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On uniqueness of numerical solution of boundary integral equations with 3-times monotone radial kernels

Zeynab Sedaghatjoo [a], Mehdi Dehghan [a,*], Hossein Hosseinzadeh [b]

[a] Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, No. 424 Hafez Avenue, Tehran, Iran

[b] Department of Mathematics, Persian Gulf University, Bushehr, Iran

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ABSTRACT

The uniqueness of solution of boundary integral equations (BIEs) is studied here when geometry of boundary and unknown functions are assumed piecewise constant. In fact we will show BIEs with 3-times monotone radial kernels have unique piecewise constant solution. In this paper nonnegative radial function $\mathcal{F}^{\delta_3}$ is introduced which has important contribution in proving the uniqueness. It can be found from the paper if δ_3 is sufficiently small then eigenvalues of the boundary integral operator are bigger than $\mathcal{F}^{\delta_3}/2$. Note that there is a smart relation between δ_3 and boundary discretization which is reported in the paper, clearly. In this article an appropriate constant c_0 is found which ensures uniqueness of solution of BIE with logarithmic kernel $\ln(c_0 r)$ as fundamental solution of Laplace equation. As a result, an upper bound for condition number of constant Galerkin BEMs system matrix is obtained when the size of boundary cells decreases. The upper bound found depends on three important issues: geometry of boundary, size of boundary cells and the kernel function. Also non-singular BIEs are proposed which can be used in boundary elements method (BEM) instead of singular ones to solve partial differential equations (PDEs). Then singular boundary integrals are vanished from BEM when the non-singular BIEs are used. Finally some numerical examples are presented which confirm the analytical results.

1. Introduction

Nowadays, the Boundary Elements Method (BEM) has received much attention from researchers. It has become an important technique in the computational solution of a number of physical problems in various fields, such as stress analysis, potential flow, fracture mechanics and acoustics [1]. Also the interested reader can see [2–5] for application on some problems.

1.1. Boundary integral equation

Let Ω be a bounded domain with Lipschitz boundary Γ, and $\mathcal{L}$ be partial differential operator defined on unknown function $u : \Omega \to \mathbb{R}$ with Dirichlet boundary condition [6,7]. Solving boundary value problem (BVP)

$$\mathcal{L}u(x) = f(x) \quad \text{for } x \in \Omega \quad \text{and} \quad u(x) = \bar{u} \quad \text{for } x \in \Gamma, \tag{1.1}$$

* Corresponding author.
E-mail addresses: z.sedaqatjoo@aut.ac.ir, zeinab.sedaghatjoo@gmail.com (Z. Sedaghatjoo), mdehghan@aut.ac.ir, mdehghan.aut@gmail.com (M. Dehghan), h_hosseinzadeh@aut.ac.ir, hosseinzadeh@pgu.ac.ir (H. Hosseinzadeh).

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is our aim. If $K(\|x-y\|)$ be fundamental solution of operator $\mathcal{L}$ [6,7], i.e.

$$\mathcal{L}K(\|x-y\|) = -\delta_{Di}(\|x-y\|), \tag{1.2}$$

when δ_{Di} is Dirac delta, via Green's identities the BVP (1.1) is replaced by the boundary integral equation (BIE)

$$\int_{\Gamma} K(\|x-y\|)q(y)\,d\Gamma_y = g(x) \quad \text{for } x \in \Gamma, \tag{1.3}$$

when the known function g is evaluated as

$$g(x) = -u(x)\int_{\Omega} \delta_{Di}(\|x-y\|)\,d\Omega_y + \int_{\Gamma} \nabla K(\|x-y\|).n(y)\,u(y)\,d\Gamma_y, \tag{1.4}$$

where $n(y)$ is outward normal vector over the boundary at y and q is flux function, i.e. $q = \nabla u.n$ [7]. If BIE (1.3) has unique solution it can be deduced that BVP (1.1) also has unique solution

$$u(x) = \int_{\Gamma} K(\|x-y\|)q(y)dy - \int_{\Gamma} \nabla K(\|x-y\|).n(y)\,u(y)dy \quad \text{for } x \in \Omega.$$

So uniqueness of solution of BIE (1.3) is an important issue in BEM. In fact BIE makes the basis of the BEM, and this method only transforms the BIE into a linear system of equations. The solvability of this system depends on the condition number of the corresponding system matrix (BEM-matrix), because if the BIE does not have a unique solution then the system of equations in the BEM may not have a unique solution, and BEM-matrix will be ill-conditioned. So, we must study the condition number when the uniqueness of the solution is considered. Unfortunately little attention has been paid to this fundamental issue [8]. For simplicity some engineers use constant boundary elements method for solving BIE (1.3) and they assume q as a piecewise constant function over the boundary [9]. This strategy is named as constant BEM. This assumption enables one to calculate boundary integrals easily and it yields a simple linear system [9]. In this paper the uniqueness of constant BEM is proved and invertibility of the corresponding system matrix is investigated. In fact we will show BIE (1.3) has unique piecewise constant solution when kernel K is a 3-times monotone radial function [10]. Some important and applicable kernels which are used as fundamental solutions in BEM are listed here:

- $-\frac{1}{2\pi}\ln(\lambda\,\|x-y\|)$ for a $\lambda > 0$ (fundamental solution of 2D Laplace equation),
- $\frac{1}{2\pi}K_0(\lambda\|x-y\|)$ for a $\lambda \neq 0$ (fundamental solution of 2D Helmholtz equation),
- $\frac{1}{4\pi}\|x-y\|^{-1}$ (fundamental solution of 3D Laplace equation),
- $\frac{1}{4\pi}\exp(-\lambda\,\|x-y\|)\|x-y\|^{-1}$ for a $\lambda \neq 0$ (fundamental solution of 3D Helmholtz equation),

where K_0 is the modified Bessel's function of the first kind and order 0. Note that all of them except the first one, are 3-times monotone radial kernels. In this paper also we will show that the fundamental solution of 2D Laplace equation can be extended to a 3-times monotone function if we get $\lambda = 1/(4\,\mathcal{D})$ when $\mathcal{D}$ is diameter of Ω.

1.2. Literature review

There are some interesting results in the literature for the uniqueness of the solution of the BIE arising from Laplace equation. Researchers in [11–13], concluded that the BIE for Laplace equation with Dirichlet boundary conditions does not have a unique solution if the scaling of the domain is inappropriate. Now a question is how we can scale the boundary such that a unique solution can be obtained. Logarithmic capacity, C_L, is introduced in [14] for BIE with logarithmic kernels that is strongly related to the Euclidean diameter (see [15]) and it is shown that if $C_L = 1$ then there is no unique solution. Logarithmic capacity can be calculated explicitly for few domains, and its upper and lower bounds are reported in [16,17].

Authors of [18–22] have demonstrated that the condition number of the BEM-matrix for the Laplace equation in 2D with Dirichlet boundary conditions may become infinitely large when $C_L = 1$. Also it can be shown that the same result is valid for the equation with mixed boundary conditions [8]. Authors of [23–25] have shown that the condition number of the BEM-matrix for the Helmholtz equation may also become infinitely large. So in all these cases, the system is ill-conditioned and solving the linear system may be very difficult.

Authors of [26] studied the condition number of BEM-matrix for solving the Stokes equations on a two-dimensional domain supplemented with Dirichlet or mixed boundary conditions. They showed the condition number of the matrix is infinitely large, and for certain critical boundary contours, boundary integral equation is not solvable. Hence, for these critical contours the Stokes equations cannot be solved with the BEM. To overcome this problem, the connected domain can be rescaled.

In [27], ill-conditioning of BEM system is analytically examined by the use of degenerate kernel scheme. Five regularization techniques, namely hypersingular formulation, the method of adding a rigid body mode, rank promotion by adding the boundary flux equilibrium (direct BEM), the Combined Helmholtz Exterior Integral Equation Formulation (CHEIEF) method and the Fichera's method (indirect BEM), are analytically studied and numerically implemented to ensure the unique solution. The authors of [27] examined the sufficient and necessary conditions of boundary integral formulation for the uniqueness of solution of 2D Laplace problem subject to the Dirichlet boundary condition. They employed the degenerate kernel in the polar and elliptic coordinates to derive the unique solution by using five regularization techniques for circles and ellipses, respectively. The Fichera's method is used in [27] and it has analytically shown that the influence matrix in the BEM has full rank.

Since fundamental solutions of BEM are singular, some singular and near singular integrals are appeared in this method where the accuracy of solution depends on them, significantly. Treatment of singular integrals arising from boundary evaluation has been intensively studied in BEM literature. Some principal strategies are developed in [28] to deal with various singular integrals. Also there are several ideas applied in BEM [29–35] which use specific ideas to obtain the boundary integrals, accurately. In this paper non-singular radial basis functions (RBFs) are introduced which can be applied as BEM's fundamental solutions. Then singular boundary integrals are vanished from BEM when this modification is used.

1.3. Outline of this paper

Some mathematical backgrounds and basic relations are mentioned in Section 2. In this section non-negative function $\mathcal{F}^{\delta_3}$ is introduced. After that, in Section 3 we will find an appropriate $\delta_3 > 0$ such that $\mathcal{F}^{\delta_3}/2$ is a lower bound of eigenvalues of boundary integral operator with kernel K in space of all piecewise constant functions. So the uniqueness of solution of boundary integral equation (1.3) is proven in this section. In Section 4 relation between δ_3 and boundary discretization will be studied when the size of boundary cells is sufficiently small. Section 5 is devoted to find an appropriate constant c_0 for which 2D Laplace fundamental solution, $-\frac{1}{2\pi}\ln(c_0\|x-y\|)$, is extendable to a 3-times monotone function over $\mathbb{R}^s$. In fact in that section we will show for $c_0 = 1/4\mathcal{D}$ where $\mathcal{D}$ is diameter of Ω, the uniqueness of solution of BIE (1.3) is guaranteed. In Section 6 new non-singular fundamental solutions are founded for BEM and corresponding boundary integral equations are derived. So singular conventional BIEs are replaced by non-singular ones. The non-singular BIEs can be solved via numerical schemes, easily. Note the uniqueness of solution of the introduced BIEs is still guaranteed. In Section 7 an upper bound for condition number of matrices of constant Galerkin BEM has been derived which shows these matrices are not ill-conditioned when the computational domain Ω is Lipschitz. Analytical results obtained in this paper are validated numerically in Section 8. In fact the upper bound achieved in Section 7 is verified by various test problems, and some PDEs are solved via non-singular BEM based on non-singular BIEs proposed in Section 6. Numerical results show the new non-singular BEM has accurate results.

2. Mathematical backgrounds and primary relations

This section is devoted to introducing some definitions and theorems which one needs to know as his/her background.

Definition 1 (*[36–38]*). A bounded domain $\Omega \subset \mathbb{R}^s$ is called Lipschitz domain if locally its boundary Γ is the graph of a Lipschitz function ψ ($\psi : \mathbb{R}^{s-1} \to \mathbb{R}$ is a Lipschitz function if it satisfies the Lipschitz condition with constant M, i.e.

$$|\psi(y') - \psi(x')| \le M\|y' - x'\|, \quad \text{for all } y',\ x' \in \mathbb{R}^{s-1}). \tag{2.1}$$

In other words a bounded domain Ω is Lipschitz domain if for every $p \in \Gamma$, up to an orthogonal change of coordinates (i.e. after a rotation), there is an open ball $B(p, \epsilon_p) \subset \mathbb{R}^s$, and a Lipschitz function ψ such that

$$\Omega \cap B(p, \epsilon_p) = \{(x', x_n) \in B(p, \epsilon_p) \mid x_n > \psi(x')\},$$
$$\Gamma \cap B(p, \epsilon_p) = \{(x', x_n) \in B(p, \epsilon_p) \mid x_n = \psi(x')\}.$$

The smallest M in which Eq. (2.1) holds will be called the bound of the Lipschitz constant over $B(p, \epsilon_p)$. By choosing finitely many balls $\{U_i\}$ covering Γ, the Lipschitz constant, L_{Lip}, for a Lipschitz domain is the biggest M such that the Lipschitz constant is bounded by it in every ball U_i. Also Lipschitz radius, r_{Lip} is defined as a positive real number where for every $x \in \Gamma$ there is an open ball U_i such that $B(x, r_{Lip}) \subseteq U_i$ [36–38].

Of course, all smooth domains are Lipschitz and polygonal domains in $\mathbb{R}^2$ and polyhedrons in $\mathbb{R}^3$ are well-known examples for non-smooth Lipschitz domains [37,38].

Definition 2 (*[39]*). Let $\Omega \subseteq \mathbb{R}^s$ be a Lipschitz domain and Γ be its boundary. We refer to $L^2(\Gamma)$ as the linear space of integrable functions on Γ which is Hilbert space with respect to the inner product

$$\langle v, w\rangle_\Gamma = \int_\Gamma v(x)w(x)\, d\Gamma.$$

Note that $L^2(\Gamma)$ is a Banach space with respect to the norm

$$\|v\|_\Gamma = \left(\int_\Gamma |v(x)|^2\, d\Gamma\right)^{1/2}.$$

If v is a real function defined over $A \subset \Gamma$, then v can be extended to the boundary function $\hat{v} : \Gamma \to \mathbb{R}$ which is evaluated as $v(x)$ for $x \in A$ and zero otherwise. In this situation $\|v\|_A$ can be defined as

$$\|v\|_A = \|\hat{v}\|_\Gamma = \left(\int_A |v(x)|^2 \, d\Gamma \right)^{1/2},$$

where the subscript A for the norm is a notation to highlight v is defined only on A [39].

Definition 3. Let $\Omega \subset \mathbb{R}^s$ be a Lipschitz domain with boundary Γ. The volume $A \subset \Omega$ is introduced here as surface of that area of Γ which lies in A, in the mathematical sense we write

$$vol(A) = (\|\chi_{A\cap\Gamma}\|_\Gamma)^2 = \int_{A\cap\Gamma} d\Gamma,$$

where χ_B is characteristic function of set B (i.e. $\chi_B(x) = 1$ for $x \in B$ and $\chi_B(x) = 0$ otherwise).

Definition 4 (*[40]*)**.** Let A be a subset of metric space X. Closure of A is the smallest closed set which contains A and is denoted by $cl(A)$ or $\bar{A}$ [40].

Definition 5 (*[41]*)**.** Let $A : X \to X$ be a bounded linear operator mapping a normed space X into itself. Then a complex number λ is called an eigenvalue of A if there exists an element $V \in X$, $V \neq 0$, such that $AV = \lambda V$. V is called an eigenelement (eigenfunction) of A. In particular case, for boundary integral operators, λ is an eigenvalue corresponding to kernel K if there exists function q where

$$\int_\Gamma K(\|x-y\|)q(y)d\Gamma_y = \lambda q(x), \quad \text{for all } x \in \Gamma,$$

when $\|t\|$ is Euclidean norm of $t \in \mathbb{R}^s$ [41].

Definition 6 (*[42]*)**.** A real-valued continuous function $\Phi : \mathbb{R}^s \to \mathbb{R}$ is called positive definite on $\mathbb{R}^s$ if

$$\sum_{i=1:N} \sum_{j=1:N} c_i \Phi(x_i - x_j) c_j \geq 0, \tag{2.2}$$

for any N pairwise different points $x_1, x_2, \dots, x_N \in \mathbb{R}^s$, and $\mathbf{C} = [c_1, \dots, c_N]^T \in \mathbb{R}^N$.

The function Φ is called strictly positive definite on $\mathbb{R}^s$ if the quadratic form (2.2) is zero only for $\mathbf{C} = \mathbf{0}$ [42].

Theorem 7 (*[43]*)**.** *Let Φ be positive definite function, then Φ also is integrally positive definite on every bounded region $\Omega \subset \mathbb{R}^s$, i.e.*

$$\int_\Omega \int_\Omega u(x)\Phi(x-y)u(y)\, dxdy \geq 0,$$

for every function $u \in L_2(\Omega)$ [43].

Definition 8 (*[42]*)**.** A function $\phi : [0, \infty) \to \mathbb{R}$ which is in $C^{k-2}(0, \infty)$, $k \geq 2$, and $(-1)^l \phi^{(l)}(r)$ is non-negative, non-increasing and convex for $l = 0, 1, 2, \dots, k-2$ is called k-times monotone [42].

Remark 9. From Definition 8 we can find if $\phi \in C^k(0, \infty)$, and $(-1)^l \phi^{(l)}(r) \geq 0$ for $l = 0, 1, 2, \dots, k$, then ϕ is k-times monotone.

Theorem 10 (*Micchelli [42]*)**.** *Let $k = [s/2] + 2$ be a positive integer. If $\phi : [0, \infty) \to \mathbb{R}$, $\phi \in C[0, \infty)$, is k-times monotone on $(0, \infty)$ but not constant, then ϕ is strictly positive definite and radial on $\mathbb{R}^s$ for any s such that $[s/2] \leq k-2$* [42].

Remark 11. Theorem 10 and Remark 9 yield if $\phi \in C^3(0, \infty)$ is 3-times monotone, so ϕ is strictly positive definite on $\mathbb{R}^2$ and $\mathbb{R}^3$.

Remark 12. Let $K : (0, \infty) \to \mathbb{R}$ be a 3-times monotone function. So for each $\delta > 0$ function $\tilde{K} : [0, \infty) \to \mathbb{R}$ defined as

$$\tilde{K}(r) := \begin{cases} \displaystyle\sum_{k=0}^{3} K^{(k)}(\delta)(r-\delta)^k/k! & \text{if } 0 \leq r \leq \delta, \\ K(r) & \text{if } \delta < r, \end{cases}$$

is also a 3-times monotone function. Superscript (k) denotes derivative of order k with respect to r.

Definition 13. Function $\mathcal{K}^\delta := K - \tilde{K}$ is a non-negative radial function which vanishes everywhere except may be on $[0, \delta)$. Furthermore, it inherits being 3-times monotone function from K. If $B(0, \delta) = B^s(0, \delta)$ be a ball with center 0 and radius δ in $\mathbb{R}^s$, then $\mathcal{F}^\delta$ defined as

$$\mathcal{F}^\delta := \int_{B^{s-1}(0,\delta)} \mathcal{K}^\delta(\|x\|)\, dx,$$

is non-negative function with respect to δ. If $\mathcal{K}^\delta(0) > 0$, then $\mathcal{F}^\delta > 0$.

$\mathcal{F}^\delta$ is calculated for some special kernels and results are reported in third column of Table 1. One can see in Table 1 that $\mathcal{F}^\delta$ decreases as $O(\delta)$ ($\simeq c\,\delta$ for a $c > 0$ when δ is sufficiently small) for singular fundamental solutions of BEM.

Remark 14. Let Ω be a bounded region in $\mathbb{R}^s$ and Γ be its boundary. Also let $q : \Gamma \to \mathbb{R}$ be a piecewise continuous function and $\tilde{K} : [0, \infty) \to \mathbb{R}$ be 3-times monotone function. So

$$\int_\Gamma \int_\Gamma q(x)\tilde{K}(\|x-y\|)\, q(y)\, d\Gamma_y d\Gamma_x = \lim_{N\to\infty} \left(\frac{vol(\Gamma)}{N}\right)^2 \sum_{i=1}^N \sum_{j=1}^N q(x_i)\tilde{K}(\|x_i - x_j\|)\, q(x_j),$$

when boundary nodes $\{x_i\}_{i=1}^N$ are selected over Γ semi-uniformly. Since quadratic form $\sum_{i=1}^N \sum_{j=1}^N q(x_i)\tilde{K}(\|x_i - x_j\|)\, q(x_j)$ is bigger than zero for every N, we can deduce

$$\int_\Gamma \int_\Gamma q(x)\tilde{K}(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x \geq 0,$$

i.e. $\tilde{K}$ is integrally positive definite.

Corollary 1. *From Remark 14 and the fact that $K = \tilde{K} + \mathcal{K}^\delta$ we have*

$$\int_\Gamma \int_\Gamma q(x)\, K(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x \geq \int_\Gamma \int_\Gamma q(x)\, \mathcal{K}^\delta(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x.$$

So a lower bound for right hand side of this equation also is true for the left one.

3. Lower bound for the smallest eigenvalue of K

Suppose Ω is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with boundary Γ. Here we suppose that Γ is approximated by straight lines and polyhedrons when Ω is in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. These boundary cells are named as boundary elements and are numerated as $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ for an integer number N. So

$$\Gamma = \cup_{i=1}^N \Gamma_i.$$

In Fig. 1 two kinds of boundary discretization (with triangle and square cells) are depicted for a region $\Omega \subset \mathbb{R}^3$. Also let $q : \Gamma \to \mathbb{R}$ be a piecewise constant function defined over the boundary, i.e. q is assumed to be fixed over each boundary element. In the mathematical sense $q(x) = \sum_{i=1}^N q_i \chi_i(x)$ when q_i is a constant number and $\chi_i(x)$ is 1 for $x \in \Gamma_i$ and 0, otherwise.

Before presenting the following lemmas, we define some parameters:

$$\begin{aligned}
\partial\Gamma_i &= \text{set of corner points of } \Gamma_i \text{ (see Fig. 1)},\\
B_i^\delta &= \cup_{x\in\partial\Gamma_i} \left(B(x,\delta) \cap \Gamma_i\right), \qquad B^\delta = \cup_{i=1}^N B_i^\delta,\\
A_i^\delta &= \Gamma_i \backslash B_i^\delta, \qquad A^\delta = \cup_{i=1}^N A_i^\delta,\\
\Gamma_{nei,i} &= \cup_{j=1:N}\{\Gamma_j \mid \Gamma_j \text{ is neighbor of } \Gamma_i \text{ i.e. } \partial\Gamma_j \cap \partial\Gamma_i \text{ is not empty}\},\\
M_{nei,i} &= \text{ number of boundary elements which are neighbors of } \Gamma_i\,,\\
M_{nei} &= \max_{i=1:n} M_{nei,i}.
\end{aligned}$$

Note that B_i^δ is an open cover for $\partial\Gamma_i$ and A_i^δ is a closed set in Γ_i, beside $B_i^\delta \cup A_i^\delta = \Gamma_i$ and $B_i^\delta \cap A_i^\delta$ is empty. So the same relations are valid for B^δ and A^δ, i.e. $B^\delta \cup A^\delta = \Gamma$ and $B^\delta \cap A^\delta$ is empty. Two boundary elements Γ_i and Γ_j are neighbors if their boundaries have common point. It is clear that each element is neighbor of itself. From definition of $\Gamma_{nei,i}$ we can deduce that there is a positive distance between Γ_i and $\Gamma\backslash\Gamma_{nei,i}$ for $i = 1, 2, \ldots, N$. The integer number M_{nei} depends on arrangement of boundary elements and remains fixed when boundary discretization has regular pattern. For example M_{nei} with respect to the boundary discretization (shown in Fig. 1) is 13 and 9 for triangle and square discretizations, respectively.

Table 1
Relation between kernel K, positive number $\mathcal{F}^\delta$, upper bound of $cond(\mathbf{A})$ and numerical results of $cond(\mathbf{A})$. $cond(\mathbf{A}) = O(N^n)$ shows $cond(\mathbf{A}) \simeq cN^n$ for a $c > 0$ when N is sufficiently large, and $\mathcal{F}^\delta = O(\delta^n)$ shows $\mathcal{F}^\delta \simeq c\delta^n$ for a $c > 0$ when δ is sufficiently small.

Dimension	$K(r)$	$\mathcal{F}^\delta$	Upper bound of $cond(\mathbf{A})$	Numerical result of $cond(\mathbf{A})$
$\mathbb{R}^2$	$-\ln(r/16)$	$O(\delta)$	$O(N)$	$O(N)$
$\mathbb{R}^2$	$K_0(r)$	$O(\delta)$	$O(N)$	$O(N)$
$\mathbb{R}^2$	$\exp(-r)$	$O(\delta^5)$	$O(N^5)$	$O(N^2)$
$\mathbb{R}^2$	$1/\sqrt{1+r}$	$O(\delta^5)$	$O(N^5)$	$O(N^2)$
$\mathbb{R}^3$	$1/r$	$O(\delta)$	$O(N^{0.5})$	$O(N^{0.5})$
$\mathbb{R}^3$	$\exp(-r)/r$	$O(\delta)$	$O(N^{0.5})$	$O(N^{0.5})$
$\mathbb{R}^3$	$\exp(-r)$	$O(\delta^6)$	$O(N^3)$	$O(N^{1.5})$
$\mathbb{R}^3$	$1/\sqrt{1+r}$	$O(\delta^6)$	$O(N^3)$	$O(N^{1.5})$

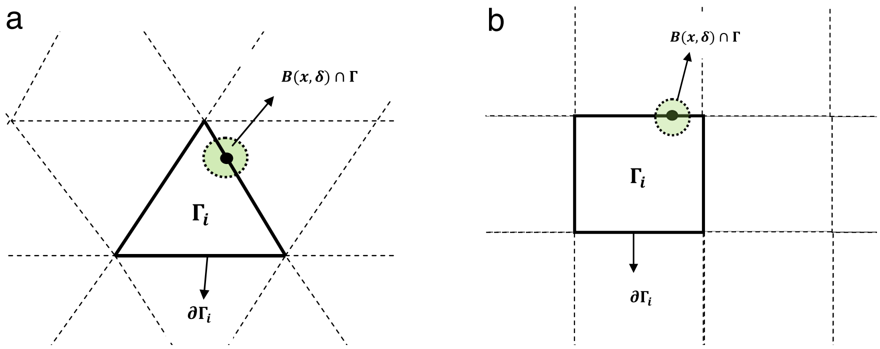

Fig. 1. Boundary element Γ_i and its corner points $\partial\Gamma_i$ for two kinds of boundary discretization, triangular and square grids. Set $B_i^\delta = \cup_{x\in\partial\Gamma_i} B(x,\delta)\cap\Gamma_i$ is an open cover for $\partial\Gamma_i$ in Γ_i.

Lemma 1. *There is a $\delta_1 > 0$ such that for $i = 1, 2, \ldots, N$ we have*

$$vol(B_i^{2\delta_1}) \le \frac{1}{4M_{nei}} vol(\Gamma_i), \qquad vol(A_i^{\delta_1}) \ge \frac{3}{4} vol(\Gamma_i).$$

Proof. Let $\Omega \subset \mathbb{R}^s$. Since Lebesgue measure of $\partial\Gamma_i$ is zero in $\mathbb{R}^{s-1}$, for each $\epsilon > 0$ there is a positive number δ such that $vol(B_i^{2\delta}) < \epsilon$ for $i = 1, 2, \ldots, N$. So δ_1 is obtained if we put $\epsilon = \frac{1}{4M_{nei}} \min\{vol(\Gamma_1), vol(\Gamma_2), \ldots, vol(\Gamma_N)\}$. For this ϵ we get

$$vol(B_i^{2\delta_1}) \le \epsilon \le \frac{1}{4M_{nei}} vol(\Gamma_i),$$

for $i = 1, 2, \ldots, N$. Beside, $M_{nei} \ge 1$ and $A_i^{2\delta_1} \subseteq A_i^{\delta_1}$ which yield

$$vol(A_i^{\delta_1}) \ge vol(A_i^{2\delta_1}) = vol(\Gamma_i) - vol(B_i^{2\delta_1}) \ge \left(1 - \frac{1}{4M_{nei}}\right) vol(\Gamma_i) \ge \frac{3}{4} vol(\Gamma_i). \quad \square$$

Corollary 2. *For piecewise constant function q defined over the boundary we have*

$$\|q\|^2_{B^{2\delta_1}} \le \frac{1}{4M_{nei}} \|q\|^2_\Gamma, \qquad \|q\|^2_{A^{\delta_1}} \ge \frac{3}{4} \|q\|^2_\Gamma.$$

Proof. Since q is piecewise constant function, from Lemma 1 we obtain

$$\|q\|^2_{B^{2\delta_1}} = \int_{B^{2\delta_1}} q(x)^2 d\Gamma = \sum_{i=1:N} q_i^2 vol(B_i^{2\delta_1}) \le \frac{1}{4M_{nei}} \sum_{i=1:N} q_i^2 vol(\Gamma_i) = \frac{1}{4M_{nei}} \|q\|^2_\Gamma,$$

and

$$\|q\|^2_{A^{\delta_1}} = \int_{A^{\delta_1}} q(x)^2 d\Gamma = \sum_{i=1:N} q_i^2 vol(A_i^{\delta_1}) \ge \frac{3}{4} \sum_{i=1:N} q_i^2 vol(\Gamma_i) = \frac{3}{4} \|q\|^2_\Gamma. \quad \square$$

Lemma 2. *There is a positive number $\delta_2 \le \delta_1$ such that for every $\delta \le \delta_2$ and $x \in A_i^{\delta_1}$, if $y \in \Gamma\backslash\Gamma_i$ then we have $\|x - y\| \ge \delta$ for $i = 1, 2, \ldots, N$.*

Proof. At first suppose $\delta_2 = \delta_1$ so we will update it later. Since $B_i^{\delta_1}$ is an open set in Γ_i, then $A_i^{\delta_1}$ and closure $\Gamma\backslash\Gamma_i$ are two disjoint compact sets in $\mathbb{R}^s$. So there is distance $r_i > 0$ between these sets for $i = 1, 2, \ldots, N$. Now update δ_2 as

$$\delta_2 = \min\{r_1, r_2, \ldots, r_N, \delta_1\}.$$

It is clear that if $x \in A_i^{\delta_1}$ and $y \in (\Gamma\backslash\Gamma_i)$ we have $\|x - y\| \geq r_i \geq \delta_2 \geq \delta$ for $i = 1, 2, \ldots, N$. □

The next corollary is obtained straightly from Lemma 2. It must be mentioned that $\delta_2 \leq \delta_1$ is highlighted in Lemma 2 because it is an essential condition for this corollary.

Corollary 3. *Let δ_2 be what was found in Lemma 2, then*

$$\min_{x\in A^{\delta_1}} \int_\Gamma \mathcal{K}^{\delta_2}(\|x-y\|)\, d\Gamma_y = \mathcal{F}^{\delta_2}.$$

Proof. If $t = x - y$, we obtain

$$\begin{aligned}\min_{x\in A^{\delta_1}} \int_\Gamma \mathcal{K}^{\delta_2}(\|x-y\|)\, d\Gamma_y &= \min_{i=1:N}\min_{x\in A_i^{\delta_1}} \int_\Gamma \mathcal{K}^{\delta_2}(\|x-y\|)\, d\Gamma_y = \min_{i=1:N}\min_{x\in A_i^{\delta_1}} \int_{\Gamma_i} \mathcal{K}^{\delta_2}(\|x-y\|)\, d\Gamma_y \\ &= \min_{i=1:N} \int_{B^{s-1}(0,\delta_2)} \mathcal{K}^{\delta_2}(\|t\|)\, dt = \mathcal{F}^{\delta_2}. \quad \square\end{aligned}$$

Lemma 3. *There is a positive number $\delta_3 \leq \delta_2$ such that*

$$\max_{x\in\Gamma} \int_\Gamma \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y \leq M_{nei}\, \mathcal{F}^{\delta_3}.$$

Proof. Sets Γ_i and closure $\Gamma\backslash\Gamma_{nei,i}$ are two disjoint closed and bounded sets in $\mathbb{R}^s$ then there is a positive distance between them, say r_i. Now if we get

$$\delta_3 = \min\{r_1, r_2, \ldots, r_N, \delta_2\},$$

we have $(B(x, \delta_3) \cap \Gamma) \subseteq \Gamma_{nei,i}$ when $x \in \Gamma_i$ for $i = 1, 2, \ldots, N$. Consequently we obtain

$$\begin{aligned}\max_{x\in\Gamma} \int_\Gamma \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y &= \max_{i=1:N}\max_{x\in\Gamma_i} \int_\Gamma \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y \\ &= \max_{i=1:N}\max_{x\in\Gamma_i} \int_{\Gamma_{nei,i}} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y \\ &\leq \max_{i=1:N}\max_{x\in\Gamma_i} \left(M_{nei} \max_{\Gamma_j\in\Gamma_{nei,i}} \int_{\Gamma_j} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y \right) \\ &= M_{nei} \max_{i=1:N}\max_{x\in\Gamma_i}\max_{\Gamma_j\in\Gamma_{nei,i}} \int_{B(x,\delta_3)\cap\Gamma_j} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y \\ &\leq M_{nei} \max_{i=1:N}\max_{x\in\Gamma_i}\max_{\Gamma_j\in\Gamma_{nei,i}} \int_{B^{s-1}(0,\delta_3)} \mathcal{K}^{\delta_3}(\|t\|)\, dt = M_{nei}\, \mathcal{F}^{\delta_3}. \quad \square\end{aligned}$$

Remark 15. Note that Corollary 3 is clearly valid for $\mathcal{K}^{\delta_3}$ i.e.

$$\min_{x\in A^{\delta_1}} \int_\Gamma \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y = \mathcal{F}^{\delta_3},$$

because $\delta_3 \leq \delta_2$.

Theorem 16. *Let $q(x)$ be a piecewise constant function on Γ, i.e. $q(x) = \sum_{i=1}^N q_i \chi_i(x)$ when χ_i is characteristic function of Γ_i. In this situation for δ_3 obtained in Lemma 3 we have*

$$\int_\Gamma \int_\Gamma q(x)\mathcal{K}^{\delta_3}(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x \geq \frac{1}{2}\mathcal{F}^{\delta_3}\|q\|_\Gamma^2. \tag{3.1}$$

Proof. Let δ_1 be determined such that B^{δ_1} and A^{δ_1} satisfy inequalities of Lemma 1. And let

$$I = \int_\Gamma \int_\Gamma q(x)\mathcal{K}^{\delta_3}(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x.$$

Since $\Gamma = A^{\delta_1} \cup B^{\delta_1}$ and $A^{\delta_1} \cap B^{\delta_1}$ is empty, we have

$$I = \int_{A^{\delta_1} \cup B^{\delta_1}} \int_{\Gamma} q(x)\mathcal{K}^{\delta_3}(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x \geq I_1 - I_2, \tag{3.2}$$

when

$$I_1 = \int_{A^{\delta_1}} \int_{\Gamma} q(x)\mathcal{K}^{\delta_3}(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x,$$
$$I_2 = \int_{B^{\delta_1}} \int_{\Gamma} |q(x)|\mathcal{K}^{\delta_3}(\|x-y\|)\, |q(y)|\, d\Gamma_y\, d\Gamma_x.$$

Now from Lemma 2 and Remark 15 we obtain

$$\begin{aligned} I_1 &= \sum_{i=1}^{N} \int_{A_i^{\delta_1}} \int_{\Gamma} q(x)\mathcal{K}^{\delta_3}(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x \\ &= \sum_{i=1}^{N} \int_{A_i^{\delta_1}} q(x)^2 \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\; d\Gamma_y\, d\Gamma_x \\ &\geq \left(\sum_{i=1}^{N} \int_{A_i^{\delta_1}} q(x)^2\, d\Gamma_x\right)\left(\min_{x\in A^{\delta_1}} \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\; d\Gamma_y\right) \geq \|q\|^2_{A^{\delta_1}}\, \mathcal{F}^{\delta_3}, \end{aligned}$$

and consequently Corollary 2 yields

$$I_1 \geq \frac{3}{4}\|q\|^2_{\Gamma}\, \mathcal{F}^{\delta_3}. \tag{3.3}$$

On the other hand, since $|q(x)|\,|q(y)| \leq \frac{1}{2}(q(x)^2 + q(y)^2)$ it can be seen

$$\begin{aligned} I_2 &\leq \frac{1}{2}\int_{B^{\delta_1}} \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, (q(x)^2 + q(y)^2)\, d\Gamma_y\, d\Gamma_x \\ &= \frac{1}{2}\int_{B^{\delta_1}} q(x)^2 \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y\, d\Gamma_x + \frac{1}{2}\int_{B^{\delta_1}} \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|) q(y)^2\, d\Gamma_y\, d\Gamma_x \\ &= \frac{1}{2}\int_{B^{\delta_1}} q(x)^2 \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y\, d\Gamma_x + \frac{1}{2}\int_{\Gamma} q(y)^2 \int_{B^{\delta_1}} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_x\, d\Gamma_y \\ &\leq \frac{1}{2}\int_{B^{\delta_1}} q(x)^2 \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y\, d\Gamma_x + \frac{1}{2}\int_{B^{2\delta_1}} q(y)^2 \int_{B^{\delta_1}} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_x\, d\Gamma_y \\ &\leq \frac{1}{2}\left(\max_{x\in\Gamma} \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y\right)\left(\int_{B^{\delta_1}} q(x)^2\, d\Gamma_x + \int_{B^{2\delta_1}} q(y)^2\, d\Gamma_y\right) \\ &\leq \left(\max_{x\in\Gamma} \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y\right) \int_{B^{2\delta_1}} q(y)^2\, d\Gamma_y, \end{aligned}$$

where Lemma 3 and Corollary 2 conclude

$$I_2 \leq M_{nei}\mathcal{F}^{\delta^3}\, \|q\|^2_{B^{2\delta_1}} \leq \frac{1}{4}\|q\|^2_{\Gamma}\mathcal{F}^{\delta^3}. \tag{3.4}$$

Finally Eqs. (3.2)–(3.4) yield

$$I \geq I_1 - I_2 \geq \frac{1}{2}\mathcal{F}^{\delta^3}\, \|q\|^2_{\Gamma}. \quad \square$$

Corollary 4. *Corollary 1 and Theorem 16 conclude boundary integral equation (1.3) has unique solution for kernels which* $\mathcal{K}^{\delta}(0) > 0$ *because*

$$\int_{\Gamma}\int_{\Gamma} q(x)K(\|x-y\|)\, q(y)\, d\Gamma_y\, d\Gamma_x \geq \frac{1}{2}\mathcal{F}^{\delta_3}\, \|q\|^2_{\Gamma} \; > 0, \tag{3.5}$$

for every piecewise constant function q defined on Γ. *Note that* $\mathcal{K}^{\delta}(0) > 0$ *implies* $\mathcal{F}^{\delta} > 0$.

4. Relation between δ_3 and boundary discretization

Positive parameter δ_3 introduced in Section 3 mainly depends on quality of boundary discretization and we are not able to express it as a function of number of boundary elements N. So we are not able to predict the rate of decreasing of $\mathcal{F}^{\delta_3}$ when N increases. In this section we will find a lower bound for δ_3 when the size of boundary elements is sufficiently small. This leads finding a lower bound for right hand side of inequality (3.5) when N is sufficiently large.

Let Ω be a polygonal and polyhedron in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Since Γ is Lipschitz boundary with Lipschitz constant and radius L_{Lip} and r_{Lip}, respectively, we assume these constants do not vary when N increases. So $\{\Gamma_i\}_{i=1}^N$ is also Lipschitz boundary with Lipschitz constant and radius L_{Lip} and r_{Lip}, respectively. The following two new parameters are used here

$$r_{\max} := \max_{r>0}\{B^{s-1}(x,r) \subseteq \Gamma_i, \text{ for every } 1 \le i \le N \text{ and a } x \in \Gamma_i\},$$

$$R_{\min} := \min_{r>0}\{\Gamma_i \subseteq B^{s-1}(x,r), \text{ for every } 1 \le i \le N \text{ and a } x \in \Gamma_i\},$$

which are characteristic parameters for size of a boundary discretization. In fact $r_{\max}$ is radius of the biggest sphere in $\mathbb{R}^{s-1}$ which is contained in every boundary elements, and $R_{\min}$ is radius of the smallest sphere in $\mathbb{R}^{s-1}$ which contains every boundary elements. Now Lemma 1 in Section 3 can be replaced by the following lemma.

Lemma 4. *Let*

$$\delta_1 \le \frac{r_{\max}}{16\sqrt{1+(L_{Lip})^2}R_{\min}}\, r_{\max}, \tag{4.1}$$

so for this δ_1 the following relations are valid

$$vol(B_i^{2\delta_1}) \le \frac{1}{4\sqrt{1+(L_{Lip})^2}}\, vol(\Gamma_i), \qquad vol(A_i^{\delta_1}) \ge \frac{3}{4}\, vol(\Gamma_i). \tag{4.2}$$

Proof. For two-dimensional problems since $r_{\max} \le R_{\min}$ we can write

$$vol(B_i^{2\delta_1}) = 4\delta_1 \le \frac{r_{\max}}{4\sqrt{1+(L_{Lip})^2}R_{\min}}\, r_{\max} \le \frac{1}{4\sqrt{1+(L_{Lip})^2}}\, r_{\max},$$

and $r_{\max} \le vol(\Gamma_i)$ for $i = 1, 2, \ldots, N$, then the first inequality of this lemma is valid.

And for three-dimensional ones the first inequality of (4.2) is obtained because

$$\begin{aligned} vol(B_i^{2\delta_1}) &= vol(\cup_{x\in\partial\Gamma_i} B(x, 2\delta_1)) \le vol(\cup_{x\in\partial\Gamma_i} B(x, R_{\min})) \le 4\pi R_{\min}\delta_1 \\ &\le \frac{\pi\, r_{\max}^2}{4\sqrt{1+(L_{Lip})^2}} \le \frac{1}{4\sqrt{1+(L_{Lip})^2}} vol(\Gamma_i). \end{aligned}$$

The second inequality of (4.2) is attained because $vol(A_i^{\delta_1}) \ge vol(A_i^{2\delta_1}) = vol(\Gamma_i) - vol(B_i^{2\delta_1})$. $\square$

Beside, Lemma 2 in Section 3 can be replaced by the following lemma when the size of boundary elements is sufficiently small.

Lemma 5. *Let $2R_{\min} < r_{Lip}$. For every $x \in A_i^{\delta_1}$ and $y \in \Gamma\backslash\Gamma_i$ we have $\|x-y\| \ge \delta_2$ when*

$$\delta_2 \le \frac{1}{\sqrt{1+(L_{Lip})^2}}\, \delta_1,$$

and $i = 1, 2, \ldots, N$.

Proof. Definition 1 and $2R_{\min} \le r_{Lip}$ yield Γ_i is contained in an open ball U_0 such that $\Gamma_i \subseteq U_0$. So after proper rotation there are Lipschitz function ψ and set $S \subset \mathbb{R}^{s-1}$ which corresponds to U_0 such that $\{(t', \psi(t')) \mid t' \in S\} = \Gamma \cap U_0$. Now if $x \in A_i^{\delta_1}$ and $y \in (\Gamma \cap U_0)\backslash\Gamma_i$ there are $x', y' \in S$ such that $x = (x', \psi(x'))$, $y = (y', \psi(y'))$ and

$$\begin{aligned} \|x-y\| &= \sqrt{\|x'-y'\|^2 + |\psi(x')-\psi(y')|^2} \ge \|x'-y'\| \ge \|x'-p'\| \\ &\ge \frac{1}{\sqrt{1+(L_{Lip})^2}}\|x-p\| \ge \frac{1}{\sqrt{1+(L_{Lip})^2}}\delta_1 \ge \delta_2, \end{aligned}$$

for a $p = (p', \psi(p')) \in \partial\Gamma_i$. Note that the third inequality is valid because

$$\begin{aligned} \|x-p\| &= \sqrt{\|x'-p'\|^2 + |\psi(x')-\psi(p')|^2} \le \sqrt{\|x'-p'\|^2 + (L_{Lip})^2\|x'-p'\|^2} \\ &\le \sqrt{1+(L_{Lip})^2}\|x'-p'\|. \quad \square \end{aligned}$$

Also the inequality presented in Lemma 3 in Section 3 can be replaced by the following lemma.

Lemma 6. *Let $2R_{\min} < r_{Lip}$. Then for $\delta_3 = \delta_2$ (presented in Lemma 5) we have*

$$\max_{x\in\Gamma} \int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y \le \sqrt{1+(L_{Lip})^2}\, \mathcal{F}^{\delta_3}.$$

Proof. Let assumptions in proof of Lemma 5 be valid, and $t = x' - y'$. Since $d\Gamma \le \sqrt{1+(L_{Lip})^2}\, dt$ for $t \in S$ we have

$$\begin{aligned}
\int_{\Gamma} \mathcal{K}^{\delta_3}(\|x-y\|)\, d\Gamma_y &\le \sqrt{1+(L_{Lip})^2} \int_S \mathcal{K}^{\delta_3}\left(\sqrt{\|x'-y'\|^2 + |\psi(x')-\psi(y')|^2}\right) dt \\
&\le \sqrt{1+(L_{Lip})^2} \int_S \mathcal{K}^{\delta_3}(\|x'-y'\|)\, dt \\
&\le \sqrt{1+(L_{Lip})^2} \int_{B^{s-1}(0,\delta_3)} \mathcal{K}^{\delta_3}(\|t\|)\, dt = \sqrt{1+(L_{Lip})^2}\mathcal{F}^{\delta_3}.
\end{aligned}$$

Note that in the above inequalities we use the fact that $\mathcal{K}^{\delta_3}(r)$ is non-decreasing radial function with respect to r. □

Theorem 17. *The positive real number*

$$\delta_3 = \frac{r_{\max}}{16(1+L_{Lip}^2)R_{\min}}\, r_{\max},$$

satisfies inequality (3.1) presented in Theorem 16 when $2R_{\min} < r_{Lip}$.

Proof. The proof of this theorem is similar to proof of Theorem 16 but one should use Lemmas 4–6 instead of Lemmas 1–3, respectively. □

5. A modification on logarithmic kernel

In this section a modification of traditional kernel of two-dimensional Laplace equation

$$\phi(r) = -1/(2\pi)\ln(r),$$

will be proposed where $r = \|x-y\|$. In fact we will show that it can be replaced by 3-times monotone radial function $\hat{\phi} : (0,\infty) \to \mathbb{R}$ which satisfies $\nabla^2\hat{\phi}(\|x-y\|) = -\delta_{Di}(x-y)$, for all $x \in \Omega$ when δ_{Di} is Dirac's delta [1]. In this situation by Theorem 16, the uniqueness of solution of boundary integral equation (1.3) for 2D Laplace equation is guaranteed.

Let $\mathcal{D}$ be diameter of bounded region Ω, so for every $x, y \in \Omega$ we have $\|x-y\| \le \mathcal{D}$. Also let $\hat{\phi}$ be modification of ϕ defined as

$$\hat{\phi}(r) = \begin{cases} \phi(r/(4\mathcal{D})) & \text{if } 0 < r \le \mathcal{D}, \\ P(r,\mathcal{D}) & \text{if } \mathcal{D} < r, \end{cases}$$

where $P(r,\mathcal{D})$ is the fractional polynomial

$$P(r,\mathcal{D}) = \frac{1}{2\pi}\left(\frac{a_0}{r} + \frac{a_1}{r-\mathcal{D}/4} + \frac{a_2}{r-\mathcal{D}/2} + \frac{a_3}{r-3\mathcal{D}/4}\right),$$

when a_0, a_1, a_2 and a_3 satisfy the following four linear equations

$$P(\mathcal{D},\mathcal{D}) = \phi(1/4), \qquad P^{(k)}(\mathcal{D},\mathcal{D}) = \phi^{(k)}(\mathcal{D}), \quad \text{for } k = 1,2,3. \tag{5.1}$$

Note that $P^{(k)}(\mathcal{D},\mathcal{D})$ is k-th derivative of $P(r,\mathcal{D})$ with respect to r at $r = \mathcal{D}$. Eqs. (5.1) ensure $\hat{\phi}$ is 3-times differentiable function over $(0,\infty)$. To check $\hat{\phi}$ is 3-times monotone function it is enough to show $(-1)^k P^{(k)}(r,\mathcal{D}) \ge 0$ for $r > \mathcal{D}$ and $k \le 3$ because for $r \le \mathcal{D}$ we have $(-1)^k\hat{\phi}^{(k)}(r) = (-1)^k\phi^{(k)}(r) \ge 0$.

From Eqs. (5.1) and definition of functions ϕ and P for $t > 0$ we have

$$t\,P(t\mathcal{D},t\mathcal{D}) = \phi(1/4), \qquad t\,P^{(k)}(t\mathcal{D},t\mathcal{D}) = \phi^{(k)}(t\mathcal{D}), \quad \text{for } k = 1,2,3,$$

and consequently $P(tr,t\mathcal{D})$ can be expressed via $P(r,\mathcal{D})$ for $r > \mathcal{D}$ as

$$tP(tr,t\mathcal{D}) = P(r,\mathcal{D}), \quad \text{for } t > 0. \tag{5.2}$$

Note that from Eq. (5.2) if we get $t = 4/\mathcal{D}$ we conclude $P(r,\mathcal{D})$ is 3-times monotone if and only if $P(r,4)$ is 3-times monotone when $r > 4$. So we will focus on

$$P(r,4) = \frac{1}{2\pi}\left(\frac{a_0}{r} - \frac{a_1}{r-1} - \frac{a_2}{r-2} + \frac{a_3}{r-3}\right),$$

where $a_0 \simeq 8.4819$, $a_1 \simeq 1.3012$, $a_2 \simeq 0.6594$ and $a_3 \simeq 0.0294$. We know

$$\lim_{r\to+\infty} P(r,4) = 0 \quad \text{and} \quad \lim_{r\to+\infty} P^{(k)}(r,4) = 0, \quad \text{for } k = 1, 2,$$

so it is enough to show $P^{(3)}(r, 4) < 0$ for $r > 4$ when

$$P^{(3)}(r,4) = -\frac{6}{2\pi}\left(\frac{a_0}{r^4} - \frac{a_1}{(r-1)^4} - \frac{a_2}{(r-2)^4} + \frac{a_3}{(r-3)^4}\right).$$

For $4 < r < 5$ it can be checked that this function obtains its maximum at $r \simeq 4.2905$ in which $P^{(3)}(r, 4) < -6/(2\pi)\,0.18 < 0$. And for $r \geq 5$ we know $r/(r-1)$, $r/(r-2)$ and $r/(r-3)$ are decreasing functions which yield

$$\begin{aligned} P^{(3)}(r,4) &= -\frac{6}{2\pi\, r^4}\left(a_0 - a_1\left(\frac{r}{r-1}\right)^4 - a_2\left(\frac{r}{r-2}\right)^4 + a_3\left(\frac{r}{r-3}\right)^4\right) \\ &\leq -\frac{6}{2\pi\, r^4}\left(a_0 - a_1\left(\frac{5}{4}\right)^4 - a_2\left(\frac{5}{3}\right)^4 + a_3\right) \leq -\frac{6}{2\pi\, r^4}(0.24) < 0. \end{aligned}$$

Then the following theorem is proved.

Theorem 18. *Let $\Omega \subset \mathbb{R}^s$ be a bounded region with diameter $\mathcal{D}$. So*

$$\phi(r/(4\mathcal{D})) = -\frac{1}{2\pi}\ln\left(\frac{r}{4\,\mathcal{D}}\right), \quad r \leq \mathcal{D}, \tag{5.3}$$

can be extended to a radial and 3-times monotone function defined on $(0, \infty)$. Then if we use function (5.3) as fundamental solution of 2D Laplace equation, the uniqueness of solution of BIE (1.3) is guaranteed as a result of Corollary 4.

6. Non-singular BIE

As can be seen from BEM's literature, fundamental solutions used in this method are singular, i.e. $K(x, y)$ is not finite at $x = y$ [44]. This yields singular boundary integrals which cannot be calculated easily via classic numerical schemes such as Gaussian quadrature rule [29,31]. In Section 3 we show that for every 3-times monotone kernel which its $\mathcal{F}^\delta$ is bigger than 0, the boundary integral equation (1.3) has unique solution. So if one uses non-singular kernel

$$G(r) := \begin{cases} \displaystyle\sum_{k=0}^{4} K^{(k)}(\delta)(r-\delta)^k/k! & \text{if } 0 \leq r \leq \delta, \\ K(r) & \text{if } \delta < r, \end{cases} \tag{6.1}$$

in BIE (1.3) instead of singular kernel K, the uniqueness of solution of BIE (1.3) is guaranteed. In fact one can check, for non-singular kernel G

$$\mathcal{F}^\delta = \frac{K^{(4)}(\delta)}{4!}\int_{B(0,\delta)^{s-1}} (\|t\| - \delta)^4\, dt = c\, K^{(4)}(\delta)\,\delta^{4+s} > 0, \tag{6.2}$$

for a positive constant c. This fact persuades us to derive BEM's formula which is based on the new non-singular fundamental solution presented in Eq. (6.1).

Let homogeneous boundary value problem (1.1) be valid. Using the Green's second identity [44] for a collocation point $x \in \Gamma$ and field point $y \in \Omega \cup \Gamma$, we have

$$\int_\Omega \mathcal{L}G(x,y)\,u(y)\,d\Omega_y + \int_\Gamma G(x,y)\,q(y)\,d\Gamma_y = \int_\Gamma \nabla G(x,y).n\,u(y)\,d\Gamma_y, \tag{6.3}$$

where $G(x, y)$ is the modified fundamental solution of operator $\mathcal{L}$ defined in Eq. (6.1). Also q is unknown flux function, i.e. $q = \nabla u.n$. Since $\mathcal{L}K(r) = 0$ for $r > 0$ we have

$$\mathcal{L}G(r) = \mathcal{L}K(r) = 0, \quad \text{for } r \geq \delta,$$

so integral equation (6.3) is replaced by

$$\int_{B(x,\delta)\cap\Omega} \mathcal{L}G(x,y)\,u(y)\,d\Omega_y + \int_\Gamma G(x,y)\,q(y)\,d\Gamma_y = \int_\Gamma \nabla G(x,y).n\,u(y)d\Gamma_y. \tag{6.4}$$

Integral equation (6.4) is non-singular because G is continuous at $x = y$. This non-singular integral equation is the base of non-singular BEM. To avoid domain integral appeared in Eq. (6.4) suppose δ is sufficiently small and u is approximated linearly on $B(x, \delta) \cap \Omega$ as $u(y) = u(x) + (y - x).n\, q(x)$ which yields

$$\int_{B(x,\delta)\cap\Omega} \mathcal{L}G(x, y)\, u(y)\, d\Omega_y \simeq \alpha(x)u(x) + \beta(x)q(x), \tag{6.5}$$

when

$$\alpha(x) = \int_{B(x,\delta)\cap\Omega} \mathcal{L}G(x, y)\, d\Omega_y, \qquad \beta(x) = \int_{B(x,\delta)\cap\Omega} \mathcal{L}G(x, y)\, (y - x).n\, d\Omega_y.$$

Note that if δ is sufficiently small we have $\mathcal{L}G(x, y) \le 0$ for $y \in B(x, \delta) \cap \Omega$ because $\mathcal{L}K(x, y) = -\delta_{Di}$. It yields $\alpha(x) \le 0$ and since $(y - x).n \le 0$ it can be deduced $\beta(x) \ge 0$.

If Eq. (6.5) is inserted in Eq. (6.4) the final boundary integral equation is obtained as

$$\beta(x)q(x) + \int_{\Gamma} G(x, y)\, q(y)\, d\Gamma_y = -\alpha(x)u(x) + \int_{\Gamma} \nabla G(x, y).n\, u(y) d\Gamma_y. \tag{6.6}$$

The uniqueness of solution of BIE (6.6) is valid because $\beta(x) \ge 0$ and $G(x, y)$ is 3-times monotone function with positive $\mathcal{F}^{\delta}$ when δ is sufficiently small, in fact if we multiply the left hand side of BIE (6.6) with $q(x)$ and integrate over Γ and use Corollary 4 and Eq. (6.2) we have

$$\int_{\Gamma} \beta(x)q(x)^2 d\Gamma_x + \int_{\Gamma}\int_{\Gamma} q(x)\, G(x, y)\, q(y)\, d\Gamma_y d\Gamma_x \ge \frac{1}{2}\mathcal{F}^{\delta}\|q\|^2_{\Gamma} > 0,$$

for piecewise constant function q defined over Γ.

7. Linear system of constant Galerkin BEM

To solve the BIE (1.3) numerically, the boundary Γ is discretized semi-uniformly. So square matrix $\mathbf{A}$ and column vector $\mathbf{b}$ are produced as

$$\mathbf{A}[i, j] = \int_{\Gamma_i}\int_{\Gamma_j} K(\|x - y\|)\, d\Gamma_x d\Gamma_y, \qquad \mathbf{b}[i] = \int_{\Gamma_i} g(x)\, d\Gamma_x, \tag{7.1}$$

for $i, j = 1, 2, \dots, N$, and unknown vector $\mathbf{q} = [q_1\ q_2\ \cdots\ q_N]^T$, where q_i is an approximation of boundary function q over Γ_i, can be obtained via solving system of linear equations $\mathbf{Aq} = \mathbf{b}$ and g is known function as is introduced in Eq. (1.4). This system is named as linear system of constant Galerkin BEM [9]. Here we are going to focus on condition number of matrix $\mathbf{A}$ (denoted $cond(\mathbf{A})$) to study ill-conditioning (or well-conditioning [45]) of system of linear equations of constant Galerkin BEM. Since the main aim is analyzing $cond(\mathbf{A})$ when N gets large, it is supposed $R_{\min}$ be sufficiently small. So Theorem 17 and Corollary 4 yield

$$\begin{aligned}\mathbf{q}^T\mathbf{A}\mathbf{q} &= \sum_{i=1:N}\sum_{j=1:N} q_i\mathbf{A}[i, j]q_j = \sum_{i=1:N}\sum_{j=1:N}\int_{\Gamma_i}\int_{\Gamma_j} q_iK(\|x - y\|)q_j\, d\Gamma_x d\Gamma_y \\ &= \int_{\Gamma}\int_{\Gamma} q(x)K(\|x - y\|)q(y)\, d\Gamma_x d\Gamma_y \ge \frac{1}{2}\mathcal{F}^{\delta_3}\|q\|^2_{\Gamma},\end{aligned} \tag{7.2}$$

for piecewise constant function $q(x) = \sum_{i=1:N} q_i\chi_{\Gamma_i}(x)$.

It can be found from inequality (7.2) that eigenvalues of $\mathbf{A}$ are non-negative. Beside, from Cauchy–Schwarz inequality we have

$$\mathbf{q}^T\mathbf{A}\mathbf{q} = \int_{\Gamma}\int_{\Gamma} q(x)K(\|x - y\|)q(y)d\Gamma_x d\Gamma_y \le \|K\|^2_{\Gamma}\ \|q\|^2_{\Gamma}, \tag{7.3}$$

when

$$\|K\|^2_{\Gamma} := \int_{\Gamma}\int_{\Gamma} K^2(\|x - y\|)d\Gamma_x d\Gamma_y.$$

Now we can see the eigenvalues of matrix $\mathbf{A}$ are restricted to right hand side of inequalities (7.2) and (7.3). Thus if eigenvalues of matrix $\mathbf{A}$ are ordered as $0 < \lambda_{\min} \le \cdots \le \lambda_{\max}$, the following corollary is valid.

Corollary 5. *Condition number of matrix* $\mathbf{A}$ *arisen from constant Galerkin BEM is bounded by*

$$cond(\mathbf{A}) = \lambda_{\max}/\lambda_{\min} \le 2\|K\|^2_{\Gamma}/\mathcal{F}^{\delta_3}. \tag{7.4}$$

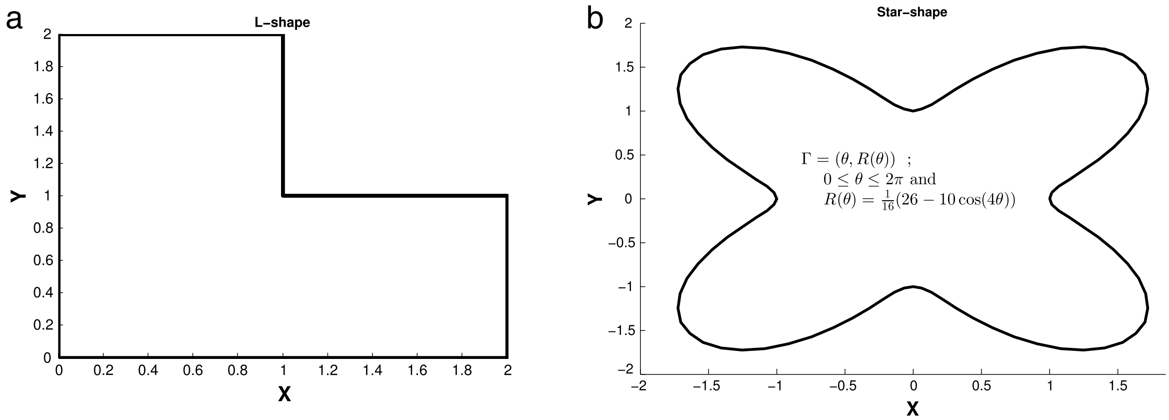

Fig. 2. Computational domains studied for 2D problems. Part (a) is L-shape produced by three square $[0, 1]^2$ and part (b) is Star-shape.

Corollary 6 (*Optimize Boundary Discretization*)**.** *From Eq. (7.4) and definition of δ_3 in Theorem 17, the upper bound for cond*$(\mathbf{A})$ *minimizes when $R_{\min} \simeq r_{\max}$. In this situation we have*

$$cond(\mathbf{A}) \le 2\|K\|_{\Gamma}^{2} / \mathcal{F}^{\frac{r_{\max}}{16(1+L_{Lip}^{2})}},$$

and consequently cond$(\mathbf{A})$ *depends to two factors $r_{\max}$ and L_{Lip}. Note that for regions with sharp corners L_{Lip} may become unbounded when $r_{\max}$ tends to zero which yields cond*$(\mathbf{A})$ *increases faster than other regions, distinctly. We test this fact numerically in Subsection 8.3.*

8. Numerical experiments

In this section analytical results obtained in Sections 6 and 7 are tested via some numerical examples. In fact in Subsection 8.1 the upper bound for $cond(\mathbf{A})$, introduced in Corollary 5 will be validated numerically for some 3-times monotone kernels defined over $\Omega \subseteq \mathbb{R}^s$. And in Subsection 8.2 the constant Galerkin BEM is implemented to solve non-singular BIE (6.6). Numerical results show the new scheme based on non-singular kernels is able to solve BVPs, accurately. Subsection 8.3 is devoted to studying condition number of BEM-matrix for some triangular regions to check Corollary 6 numerically.

8.1. Testing upper bound of $cond(\mathbf{A})$

Corollary 5 is verified here numerically by comparing rate of increasing of $cond(\mathbf{A})$ with the rate of its upper bound when number of boundary elements (N) is sufficiently large. For this purpose, six kernels are supposed and $cond(\mathbf{A})$ is calculated for them numerically. Two regions, L-shape and Star-shape are assumed here as two-dimensional domains and for three dimensional problems a L-shape and a sphere are considered. These domains are demonstrated in Figs. 2 and 4.

Note that logarithmic kernel $K(r) = -1/(2\pi)\ln(r/16)$ is assumed here as modified Laplace's kernel because diameters of the regions are less than 4 (i.e. $\mathcal{D} \le 4$) and from Theorem 18 the uniqueness of solution of BIE (1.3) is guaranteed.

The boundary of the regions is discretized semi-uniformly via N boundary elements, i.e. $R_{\min} \simeq r_{\max}$ and consequently $r_{\max}$ reduces as $O(1/N)$ and $O(1/\sqrt{N})$ for two and three-dimensional problems, respectively, when N is sufficiently large. It must be mentioned that if f and h are two functions of N, $f(N) = O(h(N))$ means $f(N) \simeq c\,h(N)$ for a constant $c > 0$ when N is sufficiently large.

From Corollary 5, since $\|K\|_{\Gamma}^{2}$, L_{Lip} are fixed, upper bound of $cond(\mathbf{A})$ depends on $r_{\max}$ so the upper bound increases as $O(1/\mathcal{F}^{1/N})$ and $O(1/\mathcal{F}^{1/\sqrt{N}})$ for two and three-dimensional domains, respectively. The analytic results for the upper bound of $cond(\mathbf{A})$ with respect to these kernels are reported in the fourth column of Table 1 and the numerical results of $cond(\mathbf{A})$ are reported in fifth column of Table 1. Numerical results are calculated via MATLAB software. In addition to, numerical results are depicted in Figs. 3 and 5 for two and three-dimensional domains, respectively. Figs. 3 and 5 are in log–log scale when N varies from 10 to 5000. Note that the powers of N in fifth column of Table 1 are slopes of the corresponding curves in Figs. 3 and 5. One can see from Table 1, condition number $\mathbf{A}$ and its upper bound increase as $O(N)$ and $O(\sqrt{N})$ for two and three-dimensional singular kernels, respectively. The analytical upper bound of $cond(\mathbf{A})$ is significantly more than the numerical results for those kernels which are non-singular. So the upper bound presented here is not perfect for continuous kernels.

8.2. Non-singular constant Galerkin BEM

To solve non-singular BIE (6.6) numerically we can use constant Galerkin BEM. Since potential function u is known over the boundary, right hand side of the BIE is supposed to be known function g. So the system of non-singular constant

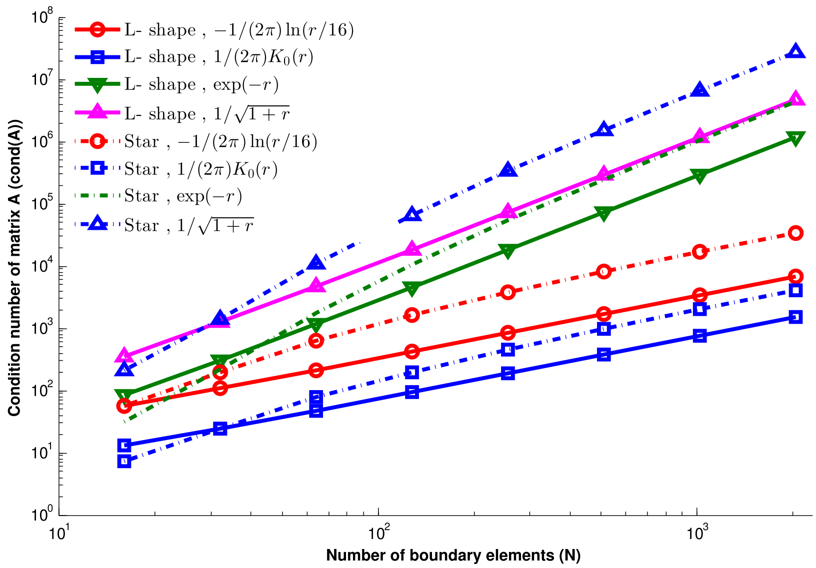

Fig. 3. Numerical results of 2D problems. Two domains, L-shape and Star-shape, are considered here. It can be seen the slope of curves with respect to singular kernels, $-1/(2\pi)\ln(r/16)$ and $1/(2\pi)K_0(r)$, tends to 1 when N gets large, but it tends to 2 for non-singular ones. This fact is reported in fifth column of Table 1.

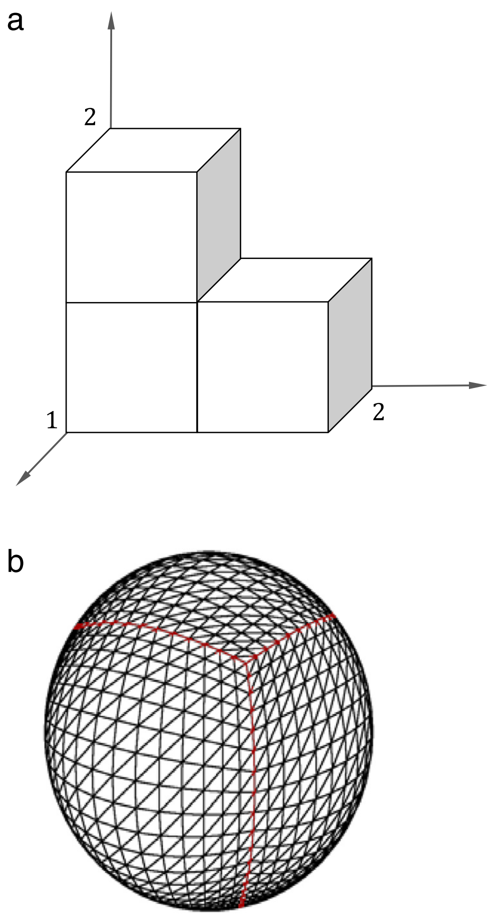

Fig. 4. Computational domains with respect to 3D problems. Part (a) is L-shape produced by 3 cube $[0, 1]^3$ and part (b) is a sphere with center 0 and radius 1 in $\mathbb{R}^3$.

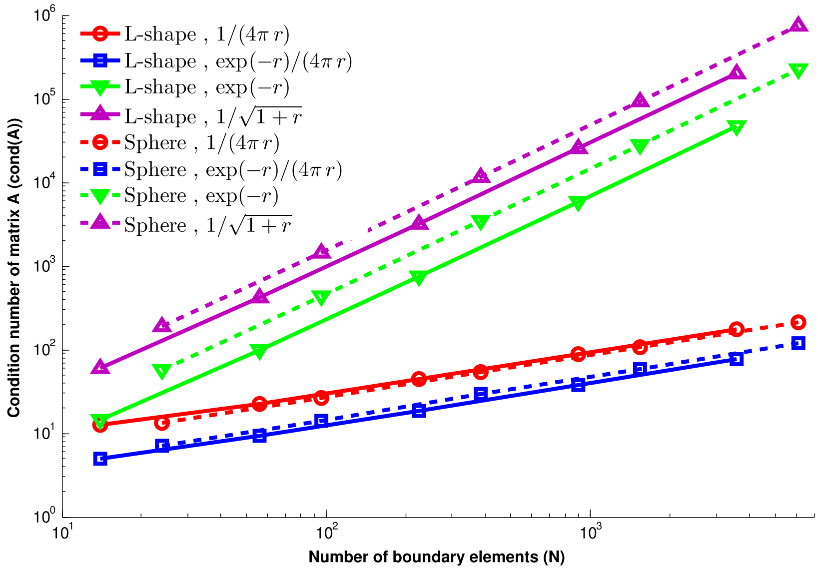

Fig. 5. Numerical results of 3D problems. Computational domains are depicted in Fig. 4. It can be seen the slope of curves with respect to singular kernels, $1/(4\pi r)$ and $1/(4\pi r)\exp(-r)$, tends to 0.5 when N gets large, but it tends to 1.5 for non-singular ones. This fact is reported in fifth column of Table 1.

Galerkin BEM

$$[-\boldsymbol{\beta} + \mathbf{G}]\mathbf{q} = \mathbf{b}, \tag{8.1}$$

is obtained where $\boldsymbol{\beta}$ and $\mathbf{G}$ are $N \times N$ diagonal and full matrices, respectively and $\mathbf{b}$ is a column vector evaluated as

$$\boldsymbol{\beta}[i,i] = \int_{\Gamma_i} \beta(x)\, d\Gamma_x, \qquad \mathbf{G}[i,j] = \int_{\Gamma_i}\int_{\Gamma_j} G(x,y)\, d\Gamma_y\, d\Gamma_x, \qquad \mathbf{b}[i] = \int_{\Gamma_i} g(x)\, d\Gamma_x,$$

for $i, j = 1, 2, \ldots, N$ where $\mathbf{q} = [q_1\, q_2 \ldots q_N]^T$ is the unknown column vector. Parameter δ in definition (6.1) is supposed to be $\delta = R_{\min}$ here.

As is mentioned in Section 6, kernel G is not singular any more and one can use any numerical scheme to calculate diagonal elements of matrix $\mathbf{G}$, easily. An efficient technique to calculate boundary integrals is transforming Cartesian coordinate to polar coordinate and approximating them via Gaussian quadrature rule. To check accuracy of numerical solution of non-singular constant Galerkin BEM (6.6), the two-dimensional BVPs

$$\nabla^2 u(x) - \lambda^2 u(x) = 0, \quad \text{for } x = (x_1, x_2) \in \Omega,$$

with Dirichlet boundary condition

$$u(x) = \bar{u} = u_{ex}(x), \quad \text{for } x = (x_1, x_2) \in \Gamma,$$

are considered when u_{ex} is $\exp(x_1)\cos(x_2)$ and $\exp(\lambda x_1)$ for Laplace ($\lambda = 0$) and Helmholtz ($\lambda \neq 0$) equations, respectively.

These BVPs are solved numerically for $\lambda = 0$, 2 and 10 when computational domains are L-shape and Star-shape depicted in Fig. 2. After solving the BVPs, the potential value is approximated at ten interior points $\{x_k\}_{k=1}^{10}$ which are selected in Ω, randomly. The second norm of numerical error at these interior points is calculated and reported in Fig. 6. It can be found from Fig. 6 that the error of non-singular constant Galerkin BEM vanishes as $O(1/N^2)$ when the number of boundary elements N gets large. Consequently, the rate of convergence of non-singular constant BEM is as accurate as conventional singular constant BEM.

8.3. Effect of corners on stability of BEM

In Corollary 6, we find that condition number of constant BEM-matrix gets large when domain Ω has a sharp corner. This fact is happened because Lipschitz constant of Ω tends to infinity when size of boundary elements tends to zero (i.e. $L_{Lip} \to \infty$ when $R_{\min} \to 0$,). In this situation Corollary 6 yields $cond(\mathbf{A})$ tends to infinity more faster than other regions which do not have sharp corner when the size of the matrix increases. Thus the corresponding system of BEM becomes ill-conditioned when the computational domain has a sharp corner.

To check the effect of sharp corners on the stability we choose four computational domains, named Region 1, Region 2, Region 3 and Region 4, presented in Fig. 7 and the $cond(\mathbf{A})$ is calculated numerically for them via MATLAB software. Numerical results for two kernels, Laplace and Helmholtz fundamental solutions, are presented in Fig. 8. In this case study

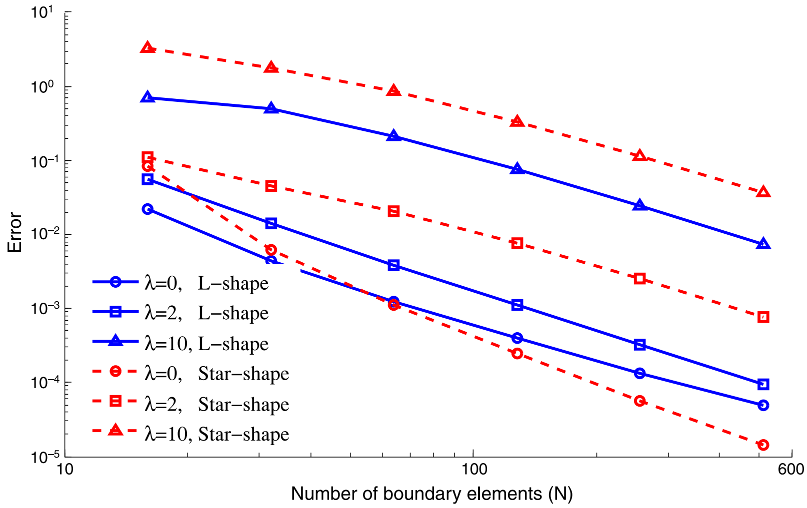

Fig. 6. Numerical results of constant BEM with non-singular kernels. It can be seen that the order of convergence of the method is nearly 2, same as conventional BEM, when N is number of boundary elements.

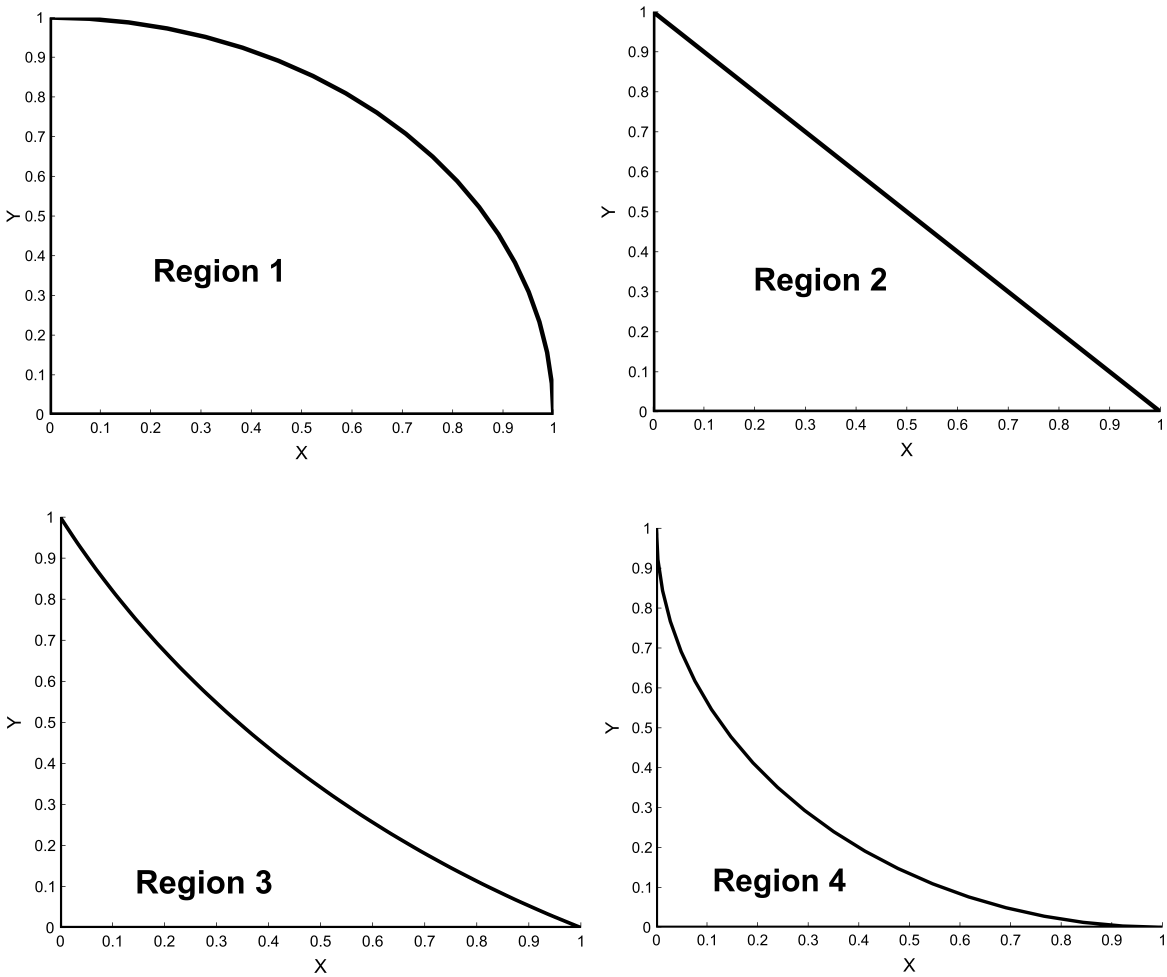

Fig. 7. Regions which are studied in Subsection 8.3. Lipschitz constants of the regions are $L_{Lip} = 1, 2.41, 4.23, \infty$, respectively.

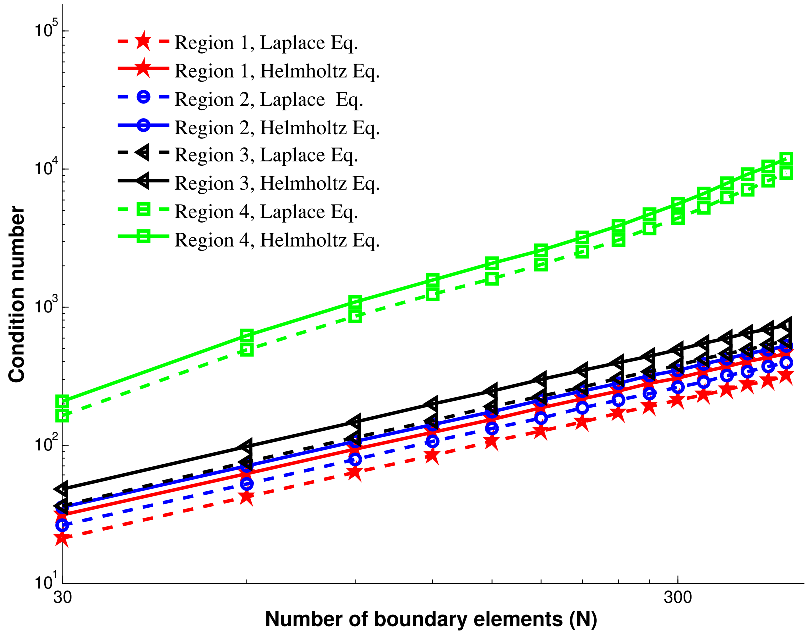

Fig. 8. Numerical results of condition number of matrix **A** for the regions which are depicted in Fig. 7. It is clear from the curves that $cond(\mathbf{A})$ for Region 4 is much bigger than the other regions because its Lipschitz constant is ∞.

the boundary is discretized semi-uniformly and $R_{\min} \simeq r_{\max}$. Lipschitz constants of these regions are $L_{Lip} = 1, 2.41, 4.23, \infty$, respectively which shows Region 4 is not Lipschitz domain. However in BEM we approximate this region with a polygon which its Lipschitz constant tends to ∞ when the size of boundary cells tends to zero. It can be found from Fig. 8 that $cond(\mathbf{A})$ for Region 4 is much bigger than the other regions, obviously. So the numerical results validate analytical results reported in Corollary 6.

9. Final remarks

In this paper the uniqueness of solution of boundary integral equations (BIEs) with 3-times monotone radial kernels is studied when the unknown boundary function is assumed to be piecewise constant. For simplicity we assumed computational domain as a polygon and polyhedron for two and three dimensional problems, respectively but the idea can be extended to any Lipschitz domains. Corresponding to the integral kernel, we find a positive lower bound for eigenvalues of the integral operator which mainly depends on Lipschitz constant of domain and the quality of boundary discretization. We show in this paper if boundary discretization is uniform then condition number of BEM-matrix gets minimum value. Also the effect of corners on the condition number of the matrix was considered and we showed BEM is not able to deal with problems defined on domains with sharp corners. We modify fundamental solution of Laplace operator (i.e. $-1/(2\pi)\ln(r)$) and we replace it by a 3-times monotone radial kernel which it guarantees the uniqueness of solution of constant BEM when we are solving Laplace equation with Dirichlet boundary conditions. Also in this paper a numerical approach is suggested to convert the singular BIEs (correspond to BEM) to non-singular ones. So singular boundary integrals remove from BEM. As a result, in this paper we showed that the condition number of matrix arisen from constant Galerkin BEM increases as rate as N and $\sqrt{N}$ for two and three dimensional problems, respectively when boundary of computational domains is discretized via N boundary elements, semi-uniformly. This fact shows although BEM-matrices are full but they are not ill-conditioned when the computational domain is Lipschitz. The authors believe the idea developed in the current paper can be extended to high-order BEM in which unknown boundary functions are approximated as piecewise polynomial functions of order $n \geq 2$.

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