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Short communication

The reciprocal of the geometric mean of many positive numbers is a Stieltjes transform

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ABSTRACT

In the paper, by the Cauchy integral formula in the theory of complex functions, an integral representation for the reciprocal of the geometric mean of many positive numbers is established. As a result, the reciprocal of the geometric mean of many positive numbers is verified to be a Stieltjes transform and, consequently, a (logarithmically) completely monotonic function.

1. Preliminaries

We recall some definitions and notion.

Recall from [1, Chapter IV] that an infinitely differentiable function f on an interval I is said to be completely monotonic on I if it satisfies $(-1)^{n-1}f^{(n-1)}(x) \ge 0$ for $x \in I$ and $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of all positive integers. Theorem 12b in [1] reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\,\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

In [2–4], it was defined implicitly and explicitly that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $(-1)^k[\ln f(x)]^{(k)} \ge 0$ on I for all $k \in \mathbb{N}$. In [5, Theorem 1.1],

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[6, Theorem 4], [3, Theorem 1], and [4, Theorem 4], it was found and verified once again that a logarithmically completely monotonic function must be completely monotonic, but not conversely.

In [7, Definition 2.1], it was defined that a Stieltjes transform is a function $f : (0, \infty) \to [0, \infty)$ which can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{u+x} \, \mathrm{d}\, \mu(u), \tag{1.1}$$

where a, b are nonnegative constants and μ is a nonnegative measure on $(0, \infty)$ such that the integral $\int_0^\infty \frac{1}{1+s} \, \mathrm{d}\, \mu(s) < \infty$. In [5, Theorem 1.2], it was proved that a positive Stieltjes transform must be a logarithmically completely monotonic function on $(0, \infty)$, but not conversely.

An infinitely differentiable function $f : I \to [0, \infty)$ is called a Bernstein function on an interval I if f' is completely monotonic on I. The Bernstein functions on $(0, \infty)$ can be characterized by [7, Theorem 3.2] which states that a function $f : (0, \infty) \to [0, \infty)$ is a Bernstein function if and only if it admits the representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xt}\right) \mathrm{d}\, \mu(t), \tag{1.2}$$

where $a, b \ge 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_0^\infty \min\{1, t\} \, \mathrm{d}\, \mu(t) < \infty$. The formula (1.2) is called the Lévy–Khintchine representation of f. In [8, pp. 161–162, Theorem 3] and [7, Proposition 5.25], it was proved that the reciprocal of a Bernstein function is logarithmically completely monotonic.

If the Lévy measure μ in (1.2) has a completely monotonic density $m(t)$ with respect to the Lebesgue measure, that is,

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xt}\right) m(t) \, \mathrm{d}\, t, \quad a, b \ge 0,$$

where $m(t)$ is a completely monotonic function on $(0, \infty)$ and satisfies $\int_0^\infty \min\{1, t\} m(t) \, \mathrm{d}\, t < \infty$, then f is said to be a complete Bernstein function on $(0, \infty)$. Theorem 7.3 in [7] reads that a (non-trivial) function f is a complete Bernstein function if and only if $\frac{1}{f}$ is a (non-trivial) Stieltjes function.

2. Motivations and main results

Theorem 4.2 in [9] states that the principal branch of the geometric mean

$$G_{a,b}(z) = \sqrt{(a+z)(b+z)}$$

for $b > a > 0$ and $z \in \mathbb{C} \setminus [-b, -a]$ has the Lévy–Khintchine representation

$$G_{a,b}(z) = G_{a,b}(0) + z + \frac{b-a}{2\pi} \int_0^\infty \frac{\rho((b-a)s)}{s} e^{-as} (1 - e^{-zs}) \, \mathrm{d}\, s,$$

where $\rho(s)$ is defined by

$$\rho(s) = \int_0^{1/2} \left(\sqrt{\frac{1}{u} - 1} - \frac{1}{\sqrt{1/u - 1}} \right) \left[1 - e^{-(1-2u)s}\right] e^{-su} \, \mathrm{d}\, u > 0.$$

Consequently, the geometric mean $G_{a,b}(t-a)$ is a complete Bernstein function on $(0, \infty)$ and $\frac{1}{G_{a,b}(t-a)}$ is a Stieltjes transform.

For $b > a$ and $z \in \mathbb{C} \setminus [-b, -a]$, Lemma 2.4 in [10] reads that the reciprocal of $G_{a,b}(z)$ can be represented as

$$h_{a,b}(z) = \frac{1}{G_{a,b}(z)} = \frac{1}{\sqrt{(z+a)(z+b)}} = \frac{1}{\pi} \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t+z} \, \mathrm{d}\, t.$$

Consequently, the function $h_{a,b}(x-a)$ is a Stieltjes transform and, consequently, a (logarithmically) completely monotonic function on $(0, \infty)$.

Let $n \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a positive sequence, that is, $a_k > 0$ for $1 \le k \le n$, and no any two are same. For $z \in \mathbb{C} \setminus [-\max\{a_k, 1 \le k \le n\}, -\min\{a_k, 1 \le k \le n\}]$ and $n \ge 2$, let $\mathbf{1} = (\overbrace{1, 1, \dots, 1}^{n})$ and

$$G_n(z\mathbf{1} + \mathbf{a}) = \left[\prod_{k=1}^n (a_k + z)\right]^{1/n}.$$

Let σ be a permutation of the set $\{1, 2, \dots, n\}$ such that $\sigma(\mathbf{a}) = \left(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}\right)$ is a rearrangement of $\mathbf{a}$ in an ascending order $a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(n)}$. Then it was obtained in [11, Theorem 1.1] that the principal branch of the geometric mean $G_n(z\mathbf{1} + \mathbf{a})$ has the Lévy–Khintchine representation

$$G_n(z\mathbf{1} + \mathbf{a}) = G_n(\mathbf{a}) + z + \int_0^\infty Q_{n,\mathbf{a}}(u)(1 - e^{-zu}) \, \mathrm{d}\, u,$$

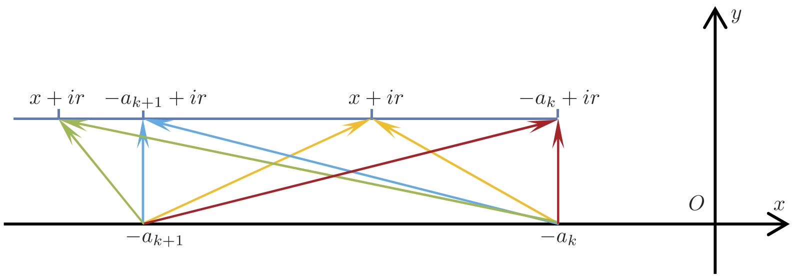

Fig. 1. The arguments of $(x + ri) - (-a_k)$ and $(x + ri) - (-a_{k+1})$.

where

$$Q_{n,\mathbf{a}}(u) = \frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}}\left|\prod_{k=1}^{n}(a_k-t)\right|^{1/n} e^{-ut}\,\mathrm{d}\,t.$$

For more information on this topic, please refer to the papers [12–18] and closely-related references therein.

The main aim of this paper is to establish by the Cauchy integral formula in the theory of complex functions an integral representation for the reciprocal $\frac{1}{G_n(z\mathbf{1}+\mathbf{a})}$ of the geometric mean $G_n(z\mathbf{1}+\mathbf{a})$ of $n > 2$ positive numbers a_k for $1 \le k \le n$. As a result, the reciprocal $\frac{1}{G_n(z\mathbf{1}+\mathbf{a})}$ of the geometric mean $G_n(z\mathbf{1}+\mathbf{a})$ is derived to be a Stieltjes transform and, consequently, a (logarithmically) completely monotonic function.

Our main results can be stated as the following theorem.

Theorem 2.1. *For $a_k < a_{k+1}$, the principal branch of the reciprocal $\frac{1}{G_n(z\mathbf{1}+\mathbf{a})}$ of the geometric mean $G_n(z\mathbf{1}+\mathbf{a})$ can be represented as*

$$\frac{1}{G_n(z\mathbf{1}+\mathbf{a})} = \frac{1}{\sqrt[n]{\prod_{k=1}^{n}(z+a_k)}} = \frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left(\frac{\ell\pi}{n}\right)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{\sqrt[n]{\prod_{k=1}^{n}|t-a_k|}}\frac{1}{t+z}\,\mathrm{d}\,t, \tag{2.1}$$

where $z \in \mathbb{C}\setminus[-a_n, -a_1]$. Consequently, the function $\frac{1}{G_n((x-a_1)\mathbf{1}+\mathbf{a})}$ is a Stieltjes transform and a (logarithmically) completely monotonic function on $(0,\infty)$.

3. Proof of Theorem 2.1

For $r > 0$ and by virtue of Fig. 1, we see that

(1) when $x \le -a_k$,
$$\frac{\pi}{2} \le \arg(x + ir + a_k) < \pi;$$
(2) when $-a_{k+1} \le x \le -a_k$,
$$0 < \arg(x + ir + a_{k+1}) \le \frac{\pi}{2}.$$

A straightforward computation gives

$$\begin{aligned}\frac{1}{G_n((x+ir)\mathbf{1}+\mathbf{a})} &= \frac{1}{\sqrt[n]{\prod_{k=1}^{n}(x+a_k+ir)}}\\ &= \frac{1}{\exp\left\{\frac{1}{n}\sum_{k=1}^{n}[\ln|x+a_k+ir| + i\arg(x+a_k+ir)]\right\}}\\ &\to \frac{1}{\exp\left[\frac{1}{n}\sum_{k=1}^{n}\ln|x+a_k|\right]}\frac{1}{\exp\left[\frac{i}{n}\sum_{k=1}^{n}\lim_{r\to0^+}\arg(x+a_k+ir)\right]}\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \dfrac{1}{\sqrt[n]{\prod_{k=1}^{n}|x+a_k|}}\dfrac{1}{\exp(i\pi)}, & x<-a_n\\[2ex] \dfrac{1}{\sqrt[n]{\prod_{k=1}^{n}|x+a_k|}}\dfrac{1}{\exp\left(\frac{i}{n}\ell\pi\right)}, & -a_{\ell+1}<x<-a_\ell,\ 1\le\ell\le n-1 \end{cases}\\
&= \begin{cases} \dfrac{1}{\sqrt[n]{\prod_{k=1}^{n}|x+a_k|}}, & x<-a_n\\[2ex] \dfrac{\cos\left(\frac{\ell}{n}\pi\right)-i\sin\left(\frac{\ell}{n}\pi\right)}{\sqrt[n]{\prod_{k=1}^{n}|x+a_k|}}, & -a_{\ell+1}<x<-a_\ell,\ 1\le\ell\le n-1 \end{cases}
\end{aligned}
$$

as $r\to0^+$. Consequently, it follows that

$$
\lim_{r\to0^+}\Im\left[\frac{1}{G_n((x+ir)\mathbf{1}+\mathbf{a})}\right]=\begin{cases} 0, & x<-a_n;\\ -\dfrac{\sin\left(\frac{\ell}{n}\pi\right)}{\sqrt[n]{\prod_{k=1}^{n}|x+a_k|}}, & -a_{\ell+1}<x<-a_\ell,\ 1\le\ell\le n-1. \end{cases} \tag{3.1}
$$

For any fixed point $z_0=x_0+iy_0\in\mathbb{C}\setminus(-\infty,-a]$, choose $r>0$ and $R>a_n>0$ such that

$$
\begin{cases} 0<r<|y_0|\le|z_0|<R, & y_0\ne0;\\ 0<r<x_0+a_1<R, & y_0=0; \end{cases}
$$

and consider the positively oriented contour $C(r,R)$ in $\mathbb{C}\setminus(-\infty,-a_1]$, which consists of the half circle $C(a_1,r):z+a_1=re^{i\theta}$ for $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and two half lines $\ell_\pm:z=x\pm ir$ for $x\le-a_1$ until they cut the circle $C(R):z=Re^{i\theta}$ for

$$
\theta\in\left(-\pi+\arctan\frac{r}{\sqrt{R^2-r^2}},\pi-\arctan\frac{r}{\sqrt{R^2-r^2}}\right)
$$

at the points $-\sqrt{R^2-r^2}\pm ir$.

Along the positively oriented contour $C(r,R)$ in $\mathbb{C}\setminus(-\infty,-a_1]$, applying the Cauchy integral formula yields

$$
\begin{aligned}
h_{n,\mathbf{a}}(z_0)&\triangleq\frac{1}{G_n(z_0\mathbf{1}+\mathbf{a})}=\frac{1}{2\pi i}\oint_{C(r,R)}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi\\
&=\frac{1}{2\pi i}\left[\int_{C(R)}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi+\int_{C(a,r)}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi+\int_{\ell_+}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi+\int_{\ell_-}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi\right].
\end{aligned}
$$

By a straightforward computation, we have

$$
\begin{aligned}
\int_{C(R)}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi&=\int_{-\pi+\arctan\frac{r}{\sqrt{R^2-r^2}}}^{\pi-\arctan\frac{r}{\sqrt{R^2-r^2}}}\frac{h_{n,\mathbf{a}}\left(Re^{i\theta}\right)}{Re^{i\theta}-z_0}iRe^{i\theta}\,\mathrm{d}\,\theta\\
&\to i\int_{-\pi}^{\pi}\lim_{R\to\infty}\frac{1}{1-z_0/Re^{i\theta}}\lim_{R\to\infty}h_{n,\mathbf{a}}\left(Re^{i\theta}\right)\mathrm{d}\,\theta,\quad R\to\infty\\
&=i\int_{-\pi}^{\pi}\lim_{R\to\infty}\frac{1}{G_n(Re^{i\theta}\mathbf{1}+\mathbf{a})}\,\mathrm{d}\,\theta\\
&=0,\\
\int_{C(a,r)}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi&=\int_{\pi/2}^{-\pi/2}\frac{ire^{i\theta}}{re^{i\theta}-a_1-z_0}\frac{1}{G_n((re^{i\theta}-a_1)\mathbf{1}+\mathbf{a})}\,\mathrm{d}\,\theta\\
&\to0,\quad r\to0^+,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\ell_+}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi+\int_{\ell_-}\frac{h_{n,\mathbf{a}}(\xi)}{\xi-z_0}\,\mathrm{d}\,\xi=\int_{-R(r)}^{-a_1}\frac{h_{n,\mathbf{a}}(x+ir)}{x+ir-z_0}\,\mathrm{d}\,x+\int_{-a_1}^{-R(r)}\frac{h_{n,\mathbf{a}}(x-ir)}{x-ir-z_0}\,\mathrm{d}\,x\\
&=\int_{-R(r)}^{-a_1}\left[\frac{h_{n,\mathbf{a}}(x+ir)}{x+ir-z_0}-\frac{h_{n,\mathbf{a}}(x-ir)}{x-ir-z_0}\right]\mathrm{d}\,x\\
&=\int_{-R(r)}^{-a_1}\frac{(x-z_0)[h_{n,\mathbf{a}}(x+ir)-h_{n,\mathbf{a}}(x-ir)]-ir[h_{n,\mathbf{a}}(x+ir)+h_{n,\mathbf{a}}(x-ir)]}{(x-z_0)^2+r^2}\,\mathrm{d}\,x\\
&=\int_{-R(r)}^{-a_1}\frac{(x-z_0)\big[h_{n,\mathbf{a}}(x+ir)-h_{n,\mathbf{a}}\big(\overline{x+ir}\big)\big]-ir\big[h_{n,\mathbf{a}}(x+ir)+h_{n,\mathbf{a}}\big(\overline{x+ir}\big)\big]}{(x-z_0)^2+r^2}\,\mathrm{d}\,x\\
&=\int_{-R(r)}^{-a_1}\frac{(x-z_0)\big[h_{n,\mathbf{a}}(x+ir)-\overline{h_{n,\mathbf{a}}(x+ir)}\big]-ir\big[h_{n,\mathbf{a}}(x+ir)+\overline{h_{n,\mathbf{a}}(x+ir)}\big]}{(x-z_0)^2+r^2}\,\mathrm{d}\,x\\
&=\int_{-R(r)}^{-a_1}\frac{2i(x-z_0)\Im[h_{n,\mathbf{a}}(x+ir)]-2ir\Re[h_{n,\mathbf{a}}(x+ir)]}{(x-z_0)^2+r^2}\,\mathrm{d}\,x\\
&\to 2i\int_{-\infty}^{-a_1}\frac{1}{x-z_0}\lim_{r\to0^+}\Im[h_{n,\mathbf{a}}(x+ir)]\,\mathrm{d}\,x\\
&=-2i\sum_{\ell=1}^{n-1}\int_{-a_{\ell+1}}^{-a_\ell}\frac{1}{x-z_0}\frac{\sin\left(\frac{\ell}{n}\pi\right)}{\sqrt[n]{\prod_{k=1}^{n}|x+a_k|}}\,\mathrm{d}\,x\quad\text{by the limit (3.1)}\\
&=2i\sum_{\ell=1}^{n-1}\int_{a_{\ell+1}}^{a_\ell}\frac{1}{-t-z_0}\frac{\sin\left(\frac{\ell}{n}\pi\right)}{\sqrt[n]{\prod_{k=1}^{n}|a_k-t|}}\,\mathrm{d}\,t\\
&=2i\sum_{\ell=1}^{n-1}\sin\left(\frac{\ell}{n}\pi\right)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{t+z_0}\frac{1}{\sqrt[n]{\prod_{k=1}^{n}|a_k-t|}}\,\mathrm{d}\,t
\end{aligned}
$$

as $r\to0^+$ and $R\to\infty$, where $R(r)=\sqrt{R^2-r^2}$. In conclusion, we obtain

$$
h_{n,\mathbf{a}}(z_0)=\frac{1}{G_n(z_0\mathbf{1}+\mathbf{a})}=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left(\frac{\ell}{n}\pi\right)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{t+z_0}\frac{1}{\sqrt[n]{\prod_{k=1}^{n}|a_k-t|}}\,\mathrm{d}\,t.
$$

The integral representation (2.1) is proved.

The rest results can be derived readily from comparing the integral representations (1.1) and (2.1) and making use of the relations, which are mentioned in Section 1 of this paper, among the Stieltjes transforms, logarithmically completely monotonic functions, and completely monotonic functions. The proof of Theorem 2.1 is complete.

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