Journal of Algebra 469 (2017) 30–76

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Birational spaces ☆

Uri Brezner

A R T I C L E   I N F O

Article history:
Received 7 October 2015
Available online 30 August 2016
Communicated by Steven Dale Cutkosky

Keywords:
Algebraic geometry

A B S T R A C T

In this paper we construct the category of *birational spaces* as the category in which the relative Riemann–Zariski spaces of [9] are naturally included. Furthermore we develop an analogue of Raynaud's theory. We prove that the category of quasi-compact and quasi-separated birational spaces is naturally equivalent to the localization of the category of pairs of quasi-compact and quasi-separated schemes with an affine schematically dominant morphism between them localized with respect to simple relative blow ups and relative normalizations.

1. Introduction

In the 1930's and 1940's Zariski studied the problem of resolution of singularities for varieties of characteristic zero, and introduced the *Riemann–Zariski space* $RZ_K(k)$ of a finitely generated field extension $k \subset K$. Namely, $RZ_K(k)$ is the space of all valuations on K/k equipped with a (natural) topology, which can also be obtained as the projective limit of all projective models of K/k [10].

Zariski's original definition $RZ_K(k)$ (for example [11, Chapter VI §17]) did not include the trivial valuation of K, in contrast we follow Temkin [9, §2] and include the trivial valuation in the Riemann–Zariski space (as a generic point).

☆ This work was supported by the Israel Science Foundation (grant No. 1018/11).
E-mail address: uri.brezner@mail.huji.ac.il.

http://dx.doi.org/10.1016/j.jalgebra.2016.08.017

Given a separated morphism $f: Y \to X$ of quasi-compact and quasi-separated (qcqs) schemes, Temkin defined the *relative Riemann–Zariski space*, $RZ_Y(X)$, as the projective limit of the underlying topological spaces of all the Y-modifications of X [8,9]. He showed that $RZ_Y(X)$ is naturally homeomorphic to the space consisting of unbounded X-valuations on Y equipped with a natural topology. Furthermore the continuous map $f: Y \to X$ naturally factors as $Y \to RZ_Y(X) \to X$.

In the present paper we study the natural category that contains all qcqs schemes and the Riemann–Zariski spaces associated to them.We adopt the valuation point of view. Our approach is to first define for given rings $A \to B$ an affinoid birational space $\mathrm{Val}(B, A)$ of unbounded A-valuations on B. General birational spaces $\mathrm{Val}(Y, X)$ are glued from affinoid ones along affinoid sub-domains.

Furthermore, we show that our construction is natural, by which we mean that it admits an analogue to Ryanaud's theory: Let R be a valuation ring of Krull dimension 1, complete with respect to the J-adic topology generated by a principal ideal $J = (\pi) \subset R$, where π is some non-zero element of the maximal ideal of R, and K the fraction field of R. On the one hand we have the category of admissible formal R-schemes, while on the other hand we have the category of rigid K-spaces. It was Raynaud [7] who suggested to view rigid spaces entirely within the framework of formal schemes. Elaborating the ideas of Raynaud, it is proved in [1] that the category of admissible formal R-schemes, localized with respect to the class of admissible formal blow ups, is naturally equivalent to the category of rigid K-spaces which are qcqs. We show that the category of pairs of qcqs-schemes with an affine, schematically dominant morphism between them, localized with respect to simple relative blow ups and relative normalizations, is naturally equivalent to the category of qcqs-birational spaces.

We note that it should be possible to further define the categories of modules and coherent modules of a birational spaces and develop a theory of cohomology of (coherent) sheaves on birational spaces. A further attestment to the "correctness" of our construction will be an analogue to Grothendieck's existence theorem [3, 5.1.4], that is a complete classification of coherent modules of $\mathrm{Val}(Y, X)$ in terms of coherent modules of X and Y. We leave this work for a future paper.

Henceforth, we restrict our attention to the case of affine, schematically dominant morphisms $f: Y \to X$ of qcqs-schemes, since it is essentially the same as assuming that $f: Y \to X$ is a separated morphism: Indeed, by Temkin's decomposition theorem [9, Theorem 1.1.3] any separated morphism $f: Y \to X$ of qcqs-schemes factors as $Y \xrightarrow{j} Z \to X$, where $j: Y \to Z$ is an affine, schematically dominant morphism and $Z \to X$ is proper. It will become clear from the construction that $\mathrm{Val}(Y, X) = \mathrm{Val}(Y, Z)$ by the valuative criterion for properness, so our results hold for separated morphisms.

The paper is organized as follows. Let $A \subset B$ be commutative rings with unity. In Section 2 we construct a topological space $\mathrm{Spa}(B, A)$, consisting of all A-valuations of B, and its subspace $\mathrm{Val}(B, A)$, consisting of unbounded valuations, which is our main interest in Section 2. We call $\mathrm{Val}(B, A)$ the affinoid birational space associated to the pair of rings $A \subset B$. We study some of the topological properties of these spaces and endow

$\mathrm{Val}(B, A)$ with two sheaves of rings $\mathcal{O}_{\mathrm{Val}(B,A)} \subset \mathcal{M}_{\mathrm{Val}(B,A)}$ both making $\mathrm{Val}(B, A)$ a locally ringed space. The main highlight of Section 3 is the proof that the functor *bir*, that takes a pair of rings $A \subset B$ to the affinoid birational space $\mathrm{Val}(B, A)$, gives rise to an anti-equivalence from the localization of the category of pairs of rings with respect to relative normalizations to the category of affinoid birational spaces. Also in Section 3, we globalize the construction by introducing general birational spaces. These are topological spaces equipped with a pair of sheaves such that the space is locally ringed with respect to both sheaves and is locally isomorphic to an affinoid birational space. Furthermore we extend *bir* to a functor from the category consisting of affine and schematically dominant morphism $Y \xrightarrow{f} X$ between qcqs schemes to the category of birational spaces. Section 5 is dedicated to the proof of

Main Theorem. *The category of pairs of qcqs schemes with an affine schematically dominant morphism between them, localized with respect to simple relative blow ups and relative normalizations, is naturally equivalent to the category of qcqs birational spaces.*

In order to achieve this goal, in Section 4 we further develop the theory of relative blow ups (called Y-blow ups of X in [9, §3.4]). In particular we prove the universal property of relative blow ups.

2. Construction of the space Val(B, A)

Throughout the paper all rings are assumed to be commutative with unity.

2.1. Valuations on rings

In this Subsection we fix terminology and collect general known facts about valuations.

Given a totally ordered abelian group Γ (written multiplicatively), we extend Γ to a totally ordered monoid $\Gamma \cup \{0\}$ by the rules

$$0 \cdot \gamma = \gamma \cdot 0 = 0 \text{ and } 0 < \gamma \qquad \forall\, \gamma \in \Gamma.$$

Definition 2.1.1. Let B be a ring and Γ a totally ordered group. A *valuation* v on B is a map $v\colon B \to \Gamma \cup \{0\}$ satisfying the conditions

- $v(1) = 1$ and $v(0) = 0$
- $v(xy) = v(x)v(y)\ \forall x, y \in B$
- $v(x + y) \le \max\{v(x), v(y)\}\ \forall x, y \in B$.

Note that $\mathfrak{p} = \ker v = \{b \in B \mid v(b) = 0\}$ is a prime ideal in B.

Remark 2.1.2. When B is a field the above definition coincides with the classical definition of a valuation with the value group written multiplicatively.

Let v be a valuation on B with kernel $\mathfrak{p}$. Denote the residue field of $\mathfrak{p}$ by $k(\mathfrak{p})$. Clearly v factors through the quotient $B/\mathfrak{p}$. Since v is multiplicative it further uniquely extends to $k(\mathfrak{p})$ and we obtain a valuation in the classical sense $\bar{v}\colon k(\mathfrak{p}) \to \Gamma \cup \{0\}$. On the other hand a prime ideal $\mathfrak{p} \in \operatorname{Spec} B$ and a valuation $\bar{v}$ on the residue field $k(\mathfrak{p})$ uniquely determine a valuation v on B with kernel $\mathfrak{p}$ by setting $v(b) = \bar{v}(\bar{b})$ where $\bar{b}$ is the image of b in $k(\mathfrak{p})$. Hence giving a valuation v on B is equivalent to giving a prime ideal $\mathfrak{p}$ and a valuation $\bar{v}$ on the residue field $k(\mathfrak{p})$.

Two valuations v_1, v_2 on B are said to be *equivalent* if $\ker v_1 = \ker v_2 = \mathfrak{p}$ and the induced valuations $\bar{v}_1, \bar{v}_2$ on $k(\mathfrak{p})$ are equivalent in the classical sense, i.e. they have the same valuation ring or, equivalently, there is an order preserving group isomorphism between their images compatible with the valuations. Every valuation v_1 on B is equivalent to a valuation v_2 such that the value group Γ is generated, as an abelian group, by $v_2(B - \mathfrak{p})$. We will usually identify equivalent valuations without mention.

With this convention a valuation v on B with kernel $\mathfrak{p}$ uniquely defines a valuation ring contained in $k(\mathfrak{p})$ by

$$R_v = \{x \in k(\mathfrak{p}) \mid \bar{v}(x) \leq 1\}.$$

Hence a valuation v on B is equivalent to a diagram

$$B \longrightarrow k(\mathfrak{p}) \hookleftarrow R_v.$$

Definition 2.1.3. Let B be a ring, A a subring and v a valuation on B. We call v an A-*valuation* on B if $v(a) \leq 1$ for every $a \in A$.

Assume v is an A-valuation with kernel $\mathfrak{p}$. From the condition $v(A) \leq 1$ the image of $A \to B \to k(\mathfrak{p})$ lies in R_v. We conclude that every A-valuation v on B uniquely defines a commutative diagram

$$\begin{tikzcd} B \arrow[r] & k(\mathfrak{p}) \\ A \arrow[u, hook] \arrow[r, "\Phi"] & R_v. \arrow[u, hook] \end{tikzcd} \tag{1}$$

Conversely any such diagram defines an A-valuation v on B and we are justified in identifying the A-valuation v on B with the 3-tuple $(\mathfrak{p}, R_v, \Phi)$.[1]

2.2. The auxiliary space Spa(B, A)

For completeness and consistency of notation we collect here results regarding valuation spectra. The main reference of this subsection is [6].

[1] Although Φ is completely determined by $B \to k(\mathfrak{p})$ we prefer to keep Φ in the notation.

Definition 2.2.1. For any pair of rings $A \subset B$ we set

$$\mathrm{Spa}(B, A) = \{A\text{-valuations on } B\}.$$

Fix a pair of rings $A \subset B$. We provide $\mathrm{Spa}(B, A)$ with a topology. For any $a, b \in B$ set

$$U_{a,b} = \{v \in \mathrm{Spa}(B, A) \mid v(a) \leq v(b) \neq 0\}.$$

The topology is the one generated by the sub-basis $\{U_{a,b}\}_{a,b \in B}$.

Given another pair of rings $A' \subset B'$ and a homomorphism of rings $\varphi \colon B \to B'$ that satisfies $\varphi(A) \subset A'$, composition with φ gives rise to the pull back map

$$\varphi^* \colon \mathrm{Spa}(B', A') \to \mathrm{Spa}(B, A).$$

Specifically given an A'-valuation $v' = (\mathfrak{p}', R_{v'}, \Phi') \in \mathrm{Spa}(B', A')$, then $v' \circ \varphi$ is a valuation on B. Since $\varphi(A) \subset A'$, $v' \circ \varphi$ is an A-valuation. So indeed $\varphi^*(v') = v' \circ \varphi \in \mathrm{Spa}(B, A)$. Clearly its kernel is $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}')$. Now, we have a commutative diagram

$$\begin{tikzcd}
 & B' \arrow[rr] & & k(\mathfrak{p}') \\
B \arrow[ur, "\varphi"] \arrow[rr] & & k(\mathfrak{p}) \arrow[ur] & \\
 & A' \arrow[uu] \arrow[rr, "\Phi'"] & & R_{v'} \arrow[uu] \\
A \arrow[uu] \arrow[ur] \arrow[rr, "\varphi^*(\Phi')"] & & R_{v' \circ \varphi} \arrow[uu] \arrow[ur] &
\end{tikzcd} \quad . \tag{2}$$

It is clear that $R_{v' \circ \varphi} = R_{v'} \cap k(\mathfrak{p})$ and that the ring map $\Phi = \varphi^*(\Phi')$ is completely determined by Φ' and φ. To conclude, φ^* takes the point $(\mathfrak{p}', R_{v'}, \Phi') \in \mathrm{Spa}(B', A')$ to the point $(\mathfrak{p}, R_{v'} \cap k(\mathfrak{p}), \Phi) \in \mathrm{Spa}(B, A)$.

Clearly $U_{\varphi(a), \varphi(b)} = {\varphi^*}^{-1}(U_{a,b})$ for any $a, b \in B$. We obtain

Lemma 2.2.2. *Let $A \subset B$ and $A' \subset B'$ be rings. For a homomorphism $\varphi \colon B \to B'$ satisfying $\varphi(A) \subset A'$ the pull back map $\varphi^* \colon \mathrm{Spa}(B', A') \to \mathrm{Spa}(B, A)$ is continuous.*

Certain subsets of $\mathrm{Spa}(B, A)$ can be naturally identified with Spa of some pair of rings.

Lemma 2.2.3. *Let $b, a_1, \ldots, a_n \in B$ and assume that $b, a_1, \ldots, a_n$ generate the unit ideal. Let φ_b be the natural homomorphism $B \to B_b$. Set $B' = B_b$ and $A' = \varphi_b(A)\left[\frac{a_1}{b}, \ldots, \frac{a_n}{b}\right]$. Then*

(1) *the pull back map* $\mathrm{Spa}\,(B', A') \to \mathrm{Spa}(B, A)$ *is injective*
(2) $\{v \in \mathrm{Spa}(B, A) \mid v(a_i) \le v(b) \quad \forall\, 1 \le i \le n\} = \mathrm{Spa}\,(B', A')$
(3) $\mathrm{Spa}\,(B', A') = \bigcap_{i=1}^{n} U_{a_i,b}$, *in particular* $\mathrm{Spa}\,(B', A')$ *is open in* $\mathrm{Spa}\,(B, A)$.

Proof. (1) Obviously $A' \subset B'$ so $\mathrm{Spa}(B', A')$ is defined. We also obtain a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\varphi_b"] & B' \\ A \arrow[u, hook] \arrow[r] & A' \arrow[u, hook] \end{tikzcd}$$

which gives rise to the pull back map $\varphi_b^*\colon \mathrm{Spa}(B', A') \to \mathrm{Spa}(B, A)$. If $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Spa}(B, A)$ is in the image and $v' = (\mathfrak{p}', R_{v'}, \Phi') \in \mathrm{Spa}(B', A')$ maps to v, then necessarily $b \notin \mathfrak{p}$ and $\mathfrak{p}' = \mathfrak{p}B'$. Hence $k(\mathfrak{p}') = k(\mathfrak{p})$, from which it follows that $R_{v'} = R_v$ and we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, k(\mathfrak{p}) \arrow[r] \arrow[d] & \mathrm{Spec}\, B' \arrow[r] & \mathrm{Spec}\, A' \arrow[d] \\ \mathrm{Spec}\, R_v \arrow[urr, "\Phi'"] \arrow[rr, "\Phi"] & & \mathrm{Spec}\, A. \end{tikzcd}$$

Since $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ is separated, Φ' is unique so $v' = (\mathfrak{p}', R_{v'}, \Phi')$ is unique.

(2) Consider $v' \in \mathrm{Spa}(B', A')$ as an element of $\mathrm{Spa}(B, A)$. As $\frac{a_i}{b} \in A'$ for all $i = 1, \dots, n$, and $b \notin \ker v'$, we see that $v'(a_i) \le v'(b) \neq 0$ for every $i = 1, \dots, n$. Hence $\mathrm{Spa}(B', A') \subset \{v \in \mathrm{Spa}(B, A) \mid v(a_i) \le v(b) \quad \forall\, 1 \le i \le n\}$.

Now let $v' \in \{v \in \mathrm{Spa}(B, A) \mid v(a_i) \le v(b) \quad \forall\, 1 \le i \le n\}$. Since $(b, a_1, \dots, a_n) = B$ there are $c_0, c_1, \dots, c_n$ in B such that $1 = c_0 b + c_1 a_1 + \ldots + c_n a_n$. Applying v' we obtain

$$1 = v'(1) = v'(c_0 b + c_1 a_1 + \ldots + c_n a_n) \le$$
$$\le \max\{v'(c_0 b), v'(c_1 a_1), \dots, v'(c_n a_n)\} \le v'(b) \max_{0 \le i \le n} \{v'(c_i)\}$$

so necessarily $v'(b) \neq 0$. Hence we can extend v' to a valuation on B' by setting $v'(\frac{b'}{b}) = \frac{v'(b')}{v'(b)}$ for any $b' \in B$. Since $v'(c) \le 1 \quad \forall\, c \in A$ we have $v'(A') \le 1$ so $v' \in \mathrm{Spa}(B', A')$. It follows that $\{v \in \mathrm{Spa}(B, A) \mid v(a_i) \le v(b) \quad \forall\, 1 \le i \le n\} \subset \mathrm{Spa}(B', A')$, hence we have equality.

(3) It is clear from the definition that

$$\{v \in \mathrm{Spa}(B, A) \mid v(a_i) \le v(b) \quad \forall\, 1 \le i \le n\} = \bigcap_{i=1}^{n} U_{a_i,b}. \quad \square$$

Definition 2.2.4. For $A' \subset B'$ as in Lemma 2.2.3 we call $\mathrm{Spa}(B', A')$ a *rational domain* of $\mathrm{Spa}(B, A)$, denote it by $\mathcal{R}(\{a_1, \ldots, a_n\}/b)$ and regard it as a subset of $\mathrm{Spa}(B, A)$.

The rational domains are similar in many ways to the basic open subsets $D(b)$ of an affine scheme $\mathrm{Spec}\, B$. Lemma 2.2.3 shows the analogue of the fact that $D(b) \subset \mathrm{Spec}\, B$ is open and canonically homeomorphic to $\mathrm{Spec}\, B_b$. The basic opens also have the properties

(1) $D(b) \cap D(c) = D(bc)$
(2) $D(b) = \emptyset \Leftrightarrow b$ is a nilpotent element
(3) $D(b^r) = D(b)$
(4) if $s \in B$ is invertible then $D(b) = D(sb)$

Here is the precise formulation of the analogues for rational domains

Lemma 2.2.5. *Let* $\mathcal{R}\left(\{a_i\}_{i=1}^n/a_0\right)$ *and* $\mathcal{R}\left(\{a'_j\}_{j=1}^m/a'_0\right)$ *be rational domains of* $\mathrm{Spa}(B, A)$. *Then*

(1) $\mathcal{R}\left(\{a_i\}_{i=1}^n/a_0\right) \cap \mathcal{R}\left(\{a'_j\}_{j=1}^m/a'_0\right) = \mathcal{R}\left(\{a_i a'_j\}_{\substack{0\le i\le n\\ 0\le j\le m}}/a_0a'_0\right)$
(2) $\mathcal{R}\left(\{a_i\}_{i=0}^n/a_0\right) = \emptyset \Leftrightarrow a_0$ *is a nilpotent element*
(3) $\mathcal{R}\left(\{a_i^r\}_{i=0}^n/a_0^r\right) = \mathcal{R}\left(\{a_i\}_{i=0}^n/a_0\right)$ *for any* $r \in \mathbb{N}$
(4) *if* $s \in B$ *is invertible then* $\mathcal{R}\left(\{a_i\}_{i=1}^n/a_0\right) = \mathcal{R}\left(\{sa_i\}_{i=1}^n/sa_0\right)$

Proof. (1) By definition both $a_0, \ldots, a_n \in B$ and $a'_0, \ldots, a'_m \in B$ generate the unit ideal. It follows that $\{a_i a'_j\}_{\substack{0\le i\le n\\ 0\le j\le m}}$ also generates the unit ideal and $\mathcal{R}\left(\{a_i a'_j\}_{i,j}/a_0a'_0\right)$ is defined. By Lemma 2.2.3

$$\mathcal{R}\left(\{a_i\}_{i=0}^n/a_0\right) = \cap_i U_{a_i,a_0} \quad , \quad \mathcal{R}\left(\{a'_j\}_{j=0}^m/a'_0\right) = \cap_j U_{a'_j,a'_0}$$

and

$$\mathcal{R}\left(\{a_i a'_j\}_{i,j}/a_0a'_0\right) = \cap_{i,j} U_{a_i a'_j, a_0 a'_0}.$$

Since

$$U_{a,b} \cap U_{c,d} = \{v \in \mathrm{Spa}(B, A) | v(ad), v(bc) \le v(bd) \neq 0\}$$

the claim follows.

(2) If a_0 is nilpotent there is some $r > 0$ such that $a_0^r = 0$. For any valuation v we have $0 = v(a_0^r) = v(a_0)^r$ so $v(a_0) = 0$. From this its follows that for any $a_1, \ldots, a_n \in B$ such that $(a_0, a_1, \ldots, a_n) = B$ we have $\mathcal{R}(\{a_1, \ldots, a_n\}/a_0) = \emptyset$. If a_0 is not nilpotent, there is a prime ideal $\mathfrak{p}$ not containing a_0. Now for any $a_1, \ldots, a_n \in B$ such that $(a_0, a_1, \ldots, a_n) = B$ the rational domain $\mathcal{R}(\{a_1, \ldots, a_n\}/a_0)$ contains the trivial valuation of $k(\mathfrak{p})$.

(3) As $a_0, \dots, a_n \in B$ generate the unit ideal so do $a_0^r, \dots, a_n^r$ and $\mathcal{R}\left(\{a_i^r\}_{i=0}^n / a_0^r\right)$ is defined. By Lemma 2.2.3

$$\mathcal{R}\left(\{a_i\}_{i=0}^n / a_0\right) = \{v \in \mathrm{Spa}(B,A) \mid v(a_i) \leq v(a_0) \quad \forall\, 1 \leq i \leq n\},$$
$$\mathcal{R}\left(\{a_i^r\}_{i=0}^n / a_0^r\right) = \{v \in \mathrm{Spa}(B,A) \mid v(a_i^r) \leq v(a_0^r) \quad \forall\, 1 \leq i \leq n\}$$

and the equality is clear.

(4) Obvious. □

Remark 2.2.6. Let $\mathcal{R}\left(\{a_i\}_{i=1}^n / a_0\right)$ and $\mathcal{R}\left(\{a'_j\}_{j=1}^m / a'_0\right)$ be rational domains of $\mathrm{Spa}(B,A)$. The intersection $\mathcal{R}\left(\{a_i\}_{i=1}^n / a_0\right) \cap \mathcal{R}\left(\{a'_j\}_{j=1}^m / a'_0\right)$ consists of all valuations v satisfying the inequalities $v(a_i a'_j) \leq v(a_0 a'_0)$ for i and j such that $ij = 0$. However, the elements $\{a_i a'_j\}_{ij=0}$ do not necessarily generate the unit ideal, for instance if $a_0 = a'_0$.

By a *rational covering* of $\mathrm{Spa}(B,A)$ we mean an open cover $\left\{\mathcal{R}(\{a_i\}_{i=1}^n / a_j)\right\}_{j=1}^n$ where $a_1, \dots, a_n \in B$ generate the unit ideal.

In [6], Huber defines the valuation spectrum of a ring B

$$\mathrm{Spv}(B) = \{\text{valuations on } B\}.$$

He provides it with the topology generated by the sub-basis consisting of sets of the form $\{v | v(a) \leq v(b) \neq 0\}$ for all $a, b \in B$. Huber proves in [6, 2.2] that $\mathrm{Spv}(B)$ is a spectral space. Clearly our $\mathrm{Spa}(B,A)$ is a subspace of Huber's $\mathrm{Spv}(B)$.

Lemma 2.2.7. *The topological space* $\mathrm{Spa}(B,A)$ *is spectral. In particular it is quasi-compact and* T_0*.*

Proof. For a spectral space X we denote by X_{cons} the same set equipped with the constructible topology i.e. the topology which has the quasi-compact open sets and their complements as an open sub-basis (cf. [5, §2] and [6, §2]). As the injection $\mathrm{Spa}(B,A) \hookrightarrow \mathrm{Spv}\, B$ is continuous it is enough to show that the image of $\mathrm{Spa}(B,A)$ is closed in $\mathrm{Spv}(B)_{\mathrm{cons}}$ (see [5, §2]). Following Huber's argument we just need to show that the set of binary relations $|$ of $\phi(SpvB)$ that satisfy $1 \mid a \ \forall a \in A$ is closed in $\phi(SpvB)$. If $|' \in \phi(SpvB) - \phi(\mathrm{Spa}(B,A))$ then there is an element $a \in A$ such that $1 \nmid' a$. The set V_a of binary relations satisfying $1 \nmid a$ contains $|'$ and is open in $\{0,1\}^{B \times B}$. Now, $V_a \cap \phi(\mathrm{Spv}\, B)$ is open in $\phi(\mathrm{Spv}\, B)$ (with the topology induced from $\{0,1\}^{B \times B}$), contains $|'$ and

$$(V_a \cap \phi(\mathrm{Spv}\, B)) \bigcap \phi(\mathrm{Spa}(B,A)) = \emptyset.$$

So $\phi(\mathrm{Spa}(B,A))$ is closed in $\phi(\mathrm{Spv}\, B)$ (with the induced topology). Hence $\mathrm{Spa}(B,A)_{\mathrm{cons}}$ is closed in $\mathrm{Spv}\, B_{\mathrm{cons}}$. □

2.3. The space Val(B, A)

We say that a valuation $v\colon B \to \Gamma \cup \{0\}$ is *bounded* if there is an element $\gamma \in \Gamma$ such that $v(b) < \gamma$ for every $b \in B$.

Definition 2.3.1. For any pair of rings $A \subset B$ we set

$$\mathrm{Val}(B, A) = \{v \in \mathrm{Spa}(B, A) \mid v \textit{ is unbounded}\}$$

equipped with the induced subspace topology from Spa(B, A).

As a subspace of a T_0 space, Val(B, A) is also a T_0 space.

For a valuation v on B with abelian group Γ we denote by $c\Gamma_v$ the convex subgroup of Γ generated by $\{v(b) \mid b \in B \quad 1 \leq v(b)\}$. For any convex subgroup Λ of Γ we define a map $v'\colon B \to \Lambda \cup \{0\}$ by $v'(b) = \begin{cases} v(b) & \text{if } v(b) \in \Lambda \\ 0 & \text{if } v(b) \notin \Lambda \end{cases}$.

A direct verification shows that v' is a valuation on B if and only if $c\Gamma_v \subset \Lambda$. The valuation v' obtained in this way is called a *primary specialization* of v associated with Λ (see [4, §1.2]). Note that $\ker v \subset \ker v'$.

Remark 2.3.2. By construction, for any valuation v on B, the primary specialization of v associated with $c\Gamma_v$ is not bounded. It follows that a valuation v is not bounded if and only if it has no primary specialization other than itself.

Lemma 2.3.3. *Let $A \subset B$ and $A' \subset B'$ be rings, and $\varphi\colon B \to B'$ a homomorphism satisfying $\varphi(A) \subset A'$. Assume $v, w \in \mathrm{Spa}(B', A')$ such that w is a primary specialization of v. Then $\varphi^*(w)$ is a primary specialization of $\varphi^*(v)$.*

Proof. Assume that $v\colon B' \to \Gamma \cup \{0\}$ and that Λ is the convex subgroup of Γ associated with w. Then for $b' \in B'$ we have $w(b') = \begin{cases} v(b') & \text{if } v(b') \in \Lambda \\ 0 & \text{if } v(b') \notin \Lambda \end{cases}$. It now follows that for $b \in B$ we have $w(\varphi(b)) = \begin{cases} v(\varphi(b)) & \text{if } v(\varphi(b)) \in \Lambda \\ 0 & \text{if } v(\varphi(b)) \notin \Lambda \end{cases}$. □

For $v \in \mathrm{Spa}(B, A)$, let P_v be the subset of all primary specializations of v. Specialization induces a partial order on P_v by the rule $u \leq w$ if u is a primary specialization of w for $u, w \in P_v$.

Proposition 2.3.4. *For any $v \in \mathrm{Spa}(B, A)$, the set P_v of primary specializations of v is totally ordered and has a minimal element.*

Proof. Let $v\colon B \to \Gamma \cup \{0\}$ be a valuation on B. Let $w\colon B \to \Lambda \cup \{0\}$ and $u\colon B \to \Delta \cup \{0\}$ be two distinct primary specializations of v. We may regard Λ and Δ as convex subgroups of Γ, so one is contained in the other. As both w and u are primary specialization of v, both Λ and Δ contain $c\Gamma_v$. Assume $\Delta \subset \Lambda$. We want to show that u is a primary specialization of w, i.e. $u(b) = \begin{cases} w(b) & \text{if } w(b) \in \Delta \\ 0 & \text{if } w(b) \notin \Delta \end{cases}$.

For any $b \in B$ if $w(b) > 1$ then $v(b) = w(b)$, hence $c\Lambda_w \subset c\Gamma_v$. Conversely if $v(b) > 1$ then since $c\Gamma_v \subset \Lambda$ we have $w(b) = v(b)$, so $c\Lambda_w = c\Gamma_v$. It follows that Δ is a convex subgroup of Λ containing $c\Lambda_w$.

For any $b \in B$, if $w(b) \in \Delta$ then $w(b) = v(b) \in \Delta$. It follows that $u(b) = v(b) = w(b)$. If $w(b) \notin \Delta$ then either $w(b) = 0$ or $0 \neq w(b) \in \Lambda$. If $w(b) = 0$ then $v(b) \notin \Lambda$, hence $v(b) \notin \Delta$ and $u(b) = 0$. If $w(b) \neq 0$ then $w(b) = v(b) \notin \Delta$, so $u(b) = 0$.

The minimal element of P_v is the primary specialization associated with $c\Gamma_v$. □

Next, we give an algebraic criterion for a valuation $v \in \mathrm{Spa}(B, A)$ to be in $\mathrm{Val}(B, A)$.

Lemma 2.3.5. *Let $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Spa}(B, A)$. Then $v \in \mathrm{Val}(B, A)$ if and only if the canonical map $B \otimes_A R_v \to k(\mathfrak{p})$ is surjective.*

Proof. Assume that $B \otimes_A R_v \to k(\mathfrak{p})$ is surjective. Since we assume that Γ is generated by the image of $B - \mathfrak{p}$, for any $1 < \gamma \in \Gamma$ there is $0 \neq f \in k(\mathfrak{p})$ satisfying $\gamma = \bar{v}(f)$. Let $b_i \in B$, $r_i \in R_v$ for $i = 1, \dots, n$ such that $\sum b_i \otimes r_i \in B \otimes_A R_v$ maps to f in $k(\mathfrak{p})$. Assume $v(b_1) = max\{v(b_i)\}$. Denote the image of b_i in $k(\mathfrak{p})$ by $\bar{b_i}$. Now, as $\bar{v}(r_i) \leq 1$, we have

$$\gamma = \bar{v}(f) = \bar{v}(\sum \bar{b_i} r_i) \leq max\{\bar{v}(\bar{b_i} r_i)\} \leq max\{\bar{v}(\bar{b_i})\} = v(b_1).$$

Hence γ does not bound v.

Conversely, for $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Val}(B, A)$ we have a diagram

$$\begin{tikzcd}
 & & k(\mathfrak{p}) \arrow[r, "\bar{v}"] & \Gamma \cup \{0\} \\
B \arrow[urr] \arrow[r] & B \otimes_A R_v \arrow[ur, dashed] & & \\
A \arrow[u] \arrow[r] & R_v. \arrow[u] \arrow[uur] & &
\end{tikzcd}$$

For any $f \in k(\mathfrak{p})$, if $\bar{v}(f) \leq 1$ then $f \in R_v$ and $1 \otimes f \in B \otimes_A R_v$ maps to $f \in k(\mathfrak{p})$. Assume $\bar{v}(f) > 1$. As $v \in \mathrm{Val}(B, A)$ we see that $\Gamma = c\Gamma_v$, so there exists $d \in B$ satisfying $\bar{v}(f) \leq v(d)$. It follows that $f/\bar{d} \in R_v$ where $\bar{d}$ is the image of d in $k(\mathfrak{p})$ and $d \otimes f/\bar{d} \in B \otimes_A R_v$ maps to $f \in k(\mathfrak{p})$. □

Remark 2.3.6. Since for any $A \subset B$ and R_v we always have

$$B \otimes_{\mathbb{Z}} R_v \twoheadrightarrow B \otimes_A R_v \ ,$$

we can replace in the Lemma $B \otimes_A R_v$ with $B \otimes_{\mathbb{Z}} R_v$.

Remark 2.3.7. Equivalently we can say that v is in $\mathrm{Val}(B, A)$ if and only if

$$\operatorname{Spec} k(\mathfrak{p}) \to SpecB \times_{\operatorname{Spec} A} \operatorname{Spec} R_v,$$

or equivalently

$$\operatorname{Spec} k(\mathfrak{p}) \to \operatorname{Spec} B \times \operatorname{Spec} R_v,$$

is a closed immersion.

As we have seen, given another pair $A' \subset B'$ and a homomorphism

$$\begin{tikzcd} B \arrow[r, "\varphi"] & B' \\ A \arrow[u, hook] \arrow[r] & A' \arrow[u, hook] \end{tikzcd}$$

composition with φ induces a map $\varphi^*\colon \mathrm{Spa}(B', A') \to \mathrm{Spa}(B, A)$. However φ^* does not necessarily restrict to a map $\mathrm{Val}(B', A') \to \mathrm{Val}(B, A)$.

Lemma 2.3.8. *In the situation described above, assume that the induced homomorphism $B \otimes_A A' \to B'$ is integral. Then composition with φ induces a map $\varphi^*\colon \mathrm{Val}(B', A') \to \mathrm{Val}(B, A)$.*

Proof. Let $v' = (\mathfrak{p}', R_{v'}, \Phi') \in \mathrm{Val}(B', A')$, set $v = \varphi^*(v')$. Then $v = (\mathfrak{p}, R_v, \Phi)$ where $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}')$, $R_v = R_{v'} \cap k(\mathfrak{p})$ and Φ is the induced map

$$\begin{tikzcd} A' \arrow[r, "\Phi'"] & R_{v'} \\ A \arrow[u] \arrow[r, dashed, "\Phi"] & R_v . \arrow[u] \end{tikzcd}$$

By Lemma 2.3.5, in order to show that $v \in \mathrm{Spa}(B, A)$ lies in $\mathrm{Val}(B, A)$ we need to show that $B \otimes_A R_v \to k(\mathfrak{p})$ is surjective. The homomorphism φ gives rise to a diagram as in (2) from which we obtain the commutative diagram

$$\begin{tikzcd} B' \otimes_{A'} R_{v'} \arrow[r, two heads] & k(\mathfrak{p}') \\ B \otimes_A R_v \arrow[r] \arrow[u] & k(\mathfrak{p}), \arrow[u] \end{tikzcd}$$

with the upper horizontal arrow surjective by Lemma 2.3.5. Let us construct a preimage of $\alpha \in k(\mathfrak{p})$. If $\overline{v}(\alpha) \leq 1$ then α is already in R_v (recall that $\overline{v}$ is the induced valuation on $k(\mathfrak{p})$). If $\overline{v}(\alpha) > 1$, then there is $b' \in B'$ such that $\overline{v}(\alpha) \leq v'(b')$ since $\bar{\varphi}(\alpha) \in k(\mathfrak{p}')$ and v' is in $\mathrm{Val}(B', A')$. As $B \otimes_A A' \to B'$ is integral we have $x_0, \ldots, x_{n-1} \in Im\,(B \otimes_A A' \to B')$ such that ${b'}^n + x_{n-1}{b'}^{n-1} + \ldots + x_0 = 0$. There is some $0 \leq i \leq n-1$ such that $v'({b'}^n) \leq v'(x_i {b'}^i)$. It follows that $v'(b') \leq v'(b')^{n-i} \leq v'(x_i)$.

Now, there are $a_1, \ldots, a_m \in A'$ and $b_1, \ldots, b_m \in B$ such that $\sum_j a_j \otimes b_j$ maps to x_i, so $v'(b') \leq max\{v'(a_j) \cdot v' \circ \varphi(b_j)\} \leq max\{v(b_j)\}$. The last inequality is due to the fact that $v'(a) \leq 1$ for every $a \in A'$. Choose $b \in \{b_1, \ldots, b_m\}$ such that $v(b) = max\{v(b_j)\}$. Now we have $\overline{v}(\alpha) \leq v(b)$. Denoting the image of b in $k(\mathfrak{p})$ by $\bar{b}$, we have $\overline{v}(\alpha) \leq \overline{v}(\bar{b})$ or in other words $\overline{v}(\alpha \bar{b}^{-1}) \leq 1$. Hence $\alpha \bar{b}^{-1} \in R_v$ and $b \otimes \alpha \bar{b}^{-1} \in B \otimes_A R_v$ maps to α. □

2.4. *Rational domains*

Set $\mathfrak{X} = \mathrm{Val}(B, A)$. For $b, a_1, \ldots, a_n \in B$ generating the unit ideal we defined a rational domain in $\mathrm{Spa}(B, A)$ as

$$\mathcal{R}(\{a_1, \ldots, a_n\}/b) = \{v \in \mathrm{Spa}(B, A) \mid v(a_i) \leq v(b)\}.$$

We call the set $\mathcal{R}(\{a_1, \ldots, a_n\}/b) \cap \mathfrak{X}$ a *rational domain* in $\mathfrak{X}$ and denote it by $\mathfrak{X}(\{a_1, \ldots, a_n\}/b)$. Obviously $\mathfrak{X}(\{a_1, \ldots, a_n\}/b) = \mathrm{Val}\left(B_b, \varphi_b(A)\left[\frac{a_1}{b}, \ldots, \frac{a_n}{b}\right]\right)$ where φ_b is the natural map $B \to B_b$.

Proposition 2.4.1. *The rational domains of $\mathfrak{X}$ form a basis for the topology.*

Proof. Let $w \in \mathfrak{X}$ and U an open neighborhood of w in $\mathrm{Spa}(B, A)$ (i.e. $U \cap \mathfrak{X}$ is an open neighborhood of w in $\mathfrak{X}$). By the definition of the topology there is a natural number N and $a_i, b_i \in B$ such that $v(a_i) \leq v(b_i) \neq 0$ for each $i = 1, \ldots, N$ such that $w \in \bigcap_i U_{a_i, b_i} \subset U$. By taking the products $\prod_i c_i$ where $c_i \in \{a_i, b_i\}$ and $b = \prod_i b_i$, replacing N with a suitable natural number, the a_i-s with the above products and shrinking U, we may assume that we have $a_1, \ldots, a_N, b \in B$ satisfying $w \in \cap_i U_{a_i, b} = U$.

As $w(b) \neq 0$ and $w \in \mathfrak{X}$ we see that $w(b) \in c\Gamma_w$. Since $w(b)^{-1}$ is not a bound of w, there exists $d \in B$ such that $w(b)^{-1} \leq w(d)$. It follows that $1 = w(1) \leq w(db)$ and

$$w \in U \cap U_{1,db} = \{v \in \mathrm{Spa}(B, A) \mid v(a_i) \leq v(b) \neq 0, 1 \leq v(db)\} =$$
$$= \mathcal{R}(\{da_1, \ldots, da_n, 1\}/db).$$

Hence

$$w \in \mathcal{R}(\{da_1, \dots, da_n, 1\}/db) \cap \mathfrak{X} = \mathfrak{X}(\{da_1, \dots, da_n, 1\}/db) \subset U \cap \mathfrak{X}.$$

It remains to show that the rational domains satisfy the intersection condition of a basis. It follows from Lemma 2.2.5 that

$$\mathfrak{X}\left(\{a_i\}_{i=0}^n/a_0\right) \cap \mathfrak{X}\left(\{a'_j\}_{j=0}^m/a'_0\right) = \mathfrak{X}\left(\{a_i a'_j\}_{\substack{0\le i\le n\\0\le j\le m}}/a_0 a'_0\right). \quad \square$$

In [9], Temkin defines the *semi-valuation ring* S_v for a valuation $v = (\mathfrak{p}, R_v, \Phi)$ on B. It is the pre-image of the valuation ring R_v in the local ring $B_\mathfrak{p}$. The valuation on B induces a valuation on S_v. We call $B_\mathfrak{p}$ the *semi-fraction ring* of S_v.

We briefly recall several properties of a semi-valuation ring (for details see [9, §2]).

Remark 2.4.2. If S_v is a semi-valuation ring with semi-fraction ring $B_\mathfrak{p}$ then

(i) the maximal ideal $\mathfrak{p}B_\mathfrak{p}$ of $B_\mathfrak{p}$ is contained in S_v.
(ii) considering v as a valuation on $B_\mathfrak{p}$ or S_v we have $\ker v = \mathfrak{p}B_\mathfrak{p}$.
(iii) $(S_v)_{\ker v} = B_\mathfrak{p}$.
(iv) $S_v/\ker v = R_v$, in particular S_v is a local ring.
(v) for any pair $g, h \in S_v$ such that $v(g) \le v(h) \ne 0$ we have $g \in hS_v$.
(vi) for any co-prime $g, h \in B_\mathfrak{p}$ (i.e. $gB_\mathfrak{p} + hB_\mathfrak{p} = B_\mathfrak{p}$), either $g \in hS_v$ or $h \in gS_v$.
(vii) the converse of (vi) is also true: for a pair of rings $C \subset D$, if for any two co-prime elements $g, h \in D$ either $g \in hC$ or $h \in gC$ then there exists a valuation on D such that C is a semi-valuation ring of v and D is its semi-fraction ring.

Remark 2.4.3. Let $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$. By the multiplicativity the valuation v extends uniquely to a valuation on $B_\mathfrak{p}$. There is a unique semi-valuation ring S_v associated to the point (namely the pull-back of R_v to $B_\mathfrak{p}$). The diagram (1) corresponding to v factors through the pair $S_v \subset B_\mathfrak{p}$ i.e.

$$\begin{tikzcd} B \arrow[r] & B_\mathfrak{p} \arrow[r] & k(\mathfrak{p}) \\ A \arrow[u, hook] \arrow[r] & S_v \arrow[u, hook] \arrow[r] & R_v. \arrow[u, hook] \end{tikzcd} \tag{3}$$

Furthermore note that $S_v = \{x \in B_\mathfrak{p} | v(x) \le 1\}$, and its maximal ideal is $\{x \in B_\mathfrak{p} | v(x) < 1\}$.

Using these notions we can reformulate Lemma 2.3.5 in terms of semi-valuation and semi-fraction rings.

Lemma 2.4.4. *Let $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Spa}(B, A)$. Then $v \in \mathrm{Val}(B, A)$ if and only if the canonical map $B \otimes_A S_v \to B_\mathfrak{p}$ is surjective.*

Proof. By Remark 2.4.2 (i) the ideal $\mathfrak{p}B_\mathfrak{p}$ lies in S_v, so $\mathfrak{p}B_\mathfrak{p}$ is in the image of $B \otimes_A S_v \to B_\mathfrak{p}$. It follows that $B \otimes_A S_v \to B_\mathfrak{p}$ is surjective if and only if $B \otimes_A S_v / B \otimes_A \mathfrak{p}B_\mathfrak{p} = B \otimes_A R_v \to B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = k(\mathfrak{p})$ is. The result follows from Lemma 2.3.5. □

Let us study pullback of valuations and primary specialization with respect to semi-valuation rings.

Lemma 2.4.5. *Let $A \subset B$ and $A' \subset B'$ be rings and $\varphi\colon B \to B'$ a ring homomorphism such that*

$$\begin{tikzcd} B \arrow[r, "\varphi"] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

commutes. For a valuation $v' \in \mathrm{Spa}(B', A')$ there is a local homomorphism between the semi-valuation ring of $\varphi^*(v')$ and the semi-valuation ring of v' induced by φ.

Proof. Let $v' = (\mathfrak{p}', R_{v'}, \Phi') \in \mathrm{Spa}(B', A')$, set $v = \varphi^*(v') = (\mathfrak{p}, R_v, \Phi)$. By applying the factorization as in diagram (3) to diagram (2) we obtain

$$\begin{tikzcd} & B' \arrow[rr] & & B'_{\mathfrak{p}'} \arrow[rr] & & k(\mathfrak{p}') \\ B \arrow[ur] \arrow[rr] & & B_\mathfrak{p} \arrow[ur] \arrow[rr] & & k(\mathfrak{p}) \arrow[ur] & \\ & A' \arrow[uu] \arrow[rr] & & S_{v'} \arrow[uu] \arrow[rr] & & R_{v'} \arrow[uu] \\ A \arrow[uu] \arrow[ur] \arrow[rr] & & S_v \arrow[uu] \arrow[ur, dashed] \arrow[rr] & & R_v. \arrow[uu] \arrow[ur] & \end{tikzcd}$$

The dashed arrow exists by the universal property of the pull back. Let x be an element of the maximal ideal of S_v. Then its image in R_v lies in the maximal ideal. Since $R_v \to R_{v'}$ is local, the image of x in $S_{v'}$ also lies in the maximal ideal. □

Given two valuations $v = (\mathfrak{p}, R_v, \Phi), w = (\mathfrak{q}, R_w, \Psi) \in \mathrm{Spa}(B, A)$ with $\mathfrak{p} \subset \mathfrak{q}$, we would like to know if w a primary specialization of v. Assume $v\colon B \to \Gamma \cup \{0\}$ and $w\colon B \to \Delta \cup \{0\}$. By replacing v and w with equivalent valuations, we may assume that $\Gamma = v(B_{\mathfrak{p}}^{\times})$ and $\Delta = w(B_{\mathfrak{q}}^{\times})$. Since $\mathfrak{p} \subset \mathfrak{q}$ we have a canonical homomorphism $j\colon B_{\mathfrak{q}} \to B_{\mathfrak{p}}$.

Lemma 2.4.6. *With the above notation, w is a primary specialization of v if and only if the canonical homomorphism $j\colon B_{\mathfrak{q}} \to B_{\mathfrak{p}}$ restricts to a local homomorphism $S_w \to S_v$ of semi-valuation rings.*

Proof. If w is a primary specialization of v, then Δ is a convex subgroup of Γ containing $c\Gamma_v$ and $w(b) = \begin{cases} v(b) & \text{if } v(b) \in \Delta \\ 0 & \text{if } v(b) \notin \Delta \end{cases}$. Now, w extends uniquely to $B_{\mathfrak{q}}$, v extends uniquely to $B_{\mathfrak{p}}$, $S_v = \{x \in B_{\mathfrak{p}} | v(x) \leq 1\}$ and $S_w = \{x \in B_{\mathfrak{q}} | w(x) \leq 1\}$ (Remark 2.4.3). As $\mathfrak{p} = \ker v \subset \mathfrak{q} = \ker w$, for every $s \notin \mathfrak{q}$ we have $w(s) = v(s)$. For any $x \in S_w$ there are $b, s \in B$, $s \notin \mathfrak{q} = \ker w$ such that $x = \frac{b}{s} \in B_{\mathfrak{q}}$. Then $j(x) = \frac{b}{s} \in B_{\mathfrak{p}}$ and $w(x) = \begin{cases} v(j(x)) & \text{if } v(b) \in \Delta \\ 0 & \text{if } v(b) \notin \Delta \end{cases}$. The only non-trivial case to consider is when $0 \neq v(b) \notin \Delta$. In this case, since $c\Gamma_v \subset \Delta$ it must be that $v(b) < \Delta$. As $v(s) = w(s) \in \Delta$ we obtain $v(b) < v(s)$ so $j(x) \in S_v$. The same argument also shows that if x is in the maximal ideal of S_w then $j(x)$ is in the maximal ideal of S_v.

Conversely assume that we have a diagram

$$\begin{tikzcd} B_{\mathfrak{q}} \arrow[r] & B_{\mathfrak{p}} \\ S_w \arrow[u, hook] \arrow[r] & S_v \arrow[u, hook] \end{tikzcd}$$

with the bottom arrow a local homomorphism. It is easy to see that $\Gamma = B_{\mathfrak{p}}^{\times}/S_v^{\times}$ and $\Delta = B_{\mathfrak{q}}^{\times}/S_w^{\times}$. Since $S_w \to S_v$ is local we obtain a homomorphism of groups $\alpha\colon \Delta \to \Gamma$. If $x = \frac{b}{s} \in B_{\mathfrak{q}} - S_w$ then its image under $B_{\mathfrak{q}} \to B_{\mathfrak{p}}$ lies in $B_{\mathfrak{p}} - S_v$, since if $w(x) > 1$ and $v(x) \leq 1$ then in particular $x^{-1} \in S_w$ and $w(x^{-1}) < 1$. By the locality of the homomorphism we have $v(x^{-1}) < 1$. But this is impossible since it would imply that $1 = v(1) = v(xx^{-1}) < 1$.

Now if $x \in B_{\mathfrak{q}}^{\times}$ with $v(x) = 1$ i.e. its image under $B_{\mathfrak{q}} \to B_{\mathfrak{p}}$ has value 1, then x is already in S_w and by locality of the homomorphism we have $w(x) = 1$. Hence α is an injection and we may regard Δ as a subgroup of Γ.

It is now clear that $w(b) = \begin{cases} v(b) & \text{if } v(b) \in \Delta \\ 0 & \text{if } v(b) \notin \Delta \end{cases}$. $\square$

There is an obvious retraction $\mathrm{Spa}(B, A) \to \mathrm{Val}(B, A)$.

Definition 2.4.7. Let $r\colon \mathrm{Spa}(B, A) \to \mathrm{Val}(B, A)$ be the map sending every valuation in $\mathrm{Spa}(B, A)$ to its minimal primary specialization.

Proposition 2.4.8. $r\colon \operatorname{Spa}(B,A) \to \operatorname{Val}(B,A)$ *is continuous.*

Proof. Let U be an open subset of $\mathfrak{X} = \operatorname{Val}(B,A)$. As the rational domains form a basis for the topology it is enough to consider the case when U is a rational domain of $\operatorname{Val}(B,A)$. Let $a_1,\dots,a_n,b \in B$ be elements generating the unit ideal. Set

$$W = \mathcal{R}(\{a_1,\dots,a_n\}/b)$$

and

$$U = \mathfrak{X}(\{a_1,\dots,a_n\}/b) = \mathcal{R}(\{a_1,\dots,a_n\}/b) \cap \mathfrak{X}.$$

We claim that $r^{-1}(U) = W$. Let $w \in \operatorname{Spa}(B,A)$ such that $r(w) \in U$. By the definition of primary specialization $r(w)(x) \le w(x)$ for every $x \in B$ and as $r(w)(b) \neq 0$ it follows that $w(b) \neq 0$. Hence $r^{-1}(U) \subset W$. Let w be a valuation in W with value group Γ and let $v = r(w)$. If $c\Gamma_w = \Gamma$ then $v = w$, that is, $w \in \operatorname{Val}(B,A)$ so $w \in W \bigcap \operatorname{Val}(B,A) = U$. If $c\Gamma_w \subsetneqq \Gamma$ then $v(a_i) \le v(b)$ since $w(a_i) \le w(b)$. It remains to show that $v(b) \neq 0$. If $v(b) = 0$ then $w(b) < c\Gamma_w$ and so $w(a_i) < c\Gamma_w$ for every $i = 1,\dots,n$. There are $c_0, c_1, \dots, c_n \in B$ such that $1 = bc_0 + a_1c_1 + \cdots + a_nc_n$ and $w(1) = 1 \in c\Gamma_w$. By convexity of $c\Gamma_w$ there is some i such that $w(a_ic_i) \in c\Gamma_w$. Thus $w(a_ic_i) \le w(bc_i) \in c\Gamma_w$, so $v(bc_i) \neq 0$. But since $v(b) = 0$ we also get $v(bc_i) = 0$, which is a contradiction. We conclude that $v(b) \neq 0$. □

Corollary 2.4.9. $\operatorname{Val}(B,A)$ *is a qcqs topological space.*

Proof. As $\operatorname{Spa}(B,A)$ is a quasi-compact space by Lemma 2.2.7 and the retraction $r\colon \operatorname{Spa}(B,A) \to \operatorname{Val}(B,A)$ is continuous, $\operatorname{Val}(B,A)$ is quasi-compact. Any rational domain can be viewed as $\operatorname{Val}(B',A')$ for suitable rings $A' \subset B'$, hence any rational domain is quasi-compact. As we saw in Proposition 2.4.1, the intersection of two rational domains is again a rational domain so in particular it is quasi-compact. Since the rational domains form a basis of the topology, any quasi-compact open subset of $\operatorname{Val}(B,A)$ is a finite union of rational domains. Now the intersection of any two quasi-compact open subsets of $\operatorname{Val}(B,A)$ is also a finite union of rational domains, thus quasi-compact and $\operatorname{Val}(B,A)$ is quasi-separated. □

Example 2.4.10 *(An Affine Scheme).* Consider $\operatorname{Val}(B,B)$. Let $v = (\mathfrak{p}, R_v, \Phi) \in \operatorname{Val}(B,B) = \mathfrak{X}$, then v is an unbounded valuation on B such that $v(B) \le 1$. The only way this could be achieved is if v is a trivial valuation (i.e. $\Gamma = \{1\}$). Hence there is a $1-1$ correspondents between points of $\operatorname{Val}(B,B)$ and prime ideals of B, that is points of $\operatorname{Spec} B$. As for the topology,

$$\mathfrak{X}(\{a_1,\dots,a_n\}/b) = \operatorname{Val}\left(B_b, \varphi_b(B)\left[\frac{a_1}{b},\dots,\frac{a_n}{b}\right]\right) = \operatorname{Val}(B_b,B_b) \leftrightarrow \operatorname{Spec} B_b = D(b)$$

So there is a homeomorphism $\mathrm{Val}(B, B) \simeq \mathrm{Spec}\, B$.

For later use we define two canonical maps of topological spaces.

Definition 2.4.11. Set

$$\sigma\colon \mathrm{Spec}\, B \to \mathfrak{X} \text{ and } \tau\colon \mathfrak{X} \to \mathrm{Spec}\, A,$$

where σ sends $\mathfrak{p} \in \mathrm{Spec}\, B$ to be the trivial valuation on $k(\mathfrak{p})$ (which is indeed in $\mathfrak{X} = \mathrm{Val}(B, A)$ since its valuation ring is $k(\mathfrak{p})$ and $B \otimes_A k(\mathfrak{p}) \to k(\mathfrak{p})$ is of course surjective), and τ sends $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$ to the image of the unique closed point of $\mathrm{Spec}\, R_v$ in $\mathrm{Spec}\, A$.

Proposition 2.4.12.

(1) *The composition* $\tau \circ \sigma\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *is the morphism corresponding to the inclusion of rings* $A \subset B$.
(2) σ *is continuous and injective.*
(3) τ *is continuous and surjective.*

Proof. (1) Given a prime $\mathfrak{p}$ in B, $\sigma(\mathfrak{p}) = (\mathfrak{p}, k(\mathfrak{p}), \Phi)$. The maximal ideal of the valuation ring $k(\mathfrak{p})$ is the zero ideal, so

$$\tau(\sigma(\mathfrak{p})) = \ker \Phi = \ker (A \to B \to k(\mathfrak{p})) = \mathfrak{p} \cap A.$$

(2) Let $U = \mathfrak{X}(\{a_1, \ldots, a_n\}/b)$ be a rational domain in $\mathfrak{X}$ and $D(b)$ a basic open set in $\mathrm{Spec}\, B$. For any $\mathfrak{p} \in D(b)$ we have $\sigma(\mathfrak{p})(b) = 1$ and $\sigma(\mathfrak{p})(a_i) = 0$ or 1, so $\sigma(D(b)) \subset \mathfrak{X}(\{a_1, \ldots, a_n\}/b)$.

Conversely if $v \in \mathfrak{X}(\{a_1, \ldots, a_n\}/b)$ and there is $\mathfrak{p} \in \mathrm{Spec}\, B$ that maps to v then we must have $\ker(v) = \mathfrak{p}$ and v is trivial on $k(\mathfrak{p})$. Hence σ is injective and $\sigma^{-1}(\mathfrak{X}(\{a_1, \ldots, a_n\}/b)) = D(b)$.

(3) Let $D(a) \subset \mathrm{Spec}\, A$ be a basic open set, $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$ and $\mathfrak{m}_v$ the maximal ideal of R_v. Then $\tau(v)$ lies in $D(a)$ if and only if $\Phi(a) \notin \mathfrak{m}_v$ or equivalently $v(a) \geq 1$. In this case $v(a) = 1$ (since $v(A) \leq 1$) and

$$v \in \mathfrak{X}(\{1\}/a) = \{w \in \mathrm{Val}(B, A) \mid w(a) = 1\}$$

so $\tau^{-1}(D(a)) = \mathfrak{X}(\{1\}/a)$.

As for surjectivity, first consider a maximal ideal $\mathfrak{q} \in \mathrm{Spec}\, A$. Since the morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is schematically dominant there is $\mathfrak{p}'' \in \mathrm{Spec}\, B$ such that $A \cap \mathfrak{p}'' \subset \mathfrak{q}$. Take $\mathfrak{p}$ to be a prime of B maximal with respect to this property. The image $i(A)$ of A under $i\colon B \to k(\mathfrak{p})$ is a subring of $k(\mathfrak{p})$. The extended ideal $i(\mathfrak{q})i(A)$ is a proper ideal, since if $1 \in i(\mathfrak{q})i(A)$ then there are $a_1, \ldots, a_n \in \mathfrak{q}$ and $b_1, \ldots, b_n \in A$ such that

$1 - \sum_i a_i b_i \in ker(i) = \mathfrak{p}$. As $1 - \sum_i a_i b_i \in A$ we have $1 - \sum_i a_i b_i \in A \cap \mathfrak{p} \subset \mathfrak{q}$, but since $\sum_i a_i b_i \in \mathfrak{q}$ we get that $1 \in \mathfrak{q}$ which is a contradiction. By [11, VI §4.4] there is a valuation ring R_v of $k(\mathfrak{p})$ containing $i(A)$ such that its maximal ideal $\mathfrak{m}_v$ contains $i(\mathfrak{q})i(A)$. This gives us a valuation $v = (\mathfrak{p}, R_v, i|_A) \in \operatorname{Spa}(B, A)$. By the retraction we obtain a valuation $v' = r(v) \in \operatorname{Val}(B, A)$ with kernel $\mathfrak{p}' \supset \mathfrak{p}$. If $\mathfrak{p} \neq \mathfrak{p}'$ then $\mathfrak{q}$ is strictly contained in $A \cap \mathfrak{p}'$, by the choice of $\mathfrak{p}$. Since $\mathfrak{q}$ is a maximal ideal we have $1 \in A \cap \mathfrak{p}'$ which is a contradiction. Hence $\mathfrak{p}' = \mathfrak{p}$ and $v' = v$ that is $v = (\mathfrak{p}, R_v, \Phi) \in \operatorname{Val}(B, A)$. Now $\tau(v) = \Phi^{-1}(\mathfrak{m}_v) \supset \mathfrak{q}$ but since $\mathfrak{q}$ is maximal we have $\tau(v) = \mathfrak{q}$.

Now, let $\mathfrak{q} \in \operatorname{Spec} A$ be some prime. Then $\mathfrak{q}A_\mathfrak{q}$ is the maximal ideal of $A_\mathfrak{q}$, and since $A_\mathfrak{q}$ is flat over A, we have $A_\mathfrak{q} \subset B_\mathfrak{q} = B \otimes_A A_\mathfrak{q}$. By the previous case there is a valuation $v \in \operatorname{Val}(B_\mathfrak{q}, A_\mathfrak{q})$ that is mapped to $\mathfrak{q}A_\mathfrak{q}$ by $\tau\colon \operatorname{Val}(B_\mathfrak{q}, A_\mathfrak{q}) \to \operatorname{Spec} A_\mathfrak{q}$ maps to. The canonical homomorphisms

$$\begin{array}{ccc} B & \longrightarrow & B_\mathfrak{q} \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_\mathfrak{q} \end{array}$$

induces by Lemma 2.3.8 a morphism $\operatorname{Val}(B_\mathfrak{q}, A_\mathfrak{q}) \to \mathfrak{X}$. As $\mathfrak{q}A_\mathfrak{q} \in \operatorname{Spec} A_\mathfrak{q}$ is pulled back to $\mathfrak{q} \in \operatorname{Spec} A$, the image of v in $\operatorname{Val}(B, A)$ is mapped by τ to $\mathfrak{q}$. $\square$

Remark 2.4.13. From the proof above we see that for any $a \in A$ we have

$$\tau^{-1}(D(a)) = \operatorname{Val}(B_a, A_a) = \mathfrak{X}(\{1\}/a),$$

and for every rational domain $\mathfrak{X}(\{a_1, \ldots, a_n\}/b) \subset \mathfrak{X}$ we have

$$\sigma^{-1}(\mathfrak{X}(\{a_1, \ldots, a_n\}/b)) = D(b)$$

Remark 2.4.14. By diagram (3) we can rephrase the definition of τ as the map that sends $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$ to the image of the unique closed point of $\operatorname{Spec} S_v$ under the map $\operatorname{Spec} S_v \to \operatorname{Spec} A$.

2.5. Rational covering

Any open cover of $\mathfrak{X}$ can be refined to a cover consisting of rational domains, since the rational domains form a basis for the topology. Furthermore there is a finite sub-cover of $\mathfrak{X}$ consisting of rational domains, as $\mathfrak{X}$ is quasi-compact. Next we show that we can always refine this cover to a rational covering, that is there are elements $T = \{a_1, \ldots, a_N\} \subset B$ generating the unit ideal such that the rational domains $\{\mathfrak{X}(T/a_j)\}_{1 \le j \le N}$ form a refinement of the finite sub-cover.

Proposition 2.5.1. *Any finite open cover of* $\mathrm{Val}(B, A)$ *consisting of rational domains can be refined to a rational covering.*

Proof. Let $\{U_i\}_{i=1}^N$ be a finite open cover of $\mathfrak{X} = \mathrm{Val}(B, A)$ consisting of rational domains, i.e. for every $i = 1, \dots, N$ we have $a_1^{(i)}, \dots, a_{n_i}^{(i)} \in B$ generating the unit ideal and

$$U_i = \mathfrak{X}(\{a_j^{(i)}\}_{1 \le j \le n_i} / a_1^{(i)}) = \{v \mid v(a_j^{(i)}) \le v(a_1^{(i)}) \quad j = 1, \dots, n_i\}.$$

For every $1 \le k \le n_i$ denote $V_{i,k} = \mathfrak{X}(\{a_j^{(i)}\}_{1 \le j \le n_i} / a_k^{(i)})$. Note that $V_{i,1} = U_i$ and $\mathfrak{X} = \cup_{k=1}^{n_i} V_{i,k}$ for each $i = 1, \dots, N$. Set

$$I = \{(r_1, \dots, r_N) \in \mathbb{N}^N \mid 1 \le r_i \le n_i \ i = 1, \dots, N\}.$$

For $(r_1, \dots, r_N) \in I$ we denote

$$V_{(r_1,\dots,r_N)} = \cap_{1 \le i \le N} V_{i,r_i} \text{ and } a_{(r_1,\dots,r_N)} = a_{r_1}^{(1)} \cdot a_{r_2}^{(2)} \cdot \dots \cdot a_{r_N}^{(N)}.$$

Note that $V_{(r_1,\dots,r_N)} = \{v \in \mathfrak{X} \mid v(a_\alpha) \le v(a_{(r_1,\dots,r_N)}) \ \forall \alpha \in I\}$ and that the elements $\{a_\alpha\}_{\alpha \in I}$ generate the unit ideal. Hence $\{V_\alpha\}_{\alpha \in I}$ is rational covering of $\mathfrak{X}$. $\square$

2.6. Sheaves on $\mathrm{Val}(B, A)$

2.6.1. $\mathcal{M}_{\mathfrak{X}}$ *and* $\mathcal{O}_{\mathfrak{X}}$

We now define two sheaves on $\mathfrak{X} = \mathrm{Val}(B, A)$, both making $\mathfrak{X}$ a locally ringed space.

Notation.

- For a pair of rings $C \subset D$ we denote the integral closure of C in D by $\mathrm{Nor}_D C$.
- For qcqs schemes Y, X and an affine morphism $f \colon Y \to X$, we denote the integral closure of $\mathcal{O}_X$ in $f_* \mathcal{O}_Y$ by $\mathrm{Nor}_{f_* \mathcal{O}_Y} (\mathcal{O}_X)$ or $\mathrm{Nor}_Y (\mathcal{O}_X)$.
- In the situation above we denote $\mathrm{Nor}_Y X = \mathbf{Spec}_X(\mathrm{Nor}_Y \mathcal{O}_X)$ and $\nu \colon \mathrm{Nor}_Y X \to X$ the canonical morphism.

Lemma 2.6.1. *Let* $A \subset B$ *be rings. Denote* $X = \mathrm{Spec}\, A$ *and* $Y = \mathrm{Spec}\, B$*. Then* $\mathfrak{X} = \mathrm{Val}(B, A) = \mathrm{Val}(B, \mathrm{Nor}_B A)$ *and the canonical map* $\tau \colon \mathfrak{X} \to X$ *factors through* $\mathrm{Nor}_Y X$*.*

Proof. The canonical morphism

$$\nu \colon \mathrm{Nor}_Y X \to X$$

is integral, hence universally closed and separated. Thus for any valuation $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$ we obtain a diagram by the valuative criterion (abusing notation and denoting by Φ both $A \to R_v$ and the induced morphism $\mathrm{Spec}\, R_v \to \mathrm{Spec}\, A$)

$$\begin{tikzcd}
\operatorname{Spec} k(\mathfrak{p}) \arrow[r] \arrow[d] & Y \arrow[r] & \operatorname{Nor}_Y X \arrow[d, "\nu"] \\
\operatorname{Spec} R_v \arrow[rr, "\Phi"] \arrow[urr, dashed, "\exists!\Phi'"] & & X.
\end{tikzcd}$$

That is, we obtain a unique $(\mathfrak{p}, R_v, \Phi') \in \operatorname{Spa}(B, \operatorname{Nor}_B A)$. As $v \in \mathfrak{X}$ the morphism $\operatorname{Spec} k(\mathfrak{p}) \to Y \times \operatorname{Spec} R_v$ is a closed immersion by Remark 2.3.7. It follows, again form Remark 2.3.7, that $(\mathfrak{p}, R_v, \Phi') \in \operatorname{Val}(B, \operatorname{Nor}_B A)$. Thus $\mathfrak{X} = \operatorname{Val}(B, \operatorname{Nor}_B A)$.

It is clear that the diagram of topological spaces

$$\begin{tikzcd}
\operatorname{Val}(B, \operatorname{Nor}_B A) \arrow[r, "\tau"] \arrow[d, equal] & \operatorname{Nor}_Y X \arrow[d, "\nu"] \\
\operatorname{Val}(B, A) \arrow[r, "\tau"] & X
\end{tikzcd}$$

commutes. □

The rational domains of $\mathfrak{X}$ with inclusions as morphisms form a category. Declaring all coverings admissible we obtain a site. As the rational domains form a basis for the topology of $\mathfrak{X}$ any sheaf on the site of rational domains uniquely extends, in a natural way, to a sheaf on the topological space $\mathfrak{X}$. Hence, it is enough to define the sheaves only over the rational domains. Let $U = \mathfrak{X}(\{a_1, \dots, a_n\}/b) = \operatorname{Val}(B', A')$ where, as before, $B' = B_b$ and $A' = \varphi_b(A)\left[\frac{a_1}{b}, \dots, \frac{a_n}{b}\right]$. We define two presheaves $\mathcal{M}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}$ on the rational domains of $\mathfrak{X}$ by the rules

$$U \mapsto \mathcal{M}_{\mathfrak{X}}(U) = B' \qquad\qquad U \mapsto \mathcal{O}_{\mathfrak{X}}(U) = \operatorname{Nor}_{B'} A'.$$

Clearly $\mathcal{O}_{\mathfrak{X}}(U) \subset \mathcal{M}_{\mathfrak{X}}(U)$.

Theorem 2.6.2. *With the above notation, the presheaves $\mathcal{M}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}$ are sheaves on the rational domains of $\mathfrak{X}$.*

Proof. Denote $Y = \operatorname{Spec} B$. We know that the sets $\{D(b)\}_{b \in B}$ form a base for the topology of Y. Recall that we defined a map $\sigma \colon Y \to \mathfrak{X}$ such that for every rational domain $\mathfrak{X}(\{a_1, \dots, a_n\}/b) \subset \mathfrak{X}$ we have $\sigma^{-1}(\mathfrak{X}(\{a_1, \dots, a_n\}/b)) = D(b)$ (Remark 2.4.13). Now, by the definition of $\mathcal{M}_{\mathfrak{X}}$, for every rational domain $U \subset \mathfrak{X}$ we have an isomorphism of rings

$$B_b = \mathcal{M}_{\mathfrak{X}}(U) \simeq \sigma_* \mathcal{O}_{\operatorname{Spec} B}(U) = \mathcal{O}_{\operatorname{Spec} B}(D(b)) = B_b.$$

Let $V \subset U \subset \mathfrak{X}$ be two rational domains of $\mathfrak{X}$. Suppose $U = \mathfrak{X}(\{a_1, \dots, a_n\}/b)$ and $V = \mathfrak{X}(\{f_1, \dots, f_m\}/g)$. Then $D(g) = \sigma^{-1}(V)$ and $D(b) = \sigma^{-1}(U)$. For any $\mathfrak{p} \in D(g)$ we have $\sigma(\mathfrak{p}) \in V \subset U$, hence $\mathfrak{p} \in D(b)$. In other words we have a diagram

$$\begin{tikzcd} D(g) \arrow[r] \arrow[d] & D(b) \arrow[r] \arrow[d] & Y \arrow[d, "\sigma"] \\ V \arrow[r] & U \arrow[r] & \mathfrak{X}. \end{tikzcd}$$

From the diagram we see that the restriction maps of $\mathcal{M}_{\mathfrak{X}}$ commute with the restriction maps of $\mathcal{O}_{\operatorname{Spec} B}$. We conclude that we have an isomorphism of presheaves between $\mathcal{M}_{\mathfrak{X}}$ and $\sigma_* \mathcal{O}_{\operatorname{Spec} B}$ as presheaves on the rational domains. Since $\mathcal{O}_{\operatorname{Spec} B}$ is a sheaf on Y, its restriction to the basic opens of Y is also a sheaf. It follows that $\mathcal{M}_{\mathfrak{X}}$ is a sheaf on the rational domains of $\mathfrak{X}$.

Let U be a rational domain in $\mathfrak{X}$ and $\{V_i\}$ an open covering of U consisting of rational domains. Let $s_i \in \mathcal{O}_{\mathfrak{X}}(V_i)$ be sections satisfying $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for every pair i, j. We already know that $\mathcal{M}_{\mathfrak{X}}$ is a sheaf so there is a unique element $s \in \mathcal{M}_{\mathfrak{X}}(U)$ such that $s|_{V_i} = s_i$ for each i. We want to show that s is in $\mathcal{O}_{\mathfrak{X}}(U)$. We may assume that $U = \mathfrak{X}$ and by Proposition 2.5.1 that $\{V_i\}$ is a rational covering corresponding to elements $b_1, \dots, b_r \in B$, that is elements generating the unit ideal of B and $V_i = \mathfrak{X}\left(\{b_j\}_j / b_i\right)$. We may further assume that none of $b_1, \dots, b_r$ are nilpotent.

We denote $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, $B_i = B_{b_i}$, $A_i = \varphi_i(A)\left[\{\frac{b_j}{b_i}\}_j\right]$ (where φ_i is the canonical homomorphism $B \to B_i$), $Y_i = \operatorname{Spec} B_i$ and $X'_i = \operatorname{Spec} A_i$. Then $V_i = \operatorname{Val}(B_i, A_i)$.

Now, $E = \sum_i A b_i$ is a finite A-module contained in B. Using the multiplication in B, we define E^d as the image of $E^{\otimes d}$ under the map $B^{\otimes d} \to B$. Then E^d is also a finite A-module contained in B for any $d \geq 1$. Denoting $E^0 = A$ we obtain a graded A-algebra $E' = \oplus_{d \geq 0} E^d$ and a morphism $X' = \operatorname{Proj}(E') \to X$. The affine charts of X' are given by $\operatorname{Spec} A'_i$, where A'_i is the zero grading part of the localization E'_{b_i}. Clearly $A'_i \subset B_i$. Denoting $I_d = \ker(E^d \to B \to B_i)$, we have $A'_i = \lim\limits_{d \to \infty} b_i^{-d}\left(E^d / I_d\right)$ which is exactly A_i. This means we have open immersions

$$\begin{tikzcd} Y_i \arrow[r, hook] \arrow[d] & Y \arrow[dr] & \\ X'_i \arrow[r, hook] & X' \arrow[r] & X. \end{tikzcd}$$

As $Y = \cup Y_i$, the schematically dominant morphisms $Y_i \to X'_i$ glue to a schematically dominant morphism $Y \to X'$ over X. Furthermore, for each i we have a commutative diagram

$$\begin{tikzcd} Y_i \arrow[r, "\sigma"] \arrow[d] & V_i \arrow[r, "\tau"] \arrow[d] & X'_i \arrow[d] \\ Y \arrow[r, "\sigma"] & \mathfrak{X} \arrow[r, "\tau"] & X. \end{tikzcd}$$

Denoting the normalization $X'' = \mathrm{Nor}_Y X'$ and taking the canonical morphism $\nu\colon X'' \to X'$, then by Lemma 2.6.1 we have a diagram over X

$$\begin{tikzcd} Y \arrow[r] \arrow[drr] & \mathfrak{X} \arrow[r, "\tau"] \arrow[dr, "\tau"] & X'' \arrow[d, "\nu"] \\ & & X'. \end{tikzcd}$$

We denote $X_i'' = \nu^{-1}(X_i')$. Now, by construction $s_i \in \mathcal{O}_{X''}(X_i'') = \tau_*\mathcal{O}_{\mathfrak{X}}(X_i'') = \mathcal{O}_{\mathfrak{X}}(V_i)$ and $s_i|_{X_i'' \cap X_j''} = s_j|_{X_i'' \cap X_j''}$ for every pair i, j. Since $\mathcal{O}_{X''}$ is a sheaf, they glue to a section $s \in \mathcal{O}_{X''}(X'') = \tau_*\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. $\square$

Remark 2.6.3. The construction above yields the same topological space and the same sheaves for both $A \subset B$ and for $\mathrm{Nor}_B A \subset B$.

Remark 2.6.4. In fact the proof shows that $\mathcal{O}_{\mathfrak{X}} = \varinjlim \tau_{\alpha *}\mathcal{O}_{X_\alpha}$, where the injective limit is over $X_\alpha = \mathrm{Proj}(\oplus E_\alpha^d) \to X$ and E_α is a finite A-module contained in B.

2.6.2. The stalks

Proposition 2.6.5. *For any point $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$, the stalk $\mathcal{M}_{\mathfrak{X},v}$ of the sheaf $\mathcal{M}_{\mathfrak{X}}$ is isomorphic to $B_{\mathfrak{p}}$ and the stalk $\mathcal{O}_{\mathfrak{X},v}$ of the sheaf $\mathcal{O}_{\mathfrak{X}}$ is isomorphic to the semi-valuation ring S_v.*

Proof. Fix a point $v = (\mathfrak{p}, R_v, \Phi) \in \mathfrak{X}$. The inclusion of sheaves $\mathcal{O}_{\mathfrak{X}} \subset \mathcal{M}_{\mathfrak{X}}$ gives an inclusion of stalks $\mathcal{O}_{\mathfrak{X},v} \subset \mathcal{M}_{\mathfrak{X},v}$. By the definition of a semi-valuation ring we have a diagram (3). For any rational domain $v \in W = \mathfrak{X}(\{a_i\}_i/b) = \mathrm{Val}(B_1, A_1)$ (we may assume that A_1 integrally closed in B_1) we have a unique factorization

$$\begin{tikzcd} B \arrow[r] & \mathcal{M}_{\mathfrak{X}}(W) = B_1 \arrow[rr] \arrow[dr, dashed] & & k(\mathfrak{p}) \\ & & B_{\mathfrak{p}} \arrow[ur] & \\ A \arrow[uu] \arrow[r] & \mathcal{O}_{\mathfrak{X}}(W) = A_1 \arrow[uu] \arrow[rr] \arrow[dr, dashed] & & R_v. \arrow[uu] \\ & & S_v \arrow[uu] \arrow[ur] & \end{tikzcd}$$

Taking direct limits we get a unique diagram for the stalks

$$\begin{tikzcd}
\mathcal{M}_{\mathfrak{X},v} \arrow[r] & B_{\mathfrak{p}} \arrow[r] & k(\mathfrak{p}) \\
\mathcal{O}_{\mathfrak{X},v} \arrow[r] \arrow[u, hook] & S_v \arrow[r] \arrow[u, hook] & R_v \arrow[u, hook]
\end{tikzcd}$$

and v induces a valuation in $\mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$. We claim that the horizontal arrows in the left square are isomorphisms.

If v is a trivial valuation then $R_v = k(\mathfrak{p})$, so $S_v = B_{\mathfrak{p}}$. It follows that for every element $b \in B$ we have $v \in \mathfrak{X}(\{b\}/1)$ and also for every element $b \notin \mathfrak{p}$ we have $v \in \mathfrak{X}(\{1\}/b)$. Hence $\mathcal{M}_{\mathfrak{X},v} = \mathcal{O}_{\mathfrak{X},v} = B_{\mathfrak{p}}$.

Assume v is not trivial. Let γ, η be co-prime elements in $\mathcal{M}_{\mathfrak{X},v}$, i.e. there are elements $\rho, \tau \in \mathcal{M}_{\mathfrak{X},v}$ such that $\gamma\rho + \eta\tau = 1$. By Remark 2.4.2 (vii) if either $\gamma \in \eta\mathcal{O}_{\mathfrak{X},v}$ or $\eta \in \gamma\mathcal{O}_{\mathfrak{X},v}$ then $\mathcal{O}_{\mathfrak{X},v}$ is a semi-valuation ring and $\mathcal{M}_{\mathfrak{X},v}$ is its semi-fraction ring. Assume this is the case. In particular both $\mathcal{M}_{\mathfrak{X},v}$ and $\mathcal{O}_{\mathfrak{X},v}$ are local rings. Denote the maximal ideal of $\mathcal{M}_{\mathfrak{X},v}$ by $\mathfrak{m}$. From Remark 2.4.2 (ii), (iii) and (iv) it follows that $\mathfrak{m} = \ker v$, $\mathfrak{m}$ lies in $\mathcal{O}_{\mathfrak{X},v}$, $\mathcal{M}_{\mathfrak{X},v} = (\mathcal{O}_{\mathfrak{X},v})_{\mathfrak{m}}$ and $\mathcal{O}_{\mathfrak{X},v}/\mathfrak{m} = R_v$. Hence $\mathcal{O}_{\mathfrak{X},v} = S_v$ and $\mathcal{M}_{\mathfrak{X},v} = B_{\mathfrak{p}}$ and we are done. It remains to show that either $\gamma \in \eta\mathcal{O}_{\mathfrak{X},v}$ or $\eta \in \gamma\mathcal{O}_{\mathfrak{X},v}$.

Clearly either $v(\gamma) \le v(\eta)$ or $v(\eta) \le v(\gamma)$. Assume $v(\gamma) \le v(\eta)$. We claim that $\gamma \in \eta\mathcal{O}_{\mathfrak{X},v}$. By Proposition 2.4.1 the intersection of a finite number of rational domains is again a rational domain. Hence there is a rational domain $U = \mathrm{Val}(B', A')$ with $g, h, r, t \in \mathcal{M}_{\mathfrak{X}}(U) = B'$ such that g, h, r, t are representatives of γ, η, ρ, τ respectively. Then $gr + ht$ is a representative of $1 \in \mathcal{M}_{\mathfrak{X},v}$. So there is a rational domain $V = \mathrm{Val}(B'', A'') \subset U$ such that $gr + ht|_V = 1 \in \mathcal{M}_{\mathfrak{X}}(V) = B''$. It follows that $g|_V, h|_V \in \mathcal{M}_{\mathfrak{X}}(V) = B''$ are representatives of γ, η and are co-prime. Furthermore v induces (canonically) a valuation on B'' which has the same valuation ring as v. Replace $g|_V, h|_V$ with g, h. Since $v(\gamma) \le v(\eta)$ it follows that $v(g) \le v(h)$. As $V = \mathrm{Val}(B'', A'')$ is a rational domain there are $a_1, \ldots, a_n, b \in B$ generating the unit ideal such that $B'' = B_b$ and $A'' = \varphi_b(A)[\frac{a_1}{b}, \ldots, \frac{a_n}{b}]$. Now $g, h \in B''$ so there is some natural number r such that $b^r g, b^r h \in B$ and $v(b^r g) \le v(b^r h)$. If $v(b) > 1$ we may shrink V to $\mathfrak{X}(\{1, a_1, \ldots, a_n\}/b)$. If $v(b) \le 1$ then there is an element $d \in B$ such that $v(b)^{-1} < v(d)$ since v is unbounded and not trivial. So $1 < v(bd)$ and $v(d) \neq 0$. It follows that $v \in \mathfrak{X}(\{1, a_1 d, \ldots, a_n d\}/bd) \subset V$. Shrinking V and replacing notation, we may assume that $V = \mathrm{Val}(B'', A'') = \mathfrak{X}(\{1, a_1, \ldots, a_n\}/b)$. Now $1 < v(b)$ so for some large enough $j \ge r$ we have $b^j g, b^j h \in B$, $v(b^j g) \le v(b^j h)$ and $1 < v(b^{j+1} h)$. The elements $1, b^j a_1, \ldots, b^j a_n, b^{j+1} g, b^{j+1} h$ generate the unit ideal of B. We have $v \in W = \mathfrak{X}\left(\{1, b^j a_1 h, \ldots, b^j a_n h, b^{j+1} g\}/b^{j+1} h\right) \subset V$ and $W = \mathrm{Val}\left(B_{b^{j+1}h}, \varphi_{b^{j+1}h}(A)\left[\frac{1}{b^{j+1}h}, \frac{a_1}{b}, \ldots, \frac{a_n}{b}, \frac{g}{h}\right]\right)$. Hence $g|_W \in h|_W \mathcal{O}_{\mathfrak{X}}(W)$ and it follows that $\gamma \in \eta\mathcal{O}_{\mathfrak{X},v}$. By the same reasoning if $v(\eta) \le v(\gamma)$ then $\eta \in \gamma\mathcal{O}_{\mathfrak{X},v}$. $\square$

Corollary 2.6.6. *For any point $\mathfrak{p} \in \operatorname{Spec} B$ the stalks of the point $\sigma(\mathfrak{p}) \in \mathfrak{X}$ are*

$$\mathcal{M}_{\mathfrak{X},\sigma(\mathfrak{p})} = \mathcal{O}_{\mathfrak{X},\sigma(\mathfrak{p})} = B_{\mathfrak{p}}.$$

3. Birational spaces

3.1. Categories of pairs

Definition 3.1.1.

(i) By a *pair of rings* (B, A) we mean a ring B and a sub-ring A.
(ii) A homomorphism of pairs of rings $\varphi\colon (B, A) \to (B', A')$ is a ring homomorphism $\varphi\colon B \to B'$ such that $\varphi(A) \subset A'$.
(iii) A homomorphism of pairs of rings $\varphi\colon (B, A) \to (B', A')$ is called *adic* if the canonical homomorphism $B \otimes_A A' \to B'$ is integral.
(iv) The *relative normalization* of a pair of rings (B, A) is the pair of rings $(B, \mathrm{Nor}_B A)$ together with the canonical homomorphism of pairs

$$\nu = id_B\colon (B, A) \to (B, \mathrm{Nor}_B A).$$

(v) By a *pair of schemes* $(Y \xrightarrow{f} X)$ or (Y, X) we mean a pair of qcqs schemes Y, X together with an affine and schematically dominant morphism $f\colon Y \to X$.
(vi) A morphism of pairs of schemes $g\colon (Y', X') \to (Y, X)$ is a pair of morphisms $g = (g_Y, g_X)$ forming a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "g_Y"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g_X"] & X. \end{tikzcd}$$

(vii) A morphism of pairs of schemes $g\colon (Y', X') \to (Y, X)$ is called *adic* if the canonical morphism of schemes $Y' \to Y \times_X X'$ is integral.
(viii) The *relative normalization* of a pair of schemes $(Y \xrightarrow{f} X)$ is the pair of schemes $(Y, \mathrm{Nor}_Y X)$ together with the canonical morphism of pairs

$$\nu = (id_Y, \nu_X)\colon (Y, \mathrm{Nor}_Y X) \to (Y, X),$$

where $\mathrm{Nor}_Y X = \mathbf{Spec}_X(\mathrm{Nor}_{f_*\mathcal{O}_Y}(\mathcal{O}_X))$ and ν_X is the canonical morphism $\mathrm{Nor}_Y X \to X$.

We denote the category of pairs of rings with their morphisms by *pa-Ring* and the category of pairs of schemes with their morphisms by *pa-Sch*.

Remark 3.1.2. As in Remark 2.3.6, a homomorphism of pairs of rings $\varphi\colon (B, A) \to (B', A')$ is adic if and only if $B \otimes_{\mathbb{Z}} A' \to B'$ is integral.

Remark 3.1.3. Since the structure morphism f of a pair of schemes $(Y \xrightarrow{f} X)$ is affine and schematically dominant we obtain embeddings of quasi-coherent sheaves of $\mathcal{O}_X$-algebras $\mathcal{O}_X \hookrightarrow \mathrm{Nor}_{f_*\mathcal{O}_Y} \mathcal{O}_X \subset f_*\mathcal{O}_Y$. It follows that $(Y, \mathrm{Nor}_Y X)$ is indeed an object of *pa-Sch*.

Definition 3.1.4. Let $(Y \xrightarrow{f} X)$ be a pair of schemes. Given an open (affine) subscheme $X' \subset X$ its preimage $Y' = f^{-1}(X')$ is an open (affine) subscheme of Y'. The restriction of f to Y' makes (Y', X') a pair of schemes. We call (Y', X') an *open (affine) sub-pair of schemes*. An *affine covering* of the pair (Y, X) is a collection of open sub-pairs $\{(Y_i, X_i)\}$ such that $\{X_i\}$ are affine and cover X (then, necessarily their preimages $\{Y_i\}$ are affine and cover Y).

Lemma 3.1.5. *Assume the elements $b, a_1, \dots, a_n \in B$ generate the unit ideal. Set $B' = B_b$. Let $\varphi_b \colon B \to B_b$ be the canonical map and denote $A' = \varphi_b(A)[\frac{a_1}{b}, \dots, \frac{a_n}{b}]$. Then the homomorphism of pairs of rings $\varphi_b \colon (B, A) \to (B', A')$ is adic.*

Proof. Since $b, a_1, \dots, a_n$ generate the unit ideal of B,

$$B \otimes_A A' = \varphi_b(B)\left[\frac{a_1}{b}, \dots, \frac{a_n}{b}\right] = B'. \quad \square$$

Proposition 3.1.6. *The category of pairs of schemes contains all fiber products.*

Proof. Let $(Y \xrightarrow{f} X)$, $(Y' \xrightarrow{f'} X')$, $(T \xrightarrow{j} Z)$ be pairs of schemes and $g \colon (Y', X') \to (Y, X)$, $h \colon (T, Z) \to (Y, X)$ morphisms of pairs. Then $f' \times j \colon Y' \times_Y T \to X' \times_X Z$ is an affine morphism. Denote the schematic image of $f' \times j$ by Z' and set $T' = Y' \times_Y T$. Denote the induced morphism $T' \to Z'$ by j'. Clearly $(T' \xrightarrow{j'} Z')$ is a pair of schemes equipped with canonical morphisms of pairs

$$(Y' \xrightarrow{f'} X') \xleftarrow{h'} (T' \xrightarrow{j'} Z') \xrightarrow{g'} (T \xrightarrow{j} Z)$$

such that $g \circ h' = h \circ g'$. Let $(W \xrightarrow{l} V)$ be a pair of schemes with morphisms of pairs

$$(Y' \xrightarrow{f'} X') \xleftarrow{p} (W \xrightarrow{l} V) \xrightarrow{q} (T \xrightarrow{j} Z)$$

such that $g \circ p = h \circ q$. By the universal property of fiber products of schemes there are unique morphisms $W \to T'$ and $V \to X' \times_X Z$. By the universal property of the scheme theoretic image $V \to X' \times_X Z$ factors uniquely through Z'. Hence there is a unique morphism of pairs $(W, V) \to (T', Z')$. $\square$

3.2. Birational spaces and the bir functor

Definition 3.2.1.

(i) A *pair-ringed space* $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ is a topological space $\mathfrak{X}$ together with a sheaf of pairs of rings $(\mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ such that both $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ are ringed spaces.

(ii) A morphism of pair-ringed spaces

$$(h, h^{\sharp}) \colon (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}})$$

is a continuous map $h \colon \mathfrak{X} \to \mathfrak{Y}$ together with a morphism of sheaves of pairs of rings

$$h^{\sharp} \colon (\mathcal{M}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}) \to (h_* \mathcal{M}_{\mathfrak{X}}, h_* \mathcal{O}_{\mathfrak{X}})$$

such that both

$$(h, h^{\sharp}) \colon (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \text{ and } (h, h^{\sharp}) \colon (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$$

are morphisms of ringed spaces.

In Section 2 we constructed a pair-ringed space $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ from a pair of rings (B, A), namely $\mathrm{Val}(B, A)$. From now on for any pair of rings (B, A) by $\mathrm{Val}(B, A)$ we mean the pair-ringed space $(\mathrm{Val}(B, A), \mathcal{M}_{\mathrm{Val}(B,A)}, \mathcal{O}_{\mathrm{Val}(B,A)})$. We will freely use the formulas for the stalks from Proposition 2.6.5.

Definition 3.2.2.

(i) An *affinoid birational space* is a pair-ringed space $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ isomorphic to $\mathrm{Val}(B, A)$ for some pair of rings (B, A).

(ii) A pair-ringed space $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ is a *birational space* if every point $x \in \mathfrak{X}$ has an open neighborhood U such that the induced subspace $(U, \mathcal{M}_{\mathfrak{X}}|_U, \mathcal{O}_{\mathfrak{X}}|_U)$ is an affinoid birational space.

(iii) A morphism of birational spaces

$$(h, h^{\sharp}) \colon (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}})$$

is a morphism of pair-ringed spaces such that the induced morphism of ringed spaces $(h, h^{\sharp}|_{\mathcal{O}_{\mathfrak{X}}}) \colon (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ is a morphism of locally ringed spaces (but not necessarily $(h, h^{\sharp}) \colon (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$; see Example 3.2.11 below).

We denote the category of affinoid birational spaces with their morphisms by *af-Birat* and the category of birational spaces with their morphisms by *Birat*.

Example 3.2.3. Given a ring B, we have a homeomorphism of topological spaces $\mathrm{Val}(B, B) \simeq \operatorname{Spec} B$ (Example 2.4.10). From Remark 2.6.6 it is clear that considered as an affinoid birational space $\mathrm{Val}(B, B)$ is exactly $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}, \mathcal{O}_{\operatorname{Spec} B})$. Given another ring B', a homomorphism of rings $B' \to B$ induces a morphism of schemes $(f, f^\sharp)\colon \operatorname{Spec} B \to \operatorname{Spec} B'$ which can also be viewed as a morphism of affinoid birational space $(f, f^\sharp)\colon \mathrm{Val}(B, B) \to \mathrm{Val}(B', B')$.

A scheme is locally isomorphic to an affine scheme so we obtain

Lemma 3.2.4.

(1) *Any scheme $(X, \mathcal{O}_X)$ can be viewed as a birational space*

$$Val(X, X) = (X, \mathcal{O}_X, \mathcal{O}_X).$$

(2) *The functor $(X, \mathcal{O}_X) \mapsto (X, \mathcal{O}_X, \mathcal{O}_X)$ from qcqs schemes to birational space is fully faithful.*

(3) *Any pair of schemes (Y, X) induces a birational space* $\mathrm{Val}(Y, X)$ *and continuous maps*

$$Y \xrightarrow{\sigma} \mathrm{Val}(Y, X) \xrightarrow{\tau} X$$

as in Definition 2.4.11.

Remark 3.2.5.

(1) For a pair of schemes (Y, X), the points of $\mathrm{Val}(Y, X)$ are 3-tuples (y, R_y, Φ) such that y is a point in Y, R_y is a valuation ring of the residue field $k(y)$ and Φ is a morphism of schemes $\operatorname{Spec} R_y \to X$ making the diagram

$$\begin{tikzcd} \operatorname{Spec} k(y) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} R_y \arrow[r, "\Phi"] & X \end{tikzcd}$$

commute, and $\operatorname{Spec} k(y) \to Y \times_X \operatorname{Spec} R_y$ is a closed immersion (cf. [9, §3]).

(2) Any affine covering $\{(Y_i, X_i)\}$ of (Y, X) gives rise to a covering of the birational space $\mathrm{Val}(Y, X)$ consisting of affinoid birational spaces $\{\mathrm{Val}(Y_i, X_i)\}$.

Lemma 3.2.6. *Let (Y, X) be a pair of schemes. Then* $\mathrm{Val}(Y, X)$ *is qcqs.*

Proof. Combine Remark 3.2.5(2) and Corollary 2.4.9. $\square$

Lemma 2.6.1 and Remark 2.6.3 easily extend from the affine case to the general one.

Corollary 3.2.7. *Let* (Y, X) *be a pair of schemes. Then* $\mathrm{Val}(Y, \mathrm{Nor}_Y X) = \mathrm{Val}(Y, X)$ *as birational spaces and the canonical map* $\tau\colon \mathrm{Val}(Y, X) \to X$ *factors through* $\mathrm{Nor}_Y X$*.*

Proof. For every affine $U \subset X$ we have $(\mathrm{Nor}_Y \mathcal{O}_X)(U) = \mathrm{Nor}_{f_* \mathcal{O}_Y(U)} \mathcal{O}_X(U)$. The result follows from Lemma 2.6.1 and Remark 2.6.3. □

Example 3.2.8. For a finitely generated field extension K/k there is an obvious natural homeomorphism from $RZ(K/k)$ to $\mathrm{Val}(K, k)$. At a valuation v the stalk is the pair of rings (K, R_v).

Let us now construct a functor bir: pa-Sch → Birat.

Construction 3.2.9. We already saw the construction of a birational space $\mathrm{Val}(Y, X)$ from a pair of schemes (Y, X). We set $\mathrm{bir}(Y, X) = (Y, X)_{\mathrm{bir}} = \mathrm{Val}(Y, X)$. By Remark 3.2.5(2) constructing a morphism of birational spaces $Val(Y_2, X_2) \to Val(Y_1, X_1)$ from a morphism of pairs $(Y_2, X_2) \to (Y_1, X_1)$ can be done locally on X and X'. For a homomorphism of pairs of rings $\varphi\colon (B_1, A_1) \to (B_2, A_2)$ we define the map of topological spaces φ_{bir} by the composition

$$\begin{tikzcd}
\mathrm{Spa}(B_2, A_2) \arrow[r, "\varphi^*"] & \mathrm{Spa}(B_1, A_1) \arrow[d, "r"] \\
\mathrm{Val}(B_2, A_2) \arrow[u, hook] \arrow[r, dashed, "\varphi_{\mathrm{bir}}"] & \mathrm{Val}(B_1, A_1)
\end{tikzcd}$$

where φ^* is the pull back map defined in section 2.2 and r is the retraction of Definition 2.4.7. We saw that both φ^* (Lemma 2.2.2) and the retraction (Lemma 2.4.8) are continuous so φ_{bir} is continuous.

For $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Val}(B_2, A_2)$ we have

$$\varphi^*(v) = v \circ \varphi \in \mathrm{Spa}(B_1, A_1)$$
$$\varphi_{\mathrm{bir}}(v) = r(v \circ \varphi) = w = (\mathfrak{q}, R_w, \Psi) \in \mathrm{Val}(B_1, A_1).$$

Since w is a primary specialization of the pullback valuation $\varphi^*(v) = v \circ \varphi$ there is a natural homomorphism of the stalks

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{Val}(B_1,A_1),w} = (B_1)_{\mathfrak{q}} \arrow[r] & (B_1)_{\varphi^{-1}(\mathfrak{p})} \arrow[r] & (B_2)_{\mathfrak{p}} = \mathcal{M}_{\mathrm{Val}(B_2,A_2),v} \\
\mathcal{O}_{\mathrm{Val}(B_1,A_1),w} = S_w \arrow[u] \arrow[r] & S_{v\circ\varphi} \arrow[u] \arrow[r] & S_v = \mathcal{O}_{\mathrm{Val}(B_2,A_2),v}. \arrow[u]
\end{tikzcd}$$

By Lemma 2.4.5 and Lemma 2.4.6 both bottom arrows are local homomorphism. We obtain a morphism of affinoid birational spaces

$$\varphi_{\mathrm{bir}} \colon (B_2, A_2)_{\mathrm{bir}} \to (B_1, A_1)_{\mathrm{bir}}.$$

It follows from Remark 2.3.2 that bir respects identity homomorphisms. As for composition again it is enough to check only the affine case, which is the subject of next lemma.

Lemma 3.2.10. *Let* $(B_1, A_1) \xrightarrow{\varphi} (B_2, A_2) \xrightarrow{\psi} (B_3, A_3)$ *be homomorphisms of pairs of rings. Then* $\varphi_{\mathrm{bir}} \circ \psi_{\mathrm{bir}} = (\psi \circ \varphi)_{\mathrm{bir}}$.

Proof. For $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Val}(B_3, A_3)$ we have $(\psi \circ \varphi)^*(v) = \varphi^*(\psi^*(v))$ as elements of $\mathrm{Spa}(B_1, A_1)$. By construction $(\psi \circ \varphi)_{\mathrm{bir}}(v)$ is a primary specialization of $(\psi \circ \varphi)^*(v)$. Also, $\psi_{\mathrm{bir}}(v)$ is a primary specialization of $\psi^*(v)$ as elements of $\mathrm{Spa}(B_2, A_2)$, and $\varphi_{\mathrm{bir}}(\psi_{\mathrm{bir}}(v))$ is a primary specialization of $\varphi^*(\psi_{\mathrm{bir}}(v))$ as elements of $\mathrm{Spa}(B_1, A_1)$. It follows from Lemma 2.3.3 that $\varphi^*(\psi_{\mathrm{bir}}(v))$ is a primary specialization of $\varphi^*(\psi^*(v))$. Hence both $\varphi_{\mathrm{bir}}(\psi_{\mathrm{bir}}(v))$ and $(\psi \circ \varphi)_{\mathrm{bir}}(v)$ are primary specializations of $(\psi \circ \varphi)^*(v)$. They are also both minimal primary specializations, since they are elements of $\mathrm{Val}(B_1, A_1)$. By Proposition 2.3.4 we have $\varphi_{\mathrm{bir}}(\psi_{\mathrm{bir}}(v)) = (\psi \circ \varphi)_{\mathrm{bir}}(v)$. $\square$

The following example shows that the homomorphism on the stalks of $\mathcal{M}$ can indeed be not local.

Example 3.2.11. Let K be an algebraically closed field. Consider $A = A' = B = K[T]$ and $B' = K[T, T^{-1}]$. Let $\varphi \colon (B, A) \to (B', A')$ be the obvious map. Clearly φ is not adic. Passing to birational spaces, we have

$$\varphi_{\mathrm{bir}} \colon \mathfrak{X}' = \mathrm{Val}(K[T, T^{-1}], K[T]) \to \mathrm{Val}(K[T], K[T]) = \mathfrak{X}.$$

As we saw in Example 3.2.3, $\mathfrak{X} = (\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1})$. Let η be the generic point of $\mathbb{A}^1$ and of $\operatorname{Spec} K[T, T^{-1}] \subset \mathbb{A}^1$. Let v be the valuation in $\mathfrak{X}'$ corresponding to the valuation ring $R_v = K[T]_{(T)} \subset K(T) = k(\eta)$. It is indeed in $\mathfrak{X}'$ as $K[T, T^{-1}] \otimes K[T]_{(T)} \to K(T)$ is surjective. Since $K[T] \otimes K[T]_{(T)} \to k(\eta) = K(T)$ is not surjective, the pullback $v \circ \varphi$ is not in $\mathfrak{X}$. Its primary specialization w is the trivial valuation on $k(\mathfrak{p}) = K$ for the ideal $\mathfrak{p} = (T)$. The stalks are $\mathcal{M}_{\mathfrak{X}', v} = K(T)$ and $\mathcal{M}_{\mathfrak{X}, w} = K[T]_{(T)}$. The induced homomorphism of stalks is the obvious injection

$$K[T]_{(T)} \to K(T)$$

which is not a local homomorphism.

Lemma 3.2.12. *Let* (Y, X), (Y', X') *be two pairs of schemes and let* $g \colon (Y', X') \to (Y, X)$ *be a morphism of pairs. Then the diagram of topological spaces*

$$\begin{tikzcd}
Y' \arrow[r, "g_Y"] \arrow[d, "\sigma'"] & Y \arrow[d, "\sigma"] \\
\mathrm{Val}(Y', X') \arrow[r, "g_{\mathrm{bir}}"] \arrow[d, "\tau'"] & \mathrm{Val}(Y, X) \arrow[d, "\tau"] \\
X' \arrow[r, "g_X"] & X
\end{tikzcd}$$

commutes.

Proof. The question is local on X and X' hence it is enough to consider only the affine case. Let (B, A), (B', A') be two pairs of rings and let $g = (g_B, g_A)\colon (B, A) \to (B', A')$ be a homomorphism of pairs. We want to show that

$$\begin{tikzcd}
\operatorname{Spec} B' \arrow[r, "\operatorname{Spec} g_B"] \arrow[d, "\sigma'"] & \operatorname{Spec} B \arrow[d, "\sigma"] \\
\mathrm{Val}(B', A') \arrow[r, "g_{\mathrm{bir}}"] \arrow[d, "\tau'"] & \mathrm{Val}(B, A) \arrow[d, "\tau"] \\
\operatorname{Spec} A' \arrow[r, "\operatorname{Spec} g_A"] & \operatorname{Spec} A
\end{tikzcd}$$

commutes.

If $\mathfrak{p} \in \operatorname{Spec} B'$, then $\sigma'(\mathfrak{p})$ is the trivial valuation on the residue field $k(\mathfrak{p})$. Similarly, $\sigma \circ \operatorname{Spec} g_B(\mathfrak{p}) \in \mathrm{Val}(B, A)$ is the trivial valuation on the residue field $k(g_B^{-1}(\mathfrak{p}))$. The homomorphism $g_B\colon B \to B'$ induces an injection $k(g_B^{-1}(\mathfrak{p})) \to k(\mathfrak{p})$, so the composition $g_{\mathrm{bir}} \circ \sigma'(\mathfrak{p})$ is just the trivial valuation on the residue field $k(g_B^{-1}(\mathfrak{p}))$.

If $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Val}(B', A')$, let $g_{bir}(v) = w = (\mathfrak{q}, R_w, \Psi)$. Denote by $\mathfrak{m}_v$ the maximal ideal of S_v and by $\mathfrak{m}_w$ the maximal ideal of S_w. The homomorphisms $\Phi\colon A' \to R_v$ and $\Psi\colon A \to R_w$ factor through S_v and S_w respectively. Denote the corresponding homomorphisms by $\Phi'\colon A' \to S_v$ and $\Psi'\colon A \to S_w$. By Remark 2.4.14 we have $\tau'(v) = \Phi'^{\,-1}(\mathfrak{m}_v)$ and $\tau(g_{\mathrm{bir}}(v)) = \Psi'^{\,-1}(\mathfrak{m}_w)$. Since the induced homomorphism $S_v \to S_w$ is local we have $g_A^{-1}(\Phi'^{\,-1}(\mathfrak{m}_v)) = \Psi'^{\,-1}(\mathfrak{m}_w)$. Hence

$$\tau(g_{\mathrm{bir}}(v)) = \Psi'^{\,-1}(\mathfrak{m}_w) = g_A^{-1}(\Phi'^{\,-1}(\mathfrak{m}_v)) = \operatorname{Spec} g_A(\tau'(v)). \quad \square$$

The functor bir: *pa-Rings* $\to$ *af-Birat* is essentially surjective by the definition of the category of affinoid birational spaces. Given two pairs of rings $(B, A), (B', A')$ and a morphism of affinoid birational spaces $h\colon \mathrm{Val}(B, A) \to \mathrm{Val}(B', A')$, by taking global sections we obtain a homomorphism of pairs of rings φ

$$\begin{tikzcd} B' \arrow[r, "\varphi"] & B \\ \mathrm{Nor}_{B'} A' \arrow[u, hook] \arrow[r] & \mathrm{Nor}_B A. \arrow[u, hook] \end{tikzcd}$$

Composition with the inclusion $A' \subset \mathrm{Nor}_{B'} A'$ gives only a homomorphism of the pairs $\varphi\colon (B', A') \to (B, \mathrm{Nor}_B A)$, not of the original pairs (B, A) and (B', A'). As $\mathrm{Val}(B, \mathrm{Nor}_B A) = \mathrm{Val}(B, A)$ and any homomorphism $(B', A') \to (B, A)$ uniquely extends to a homomorphism $(B', \mathrm{Nor}_{B'} A') \to (B, \mathrm{Nor}_B A)$, this discrepancy is superficial in some sense.

Theorem 3.2.13. *The functor* bir *is an anti-equivalence of categories between the category of pairs of rings* pa-Rings *localized at the class of relative normalizations and the category of affinoid birational spaces* af-Birat*. The functor of global sections is a quasi-inverse.*

Proof. Denote the class of relative normalization homomorphisms by M. By Lemma 2.6.1, bir factors through *pa-Rings*$_M$ and by definition of *af-Birat* it is essentially surjective. It remains to show that

$$\mathrm{bir}\colon \textit{pa-Rings}_M \to \textit{af-Birat}$$

is full and faithful.

We start by proving fullness. Let $(B', A'), (B, A)$ be two pairs of rings and h a morphism of affinoid birational spaces $h\colon (B, A)_{\mathrm{bir}} \to (B', A')_{\mathrm{bir}}$. We may assume that A' and A are integrally closed in B' and B respectively. Taking global sections we obtain a diagram

$$\begin{tikzcd} B' \arrow[r, "\varphi"] & B \\ A' \arrow[u, hook] \arrow[r] & A \arrow[u, hook] \end{tikzcd}$$

i.e. φ is a homomorphism of pairs of rings $\varphi\colon (B', A') \to (B, A)$. We claim that $\varphi_{\mathrm{bir}} = h$. Given a valuation $v = (\mathfrak{p}, R_v, \Phi) \in \mathrm{Val}(B, A)$ we denote

$$h(v) = w = (\mathfrak{q}, R_w, \Psi) \in \mathrm{Val}(B', A').$$

Passing to stalks we obtain a diagram of pairs of rings

$$\begin{tikzcd} (B', A') \arrow[r, "\varphi"] \arrow[d] & (B, A) \arrow[d] \\ (B'_{\mathfrak{q}}, S_w) \arrow[r, "h_v^\sharp"] & (B_{\mathfrak{p}}, S_v). \end{tikzcd}$$

Let $\mathfrak{n} = \mathfrak{p}B_\mathfrak{p}$ denote the maximal ideal of $B_\mathfrak{p}$, $\mathfrak{n}' = \mathfrak{q}B'_\mathfrak{q}$ the maximal ideal of $B'_\mathfrak{q}$, $\mathfrak{m}_v$ the maximal ideal of S_v and $\mathfrak{m}_w$ the maximal ideal of S_w. Since $B'_\mathfrak{q}$ is a local ring ${h_v^\sharp}^{-1}(\mathfrak{n}) \subset \mathfrak{n}'$. Pulling back to B' we see that $\varphi^{-1}(\mathfrak{p}) \subset \mathfrak{q}$. Hence the bottom line of the above diagram can be factored as

$$\begin{tikzcd}
B'_\mathfrak{q} \arrow[r] & B'_{\varphi^{-1}(\mathfrak{p})} = (B'_\mathfrak{q})_{{h_v^\sharp}^{-1}(\mathfrak{n})} \arrow[r] & B_\mathfrak{p} \\
S_w \arrow[r] \arrow[u] & S_{v\circ\varphi} \arrow[r] \arrow[u] & S_v . \arrow[u]
\end{tikzcd}$$

By Lemma 2.4.5 and the definition of morphisms of birational spaces, we see that the bottom left arrow is a local homomorphism. By Lemma 2.4.6 w is a primary specialization of the pullback valuation $v \circ \varphi$. As w is already in $\mathrm{Val}(B', A')$ it has no primary specialization other than itself. By Proposition 2.3.4 the primary specializations of $v \circ \varphi$ are linearly ordered and we conclude that $r(v \circ \varphi) = w$ where r is the retraction, or in other words $\varphi_{\mathrm{bir}}(v) = h(v)$.

As for faithfulness, given two homomorphisms of pairs of rings $\varphi, \psi \colon (B', A') \to (B, A)$ such that $\varphi_{\mathrm{bir}} = \psi_{\mathrm{bir}}$ we obtain a diagram

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, shift left, "\operatorname{Spec}\varphi"] \arrow[r, shift right, "\operatorname{Spec}\psi"'] \arrow[d, "\sigma"] & \operatorname{Spec} B' \arrow[d, "\sigma'"] \\
\mathrm{Val}(B, A) \arrow[r, shift left, "\varphi_{\mathrm{bir}}"] \arrow[r, shift right, "\psi_{\mathrm{bir}}"'] & \mathrm{Val}(B', A').
\end{tikzcd}$$

It follows from Proposition 2.4.12(2), Corollary 2.6.6, the definition of a morphism of birational spaces and Lemma 3.2.12 that $\operatorname{Spec}\varphi = \operatorname{Spec}\psi$ as morphisms of schemes $\operatorname{Spec} B \to \operatorname{Spec} B'$. Hence $\varphi = \psi$ as homomorphisms of rings $B' \to B$. As A' and A are integrally closed in B' and B respectively, we also have that $\varphi = \psi$ as homomorphisms of pairs of rings $(B', A') \to (B, A)$. $\square$

Since a birational space is locally affinoid we obtain the following result

Corollary 3.2.14. *Let $\mathfrak{X}$ be a birational space and (B, A) a pair of rings. There is a natural bijection*

$$Hom_{Birat}(\mathfrak{X}, (B, A)_{\mathrm{bir}}) \simeq Hom_{pa\text{-}Rings}((B, \mathrm{Nor}_B A), \Gamma(\mathcal{M}_\mathfrak{X}, \mathcal{O}_\mathfrak{X}))\,.$$

Proposition 3.2.15. *The category of birational spaces contains all fiber products and the functor* bir *respects fiber products.*

Proof. Consider the full subcategory of *pa-Rings* consisting of all normalized pairs of rings, i.e. all pairs of rings $A \subset B$ such that $A = \mathrm{Nor}_B A$. Denote this category $\mathcal{C}$.

Note that this category contains all pushouts and that the pushout of the arrows $(B_2, A_1) \longleftarrow (B_1, A_1) \longrightarrow (B_3, A_3)$ is the pair (B, A) where $B = B_2 \otimes_{B_1} B_3$ and A is the integral closure of the image of $A_2 \otimes_{A_1} A_3$ in B. The embedding of $\mathcal{C}$ in *pa-Rings* has a left adjoint given by sending a pair of rings (B, A) to its relative normalization $(B, \operatorname{Nor}_B A)$. Hence $\mathcal{C}$ is a reflective subcategory of *pa-Rings* and is equivalent to the localization of *pa-Rings* with respect to relative normalizations. By Proposition 3.1.6 bir takes pushouts in *pa-Rings* to fiber products in *af-Birat*. It follows from the anti-equivalence of Theorem 3.2.13 that the category of affinoid birational spaces contains all fiber products. From Corollary 3.2.14 we see that a fiber product in *af-Birat* is also a fiber product in *Birat*. As a birational space is locally affinoid *Birat* contains all fiber products. Furthermore, since an affine covering (Y_i, X_i) of a pair of schemes (Y, X) induces an affinoid covering $\operatorname{Val}(Y_i, X_i) = (Y_i, X_i)_{\text{bir}}$ of $\operatorname{Val}(Y, X) = (Y, X)_{\text{bir}}$, the functor $\text{bir}\colon$ *pa-Sch* $\to$ *Birat* respects fiber products. $\square$

Definition 3.2.16. Let $\mathfrak{X}$ be a birational space and (Y, X) a pair of schemes. If $(Y, X)_{\text{bir}} = \mathfrak{X}$ we say that (Y, X) is a *scheme model* of $\mathfrak{X}$. Given another scheme model (Y', X') of $\mathfrak{X}$, if there is a morphism of pairs of schemes $g\colon (Y', X') \to (Y, X)$ such that g_{bir} is the identity we say that (Y', X') *dominates* (Y, X).

3.3. Characterization of adic morphisms

In this section we collect results regarding adic morphisms. These results are not used later on.

Lemma 3.3.1.

- (i) *Composition of adic morphisms of pairs of schemes is adic.*
- (ii) *Let $g\colon (Y', X') \to (Y, X)$ and $h\colon (Y'', X'') \to (Y', X')$ be morphisms of pairs of schemes. If $h \circ g$ is adic and the induced morphism $X'' \times_{X'} Y' \to X'' \times_X Y$ is separated (e.g. if $g_Y\colon Y' \to Y$ is separated) then h is adic.*
- (iii) *Adic morphisms are stable under base change.*
- (iv) *Let $g\colon (Y', X') \to (Y, X)$ be a morphism of pairs of schemes and $\{(V_i, U_i)\}$ an affine covering of (Y, X). If all the restrictions $(g_Y^{-1}(V_i), g_X^{-1}(U_i)) \to (V_i, U_i)$ are adic, then g is adic.*

We omit the proof since it is a direct application of the same results for integral morphisms.

Let X be a scheme and $x \in X$ a point. We know that there are canonical morphisms $\operatorname{Spec} k(x) \to \operatorname{Spec} \mathcal{O}_{X,x} \to X$ where the image of (the singleton) $\operatorname{Spec} k(x)$ in X is x and the image of $\operatorname{Spec} \mathcal{O}_{X,x}$ is the set of generalizations of x. With this in mind, let $\mathfrak{X}$ be a birational space, $v \in \mathfrak{X}$ a valuation and $\mathfrak{m}$ the maximal ideal of $\mathcal{M}_{\mathfrak{X},v}$. There are canonical homomorphisms of pairs of rings

$$\Gamma(\mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \to (\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v}) \to (k(v), R_v),$$

where $k(v) = \mathcal{M}_{\mathfrak{X},v}/\mathfrak{m}$, which give rise to canonical morphisms of birational spaces

$$\mathrm{Val}(k(v), R_v) \to \mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v}) \to \mathfrak{X}.$$

Note that the homomorphism $(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v}) \to (k(v), R_v)$ is adic.

Lemma 3.3.2. *Let $\mathfrak{X}$ be a birational space and $v \in \mathfrak{X}$ a valuation. The canonical map $\mathrm{Val}(k(v), R_v) \to \mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$ is injective and its image is the set of $\mathcal{O}_{\mathfrak{X},v}$-valuations on $\mathcal{M}_{\mathfrak{X},v}$ whose kernel is the maximal ideal of $\mathcal{M}_{\mathfrak{X},v}$.*

Proof. Note that a point of $\mathrm{Val}(k(v), R_v)$ corresponds to a valuation ring $R_v \subset T \subset k(v)$, and the generalizing relation of points corresponds to the inclusion relation of valuation rings. Since $(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v}) \to (k(v), R_v)$ is adic, the image of T in $\mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$ is the outer square of the diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathfrak{X},v} \arrow[r] & k(v) \arrow[r, equal] & k(v) \\
\mathcal{O}_{\mathfrak{X},v} \arrow[u] \arrow[r] & R_v \arrow[u] \arrow[r] & T \arrow[u]
\end{tikzcd}$$

obtained by composition. It follows that only a valuation $w \in \mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$ whose kernel is the unique maximal ideal of $\mathcal{M}_{\mathfrak{X},v}$ can be in the image. Furthermore for any such w, $\mathcal{O}_{\mathfrak{X},v} \to T$ factors uniquely through R_v. Hence w is in the image and its preimage consists of one point. □

Lemma 3.3.3. *Let $\mathfrak{X}$ be a birational space. Let $v, v' \in \mathfrak{X}$ be points of $\mathfrak{X}$. Then v' is a generalization of v if and only if v' is in the image of the canonical morphism $\mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v}) \to \mathfrak{X}$.*

Proof. Since every point of $\mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$ is a generalization of the unique closed point, which maps to v, we see that the image of $\mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$ in $\mathfrak{X}$ consists of generalizations of v. Conversely, suppose that v' is a generalization of v. Choose an affinoid open neighborhood $U = \mathrm{Val}(B, A)$ of v. Then $v' \in U$. As there are no primary specializations in $\mathrm{Val}(B, A)$, it follows from [4, Remark 1.2.2 and Proposition 1.2.4] that $\ker v' = \ker v$ and that the valuation ring of v' is a localization of the valuation ring of v at a prime ideal. Denote by R and R' the valuation rings of v and v' respectively. The valuation in $\mathrm{Val}(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v})$ that corresponds to the diagram

$$\begin{tikzcd} \mathcal{M}_{\mathfrak{X},v} \arrow[rr] & & k(v) \\ \mathcal{O}_{\mathfrak{X},v} \arrow[u] \arrow[r] & R \arrow[r] & R' \arrow[u] \end{tikzcd}$$

maps to v'. □

Corollary 3.3.4. *The image of the canonical map $\sigma\colon Y \to \mathrm{Val}(Y,X)$ is a dense subset.*

Proof. For every $v \in \mathrm{Val}(Y,X)$ the generic point of $\mathrm{Val}\left(\mathcal{M}_{\mathfrak{X},v}, \mathcal{O}_{\mathfrak{X},v}\right)$ is a trivial valuation which is in the image of σ. □

Let $g\colon (Y_2,X_2) \to (Y_1,X_1)$ be a morphism of pairs of schemes. Considering Y_1 as the birational space $Y_1 = \mathrm{Val}(Y_1,Y_1)$ we can form the fiber product of the maps $\mathrm{Val}(Y_2,X_2) \xrightarrow{g_{bir}} \mathrm{Val}(Y_1,X_1) \xleftarrow{\sigma_1} Y_1$. It is $\mathrm{Val}(Y,X)$ where $Y = Y_2 \times_{Y_1} Y_1 = Y_2$ and X is the relative normalization of the scheme theoretic image of Y in $X_2 \times_{X_1} Y_1$. Denoting the scheme theoretic image of Y_2 in $X_2 \times_{X_1} Y_1$ by X_2' we have a pull back square

$$\begin{tikzcd} \mathrm{Val}(Y_2,X_2') \arrow[r] \arrow[d] & Y_1 \arrow[d, "\sigma_1"] \\ \mathrm{Val}(Y_2,X_2) \arrow[r, "g_{\mathrm{bir}}"] & \mathrm{Val}(Y_1,X_1) \end{tikzcd}$$

(since $\mathrm{Val}(Y_2,X_2') = \mathrm{Val}(Y_2, \mathrm{Nor}_{Y_2} X_2')$). It is now immediate that

Proposition 3.3.5. *A morphism of pairs of schemes $g\colon (Y_2,X_2) \to (Y_1,X_1)$ is adic if and only if the canonical diagram*

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "\sigma_2"] & Y_1 \arrow[d, "\sigma_1"] \\ \mathrm{Val}(Y_2,X_2) \arrow[r, "g_{\mathrm{bir}}"] & \mathrm{Val}(Y_1,X_1) \end{tikzcd}$$

is a pullback square.

Lemma 3.3.6. *Let $g\colon (Y_2,X_2) \to (Y_1,X_1)$ be an adic morphism of pairs of schemes.*

(1) *The morphism of ringed spaces*

$$(g,g^\sharp)\colon (\mathrm{Val}(Y_2,X_2), \mathcal{M}_{\mathrm{Val}(Y_2,X_2)}) \to (\mathrm{Val}(Y_1,X_1), \mathcal{M}_{\mathrm{Val}(Y_1,X_1)})$$

is a morphism of locally ringed spaces.

(2) *For every* $v \in \mathrm{Val}(Y_2, X_2)$ *the induced homomorphism of stalks*

$$(\mathcal{M}_{\mathrm{Val}(Y_1,X_1),g_{\mathrm{bir}}(v)}, \mathcal{O}_{\mathrm{Val}(Y_1,X_1),g_{\mathrm{bir}}(v)}) \to (\mathcal{M}_{\mathrm{Val}(Y_2,X_2),v}, \mathcal{O}_{\mathrm{Val}(Y_2,X_2),v})$$

is adic.

Proof. It is enough to prove for the affine case. Assume $Y_i = \operatorname{Spec} B_i$, $X_i = \operatorname{Spec} A_i$, $A_i \subset B_i$ integrally closed in B_i, $i = 1, 2$ and $g\colon (B_1, A_1) \to (B_2, A_2)$ is adic (abusing notation and using g for the homomorphism of pairs of rings and the corresponding morphism of pairs of schemes).

(1) By Lemma 2.3.8 the pullback of every valuation in $\mathrm{Val}(B_2, A_2)$ is already in $\mathrm{Val}(B_1, A_1)$. Hence g_{bir} sends $v \in \mathrm{Val}(B_2, A_2)$ to $v \circ g \in \mathrm{Val}(B_1, A_1)$. The induced homomorphism of stalks is the canonical map $(B_1)_{g^{-1}(\mathfrak{p})} \to (B_2)_{\mathfrak{p}}$ which is local.

(2) Let $v \in \mathfrak{X}_2 = \mathrm{Val}(B_2, A_2)$ with image $w \in \mathfrak{X}_1 = \mathrm{Val}(B_1, A_1)$. Let $\mathfrak{p} = \ker v$ and $\mathfrak{q} = \ker w$. We claim that the induced homomorphism $((B_1)_{\mathfrak{q}}, S_w) \to ((B_2)_{\mathfrak{p}}, S_v)$ is adic, where S_w and S_v are the semi-valuation rings of w and v. As in (1) above, $\mathfrak{q} = g^{-1}(\mathfrak{p})$. We have a commutative diagram

$$\begin{tikzcd}
B_1 \arrow[r] \arrow[d] & B_1 \otimes_{A_1} A_2 \arrow[r, "\alpha"] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_2 \otimes_{A_2} S_v \arrow[dl, "\gamma"'] \arrow[ddl] \\
(B_1)_{\mathfrak{q}} \arrow[r] & (B_1)_{\mathfrak{q}} \otimes_{S_w} S_v \arrow[r, "\delta"] \arrow[dr, "\epsilon"] & (B_2)_{\mathfrak{p}} & \\
 & & (B_1)_{\mathfrak{q}} \otimes_{S_w} S_v \otimes_{B_1 \otimes_{A_1} A_2} B_2. \arrow[u, "\zeta"] &
\end{tikzcd}$$

The claim is that δ is integral. Since g is adic α is integral. Hence ϵ is integral as a base change of an integral homomorphism. Since $v \in \mathfrak{X}_2$, by Lemma 2.4.4, γ is surjective. It follows that ζ is surjective and $\delta = \zeta \circ \epsilon$ is integral. $\square$

Theorem 3.3.7. *Let* $g\colon (Y_2, X_2) \to (Y_1, X_1)$ *be a morphism of pairs of schemes. Then* g *is adic if and only if for any morphism of pairs of schemes* $h\colon (T_1, Z_1) \to (Y_1, X_1)$ *the induced morphism of ringed spaces*

$$(\tilde{g}_{\mathrm{bir}}, \tilde{g}^{\sharp}_{\mathrm{bir}})\colon (\mathrm{Val}(T_2, Z_2), \mathcal{M}_{\mathrm{Val}(T_2,Z_2)}) \to (\mathrm{Val}(T_1, Z_1), \mathcal{M}_{\mathrm{Val}(T_1,Z_1)})$$

is a morphism of locally ringed spaces, where

$$\tilde{g}\colon (T_2, Z_2) = (Y_2, X_2) \times_{(Y_1,X_1)} (T_1, Z_1) \to (T_1, Z_1)$$

is the pullback of pairs of schemes.

Proof. Assume g is adic. As adic morphisms are stable under base change (Lemma 3.3.1(iii)) the claim follows from Lemma 3.3.6(1).

For the opposite direction, by Lemma 3.3.1(iv) it is enough to consider only the affine case. Let $g\colon (B_1, A_1) \to (B_2, A_2)$ be a homomorphism satisfying the assumption. We want to show that g is adic i.e. that $B_1 \otimes_{A_1} A_2 \to B_2$ is integral. This is the same as showing that $A'_2 \to B_2$ is integral where A'_2 is the image of $B_1 \otimes_{A_1} A_2$ in B_2. Let R be a valuation ring with fraction ring K and a homomorphism $\Phi\colon A'_2 \to R$ such that the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} B_2 \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} A'_2 \end{tikzcd}$$

commutes. As the assumption is stable under base change we may assume that this diagram is in fact a point of $\operatorname{Val}(B_2, A'_2)$ (if not take the fiber product with $\operatorname{Val}(R, R)$). Let $w = (\mathfrak{p}, R, \Phi) \in \operatorname{Val}(B_2, A'_2)$ be this point. By Proposition 3.2.15

$$\begin{tikzcd} \operatorname{Val}(B_2, A'_2) \arrow[r] \arrow[d] & \operatorname{Spec} B_1 \arrow[d, "\sigma_1"] \\ \operatorname{Val}(B_2, A_2) \arrow[r, "g_{\mathrm{bir}}"] & \operatorname{Val}(B_1, A_1) \end{tikzcd}$$

is a pullback diagram in the category of birational spaces. Hence w is a point of $\operatorname{Val}(B_2, A_2)$ and $g_{\mathrm{bir}}(w)$ is a trivial valuation of B_1. Denote $g_{\mathrm{bir}}(w) = (\mathfrak{q}, k(\mathfrak{q}), \Psi)$. By assumption the induced homomorphism

$$(B_1)_{\mathfrak{q}} = \mathcal{M}_{\operatorname{Val}(B_1,A_1),g_{bir}(w)} \to \mathcal{M}_{\operatorname{Val}(B_2,A_2),w} = (B_2)_{\mathfrak{p}}$$

is local. It follows that $\mathfrak{q} = g^{-1}(\mathfrak{p})$. Since

$$S_{g_{\mathrm{bir}}(w)} = \mathcal{O}_{\operatorname{Val}(B_1,A_1),g_{\mathrm{bir}}(w)} \to \mathcal{O}_{\operatorname{Val}(B_2,A_2),w} = S_w$$

is (always) local and $S_{g_{\mathrm{bir}}(w)} = (B_1)_{\mathfrak{q}}$ as $g_{\mathrm{bir}}(w)$ is trivial, we obtain a local homomorphism $(B_1)_{\mathfrak{q}} \to S_w \to R = S_w/\mathfrak{p}(B_2)_{\mathfrak{p}}$. Hence the maximal ideal of S_w is in fact $\mathfrak{p}(B_2)_{\mathfrak{p}}$. In other words for any $b \in (B_2)_{\mathfrak{p}}$, if $w(b) < 1$ then $w(b) = 0$. So $S_w = (B_2)_{\mathfrak{p}}$ and w is a trivial valuation of B_2. This means that $\operatorname{Val}(B_2, A'_2) \to \operatorname{Spec} B_1$ factors through $\operatorname{Spec} B_2$. Hence $\operatorname{Val}(B_2, A'_2) = \operatorname{Spec} B_2$ as birational spaces and g is adic by Proposition 3.3.5. $\square$

For a morphism of pairs of schemes $g\colon (Y_2, X_2) \to (Y_1, X_1)$ it is not enough that

$$(g_{\mathrm{bir}}, g^{\sharp}_{\mathrm{bir}})\colon (\operatorname{Val}(Y_2, X_2), \mathcal{M}_{\operatorname{Val}(Y_2,X_2)}) \to (\operatorname{Val}(Y_1, X_1), \mathcal{M}_{\operatorname{Val}(Y_1,X_1)})$$

is a morphism of locally ringed spaces for g to be adic as the next example shows.

Example 3.3.8. Let k be a field. Set $\mathfrak{X} = \mathrm{Val}(k,k)$, $\mathfrak{X}' = \mathrm{Val}(k(T), k[T]_{(T)})$ and $h\colon \mathfrak{X}' \to \mathfrak{X}$ the map induced by the obvious injection. $\mathfrak{X}$ consists of one point x with stalk $\mathcal{M}_{\mathfrak{X},x} = k$. $\mathfrak{X}'$ consists of two points x'_0 corresponding to the trivial valuation on $k(T)$ and x'_1 corresponding to the valuation ring $k[T]_{(T)}$, both mapping to x. The stalks are $\mathcal{M}_{\mathfrak{X}',x'_0} = \mathcal{M}_{\mathfrak{X}',x'_1} = k(T)$. The map h is clearly not adic but both induced maps of stalks $\mathcal{M}_{\mathfrak{X},x} \to \mathcal{M}_{\mathfrak{X}',x'_i}$ are an injection of fields which is local.

4. Relative blow ups

In order to prove an analogue of Raynaud's theory we need an analogue of admissible formal blow ups. For this purpose we introduce and study relative blow ups.

4.1. Modifications

Let $(Y \xrightarrow{f} X)$ be a pair of schemes. A Y-modification of X is a factorization of f into a schematically dominant morphism $h\colon Y \to Z$ and a proper morphism $g\colon Z \to X$ (see [8, §3.3]). Since f is affine (by the definition of a pair of schemes) and g is proper, h is also affine. Hence $(Y \xrightarrow{h} Z)$ is also a pair of schemes. In terms of pairs of schemes, a Y-modification of X is a morphism of pairs $(g_Y, g_X)\colon (Y', X') \to (Y, X)$ such that g_Y is an isomorphism and g_X is proper.

Proposition 4.1.1. *Let (Y, X) be a pair of schemes. Any Y-modification of X dominates (Y, X) as scheme models of* $\mathrm{Val}(Y, X)$.

Proof. Let $(g_Y, g_X)\colon (Y', X') \to (Y, X)$ be a Y-modification of X. By the valuative criterion of properness $g_{\mathrm{bir}}\colon \mathrm{Val}(Y', X') \to \mathrm{Val}(Y, X)$ is topologically the identity. It is also clear that $g^\sharp_{\mathrm{bir}}\colon \mathcal{M}_{\mathrm{Val}(Y,X)} \to g_{\mathrm{bir}\,*}\mathcal{M}_{\mathrm{Val}(Y',X')}$ is the identity. For every point $v = (y, R_y, \Phi) \in \mathrm{Val}(Y, X)$ the stalk of $\mathcal{O}_{\mathrm{Val}(Y,X)}$ is the pullback of the maps

$$\mathcal{M}_{\mathrm{Val}(Y,X),v} \longrightarrow k(y) \longleftarrow R_y,$$

and the same is true for $v' \in \mathrm{Val}(Y', X')$, so $\mathcal{O}_{\mathrm{Val}(Y,X)} \to g_{\mathrm{bir}\,*}\mathcal{O}_{\mathrm{Val}(Y',X')}$ is an isomorphism and must be the identity as a restriction of $\mathcal{M}_{\mathrm{Val}(Y,X)} \to \mathcal{M}_{\mathrm{Val}(Y',X')}$. $\square$

4.2. Construction of a relative blow up

Let $(Y \xrightarrow{f} X)$ be a pair of schemes and $\mathcal{E}$ a finite quasi-coherent $\mathcal{O}_X$-module (i.e. locally generated by finitely many sections) contained in $f_*\mathcal{O}_Y$. We construct a morphism of pairs $\pi_{\mathcal{E}} = (\pi_Y, \pi_X)\colon (Y_{\mathcal{E}}, X_{\mathcal{E}}) \to (Y, X)$ with π_X projective. If we further assume that $\mathcal{E}$ contains the image of $\mathcal{O}_X$ (in $f_*\mathcal{O}_Y$) then we obtain a modification $\pi_{\mathcal{E}} = (id_Y, \pi_{\mathcal{E}})\colon (Y, X_{\mathcal{E}}) \to (Y, X)$.

Construction 4.2.1. Using the multiplication of the $\mathcal{O}_X$-algebra $f_*\mathcal{O}_Y$, for every $d \geq 0$ we define the d-th power $\mathcal{O}_X$-module $\mathcal{E}^d$ to be the image of $\mathcal{E}^{\otimes d} \to f_*\mathcal{O}_Y^{\otimes d} \to f_*\mathcal{O}_Y$, where the tensor product is over $\mathcal{O}_X$. The graded $\mathcal{O}_X$-module $\oplus_{d\geq 0}\mathcal{E}^d$ is quasi-coherent and has a structure of a graded $\mathcal{O}_X$-algebra. Denote by $\mathcal{I}$ the image of $\mathcal{E} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y$ (the ideal in $f_*\mathcal{O}_Y$ generated by $\mathcal{E}$). It is a finitely generated sheaf of ideals in $f_*\mathcal{O}_Y$. Note that $\mathcal{I}^d$, the d-th power of the ideal $\mathcal{I}$, is the image of the multiplication map $\mathcal{E}^d \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y$. By the construction are natural homomorphisms of graded $\mathcal{O}_X$-algebras

$$\oplus_{d\geq 0}\mathcal{E}^d \to \oplus_{d\geq 0}(\mathcal{E}^d \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y) \to \oplus_{d\geq 0}\mathcal{I}^d.$$

Note that the second homomorphism is surjective. Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{Proj}_X\left(\oplus_{d\geq 0}\mathcal{I}^d\right) \arrow[r, hook] & \mathbf{Proj}_X\left(\oplus_{d\geq 0}\mathcal{E}^d \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y\right) \arrow[r] \arrow[d] & \mathbf{Proj}_X\left(\oplus_{d\geq 0}\mathcal{E}^d\right) \arrow[d] \\
& Y \arrow[r, "f"] & X.
\end{tikzcd}$$

where the top left arrow is a closed immersion [2, 3.6.2] and the square is a pullback[2, 3.5.3]. Set $X_{\mathcal{E}} = \mathbf{Proj}_X\left(\oplus_{d\geq 0}\mathcal{E}^d\right)$, $Y_{\mathcal{E}} = \mathbf{Proj}_X\left(\oplus_{d\geq 0}\mathcal{I}^d\right)$. Denote the natural projective morphisms $X_{\mathcal{E}} \to X$, $Y_{\mathcal{E}} \to Y$ by π_X, π_Y respectively and by $f_{\mathcal{E}}$ the affine morphism which is the composition of the two top arrows in the diagram.

Let $U \subset X$ be an open affine subscheme such that $\mathcal{E}|_U$ is generated by finitely many global sections. Assume $U = \operatorname{Spec} A$ and set $E = \Gamma(U, \mathcal{E})$, $B = \Gamma(U, f_*\mathcal{O}_Y)$ and $I = \Gamma(U, \mathcal{I})$. It follows that $V = f^{-1}(U) = \operatorname{Spec} B$ and $U_E = \operatorname{Proj}\left(\oplus_{d\geq 0}E^d\right)$, $V_E = \operatorname{Proj}\left(\oplus_{d\geq 0}I^d\right)$ are open subschemes of $X_{\mathcal{E}}$ and $Y_{\mathcal{E}}$ respectively over U. Let $s_1, \ldots, s_r$ be generators of the A-module E. The corresponding sections of degree 1 in $\oplus_{d\geq 0}E^d$ and their images in $\oplus_{d\geq 0}I^d$ are also generators of the A-algebras $\oplus_{d\geq 0}E^d$ and $\oplus_{d\geq 0}I^d$ respectively. For each i there are affine and schematically dominant open immersions $(U_E)_{s_i} \to U_E$ and $(V_E)_{s_i} \to V_E$ [2, 3.1.4], fitting into commutative diagrams

$$\begin{tikzcd}
(V_E)_{s_i} \arrow[r] \arrow[d] & V_E \arrow[d, "f_{\mathcal{E}}"] \\
(U_E)_{s_i} \arrow[r] & U_E,
\end{tikzcd}$$

and these immersions are compatible on the intersections. Furthermore, $(U_E)_{s_i} = \operatorname{Spec}\left(\oplus_{d\geq 0}E^d\right)_{(s_i)}$, where $\left(\oplus_{d\geq 0}E^d\right)_{(s_i)}$ is the degree 0 part of the localization of $\oplus_{d\geq 0}E^d$ with respect to s_i. Similarly $(V_E)_{s_i} = \operatorname{Spec}\left(\oplus_{d\geq 0}I^d\right)_{(s_i)}$, and the restriction of $f_{\mathcal{E}}$ to $(V_E)_{s_i}$ corresponds to the homomorphism $\left(\oplus_{d\geq 0}E^d\right)_{(s_i)} \to \left(\oplus_{d\geq 0}I^d\right)_{(s_i)}$. Since $E^d \to I^d$

is injective for all $d \geq 0$, the graded homomorphism $\oplus_{d\geq 0} E^d \to \oplus_{d\geq 0} I^d$ is injective hence $\left(\oplus_{d\geq 0} E^d\right)_{(s_i)} \to \left(\oplus_{d\geq 0} I^d\right)_{(s_i)}$ is injective as well. Thus $(V_E)_{s_i} \to (U_E)_{s_i}$ is schematically dominant. As $U_E = \cup_i (U_E)_{s_i}$, the morphism $f_{\mathcal{E}}|_{V_E} : V_E \to U_E$ is schematically dominant as well. Hence we conclude that the map $f_{\mathcal{E}} : Y_{\mathcal{E}} \to X_{\mathcal{E}}$ is affine and dominant, i.e. $(Y_{\mathcal{E}} \overset{f_{\mathcal{E}}}{\to} X_{\mathcal{E}})$ is a pair of schemes and $\pi_{\mathcal{E}} = (\pi_Y, \pi_X)$ is a morphism of pairs.

If we also assume that $\mathcal{E}$ contains the image of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$ then $\mathcal{I} = f_*\mathcal{O}_Y$ thus $Y_{\mathcal{E}} = Y$. In other words $\pi_{\mathcal{E}} : (Y, X_{\mathcal{E}}) \to (Y, X)$ is a Y-modification of X.

Definition 4.2.2. We call the morphism of pairs $\pi_{\mathcal{E}} : (Y_{\mathcal{E}}, X_{\mathcal{E}}) \to (Y, X)$ constructed above the *relative blow up* of (Y, X) with respect to the module $\mathcal{E}$. If $\mathcal{E}$ contains the image of $\mathcal{O}_X$ we call the modification $\pi_{\mathcal{E}} : (Y, X_{\mathcal{E}}) \to (Y, X)$ a *simple relative blow up.*

Remark 4.2.3. Note that $Y_{\mathcal{E}}$ is just a usual blow up of Y. Explicitly, the quasi-coherent ideal $\mathcal{I} \subset f_*\mathcal{O}_Y$ corresponds to a quasi-coherent ideal $\widetilde{\mathcal{I}} \subset \mathcal{O}_Y$ ([2, 1.4.10]) which in turn defines a closed subscheme $Z \subset Y$ and $Y_{\mathcal{E}} = Bl_Z(Y)$.

As a special case of Proposition 4.1.1 we have

Corollary 4.2.4. *Let (Y, X) be a pair of schemes. Any simple relative blow up of (Y, X) dominates (Y, X) as scheme models of* $\mathrm{Val}(Y, X)$.

4.3. Properties of relative blow ups

Let $h : (Y', X') \to (Y, X)$ be a morphism of pairs of schemes, i.e. a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "h_Y"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "h_X"] & X \end{tikzcd}$$

and let $\mathcal{E}$ be a finite quasi-coherent $\mathcal{O}_X$-module contained in $f_*\mathcal{O}_Y$. Then $h_X^*(\mathcal{E})$ is a finite quasi-coherent $h_X^*(\mathcal{O}_X) = \mathcal{O}_{X'}$-module. The inclusion $\mathcal{E} \subset f_*\mathcal{O}_Y$ induces a homomorphism $h_X^*(\mathcal{E}) \to h_X^*(f_*\mathcal{O}_Y)$ of $\mathcal{O}_{X'}$-modules. The morphism of schemes $h_Y : Y' \to Y$ is equipped with a homomorphism of sheaves $\mathcal{O}_Y \to h_{Y*}\mathcal{O}_{Y'}$ on Y. Pushing forward to X and then pulling back to X' the homomorphism of $\mathcal{O}_Y$-modules $\mathcal{O}_Y \to h_{Y*}\mathcal{O}_{Y'}$ we obtain a homomorphism of $\mathcal{O}_{X'}$-modules $h_X^* f_*\mathcal{O}_Y \to h_X^*(f \circ h_Y)_*\mathcal{O}_{Y'}$. Also, from the adjunction $h_X^* \circ h_{X*} \to Id$, we have a natural homomorphism

$$h_X^*(f \circ h_Y)_*\mathcal{O}_{Y'} = h_X^*(h_X \circ f')_*\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{Y'}.$$

Composing, we obtain a homomorphism of $\mathcal{O}_{X'}$-modules $h_X^*(\mathcal{E}) \to f'_*\mathcal{O}_{Y'}$.

Definition 4.3.1. We call the sheaf theoretic image of the above morphism the *inverse image module*[2] of $\mathcal{E}$ (with respect to the morphism of pairs h) and denote it $h^{-1}(\mathcal{E})$.

The inverse image module $h^{-1}(\mathcal{E})$ is a finite quasi-coherent $\mathcal{O}_{X'}$-module contained in $f'_*\mathcal{O}_{Y'}$. If furthermore $\mathcal{E}$ contains $f^\sharp(\mathcal{O}_X)$ then the homomorphism $f'^\sharp\colon \mathcal{O}_{X'} \to f'_*\mathcal{O}_{Y'}$ factors through $h^{-1}(\mathcal{E})$. Since $f'^\sharp$ is injective $h^{-1}(\mathcal{E})$ contains $f'^\sharp(\mathcal{O}_{X'})$.

Proposition 4.3.2. *Let $\pi_\mathcal{E}\colon (Y_\mathcal{E}, X_\mathcal{E}) \to (Y, X)$ be a relative blow up of (Y, X) with respect to the module $\mathcal{E}$. Then the inverse image module $\pi_\mathcal{E}^{-1}(\mathcal{E})$ is an invertible sheaf on $X_\mathcal{E}$.*

Proof. We have $\pi_\mathcal{E}\colon (Y_\mathcal{E}, X_\mathcal{E}) \to (Y, X)$

$$\begin{array}{ccc} Y_\mathcal{E} & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle f_\mathcal{E}}\downarrow & & \downarrow{\scriptstyle f} \\ X_\mathcal{E} & \xrightarrow{\pi_X} & X. \end{array}$$

The inverse image module $\pi_\mathcal{E}^{-1}(\mathcal{E})$ is the image of $\pi_X^*(\mathcal{E})$ in $f_{\mathcal{E}*}\mathcal{O}_Y$. The question is local on X so we assume that $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $A \subset B$ and $E = \mathcal{E}(X) = \sum_{i=0}^n Ab_i$ with $b_0, \dots, b_n \in B$. Denote $A_i = \varphi_i(A)\left[\left\{\frac{b_j}{b_i}\right\}_j\right] \subset B_{b_i}$ (φ_i the canonical homomorphism $B \to B_{b_i}$). Then $X_\mathcal{E} = \operatorname{Proj}\left(\oplus_{d\ge 0}E^d\right)$. As we saw in Theorem 2.6.2 the affine charts are $(X_\mathcal{E})_{b_i} = \operatorname{Spec} A_i$. Denote by E_i the image of $E \otimes_A A_i$ in B_{b_i} under the map induced by multiplication. Since $\frac{b_j}{b_i} \in A_i$ we see that $b_j = \frac{b_j}{b_i} \cdot b_i \in E_i$. So E_i is generated by the single element b_i over A_i. In other words, multiplication by b_i gives an isomorphism of modules $\mathcal{O}_{X_\mathcal{E}}|_{(X_\mathcal{E})_{b_i}} \xrightarrow{\sim} \pi_\mathcal{E}^{-1}(\mathcal{E})|_{(X_\mathcal{E})_{b_i}}$. □

Proposition 4.3.3 *(Universal property). Let (Y, X) be a pair of schemes with $\mathcal{E}$ as above. Let $Y' \xrightarrow{f'} X'$ be another pair and $h = (h_Y, h_X)\colon (Y', X') \to (Y, X)$ a morphism of pairs. If $h^{-1}(\mathcal{E})$ is invertible on X' then h factors uniquely through $\pi_\mathcal{E}$.*

Proof. By construction $h_X^*(\mathcal{E}) \to h^{-1}(\mathcal{E})$ is a surjection to an invertible $\mathcal{O}_{X'}$-module. This extends to a map $h_X^*\left(\oplus_{d\ge 0}\mathcal{E}^d\right) \to \oplus_{d\ge 0}\left(h^{-1}(\mathcal{E})\right)^d$ of graded $\mathcal{O}_{X'}$-algebras. Note that $\left(h^{-1}(\mathcal{E})\right)^d = \left(h^{-1}(\mathcal{E})\right)^{\otimes d}$ since $h^{-1}(\mathcal{E})$ is invertible. Let $\mathcal{I}'$ be the ideal of $f'_*\mathcal{O}_{Y'}$ generated by $h^{-1}(\mathcal{E})$, that is the image of $h^{-1}(\mathcal{E}) \otimes_{\mathcal{O}_{X'}} f'_*\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{Y'}$. Then $\mathcal{I}'$ is the ideal $h^{-1}(\mathcal{I}) = h_X^*(\mathcal{I})\, f'_*\mathcal{O}_{Y'}$, where $\mathcal{I}$ is the ideal of $f_*\mathcal{O}_Y$ generated by $\mathcal{E}$. As $h^{-1}(\mathcal{E})$ is invertible, $\mathcal{I}'$ is locally generated by single element. We obtain a commutative diagram of quasi-coherent graded $\mathcal{O}_{X'}$-algebras

[2] By analogy with the inverse image ideal of the usual blow up.

$$\begin{tikzcd}
h_X^* \left(\oplus_{d\geq 0}\mathcal{I}^d\right) \arrow[r] & \oplus_{d\geq 0}\mathcal{I}'^d \\
h_X^* \left(\oplus_{d\geq 0}\mathcal{E}^d\right) \arrow[r] \arrow[u] & \oplus_{d\geq 0}\left(h^{-1}\left(\mathcal{E}\right)\right)^d \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are surjections and the top row consists of $f'_*\mathcal{O}_{Y'}$-algebras. The follows from [2, 3.2.4 and 3.7.1]. □

Lemma 4.3.4. *Let (Y, X) be a pair of schemes. The families of relative blow ups and simple relative blow ups of (Y, X) are filtered.*

Proof. Let $\mathcal{E}'$ and $\mathcal{E}''$ be two finite quasi-coherent $\mathcal{O}_X$-modules contained in $f_*\mathcal{O}_Y$. Then $\mathcal{E} = \mathcal{E}' \cdot \mathcal{E}''$ is also a finite quasi-coherent $\mathcal{O}_X$-module contained in $f_*\mathcal{O}_Y$. Note that $\mathcal{E}$ is the image of

$$\mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{E}'' \to f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y.$$

If, furthermore, both $\mathcal{E}'$ and $\mathcal{E}''$ contain $f^\sharp(\mathcal{O}_X)$ then we have

$$\mathcal{O}_X \simeq \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \to \mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{E}'' \to \mathcal{E}' \cdot \mathcal{E}'' = \mathcal{E} \subset f_*\mathcal{O}_Y.$$

Since $f^\sharp$ is injective $\mathcal{E}$ contains $f^\sharp(\mathcal{O}_X)$. Consider the relative blow up $(Y_\mathcal{E}, X_\mathcal{E})$. By Proposition 4.3.2, the inverse image module $\pi_\mathcal{E}^{-1}(\mathcal{E}) = \pi_\mathcal{E}^{-1}(\mathcal{E}') \cdot \pi_\mathcal{E}^{-1}(\mathcal{E}'')$ is an invertible sheaf on $X_\mathcal{E}$. Hence $\pi_\mathcal{E}^{-1}(\mathcal{E}')$ and $\pi_\mathcal{E}^{-1}(\mathcal{E}'')$ are also invertible on $X_\mathcal{E}$. By the universal property of the relative blow up (Proposition 4.3.3) $(Y_\mathcal{E}, X_\mathcal{E}) \to (Y, X)$ factors through both $(Y_{\mathcal{E}'}, X_{\mathcal{E}'}) \to (Y, X)$ and $(Y_{\mathcal{E}''}, X_{\mathcal{E}''}) \to (Y, X)$. □

Lemma 4.3.5. *Let (Y, X) be a pair of schemes. Assume that X' is an open subscheme of X. Denote $Y' = f^{-1}(X')$. Then a (simple) relative blow up of (Y', X') extends to a (simple) relative blow up of (Y, X).*

Proof. In [9, Corollary 3.4.4] Temkin shows that a simple relative blow up (Y-blow up of X in his terminology) of (Y', X') extends to a simple relative blow up of (Y, X). The argument also applies to a general relative blow up. □

Combining the Lemmas 4.3.4 and 4.3.5 we obtain

Corollary 4.3.6. *Let (Y, X) be a pair of schemes with open sub-pairs of schemes $(Y_1, X_1), \ldots, (Y_n, X_n)$. For each $i = 1, \ldots, n$ let $(Y_i, X_{i\mathcal{E}_i}) \to (Y_i, X_i)$ be simple relative blow up. Then*

(1) *each simple relative blow up $(Y_i, X_{i\mathcal{E}_i}) \to (Y_i, X_i)$ extends to a simple relative blow up $(Y, X_{\mathcal{E}'_i}) \to (Y, X)$.*

(2) *there is a simple relative blow up* $(Y, X_{\mathcal{E}}) \to (Y, X)$ *which factors through each* $(Y, X_{\mathcal{E}'_i}) \to (Y, X)$.

Lemma 4.3.7. *Given a quasi-compact open subset* $\mathfrak{U} \subset \mathfrak{X} = \mathrm{Val}(Y, X)$, *there exists a simple relative blow up* $(Y, X_{\mathcal{E}}) \to (Y, X)$ *and an open subscheme* $U \subset X_{\mathcal{E}}$ *such that* $\mathfrak{U} = \tau^{-1}(U) = \mathrm{Val}(f_{\mathcal{E}}^{-1}(U), U)$.

Proof. Since $\mathfrak{U}$ is quasi-compact and open, by Corollary 4.3.6 it suffices to show for any $v \in \mathfrak{U}$ there exists a simple relative blow up $(Y, X_{\mathcal{E}}) \to (Y, X)$ and an open subscheme $U \subset X_{\mathcal{E}}$ such that $v \in \tau^{-1}(U) = \mathrm{Val}(f_{\mathcal{E}}^{-1}(U), U) \subset \mathfrak{U}$. Fix $v \in \mathfrak{U}$. Let $\mathfrak{X}'$ be an open affinoid subspace of $\mathfrak{X}$ containing v. Then $v \in \mathfrak{U} \cap \mathfrak{X}'$, which by Lemma 3.2.6 is also quasi-compact and open. Hence we can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Furthermore, since the rational domains form a basis of the topology of $\mathrm{Val}(B, A)$ (Proposition 2.4.1) and each rational domain is quasi-compact(Corollary 2.4.9), we can assume that $\mathfrak{U}$ is a rational domain. Let $\mathfrak{U} = \mathfrak{X}(\{a_1, \dots, a_n\}/b)$. Then v satisfies the inequalities $v(a_i) \leq v(b)$ for all i and $v(b) \neq 0$. Since v is unbounded $v(b)^{-1}$ is not a bound of v, so there exists $d \in B$ such that $v(b)^{-1} \leq v(d)$. It follows that $v(a_i d) \leq v(bd)$ for all i and $1 \leq v(bd)$. Let $\mathcal{E}$ be the $\mathcal{O}_X$-module, contained in $f_*\mathcal{O}_Y$, generated by the global sections $1, a_1 d, \dots, a_n d, bd$. The open chart $U = (X_{\mathcal{E}})_{bd} \subset X_{\mathcal{E}}$ satisfies $v \in \tau^{-1}(U) \subset \mathfrak{U}$. $\square$

Taken together with Corollary 4.3.6, we immediately obtain

Corollary 4.3.8. *Let* (Y, X) *be a pair of schemes and let* Ω *be a finite family of quasi-compact open subspaces of the associated birational space* $(Y, X)_{\mathrm{bir}}$. *Then there is a simple relative blow up* $(Y, X_{\mathcal{E}}) \to (Y, X)$ *together with a family* $\widetilde{\Omega}$ *of open subschemes of* $X_{\mathcal{E}}$ *such that the associated family* $\widetilde{\Omega}_{\mathrm{bir}}$ *coincides with* Ω. *Furthermore if* Ω *covers* $(Y, X)_{\mathrm{bir}}$, *the family* $\widetilde{\Omega}$ *covers* $X_{\mathcal{E}}$.

5. Birational spaces in terms of pairs of schemes

5.1. Statement of theorem

By Corollaries 4.2.4 and 3.2.7 the functor bir takes simple relative blow ups and relative normalizations to isomorphisms of birational spaces. Hence we may regard bir as a functor from the localization of the category of pairs of schemes, with respect to the class of simple relative blow ups and relative normalizations, to the category of birational spaces. By Lemma 3.2.6 any birational space in the essential image of bir is qcqs. This section is dedicated to proving that

Theorem 5.1.1. *The* bir *functor provides an equivalence of categories between the localization of the category of pairs of schemes, with respect to the class of simple relative blow ups and relative normalizations, and the category of qcqs birational spaces.*

The proof splits to checking that bir is faithful, full and essentially surjective. These checks are Proposition 5.2.1, Theorem 5.3.1 and Proposition 5.4.1 below.

In light of Proposition 4.1.1 it would appear that we need to invert all modifications and not just simple relative blow ups. However by [9, Corollary 3.4.8] any modification of (Y, X) is dominated by a simple relative blow up of (Y, X). It follows that the category of pairs of schemes localized with respect to the class of simple relative blow ups (and relative normalizations) is equivalent to the category of pairs of schemes localized with respect to the class of modifications (and relative normalizations).

5.2. Faithfulness

We start with proving that *bir* is faithful.

Proposition 5.2.1. *Let $(Y \xrightarrow{f} X)$ and $(Y' \xrightarrow{f'} X')$ be pairs of schemes such that $X = \mathrm{Nor}_Y X$ and $X' = \mathrm{Nor}_{Y'} X'$. Denote $\mathfrak{X} = (Y, X)_{\mathrm{bir}}$ and $\mathfrak{X}' = (Y', X')_{\mathrm{bir}}$. Let $g_1, g_2 \colon (Y, X) \to (Y', X')$ be two morphisms of pairs of schemes. If $g_{1\,\mathrm{bir}} = g_{2\,\mathrm{bir}}$ as morphisms of the birational spaces $\mathfrak{X} \to \mathfrak{X}'$ then $g_1 = g_2$.*

Proof. Denote $g_{1,bir} = g_{2,bir} = h$. We have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "g_{1,Y}"] \arrow[r, shift right, "g_{2,Y}"'] \arrow[d, "\sigma"] & Y' \arrow[d, "\sigma"] \\ \mathfrak{X} \arrow[r, "h"] \arrow[d, "\tau"] & \mathfrak{X}' \arrow[d, "\tau"] \\ X \arrow[r, shift left, "g_{1,X}"] \arrow[r, shift right, "g_{2,X}"'] & X'. \end{tikzcd}$$

It follows from Proposition 2.4.12 and Lemma 3.2.12 that g_1 and g_2 agree on the underlying topological spaces (both Y and X). Denote the topological part of g_1 and g_2 by $g = (g_Y, g_X)$. Furthermore, as in the faithfulness part of Theorem 3.2.13, $g_{1,Y}$ and $g_{2,Y}$ agree as morphisms of schemes.

Let $\{(Y_i', X_i')\}_{i \in I}$ be an open affine covering of (Y', X'). Denote the topological pull back of each (Y_i', X_i') through g by (Y_i, X_i). For every $i \in I$ there is an open affine covering $\{(Y_{ij}, X_{ij})\}_{j \in J_i}$ of (Y_i, X_i). It is enough to show that the restrictions $g_{1,X}|_{X_{ij}}$ and $g_{2,X}|_{X_{ij}}$ agree as morphisms of schemes $X_{ij} \to X_i'$ for each $j \in J_i$ and $i \in I$. Hence we may assume that (Y', X') and (Y, X) are affine pairs. This was already proved in the faithfulness part of Theorem 3.2.13. $\square$

5.3. Fullness

Next we prove that *bir* is full.

Theorem 5.3.1. *Let (Y, X) and (Y', X') be pairs of schemes and let $h\colon (Y', X')_{\mathrm{bir}} \to (Y, X)_{\mathrm{bir}}$ be a morphism of birational spaces. Then there exist a simple relative blow up $(Y', X'_{\mathcal{E}}) \to (Y', X')$ and a morphism of pairs $k\colon (Y', \mathrm{Nor}_{Y'} X'_{\mathcal{E}}) \to (Y, X)$ such that $k_{\mathrm{bir}} = h \circ g_{\mathrm{bir}}$, where g is the morphism $(Y', \mathrm{Nor}_{Y'} X'_{\mathcal{E}}) \to (Y', X'_{\mathcal{E}}) \to (Y', X')$. In particular h is isomorphic to k_{bir}.*

Proof. Consider the affine case. Let (B, A) and (B', A') be two pairs of rings. By Theorem 3.2.13 the morphism between the associated birational spaces $h\colon (B', A')_{\mathrm{bir}} \to (B, A)_{\mathrm{bir}}$ is given by a morphism of the pairs of rings $(B, A) \to (B', \mathrm{Nor}_{B'} A') = (B', A')$. We see that the required simple relative blow up is just the identity.

For the general case, denote $\mathfrak{X} = (Y, X)_{\mathrm{bir}}$ and $\mathfrak{X}' = (Y', X')_{\mathrm{bir}}$. An affine covering $\{(Y_i, X_i)\}$ of (Y, X) gives an affinoid covering $\{\mathfrak{X}_i\}$ of $\mathfrak{X}$. Each preimage $h^{-1}(\mathfrak{X}_i)$ can also be covered by finitely many open affinoid birational subspaces. Hence by Corollary 4.3.8, refining the coverings in a suitable way and replacing (Y', X') with a suitable simple relative blow up, we may assume that we have coverings $\{\mathfrak{X}_i\}$ of $\mathfrak{X}$ and $\{\mathfrak{X}'_i\}$ of $\mathfrak{X}'$ consisting of finitely many open affinoid birational subspaces such that $h(\mathfrak{X}'_i) \subset \mathfrak{X}_i$ for all i and they are represented by affine open coverings $\{(Y_i, X_i)\}$ of (Y, X) and $\{(Y'_i, X'_i)\}$ of (Y', X').

By the affine case we obtain for every i a simple relative blow up $g_i\colon (Y'_i, X'_{i\mathcal{E}_i}) \to (Y'_i, X'_i)$ and morphism $k_i\colon (Y'_i, X'_{i\mathcal{E}_i}) \to (Y_i, X_i)$ satisfying $k_{i\,\mathrm{bir}} = h|_{\mathfrak{X}'_i} \circ g_{i\,\mathrm{bir}}$. By Corollary 4.3.6, there is a simple relative blow up $g\colon (Y', X'_{\mathcal{E}}) \to (Y', X')$ such that the pairs $(Y'_i, X'_{i\mathcal{E}_i})$ are open sub-pairs of $(Y', X'_{\mathcal{E}})$ and form a covering. It follows from Proposition 5.2.1 that we can glue the k_i and get a morphism $k\colon (Y', X'_{\mathcal{E}}) \to (Y, X)$ such that $k_{\mathrm{bir}} = h \circ g_{\mathrm{bir}}$. □

Corollary 5.3.2. *Let the pairs of schemes (Y, X) and (Y', X') both be scheme models for the same birational space $\mathfrak{X}$. Then there is another scheme model (Y'', X'') of $\mathfrak{X}$ that dominates both via simple relative blow ups and perhaps a relative normalization.*

Proof. We replace (Y, X) and (Y', X') with $(Y, \mathrm{Nor}_Y X)$ and $(Y', \mathrm{Nor}_{Y'} X')$ respectively. We have an isomorphism $h\colon (Y', X')_{\mathrm{bir}} \xrightarrow{\sim} (Y, X)_{\mathrm{bir}}$. As Y is embedded in the subset of $\mathfrak{X} = (Y, X)_{\mathrm{bir}}$ of points v such that $\mathcal{M}_{\mathfrak{X},v} = \mathcal{O}_{\mathfrak{X},v}$, its scheme structure is determined by $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The same is true for Y', so h induces an isomorphism $Y \simeq Y'$. We assume that $Y' = Y$.

Applying Theorem 5.3.1 we obtain a simple relative blow up $g\colon (Y, X'_{\mathcal{E}}) \to (Y', X')$ and a morphism of pairs $k\colon (Y', X'_{\mathcal{E}}) \to (Y, X)$ such that $k_{\mathrm{bir}} = h \circ g_{\mathrm{bir}}$, in particular $k_{\mathrm{bir}}\colon (Y', X'_{\mathcal{E}})_{\mathrm{bir}} \to (Y, X)_{\mathrm{bir}}$ is also an isomorphism. Using Theorem 5.3.1 again for k_{bir}^{-1} we obtain a simple relative blow up $j\colon (Y, X_{\mathcal{F}}) \to (Y, X)$ and a morphism of pairs $q\colon (Y, X_{\mathcal{F}}) \to (Y', X'_{\mathcal{E}})$ such that $q_{\mathrm{bir}} = k_{\mathrm{bir}}^{-1} \circ j_{\mathrm{bir}}$.

$$\begin{tikzcd}
(Y, X'_{\mathcal{E}}) \arrow[d, "g"] \arrow[dr, "k"] & (Y, X_{\mathcal{F}}) \arrow[l, "q"'] \arrow[d, "j"] \\
(Y', X') & (Y, X).
\end{tikzcd}$$

As $j = k \circ q \colon (Y, X_{\mathcal{F}}) \to (Y, X)$ is a simple relative blow up, $q \colon (Y, X_{\mathcal{F}}) \to (Y', X'_{\mathcal{E}})$ is a simple relative blow up too. Thus the composition of simple relative blow ups $g \circ q \colon (Y, X_{\mathcal{F}}) \to (Y', X')$ is also a simple relative blow up. So $(Y, X_{\mathcal{F}})$ is the required scheme model. □

5.4. *Essential surjectivity*

Finally, let us prove that bir is essentially surjective.

Proposition 5.4.1. *Every qcqs birational space $\mathfrak{X}$ has a scheme model.*

Proof. Consider a qcqs birational space $\mathfrak{X}$. We want to show that there is a pair of schemes (Y, X) satisfying $(Y, X)_{\mathrm{bir}} \simeq \mathfrak{X}$. We proceed by induction on the number of open birational spaces which cover $\mathfrak{X}$ and have scheme models. As $\mathfrak{X}$ is quasi-compact, it is enough to consider only the case of a covering consisting of two affinoid subspaces.

Assume that $\mathfrak{X}$ is covered by two quasi-compact open subspaces $\mathfrak{U}_1$ and $\mathfrak{U}_2$, which admit scheme models (V_1, U_1) and (V_2, U_2). Set $\mathfrak{W} = \mathfrak{U}_1 \cap \mathfrak{U}_2$. Since $\mathfrak{X}$ is quasi-separated, an application of Corollary 4.3.8 shows that, after blowing-up, we may assume that the open immersions $\mathfrak{W} \subset \mathfrak{U}_1$ and $\mathfrak{W} \subset \mathfrak{U}_2$ are represented by open immersions of sub-pairs $(T', W') \subset (V_1, U_1)$ and $(T'', W'') \subset (V_2, U_2)$. Now, using Corollary 5.3.2, we can dominate the scheme models (T', W') and (T'', W'') by a third scheme model (T, W) of $\mathfrak{W}$. Using Corollary 4.3.6 we extend the corresponding blow ups to (V_1, U_1) and (V_2, U_2), so we may view (T'', W'') as an open sub-pair of (V_1, U_1) and (V_2, U_2). Gluing both along W yields the required scheme model (Y, X) of $\mathfrak{X}$. □

References

[1] S. Bosch, W. Lütkebohmert, Formal and rigid geometry. I. Rigid spaces, Math. Ann. 295 (2) (1993) 291–317.

[2] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes, Publ. Math. Inst. Hautes Études Sci. (8) (1961) 222.

[3] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents. I, Publ. Math. Inst. Hautes Études Sci. (11) (1961) 167.

[4] R. Huber, M. Knebusch, On valuation spectra, Contemp. Math. 155 (1994) 167–206.

[5] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969) 43–60.

[6] R. Huber, Continuous valuations, Math. Z. 212 (1993) 455–477.

[7] M. Raynaud, Géométrie analytique rigide d'après Tate, Kiehl... Table ronde d'analyse non archimedienne, Bull. Soc. Math. France Mém. 39–40 (1974) 319–327.

[8] M. Temkin, Stable modification of relative curves, J. Algebraic Geom. 19 (2010) 603–677.

[9] M. Temkin, Relative Riemann–Zariski spaces, Israel J. Math. 185 (2011) 1–42.
[10] O. Zariski, The compactness of the Riemann manifold of an abstract field of algebraic functions, Bull. Amer. Math. Soc. 50 (2) (1944) 683–691.
[11] O. Zariski, P. Samuel, Commutative Algebra, vol. 2, van Nostrand, Princeton, 1960.