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A counterexample on outer inverses in semigroups

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ABSTRACT

We construct a counterexample to the following conjecture, which was proposed recently by Bapat et al. 'If a is a regular element in a semigroup S, and x is an outer inverse of a, then a has a reflexive generalized inverse y which dominates x with respect to the minus order on S.'

The aim of this note is to provide a counterexample to the statement in the abstract, which is the content of Conjecture 1 in the recent paper [1]. In the basic terms, this statement reads as follows: For any elements a, b, x satisfying $xax = x$, $aba = a$, there are elements α, y such that

$$aya = a, \quad x\alpha = y\alpha, \tag{1}$$

and some additional equalities hold.

Let S be the monoid presented by generators a, b, x and relations $xax = x$, $aba = a$. Let us explain what is S in more elementary terms. For arbitrary strings u, v of letters

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a, b, x, we say that the strings uav, $uabav$ (as well as uxv, $uxaxv$, respectively) can be obtained from each other by an *elementary substitution*. We think of elements in S as such strings, we multiply them by concatenation, and we identify two strings if and only if one of them can be obtained from the other by a sequence of elementary substitutions.

We have $xax = x$, $aba = a$ immediately, and we assume that $\alpha, y \in S$ are such that the equalities (1) are true. Since the elementary substitutions respect the property of strings to contain x, the first equality of (1) shows that y does not contain x. Similarly, the elementary substitutions respect the first letters, so the second equality of (1) shows that y is the empty word. We get $x\alpha = \alpha$, which is impossible because the elementary substitutions preserve the parity of the length of words.

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Reference

[1] R.B. Bapat, S.K. Jain, K.M.P. Karantha, M.D. Raj, Outer inverses: characterization and applications, Linear Algebra Appl. (2016), http://dx.doi.org/10.1016/j.laa.2016.06.045, in press.