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Research paper

Stability analysis for impulsive fractional hybrid systems via variational Lyapunov method

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ABSTRACT

This paper investigates the stability properties for a class of impulsive Caputo fractional-order hybrid systems with impulse effects at fixed moments. By utilizing the variational Lyapunov method, a fractional variational comparison principle is established. Some stability and instability criteria in terms of two measures are obtained. These results generalize the known ones, extending the corresponding theory of impulsive fractional differential systems. An example is given to demonstrate their effectiveness.

1. Introduction

In recent years, fractional calculus has played a vital role in different areas such as physics, control theory, electrical circuits, and fractal media [1–3]. Fractional-order models have been found to be an excellent tool for describing complex real-world systems with hereditary and memory properties. Consequently, the study of fractional-order systems has attracted considerable attention, and significant progress has been made [4–7], particularly in stability theory [8–12].

Numerous real models or modern complex engineering systems usually influence each other mutually and physically. By operating a number of constraints with information and communication networks, they become highly interconnected and interdependent. Normally, we need to use a sequence of abstract decision-making units to decide operation mode and to specify sub controllers to be activated, which will make controlled dynamical systems more complex and to be hierarchical. These multi-echelon systems are classified as hybrid systems [13]. Sometimes, the systems need to be switched to a new set of differential equations with momentary perturbations in form of impulses. Such systems are defined as impulsive hybrid systems (IHSs). IHSs have a wide range of applications in practical situations, such as computer science, mathematical programming, modeling and simulation [14,15]. For more details about such systems, one can be referred to the references [16–19].

Stability analysis is one of the most essential and fundamental issues for control systems including IHSs. It must be noticed that much attention has been mainly focused on IHSs of integer order. However, few theoretical results for stability analysis of impulsive fractional hybrid systems (IFHSs) are reported in the literature. In [20], the authors presented some existence results of a two-point boundary value problem for nonlinear impulsive fractional hybrid differential equations. In [21], fixed point theorems were used to investigate the problem of the existence of solutions for IHSs involving Caputo-

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Hadamard type fractional multi-orders with nonlinear integral boundary conditions. Despite the great potential applications, the stability theory of IFHSs has not yet been fully developed, which inspires us to conduct the current work.

As a commonly-used approach, the Lyapunov direct method provide a convenient way to obtain sufficient conditions for the stability of a system without explicitly requiring the behaviors of solutions. However, if we treat the hybrid term as a perturbation of IFHSs and use the Lyapunov direct method to judge the stability, dynamic characteristics of the perturbation often lead to a more complex structure of derivative of the Lyapunov function and complicated calculation. Even in some cases, it will fail due to an indefinite derivative of the Lyapunov function. Moreover, boundedness or precise measurement of perturbation is required in certain circumstances.

An alternative method to study the stability property of IFHSs is the variational Lyapunov method. In perturbation theory, the variational Lyapunov method [22–24] is proved to be an effective instrument in the investigation of perturbed systems. This technique combines the method of variation of parameters and the Lyapunov second method to provide a flexible mechanism for studying the effect of perturbations on differential systems. The advantage is that it is not necessary for the perturbations to be measured by means of a norm. Additionally, Lyapunov-like functions and differential inequalities are employed to connect the solutions of the systems with and without perturbation in terms of the maximal solution of a comparison problem. This method has been promoted from integer-order systems to fractional-order systems [25].

Due to the flexibility and strength of the variational Lyapunov method, this paper aims to extend the promoted variational Lyapunov method to the area of IFHSs and to give stability conditions of the systems. Most previous studies about stability analysis for fractional systems are concerned with stability or asymptotical stability in the sense of Lyapunov [11,12], Mittag-Leffler stability [8–10], and practical stability [26,27]. However, few results on stability analysis in terms of two measures [28] are available. In [29], practical stability in terms of two measures with initial time difference for nonlinear fractional equations was studied. The concepts of stability in terms of two measures can unify many known stability notions such as Lyapunov stability, eventual stability and partial stability in a single setup, and offer a general framework for investigation of stability theory in the qualitative analysis [28].

Motivated by the aforementioned discussions, we investigate the stability properties in terms of two measures for a class of IHSs involving Caputo's fractional order and impulse effects at fixed moments via the promoted variational Lyapunov method. The main contributions of the paper can be summarized as follows. Comparing with the conventional way, we treat the hybrid term of IFHSs as a perturbation to avoid considering the specific characteristics of the hybrid term directly. Then, a fractional variational comparison principle is established by utilizing differential inequalities and the Lyapunov-like function in Caputo's sense, which is an extension of the commonly-used one for impulsive fractional systems and provides a link between the perturbed system and the unperturbed one. Furthermore, based on the comparison principle, some stability properties in terms of two different measures are discussed for the system under consideration. Finally, an example is given to illustrate the effectiveness of the results.

The rest of paper is organized as follows. In Section 2, some preliminaries are introduced, and the research problem is formulated. The fractional variational comparison principle is established and corresponding results are shown in Section 3. Section 4 discusses some stability and instability properties in terms of two measures for the systems. In Section 5, an example is presented to illustrate the results. Finally, conclusions are made in Section 6.

2. Systems description and preliminaries

In this section, some useful notations, definitions and lemmas are presented.

Definition 2.1. [4] The fractional integral of order α for a function $x(t)$ is defined as

$$ {}_{t_0}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau, $$

where $t \geq t_0$ and $\alpha > 0$, $\Gamma(\,\cdot\,)$ is the Gamma function.

Definition 2.2. [4] The Riemann–Liouville derivative of fractional order α of function $x(t)$ is given as

$$ {}_{t_0}^{RL}D_t^{\alpha}x(t) = \frac{d^n}{dt^n}\left({}_{t_0}D_t^{-(n-\alpha)}x(t)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x(\tau)d\tau\right), $$

in which $n-1 < \alpha < n \in \mathbb{Z}^+$.

Definition 2.3. [4] The Caputo derivative of fractional order α of function $x(t)$ is defined as

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = {}_{t_0}D_t^{-(n-\alpha)}\left(\frac{d^n}{dt^n}x(t)\right) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau, $$

in which $n-1 < \alpha < n \in \mathbb{Z}^+$.

Lemma 2.4. [30] *If $x(t) \in C^m[t_0, \infty)$ and $m-1 < \alpha < m$, $m \in \mathbb{Z}^+$, $m = 1$, then*

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = {}_{t_0}^{RL}D_t^{\alpha}\left(x(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}x^{(k)}(0)\right). $$

Definition 2.5. [25] $m \in C_p([t_0, T], \mathbb{R})$ means that $m \in C([t_0, T], \mathbb{R})$ and $(t-t_0)^p m(t) \in C([t_0, T], \mathbb{R})$ with $p+q=1$.

Definition 2.6. [25] For $m \in C_p([t_0, T], \mathbb{R})$, the Riemann–Liouville derivative of $m(t)$ is defined as

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t-\tau)^{p-1} m(\tau) d\tau, $$

with $p+q=1$.

Lemma 2.7. [25] *Let $m \in C_p([t_0, T], \mathbb{R})$. Suppose that for any $t_1 \in [t_0, T]$, if $m(t_1)=0$ and $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$ {}_{t_0}^{RL}D_t^q m(t_1) \geqslant 0, $$

with $p+q=1$.

Based on the lemma above, a similar result can be acquired as follows, which will be used in the next section.

Lemma 2.8. *Let $m \in C^1([t_0, T], \mathbb{R})$. Suppose that for any $t_1 \in [t_0, T]$, if $m(t_1)=0$ and $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$ {}_{t_0}^{C}D_t^q m(t_1) \geqslant 0, $$

where ${}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(p)} \int_{t_0}^{t} (t-\tau)^{p-1} x'(\tau) d\tau$ and $p+q=1$.

Proof. Since $m \in C^1([t_0, T], \mathbb{R})$, then $m(t)$ is continuous on $[t_0, T]$ and $(t-t_0)^p m(t)$ is also continuous on $[t_0, T]$. Thus, we have $m \in C_p([t_0, T], \mathbb{R})$ based on the Definition 2.5. By Definition 2.2, Lemma 2.4 and 2.7, for $0 < q < 1$, we have

$$ {}_{t_0}^{C}D_t^q m(t) = {}_{t_0}^{RL}D_t^q m(t) - \frac{m(t_0)}{\Gamma(p)} (t-t_0)^{p-1}, \tag{2.1} $$

and

$$ {}_{t_0}^{RL}D_t^q m(t_1) \geqslant 0. $$

Replace t by t_1 in (2.1), then

$$ {}_{t_0}^{C}D_t^q m(t_1) = {}_{t_0}^{RL}D_t^q m(t_1) - \frac{m(t_0)}{\Gamma(p)} (t_1-t_0)^{p-1}. $$

For any $t_0 \le t < t_1$, since $m(t_0) < 0$ and ${}_{t_0}^{RL}D_t^q m(t_1) \geqslant 0$, $(t_1-t_0)^{p-1} > 0$, we obtain

$$ {}_{t_0}^{C}D_t^q m(t_1) \geqslant 0. $$

The proof is completed. □

Consider the impulsive fractional hybrid system with impulse effects at fixed moments,

$$ \begin{cases} {}_{t_0}^{C}D_t^\alpha x(t) = f(t, x, \lambda_k(x_k)), t \in (t_k, t_{k+1}], \\ x\left(t_k^+\right) = x_k^+, \quad x_k^+ = x_k + I_k(x_k), k = 1, 2, \cdots, \\ x_k = x(t_k), \quad x_0\left(t_0^+\right) = x_0, \end{cases} \tag{2.2} $$

and the fractional-order differential system

$$ \begin{cases} {}_{t_0}^{C}D_t^\alpha y(t) = F(t, y), \\ y\left(t_0^+\right) = x_0, \end{cases} \tag{2.3} $$

where ${}_{t_0}^{C}D_t^\alpha$ denotes Caputo fractional derivative of order α, $0 < \alpha < 1$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t, x, \lambda_k(x_k)) = F(t, x(t)) + R(t, x, \lambda_k(x_k))$, $R(t, x, \lambda_k(x_k)) \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $\lambda_k \in C(\mathbb{R}^n, \mathbb{R}^m)$, $\triangle x(t_k) = x(t_k^+) - x(t_k)$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, \cdots$, $t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ are impulsive moments, and $t_k \to \infty$, $k \to \infty$.

Let $x_0 \in \mathbb{R}^n$. Denote by $x(t) = x(t; t_0, x_0)$ the solution to system (2.2) and $y(t) = y(t; t_0, x_0)$ the solution to system (2.3) respectively, satisfying the initial conditions

$$ x(t_0^+; t_0, x_0) = x_0, \quad y(t_0^+; t_0, x_0) = x_0. $$

In general, the solutions $x(t) = x(t; t_0, x_0)$ are piecewise continuous functions with points of discontinuity of first type at which they are left continuous, that is, at the moments t_k, $k = 1, 2, \cdots$, the following relations are satisfied [31]:

$x(t_k^-) = x(t_k)$ and $x(t_k^+) = x(t_k) + I_k(x(t_k))$,where $x(t_k^-) = \lim_{\varepsilon \to 0^-} x(t_k - \varepsilon)$ and $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$.

Without loss of generality, assume that the functions $f, F, I_k, k = 1, 2, \cdots$ are smooth enough to guarantee the existence, uniqueness and continuability of the solutions of systems (2.2) and (2.3).

Moreover, the following assumption [25] is needed relative to system (2.3).

Assumption (H) The solutions $y(t) = y(t; t_0, x_0)$ of system (2.3) that exist for all $t \geq t_0$, are unique and depend continuously on the initial data, and $||y(t;\ t_0,\ x_0)||$ is locally Lipschitzian in x_0.

Let $G_k = (t_{k-1}, t_k) \times \mathbb{R}^n, k = 1, 2, \cdots$ and $G = \bigcup_{k=1}^{\infty} G_k$. In the further considerations, the piecewise continuous auxiliary functions [31] are used, which are similar to the classical Lyapunov functions.

Definition 2.9. [32] A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to the class ν_0', if:

(1) $V(t, x)$ is continuous in G and locally Lipschitz continuous with respect to x on each of the sets G_k, $k = 1, 2, \cdots$.
(2) For each $k = 1, 2, \cdots$ and $x \in \mathbb{R}^n$, there exist the finite limits

$$V(t_k^-, x) = \lim_{\substack{t \to t_k \\ t < t_k}} V(t, x), \qquad V(t_k^+, x) = \lim_{\substack{t \to t_k \\ t > t_k}} V(t, x)$$

and the following equality is valid

$$V(t_k^-, x) = V(t_k, x).$$

For a function $V \in \nu_0'$, the Dini-like fractional order derivative in Caputo's sense is defined as follows. The specific formulation can be seen in [25].

Definition 2.10. [25] Given a function $V \in \nu_0'$. For any fixed $t \geq t_0$, any arbitrary point $s \in (t_0,\ t]$, and $x \in \mathbb{R}^n$, the Caputo fractional Dini derivative of the Lyapunov function $V(s,\ y(t;\ s,\ x))$ is given by

$$\begin{aligned} &{}^{C}D_+^{\alpha} V(s, y(t; s, x)) \\ &= \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} [V(s, y(t; s, x)) - V(s-h, y(t; s-h, x - h^{\alpha} F(s, x)))], \end{aligned}$$

where $V(s-h, y(t; s-h, x-h^{\alpha}F(s,x))) = \sum_{r=1}^{n} (-1)^{r+1} \alpha_{C_r} V(s - rh, y(t; s - rh, x - h^{\alpha} F(s, x)))$, and α_{C_r} represents the binomial coefficients in the definition of the Grünwald–Letnikov fractional derivative.

For convenience, the following classes of functions are introduced.
$\mathcal{K} = \{u \in C(\mathbb{R}_+, \mathbb{R}_+), u(0) = 0, u \text{ is strictly increasing}\}$,
$PC = \left\{u : \mathbb{R}_+ \to \mathbb{R}_+, \text{ continuous on } (t_k, t_{k+1}] \text{ and } \lim_{t \to t_k^+} u(t) = u(t_k^+) \text{ exists }\right\}$,
$PC\mathcal{K} = \{u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, \forall s \in \mathbb{R}_+, u(\cdot, s) \in PC, \forall t \in \mathbb{R}_+, u(t, \cdot) \in \mathcal{K}\}$,
$\Gamma = \left\{h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+, \forall x \in \mathbb{R}^n, h(\cdot, x) \in PC, \forall t \in \mathbb{R}_+, h(t, \cdot) \in C(\mathbb{R}^n, \mathbb{R}_+), \text{ and } \inf_x h(t, x) = 0\right\}$,
$\Gamma_0 = \left\{h \in \Gamma : \sup_{\mathbb{R}_+} h(t, x) \text{ exists for } x \in \mathbb{R}^n\right\}$.

Here, we shall state the relationships between the two measures $h_0(t,\ x)$ and $h(t,\ x)$ (or h_0 and h for short), some concepts of stability in terms of two measures of system (2.2), and some features of the function $V(t,\ x)$ in terms of the two measures h_0 and h.

Definition 2.11. [23] Let $h_0,\ h \in \Gamma$. h_0 is finer than h, that is, if there exists a $\delta > 0$ and a function $\phi \in PC\mathcal{K}$ such that $h(t,\ x) \leq \phi(t,\ h_0(t,\ x))$ whenever $h_0(t,\ x) < \delta$. If $\phi \in \mathcal{K}$, then we say that h_0 is uniformly finer than h.

Definition 2.12. [23] Let $V \in \nu_0'$, $h_0,\ h \in \Gamma$. $V(t,\ x)$ is said to be

(1) h-positive definite, if there exists a function $b \in \mathcal{K}$ and a constant $\rho > 0$ such that $h(t,\ x) < \rho$ implies $b(h(t,\ x)) \leq V(t,\ x)$;
(2) weakly h_0-decrescent, if there exists a $\delta > 0$ and a function $a \in PC\mathcal{K}$ such that $h_0(t,\ x) < \delta$ implies $V(t,\ x) \leq a(h_0(t,\ x))$;
(3) h_0-decrescent if $a \in \mathcal{K}$ in (2).

Definition 2.13. [22] Let $h_0,\ h \in \Gamma$. The system is said to be $(h_0,\ h)$-stable, if for any given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0,\ x_0) < \delta$ implies that $h(t,\ x(t;\ t_0,\ x_0)) < \varepsilon$, $t \geq t_0$.

Definition 2.14. [28] Let $h_0,\ h \in \Gamma$. The system is said to be $(h_0,\ h)$-attractive, if for each $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exist positive constants $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \varepsilon)$ such that $h_0(t_0,\ x_0) < \delta_0$ implies $h(t,\ x(t)) < \varepsilon$, $t \geqslant t_0 + T$.

Definition 2.15. [22] Let $h_0,\ h \in \Gamma$. We define $\tilde{h}_0(x) = \sup_{\mathbb{R}_+} h_0(t_0, x)$, $\tilde{h}_1(x) = \sup_{\mathbb{R}_+} h_1(t_0, x)$. Then we say that the system (2.2) is $(\tilde{h}_0, \tilde{h}_1)$-strictly stable if given $\varepsilon_1 > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(\varepsilon_1)$ such that $\tilde{h}_0(x_0) \leqslant \delta_1$ implies $\tilde{h}_0(y(t; t_0, x_0)) < \varepsilon_1$, $t \geq t_0$ and for every $\delta_2 \leq \delta_1$, there exists an $\varepsilon_2 \leq \delta_2$ such that $\delta_2 \leqslant \tilde{h}_1(x_0)$ implies $\varepsilon_2 < \tilde{h}_1(t, y(t; t_0, x_0))$, $t \geq t_0$.

Based on the definitions mentioned above, it is easy to formulate other kinds of $(h_0,\ h)$-stability for system (2.2). For more discussions of the concepts of two measures, one can be referred to the references [28,33].

Remark 2.16. If $h_0(t, x) = h(t, x) = ||x||$, then the Definition 2.13 can be reduced to the well known concepts of stability of system (2.2) in the Lyapunov sense.

Remark 2.17. By choosing suitable formulation of the two measures (h_0, h), the concepts in terms of two measures (h_0, h) [28] enable us to unify a variety of stability notions, such as eventual stability, partial stability, relative stability, and conditional stability, which have been found in the literature but would otherwise be treated separately.

3. Fractional variational comparison principle

In this section, a fractional variational comparison principle is established, which can connect the solutions of the perturbed system and the solutions of the unperturbed one through the maximal or minimal solution of a fractional scalar comparison system. As derivative results, some corollaries are obtained.

For the sake of our later proofs, we present the following comparison lemma.

Lemma 3.1. *Let $m \in C^1 \in ([t_0, \infty], \mathbb{R}_+)$, and*

$$ {}_{t_0}^{C}D_t^{\alpha} m(t) \leqslant (\geqslant) g(t, m(t), \sigma(m(t_0))), $$

where $g \in C([t_0, \infty] \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $\sigma \in C(\mathbb{R}_+, \mathbb{R})$, and $g(t, u, v)$ is nondecreasing in v, $\sigma(u)$ is nondecreasing in u. Assume that $\eta(t) = \eta(t; t_0, u_0)$ is the maximal (minimal) solution of the initial value problem (IVP) for the fractional equation

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha} u(t) = g(t, u, \sigma(u_0)), \\ u(t_0) = u_0, \end{cases} \tag{3.1} $$

existing on $[t_0, \infty)$. If $m(t_0) \le (\ge) u_0$, then

$$ m(t) \leqslant (\geqslant) \eta(t), \quad t \in [t_0, \infty]. \tag{3.2} $$

Proof. Case I: Suppose that all the inequalities are $\le$. Since $u(t; t_0, u_0, \varepsilon) \to \eta(t; t_0, u_0)$ uniformly with $\varepsilon \to \infty$, to prove (3.2), it is enough to prove that

$$ m(t) < u(t; t_0, u_0, \varepsilon), \quad t \in [t_0, \infty], \tag{3.3} $$

where $u(t; t_0, u_0, \varepsilon)$ is any solution of IVP for

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha} u(t) = g(t, u, \sigma(u_0)) + \varepsilon, \\ u(t_0) = u_0 + \varepsilon, \end{cases} \tag{3.4} $$

and ε being an arbitrary small number.

If (3.3) is not true, then there would exist a $t_1 > t_0$ such that $m(t_1) = u(t_1; t_0, u_0, \varepsilon)$, and for $t_0 \le t < t_1$, $m(t) < u(t; t_0, u_0, \varepsilon)$.

Now by Lemma 2.8 and the linearity of fractional operator [34], we have

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} m(t_1) &\ge {}_{t_0}^{C}D_t^{\alpha} u(t_1; t_0, u_0, \varepsilon) \\ &= g(t_1, u(t_1; t_0, u_0, \varepsilon), \sigma(u_0)) + \varepsilon. \end{aligned} \tag{3.5} $$

Since $g(t, u, v)$ is nondecreasing in v, $\sigma(u)$ is nondecreasing in u, $m(t_1) = u(t_1; t_0, u_0, \varepsilon)$ and $m(t_0) \le u(t_0)$, this implies, from the preceding considerations, that

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} m(t_1) &\ge {}_{t_0}^{C}D_t^{\alpha} u(t_1; t_0, u_0, \varepsilon) \\ &= g(t_1, u(t_1; t_0, u_0, \varepsilon), \sigma(u_0)) + \varepsilon \\ &\ge g(t_1, m(t_1), \sigma(m(t_0))) + \varepsilon \\ &> g(t_1, m(t_1), \sigma(m(t_0))), \end{aligned} \tag{3.6} $$

which contradicts with ${}_{t_0}^{C}D_t^{\alpha} m(t_1) \leqslant g(t_1, m(t_1), \sigma(m(t_0)))$. Hence (3.3) is valid.

Case II: For all the inequalities are $\ge$. Set $\widetilde{m}(t) = -m(t)$. Similarly, to prove $m(t) \ge \eta(t)$, we need to prove that

$$ \widetilde{m}(t) < -u(t; t_0, u_0, \varepsilon), \quad t \in [t_0, \infty], \tag{3.7} $$

where $u(t; t_0, u_0, \varepsilon)$ is any solution of IVP for

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha} u(t) = g(t, u, \sigma(u_0)) - \varepsilon, \\ u(t_0) = u_0 - \varepsilon, \end{cases} \tag{3.8} $$

and ε being an arbitrary small number, since $u(t; t_0, u_0, \varepsilon) \to \eta(t; t_0, u_0)$ uniformly with $\varepsilon \to \infty$.

If (3.7) does not hold, then there would exist a $t_1 > t_0$ such that $\widetilde{m}(t_1) = -u(t_1; t_0, u_0, \varepsilon)$, and for $t_0 \le t < t_1$, $\widetilde{m}(t) < -u(t; t_0, u_0, \varepsilon)$. It follows from Lemma 2.8 and the linearity of fractional operator [34] that

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} \widetilde{m}(t_1) &\ge -{}_{t_0}^{C}D_t^{\alpha} u(t_1; t_0, u_0, \varepsilon) \\ &= -g(t_1, u(t_1; t_0, u_0, \varepsilon), \sigma(u_0)) + \varepsilon. \end{aligned} \tag{3.9} $$

According to the monotonicity of functions g and σ and $-\widetilde{m}(t_0) = m(t_0) \geqslant u(t_0)$, we can obtain that

$$\sigma(-\widetilde{m}(t_0)) \leqslant \sigma(u(t_0))$$

and

$$-g(t_1, u(t_1; t_0, u_0, \varepsilon), \sigma(u_0)) \geqslant -g(t_1, u(t_1; t_0, u_0, \varepsilon), \sigma(-\widetilde{m}(t_0))).$$

Since $\widetilde{m}(t_1) = -u(t_1; t_0, u_0, \varepsilon)$, this implies that

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}\tilde{m}(t_1) &\geq -{}_{t_0}^{C}D_t^{\alpha}u(t_1; t_0, u_0, \varepsilon) \\ &= -g(t_1, u(t_1; t_0, u_0, \varepsilon), \sigma(u_0)) + \varepsilon \\ &\geq -g(t_1, -\tilde{m}(t_1), \sigma(-\tilde{m}(t_0))) + \varepsilon, \end{aligned} \tag{3.10}$$

that is,

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}m(t_1) &\leq g(t_1, -\tilde{m}(t_1), \sigma(-\tilde{m}(t_0))) - \varepsilon \\ &< g(t_1, -\tilde{m}(t_1), \sigma(-\tilde{m}(t_0))), \end{aligned} \tag{3.11}$$

which shows a contradiction with ${}_{t_0}^{C}D_t^{\alpha}m(t_1) \geqslant g(t_1, m(t_1), \sigma(m(t_0)))$. Thus, (3.7) is valid and the proof is completed. □

Remark 3.2. The discussions of the existence of extremal solutions for initial value problems (3.4) and (3.8) can be seen in [6], [35] and [36], which present an effective and reasonable result. Also, the conclusion that $u(t;\, t_0,\, u_0,\, \varepsilon) \to \eta(t;\, t_0,\, u_0)$ uniformly with $\varepsilon \to \infty$ can be obtained in the references mentioned above.

Consider the following scalar comparison hybrid system

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}u(t) = g(t, s, u, \sigma_k(u_k)), \quad t_0 \leq t, \quad t \in (t_k, t_{k+1}], \\ u\left(t_k^+\right) = u_k^+, \quad u_k^+ = \psi_k(u_k), \\ u_k = u(t_k), \quad \psi_0(u_0) = u_0, \quad u\left(t_0^+\right) = u_0, \end{cases} \tag{3.12}$$

where $g : \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous on $(t_k, t_{k+1}]$ in s, $k = 0, 1, 2, \cdots$, for each $t \in \mathbb{R}_+$ and for all $k \in \mathbb{N}$

$$\lim_{(t,s,u)\to(t,t_k^+,w)} g(t, s, u, \sigma_k(u_k)) = g(t, t_k^+, w, \sigma_k(w_k)),$$

where $w_k = w(t_k)$, and for any $(t,\, s,\, u,\, v)$, $g(t,\, s,\, u,\, v)$ is nondecreasing in v, $\psi_k(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\psi_k(u)$ is nondecreasing in u, $\sigma_k(u) \in C(\mathbb{R}_+, \mathbb{R})$ and $\sigma_k(u)$ is nondecreasing in u.

We now acquire the fractional comparison theorem as follows.

Theorem 3.3. *Assume that assumption (H) holds. Suppose that*

(1) $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $V \in \nu_0'$, *$V(t,\, x)$ is locally Lipschitzian in x with Lipschitz constant $L > 0$ and for $t_0 \leq s \leq t$, $x \in S(\rho)$,*

$${}^{C}D_+^{\alpha}V(s, y(t; s, x)) \leqslant g(t, s, V(s, y(t; s, x)), \sigma_k(V(t_k, y(t; t_k, x(t_k))))),$$

where $s \neq t_k$, $k = 1, 2, \cdots$, $S(\rho) = \left\{x \in \mathbb{R}^n \middle| ||x|| < \rho\right\}$;

(2) $V(s, y(t; s, x + I_k(x(s)))) \leqslant \psi_k(V(s, y(t; s, x)))$, $s = t_k$, $k = 1, 2, \cdots$;

(3) $\psi_k(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $k = 1, 2, \cdots$ *is nondecreasing in u;*

(4) *The function $g : \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous in each of the sets $(t_k, t_{k+1}]$ in s, $k = 0, 1, 2, \cdots$, and for any $(t,\, s,\, u,\, v)$, $g(t,\, s,\, u,\, v)$ is nondecreasing in v, $\sigma_k(u) \in C(\mathbb{R}_+, \mathbb{R})$ and $\sigma_k(u)$ is nondecreasing in u;*

(5) *The maximal solution $r(t;\, s,\, t_0,\, u_0)$ with initial value $(t_0,\, u_0)$ of the scalar system (3.12) is defined in the interval $[t_0,\, \infty)$.*

If $V(t_0,\, y(t;\, t_0,\, x_0)) \leq u_0$, then we have

$$V(s, y(t; s, x)) \leqslant r(t; s, t_0, u_0), \quad t_0 \leqslant s \leqslant t. \tag{3.13}$$

Specifically, if $s = t$, then

$$V(t, x(t; t_0, x_0)) \leqslant \tilde{r}(t; t_0, u_0),$$

where $\tilde{r}(t; t_0, u_0) = r(t; t, t_0, u_0)$.

Proof. Let $x_0 \in \mathbb{R}^n$ and $x(t) = x(t; t_0, x_0)$ be any solution of system (2.2) such that $||x_0|| < \rho$. Let $y(t) = y(t; s, x(s))$ be any solution of system (2.3) with the initial value $(s,\, x(s))$ in the interval $(t_0,\, t]$ and $V(t_0,\, y(t;\, t_0,\, x_0)) \leq u_0$. Set $m(t, s) = V(s, y(t; s, x(s)))$. For $k = 1, 2, \cdots$, $s \neq t_k$, and for $h > 0$ small enough, we have the following result.

Since for each $(t,\, s)$, $V(t,\, x)$ and $||y(t;\, s,\, x)||$ are locally Lipschitzian in x with Lipschitz constants L and M, respectively. Then, for any fixed t, we have

$$\begin{aligned}
& m(t,s) - \sum_{r=1}^{n} (-1)^{r+1} \alpha_{C_r} m(t, s-rh) \\
& = V(s, y(t; s, x)) - \sum_{r=1}^{n} (-1)^{r+1} \alpha_{C_r} V(s-rh, y(t; s-rh, S(x, h, r, \alpha))) \\
& \le V(s, y(t; s, x)) - \sum_{r=1}^{n} (-1)^{r+1} \alpha_{C_r} V(s-rh, y(t; s-rh, x - h^{\alpha} F(s, x))) \\
& \quad + LM \sum_{r=1}^{n} \alpha_{C_r} \epsilon(h^{\alpha}),
\end{aligned}$$

where $L > 0$, $M > 0$, $S(x, h, r, \alpha) = x(s) - h^{\alpha} F(s, x) - \epsilon(h^{\alpha})$ with $\frac{\epsilon(h^{\alpha})}{h^{\alpha}} \to 0$ as $h \to 0$. The details about the form of $S(x, h, r, \alpha)$ can be seen in [25].

The proof of the inequality above is not a new one, which is similar to that of Theorem 4.2 in [25], so we omit it here. Dividing through h^{α} by two sides and taking limits as $h \to 0^+$, we get

$$\begin{aligned}
{}^{C}D_{+}^{\alpha} m(t,s) &= \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \Big(m(t,s) - \sum_{r=1}^{n} (-1)^{r+1} \alpha_{C_r} m(t, s-rh) \Big) \\
&\leqslant \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \Big(V(s, y(t; s, x)) \\
&\quad - \sum_{r=1}^{n} (-1)^{r+1} \alpha_{C_r} V(s-rh, y(t; s-rh, x - h^{\alpha} F(s, x))) \Big) \\
&\quad + LM \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \Big(\sum_{r=1}^{n} \alpha_{C_r} \epsilon(h^{\alpha}) \Big).
\end{aligned}$$

The right side does approach to 0 as $h \to 0^+$ and hence

$$\begin{aligned}
{}^{C}D_{+}^{\alpha} m(t,s) &\leqslant {}^{C}D_{+}^{\alpha} V(s, y(t; s, x)) \\
&\leqslant g(t, s, V(s, y(t; s, x)), \sigma_k(V(t_k, y(t; t_k, x(t_k))))) \\
&= g(t, s, m(t,s), \sigma_k(m(t, t_k))),
\end{aligned} \tag{3.14}$$

where $s \neq t_k$, $k = 1, 2, \cdots$.

To prove (3.13), firstly, we should prove that

$$V(s, y(t; s, x)) \leqslant r_0(t; s, t_0, u_0^+), \quad t_0 \leqslant s \leqslant t, \quad s \in (t_0, t_1],$$

where $r_0(t; s, t_0, u_0^+)$ is the maximal solution of the equation ${}_{t_0}^{C}D_t^{\alpha} u(t) = g(t, s, u, \sigma_0(u_0^+))$ with $u(t_0) = u_0 = u_0^+$ in the interval $(t_0, t_1]$.

Then on $(t_0, t_1]$, it follows from (3.14) that

$$\begin{aligned}
{}^{C}D_{+}^{\alpha} m(t,s) &\leqslant g(t, s, V(s, y(t; s, x)), \sigma_0(V(t_0, y(t; t_0, x_0)))) \\
&= g(t, s, m(t,s), \sigma_0(m(t, t_0))).
\end{aligned}$$

Since $m(t, t_0) = V(t_0, y(t; t_0, x_0)) \leqslant u_0^+$, due to the monotonic character of function σ_0 in u and $g(t, s, u, v)$ in v, by applying Lemma 3.1 with appropriate modifications, we obtain

$$m(t,s) \leqslant r_0(t; s, t_0, u_0^+), \quad s \leqslant t, \quad s \in (t_0, t_1].$$

That is

$$V(s, y(t; s, x)) \leqslant r_0(t; s, t_0, u_0^+), \quad t_0 \leqslant s \leqslant t.$$

Specially, we have $m(t, t_1) \leqslant r_0(t; t_1, t_0, u_0^+)$ with $s = t_1$. Then, applying Condition (2) and the fact that $\psi_k(u)$ is nondecreasing in u, we obtain that

$$\begin{aligned}
m\big(t, t_1^+\big) &= V\Big(t_1^+, y\big(t; t_1^+, x(t_1^+)\big)\Big) \\
&= V\Big(t_1^+, y\big(t; t_1^+, x(t_1) + I_1(x(t_1))\big)\Big) \\
&\le \psi_1(V(t_1, y(t; t_1, x(t_1)))) \\
&= \psi_1(m(t, t_1)) \\
&\le \psi_1\Big(r_0\big(t; t_1, t_0, u_0^+\big)\Big) = r_0\Big(t; t_1^+, t_0, u_0^+\Big) = u_1^+.
\end{aligned}$$

Then on $(t_1, t_2]$, by (3.14), we have

$$\begin{cases} {}^{C}D_{+}^{\alpha} m(t,s) \le g(t,s,m(t,s),\sigma_1(m(t,t_1))), \\ m\left(t,t_1^+\right) \le u_1^+. \end{cases} \tag{3.15}$$

It also implies due to the monotonic character of σ_1 in u and $g(t, s, u, v)$ in v, by Lemma 3.1 with appropriate modifications, that

$$m(t,s) \leqslant r_1(t;s,t_1,u_1^+), \quad s \leqslant t, \quad s \in (t_1,t_2],$$

and

$$V(s,y(t;s,x)) \leqslant r_1(t;s,t_1,u_1^+), \quad t_0 \leqslant s \leqslant t,$$

where $r_1(t;s,t_1,u_1^+)$ is the maximal solution of the equation ${}_{t_0}^{C}D_t^{\alpha} u(t) = g(t,s,u,\sigma_1(u_1^+))$ with $u_1^+ = \psi_1(r_0(t;t_1,t_0,u_0^+)) = r_0(t;t_1^+,t_0,u_0^+)$ in the interval $(t_1, t_2]$.

Repeating the procedure above, generally, let $s \in (t_{k-1}, t_k]$, $k > 1$,

$$V(s,y(t;s,x)) \leqslant r_{k-1}(t;s,t_{k-1},u_{k-1}^+), \quad t_0 \leqslant s \leqslant t, \tag{3.16}$$

where $r_{k-1}(t;s,t_{k-1},u_{k-1}^+)$ is the maximal solution of the equation ${}_{t_0}^{C}D_t^{\alpha} u(t) = g(t,s,u,\sigma_{k-1}(u_{k-1}^+))$ with $u_{k-1}^+ = \psi_{k-1}(r_{k-2}(t;t_{k-1},t_{k-2},u_{k-2}^+)) = r_{k-2}(t;t_{k-1}^+,t_{k-2},u_{k-2}^+)$ in the interval $(t_{k-1}, t_k]$.

Next, we should prove that

$$V(s,y(t;s,x)) \leqslant r_k(t;s,t_k,u_k^+), \quad t_0 \leqslant s \leqslant t$$

with $s \in (t_k, t_{k+1}]$.

By (3.16), when $s = t_k$, we have

$$m(t,t_k) = V(t_k, y(t;t_k,x(t_k))) \leqslant r_{k-1}(t;t_k,t_{k-1},u_{k-1}^+).$$

It follows from Conditions (2) and (3) that

$$\begin{aligned} m\left(t,t_k^+\right) &= V\left(t_k^+, y\left(t;t_k^+,x\left(t_k^+\right)\right)\right) \\ &= V\left(t_k^+, y\left(t;t_k^+,x(t_k)+I_k(x(t_k))\right)\right) \\ &\le \psi_k(V(t_k,y(t;t_k,x(t_k)))) \\ &= \psi_k(m(t,t_k)) \\ &\le \psi_k\left(r_{k-1}\left(t;t_k,t_{k-1},u_{k-1}^+\right)\right) = r_{k-1}\left(t;t_k^+,t_{k-1},u_{k-1}^+\right) = u_k^+. \end{aligned}$$

Then on $(t_k, t_{k+1}]$, we have

$$\begin{cases} {}^{C}D_{+}^{\alpha} m(t,s) \leqslant g(t,s,m(t,s),\sigma_k(m(t,t_k))), \\ \quad m(t,t_k^+) \leqslant u_k^+. \end{cases} \tag{3.17}$$

Similarly, we obtain

$$m(t,s) \leqslant r_k(t;s,t_k,u_k^+), \quad s \leqslant t, \quad s \in (t_k,t_{k+1}],$$

and

$$V(s,y(t;s,x)) \leqslant r_k(t;s,t_k,u_k^+), \quad t_0 \leqslant s \leqslant t,$$

where $r_k(t;s,t_k,u_k^+)$ is the maximal solution of the equation ${}_{t_0}^{C}D_t^{\alpha} u(t) = g(t,s,u,\sigma_k(u_k^+))$ with $u_k^+ = \psi_k(r_{k-1}(t;t_k,t_{k-1},u_{k-1}^+)) = r_{k-1}(t;t_k^+,t_{k-1},u_{k-1}^+)$ in the interval $(t_{k-1}, t_k]$.

Thus,

$$m(t,s) = V(s,y(t;s,x)) \leqslant r_k(t;s,t_k,u_k^+), \quad k = 0,1,2,\cdots,$$

and

$$V(t,x(t;t_0,x_0)) \leqslant r_k(t;t_k,u_k^+), \quad k = 0,1,2,\cdots$$

with $s = t$.

Generally, for any $s \in [t_0, t]$, we have the following equalities

$$\tilde{r}(t; s, t_0, u_0) = \begin{cases} r_0(t; s, t_0, u_0^+), & s \in (t_0, t_1], \\ r_1(t; s, t_1, u_1^+), & s \in (t_1, t_2], \\ \vdots & \vdots \\ r_k(t; s, t_k, u_k^+), & s \in (t_k, t_{k+1}], \\ \vdots & \vdots \end{cases} \tag{3.18}$$

where $\tilde{r}(t; s, t_0, u_0)$ is the maximal solution of equation ${}^C_{t_0}D^\alpha_t u(t) = g(t, s, u, \sigma_k(u_k^+))$ in the interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \cdots$ for which $u_k^+ = \psi_k(r_{k-1}(t; t_k, t_{k-1}, u_{k-1}^+)) = r_{k-1}(t; t_k^+, t_{k-1}, u_{k-1}^+)$, $k = 1, 2, \cdots$ and $u_0^+ = u_0$.

Thus, we can get that

$$m(t, s) = V(s, y(t; s, x)) \leqslant r(t; s, t_0, u_0), \quad t_0 \leqslant s \leqslant t.$$

Specifically, we have

$$V(t, x(t)) \leqslant \tilde{r}(t; t_0, u_0),$$

with $s = t$, where $\tilde{r}(t; t_0, u_0) = r(t; t, t_0, u_0)$. □

Remark 3.4. In Theorem 3.3, the variational Lyapunov function $V(t, x)$ plays the role of connecting the solutions of systems (2.2), (2.3) and (3.12). Based on this, the stability properties of system (2.2) can be obtained by the corresponding stability properties of system (2.3).

Remark 3.5. In Theorem 3.3, if Conditions (1) and (2) are replaced with

(1) ${}^CD^\alpha_+ V(s, y(t; s, x)) \geqslant g(t, s, V(s, y(t; s, x)), \sigma_k(V(t_k, y(t; t_k, x(t_k)))))$,
(2) $V(s, y(t; s, x + I_k(x(s)))) \geqslant \psi_k(V(s, y(t; s, x)))$.

and $r(t; s, t_0, u_0)$ represents the minimal solution with initial value (t_0, u_0) of the scalar system (3.12) defined in the interval $[t_0, \infty)$, while other conditions remain unchanged, then, we can obtain that if $V(t_0, y(t; t_0, x_0)) \geq u_0$, then for $t_0 \leq s \leq t$, we have $V(s, y(t; s, x)) \geq r(t; s, t_0, u_0)$. Specifically, if $s = t$, then $V(t, x(t; t_0, x_0)) \geqslant \tilde{r}(t; t_0, u_0)$, where $\tilde{r}(t; t_0, u_0) = r(t; t, t_0, u_0)$. The proof of this case is similar to that of Theorem 3.3 by setting $m(t, s) = -V(s, y(t; s, x(s)))$ and using Lemma 3.1 under the condition that the inequalities are all $\geq$.

Corollary 3.6. *In Theorem 3.3, if for all k, $F(t, y) = 0$, then*

$$V(t, x(t; t_0, x_0)) \leqslant \tilde{r}(t; t_0, y(t; t_0, x_0))$$

with $V(t_0, x_0) \leq u_0$. In fact, in this case, $y(t_0; t_0, x_0) = x_0$, the definition of the Caputo fractional Dini derivative reduces to

$$ {}^CD^\alpha_+ V(s, x) = \lim_{h \to 0^+} \sup \frac{1}{h^\alpha}[V(s, x) - V(s - h, x - h^\alpha F(s, x))]. \tag{3.19}$$

Remark 3.7. Note that Corollary 3.6 is the Theorem 3.1 of [32] when we set $s = t$ in equation (3.19). So, the result of [32] can be regarded as a special case of Theorem 3.3.

For some special cases of the function g, the following corollaries are derived from Theorem 3.3. The proofs are straightforward, so we omit them here.

Corollary 3.8. *In Theorem 3.3 and due to Remark 3.5, in the case $g(t, s, u, \sigma_k(u_k)) = 0$, $u_0 = V(t_0, y(t; t_0, x_0))$ and for all k, $\psi_k(u_k) = u$, we have*

$$V(t, x(t; t_0, x_0)) \leqslant (\geqslant) V(t_0, y(t; t_0, x_0)).$$

If $V(t, x) = ||x||$, then

$$||x(t; t_0, x_0)|| \leqslant (\geqslant) ||y(t; t_0, x_0)||.$$

Corollary 3.9. *In Theorem 3.3, in the case $g(t, s, u, \sigma_k(u_k)) = \beta u$, $u_0 = V(t_0, y(t; t_0, x_0))$ and for all k, $\psi_k(u_k) = u$, we have*

$$V(t, x(t; t_0, x_0)) \leqslant V(t_0, y(t; t_0, x_0)) E_\alpha(\beta(t - t_0)^\alpha), \quad t \in [t_0, \infty).$$

Corollary 3.10. *Assume that assumption (H) holds. In Theorem 3.3, suppose that*

(1) ${}^CD^\alpha_+ V(s, y(t; s, x)) \leqslant -c(h_1(s, y(t; s, x)))\lambda(s, \sigma_k)$, *where $\lambda(t, \sigma_k) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is integrable and $c \in \mathcal{K}$, $h_1 \in \Gamma_0$, $s \neq t_k$;*
(2) $V(t_k^+, y(t; t_k^+, x(t_k^+))) \leqslant V(t_k, y(t; t_k, x_k))$, $k = 1, 2, \cdots$.

Then for $t \geq t_0$, we have

(1) *if* $k=0,\ t\in(t_0,\ t_1]$, *then*

$$V(t,x(t;t_0,x_0))\leqslant V(t_0,y(t;t_0,x_0))\\ -\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_0)d\tau,$$

(2) *if* $k=1,2,\cdots,\ t\in(t_k,t_{k+1}]$, *then*

$$V(t,x(t;t_0,x_0))\leq V(t_0,y(t;t_0,x_0))\\ -\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_{i-1})d\tau\\ -\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_k)d\tau.$$

Proof. Let $t_0<s\leq t_1,\ s\leq t$. Set

$$W(s,y(t;s,x))\\ =V(s,y(t;s,x))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{s}(s-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_0)d\tau\\ =V(s,y(t;s,x))+{}_{t_0}D_s^{-\alpha}c(h_1(s,y(t;s,x)))\lambda(s,\sigma_0).$$

By using Lemma 2.4, we get

$$^{C}D_{+}^{\alpha}W(s,y(t;s,x))\\ ={}^{C}D_{+}^{\alpha}V(s,y(t;s,x))+{}_{t_0}^{C}D_s^{\alpha}\,{}_{t_0}D_s^{-\alpha}c(h_1(s,y(t;s,x)))\lambda(s,\sigma_0)\\ \leqslant -c(h_1(s,y(t,s,x)))\lambda(s,\sigma_0)+c(h_1(s,y(t;s,x)))\lambda(s,\sigma_0)\\ \leqslant 0.$$

On the other hand,

$$W(t_0,y(t;t_0,x_0))=V(t_0,y(t;t_0,x_0))=u_0$$

and

$$W(t_1^+,y(t;t_1^+,x(t_1^+)))\\ =V(t_1^+,y(t;t_1^+,x(t_1^+)))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_0)d\tau\\ \leqslant V(t_1,y(t;t_1,x_1))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_0)d\tau\\ =W(t_1,y(t;t_1,x(t_1))).$$

By Corollary 3.6, we have

$$W(s,y(t;s,x))\leqslant W(t_0,y(t;t_0,x_0)),$$

which implies, by definition of W,

$$V(t,x(t;t_0,x_0))\leq V(t_0,y(t;t_0,x_0))\\ -\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_0)d\tau$$

with $s=t$.

If $s\in(t_1,\ t_2],\ s\leq t$, set

$$W(s,y(t;s,x))\\ =V(s,y(t;s,x))+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_0)d\tau\\ +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{s}(s-\tau)^{\alpha-1}c(h_1(\tau,y(t;\tau,x)))\lambda(\tau,\sigma_1)d\tau.$$

Similarly, we get

$$^{C}D_{+}^{\alpha}W(s, y(t; s, x)) \leqslant 0,$$

and

$$\begin{aligned}
&W\left(t_1^+, y\left(t; t_1^+, x\left(t_1^+\right)\right)\right) \\
&= V\left(t_1^+, y\left(t; t_1^+, x\left(t_1^+\right)\right)\right) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}c(h_1(\tau, y(t; \tau, x)))\lambda(\tau, \sigma_0)d\tau \\
&\leq V(t_1, y(t; t_1, x_1)) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}c(h_1(\tau, y(t; \tau, x)))\lambda(\tau, \sigma_0)d\tau \\
&= W(t_1, y(t; t_1, x(t_1))) \\
&\leq r(t; t_1, t_0, u_0) = u_1^+ = u_0 = W(t_0, y(t; t_0, x_0)),
\end{aligned}$$

where $r(t;\ t_1,\ t_0,\ u_0)$ is the maximal solution of system (3.12) with $g(t, s, u, \sigma_k(u_k)) = 0$ and initial condition $u_0 = V(t_0, y(t; t_0, x_0))$ in the interval $(t_1,\ t_2]$.

Also, we can prove that

$$W(t_2^+, y(t; t_2^+, x(t_2^+))) \leqslant W(t_2, y(t; t_2, x(t_2))).$$

By the similar proof of Theorem 3.3, based on Lemma 3.1, Corollary 3.8 and the definition of W, when $s = t$, we have

$$\begin{aligned}
V(t, x(t; t_0, x_0)) \leq{}& V(t_0, y(t; t_0, x_0)) \\
&- \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}c(h_1(\tau, y(t; \tau, x)))\lambda(\tau, \sigma_0)d\tau \\
&- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-\tau)^{\alpha-1}c(h_1(\tau, y(t; \tau, x)))\lambda(\tau, \sigma_1)d\tau.
\end{aligned}$$

If $t \in (t_k, t_{k+1}]$, by repeating the procedure above, the result can be obtained. □

4. Stability in terms of two measures

The concepts of stability in terms of two measures can unify many known stability notions such as Lyapunov stability, eventual stability, and partial stability in a single setup, and offer a general framework for investigation of stability theory. The significance of the study of the stability theory in terms of two measures is demonstrated in many studies [28,37–39]. The aim of this section is to discuss the stability properties in terms of two measures of impulsive hybrid systems with fractional order by applying the fractional variational comparison principle obtained in Section 3. Several stability and instability criteria for system (2.2) are derived.

Let $\rho > 0$ and $h \in \Gamma_0$. Define $S(h, \rho) = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t, x) < \rho\}$. Throughout this section, $h(t,\ x)$ and $V(t,\ x)$ stand for $h(t,\ x(t))$ and $V(t,\ x(t))$, respectively.

In the following theorems, we present the results of stability, uniform stability, and unstability properties in terms of two measures for system (2.2).

Theorem 4.1. *Assume that*

(1) h_0, $h \in \Gamma_0$ *and* h_0 *is finer than* h*;*
(2) $V \in \nu_0'$, $V(t,\ x)$ *and* $y(t;\ s,\ x)$ *are locally Lipschitzian in* x *for each* $(t,\ x)$*;*

$$^{C}D_{+}^{\alpha}V(s, y(t; s, x)) \leqslant 0,$$

where $s \neq t_k$*;*
(3) $V(s, y(t; s, x + I_k(x(s)))) \leqslant V(s, y(t; s, x))$, $s = t_k$, $k = 1, 2, \cdots$*;*
(4) $V(t,\ x)$ *is* h*-positive definite on* $S(h,\ \rho)$ *and weakly* h_0*-decrescent, where* $(t,\ x) \in S(h,\ \rho)$, $\rho > 0$*;*
(5) *There exists a* ρ_0, $0 < \rho_0 < \rho$ *such that* $h(t_k,\ x) < \rho_0$ *implies* $h(t_k^+, x + I_k(x)) < \rho$.

Then, the $(\tilde{h}_0, \tilde{h}_0)$*-stability of system (2.3) implies the corresponding* $(h_0,\ h)$*-stability of system (2.2).*

Proof. Let $0 < \varepsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$ be given, and $u_0 = V(t_0, y(t; t_0, x_0))$. Since h_0 is finer than h, there exist a $\delta_1 > 0$ and a function $\phi \in PC\mathcal{K}$ such that

$$h(t, x) \leqslant \phi(t, h_0(t, x)), \quad (t, x) \in S(h_0, \delta_1). \tag{4.1}$$

whenever $h_0(t,\ x) < \delta_1$ with $\phi(t,\ \delta_1) \leq \rho$ for $t \geq t_0$.

Since $V(t,\ x)$ is h-positive definite on $S(h,\ \rho)$, there exists a function $b \in \mathcal{K}$ such that

$$b(h(t, x)) \leqslant V(t, x), \quad (t, x) \in S(h, \rho). \tag{4.2}$$

Because $V(t, x)$ is weakly h_0-decrescent, there exist a $\delta_0 > 0$ and a function $a \in PC\mathcal{K}$ such that

$$V(t,x) \leqslant a(t,h_0(t,x)), \quad (t,x) \in S(h_0,\delta_0). \tag{4.3}$$

Due to the property of function a, choose $\eta = \eta(t_0, \varepsilon) < \min\{\rho, \delta_0, \delta_1\}$ such that for $t \geq t_0$,

$$a(t_0, h_0(t_0, y(t))) < b(\varepsilon), \tag{4.4}$$

whenever $h_0(t_0, y(t)) < \eta$.

Assume that system (2.3) is $(\tilde{h}_0, \tilde{h}_0)$-stable. Then, for the η chosen above, there exists a $\delta = \delta(t_0, \eta) > 0$ ($\delta < \eta$) such that $h_0(t_0, x_0) < \delta$ implies

$$h_0(t_0, y(t)) < \eta, \quad t \geqslant t_0, \tag{4.5}$$

where $y(t) = y(t; t_0, x_0)$ is any solution of system (2.3).

Suppose that $x(t) = x(t; t_0, x_0)$ is any solution of system (2.2) with $h_0(t_0, x_0) < \delta$. In views of (4.1)–(4.5) and the choice of δ, note that

$$b(h(t_0,x_0)) \leqslant V(t_0,x_0) \leqslant a(t_0,h_0(t_0,x_0)) < b(\varepsilon),$$

which implies $h(t_0, x_0) < \varepsilon$.

We claim that with this δ, system (2.2) is (h_0, h)-stable. That is

$$h(t,x) < \varepsilon, \quad t \geqslant t_0,$$

with $h_0(t_0, x_0) < \delta$.

If it does not hold, then there would exist a solution $x(t) = x(t; t_0, x_0)$ of (2.2) with $h_0(t_0, x_0) < \delta$ and a $t^* > t_0$ such that $t_k < t^* \leqslant t_{k+1}$ for some k, satisfying

$h(t^*, x(t^*)) \geq \varepsilon$ and $h(t, x(t)) < \varepsilon, \quad t \in [t_0, t_k]$.

Since $0 < \varepsilon < \rho_0$, by Condition (5), we have

$$h(t_k^+, x(t_k^+)) < \rho.$$

Hence, we can find a $t' \in (t_k, t^*)$ such that

$\varepsilon \leqslant h(t', x(t')) < \rho$ and $h(t, x(t)) < \rho, \quad t \in [t_0, t']$.

It follows from Condition (1), Condition (2) and Corollary 3.8, that

$$V(t,x(t)) \leqslant V(t_0,y(t)), \quad t \in [t_0,t'].$$

By using Condition (4), and (4.2)–(4.5), we have

$$b(\varepsilon) \leqslant b(h(t',x(t'))) \leqslant V(t',x(t')) \leqslant V(t_0,y(t';t_0,x_0)) \leqslant a(t_0,h_0(t_0,y(t'))) < b(\varepsilon),$$

which is a contradiction. Thus, system (2.2) is (h_0, h)-stable. The proof is completed. □

Theorem 4.2. *In Theorem 4.1, if Conditions (1) and (4) are replaced with*

(1) *h_0 is uniformly finer than h;*
(2) *$V(t, x)$ is h-positive definite on $S(h, \rho)$ and h_0-decrescent, where $\rho > 0$.*

Then, the $(\tilde{h}_0, \tilde{h}_0)$-uniform stability of system (2.3) implies (h_0, h)-uniform stability of system (2.2).

Proof. Since $V(t, x(t))$ is h_0-decrescent, then there exist a $\delta_1 > 0$ and a function $a \in \mathcal{K}$ such that $h_0(t, x) < \delta_1$ implies

$$V(t,x) \leqslant a(h_0(t,x)).$$

For any given $0 < \varepsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$, by the property of function a, choose $\eta > 0$ such that $a(\eta) < b(\varepsilon)$ and $\eta < \delta_1$.

Assume that system (2.3) is $(\tilde{h}_0, \tilde{h}_0)$-uniformly stable. Then there exists a $\delta = \delta(\eta) > 0$ such that for any given $t_0 \in \mathbb{R}_+$ and for $\eta > 0$ chosen above, $h_0(t_0, x_0) < \delta$ implies

$$h_0(t_0,y(t;t_0,x_0)) < \eta, \quad t \geqslant t_0,$$

where $y(t;\ t_0,\ x_0)$ is any solution of system (2.3).

Suppose that $x(t) = x(t; t_0, x_0)$ is any solution of system (2.2) with $h_0(t_0, x_0) < \delta$. By a similar proof of Theorem 4.1, we can obtain that $h(t, x(t)) < \varepsilon$, $t \geq t_0$, where δ is independent of t_0. This completes the proof of (h_0, h)-uniform stability of system (2.2). □

Theorem 4.3. *Let $h_0, h \in \Gamma_0$. Assume that*

(1) *$V \in \nu_0'$, $V(t, x)$ and $y(t; s, x)$ are locally Lipschitzian in x for each (t, x);*

$$^{C}D_+^{\alpha}V(s,y(t;s,x)) \geqslant 0,$$

where $s \neq t_k$;

(2) *for all* $k \in \mathbb{Z}^+$, $s = t_k$ *and* $(t_k, x) \in S(h, \rho)$,

$$V(s, y(t; s, x + I_k(x(s)))) \geqslant V(s, y(t; s, x));$$

(3) $V(t, x)$ *is* h*-decrescent on* $S(h, \rho)$ *and* h_0*-positive definite.*

Then, the $(\tilde{h}_0, \tilde{h}_0)$*-unstability of system (2.3) implies* (h_0, h)*-unstability of system (2.2).*

Proof. Suppose that system (2.3) is $(\tilde{h}_0, \tilde{h}_0)$-unstable. Then, there exists a $\varepsilon_0 > 0$ such that for any $\delta^* > 0$, $h_0(t_0, y(t; t_0, x_0)) \geq \varepsilon_0$ with $h_0(t_0, x_0) < \delta^*$. Thus, for some $t^* > t_0$, there exists a solution $y(t; t_0, x_0)$ of (2.3) with $h_0(t_0, x_0) < \delta^*$ such that $h_0(t_0, y(t^*; t_0, x_0)) = \varepsilon_0$.

We claim that system (2.2) is (h_0, h)-unstable. If it is not true, then for the above $\varepsilon_0 > 0$ and $\delta^* > 0$ such that $h_0(t_0, x_0) < \delta^*$ implies $h(t, x(t)) < a^{-1}(b(\varepsilon_0))$, $t \geq t_0$. From Condition (1), Condition (2) and Corollary 3.8, we get

$V(t, x) \geq V(t_0, y(t; t_0, x_0)), \quad t \geq t_0$.

Since $V(t, x(t))$ is h-decrescent on $S(h, \rho)$ and h_0-positive definite, there exist a $\delta_1 > 0$, $\delta_1 = \max\{\varepsilon_0, a^{-1}(b(\varepsilon_0))\} < \rho$ and functions $a, b \in \mathcal{K}$ such that

$$V(t, x(t)) \leqslant a(h(t, x)), \quad (t, x) \in S(h, \delta_1), \tag{4.6}$$

and

$$b(h_0(t, x)) \leqslant V(t, x(t)), \quad (t, x) \in S(h_0, \delta_1). \tag{4.7}$$

Thus, we have

$$\begin{aligned} b(\varepsilon_0) = a\big(a^{-1}(b(\varepsilon_0))\big) &> a(h(t^*, x(t^*))) \\ &\geq V(t^*, x(t^*)) \\ &\geq V(t_0, y(t^*; t_0, x_0)) \\ &\geq b(h_0(t_0, y(t^*; t_0, x_0))) = b(\varepsilon_0), \end{aligned}$$

which is a contradiction. Thus, system (2.2) is (h_0, h)-unstable. The proof is completed. □

The following theorem presents the relationship between the strict uniform stability of system (2.3) and the uniformly asymptotic stability of system (2.2). The modified concepts of strict stability [40] in terms of two measures promoted in [22] are applied to obtain the sufficient condition of asymptotic stability property.

Theorem 4.4. *Assume that*

(1) $h_0, h_1, h \in \Gamma_0$. h_0 *is uniformly finer than* h_1 *and* h; h_1 *is uniformly finer than* h;
(2) $V \in \nu_0'$ *and* $y(t; s, x)$ *are locally Lipschitzian in* x *for each* (t, x);
(3) $V(t, x)$ *is* h*-positive definite and* h_1*-decrescent;*
(4) ${}^{C}D_+^{\alpha}V(s, y(t; s, x)) \leqslant -c(h_1(s, y(t; s, x)))\lambda(s, \sigma_k)$, $s \neq t_k$, $t_0 \leq s \leq t$, $(s, x) \in S(h, \rho)$, *where* $c \in \mathcal{K}$, $\lambda(t, \sigma_k) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ *is integrable and satisfying:*

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{N} \int_{a_i}^{b_i} (b_i - t)^{\alpha-1} \lambda(t, \sigma_i) dt = \infty, \quad (N \to \infty);$$

(5) $V(s, y(t; s, x + I_k(x(s)))) \leqslant V(s, y(t; s, x))$, $s = t_k$, $k = 1, 2, \cdots$;
(6) *There exists a* $\rho_0 \in (0, \rho)$ *such that* $h(t_k, x) < \rho_0$ *implies* $h(t_k^+, x + I_k(x)) < \rho$.

Then, the $(\tilde{h}_0, \tilde{h}_1)$*-strict uniform stability of system (2.3) implies* (h_0, h)*-uniformly asymptotic stability of system (2.2).*

Proof. Since $V(t, x)$ is h-positive definite, there exist a $\rho_0 \in (0, \rho)$ and a function $b \in \mathcal{K}$ such that

$$b(h(t, x)) \leqslant V(t, x), \quad (t, x) \in S(h, \rho_0). \tag{4.8}$$

Also, since $V(t, x)$ is h_0-decrescent, there exist a $\delta^* > 0$ and a function $a \in \mathcal{K}$ such that

$$V(t, x) \leqslant a(h_1(t, x)), \quad (t, x) \in S(h_1, \delta^*). \tag{4.9}$$

From Theorem 4.2, it can be proven that system (2.2) is (h_0, h)-uniformly stable. That is, for any given $\varepsilon = \rho_0 > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\varepsilon) = \delta(\rho_0) > 0$ such that

$$h(t, x(t)) < \rho_0, \quad t \geqslant t_0, \tag{4.10}$$

with $h_0(t_0, x_0) < \delta$.

Suppose that system (2.3) is $(\tilde{h}_0, \tilde{h}_1)$-strictly uniformly stable. Then, in view of Definition 2.15, for any given $\varepsilon_1 > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that for all $t \geq t_0$,

$$h_0(t_0, y(t; t_0, x_0)) < \varepsilon_1 \tag{4.11}$$

with $h_0(t_0, x_0) < \delta_1$. And for every $\delta_2 \le \delta_1$, whenever $h_1(s, x(s)) \ge \delta_2$, there exists a $\varepsilon_2 \le \delta_2$ such that

$$h_1(s, y(t; s, x(s))) > \varepsilon_2, \qquad t_0 \leqslant s \leqslant t. \tag{4.12}$$

From Condition (1), there exists a $\phi_1 \in \mathcal{K}$ such that for all $t \ge t_0$

$$h_1(t_0, y(t; t_0, x_0)) \leqslant \phi_1(h_0(t_0, y(t; t_0, x_0))) < \phi_1(\varepsilon_1) < \delta^*, \tag{4.13}$$

with $h_0(t_0, y(t;\ t_0, x_0)) < \varepsilon_1$. Thus, by the property of function a, we obtain

$$a(h_1(t_0, y(t; t_0, x_0))) < a(\phi_1(\varepsilon_1)) < a(\delta^*). \tag{4.14}$$

Take $\delta' = \min\{\delta_1, \delta^*, \delta\}$. Due to the property of function $\lambda(t, \sigma_k)$, there exist $M > 0$ and $T = T(\varepsilon) > 0$, where γ is a constant, $0 < \gamma \leqslant a_i - t_i$ and $a_i < t_{i+1}$, such that

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{T} \int_{t_i}^{\alpha_i} (\alpha_i - t)^{\alpha-1} \lambda(t, \sigma_i) dt \geqslant M.$$

Next, we should prove that for δ' and $T = T(\varepsilon)$ above, system (2.2) is (h_0, h)-uniformly attractive. Choosing $\eta > 0$ and $\eta > \delta_2$, we assume that for any $t \in [t_0, t_0 + T]$, the inequality

$$h_1(t, x(t)) \geqslant \eta \tag{4.15}$$

holds whenever $h_0(t_0, x_0) < \delta'$.

Suppose that there exist $2T - 1$ impulsive points between t_0 and $t_0 + T$. Set $t_i = t_0 + \frac{i}{2}$, $i = 1, 2, \cdots, 2T - 1$. By Conditions (3)-(6) and Corollary 3.8, we have

$$\begin{aligned} V(t, x(t; t_0, x_0)) \le{}& V(t_0, y(t; t_0, x_0)) \\ & - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} c(h_1(\tau, y(t; \tau, x))) \lambda(\tau, \sigma_{i-1}) d\tau \\ & - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - \tau)^{\alpha-1} c(h_1(\tau, y(t; \tau, x))) \lambda(\tau, \sigma_k) d\tau. \end{aligned}$$

Setting $t = t_0 + T$, $M = \frac{a(\delta^*)+1}{c(\varepsilon_2)}$, from (4.9), (4.11)–(4.14), we have

$$\begin{aligned} V(t_0 + T, x(t_0 + T)) \le{}& V(t_0, y(t_0 + T; t_0, x_0)) \\ & - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{2T-1} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} c(h_1(\tau, y(t; \tau, x))) \lambda(\tau, \sigma_{i-1}) d\tau \\ & - \frac{1}{\Gamma(\alpha)} \int_{t_{2T-1}}^{t_0+T} (t_0 + T - \tau)^{\alpha-1} c(h_1(\tau, y(t; \tau, x))) \lambda(\tau, \sigma_{2T-1}) d\tau \\ \le{}& a(h_1(t_0, y(t_0 + T))) - \frac{c(\varepsilon_2)}{\Gamma(\alpha)} \sum_{i=1}^{2T-1} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} \lambda(\tau, \sigma_{i-1}) d\tau \\ & - \frac{c(\varepsilon_2)}{\Gamma(\alpha)} \int_{t_{2T-1}}^{t_0+T} (t_0 + T - \tau)^{\alpha-1} \lambda(\tau, \sigma_{2T-1}) d\tau \\ \le{}& a(h_1(t_0, y(t_0 + T))) - \frac{c(\varepsilon_2)}{\Gamma(\alpha)} \sum_{i=1}^{T} \int_{t_0+i-1}^{t_0+i-\frac{1}{2}} \left[\left(t_0 + i - \frac{1}{2} \right) - \tau \right]^{\alpha-1} \lambda(\tau, \sigma_i) d\tau \\ <{}& a(\delta^*) - c(\varepsilon_2) M \\ ={}& a(\delta^*) - c(\varepsilon_2) \frac{a(\delta^*) + 1}{c(\varepsilon_2)} < 0. \end{aligned}$$

However, from (4.8), we get $0 < b(h(t_0 + T, x(t_0 + T))) \leqslant V(t_0 + T, x(t_0 + T))$, which shows a contradiction.

Thus, in any case, we have $h_1(t, x(t)) < \eta$ for $t \geqslant t_0 + T$ whenever $h_0(t_0, x_0) < \delta'$, which implies that the system (2.2) is (h_0, h_1)-uniformly asymptotically stable.

Since h_1 is uniformly finer than h, there exist a function $\phi \in \mathcal{K}$ and a $\tilde{\varepsilon} > 0$ such that

$$h(t, x(t)) \leqslant \phi(h_1(t, x(t))) < \phi(\eta)$$

whenever $h_0(t_0, x_0) < \delta'$ with $\phi(\eta) < \tilde{\varepsilon}$ for $t \ge t_0$. This completes the proof. □

In the following theorem, we state a more general stability criteria combining the comparison system (3.12).

Theorem 4.5. *Assume that*

(1) h_0, $h \in \Gamma_0$; *h_0 is finer than h;*

(2) $V \in \nu_0'$ *and* $y(t;\, s,\, x)$ *are locally Lipschitzian in* x *for each* $(t,\, x)$,

$$^{C}D_{+}^{\alpha}V(s, y(t;s,x)) \leqslant g(t, s, V(s, y(t;s,x)), \sigma_k(V(t_k, y(t;t_k,x_k)))),$$

where $s \neq t_k$, $k = 1, 2, \cdots$, $g : \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ *is continuous in each of the sets* $(t_k, t_{k+1}]$ *in* s, $k = 0, 1, 2, \cdots$, *for each* $t \in \mathbb{R}_+$ *and for all* $k \in \mathbb{N}$

$$\lim_{(t,s,u)\to(t,t_k^+,w)} g(t, s, u, \sigma_k(u_k)) = g(t, t_k^+, w, \sigma_k(w_k))$$

and for any $(t,\, s,\, u,\, v)$, $g(t,\, s,\, u,\, v)$ *is nondecreasing in* v, $\sigma_k(u) \in C(\mathbb{R}_+, \mathbb{R})$ *and* $\sigma_k(u)$ *is nondecreasing in* u;

(3) $V(t,\, x)$ *is* h*-positive definite on* $S(h,\, \rho)$ *and weakly* h_0*-decrescent, where* $(t,\, x) \in S(h,\, \rho)$, $\rho > 0$;

(4) $V(s, y(t;s, x + I_k(x(s)))) \leqslant \psi_k(V(s, y(t;s,x)))$, $s = t_k$, $k = 1, 2, \cdots$, *where* $\psi_k(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $k = 1, 2, \cdots$ *is nondecreasing in* u;

(5) *There exists a* $\rho_0 \in (0,\, \rho)$ *such that* $h(t_k,\, x) < \rho$ *implies* $h(t_k^+, x + I_k(x)) < \rho$;

(6) *The system* (2.3) *is* $(\tilde{h}_0,\, \tilde{h}_0)$*-stable.*

Then, for the stability properties of the trivial solution of system (3.12), *we can get the corresponding stability properties of system* (2.2)

Proof. Suppose that the initial solution of (3.12) is stable. Let $0 < \varepsilon < \rho_0$ and $t_0 \in \mathbb{R}_+$ be given. Then given $b(\varepsilon) > 0$, there exists a $\delta^* = \delta^*(t_0, \varepsilon) > 0$ such that

$$|u(t;t_0,u_0)| < b(\varepsilon), \quad t \geqslant t_0,$$

whenever $|u_0| < \delta^*$, where $u(t;\, t_0,\, u_0)$ is any solution of (2.3) with initial value $(t_0,\, u_0)$ when $t = s$.

Let $V(t_0, y(t;t_0,x_0)) = |u_0|$. By using this $\delta^*(\delta^* < b(\varepsilon))$ in place of $b(\varepsilon)$ in proof of Theorem 4.1, we can find a $\delta = \delta(t_0, \varepsilon) > 0$ as before. Then, from Conditions (1) and (3), an argument similar to the proof of Theorem 4.1 shows that

$$b(h(t_0,x_0)) \leqslant V(t_0,x_0) \leqslant a(t_0, h_0(t_0,x_0)) < \delta^* < b(\varepsilon).$$

Thus, we get $h(t_0,\, x_0) < \varepsilon$.

We claim that $h(t,\, x(t)) < \varepsilon$, $t \geq t_0$, whenever $h_0(t_0,\, x_0) < \delta$. If it is not true, then there exist a solution $x(t) = x(t;t_0,x_0)$ of system (2.2) and a $t_1 > t_0$, $t_1 \in (t_k, t_{k+1}]$ for some k such that

$h(t_1,\, x(t_1)) \geq \varepsilon$ and $h(t,\, x(t)) < \varepsilon$, $\quad t \in [t_0,\, t_k]$.

Since $0 < \varepsilon < \rho_0$, by Condition (5), $h(t_k^+, x(t_k^+)) < \rho$, there exists a $\hat{t} > 0$, $\hat{t} \in (t_k, t_1)$ such that

$\varepsilon \leqslant h(\hat{t}, x(\hat{t})) < \rho$ and $h(t,\, x(t)) < \rho$, $\quad t \in [t_0, \hat{t}]$.

By Conditions (2)–(4) and Theorem 3.3, we have

$$V(t,x(t)) \leqslant r_0(t;t_0,u_0)), \quad t \in [t_0,\hat{t}],$$

where $r_0(t;\, t_0,\, u_0)$ is the maximum solution of (3.12) with initial value $(t_0,\, u_0)$ when $t = s$ and

$$|u_0| = V(t_0, y(t;t_0,x_0)) \leqslant a(t_0, h_0(t_0, y(t;t_0,x_0))) < \delta^*.$$

Since $V(t,\, x)$ is h-positive, then we obtain

$$b(\varepsilon) \leqslant b(h(\hat{t}, x(\hat{t}))) \leqslant V(\hat{t}, x(\hat{t})) \leqslant r_0(\hat{t};t_0,u_0) < b(\varepsilon),$$

which leads to a contradiction. Hence, system (2.2) is $(h_0,\, h)$-stable.

Next, we prove that the asymptotic stability of the trivial solution of system (3.12) implies $(h_0,\, h)$-asymptotic stability of system (2.2).

Suppose that the trivial solution of (3.12) is asymptotically stable, which implies stability and attractivity. By the proof above, system (2.2) is $(h_0,\, h)$-stable. Taking $\varepsilon = \rho$, $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(t_0, \varepsilon) > 0$ such that $h(t,\, x(t)) < \rho$, $t \geq t_0$ with $h_0(t_0,\, x_0) < \delta_0$. Corresponding to $b(\varepsilon)$, there exist $\delta_0^* = \delta_0^*(t_0, \varepsilon) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that $|u_0| \leqslant \delta_0^*$ implies that

$$u(t;t_0,u_0) < b(\varepsilon), \quad t \geqslant t_0 + T,$$

which also means that $u(t;\, t_0,\, u_0) \to 0$, $t \to \infty$ with $|u_0| \leqslant \delta_0^*$.

Taking $\delta_1 = \min\{\delta_0^*, \delta_0\}$, we have

$$h(t,x(t)) < \rho, \quad t \geqslant t_0$$

with $h_0(t_0,\, x_0) < \delta_1$.

We assert that $h(t,\, x(t)) < \varepsilon$, $t \geqslant t_0 + T$ whenever $h_0(t_0,\, x_0) < \delta_1 < \delta$. If it is not true, then there would exist a solution $x(t) = x(t;t_0,x_0)$ of (2.2) and a divergent sequence $\{t^{(m)}\}$, $t^{(m)} \to \infty (m \to \infty)$, $t^{(m)} \geqslant t_0 + T$ such that

$h(t^{(m)},\, x(t^{(m)})) \geq \varepsilon$ and $h(t,\, x(t)) < \varepsilon$, $\quad t \in [t_0,\, t^{(m)}]$.

Since $h_0(t_0,\, x_0) < \delta_1$ and system (2.2) is $(h_0,\, h)$-stable, we have

$\varepsilon \leq h(t^{(m)},\, x(t^{(m)})) < \rho$ and $h(t,\, x(t)) < \varepsilon$, $\quad t \in [t_0,\, t^{(m)}]$.

By Conditions (2)–(4) and Theorem 3.3, we get

$$V(t, x(t)) \le r_0(t; t_0, u_0), \quad t \in \left[t_0, t^{(m)}\right],$$

where $r_0(t;\, t_0,\, u_0)$ is the maximum solution of (3.12). By the similar proof as before, we have $|u_0| < \delta_0^*$. Thus,

$$r_0(t; t_0, u_0) \to 0, \quad t \to \infty.$$

Then

$$b(\varepsilon) \leqslant b(h(t^{(m)}, x(t^{(m)}))) \leqslant V(t^{(m)}, x(t^{(m)})) \leqslant r_0(t^{(m)}; t_0, u_0),$$

where $r_0(t^{(m)};\, t_0,\, u_0)$ approaches 0 with $m \to \infty$, which leads to a contradiction. Thus, system (2.2) is $(h_0,\, h)$ attractive. This completes the proof of $(h_0,\, h)$-asymptotic stability of system (2.2). □

Remark 4.6. Note that $a \in PC\mathcal{K}$, one can only get nonuniform $(h_0,\, h)$-stability properties for (2.2) even when we assume uniform stability properties of (3.12) as well as (2.3). If $a \in \mathcal{K}$, then uniform $(h_0,\, h)$-stability properties can be obtained whenever we assume the corresponding notion for (3.12) and (2.3).

Remark 4.7. Theorem 4.5 presents the relationship of stability properties among systems (2.2), (2.3) and (3.12).

5. An example

In this section, we discuss the following example to demonstrate our theoretical results obtained in Section 4.

Let $0 < \alpha < 1$. Considering the following scalar impulsive fractional hybrid equation

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = -ax(t) + b_k x(t_k), & t \neq t_k, \quad t \geq t_0, \\ \Delta x(t_k) = c_k x(t_k), & t = t_k, \quad k = 1, 2, \cdots, \\ x\left(t_0^+\right) = x_0, \end{cases} \tag{5.1}$$

and the fractional differential equation

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} y(t) = -ay(t), \\ y\left(t_0^+\right) = x_0, \end{cases} \tag{5.2}$$

where $a > 0$, $-1 < c_k \leqslant 0$, $b_k \in \mathbb{R}$, $k = 1, 2, \cdots$, are constants, $t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ are impulsive moments and $t_k \to \infty$, $k \to \infty$.

Denote by $x(t) = x(t; t_0, x_0)$ the solution to system (5.1) and $y(t) = y(t; t_0, x_0)$ the solution to system (5.2) respectively. It is straightforward to show from (5.2) that

$$y(t; t_0, x_0) = E_{\alpha,1}(-a(t - t_0)^{\alpha}) x_0 \tag{5.3}$$

and

$$y(t; s, x(s)) = E_{\alpha,1}(-a(t - s)^{\alpha}) x(s), \tag{5.4}$$

where $E_{\alpha,\, \beta}(z)$ represents the Mittag-Leffler function with two parameters. For the specific formulation and properties of the Mittag-Leffler function, we refer the reader to the references [4,34]. An easy induction gives that (5.2) is $(\tilde{h}_0, \tilde{h}_0)$-stable if $a > 0$ for any $t_0 \in \mathbb{R}_+$ and $t \geq t_0$.

Let $V(x(t)) = |x(t)|$ and $h_0(t, x) = h(t, x) = |x(t)|$ for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Obviously, h_0 is uniformly finer than h when choosing the function ϕ as $\phi(u) = |u|^p$, where p is a constant satisfying $p > 1$ and $u \in \mathbb{R}_+$. It is also obvious that V is h-positive and h_0-decrescent. Thus, Conditions (1) and (4) are satisfied in Theorem 4.1. For a given ρ_0, $0 < \rho_0 < \rho$, $h(t_k,\, x(t_k)) < \rho$ implies $h(t_k^+, x(t_k) + c_k x(t_k)) < \rho$ by $-1 < c_k \leqslant 0$, which shows that Condition (5) holds. By a direct calculation and the proof of Corollary 4.1 in [41], for $t \geq t_0$ and $s \neq t_k$, we get

$$\begin{aligned} {}^{C}D_+^{\alpha} V(s, y(t; s, x)) &= {}^{C}D_+^{\alpha} \left| E_{\alpha,1}\left(-a(t - s)^{\alpha}\right) x(s) \right| \\ &= {}^{C}D_+^{\alpha} |y(t; s, x(s))| \\ &= \operatorname{sgn}(y(t; s, x(s)))^{C} D_+^{\alpha} y(t; s, x(s)) \\ &= \operatorname{sgn}(y(t; s, x(s)))_{t_0}^{C} D_t^{\alpha} y(t; s, x(s)) \\ &= \operatorname{sgn}(y(t; s, x(s)))(-ay(t; s, x(s))) \\ &= -a|y(t; s, x(s))| \leq 0. \end{aligned}$$

On the other hand, when $s = t_k$, we have

$$\begin{aligned} V\left(t_k^+, y\left(t; t_k^+, x\left(t_k^+\right)\right)\right) &= |E_{\alpha,1}\left(-a\left(t-t_k^+\right)^{\alpha}\right)x\left(t_k^+\right)| \\ &= |E_{\alpha,1}\left(-a\left(t-t_k^+\right)^{\alpha}\right)(1+c_k)x(t_k)| \\ &\le |E_{\alpha,1}\left(-a\left(t-t_k^+\right)^{\alpha}\right)x(t_k)| \\ &\le |E_{\alpha,1}\left(-a(t-t_k)^{\alpha}\right)x(t_k)| \\ &= V(t_k, y(t; t_k, x(t_k))). \end{aligned}$$

Thus, Conditions (2) and (3) hold.

By the previous discussions, all conditions of Theorem 4.1 are satisfied. Consequently, if $a > 0$ for any $t_0 \in \mathbb{R}_+$ and $t \ge t_0$, then it follows from Theorem 4.1 that (5.1) is (h_0, h)-stable.

6. Conclusions

This paper has employed the promoted variational Lyapunov method to analyze the stability properties for a class of impulsive hybrid systems with Caputo's fractional order $0 < \alpha < 1$. An extended fractional variational comparison principle was established by using differential inequalities and the Lyapunov-like function in Caputo's sense. Based on the obtained comparison theorem, some stability conditions in terms of two different measures for IFHSs was presented, which generalizes the corresponding stability theory. An example was given to illustrate the validity of the theoretical results.

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