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# Rank nonincreasing linear maps preserving the determinant of tensor product of matrices ☆

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Let $l, m_1, m_2, \dots m_l \geq 2$ be positive integers. We describe some linear maps $\phi : M_{m_1 \dots m_l}(\mathbb{F}) \to M_{m_1 \dots m_l}(\mathbb{F})$ satisfying

$$\det(\phi(A_1 \otimes \dots \otimes A_l)) = \det(A_1 \otimes \dots \otimes A_l),$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \dots, l$.
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## 1. Introduction

Let $M_n(\mathbb{F})$ be the linear space of $n$-square matrices over a field $\mathbb{F}$. For $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$, we denote by $A \otimes B \in M_{mn}(\mathbb{F})$ their tensor product (also known as

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the Kronecker product). Throughout this note, $O_n$ denotes the $n$ by $n$ null matrix, and $E_{i,j}^{(n)}$ denotes the $n$ by $n$ matrix whose all entries are equal to zero except for the $(i,j)$-th entry which is one. The rank of a matrix $A$ is denoted by $\rho(A)$.

In a wide variety of pure or applied studies the tensor product of matrices plays a fundamental role. For instance, in quantum physics, the quantum states of a system with $n$ physical states are represented as $n$ by $n$ positive semi-definite matrices with trace one, and if $A$ and $B$ are two states of two quantum system, then $A \otimes B$ describes the joint (bipartite) system. Recent studies in quantum information theory require the description of linear maps that preserve certain properties of tensor product of matrices (see [3]). Therefore, many papers have been appearing in the literature with the characterization of linear maps that preserve certain properties of the tensor product of matrices, such as norms, the rank, the spectrum, the spectral radius, the numerical radius or idempotency (see [1,2,4–7]).

Let $l, m_1, m_2, \dots m_l \geq 2$ be positive integers. Our main goal is to describe linear maps $\phi : M_{m_1 \dots m_l}(\mathbb{F}) \to M_{m_1 \dots m_l}(\mathbb{F})$ which satisfy

$$\det(\phi(A_1 \otimes \dots \otimes A_l)) = \det(A_1 \otimes \dots \otimes A_l), \tag{1}$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \dots, l$, with the assumption that $\phi$ do not increase the rank of any matrix $A_1 \otimes \dots \otimes A_l$.

## 2. Main result and proofs

We call a linear map $\pi$ on $M_{m_1 \dots m_l}(\mathbb{F})$ *canonical*, if

$$\pi(A_1 \otimes \dots \otimes A_l) = \psi_1(A_1) \otimes \dots \otimes \psi_l(A_l),$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \dots, l$, where $\psi_k : M_{m_k}(\mathbb{F}) \to M_{m_k}(\mathbb{F})$, $k = 1, \dots, l$, is either the identity map, $\psi_k(X) = X$ or the transposition map $\psi_k(X) = X^T$. In this case, we write $\pi = \psi_1 \otimes \dots \otimes \psi_l$.

Our main theorem reads as follows:

**Theorem 1.** *Let $\phi : M_{m_1 \dots m_l}(\mathbb{F}) \to M_{m_1 \dots m_l}(\mathbb{F})$ be a linear map which does not increase the rank of any matrix $A_1 \otimes \dots \otimes A_l \in M_{m_1 \dots m_l}(\mathbb{F})$. Then, $\phi$ satisfies (1) if and only if there are invertible matrices $U, V \in M_{m_1 \dots m_l}(\mathbb{F})$ with $\det(UV) = 1$, and a canonical map $\pi$ on $M_{m_1 \dots m_l}(\mathbb{F})$ such that*

$$\phi(A_1 \otimes \dots \otimes A_l) = U\pi(A_1 \otimes \dots \otimes A_l)V,$$

*for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \dots, l$.*

For proving our main results we will need some auxiliary results.

In a recent work, [6], Lim described the linear maps that preserve the rank of the tensor product of matrices over an arbitrary field $\mathbb{F}$:

**Theorem 2.** [6] *Let $\phi : M_{m_1 \dots m_l}(\mathbb{F}) \to M_{m_1 \dots m_l}(\mathbb{F})$ be a linear map. Then, $\phi$ satisfies the following two conditions:*

1. $\rho(\phi(A_1 \otimes \dots \otimes A_l)) = m_1 \dots m_l$, *whenever* $\rho(A_1 \otimes \dots \otimes A_l) = m_1 \dots m_l$;
2. $\rho(\phi(A_1 \otimes \dots \otimes A_l)) = 1$, *whenever* $\rho(A_1 \otimes \dots \otimes A_l) = 1$,

*if and only if there are invertible matrices $P, Q \in M_{m_1 \dots m_l}(\mathbb{F})$ and a canonical map $\pi$ on $M_{m_1 \dots m_l}(\mathbb{F})$ such that*

$$\phi(A_1 \otimes \dots \otimes A_l) = P\pi(A_1 \otimes \dots \otimes A_l)Q,$$

*for any $A_i \in M_{m_i}(\mathbb{F})$, $i = 1, \dots, l$.* $\square$

To prove Theorem 1 we need the following lemma:

**Lemma 1.** *Let $\phi : M_{m_1 \dots m_l}(\mathbb{F}) \to M_{m_1 \dots m_l}(\mathbb{F})$ be a linear map which does not increase the rank of any matrix $A_1 \otimes \dots \otimes A_l \in M_{m_1 \dots m_l}(\mathbb{F})$. If $\phi$ satisfies (1), then $\rho(\phi(A_1 \otimes \dots \otimes A_l)) = 1$, whenever $\rho(A_1 \otimes \dots \otimes A_l) = 1$.*

**Proof.** Assume that, for each $i = 1, \dots, l$, there is a matrix $A_i \in M_{m_i}(\mathbb{F})$ such that $\rho(A_i) = 1$, and

$$\phi\left(A_1 \otimes \dots \otimes A_l\right) = O_{m_1 \dots m_l}.$$

For each $t = 1, \dots, l$, and for each $j = 1, \dots, m_t$, let $A_{t,j} \in M_{m_t}(\mathbb{F})$ be defined by $A_{t,1} = A_t$, and for $j \geq 2$, let $A_{t,j}$ be rank one matrices satisfying $\rho(A_{t,1} + \dots + A_{t,m_t}) = m_t$. Let

$$B = \left( \sum_{j_1=1}^{m_1} A_{1,j_1} \right) \otimes \dots \otimes \left( \sum_{j_l=1}^{m_l} A_{l,j_l} \right).$$

As $\rho(B) = m_1 \dots m_l$ then $\det(B) \neq 0$, and by (1), we have that $\det(\phi(B)) \neq 0$. So

$$\begin{aligned} \rho\left(\phi\left(B\right)\right) &= \rho\left(\left( \sum_{j_1=1}^{m_1} A_{1,j_1} \right) \otimes \dots \otimes \left( \sum_{j_l=1}^{m_l} A_{l,j_l} \right)\right) \\ &= m_1 \dots m_l. \end{aligned}$$

Since

$$\phi\left(B\right) = \sum_{j_1=1}^{m_1} \dots \sum_{j_l=1}^{m_l} \phi\left(A_{1,j_1} \otimes \dots \otimes A_{l,j_l}\right),$$

we have that

$$\rho\left(\phi\left(B\right)\right) \leq \sum_{j_1=1}^{m_1} \ldots \sum_{j_l=1}^{m_l} \rho\left(\phi\left(A_{1,j_1} \otimes \ldots \otimes A_{l,j_l}\right)\right).$$

Since

$$\rho\left(\phi\left(B\right)\right) = m_1 \ldots m_l,$$

we conclude that

$$\sum_{j_1=1}^{m_1} \ldots \sum_{j_l=1}^{m_l} \rho\left(\phi\left(A_{1,j_1} \otimes \ldots \otimes A_{l,j_l}\right)\right) \geq m_1 \ldots m_l.$$

But we know that

$$\rho\left(\phi\left(A_{1,j_1} \otimes \ldots \otimes A_{l,j_l}\right)\right) \leq 1,$$

for all $t = 1, \ldots, l$, and for all $j_t = 1, \ldots, m_t$, and

$$\rho\left(\phi\left(A_{1,1} \otimes \ldots \otimes A_{l,1}\right)\right) = 0.$$

Therefore

$$\sum_{j_1=1}^{m_1} \ldots \sum_{j_l=1}^{m_l} \rho\left(\phi\left(A_{1,j_1} \otimes \ldots \otimes A_{l,j_l}\right)\right) \leq \left(m_1 \ldots m_l\right) - 1,$$

which is a contradiction. □

**Proof of Theorem 1.** The sufficiency is obvious. Assume now that

$$\phi : M_{m_1 \ldots m_l}(\mathbb{F}) \to M_{m_1 \ldots m_l}(\mathbb{F})$$

is a linear map which does not increase the rank of any matrix $A_1 \otimes \ldots \otimes A_l$, $A_i \in M_{m_i}(\mathbb{F})$, $i = 1, \ldots, l$, and satisfies (1). Then $\rho(\phi(A_1 \otimes \ldots \otimes A_l)) = m_1 \ldots m_l$, whenever $\rho(A_1 \otimes \ldots \otimes A_l) = m_1 \ldots m_l$, and using Lemma 1, we conclude that $\rho(\phi(A_1 \otimes \ldots \otimes A_l)) = 1$, whenever $\rho(A_1 \otimes \ldots \otimes A_l) = 1$. As we proved that $\phi$ satisfies the two conditions of Theorem 2, the result follows. □

## 3. Concluding remarks

It is worth to point out that the maps described in our Main Theorem are invertible. However they may not preserve the determinant of all matrices in $M_{m_1 \ldots m_l}(\mathbb{F})$.

**Example 1.** Let $l = 2$ and $m_1 = m_2 = 2$ and let $\phi : M_{m_1 m_2}(\mathbb{F}) \to M_{m_1 m_2}(\mathbb{F})$ such that

$$\phi(A_1 \otimes A_2) = A_1^T \otimes A_2.$$

Let

$$X = E_{11}^{(2)} \otimes E_{11}^{(2)} + E_{12}^{(2)} \otimes E_{12}^{(2)} + E_{21}^{(2)} \otimes E_{21}^{(2)} + E_{22}^{(2)} \otimes E_{22}^{(2)},$$

and

$$Y = E_{11}^{(2)} \otimes E_{11}^{(2)} + E_{12}^{(2)} \otimes E_{21}^{(2)} + E_{21}^{(2)} \otimes E_{12}^{(2)} + E_{22}^{(2)} \otimes E_{22}^{(2)}.$$

If

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

it is clear that

$$Y = \phi(X).$$

However,

$$\det(X) = 0 \quad \text{and} \quad \det(Y) = \det(\phi(X)) = -1. \quad \square$$

Our Main Theorem does not describe all linear maps that satisfy (1). In fact, there are linear maps satisfying (1) which are neither invertible nor rank nonincreasing.

**Example 2.** Let $m_1 = m_2 = 4$ and let $\phi : M_{16}(\mathbb{F}) \to M_{16}(\mathbb{F})$ such that

$$\phi(A \otimes B) = I_4 \otimes AB.$$

Then $\phi$ satisfies (1) but $\phi$ is neither invertible nor rank nonincreasing. In fact, if $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, we have that

$$\rho(\phi((E_{i,j}^{(4)} \otimes E_{j,i}^{(4)})) = \rho(I_4 \otimes E_{i,i}^{(4)}) = 4 > \rho(E_{i,j}^{(4)} \otimes E_{j,i}^{(4)}),$$

and

$$\phi(E_{i,j}^{(4)} \otimes E_{i,j}^{(4)}) = O_4.$$

Moreover $\phi$ is not a rank increasing map, since

$$\rho(\phi(E_{i,j}^{(4)} \otimes E_{i,j}^{(4)})) < \rho(E_{i,j}^{(4)} \otimes E_{i,j}^{(4)}).$$

Note that this linear map $\phi$ does not send nonzero decomposable tensors into nonzero decomposable tensors. □

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## References

[1] Ajda Fosner, Zejun Huang, Chi-kwong Li, Nung-Sing Sze, Linear maps preserving Ky Fan norms and Schatten norms of tensor products of matrices, SIAM J. Matrix Anal. Appl. (2) (2013) 673–685.
[2] Ajda Fosner, Zejun Huang, Chi-kwong Li, Nung-Sing Sze, Linear maps preserving numerical radius of tensor products of matrices, J. Math. Anal. Appl. 407 (2013) 183–189.
[3] Ajda Fosner, Zejun Huang, Chi-kwong Li, Nung-Sing Sze, Linear preserves and quantum information science, Linear Multilinear Algebra (10) (2013) 1377–1390.
[4] Ajda Fosner, Zejun Huang, Chi-kwong Li, Yiu-Tung Poon, Nung-Sing Sze, Linear maps preserving the higher numerical ranges of tensor products of matrices, Linear Multilinear Algebra (6) (2014) 776–791.
[5] Boadong Zeng, Jinli Xu, Ajda Fosner, Linear maps preserving rank of tensor products of matrices, Linear Multilinear Algebra (2) (2015) 366–376.
[6] Ming-Hauat Lim, A note on linear preservers of certain ranks of tensor products of matrices, Linear Multilinear Algebra (7) (2015) 1442–1447.
[7] Boadong Zeng, Jinli Xu, Ajda Fosner, Linear maps preserving idempotents of tensor products of matrices, Linear Algebra Appl. 470 (2015) 25–39.