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The problem of the nonlinear diffusive predator–prey model with the same biotic resource

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In recent years, the research on the diffusive predator–prey model has attracted much attention. In these models, the carrying capacity is considered as a constant. In 2013, H. M. Safuan investigated the system of a predator and prey that shares the same biotic resource, where the carrying capacity is a function of the time. The spatial component of ecological interactions has been recognized as an important factor. So, we will discuss the problem of the nonlinear diffusive predator–prey model with the same biotic resource. This model is the system of the nonlinear partial differential equations with zero-flux boundary condition. The main objective of the present paper is to investigate the existence and uniqueness of the solution of this model. In this paper, we also obtain that there is a unique solution of the nonlinear partial differential equations with Dirichlet boundary condition.

1. Introduction

In recent years, the research on the diffusive predator–prey model has attracted much attention [1–15]. For example, in [4,6,8], the authors discussed the diffusive Leslie–Gower predator–prey system, and in [11,15], they discussed the diffusive Lotka–Volterra model. In these models, the carrying capacity is considered as a constant. In [16], H. M. Safuan investigated the system of a predator and prey that share the same biotic resource.

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\left(1-\frac{x}{pz}\right)-axy, \\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{qz}\right)+bxy, \\ \frac{dz}{dt} &= z\left(c-dx-ey\right), \end{aligned} \tag{1.1}$$

where functions $x(t), y(t), z(t)$ are populations of prey, predator and biotic resource, respectively; and $r_1, r_2, a, b, c, d, e, p, q$ are positive constants. In this model, the carrying capacities of both the predator and

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prey depend on the amount of resource. The spatial component of ecological interactions has been recognized as an important factor. Considering the natural diffusion and inhibitory effect, many researchers extend the predator–prey model of ODE to the corresponding diffusive predator–prey model by incorporating the diffusion terms [4,6,8–10].

In this paper, we extend the predator–prey model (1.1) to the following the nonlinear diffusive predator–prey model.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = d_1 \Delta u + r_1 u\left(1-\frac{u}{pw}\right) - auv && \text{for all } t>0,\ x\in\Omega,\\
&\frac{\partial v}{\partial t} = d_2 \Delta v + r_2 v\left(1-\frac{v}{qw}\right) + buv && \text{for all } t>0,\ x\in\Omega,\\
&\frac{\partial w}{\partial t} = w\left(c-du-ev\right) && \text{for all } t>0,\ x\in\Omega,\\
&\frac{\partial}{\partial \nu} u\left(t,x\right)=0, \qquad \frac{\partial}{\partial \nu} v\left(t,x\right)=0 && \text{for all } t>0,\ x\in\partial\Omega,\\
&u\left(0,x\right)=u_0\left(x\right), \qquad v\left(0,x\right)=v_0\left(x\right), \qquad w\left(0,x\right)=w_0\left(x\right) && \text{for all } x\in\Omega,
\end{aligned} \tag{1.2}$$

where Ω is a bounded open set in $\mathbb{R}^n, \partial\Omega$ is C^1-class, and u_0, v_0, w_0 are Holder continuous functions on Ω. The main objective of present paper is to investigate the existence and uniqueness of solution for the nonlinear partial differential equations with zero-flux boundary conditions. In the problem (1.2), functions $u\left(t,x\right), v\left(t,x\right), w\left(t,x\right)$ are the densities of prey, predator and biotic resource at position x and time t, respectively. Positive constants d_1, d_2 are the diffusion coefficients of prey and predator, respectively. The practical significance of positive constants r_1, r_2, a, b, c, d, e, p and q are described below.

(1) r_1 and r_2 are growth rates of the prey and the predator, respectively.
(2) a is the consumption rate of the prey by the predator in the interaction of the two species.
(3) b is the conversion efficiency of the predator by the prey in the interaction of the two species.
(4) c is the growth rate of the biotic resource.
(5) d is the consumption rate of the biotic resource by the prey.
(6) e is the consumption rate of the biotic resource by the predator.
(7) p is the maximum ratio of consumption of the biotic resource by the prey.
(8) q is the maximum ratio of consumption of the biotic resource by the predator.

On the other hand, the zero-flux boundary condition indicates that predator–prey system is self-contained with zero population flux across the boundary.

In this paper, we will prove the existence and uniqueness of solution for this system of the nonlinear partial differential equations by the methods of the upper and lower solutions [17] and the semigroup theory [18,19]. Moreover, we will prove that the problem (1.2) has a unique solution. In the rest of this section, we will introduce the concept of the upper and lower solutions. In Section 2, under the assumption of the existence of the upper solution $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$ and lower solution $\hat{\mathbf{u}} = (\hat{u}, \hat{v}, \hat{w})$ of the problem (1.2), we will show the existence and uniqueness of solution of the problem (1.2) on the sector $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}}\rangle \equiv \left\{ \mathbf{u} = (u, v, w) \in C\left(\bar{D}_T\right) : \hat{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}} \right\}$. In Section 3, we will give a pair of upper and lower solutions of the problem (1.2) on $[0, T] \times \overline{\Omega}$, where T is an arbitrary positive number. Then we prove the existence and uniqueness of the solution (u, v, w) of the problem (1.2) on $[0, T] \times \overline{\Omega}$. On the other hand, we also obtain that the nonlinear partial differential equations (1.2) with Dirichlet boundary condition

$$u\left(t,x\right)=0, v\left(t,x\right)=0 \quad \text{for all } t>0, x\in\partial\Omega,$$

has a unique solution on $[0,T]\times\overline{\Omega}$, where T is an arbitrary positive number.

To simplify the notations of the problem (1.2), we set $u_1=u, u_2=v, u_3=w, u_{1,0}=u_0, u_{2,0}=v_0, u_{3,0}=w_0, L_1=d_1\Delta, L_2=d_2\Delta, L_3=0, B=\partial/\partial\nu$, and

$$\begin{aligned} f_1\left(u_1,u_2,u_3\right) &= r_1u_1-\frac{r_1u_1^2}{pu_3}-au_1u_2,\\ f_2\left(u_1,u_2,u_3\right) &= r_2u_2-\frac{r_2u_2^2}{qu_3}+bu_1u_2,\\ f_3\left(u_1,u_2,u_3\right) &= cu_3-du_1u_3-eu_2u_3. \end{aligned} \tag{1.3}$$

Suppose that T is an arbitrary positive number and $D_T=(0,T]\times\Omega, S_T=(0,T]\times\partial\Omega$. We consider the system

$$\begin{array}{ll} \left(u_i\right)_t-L_iu_i=f_i\left(u_1,u_2,u_3\right) & \text{in } D_T,\ i=1,2,3,\\ Bu_i\left(t,x\right)=0 & \text{on } S_T,\ i=1,2,\\ u_i\left(0,x\right)=u_{i,0}\left(x\right) & \text{in } \Omega,\ i=1,2,3. \end{array} \tag{1.4}$$

We set $J_1=J_2=\{\,x\in\mathbb{R}^n: x_1\ge 0,\ldots,x_n\ge 0\,\}$ and $J_3=\{\,x\in\mathbb{R}^n: x_1>0,\ldots,x_n>0\,\}$. Then

$$\begin{array}{ll} \dfrac{\partial}{\partial u_2}f_1=-au_1\le 0, & \dfrac{\partial}{\partial u_3}f_1=\dfrac{r_1u_1^2}{pu_3^2}\ge 0,\\ \dfrac{\partial}{\partial u_1}f_2=bu_2\ge 0, & \dfrac{\partial}{\partial u_3}f_2=\dfrac{r_2u_2^2}{qu_3^2}\ge 0,\\ \dfrac{\partial}{\partial u_1}f_3=-du_3<0, & \dfrac{\partial}{\partial u_2}f_3=-eu_3<0, \end{array}$$

for all $(u_1,u_2,u_3)\in J_1\times J_2\times J_3$. This implies that for all $(u_1,u_2,u_3)\in J_1\times J_2\times J_3$,

f_1 is monotone nonincreasing in u_2, and monotone nondecreasing in u_3,
f_2 is monotone nondecreasing in u_1, and monotone nondecreasing in u_3,
f_3 is monotone nonincreasing in u_1, and monotone nonincreasing in u_2.

Let $\mathbf{u}=(u_1,u_2,u_3)$ and $f_i\left(\mathbf{u}\right)=f_i\left(u_i,[\mathbf{u}]_{a_i},[\mathbf{u}]_{b_i}\right), i=1,2,3,$ where $a_i+b_i=2$, and f_i is monotone nondecreasing in $[\mathbf{u}]_{a_i}$, and monotone nonincreasing in $[\mathbf{u}]_{b_i}$. Then we have the following definition of coupled upper and lower solutions of the system (1.4) (see e.g. [17]).

Definition 1.1. A pair of functions $\tilde{\mathbf{u}}=(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3), \hat{\mathbf{u}}=(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ are called coupled upper and lower solutions of the system (1.4) if $\tilde{u}_1,\tilde{u}_2,\hat{u}_1,\hat{u}_2\in C\left(\bar{D}_T\right)\cap C^{1,2}\left(D_T\right), \tilde{u}_3,\hat{u}_3\in C\left(\bar{D}_T\right)\cap C^{1,0}\left(D_T\right)$, and $\tilde{\mathbf{u}}\ge\hat{\mathbf{u}}$ (i.e., $\tilde{u}_i\ge\hat{u}_i, i=1,2,3$) with $\hat{u}_3\left(t,x\right)>0$ in $\bar{D}_T=[0,T]\times\bar{\Omega}$, and

$$\begin{array}{ll} \left(\tilde{u}_i\right)_t-L_i\tilde{u}_i-f_i(\tilde{u}_i,[\tilde{\mathbf{u}}]_{a_i},[\hat{\mathbf{u}}]_{b_i})\ge 0, & \text{in } D_T,\ i=1,2,3,\\ \left(\hat{u}_i\right)_t-L_i\hat{u}_i-f_i(\hat{u}_i,[\hat{\mathbf{u}}]_{a_i},[\tilde{\mathbf{u}}]_{b_i})\le 0, & \text{in } D_T,\ i=1,2,3,\\ B\tilde{u}_i\left(t,x\right)\ge 0\ge B\hat{u}_i\left(t,x\right), & \text{on } S_T,\ i=1,2\\ \tilde{u}_i\left(0,x\right)\ge u_{i,0}\left(x\right)\ge\hat{u}_i\left(0,x\right), & \text{in } \Omega,\ i=1,2,3. \end{array} \tag{1.5}$$

For a given pair of coupled upper and lower solutions $\tilde{\mathbf{u}},\hat{\mathbf{u}}$, we define the sector

$$\langle\hat{\mathbf{u}},\tilde{\mathbf{u}}\rangle\equiv\left\{\mathbf{u}=(u_1,u_2,u_3)\in C\left(\bar{D}_T\right):\hat{\mathbf{u}}\le\mathbf{u}\le\tilde{\mathbf{u}}\right\}.$$

Assume that functions $\underline{c}_1 = (2r_1\tilde{u}_1/p\hat{u}_3) + a\tilde{u}_2 - r_1, \underline{c}_2 = (2r_2\tilde{u}_2/q\hat{u}_3) - b\hat{u}_1 - r_2$, and $\underline{c}_3 = d\tilde{u}_1 + e\tilde{u}_2 - c$. Hence $\underline{c}_i \in C\left(\bar{D}_T\right)$ for each $i = 1, 2, 3$, and

$$\begin{aligned} f_1(u_1, u_2, u_3) - f_1(v_1, u_2, u_3) &= r_1(u_1 - v_1) - \frac{r_1}{pu_3}(u_1 - v_1)(u_1 + v_1) - a(u_1 - v_1)u_2 \\ &= \left(r_1 - \frac{r_1}{pu_3}(u_1 + v_1) - au_2\right)(u_1 - v_1) \\ &\geq \left(r_1 - \frac{2r_1\tilde{u}_1}{p\hat{u}_3} - a\tilde{u}_2\right)(u_1 - v_1) \\ &= -\underline{c}_1(u_1 - v_1) \quad \text{for } \hat{u}_1 \leq v_1 \leq u_1 \leq \tilde{u}_1. \end{aligned}$$

Similarly, we have

$$f_1(u_1, u_2, u_3) - f_1(u_1, v_2, u_3) \geq -\underline{c}_2(u_2 - v_2) \quad \text{for } \hat{u}_2 \leq v_2 \leq u_2 \leq \tilde{u}_2,$$

and

$$f_3(u_1, u_2, u_3) - f_3(u_1, u_2, v_3) \geq -\underline{c}_3(u_3 - v_3) \quad \text{for } \hat{u}_3 \leq v_3 \leq u_3 \leq \tilde{u}_3.$$

Thus we prove that there are $\underline{c}_i \in C\left(\bar{D}_T\right), i = 1, 2, 3$, such that

$$f_i\left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) - f_i\left(v_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) \geq -\underline{c}_i(u_i - v_i) \tag{1.6}$$

for $\hat{u}_i \leq v_i \leq u_i \leq \tilde{u}_i, i = 1, 2, 3$. Let functions $\bar{c}_1, \bar{c}_2$, and, $\bar{c}_3$ be defined by

$$\bar{c}_1 = r_1 - \frac{2r_1}{p\tilde{u}_3}\hat{u}_1 - a\hat{u}_2, \qquad \bar{c}_2 = r_2 - \frac{2r_2}{q\tilde{u}_3}\hat{u}_2 + b\tilde{u}_1 \quad \text{and} \quad \bar{c}_3 = c - d\hat{u}_1 - e\hat{u}_2.$$

Hence $\bar{c}_i \in C\left(\bar{D}_T\right)$ for each $i = 1, 2, 3$, and

$$\begin{aligned} f_1(u_1, u_2, u_3) - f_1(v_1, u_2, u_3) &\leq \bar{c}_1(u_1 - v_1) \quad \text{for } \hat{u}_1 \leq v_1 \leq u_1 \leq \tilde{u}_1, \\ f_2(u_1, u_2, u_3) - f_2(u_1, v_2, u_3) &\leq \bar{c}_2(u_2 - v_2) \quad \text{for } \hat{u}_2 \leq v_2 \leq u_2 \leq \tilde{u}_2, \\ f_3(u_1, u_2, u_3) - f_3(u_1, u_2, v_3) &\leq \bar{c}_3(u_3 - v_3) \quad \text{for } \hat{u}_3 \leq v_3 \leq u_3 \leq \tilde{u}_3. \end{aligned}$$

Let $K_{i,i} = |\bar{c}_i| + |\underline{c}_i|$ on $\bar{D}_T$ for each $i = 1, 2, 3$, and

$$\begin{aligned} K_{1,2} &= a\tilde{u}_1, \qquad K_{1,3} = \frac{r_1\tilde{u}_1^2}{p\hat{u}_3^2}, \\ K_{2,1} &= b\tilde{u}_2, \qquad K_{2,3} = \frac{r_2\tilde{u}_2^2}{q\hat{u}_3^2}, \\ K_{3,1} &= d\tilde{u}_3, \qquad K_{3,2} = e\tilde{u}_3 \end{aligned}$$

on $\bar{D}_T$, and $K_i = K_{i,1} + K_{i,2} + K_{i,3}$ for each $i = 1, 2, 3$. Then $K_{i,j} \in C\left(\bar{D}_T\right)$, and $K_i \in C\left(\bar{D}_T\right)$ for each $i, j = 1, 2, 3$, and so $K_{i,j}$ and K_i are bounded functions in $\bar{D}_T$. Since

$$\begin{aligned} -a\tilde{u}_1 \leq \frac{\partial f_1}{\partial u_2} = -au_1 \leq 0, &\qquad \frac{r_1\tilde{u}_1^2}{p\hat{u}_3^2} \geq \frac{\partial f_1}{\partial u_3} = \frac{r_1u_1^2}{pu_3^2} \geq 0, \\ b\tilde{u}_2 \geq \frac{\partial f_2}{\partial u_1} = bu_2 \geq 0, &\qquad \frac{r_2\tilde{u}_2^2}{q\hat{u}_3^2} \geq \frac{\partial f_2}{\partial u_3} = \frac{r_2u_2^2}{qu_3^2} \geq 0, \\ -d\tilde{u}_3 \leq \frac{\partial f_3}{\partial u_1} = -du_3 < 0, &\qquad -e\tilde{u}_3 \leq \frac{\partial f_3}{\partial u_2} = -eu_3 < 0 \end{aligned}$$

on $\langle\hat{\mathbf{u}}, \tilde{\mathbf{u}}\rangle$, we have for all $(u_1, u_2, u_3), (v_1, v_2, v_3) \in \langle\hat{\mathbf{u}}, \tilde{\mathbf{u}}\rangle$,

$$\begin{aligned} |f_i(u_1, u_2, u_3) - f_i(v_1, v_2, v_3)| &\leq K_{i,1}|u_1 - v_1| + K_{i,2}|u_2 - v_2| + K_{i,3}|u_3 - v_3| \\ &\leq K_i(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|) \end{aligned}$$

for each $i = 1, 2, 3$. This inequality shows that f_i satisfies the Lipschitz condition for $\mathbf{u} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle, i = 1, 2, 3$. Moreover, f_i is a holder continuous function on $(t, x) \in \bar{D}_T, i = 1, 2, 3$.

Let $F_i\left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) = f_i\left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) + \underline{c}_i u_i, i = 1, 2, 3$. Then the differential equations in system (1.4) can be written as

$$(u_i)_t - L_i u_i + \underline{c}_i u_i = F_i\left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) \quad \text{in } D_T, i = 1, 2, 3.$$

From Lemma 8.1 in [17], we have the following lemma.

Lemma 1.2. *For each $\mathbf{u} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, we denote that $F_i(\mathbf{u})(t, x) = F_i(\mathbf{u}(t, x))$ on $\bar{D}_T, i = 1, 2, 3$. If $\mathbf{u} \in C^\alpha(D_T)$, and $\alpha \in (0, 1)$, then the function $F_i(\mathbf{u})$ is Holder continuous in D_T for every $i = 1, 2, 3$.*

Moreover, if $\mathbf{u}, \mathbf{v} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ satisfy that $\mathbf{u} \geq \mathbf{v}$, then

$$F_i\left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{v}]_{b_i}\right) - F_i\left(v_i, [\mathbf{v}]_{a_i}, [\mathbf{u}]_{b_i}\right) \geq 0 \quad \textit{for all } i = 1, 2, 3.$$

2. Existence and uniqueness of the solution on $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$

In this section, we always assume that the upper solution $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and lower solution $\hat{\mathbf{u}} = \left(\hat{u}_1, \hat{u}_2, \hat{u}_3\right)$ of the system (1.4) exist. Let $A_i u_i = (u_i)_t - L_1 u_i + \underline{c}_i u_i$ for all $i = 1, 2, 3$. We choose $\bar{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ and $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ as two initial iterations and construct the maximal and minimal sequences $\bar{\mathbf{u}}^{(k)} = \left(\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \bar{u}_3^{(k)}\right)$, $\underline{\mathbf{u}}^{(k)} = \left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \underline{u}_3^{(k)}\right)$ from the iteration process

$$\begin{aligned} A_i \bar{u}_i^{(k)} &= F_i\left(\bar{u}_i^{(k-1)}, \left[\bar{\mathbf{u}}^{(k-1)}\right]_{a_i}, \left[\underline{\mathbf{u}}^{(k-1)}\right]_{b_i}\right) \quad \text{in } D_T, i = 1, 2, 3, \\ A_i \underline{u}_i^{(k)} &= F_i\left(\underline{u}_i^{(k-1)}, \left[\underline{\mathbf{u}}^{(k-1)}\right]_{a_i}, \left[\bar{\mathbf{u}}^{(k-1)}\right]_{b_i}\right) \quad \text{in } D_T, i = 1, 2, 3, \end{aligned}$$

the boundary and initial conditions are given by

$$\begin{aligned} &B\bar{u}_i^{(k)}(t, x) = B\underline{u}_i^{(k)}(t, x) = 0 \quad \text{on } S_T, i = 1, 2, \\ &\bar{u}_i^{(k)}(0, x) = \underline{u}_i^{(k)}(0, x) = u_{i,0}(x) \quad \text{in } \Omega, i = 1, 2, 3. \end{aligned}$$

Before proving the monotone property of the maximal and minimal sequences we state the following positive lemmas which were given in [17].

Positivity Lemma 2.1. *Let $u \in C\left(\bar{D}_T\right) \cap C^{1,2}(D_T)$ be such that*

$$\begin{aligned} &\frac{\partial u}{\partial t} - \alpha \Delta u + \beta u \geq 0 \quad \textit{for all } 0 < t \leq T, x \in \Omega, \\ &\frac{\partial}{\partial \nu} u(t, x) \geq 0 \quad \textit{for all } 0 < t \leq T, x \in \partial\Omega, \\ &u(0, x) \geq 0 \quad \textit{for } x \in \Omega, \end{aligned}$$

where $\alpha > 0$ and $\beta = \beta(t, x)$ is a bounded function in $D_T = (0, T] \times \Omega$. Then $u(t, x) \geq 0$ in D_T.

Moreover $u(t, x) > 0$ in D_T unless it is identically zero in D_T.

Positivity Lemma 2.2. *Let $u \in C\left(\bar{D}_T\right) \cap C^{1,2}(D_T)$ be such that*

$$\begin{aligned} &\frac{\partial u}{\partial t} + \beta u \geq 0 \quad \textit{for } 0 < t \leq T, x \in \Omega, \\ &u(0, x) \geq 0 \quad \textit{for } x \in \Omega, \end{aligned}$$

where $\beta = \beta(t,x)$ *is a bounded function in* $D_T = (0,T] \times \Omega$. *Then* $u(t,x) \geq 0$ *in* D_T.

Moreover $u(t,x) > 0$ *in* D_T *unless it is identically zero in* D_T.

Theorem 2.3. *Suppose that* $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ *and* $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ *are Holder continuous in* x, *uniformly in* $\bar{D}_T$, *and* $u_{1,0}$, $u_{2,0}$ *are Holder continuous on the domain* $\overline{\Omega}$ *satisfying the boundary condition at* $t = 0$ *and* $u_{3,0}$ *is Holder continuous on the domain* $\overline{\Omega}$. *Then the maximal and minimal sequences* $\left\{ \bar{\mathbf{u}}^{(k)} \right\}, \left\{ \underline{\mathbf{u}}^{(k)} \right\}$ *are well-defined on* $\bar{D}_T$, *and they possess the monotone property*

$$\hat{\mathbf{u}} \leq \underline{\mathbf{u}}^{(k)} \leq \underline{\mathbf{u}}^{(k+1)} \leq \bar{\mathbf{u}}^{(k+1)} \leq \bar{\mathbf{u}}^{(k)} \leq \tilde{\mathbf{u}} \quad \text{in } \bar{D}_T \tag{2.1}$$

for every k.

Moreover, for each integer $k, \bar{\mathbf{u}}^{(k)}$ *and* $\underline{\mathbf{u}}^{(k)}$ *are coupled upper and lower solutions of the system* (1.4).

Proof. Since $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ are Holder continuous in x, uniformly in $\bar{D}_T$, we have $F_1\left(\bar{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right), F_1\left(\underline{u}_1^{(0)}, \bar{u}_2^{(0)}, \underline{u}_3^{(0)}\right), F_2\left(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}, \bar{u}_3^{(0)}\right)$ and $F_2\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \underline{u}_3^{(0)}\right)$ are Holder continuous in x, uniformly in $\bar{D}_T$. From Theorem 2.1.2 in [17], the systems

$$\begin{aligned}
&A_1\bar{u}_1^{(1)} = F_1\left(\bar{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right), \qquad A_1\underline{u}_1^{(1)} = F_1\left(\underline{u}_1^{(0)}, \bar{u}_2^{(0)}, \underline{u}_3^{(0)}\right),\\
&A_2\bar{u}_2^{(1)} = F_2\left(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}, \bar{u}_3^{(0)}\right), \qquad A_2\underline{u}_2^{(1)} = F_2\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \underline{u}_3^{(0)}\right),\\
&B\bar{u}_i^{(1)}(t,x) = B\underline{u}_i^{(1)}(t,x) = 0 \quad \text{on } S_T, i = 1,2,\\
&\bar{u}_i^{(1)}(0,x) = \underline{u}_i^{(1)}(0,x) = u_{i,0}(x) \quad \text{in } \Omega, i = 1,2,
\end{aligned}$$

have a unique solution $\bar{u}_1^{(1)}, \underline{u}_1^{(1)}, \bar{u}_2^{(1)}$ and $\underline{u}_2^{(1)}$ on $\bar{D}_T$. Moreover, $\bar{u}_1^{(1)}, \underline{u}_1^{(1)}, \bar{u}_2^{(1)}$ and $\underline{u}_2^{(1)}$ are Holder continuous in x, uniformly in $\bar{D}_T$. Since $\underline{c}_3$ and $F_3\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right)$ are continuous functions on $\bar{D}_T$, for any fixed $x \in \overline{\Omega}$, the linear differential equation

$$A_3\bar{u}_3^{(1)} = F_3\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right), \quad \text{i.e.,} \ \frac{d}{dt}\bar{u}_3^{(1)} + \underline{c}_3\bar{u}_3^{(1)} = F_3\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right),$$

with the initial condition $\bar{u}_3^{(1)}(0,x) = u_{3,0}(x)$ has a unique solution $\bar{u}_3^{(1)}(t,x)$ on $[0,T]$, and

$$\bar{u}_3^{(1)}(t,x) = u_{3,0}(x)\, e^{-\int_0^t \underline{c}_3(\tau,x)d\tau} + e^{-\int_0^t \underline{c}_3(\tau,x)d\tau} \int_0^t e^{\int_0^s \underline{c}_3(\tau,x)d\tau} F_3\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right)(s,x)\, ds.$$

Since $\underline{c}_3$ is a bounded function on $\overline{\Omega}$, and $u_{3,0}$ is Holder continuous on $\overline{\Omega}$, and $F_3\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \bar{u}_3^{(0)}\right)$ is Holder continuous in x, uniformly in $\bar{D}_T$, we have $\bar{u}_3^{(1)}$ is Holder continuous in x, uniformly in $\bar{D}_T$. Similarly, $\underline{u}_3^{(1)}$ exists, and it is Holder continuous in x, uniformly in $\bar{D}_T$,

$$\underline{u}_3^{(1)}(t,x) = u_{3,0}(x)\, e^{-\int_0^t \underline{c}_3(\tau,x)d\tau} + e^{-\int_0^t \underline{c}_3(\tau,x)d\tau} \int_0^t e^{\int_0^s \underline{c}_3(\tau,x)d\tau} F_3\left(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}, \underline{u}_3^{(0)}\right)(s,x)\, ds.$$

Hence $\bar{\mathbf{u}}^{(1)} = \left(\bar{u}_1^{(1)}, \bar{u}_2^{(1)}, \bar{u}_3^{(1)}\right)$ and $\underline{\mathbf{u}}^{(1)} = \left(\underline{u}_1^{(1)}, \underline{u}_2^{(1)}, \underline{u}_3^{(1)}\right)$ are Holder continuous in x, uniformly in $\bar{D}_T$.

Let $\bar{y}_1^{(0)} = \bar{u}_1^{(0)} - \bar{u}_1^{(1)}, \bar{y}_2^{(0)} = \bar{u}_2^{(0)} - \bar{u}_2^{(1)}$, and $\bar{y}_3^{(0)} = \bar{u}_3^{(0)} - \bar{u}_3^{(1)}$. Then

$$\begin{aligned}
A_1\bar{y}_1^{(0)} &= A_1\bar{u}_1^{(0)} - A_1\bar{u}_1^{(1)} = A_1\tilde{u} - F_1(\tilde{u}_1, \hat{u}_2, \tilde{u}_3)\\
&= (\tilde{u}_1)_t - d_1\Delta\tilde{u}_1 - f_1(\tilde{u}_1, \hat{u}_2, \tilde{u}_3) \geq 0,\\
A_2\bar{y}_2^{(0)} &= A_2\bar{u}_2^{(0)} - A_2\bar{u}_2^{(1)} = A_2\tilde{u}_2 - F_2(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\\
&= (\tilde{u}_2)_t - d_2\Delta\tilde{u}_2 - f_2(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \geq 0,
\end{aligned}$$

$$\begin{aligned}
A_3\bar{y}_3^{(0)} &= A_3\bar{u}_3^{(0)} - A_3\bar{u}_3^{(1)} = A_3\tilde{u} - F_3(\hat{u}_1, \hat{u}_2, \tilde{u}_3) \\
&= (\tilde{u}_3)_t - f_3(\hat{u}_1, \hat{u}_2, \tilde{u}_3) \geq 0,
\end{aligned}$$

$$B\bar{y}_i^{(0)}(t,x) = B\bar{u}_i^{(0)}(t,x) - B\bar{u}_i^{(1)}(t,x) = B\tilde{u}_i(t,x) \geq 0 \quad \text{on } S_T, i = 1,2,$$

$$\bar{y}_i^{(0)}(0,x) = \bar{u}_i^{(0)}(0,x) - \bar{u}_i^{(1)}(0,x) = \tilde{u}_i(0,x) - u_{i,0}(x) \geq 0 \quad \text{in } \Omega, i = 1,2,3.$$

By Positivity Lemma, $\bar{y}_1^{(0)} \geq 0, \bar{y}_2^{(0)} \geq 0$, and $\bar{y}_3^{(0)} \geq 0$ on $\bar{D}_T$ which leads to

$$\bar{u}_1^{(0)} \geq \bar{u}_1^{(1)}, \qquad \bar{u}_2^{(0)} \geq \bar{u}_2^{(1)}, \quad \text{and} \quad \bar{u}_3^{(0)} \geq \bar{u}_3^{(1)}.$$

A similar reasoning using the property of a lower solution gives $\underline{u}_1^{(0)} \leq \underline{u}_1^{(1)}, \underline{u}_2^{(0)} \leq \underline{u}_2^{(1)}$, and $\underline{u}_3^{(0)} \leq \underline{u}_3^{(1)}$.

Let $z_1^{(1)} = \bar{u}_1^{(1)} - \underline{u}_1^{(1)}, z_2^{(1)} = \bar{u}_2^{(1)} - \underline{u}_2^{(1)}$, and $z_3^{(1)} = \bar{u}_3^{(1)} - \underline{u}_3^{(1)}$. Since

$$F_i\left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{v}]_{b_i}\right) - F_i\left(v_i, [\mathbf{v}]_{a_i}, [\mathbf{u}]_{b_i}\right) \geq 0$$

for all $\mathbf{u}, \mathbf{v} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ satisfying $\mathbf{u} \geq \mathbf{v}$, and $i = 1,2,3$, this implies that

$$\begin{aligned}
F_1(\xi_1, \eta_2, \tau_1) - F_1(\xi_2, \eta_1, \tau_2) &\geq 0, \\
F_2(\xi_1, \eta_1, \tau_1) - F_2(\xi_2, \eta_2, \tau_2) &\geq 0, \\
F_3(\xi_2, \eta_2, \tau_1) - F_3(\xi_1, \eta_1, \tau_2) &\geq 0,
\end{aligned}$$

for any $(\xi_1, \eta_1, \tau_1), (\xi_2, \eta_2, \tau_2) \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ with $(\xi_1, \eta_1, \tau_1) \geq (\xi_2, \eta_2, \tau_2)$. Hence

$$\begin{aligned}
&A_1 z_1^{(1)} = A_1\bar{u}_1^{(1)} - A_1\underline{u}_1^{(1)} = F_1(\tilde{u}_1, \hat{u}_2, \tilde{u}_3) - F_1(\hat{u}_1, \tilde{u}_2, \hat{u}_3) \geq 0, \\
&A_2 z_2^{(1)} = A_2\bar{u}_2^{(1)} - A_2\underline{u}_2^{(1)} = F_2(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) - F_2(\hat{u}_1, \hat{u}_2, \hat{u}_3) \geq 0, \\
&A_3 z_3^{(1)} = A_3\bar{u}_3^{(1)} - A_3\underline{u}_3^{(1)} = F_3(\hat{u}_1, \hat{u}_2, \tilde{u}_3) - F_3(\tilde{u}_1, \tilde{u}_2, \hat{u}_3) \geq 0, \\
&B{z_i}^{(1)}(t,x) = B\bar{u}_i^{(1)}(t,x) - B\underline{u}_i^{(1)}(t,x) = 0 \quad \text{on } S_T, i = 1,2, \\
&{z_i}^{(1)}(0,x) = \bar{u}_i^{(1)}(0,x) - \underline{u}_i^{(1)}(0,x) = u_{i,0}(x) - u_{i,0}(x) = 0 \quad \text{in } \Omega, i = 1,2,3.
\end{aligned}$$

By Positivity Lemma, $z_1^{(1)} \geq 0, z_2^{(1)} \geq 0$, and $z_3^{(1)} \geq 0$ on $\bar{D}_T$ which leads to

$$\bar{u}_1^{(1)} \geq \underline{u}_1^{(1)}, \qquad \bar{u}_2^{(1)} \geq \underline{u}_2^{(1)}, \quad \text{and} \quad \bar{u}_3^{(1)} \geq \underline{u}_3^{(1)}.$$

This shows that $\underline{\mathbf{u}}^{(0)} \leq \underline{\mathbf{u}}^{(1)} \leq \bar{\mathbf{u}}^{(1)} \leq \bar{\mathbf{u}}^{(0)}$, that is,

$$\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}, \underline{u}_3^{(0)}\right) \leq \left(\underline{u}_1^{(1)}, \underline{u}_2^{(1)}, \underline{u}_3^{(1)}\right) \leq \left(\bar{u}_1^{(1)}, \bar{u}_2^{(1)}, \bar{u}_3^{(1)}\right) \leq \left(\bar{u}_1^{(0)}, \bar{u}_2^{(0)}, \bar{u}_3^{(0)}\right).$$

The existence of maximal and minimal sequences $\bar{\mathbf{u}}^{(k)}, \underline{\mathbf{u}}^{(k)}$ and the monotone property (2.1) follow by an induction argument. The assertion of this theorem seems to be true.

In view of the monotone property the pointwise (and componentwise) limits

$$\lim_{k\to\infty} \bar{\mathbf{u}}^{(k)}(t,x) = \bar{\mathbf{u}}(t,x), \quad \text{and} \quad \lim_{k\to\infty} \underline{\mathbf{u}}^{(k)}(t,x) = \underline{\mathbf{u}}(t,x) \tag{2.2}$$

exist and satisfy the relation $\hat{\mathbf{u}} \leq \underline{\mathbf{u}} \leq \bar{\mathbf{u}} \leq \tilde{\mathbf{u}}$ in $\bar{D}_T$. To show that the system (1.4) has a unique solution in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ we should prove that $\bar{\mathbf{u}} = \underline{\mathbf{u}}$ in D_T.

Theorem 2.4. *Let $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ be coupled upper and lower solutions of the system (1.4). Then there exists a unique solution $\mathbf{u}^*$ to the system (1.4) and $\mathbf{u}^* \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.*

Moreover, the sequences $\bar{\mathbf{u}}^{(k)}, \underline{\mathbf{u}}^{(k)}$ given by the iteration process with initial iterations $\bar{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ and $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ both converge monotonically to $\mathbf{u}^*$.

Proof. By (2.2) and the continuity of F_i $(i = 1, 2, 3)$, the pointwise limits

$$\begin{aligned}
\lim_{k\to\infty} F_1\left(\bar{u}_1^{(k)}, \underline{u}_2^{(k)}, \bar{u}_3^{(k)}\right) &= F_1\left(\bar{u}_1, \underline{u}_2, \bar{u}_3\right),\\
\lim_{k\to\infty} F_1\left(\underline{u}_1^{(k)}, \bar{u}_2^{(k)}, \underline{u}_3^{(k)}\right) &= F_1\left(\underline{u}_1, \bar{u}_2, \underline{u}_3\right),\\
\lim_{k\to\infty} F_2\left(\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \bar{u}_3^{(k)}\right) &= F_2\left(\bar{u}_1, \bar{u}_2, \bar{u}_3\right),\\
\lim_{k\to\infty} F_2\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \underline{u}_3^{(k)}\right) &= F_2\left(\underline{u}_1, \underline{u}_2, \underline{u}_3\right),\\
\lim_{k\to\infty} F_3\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \bar{u}_3^{(k)}\right) &= F_3\left(\underline{u}_1, \underline{u}_2, \bar{u}_3\right),\\
\lim_{k\to\infty} F_3\left(\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \underline{u}_3^{(k)}\right) &= F_3\left(\bar{u}_1, \bar{u}_2, \underline{u}_3\right)
\end{aligned}$$

exist. Using the same regularity argument as in the proof of Theorem 8.3.1 in [17] the limits $\bar{u}_1, \underline{u}_1, \bar{u}_2, \underline{u}_2$ satisfy the coupled equations

$$\begin{aligned}
\left(\bar{u}_1\right)_t - d_1 \Delta \bar{u}_1 &= f_1\left(\bar{u}_1, \underline{u}_2, \bar{u}_3\right), \qquad \left(\underline{u}_1\right)_t - d_1 \Delta \underline{u}_1 = f_1\left(\underline{u}_1, \bar{u}_2, \underline{u}_3\right),\\
\left(\bar{u}_2\right)_t - d_2 \Delta \bar{u}_2 &= f_2\left(\bar{u}_1, \bar{u}_2, \bar{u}_3\right), \qquad \left(\underline{u}_2\right)_t - d_2 \Delta \underline{u}_2 = f_2\left(\underline{u}_1, \underline{u}_2, \underline{u}_3\right)
\end{aligned}$$

and the boundary and initial conditions

$$\begin{aligned}
&\frac{\partial}{\partial \nu}\bar{u}_i(t,x) = \frac{\partial}{\partial \nu}\underline{u}_i(t,x) = 0 \quad \text{for all } 0 < t \le T, x \in \partial\Omega, i = 1, 2,\\
&\bar{u}_i(0,x) = \underline{u}_i(0,x) = u_{i,0}(x) \quad \text{for all } x \in \Omega, i = 1, 2.
\end{aligned}$$

Since F_3 satisfies the Lipschitz condition on $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, we have

$$\lim_{k\to\infty} F_3\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \bar{u}_3^{(k)}\right) = F_3\left(\underline{u}_1, \underline{u}_2, \bar{u}_3\right) \quad \text{uniformly on } \bar{D}_T,$$

and so

$$\bar{u}_3(t,x) = u_{3,0}(x)\, e^{-\int_0^t \underline{c}_3(\tau,x)d\tau} + e^{-\int_0^t \underline{c}_3(\tau,x)d\tau} \int_0^t e^{\int_0^s \underline{c}_3(\tau,x)d\tau} F_3\left(\underline{u}_1, \underline{u}_2, \bar{u}_3\right)(s,x)\, ds.$$

This implies that $\left(\bar{u}_3\right)_t = f_3\left(\underline{u}_1, \underline{u}_2, \bar{u}_3\right)$, and $\bar{u}_3(0,x) = u_{3,0}(x)$ for all $x \in \Omega$. Similarly, $\left(\underline{u}_3\right)_t = f_3\left(\bar{u}_1, \bar{u}_2, \underline{u}_3\right)$, and $\underline{u}_3(0,x) = u_{3,0}(x)$ for all $x \in \Omega$.

Since Ω is a bounded open set in $\mathbb{R}^n$ and $\partial\Omega$ is a C^1-class, the operator $L_1 = d_1\Delta$ is the generator of a C_0-semigroup $S_1(t)$ of contractions on $L^2(\Omega)$, and the domain of L_1 is

$$D\left(L_1\right) = \left\{ u \in H^2(\Omega) : \frac{\partial}{\partial \nu} u = 0 \quad \text{on } \partial\Omega \right\}.$$

(See Theorem 4.2.2 in [20].) Let $\bar{u}_i(t)(x) = \bar{u}_i(t,x)$ and $\underline{u}_i(t)(x) = \underline{u}_i(t,x)$ for all $t \in [0,T]$ and $x \in \bar{\Omega}$. Suppose that $g(s) = S_1(t-s)\bar{u}_1(s)$ for all $0 \le s \le t$. Since $\frac{\partial}{\partial \nu}\bar{u}_1(t,x) = 0$ for all $t > 0, x \in \partial\Omega$, and $\bar{u}_1(t,\cdot) \in H^2(\Omega)$, we have $\bar{u}_1(t) \in D\left(L_1\right)$ and

$$\begin{aligned}
\frac{d}{ds}g(s) &= \frac{d}{ds}S_1(t-s)\bar{u}_1(s) + S_1(t-s)\frac{d}{ds}\bar{u}_1(s)\\
&= -L_1 S_1(t-s)\bar{u}_1(s) + S_1(t-s)L_1\bar{u}_1(s) + S_1(t-s) f_1\left(\bar{u}_1(s), \underline{u}_2(s), \bar{u}_3(s)\right)\\
&= S_1(t-s) f_1\left(\bar{u}_1(s), \underline{u}_2(s), \bar{u}_3(s)\right).
\end{aligned}$$

Therefore,

$$g(t) - g(0) = \int_0^t S_1(t-s) f_1\left(\bar{u}_1(s), \underline{u}_2(s), \bar{u}_3(s)\right) ds$$

and so

$$\bar{u}_1(t) = S_1(t)\, u_{1,0} + \int_0^t S_1(t-s)\, f_1(\bar{u}_1(s), \underline{u}_2(s), \bar{u}_3(s))\, ds.$$

Similarly, we obtain that

$$\begin{aligned}
\underline{u}_1(t) &= S_1(t)\, u_{1,0} + \int_0^t S_1(t-s)\, f_1(\underline{u}_1(s), \bar{u}_2(s), \underline{u}_3(s))\, ds, \\
\bar{u}_2(t) &= S_2(t)\, u_{2,0} + \int_0^t S_2(t-s)\, f_2(\bar{u}_1(s), \bar{u}_2(s), \bar{u}_3(s))\, ds, \\
\underline{u}_2(t) &= S_2(t)\, u_{2,0} + \int_0^t S_2(t-s)\, f_2(\underline{u}_1(s), \underline{u}_2(s), \underline{u}_3(s))\, ds, \\
\bar{u}_3(t) &= u_{3,0} + \int_0^t f_3(\underline{u}_1(s), \underline{u}_2(s), \bar{u}_3(s))\, ds, \\
\underline{u}_3(t) &= u_{3,0} + \int_0^t f_3(\bar{u}_1(s), \bar{u}_2(s), \underline{u}_3(s))\, ds
\end{aligned}$$

where the operator $L_2 = d_2 \Delta$ is the generator of a C_0-semigroup $S_2(t)$ of contractions on $L^2(\Omega)$, and the domain of L_2 is

$$D(L_2) = \left\{ u \in H^2(\Omega) : \frac{\partial}{\partial \nu} u = 0 \text{ on } \partial\Omega \right\}.$$

Hence

$$\begin{aligned}
\| \bar{u}_1(t) - \underline{u}_1(t) \| &\le \int_0^t \| S_1(t-s) \| \, \| f_1(\bar{u}_1(s), \underline{u}_2(s), \bar{u}_3(s)) - f_1(\underline{u}_1(s), \bar{u}_2(s), \underline{u}_3(s)) \| \, ds \\
&\le \bar{K}_1 \int_0^t (\| \bar{u}_1(s) - \underline{u}_1(s) \| + \| \bar{u}_2(s) - \underline{u}_2(s) \| + \| \bar{u}_3(s) - \underline{u}_3(s) \|)\, ds,
\end{aligned}$$

where $\bar{K}_1 = \max_{(t,x) \in \bar{D}_T} K_1(t,x)$. Similarly, we obtain that

$$\| \bar{u}_2(t) - \underline{u}_2(t) \| \le \bar{K}_2 \int_0^t (\| \bar{u}_1(s) - \underline{u}_1(s) \| + \| \bar{u}_2(s) - \underline{u}_2(s) \| + \| \bar{u}_3(s) - \underline{u}_3(s) \|)\, ds,$$

and

$$\| \bar{u}_3(t) - \underline{u}_3(t) \| \le \bar{K}_3 \int_0^t (\| \bar{u}_1(s) - \underline{u}_1(s) \| + \| \bar{u}_2(s) - \underline{u}_2(s) \| + \| \bar{u}_3(s) - \underline{u}_3(s) \|)\, ds$$

where $\bar{K}_2 = \max_{(t,x) \in \bar{D}_T} K_2(t,x)$ and $\bar{K}_3 = \max_{(t,x) \in \bar{D}_T} K_3(t,x)$. This implies that

$$\begin{aligned}
\| \bar{u}_1(t) - \underline{u}_1(t) \| + \| \bar{u}_2(t) - \underline{u}_2(t) \| + \| \bar{u}_3(t) - \underline{u}_3(t) \| &\le \left(\bar{K}_1 + \bar{K}_2 + \bar{K}_3 \right) \int_0^t (\| \bar{u}_1(s) - \underline{u}_1(s) \| \\
&\quad + \| \bar{u}_2(s) - \underline{u}_2(s) \| + \| \bar{u}_3(s) - \underline{u}_3(s) \|) ds.
\end{aligned}$$

Hence by Gronwall's inequality,

$$\| \bar{u}_1(t) - \underline{u}_1(t) \| + \| \bar{u}_2(t) - \underline{u}_2(t) \| + \| \bar{u}_3(t) - \underline{u}_3(t) \| = 0.$$

Hence $\bar{u}_1(t) = \underline{u}_1(t) \in H^2(\Omega), \bar{u}_2(t) = \underline{u}_2(t) \in H^2(\Omega), \bar{u}_3(t) = \underline{u}_3(t) \in L^2(\Omega)$ for all $t \in [0,T]$. Since $\bar{u}_1, \underline{u}_1, \bar{u}_2, \underline{u}_2, \bar{u}_3, \underline{u}_3 \in C\left(\bar{D}_T\right)$, we have

$$\bar{u}_1(t,x) = \underline{u}_1(t,x), \bar{u}_2(t,x) = \underline{u}_2(t,x), \bar{u}_3(t,x) = \underline{u}_3(t,x)$$

for all $t \in [0,T]$ and $x \in \bar{\Omega}$. The proof of this theorem is completed.

3. Existence and uniqueness of problem (1.2)

First, we prove that the existence of the upper and lower solutions of the problem (1.2) on $\bar{D}_T = [0,T] \times \bar{\Omega}$. It implies that the existence of the solution of the problem (1.2) on $\bar{D}_T$. Next, we prove that the uniqueness of the solution of the problem (1.2) on $\bar{D}_T$. Therefore, there exists a unique solution of the problem (1.2) on $\bar{D}_T$, where T is an arbitrary positive number.

Theorem 3.1. *Suppose that constants $\varepsilon, \alpha, \beta, M, N$ and K satisfy*

$$0 < \varepsilon \le \min_{x\in\bar{\Omega}} u_{3,0}(x), \qquad \alpha \ge \| u_{3,0} \|_\infty, \qquad \beta \ge c, \qquad M \ge \max\left\{ \| u_{1,0} \|_\infty ,\ p\alpha e^{\beta T} \right\},$$
$$N \ge \max\left\{ \| u_{2,0} \|_\infty ,\ \frac{r_2 + bM}{r_2} q\alpha e^{\beta T},\ \frac{1}{e}(c - dM) \right\} \quad \text{and} K \ge dM + eN - c.$$

Then a pair of functions $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \left(M, N, \alpha e^{\beta t}\right), (\hat{u}_1, \hat{u}_2, \hat{u}_3) = \left(0, 0, \varepsilon e^{-Kt}\right)$ are coupled upper and lower solutions of the system (1.4) on $\bar{D}_T$.

Moreover, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ are Holder continuous in x, uniformly in $\bar{D}_T$.

Proof. Since $\tilde{u}_1 = M, \hat{u}_1 = 0, \tilde{u}_2 = N$ and $\hat{u}_2 = 0$, we have

$$\hat{u}_1(t,x) = 0 \le M = \tilde{u}_1(t,x), \quad \text{and} \quad \hat{u}_2(t,x) = 0 \le N = \tilde{u}_2(t,x).$$

Since $N \ge (c - dM)/e$, we have $K \ge dM + eN - c \ge 0$ and so

$$\hat{u}_3(t,x) = \varepsilon e^{-Kt} \le \varepsilon \le \min_{x\in\bar{\Omega}} u_{3,0}(x) \le \max_{x\in\bar{\Omega}} u_{3,0}(x) \le \alpha \le \alpha e^{\beta t} = \tilde{u}_3(t,x).$$

Hence $(\hat{u}_1, \hat{u}_2, \hat{u}_3) \le (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$.

Since $M \ge p\alpha e^{\beta T}$, we have

$$\begin{aligned}
(\tilde{u}_1)_t - d_1 \Delta \tilde{u}_1 - f_1(\tilde{u}_1, \hat{u}_2, \tilde{u}_3) &= -r_1 M + \frac{r_1}{p} M^2 \left(\frac{1}{\alpha} e^{-\beta t} \right) \\
&= r_1 M \left(-1 + \frac{1}{p\alpha} e^{-\beta t} M \right) \\
&\ge r_1 M \left(-1 + \frac{1}{p\alpha} e^{-\beta T} M \right) \\
&\ge r_1 M \left(-1 + \frac{1}{p\alpha} e^{-\beta T} \left(p\alpha e^{\beta T} \right) \right) \\
&= 0.
\end{aligned}$$

Since $\hat{u}_1 = 0$ and $f_1(u_1, u_2, u_3) = r_1 u_1 - \left(r_1 u_1^2 / p u_3 \right) - a u_1 u_2$, we have

$$(\hat{u}_1)_t - d_1 \Delta \hat{u}_1 - f_1(\hat{u}_1, \tilde{u}_2, \hat{u}_3) = 0.$$

Since $N \ge ((r_2 + bM)/r_2) q\alpha e^{\beta T}$, we have

$$\begin{aligned}
(\tilde{u}_2)_t - d_2 \Delta \tilde{u}_2 - f_2(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) &= -r_2 N + \frac{r_2}{q} N^2 \left(\frac{1}{\alpha} e^{-\beta t} \right) - bMN \\
&\ge N \left(-r_2 + \frac{r_2}{q\alpha} e^{-\beta T} N - bM \right) \\
&\ge N \left(-r_2 + \frac{r_2}{q\alpha} e^{-\beta T} \frac{r_2 + bM}{r_2} q\alpha e^{\beta T} - bM \right) \\
&= N(-r_2 + (r_2 + bM) - bM) \\
&= 0.
\end{aligned}$$

Since $\hat{u}_2 = 0$ and $f_2(u_1, u_2, u_3) = r_2 u_2 - \left(r_2 u_2^2 / q u_3\right) + b u_1 u_2$, we have

$$(\hat{u}_2)_t - d_2 \Delta \hat{u}_2 - f_2(\hat{u}_1, \hat{u}_2, \hat{u}_3) = 0.$$

Since $\beta \geq c$, we have

$$(\tilde{u}_3)_t - f_3(\hat{u}_1, \hat{u}_2, \tilde{u}_3) = \alpha\beta e^{\beta t} - c\alpha e^{\beta t} = (\beta - c)\,\alpha e^{\beta t} \geq 0.$$

Since $N \geq (c - dM)/e$ and $K \geq dM + eN - c \geq 0$, we have

$$\begin{aligned}(\hat{u}_3)_t - f_3(\tilde{u}_1, \tilde{u}_2, \hat{u}_3) &= -\varepsilon K e^{-Kt} - c\varepsilon e^{-Kt} + dM\varepsilon e^{-Kt} + eN\varepsilon e^{-Kt}\\ &= (-K - c + dM + eN)\,\varepsilon e^{-Kt}\\ &\leq (-(dM + eN - c) - c + dM + eN)\,\varepsilon e^{-Kt}\\ &= 0.\end{aligned}$$

Since $\tilde{u}_1 = M, \hat{u}_1 = 0, \tilde{u}_2 = N$ and $\hat{u}_2 = 0$, we have

$$\frac{\partial}{\partial \nu}\hat{u}_1(t,x) = \frac{\partial}{\partial \nu}\tilde{u}_1(t,x) = 0, \qquad \frac{\partial}{\partial \nu}\hat{u}_2(t,x) = \frac{\partial}{\partial \nu}\tilde{u}_2(t,x) = 0$$

for all $t > 0, x \in \partial\Omega$. Since $M \geq \| u_{1,0} \|_\infty = \max\left\{ u_{1,0}(x) : x \in \bar{\Omega} \right\}$ and $\hat{u}_1 = 0$, we have

$$\hat{u}_1(0,x) = 0 \leq u_{1,0}(x) \leq \max\left\{ u_{1,0}(x) : x \in \bar{\Omega} \right\} \leq M = \tilde{u}_1(0,x).$$

Since $N \geq \| u_{2,0} \|_\infty = \max\left\{ u_{2,0}(x) : x \in \bar{\Omega} \right\}$ and $\hat{u}_2 = 0$, we have

$$\hat{u}_2(0,x) = 0 \leq u_{2,0}(x) \leq \max\left\{ u_{2,0}(x) : x \in \bar{\Omega} \right\} \leq N = \tilde{u}_2(0,x).$$

Since $\alpha \geq \| u_{3,0} \|_\infty = \max\left\{ u_{3,0}\ (x) : x \in \bar{\Omega} \right\}$ and $0 < \varepsilon \leq \min\left\{ u_{3,0}(x) : x \in \bar{\Omega} \right\}$, we have

$$\hat{u}_3(0,x) = \varepsilon \leq \min_{x \in \bar{\Omega}} u_{3,0}(x) \leq u_{3,0}(x) \leq \max_{x \in \bar{\Omega}} u_{3,0}(x) \leq \alpha = \tilde{u}_3(0,x).$$

Hence a pair of functions $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \left(M, N, \alpha e^{\beta t}\right), (\hat{u}_1, \hat{u}_2, \hat{u}_3) = \left(0, 0, \varepsilon e^{-Kt}\right)$ are coupled upper and lower solutions of the system (1.4) on $\bar{D}_T$. The proof of this theorem is completed.

Now, we prove the limit of the maximal and minimal sequences $\bar{u}^{(k)} = \left(\bar{u}_1^{(k)}, \bar{u}_2^{(k)}, \bar{u}_3^{(k)}\right), \underline{\mathbf{u}}^{(k)} = \left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \underline{u}_3^{(k)}\right)$ with initial iterations $\bar{\mathbf{u}}^{(0)} = \left(M, N, \alpha e^{\beta t}\right)$ and $\underline{\mathbf{u}}^{(0)} = \left(0, 0, \varepsilon e^{-Kt}\right)$ is a unique solution of the system (1.4) on $[0,T] \times \bar{\Omega}$.

Theorem 3.2. *The system (1.4) has a unique solution (u_1, u_2, u_3) on $[0,T] \times \bar{\Omega}$, and*

$$\left(0, 0, \varepsilon e^{-Kt}\right) \leq (u_1, u_2, u_3) \leq \left(M, N, \alpha e^{\beta t}\right),$$

where constants $\varepsilon, \alpha, \beta, M, N$ and K are defined as in Theorem 3.1.

Proof. Suppose that (u_1, u_2, u_3) and (v_1, v_2, v_3) are solutions of the system (1.4) on $[0,T] \times \bar{\Omega}$. Then there are positive numbers ε_0 and M_0 such that

$$(0, 0, \varepsilon_0) \leq (u_1, u_2, u_3), (v_1, v_2, v_3) \leq (M_0, M_0, M_0).$$

Moreover, there are constants $\tilde{K}_i, i = 1, 2, 3$, such that

$$| f_i(u_1, u_2, u_3) - f_i(v_1, v_2, v_3) | \leq \tilde{K}_i (| u_1 - v_1 | + | u_2 - v_2 | + | u_3 - v_3 |).$$

Since

$$\begin{aligned}&(u_1)_t - d_1 \Delta u_1 = f_1(u_1, u_2, u_3), \qquad (v_1)_t - d_1 \Delta v_1 = f_1(v_1, v_2, v_3),\\ &(u_2)_t - d_2 \Delta u_2 = f_2(u_1, u_2, u_3), \qquad (v_2)_t - d_2 \Delta v_2 = f_2(v_1, v_2, v_3),\\ &(u_3)_t = f_3(u_1, u_2, u_3), \qquad (v_3)_t = f_3(v_1, v_2, v_3),\end{aligned}$$

and the boundary and initial conditions

$$\frac{\partial}{\partial \nu}u_i(t,x) = \frac{\partial}{\partial \nu}v_i(t,x) = 0 \quad \text{for all } t>0, x\in\partial\Omega, i=1,2,$$
$$u_i(0,x) = v_i(0,x) = u_{i,0}(x) \quad \text{for all } x\in\Omega, i=1,2,3.$$

Let $u_i(t)(x) = u_i(t,x)$ and $v_i(t)(x) = v_i(t,x)$ for all $t\in[0,T]$ and $x\in\bar{\Omega}$. By similar proof of Theorem 2.4, we obtain

$$\begin{aligned} &\| u_1(t) - v_1(t) \| + \| u_2(t) - v_2(t) \| + \| u_3(t) - v_3(t) \| \\ &\quad \le (\tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3) \int_0^t (\| u_1(s) - v_1(s) \| + \| u_2(s) - v_2(s) \| + \| u_3(s) - v_3(s) \|) ds. \end{aligned}$$

Hence by Gronwall's inequality, we have $u_i(t,x) = v_i(t,x), i=1,2,3$, for all $t\in[0,T]$ and $x\in\bar{\Omega}$. From Theorems 2.4 and 3.1, we obtain the system (1.4) has a unique solution (u_1,u_2,u_3) on $[0,T]\times\bar{\Omega}$, and

$$\left(0,0,\varepsilon e^{-Kt}\right) \le (u_1,u_2,u_3) \le \left(M,N,\alpha e^{\beta t}\right).$$

From [19], the operator $L_i = d_i\Delta$ is the generator of a C_0-semigroup on $L^2(\Omega)$, and the domain of L_i is

$$D(L_i) = \left\{ u\in H^2(\Omega) : u=0 \text{ on } \partial\Omega \right\},$$

for every $i=1,2$. Using the same arguments, we can obtain the following theorem.

Theorem 3.3. *The nonlinear partial differential equations in the problem (1.2) with Dirichlet boundary condition*

$$u(t,x)=0,\ v(t,x)=0 \quad \textit{for all } t>0, x\in\partial\Omega,$$

has a unique solution (u,v,w) on $[0,T]\times\overline{\Omega}$, where T is an arbitrary positive number.

Moreover, one may have that

$$\left(0,0,\varepsilon e^{-Kt}\right) \le (u,v,w) \le \left(M,N,\alpha e^{\beta t}\right) \quad \textit{on } [0,T]\times\bar{\Omega},$$

where constants $\varepsilon,\alpha,\beta,M,N$ and K satisfy

$$0<\varepsilon\le\min_{x\in\bar{\Omega}} w_0(x), \qquad \alpha\ge\| w_0 \|_\infty, \qquad \beta\ge c, \qquad M\ge\max\left\{ \| u_0 \|_\infty,\ p\alpha e^{\beta T} \right\},$$
$$N\ge\max\left\{ \| v_0 \|_\infty,\ \frac{r_2+bM}{r_2}q\alpha e^{\beta T},\ \frac{1}{e}(c-dM) \right\} \quad \textit{and} \quad K\ge dM+eN-c.$$

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