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Constructions of ϵ-mono-components and mathematical analysis on signal decomposition algorithm

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A B S T R A C T

The concept of mono-component is widely used in non-stationary signal processing and time-frequency analysis. A special class of mono-components, called ϵ-mono-components, were proposed in our recent publication. It was illustrated that this model coincides with the intuition of human beings on the physical mono-components very well provided that the parameter ϵ is sufficiently small. It is then very meaningful to construct desired ϵ-mono-components and design algorithms to decompose and represent non-stationary signals adaptively. This paper studies the constructions of ϵ-mono-components and makes mathematical analysis on an adaptive signal decomposition algorithm based on ϵ-mono-components.

1. Introduction

The Fourier transform is a powerful tool for stationary signal processing. By taking the Fourier transform, a signal is decomposed into a band of sinusoidal waves of different frequencies unchanged over time. However, for the non-stationary signals, the frequencies of the signals do vary over time. To study the instantaneous oscillating behavior of a non-stationary signal, the concept of instantaneous frequency (IF) was introduced and plays a key role in non-stationary signal processing and time-frequency analysis. In 1946, Gabor proposed the definition of IF by means of the Hilbert transform [6]. Suppose $x(t)$ is a real-valued 2π-periodic function. To define its IF, one first constructs the analytic signal $s(t) := x(t) + iHx(t)$ in which H is the Hilbert transform defined by the Cauchy principal value of the singular integral

$$Hx(t) := \text{p.v.}\frac{1}{2\pi}\int_0^{2\pi} x(t-s)\cot\frac{s}{2}\,ds. \tag{1.1}$$

Rewriting the analytic signal in the polar form $s(t) = \rho(t)e^{i\theta(t)}$, we call $\rho(t)$ and $\theta(t)$ the analytic amplitude and phase of $x(t)$ (or $s(t)$) respectively. The phase derivative $\theta'(t)$, if exists, is consequently called the IF of $x(t)$ (or $s(t)$).

As discussed in our recent publication [12], the above concept of IF makes sense for physical monocomponent signals only. That is, the signal should contain only one frequency or a narrow range of frequencies at any time [1,2,4,8,13,16]. It is a fundamental question and remains open in the past several decades to establish a rigorous mathematical model for mono-component signals. In 1998, an empirical and practical model for monocomponent signals was proposed, which is called the

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intrinsic mode function (IMF) [9,14,30]. The basic requirement of an IMF is that the upper and lower envelopes defined by the cubic spline interpolation are symmetrical with respect to the time axis. But strictly speaking, this symmetrical condition is nearly impossible to be met since the upper and lower envelopes are obtained through different interpolation points respectively. Moreover, the upper/lower envelopes cannot be regarded as the amplitudes of a signal due to the unavoidable undershoots [9,31] caused by the spline interpolation. In addition, it has been shown that an IMF may contain negative IF [25].

As the advent of the EMD (empirical mode decomposition) at the end of the last century [9,14], the study on mono-component signals has become a hot spot of research. Based on the basic assumption that a monocomponent signal has nonnegative IF, Qian et al. introduced the following concept and termed as 'mono-component' in [17–19,22].

Definition 1. Let $s(t) = \rho(t)e^{i\theta(t)}$ be an analytic signal, where $\rho(t)$ and $\theta(t)$ are respectively the analytic amplitude and phase. If $\theta(t)$ is differentiable and $\theta'(t) \geq 0$, then $s(t)$ (or: its real part $x(t) = \rho(t)\cos\theta(t)$) is called a mono-component and denoted by $s \in \mathcal{M}$ (or: $x \in \mathcal{M}$).

The condition $\theta'(t) \geq 0$ in Definition 1 is presented to guarantee the nonnegativity of the IF since the frequency, which is a quantity to represent the velocity of oscillation of a signal, is nonnegative from the physical point of view. It has been shown that an analytic signal is the nontangential boundary value of a holomorphic function in the Hardy space [7,17]. Therefore, mono-components can be studied through the classical theory of the Hardy space. Conditions for $\rho(t)e^{i\theta(t)}$ to be an analytic signal or a mono-component were studied deeply in [10,17,19,26,29,32]. Several important families of mono-components with nonlinear phases were constructed based on the Blaschke products and the factorization theorem for functions in the Hardy space [17–19,22,28].

The amplitude of a mono-component may not always coincide with its physical amplitude which is the maximum displacement or distance moved by a point on a vibrating body or wave measured from its equilibrium position [5,12]. By analyzing and observing some typical mono-components, a special class of mono-components, called ϵ-mono-components, were presented in our recent publication [12], in which the parameter ϵ is employed to measure the consistency between the analytic amplitude and the physical one. Before the definition of ϵ-mono-components, we introduce two function spaces as follows [12]:

$$C_{\rightarrow}(\mathbb{R}) := \{\phi \in C(\mathbb{R}) : \phi \text{ is increasing and } \lim_{t\to\pm\infty} \phi(t) = \pm\infty\},$$
$$C_{\uparrow}(\mathbb{R}) := \{\theta \in C(\mathbb{R}) : \theta \text{ is strictly increasing and } \theta(t+2\pi) - \theta(t) \in 2\pi\mathbb{Z},\ \forall t \in \mathbb{R}\},$$

where $\mathbb{R}$ denotes the set of all the real numbers and $\mathbb{Z}$ the set of all the integers, and $C(\mathbb{R})$ denotes the set of all the continuous functions on $\mathbb{R}$. It holds obviously that $C_{\uparrow}(\mathbb{R}) \subset C_{\rightarrow}(\mathbb{R})$. Below is the definition of ϵ-mono-components [12]:

Definition 2. Given $\epsilon > 0$, a signal $s(t) = \rho(t)e^{i\theta(t)} \in \mathcal{M}$ is said to be an ϵ-mono-component and denoted by $s \in \mathcal{M}_\epsilon$ if $\theta \in C_{\uparrow}(\mathbb{R}),\ \rho(t) \geq 0,\ \forall t \in \mathbb{R}$, and $\rho(t)$ can be expressed as

$$\rho(t) = \lambda_0 + \sum_{j=1}^{N} \lambda_j \cos\phi_j(t)$$

for some $\lambda_0, \lambda_j \in \mathbb{R},\ \phi_j \in C_{\rightarrow}(\mathbb{R}),\ j = 1, \ldots, N$, satisfying

$$\phi_j'(t) \leq \epsilon\theta'(t), \qquad \text{a. e.}\, t \in \mathbb{R}, \qquad j = 1, 2, \ldots, N. \tag{1.2}$$

The condition $s(t) = \rho(t)e^{i\theta(t)} \in \mathcal{M}$ in Definition 2 guarantees $\rho(t)$ and $\theta(t)$ are respectively the analytic amplitude and phase of the signal, and the IF of the signal has nonnegative values. The condition (1.2) guarantees that the amplitude oscillates much more slowly than the phase part at any time provided that the parameter ϵ is sufficiently small.

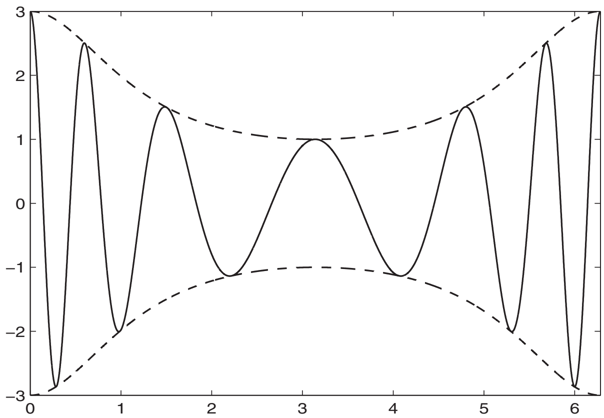
In this paper, we first study the constructions of ϵ-mono-components. Then, two mathematical issues, the convergence property and the solution of the minimization problem of the adaptive signal decomposition algorithm presented in [12], are studied. The rest of this paper is organized as follows. The constructions of ϵ-mono-components for a special class of phases and for a given amplitude are studied respectively in Sections 2 and 3. In Section 4, we make mathematical analysis on the adaptive signal decomposition algorithm presented in [12], numerical experiments are also given in this section. Discussion of ϵ-mono-components on the real line is given in Section 5. Finally, Section 6 is the conclusion of the paper.

2. Construction of ϵ-mono-components for a special class of phases

There is a class of fundamental unimodular mono-components called nonlinear Fourier atoms [11,17,18]. A nonlinear Fourier atom $e^{i\theta_a(t)}$ is defined through the boundary value of the Möbius transform, that is,

$$e^{i\theta_a(t)} := \tau_a(e^{it}) = \frac{e^{it} - a}{1 - \bar{a}e^{it}}, \quad t \in [0, 2\pi), \quad 0 \leq |a| < 1. \tag{2.1}$$

In the following we construct ϵ-mono-component $\rho(t)e^{i\theta(t)}$ with the phase $\theta(t) = n\theta_a(t)$.



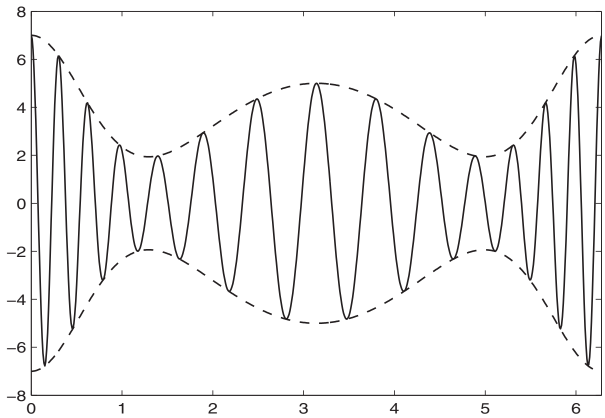


Fig. 1. The plots of the real part of the signal $s(t)$ (solid line) and its analytic amplitude (dashed line). Left: $s(t) = [2 + \cos\theta_{0.3}(t)]e^{i6\theta_{0.3}(t)} \in \mathcal{M}_{1/6}$; right: $s(t) = [4 + \cos(\theta_{0.2}(t)) + 2\cos(2\theta_{0.2}(t))]e^{i14\theta_{0.2}(t)} \in \mathcal{M}_{1/7}$.

Lemma 2.1. *Let $\rho(t) > 0$, $a \in [0, 1)$, and $\theta_a(t)$ be defined by (2.1). Then $\rho(t)e^{in\theta_a(t)}$ is an analytic signal if and only if $\rho(t)$ has the form*

$$\rho(t) = \lambda_0 + \sum_{k=1}^{n} \lambda_k \cos[k\theta_a(t) + \phi_k], \tag{2.2}$$

where $\lambda_0, \ldots, \lambda_n$ and $\phi_1, \ldots, \phi_n$ are real constants.

Proof. Firstly, suppose $\rho(t)$ has the form (2.2). We note that $e^{i\theta_a(t)}$ is the boundary value of the Möbius transform $\tau_a(z)$, then $\cos[k\theta_a(t)]e^{in\theta_a(t)}$ is the boundary value of $[\tau_a^{n+k}(z) + \tau_a^{n-k}(z)]/2$, which implies that $\cos[k\theta_a(t)]e^{in\theta_a(t)}$ is an analytic signal. Similarly, $\sin[k\theta_a(t)]e^{in\theta_a(t)}$ is also an analytic signal. Finally, the linear combination $\rho(t)e^{in\theta_a(t)}$ is an analytic signal.

Conversely, suppose $\rho(t)e^{in\theta_a(t)}$ is an analytic signal, then by Qian et al. [22], $\rho(t)$ is the real part of the boundary value of a holomorphic function with the form

$$\frac{\sum_{k=1}^{n} c_k z^k}{(1 - az)^n} + c_0, \tag{2.3}$$

where $c_0, \ldots, c_n$ are complex constants. It's obvious that (2.3) can be expressed as a linear combination of functions

$$1, \ \frac{1}{1 - az}, \ \frac{1}{(1 - az)^2}, \ \ldots, \ \frac{1}{(1 - az)^n},$$

and consequently, by Qian [20] and Qian and Wang [23], can be further expressed as a linear combination of functions

$$1, \ \frac{z - a}{1 - az}, \ \left(\frac{z - a}{1 - az}\right)^2, \ \ldots, \ \left(\frac{z - a}{1 - az}\right)^n.$$

Therefore, there exist complex constants $d_0, \ldots, d_n$ such that

$$\rho(t) = \sum_{k=0}^{n} \left[d_k \left(\frac{e^{it} - a}{1 - ae^{it}}\right)^k + \overline{d_k \left(\frac{e^{it} - a}{1 - ae^{it}}\right)^k} \right],$$

which can be further rewritten as (2.2). The proof is completed. □

Based on Lemma 2.1, we immediately have the following theorem (Ref. [12, Theorem III.1]).

Theorem 2.2. *Let $\rho(t) > 0$, $a \in [0, 1)$, and $\theta_a(t)$ be defined by (2.1). Then $\rho(t)e^{in\theta_a(t)}$ is an ϵ-mono-component if and only if $\rho(t)$ has the form*

$$\rho(t) = \lambda_0 + \sum_{k=1}^{m} \lambda_k \cos[k\theta_a(t) + \phi_k],$$

where $m \le n\epsilon$, $\lambda_0, \ldots, \lambda_m$ and $\phi_1, \ldots, \phi_m$ are real constants.

Fig. 1 shows two examples of ϵ-mono-components with $\epsilon = 1/6$ and $1/7$.

3. Construction of ϵ-mono-components for a given amplitude

An interesting question is: given an amplitude $\rho(t)$ and a constant $\epsilon > 0$, is there a phase $\theta(t)$ such that $\rho(t)e^{i\theta(t)}$ is an ϵ-mono-component? This section attempts to answer this question.

As preliminary, we introduce some notations. Let $L^\infty(\mathbb{R})$ stand for the space of all the bounded functions defined on $\mathbb{R}$ endowed with the norm $\|f\|_{L^\infty(\mathbb{R})} = \text{ess sup}_{t\in\mathbb{R}}|f(t)|$. Set $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, the quotient set of $\mathbb{R}$ divided by $2\pi\mathbb{Z}$. For any positive integer k, let $C^k(\mathbb{T})$ be the set of all the k-times continuously differentiable functions on $\mathbb{T}$, and $C(\mathbb{T})$ the set of all the continuous functions on $\mathbb{T}$.

The following lemma shows that any positive and twice continuously differentiable function can be expressed as the form of the amplitude in Definition 2.

Lemma 3.1. *Let $\rho \in C^2(\mathbb{T})$ be not a constant function, $t_0 \in [0, 2\pi)$ satisfy $\rho'(t_0) = 0$ and $\rho''(t_0) \neq 0$. Then there exists $\phi \in C_\rightarrow(\mathbb{R})$ satisfying $\phi' \in L^\infty(\mathbb{R})$ such that*

$$\rho(t) = \rho(t_0) + K[\cos\phi(t) + \cos\phi(t-2\pi)], \quad \forall t \in \mathbb{R}, \tag{3.1}$$

where $K := \frac{1}{4}\int_{\mathbb{T}}|\rho'(x)|dx > 0$.

Proof. For any $\epsilon > 0$, it is easy to verify that

$$f_i(t) := 1 - \frac{1}{2K}\left(\int_{t_0}^{t}|\rho'(x)|dx - (-1)^{i+1}[\rho(t) - \rho(t_0)]\right), \quad \forall t \in [t_0, t_0 + 2\pi], \quad i = 1, 2,$$

where $K := \frac{1}{4}\int_{\mathbb{T}}|\rho'(x)|dx > 0$, are decreasing functions, ranging from 1 to -1 as t goes from t_0 to $t_0 + 2\pi$, and satisfy

$$f_i'(t) = -\frac{1}{2K}[|\rho'(t)| - (-1)^{i+1}\rho'(t)] \le 0, \quad i = 1, 2,$$

and

$$\rho(t) = \rho(t_0) + K[f_1(t) - f_2(t)], \quad \forall t \in [t_0, t_0 + 2\pi].$$

We define

$$\phi_i(t) := \arccos(f_i(t)), \quad \forall t \in [t_0, t_0 + 2\pi], \qquad i = 1, 2,$$

then

$$\rho(t) = \rho(t_0) + K[\cos\phi_1(t) - \cos\phi_2(t)], \quad \forall t \in [t_0, t_0 + 2\pi].$$

It is easy to see that ϕ_1, ϕ_2 are two increasing functions ranging from 0 to π as t goes from t_0 to $t_0 + 2\pi$.

Then, we prove that ϕ_1', ϕ_2' are bounded on $[t_0, t_0 + 2\pi]$. If there exists $\delta > 0$ such that $\rho'(t) < 0$, $t \in (t_0, t_0 + \delta)$, we have

$$f_1'(t) = -\frac{1}{2K}[|\rho'(t)| - \rho'(t)] = \frac{1}{K}\rho'(t) < 0, \quad t \in (t_0, t_0 + \delta),$$

so there holds

$$\lim_{t\to t_0^+}[\phi_1'(t)]^2 = \lim_{t\to t_0^+}\frac{[f_1'(t)]^2}{1 - f_1^2(t)} = \lim_{t\to t_0^+}\frac{2f_1'(t)f_1''(t)}{-2f_1(t)f_1'(t)} = -f_1''(t_0 + 0) = -\frac{1}{K}\rho''(t_0);$$

Otherwise, there exists $\delta > 0$ such that $\rho'(t) > 0$, $t \in (t_0, t_0 + \delta)$, then $f_1(t) = 1$ and $\phi_1(t) = 0$, $t \in (t_0, t_0 + \delta)$, which implies $\lim_{t\to t_0^+}\phi_1'(t) = 0$. Consequently, $\lim_{t\to t_0^+}|\phi_1'(t)| < +\infty$. Similarly, we have $\lim_{t\to(t_0+2\pi)^-}|\phi_1'(t)| < +\infty$. Therefore, ϕ_1' is bounded on $[t_0, t_0 + 2\pi]$. Similarly, we can prove ϕ_2' is bounded on $[t_0, t_0 + 2\pi]$.

Finally, we construct $\phi \in C_\rightarrow(\mathbb{R})$ satisfying $\phi' \in L^\infty(\mathbb{R})$ such that (3.1) holds. Let

$$\phi(2k\pi + t) := k\pi + \begin{cases} \phi_1(t), & \text{if } k \text{ is even,} \\ \phi_2(t), & \text{if } k \text{ is odd,} \end{cases} \quad \forall t \in [t_0, t_0 + 2\pi), \ k \in \mathbb{Z}, \quad i = 1, 2.$$

It can be verified that ϕ is an increasing and continuous function defined on $\mathbb{R}$, and satisfies

$$\phi(4k\pi + t) = 2k\pi + \phi(t) \ \ \forall t \in \mathbb{R} \quad \text{and} \quad \lim_{t\to\pm\infty}\phi(t) = \pm\infty,$$

which imply that $\phi \in C_\rightarrow(\mathbb{R})$ and $\phi' \in L^\infty(\mathbb{R})$.

For any even k and $t \in [t_0, t_0 + 2\pi)$ we have that

$$\begin{aligned}\cos\phi(2k\pi + t) + \cos\phi(2k\pi + t - 2\pi) &= \cos(k\pi + \phi_1(t)) + \cos((k-1)\pi + \phi_2(t)) \\ &= \cos\phi_1(t) - \cos\phi_2(t).\end{aligned}$$

Similarly, the above equality can be shown to hold for odd k. Thus, $\cos\phi(t) + \cos\phi(t - 2\pi)$ is a 2π-periodic function on $\mathbb{R}$ and (3.1) holds for $t \in [t_0, t_0 + 2\pi)$. By the periodicity of $\rho(t)$ and $\cos\phi(t) + \cos\phi(t - 2\pi)$ we conclude that (3.1) holds for all $t \in \mathbb{R}$. □

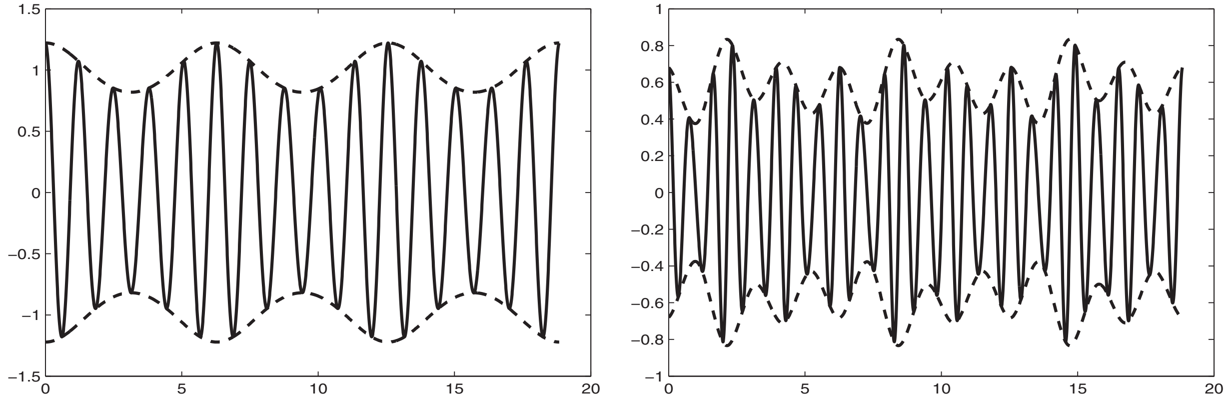


Fig. 2. The plots of the signal $x(t)=\rho(t)\cos\theta(t)$ (solid line) and its analytic amplitude (dotted line). Left: $\rho(t)=e^{0.2\cos t}$, $\theta(t)=5t+H(\ln\rho)(t)$; right: $\rho(t)$ is the cubic spline curve of a set of random data, $\theta(t)=8t+H(\ln\rho)(t)$.

Based on the above lemma, we prove the following main theorem of this section.

Theorem 3.2. *Let $\rho\in C^2(\mathbb{T})$, $\rho(t)>0$, then for any $\epsilon>0$ there exists a phase $\theta(t)$ such that $\rho(t)e^{i\theta(t)}\in\mathcal{M}_\epsilon$.*

Proof. Since $\ln\rho\in C^2(\mathbb{T})$, by Pandey [15, p. 95] there holds

$$[H(\ln\rho)]'=H[(\ln\rho)']=H(\rho'/\rho)\in C(\mathbb{T}).$$

By Lemma 3.1, there exists $\phi\in C_{\rightarrow}(\mathbb{R})$ satisfying $\phi'\in L^\infty(\mathbb{R})$ such that

$$\rho(t)=\rho(t_0)+K[\cos\phi(t)+\cos\phi(t-2\pi)],\quad \forall t\in\mathbb{R},$$

where $t_0\in[0,2\pi)$ and $K:=\frac{1}{4}\int_{\mathbb{T}}|\rho'(x)|dx$. Choose a positive integer M such that

$$M\geq\|(H(\ln\rho))'\|_{L^\infty(\mathbb{R})}+\epsilon^{-1}\|\phi'\|_{L^\infty(\mathbb{R})}$$

and let $\theta(t):=Mt+H(\ln\rho)(t)$. It is easy to see that $0\leq\phi'(t),\ \phi'(t-2\pi)\leq\epsilon\,\theta'(t)$ a. e.$t\in\mathbb{R}$. On the other hand, by Tan et al. [27, Theorem 2.1] we know that $\rho(t)\exp\{iH(\ln\rho)(t)\}$ is an analytic signal, which implies that $\rho(t)e^{i\theta(t)}$ is also an analytic signal. So $\rho(t)e^{i\theta(t)}$ is an ϵ-mono-component. The proof ends. □

Fig. 2 shows two examples of ϵ-mono-components which are constructed from two given amplitudes, respectively.

4. Mathematical analysis on the signal decomposition algorithm based on ϵ-mono-components

This section investigates a signal decomposition algorithm based on ϵ-mono-components. To represent signals with ϵ-mono-components effectively, the dictionary used as the basic wave atoms for decomposition should contain as many ϵ-mono-components as possible. Meantime, too many wave atoms may cause a heavy computational burden. The set of all the IMFs can be regarded as the dictionary for EMD since signals can be decomposed into IMFs with EMD [9,14]. As an important theoretic contribution in this field, several types of AFD (adaptive Fourier decomposition) were proposed recently, which employ the functions in the Takenaka-Malmquist system as the dictionary [20,21,23,24]. Different frame or dictionary provides different decomposition, which characterizes the signal from different aspect. In this section, we consider the signal decomposition with the dictionary of the ϵ-mono-components containing time-dependent amplitudes but constant IFs. Let $L^2(\mathbb{T})$ stand for the space of all the square integrable functions defined on $\mathbb{T}$ endowed with the following norm

$$\|f\|_{L^2(\mathbb{T})}:=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2dt\right)^{1/2}. \tag{4.1}$$

For simplicity, we use the L^2-norm in the following.

4.1. Signal decomposition algorithm and its convergence

In our algorithm, we consider the signal decomposition with the dictionary of the ϵ-mono-components containing time-dependent amplitudes but constant IFs. Let M be a positive integer and $\epsilon\in(0,1)$. Let the dictionary $\mathcal{D}$ consist of atoms

$$f_m(t):=\begin{cases}\left(L_m+\sum_{k=1}^{[m\epsilon]}(a_k\cos(kt)+b_k\sin(kt))\right)\cos(mt+\phi_m), & 0\leq m\leq M,\\ L_m\cos(mt+\phi_m), & M+1\leq m\leq M+[M\epsilon],\end{cases} \tag{4.2}$$

where $[m\epsilon]$ stands for the largest integer not exceeding $m\epsilon$.

Using this dictionary, we introduce the following signal decomposition algorithm based on ϵ-mono-components [12].

Algorithm 1. Given $f \in L^2(\mathbb{T})$, let M, N be positive integers and ϵ, $\delta \in (0, 1)$.

Step 1: Let $r = f$, $n = 1$.

Step 2: If $\|r\|/\|f\| < \delta$ or $n > N$, go to Step 3; else, solve the minimization problem

$$\tilde{x} = \text{argmin}_{x \in \mathcal{D}} \|r - x\|, \tag{4.3}$$

and let $r = r - \tilde{x}$, $x_n = \tilde{x}$, $n = n + 1$. Go back to Step 2.

Step 3: Stop the program and the decomposition of f is $f \approx \Sigma_n x_n$ and each item x_n is an ϵ-mono-component.

Remark. Algorithm 1 decomposes the given signal f adaptively. At each step, it selects the best approximation element $\tilde{x}$ from the dictionary $\mathcal{D}$ by solving the minimization problem (4.3). The component $\tilde{x}$ depends on f adaptively. Generally, the decomposition may not be unique since (4.3) may have more than one solution. However, if, for instance, the algorithm is arranged to select the $\tilde{x}$ that comes up first in the program, then the decomposition is unique.

In the following, we study the convergence property of Algorithm 1. Suppose $f(t)$ has the Fourier expansion

$$f(t) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k \cos(kt) + d_k \sin(kt)).$$

Let $f_1(t) = c_0/2 + \sum_{k=1}^{M+[M\epsilon]}(c_k \cos(kt) + d_k \sin(kt))$, which represents the finite low-frequency components of $f(t)$. There holds $\|f\|^2 = \|f_1\|^2 + \|f - f_1\|^2$. We denote by x_n the solution of (4.3) in the nth step of Algorithm 1 and use r_n to express the residue $r_n = f - \sum_{k=1}^{n} x_k$, there holds

$$\|r_{n+1}\| = \min_{x \in \mathcal{D}} \|r_n - x\|. \tag{4.4}$$

Then, we have the following convergence theorem for Algorithm 1.

Theorem 4.1. $\lim_{n\to\infty} r_n = f - f_1$.

Proof. We denote $\tilde{r}_n = f_1 - \sum_{k=1}^{n} x_k$, then $\tilde{r}_n$ contains only finite low-frequency components. There holds $r_n = f - \sum_{k=1}^{n} x_k = \tilde{r}_n + (f - f_1)$, which implies that $\tilde{r}_n$ represents the finite low-frequency components of r_n, and $f - f_1$ represents the infinite high-frequency components of r_n. In the following we just need to prove $\lim_{n\to\infty} \tilde{r}_n = 0$. It's easy to verify that (4.4) is equivalent to

$$\|\tilde{r}_{n+1}\| = \min_{x \in \mathcal{D}} \|\tilde{r}_n - x\|. \tag{4.5}$$

By (4.5) and the fact that $0 \in \mathcal{D}$, we have $\|\tilde{r}_{n+1}\| \le \|\tilde{r}_n\|$, which means $\{\|\tilde{r}_n\|\}_{n=1}^{\infty}$ is a decreasing sequence with a low bound 0, so $\lim_{n\to\infty} \|\tilde{r}_n\| = \tilde{r}$ exists.

Suppose $\tilde{r}_n$ has the Fourier expansion $\tilde{r}_n(t) = c_0^{(n)}/2 + \sum_{k=1}^{M+[M\epsilon]}(c_k^{(n)} \cos(kt) + d_k^{(n)} \sin(kt))$. Since $c_0^{(n)}/2 \in \mathcal{D}$, $c_k^{(n)} \cos(kt) + d_k^{(n)} \sin(kt) \in \mathcal{D}$, $1 \le k \le M + [M\epsilon]$, by (4.5) we have

$$\begin{aligned}
&\|\tilde{r}_{n+1}\|^2 \le \|\tilde{r}_n - c_0^{(n)}/2\|^2 = \|\tilde{r}_n\|^2 - |c_0^{(n)}|^2/4, \\
&\|\tilde{r}_{n+1}\|^2 \le \|\tilde{r}_n - (c_k^{(n)} \cos(kt) + d_k^{(n)} \sin(kt))\|^2 = \|\tilde{r}_n\|^2 - (|c_k^{(n)}|^2 + |d_k^{(n)}|^2)/2.
\end{aligned}$$

Sum up the above $M + [M\epsilon] + 1$ inequalities we obtain

$$\begin{aligned}
(M + [M\epsilon] + 1)\|\tilde{r}_{n+1}\|^2 &\le (M + [M\epsilon] + 1)\|\tilde{r}_n\|^2 - \frac{1}{4}|c_0^{(n)}|^2 - \frac{1}{2}\sum_{k=1}^{M+[M\epsilon]}(|c_k^{(n)}|^2 + |d_k^{(n)}|^2) \\
&= (M + [M\epsilon])\|\tilde{r}_n\|^2.
\end{aligned}$$

Then, let $n \to \infty$ we have $(M + [M\epsilon] + 1)\tilde{r}^2 \le (M + [M\epsilon])\tilde{r}^2$, which implies $\tilde{r} = 0$. Therefore, $\lim_{n\to\infty} \tilde{r}_n = 0$, and consequently, $\lim_{n\to\infty} r_n = f - f_1$. The proof ends. □

Remark. Since $f \in L^2(\mathbb{T})$, for sufficiently large integer M there holds $f \approx f_1$, then $\lim_{n\to\infty} r_n \approx 0$ and for sufficiently large n there holds $r_n \approx 0$, and consequently, $f \approx \sum_{k=1}^{n} x_k$. That is to say, by Algorithm 1 any signal can be decomposed into a sum of ϵ-mono-components.

4.2. Selection of the best approximation element

A key step in Algorithm 1 is to solve the minimization problem (4.3) (i.e., the minimization problem Eq. (v.2) of Algorithm 1 in [12]). We did not discuss in detail on how to solve it due to limited space there. This section is devoted to this question.

To solve (4.3), we firstly select $g_m \in \mathcal{D}_m$ such that

$$g_m = \text{argmin}_{f_m \in \mathcal{D}_m} \|r - f_m\|, \quad 0 \le m \le M + [M\epsilon], \tag{4.6}$$

where $\mathcal{D}_m$ is the set of all the ϵ-mono-components with frequency m defined in (4.2), then we obtain the best approximation element $\tilde{x} = g_{\tilde{m}}$ where

$$\tilde{m} = \mathrm{argmin}_{0\le m\le M+[M\epsilon]} \|r - g_m\|.$$

This means that in order to solve (4.3), the key step is to solve (4.6). In this section, we study the selection of the best approximation element g_m in the minimization problem (4.6). Suppose $r(t)$ in (4.6) has the Fourier expansion

$$r(t) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k\cos(kt) + d_k\sin(kt)).$$

We firstly consider the case $0 \le m \le M$. Hereinafter, m, ϵ, c_k, d_k are prescribed constants, and L_m, a_k, b_k, ϕ_m are unknown variables. According to (4.2), a direct computation shows that

$$\begin{aligned}
2\|r - f_m\|^2 = {} & [L_m - (c_m\cos\phi_m - d_m\sin\phi_m)]^2 \\
& + \frac{1}{2}\sum_{k=1}^{[m\epsilon]}[a_k - (c_{m+k}\cos\phi_m + c_{m-k}\cos\phi_m - d_{m+k}\sin\phi_m - d_{m-k}\sin\phi_m)]^2 \\
& + \frac{1}{2}\sum_{k=1}^{[m\epsilon]}[b_k - (c_{m+k}\sin\phi_m - c_{m-k}\sin\phi_m + d_{m+k}\cos\phi_m - d_{m-k}\cos\phi_m)]^2 \\
& + e(\phi_m),
\end{aligned} \tag{4.7}$$

where $e(\phi_m)$ depends only on ϕ_m and has the form

$$e(\phi_m) = A_m\sin(2\phi_m) + B_m\cos(2\phi_m) + D_m,$$

in which

$$\begin{aligned}
A_m &= c_m d_m + \sum_{k=1}^{[m\epsilon]}(c_{m+k}d_{m-k} + c_{m-k}d_{m+k}), \\
B_m &= -\frac{1}{2}c_m^2 + \frac{1}{2}d_m^2 - \sum_{k=1}^{[m\epsilon]}(c_{m+k}c_{m-k} - d_{m+k}d_{m-k}),
\end{aligned}$$

and D_m is a constant not depending on L_m, a_k, b_k, ϕ_m.

In order to minimize (4.7), we just need to minimize the last term $e(\phi_m)$ and then set the other three terms to be zero. To minimize $e(\phi_m)$, we rewrite it as

$$e(\phi_m) = \sqrt{A_m^2 + B_m^2}\,\sin(2\phi_m + \varphi_m) + D_m, \tag{4.8}$$

where

$$\varphi_m = \begin{cases} \arctan(B_m/A_m), & A_m > 0, \\ \arctan(B_m/A_m) + \pi, & A_m < 0, \\ \pi/2, & A_m = 0,\ B_m > 0, \\ 3\pi/2, & A_m = 0,\ B_m < 0. \end{cases}$$

To minimize $e(\phi_m)$, we just need to set $\sin(2\phi_m + \varphi_m) = -1$ in (4.8) and obtain

$$\phi_m = \frac{3\pi}{4} - \frac{\varphi_m}{2}.$$

Then, according to (4.7), we obtain L_m, a_k, b_k as follows

$$\begin{aligned}
L_m &= c_m\cos\phi_m - d_m\sin\phi_m, \\
a_k &= c_{m+k}\cos\phi_m + c_{m-k}\cos\phi_m - d_{m+k}\sin\phi_m - d_{m-k}\sin\phi_m, \\
b_k &= c_{m+k}\sin\phi_m - c_{m-k}\sin\phi_m + d_{m+k}\cos\phi_m - d_{m-k}\cos\phi_m.
\end{aligned}$$

The other case $M+1 \le m \le M+[M\epsilon]$, where $f_m(t)$ is a sinusoidal function $f_m(t) = L_m\cos(mt + \phi_m)$, can be handled similarly and more easily. We omit here.

4.3. Numerical experiments

In this section, we give some numerical experiments for Algorithm 1. As the first example, let us consider the signal

$$f(t) = 1 + \cos(2t) + (1 + 0.4\cos(t))\cos(5t+2) + (1 + 0.3\cos(t) + 0.4\sin(2t))\cos(10t+3). \tag{4.9}$$

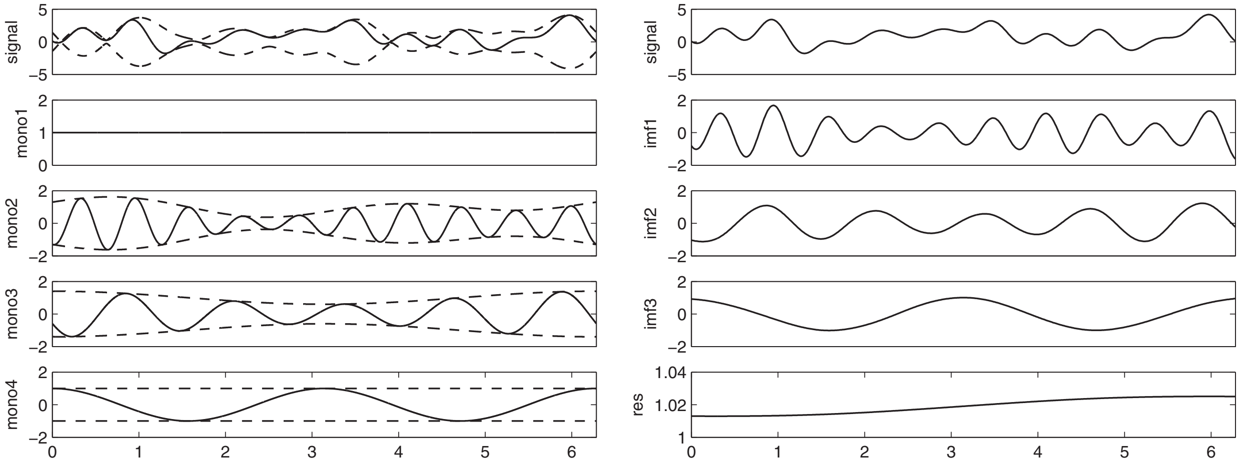


Fig. 3. The decomposition of the signal $f(t) = 1 + \cos(2t) + (1 + 0.4\cos(t))\cos(5t + 2) + (1 + 0.3\cos(t) + 0.4\sin(2t))\cos(10t + 3)$. The left lists the result by Algorithm 1. From top to bottom are respectively the signal $f(t)$ and its four ϵ-mono-components obtained. The signal and the ϵ-mono-components are plotted with solid lines and all the analytic amplitudes are with dashed lines. The right, from top to bottom, are respectively the signal $f(t)$ and its three IMFs and the residue by the EMD.

Fig. 3 (Left-top) shows the signal $f(t)$ and its analytic amplitude. By Algorithm 1 with $M = 20$, $\epsilon = 1/3$ and $\delta = 0.01$, $f(t)$ is decomposed into four ϵ-mono-components, as shown in the 2nd–5th rows of the left column of Fig. 3. It is verified that these four ϵ-mono-components are exactly its constituents (see Eq. (4.9))

$$\begin{aligned} &x_1(t) = 1, \quad x_2(t) = (1 + 0.3\cos(t) + 0.4\sin(2t))\cos(10t + 3), \\ &x_3(t) = (1 + 0.4\cos(t))\cos(5t + 2), \quad x_4(t) = \cos(2t). \end{aligned} \tag{4.10}$$

To compare our result with the EMD, the decomposition of EMD is shown on the right column of Fig. 3. It is easy to see that the first, second and third IMFs are respectively almost the same with our second, third and fourth ϵ-mono-components, that is, $x_2(t)$, $x_3(t)$ and $x_4(t)$. The residue differs slightly from $x_1(t) = 1$, which is caused by the computational error and the stop criteria of the EMD.

As the second example, we add white Gaussian noise to $f(t)$ with SNR = 20 to get a noisy signal $\tilde{f}(t)$. Fig. 4 shows the experimental result, in which the result by Algorithm 1 is listed on the left column and that by the EMD on the right column. By Algorithm 1 with $M = 50$, $\epsilon = 1/3$ and $\delta = 0.1$, $\tilde{f}(t)$ is decomposed into four ϵ-mono-components, which are verified to equal exactly to the four constituents given by Eq. (4.10), and an error, which is just the noise. Similar to the last experiment, we conduct the EMD for $\tilde{f}(t)$, the signal $\tilde{f}(t)$ and all the IMFs and the residue obtained are listed from top to bottom on the right column of Fig. 4. It is observed that the 7th–9th IMFs and the residue are respectively the constituents $x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_1(t)$ approximately. The first six IMFs are mainly produced by the noise.

4.4. Dictionary atoms with time-dependent instantaneous frequencies

In Algorithm 1, each atom in the dictionary is required to have a linear phase. Under the linear assumption, we can solve the minimization problem (4.3) efficiently by using the following orthogonality

$$\langle e^{ikt}, e^{ilt}\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} e^{ikt}\cdot e^{-ilt} = \delta_{k,l}, \quad k, l \in \mathbb{Z},$$

where $\delta_{k,l}$ is the Kronecker symbol defined as 1 for $k = l$ and 0 for $k \neq l$.

Now we consider the case that each atom in the dictionary has a nonlinear phase. Let $\theta(t)$ be a nonlinear function satisfying $\theta'(t) > 0$ and $\theta(t + 2\pi) - \theta(t) = 2\pi$ (for example, we can choose $\theta(t)$ to be the phase $\theta_a(t)$ of the nonlinear Fourier atom defined in (2.1)), and the corresponding dictionary $\mathcal{D}^\theta$ consists of atoms

$$f_m^\theta(t) := \begin{cases} \left(L_m + \sum_{k=1}^{[m\epsilon]}(a_k\cos(k\theta(t)) + b_k\sin(k\theta(t)))\right)\cos(m\theta(t) + \phi_m), & 0 \le m \le M; \\ L_m\cos(m\theta(t) + \phi_m), & M + 1 \le m \le M + [M\epsilon]. \end{cases}$$

Obviously, each atom in $\mathcal{D}^\theta$ has a nonlinear phase and consequently a time-dependent IF. Similarly, we can solve the minimization problem (4.3) by using the following weighted orthogonality (with the weight $\omega(t) := \theta'(t)$)

$$\langle e^{ik\theta(t)}, e^{il\theta(t)}\rangle_{L^2_\omega(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta(t)}\cdot e^{-il\theta(t)}\cdot\theta'(t)dt = \delta_{k,l}, \quad k, l \in \mathbb{Z}.$$

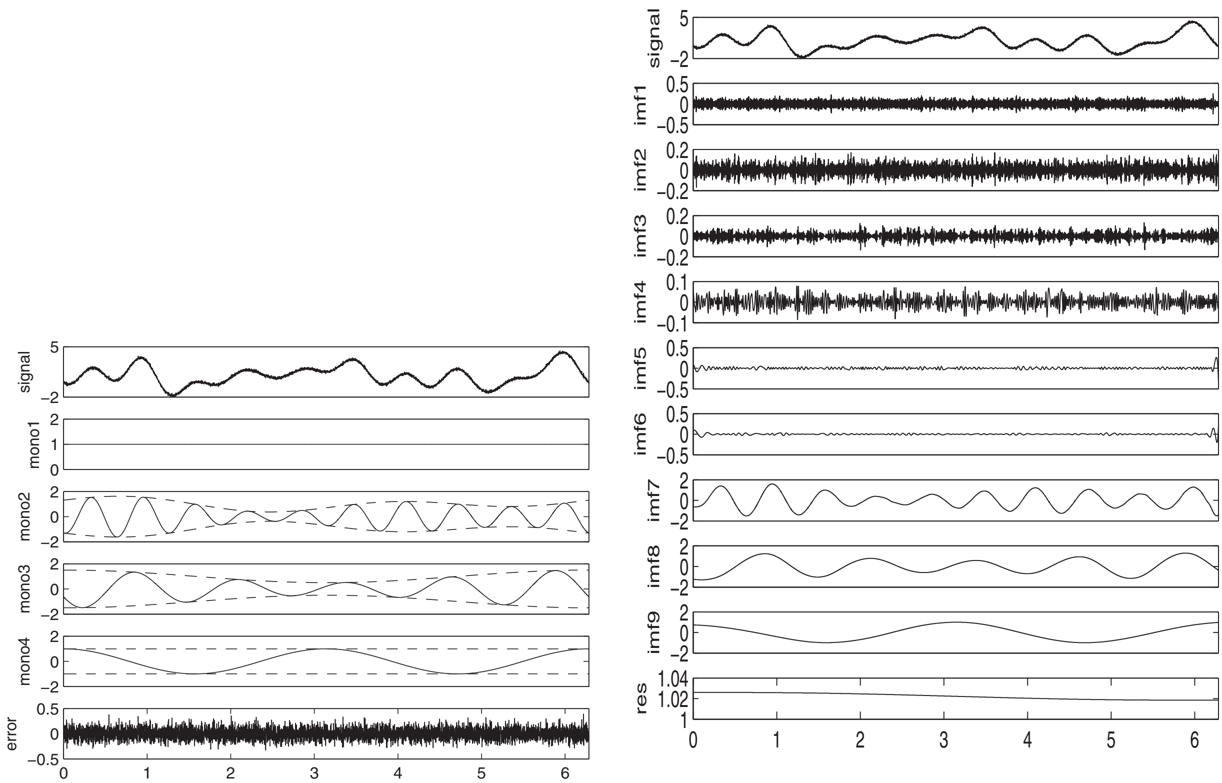


Fig. 4. The decomposition of the noisy signal $\tilde{f}(t)$. The left lists the result by Algorithm 1. From top to bottom are respectively the signal $\tilde{f}(t)$ and its four ϵ-mono-components and an error. The right, from top to bottom, are respectively the signal $\tilde{f}(t)$ and its nine IMFs and the residue by the EMD.

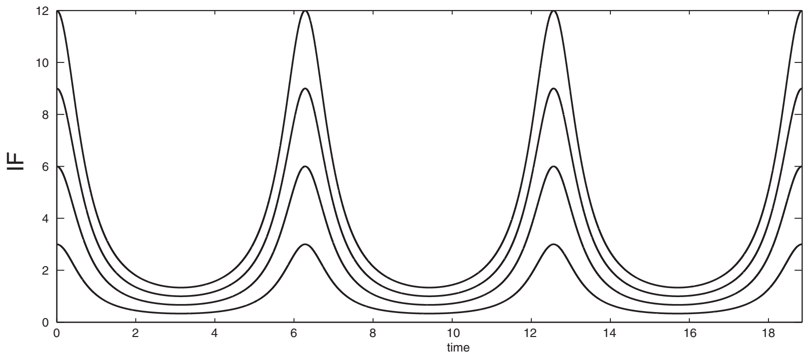


Fig. 5. From bottom to top are the IFs of $f_m^{\theta}(t)$ for $m = 1, 2, 3, 4$ respectively, where the nonlinear phase $\theta(t)$ is the phase $\theta_{0.5}(t)$ of the nonlinear Fourier atom defined in (2.1) corresponding to $a = 0.5$. It is shown that the IFs are separated from each other at any time t.

So, we can decompose a signal into a sum of ϵ-mono-components containing time-dependent IFs by using the dictionary $\mathcal{D}^{\theta}$. Moreover, the IFs of decomposed components are separated from each other at any time, as shown in Fig. 5.

5. The ϵ-mono-components on the real line

In this section we turn to consider the case of non-periodic signals. The ϵ-mono-components on the real line are defined as follows [12]:

Definition 3. Given $\epsilon \in (0, 1]$, a signal $s(t) = \rho(t)e^{i\theta(t)} \in \mathcal{M}$ is said to be an ϵ-mono-component on the real line if $\theta \in C_{\uparrow}(\mathbb{R})$, $\rho(t) \geq 0\,(\forall t \in \mathbb{R})$ and can be expressed as

$$\rho(t) = \left(\lambda_0 + \sum_{j=1}^{N} \lambda_j \cos\phi_j(t)\right)\gamma(t)$$

for some $\lambda_0, \lambda_j \in \mathbb{R}$, $\phi_j \in C_{\rightarrow}(\mathbb{R})$, $j = 1, \ldots, N$, satisfying

$$\phi_j'(t) \le \epsilon\theta'(t), \quad \forall t \in \mathbb{R}, \quad j = 1, 2, \ldots, N,$$

and $\gamma \in C(\mathbb{R})$ is decreasing on $[0, \infty)$, satisfying $\gamma(t) = \gamma(-t) \ge 0$, $\forall t \in \mathbb{R}$.

It is well-known that periodic and non-periodic analytic signals are the nontangential boundary values of analytic functions in the unit disk $\mathbb{U}$ and upper half plane $\mathbb{C}_+$, respectively. It is known that there is an isomorphism between these two kinds of analytic functions [3, p. 19]

$$Tf := \frac{1}{\sqrt{\pi}} \frac{1}{1 - iz} (f \circ K), \tag{5.11}$$

where

$$K(z) := \frac{i - z}{i + z}, \quad z \in \mathbb{C}_+$$

is the Cayley transformation, which is a conformal mapping from $\mathbb{C}_+$ to $\mathbb{U}$. With this mapping, almost all the results for periodic analytic signals can be extended into the non-periodic cases.

6. Conclusion

The paper studies a special class of mono-components called ϵ-mono-components, where the amplitude of the signal oscillates much more slowly than the phase part at any instant as long as the parameter ϵ is sufficiently small. The main contributions of this paper include: (1) The constructions of ϵ-mono-components are studied; (2) an adaptive signal decomposition algorithm based on ϵ-mono-components is analyzed.

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