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Representation of solutions of delayed difference equations with linear parts given by pairwise permutable matrices via $\mathcal{Z}$-transform

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ABSTRACT

In the present paper, a system of nonhomogeneous linear difference equations with any finite number of constant delays and linear parts given by pairwise permutable matrices is considered. Representation of its solution is derived in a form of a matrix polynomial using the $\mathcal{Z}$-transform. So the recent results for one and two delays, and an inductive formula for multiple delays are unified. The representation is suitable for theoretical as well as practical computations.

1. Introduction

Recently, Khusainov and Diblík [10] derived a solution of a difference equation with a delay using a matrix polynomial. This representation lead to results on controllability, observability, stability etc. (see e.g. [5,12–14]). Motivated by these applications, results on representation of a solution of a difference equation with more than one delay appeared. However, these contain an explicit formula for two delays [6] or an inductively built formula for the case of $n \in \mathbb{N}$ delays [11]. Although the latter result was successfully applied in the same paper to prove results on exponential stability, for practical computations it seems to be not very suitable. The aim of this paper is to provide a representation of a solution of a nonhomogeneous linear difference equation with multiple delays in a closed form, and so to generalize results of [6,10]. The results of this paper may also help to explore the stability properties of delayed difference equations [3,4].

To achieve our goal we use the $\mathcal{Z}$-transform and its inverse. By this way, we also avoid investigating particular cases of the time k resulting in vast proofs as in [6]. We note that as in [6,11] we suppose that the linear parts are given by pairwise permutable matrices. This allows to change the order when multiplying matrices.

Throughout the paper we denote Θ and $\mathbb{I}$ the $N \times N$ zero and identity matrix, respectively, $\mathbb{Z}_a^b := \{a, a+1, \dots, b\}$ for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a \leq b$, and $\mathbb{Z}_a^b = \emptyset$ if $a > b$. We suppose the properties of an empty sum $\sum_{i=a}^b z(i) = 0$ and an empty product $\prod_{i=a}^b z(i) = 1$ for integers $a > b$, where $z(i)$ is a given function which does not have to be defined for each $i \in \mathbb{Z}_b^a$ in this case. Standardly, $\Delta x(k) := x(k+1) - x(k)$ is the forward difference operator, and $\lfloor \cdot \rfloor$ denotes the floor function. We shall denote $\|v\| = \max_{i \in \mathbb{Z}_1^N} |v^i|$ the maximum norm of a vector $v = (v^1, \dots, v^N) \in \mathbb{R}^N$, and $\|B\|$ an induced matrix norm of an $N \times N$ matrix B.

Let us recall the above-mentioned results for one, two and multiple delays.

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Theorem 1.1 (see [10]). *Let $m \geq 1$, B be a constant $N \times N$ matrix, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ and $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^N$ be given functions. Solution $x(k)$ of the Cauchy problem consisting of the equation*

$$\Delta x(k) = Bx(k-m) + f(k), \qquad k \geq 0 \tag{1.1}$$

and the initial condition

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^0 \tag{1.2}$$

has the form

$$x(k) = \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1)$$

for $k \in \mathbb{Z}_{-m}^\infty$ where e_m^{Bk} is the discrete delayed matrix exponential given by

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta, & \text{if } k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ \mathbb{I} + \sum_{j=1}^{l} B^j \binom{k-(j-1)m}{j}, & \text{if } k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)},\ l \in \mathbb{Z}_0^\infty. \end{cases} \tag{1.3}$$

It was proved in [10] that e_m^{Bk} is a matrix solution of the equation

$$\Delta X(k) = BX(k-m), \qquad k \in \mathbb{Z}_{-m}^\infty, \tag{1.4}$$

satisfying the initial condition

$$X(k) = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ \mathbb{I}, & k \in \mathbb{Z}_{-m}^0. \end{cases} \tag{1.5}$$

Theorem 1.2 (see [6]). *Let $m_2 > m_1 \geq 1$, B_1 and B_2 be permutable, i.e. $B_1B_2 = B_2B_1$, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ and $\varphi : \mathbb{Z}_{-m_2}^0 \to \mathbb{R}^N$ be given functions. Solution $x(k)$ of the Cauchy problem*

$$\begin{aligned} \Delta x(k) &= B_1x(k-m_1) + B_2x(k-m_2) + f(k), \qquad k \geq 0 \\ x(k) &= \varphi(k), \qquad k \in \mathbb{Z}_{-m_2}^0 \end{aligned}$$

has the form

$$x(k) = \sum_{j=0}^{m_2} \tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(k+j)} w_j + \sum_{j=1}^{k} \tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(k-j)} f(j-1), \qquad k \in \mathbb{Z}_{-m_2}^\infty$$

where

$$\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2k} = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-1}, \\ \mathbb{I}, & k \in \mathbb{Z}_0^{m_1}, \\ \mathbb{I} + \sum_{i=0}^{\lfloor \frac{k-1}{m_1+1} \rfloor} \sum_{j=0}^{\lfloor \frac{k-1}{m_2+1} \rfloor} B_1^i B_2^j & \\ \quad \times \binom{i+j}{i} \left[B_1 \binom{k-m_1(i+1)-m_2j}{i+j+1} + B_2 \binom{k-m_1i-m_2(j+1)}{i+j+1} \right], & k \in \mathbb{Z}_{m_1+1}^\infty \end{cases}$$

and

$$w_k = \begin{cases} \Delta\varphi(-k-1) - \Delta\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(-k+m_2-1)}\varphi(-m_2) & \\ - \sum_{j=-m_2}^{-k-m_1-2} \Delta\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(-k-j-2)}\Delta\varphi(j), & k \in \mathbb{Z}_0^{m_2-m_1-1} \\ \Delta\varphi(-k-1), & k \in \mathbb{Z}_{m_2-m_1}^{m_2-1}, \\ \varphi(-k), & k = m_2. \end{cases}$$

From [6] we know that $\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2k}$ is a matrix solution of the equation

$$\Delta X(k) = B_1X(k-m_1) + B_2X(k-m_2), \qquad k \in \mathbb{Z}_0^\infty$$

and, clearly, it satisfies

$$X(k) = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-1}, \\ \mathbb{I}, & k = 0. \end{cases} \tag{1.6}$$

Theorem 1.3 (see [11]). *Let $n > 1$, $m_1, \dots, m_n \geq 1$, $m = \max\{m_1, \dots, m_n\}$, $B_1, \dots, B_n$ be pairwise permutable $N \times N$ matrices, i.e. $B_iB_j = B_jB_i$ for each $i, j = 1, \dots, n$, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ and $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^N$ be given functions. Then the solution of the nonhomogeneous initial Cauchy problem consisting of the equation*

$$\Delta x(k) = B_1 x(k - m_1) + \cdots + B_n x(k - m_n) + f(k), \qquad k \geq 0, \tag{1.7}$$

and initial condition(1.2) has the form

$$\begin{aligned} x(k) = {} & \mathcal{Y}(k+m)\varphi(-m) + \sum_{j=-m+1}^{0} \mathcal{Y}(k-j)\Delta\varphi(j-1) \\ & - \sum_{i=1}^{n} B_i \sum_{j=-m}^{-1-m_i} \mathcal{Y}(k-1-m_i-j)\varphi(j) + \sum_{j=1}^{k} \mathcal{Y}(k-j)f(j-1) \end{aligned} \tag{1.8}$$

for $k \in \mathbb{Z}_{-m}^\infty$, where $\mathcal{Y}(k) = \mathrm{e}_{m_1,\dots,m_n}^{B_1,\dots,B_n(k-m_n)}$ and $\mathrm{e}_{m_1,\dots,m_j}^{B_1,\dots,B_jk}$ is the discrete multidelayed matrix exponential given by

$$\mathrm{e}_{m_1,\dots,m_j}^{B_1,\dots,B_jk} = \begin{cases} \Theta, & \text{if } k \in \mathbb{Z}_{-\infty}^{-m_j-1}, \\ \mathcal{X}_{j-1}(k+m_j) + B_j \sum\limits_{i_1=1}^{k} \mathcal{X}_{j-1}(k-i_1)\mathcal{X}_{j-1}(i_1-1) + \cdots & \\ \cdots + B_j^l \sum\limits_{\substack{i_1=(l-1)\\ \times(m_j+1)+1}}^{k} \sum\limits_{\substack{i_2=(l-1)\\ \times(m_j+1)+1}}^{i_1} \cdots \sum\limits_{\substack{i_l=(l-1)\\ \times(m_j+1)+1}}^{i_{l-1}} \mathcal{X}_{j-1}(k-i_1) & \\ \times \prod\limits_{s=1}^{l-1} \mathcal{X}_{j-1}(i_s - i_{s+1})\mathcal{X}_{j-1}(i_l-(l-1)(m_j+1)-1), & \\ & \text{if } k \in \mathbb{Z}_{(l-1)(m_j+1)+1}^{l(m_j+1)},\ l \in \mathbb{Z}_0^\infty \end{cases} \tag{1.9}$$

with $\mathcal{X}_{j-1}(k) = \mathrm{e}_{m_1,\dots,m_{j-1}}^{B_1,\dots,B_{j-1}(k-m_{j-1})}$ for each $j = 2, \dots, n$ and e_m^{Bk} given by (1.3).

Here we know [11] that $\mathcal{Y}(k)$ is a matrix solution of the equation

$$\Delta X(k) = B_1 X(k - m_1) + \cdots + B_n X(k - m_n), \qquad k \in \mathbb{Z}_0^\infty \tag{1.10}$$

satisfying (1.6). This can be also shown by using Theorem 1.3 and considering (1.7) as a matrix equation with an appropriate initial condition (see Remark 3.5.2 below).

The present paper is organized as follows. In the next section, we conclude some known or easily proved results. Section 3 contains the main results on the representation of a solution of a nonhomogeneous linear difference equation with one or multiple delays and the linear parts given by pairwise permutable matrices. We close this section with an example of a scalar equation.

2. Preliminary results

This section contains basic results which will be in force in the main section.

In the following we use the lower index to emphasize on which variable the difference operator is applied, e.g. $\Delta_k f(k, j) = f(k+1, j) - f(k, j)$.

Lemma 2.1. *Let $a, b \in \mathbb{Z}$, $a < b$. For given functions f, g the following holds*

$$\Delta_k f(k-j) = -\Delta_j f(k+1-j),$$

$$\sum_{j=a}^{b} f(j)\Delta g(j) = f(b)g(b+1) - f(a)g(a) - \sum_{j=a+1}^{b} \Delta f(j-1)g(j),$$

$$\Delta_k \left[\sum_{j=1}^{k} f(k, j) \right] = f(k+1, k+1) + \sum_{j=1}^{k} \Delta_k f(k, j).$$

Proof. All the statements follow directly from the definition of the difference operator, e.g.

$$\Delta_k \left[\sum_{j=1}^{k} f(k, j) \right] = \sum_{j=1}^{k+1} f(k+1, j) - \sum_{j=1}^{k} f(k, j) = f(k+1, k+1) + \sum_{j=1}^{k} \Delta_k f(k, j).$$

□

Note that by the latter identity,

$$\Delta_k\left[\sum_{j=-a}^{k-a-1} f(k,j)\right] = f(k+1,k-a) + \sum_{j=-a}^{k-a-1}\Delta_k f(k,j)$$

for $a \in \mathbb{Z}$.

Definition 2.2. A function $f = (f^1, \dots, f^N) : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ is exponentially bounded if there are constants $c_1, c_2 \geq 0$ such that $\|f(k)\| \leq c_1 c_2^k$ for all $k \in \mathbb{Z}_0^\infty$.

Definition 2.3. Let $f : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ be a given function. The unilateral $\mathcal{Z}$-transform of the function f is defined as

$$\mathcal{Z}\{f(k)\}(z) = \sum_{k=0}^{\infty}\frac{f(k)}{z^k} \tag{2.1}$$

for $z \in \mathbb{R}$.

Note that if f is exponentially bounded, then $\mathcal{Z}\{f(k)\}(z)$ exists for all z sufficiently large (the sum in (2.1) converges).
In the rest of the paper we shall use capital letters to denote the $\mathcal{Z}$-transform of a function, e.g. $F(z) = \mathcal{Z}\{f(k)\}(z)$ and $f(k) = \mathcal{Z}^{-1}\{F(z)\}(k)$. We note that the $\mathcal{Z}$-transform is considered component-wisely, i.e. the $\mathcal{Z}$-transform of a vector-valued function is a vector of $\mathcal{Z}$-transformed coordinates.

Lemma 2.4. *The following identities hold true for sufficiently large $z \in \mathbb{R}$ and exponentially bounded functions f, g:*

1. $\mathcal{Z}\{af(k) + bg(k)\} = a\mathcal{Z}\{f(k)\} + b\mathcal{Z}\{g(k)\}$ *for constants* $a, b \in \mathbb{R}$,
2. $\mathcal{Z}^{-1}\{F(z)G(z)\}(k) = (f * g)(k)$ *for convolution operator* $*$ *given by*

$$(f*g)(k) = \sum_{j=0}^{k} f(j)g(k-j),$$

3. $\mathcal{Z}^{-1}\{z^{-\alpha}\}(k) = \delta_\alpha(k)$ *for* $\alpha \in \mathbb{Z}_0^\infty$ *where* δ_α *is the Kronecker delta,*

$$\delta_\alpha(k) = \begin{cases} 1, & k = \alpha, \\ 0, & k \neq \alpha. \end{cases}$$

In particular, $\mathcal{Z}^{-1}\{1\}(k) = \delta_0(k)$,

4. $\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\}(k) = \sigma(k)$ *for* $z > 1$ *where* σ *is the step function defined as*

$$\sigma(k) = \begin{cases} 1, & k \in \mathbb{Z}_0^\infty, \\ 0, & k \notin \mathbb{Z}_0^\infty. \end{cases}$$

Proof. Linearity and convolution theorem are proved e.g. in [7]. The third statement is obvious, since $\sum_{k=0}^{\infty}\delta_\alpha(k)z^{-k} = z^{-\alpha}$. The last statement follows from the identity $\frac{z}{z-1} = \frac{1}{1-z^{-1}} = \sum_{k=0}^{\infty} z^{-k}$ for $z > 1$. □

Lemma 2.5. *The following identities hold true for sufficiently large $z \in \mathbb{R}$:*

1. $\mathcal{Z}^{-1}\{F_1(z)F_2(z)\dots F_n(z)\}(k) = (f_1 * f_2 * \dots * f_n)(k)$ *for* $n \in \mathbb{Z}_2^\infty$ *and exponentially bounded functions* $f_1, f_2, \dots, f_n$,
2. $\mathcal{Z}^{-1}\{(z-1)^{-j}\}(k) = \binom{k-1}{j-1}\sigma(k-j)$ *for* $j \in \mathbb{N}$.

Proof. The first statement follows from Lemma 2.4 using the associativity of the convolution.
To prove the second statement, we consider $z > 1$. Then we $(j-1)$ times integrate $(z-1)^{-j}$, expand the result into a geometric series with the quotient z^{-1} and, consequently, differentiate $(j-1)$ times:

$$\begin{aligned}
(z-1)^{-j} &= -\frac{1}{j-1}\left((z-1)^{-j+1}\right)' = \dots = \frac{(-1)^{j-1}}{(j-1)!}\left((z-1)^{-1}\right)^{(j-1)} \\
&= \frac{(-1)^{j-1}}{(j-1)!}\left(\sum_{i=0}^{\infty} z^{-(i+1)}\right)^{(j-1)} = \frac{(-1)^{j-2}}{(j-1)!}\left(\sum_{i=0}^{\infty}(i+1)z^{-(i+2)}\right)^{(j-2)} \\
&= \dots = \frac{1}{(j-1)!}\sum_{i=0}^{\infty}\frac{(i+1)(i+2)\dots(i+j-1)}{z^{i+j}} \\
&= \sum_{i=0}^{\infty}\binom{i+j-1}{j-1}z^{-i-j} = \sum_{k=j}^{\infty}\binom{k-1}{j-1}z^{-k}.
\end{aligned} \tag{2.2}$$

In other words, $(z-1)^{-j} = \sum_{k=0}^{\infty} a_k z^{-k}$ for

$$a_k = \begin{cases} 0, & k \in \mathbb{Z}_0^{j-1}, \\ \binom{k-1}{j-1}, & k \in \mathbb{Z}_j^{\infty} \end{cases}$$

or, shortly, $a_k = \binom{k-1}{j-1}\sigma(k-j)$. □

We enclose this section with a result proving that a solution of a delayed linear nonhomogeneous difference equation is exponentially bounded provided that its right-hand side is exponentially bounded.

Lemma 2.6. *Let $n > 1$, $m_1, \dots, m_n \geq 1$, $m = \max\{m_1, \dots, m_n\}$, $B_1, \dots, B_n$ be $N \times N$ matrices, $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^N$ an exponentially bounded function, and $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^N$ a given function. Then a solution $x(k)$ of the initial value problem (1.7), (1.2) is exponentially bounded.*

Proof. Solution $x(k)$ satisfies the corresponding sum equation

$$x(k) = \varphi(0) + \sum_{j=0}^{k-1}\sum_{i=1}^{n} B_i x(j-m_i) + \sum_{j=0}^{k-1} f(j), \qquad k \geq 0.$$

Without any loss of generality we can assume that $c_1 \geq 0$, $c_2 \geq 1$ are such that $\|f(k)\| \leq c_1 c_2^k$ for each $k \in \mathbb{Z}_0^{\infty}$. Consequently, denoting $\|\varphi\| := \max_{k \in \mathbb{Z}_{-m}^0} \|\varphi(k)\|$ and $b := \max_{i \in \mathbb{Z}_0^n} \|B_i\|$,

$$\begin{aligned} \|x(k)\| &\leq \|\varphi\| + b\sum_{j=0}^{k-1}\sum_{i=1}^{n} \|x(j-m_i)\| + c_1 \sum_{j=0}^{k-1} c_2^j \\ &\leq (1+bnm)\|\varphi\| + bn\sum_{j=0}^{k-1}\|x(j)\| + \frac{c_1(c_2^k - 1)}{c_2 - 1} \leq C + \frac{c_1 c_2^k}{c_2 - 1} + bn\sum_{j=0}^{k-1}\|x(j)\| \end{aligned}$$

with $C = (1+bnm)\|\varphi\|$, for each $k \in \mathbb{Z}_0^{\infty}$. Applying the discrete Gronwall inequality [2, Theorem 4.1.1], one obtains

$$\|x(k)\| \leq C + \frac{c_1 c_2^k}{c_2 - 1} + bn\sum_{j=0}^{k-1}\left(C + \frac{c_1 c_2^j}{c_2 - 1}\right)(1+bn)^{k-j-1} \leq C + \frac{c_1 c_2^k}{c_2 - 1} + bn(1+bn)^k\left(Ck + \frac{c_1 c_2^k}{(c_2-1)^2}\right).$$

Clearly, there are constants $a_1, a_2 \geq 0$ such that the right-hand side of the latter inequality is less than or equal to $a_1 a_2^k$ for each $k \in \mathbb{Z}_0^{\infty}$ what was to be proved. □

3. Main results

This section is devoted to the main results of this paper. First we consider the case of one delay.

Theorem 3.1. *Let $m \geq 1$, B be a constant $N \times N$ matrix, $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^N$ a given exponentially bounded function, and $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^N$ a given function. Solution $x(k)$ of the initial value problem (1.1), (1.2) has the form*

$$x(k) = \begin{cases} \varphi(k), & k \in \mathbb{Z}_{-m}^0, \\ \mathcal{A}(k-m)\varphi(0) + B\sum_{j=-m}^{-1}\mathcal{A}(k-1-2m-j)\varphi(j) & \\ \quad + \sum_{j=1}^{k}\mathcal{A}(k-m-j)f(j-1), & k \in \mathbb{N} \end{cases} \tag{3.1}$$

where

$$\mathcal{A}(k) = \sum_{i=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} B^i \binom{k-m(i-1)}{i}. \tag{3.2}$$

Proof. We apply the $\mathcal{Z}$-transform to equation (1.1) to get

$$\sum_{k=0}^{\infty}\frac{x(k+1)}{z^k}-\sum_{k=0}^{\infty}\frac{x(k)}{z^k}=B\sum_{k=0}^{\infty}\frac{x(k-m)}{z^k}+\sum_{k=0}^{\infty}\frac{f(k)}{z^k}$$
$$z(X(z)-\varphi(0))-X(z)=\frac{B}{z^m}\left(X(z)+\sum_{k=-m}^{-1}\frac{\varphi(k)}{z^k}\right)+F(z)$$
$$(z-1)\left(\mathbb{I}-\frac{B}{z^m(z-1)}\right)X(z)=z\varphi(0)+B\sum_{k=-m}^{-1}\frac{\varphi(k)}{z^{k+m}}+F(z).$$

Here the existence of $F(z)$ and $X(z)$ is assured by the exponential boundedness of f and Lemma 2.6. From theory of matrices we know (see e.g. [15, Proposition 7.5]) that

$$\left(\mathbb{I}-\frac{B}{z^m(z-1)}\right)^{-1}=\sum_{j=0}^{\infty}\left(\frac{B}{z^m(z-1)}\right)^j$$

on suppose that $z\in\mathbb{R}$ is large enough so that $\|B\|<z^m(z-1)$. Then

$$X(z)=\frac{1}{z-1}\left(\sum_{j=0}^{\infty}\left(\frac{B}{z^m(z-1)}\right)^j\right)\left(z\varphi(0)+B\sum_{k=-m}^{-1}\frac{\varphi(k)}{z^{k+m}}+F(z)\right)$$

for sufficiently large z or, taking the inverse $\mathcal{Z}$-transform,

$$x(k)=A_0(k)+B\sum_{j=-m}^{-1}A_j(k)+A_f(k)$$

where

$$A_0(k)=\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\left(\sum_{j=0}^{\infty}\left(\frac{B}{z^m(z-1)}\right)^j\right)\varphi(0)\right\}(k),$$
$$A_j(k)=\mathcal{Z}^{-1}\left\{\frac{1}{z^{j+m}(z-1)}\left(\sum_{i=0}^{\infty}\left(\frac{B}{z^m(z-1)}\right)^i\right)\varphi(j)\right\}(k),\quad j\in\mathbb{Z}_{-m}^{-1},$$
$$A_f(k)=\mathcal{Z}^{-1}\left\{\frac{1}{z-1}\left(\sum_{j=0}^{\infty}\left(\frac{B}{z^m(z-1)}\right)^j\right)F(z)\right\}(k).$$

Application of Lemmas 2.4 and 2.5 yields

$$\begin{aligned}\sum_{j=0}^{\infty}\mathcal{Z}^{-1}&\left\{\frac{z}{z-1}\left(\frac{B}{z^m(z-1)}\right)^j\varphi(0)\right\}(k)=\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\}(k)\varphi(0)\\&+\sum_{j=1}^{\infty}\left(\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\}*\mathcal{Z}^{-1}\{z^{-mj}\}*\mathcal{Z}^{-1}\{(z-1)^{-j}\}\right)(k)B^j\varphi(0)\\&=\sigma(k)\varphi(0)+\sum_{j=1}^{\infty}\left(\sigma*\delta_{mj}*\binom{\cdot-1}{j-1}\sigma(\cdot-j)\right)(k)B^j\varphi(0).\end{aligned}\tag{3.3}$$

Next, since

$$(\sigma*\delta_{mj})(k)=\sum_{i=0}^{k}\delta_{mj}(i)\sigma(k-i)=\sigma(k-mj)$$

and

$$\begin{aligned}&\left(\sigma(\cdot-mj)*\binom{\cdot-1}{j-1}\sigma(\cdot-j)\right)(k)\\&=\sum_{i=0}^{k}\sigma(i-mj)\sigma(k-i-j)\binom{k-i-1}{j-1}=\sum_{i=mj}^{k-j}\binom{k-i-1}{j-1}\sigma(k-(m+1)j)\\&=\sum_{i=0}^{k-(m+1)j}\binom{i+j-1}{j-1}\sigma(k-(m+1)j)=\binom{k-mj}{j}\sigma(k-(m+1)j)\end{aligned}$$

(for the last equality in the latter identity see e.g. [8, 0.15.1]), from (3.3) one obtains

$$\begin{aligned}
&\mathcal{Z}\left\{\sum_{j=0}^{\infty}\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\left(\frac{B}{z^m(z-1)}\right)^j\varphi(0)\right\}(k)\right\}(z)\\
&=\mathcal{Z}\left\{\sum_{j=0}^{\infty}\binom{k-mj}{j}\sigma(k-(m+1)j)B^j\varphi(0)\right\}(z)\\
&=\sum_{k=0}^{\infty}\frac{1}{z^k}\sum_{j=0}^{\infty}\binom{k-mj}{j}\sigma(k-(m+1)j)B^j\varphi(0)\\
&=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{z^k}\binom{k-mj}{j}\sigma(k-(m+1)j)\right)B^j\varphi(0)\\
&=\sum_{j=0}^{\infty}\left(\sum_{k=(m+1)j}^{\infty}\frac{1}{z^k}\binom{k-mj}{j}\right)B^j\varphi(0)\\
&=\sum_{j=0}^{\infty}\left(\sum_{k=(m+1)j}^{\infty}\frac{1}{z^{k-mj}}\binom{k-mj}{j}\right)\frac{B^j\varphi(0)}{z^{mj}}\\
&=\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\frac{1}{z^k}\binom{k}{j}\right)\frac{B^j\varphi(0)}{z^{mj}}=\frac{z}{z-1}\left(\sum_{j=0}^{\infty}\frac{B^j}{z^{mj}(z-1)^j}\right)\varphi(0)=\mathcal{Z}\{A_0(k)\}(z)
\end{aligned}$$

where the penultimate identity follows from (2.2). Now we apply the uniqueness of $\mathcal{Z}^{-1}$ (see [7]) and the above computation to derive

$$\begin{aligned}
A_0(k)&=\sum_{j=0}^{\infty}\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\left(\frac{B}{z^m(z-1)}\right)^j\varphi(0)\right\}(k)\\
&=\sum_{j=0}^{\infty}\binom{k-mj}{j}\sigma(k-(m+1)j)B^j\varphi(0)\\
&=\sum_{j=0}^{\left\lfloor\frac{k}{m+1}\right\rfloor}\binom{k-mj}{j}B^j\varphi(0)=\mathcal{A}(k-m)\varphi(0).
\end{aligned}$$

Similarly, for any $j\in\mathbb{Z}_{-m}^{-1}$, we derive

$$\begin{aligned}
A_j(k)&=\sum_{i=0}^{\infty}\left(\mathcal{Z}^{-1}\{z^{-j-(i+1)m}\}*\mathcal{Z}^{-1}\{(z-1)^{-i-1}\}\right)(k)B^i\varphi(j)\\
&=\sum_{i=0}^{\infty}\left(\delta_{j+(i+1)m}*\binom{\cdot-1}{i}\sigma(\cdot-1-i)\right)(k)B^i\varphi(j)\\
&=\sum_{i=0}^{\infty}\left(\sum_{s=0}^{k}\delta_{j+(i+1)m}(s)\sigma(k-s-1-i)\binom{k-s-1}{i}\right)B^i\varphi(j)\\
&=\sum_{i=0}^{\infty}\binom{k-j-(i+1)m-1}{i}\sigma(k-j-(i+1)(m+1))B^i\varphi(j)\\
&=\sum_{i=0}^{\left\lfloor\frac{k-j}{m+1}\right\rfloor-1}\binom{k-j-(i+1)m-1}{i}B^i\varphi(j)=\mathcal{A}(k-1-2m-j)\varphi(j),
\end{aligned}$$

and

$$\begin{aligned}
A_f(k)&=\sum_{j=0}^{\infty}B^j\left(\mathcal{Z}^{-1}\{z^{-mj}\}*\mathcal{Z}^{-1}\{(z-1)^{-j-1}\}*f\right)(k)\\
&=\sum_{j=0}^{\infty}B^j\left(\delta_{mj}*\binom{\cdot-1}{j}\sigma(\cdot-1-j)*f\right)(k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} B^j \left(\binom{\cdot - mj - 1}{j} \sigma(\cdot - (m+1)j - 1) * f \right)(k) \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{k} B^j \binom{k-i-mj-1}{j} \sigma(k-i-(m+1)j-1) f(i) \\
&= \sum_{i=0}^{k-1} \sum_{j=0}^{\left\lfloor \frac{k-i-1}{m+1} \right\rfloor} \binom{k-i-mj-1}{j} B^j f(i) \\
&= \sum_{i=1}^{k} \sum_{j=0}^{\left\lfloor \frac{k-i}{m+1} \right\rfloor} \binom{k-i-mj}{j} B^j f(i-1) = \sum_{i=1}^{k} \mathcal{A}(k-m-i) f(i-1).
\end{aligned}
$$

Summarizing, we obtain the statement of the theorem. □

Remark 3.2. Above we presented a constructive proof of Theorem 3.1.

Another way how to prove this result is to use the known Theorem 1.1. Here, one applies Lemma 2.1 to get

$$
\begin{aligned}
&\sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\varphi(j-1) \\
&\quad = \mathrm{e}_m^{B(k-m)}\varphi(0) - \mathrm{e}_m^{B(k-1)}\varphi(-m) - \sum_{j=-m+2}^{0} \Delta_j \mathrm{e}_m^{B(k-m-j+1)}\varphi(j-1) \\
&\quad = \mathrm{e}_m^{B(k-m)}\varphi(0) - \mathrm{e}_m^{B(k-1)}\varphi(-m) + \sum_{j=-m+2}^{0} \Delta_k \mathrm{e}_m^{B(k-m-j)}\varphi(j-1) \\
&\quad = \mathrm{e}_m^{B(k-m)}\varphi(0) - \mathrm{e}_m^{B(k-1)}\varphi(-m) + \sum_{j=-m+1}^{-1} \Delta_k \mathrm{e}_m^{B(k-m-j-1)}\varphi(j).
\end{aligned}
$$

Hence by Theorem 1.1,

$$
\begin{aligned}
x(k) &= \mathrm{e}_m^{B(k-m)}\varphi(0) + \sum_{j=-m}^{-1} \Delta_k \mathrm{e}_m^{B(k-m-j-1)}\varphi(j) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) \\
&= \mathrm{e}_m^{B(k-m)}\varphi(0) + B \sum_{j=-m}^{-1} \mathrm{e}_m^{B(k-2m-j-1)}\varphi(j) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1)
\end{aligned}
$$

for $k \in \mathbb{Z}_0^\infty$. Here we used the property of e_m^{Bk} described after Theorem 1.1, since $k-m-j-1 \in \mathbb{Z}_{-m}^\infty$. Now it remains to show that $\mathcal{A}(k) = \mathrm{e}_m^{Bk}$ for $k \in \mathbb{Z}_{-m}^\infty$. But this is immediate, since $k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}$ for $l \in \mathbb{Z}_0^\infty$ if and only if

$$
l = \left\lfloor \frac{k-1}{m+1} \right\rfloor + 1 = \left\lfloor \frac{k+m}{m+1} \right\rfloor.
$$

Indeed, for this l,

$$
(l-1)(m+1)+1 = \left\lfloor \frac{k-1}{m+1} \right\rfloor (m+1) + 1 \le k
$$

and

$$
l(m+1) = \left\lfloor \frac{k+m}{m+1} \right\rfloor (m+1) = \left\lceil \frac{k}{m+1} \right\rceil (m+1) \ge k
$$

for the ceiling function $\lceil \cdot \rceil$, since $\left\lfloor \frac{a}{b} \right\rfloor = \left\lceil \frac{a-b+1}{b} \right\rceil$ for any $a \in \mathbb{Z}$, $b \in \mathbb{N}$ [9, Exercise 3.12]. On the other side, if $k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)}$ for some $l \in \mathbb{Z}_0^\infty$, then $l \le \frac{k+m}{m+1}$ and $\frac{k}{m+1} \le l$, which means that $l \le \left\lfloor \frac{k+m}{m+1} \right\rfloor$ and $\left\lceil \frac{k}{m+1} \right\rceil \le l$, i.e. $l = \left\lfloor \frac{k+m}{m+1} \right\rfloor$.

From the proof of Theorem 1.1 as well as from the preceding remark one can feel that the existence of the image of f and x under the $\mathcal{Z}$-transform is not necessary. We prove that this is true, and the assumption on the exponential boundedness of the function f can be omitted.

Corollary 3.3. *The statement of Theorem 3.1 remains valid if f is not exponentially bounded.*

Proof. Using Remark 3.2 and the property of e_m^{Bk}; or simply setting

$$\varphi(k) = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ \mathbb{I}, & k \in \mathbb{Z}_{-m}^{0}, \end{cases}$$

$f \equiv \Theta$ in (1.1), (1.2), one can see that $\mathcal{A}(k)$ given by (3.2) is a matrix solution of the equation

$$\Delta X(k) = BX(k-m), \qquad k \geq 0$$

satisfying (1.5). Clearly, for $k \in \mathbb{Z}_{-m}^{-1}$,

$$\Delta\mathcal{A}(k) = \mathcal{A}(k+1) - \mathcal{A}(k) = \mathbb{I} - \mathbb{I} = \Theta = B\mathcal{A}(k-m).$$

Hence, $\mathcal{A}(k)$ fulfills (1.4), (1.5). Therefore, taking the difference of (3.1) for $k \in \mathbb{Z}_m^{\infty}$ and applying Lemma 2.1, we get

$$\begin{aligned}\Delta x(k) &= \Delta_k\mathcal{A}(k-m)\varphi(0) + B\sum_{j=-m}^{-1}\Delta_k\mathcal{A}(k-1-2m-j)\varphi(j) \\ &\quad + \sum_{j=1}^{k}\Delta_k\mathcal{A}(k-m-j)f(j-1) + \mathcal{A}(-m)f(k) \\ &= B\mathcal{A}(k-2m)\varphi(0) + B^2\sum_{j=-m}^{-1}\mathcal{A}(k-1-3m-j)\varphi(j) \\ &\quad + B\sum_{j=1}^{k}\mathcal{A}(k-2m-j)f(j-1) + f(k) = Bx(k-m) + f(k).\end{aligned}$$

That means that $x(k)$ of (3.1) solves (1.1) for $k \in \mathbb{Z}_m^{\infty}$.

If $k \in \mathbb{Z}_0^{m-1}$, then $k - m \in \mathbb{Z}_{-m}^{-1}$ and

$$\mathcal{A}(k-1-2m-j) = \begin{cases} \Theta, & j \in \mathbb{Z}_{k-m}^{-1}, \\ \mathbb{I}, & j \in \mathbb{Z}_{-m}^{k-m-1}. \end{cases}$$

Thus (3.1) gives

$$x(k) = \varphi(0) + B\sum_{j=-m}^{k-m-1}\varphi(j) + \sum_{j=1}^{k}f(j-1)$$

and

$$\Delta x(k) = B\varphi(k-m) + f(k)$$

for $k \in \mathbb{Z}_0^{m-1}$. The proof is finished. □

In the next result we investigate the case of multiple delays. We use the multinomial coefficient [1] defined as

$$\binom{a}{b_1, \dots, b_m} = \frac{a!}{b_1! \dots b_m!}$$

for $m \in \mathbb{N}$ and $a, b_1, \dots, b_m \in \mathbb{Z}_0^{\infty}$.

Theorem 3.4. *Let $n > 1$, $m_1, \dots, m_n \geq 1$, $m = \max\{m_1, \dots, m_n\}$, $B_1, \dots, B_n$ be pairwise permutable $N \times N$ matrices, $f : \mathbb{Z}_0^{\infty} \to \mathbb{R}^N$ an exponentially bounded function, and $\varphi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^N$ a given function. Solution $x(k)$ of the initial value problem (1.7), (1.2) has the form*

$$x(k) = \begin{cases} \varphi(k), & k \in \mathbb{Z}_{-m}^{0}, \\ \mathcal{B}(k)\varphi(0) + \sum_{j=1}^{n}B_j\sum_{i=-m_j}^{-1}\mathcal{B}(k-i-m_j-1)\varphi(i) & \\ \quad + \sum_{j=1}^{k}\mathcal{B}(k-j)f(j-1), & k \in \mathbb{N} \end{cases} \tag{3.4}$$

where

$$\mathcal{B}(k)=\sum_{\substack{i_1,\dots,i_n\geq 0\\ \sum_{j=1}^{n}(m_j+1)i_j\leq k}}\binom{k-\sum_{j=1}^{n}m_ji_j}{i_1,\dots,i_n,k-\sum_{j=1}^{n}(m_j+1)i_j}\prod_{j=1}^{n}B_j^{i_j}. \tag{3.5}$$

Proof. Since the proof is analogous to the proof of Theorem 3.1, we shall skip some details. Applying the $\mathcal{Z}$-transform to equation (1.7) gives

$$(z-1)X(z)-z\varphi(0)=\sum_{j=1}^{n}\frac{B_j}{z^{m_j}}\left(X(z)+\sum_{i=-m_j}^{-1}\frac{\varphi(i)}{z^i}\right)+F(z)$$

$$(z-1)\left(\mathbb{I}-\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right)X(z)=z\varphi(0)+\sum_{j=1}^{n}B_j\sum_{i=-m_j}^{-1}\frac{\varphi(i)}{z^{i+m_j}}+F(z).$$

Consequently, for z such large that

$$\left\|\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right\|<1$$

we get

$$x(k)=\mathcal{Z}^{-1}\left\{\frac{1}{z-1}\left(\mathbb{I}-\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right)^{-1}\times\left(z\varphi(0)+\sum_{j=1}^{n}B_j\sum_{i=-m_j}^{-1}\frac{\varphi(i)}{z^{i+m_j}}+F(z)\right)\right\}(k).$$

Hence, using

$$\left(\mathbb{I}-\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right)^{-1}=\sum_{i=0}^{\infty}\left(\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right)^{i},$$

we arrive at

$$x(k)=A_0(k)+\sum_{j=1}^{n}B_j\sum_{i=-m_j}^{-1}A_{ij}(k)+A_f(k)$$

where

$$A_0(k)=\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\sum_{i=0}^{\infty}\left(\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right)^{i}\varphi(0)\right\}(k),$$

$$A_{ij}(k)=\mathcal{Z}^{-1}\left\{\frac{1}{z^{i+m_j}(z-1)}\sum_{p=0}^{\infty}\left(\sum_{q=1}^{n}\frac{B_q}{z^{m_q}(z-1)}\right)^{p}\varphi(i)\right\}(k)$$

for $i\in\mathbb{Z}_{-m_j}^{-1}$, $j\in\mathbb{Z}_1^n$, and

$$A_f(k)=\mathcal{Z}^{-1}\left\{\frac{1}{z-1}\sum_{i=0}^{\infty}\left(\sum_{j=1}^{n}\frac{B_j}{z^{m_j}(z-1)}\right)^{i}F(z)\right\}(k).$$

The multinomial theorem

$$\left(\sum_{j=1}^{m}a_j\right)^{b}=\sum_{\substack{b_1,\dots,b_m\geq 0\\ \sum_{i=1}^{m}b_i=b}}\binom{b}{b_1,\dots,b_m}\prod_{j=1}^{m}a_j^{b_j}$$

for any real numbers $a_1, \dots, a_m$ and $b \in \mathbb{Z}_0^\infty$, along with Lemmas 2.4, 2.5 results in

$$\begin{aligned}
A_0(k) &= \sum_{i=0}^{\infty} \sum_{\substack{i_1,\dots,i_n \ge 0 \\ \sum_{j=1}^n i_j = i}} \binom{i}{i_1,\dots,i_n} \mathcal{Z}^{-1}\left\{ z^{1-\sum_{j=1}^n m_j i_j}(z-1)^{-1-i}\right\}(k) \left(\prod_{j=1}^n B_j^{i_j}\right)\varphi(0) \\
&= \sum_{i_1,\dots,i_n \ge 0} \binom{\sum_{j=1}^n i_j}{i_1,\dots,i_n} \left(\delta_{\sum_{j=1}^n m_j i_j - 1} * \binom{\cdot - 1}{\sum_{j=1}^n i_j} \sigma\left(\cdot - 1 - \sum_{j=1}^n i_j\right)\right)(k) \left(\prod_{j=1}^n B_j^{i_j}\right)\varphi(0) \\
&= \sum_{i_1,\dots,i_n \ge 0} \binom{\sum_{j=1}^n i_j}{i_1,\dots,i_n} \sigma\left(k - \sum_{j=1}^n m_j i_j + 1\right) \\
&\quad \times \binom{k - \sum_{j=1}^n m_j i_j}{\sum_{j=1}^n i_j} \sigma\left(k - \sum_{j=1}^n (m_j+1) i_j\right) \left(\prod_{j=1}^n B_j^{i_j}\right)\varphi(0) \\
&= \sum_{i_1,\dots,i_n \ge 0} \binom{k - \sum_{j=1}^n m_j i_j}{i_1,\dots,i_n, k - \sum_{j=1}^n (m_j+1) i_j} \sigma\left(k - \sum_{j=1}^n (m_j+1) i_j\right) \left(\prod_{j=1}^n B_j^{i_j}\right)\varphi(0) \\
&= \sum_{\substack{i_1,\dots,i_n \ge 0 \\ \sum_{j=1}^n (m_j+1) i_j \le k}} \binom{k - \sum_{j=1}^n m_j i_j}{i_1,\dots,i_n, k - \sum_{j=1}^n (m_j+1) i_j} \left(\prod_{j=1}^n B_j^{i_j}\right)\varphi(0) = \mathcal{B}(k)\varphi(0).
\end{aligned}$$

Next,

$$\begin{aligned}
A_{ij}(k) &= \sum_{p=0}^{\infty} \sum_{\substack{p_1,\dots,p_n \ge 0 \\ \sum_{q=1}^n p_q = p}} \binom{p}{p_1,\dots,p_n} \mathcal{Z}^{-1}\left\{ z^{-i-m_j-\sum_{q=1}^n m_q p_q}(z-1)^{-1-p}\right\}(k) \left(\prod_{q=1}^n B_q^{p_q}\right)\varphi(i) \\
&= \sum_{p_1,\dots,p_n \ge 0} \binom{\sum_{q=1}^n p_q}{p_1,\dots,p_n} \left(\delta_{i+m_j+\sum_{q=1}^n m_q p_q} * \binom{\cdot - 1}{\sum_{q=1}^n p_q} \sigma\left(\cdot - 1 - \sum_{q=1}^n p_q\right)\right)(k) \left(\prod_{q=1}^n B_q^{p_q}\right)\varphi(i) \\
&= \sum_{p_1,\dots,p_n \ge 0} \binom{\sum_{q=1}^n p_q}{p_1,\dots,p_n} \sigma\left(k - i - m_j - \sum_{q=1}^n m_q p_q\right) \\
&\quad \times \binom{k - i - m_j - \sum_{q=1}^n m_q p_q - 1}{\sum_{q=1}^n p_q} \sigma\left(k - i - m_j - \sum_{q=1}^n (m_q+1) p_q - 1\right) \left(\prod_{q=1}^n B_q^{p_q}\right)\varphi(i) \\
&= \sum_{p_1,\dots,p_n \ge 0} \binom{k - i - m_j - \sum_{q=1}^n m_q p_q - 1}{p_1,\dots,p_n, k - i - m_j - \sum_{q=1}^n (m_q+1) p_q - 1} \\
&\quad \times \sigma\left(k - i - m_j - \sum_{q=1}^n (m_q+1) p_q - 1\right) \left(\prod_{q=1}^n B_q^{p_q}\right)\varphi(i) \\
&= \sum_{\substack{p_1,\dots,p_n \ge 0 \\ \sum_{q=1}^n (m_q+1) p_q \le k-i-m_j-1}} \binom{k - i - m_j - \sum_{q=1}^n m_q p_q - 1}{p_1,\dots,p_n, k - i - m_j - \sum_{q=1}^n (m_q+1) p_q - 1} \left(\prod_{q=1}^n B_q^{p_q}\right)\varphi(i) \\
&= \mathcal{B}(k - i - m_j - 1)\varphi(i)
\end{aligned}$$

for any $i \in \mathbb{Z}_{-m_j}^{-1}$, $j \in \mathbb{Z}_1^n$. Finally,

$$\begin{aligned}
A_f(k) &= \sum_{i=0}^{\infty} \sum_{\substack{i_1,\dots,i_n \ge 0 \\ \sum_{j=1}^n i_j = i}} \binom{i}{i_1,\dots,i_n} \mathcal{Z}^{-1}\left\{ z^{-\sum_{j=1}^n m_j i_j}(z-1)^{-1-i} \left(\prod_{j=1}^n B_j^{i_j}\right) F(z)\right\}(k) \\
&= \sum_{i_1,\dots,i_n \ge 0} \binom{\sum_{j=1}^n i_j}{i_1,\dots,i_n} \left(\delta_{\sum_{j=1}^n m_j i_j} * \binom{\cdot - 1}{\sum_{j=1}^n i_j} \sigma\left(\cdot - 1 - \sum_{j=1}^n i_j\right) * \left(\prod_{j=1}^n B_j^{i_j}\right) f\right)(k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_1,\dots,i_n\ge 0} \binom{\sum_{j=1}^n i_j}{i_1,\dots,i_n} \left(\sigma\left(\cdot - \sum_{j=1}^n m_j i_j \right) \binom{\cdot - \sum_{j=1}^n m_j i_j - 1}{\sum_{j=1}^n i_j} \right.\\
&\quad \left. \times \sigma\left(\cdot - \sum_{j=1}^n (m_j+1) i_j - 1 \right) * \left(\prod_{j=1}^n B_j^{i_j} \right) f \right)(k)\\
&= \sum_{i_1,\dots,i_n\ge 0} \binom{\sum_{j=1}^n i_j}{i_1,\dots,i_n} \sum_{q=0}^{k} \binom{k-q-\sum_{j=1}^n m_j i_j - 1}{\sum_{j=1}^n i_j}\\
&\quad \times \sigma\left(k-q-\sum_{j=1}^n (m_j+1) i_j - 1 \right) \left(\prod_{j=1}^n B_j^{i_j} \right) f(q).
\end{aligned}
$$

Note that the inner sum vanishes when $q = k$ because of the σ operator. Next, we shift the inner sum $q+1 \mapsto q$ and change the order of sums to obtain

$$
\begin{aligned}
A_f(k) &= \sum_{q=1}^{k} \sum_{\substack{i_1,\dots,i_n\ge 0\\ \sum_{j=1}^n (m_j+1) i_j \le k-q}} \binom{\sum_{j=1}^n i_j}{i_1,\dots,i_n} \binom{k-q-\sum_{j=1}^n m_j i_j}{\sum_{j=1}^n i_j} \left(\prod_{j=1}^n B_j^{i_j} \right) f(q-1)\\
&= \sum_{q=1}^{k} \sum_{\substack{i_1,\dots,i_n\ge 0\\ \sum_{j=1}^n (m_j+1) i_j \le k-q}} \binom{k-q-\sum_{j=1}^n m_j i_j}{i_1,\dots,i_n, k-q-\sum_{j=1}^n (m_j+1) i_j} \left(\prod_{j=1}^n B_j^{i_j} \right) f(q-1)\\
&= \sum_{q=1}^{k} \mathcal{B}(k-q) f(q-1).
\end{aligned}
$$

This proves the statement. □

Remark 3.5.

1. Note that if $n = 1$, the matrix function $\mathcal{B}$ of (3.5) becomes the matrix function $\mathcal{A}(\cdot - m)$ for $\mathcal{A}$ given by (3.2) with $B = B_1$, $m = m_1$.
2. Considering (1.7) as a matrix equation with $f \equiv \Theta$ and the initial condition (1.6), one can see that $\mathcal{B}(k)$ solves (1.10). That means that $\mathcal{B}(k) = \mathcal{Y}(k) = \mathrm{e}_{m_1,\dots,m_n}^{B_1,\dots,B_n(k-m_n)}$ for any $k \in \mathbb{Z}$. In particular, if $n = 2$, $\mathcal{B}(k) = \tilde{\mathrm{e}}_{m_1 m_2}^{B_1 B_2 k}$ for any $k \in \mathbb{Z}$.

   As for the case of one delay, the assumption on exponential boundedness of f can be omitted.

Corollary 3.6. *The statement of Theorem 3.4 remains valid if f is not exponentially bounded.*

Proof. We only have to show that $x(k)$ of (3.4) solves (1.7). Let $k \in \mathbb{Z}_0^\infty$ be arbitrary and fixed. Denote $M \subset \{1,\dots,n\}$ the (possibly empty) set of all indices j such that $k < m_j$ if and only if $j \in M$. Then (3.4) states that

$$
\begin{aligned}
x(k) &= \mathcal{B}(k)\varphi(0) + \sum_{j\in M} B_j \sum_{i=-m_j}^{k-m_j-1} \mathcal{B}(k-i-m_j-1)\varphi(i)\\
&\quad + \sum_{M\not\ni j=1}^{n} B_j \sum_{i=-m_j}^{-1} \mathcal{B}(k-i-m_j-1)\varphi(i) + \sum_{j=1}^{k} \mathcal{B}(k-j) f(j-1).
\end{aligned}
$$

Now, we take the difference and apply Lemma 2.1 and Remark 3.5.2 to obtain

$$
\begin{aligned}
\Delta x(k) &= \sum_{q=1}^{n} B_q \mathcal{B}(k-m_q)\varphi(0)\\
&\quad + \sum_{j\in M} B_j \left(\mathcal{B}(0)\varphi(k-m_j) + \sum_{i=-m_j}^{k-m_j-1} \sum_{q=1}^{n} B_q \mathcal{B}(k-i-m_j-m_q-1)\varphi(i) \right)\\
&\quad + \sum_{M\not\ni j=1}^{n} B_j \sum_{i=-m_j}^{-1} \sum_{q=1}^{n} B_q \mathcal{B}(k-i-m_j-m_q-1)\varphi(i) + \mathcal{B}(0) f(k)\\
&\quad + \sum_{j=1}^{k} \sum_{q=1}^{n} B_q \mathcal{B}(k-m_q-j) f(j-1)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{M\not\ni q=1}^{n} B_q\mathcal{B}(k-m_q)\varphi(0)\\
&\quad + \sum_{j\in M} B_j\left(\varphi(k-m_j) + \sum_{M\not\ni q=1}^{n}\sum_{i=-m_j}^{k-m_j-1} B_q\mathcal{B}(k-i-m_j-m_q-1)\varphi(i)\right)\\
&\quad + \sum_{M\not\ni j=1}^{n} B_j \sum_{M\not\ni q=1}^{n}\sum_{i=-m_j}^{-1} B_q\mathcal{B}(k-i-m_j-m_q-1)\varphi(i) + f(k)\\
&\quad + \sum_{M\not\ni q=1}^{n}\sum_{j=1}^{k} B_q\mathcal{B}(k-m_q-j)f(j-1)\\
&= \sum_{j\in M} B_j\varphi(k-m_j)\\
&\quad + \sum_{M\not\ni q=1}^{n} B_q\left(\mathcal{B}(k-m_q)\varphi(0) + \sum_{j=1}^{n} B_j\sum_{i=-m_j}^{-1}\mathcal{B}(k-i-m_j-m_q-1)\varphi(i)\right.\\
&\quad \left. + \sum_{j=1}^{k}\mathcal{B}(k-m_q-j)f(j-1)\right) + f(k).
\end{aligned}
\tag{3.6}
$$

If $M=\{1,\dots,n\}$, the above identity says

$$
\Delta x(k) = \sum_{j=1}^{n} B_j x(k-m_j) + f(k) \tag{3.7}
$$

for $0\le k<\min\{m_1,\dots,m_n\}$, since $x(k-m_j)=\varphi(k-m_j)$ for each $j=1,\dots,n$. Now, if $k\in\mathbb{Z}_0^\infty$ is such that $m_{j_0}\le k<\min_{j\in M}\{m_j,2m_{j_0}\}$ for some $j_0\in\{1,\dots,n\}$ and $M=\{1,\dots,n\}\backslash\{j_0\}$, then $0\le k-m_{j_0}<m_{j_0}$ and we apply the previous case to show again that (3.6) yields (3.7) for $m_{j_0}\le k<\min_{j\in M}\{m_j,2m_{j_0}\}$. Similarly, one can prove the statement for arbitrary $k\in\mathbb{Z}_0^\infty$, i.e. the proof follows from (3.6) by induction with respect to k. □

Finally, we prove a result for a more general non-delayed term.

Theorem 3.7. *Let $n>1$, $m_1,\dots,m_n\ge 1$, $m=\max\{m_1,\dots,m_n\}$, A and $B_1,\dots,B_n$ be pairwise permutable $N\times N$ matrices, $\det A\neq 0$, $f:\mathbb{Z}_0^\infty\to\mathbb{R}^N$ and $\varphi:\mathbb{Z}_{-m}^0\to\mathbb{R}^N$ be given functions. Solution $x(k)$ of the initial value problem consisting of the equation*

$$
x(k+1) = Ax(k) + B_1x(k-m_1)+\dots+B_nx(k-m_n)+f(k),\quad k\ge 0 \tag{3.8}
$$

and initial condition(1.2)has the form

$$
x(k)=\begin{cases}
\varphi(k), & k\in\mathbb{Z}_{-m}^0,\\
\widetilde{\mathcal{B}}(k)\varphi(0)+\sum_{j=1}^{n}B_j\sum_{i=-m_j}^{-1}\widetilde{\mathcal{B}}(k-i-m_j-1)\varphi(i) & \\
\quad +\sum_{j=1}^{k}\widetilde{\mathcal{B}}(k-j)f(j-1), & k\in\mathbb{N}
\end{cases}
$$

where

$$
\widetilde{\mathcal{B}}(k) = A^k \sum_{\substack{i_1,\dots,i_n\ge 0\\ \sum_{j=1}^n(m_j+1)i_j\le k}} \binom{k-\sum_{j=1}^n m_ji_j}{i_1,\dots,i_n,k-\sum_{j=1}^n(m_j+1)i_j}\prod_{j=1}^n\widetilde{B}_j^{i_j}
$$

and $\widetilde{B}_i=B_iA^{-1-m_i}$ for $i=1,\dots,n$.

Proof. Let us take $x(k)=A^ky(k)$. Eq. (3.8) becomes

$$
\Delta y(k)=\widetilde{B}_1y(k-m_1)+\dots+\widetilde{B}_ny(k-m_n)+\tilde{f}(k),\qquad k\in\mathbb{Z}_0^\infty
$$

with $\tilde{f}(k)=A^{-1-k}f(k)$ for each $k\in\mathbb{Z}_0^\infty$. We also obtain the new initial condition

$$
y(k)=A^{-k}\varphi(k)=:\widetilde{\varphi}(k),\qquad k\in\mathbb{Z}_{-m}^0.
$$

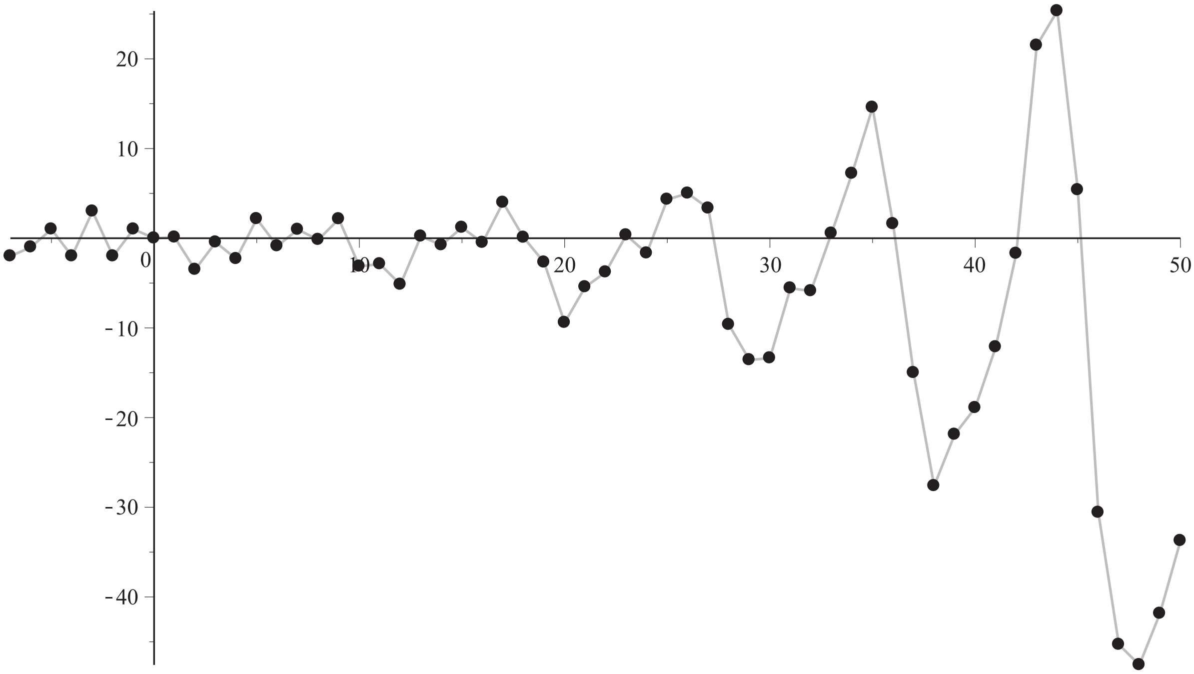

Fig. 1. The solution of (3.9), (3.11) (for a better orientation we added a line subsequently connecting the discrete values of the solution).

Applying Corollary 3.6 to this initial value problem, we obtain

$$y(k) = \mathcal{B}(k)\widetilde{\varphi}(0) + \sum_{j=1}^{n} \widetilde{B}_j \sum_{i=-m_j}^{-1} \mathcal{B}(k-i-m_j-1)\widetilde{\varphi}(i)$$

$$+ \sum_{j=1}^{k} \mathcal{B}(k-j)\tilde{f}(j-1), \qquad k \in \mathbb{N}$$

for $\mathcal{B}$ given by (3.5) with $\widetilde{B}_j$ instead of B_j. When one returns to $x(k)$, the statement is proved. $\square$

Now, we present an application of the results derived. For practical computations, we rewrite the sum in (3.5) into a form of iterated sums,

$$\sum_{\substack{i_1,\ldots,i_n \geq 0 \\ \sum_{j=1}^{n}(m_j+1)i_j \leq k}} = \sum_{i_1=0}^{\left\lfloor \frac{k}{m_1+1} \right\rfloor} \sum_{i_2=0}^{\left\lfloor \frac{k-(m_1+1)i_1}{m_2+1} \right\rfloor} \cdots \sum_{i_n=0}^{\left\lfloor \frac{k-\sum_{j=1}^{n}(m_j+1)i_j}{m_n+1} \right\rfloor}.$$

Now, a solution $x(k)$ given by (3.4) can be easily computed by hand or using a software.

Example 3.8. Let us consider the following scalar equation

$$\Delta x(k) = -0.5x(k-1) + 0.3x(k-3) - 0.3x(k-4) \tag{3.9}$$

$$-0.1x(k-6) + x(k-7) + (-1)^k, \qquad k \in \mathbb{Z}_0^{\infty} \tag{3.10}$$

with the initial condition

$$(x(-7), \ldots, x(0)) = (-2, -1, 1, -2, 3, -2, 1, 0). \tag{3.11}$$

Using Theorem 3.4 we computed the solution depicted in Fig. 1. For instance, $x(50) \doteq -33.739$.

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