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Research paper

Implicit Euler approximation of stochastic evolution equations with fractional Brownian motion

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ABSTRACT

This work was intended as an attempt to motivate the approximation of quasi linear evolution equations driven by infinite-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The spatial approximation method is based on Galerkin and the temporal approximation is based on implicit Euler scheme. An error bound and the convergence of the numerical method are given. The numerical results show usefulness and accuracy of the method.

1. Introduction

Models based on stochastic partial differential equations (SPDEs) play a prominent role in a range of application areas, including chemistry, physics, biology, economics, finance, microelectronics, and nowadays also nanotechnology.

Very recently, stochastic equations in infinite dimensions with fractional Brownian motion as a driving process has become an important subject of interest of many researchers. These objects are interesting because they model a type of random disturbances with features different from those of Wiener process (like the long-range dependence). They also provide new mathematical challenges because large parts of the standard stochastic analysis tools are not applicable for them.

Fractional Brownian motion indexed by a Hurst parameter $H \in (0, 1)$ is a Gaussian process. It has been introduced by Hurst [8] to model the long term storage capacity of reservoirs along the Nile river. This process seems to be applicable in modeling real phenomena based especially in empirical data such as economic data (see e.g. [3,9]) telecommunication traffic (see e.g. [13,18]) and others.

Since a fractional Brownian motion with $H \neq \frac{1}{2}$ is not a semi martingale, the standard stochastic calculus and the classical stochastic integration theory cannot be used for them. In recent years there have been various developments of stochastic calculus for this process, especially for $H \in (\frac{1}{2}, 1)$(see e.g. [1,2,19]). One of the main obstacles in the stochastic calculus for fractional Brownian motion is the concept of stochastic integral which has been discussed by [1,19] and [6].

Finite dimensional stochastic differential equations driven by fractional Brownian motion have been treated for instance in [14] and [16]. In [10] for these kinds of equations, existence and uniqueness results are given. Theory of SPDEs driven by fractional Brownian motion also has been studied recently. For example Linear and semilinear stochastic equations in a Hilbert space with an infinite dimensional fractional Brownian motion are considered in [4] and [5].

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Numerical solutions of stochastic differential equations (SDEs) driven by fractional Brownian motion have been investigated via several schemes such as Milestein and Euler scheme by [11] and [15] . But numerical approximation for SPDEs with fractional Brownian motion have been very little studied. In [12] the explicit Euler scheme for SPDEs with fractional Brownian motion has been proposed and the rate of convergence of the approximation method is established.

But as we know there are lots of equations which explicit methods are not applicable for them, for example for stiff systems in many cases we have to concentrate on implicit methods. The main idea of this article is to propose an implicit method for solving numerical solution of SPDEs driven by infinite dimensional fractional Brownian motions. For this aim for spatial discretization we apply Galerkin method and for time discretization the implicit Euler scheme will be used and then the convergence rate of the method will be proved.

The rest of this article is organized as follows. In Section 2 we define the fractional Brownian motion and the stochastic fractional integral. The basic setting and the assumptions are presented in Section 3 and in Subsection 3.1 a brief exposition of Galerkin Method is introduced. The numerical scheme and its rate of convergence theorem which are the main results of this article are given in Section 4 . Finally in Section 5 a method(Cholesky methods) for simulating of fractional Brownian motions is described and the main result of the paper is illustrated with some examples.

2. Preliminary

In this section, we review some of the standard facts on the fractional calculus. Let $T \in (0, \infty)$ be a real number and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Assume $U = (U, \langle ., . \rangle, |.|_U)$ and $V = (V, \langle ., . \rangle, |.|_V)$ are separable Hilbert spaces and $\mathcal{L}_2 := \mathcal{L}_2(U, V)$ be the family of linear Hilbert-Schmidt operators from U to V with $|T|_{\mathcal{L}_2} = (\sum_{k=1}^{\infty} |Te_k|^2)^{\frac{1}{2}}$, where $\{e_k, k \in \mathbb{N}\}$ is a set of complete orthonormal basis in U. The following definition provides an infinite-dimensional analogue of a fractional Brownian motion in a finite-dimensional space with Hurst parameter $H \in (\frac{1}{2}, 1)$[4].

Definition 2.1. A $U-$valued Gaussian process $(B^H(t), t \in \mathbb{R})$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ if

(i) $E(B_t^H) = 0$,
(ii) $E(B_t^H B_s^H) = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$.

If $H = \frac{1}{2}$, then the corresponding fractional Brownian motion is the usual standard Brownian motion.

A standard (cylindrical) fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ in the Hilbert space U is defined by

$$B^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) e_n, \tag{2.1}$$

where $\{\beta_n^H(t), n \in \mathbb{N}, t \in \mathbb{R}\}$ is a sequence of independent, real-valued standard fractional Brownian motions each with the same Hurst parameter H.

Let $p > \frac{1}{H}$ be an arbitrary but fixed number and G be a function such that for each $x \in U$, $G(.)x \in L^p([0, T], V)$ and

$$\int_0^T \int_0^T |G(s)|_{\mathcal{L}_2} |G(r)|_{\mathcal{L}_2} |s-r|^{2H-2} dsdr < \infty, \tag{2.2}$$

then the stochastic integral is defined as

$$I(G, 0, T) = \int_0^T G dB^H = \sum_{n=1}^{\infty} \int_0^T Ge_n d\beta_n^H. \tag{2.3}$$

The sequence of random variables $(\int_0^T Ge_n d\beta_n^H, n \in \mathbb{N})$ are mutually independent Gaussian random variables, such that

$$\begin{aligned} \mathbb{E}|I(G, 0, T)|_V^2 &= \sum_{n=1}^{\infty} \mathbb{E}\Big| \int_0^T Ge_n d\beta_n^H \Big|_V^2 \\ &\leq H(2H-1) \sum_{n=1}^{\infty} \int_0^T \int_0^T \langle G(s)e_n, G(r)e_n \rangle_V |s-r|^{2H-2} drds \\ &\leq H(2H-1) \int_0^T \int_0^T |G(s)|_{\mathcal{L}_2(U,V)} |G(r)|_{\mathcal{L}_2(U,V)} |s-r|^{2H-2} dsdr < \infty. \end{aligned} \tag{2.4}$$

3. Setting of the problem

Let $U = V = L^2(0, 1)$, equipped with the norm $\|.\|$ and inner product $< ., . >$. Fix γ, θ, ρ and θ_σ, such that $\gamma > 0, 0 \leq \theta \leq \frac{1}{2}, 0 \leq \rho < \min(\frac{1}{2}, \gamma)$, and $\rho + \theta_\sigma \leq \theta$.

Through the paper let the following assumptions are satisfied.

A1. Let $(\lambda_n)_{n\in\mathbb{N}} \subset (0, \infty)$ be an increasing sequence of real numbers and let $(e_n)_{n\in\mathbb{N}} \subset U$ be an orthonormal basis of U. Assume that the Laplacian operator A: $Dom(A) \subset U \to U$ is the infinitesimal generator of an analytic semigroup $T_A(t) = e^{At}$, such that

$$Av = \sum_{n=1}^{\infty} -\lambda_n \langle e_n, v\rangle e_n,$$

and let $u_0 \in D((-A)^\gamma)$ where $D((-A)^\gamma)$ is the domain of the fractional power $(-A)^\gamma$ equipped with the norm $\|u\|_\gamma = \|(-A)^\gamma u\|$ for $u \in D((-A)^\gamma)$.

As we know the eigenfunctions, i.e., e_n and the eigenvalues, i.e., λ_n of the Laplacian operator in one dimension in $L^2(0, 1)$ are given by

$$\lambda_n = n^2\pi^2, \qquad e_n = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \cdots .$$

A2. Assume $f(t, .)$: $U \to U$

(i) is a Lipschitz continuous function such that satisfies the linear growth condition, i.e.,

$$\begin{aligned} &\|f(t,x) - f(t,y)\|_\delta \le C\|x - y\|_\delta \quad \text{for} \quad \delta \in [\rho, \gamma],\ x, y \in D((-A)^\gamma), \\ &\|f(t,x)\|_\delta^2 \le K(1 + \|x\|_\delta^2) \quad \text{for} \quad \delta \in [\rho, \gamma],\ x, y \in D((-A)^\gamma), \end{aligned} \tag{3.1}$$

and

(ii) is *Hölder* continuous in time with the exponent $\min(\frac{1}{2}, \gamma - \rho - \theta_\sigma)$ in the sense of

$$\|f(t,x) - f(s,x)\|_\rho \le C|t - s|^{\min(\frac{1}{2}, \gamma-\rho-\theta_\sigma)}\|x\|_{\min(1+\rho+\theta+\theta_\sigma, \gamma+\theta)}. \tag{3.2}$$

A3. Let $\sigma \in L(U, U)$ be a linear operator such that for $0 < \theta < H$ and $\theta < \gamma$

$$\|(-A)^{-\theta+\gamma}\sigma\|_{\mathcal{L}_2} < \infty. \tag{3.3}$$

A4. Assume there exists $\theta > 0$ such that for $\delta \in [p, \gamma]$

$$\int_0^T \int_0^T x^{-(\theta+\delta)} y^{-(\theta+\delta)} \|T_A(x)\sigma\|_{\mathcal{L}_2} \|T_A(y)\sigma\|_{\mathcal{L}_2} |x - y|^{2H-2} dxdy < \infty. \tag{3.4}$$

Consider the linear equation

$$\begin{aligned} &du(t) = (Au(t) + f(t, u(t)))dt + \sigma(t)dB^H(t), \\ &u(0) = u_0, \end{aligned} \tag{3.5}$$

where $B^H(t)$ is a standard fractional Brownian motion in U. The mild solution (also known as evolution solution) of (3.5), for $t \in [0, T]$ is defined by

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(s, u(s))ds + \int_0^t e^{A(t-s)}\sigma dB_s^H, \tag{3.6}$$

where the stochastic integral on the right hand side of (3.6) is the stochastic integral with respect to fractional Brownian motions.

The following theorem shows the existences and uniqueness of the mild solution of Eq. (3.5).

Theorem 3.1. *[12] Let Assumptions* $A1 - A4$ *are satisfied, then for* $u(0) = u_0 \in Dom((-A)^\gamma)$, *Eq. (3.5) has a unique mild solution* $(u(t))$, $t \in [0, T]$.

Since in most cases the mild solution is not explicitly known, our goal is to obtain it's numerical approximation. In order to be able to calculate such a numerical approximation on a computer both the time interval [0, T] and the infinite dimensional $\mathbb{R}$-Hilbert space U have to be discretized. In the next subsection spatial discretization is done by Galerkin method and in Section 4 the temporal discretization will be proposed.

3.1. Space discretization

The Galerkin approximation is done by only taking a finite number of basis functions. Let $U_N = span\{e_i; 1 \le i \le N\}$, the Galerkin projection P_N: $U \to U_N$ on $u = \sum_{i=1}^{\infty} u_i e_i$ is defined as

$$u_N := P_N u = \sum_{i=1}^{N} \langle e_i, u\rangle e_i. \tag{3.7}$$

Now, we truncate or project the initial random variable u_0, the nonlinearity f and the fractional Brownian motion $B^H(t)$ by

$$u_0^N = P_N u_0, \quad f_N(t,u(t)) = (P_N f)(t,u(t)), \quad B_N^H(t) = P_N B^H(t) = \sum_{i=1}^{N} e_i \beta_i^H(t), \tag{3.8}$$

for all $u \in U$.

Using these notations, the Galerkin approximation leads to the following system of stochastic differential equations

$$\begin{aligned} du_N(t) &= (A_N u_N(t) + f_N(t,u(t)))dt + \sigma_N dB_N^H(t), \ \ t \in [0,T], \\ u_N(0) &= P_N u_0, \end{aligned} \tag{3.9}$$

where σ_N and A_N are bounded linear operators defined on U_N such that $\sigma_N = P_N\sigma$ and $A_N = P_N A$ respectively.

4. Main theorem

In this section we obtain the numerical solution of (3.9) by an implicit method, then we will investigate the convergence of the proposed numerical method.

For time discretization of (3.9), we apply the implicit Euler method. For this aim let $\Delta t = \frac{T}{N}, N \in \mathbb{N}$ be the time step size corresponding to the space U_N, and define $\mathcal{F}/\mathcal{B}(U_N)$ measurable mapping $v_N^k : \Omega \to U_N$ as the approximation of $u_N(k\Delta t)$, by

$$\begin{aligned} v_N^{k+1} &= (1 - \Delta t A_N)^{-1}\left\{v_N^k + \Delta t P_N f(k\Delta t, v_N^k) + \sigma_N \delta_k B_N^H\right\}, \quad k = 0,1,\cdots,N-1 \\ v_N^0 &= P_N u_0, \end{aligned} \tag{4.1}$$

where $\delta_k B_N^H = B_N^H((k+1)\Delta t) - B_N^H(k\Delta t), \quad k = 0,1,\cdots,N-1$. Between the points $t = k\Delta t$ and $t = (k+1)\Delta t$ the solution can be interpolated linearly, that is $v_N(t) = v_N^k + \frac{v_N^{k+1} - v_N^k}{\Delta t}(t - k\Delta t)$ for $t \in (k\Delta t, (k+1)\Delta t)$.

Eq. (4.1) can be written as

$$v_N^k = G_N^k P_N u_0 + \Delta t \sum_{i=0}^{k-1} G_N^{k-i-1} P_N f(i\Delta t, v_N^i) + \sum_{i=0}^{k-1} G_N^{k-i-1}\sigma_N \delta_i B_N^H, \tag{4.2}$$

where

$$G_N = (I - \Delta t A_N)^{-1}. \tag{4.3}$$

In order to prove the main theorem which is related to the convergence of the numerical solution given by (4.2), we need the following lemmas.

Lemma 4.1. *If A be the Laplacian operator in one dimension then the following inequalities hold*

$$\begin{aligned} &(i) \ \ \|(I - P_N)x\|^2 \le \phi_\gamma(N)\|x\|_\gamma^2, \qquad x \in D(-A)^\gamma, \\ &(ii) \ \ \|A(I - P_N)x\|^2 \le \phi_\gamma(N)\|A_n\|^2\|x\|_\gamma^2, \quad x \in D(-A)^\gamma, \end{aligned}$$

where

$$\phi_\gamma(N) = \frac{1}{(\pi(N+1))^{4\gamma}}.$$

Proof. By assumption $A1$:

$$Av = \sum_{i=1}^{\infty} -\lambda_i \langle e_i, v\rangle e_i, \qquad \lambda_i = \pi^2 i^2, \tag{4.4}$$

for every $v \in U$, therefore

$$\begin{aligned} \|(I - P_N)x\|^2 &\le \sum_{i=N+1}^{\infty} \langle e_i, x\rangle^2 \le \frac{1}{\lambda_{N+1}^{2\gamma}} \sum_{i=N+1}^{\infty} \lambda_i^{2\gamma}\langle e_i, x\rangle^2 \\ &= \frac{1}{(\pi(N+1))^{4\gamma}} \sum_{i=N+1}^{\infty} \langle (-\Delta)^\gamma e_i, x\rangle^2 \\ &\le \phi_\gamma(n)\|x\|_\gamma^2. \end{aligned} \tag{4.5}$$

Similarity, the proof of the second part is as follows

$$\begin{aligned}\|A(I-P_N)x\|^2 &= \sum_{i=N+1}^{\infty} \lambda_i^2 \langle e_i, x\rangle^2 \leq \lambda_{N+1}^2 \sum_{i=N+1}^{\infty} \langle e_i, x\rangle^2 \\ &\leq \lambda_{N+1}^{2-2\gamma} \sum_{i=N+1}^{\infty} \langle (-\Delta)^{\gamma} e_i, x\rangle^2 \\ &= \frac{1}{(\pi(N+1))^{4\gamma}} \|A_N\|^2 \|x\|_{\gamma}^2 = \phi_{\gamma}(N) \|A_N\|^2 \|x\|_{\gamma}^2.\end{aligned}$$

□

Lemma 4.2. [7] *Fix γ, and ρ such that $0 \leq \rho < \gamma$. Let $G_\tau(A) = (I-\tau A)^{-1}$, then for every $x \in U$ we have*

$$\|[G_\tau(A)^k - T(k\tau)]x\|_\rho \leq C\tau^{\min(1,\gamma-\rho)} \|x\|_{\min(1+\rho,\gamma)},$$

$k \geq 2$, and

$$\|[G_\tau - T(\tau)]x\|_\rho \leq C\tau^{\min(1,\gamma-\rho)} \|x\|_{\min(1+\rho,\gamma)}.$$

Lemma 4.3. [7] *The bounded operator G_N satisfies the following inequalities*

$$\begin{aligned}&\|G_N^k\| \leq Me^{\tilde{w}\Delta t k} \text{ for } k \geq 1,\\ &\|G_N\|^2 + \Delta t \|(-A_N)^{\theta_\sigma}\|^2 \leq 1 \text{ if } \theta_\sigma > 0,\end{aligned} \tag{4.6}$$

for some $M < \infty$ and $\tilde{w} \in \mathbb{R}$.

Theorem 4.4. *Let u be the solution of the Eq. (3.5), then under the assumptions A1–A4, the priori error of the implicit Euler scheme for arbitrary $\epsilon > 0$, can be estimated at $t = k\Delta t$ by*

$$\mathbb{E}[\|v_N^k - u(k\Delta t)\|_\rho^2] \leq C_1 \Delta t^{\min(1,2(\gamma-\rho-\theta_\sigma))} + C_2\phi_{\gamma-\rho}^2(N) + C_3\phi_{\gamma-\epsilon}^2(N) + C_4\phi_\gamma^2(N),$$

where C_1, C_2, C_3, C_4 are some positive constants.

Proof. We shall write Eq. (4.2) as

$$v_N^k = v_N^{[t]^k} = G_N^{[t]^k} P_N u_0 + \int_0^{[t]} G_N^{[t]^k-1-[s]^k} P_N f([s], v_N([s]^k)) ds + \int_0^{[t]} G_N^{[t]^k-1-[s]^k} P_N \sigma dB_N^H(s),$$

where the notations $[t]$, $[t]^k$ are defined by $[t] := [\frac{t}{\Delta t}]\Delta t$ and $[t]^k := [\frac{t}{\Delta t}]$.

By using Eq. (3.6) we obtain

$$u(k\Delta t) - v_N^k = \sum_{i=0}^{4} I_i(t), \tag{4.7}$$

where

$$\begin{aligned}I_0(t) &= \int_0^{[t]} \left[e^{A_N([t]-[s]-\Delta t)} - G_N^{[t]^k-1-[s]^k}\right] P_N \sigma P_N dB^H(s),\\ I_1(t) &= \int_0^{[t]} \left[e^{A([t]-s)} - e^{A_N([t]-[s]-\Delta t)}\right] P_N \sigma P_N dB^H(s),\\ I_2(t) &= \int_0^{[t]} e^{A([t]-s)} \left[\sigma - P_N \sigma P_N\right] dB^H(s),\\ I_3(t) &= \int_0^{[t]} \left\{e^{A([t]-s)} f(s, u(s)) - G_N^{[t]^k-1-[s]^k} P_N f([s], v_n^{[s]^k})\right\} ds,\\ I_4(t) &= e^{A([t])} u_0 - G_N^{[t]^k} P_N u_0.\end{aligned}$$

Let $r(t) = u(t) - v_N(t)$, with $t = k\Delta t$ for some $k \in \mathbb{N}$. By substituting (4.7) we have

$$\mathbb{E}[\|r([t])\|_\rho^2] \leq C \sum_{i=0}^{4} \mathbb{E}[\|I_i([t])\|_\rho^2].$$

By the definition of stochastic integral with respect to FBM with $H > \frac{1}{2}$, we get

$$\begin{aligned}\mathbb{E}\Big[\|I_0([t]_n)\|_\rho^2\Big] &= \sum_{l=1}^{\infty}\int_0^{[t]}\int_0^{[t]}\langle(-A)^\rho[e^{([t]-[u]-\Delta t)A_N} - G_N^{[t]^l-[u]^l-1}]P_N\sigma P_N e_l,\\ &\quad (-A)^\rho[e^{([t]-[v]-\Delta t)A_N} - G_N^{[t]^l-[v]^l-1}]P_N\sigma P_N e_l\rangle|u-v|^{2H-2}dudv\\ &\le \sum_{l=1}^{\infty}\sum_{i=1}^{[t]^l-1}\sum_{j=1}^{[t]^l-1}\int_{i\Delta t}^{(i+1)\Delta t}\int_{j\Delta t}^{(j+1)\Delta t}\Big\|[e^{([t]^l-i-1)\Delta tA_N} - G_N^{[t]^l-i-1}]P_N\sigma P_N e_l\Big\|_\rho\\ &\quad\times\Big\|[e^{([t]^l-j-1)\Delta tA_N} - G_N^{[t]^l-j-1}]P_N\sigma P_N e_l\Big\|_\rho|u-v|^{2H-2}dudv.\end{aligned}$$

If we define $\Psi_k = G_N^k - e^{A_N(k\Delta t)}$ for $k \ge 1$, then by induction, we have

$$\Psi_k(x) = \sum_{i=0}^{k-1} e^{A_N((k-i-1)\Delta t)}G_N^i\Psi_1(x),$$

where an application of Pazy [17] yields

$$\Psi_1 = G_N - e^{A_N\Delta t} = G_N\int_0^{\Delta t}\int_0^{\Delta t-s}A_N^2e^{A_Ns}e^{A_Nr}drds,$$

and by Lemma 4.2 we conclude that

$$\|(A_N)^{-1}\Psi_1x\|_\rho \le \Delta t^{\min(\gamma-\rho,1)}\|(-A_N)^\theta x\|_{\min(1+\rho+\theta,\gamma)}.$$

Therefore

$$\begin{aligned}\|\Psi_k(x)\|_\rho &= \Big\|\sum_{i=0}^{k-1}e^{A_N((k-i-1)\Delta t)}G_N^i\Psi_1(x)\Big\|_\rho\\ &\le C\Delta t^{\min(1,\gamma-\rho)}\Big\|(-A_N)^\theta x\Big\|_{\min(1+\rho+\theta,\gamma)}.\end{aligned} \tag{4.8}$$

From (4.8), it follows that

$$\begin{aligned}\mathbb{E}\Big[\|I_0([t])\|_\rho^2\Big] &\le \sum_{l=1}^{\infty}\sum_{i=1}^{[t]^l-1}\sum_{j=1}^{[t]^l-1}\|\Psi_iP_N\sigma P_Ne_l\|_\rho\|\Psi_jP_N\sigma P_Ne_l\|_\rho\times\int_{i\Delta t}^{(i+1)\Delta t}\int_{j\Delta t}^{(j+1)\Delta t}|u-v|^{2H-2}dudv\\ &\le C(H)\Delta t^{2\min(1,\gamma-\rho)}\sum_{l=1}^{\infty}\|(-A_N)^{-\theta}P_N\sigma P_Ne_l\|_{\min(\gamma,1+\rho+\theta)}\\ &\le C(H)\Delta t^{2\min(1,\gamma-\rho)}\Big\|(-A_N)^{-\theta+\min(\gamma,1+\rho+\theta)}\sigma\Big\|_{\mathcal{L}_2}.\end{aligned} \tag{4.9}$$

Therefore by (3.3) we derive

$$\mathbb{E}\Big[\|I_0([t])\|_\rho^2\Big] \le C(H)\Delta t^{2\min(1,\gamma-\rho)}. \tag{4.10}$$

For estimating $\mathbb{E}[\|I_1([t])\|_\rho^2]$ by using $e^{tA}P_N = e^{tA_N}P_N$, we have

$$\begin{aligned}\mathbb{E}[\|I_1(\Delta t)\|_\rho^2] &= \mathbb{E}\Big\|\int_0^{[t]}e^{A_N([t]-[s]-\Delta t)}(e^{A_N([s]+\Delta t-s)}-I)P_N\sigma P_NdB^H(s)\Big\|_\rho^2\\ &= C(H)\sum_{i=1}^{\infty}\int_0^{[t]}\int_0^{[t]}\langle(-A)^\rho e^{A_N([t]-[u]-\Delta t)}(e^{A_N([u]+\Delta t-u)}-I)P_N\sigma P_Ne_i,\\ &\quad (-A)^\rho e^{A_N([t]-[v]-\Delta t)}(e^{A_N([v]+\Delta t-v)}-I)P_N\sigma P_Ne_i\rangle|u-v|^{2H-2}dudv\\ &\le C(H)\sum_{l=1}^{\infty}\sum_{i=1}^{[t]^l-1}\sum_{j=1}^{[t]^l-1}\Big\|e^{([t]^l-i-1)\Delta tA_N}\Big\|\times\Big\|e^{([t]^l-j-1)\Delta tA_N}\Big\|\times\int_{i\Delta t}^{(i+1)\Delta t}\int_{j\Delta t}^{(j+1)\Delta t}\\ &\quad \Big\|(e^{(i\Delta t-\Delta t-u)}-I)P_N\sigma P_Ne_i\Big\|_\rho\Big\|(e^{(j\Delta t-\Delta t-u)}-I)P_N\sigma P_Ne_l\Big\|_\rho|u-v|^{2H-2}dudv.\end{aligned}$$

By using [17], and the fact that $(e^{At}-I)x = \int_0^t e^{As}xds$ and by repeating the previous argument for estimating $\mathbb{E}[\|I_0([t])\|_\rho^2]$ we have

$$\mathbb{E}[\|I_1([t])\|_\rho^2] \le C(H)\Delta t^{2\min(1,\gamma-\rho-\theta)}\|(-A)^{-\theta+\min(\gamma,1+\rho)}\sigma\|_{\mathcal{L}_2}. \tag{4.11}$$

For estimating $\mathbb{E}[\|I_2([t])\|_\rho^2]$ we have

$$\begin{aligned}\mathbb{E}[\|I_2([t])\|_\rho^2] &= \mathbb{E}\Big\|\int_0^{[t]}(-A)^\rho e^{A([t]-s)}(\sigma - P_N\sigma P_N)dB^H(s)\Big\|^2\\ &= C(H)\sum_{i=1}^{\infty}\int_0^{[t]}\int_0^{[t]}\langle(-A)^\rho e^{A([t]-u)}\sigma(I-P_N)e_i, (-A)^\rho e^{A([t]-v)}\sigma(I-P_N)e_i\rangle|u-v|^{2H-2}dudv\\ &= C(H)\sum_{i=1}^{\infty}\|(-A)^{\rho-\theta}\sigma(I-P_N)e_i\|^2\times\int_0^{[t]}\int_0^{[t]}\|(-A)^\theta e^{A([t]-u)}\|.\|(-A)^\theta e^{A([t]-v)}\|\times|u-v|^{2H-2}dudv.\end{aligned}$$

From [17] and by a similar argument, as the proof of I_0 and I_1, the following inequality is satisfied for I_2:

$$\mathbb{E}[\|I_2([t])\|_\rho^2] \le C(H)\phi_{\gamma-\rho}^2(N)\|(-A)^{-(\theta-\gamma)}\sigma\|_{\mathcal{L}_2}, \tag{4.12}$$

which from (3.3) it yields

$$\mathbb{E}[\|I_2([t])\|_\rho^2] \le C(H)\phi_{\gamma-\rho}^2(N). \tag{4.13}$$

From [7] we have the following two inequalities

$$E[\|I_3([t])\|_\rho^2] \le C_1\left[\phi_{\gamma-\epsilon}^2(N) + \phi_\gamma^2(N)\|u_0\|_\gamma^2\right] + C_2\Delta t^{2\min(1,\gamma-\rho)}\|u_0\|_\gamma^2, \tag{4.14}$$

and

$$E[\|I_4([t])\|_\rho^2] \le C\Big\{\Delta t^{\min(1,2(\gamma-\rho-\theta_\sigma))} + \Delta t^{2\min(1,\gamma-\rho)} + \int_0^{[t]}\mathbb{E}[\|r([s])\|_\rho^2]ds + \phi_{\gamma-\epsilon}^2(N) + \phi_\gamma^2(N)\Big\}. \tag{4.15}$$

Eqs. (4.10) –(4.15) yield

$$\mathbb{E}[\|r([t])\|_\rho^2] \le C_1\Delta t^{\min(1,2(\gamma-\rho-\theta_\sigma))} + C_2\phi_{\gamma-\rho}^2(N) + C_3\phi_{\gamma-\epsilon}^2(N) + C_4\phi_\gamma^2(N) + C\int_0^{[t]}E[\|r([s])\|_\rho^2]ds.$$

Now, by using a discrete version of the Gronwall's lemma we conclude

$$\begin{aligned}\mathbb{E}[\|v_N^k - u(k\Delta t)\|_\rho^2] \le\, & C_1\Delta t^{\min(1,2(\gamma-\rho-\theta_\sigma))} + C_2\phi_{\gamma-\rho}^2(N)\\ &+ C_3\phi_{\gamma-\epsilon}^2(N) + C_4\phi_\gamma^2(N).\end{aligned}$$

□

5. Simulation of fractional Brownian motion

This section is devoted to the simulation of fractional Brownian motions. Since few numerical results are known about queueing systems with a fractional Brownian motion-type process, it is important to know how this process can be simulated. In this section we describe a method for simulating fractional Brownian motions, which is called Cholesky methods.

5.1. The Cholesky method

If $H = \frac{1}{2}$, then the corresponding fractional Brownian motion is the usual standard Brownian motion, therefore in this case, the process has independent increments. But if $H \neq \frac{1}{2}$, it does not have independent increments. More precisely, according to the Definition 2.1 in Section 2 we know that the covariance between $B^H(t+h) - B^H(t)$ and $B^H(s+h) - B^H(s)$ with $s+h \le t$ and $t-s = nh$ is as the following

$$Cov(X_k, X_{k+n}) = \frac{1}{2}h^{2H}\left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right]. \tag{5.1}$$

Therefore the main part of simulating fractional Brownian motions, is generating the increments of this process, for this aim suppose $0 \le t_1 \le t_2 \ldots \le t_{n-1} \le t_n = 1$, and consider the vector Z as the following

$$Z = (B^H(t_1), B^H(t_2) - B^H(t_1), B^H(t_3) - B^H(t_2), \ldots, B^H(t_n) - B^H(t_{n-1})).$$

The vector Z has mean zero normal distribution with the covariance matrix Σ, i,e. $Z \sim N(0, \Sigma)$ such that

$$\begin{aligned}\Sigma_{i,j} &= [E(B^H(t_{i+1}) - B^H(t_i), B^H(t_{j+1}) - B^H(t_j))]\\ &= \frac{1}{2}\left[(t_{j+1} - t_i)^{2H} - 2(t_j - t_i)^{2H} + (t_j - t_{i+1})^{2H}\right] \qquad i, j = 0, \ldots, n\end{aligned} \tag{5.2}$$

Now, For simulating Z, assume C is an $n \times n$ matrix and let $V = (V_1, \ldots, V_N)^T$ with $V_i \sim N(0, 1)$ for $i = 1, \ldots, N$. If we can find C such that $C^TC = \Sigma$, then obviously $C^TV \sim N(0, C^TC)$ and by putting $Z = C^TV$, Z is simulated.

Because Σ is a symmetric positive definite matrix, therefore it has Cholesky decomposition and by this decomposition Z can be obtained clearly.

6. Numerical results

In this section we consider the numerical solution of some Stochastic fractional differential equations by the numerical scheme (4.1).

Example 1. Let $\alpha > \frac{1}{4}$ and consider the following stochastic evolution equation:

$$\begin{aligned} &du(t) = (\Delta u(t) + f(t, u(t)))dt + (I - \Delta)^{-\alpha} dB^H(t), \\ &u(0) = 0, \end{aligned} \tag{6.1}$$

where

$$f(t, x) = \frac{5}{2} t^2 \Big[3[1 + (2\pi)^2] \sin(2\pi x) - [1 + (6\pi)^2] \sin(6\pi x)\Big] + 2t \sin^3(2\pi x) - 10x.$$

Clearly, $(I - \Delta)^{-\alpha} \in L_2(U, U)$ and the operator $(-\Delta)$ is a nonnegative self-adjoint operator on $L^2([0,\ 1])$. Obviously Assumption $A2$ is satisfied for $\gamma = 1, 0 \le \rho < \frac{1}{2}, 0 \le \theta \le \frac{1}{2}$ and $\rho + \theta_\sigma \le \theta$.

The eigenvalues and eigenfunctions of the Laplacian operator given by $(\lambda_n)_{n\in\mathbb{N}} \subset (0, \infty)$ and $(e_n)_{n\in\mathbb{N}} \subset U$ are

$$\lambda_n = \pi^2 n^2, \qquad e_n(x) = \sqrt{2} \sin(n\pi x), n = 1, \cdots.$$

Therefore the Galerkin operator P_N: $U \to U_n$ is given by

$$(P_N(u))(x) = \sum_{n=1}^{N} 2 \sin(n\pi x) \int_0^1 \sin(n\pi s) u(s) ds, \tag{6.2}$$

for all $x \in (0, 1)$, $u \in U$ and $N \in \mathbb{N}$.

By applying Galerkin approximation for the spatial discretization of (6.1) we have the following system of stochastic fractional differential equations

$$\begin{aligned} &du_N(t) = (P_N \Delta u_N(t) + P_N f(t, u(t)))dt + (I - \Delta_N)^{-\alpha} P_N dB^H(t),\ \ t \in [0, T], \\ &u_N(0) = P_N u_0. \end{aligned} \tag{6.3}$$

Now for time discretization of (6.3), we apply the implicit Euler method (4.1), therefore

$$\begin{aligned} &v_N^{k+1} = (1 - \Delta t A_N)^{-1} \Big\{ v_N^k + \Delta t P_N f(k\Delta t, v_N^k) + (I - \Delta_N)^{-\alpha} \delta_k B_N^H \Big\}, \quad k = 0, 1, \cdots, N-1, \\ &v_N^0 = P_N u_0. \end{aligned} \tag{6.4}$$

In order to describe the implementation of numerical scheme (6.4), we use the $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable mappings $v_N^{k,j} =< v_N^k, e_j >_U$. The orthogonality of the basis functions implies that each of the Fourier coefficients evolves independently to the others, so for the $j-$th Fourier mode the space-time discretization of (6.1) simply becomes

$$v_N^{k+1,j} = \frac{1}{1 + \Delta t \lambda_j} [v_N^{k,j} + \Delta t \langle f(v_N^k, t^k), e_j \rangle + (1 + \lambda_j)^{-\alpha} \delta_k \beta_j^H], \quad k = 0, 1, \cdots, N-1.$$

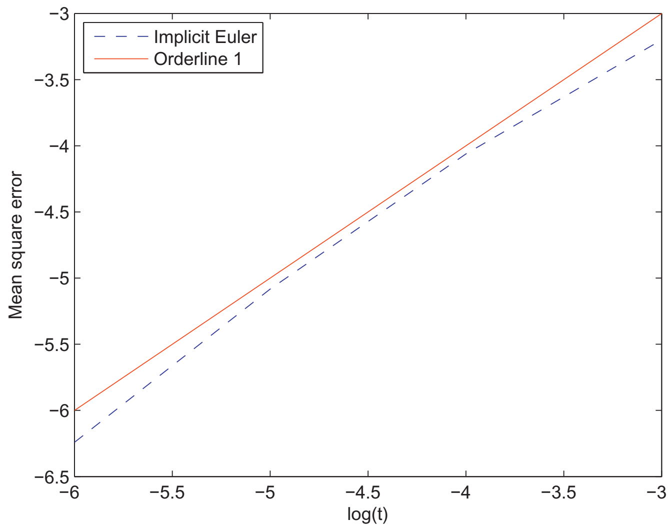

Fig. 1. Mean square approximation error of linear implicit Euler method of Example 1.

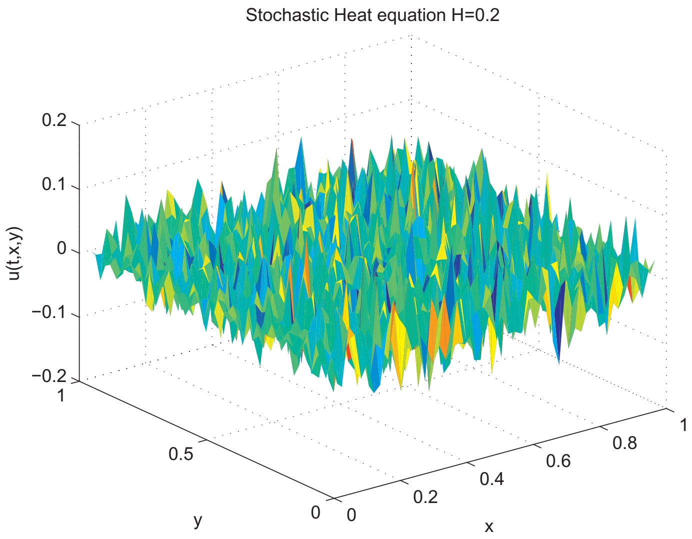

Fig. 2. $u(t, x, y)$, $(x, y) \in [0, 1]^2$, for $H \in \{0.2, 0.8\}$ and $\Delta t = 0.01$ for the Example 2.

where $\delta_k \beta_j^H = \beta_j^H((k+1)\Delta t) - \beta_j^H(k\Delta t)$. By choosing $\rho = 0$, by the fact that for this example $\gamma = 1, 0 \leq \theta \leq \frac{1}{2}$ and $\theta_\sigma \leq \theta$, from the Theorem 4.4 we get

$$\mathbb{E}[\|v_N^k - u(k\Delta t)\|^2] \leq C\Delta t. \tag{6.5}$$

In Fig. 1 the mean square approximation error $\mathbb{E}[\|v_N^k - u(k\Delta t)\|^2]$ of Example 1 has been plotted against the step size $\Delta t \in \{2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}\}$. As a replacement of the unknown solution, we have used the numerical approximation for Δt sufficiently small, here we have used $\Delta t = 2^{-10}$. Fig. 1 confirms that, as expected from Theorem 4.4, the error decrease with slope 1. It should be mentioned that for solving this equation the explicit Euler method which was introduced in [12], can not be used because for that case for the stability condition we need to choose Δt very small. For example, by using $\Delta t = 2^{-10}$ it was seen that the solution is divergent, while by using the implicit Euler method (4.1), even we can use $\Delta t = 2^{-3}$.

Example 2. Let $U = L^2([0, 1] \times [0, 1])$ and A be the Laplacian operator in two dimensions with Dirichlet boundary conditions. We consider the following stochastic equation

$$\begin{aligned} du(t) &= Au(t)dt + dB^H(t), \\ u(0) &= 0. \end{aligned} \tag{6.6}$$

In this case, we have

$$\lambda_{n,m} = \pi^2(n^2 + m^2), \quad e_{n,m}(x) = 2\sin(n\pi x)\sin(m\pi y),$$

where $n, m \in \mathbb{N}$ and $(x, y) \in [0, 1]^2$. The mild solution of this equation is as follows

$$u(t) = \int_0^t e^{A(t-s)} dB^H(s).$$

In Fig. 2 we have plotted $u(t)$ for $x, y \in [0, 1]$ and $H \in \{0.2, 0.8\}$ by the numerical method (4.1).

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