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Extended shift-splitting preconditioners for saddle point problems☆

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ABSTRACT

In this paper we consider to solve the linear systems of the saddle point problems by preconditioned Krylov subspace methods. The preconditioners are based on a special splitting of the saddle point matrix. The convergence theory of this class of the extended shift-splitting preconditioned iteration methods is established. The spectral properties of the preconditioned matrices are analyzed. Numerical implementations show that the resulting preconditioners lead to fast convergence when they are used to precondition Krylov subspace iteration methods such as GMRES.

1. Introduction

Suppose that $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank, and $m \geq n$. Denote by $B^\top$ the transpose of the matrix B. Then the nonsingular saddle point problem is of the form

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{1.1}$$

where $p \in R^m$ and $q \in R^n$ are two given vectors. Under the above assumptions, the existence and uniqueness of the solution of linear equations (1.1) are guaranteed.

Many practical problems arising from scientific computation and engineering applications require to solve saddle point problem (1.1). For example, computational fluid dynamics [1–4], optimal control [5], the finite element approximation for solving the Navier–Stokes equation [6], constrained optimization [7], the Lagrange-type methods for constrained nonconvex optimization problems [8], weighted least-squares problems [9–11], electronic networks [12], mixed finite element of elliptic PDEs, element-free Galerkin method and so forth; see [13–18] and the references therein. For these practical problems, in general, both A and B in (1.1) are large sparse matrices. Iterative methods become more attractive than direct methods for solving the nonsingular saddle point problem (1.1), although direct methods play an important role in the form of preconditioners embedded in an iterative framework, see [11,19]. Numerical iteration methods for the saddle point problem (1.1) have been studied in many papers, including Uzawa-type methods [3,10,11,19–23], SOR-like methods

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[24–31], RPCG iteration methods [32,33], iterative null space methods [34,35], HSS-type methods [36–40], block triangular and skew-Hermitian splitting methods for positive-definite linear systems [41]. The linear system (1.1) also can be solved using Krylov subspace methods [2,15]. The Krylov subspace methods are more efficient than the stationary iterative methods in general [16]. However, Krylov subspace methods tend to converge slowly when applied to the saddle point problem (1.1), and good preconditioners are key ingredients for the success of Krylov subspace methods in the application. Fortunately, a variety of preconditioners have been proposed and studied in many papers; see [42–44] and their references therein.

In this paper, we introduce a so-called extended shift-splitting (ESS) for the coefficient matrix of saddle point problem (1.1), which produces a class of preconditioners. Based on the ESS splitting, an unconditional convergent fixed-point iteration is proposed, and we call this fixed-point iteration ESS iteration method. Also, the ESS preconditioners we obtained are used in the Krylov subspace methods. For the obtained ESS method, the characteristics of the eigenvalues and eigenvectors of the iteration matrix of this new method are analyzed. Moreover, we also analyze the spectral property of the corresponding preconditioned matrix.

The rest of this paper is organized as follows. In Section 2, the ESS preconditioners for the saddle point problems are introduced and some special ESS preconditioners are analyzed. The convergence properties of the ESS iteration method are studied in Section 3. Moreover, we also study the spectral property of the corresponding preconditioned matrix in the last of this section. In Section 4, some numerical experiments are given to show the feasibility and effectiveness of the ESS preconditioners for the saddle point problems. Finally, we give some brief concluding remarks in Section 5.

Here and in the sequel, for a matrix $C^{m\times n}$, we denote the transpose and the rank of C by $C^\top$ and rank(C), respectively. I is the identity matrix with proper dimension. The 0-matrix is denoted by O. Moreover, the spectral radius of C is denoted by $\rho(C)$. $\|\cdot\|_2$ denotes the l_2 norm of the corresponding vector.

2. The ESS preconditioners

In this section, we present the ESS preconditioners for solving the saddle point problem (1.1), and introduce some special ESS preconditioners.

Denote

$$\mathcal{K} = \begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad g = \begin{pmatrix} p \\ -q \end{pmatrix},$$

then (1.1) can be rewritten as

$$\mathcal{K} z = g. \tag{2.1}$$

For the coefficient matrix $\mathcal{K}$ of Eq. (2.1), we make the following ESS splitting (**ESS**):

$$\mathcal{K} = \frac{1}{2}\begin{pmatrix} Q_1 + A & B \\ -B^\top & Q_2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} Q_1 - A & -B \\ B^\top & Q_2 \end{pmatrix} = \mathcal{M} - \mathcal{N}, \tag{2.2}$$

where Q_1 and Q_2 are two symmetric positive definite matrices.

Obviously, because Q_1 and Q_2 are two symmetric positive definite matrices, we can see that $\mathcal{M}$ is a nonsingular matrix. By the special splitting (2.2) for the coefficient matrix $\mathcal{K}$ of Eq. (2.1), the following ESS iteration method can be defined for solving the saddle point problem (1.1):

The ESS method for saddle point problems. Let Q_1 and Q_2 be two symmetric positive definite matrices. Given initial vectors $x^{(0)} \in R^m$, $y^{(0)} \in R^n$. For $k = 0, 1, 2, \ldots$ until the iteration sequence $\{(x^{(k)^\top}, y^{(k)^\top})^\top\}$ converges to the exact solution of the saddle point problem (1.1), compute

$$\frac{1}{2}\begin{pmatrix} Q_1 + A & B \\ -B^\top & Q_2 \end{pmatrix}\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} Q_1 - A & -B \\ B^\top & Q_2 \end{pmatrix}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ -q \end{pmatrix}. \tag{2.3}$$

It is easy to see that the iteration matrix of the above ESS method is

$$\mathcal{T} = \mathcal{M}^{-1}\mathcal{N} = \begin{pmatrix} Q_1 + A & B \\ -B^\top & Q_2 \end{pmatrix}^{-1}\begin{pmatrix} Q_1 - A & -B \\ B^\top & Q_2 \end{pmatrix}. \tag{2.4}$$

The splitting preconditioner that corresponds to the ESS iteration (2.3) is given by

$$\mathcal{P}_{ESS} = \frac{1}{2}\begin{pmatrix} Q_1 + A & B \\ -B^\top & Q_2 \end{pmatrix},$$

which is called the **ESS preconditioner** for the saddle point matrix $\mathcal{K}$.

With different choices of matrices Q_1 and Q_2, we can easily get a series of splitting preconditioners for the saddle point problem (1.1).

Case 1. If $Q_1 = O$ and $Q_2 = \alpha I$, then the ESS preconditioner reduces to the local shift-splitting preconditioner in [16]:

$$\mathscr{P}_{LSS} = \frac{1}{2}\begin{pmatrix} A & B \\ -B^\top & \alpha I \end{pmatrix}.$$

Case 2. If $Q_1 = Q_2 = \alpha I$ ($\alpha > 0$), then the ESS preconditioner becomes the shift-splitting preconditioner in [16]:

$$\mathscr{P}_{SS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B \\ -B^\top & \alpha I \end{pmatrix}.$$

Case 3. If $Q_1 = \alpha I$ and $Q_2 = \beta I$ ($\alpha > 0, \beta > 0$), then the ESS preconditioner yields the generalized shift-splitting preconditioner in [45]:

$$\mathscr{P}_{GSS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B \\ -B^\top & \beta I \end{pmatrix}.$$

At each step of the ESS iteration or applying the ESS preconditioner $\mathscr{P}_{ESS}$ within a Krylov subspace method, we need to solve a linear system with $\mathscr{P}_{ESS}$ as the coefficient matrix. That is to say, we need to solve linear systems of the form

$$\frac{1}{2}\begin{pmatrix} Q_1 + A & B \\ -B^\top & Q_2 \end{pmatrix} z = r$$

for a given vector r at each step. Since the matrix $\mathscr{P}_{ESS}$ has the following matrix factorization

$$\mathscr{P}_{ESS} = \frac{1}{2}\begin{pmatrix} I & BQ_2^{-1} \\ O & I \end{pmatrix}\begin{pmatrix} A + Q_1 + BQ_2^{-1}B^\top & O \\ O & Q_2 \end{pmatrix}\begin{pmatrix} I & O \\ -Q_2^{-1}B^\top & I \end{pmatrix}. \tag{2.5}$$

Let $r = (r_1^\top, r_2^\top)^\top$ and $z = (z_1^\top, z_2^\top)^\top$, where $r_1, z_1 \in R^m$ and $r_2, z_2 \in R^n$. Then by (2.5), we have

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2\begin{pmatrix} I & O \\ Q_2^{-1}B^\top & I \end{pmatrix}\begin{pmatrix} A + Q_1 + BQ_2^{-1}B^\top & O \\ O & Q_2 \end{pmatrix}^{-1}\begin{pmatrix} I & -BQ_2^{-1} \\ O & I \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{2.6}$$

Hence, by making use of (2.6), we can obtain the following detailed algorithmic description of the ESS iteration method.

Algorithm 2.1. For a given vector $r = (r_1^\top, r_2^\top)^\top$, then the vector $z = (z_1^\top, z_2^\top)^\top$ can be computed by (2.6) according the following steps:

- Step 1: $t_1 = r_1 - BQ_2^{-1}r_2$;
- Step 2: solve $(A + Q_1 + BQ_2^{-1}B^\top)z_1 = 2t_1$;
- Step 3: $z_2 = Q_2^{-1}(B^\top z_1 + 2r_2)$.

From Algorithm 2.1, we can see that at each iteration, it is required to solve a linear system with the coefficient matrix $A + Q_1 + BQ_2^{-1}B^\top$. However, this may be very costly and impractical in actual implementations since the sparsity pattern of $A + Q_1 + BQ_2^{-1}B^\top$ is more complicated than A. Fortunately, the matrix $A + Q_1 + BQ_2^{-1}B^\top$ is symmetric positive definite (because Q_1 and Q_2 are two symmetric positive definite matrices). Therefore, in practical implementations, we can employ the conjugate gradient (CG) or the preconditioned conjugate gradient (PCG) method to solve the system of linear equations with the coefficient matrix $A + Q_1 + BQ_2^{-1}B^\top$ by a prescribed accuracy. Besides, the sub-linear systems with the coefficient matrix $A + Q_1 + BQ_2^{-1}B^\top$ can be solved by some direct methods, such as the Cholesky or LU factorization in combination with AMD or column AMD reordering.

3. Convergence analysis for the ESS method

In this section, the unconditional convergence of the ESS iteration is presented. Moreover, the spectral property of the corresponding preconditioned matrix is studied in the last part of this section. For the splitting (2.2), it is well known that the associated iteration (2.3) is convergent if and only if the spectral radius of the iterative matrix $\mathcal{T}$ satisfies $\rho(\mathcal{T}) < 1$.

If we let λ be an eigenvalue of the iteration matrix $\mathcal{T}$ of the ESS method and $\binom{u}{v}$ be the corresponding eigenvector, then we have

$$\mathcal{T}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}, \text{ which implies } N\begin{pmatrix} u \\ v \end{pmatrix} = \lambda M\begin{pmatrix} u \\ v \end{pmatrix}.$$

Or equivalently,

$$\begin{pmatrix} Q_1 - A & -B \\ B^\top & Q_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} Q_1 + A & B \\ -B^\top & Q_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

then we can obtain

$$\begin{cases} (\lambda - 1)Q_1 u + (\lambda + 1)Au + (\lambda + 1)Bv = 0, \\ (\lambda + 1)B^\top u + (1 - \lambda)Q_2 v = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $A \in R^{m\times m}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices. If λ is an eigenvalue of the iteration matrix $\mathcal{T}$ of the ESS iteration method, then $\lambda \neq \pm 1$.*

Proof. Let $(u^*, v^*)^*$, with $u \in C^m$ and $v \in C^n$ being two complex vectors, be the corresponding eigenvector of λ. If $\lambda = 1$, then from (3.1) we have

$$Au + Bv = 0, \qquad B^\top u = 0,$$

which implies

$$\begin{pmatrix} A & B \\ B^\top & O \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{3.2}$$

Since the coefficient matrix of (3.2) is nonsingular, we have $u = v = 0$, which contradicts the assumption that $(u^*, v^*)^*$ is an eigenvector of the iteration matrix $\mathcal{T}$. So $\lambda \neq 1$.

Similarly, if $\lambda = -1$, then from (3.1) we get

$$Q_1 u = 0 \quad \text{and} \quad Q_2 v = 0.$$

Hence, we have $u = v = 0$ because Q_1 and Q_2 are symmetric positive definite matrices. This also contradicts that $(u^*, v^*)^*$ is an eigenvector of $\mathcal{T}$. So $\lambda \neq -1$ holds true. This completes the proof. □

Lemma 3.2. *Let $A \in R^{m\times m}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices, and satisfy $AQ_1 = Q_1A$. Assume λ is an eigenvalue of the iteration matrix $\mathcal{T}$ of the ESS method and $z = (u^*, v^*)^* \in C^{m+n}$, with $u \in C^m$ and $v \in C^n$ being two complex vectors, is the corresponding eigenvector. Then $u \neq 0$. Moreover, if $v = 0$, then $|\lambda| < 1$.*

Proof. If $u = 0$, then it follows from the first equality of (3.1) that $(\lambda + 1)Bv = 0$. By Lemma 3.1 we know that $\lambda \neq -1$. Thus we have $Bv = 0$. Because B is full column rank, we get $v = 0$, which contradicts with the assumption that $z = (u^*, v^*)^*$ is an eigenvector. Thus $u \neq 0$.

If $v = 0$, then from the first equality of (3.1), we can obtain

$$(Q_1 + A)^{-1}(Q_1 - A)u = \lambda u.$$

Note $u \neq 0$, we can see that λ is an eigenvalue of matrix $(Q_1 + A)^{-1}(Q_1 - A)$. Because A and Q_1 are symmetric positive definite matrices and satisfy $AQ_1 = Q_1A$, we know that there exists an invertible matrix P which makes

$$P^{-1}AP = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_m) \quad \text{and} \quad P^{-1}Q_1P = \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_m),$$

where $\mu_i > 0,\ \xi_i > 0\ (i = 1, 2, \ldots, m)$. So the eigenvalues of matrix $(Q_1 + A)^{-1}(Q_1 - A)$ are

$$\frac{\xi_i - \mu_i}{\xi_i + \mu_i}, \quad i = 1, 2, \ldots, m.$$

This implies $|\lambda| < 1$. The proof is completed. □

Theorem 3.1. *Let $A \in R^{m\times m}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices, and satisfy $AQ_1 = Q_1A$. Assume λ is an eigenvalue of the iteration matrix $\mathcal{T}$ of the ESS method and $z = (u^*, v^*)^* \in C^{m+n}$, with $u \in C^m$ and $v \in C^n$ being two complex vectors, is the corresponding eigenvector. Denote*

$$\alpha := \frac{u^*Q_1u}{u^*u}, \qquad \beta := \frac{u^*Au}{u^*u}, \qquad \gamma := \frac{u^*BQ_2^{-1}B^\top u}{u^*u}. \tag{3.3}$$

Then λ satisfies the following quadratic equation

$$(\alpha + \beta + \gamma)\lambda^2 - (2\alpha - 2\gamma)\lambda + (\alpha - \beta + \gamma) = 0. \tag{3.4}$$

Proof. From Lemma 3.1, we can see $\lambda \neq 1$, so it follows from the second equality of (3.1) that

$$v = \frac{\lambda + 1}{\lambda - 1} Q_2^{-1}B^\top u.$$

By substituting this relationship into the first equality in (3.1) we get

$$(\lambda - 1)Q_1u + (\lambda + 1)Au + \frac{(\lambda + 1)^2}{\lambda - 1} BQ_2^{-1}B^\top u = 0.$$

By Lemma 3.2, we know that $u \neq 0$. Multiplying $\lambda - 1$ as well as premultiplying $\frac{u^*}{u^*u}$ on both sides of the above equality we immediately obtain

$$(\lambda - 1)^2 \frac{u^* Q_1 u}{u^* u} + (\lambda^2 - 1)\frac{u^* A u}{u^* u} + (\lambda + 1)^2 \frac{u^* B Q_2^{-1} B^\top u}{u^* u} = 0.$$

So we have

$$\alpha(\lambda - 1)^2 + \beta(\lambda^2 - 1) + \gamma(\lambda + 1)^2 = 0.$$

Now, after rearranging we easily know that λ is a root of the quadratic equation (3.4). The proof is completed. □

The above two lemmas characterize the property about the eigenvalues and the eigenvectors of the iteration matrix $\mathcal{T}$ of the ESS method. Moreover, from Theorem 3.1, we can get the following result.

Corollary 3.1. *From Eq. (3.4) in Theorem 3.1, we can give the specific expression of the eigenvalue λ for the iteration matrix $\mathcal{T}$ of the ESS method when the conditions of Theorem 3.1 are satisfied. That is,*

$$\lambda = \frac{\alpha - \gamma \pm \sqrt{\beta^2 - 4\alpha\gamma}}{\alpha + \beta + \gamma}.$$

Lemma 3.3 ([31]). *Both roots of the real quadratic equation $x^2 - px + q = 0$ are less than 1 in modulus if and only if $|q| < 1$ and $|p| < 1 + q$.*

With Theorem 3.1 and Lemma 3.3, we obtain the following important theorem which shows the unconditional convergence of the ESS iteration method.

Theorem 3.2. *Assume $A \in R^{m\times m}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices, and satisfy $AQ_1 = Q_1 A$. Then the ESS iteration converges to the unique solution of the saddle point problem (1.1).*

Proof. Assume λ is an eigenvalue of the iteration matrix $\mathcal{T}$ of the ESS method and $z = (u^*, v^*)^* \in C^{m+n}$, with $u \in C^m$ and $v \in C^n$ being two complex vectors, is the corresponding eigenvector. Then from Theorem 3.1, we know that λ satisfies the real quadratic equation (3.4). Because A, Q_1 and Q_2 are symmetric positive definite matrices and $\mathrm{rank}(B) = n$, we have

$$\alpha > 0, \qquad \beta > 0, \qquad \gamma \geq 0,$$

where α, β, γ are denoted by (3.3). This implies $\alpha + \beta + \gamma > 0$. So we can get

$$\lambda^2 - \frac{2\alpha - 2\gamma}{\alpha + \beta + \gamma}\lambda + \frac{\alpha - \beta + \gamma}{\alpha + \beta + \gamma} = 0. \tag{3.5}$$

By making use of Lemma 3.3, both roots λ of the real quadratic equation (3.5) satisfy $|\lambda| < 1$ if and only if

$$\left| \frac{\alpha - \beta + \gamma}{\alpha + \beta + \gamma} \right| < 1 \tag{3.6}$$

and

$$\left| \frac{2\alpha - 2\gamma}{\alpha + \beta + \gamma} \right| < 1 + \frac{\alpha - \beta + \gamma}{\alpha + \beta + \gamma}. \tag{3.7}$$

It is easy to check that (3.6) and (3.7) hold for all $\alpha > 0,\ \beta > 0,\ \gamma > 0$.

Moreover, if $\alpha > 0,\ \beta > 0,\ \gamma = 0$, then there is a $u \neq 0$ such that $B^\top u = 0$. From the second equality of (3.1) and Lemma 3.1, we have $Q_2 v = 0$. Because Q_2 is a symmetric positive definite matrix, we have $v = 0$. Hence, by Lemma 3.2 we obtain $|\lambda| < 1$. Thus $\rho(\mathcal{T}) < 1$. Therefore, the ESS iteration converges to the unique solution of the saddle point problem (1.1). The proof is completed. □

Besides using as a solver, the ESS iteration can also be used as a preconditioner to accelerate the Krylov subspace methods such as generalized minimal residual (GMRES) iteration method [46]. In the rest of this section, we study the spectral properties of the preconditioned matrix.

Theorem 3.3. *Assume $A \in R^{m\times m}$ be symmetric positive definite, and $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices, and satisfy $AQ_1 = Q_1 A$. Assume $\widetilde{\lambda}$ is an eigenvalue of the preconditioned matrix $\mathscr{P}_{ESS}^{-1}\mathcal{K}$, then $\widetilde{\lambda}$ satisfies the following quadratic equation*

$$(\alpha + \beta + \gamma)\widetilde{\lambda}^2 - (2\beta + 4\gamma)\widetilde{\lambda} + 4\gamma = 0. \tag{3.8}$$

Moreover, the real parts of $\widetilde{\lambda}$ are all nonnegative, i.e., the preconditioned matrix $\mathcal{M}^{-1}\mathcal{K}$ are positive semi-stable.

Proof. From Eqs. (2.2) and (2.4), we can obtain

$$\mathscr{P}_{ESS}^{-1}\mathcal{K} = \mathscr{P}_{ESS}^{-1}(\mathscr{P}_{ESS} - \mathcal{N}) = I - \mathscr{P}_{ESS}^{-1}\mathcal{N} = I - \mathcal{T},$$

then it holds

$$\lambda(\mathcal{M}^{-1}\mathcal{K}) = 1 - \lambda(\mathcal{T}),$$

or equivalently,

$$\widetilde{\lambda} = 1 - \lambda, \tag{3.9}$$

where λ is an eigenvalue of the iteration matrix $\mathcal{T}$.

From the results in Theorem 3.1, we know λ satisfies the quadratic equation (3.4). Then by substituting the relationship (3.9) into (3.4), we immediately obtain that $\widetilde{\lambda}$ is a root of the quadratic equation (3.8).

Now, we prove that the preconditioned matrix $\mathscr{P}_{ESS}^{-1}\mathcal{K}$ are positive semi-stable. From the result in Corollary 3.1, we can get

$$\lambda = \frac{\alpha - \gamma \pm \sqrt{\beta^2 - 4\alpha\gamma}}{\alpha + \beta + \gamma}.$$

By making use of (3.9), we can obtain

$$\widetilde{\lambda} = \frac{\beta + 2\gamma \pm \sqrt{\beta^2 - 4\alpha\gamma}}{\alpha + \beta + \gamma}. \tag{3.10}$$

(1) If $\Delta = \beta^2 - 4\alpha\gamma < 0$, then $\widetilde{\lambda}$ is complex, since $\beta + 2\gamma > 0$, it is easy to see that $\mathrm{Re}(\widetilde{\lambda}) > 0$, where $\mathrm{Re}(\cdot)$ denotes the real part of $(\cdot)$.

(2) If $\Delta = \beta^2 - 4\alpha\gamma \geq 0$, then $\widetilde{\lambda}$ is real, from (3.10) we only need to consider the following case

$$\widetilde{\lambda} = \frac{\beta + 2\gamma - \sqrt{\beta^2 - 4\alpha\gamma}}{\alpha + \beta + \gamma}.$$

Since

$$(\beta + 2\gamma)^2 - (\beta^2 - 4\alpha\gamma) = 4(\gamma^2 + \alpha\gamma + \beta\gamma) \geq 0,$$

then we have $\mathrm{Re}(\widetilde{\lambda}) \geq 0$. This completes the proof. □

4. Numerical experiments

In this section, we present some numerical experiments to illustrate the effectiveness of the ESS preconditioners for the saddle point problems. In practical computations, we use left preconditioning with restarted GMRES(m) as the Krylov subspace method. Here, the integer m in GMRES(m) denotes that the algorithm is restarted after every m iterations. In this paper, we take $m = 20$. In these examples, the numerical results are compared with the GMRES(m) method without preconditioning and the GMRES(m) method with the shift-splitting preconditioner [16] and the generalized shift-splitting preconditioner [45]. In fact, the authors in [16,45] have proved that the shift-splitting and the generalized shift-splitting preconditioners are better than the well-known HSS (Hermitian and Skew-Hermitian Splitting) preconditioner which has been proved to be very efficient for solving the Stokes problem.

We will report the number of iterations (denoted by "IT"), elapsed CPU time in seconds (denoted by "CPU") and the norm of absolute residual vectors (denoted as "RES") defined by

$$\mathrm{RES} = \sqrt{\|p - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^\top x^{(k)}\|_2^2}.$$

In our actual implementations, we choose the right-hand-side vector $(p^\top, q^\top)^\top \in R^{m+n}$ such that an exact solution of the saddle point problem (1.1) is $(x_*^\top, y_*^\top)^\top = (1, 1, \ldots, 1)^\top \in R^{m+n}$. The iteration schemes are started from the zero vector and terminated if the current iterations satisfy $\mathrm{ERR} \leq 10^{-6}$ or the maximal iteration step is larger than 1600, where

$$\mathrm{ERR} = \frac{\sqrt{\|p - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^\top x^{(k)}\|_2^2}}{\sqrt{\|p\|_2^2 + \|q\|_2^2}}.$$

Table 1
Numerical results for Example 4.1 with $\nu = 0.1$.

l	$\mathscr{P}$			$\mathscr{P}_{SS}$			$\mathscr{P}_{GSS}$		
	IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
8	126	0.0392	9.67e−007	25	0.0066	3.10e−008	25	0.0059	1.45e−007
16	277	0.4838	9.86e−007	25	0.2327	3.48e−008	25	0.2309	2.01e−007
24	509	3.8639	9.82e−007	25	2.2621	3.70e−008	25	2.1551	2.22e−007
32	813	18.8838	9.83e−007	25	11.6960	3.75e−008	25	12.1703	2.29e−007
l	$\mathscr{P}_{ESS}$ (Case 1)			$\mathscr{P}_{ESS}$ (Case 2)			$\mathscr{P}_{ESS}$(Case 3)		
	IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
8	24	0.0048	2.40e−007	24	0.0047	6.65e−009	24	0.0035	2.18e−007
16	23	0.1699	2.80e−007	23	0.1702	4.14e−009	24	0.1833	7.75e−008
24	23	1.8873	2.94e−007	23	1.7741	2.92e−009	24	1.7534	4.14e−007
32	23	9.7191	2.95e−007	23	9.4043	2.24e−009	25	10.2234	1.61e−007

All experiments are performed in MATLAB (version 7.4.0.336 (R2012b)) with machine precision 10^{-16}, and all experiments are implemented on a personal computer with 8.00G memory and Win10 operating system.

Example 4.1. In this test, we consider the following Stokes equation:

$$-\nu\Delta \boldsymbol{u} + \nabla \boldsymbol{f} = f, \quad \text{in } \Omega \tag{4.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega \tag{4.2}$$

$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega \tag{4.3}$$

$$\int_{\Omega} p(x)dx = 0. \tag{4.4}$$

Here $\Omega = [0, 1] \times [0, 1] \subset R^2$, $\partial\Omega$ is the boundary of Ω, ν stands for the viscosity scalar, Δ denote the componentwise Laplace operator, $\boldsymbol{u}$ and p represent the velocity and pressure of fluid, respectively.

Firstly, discrete Laplace operator with center difference scheme, and also discrete the pressure and continuity parts with the first-order forward difference scheme. Then we can obtain the linear system like the form (1.1) with nonsingular coefficient matrix of the following matrix blocks

$$A = \begin{pmatrix} I \otimes T + T \otimes I & O \\ O & I \otimes T + T \otimes I \end{pmatrix} \in R^{2l^2 \times 2l^2}$$

and

$$B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2l^2 \times l^2},$$

with

$$T = \frac{\nu}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in R^{l\times l} \quad \text{and} \quad F = \frac{1}{h} \cdot \text{tridiag}(-1, 1, 0) \in R^{l\times l}$$

being tridiagonal matrices. Here, $m = 2l^2$ and $n = l^2$. Hence, the total number of variables is $m + n = 3l^2$. Moreover, $\otimes$ denotes the Kronecker product and $h = \frac{1}{l+1}$ is the discretization mesh size. This problem is from Example 1 in [22]; see also ([24], Example 5.1). Furthermore, we consider four different choices of the preconditioning matrices Q_1 and Q_2:

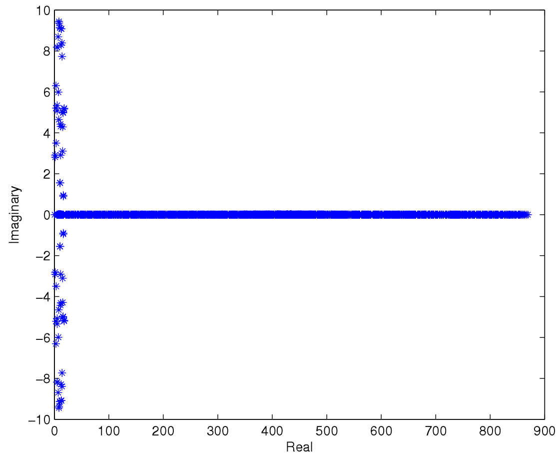
- Case 1: $Q_1 = a_1 I \in R^{m\times m}$, $Q_2 = a_2\text{tridiag}(B^\top A^{-1}B)$, $a_1 = a_2 = 0.01$;
- Case 2: $Q_1 = b_1 A$, $Q_2 = b_2 B^\top\text{tridiag}(A)B$, $b_1 = 0.01$, $b_2 = 0.001$;
- Case 3: $Q_1 = c_1 A$, $Q_2 = c_2 B^\top B$, $c_1 = 0.01$, $c_2 = 0.001$;
- Case 4: $Q_1 = d_1 I \in R^{m\times m}$, $Q_2 = d_2 B^\top B$, $c_1 = 0.01$, $c_2 = 0.001$.

The parameter α in the shift-splitting and the generalized shift-splitting preconditioners is taken as 0.1, and the parameter β of the generalized shift-splitting preconditioners is taken as 0.2. For this example, we test two ν, i.e., $\nu = 0.1, 1$. For each ν, four different l are used, i.e., $l = 8, 16, 24, 32$. In Tables 1 and 2, we list numerical results for the Stokes problem on different uniform grids with $\nu = 0.1$ and $\nu = 1$, respectively. In the tables, $\mathscr{P}$ denotes the GMRES method without preconditioning, $\mathscr{P}_{SS}$, $\mathscr{P}_{GSS}$ and $\mathscr{P}_{ESS}$ denote the GMRES method with the left shift-splitting preconditioning, the left generalized shift-splitting preconditioning and the left ESS preconditioning, respectively.

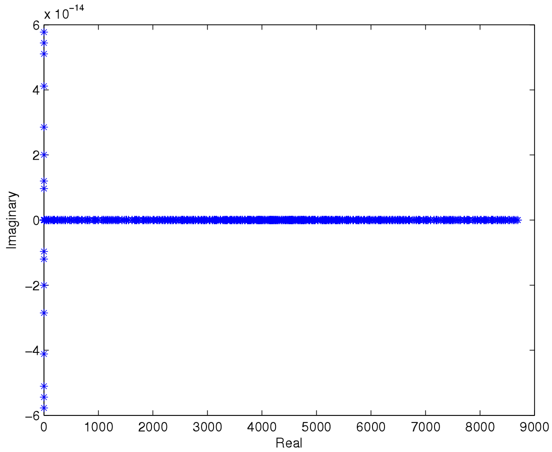
From Tables 1 and 2, we can see that the GMRES method converges very slowly for solving Example 4.1. If the shift-splitting preconditioner, the generalized shift-splitting preconditioner and the ESS preconditioners are used, the corresponding preconditioned GMRES methods converge very fast. From the numerical results, we can find that the shift-splitting and the extended shift-splitting preconditioners are less efficient than the ESS preconditioners. The ESS

Table 2
Numerical results for Example 4.1 with $\nu = 1$.

l	$\mathscr{P}$			$\mathscr{P}_{SS}$			$\mathscr{P}_{GSS}$		
	IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
8	262	0.0859	9.74e−007	25	0.0063	4.54e−007	26	0.0088	8.54e−008
16	979	1.9136	9.89e−007	25	0.3175	4.65e−007	26	0.3854	1.45e−007
24	3021	24.7024	9.78e−007	25	2.5907	2.78e−007	26	3.1491	1.89e−007
32	4096	99.3035	9.98e−007	25	11.7959	3.10e−007	26	13.7312	2.16e−007
l	$\mathscr{P}_{ESS}$ (Case 1)			$\mathscr{P}_{ESS}$ (Case 2)			$\mathscr{P}_{ESS}$(Case 4)		
	IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
8	24	0.0062	3.20e−009	23	0.0045	8.65e−007	25	0.0054	5.99e−007
16	24	0.2599	4.53e−009	23	0.2205	4.41e−007	26	0.3431	8.96e−007
24	24	2.2536	5.67e−009	23	1.6931	2.79e−007	27	3.4270	7.83e−007
32	24	9.3233	6.53e−009	23	7.5361	1.98e−007	28	10.8985	6.01e−007



(a) $\nu = 0.1$.



(b) $\nu = 1$.

Fig. 1. Eigenvalue distribution of the saddle point matrix for Example 4.1 when $l = 32$.

preconditioners are much efficient than the shift-splitting and the generalized shift-splitting preconditioners because the CPU time and the RES of the ESS preconditioners are less than those of the other two preconditioners.

In Fig. 1, we depict the eigenvalue distributions of the saddle point matrix $\mathcal{K}$. Fig. 1(a) and Fig. 1(b) stand for $\nu = 0.1$ and $\nu = 1$, respectively. Moreover, Figs. 2 and 3 plot the eigenvalue distribution of the preconditioned matrices $\mathcal{M}^{-1}\mathcal{K}$ when $l = 32$. "SS", "GSS" and "ESS" stand for the shift-splitting preconditioned matrix $\mathscr{P}_{SS}^{-1}\mathcal{K}$, the generalized shift-splitting preconditioned matrix $\mathscr{P}_{GSS}^{-1}\mathcal{K}$ and the ESS preconditioned matrix $\mathscr{P}_{ESS}^{-1}\mathcal{K}$, respectively. Fig. 1 shows that the eigenvalues of the preconditioned matrices are more cluster than those of the saddle point matrix $\mathcal{K}$. Besides, from Figs. 2 and 3, we can also see that all the eigenvalues of the ESS preconditioned matrix $\mathscr{P}_{ESS}^{-1}\mathcal{K}$ are more clustered than those of $\mathscr{P}_{SS}^{-1}\mathcal{K}$ and $\mathscr{P}_{GSS}^{-1}\mathcal{K}$.

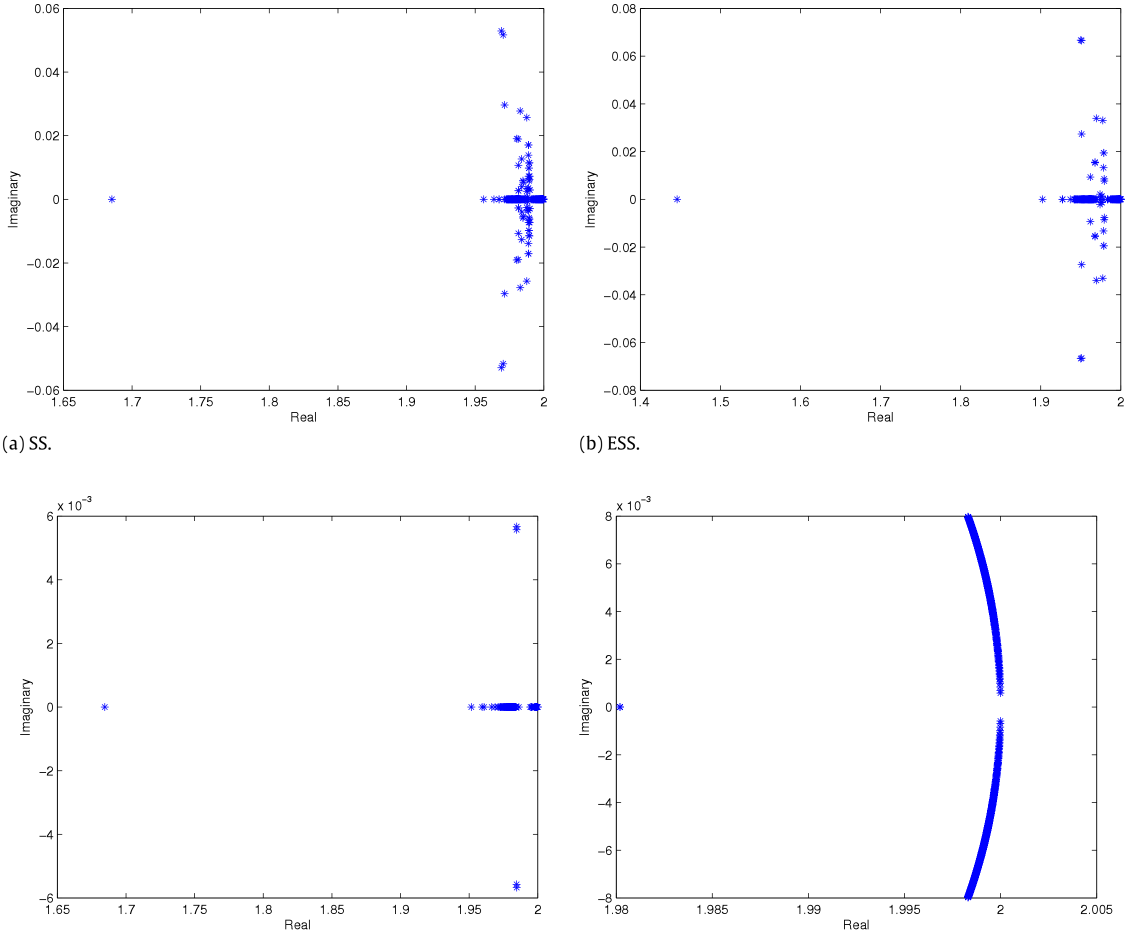
Example 4.2. In the second test, we consider the saddle point problem (1.1) with matrices $A \in R^{m\times m}$ and $B \in R^{m\times n}$ defied as follows:

$$A = (a_{i,j})_{m\times m} = \begin{cases} i+1, & i=j, \\ 1, & |i-j|=1, \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad B = (b_{i,j})_{m\times n} = \begin{cases} j, & i=j+m-n, \\ 0, & \text{otherwise}. \end{cases}$$

This nonsingular saddle point problem is from the literature [22,47]. Furthermore, we consider the following three different choices for the preconditioning matrices Q_1 and Q_2:

- Case 5: $Q_1 = a_1 I \in R^{m\times m}$, $Q_2 = a_2 \mathrm{tridiag}(B^\top A^{-1} B)$, $a_1 = a_2 = 0.01$;
- Case 6: $Q_1 = b_1 A$, $Q_2 = b_2 B^\top \mathrm{tridiag}(A) B$, $b_1 = 0.01,\ b_2 = 0.001$;
- Case 7: $Q_1 = c_1 A$, $Q_2 = c_2 B^\top B$, $c_1 = 0.01,\ c_2 = 0.001$.

The parameter α in the shift-splitting and the generalized shift-splitting preconditioners is taken as 0.1, and the parameter β of the generalized shift-splitting preconditioners is taken as 0.2. In Table 3, we list numerical results (IT, CPU, RES) for the saddle point problem on different m and n. In the table, $\mathscr{P}$ denotes the GMRES method without preconditioning,



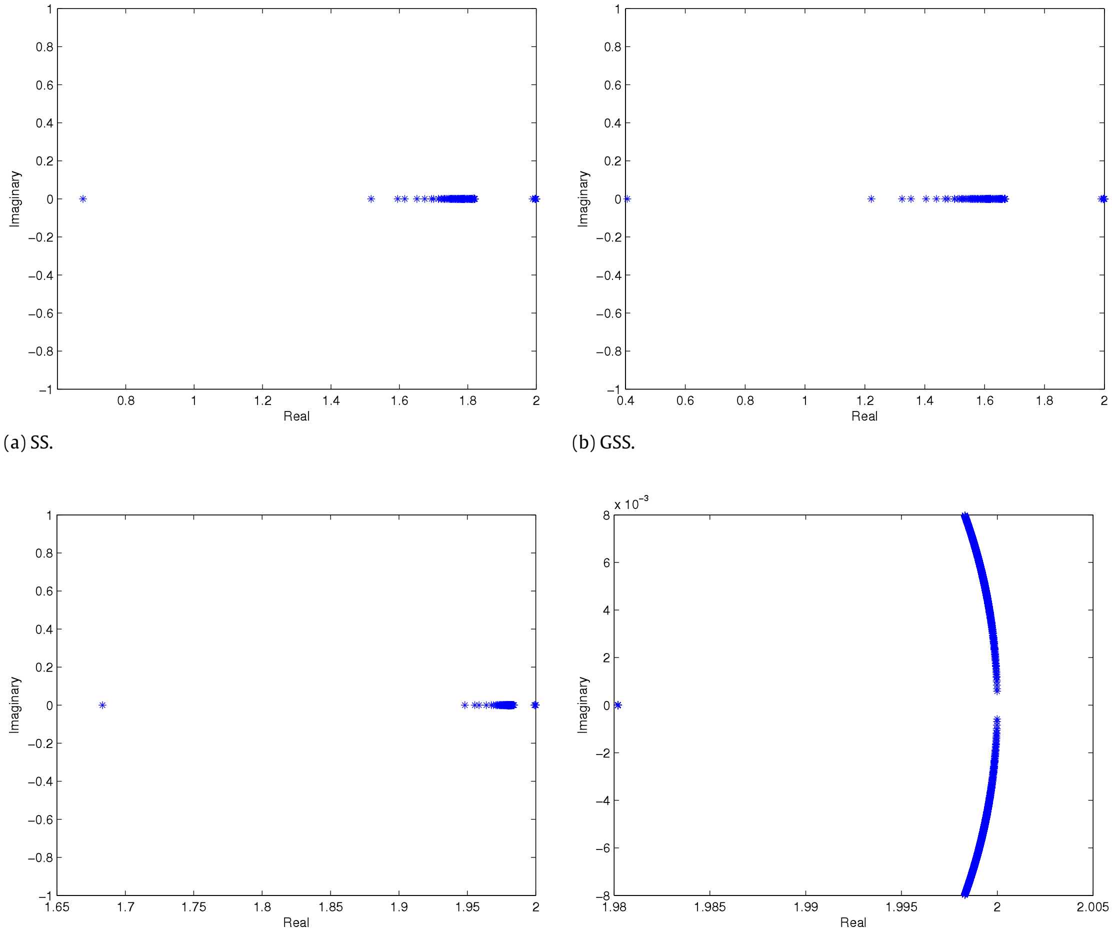
(a) SS. (b) ESS.

(c) ESS (Case 1). (d) ESS (Case 2).

Fig. 2. Eigenvalue distribution of the preconditioned saddle point matrices for Example 4.1 when $\nu = 0.1,\ l = 32$.

Table 3
Numerical results of different methods for Example 4.2.

m	n	$\mathcal{P}$			$\mathcal{P}_{SS}$			$\mathcal{P}_{GSS}$		
		IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
50	40	300	0.2047	9.78e−007	26	0.0185	4.71e−007	27	0.0052	5.84e−007
200	150	951	0.2802	9.99e−007	28	0.0337	2.14e−007	30	0.0421	1.51e−007
300	200	941	0.2765	9.88e−007	29	0.0537	5.73e−007	32	0.0719	1.75e−007
400	300	1053	0.8675	9.98e−007	29	0.1184	3.45e−007	30	0.1451	6.14e−007
800	600	1027	3.5669	9.98e−007	30	0.7500	7.30e−007	31	0.7301	9.30e−007
1 000	800	1414	7.9206	9.98e−007	30	1.1128	7.30e−007	29	0.9683	9.45e−007
15 000	10 000	–	–	–	29	2.0312	9.21e−007	31	1.8983	9.32e−007
200 000	150 000	–	–	–	30	2.1226	5.78e−007	31	2.2564	8.99e−007
m	n	$\mathcal{P}_{MS}$ (Case 5)			$\mathcal{P}_{MS}$ (Case 6)			$\mathcal{P}_{MS}$ (Case 7)		
		IT	CPU	RES	IT	CPU	RES	IT	CPU	RES
50	40	23	0.0025	1.51e−007	24	0.0013	1.45e−010	24	0.0032	1.23e−007
200	150	23	0.0150	2.65e−008	23	0.0148	4.34e−007	25	0.0214	3.96e−008
300	200	23	0.0205	1.64e−008	23	0.0207	3.52e−007	25	0.0338	7.50e−008
400	300	23	0.0472	1.01e−008	23	0.0502	2.19e−007	25	0.0706	1.85e−007
800	600	23	0.2268	3.75e−009	23	0.2249	1.10e−007	25	0.3299	6.13e−007
1 000	800	23	0.3796	2.58e−009	23	0.3873	7.62e−008	26	0.6716	1.31e−007
15 000	10 000	23	0.5039	7.90e−007	23	0.4995	6.99e−007	26	0.7215	2.45e−007
200 000	150 000	23	0.6590	6.30e−007	23	0.6528	4.53e−007	25	0.7902	3.56e−007



(a) SS. (b) GSS.

(c) ESS (Case 1). (d) ESS (Case 2).

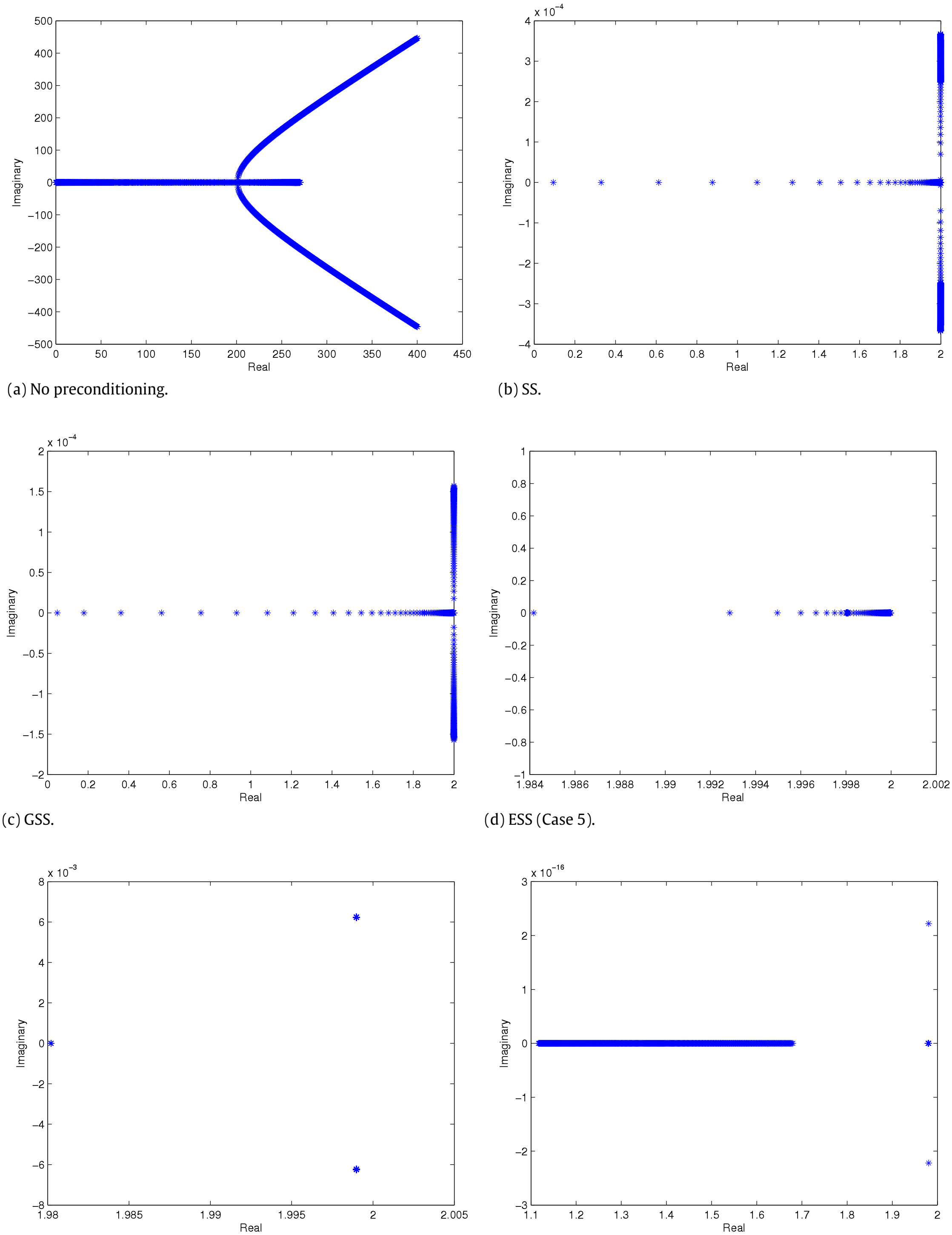
Fig. 3. Eigenvalue distribution of the preconditioned saddle point matrices for Example 4.1 when $\nu = 1,\ l = 32$.

$\mathscr{P}_{SS}$, $\mathscr{P}_{ESS}$ and $\mathscr{P}_{ESS}$ denote the GMRES method with the left shift-splitting preconditioning, the left generalized shift-splitting preconditioning and the left ESS preconditioning, respectively.

In Fig. 4, we depict the eigenvalue distributions of the saddle point matrix $\mathcal{K}$ and the preconditioned matrices $\mathcal{M}^{-1}\mathcal{K}$ when $m = 800, n = 600$. "No preconditioning", "SS", "GSS" and "ESS" stand for the original saddle point matrix $\mathcal{K}$, the shift-splitting preconditioned matrix $\mathscr{P}_{SS}^{-1}\mathcal{K}$, the generalized shift-splitting preconditioned matrix $\mathscr{P}_{GSS}^{-1}\mathcal{K}$ and the ESS preconditioned matrix $\mathscr{P}_{ESS}^{-1}\mathcal{K}$, respectively. As we can see from Fig. 4 that the eigenvalues of the preconditioned matrices are more cluster than those of the saddle point matrix $\mathcal{K}$. Besides, from Fig. 4, we can also see that all the eigenvalues of the ESS preconditioned matrix $\mathscr{P}_{ESS}^{-1}\mathcal{K}$ are more clustered than those of $\mathscr{P}_{SS}^{-1}\mathcal{K}$ and $\mathscr{P}_{GSS}^{-1}\mathcal{K}$. This further illustrates that the ESS preconditioners are effective preconditioners for solving the large and sparse saddle point problems.

5. Conclusions

For solving the large sparse saddle point problems, a class of extended shift-splitting (ESS) preconditioners are proposed and studied in this paper. The ESS preconditioners can be derived from a special splitting of the coefficient matrix of saddle point problem (1.1). The unconditionally convergent property of the ESS iteration method is proved and some characteristics about the eigenvalues and the eigenvectors of the iteration matrix $\mathcal{T}$ of the ESS method are presented. Moreover, the special properties of the preconditioned matrices are studied in this paper. The efficiency of the ESS preconditioners for the saddle point problems is illustrated by some numerical examples.



(a) No preconditioning.

(b) SS.

(c) GSS.

(d) ESS (Case 5).

(e) ESS (Case 6).

(f) ESS (Case 7).

Fig. 4. Eigenvalue distribution of the saddle point matrix and the preconditioned saddle point matrices for Example 4.2 when $m = 800$, $n = 600$.

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