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The Hurwitz form of a projective variety

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ABSTRACT

The Hurwitz form of a variety is the discriminant that characterizes linear spaces of complementary dimension which intersect the variety in fewer than degree many points. We study computational aspects of the Hurwitz form, relate this to the dual variety and Chow form, and show why reduced degenerations are special on the Hurwitz polytope.

1. Introduction

Many problems in applied algebraic geometry can be expressed as follows. We are given a fixed irreducible variety X in complex projective space $\mathbb{P}^n$ of dimension $d \geq 1$ and degree p. Suppose X is defined over the field $\mathbb{Q}$ of rational numbers. We consider various linear subspaces $L \subset \mathbb{P}^n$ of complementary dimension $n-d$, usually defined over the real numbers in floating point representation. The goal is to compute the intersection $L \cap X$ as accurately as possible. If the subspace L is generic then $L \cap X$ consists of p distinct points with complex coordinates in $\mathbb{P}^n$. How many of the p points are real depends on the specific choice of L.

In this paper we study the discriminant associated with this family of polynomial systems. The precise definition is as follows. Let $\mathrm{Gr}(d, \mathbb{P}^n)$ denote the Grassmannian of codimension d subspaces in $\mathbb{P}^n$, and let $\mathcal{H}_X \subset \mathrm{Gr}(d, \mathbb{P}^n)$ be the subvariety consisting of all subspaces L such that $L \cap X$ does not consist of p reduced points. The *sectional genus* of X, denoted $g = g(X)$, is the arithmetic genus of the curve $X \cap L'$ where $L' \subset \mathbb{P}^n$ is a general subspace of codimension $d-1$. If X is regular in

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codimension 1 then the curve $X \cap L'$ is smooth (by Bertini's Theorem) and g is its geometric genus. The following result describes our object.

Theorem 1.1. *Let X be an irreducible subvariety of $\mathbb{P}^n$ having degree $p \geq 2$ and sectional genus g. Then $\mathcal{H}_X$ is an irreducible hypersurface in the Grassmannian $\mathrm{Gr}(d, \mathbb{P}^n)$, defined by an irreducible element Hu_X in the coordinate ring of $\mathrm{Gr}(d, \mathbb{P}^n)$. If the singular locus of X has codimension at least 2 then the degree of Hu_X in Plücker coordinates equals $2p + 2g - 2$.*

The polynomial Hu_X defined here is the *Hurwitz form* of X. The name was chosen as a reference to the Riemann–Hurwitz formula, which says that a curve of degree p and genus g has $2p + 2g - 2$ ramification points when mapped onto $\mathbb{P}^1$. We say that $\mathrm{Hdeg}(X) := \deg(\mathrm{Hu}_X) = 2p + 2g - 2$ is the *Hurwitz degree* of X. When X is defined over $\mathbb{Q}$ then so is $\mathcal{H}_X$. Since X is irreducible, its Hurwitz form Hu_X is irreducible. With the geometric definition we have given, Hu_X is unique up to sign, when written in Stiefel coordinates on $\mathrm{Gr}(d, \mathbb{P}^n)$, if we require it to have relatively prime integer coefficients. When written in Plücker coordinates, $\pm\mathrm{Hu}_X$ is unique only modulo the ideal of quadratic Plücker relations.

The Hurwitz form Hu_X belongs to the family of *higher associated hypersurfaces* described by Gel'fand, Kapranov and Zelevinsky in Gel'fand et al. (1994, Section 3.2.E). These hypersurfaces interpolate between the Chow form Ch_X and the X-discriminant. The latter is the equation of the dual variety X^*. In that setting, the Hurwitz form Hu_X is only one step away from the Chow form Ch_X. An important result in Gel'fand et al. (1994, Section 4.3.B) states that these higher associated hypersurfaces are precisely the coisotropic hypersurfaces in $\mathrm{Gr}(d, \mathbb{P}^n)$, so their defining polynomials are governed by the Cayley–Green–Morrison constraints for integrable distributions.

This article is organized as follows. In Section 2 we discuss examples, basic facts, and we derive Theorem 1.1. Section 3 concerns the Hurwitz polytope whose vertices correspond to the initial Plücker monomials of the Hurwitz form. We compare this to the Chow polytope of Kapranov et al. (1992). In Section 4 we define the Hurwitz form of a reduced cycle, and we show that this is compatible with flat families. As an application we resolve problems (4) and (5) in Sanyal et al. (2013, §7).

2. Basics

We begin with examples that illustrate Hurwitz forms in computational algebraic geometry.

Example 2.1 *(Curves).* If X is a curve in $\mathbb{P}^n$, so $d = 1$, then $\mathcal{H}_X = X^*$ is the hypersurface dual to X, and Hu_X is the X-discriminant. For instance, if X is the rational normal curve in $\mathbb{P}^n$ then Hu_X is the discriminant of a polynomial of degree n in one variable. For a curve X in the plane ($n = 2$), the Hurwitz form is the polynomial that defines the dual curve X^*, so the Hurwitz degree $\mathrm{Hdeg}(X)$ is the degree of X^*, which is $p(p-1)$ if X is nonsingular. $\diamond$

Example 2.2 *(Hypersurfaces).* Suppose that X is a hypersurface in $\mathbb{P}^n$, so $d = n - 1$, with defining polynomial $f(x_0, x_1, \ldots, x_n)$. The Grassmannian $\mathrm{Gr}(n-1, \mathbb{P}^n)$ is a manifold of dimension $2n - 2$ in the projective space $\mathbb{P}^{\binom{n+1}{2}-1}$ with dual Plücker coordinates $q_{01}, q_{02}, \ldots, q_{n-1,n}$. We can compute Hu_X by first computing the discriminant of the univariate polynomial function $t \mapsto f(u_0 + tv_0, u_1 + tv_1, \ldots, u_n + tv_n)$, then removing extraneous factors, and finally expressing the result in terms of 2×2-minors via $q_{ij} = u_i v_j - u_j v_i$.

We can make this explicit when $p = 2$. Let M be a symmetric $(n+1) \times (n+1)$-matrix of rank ≥ 2 and X the corresponding quadric hypersurface in $\mathbb{P}^n$. We write $\wedge_2 M$ for the second exterior power of M, and $Q = (q_{01}, q_{02}, \ldots, q_{n-1,n})$ for the row vector of dual Plücker coordinates. With this notation, the Hurwitz form is the following quadratic form in the q_{ij}:

$$\mathrm{Hu}_X \;=\; Q \cdot (\wedge_2 M) \cdot Q^t. \tag{1}$$

For a concrete example let $n = 3$ and consider the quadric surface $X = V(x_0x_3 - x_1x_2)$. Then $\mathrm{Hdeg}(X) = 2$ and the Hurwitz form equals $\mathrm{Hu}_X = q_{03}^2 + q_{12}^2 + 2q_{03}q_{12} - 4q_{02}q_{13}$. When expressed in terms of Stiefel coordinates u_i, v_j, this is precisely the *hyperdeterminant* of format $2 \times 2 \times 2$. This is explained by Proposition 2.4, with $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $Y = X \times \mathbb{P}^1 \subset \mathbb{P}^7$. ◇

Example 2.3 *(Toric Varieties).* Consider a toric variety $X_A \subset \mathbb{P}^n$, defined by a rank d matrix $A \in \mathbb{Z}^{d \times (n+1)}$ with $(1, 1, \dots, 1)$ in its row space. Then Hu_{X_A} is the *mixed discriminant* (Cattani et al., 2013) of d Laurent polynomials in d variables with the same support A. If A is a unit square then Hu_{X_A} is the hyperdeterminant seen above and in Cattani et al. (2013, Example 2.3). If X_A is the kth Veronese embedding of $\mathbb{P}^2$, a surface of degree $p = k^2$ in $\mathbb{P}^{\binom{k+2}{2}-1}$, then Hu_{X_A} is the classical *tact invariant* which vanishes whenever two plane curves of degree k are tangent. Its degree is $\mathrm{Hdeg}(X_A) = 3k^2 - 3k$. An explicit formula for $k = 2$ will be displayed in Example 2.7. ◇

Our goal is to develop tools for writing the Hurwitz form Hu_X explicitly as a polynomial. There are four different coordinate systems for doing so, depending on how the subspace $L \in \mathrm{Gr}(d, \mathbb{P}^n)$ is expressed. If L is the kernel of a $d \times (n+1)$-matrix then the entries of that matrix are the *primal Stiefel coordinates* and its maximal minors are the *primal Plücker coordinates*, denoted $p_{i_1 i_2 \cdots i_d}$. If L is the row space of an $(n+1-d) \times (n+1)$-matrix then the entries of that matrix are the *dual Stiefel coordinates* and its maximal minors are the *dual Plücker coordinates*, denoted $q_{j_0 j_1 \cdots j_{n-d}}$, as in Example 2.2. In practice, one uses primal coordinates when $d = \dim(X)$ is small, and one uses dual coordinates when $n - d = \mathrm{codim}(X)$ is small. The same conventions are customary for writing *Chow forms* (Dalbec and Sturmfels, 1995; Kapranov et al., 1992; Sturmfels, 1992).

The hypersurface defined by the Chow form of X is called the *associated variety* in Gel'fand et al. (1994), Weyman and Zelevinsky (1994). In these references, the Chow form is constructed as the dual of a Segre product. This is the *Cayley trick* of elimination theory. We now do the same for the Hurwitz form of X.

Proposition 2.4. *Let X be an irrreducible variety of dimension d and degree $p \geq 2$ in $\mathbb{P}^n$, and consider $Y = X \times \mathbb{P}^{d-1}$ in its Segre embedding in $\mathbb{P}^{d(n+1)-1}$. The dual variety Y^* is a hypersurface in the dual $\mathbb{P}^{d(n+1)-1}$. The Hurwitz form Hu_X of the given variety X, when written in the $d(n+1)$ primal Stiefel coordinates, is equal to the defining polynomial of Y^*.*

Proof. This is analogous to Theorem 2.7 in Section 3.2.D of Gel'fand et al. (1994). While that particular statement concerns the case of the Chow form, the discussion in Gel'fand et al. (1994, Section 3.2.E) ensures that the result is the same for higher associated hypersurfaces, such as the Hurwitz form. □

We next prove the statements about dimension and degree of $\mathcal{H}_X$ given in the introduction. The idea is to reduce to the special case of curves, as discussed in Example 2.1.

Proof of Theorem 1.1. Applying Corollary 5.9 in Gel'fand et al. (1994, Section 1.5.D) to the representation in Proposition 2.4, we find that $\mathcal{H}_X$ is a hypersurface if and only if $\mathrm{codim}(X^*) \leq d$, or $\dim(X^*) \geq n - d$. In light of Corollary 1.2 in Gel'fand et al. (1994, Section 1.1.A), this happens if and only if X is not a linear space. So, since we assumed $p \geq 2$, this means that $\mathcal{H}_X$ is a hypersurface.

Write $L = L' \cap H$ where H is a varying hyperplane and L' is a fixed generic linear subspace of codimension $d - 1$ in $\mathbb{P}^n$. The codimension d subspace L is a point in $\mathcal{H}_X$ if and only if the zero-dimensional scheme $X \cap L = X \cap (L' \cap H) = (X \cap L') \cap H$ is not reduced. This happens if and only if H is tangent to the curve $X \cap L'$ if and only if H is a point in the projective variety dual to $X \cap L'$. This curve is smooth and irreducible, by Bertini's Theorem, and it has degree p and genus g. A classical result states this dual to $X \cap L'$ is a hypersurface of degree $2p + 2g - 2$; see, for instance, the paragraph after Theorem 2.14 in Gel'fand et al. (1994, Section 2.2.B). Hence $\mathcal{H}_X$ is a hypersurface of that same degree in $\mathrm{Gr}(d, \mathbb{P}^n)$. □

One motivation for studying the Hurwitz form Hu_X comes from the analysis of numerical algorithms for computing $L \cap X$. An appropriate tubular neighborhood around $\mathcal{H}_X$ in $\mathrm{Gr}(d, \mathbb{P}^n)$ is the locus where the homotopy methods of numerical algebraic geometry run into trouble. This is quantified by the *condition number* of the algebraic function $L \mapsto L \cap X$. The quantity $\mathrm{Hdeg}(X) = 2p + 2g - 2$ is crucial for bounding that condition number (cf. Bürgisser and Cucker, 2013).

Example 2.5. This article was inspired by a specific application to multiview geometry in computer vision, studied in Agarwal et al. (in press, §3). The *variety of essential matrices* is a subvariety X in the projective space $\mathbb{P}^8$ of 3×3-matrices. Its real points are the rank 2 matrices whose two nonzero singular values coincide. We have $d = 5$, $p = 10$, $g = 6$. This implies that $\mathrm{Hdeg}(X) = 30$, by Theorem 1.1, so $\mathcal{H}_X$ is a hypersurface in the Grassmannian $\mathrm{Gr}(5, \mathbb{P}^8)$ whose defining polynomial Hu_X has degree 30 in the $\binom{9}{3} = 84$ Plücker coordinates. $\diamond$

Remark 2.6. Let $h(m)$ be the Hilbert polynomial of $X \subset \mathbb{P}^n$. If X is Cohen–Macaulay then we can read off the Hurwitz degree from the $(d-1)$st derivative of the Hilbert polynomial:

$$\mathrm{Hdeg}(X) \;=\; 2p + 2g - 2 \;=\; 2 \cdot |h^{(d-1)}(-1)|.$$

Indeed, if $d = 1$ then this comes from the familiar formula $h(m) = pm + (1 - g)$ for the Hilbert polynomial of a projective curve. For $d \geq 2$, the Cohen–Macaulay hypothesis ensures that the numerator of the Hilbert series remains the same under generic hyperplane sections, and the Hilbert polynomial is transformed under such sections by taking the derivative.

In `Macaulay2` (Grayson and Stillman, 1996), we can compute the integer $|h^{(d-1)}(-1)| = \mathrm{Hdeg}(X)/2$ from the ideal `I` of X in two possible ways: take the coefficient of $\mathtt{P}_{d-1}$ in `hilbertPolynomial(I)`, or add the last two entries in `genera(I)`. For instance, for the variety in Example 2.5, the former command gives $\mathtt{6*P_3 - 15*P_4 + 10*P_5}$ and the latter command gives `{0,0,0,0,6,9}`.

If the parameters d, n, p, g of the given variety X are small enough, then we can use computer algebra to determine the Hurwitz form Hu_X, and to write it as an explicit polynomial in the Plücker coordinates on $\mathrm{Gr}(d, \mathbb{P}^n)$. The following example will serve as an illustration:

Example 2.7. Let X be the Veronese surface in $\mathbb{P}^5$, defined by the parametrization $(x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2)$. Here $n = 5$, $d = 2$, $p = 4$, $g = 0$. Following Example 2.3, the Hurwitz form of X is the classical tact invariant for $k = 2$. We have the explicit formula

$$\begin{gathered}
\mathrm{Hu}_X \;=\; 4p_{01}p_{14}^2p_{15}^3 - 4p_{13}p_{12}^2p_{15}^3 - 4p_{34}p_{24}^2p_{04}^3 + 4p_{45}p_{14}^2p_{04}^3 - 4p_{25}p_{12}^2p_{23}^3 + 4p_{02}p_{24}^2p_{23}^3 + p_{12}^2p_{14}^2p_{24}^2 \\
+ p_{04}^2p_{14}^2p_{24}^2 + p_{12}^2p_{14}^2p_{15}^2 + p_{12}^2p_{23}^2p_{24}^2 - 16p_{01}p_{13}p_{15}^4 - 16p_{02}p_{25}p_{23}^4 - 16p_{34}p_{45}p_{04}^4 + 256p_{03}^2p_{05}^2p_{35}^2 \\
-12p_{04}^2p_{24}^3p_{13} - 8p_{23}p_{04}^2p_{14}p_{24}^2 + 18p_{04}p_{24}^3p_{14}p_{03} - 8p_{23}^2p_{04}^2p_{24}^2 + 36p_{24}^3p_{03}p_{23}p_{04} - 27p_{24}^4p_{03}^2 - 4p_{02}p_{14}^3p_{24}^2 - 10p_{23}p_{14}p_{12}p_{24}^2p_{04} \\
+2p_{14}^2p_{12}p_{24}^2p_{04} + 12p_{02}p_{23}p_{14}^2p_{24}^2 + 18p_{24}^3p_{14}p_{03}p_{12} - 12p_{04} + p_{24}^3p_{12}p_{13} - 18p_{23}p_{24}^3p_{03}p_{12} + 20p_{23}^2p_{12}p_{24}^2p_{04} - 12p_{23}^2p_{02}p_{24}^2p_{14} \\
+2p_{23}p_{12}^2p_{24}^2p_{14} - 4p_{12}^2p_{24}^3p_{13} - 8p_{04}^3p_{14}p_{24}p_{35} + 48p_{04}^3p_{24}p_{34}p_{15} + 48p_{04}^3p_{14}p_{34}p_{25} - 80p_{04}^2p_{14}p_{24}p_{34}p_{05} + 8p_{04}^3p_{24}p_{23}p_{35} \\
+48p_{04}^3p_{23}p_{34}p_{25} - 120p_{04}^2p_{24}p_{03}p_{34}p_{25} + 144p_{05}p_{24}^2p_{03}p_{04}p_{34} - 40p_{04}^2p_{24}p_{23}p_{05}p_{34} - 12p_{14}^3p_{04}^2p_{25} + 18p_{14}^3p_{04}p_{24}p_{05} \\
-8p_{04}^2p_{14}^2p_{24}p_{15} + 28p_{04}^2p_{24}p_{14}p_{23}p_{15} + 4p_{04}^2p_{24}p_{14}p_{25}p_{13} - 52p_{04}p_{14}^2p_{24}p_{23}p_{05} - 52p_{04}p_{14}p_{03}p_{24}^2p_{15} + 120p_{05}p_{23}p_{14}p_{24}^2p_{03} \\
+36p_{04}^2p_{24}^2p_{15}p_{13} + 36p_{14}^2p_{23}p_{04}^2p_{25} + 72p_{24}^3p_{15}p_{03}^2 + 216p_{24}p_{15}p_{23}^2p_{04}^2 - 144p_{24}^3p_{13}p_{05}p_{03} + 368p_{24}^2p_{13}p_{05}p_{23}p_{04} - 6p_{05}p_{14}^2p_{24}^2p_{03} \\
-264p_{04}^2p_{24}p_{25}p_{13}p_{23} + 120p_{04}p_{24}^2p_{25}p_{13}p_{03} - 264p_{04}p_{23}^2p_{14}p_{24}p_{05} - 304p_{24}^2p_{15}p_{03}p_{23}p_{04} + 96p_{14}p_{23}^2p_{04}^2p_{25} + 144p_{03}^2p_{23}p_{24}^2p_{25} \\
-16p_{23}^3p_{04}p_{24}p_{05} - 24p_{05}p_{23}^2p_{24}^2p_{03} - 160p_{03}p_{23}^2p_{24}p_{25}p_{04} + 48p_{23}^3p_{04}^2p_{25} - 2p_{04}p_{14}^2p_{12}p_{24}p_{15} + 12p_{15}p_{02}p_{14}^3p_{24} + 12p_{14}^3p_{12}p_{04}p_{25} \\
-18p_{14}^3p_{12}p_{24}p_{05} + 50p_{15}p_{14}p_{24}p_{12}p_{23}p_{04} - 46p_{14}p_{24}p_{12}p_{04}p_{25}p_{13} - 46p_{14}^2p_{23}p_{12}p_{05}p_{24} + 8p_{14}^2p_{23}p_{12}p_{04}p_{25} - 8p_{23}p_{15}p_{02}p_{14}^2p_{24} \\
-8p_{04}p_{12}p_{24}^2p_{15}p_{13} - 46p_{15}p_{14}p_{24}^2p_{03}p_{12} + 92p_{14}p_{24}^2p_{12}p_{05}p_{13} + 52p_{15}p_{24}p_{23}^2p_{12}p_{04} - 58p_{15}p_{24}^2p_{23}p_{03}p_{12} + 6p_{03}p_{24}^2p_{12}p_{25}p_{13} \\
-36p_{23}^2p_{12}p_{14}p_{04}p_{25} - 4p_{23}^2p_{12}p_{14}p_{05}p_{24} - 36p_{23}^2p_{02}p_{15}p_{14}p_{24} + 52p_{13}p_{23}p_{24}p_{12}p_{25}p_{04} - 40p_{23}^3p_{12}p_{05}p_{24} + 48p_{23}^3p_{02}p_{15}p_{24} \\
-48p_{23}^3p_{12}p_{04}p_{25} + 80p_{23}^2p_{24}p_{12}p_{25}p_{03} - 4p_{14}^3p_{12}^2p_{25} + 2p_{14}^2p_{12}^2p_{15}p_{24} + 18p_{12}^2p_{14}p_{24}p_{25}p_{13} - 12p_{12}^2p_{15}p_{24}^2p_{13} - 12p_{23}p_{12}^2p_{14}^2p_{25} \\
+4p_{23}p_{12}^2p_{14}p_{15}p_{24} + 18p_{23}p_{12}^2p_{24}p_{25}p_{13} - 12p_{23}^2p_{12}^2p_{14}p_{25} + 2p_{23}^2p_{12}^2p_{15}p_{24} + 128p_{04}^3p_{34}p_{05}p_{35} - 128p_{04}^2p_{34}^2p_{05}^2 + 16p_{04}^4p_{35}^2 \\
+48p_{04}^3p_{15}^2p_{34} - 16p_{04}^3p_{14}p_{15}p_{35} + 144p_{14}^2p_{04}p_{05}^2p_{34} - 24p_{14}^2p_{04}^2p_{05}p_{35} - 160p_{04}^2p_{14}p_{15}p_{34}p_{05} - 32p_{04}^3p_{25}p_{13}p_{35} - 32p_{04}^3p_{15}p_{23}p_{35}
\end{gathered}$$

$$\begin{gathered}
+416p_{04}p_{24}p_{15}p_{03}p_{34}p_{05}+128p_{04}^2p_{05}p_{14}p_{23}p_{35}-352p_{04}^2p_{25}p_{03}p_{34}p_{15}+160p_{04}^2p_{25}p_{13}p_{34}p_{05}-384p_{04}^2p_{15}p_{23}p_{34}p_{05}-96p_{05}^2p_{13}p_{24}p_{04}p_{34} \\
+320p_{23}p_{14}p_{05}^2p_{04}p_{34}-192p_{05}^2p_{03}p_{24}p_{14}p_{34}+384p_{04}p_{03}^2p_{25}^2p_{34}-288p_{03}^2p_{25}p_{24}p_{34}p_{05}+64p_{04}^2p_{25}p_{03}p_{23}p_{35}-96p_{05}^2p_{03}p_{23}p_{24}p_{34} \\
-128p_{23}^2p_{04}^2p_{05}p_{35}-224p_{04}p_{25}p_{03}p_{23}p_{34}p_{05}+160p_{05}^2p_{23}^2p_{04}p_{34}-27p_{14}^4p_{05}^2+36p_{14}^3p_{04}p_{05}p_{15}-8p_{04}^2p_{14}^2p_{15}^2-72p_{14}^3p_{05}^2p_{23} \\
-24p_{14}^2p_{13}p_{04}p_{05}p_{25}+96p_{04}^2p_{15}^2p_{24}p_{13}+144p_{05}^2p_{14}^2p_{24}p_{13}-152p_{15}^2p_{04}^2p_{14}p_{23}+208p_{04}p_{14}^2p_{15}p_{23}p_{05}+104p_{15}^2p_{04}p_{14}p_{24}p_{03} \\
-24p_{14}^2p_{15}p_{05}p_{24}p_{03}+104p_{04}^2p_{14}p_{15}p_{25}p_{13}-368p_{04}p_{14}p_{15}p_{05}p_{24}p_{13}+336p_{15}^2p_{03}p_{24}p_{23}p_{04}+336p_{04}p_{15}p_{23}^2p_{14}p_{05} \\
+336p_{13}p_{05}p_{15}p_{24}^2p_{03}+336p_{04}p_{25}p_{13}^2p_{05}p_{24}-224p_{15}p_{04}p_{25}p_{13}p_{24}p_{03}-336p_{13}p_{05}p_{15}p_{24}p_{23}p_{04}-112p_{23}p_{05}p_{15}p_{14}p_{24}p_{03} \\
-224p_{04}p_{25}p_{13}p_{05}p_{23}p_{14}+336p_{05}^2p_{24}p_{13}p_{23}p_{14}-56p_{04}^2p_{25}^2p_{13}^2-336p_{05}^2p_{24}^2p_{13}^2-56p_{05}^2p_{14}^2p_{23}^2-56p_{15}^2p_{03}^2p_{24}^2 \\
-280p_{23}^2p_{15}^2p_{04}^232p_{14}p_{05}^2p_{23}^3+320p_{23}p_{25}^2p_{03}p_{04}p_{13}+256p_{23}^2p_{15}p_{03}p_{25}p_{04}-128p_{23}p_{15}p_{03}^2p_{25}p_{24}-288p_{23}^2p_{13}p_{04}p_{05}p_{25} \\
-64p_{23}^2p_{03}p_{05}p_{24}p_{15}+96p_{23}p_{25}p_{03}p_{05}p_{24}p_{13}-64p_{23}^2p_{05}^2p_{24}p_{13}-192p_{03}^2p_{24}p_{25}^2p_{13}+128p_{23}^3p_{03}p_{05}p_{25}+16p_{23}^4p_{05}^2-128p_{03}^2p_{23}^2p_{25}^2 \\
-12p_{14}^3p_{15}^2p_{02}+18p_{14}^3p_{12}p_{05}p_{15}-8p_{14}^2p_{15}^2p_{12}p_{04}-52p_{14}p_{15}p_{13}p_{12}p_{25}p_{04}-36p_{14}^2p_{15}^2p_{02}p_{23}+36p_{15}^2p_{04}p_{12}p_{24}p_{13}-16p_{15}^2p_{14}p_{12}p_{23}p_{04} \\
+4p_{15}^2p_{14}p_{24}p_{03}p_{12}-6p_{14}^2p_{13}p_{12}p_{05}p_{25}+58p_{05}p_{12}p_{15}p_{14}^2p_{23}+80p_{12}p_{23}^2p_{05}p_{15}p_{14}+40p_{23}p_{13}p_{15}p_{12}p_{25}p_{04}+120p_{24}p_{15}p_{03}p_{12}p_{25}p_{13} \\
+96p_{15}^2p_{23}^2p_{14}p_{02}+144p_{15}^2p_{23}p_{24}p_{03}p_{12}-144p_{12}p_{13}^2p_{25}p_{05}p_{24}+24p_{23}p_{12}p_{13}p_{14}p_{05}p_{25}-160p_{15}^2p_{23}^2p_{12}p_{04}+72p_{12}p_{13}^2p_{25}^2p_{04} \\
-224p_{23}p_{12}p_{13}p_{15}p_{05}p_{24}+56p_{05}p_{15}p_{23}^3p_{12}+120p_{12}p_{23}^2p_{13}p_{05}p_{25}-16p_{23}^2p_{15}p_{12}p_{25}p_{03}-144p_{23}p_{12}p_{25}^2p_{03}p_{13}-27p_{12}^2p_{13}^2p_{25}^2 \\
-48p_{02}p_{23}^3p_{15}^2+2p_{15}^2p_{12}^2p_{14}p_{23}-12p_{12}^2p_{15}^2p_{24}p_{13}+18p_{12}^2p_{13}p_{14}p_{15}p_{25}+p_{15}^2p_{12}^2p_{23}^2+18p_{23}p_{12}^2p_{13}p_{15}p_{25}-256p_{05}^2p_{04}p_{34}p_{03}p_{35} \\
+256p_{34}^2p_{03}p_{05}^3-128p_{05}p_{04}^2p_{03}p_{35}^2-96p_{14}p_{13}p_{04}p_{05}^2p_{35}-32p_{15}^2p_{04}^2p_{03}p_{35}-288p_{05}^3p_{14}p_{13}p_{34}-96p_{05}^2p_{15}p_{03}p_{14}p_{34} \\
+64p_{05}^2p_{15}p_{04}p_{13}p_{34}+256p_{05}p_{15}p_{04}^2p_{13}p_{35}+256p_{15}^2p_{04}p_{03}p_{34}p_{05}-128p_{05}^3p_{23}p_{13}p_{34}-128p_{05}p_{15}p_{03}p_{23}p_{35}p_{04} \\
-128p_{05}p_{25}p_{03}p_{13}p_{35}p_{04}+256p_{05}^2p_{25}p_{03}p_{13}p_{34}+256p_{05}^2p_{15}p_{03}p_{23}p_{34}-128p_{03}^2p_{25}p_{15}p_{34}p_{05}+256p_{03}^2p_{25}p_{15}p_{35}p_{04} \\
-128p_{13}p_{23}p_{04}p_{05}^2p_{35}-256p_{03}^3p_{25}^2p_{35}+128p_{05}^2p_{03}p_{23}^2p_{35}+256p_{03}^2p_{23}p_{25}p_{05}p_{35}+48p_{15}^3p_{04}^2p_{13}-160p_{14}p_{15}^2p_{04}p_{13}p_{05} \\
-24p_{14}^2p_{15}^2p_{05}p_{03}-16p_{14}p_{15}^3p_{04}p_{03}+144p_{14}^2p_{13}p_{05}^2p_{15}-96p_{05}^2p_{13}^2p_{15}p_{24}-192p_{13}^2p_{14}p_{05}^2p_{25}+64p_{15}^3p_{03}p_{23}p_{04} \\
+160p_{13}p_{15}^2p_{04}p_{23}p_{05}-352p_{13}p_{15}^2p_{04}p_{25}p_{03}-32p_{14}p_{13}p_{15}p_{05}^2p_{23}+160p_{15}^2p_{13}p_{05}p_{24}p_{03}-128p_{15}^2p_{03}p_{05}p_{23}p_{14} \\
+416p_{05}p_{13}^2p_{15}p_{04}p_{25}-32p_{15}^3p_{03}^2p_{24}+384p_{03}^2p_{15}p_{25}^2p_{13}-288p_{13}^2p_{05}p_{03}p_{25}^2-64p_{03}p_{15}^2p_{23}^2p_{05}-64p_{03}^2p_{15}^2p_{23}p_{25}-64p_{05}^2p_{13}p_{15}p_{23}^2 \\
+384p_{13}^2p_{23}p_{05}^2p_{25}-256p_{05}p_{13}p_{15}p_{23}p_{25}p_{03}-80p_{14}p_{13}p_{12}p_{15}^2p_{05}+48p_{13}p_{14}p_{15}^3p_{02}+48p_{15}^3p_{13}p_{12}p_{04}-8p_{14}p_{15}^3p_{12}p_{03} \\
-48p_{13}p_{15}^3p_{02}p_{23}-8p_{15}^3p_{03}p_{12}p_{23}-120p_{15}^2p_{13}p_{12}p_{25}p_{03}+16p_{12}p_{13}p_{15}^2p_{05}p_{23}+144p_{13}^2p_{12}p_{15}p_{05}p_{25} \\
+256p_{05}^3p_{13}^2p_{35}-256p_{13}p_{03}p_{05}^2p_{15}p_{35}-128p_{03}^2p_{15}^2p_{05}p_{35}+128p_{15}^3p_{03}p_{13}p_{05}+16p_{15}^4p_{03}^2-128p_{05}^2p_{13}^2p_{15}^2
\end{gathered}$$

The first 14 monomials are the initial terms with respect to the torus action on $\mathrm{Gr}(2,\mathbb{P}^5)$. In addition to these 14 weight components, there are 45 other weight components from the non-initial terms. Substituting $p_{ij} = a_i b_j - a_j b_i$ in Hu_X gives a sum of 3210 monomials of bidegree $(6,6)$. This is the tact invariant (cf. Example 2.3) of two ternary quadrics. $\diamond$

Clearly, as the parameters d, p, n, g increase, it soon becomes prohibitive to compute such an explicit expansion for Hu_X. We can still hope to compute the initial terms from various initial monomial ideals of X. This will be studied in the next section. Another possibility is to use *numerical algebraic geometry* to compute and represent the Hurwitz form Hu_X. We illustrate the capabilities of `Bertini` (Bates et al., 2013) with a non-trivial example from computer vision.

Example 2.8. The *calibrated trifocal variety* X has dimension $d = 11$ and it lives in $\mathbb{P}^{26}$. This extends the variety of essential matrices in Example 2.5 from two cameras to three cameras. It is obtained from the parametrization of the trifocal variety in Aholt and Oeding (2014) by specializing each of the three camera matrices to have a rotation matrix in its left 3×3-block. Computations in `Bertini` (Bates et al., 2013) carried out by Joe Kileel and Jon Hauenstein revealed that $p = 4912$ and $g = 13569$ respectively. The details of this computation, and the equations known to vanish on X, will be described elsewhere. Hence for this variety, we have $\mathrm{Hdeg}(X) = 36960$. $\diamond$

3. The Hurwitz polytope

An important combinatorial invariant of hypersurface in a projective space $\mathbb{P}^n$ is its Newton polytope. This generalizes to hypersurfaces in a Grassmannian $\mathrm{Gr}(d,\mathbb{P}^n)$ if we take the weight polytope with respect to the action of the $(n+1)$-dimensional torus on $\mathrm{Gr}(d,\mathbb{P}^n)$. This torus action corresponds to the $\mathbb{Z}^{n+1}$-grading on the primal Plücker coordinates by

$$\deg(p_{i_1 i_2 \cdots i_d}) \;=\; \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \cdots + \mathbf{e}_{i_d}.$$

The weight polytope of the Chow form of a variety $X \subset \mathbb{P}^n$ was studied in Kapranov et al. (1992). It is known as the *Chow polytope*. In this section we study the analogous polytope for the Hurwitz form.

We define the *Hurwitz polytope*, denoted HuPo_X, of a projective variety $X \subset \mathbb{P}^n$ to be the weight polytope of the Hurwitz form Hu_X. This definition is to be understood as follows. The coordinate ring of $\mathrm{Gr}(d, \mathbb{P}^n)$ is $\mathbb{Z}^{n+1}$-graded. The Hurwitz form Hu_X is an element of that ring, so it decomposes uniquely into $\mathbb{Z}^{n+1}$-graded components. The Hurwitz polytope HuPo_X is the convex hull in $\mathbb{R}^{n+1}$ of all degrees that appear in this decomposition.

Example 3.1. For the Segre variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, the Hurwitz form is invariant under passing from dual Plücker coordinates (as in Example 2.2) to primal Plücker coordinates. The weights occurring in $\mathrm{Hu}_X = p_{03}^2 + p_{12}^2 + (2p_{03}p_{12} - 4p_{02}p_{13})$ are the following three points:

$$\mathrm{degree}(p_{03}^2) = (2,0,0,2), \mathrm{degree}(p_{12}^2) = (0,2,2,0) \quad \text{and} \quad \mathrm{degree}(p_{02}p_{13}) = (1,1,1,1).$$

Their convex hull in $\mathbb{R}^4$ is a line segment, and this is the Hurwitz polytope HuPo_X. We note that the Chow form of our surface equals $\mathrm{Ch}_X = p_0p_3 - p_1p_2$, in dual Plücker coordinates. The line segment HuPo_X is twice the line segment $\mathrm{ChPo}_X = \mathrm{conv}\{(1,0,0,1),(0,1,1,0)\}$. The latter is the Chow polytope of X. $\diamond$

Integer vectors $\mathbf{w} \in \mathbb{Z}^{n+1}$ represent one-parameter subgroups of the torus action on $\mathbb{P}^n$. Their action on subvarieties X is compatible with the construction of Hurwitz forms:

$$\mathrm{Hu}_{\epsilon^{\mathbf{w}}X}(p) \;=\; \mathrm{Hu}_X(\epsilon^{-\mathbf{w}}p). \tag{2}$$

Here ϵ is a parameter and $\epsilon^{\mathbf{w}} = (\epsilon^{w_0}, \epsilon^{w_1}, \ldots, \epsilon^{w_n})$, and the action is the same as that on Chow forms seen in Dalbec and Sturmfels (1995, §2.3). For $\epsilon \to 0$, the polynomial in (2) is the *initial form* $\mathrm{in}_{\mathbf{w}}(\mathrm{Hu}_X)$. For generic $\mathbf{w}$, this initial form is fixed under the action of the torus $(\mathbb{C}^*)^{n+1}$ on $\mathrm{Gr}(d, \mathbb{P}^n)$, and it can hence be expressed as a monomial in the Plücker coordinates. The following example illustrates these *initial monomials* and how they determine the Hurwitz polytope.

Example 3.2. Let X be the Veronese surface in $\mathbb{P}^5$. Its Hurwitz form Hu_X was displayed explicitly in Example 2.7. The Hurwitz polytope HuPo_X is 3-dimensional and it has 14 vertices, namely the weights of the first 14 Plücker monomials, which were shown in larger font. The weights of these monomials, in the given order, are the following 14 points in $\mathbb{R}^6$:

(160023), (062103), (302160), (320061), (026301), (106320), (044040),
(222060), (062022), (026220), (160104), (106401), (400161), (400404).

The Chow form and Chow polytope of the Veronese surface X were displayed explicitly in Sturmfels (1992, Section 2, page 270). This should be compared to the Hurwitz form and Hurwitz polytope. The initial monomials of Ch_X correspond to the 14 maximal toric degenerations of X, one for each of the 14 triangulations of the triangle $2\Delta = \mathrm{conv}\{(200),(020),(002)\}$. The Chow polytope ChPo_X is *normally equivalent* to the Hurwitz polytope HuPo_X. This means that both polytopes share the same normal fan in $\mathbb{R}^6$. In particular, they have the same combinatorial type. That type is the 3-*dimensional associahedron*. The correspondence between the initial monomials of Ch_X and those of Hu_X is as follows, up to symmetry:

Ch_X	$p_{012}p_{124}p_{134}p_{345}$	$p_{014}p_{024}p_{134}p_{245}$	$p_{015}^2p_{134}p_{145}$	$p_{015}^2p_{135}^2$	p_{035}^4
Hu_X	$p_{12}^2p_{14}^2p_{24}^2$	$p_{04}^2p_{14}^2p_{24}^2$	$p_{01}p_{14}^2p_{15}^3$	$p_{01}p_{13}p_{15}^4$	$p_{03}^2p_{05}^2p_{35}^2$

The first two degenerations are reduced unions of four coordinate planes in $\mathbb{P}^5$, corresponding to the unimodular triangulations of 2Δ. The last degeneration is a plane of multiplicity four, corresponding to the trivial triangulation of 2Δ into a single large triangle $\{0,3,5\}$. $\diamond$

Example 3.2 raises the question whether the vertices of the Hurwitz polytope and the Chow polytope are always in bijection. The answer is “no” as the following example shows.

Example 3.3 *(Plane conics).* Let X be a plane conic in $\mathbb{P}^2$ defined by the quadratic form

$$\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{01} & m_{11} & m_{12} \\ m_{02} & m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, \tag{3}$$

where the m_{ij} are scalars in $\mathbb{Q}$. This equation agrees with the Chow form of X, so generally the Chow polytope is the triangle $\mathrm{Ch}_X = 2\Delta$. According to Example 2.1, the hypersurface $\mathcal{H}_X$ is the dual conic X^*. Hence the Hurwitz form is the quadratic form of the adjoint

$$\mathrm{Hu}_X = \begin{pmatrix} p_0 & p_1 & p_2 \end{pmatrix} \begin{pmatrix} m_{11}m_{22} - m_{12}^2 & m_{12}m_{02} - m_{01}m_{22} & m_{01}m_{12} - m_{11}m_{02} \\ m_{12}m_{02} - m_{01}m_{22} & m_{00}m_{22} - m_{02}^2 & m_{01}m_{02} - m_{00}m_{12} \\ m_{01}m_{12} - m_{11}m_{02} & m_{01}m_{02} - m_{00}m_{12} & m_{00}m_{11} - m_{01}^2 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}. \tag{4}$$

This agrees with formula (1) if we set $p_0 = q_{12}$, $p_1 = -q_{02}$ and $p_2 = q_{01}$. The Hurwitz polytope HuPo_X is a subpolytope of the triangle $2\Delta = \mathrm{conv}\{(200), (020), (002)\}$. There are several combinatorial possibilities, depending on which matrix entries in (4) are zero. For instance, if $m_{11}m_{22} = m_{12}^2$ but the m_{ij} are otherwise generic then HuPo_X is a quadrilateral. Similarly if $m_{11} = 0$ then ChPo_X is quadrilateral and HuPo_X is a triangle. Hence, there is generally no map from the set of vertices of one polytope to the vertices of the other. ◇

A geometric explanation for this example is given by the analysis due to Katz (1986) of the duality of plane curves under degenerations. Suppose that X_ϵ is a plane curve defined by a ternary form $f^q + \epsilon g$ where f and g are general ternary forms of degree r and $p = qr$ respectively. Here ϵ is a parameter. For $\epsilon \neq 0$, the curve X_ϵ is smooth, and the Hurwitz form Hu_{X_ϵ} defines the dual curve, of degree $p(p-1) = q^2r^2 - qr$. Now consider the limit $\epsilon \to 0$. The limit curve X_0 is $V(f)$ with multiplicity q, with no trace of $V(g)$, but the limit of the Hurwitz form remembers the points in the intersection $V(f, g)$.

Proposition 3.4. (Katz, 1986, Proposition 1.2) *The constant term of* Hu_ϵ *with respect to* ϵ *equals*

$$\mathrm{Hu}_{X_\epsilon}|_{\epsilon=0} \quad = \quad (\mathrm{Hu}_{V(f)})^q \cdot \prod_{u \in V(f,g)} (\mathrm{Ch}_u)^{q-1}. \tag{5}$$

Note that the variety $V(f, g)$ consists of $rp = qr^2$ points $u = (u_0 : u_1 : u_2)$. For each of these, the linear form $\mathrm{Ch}_u = u_0p_0 + u_1p_1 + u_2p_2$ appears as a factor. The Hurwitz form $\mathrm{Hu}_{V(f)}$ of the curve $V(f)$ has degree $r(r-1)$. Hence the right hand side in (5) has degree

$$r(r-1)q + (qr^2)(q-1) \quad = \quad p(p-1).$$

Proposition 3.4 is indicative of what happens to the limit of the Hurwitz form Hu_{X_ϵ} when a family of irreducible varieties X_ϵ degenerates to a non-reduced cycle or scheme X_0. The limit $\mathrm{Hu}_{X_\epsilon}|_{\epsilon=0}$ remembers information about the family that cannot be recovered from X_0.

4. Reduced degenerations

In this section we consider families X_ϵ whose limit object X_0 is a reduced cycle, and we show that the indeterminacy seen in Proposition 3.4 no longer happens. Instead, we will have

$$\mathrm{Hu}_{X_\epsilon}|_{\epsilon=0} = \mathrm{Hu}_{X_0}. \tag{6}$$

To make sense of this identity, we now define the Hurwitz form of a *reduced cycle* Y in $\mathbb{P}^n$. Let $Y = \sum_{i=1}^{l} Y_i$ where $Y_1, \ldots, Y_l$ are distinct irreducible d-dimensional subvarieties in $\mathbb{P}^n$. Let $Z_1, \ldots, Z_m$ be the distinct irreducible varieties of dimension $d-1$ that arise as components in the pairwise intersections $Y_i \cap Y_k$. We write ν_j for the multiplicity of the one-dimensional local ring $\mathcal{O}_{Y,Z_j}$ at its maximal ideal. If all pairwise intersections $Y_i \cap Y_k$ are transverse then ν_j simply counts the number of components Y_i of Y that contain Z_j.

We now define the *Hurwitz form* of the reduced cycle Y as follows:

$$\mathrm{Hu}_Y \;=\; \prod_{i=1}^{l} \mathrm{Hu}_{Y_i} \cdot \prod_{j=1}^{m} (\mathrm{Ch}_{Z_j})^{2\nu_j - 2}. \tag{7}$$

Here Hu_{Y_i} is the Hurwitz form of a d-dimensional variety, while Ch_{Z_j} is the Chow form of a $(d-1)$-dimensional variety, so they both define hypersurfaces in the same Grassmannian $\mathrm{Gr}(d, \mathbb{P}^n)$. Our main result in this section states that this is the correct definition for limits.

Theorem 4.1. *Let (X_ϵ) be a flat family of subschemes in $\mathbb{P}^n$, where the general fiber (for $\epsilon \neq 0$) is irreducible and regular in codimension* 1*, and the special fiber (for $\epsilon = 0$) is reduced and each of its irreducible components is regular in codimension* 1*. The Hurwitz form of X_ϵ satisfies the identity (6) with the Hurwitz form Hu_{X_0} of the special fiber defined as in (7).*

Proof. By intersecting with a general linear subspace L' of codimension $d-1$ in $\mathbb{P}^n$, as in the proof of Theorem 1.1, we reduce the statement to the $d = 1$ case, when (X_ϵ) is a family of curves. In that case, the Hurwitz form Hu_{X_ϵ} is the polynomial that defines the hypersurface $(X_\epsilon)^*$ dual to the curve X_ϵ. Here we are tacitly assuming that X is a curve of degree ≥ 2.

We first consider the case of planar curves ($n = 2$). Here, our result follows from the *General Class Formula* in Fischer (2001, Section A.5.4). Indeed, the Z_i correspond to singular points on the curve, and the ν_i are the degrees of the singularities in the sense of Fischer (2001, Section A.5.2).

For the general case ($n \geq 3$), we consider a random projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, and we examine the degeneration of the irreducible plane curve $\pi(X_\epsilon)$ to its limit $\pi(X_0)$. The equation of the dual curve $\pi(X_\epsilon)^*$ is an irreducible polynomial. In particular, the singularities of $\pi(X_\epsilon)$ that were acquired by the projection do not appear as factors. By Gel'fand et al. (1994, Proposition I.1.4.1, page 31), the equation of $\pi(X_\epsilon)^*$ is found by restricting the irreducible polynomial Hu_{X_ϵ} to a general plane $\mathbb{P}^2$ in $\mathbb{P}^n$, namely the plane dual to the base locus of π. As ϵ approaches 0, we obtain the restriction of Hu_{X_0} to the same plane. This now factors as in the previous paragraph, with one reduced factor for each irreducible component of X_0 or $\pi(X_0)$, and non-reduced factors for the codimension 1 intersections of these components. □

Remark 4.2. Theorem 4.1 requires the hypothesis that the components of X_0 are regular in codimension 1. For instance, suppose that (X_ϵ) is a family of smooth cubic curves in $\mathbb{P}^2$ and X_0 is a nodal cubic. Then H_{X_0} has degree 4 but the left hand side of (6) has degree 6.

Remark 4.3. Formula (7) can be viewed a generalization of the following familiar equation relating the resultant of two polynomials in one variable to the discriminant of their product:

$$\mathrm{Discrim}(f_1 \cdot f_2) \;=\; \mathrm{Discrim}(f_1) \cdot \mathrm{Discrim}(f_2) \cdot \mathrm{Res}(f_1, f_2)^2.$$

For an extension to linear sections of toric varieties see Dickenstein et al. (2014, Corollary 6).

An important special case of Theorem 4.1 arises when the limit cycle $Y = X_0$ is an arrangement of linear subspaces of dimension d in $\mathbb{P}^n$. Here, the first factor on the right hand side in (7) disappears, and the Hurwitz form of the subspace arrangement Y equals

$$\mathrm{Hu}_Y \;=\; \prod_{Z} (\mathrm{Ch}_Z)^{2\nu(Z)-2}. \tag{8}$$

In this formula, Z runs over all strata of dimension $d-1$ in the subspace arrangement Y, and $\nu(Z)$ is the number of d-planes Y_i that contain Z.

Of particular interest is the scenario when Y consists of coordinate subspaces in $\mathbb{P}^n$. Here, Y can be identified with a simplicial complex of dimension d on the vertex set $\{0, 1, \ldots, n\}$. The product (8) is over all $(d-1)$-simplices Z. These correspond to coordinate planes

$$\mathrm{span}(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_d}) \;=\; V(x_{j_0}, x_{j_1}, \ldots, x_{j_{n-d}}).$$

These are indexed by set partitions $\{0, 1, \ldots, n\} = \{i_1, i_2, \ldots, i_d\} \cup \{j_0, j_1, \ldots, j_{n-d}\}$, and their Chow forms are just Plücker variables, as in Sturmfels (1992, Proposition 3.4).

$$p_Z \;:=\; p_{i_1 j_2 \cdots j_d} \;=\; q_{j_0 j_1 \cdots j_{n-d}} \;=\; \mathrm{Ch}_Z. \tag{9}$$

This situation arises whenever the ideal of a projective variety has a Gröbner basis with a squarefree initial ideal. This is now the Stanley–Reisner ideal of the simplicial complex Y.

Corollary 4.4. *Let I be a homogeneous prime ideal in $K[x_0, x_1, \ldots, x_n]$ and suppose $M = \mathrm{in}_{\mathbf{w}}(I)$ is a squarefree initial monomial ideal. The initial form $\mathrm{in}_{\mathbf{w}}(\mathrm{Hu}_{V(I)})$ of the Hurwitz form equals the monomial $\prod_Z p_Z^{2\nu(Z)-2}$ where Z runs over codimension 1 simplices in $V(M)$.*

Proof. Using (8) and the identification (9), this is a direct consequence of Theorem 4.1. $\square$

This corollary allows us to compute the Hurwitz polytope and initial terms of the Hurwitz form for any homogeneous ideal whose initial monomial ideals are all squarefree. One situation where this holds is the ideal generated by the maximal minors of a rectangular matrix of unknowns, by Bernstein and Zelevinsky (1993), Sturmfels and Zelevinsky (1993). Here the initial ideals correspond to the *coherent matching fields* in Sturmfels and Zelevinsky (1993), and each of these determines an initial Plücker monomial in the Hurwitz form for the determinantal variety of maximal minors. We illustrate this with an example.

Example 4.5 *(Ideals of Maximal Minors).* The four 3×3-minors of the 3×4-matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & x_{11} \end{pmatrix}$$

form a universal Gröbner basis for the ideal I they generate Sturmfels and Zelevinsky (1993, Theorem 7.2). The variety $X = V(I) \subset \mathbb{P}^{11}$ represents the 3×4-matrices of rank ≤ 2. Here, $n = 11$, $d = 9$, $p = 6$, and $g = 3$, so the Hurwitz degree equals $\mathrm{Hdeg}(X) = 16$. The determinantal ideal I has 96 initial monomial ideals, all squarefree, in two symmetry classes. Using Corollary 4.4, we read off the corresponding initial Plücker monomials of degree 16 in the Hurwitz form Hu_X:

	initial monomial ideal		initial Plücker monomial in Hu_X
72 ideals like	$\langle x_2x_5x_8, x_3x_5x_8, x_3x_6x_8, x_3x_6x_9 \rangle$	$\rightsquigarrow$	$q_{235}^2 q_{238}^2 q_{358}^2 q_{356}^2 q_{368}^2 q_{389}^2 q_{568}^2 q_{689}^2$
24 ideals like	$\langle x_0x_6x_9, x_3x_4x_9, x_3x_6x_8, x_3x_6x_9 \rangle$	$\rightsquigarrow$	$q_{036}^2 q_{039}^2 q_{346}^2 q_{389}^2 q_{469}^2 q_{689}^2 q_{369}^4$

So, while the Hurwitz form Hu_X is a huge polynomial that is hard to compute explicitly, it is easy to write down the 96 initial Plücker monomials of Hu_X, one for each vertex of the Hurwitz polytope HuPo_X. This polytope has dimension 6, it is simple, and has the same normal fan as the Chow polytope ChPo_X. By Sturmfels and Zelevinsky (1993, Theorem 2.8), this is the *transportation polytope* of nonnegative 3×4-matrices whose rows sum to 4 and whose columns sum to 3. $\diamond$

Here is another important class of varieties all of whose initial ideals are squarefree.

Example 4.6 *(Reciprocal Linear Spaces).* Fix a $(d+1) \times (n+1)$-matrix A of rank $d+1$ with entries in $\mathbb{Q}$, and let X be the reciprocal of the row space of A. Thus X is the Zariski closure in $\mathbb{P}^n$ of the set of points $(u_0 : u_1 : \cdots : u_n)$ with all coordinates nonzero and such that $(u_0^{-1}, u_1^{-1}, \ldots, u_n^{-1}) \in$

rowspace(A). Proudfoot and Speyer (2006) show that the circuits of A define a universal Gröbner basis for the ideal of X, and each initial monomial ideal corresponds to the *broken circuit complex* of A under some ordering of $\{0, 1, \ldots, n\}$. For example, if A is generic, in the sense that all maximal minors of A are nonzero, then $p = \binom{n}{d}$, and the p facets of the broken circuit complex are $\{0, i_1, \ldots, i_d\}$ where $1 \le i_1 < \cdots < i_d \le n$. The corresponding initial monomial of the Hurwitz form is

$$\prod (p_{0 i_2 \cdots i_d})^{2(n-d)},$$

where the product is over all $(d-1)$-element subsets $\{i_2, \ldots, i_d\}$ of $\{1, \ldots, n\}$. Therefore,

$$\mathrm{Hdeg}(X) \;=\; 2\binom{n}{d-1}(n-d).$$

If A is not generic then $\mathrm{Hdeg}(X)$ is given by a matroid invariant that appears in De Loera et al. (2012) and Sanyal et al. (2013).

Our study of the Hurwitz form furnishes the answer to Question 4 in Sanyal et al. (2013, §7, p. 706): *How is the entropic discriminant related to the Gauss curve of the central curve?* The central curve of A is essentially the linear section $X \cap L'$, a smooth curve of degree p and genus g, and its Gauss curve has degree $\mathrm{Hdeg}(X) = 2p + 2g - 2$, by the generalized Plücker formula. The Hurwitz form Hu_X has that same degree, and hence so does the entropic discriminant:

Corollary 4.7. *The entropic discriminant of the matrix A equals $(b_1 b_2 \cdots b_d)^{2\binom{n}{d-1}(d-n)}$ times the Hurwitz form in primal Stiefel coordinates of $X = \mathrm{rowspace}(A)^{-1}$ evaluated at the matrix*

$$\begin{pmatrix} b_1 & -b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & -b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & -b_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & b_d & -b_{d-1} \end{pmatrix} \cdot A. \tag{10}$$

We follow (Sanyal et al., 2013) in using coordinates $b_0, b_1, \ldots, b_d$ for right hand side vectors of the matrix A.

Proof. The point $(b_0{:}b_1{:}\cdots{:}b_d) \in \mathbb{P}^d$ lies in the entropic discriminant of A if and only if the row space of (10) intersects X. The Plücker coordinate vector of that row space can be written as a linear expression in $(b_0, b_1, \ldots, b_d)$. It is obtained by removing the common factor $b_1 b_2 \cdots b_{d-1}$ from the maximal minors. This extraneous factor has exponent $-\mathrm{Hdeg}(X)$. $\square$

For example, let $d = 2$, $n = 4$, and consider the surface $X \subset \mathbb{P}^4$ defined by the matrix

$$A \;=\; \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Explicitly, $X = V(x_0x_2 - x_0x_4 - x_2x_4, x_0x_1 - x_0x_3 - x_1x_3)$, and $\mathrm{Hdeg}(X) = 8$. The Hurwitz form (in primal Stiefel coordinates) is a homogeneous polynomial Hu_X with 46958 terms of degree 16 in the ten entries of a 2×5-matrix of unknowns. If we substitute (10) into Hu_X then we obtain b_1^8 times the sum of squares listed explicitly in Sanyal et al. (2013, Example 1, page 679).

We close with the remark that Corollary 4.7 can be extended to also answer Question 5 in Sanyal et al. (2013, §7, p. 706), as that pertains to translating the variety X via the torus action on $\mathbb{P}^n$. The "Varchenko discriminant" sought in that question is obtained by multiplying the matrix (10) on the right with the diagonal matrix $\mathrm{diag}(c_0, c_1, \ldots, c_n)$ before substituting into Hu_X. Finally, it should be possible to resolve also Question 2 in Sanyal et al. (2013, §7, p. 705), via the description of the Hurwitz polytope that is implicit in Example 4.6. We leave this to a future project.

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