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Research paper

A new fractional wavelet transform

Hongzhe Dai*, Zhibao Zheng, Wei Wang

School of Civil Engineering, Harbin Institute of Technology, Harbin, China

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ABSTRACT

The fractional Fourier transform (FRFT) is a potent tool to analyze the time-varying signal. However, it fails in locating the fractional Fourier domain (FRFD)-frequency contents which is required in some applications. A novel fractional wavelet transform (FRWT) is proposed to solve this problem. It displays the time and FRFD-frequency information jointly in the time-FRFD-frequency plane. The definition, basic properties, inverse transform and reproducing kernel of the proposed FRWT are considered. It has been shown that an FRWT with proper order corresponds to the classical wavelet transform (WT). The multiresolution analysis (MRA) associated with the developed FRWT, together with the construction of the orthogonal fractional wavelets are also presented. Three applications are discussed: the analysis of signal with time-varying frequency content, the FRFD spectrum estimation of signals that involving noise, and the construction of fractional Harr wavelet. Simulations verify the validity of the proposed FRWT.

1. Introduction

Fourier transform (FT) is one of the most valuable and frequently used tools in signal processing and analysis. For FT, a signal can be represented either in the time or in the frequency domain, and it can be viewed as the time-frequency representation of a signal. However, the Fourier coefficients define the average spectral content over the entire duration of the signal and it does not give any information about the occurrence of the frequency component at a particular time and is not applicable for non-stationary signals. A straightforward approach to overcome this problem is to perform the FT on a block by block basis rather than to process the entire signal at once, which is named the short-time Fourier transform (STFT) [1]. Although STFT has rectified almost all the limitations of FT, but still in some cases STFT is also not applicable as in the case of real signals having low frequencies of long duration and high frequencies of short duration. Such signals could be better described by a transform which has a high time resolution for short-lived high-frequency phenomena, and has high frequency resolution for long-lasting low-frequency phenomena. In these types of situations, wavelet transform (WT) can provide a better description of the signal instead of the STFT. Since wavelets have special ability to analyze signal in both time and frequency domain simultaneously, and can easily detect the local properties of a signal, WT is widely used to analyze transient and non-stationary signals.

Recently, researchers have come up with the new transform namely fractional Fourier transform (FRFT). FRFT is the generalization of FT since it can be viewed as the rotation through an angle α of FT. Like FT corresponds to a rotation in the time-frequency plane over an angle $\alpha = \pi/2$, the FRFT corresponds to a rotation over an arbitrary angle $\alpha = p \times \pi/2$ with $p \in \mathbb{R}$. That is to say, FRFT is the representation of a signal in the fractional Fourier domain (FRFD) [2]. Although the

* Corresponding author. Fax: 86-451-86282704.
E-mail address: hzdai@hit.edu.cn (H. Dai).

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FRFT has a number of attractive properties, the fractional Fourier representation of a signal only provides overall FRFD-frequency content with no indication about the occurrence of the FRFD spectral component at a particular time. Since the FRFT uses a global kernel like FT, it fails in locating the FRFD spectral contents which is required in some applications. So the representation combining the time and FRFD-frequency information should be developed which is termed the time-FRFD-frequency representation (TFFR). The short-time FRFT (STFRFT) is one of such approaches. The idea is segmenting the signal by using a time-localized window and performing FRFT spectral analysis for each segment [3]. Since the FRFT is computed for each windowed segment of the signal, the STFRFT can provide the time and FRFD spectral information jointly in the time-FRFD plane. Another kind of TFFR method is the fractional WT (FRWT). The concept of FRWT was initially proposed in [4], where FRFT is firstly used to derive the fractional spectrum of a signal and WT is then performed on the obtained fractional spectrum. Since the fractional spectrum derived by the FRFT only represents the FRFD-frequency over the entire duration of the signal, the FRWT defined in [4] actually fails in obtaining the information of the local property of the signal. In [5], a fractional wave packet transform was developed and the basic idea is to introduce the wavelet basis function to FRFT. More recently, a new FRWT was proposed in [6] based on the concept of fractional convolution. The multiresolution analysis (MRA) associated with this kind of FRWT was then given in [7] by the same authors. Since this kind of FRWT analyze the signal in time-frequency-FRFD domain, its physical meaning requires deeper interpretation. Another kind of FRWT which was developed in [8] solves the issue in [6] since the analysis only involves time-FRFD domain. However, the MRA associated with this kind of FRWT is not addressed. In [9], WT and FRWT are, respectively, used for the simultaneous spectral analysis of a binary mixture system.

The purpose of this paper is to define a new type of FRWT that has more elegant mathematical properties and is more general than the transforms defined in [6] and [8]. The new FRWT displays the time and FRFD-frequency information jointly in the time-FRFD-frequency plane. The remainder of the paper is organized as follows. In Section 2, preliminaries about FRFT and WT are given. In Section 3, the theoretical framework of FRWT is established, including its definition, properties, inverse transform and reproducing kernel equation. In Section 4, the MRA associated with the developed FRWT, together with the construction of the orthogonal fractional wavelets are described. Three applications are discussed in Section 5, including the process of non-stationary signal and the construction of fractional Harr wavelets. The last section concludes this paper and presents its future directions.

2. Preliminaries

2.1. Fractional fourier transform

Mathematically, the α-order FRFT of a signal $x(t) \in L^2(\mathbb{R})$ is defined as

$$X_\alpha(u) = F^\alpha[x(t)] = \int_{-\infty}^{+\infty} x(t) K_\alpha(t,u)dt \tag{1}$$

where the transform kernel is given by

$$K_\alpha(t,u) = \begin{cases} A_\alpha e^{\frac{j}{2}(t^2+u^2)\cot\alpha - jtu\csc\alpha}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k+1)\pi \end{cases} \tag{2}$$

and A_α is given by

$$A_\alpha = \sqrt{\frac{1-j\cot\alpha}{2\pi}} \tag{3}$$

The inverse FRFT is

$$x(t) = \int_{-\infty}^{+\infty} X_\alpha(u) K_\alpha^*(t,u)du \tag{4}$$

The definition implies that the FRFT is the decomposition into the chirp bases $\{K_\alpha^*(t,u)\}$. So a proper order FRFT of the chirp signal is an impulse. The argument u represents a new physical quantity extended from the frequency concept and is termed the FRFD-frequency, so the FRFT can also be interpreted as the FRFD-spectrum [10]. The Parseval identity of FRFT, which will be useful in this paper, is given by

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X_\alpha(u)Y_\alpha^*(u)du \tag{5}$$

2.2. Wavelet transform

The continuous wavelet transform of a signal $x(t) \in L^2(\mathbb{R})$ is defined as

$$WT_x(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\varphi^*\left(\frac{t-b}{a}\right)dt \tag{6}$$

where $\varphi(t)$ is the mother wavelet, $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ are scale and shift of the mother wavelet along the t domain. Both a and b are continuous parameters within the respective domain. Time and frequency localization properties of the wavelet transform depend on the value of the scale a. As a approaches zero or $+\infty$, the compressed or dilated wavelet $\varphi_{a,b}(t) = a^{-1/2}\varphi((t-b)/a)$ provides increasingly sharper or coarser time resolution, with the corresponding $WT_x(a, b)$ displaying the small-scale/high- frequency or large-scale/low-frequency features of $x(t)$, at various locations b. From the set of coefficients $WT_x(a, b)$ the signal $x(t)$ can be reconstructed in the form

$$x(t) = \frac{1}{\pi C_\varphi}\int_0^\infty a^{-2}\int_{-\infty}^{\infty} WT_x(a,b)\varphi_{a,b}(t)dadb \tag{7}$$

with

$$C_\varphi = \int_0^\infty |\omega|^{-1}|\Phi(\omega)|^2 d\omega < \infty \tag{8}$$

where $\Phi(\omega)$ is the FT of $\varphi(t)$. Notice that Eq. (8) corresponds to a certain number of sub-conditions on $\varphi(t)$, as detailed in [11].

3. The proposed fractional wavelet transform

3.1. Definition of the new FRWT

Motivated by the basic properties of FRFT and WT, the α-order FRWT of a signal $x(t) \in L^2(\mathbb{R})$is defined as

$$\begin{aligned} W_x^\alpha(a,b) &= e^{-\frac{j}{2}b^2\cot\alpha}\left\{\left[x(t)e^{\frac{j}{2}t^2\cot\alpha}\right] * \left[\frac{1}{\sqrt{a}}\varphi\left(-\frac{t}{a}\right)e^{\frac{j}{2}\left(\frac{t}{a}\right)^2\cot\alpha}\right]^*\right\} \\ &= e^{-\frac{j}{2}b^2\cot\alpha}\left\langle x(t)e^{\frac{j}{2}t^2\cot\alpha}, \varphi_{a,b}(t)e^{\frac{j}{2}\left(\frac{t-b}{a}\right)^2\cot\alpha}\right\rangle = \int_{-\infty}^{+\infty} x(t)\varphi^*_{\alpha,a,b}(t)dt \end{aligned} \tag{9}$$

where the α-order fractional wavelet $\varphi_{\alpha,\,a,\,b}(t)$ is defined by multiplying the conventional wavelet $\varphi_{a,\,b}(t)$ with a chirp as

$$\varphi_{\alpha,a,b}(t) = e^{-\frac{j}{2}\left(t^2-b^2-\left(\frac{t-b}{a}\right)^2\right)\cot\alpha}\varphi_{a,b}(t) \tag{10}$$

Thus, the FRWT as described in Eq. (9) can be rewritten as

$$W_x^\alpha(a,b) = \langle x(t), \varphi_{\alpha,a,b}(t)\rangle \tag{11}$$

Clearly, the new FRWT is the inner product of the signal $x(t)$ and the α-order fractional wavelet $\varphi_{\alpha,\,a,\,b}(t)$. Noted that when $\alpha = \pi/2$, the FRWT reduces to the conventional WT because $\varphi_{\alpha,\,a,\,b}(t)$ coincides with the wavelet $\varphi_{a,\,b}(t)$. In this regard, the FRWT is the generalization of WT.

Based on Eqs. (5) and (9), we obtain the definition of α-order FRWT in the FRFD as

$$W_x^\alpha(a,b) = \sqrt{\frac{2\pi a}{1+j\cot\alpha}}\int_{-\infty}^{+\infty} e^{-\frac{j}{2}a^2u^2\cot\alpha}X_\alpha(u)\Phi_\alpha^*(au)K_{-\alpha}(u,b)du \tag{12}$$

where $X_\alpha(u)$ and $\Phi_\alpha(au)$ are the α-order FRFT of $x(t)$ and $\varphi(a/t)$, respectively. The derivation of Eq. (12) can be found in Appendix A.

According to Eq. (9), we know that if $\varphi_{\alpha,\,a,\,b}(t)$ is supported in the time domain, then $W_x^\alpha(a,b)$ will be also supported in the time domain. Similarly, Eq. (12) states that each fractional wavelet component is a scaled bandpass filter in the FRFD, and thus the multiplication of $X_\alpha(u)$ and $\Phi_\alpha(au)$ can provide the local properties of $x(t)$ in the FRFD. This implies that the FRWT extend the time-frequency domain in the conventional WT to the time-FRFD domain. To be more specific, suppose that $\varphi(t)$ is the window function with center E_φ and radius Δ_φ in the time domain, and that $\Phi_\alpha(u)$ is the window function with center E_{Φ_α} and radius Δ_{Φ_α} in the FRFD. Then, the center and the radius of the window function $\varphi_{\alpha,\,a,\,b}(t)$ are respectively computed as

$$E[\varphi_{\alpha,a,b}(t)] = \frac{\int_{-\infty}^{+\infty} t|\varphi_{\alpha,a,b}(t)|^2dt}{\int_{-\infty}^{+\infty}|\varphi_{\alpha,a,b}(t)|^2dt} = \frac{\int_{-\infty}^{+\infty} t|\varphi_{a,b}(t)|^2dt}{\int_{-\infty}^{+\infty}|\varphi_{a,b}(t)|^2dt} = E[\varphi_{a,b}(t)] = b + aE_\varphi \tag{13}$$

$$\Delta[\varphi_{\alpha,a,b}(t)] = \sqrt{\frac{\int_{-\infty}^{+\infty}\left(t-E\left[\varphi_{a,b,\alpha}(t)\right]\right)^2|\varphi_{a,b,\alpha}(t)|^2dt}{\int_{-\infty}^{+\infty}|\varphi_{a,b,\alpha}(t)|^2dt}} = \Delta\left[\varphi_{a,b}(t)\right] = a\Delta_\varphi \tag{14}$$

According to the scaling property of FRFT, we have

$$F^\alpha\left[\varphi_{\alpha,a,b}(at)\right] = kg(u)\frac{\sqrt{a}}{A_\beta}e^{-\frac{j}{2}a^2(ru)^2\cot\beta}\Phi_\beta(ru)K_\beta(ru,b) \tag{15}$$

where $k = \sqrt{\frac{1-j\cot\alpha}{a^2-j\cot\alpha}}$, $g(u) = e^{j\frac{u^2\cot\alpha}{2}(1-\frac{\cos^2\beta}{\cos^2\alpha})\cot\alpha}$, $r = \frac{\sin\beta}{\sin\alpha}$ and $\beta = \arctan(a^2\tan\alpha)$. Accordingly, we have

$$E[\Phi_\alpha(au)] = kE[\Phi_\beta(ru)] \tag{16}$$

$$E[\Phi_\alpha(au\sin\alpha)] = kE[\Phi_\beta(u\sin\beta)] \tag{17}$$

$$\Delta[\Phi_\alpha(au\sin\alpha)] = k\Delta[\Phi_\beta(u\sin\beta)] \tag{18}$$

Thus, the Q-factor of $\Phi_\alpha(u)$ is obtained as the ratio

$$Q = \frac{\Delta[\Phi_\alpha(au)]}{E[\Phi_\alpha(au)]} = \frac{\Delta\left[\Phi_\beta(ru)\right]}{E\left[\Phi_\beta(ru)\right]} \tag{19}$$

Eq. (19) indicates that the value of the Q-factor remains unchanged when the window function is analyzed between the α-order FRFD and β-order FRFD. This is the constant Q-property of the FRWT. Particularly, the constant Q-property of FRWT reduces to that of the conventional WT with $\alpha = \pi/2$.

Thus, the FRWT gives local information of signal $x(t)$ in the time window as

$$[b + aE_\varphi - a\Delta_\varphi, b + aE_\varphi + a\Delta_\varphi] \tag{20}$$

and gives the local FRFD spectrum information in the FRFD as

$$\left[kE_{\Phi_\beta} - k\Delta_{\Phi_\beta}, kE_{\Phi_\beta} + k\Delta_{\Phi_\beta}\right] \tag{21}$$

with the constant window area

$$2a\Delta_\varphi \times 2k\Delta_{\Phi_\beta} = 4ak\Delta_\varphi\Delta_{\Phi_\beta} \tag{22}$$

in the time-FRFD-frequency plane. From Eqs. (21) and (22), one can see how the time-FRFD-frequency resolution changes with the width and the height of the window. The area of the window depends only on the mother wavelet $\varphi(t)$ and the angle α, and is independent of the parameters a and b. When $\alpha = \pi/2$, the window area of FRWT coincides with that of traditional WT. With the given angle α, the shape of the window varies with a. Like the case in WT, the window automatically narrows for detecting high FRFD-frequency phenomena (i.e. small $a > 0$), and widens for investigating low FRFD- frequency behavior (i.e. small $a > 0$).

3.2. Properties of the new FRWT

Similar as the WT, the proposed FRWT possess the following properties.

(1) *Linearity:* It is obvious that the FRWT is linear, i.e.,

$$W_z^\alpha(a,b) = k_1W_x^\alpha(a,b) + k_2W_y^\alpha(a,b) \tag{23}$$

where $z(t) = k_1x(t) + k_2y(t)$, k_1 and k_2 are arbitrary constant coefficients. This property indicates that the FRWT satisfies the superposition principle, which is favorable to the analysis of multicomponent signals.

Theorem 1. *If the mother wavelet $\varphi(t)$ satisfies the "admissibility" condition:*

$$C_\Phi = \int_0^\infty \frac{|\Phi_\alpha(s)|^2}{|s|}ds < \infty \tag{24}$$

where $\Phi_\alpha(s)$ is the α-order FRFT of $\varphi(t)$. Then

$$\int_0^\infty\int_{-\infty}^{+\infty} W_x^\alpha(a,b)\left[W_y^\alpha(a,b)\right]^*\frac{da}{a^2}db = 2\pi\sin\alpha C_\Phi\langle x(t), y(t)\rangle \tag{25}$$

The proof of this theorem can be found in Appendix B.

(2) *Parseval Identity:* If the mother wavelet $\varphi(t)$ satisfies Eq. (24), Then

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi\sin\alpha C_\Phi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{a^2}|W_x^\alpha(a,b)|^2dadb \tag{26}$$

Obviously, by setting $y(t) = x(t)$ in Eq. (25), Eq. (26) can be easily derived. This property states that the α-order fractional scalogram shows how the energy of the signal is distributed in the α-order time-FRFD-frequency plane. This is the generalization of the fact that the scalogram of WT measures the energy of the signal in the time-frequency plane.

3.3. Inverse FRWT and reproducing kernel

Theorem 2. *If the mother wavelet $\varphi(t)$ satisfies the "admissibility" condition as in Eq. (24). Then, the inverse formula of FRWT is*

$$x(t) = \frac{1}{2\pi \sin\alpha C_\Phi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{a^2} W_x^\alpha(a, b) \varphi_{a,b,\alpha}(t) dadb \tag{27}$$

The proof of Theorem 2 can be easily obtained by setting $y(t) = \delta(t)$ in Theorem 1. It is noted that when $\alpha = \pi/2$, the admissibility condition of FRWT in Eq. (24) coincides with that of the conventional WT in Eq. (8), and hence Eq. (24) is the generalization of Eq. (8). The significance of Eq. (24) lies in that the functions that are not satisfied with Eq. (8) may be satisfied Eq. (24) under some special angle, and hence the admissibility condition of FRWT make it possible for selection more mother wavelets and meanwhile make it easy for constructing mother wavelets.

Theorem 3. *Suppose that (a_0, b_0) is an arbitrary point on the (a, b) plane, the sufficient and necessary condition that the function $W_x^\alpha(a, b)$ is the FRWT of some function under some angle α is that $W_x^\alpha(a, b)$ has to satisfy the following reproducing kernel equation*

$$W_x^\alpha(a_0, b_0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_x^\alpha(a, b) K_{\varphi_\alpha}(a_0, b_0; a, b) \frac{da}{a^2} db \tag{28}$$

where $W_x^\alpha(a_0, b_0)$ is the value of function $W_x^\alpha(a, b)$ on point (a_0, b_0), and $K_{\varphi_\alpha}(a_0, b_0; a, b)$ is called the reproducing kernel which needs to satisfy

$$K_{\varphi_\alpha}(a_0, b_0; a, b) = \frac{1}{2\pi \sin\alpha C_\Phi} \int_{-\infty}^{+\infty} \varphi_{a,b,\alpha}(t) \varphi^*_{a_0,b_0,\alpha}(t) dt \tag{29}$$

Proof. From Eq. (9) and Eq. (28), we have

$$W_x^\alpha(a, b) = \int_{-\infty}^{+\infty} x(t) \varphi^*_{\alpha,a,b}(t) dt = \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi \sin\alpha C_\Phi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_x^\alpha(a, b) \varphi_{a,b,\alpha}(t) \frac{da}{a^2} db \right] \varphi^*_{a,b,\alpha}(t) dt \tag{30}$$

By setting $(a, b) = (a_0, b_0)$ in Eq. (30), we have

$$\begin{aligned} W_x^\alpha(a_0, b_0) &= \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi \sin\alpha C_\Phi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_x^\alpha(a, b) \varphi_{a,b,\alpha}(t) \frac{da}{a^2} db \right] \varphi^*_{a_0,b_0,\alpha}(t) dt \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_x^\alpha(a, b) \left[\frac{1}{2\pi \sin\alpha C_\Phi} \int_{-\infty}^{+\infty} \varphi_{a,b,\alpha}(t) \varphi^*_{a_0,b_0,\alpha}(t) dt \right] \frac{da}{a^2} db \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_x^\alpha(a, b) K_{\varphi_\alpha}(a_0, b_0; a, b) \frac{da}{a^2} db \end{aligned} \tag{31}$$

This completes the proof. From Theorem 3, we know that the value of $W_x^\alpha(a_0, b_0)$ can be represented by $W_x^\alpha(a, b)$ on the set of grids (a, b). This indicates that there exists redundant information when reconstructing $x(t)$ with Eq. (27). Therefore the redundant information can be eliminated when using $W_x^\alpha(a, b)$ to recover the signal.

4. Multiresolution analysis and orthogonal fractional wavelets

4.1. Multiresolution analysis associated with FRWT

Similar to the conventional wavelets, any fractional wavelet gives rise to some decomposition of $L^2(\mathbb{R})$ into a direct sum of closed subspaces $W_m^\alpha,\ m \in \mathbb{Z}$; in the sense that each subspace W_m is the closure in $L^2(\mathbb{R})$ of the linear span of the collection of functions $\varphi_{\alpha,m,n}(t),\ n \in \mathbb{Z}$. Hence, the corresponding subspaces

$$V_m^\alpha := \cdots + W_{m-2}^\alpha + W_{m-1}^\alpha, \quad m \in \mathbb{Z} \tag{32}$$

form a nested sequence of subspaces of $L^2(\mathbb{R})$, whose union is dense in $L^2(\mathbb{R})$ and whose intersection is the null space $\{0\}$. The spaces $V_m^\alpha,\ m \in \mathbb{Z}$ are generated, in the same manner as φ generates the spaces W_m^α, by the so called fractional scaling function $\phi_{\alpha,\,m,\,n}(t)$. In particular, the collection of functions $\phi_\alpha(t-n),\ \ n \in \mathbb{Z}$ is to form a Riesz basis of V_0^α; and hence, ϕ generates an MRA $\{V_m^\alpha\}$ of $L^2(\mathbb{R})$.

Definition 1. An orthogonal MRA of the FRWT is defined as a sequence of closed subspaces $\{V_m^\alpha\}_{m\in\mathbb{Z}} \in L^2(\mathbb{R})$ such that

(1) $V_m^\alpha \subseteq V_{m+1}^\alpha,\ \ m \in \mathbb{Z}$;

(2) $x(t) \in V_m^\alpha \Leftrightarrow x(2t) e^{\frac{j}{2}((2t)^2 - t^2)\cot\alpha} \in V_{m+1}^\alpha,\ \ m \in \mathbb{Z}$;

(3) $\bigcap_{m\in\mathbb{Z}} V_m^\alpha = \{0\}$; $\quad clos_{L^2}(\bigcup_{m\in\mathbb{Z}} V_m^\alpha) = L^2(\mathbb{R})$;

(4) There exists a function $\phi(t)$ such that $\{\phi_{\alpha,0,n}(t) = \phi(t-n)e^{-j(tn+n^2)\cot\alpha}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of subspace V_0^α.

The function $\phi(t)$ in Definition 1 is called scaling function. If condition (4) in Definition 1 is relaxed by assuming that the set of functions $\{\phi_{\alpha,0,n}(t)\}_{n\in\mathbb{Z}}$ is to form a Riesz basis of V_0^α, then $\phi(t)$ generates a generalized MRA $\{V_m^\alpha\}$of $L^2(\mathbb{R})$. Furthermore, if $\phi(t)$ is a scaling function that generates an orthonormal MRA $\{V_m^\alpha\}$ of $L^2(\mathbb{R})$, then

$$\phi_{\alpha,m,n}(t) = 2^{\frac{m}{2}}\phi(2^m t-n)e^{-\frac{j}{2}\left(t^2-\left(\frac{n}{2^m}\right)^2-(2^m t-n)^2\right)\cot\alpha} \tag{33}$$

is the orthonormal basis of $\{V_m^\alpha\}$.

The following gives a useful Lemma for deriving the theorem of MRA that associated with FRWT.

Lemma 1. *Assume that* $\{\phi_{\alpha,0,n}(t) = \phi(t-n)e^{-j(tn+n^2)\cot\alpha}\}_{n\in\mathbb{Z}}$ *is an orthonormal basis of subspace* V_0^α. *Then*

$$\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2 = \frac{1}{\sin\alpha} \tag{34}$$

where $\Theta_\alpha(u)$ *is FRFT of* $\phi(t)$. *The proof of the lemma can be found in Appendix C.*

Theorem 4. *If scaling function* $\phi(t)$ *generates a generalized MRA* $\{V_m^\alpha\}$ *of* $L^2(\mathbb{R})$, *let*

$$\Omega_\alpha(u)=\frac{\Theta_\alpha(u)}{\sqrt{2\pi\sin\alpha\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2}} \tag{35}$$

Then, $\omega(t)$, *which is the inverse FRFT of* $\Omega_\alpha(u)$, *is an orthogonal scaling function that generates the orthogonal MRA* $\{V_m^\alpha\}$ *of* $L^2(\mathbb{R})$.

Proof. Since $\phi(t)$ generates a generalized MRA $\{V_m^\alpha\}$, $\{\phi_{\alpha,0,n}(t)\}_{n\in\mathbb{Z}}$ form a Riesz basis of V_0^α. Hence, we have

$$\omega(t) = \sum_n c_n\phi(t-n)e^{-j(tn+n^2)cot\alpha} \tag{36}$$

By taking the FRFT on both sides of Eq. (36) we obtain

$$\begin{aligned}
\Omega_\alpha(u) &= \sum_{n\in\mathbb{Z}} c_n A_\alpha\int_{-\infty}^{+\infty}\phi(t-n)e^{-j(tn+n^2)\cot\alpha}e^{\frac{1}{2}j(t^2+u^2)\cot\alpha-jtu\csc\alpha}du\\
&= \sum_{n\in\mathbb{Z}} c_n A_\alpha\int_{-\infty}^{+\infty}\phi(t-n)e^{\frac{1}{2}j((t-n)^2+u^2)\cot\alpha-j(t-n)u\csc\alpha+\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}du\\
&= \sum_{n\in\mathbf{Z}} c_n e^{\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}A_\alpha\int_{-\infty}^{+\infty}\phi(t-n)e^{\frac{1}{2}j((t-n)^2+u^2)\cot\alpha-j(t-n)u\csc\alpha}du\\
&= C(u)\Theta_\alpha(u)
\end{aligned} \tag{37}$$

where $C(u) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}$ is $2\pi\sin\alpha$-periodic function. On the other hand, if $\{\omega(t-n)e^{-jtncot\alpha}\}_{n\in\mathbb{Z}}$ forms an orthogonal MRA $\{V_m^\alpha\}$, from Lemma 1, we have

$$\begin{aligned}
\frac{1}{\sin\alpha} &= \sum_{k\in\mathbb{Z}}|\Omega_\alpha(u+2k\pi\sin\alpha)|^2 = \sum_{k\in\mathbb{Z}}|C(u+2k\pi\sin\alpha)|^2|\Theta_\alpha(u+2k\pi\sin\alpha)|^2\\
&= |C(u)|^2\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2
\end{aligned} \tag{38}$$

which yields

$$C(u)=\frac{1}{\sqrt{2\pi\sin\alpha\sum_{k\in\mathbf{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2}} \tag{39}$$

From Eqs. (37) and (39), we have

$$\Omega_\alpha(u)=C(u)\Theta_\alpha(u) = \frac{\Theta_\alpha(u)}{\sqrt{2\pi\sin\alpha\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2}}$$

as in Eq. (35), this completes the proof of Theorem 4.

4.2. Construction of orthogonal fractional wavelets

Once the MRA associated with FRWT is developed, the orthogonal fractional wavelets can be constructed. We firstly need to define W_m^α such that V_m^α and W_m^α are complementary subspaces of V_{m+1}^α, in the sense that

$$W_m^\alpha \perp V_m^\alpha \ \text{ and } \ V_{m+1}^\alpha = W_m^\alpha \oplus V_m^\alpha, \tag{40}$$

Followed by Definition 1, W_m^α possess the following properties

(1) $W_m^\alpha \perp W_n^\alpha, \quad \forall m \neq n$
(2) $L^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} W_m^\alpha$
(3) $g(t) \in W_m^\alpha \Leftrightarrow g(2t)e^{\frac{j}{2}((2t)^2 - t^2)\cot\alpha} \in W_{m+1}^\alpha, \ \ \forall m \in \mathbb{Z}$

Note that condition (2) means that the orthonormal basis for $L^2(\mathbb{R})$ can be constructed by finding out an orthonormal basis for the subspace W_m^α. Condition (3) implies that the orthogonal fractional wavelet can be constructed as long as orthonormal basis for W_0^α is found. Therefore, the main concern is to construct a function $\varphi(t)$ such that $\{\varphi_{\alpha,0,n}(t)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for W_0^α.

Since $\phi_{\alpha,0,0}(t) \in V_0^\alpha \subseteq V_1^\alpha$, and $\{\phi_{\alpha,1,n}(t)\}_{n\in\mathbb{Z}}$ is orthonormal basis of V_1^α, there must exist coefficient $\{h[n]\}_{n\in\mathbb{Z}}$ such that

$$\phi_{\alpha,0,0}(t) = \sum_n h[n]\phi_{\alpha,1,n}(t) \tag{41}$$

Eq. (41) is called the fractional scaling equation, which can be simplified as

$$\phi(t) = \sum_n h[n]\sqrt{2}\phi(2t-n)e^{-\frac{j}{2}\left(t^2 - \left(\frac{n}{2}\right)^2 - (2t-n)^2\right)\cot\alpha} \tag{42}$$

and the coefficient can be solved as

$$h[n] = \sqrt{2}\int_{-\infty}^{+\infty} \phi(t)\phi^*(2t-n)e^{\frac{j}{2}\left(t^2 - \left(\frac{n}{2}\right)^2 - (2t-n)^2\right)\cot\alpha}dt \tag{43}$$

By taking the FRFT on both sides of Eq. (42), we have

$$\begin{aligned}
\Theta_\alpha(u) &= \sum_{n\in\mathbb{Z}} h[n]\sqrt{2}A_\alpha \int_{-\infty}^{+\infty} \phi(2t-n)e^{\frac{1}{2}j\left((2t-n)^2 + u^2 + \left(\frac{n}{2}\right)^2\right)\cot\alpha - jtu\csc\alpha}dt \\
&= \frac{1}{\sqrt{2}}e^{\frac{1}{2}j\frac{3}{4}u^2\cot\alpha}\sum_{n\in\mathbb{Z}} h[n]e^{\frac{1}{2}j\left(\frac{n}{2}\right)^2\cot\alpha - jn\frac{u}{2}\csc\alpha}A_\alpha\int_{-\infty}^{+\infty}\phi(2t-n)e^{\frac{1}{2}j\left((2t-n)^2+\left(\frac{u}{2}\right)^2\right)\cot\alpha - j(2t-n)\frac{u}{2}\csc\alpha}d(2t-n) \\
&= \frac{1}{\sqrt{2}}e^{\frac{1}{2}j\frac{3}{4}u^2\cot\alpha}\sum_{n\in\mathbb{Z}} h[n]e^{\frac{1}{2}j\left(\frac{n}{2}\right)^2\cot\alpha - jn\frac{u}{2}\csc\alpha}\Theta_\alpha\left(\frac{u}{2}\right) \\
&= \frac{1}{\sqrt{2}}e^{\frac{3}{2}j\left(\frac{u}{2}\right)^2\cot\alpha}C_\alpha\left(\frac{u}{2}\right)\Theta_\alpha\left(\frac{u}{2}\right)
\end{aligned} \tag{44}$$

where

$$C_\alpha(u) = \sum_{n\in\mathbb{Z}} h[n]e^{\frac{1}{2}j\left(\frac{n}{2}\right)^2\cot\alpha - jn\frac{u}{2}\csc\alpha} \tag{45}$$

Here, assume that $C_\alpha(u)$ is the discrete time FRFT of sequence $f[n]$, that is

$$C_\alpha(u) = e^{-2ju^2\cot\alpha}\sum_{n\in\mathbb{Z}} f[n]K_\alpha(n,u) = e^{-2ju^2\cot\alpha}\tilde{F}_\alpha(u) \tag{46}$$

If the assumption Eq. (46) holds, then from Eqs. (45) and (46) we have

$$\sum_{n\in\mathbb{Z}} h[n]e^{\frac{1}{2}j\left(\frac{n}{2}\right)^2\cot\alpha - jn\frac{u}{2}\csc\alpha} = e^{-2ju^2\cot\alpha}\tilde{F}_\alpha(u) \tag{47}$$

which is equivalent to

$$\frac{1}{A_\alpha}\sum_{n\in\mathbb{Z}} h[n]A_\alpha e^{\frac{1}{2}j\left(\left(\frac{n}{2}\right)^2 + (2u)^2\right)\cot\alpha - j\frac{n}{2}(2u)\csc\alpha} = \tilde{F}_\alpha(u) \tag{48}$$

so that

$$\tilde{F}_\alpha(u) = \frac{1}{A_\alpha}\sum_{n\in\mathbb{Z}} h[n]K_\alpha\left(\frac{n}{2}, 2u\right) \tag{49}$$

As a consequence, we show the rationality of assumption Eq. (46). Thus, by defining

$$\begin{aligned}\Lambda_\alpha(u) &= \frac{1}{\sqrt{2}} e^{\frac{3}{2}ju^2\cot\alpha} C_\alpha(u) = \frac{1}{\sqrt{2}} e^{\frac{3}{2}ju^2\cot\alpha} e^{-2ju^2\cot\alpha} \sum_{n\in\mathbb{Z}} f[n] K_\alpha(n,u) \\ &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} f[n] A_\alpha e^{\frac{1}{2}jn^2\cot\alpha - jnu\csc\alpha}\end{aligned} \tag{50}$$

Eq. (44) can be rewritten as

$$\Theta_\alpha(u)=\Lambda_\alpha\left(\frac{u}{2}\right)\Theta_\alpha\left(\frac{u}{2}\right) \tag{51}$$

It is noted that $\Lambda_\alpha(u)$ is a $2k\pi\sin\alpha$-periodic function since we have

$$\begin{aligned}\Lambda_\alpha(u+2k\pi\sin\alpha) &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} f[n] A_\alpha e^{\frac{1}{2}jn^2\cot\alpha - jn(u+2k\pi\sin\alpha)\csc\alpha} \\ &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} f[n] A_\alpha e^{\frac{1}{2}jn^2\cot\alpha - jnu\csc\alpha} = \Lambda_\alpha(u)\end{aligned} \tag{52}$$

On the other hand, from Lemma 1 and Eq. (51), we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} |\Theta_\alpha(u+2k\pi\sin\alpha)|^2 \\ &= \sum_{k\in\mathbb{Z}} \left|\Lambda_\alpha\left(\frac{u}{2}+k\pi\sin\alpha\right)\right|^2 \left|\Theta_\alpha\left(\frac{u}{2}+k\pi\sin\alpha\right)\right|^2 \\ &= \sum_{l\in\mathbb{Z}} \left|\Lambda_\alpha\left(\frac{u}{2}+2l\pi\sin\alpha\right)\right|^2 \left|\Theta_\alpha\left(\frac{u}{2}+2l\pi\sin\alpha\right)\right|^2 + \sum_{l\in\mathbb{Z}} \left|\Lambda_\alpha\left(\frac{u}{2}+(2l+1)\pi\sin\alpha\right)\right|^2 \left|\Theta_\alpha\left(\frac{u}{2}+(2l+1)\pi\sin\alpha\right)\right|^2 \\ &= \left|\Lambda_\alpha\left(\frac{u}{2}\right)\right|^2 \sum_{l\in\mathbb{Z}} \left|\Theta_\alpha\left(\frac{u}{2}+2l\pi\sin\alpha\right)\right|^2 + \left|\Lambda_\alpha\left(\frac{u}{2}+\pi\sin\alpha\right)\right|^2 \sum_{l\in\mathbb{Z}} \left|\Theta_\alpha\left(\frac{u}{2}+(2l+1)\pi\sin\alpha\right)\right|^2 \\ &= \left|\Lambda_\alpha\left(\frac{u}{2}\right)\right|^2 \frac{1}{\sin\alpha} + \left|\Lambda_\alpha\left(\frac{u}{2}+\pi\sin\alpha\right)\right|^2 \frac{1}{\sin\alpha} = \frac{1}{\sin\alpha}\end{aligned} \tag{53}$$

It follows from Eq. (53) that

$$\left|\Lambda_\alpha(\frac{u}{2})\right|^2 + \left|\Lambda_\alpha(\frac{u}{2}+\pi\sin\alpha)\right|^2 = 1 \tag{54}$$

which is equivalent to

$$|\Lambda_\alpha(u)|^2 + |\Lambda_\alpha(u+\pi\sin\alpha)|^2 = 1 \tag{55}$$

Similar to the derivation of Eqs. (41), (51) and (55), since $\varphi_{\alpha,0,0}(t) \in W_0^\alpha \subseteq V_1^\alpha$, there must exist coefficient $\{g[n]\}_{n\in\mathbb{Z}}$ such that

$$\varphi_{\alpha,0,0}(t) = \sum_n g[n]\phi_{\alpha,1,n}(t) \tag{56}$$

Eq. (56) is called the fractional wavelet equation, which can be simplified as

$$\varphi(t) = \sum_n g[n]\sqrt{2}\phi(2t-n) e^{-\frac{j}{2}\left(t^2-\left(\frac{n}{2}\right)^2-(2t-n)^2\right)\cot\alpha} \tag{57}$$

By taking the FRFT on both sides of Eq. (57) and applying the similar steps as in Eqs. (44)~(55), we have

$$\Theta_\alpha(u)=\Gamma_\alpha\left(\frac{u}{2}\right)\Phi_\alpha\left(\frac{u}{2}\right) \tag{58}$$

where

$$\Gamma_\alpha(u)=\frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d[n] A_\alpha e^{\frac{1}{2}jn^2\cot\alpha - jnu\csc\alpha} \tag{59}$$

where $d[n]$ and $g[n]$ has the relationship as

$$\tilde{D}_\alpha(u) = \frac{1}{A_\alpha} \sum_{n\in\mathbb{Z}} g[n] K_\alpha\left(\frac{n}{2}, 2u\right) \tag{60}$$

where $\tilde{D}_\alpha(u)$ is the discrete time FRFT of sequence $d[n]$. $\Gamma_\alpha(u)$ is also $2k\pi\sin\alpha$-periodic function which satisfy

$$|\Gamma_\alpha(u)|^2 + |\Gamma_\alpha(u+\pi\sin\alpha)|^2 = 1 \tag{61}$$

From Appendix C, we have

$$\begin{aligned}
\langle \phi_{\alpha,0,n}(t), \varphi_{\alpha,0,m}(t)\rangle &= \langle F^{\alpha}[\theta_{\alpha,0,n}(t)](u),\ \ F^{\alpha}[\varphi_{\alpha,0,m}(t)](u)\rangle \\
&= \left\langle e^{\frac{1}{2}jn^2\cot\alpha - jnu\csc\alpha}\Theta_{\alpha}(u), e^{\frac{1}{2}jm^2\cot\alpha - jmu\csc\alpha}\Phi_{\alpha}(u)\right\rangle \\
&= \int_{-\infty}^{+\infty} e^{\frac{1}{2}j\left(n^2-m^2\right)\cot\alpha - j(n-m)u\csc\alpha}\Theta_{\alpha}(u)\Phi_{\alpha}^{*}(u)du \\
&= e^{\frac{1}{2}j\left(n^2-m^2\right)\cot\alpha}\int_{-\infty}^{+\infty}\Lambda_{\alpha}(\frac{u}{2})\Gamma_{\alpha}^{*}(\frac{u}{2})\left|\Theta_{\alpha}\left(\frac{u}{2}\right)\right|^2 e^{-j(n-m)u\csc\alpha}du = \delta_{n,m}
\end{aligned} \tag{62}$$

Since when $n \neq m$,

$$\int_{-\infty}^{+\infty}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right)\left|\Phi_{\alpha}\left(\frac{u}{2}\right)\right|^2 e^{-j(n-m)u\csc\alpha}du = 0 \tag{63}$$

Hence, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right)\left|\Phi_{\alpha}\left(\frac{u}{2}\right)\right|^2 e^{-jnu\csc\alpha}du \\
&= \sum_{k\in\mathbb{Z}}\int_{0}^{4\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}+2k\pi\sin\alpha\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}+2k\pi\sin\alpha\right)\left|\Phi_{\alpha}\left(\frac{u}{2}+2k\pi\sin\alpha\right)\right|^2 e^{-jn(u+4k\pi\sin\alpha)sc\alpha}du \\
&= \int_{0}^{4\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}(\frac{u}{2})\sum_{k\in\mathbb{Z}}\left|\Phi_{\alpha}\left(\frac{u}{2}+2k\pi\sin\alpha\right)\right|^2 e^{-jnusc\alpha}du \\
&= \frac{1}{2\pi\sin\alpha}\int_{0}^{4\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right)e^{-jnusc\alpha}du \\
&= \frac{1}{2\pi\sin\alpha}\left[\int_{0}^{2\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right)e^{-jnusc\alpha}du + \int_{2\pi\sin\alpha}^{4\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right)e^{-jnusc\alpha}du\right] \\
&= \frac{1}{2\pi\sin\alpha}\left[\int_{0}^{2\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right)e^{-jnusc\alpha}du + \int_{0}^{2\pi\sin\alpha}\Lambda_{\alpha}\left(\frac{u}{2}+\pi\sin\alpha\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}+\pi\sin\alpha\right)e^{-jn(u+2\pi\sin\alpha)sc\alpha}du\right] \\
&= \frac{1}{2\pi\sin\alpha}\int_{0}^{2\pi\sin\alpha}\left[\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right) + \Lambda_{\alpha}\left(\frac{u}{2}+\pi\sin\alpha\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}+\pi\sin\alpha\right)\right]e^{-jnusc\alpha}du = 0
\end{aligned} \tag{64}$$

Eq. (64) leads to

$$\Lambda_{\alpha}\left(\frac{u}{2}\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}\right) + \Lambda_{\alpha}\left(\frac{u}{2}+\pi\sin\alpha\right)\Gamma_{\alpha}^{*}\left(\frac{u}{2}+\pi\sin\alpha\right) = 0 \tag{65}$$

which is equivalent to

$$\Lambda_{\alpha}(u)\Gamma_{\alpha}^{*}(u) + \Lambda_{\alpha}(u+\pi\sin\alpha)\Gamma_{\alpha}^{*}(u+\pi\sin\alpha) = 0 \tag{66}$$

If we define

$$M_{\alpha}(u) = \begin{bmatrix}\Lambda_{\alpha}(u) & \Lambda_{\alpha}(u+\pi\sin\alpha) \\ \Gamma_{\alpha}(u) & \Gamma_{\alpha}(u+\pi\sin\alpha)\end{bmatrix} \tag{67}$$

Then, Eq. (66) can be rewritten as

$$M_{\alpha}(u)M_{\alpha}^{H}(u) = \mathbf{I} \tag{68}$$

where $M_{\alpha}^{H}(u)$ is the conjugate transpose matrix of $M_{\alpha}(u)$, and $\mathbf{I}$ is the identity matrix.

Observe Eqs. (55), (61) and (68), we find they are very similar with those as in the traditional WT. When $\alpha = \pi/2$, they coincide with

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 \tag{69}$$

$$|G(\omega)|^2 + |G(\omega+\pi)|^2 = 1 \tag{70}$$

$$H(\omega)G^{*}(\omega) + H(\omega+\pi)G^{*}(\omega+\pi) = 0 \tag{71}$$

where $H(\omega)$ and $G(\omega)$ are the well-known quadrature mirror filters in WT. In this sense, the construction of orthogonal fractional wavelets is the generalization of that of the traditional wavelets.

Notice that if Eq. (68) holds, then

$$(\Gamma_\alpha(u),\ \ \Gamma_\alpha(u+\pi\sin\alpha)) = (\lambda(u)\Lambda^*_\alpha(u+\pi\sin\alpha), -\lambda(u)\Lambda^*_\alpha(u)) \tag{72}$$

is the solution of Eq. (68), where $\lambda(u)$ is arbitrary function which needs to satisfy

$$\Gamma_\alpha(u+\pi\sin\alpha) = \lambda(u+\pi\sin\alpha)\Lambda^*_\alpha(u+2\pi\sin\alpha) = \lambda(u+\pi\sin\alpha)\Lambda^*_\alpha(u) = -\lambda(u)\Lambda^*_\alpha(u) \tag{73}$$

From Eq. (73), we have

$$\lambda(u)+\lambda(u+\pi\sin\alpha)=0 \tag{74}$$

If we let $u=u+\pi\sin\alpha$, then

$$\lambda(u+\pi\sin\alpha)+\lambda(u+2\pi\sin\alpha)=0 \tag{75}$$

which is equivalent to

$$\lambda(u)=\lambda(u+2\pi\sin\alpha) \tag{76}$$

Eq. (76) indicates that $\lambda(u)$ is a $2\pi\sin\alpha$-periodic function, which can be expanded in the Fourier series as

$$\lambda(u)=\sum_{k\in\mathbb{Z}} c[k]e^{-jku\csc\alpha} \tag{77}$$

Since the coefficients $c[k]$ in Eq. (77) can be solved as

$$c[k]=\frac{1}{2\pi\sin\alpha}\left(\int_0^{\pi\sin\alpha}\lambda(u)e^{-jku\csc\alpha}du+\int_{\pi\sin\alpha}^{2\pi\sin\alpha}\lambda(u)e^{-jku\csc\alpha}du\right)=\frac{1-(-1)^k}{2\pi\sin\alpha}\int_0^{\pi\sin\alpha}\lambda(u)e^{-jku\csc\alpha}du \tag{78}$$

$\lambda(u)$ can be reformulated as

$$\lambda(u)=\sum_{k\in\mathbb{Z}} c[2k+1]e^{-j(2k+1)u\csc\alpha}=e^{-ju\csc\alpha}\xi(2u) \tag{79}$$

where $\xi(u)=\sum_{n\in\mathbb{Z}} p[2n+1]e^{-jnu\csc\alpha}$.

On the other hand, from Eq. (52), we have

$$\begin{aligned}\Lambda_\alpha(u+\pi\sin\alpha) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} f[n]A_\alpha e^{\frac{1}{2}jn^2\cot\alpha-jn(u+\pi\sin\alpha)\csc\alpha}\\ &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} f[n]A_\alpha e^{\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}(\cos n\pi - j\sin n\pi)=\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}(-1)^n f[n]A_\alpha e^{\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}\end{aligned} \tag{80}$$

Thus, from Eqs. (72), (79) and (80), it is very easy to compute

$$\Gamma_\alpha(u)=\lambda(u)\Lambda^*_\alpha(u+\pi\sin\alpha)=e^{-ju\csc\alpha}\xi(2u)\sum_{n\in\mathbb{Z}}(-1)^n f^*[n]e^{-\frac{1}{2}jn^2\cot\alpha+jnu\csc\alpha} \tag{81}$$

Compare Eq. (59) with Eq. (81), we know that

$$\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d[n]A_\alpha e^{\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}=e^{-ju\csc\alpha}\xi(2u)\sum_{n\in\mathbb{Z}}(-1)^n f^*[n]e^{-\frac{1}{2}jn^2\cot\alpha+jnu\csc\alpha} \tag{82}$$

where $\xi(u)$ can be formulated as $\xi(u)=\sum_{n\in\mathbb{Z}} q[n]e^{-jnu\csc\alpha}$. In particular, if we set $\xi(u)=1$, then Eq. (82) leads to

$$\sum_{n\in\mathbb{Z}} d[n]e^{\frac{1}{2}jn^2\cot\alpha-jnu\csc\alpha}=e^{-ju\csc\alpha}\sum_{n\in\mathbb{Z}}(-1)^n f^*[n]e^{-\frac{1}{2}jn^2\cot\alpha+jnu\csc\alpha} \tag{83}$$

which is equivalent to

$$\begin{aligned} d[k]e^{\frac{1}{2}jk^2\cot\alpha} &= \int_{-\infty}^{\infty} e^{-ju\csc\alpha}\sum_{n\in\mathbb{Z}}(-1)^n f^*[n]e^{-\frac{1}{2}jn^2\cot\alpha+jnu\csc\alpha}e^{jku\csc\alpha}du\\ \Leftrightarrow d[k]e^{\frac{1}{2}jk^2\cot\alpha} &= \sum_{n\in\mathbb{Z}}(-1)^n f^*[n]e^{-\frac{1}{2}jn^2\cot\alpha}\int_{-\infty}^{\infty} e^{-j(1-n-k)u\csc\alpha}du\\ \Leftrightarrow d[k]e^{\frac{1}{2}jk^2\cot\alpha} &= \sum_{n\in\mathbb{Z}}(-1)^n f^*[n]e^{-\frac{1}{2}jn^2\cot\alpha}\delta_{1-n-k}\\ \Leftrightarrow d[k]e^{\frac{1}{2}jk^2\cot\alpha} &= (-1)^{1-k}f^*[1-k]e^{-\frac{1}{2}j(1-k)^2\cot\alpha}\end{aligned} \tag{84}$$

From Eq. (84), we obtain the relationship between coefficient $d[n]$ and $f[n]$ as

$$d[n]e^{\frac{1}{2}jn^2\cot\alpha}=(-1)^{1-n}\left[f[1-n]e^{\frac{1}{2}j(1-n)^2\cot\alpha}\right]^* \tag{85}$$

If we set $d_0[n]=d[n]e^{\frac{1}{2}jn^2\cot\alpha}$ and $f_0[n]=f[n]e^{\frac{1}{2}jn^2\cot\alpha}$, then Eq. (85) can be rewritten as

$$d_0[n]=(-1)^{1-n}f_0^*[1-n] \tag{86}$$

To summarize, the procedure to compute the coefficient $g[n]$ is as follows:

Step 1: Compute $h[n]$ based on the fractional scaling function as in Eq. (43), and then compute $\tilde{F}_\alpha(u)$ according to Eq. (49).

Step 2: Compute coefficients $f[n]$ by taking the discrete time IFRFT on $\tilde{F}_\alpha(u)$ as

$$f[n]=\int_0^{2\pi\sin\alpha}\tilde{F}_\alpha(u)K_\alpha^*(u,n)du \tag{87}$$

Step 3: Compute $d[n]$ based on the obtained $f[n]$ by using Eq. (85), and take the discrete time FRFT on $d[n]$ to compute $\tilde{D}_\alpha(u)$ as

$$\tilde{D}_\alpha(u)=\sum_{n\in\mathbb{Z}}d[n]K_\alpha(u,n) \tag{88}$$

Step 4: From Eq. (60), $g[n]$ is computed as

$$g[n]=\int_0^{2\pi\sin\alpha}\tilde{D}_\alpha(u)K_\alpha^*(2u,\frac{n}{2})du \tag{89}$$

This completes the construction of the orthogonal fractional wavelets. It is important to note that there will exist different fractional wavelet for different choice of function $\xi(u)$ in Eq. (82). However, we find that when $\xi(u)=e^{-jmu\csc\alpha}$, $m\in\mathbb{Z}$, we have

$$d[n]e^{\frac{1}{2}jn^2\cot\alpha}=(-1)^{m+1-n}\left[f[m+1-n]e^{\frac{1}{2}j(m+1-n)^2\cot\alpha}\right]^* \tag{90}$$

which is equivalent to

$$d_0[n]=(-1)^{1+m-n}f_0^*[1+m-n] \tag{91}$$

Since Eq. (91) is actually the same with Eq. (85), the unique orthogonal fractional wavelet can be constructed with this case.

5. Applications

Because the FRWT is the generalization of the classical WT, many of the applications of the WT can be extended for the FRWT. Two examples are presented in the following. First, the proposed FRWT is applied to a typical signal with time-varying frequency content. Second, the FRWT is used to estimate the FRFD spectrum of signals that involving noise. At last, the construction of fractional Harr wavelets is presented. Comparisons with results from a standard implementation of WT are provided.

5.1. Analysis of signals with time-varying frequency content

Consider a signal which is analyzed in [12], given as

$$f(t)=A_1(t)\cos(\omega_1 t)+B_2(t)\sin(\omega_2 t) \tag{92}$$

where $A_1(t)$ and $B_2(t)$ are two modulating functions

$$A_1(t)=\frac{\alpha_1}{\alpha_2-\alpha_1}\exp\left(\frac{\alpha_2}{\alpha_2-\alpha_1}\ln\left(\frac{\alpha_2}{\alpha_1}\right)\right)(\exp(-\alpha_1 t)-\exp(-\alpha_2 t)) \tag{93}$$

$$B_2(t)=\frac{\beta_1}{\beta_2-\beta_1}\exp\left(\frac{\beta_2}{\beta_2-\beta_1}\ln\left(\frac{\beta_2}{\beta_1}\right)\right)(\exp(-\beta_1 t)-\exp(-\beta_2 t)) \tag{94}$$

The following parameters are assumed: $\omega_1=5.0\,\mathrm{rad/s}$, $\omega_2=1.0\,\mathrm{rad/s}$, $\alpha_1=0.5$, $\alpha_2=1.0$, $\beta_1=0.1$, $\beta_2=10.0$. The signal that defined in Eq. (92) is illustrated in Fig. 1. Notice that the frequency content varies with time, with frequency ω_2 that prevails as the first modulating function $A_1(t)$ vanishes.

A real Morlet wavelet family is firstly considered, with mother wavelet

$$\varphi(t)=exp(-t^2/2)cos(5t) \tag{95}$$

Fig. 2 shows the modulus of the WT, $|W_f(a,b)|$, for various scales a and shifts b. It is noted that for such real Morlet wavelet, the WT of signal can be obtained in a closed form. Therefore, the solution of this example can be useful as benchmark solutions for the developed FRWT. The modulus of the FRWT, $|W_f^\alpha(a,b)|$, of the signal is shown in Fig. 3. Here, we

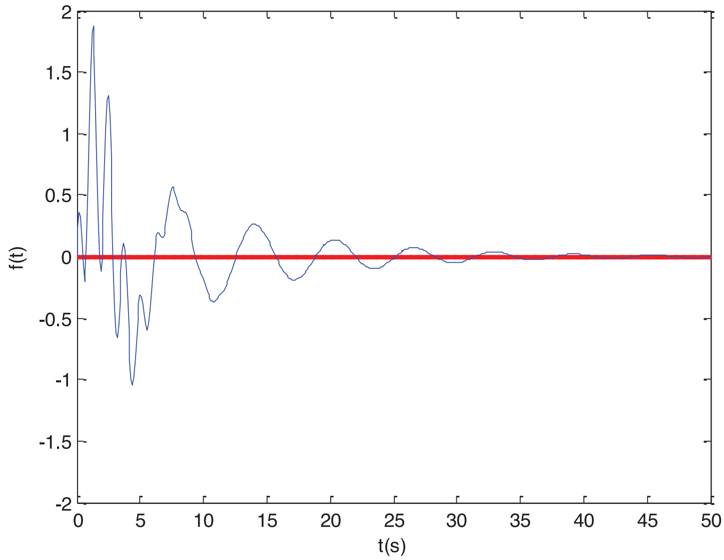

Fig. 1. Signal $f(t)$ in Eq. (92).

still use the mother wavelet in Eq. (95) but adopt the fractional order as $\alpha = \pi/4$. From Figs. 2 and 3, it is find that the resolution of the FRWT is obviously improved when compared with that of the WT, thus the superior resolution of the FRWT is demonstrated via fractional scalogram. Since the FRFD-spectrum obtained by FRWT is actually not the classical frequency, it can reveal more information of the signal by changing the value of α, the new FRWT has great potential for dealing with complex time-varying signal. It should be also noted that the fractional scalogram in Fig. 3 is obtained with the value $\alpha = \pi/4$, and thus the angle between the two axis, which represent the time and the FRFD spectrum, equals $\pi/4$. For the purpose of intuitively illustration, the rotation of coordinates is performed here to make the two axis orthogonal, and to keep the provided information unchanged.

5.2. FRFD spectrum estimation of signals involving noise

Consider the following signal with multi-frequency components and the chirp-type signal, with the analytical form

$$(a) \quad f(t) = \sin(100t) + \sin(200t) + \sin(300t) + e(t) \tag{96}$$

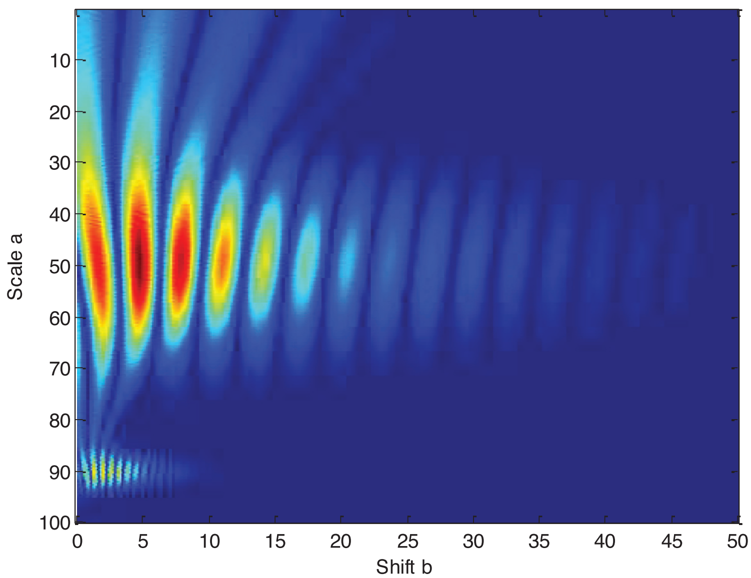

Fig. 2. Wavelet transform modulus $|W_f(a, b)|$ of signal $f(t)$.

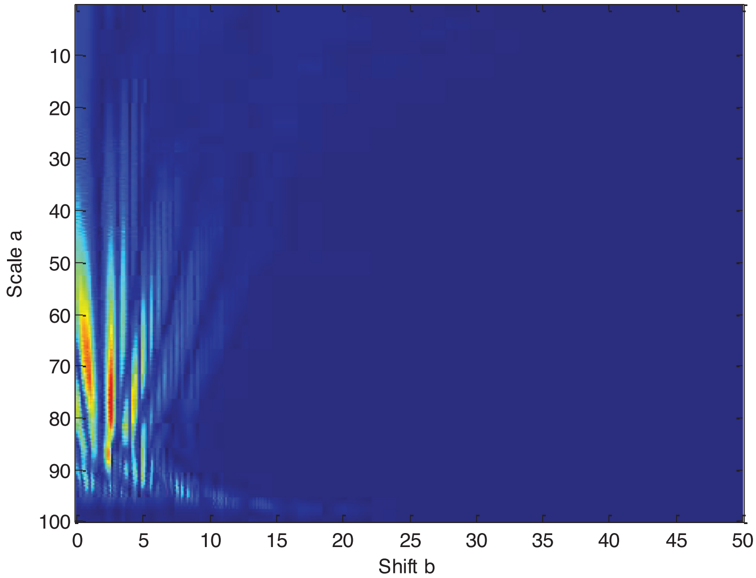

Fig. 3. Fractional wavelet transform modulus $|W_f^\alpha(a, b)|$ of signal $f(t)$.

$$(b) \quad f(t) = \sin(100t^2) + e(t) \tag{97}$$

where $e(t)$ is white Gaussian noise. The signals (a) and (b) are graphically described in Figs. 4(a), and 5(a), respectively. In this example, the WT and FRWT are used to estimate the spectrum and FRFD spectrum of the above signals, as shown in Figs. 4 and 5, respectively. The mother wavelet as defined in Eq. (95) is used to construct the fractional wavelet with order $\alpha = \pi/4$.

From Fig. 4, it can be seen that both WT and FRWT capture the three frequency components of the signal as given in Eq. (96). However, the frequency that induced by Gaussian noise is also identified by WT, as shown in Fig. 4(b). Although the resolution obtained by FRWT is not as good as that from WT, the FRFD spectrum correctly detects the real frequency components of the signal, demonstrating the advantage of the FRWT for signal denoising. It is noted that there exists shift between the FRFD spectrum and the true frequency of the signal, this is because the FRFD spectrum is actually not same with the classical spectrum.

For the chirp-type signal as given in Eq. (97), it is clear that FRWT correctly captures the frequency of the signal, as shown in Fig. 5(c), while the spectrum estimated by WT contains some other frequency contents that induced by Gaussian noise (See Fig. 5(b)), which makes the result confused. This example illustrates the advantage of FRWT for analyzing chirp-type signal. Since the chirp is included in the construction of the fractional wavelet, as is shown in Eq. (10), the FRWT has potential for dealing with this kind of signal.

5.3. Construction of fractional harr wavelets

The construction of wavelets is an important problem in classical WT theory, the example focus on this issue. If we take the fractional scaling function as $\phi_{\alpha,0,0}(t) = \chi_{[0,1)}(t)$, which is the same as the scaling function of classical Harr wavelet. Then the coefficients $g[n]$ can be obtained by following Step 1-Step 4 as given in Section 4.2. Figs. 6 and 7 describe the constructed fractional Harr wavelets with different fractional orders as $\alpha = \pi/4$ and $\alpha = \pi/2$, respectively. For the case $\alpha = \pi/4$, $g[n]$ are computed as

$$g[n] = \begin{cases} -0.0261 + 0.0432i, & n = 0 \\ 0.0341 + 0.0555i, & n = 1 \\ 0, & \text{others} \end{cases} \tag{98}$$

and the corresponding fractional Harr wavelet is obtained as

$$\varphi_{\alpha,0,0}(t) = \begin{cases} (-0.0261 + 0.0432i)e^{\frac{3}{2}jt^2}, & 0 \le t \le \frac{1}{2} \\ (0.0341 + 0.0555i)e^{\frac{1}{2}j\left(3t^2 - 4t + \frac{3}{4}\right)}, & \frac{1}{2} < t \le 1 \\ 0, & \text{others} \end{cases} \tag{99}$$

It can be seen from Fig. 6 that there exists three parts in the fractional Harr wavelet with $\alpha = \pi/4$, including the real part, the imaginary part and the amplitude part. This is due to the introduction of complex chirp in the construction of

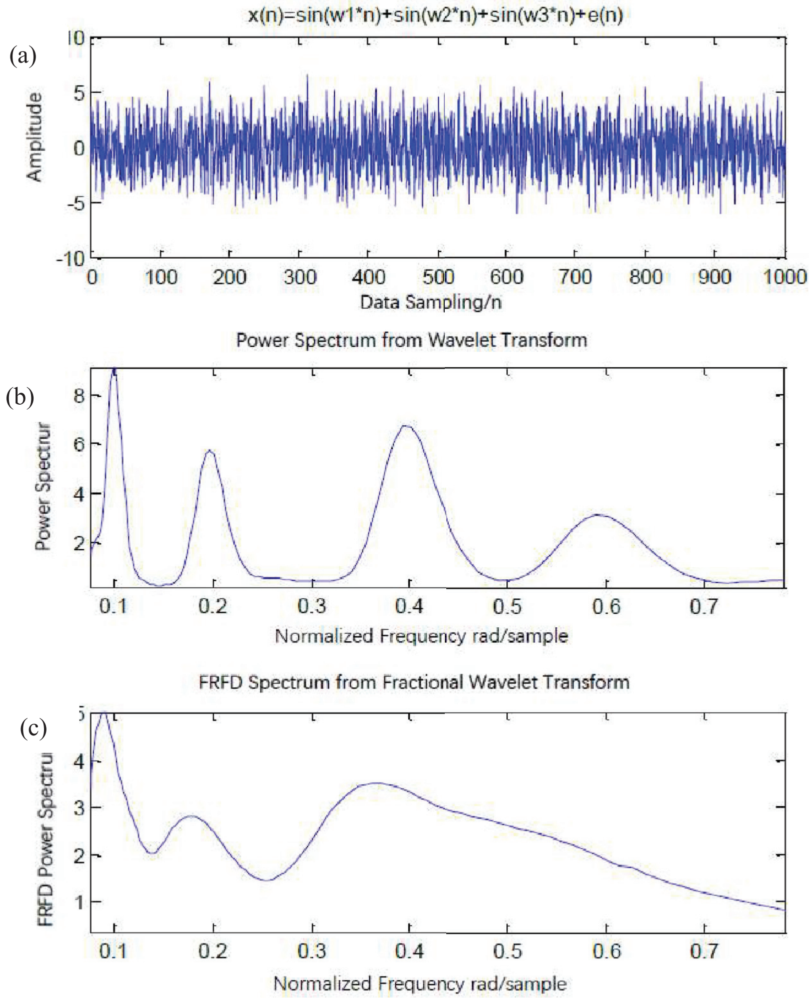

Fig. 4. Spectrum estimation of signal (a).

fractional wavelets. Compared with the classical Harr wavelet, the selection of the fractional Harr wavelet is more flexible by choosing various real part, the imaginary part and the amplitude part, and the different values of α. When $\alpha = \pi/2$, the fractional Harr wavelet reduces to the classical Harr wavelet, as shown in Fig. 7. In this case, the imaginary part of the fractional Harr wavelets becomes zero.

6. Conclusion

In this paper, we have derived a novel fractional wavelet transform (FRWT). It displays the time and FRFD-frequency information jointly in the time-FRFD-frequency plane, and it is the generalization of the classical wavelet transform. We also discuss the properties, inverse transform and reproducing kernel of the FRWT. By analyzing a typical signal with time-varying frequency content, the developed FRWT appears better resolution when compared with the classical WT. The de-noising and the resolution of the FRWT is also demonstrated by estimating the FRFD spectrum of signals that involving noise. The multiresolution analysis associated with the FRWT has been also derived and the procedure for constructing orthogonal fractional wavelet is presented. A fractional Harr wavelet is constructed with various value of fractional order. It has been shown that the selection of fractional Harr wavelet is more flexible when compared with classical Harr wavelet. Therefore, one can choose different fractional Harr wavelet to deal with specific problem to improve the resolution of the solution.

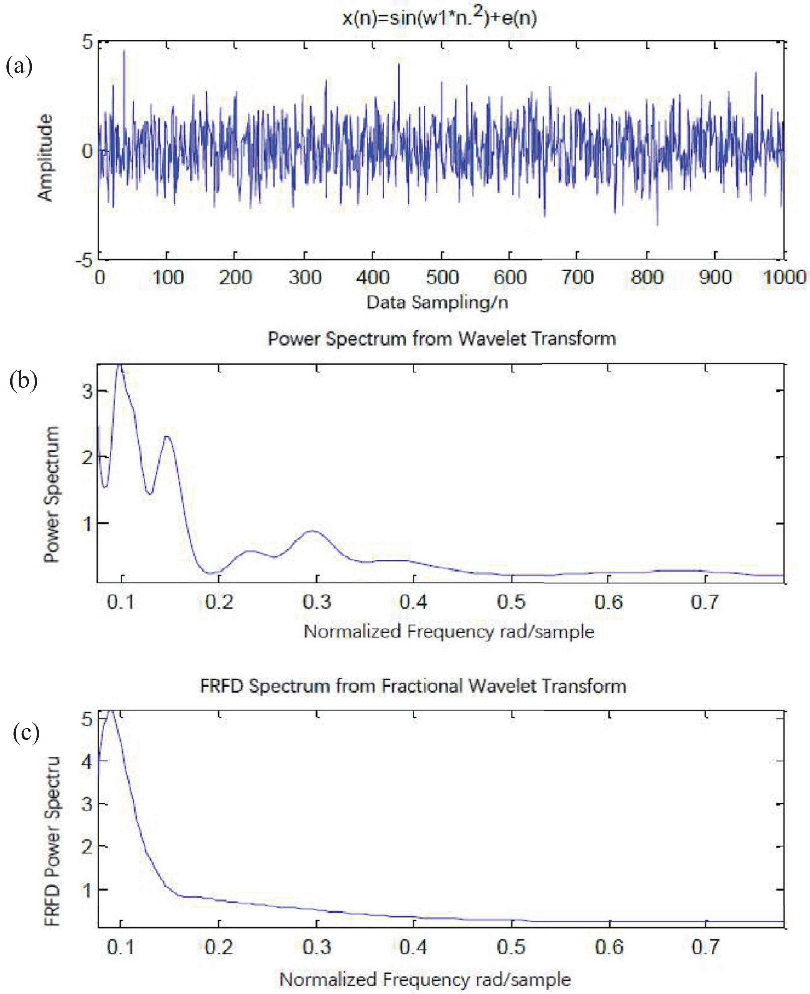

Fig. 5. Spectrum estimation of signal (*b*).

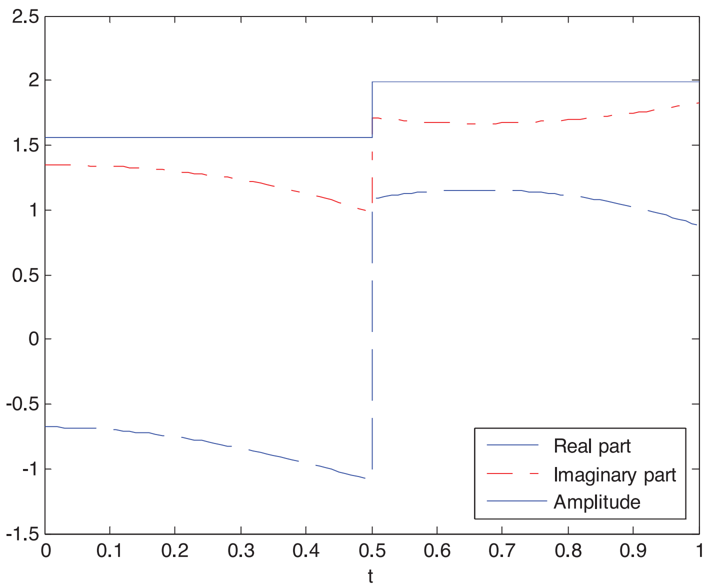

Fig. 6. Fractional Harr wavelet with $\alpha = \pi/4$.

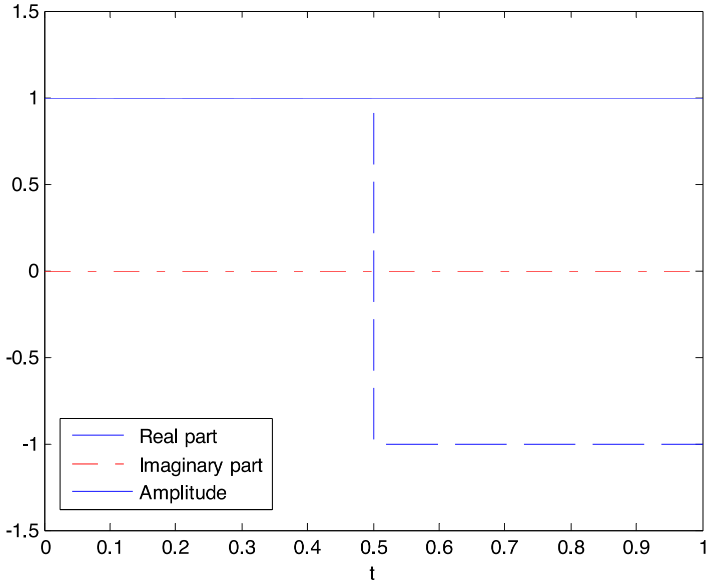

Fig. 7. Fractional Harr wavelet with $\alpha = \pi/2$.

Acknowledgments

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Appendix A

By taking the FRFT on both sides of Eq. (10), we have

$$\begin{aligned}
F_\alpha\left(\varphi_{\alpha,a,b}\right)(u) &= \int_{-\infty}^{+\infty} e^{-\frac{j}{2}\left(t^2-b^2-\left(\frac{t-b}{a}\right)^2\right)\cot\alpha} \frac{1}{\sqrt{a}}\varphi\left(\frac{t-b}{a}\right)K_\alpha(u,t)dt \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{a}}\varphi\left(\frac{t-b}{a}\right)A_\alpha e^{\frac{j}{2}\left(b^2+\left(\frac{t-b}{a}\right)^2+u^2\right)\cot\alpha - jtu\csc\alpha}dt \\
&= \frac{\sqrt{a}}{A_\alpha}e^{-\frac{j}{2}a^2u^2\cot\alpha -} \int_{-\infty}^{+\infty} \varphi\left(\frac{t-b}{a}\right)A_\alpha e^{\frac{j}{2}(b^2+u^2)\cot\alpha - jbu\csc\alpha} \times A_\alpha e^{\frac{j}{2}\left(\left(\frac{t-b}{a}\right)^2+a^2u^2\right)\cot\alpha - j\frac{t-b}{a}au\csc\alpha} d\frac{t-b}{a} \\
&= \frac{\sqrt{a}}{A_\alpha}e^{-\frac{j}{2}a^2u^2\cot\alpha}\int_{-\infty}^{\infty}\varphi(s)K_\alpha(au,s)K_\alpha(u,b)ds
\end{aligned} \tag{100}$$

From the Parseval identity of FRFT in Eq. (5) and substituting Eq. (100) into Eq. (11), we have

$$\begin{aligned}
W_x^\alpha(a,b) &= \langle X_\alpha(u), F_\alpha\left(\varphi_{\alpha,a,b}\right)(u)\rangle = \left\langle X_\alpha(u), \frac{\sqrt{a}}{A_\alpha}e^{-\frac{j}{2}a^2u^2\cot\alpha}\Phi_\alpha(au)K_\alpha(u,b)\right\rangle \\
&= \sqrt{\frac{2\pi a}{1+j\cot\alpha}}\int_{-\infty}^{+\infty} e^{\frac{j}{2}a^2u^2\cot\alpha}X_\alpha(u)\Phi_\alpha^*(au)K_{-\alpha}(u,t)du
\end{aligned} \tag{101}$$

Appendix B

From Eq. (12), we have

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^{+\infty} W_x^\alpha(a,b)\left[W_y^\alpha(a,b)\right]^* \frac{da}{a^2}db \\
&= \int_0^\infty \int_{-\infty}^{+\infty} \left[\sqrt{\frac{2\pi a}{1+j\cot\alpha}}\int_{-\infty}^{+\infty} e^{-\frac{j}{2}a^2u^2\cot\alpha}X_\alpha(u)\Phi_\alpha^*(au)K_{-\alpha}(u,t)du\right]
\end{aligned}$$

$$\left[\sqrt{\frac{2\pi a}{1+j\cot\alpha}}\int_{-\infty}^{+\infty} e^{-\frac{j}{2}a^2u'2\cot\alpha}Y_\alpha(u')\Phi_\alpha^*(au')K_{-\alpha}(u',t)du'\right]^*\frac{da}{a^2}db$$
$$=\int_0^{\infty}\int_{-\infty}^{+\infty}2\pi\sin\alpha\frac{da}{a}\int_{-\infty}^{+\infty}e^{-\frac{j}{2}a^2\left(u^2-u'2\right)\cot\alpha}X_\alpha(u)Y_\alpha^*(u')\Phi_\alpha^*(au)\Phi_\alpha(au')dudu'\int_{-\infty}^{+\infty}K_\alpha^*(u',t)K_\alpha(u,t)db$$
$$=\int_0^{\infty}\int_{-\infty}^{+\infty}2\pi\sin\alpha\frac{da}{a}\int_{-\infty}^{+\infty}e^{-\frac{j}{2}a^2\left(u^2-u'2\right)\cot\alpha}X_\alpha(u)Y_\alpha^*(u')\Phi_\alpha^*(au)\Phi_\alpha(au')\delta(u-u')dudu$$
$$=\int_0^{\infty}\int_{-\infty}^{+\infty}2\pi\sin\alpha\frac{da}{a}X_\alpha(u)Y_\alpha^*(u)|\Phi_\alpha(au)|^2du=2\pi\sin\alpha\int_{-\infty}^{+\infty}\int_0^{\infty}\frac{|\Phi_\alpha(au)|^2}{au}d(au)X_\alpha(u)Y_\alpha^*(u)du$$
$$=2\pi\sin\alpha\int_0^{\infty}\frac{|\Phi_\alpha(s)|^2}{s}ds\int_{-\infty}^{+\infty}X_\alpha(u)Y_\alpha^*(u)du=2\pi\sin\alpha C_\Phi\langle X_\alpha(u),Y_\alpha(u)\rangle=2\pi\sin\alpha C_\Phi\langle x(t),y(t)\rangle \tag{102}$$

Appendix C

The FRFT of $\phi_{\alpha,\,0,\,n}(t)$ is

$$\begin{aligned}F_\alpha(\phi_{\alpha,0,n}(t))(u)&=\int_{-\infty}^{+\infty}\phi(t-n)e^{-\frac{j}{2}\left(t^2-n^2-(t-n)^2\right)\cot\alpha}K_\alpha(u,t)du\\&=A_\alpha\int_{-\infty}^{+\infty}\phi(t-n)e^{\frac12 j\left((t-n)^2+u^2\right)\cot\alpha-jtu\csc\alpha+\frac12 jn^2\cot\alpha}du\\&=e^{\frac12 jn^2\cot\alpha-jnu\csc\alpha}A_\alpha\int_{-\infty}^{+\infty}\phi(t-n)e^{\frac12 j\left((t-n)^2+u^2\right)\cot\alpha-j(t-n)u\csc\alpha}du\\&=e^{\frac12 jn^2\cot\alpha-jnu\csc\alpha}\Theta_\alpha(u)\end{aligned}\tag{103}$$

From the Parseval identity of FRFT in Eq. (5) and the formula of FRFT of $\phi_{\alpha,\,0,\,n}(t)$, we have

$$\begin{aligned}\langle\phi_{\alpha,0,n}(t),\phi_{\alpha,0,m}(t)\rangle&=\langle F_\alpha(\phi_{\alpha,0,n}(t))(u),F_\alpha(\phi_{\alpha,0,m}(t))(u)\rangle\\&=\left\langle e^{\frac12 jn^2\cot\alpha-jnu\csc\alpha}\Theta_\alpha(u),e^{\frac12 jm^2\cot\alpha-jmu\csc\alpha}\Theta_\alpha(u)\right\rangle\\&=\int_{-\infty}^{+\infty}e^{\frac12 j\left(n^2-m^2\right)\cot\alpha-j(n-m)u\csc\alpha}|\Theta_\alpha(u)|^2du\\&=e^{\frac12 j\left(n^2-m^2\right)\cot\alpha}\int_{-\infty}^{+\infty}e^{-j(n-m)u\csc\alpha}|\Theta_\alpha(u)|^2du\end{aligned}\tag{104}$$

Since $\phi_{\alpha,\,0,\,n}(t)$ is an orthonormal basis of subspace V_0^α, we have

$$\langle\phi_{\alpha,0,n}(t),\phi_{\alpha,0,m}(t)\rangle=\delta_{n,m},\quad\forall n,m \tag{105}$$

If we set $n=n-m$, then it follows from Eqs. (104) and (105) that

$$\int_{-\infty}^{+\infty}e^{-jnu\csc\alpha}|\Theta_\alpha(u)|^2du=\delta_{n0} \tag{106}$$

which leads to

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\int_0^{2\pi\sin\alpha}e^{-jn(u+2k\pi\sin\alpha)\csc\alpha}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2du\\&=\int_0^{2\pi\sin\alpha}e^{-jnu\csc\alpha}\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2du=\delta_{n0}\end{aligned}\tag{107}$$

Let $F(u)=\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2k\pi\sin\alpha)|^2$, then we have

$$F(u+2\pi\sin\alpha)=\sum_{k\in\mathbb{Z}}|\Theta_\alpha(u+2(k+1)\pi\sin\alpha)|^2 \tag{108}$$

If we set $k'=k+1$ in Eq. (108), then we find

$$F(u+2\pi\sin\alpha)=\sum_{k'\in\mathbb{Z}}\left|\Theta_\alpha\left(u+2k'\pi\sin\alpha\right)\right|^2=F(u) \tag{109}$$

which indicates that $F(u)$ is $2k\pi\sin\alpha$-periodic function. Therefore, from Eq. (107), we have

$$\int_0^{2\pi\sin\alpha}e^{-jnu\csc\alpha}duF(u)=\delta_{n,0}=\delta_{n\csc\alpha,0} \tag{110}$$

If we set $n' = n \csc\alpha$ in Eq. (110), then

$$\int_0^{2\pi \sin\alpha} F(u) e^{-jn'u} du = \delta_{n',0} \tag{111}$$

by dividing $2\pi \sin\alpha$ on both sides of Eq. (111), we have

$$\frac{1}{2\pi \sin\alpha} \int_0^{2\pi \sin\alpha} F(u) e^{-jn'u} du = \frac{1}{2\pi \sin\alpha} \delta_{n',0} \tag{112}$$

which leads to

$$F(u) = 2\pi F^{-1}\left(\frac{1}{2\pi \sin\alpha} \delta_{n',0}\right)(u) = \frac{1}{\sin\alpha} \tag{113}$$

Therefore, we have

$$\sum_{k\in\mathbb{Z}} |\Theta_\alpha(u + 2k\pi \sin\alpha)|^2 = \frac{1}{\sin\alpha} \tag{114}$$

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