g^+(x,y)\right)\mathrm{d}t \\ &= \varepsilon \int_{\widehat{AA_\varepsilon}} \left(H^+_x(x,y)f^+(x,y) + H^+_y(x,y)g^+(x,y)\right)\mathrm{d}t \\ &= \varepsilon \int_{\widehat{AA_\varepsilon}} f^+(x,y)\left(-R^+(x,y)\mathrm{d}y + \varepsilon g^+(x,y)R^+(x,y)\mathrm{d}t\right) \\ &\quad + \varepsilon \int_{\widehat{AA_\varepsilon}} g^+(x,y)\left(R^+(x,y)\mathrm{d}x + \varepsilon f^+(x,y)R^+(x,y)\mathrm{d}t\right) \\ &= \varepsilon \int_{\widehat{AA_1}} R^+(x,y)g^+(x,y)\mathrm{d}x - R^+(x,y)f^+(x,y)\mathrm{d}y + O(\varepsilon^2). \end{aligned} \tag{15}$$

On one hand, from (15), we obtain

$$\left.\frac{\partial l_4}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_{\widehat{AA_1}} R^+(x,y)g^+(x,y)\mathrm{d}x - R^+(x,y)f^+(x,y)\mathrm{d}y. \tag{16}$$

On the other hand, note that $A_\varepsilon(h) = (0, a_\varepsilon(h))$, from (14), we get

$$\begin{aligned}\left.\frac{\partial l_4}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial \left(H^+\left(A_\varepsilon(h)\right) - H^+(A(h))\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= H_y^+\left(A_1(h)\right)\left.\frac{\partial a_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0}.\end{aligned} \tag{17}$$

From (16) and (17), we obtain

$$\left.\frac{\partial a_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{H_y^+\left(A_1(h)\right)}\int_{\widehat{AA_1}} R^+(x,y)g^+(x,y)\mathrm{d}x - R^+(x,y)f^+(x,y)\mathrm{d}y. \tag{18}$$

Similarly, we have

$$l_2 = \varepsilon\int_{\widehat{A_1A}} R^-(x,y)g^-(x,y)\mathrm{d}x - R^-(x,y)f^-(x,y)\mathrm{d}y + O(\varepsilon^2). \tag{19}$$

It is obvious that

$$\left.\frac{\partial l_2}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_{\widehat{A_1A}} R^-(x,y)g^-(x,y)\mathrm{d}x - R^-(x,y)f^-(x,y)\mathrm{d}y. \tag{20}$$

Note that $A_\varepsilon(h) = (0, a_\varepsilon(h)), B_\varepsilon(h) = (0, b_\varepsilon(h))$, then we have

$$\begin{aligned}\left.\frac{\partial l_2}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial \left(H^-\left(B_\varepsilon(h)\right) - H^-\left(A_\varepsilon(h)\right)\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= H_y^-(A(h))\left.\frac{\partial b_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0} - H_y^-(A_1(h))\left.\frac{\partial a_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0}.\end{aligned} \tag{21}$$

From (18), (20) and (21), we obtain

$$\begin{aligned}\left.\frac{\partial b_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{H_y^-(A(h))}&\left(\int_{\widehat{A_1A}} R^-(x,y)g^-(x,y)\mathrm{d}x - R^-(x,y)f^-(x,y)\mathrm{d}y\right. \\ &\left.+\frac{H_y^-(A_1(h))}{H_y^+(A_1(h))}\int_{\widehat{AA_1}} R^+(x,y)g^+(x,y)\mathrm{d}x - R^+(x,y)f^+(x,y)\mathrm{d}y\right).\end{aligned} \tag{22}$$

Secondly, from (13) and (18), we have

$$\begin{aligned}\left.\frac{\partial l_3}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial \left(H^-\left(A_\varepsilon(h)\right) - H^+\left(A_\varepsilon(h)\right)\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= H_y^-\left(A_1(h)\right)\left.\frac{\partial a_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0} - H_y^+\left(A_1(h)\right)\left.\frac{\partial a_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= \left(\frac{H_y^-\left(A_1(h)\right)}{H_y^+\left(A_1(h)\right)} - 1\right)\int_{\widehat{AA_1}} R^+(x,y)g^+(x,y)\mathrm{d}x - R^+(x,y)f^+(x,y)\mathrm{d}y.\end{aligned} \tag{23}$$

Similarly, from (13) and (22), we get

$$\begin{aligned}\left.\frac{\partial l_1}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial \left(H^+\left(B_\varepsilon(h)\right) - H^-\left(B_\varepsilon(h)\right)\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= H_y^+\left(A(h)\right)\left.\frac{\partial b_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0} - H_y^-\left(A(h)\right)\left.\frac{\partial b_\varepsilon(h)}{\partial \varepsilon}\right|_{\varepsilon=0}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{H_y^+(A(h))}{H_y^-(A(h))} - 1\right) \left(\int_{\widehat{A_1 A}} R^-(x,y) g^-(x,y) \mathrm{d}x - R^-(x,y) f^-(x,y) \mathrm{d}y \right. \\
&\qquad \left. + \frac{H_y^-(A_1(h))}{H_y^+(A_1(h))} \int_{\widehat{A A_1}} R^+(x,y) g^+(x,y) \mathrm{d}x - R^+(x,y) f^+(x,y) \mathrm{d}y \right).
\end{aligned} \tag{24}
$$

Finally, from (9), we can obtain

$$
\begin{aligned}
\widetilde{M}_1(h) &= \left.\frac{\partial \left(H^+(B_\varepsilon(h)) - H^+(A(h))\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\
&= \left.\frac{\partial \left(l_1 + l_2 + l_3 + l_4\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\
&= \frac{H_y^+(A(h))}{H_y^-(A(h))} \left(\frac{H_y^-(A_1(h))}{H_y^+(A_1(h))} \int_{L_h^+} R^+(x,y) g^+(x,y) \mathrm{d}x - R^+(x,y) f^+(x,y) \mathrm{d}y \right. \\
&\quad \left. + \int_{L_h^-} R^-(x,y) g^-(x,y) \mathrm{d}x - R^-(x,y) f^-(x,y) \mathrm{d}y \right)
\end{aligned}
$$

by (16), (20), (23) and (24). This completes the proof of Theorem 1.

3. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. According to Theorem 1, we only need to estimate the number of simple zeros of $\widetilde{M}_1(h), h \in (0, +\infty)$. At first, we should deduce the first order Melnikov function (10) of system (11). Since $H^+(x,y) \equiv H^-(x,y)$ and $R^+(x,y) \equiv R^-(x,y)$, it is obvious that

$$
\begin{aligned}
\widetilde{M}_1(h) &= \int_{L_h^+} R(x,y) Q^+(x,y) \mathrm{d}x - R(x,y) P^+(x,y) \mathrm{d}y \\
&\quad + \int_{L_h^-} R(x,y) Q^-(x,y) \mathrm{d}x - R(x,y) P^-(x,y) \mathrm{d}y.
\end{aligned} \tag{25}
$$

Let

$$
x = h \cos\theta e^{\cos^2\theta}, \qquad y = h \sin\theta e^{\cos^2\theta}, \quad h \in (0, +\infty), \tag{26}
$$

then

$$
R(x,y) = 2h^{-2} e^{-4\cos^2\theta}. \tag{27}
$$

Substitute (26) and (27) into (25), we have

$$
\begin{aligned}
\widetilde{M}_1(h) &= 2 \sum_{i+j=0}^{n} q_{i,j} h^{i+j-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^i\theta \sin^{j+1}\theta \left(1 + 2\cos^2\theta\right) e^{(i+j-3)\cos^2\theta} \mathrm{d}\theta \\
&\quad + 2 \sum_{i+j=0}^{n} p_{i,j} h^{i+j-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{i+1}\theta \sin^j\theta \left(1 - 2\sin^2\theta\right) e^{(i+j-3)\cos^2\theta} \mathrm{d}\theta \\
&= \sum_{i=0}^{n} h^{i-1} \left(\bar{p}_i + \bar{q}_i\right)
\end{aligned} \tag{28}
$$

where

$$\begin{aligned}\bar{p}_i &= 4\sum_{j=0}^{\left[\frac{i}{2}\right]} p_{i-2j,2j}\int_0^{\frac{\pi}{2}} \cos^{i-2j+1}\theta\sin^{2j}\theta\left(1-2\sin^2\theta\right)e^{(i-3)\cos^2\theta}\mathrm{d}\theta,\\ \bar{q}_i &= 4\sum_{j=0}^{\left[\frac{i-1}{2}\right]} q_{i-2j-1,2j+1}\int_0^{\frac{\pi}{2}} \cos^{i-2j-1}\theta\sin^{2j+2}\theta\left(1+2\cos^2\theta\right)e^{(i-3)\cos^2\theta}\mathrm{d}\theta,\end{aligned} \tag{29}$$

and

$$\begin{aligned}p_{i-2j,2j} &= p^+_{i-2j,2j} + (-1)^{i+1}p^-_{i-2j,2j},\\ q_{i-2j-1,2j+1} &= q^+_{i-2j-1,2j+1} + (-1)^{i+1}q^-_{i-2j-1,2j+1}.\end{aligned} \tag{30}$$

Note that $p^+_{i,j}, p^-_{i,j}, q^+_{i,j}, q^-_{i,j}$ are arbitrary coefficients, then we can choose $\bar{p}_i$ and $\bar{q}_i$ independently. Since $h\widetilde{M}_1(h)$ is a polynomial function in the variable h of degree n with independent coefficients, it is obvious that $\widetilde{M}_1(h)$ has at most n zeros in $(0,+\infty)$ and this upper bound is sharp. This completes the proof of Theorem 3.

Note that if $p^+_{i,j} = p^-_{i,j}$ and $q^+_{i,j} = q^-_{i,j}$, then $p_{2i-2j,2j} = q_{2i-2j-1,2j+1} = 0$ by (30), thus $\bar{p}_{2i} = \bar{q}_{2i} = 0$. From (28), we have

$$\widetilde{M}_1(h) = \sum_{i=0}^{[(n-1)/2]} h^{2i}(\bar{p}_{2i+1} + \bar{q}_{2i+1}).$$

It is obvious that $\widetilde{M}_1(h)$ has at most $[(n-1)/2]$ zeros in $(0,+\infty)$ since $\bar{p}_{2i+1}$ and $\bar{q}_{2i+1}$ can be chosen arbitrarily, which means that smooth system (11) has at most $[(n-1)/2]$ limit cycles bifurcating from the periodic annulus of the origin of the unperturbed system $(11)|_{\varepsilon=0}$. Our result coincides with the result of [20].

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References

[1] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, Int. J. Bifurcation Chaos 13 (2003) 47–106.
[2] C. Li, W. Li, Weak Hilbert's 16th problem and the relative reseach, Adv. Math. 39 (2010) 513–526.
[3] M. Han, V.G. Romanovski, On the number of limit cycles of polynomial Liénard systems, Nonlinear Anal. RWA 14 (2013) 1655–1668.
[4] J. Llibre, A.C. Mereu, M.A. Teixeira, Limit cycles of the generalized polynomial Liénard differential equations, Math. Proc. Cambridge Philos. Soc. 148 (2010) 363–383.
[5] Y. Xiong, M. Han, New lower bounds for the Hilbert number of polynomial systems of Liénard type, J. Differential Equations 257 (2014) 2565–2590.
[6] L. Gavrilov, The infinitesimal 16th Hilbert problem in the quadratic case, Invent. Math. 143 (2001) 449–497.
[7] H. Liang, K. Wu, Y. Zhao, Quadratic perturbation of a class of quadratic reversible center of genus one, Sci. China Math. 56 (2013) 577–596.
[8] Y. Liu, J. Li, Z^2-equivariant cubic system which yields 13 limit cycles, Acta Math. Appl. Sin. Engl. Ser. 30 (2014) 781–800.
[9] B. Coll, A. Gasull, R. Prohens, Bifurcation of limit cycles from two families of centers, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 12 (2005) 275–287.
[10] I.D. Iliev, C. Li, J. Yu, Bifurcations of limit cycles from quadratic non-Hamiltonian systems with two centres and two unbounded heteroclinic loops, Nonlinearity 18 (2005) 305–330.
[11] M. di Bernardo, ETCS: Bifurcations in nonsmooth dynamic systems, SIAM Rev. 50 (2008) 629–701.
[12] C. Buzzi, C. Pessoa, J. Torregrosa, Piecewise linear perturbations of a linear center, Discrete Contin. Dyn. Syst. 33 (2013) 3915–3936.
[13] F. Liang, M. Han, V.G. Romanovski, Bifurcation of limit cycles by perturbing piecewise linear Hamiltonian system with a homoclinic loop, Nonlinear Anal. 75 (2012) 4355–4374.

[14] J. Llibre, A.C. Mereu, Limit cycles for discontinuous quadratic differential systems with two zones, J. Math. Anal. Appl. 413 (2014) 763–775.
[15] X. Liu, M. Han, Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems, Int. J. Bifurcation Chaos 20 (2010) 1379–1390.
[16] R.M. Martins, A.C. Mereu, Limit cycles in discontinuous classical Liénard equations, Nonlinear Anal. RWA 20 (2014) 67–73.
[17] J. Yang, M. Han, W. Huang, On Hopf bifurcations of piecewise planar Hamiltonian systems, J. Differential Equations 250 (2011) 1026–1051.
[18] J. Llibre, A.C. Mereu, D.D. Novaes, Averaging theory for discontinuous piecewise differential systems, J. Differential Equations 258 (2015) 4007–4032.
[19] A. Buică, J. Giné, J. Llibre, A second order analysis of the Periodic solutions for nonlinear periodic differential systems with a small parameter, Physica D 241 (2012) 528–533.
[20] J. Llibre, M.A. Teixeira, J. Torregrosa, Limit cycles bifurcating from a k-dimensional isochronous center contained in $\mathbb{R}^n$ with $k \leqslant n$, Math. Phys. Anal. Geom. 10 (2007) 237–249.
[21] S. Li, Y. Zhao, Z. Sun, On the limit cycles of planar polynomial system with non-rational first integral via averaging method at any order, Appl. Math. Comput. 256 (2015) 876–880.