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Some novel approaches on state estimation of delayed neural networks☆

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ABSTRACT

This paper studies the issue of state estimation for a class of neural networks (NNs) with time-varying delay. A novel Lyapunov-Krasovskii functional (LKF) is constructed, where triple integral terms are used and a secondary delay-partition approach (SDPA) is employed. Compared with the existing delay-partition approaches, the proposed approach can exploit more information on the time-delay intervals. By taking full advantage of a modified Wirtinger's integral inequality (MWII), improved delay-dependent stability criteria are derived, which guarantee the existence of desired state estimator for delayed neural networks (DNNs). A better estimator gain matrix is obtained in terms of the solution of linear matrix inequalities (LMIs). In addition, a new activation function dividing method is developed by bringing in some adjustable parameters. Three numerical examples with simulations are presented to demonstrate the effectiveness and merits of the proposed methods.

1. Introduction

Neural networks (NNs) have attracted a great deal of attention in recent years due to their important applications in many applied fields such as pattern recognition, signal processing, adaptive control, and combinatorial optimization [6,11,13,20,40,43,46]. Their dynamical behaviors, such as stability, attraction, and oscillation, have become hot research topics studied by numerous researchers around the globe. But in practical applications, stability is a key property needed in the design of NNs.

In the implementation of NNs, it is often inevitable to introduce time delay in the signals transmitted among neurons [16,47,48,52]. Hence it is practical to study delayed neural networks (DNNs). DNNs have achieved high recognition for speech data and have the ability to tolerate the time lag caused by variation in the phoneme extraction position (time-shifting invariance) [18]. It is also used to capture the temporal relationship between predictions on continuous instances of facial expression video recording [27]. However, time delay may cause instability, oscillation, or poor performance of NNs. Therefore, the stability problem of DNNs has been recognized as an important issue. Numerous important and interesting research

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results have been developed in [3–5,21,38,41]. Based on a linear matrix inequality (LMI) approach, the global asymptotical stability analysis for a class of stochastic NNs with mixed time delays was investigated in [37]. By making full use of the information of neuron activation function, a new exponential criterion was proposed in [39]. The authors in [42] discussed the problem for robust exponential stability for a class of uncertain stochastic NNs with multiple delays by using free-weighing matrix method. The issue of dissipativity analysis of memristor-based complex-valued NNs with time-varying delays was studied in [26].

Although DNNs have been used as viable network models the neuron states are seldom fully available in the network outputs, especially in relatively large-scale NNs. Consequently, neuron state estimation becomes an important research topic in many practical applications. Paper [36] studied the state estimation problem for NNs with time-varying delay through available output measurements. Since then, the investigation of the state estimation in the case of DNNs has gained rapid development. A number of available measurements and kinds of valid methods have been proposed in [8–10,12,45,51]. For instance, a delay-dependent criterion was developed to estimate the neuron states through available output measurements and the free-weighting matrix approach in [8]. However, the obtained stability criterion in [8] was presented in terms of a matrix inequality, rather than an LMI, which corresponds to a nonlinear programming problem and is generally very difficult to deal with. Paper [9] considered the robust state estimation problem for a class of uncertain DNNs based on a bounding technique. Different from the assumptions in [8], the boundedness of the time-varying delay was only required by defining a new LKF in [10]. New delay-dependent stability criteria for DNNs were obtained by using a delay-partition approach (DPA) in [12,45,50,51]. The advantage of this approach is that less conservative stability criteria can be achieved without introducing any slack variables.

In order to reduce ulteriorly the conservatism of stability criteria for DNNs, the reciprocally convex optimization technique [31,32] was fully applied to the DPA in [7,42]. Alternatively, inspired by the above division, the activation function dividing approach was proposed to study the problem of delay-dependent stability criteria for DNNs in [19,24]. By introducing a tuning parameter, this approach in [19] was modified for investigating an extended dissipative analysis of DNNs in [24]. Besides, other effective methods were utilized, such as a suitable LKF including double and triple integral terms [23], zero equalities and reciprocally convex approach [25], free-weighting matrix technique [17,33], a new convex combination technique [1,49] and a decoupling technique [14]. However, these results appear to have some common shortcomings. On the one side, the relationship between time-varying delay and each subinterval is not taken sufficiently into account. On the other side, some useful integral terms and more information of neuron activation functions are not well utilized, see [9,10,12,51], which may obtain a smaller time-delay upper bound to a certain extent.

Motivated by the proceeding discussion, we investigate, in this paper, the state estimation problem of NNs with time-varying delay and establish some less conservative results by using some more effective methods and novel approaches. The main contribution of this paper lies in the following three aspects. In the first place, different from the existing methods in [19,24], we propose a general bounding partitioning method of activation function by introducing n variable parameters, which plays a key role in obtaining less conservative stability conditions. Moreover, the methods in references [19,24] can be considered as special cases of the proposed approaches in this paper. In the second place, in order to obtain new stability results, a more general SDPA is proposed for constructing an augmented LKF, which is not used in [19,24]. The total interval of the time-varying delay is divided into two alterable subintervals, and then each subinterval is further divided into two variable parts. Compared with the approaches in [7,12,34,45,50,51], the proposed approach is able to take full account of the relationship between time-varying delay and each subinterval. In the third place, by using a MWII, which is less conservative than the celebrated Jensen's inequality used in [7,12,19,24,34,45,50,51], the state desired estimator can be achieved by solving a set of LMIs. Finally, three examples are given to demonstrate the effectiveness and advantages of the developed results.

Notations: Notations used in this paper are fairly standard: Let $\mathbb{R}$ be the real line, $\mathbf{I}$ the identity matrix of appropriate dimensions, $\mathbf{A}^T$ the matrix transposition of the matrix $\mathbf{A}$, $\mathbf{B}^{-1}$ the inverse matrix of the matrix $\mathbf{B}$. By $\mathbf{X}\succ 0$ (respectively $\mathbf{X}\succeq 0$), for $\mathbf{X}\in\mathbb{R}^{n\times n}$, we mean that the matrix $\mathbf{X}$ is real symmetric positive definite (respectively, positive semi-definite); $diag\{r_1, \cdots, r_n\}$ diagonal matrix with diagonal elements $r_i, i=1,\cdots,n$, the symbol $*$ represents the elements below the main diagonal of a symmetric matrix, $\overrightarrow{\mathbf{S}}$ is defined as $\overrightarrow{\mathbf{S}}=\mathbf{S}+\mathbf{S}^T$.

2. Preliminaries

Consider the following NNs with time-varying delay:

$$\dot{\mathbf{x}}(t) = -\mathbf{W}_0\mathbf{x}(t) + \mathbf{W}_1\mathbf{g}(\mathbf{x}(t)) + \mathbf{W}_2\mathbf{g}(\mathbf{x}(t-d(t))) + \mathbf{J},$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \tilde{\mathbf{g}}(t,\mathbf{x}(t)), \tag{1}$$

where $\mathbf{x}(t)=[x_1(t),\cdots,x_n(t)]^T\in\mathbb{R}^n$ is the neuron state vector, $\mathbf{g}(\mathbf{x}(t))=[g_1(x_1(t)),\cdots,g_n(x_n(t))]^T\in\mathbb{R}^n$ is the neuron activation function, and $\mathbf{J}=[J_1,\cdots,J_n]^T\in\mathbb{R}^n$ is an external constant input vector. $\mathbf{W}_0=diag\{w_{01},\cdots,w_{0n}\}\succ 0$, $\mathbf{W}_1\in\mathbb{R}^{n\times n}$ is the interconnection weight matrix, $\mathbf{W}_2\in\mathbb{R}^{n\times n}$ is the delayed interconnection weight matrix, and $\mathbf{C}\in\mathbb{R}^{m\times n}$ is the output weight matrix. $\mathbf{y}(t)\in\mathbb{R}^m$ is the measurement output of the networks, $\tilde{\mathbf{g}}(t,\mathbf{x}(t)):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ is the neuron-dependent nonlinear disturbance on the network outputs, and $d(t)$ is the time-varying discrete delay.

Assumption A. The function $d(t)$ is differentiable and satisfies the following inequalities:

$$0 \leq d(t) \leq d, \qquad \dot{d}(t) \leq \mu, \tag{2}$$

where d and μ are constants.

Assumption B. There exist constants γ_s^-, γ_s^+, and F_l, the neuron activation function $g_s(\cdot)$ and the neuron-dependent nonlinear disturbances $\tilde{g}_s(t, \mathbf{x}(t))$ satisfy the following conditions:

$$\gamma_s^- \leq \frac{g_i(\alpha) - g_i(\beta)}{\alpha - \beta} \leq \gamma_s^+, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \alpha \neq \beta, \quad s = 1, \cdots, n, \tag{3}$$

$$|\tilde{g}_l(t, \mathbf{a}) - \tilde{g}_l(t, \mathbf{b})| \leq F_l |\mathbf{a} - \mathbf{b}|, \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \quad l = 1, \cdots, m, \tag{4}$$

where F_l is called the Lipschitz constant. It should be note that the nonlinearity condition given in (4) has been frequently used in many papers [35]. For simplicity, let $\mathbf{\Gamma}^+ = diag\{\gamma_1^+, \cdots, \gamma_n^+\}$, $\mathbf{\Gamma}^- = diag\{\gamma_1^-, \cdots, \gamma_n^-\}$, $\mathbf{\Gamma} = diag\{\ \gamma_1, \cdots, \gamma_n\}$ $(\gamma_i = max\{|\gamma_i^+|, |\gamma_i^-|\}, i = 1, \cdots, n)$, and $\mathbf{F} = diag\{F_1, \cdots, F_m\}$.

Consider the following full-order state estimator for DNNs (1) (see references [8,9,36]):

$$\dot{\hat{\mathbf{x}}}(t) = -\mathbf{W}_0\hat{\mathbf{x}}(t) + \mathbf{W}_1\mathbf{g}(\hat{\mathbf{x}}(t)) + \mathbf{W}_2\mathbf{g}(\hat{\mathbf{x}}(t - d(t))) + \mathbf{J} + \mathbf{K}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \tilde{\mathbf{g}}(t, \hat{\mathbf{x}}(t))), \tag{5}$$

where $\hat{\mathbf{x}}(t) \in \mathbb{R}^n$ is the estimation of the neuron state, $\mathbf{K} \in \mathbb{R}^{n\times m}$ is the state estimator gain matrix to be determined.

Define the error vectors by $\mathbf{r}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, $\mathbf{f}(\mathbf{r}(t)) = \mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\hat{\mathbf{x}}(t))$ and $\mathbf{h}(t, \mathbf{r}(t)) = \tilde{\mathbf{g}}(t, \mathbf{x}(t)) - \tilde{\mathbf{g}}(t, \hat{\mathbf{x}}(t))$, then the error-state system can be expressed by

$$\dot{\mathbf{r}}(t) = -(\mathbf{W}_0 + \mathbf{KC})\mathbf{r}(t) + \mathbf{W}_1\mathbf{f}(\mathbf{r}(t)) + \mathbf{W}_2\mathbf{f}(\mathbf{r}(t - d(t))) - \mathbf{Kh}(t, \mathbf{r}(t)), \tag{6}$$

where $\mathbf{r}(t) = [r_1(t), \cdots, r_n(t)] \in \mathbb{R}^n$ is the error neuron state vector, $\mathbf{f}(\mathbf{r}(t)) = [f_1(r_1(t)), \cdots, f_n(r_n(t))] \in \mathbb{R}^n$ is the error neuron activation function, and $\mathbf{h}(t, \mathbf{r}(t)) = [h_1(t, \mathbf{r}(t)), \cdots, h_m(t, \mathbf{r}(t))] \in \mathbb{R}^m$ is the error neuron-dependent nonlinear disturbance on the network outputs. Then the error-state system (6) can be directly obtained from (1) and (5), see reference [36].

From Assumption B, we obtain the followings inequalities:

$$\gamma_s^- \leq \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \leq \gamma_s^+, \quad f_i(0) = 0, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \alpha \neq \beta, \quad s = 1, \cdots, n, \tag{7}$$

$$|h_l(t, \mathbf{a}) - h_l(t, \mathbf{b})| \leq F_l |\mathbf{a} - \mathbf{b}|, \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \quad l = 1, \cdots, m, \tag{8}$$

where $f_s(\cdot)$ is the error neuron activation function, $h_l(\cdot)$ is the error neuron-dependent nonlinear disturbance on the network outputs. It should be noted that $\mathbf{r}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) = \mathbf{0}$ implies $\mathbf{f}(\mathbf{r}(t)) = \mathbf{g}(\mathbf{x}(t)) - \mathbf{g}(\hat{\mathbf{x}}(t)) = \mathbf{0}$. Hence $f_i(0) = 0$, $\forall i$.

We shall formulate some practically computable criteria to check global stability of the error-state system (6). The following lemmas will be used in deriving the criteria.

Lemma 1. *[22] Let $\mathbf{r}(t) \in \mathbb{R}^n$ have continuous derivative $\dot{\mathbf{r}}(t)$ in the interval $[0, d]$. Then for any positive definite matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$, any scalar $d > 0$, the following inequality holds:*

$$-\int_{t-d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}\dot{\mathbf{r}}(s)ds \leq -\frac{2}{d}\begin{bmatrix} \frac{1}{d}\int_{t-d}^{t}\mathbf{r}(s)ds \\ \mathbf{r}(t-d) \end{bmatrix}^T \begin{bmatrix} \mathbf{R} & -\mathbf{R} \\ -\mathbf{R} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \frac{1}{d}\int_{t-d}^{t}\mathbf{r}(s)ds \\ \mathbf{r}(t-d) \end{bmatrix}.$$

Lemma 2. *[29] For any positive definite matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$, and for a differentiable signal $\mathbf{r}(t)$: $[\alpha, \beta] \to \mathbf{R}^n$, the following inequality holds:*

$$-(\beta - \alpha)\int_{\alpha}^{\beta} \dot{\mathbf{r}}^T(s)\mathbf{R}\dot{\mathbf{r}}(s)ds \leq -\begin{bmatrix} \mathbf{r}(\alpha) \\ \mathbf{r}(\beta) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathbf{r}(s)ds \end{bmatrix}^T \begin{bmatrix} 4\mathbf{R} & 2\mathbf{R} & -6\mathbf{R} \\ * & 4\mathbf{R} & -6\mathbf{R} \\ * & * & 12\mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{r}(\alpha) \\ \mathbf{r}(\beta) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathbf{r}(s)ds \end{bmatrix},$$

where α and β are constants.

Lemma 3. *[15] For a given scalar $d > 0$, a differentiable signal $\mathbf{r}(t)$: $[\alpha, \beta] \to \mathbf{R}^n$, any positive definite matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$, an arbitrary matrix $\mathbf{X}$ with compatible dimension, and any vector function $\boldsymbol{\xi}(t)$, the following inequality holds for $\forall\theta \in [-d, 0]$:*

$$-\int_{-d}^{0}\int_{t+\theta}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}\dot{\mathbf{r}}(s)dsd\theta \leq \frac{d^2}{2}\xi^T(t)\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}\xi(t) + 2\xi^T(t)\mathbf{X}^T\left[d\mathbf{r}(t) - \int_{t-d}^{t}\mathbf{r}(s)ds\right].$$

where $\boldsymbol{\xi}^T(t)$ is defined in Appendix A.

Remark 1. It should be noted that the above inequality is correct for any vector function $\boldsymbol{\xi}(t)$. Furthermore, there is no restriction imposed on the matrix $\mathbf{X}$. It can be seen that the double-integral inequality in [28, Lemma 1] is a special case of that in [15] since $\xi^T(t) = [\mathbf{r}^T(t), \int_{t-d}^t \mathbf{r}^T(s)ds]^T$ and $\mathbf{X} = [-\frac{2}{d}\mathbf{R}, \frac{2}{d^2}\mathbf{R}]$. Besides, this integral inequality in [15] can be also regarded as an extension of the free-weighting matrix technique in [2] to the double-integral case. Note that the detailed proof of Lemma 3 is provided in Appendix C in [15].

For the sake of simplicity, the notations for some vectors are defined in Appendix A.

3. Main results

In this section, we will develop an effective algorithm to design a suitable delay-dependent state estimator for DNNs (1). Let us begin by establishing a criterion on global asymptotical stability of the error-state system (6).

Theorem 3.1. *Set $0 \le d(t) \le \sigma d$, for given scalars $d > 0$, $0 < \sigma < 1$ and μ, the error-state system (6) under Assumptions A and B is globally asymptotically stable if there exists $\mathbf{P} \succ 0$, $\mathbf{Z}_i \succ 0$ $(i = 1, 2)$, $\mathbf{R}_i \succ 0$ $(i = 1, \cdots, 8)$, arbitrary positive diagonal matrices $\mathbf{\Delta} = diag\{\delta_1, \cdots, \delta_n\}$ $(0 < \delta_i < 1)$, $\mathbf{R} = diag\{r_1, \cdots, r_n\} \succ 0$, $\mathbf{H} = diag\{h_1, \cdots, h_n\} \succ 0$, $\mathbf{G} = diag\{g_1, \cdots, g_n\} \succ 0$, $\mathbf{M}_i^1 = diag\{m_{i1}^1, \cdots, m_{in}^1\} \succ 0$, and $\mathbf{M}_i^2 = diag\{m_{i1}^2, \cdots, m_{in}^2\} \succ 0 (i = 1, 2, \cdots, 7)$; for a scalar $\epsilon > 0$ and any constants x, y, l, m; for arbitrary matrices $\mathbf{X}_i$, $\mathbf{Y}_i$ $(i = 1, \cdots, 16)$ and $\mathbf{T}$ with appropriate dimensions, such that the following LMI hold:*

$$\begin{bmatrix} \mathbf{\Xi} + \mathbf{\Pi}^i & \sigma d \mathbf{X}^T & \varsigma d \mathbf{Y}^T \\ * & -2\mathbf{R}_7 & 0 \\ * & * & -2\mathbf{R}_8 \end{bmatrix} \preceq 0, \ (i = 1, 2) \tag{9}$$

where $\mathbf{\Xi}$ and $\mathbf{\Pi}^i$ are defined in Appendix B. Moreover, the gain matrix of state estimator is given by $\mathbf{K} = \mathbf{T}^{-1}\mathbf{U}$.

Proof. For the error-state system (6), consider the following augmented LKF:

$$\mathbf{V}(t, \mathbf{r}_t) = \mathbf{V}_1(t, \mathbf{r}_t) + \mathbf{V}_2(t, \mathbf{r}_t) + \mathbf{V}_3(t, \mathbf{r}_t) + \mathbf{V}_4(t, \mathbf{r}_t) + \mathbf{V}_5(t, \mathbf{r}_t), \tag{10}$$

where

$$\mathbf{V}_1(t, \mathbf{r}_t) = \mathbf{r}^T(t)\mathbf{P}\mathbf{r}(t) + 2\left[\sum_{i=1}^n r_i \int_0^{r_i(t)} (f_i(s) - \gamma_i^- s)ds + \sum_{i=1}^n h_i \int_0^{r_i(t)} (\gamma_i^+ s - f_i(s))ds + \sum_{i=1}^n g_i \int_0^{r_i(t)} (f_i(s) + \gamma_i s)ds\right], \tag{11}$$

$$\mathbf{V}_2(t, \mathbf{r}_t) = \int_{t-\sigma d}^t \mathbf{r}^T(s)\mathbf{R}_1\mathbf{r}(s)ds + \int_{t-d}^{t-\sigma d} \mathbf{r}^T(s)\mathbf{R}_2\mathbf{r}(s)ds + \int_{t-\sigma d(t)}^t \mathbf{r}^T(s)\mathbf{R}_3\mathbf{r}(s)ds + \int_{t-\sigma d-\varsigma d(t)}^{t-\sigma d} \mathbf{r}^T(s)\mathbf{R}_4\mathbf{r}(s)ds, \tag{12}$$

$$\mathbf{V}_3(t, \mathbf{r}_t) = \int_{-\sigma d}^0 \int_{t+\theta}^t \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)dsd\theta + \int_{-d}^{-\sigma d} \int_{t+\theta}^t \dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)dsd\theta, \tag{13}$$

$$\mathbf{V}_4(t, \mathbf{r}_t) = \int_{t-\sigma d}^t \mathbf{f}^T(\mathbf{r}(s))\mathbf{Z}_1\mathbf{f}(\mathbf{r}(s))ds + \int_{t-d}^{t-\sigma d} \mathbf{f}^T(\mathbf{r}(s))\mathbf{Z}_2\mathbf{f}(\mathbf{r}(s))ds, \tag{14}$$

$$\mathbf{V}_5(t, \mathbf{r}_t) = \int_{-\sigma d}^0 \int_\theta^0 \int_{t+\lambda}^t \dot{\mathbf{r}}^T(s)\mathbf{R}_7\dot{\mathbf{r}}(s)dsd\lambda d\theta + \int_{-d}^{-\sigma d} \int_\theta^0 \int_{t+\lambda}^t \dot{\mathbf{r}}^T(s)\mathbf{R}_8\dot{\mathbf{r}}(s)dsd\lambda d\theta. \tag{15}$$

Taking the derivative of $\mathbf{V}(t, \mathbf{r}_t)$ along the trajectory of the error-state system (6), we obtain the following:

$$\dot{\mathbf{V}}(t, \mathbf{r}_t) = \dot{\mathbf{V}}_1(t, \mathbf{r}_t) + \dot{\mathbf{V}}_2(t, \mathbf{r}_t) + \dot{\mathbf{V}}_3(t, \mathbf{r}_t) + \dot{\mathbf{V}}_4(t, \mathbf{r}_t) + \dot{\mathbf{V}}_5(t, \mathbf{r}_t), \tag{16}$$

with

$$\begin{aligned} \dot{\mathbf{V}}_1(t, \mathbf{r}_t) &= 2\mathbf{r}^T(t)\mathbf{P}\dot{\mathbf{r}}(t) + 2[\sum_{i=1}^n r_i(f_i(r_i(t)) - \gamma_i^- r_i(t))\dot{r}_i(t) \\ &\quad + \sum_{i=1}^n h_i(\gamma_i^+ r_i(t) - f_i(r_i(t)))\dot{r}_i(t) + \sum_{i=1}^n g_i(f_i(r_i(t)) + \gamma_i r_i(t))\dot{r}_i(t)] \\ &= 2\mathbf{r}^T(t)\mathbf{P}\dot{\mathbf{r}}(t) + 2[\mathbf{f}^T(\mathbf{r}(t))(\mathbf{R} - \mathbf{H} + \mathbf{G}) + \mathbf{r}^T(t)(\mathbf{\Gamma}^+\mathbf{H} - \mathbf{\Gamma}^-\mathbf{R} + \mathbf{G}\mathbf{\Gamma})]\dot{\mathbf{r}}(t), \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{\mathbf{V}}_2(t, \mathbf{r}_t) &= \mathbf{r}^T(t)(\mathbf{R}_1 + \mathbf{R}_3)\mathbf{r}(t) - (1 - \sigma\dot{d}(t))\mathbf{r}^T(t - \sigma d(t))\mathbf{R}_3\mathbf{r}(t - \sigma d(t)) - \mathbf{r}^T(t - \sigma d)(\mathbf{R}_1 - \mathbf{R}_2 - \mathbf{R}_4) \\ &\quad \times \mathbf{r}(t - \sigma d) - \mathbf{r}^T(t - d)\mathbf{R}_2\mathbf{r}(t - d) - (1 - \varsigma\dot{d}(t))\mathbf{r}^T(t - \sigma d - \varsigma d(t))\mathbf{R}_4\mathbf{r}(t - \sigma d - \varsigma d(t)) \\ &\le \mathbf{r}^T(t)(\mathbf{R}_1 + \mathbf{R}_3)\mathbf{r}(t) - (1 - \sigma\mu)\mathbf{r}^T(t - \sigma d(t))\mathbf{R}_3\mathbf{r}(t - \sigma d(t)) - \mathbf{r}^T(t - \sigma d)(\mathbf{R}_1 - \mathbf{R}_2 - \mathbf{R}_4) \\ &\quad \times \mathbf{r}(t - \sigma d) - \mathbf{r}^T(t - d)\mathbf{R}_2\mathbf{r}(t - d) - (1 - \varsigma\mu)\mathbf{r}^T(t - \sigma d - \varsigma d(t))\mathbf{R}_4\mathbf{r}(t - \sigma d - \varsigma d(t)), \end{aligned} \tag{18}$$

$$\begin{aligned}\dot{\mathbf{V}}_3(t,\mathbf{r}_t) &= \dot{\mathbf{r}}^T(t)(\sigma d\mathbf{R}_5+\varsigma d\mathbf{R}_6)\dot{\mathbf{r}}(t)-\int_{t-\sigma d}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds-\int_{t-d}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds\\ &= \dot{\mathbf{r}}^T(t)(\sigma d\mathbf{R}_5+\varsigma d\mathbf{R}_6)\dot{\mathbf{r}}(t)-\int_{t-d(t)}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds-\int_{t-\sigma d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds-\int_{t-d}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds. \end{aligned}\tag{19}$$

Next, we deal with the last three integral terms in (19). It can be seen from Lemmas 1 and 2 that the following inequalities hold:

$$\begin{aligned}&-\int_{t-d(t)}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds=-\frac{1}{2}\int_{t-d(t)}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds-\frac{1}{2}\int_{t-d(t)}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds\\ &\le-\frac{1}{\sigma d}\begin{bmatrix}\mathbf{r}(t)\\ \mathbf{r}(t-d(t))\\ \varpi_1(t)\end{bmatrix}^T\begin{bmatrix}2\mathbf{R}_5 & \mathbf{R}_5 & -3\mathbf{R}_5\\ \mathbf{R}_5 & 3\mathbf{R}_5 & -4\mathbf{R}_5\\ -3\mathbf{R}_5 & -4\mathbf{R}_5 & 7\mathbf{R}_5\end{bmatrix}\begin{bmatrix}\mathbf{r}(t)\\ \mathbf{r}(t-d(t))\\ \varpi_1(t)\end{bmatrix},\end{aligned}\tag{20}$$

$$\begin{aligned}&-\int_{t-\sigma d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds=-\frac{1}{2}\int_{t-\sigma d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds-\frac{1}{2}\int_{t-\sigma d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds\\ &\le-\frac{1}{\sigma d}\begin{bmatrix}\mathbf{r}(t-d(t))\\ \mathbf{r}(t-\sigma d)\\ \varpi_2(t)\end{bmatrix}^T\begin{bmatrix}2\mathbf{R}_5 & \mathbf{R}_5 & -3\mathbf{R}_5\\ \mathbf{R}_5 & 3\mathbf{R}_5 & -4\mathbf{R}_5\\ -3\mathbf{R}_5 & -4\mathbf{R}_5 & 7\mathbf{R}_5\end{bmatrix}\begin{bmatrix}\mathbf{r}(t-d(t))\\ \mathbf{r}(t-\sigma d)\\ \varpi_2(t)\end{bmatrix},\end{aligned}\tag{21}$$

$$\begin{aligned}&-\int_{t-d}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds=-\frac{1}{2}\int_{t-d}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds-\frac{1}{2}\int_{t-d}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds\\ &\le-\frac{1}{\varsigma d}\begin{bmatrix}\mathbf{r}(t-\sigma d)\\ \mathbf{r}(t-d)\\ \varpi_3(t)\end{bmatrix}^T\begin{bmatrix}2\mathbf{R}_6 & \mathbf{R}_6 & -3\mathbf{R}_6\\ \mathbf{R}_6 & 3\mathbf{R}_6 & -4\mathbf{R}_6\\ -3\mathbf{R}_6 & -4\mathbf{R}_6 & 7\mathbf{R}_6\end{bmatrix}\begin{bmatrix}\mathbf{r}(t-\sigma d)\\ \mathbf{r}(t-d)\\ \varpi_3(t)\end{bmatrix},\end{aligned}\tag{22}$$

$$\dot{\mathbf{V}}_4(t,\mathbf{r}_t)=\mathbf{f}^T(\mathbf{r}(t))\mathbf{Z}_1\mathbf{f}(\mathbf{r}(t))-\mathbf{f}^T(\mathbf{r}(t-\sigma d))(\mathbf{Z}_1-\mathbf{Z}_2)\mathbf{f}(\mathbf{r}(t-\sigma d))-\mathbf{f}^T(\mathbf{r}(t-d))\mathbf{Z}_2\mathbf{f}(\mathbf{r}(t-d)),\tag{23}$$

$$\dot{\mathbf{V}}_5(t,\mathbf{r}_t)=\dot{\mathbf{r}}^T(t)\left[\frac{(\sigma d)^2}{2}\mathbf{R}_7+\frac{(\varsigma d)^2}{2}\mathbf{R}_8\right]\dot{\mathbf{r}}(t)-\int_{-\sigma d}^{0}\int_{t+\theta}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_7\dot{\mathbf{r}}(s)dsd\theta-\int_{-d}^{-\sigma d}\int_{t+\theta}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_8\dot{\mathbf{r}}(s)dsd\theta.\tag{24}$$

Then we handle the last two integral terms of (24) for arbitrary matrices $\mathbf{X}_i$ and $\mathbf{Y}_i (i=1,\cdots,16)$ with appropriate dimensions. It can be seen from Lemma 3 that the following inequalities hold:

$$-\int_{-\sigma d}^{0}\int_{t+\theta}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_7\dot{\mathbf{r}}(s)dsd\theta\le\frac{(\sigma d)^2}{2}\xi^T(t)\mathbf{X}^T\mathbf{R}_7^{-1}\mathbf{X}\xi(t)+2\xi^T(t)\mathbf{X}^T\left[\sigma d\mathbf{r}(t)-\int_{t-\sigma d}^{t}\mathbf{r}(s)ds\right],\tag{25}$$

$$-\int_{-d}^{-\sigma d}\int_{t+\theta}^{t}\dot{\mathbf{r}}^T(s)\mathbf{R}_8\dot{\mathbf{r}}(s)dsd\theta\le\frac{(\varsigma d)^2}{2}\xi^T(t)\mathbf{Y}^T\mathbf{R}_8^{-1}\mathbf{Y}\xi(t)+2\xi^T(t)\mathbf{Y}^T\left[\varsigma d\mathbf{r}(t-\sigma d)-\int_{t-d}^{t-\sigma d}\mathbf{r}(s)ds\right],\tag{26}$$

where $\mathbf{X}=[\mathbf{X}_1,\cdots,\mathbf{X}_{16}],\mathbf{Y}=[\mathbf{Y}_1,\cdots,\mathbf{Y}_{16}],\ \varsigma=1-\sigma$, and $\xi^T(t)$ is defined in Appendix A.

In order to improve the feasible region of the criteria, according to (6), for any constants x, y, l, m and arbitrary matrix $\mathbf{T}$ with appropriate dimensions, the following equation holds:

$$\begin{aligned}0&=2\left[\dot{\mathbf{r}}^T(t)x+\mathbf{r}^T(t)y+\mathbf{f}^T(\mathbf{r}(t))l+\mathbf{f}^T(\mathbf{r}(t-d(t)))m\right]\mathbf{T}\\ &\times[-\dot{\mathbf{r}}(t)-(\mathbf{W}_0+\mathbf{KC})\mathbf{r}(t)+\mathbf{W}_1\mathbf{f}(\mathbf{r}(t))+\mathbf{W}_2\mathbf{f}(\mathbf{r}(t-d(t)))-\mathbf{Kh}(t,\mathbf{r}(t))].\end{aligned}\tag{27}$$

It should be noted that some improved stable conditions have been obtained by dividing the range of activation function into two equal subintervals in [19,24]. In order to reduce further the conservatism of the stability results, we propose a modified bounding of activation functions by introducing n adjustable δ_i parameters $(0<\delta_i<1, i=1,\cdots,n)$. That is: dividing the bounding of activation function $\gamma_i^-\le\frac{f_i(\alpha)}{\alpha}\le\gamma_i^+$ into two different subintervals $\gamma_i^-\le\frac{f_i(\alpha)}{\alpha}\le\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-)$ and $\gamma^-+\delta_i(\gamma_i^+-\gamma_i^-)\le\frac{f_i(\alpha)}{\alpha}\le\gamma_i^+$, based on the work reported in [19,24].

From (7), the following condition on the activation function holds:

$$Case I:\gamma_i^-\le\frac{f_i(\alpha)-f_i(\beta)}{\alpha-\beta}\le\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-),\forall\alpha,\beta\in\mathbb{R},\alpha\ne\beta,i=1,\cdots,n.\tag{28}$$

When $\beta = 0$, $\gamma_i^- \le \frac{f_i(\alpha)}{\alpha} \le \gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-)(i = 1, \cdots, n)$ is true, then the following condition holds:

$$[f_i(\alpha) - \alpha\gamma_i^-][(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))\alpha - f_i(\alpha)] \ge 0, i = 1, \cdots, n. \tag{29}$$

From (29), for any positive diagonal matrices $\mathbf{M}_j^1 = diag\{m_{j1}^1, \cdots, m_{jn}^1\}(j = 1, 2, 3, 4)$ defined in Theorem 3.1, the following inequalities hold:

$$\Theta_1^1(t) = \sum_{i=1}^{n} \left[f_i(r_i(t)) - r_i(t)\gamma_i^-\right] m_{1i}^1 \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))r_i(t) - f_i(r_i(t))\right] \ge 0, \tag{30}$$

$$\Theta_2^1(t) = \sum_{i=1}^{n} \left[f_i(r_i(t - d(t))) - r_i(t - d(t))\gamma_i^-\right] m_{2i}^1 \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))r_i(t - d(t)) - f_i(r_i(t - d(t)))\right] \ge 0, \tag{31}$$

$$\Theta_3^1(t) = \sum_{i=1}^{n} \left[f_i(r_i(t - \sigma d)) - r_i(t - \sigma d)\gamma_i^-\right] m_{3i}^1 \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))r_i(t - \sigma d) - f_i(r_i(t - \sigma d))\right] \ge 0, \tag{32}$$

$$\Theta_4^1(t) = \sum_{i=1}^{n} \left[f_i(r_i(t - d)) - r_i(t - d)\gamma_i^-\right] m_{4i}^1 \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))r_i(t - d) - f_i(r_i(t - d))\right] \ge 0, \tag{33}$$

where $\Theta_1^1(t)$, $\Theta_2^1(t)$, $\Theta_3^1(t)$ and $\Theta_4^1(t)$ are defined in Appendix C.

When $\beta \ne 0$, the condition in (7) is equivalent to:

$$[f_i(\alpha) - f_i(\beta) - (\alpha - \beta)\gamma_i^-][(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))(\alpha - \beta) - (f_i(\alpha) - f_i(\beta))] \ge 0. \tag{34}$$

Thus, for any positive diagonal matrices $\mathbf{M}_j^1 = diag\{m_{j1}^1, \cdots, m_{jn}^1\}(j = 5, 6, 7)$ defined in Theorem 3.1, it yields that

$$\begin{aligned}\Theta_5^1(t) = &\sum_{i=1}^{n} \left[f_i(r_i(t)) - f_i(r_i(t - d(t))) - (r_i(t) - r_i(t - d(t)))\gamma_i^-\right] m_{5i}^1 \\ &\times \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))(r_i(t) - r_i(t - d(t))) - (f_i(r_i(t)) - f_i(r_i(t - d(t))))\right] \ge 0,\end{aligned} \tag{35}$$

$$\begin{aligned}\Theta_6^1(t) = &\sum_{i=1}^{n} \left[f_i(r_i(t - d(t))) - f_i(r_i(t - \sigma d)) - (r_i(t - d(t)) - r_i(t - \sigma d))\gamma_i^-\right] m_{6i}^1 \\ &\times \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))(r_i(t - d(t)) - r_i(t - \sigma d)) - (f_i(r_i(t - d(t))) - f_i(r_i(t - \sigma d)))\right] \ge 0,\end{aligned} \tag{36}$$

$$\begin{aligned}\Theta_7^1(t) = &\sum_{i=1}^{n} \left[f_i(r_i(t - \sigma d)) - f_i(r_i(t - d)) - (r_i(t - \sigma d) - r_i(t - d))\gamma_i^-\right] m_{7i}^1 \\ &\times \left[(\gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-))(r_i(t - \sigma d) - r_i(t - d)) - (f_i(r_i(t - \sigma d)) - f_i(r_i(t - d)))\right] \ge 0,\end{aligned} \tag{37}$$

where $\Theta_5^1(t)$, $\Theta_6^1(t)$ and $\Theta_7^1(t)$ are also defined in Appendix C.

On the other hand, from (8), for any $\varepsilon > 0$, the following inequality holds:

$$\varepsilon\mathbf{h}^T(t, \mathbf{r}(t))\mathbf{h}(t, \mathbf{r}(t)) \le \varepsilon\mathbf{r}^T(t)\mathbf{F}^T\mathbf{F}\mathbf{r}(t). \tag{38}$$

When $\gamma_i^- \le \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \le \gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-)$, an upper bound of $\dot{\mathbf{V}}(t, \mathbf{r}_t)$ can be obtained using (17)–(38),

$$\dot{\mathbf{V}}(t, \mathbf{r}_t) \le \xi^T(t)(\mathbf{\Xi} + \mathbf{\Pi}^1 + \mathbf{\Sigma})\xi(t), \tag{39}$$

where $\mathbf{\Sigma} = \frac{(\sigma d)^2}{2}\mathbf{R}_7 + \frac{(\varsigma d)^2}{2}\mathbf{R}_8$, $\mathbf{\Xi}$ and $\mathbf{\Pi}^1$ are defined in Appendix B.

$$CaseII: \gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-) \le \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \le \gamma_i^+, \tag{40}$$

For this case, using the same procedure used in case I, we have

$$\dot{\mathbf{V}}(t, \mathbf{r}_t) \le \xi^T(t)(\mathbf{\Xi} + \mathbf{\Pi}^2 + \mathbf{\Sigma})\xi(t), \tag{41}$$

where $\mathbf{M}_j^2 = diag\{m_{j1}^2, \cdots, m_{jn}^2\}(j = 1, \cdots, 7)$ defined in Theorem 3.1, $\mathbf{\Pi}^2$ is defined in Appendix B.

Finally, when $\gamma_i^- \le \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \le \gamma_i^+$, we obtain

$$\dot{\mathbf{V}}(t, \mathbf{r}_t) \le \xi^T(t)(\mathbf{\Xi} + \mathbf{\Pi}^i + \mathbf{\Sigma})\xi(t), i = 1, 2. \tag{42}$$

By using Schur complement in [2], $\boldsymbol{\Xi} + \boldsymbol{\Pi}^i + \boldsymbol{\Sigma} \prec 0$ $(i = 1, 2)$ is equivalent to (9), then $\dot{\mathbf{V}}(t, \mathbf{r}_t) \leq -\epsilon \parallel \mathbf{r}(t) \parallel^2$ holds for any sufficiently small $\epsilon > 0$. Therefore, this implies that error-state system (6) is globally asymptotically stable for $0 \leq d(t) \leq \sigma d$. The proof is thus complete. □

Remark 2. In the area of stability analysis for DNNs, the DPA is often used in order to improve the feasible region of the stability criteria, see [7,30,34,44,50]. Inspired by the approach, Kwon et al. [19] proposed firstly a new method to investigate the problem of delay-dependent stability criteria for NNs with discrete time-varying delays. This new method is to divide the activation function bounding $\gamma_i^- \leq \frac{f_i(u)}{u} \leq \gamma_i^+$ into two equal subintervals $\gamma_i^- \leq \frac{f_i(u)}{u} \leq \frac{\gamma_i^+ + \gamma_i^-}{2}$ and $\frac{\gamma_i^+ + \gamma_i^-}{2} \leq \frac{f_i(u)}{u} \leq \gamma_i^+$ $(i = 1, \cdots, n)$. Compared with those of Theorem 1, it is shown that Theorem 2 using this method improves significantly the feasible region of the stability criterion through three numerical examples.

Remark 3. Different from the method given in [19], by introducing a new parameter ρ, the authors in [16] divided the activation function bounding $\gamma_i^- \leq \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \leq \gamma_i^+$ into two unequal subintervals $\gamma_i^- \leq \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \leq \gamma_i^- + \rho(\gamma_i^+ - \gamma_i^-)$ and $\gamma_i^- + \rho(\gamma_i^+ - \gamma_i^-) \leq \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \leq \gamma_i^+ (i = 1, \cdots, n)$. It should be noted that the Kwon's approach in [19] is a special case of that in [24] with $\rho = \frac{1}{2}$. By using the modified approach, the NNs with time-varying delays in the sense of the extended dissipativity is investigated in [36]. Moreover, two numerical examples are presented to show the effectiveness and less conservatism of the proposed method in [24].

Remark 4. In this study, stimulated by the existing methods in [19,24], we investigate the state estimation problem of DNNs by introducing a more general and complete bounding-partitioning method of activation function. Unlike the existing methods in [19,24], this bounding-partitioning method is developed by introducing the n tuning parameters δ_i $(0 < \delta_i < 1, i = 1, \cdots, n)$. This bounding parting method is to divide the activation function bounding $\gamma_i^- \leq \frac{f_i(\alpha)}{\alpha} \leq \gamma_i^+$ into two alterable subintervals $\gamma_i^- \leq \frac{f_i(\alpha)}{\alpha} \leq \gamma_i^- + \delta_i(\gamma_i^+ - \gamma_i^-)$ and $\gamma^- + \delta_i(\gamma_i^+ - \gamma_i^-) \leq \frac{f_i(\alpha)}{\alpha} \leq \gamma_i^+$.

Remark 5. It is worth pointing out that those methods given in [19,24] are special cases of our method. For example, if we let $\delta_1 = \cdots = \delta_n = \frac{1}{2}$, this method reduces to the method given in [19]. And if we let $\delta_1 = \cdots = \delta_n = \rho$, this method also becomes the method given in [24]. Thus, the proposed method in this paper is more general and less conservative.

Theorem 3.2. *Set $\sigma d \leq d(t) \leq d$, for given scalars $d > 0$, $0 < \sigma < 1$ and μ, the error-state system (6) under Assumptions A and B is globally asymptotically stable if there exists $\mathbf{P} \succ 0$, $\mathbf{Z}_i \succ 0$ $(i = 1, 2)$, $\mathbf{R}_i \succ 0 (i = 1, \cdots, 8)$, arbitrary positive diagonal matrices $\boldsymbol{\Delta} = diag\{\delta_1, \cdots, \delta_n\}(0 < \delta_i < 1)$, $\mathbf{R} = diag\{r_1, \cdots, r_n\} \succ 0$, $\mathbf{H} = diag\{h_1, \cdots, h_n\} \succ 0$, $\mathbf{G} = diag\{g_1, \cdots, g_n\} \succ 0$, $\mathbf{M}_i^1 = diag\{m_{i1}^1, \cdots, m_{in}^1\} \succ 0$, and $\mathbf{M}_i^2 = diag\{m_{i1}^2, \cdots, m_{in}^2\} \succ 0 (i = 1, 2, \cdots, 7)$, for a scalar $\epsilon > 0$ and any constants x, y, l, m; for arbitrary matrices $\mathbf{X}_i$, $\mathbf{Y}_i$ $(i = 1, \cdots, 16)$ and $\mathbf{T}$ with appropriate dimensions, such that the following LMI holds:*

$$\begin{bmatrix} \tilde{\boldsymbol{\Xi}} + \boldsymbol{\Pi}^i & \sigma d \mathbf{X}^T & \varsigma d \mathbf{Y}^T \\ * & -2\mathbf{R}_7 & 0 \\ * & * & -2\mathbf{R}_8 \end{bmatrix} \preceq 0, \ (i = 1, 2) \tag{43}$$

where $\tilde{\boldsymbol{\Xi}}$ is defined in Appendix D. Other terms are the same as the terms defined in Theorem 3.1.

Proof. We choose the same LKF (10) as in Theorem 3.1 for the error-state system (6).

When $\sigma d \leq d(t) \leq d$, it yields

$$\begin{aligned} \dot{\mathbf{V}}_3(t, \mathbf{r}_t) &= \dot{\mathbf{r}}^T(t)(\sigma d \mathbf{R}_5 + \varsigma d \mathbf{R}_6)\dot{\mathbf{r}}(t) - \int_{t-\sigma d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds - \int_{t-d}^{t-\sigma d} \dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds \\ &= \dot{\mathbf{r}}^T(t)(\sigma d \mathbf{R}_5 + \varsigma d \mathbf{R}_6)\dot{\mathbf{r}}(t) - \int_{t-\sigma d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds \\ &\quad - \int_{t-d(t)}^{t-\sigma d} \dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds - \int_{t-d}^{t-d(t)} \dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds. \end{aligned} \tag{44}$$

Now, we deal with the last three integral terms in (44). From Lemmas 1 and 2 the following inequalities hold:

$$\begin{aligned} &-\int_{t-\sigma d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds = -\frac{1}{2}\int_{t-\sigma d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds - \frac{1}{2}\int_{t-\sigma d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds \\ &\leq -\frac{1}{\sigma d}\begin{bmatrix} \mathbf{r}(t) \\ \mathbf{r}(t-\sigma d) \\ \varpi_4(t) \end{bmatrix}^T \begin{bmatrix} 2\mathbf{R}_5 & \mathbf{R}_5 & -3\mathbf{R}_5 \\ \mathbf{R}_5 & 3\mathbf{R}_5 & -4\mathbf{R}_5 \\ -3\mathbf{R}_5 & -4\mathbf{R}_5 & 7\mathbf{R}_5 \end{bmatrix} \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{r}(t-\sigma d) \\ \varpi_4(t) \end{bmatrix}, \end{aligned} \tag{45}$$

$$-\int_{t-d(t)}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds=-\frac{1}{2}\int_{t-d(t)}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds-\frac{1}{2}\int_{t-d(t)}^{t-\sigma d}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds$$
$$\leq-\frac{1}{\varsigma d}\begin{bmatrix}\mathbf{r}(t-\sigma d)\\ \mathbf{r}(t-d(t))\\ \varpi_5(t)\end{bmatrix}^T\begin{bmatrix}2\mathbf{R}_6 & \mathbf{R}_6 & -3\mathbf{R}_6\\ \mathbf{R}_6 & 3\mathbf{R}_6 & -4\mathbf{R}_6\\ -3\mathbf{R}_6 & -4\mathbf{R}_6 & 7\mathbf{R}_6\end{bmatrix}\begin{bmatrix}\mathbf{r}(t-\sigma d)\\ \mathbf{r}(t-d(t))\\ \varpi_5(t)\end{bmatrix}, \tag{46}$$

$$-\int_{t-d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds=-\frac{1}{2}\int_{t-d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds-\frac{1}{2}\int_{t-d}^{t-d(t)}\dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds$$
$$\leq-\frac{1}{\varsigma d}\begin{bmatrix}\mathbf{r}(t-d(t))\\ \mathbf{r}(t-d)\\ \varpi_6(t)\end{bmatrix}^T\begin{bmatrix}2\mathbf{R}_6 & \mathbf{R}_6 & -3\mathbf{R}_6\\ \mathbf{R}_6 & 3\mathbf{R}_6 & -4\mathbf{R}_6\\ -3\mathbf{R}_6 & -4\mathbf{R}_6 & 7\mathbf{R}_6\end{bmatrix}\begin{bmatrix}\mathbf{r}(t-d(t))\\ \mathbf{r}(t-d)\\ \varpi_6(t)\end{bmatrix}, \tag{47}$$

Moreover, based on the method used in Theorem 3.1, the following conditions on the activation function hold:

$$Case I: \gamma_i^-\leq\frac{f_i(\alpha)-f_i(\beta)}{\alpha-\beta}\leq\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-), \tag{48}$$

Choose $\beta=0$, then we get

$$[f_i(\alpha)-\alpha\gamma_i^-][(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))\alpha-f_i(\alpha)]>0, \tag{49}$$

From (49), for any positive diagonal matrices $M_j^1=diag\{m_{j1}^1,\cdots,m_{jn}^1\}$ $(j=1,2,3,4)$ defined in Theorem 3.2, it yields

$$\Theta_1^2(t)=\sum_{i=1}^n\big[f_i(r_i(t))-r_i(t)\gamma_i^-\big]m_{1i}^1\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))r_i(t)-f_i(r_i(t))\big]=\Theta_1^1(t)\geq0, \tag{50}$$

$$\Theta_2^2(t)=\sum_{i=1}^n\big[f_i(r_i(t-\sigma d))-r_i(t-\sigma d)\gamma_i^-\big]m_{2i}^1\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))r_i(t-\sigma d)-f_i(r_i(t-\sigma d))\big]\geq0, \tag{51}$$

$$\Theta_3^2(t)=\sum_{i=1}^n\big[f_i(r_i(t-d(t)))-r_i(t-d(t))\gamma_i^-\big]m_{3i}^1\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))r_i(t-d(t))-f_i(r_i(t-d(t)))\big]\geq0, \tag{52}$$

$$\Theta_4^2(t)=\sum_{i=1}^n\big[f_i(r_i(t-d))-r_i(t-d)\gamma_i^-\big]m_{4i}^1\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))r_i(t-d)-f_i(r_i(t-d))\big]=\Theta_4^1(t)\geq0, \tag{53}$$

where $\Theta_2^2(t)$ and $\Theta_3^2(t)$ are defined in Appendix E.

Similarly, (48) is equivalent to:

$$[f_i(\alpha)-f_i(\beta)-(\alpha-\beta)\gamma_i^-][(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))(\alpha-\beta)-(f_i(\alpha)-f_i(\beta))]>0. \tag{54}$$

Thus, for any positive diagonal matrices $M_j^1=diag\{m_{j1}^1,\cdots,m_{jn}^1\}$ $(j=5,6,7)$ defined in Theorem 3.2, it yields

$$\begin{aligned}\Theta_5^2(t)=&\sum_{i=1}^n\big[f_i(r_i(t))-f_i(r_i(t-\sigma d))-(r_i(t)-r_i(t-\sigma d))\gamma_i^-\big]m_{5i}^1\\&\times\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))(r_i(t)-r_i(t-\sigma d))-(f_i(r_i(t))-f_i(r_i(t-\sigma d)))\big]\geq0,\end{aligned} \tag{55}$$

$$\begin{aligned}\Theta_6^2(t)=&\sum_{i=1}^n\big[f_i(r_i(t-\sigma d))-f_i(r_i(t-d(t)))-(r_i(t-\sigma d)-r_i(t-d(t)))\gamma_i^-\big]m_{6i}^1\\&\times\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))(r_i(t-\sigma d)-r_i(t-d(t)))-(f_i(r_i(t-\sigma d))-f_i(r_i(t-d(t))))\big]\geq0,\end{aligned} \tag{56}$$

$$\begin{aligned}\Theta_7^2(t)=&\sum_{i=1}^n\big[f_i(r_i(t-d(t)))-f_i(r_i(t-d))-(r_i(t-d(t))-r_i(t-d))\gamma_i^-\big]m_{7i}^1\\&\times\big[(\gamma_i^-+\delta_i(\gamma_i^+-\gamma_i^-))(r_i(t-d(t))-r_i(t-d))-(f_i(r_i(t-d(t)))-f_i(r_i(t-d)))\big]\geq0,\end{aligned} \tag{57}$$

where $\Theta_5^2(t)$, $\Theta_6^2(t)$ and $\Theta_7^2(t)$ are defined in Appendix E.

When $\gamma_i^- \le \frac{f_i(\alpha)-f_i(\beta)}{\alpha-\beta} \le \gamma_i^- + \sigma_i(\gamma_i^+ - \gamma_i^-)$, combining (17)–(18), (23)–(27), and (44)–(57), we obtain

$$\dot{\mathbf{V}}(t, \mathbf{r}_t) \le \tilde{\xi}^T(t)(\tilde{\boldsymbol{\Xi}} + \boldsymbol{\Pi}^1 + \boldsymbol{\Sigma})\tilde{\xi}(t). \tag{58}$$

Using the same argument as Case I for Case II, we get

$$\dot{\mathbf{V}}(t, \mathbf{r}_t) \le \tilde{\xi}^T(t)(\tilde{\boldsymbol{\Xi}} + \boldsymbol{\Pi}^2 + \boldsymbol{\Sigma})\tilde{\xi}(t). \tag{59}$$

The rest of the proof is the same as that of Theorem 1 and thus is omitted. □

Remark 6. Stimulated by the existing DPAs in [12,45,51], we propose a novel SDPA to obtain less conservative stability criteria, which may contain more information on the relationship between time-varying delay $d(t)$ and each subinterval. The proposed method is more general and more effective than those given in [12,45,51].

Remark 7. In [7,50], the subinterval $[(k-1)h, kh]$ is only divided into two parts $[(k-1)h, d(t)]$ and $[d(t), kh]$. In [40], unlike some existing methods in [7,50], the subinterval $[(k-1)h, kh]$ is not only divided into two parts $[(k-1)h, d(t)]$ and $[d(t), kh]$, but also divided into another two parts $[(k-1)h, (k-1)h+\rho(t)]$ and $[(k-1)h+\rho(t), kh]$. Thus, the stability criteria derived in [34] may be less conservative.

Remark 8. Our method is different from that given in [7,12,30,34,44,45,50,51]. By introducing a tuning parameter σ $(0 < \sigma < 1)$, we divide the time-delay interval $[0, d]$ into two unequal subintervals $[0, \sigma d] \cup [\sigma d, d]$. Then for each subinterval, we use two different DPAs to take full account of the relationship between time-varying delay $d(t)$ and each subinterval. We first introduce two new variables $\sigma d(t)$ and $\sigma d + \varsigma d(t)$ $(\sigma + \varsigma = 1)$, and then divide $[0, \sigma d]$ and $[\sigma d, d]$ into $[0, \sigma d(t)] \cup [\sigma d(t), \sigma d]$ and $[\sigma d, \sigma d + \varsigma d(t)] \cup [\sigma d + \varsigma d(t), d]$, respectively. Note that $\forall t > 0$, $d(t)$ belongs to $[0, \sigma d]$ or $[\sigma d, d]$, that is also to say, $[0, \sigma d]$ and $[\sigma d, d]$ are divided into $[0, d(t)] \cup [d(t), \sigma d]$ or $[\sigma d, d(t)] \cup [d(t), d]$ once again. Hence, the relationship between the time-varying delay $d(t)$ and each subinterval is further employed, which may result in much less conservative results. Moreover, when $\sigma = \frac{1}{2}$ and $k = 2$, the method in [34] becomes the special case of the proposed method in this paper.

Remark 9. In order to realize the secondary delay-partitioning method, the LKF $\mathbf{V}_2(t, \mathbf{r}_t)$ is constructed. The last two terms in $\mathbf{V}_2(t, \mathbf{r}_t)$ are useful for reducing the conservatism of the proposed results. Moreover, it should be pointed out that the useful term $\int_{t-d}^{t-d(t)} \dot{\mathbf{r}}^T(s)\mathbf{R}\dot{\mathbf{r}}(s)ds$ was ignored in [9,10,30] when estimating the upper bound of the derivative of LKF. In addition, the methods to design the estimator gain matrices in [8–10,12] are very simple. Hence, the stability results proposed in [8–10,12] are conservative.

Remark 10. The less conservative results given in Theorems 3.1 and 3.2 may come from dealing with the integral terms of $-\int_{t-d(t)}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds$, $-\int_{t-\sigma d}^{t-d(t)} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds$, $-\int_{t-d}^{t-\sigma d} \dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds$, $-\int_{t-\sigma d}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds$, $-\int_{t-d(t)}^{t-\sigma d} \dot{\mathbf{r}}^T(s)\mathbf{R}_5\dot{\mathbf{r}}(s)ds$ and $-\int_{t-d}^{t-d(t)} \dot{\mathbf{r}}^T(s)\mathbf{R}_6\dot{\mathbf{r}}(s)ds$ by using a new combined optimization method, which is proposed via Lemmas 1 and 2. This merit of combined optimization method lies in taking full advantage of the relationship between $\mathbf{r}(t)$, $\mathbf{r}(t-d(t))$ and $\frac{1}{d(t)}\int_{t-d(t)}^{t} \mathbf{r}(s)ds$, $\mathbf{r}(t-d(t))$, $\mathbf{r}(t-\sigma d)$ and $\frac{1}{(\sigma d - d(t))}\int_{t-\sigma d}^{t-d(t)} \mathbf{r}(s)ds$, $\mathbf{r}(t-\sigma d)$, $\mathbf{r}(t-d)$ and $\frac{1}{(1-\sigma)d}\int_{t-d}^{t-\sigma d} \mathbf{r}(s)ds$, $\mathbf{r}(t)$, $\mathbf{r}(t-\sigma d)$ and $\frac{1}{\sigma d}\int_{t-\sigma d}^{t} \mathbf{r}(s)ds$, $\mathbf{r}(t-\sigma d)$, $\mathbf{r}(t-d(t))$ and $\frac{1}{(d(t)-\sigma d)}\int_{t-d(t)}^{t-\sigma d} \mathbf{r}(s)ds$, $\mathbf{r}(t-d(t))$, $\mathbf{r}(t-d)$ and $\frac{1}{d-d(t)}\int_{t-d}^{t-d(t)} \mathbf{r}(s)ds$.

Remark 11. In order to fully illustrate the differences between the proposed method given in this paper and the methods given in [6,33], we shall give further explanation and analysis in Appendix F.

Remark 12. Motivated by the method in [15,28], we introduce two triple integral terms $\int_{-\sigma d}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_7\dot{\mathbf{r}}(s)dsd\lambda d\theta$ and $\int_{-d}^{-\sigma d}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_8\dot{\mathbf{r}}(s)dsd\lambda d\theta$ in LKF (15), which was not taken into account in [8–10,12,36,51]. It should be noted that in the proof of Theorem 3.1, the two terms $\int_{-\sigma d}^{0}\int_{t+\theta}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_7\dot{\mathbf{r}}(s)dsd\theta$ and $\int_{-d}^{-\sigma d}\int_{t+\theta}^{t} \dot{\mathbf{r}}^T(s)\mathbf{R}_8\dot{\mathbf{r}}(s)dsd\theta$ in the time derivative of $\mathbf{V}(t, \mathbf{r}(t))$ is estimated by using Lemma 3, which are less conservative than those in [8–10,12,36,51].

Remark 13. In many applications, especially in stability problems, maximal allowable time-delay upper bound $\bar{d}$ ensures the error-state system (6) is stable for any d. In Theorems 3.1 and 3.2 with fixed values μ, x, y, l and m, the optimal value can be obtained through following optimization procedure:

$$\begin{cases} \textbf{Maximize} \quad d \quad\quad for \quad 0 < \sigma < 1, 0 < \delta_i < 1, i = 1, \cdots, n, \\ \quad \textbf{Respect} \quad to \quad\quad (9) \quad or \quad (43). \end{cases} \tag{60}$$

Inequality (60) is a convex optimization problem that can be solved.

4. Numerical examples

In this section, three numerical simulation examples are presented to illustrate the merits and effectiveness of the our results derived in the previous section.

Table 1
Maximal allowable time-delay upper bounds d for different values μ in Example 1.

Method	0.3	0.5	0.8	1.2
[9]	0.6541	0.5128	0.1146	Infeasible
[10]	0.5941	0.5941	0.5941	0.5941
Theorem 1 (k = 2, l = 2) in [12]	1.1763	1.1261	0.9701	0.9560
Theorem 1 (k = 2, l = 3) in [12]	1.1763	1.1261	0.9724	0.9579
Theorem 1 (k = 3, l = 4) in [12]	1.1764	1.1261	0.9727	0.9580
Theorem 1 (k = 5,l = 5) in [12]	1.1764	1.1261	0.9728	0.9582
Theorem 3.1	1.2135	1.1618	1.0937	1.0716
Theorem 3.2	1.2512	1.1965	1.1328	1.0989

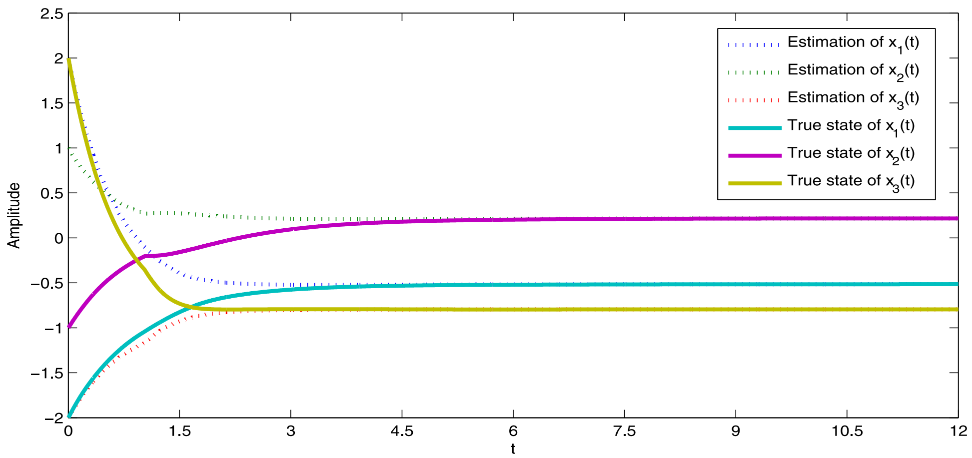

Fig. 1. Responses of the true state $\mathbf{x}(t)$ and its estimation $\hat{\mathbf{x}}(t)$ in the plane for $d = 1.2135$ in Example 1.

Example 1. To demonstrate the superiority of our methods, we consider the DNNs in (6) with the following parameters given in [9,10,12],

$$\mathbf{W}_0 = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.6 \end{bmatrix}, \mathbf{W}_1 = \begin{bmatrix} 0.2 & 0.4 & -0.3 \\ 0 & 0.4 & 0.2 \\ 0.1 & -0.5 & -0.2 \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} -0.5 & 0.4 & 0 \\ 0.2 & 0.4 & -0.3 \\ 0.1 & 0.3 & -0.7 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T.$$

Let $\mathbf{\Gamma} = 0.4 * \mathbf{I}$, $\mathbf{F} = \mathbf{I}$ and $\mathbf{J} = [-1, 0.2, -1.5]^T$. To the best of our knowledge, the best results in [12] of the delay bounds for guaranteeing stability of system (6), when μ is 0.3, 0.5, 0.8 and 1.2, are 1.1764, 1.1261, 0.9728 and 0.9582, respectively. However, by solving the LMIs (9), the delay bounds, when μ is 0.3, 0.5, 0.8 and 1.2, are 1.2135, 1.1261, 1.0937 and 1.0716, respectively. Moreover, Table 1 gives a more detailed comparison of the upper bounds of the time-varying delay $d(t)$ obtained by different methods. According to Table 1, it can be seen that the proposed stability criteria in this paper give much less conservative results than those methods without making good use of the information of the activation function. Therefore, it is clear to see that our method is more effective than those given in the ones reported recently.

When $\mu = 0.3$, $\sigma = 0.8147$, $\delta_1 = 0.9058$, $\delta_2 = 0.1270$, $\delta_3 = 0.9134$, $x = 0.0975$, $y = 0.2785$, $l = 0.5469$ and $m = 0.9575$, we obtain the corresponding upper bound $d = 1.2135$ by Theorem 3.1. By solving the LMIs in Theorem 3.1, the estimator gain matrix is obtained as $\mathbf{K} = \mathbf{T}^{-1}\mathbf{U} = [0.0895, 0.0867, 0.0219]^T$.

Furthermore, the activation functions are assumed to be $g_1(x(s)) = 0.2(|x(s) + 1| - |x(s) - 1|)$, $g_2(x(s)) = 0.2(|x(s) + 1| - |x(s) - 1|)$, and $g_3(x(s)) = 0.2(|x(s) + 1| - |x(s) - 1|)$, and the nonlinear disturbance is taken as $\tilde{\mathbf{g}}(t, \mathbf{x}(t)) = cos(\mathbf{x}(t))$. Fig. 1 represents the true state $\mathbf{x}(t)$ and its estimation $\hat{\mathbf{x}}(t)$ with initial conditions $[-2, -1, 2]^T$ and $[2, 1, -2]^T$, respectively, the response of the error states $\mathbf{r}(t)$ with initial condition $[-3, -1, 3]^T$ is also given in Fig. 2. Figs. 3–4 clearly show that the error system (6) is globally asymptotically stable, which verifies the effectiveness of our proposed methods.

Example 2. Next, we consider the NNs with time-varying delay in (6) with parameters listed in [45] and provide the comparison results between our method and the existing methods. Parameters are given as follows:

$$\mathbf{W}_0 = \begin{bmatrix} 3.6 & 0 & 0 \\ 0 & 4.2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \mathbf{W}_1 = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.1 & 0.3 & -0.2 \\ -0.2 & 0.1 & 0.4 \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} 0.1 & 1 & 0.2 \\ -0.1 & 0.2 & 0.1 \\ 0.2 & -0.1 & 0.4 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

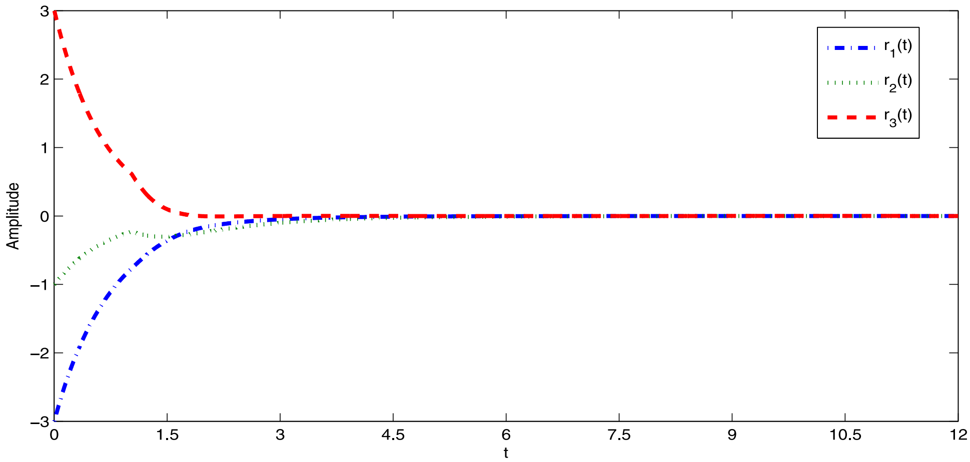

Fig. 2. State trajectories of $\mathbf{r}(t)$ in the plane for $d = 1.2135$ in Example 1.

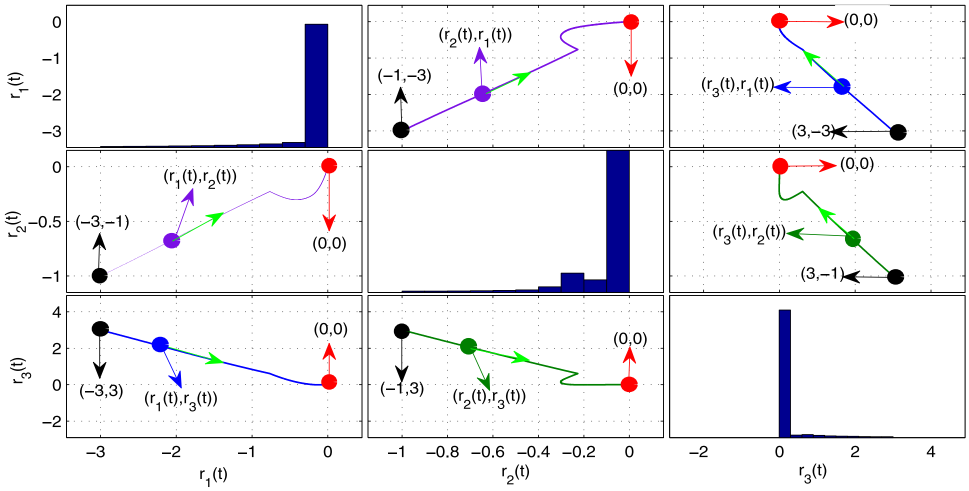

Fig. 3. Phase trajectory of model of $\mathbf{r}(t)$ in the plane for $d = 1.2135$ in Example 1.

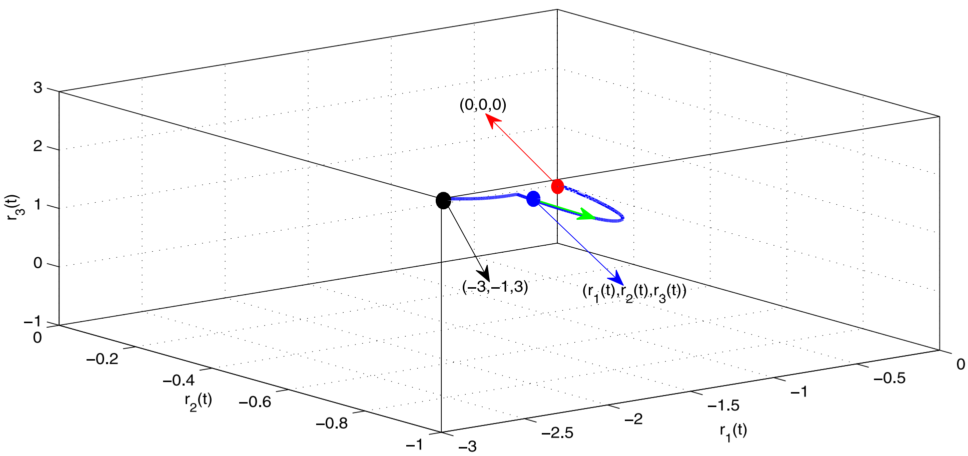

Fig. 4. Phase trajectory of model of $\mathbf{r}(t)$ in the space for $d = 1.2135$ in Example 1.

Table 2
The maximal allowable time-delay upper bounds d in Example 2.

Method	[45]	Theorem 3.1	Theorem 3.2
d	0.80	0.88	0.93
K	$\begin{bmatrix} 0.1538 & 0.0617 & 0.0354 \\ 0.0842 & 0.1094 & -0.0802 \\ 0.0524 & -0.0866 & 0.2748 \end{bmatrix}$	$\begin{bmatrix} 0.2908 & -0.0241 & -0.0012 \\ -0.0720 & 0.2779 & -0.0065 \\ -0.0258 & -0.0078 & 0.2967 \end{bmatrix}$	$\begin{bmatrix} 0.3214 & -0.0367 & 0.0035 \\ -0.0740 & 0.2962 & -0.0184 \\ -0.0241 & -0.0136 & 0.3265 \end{bmatrix}$

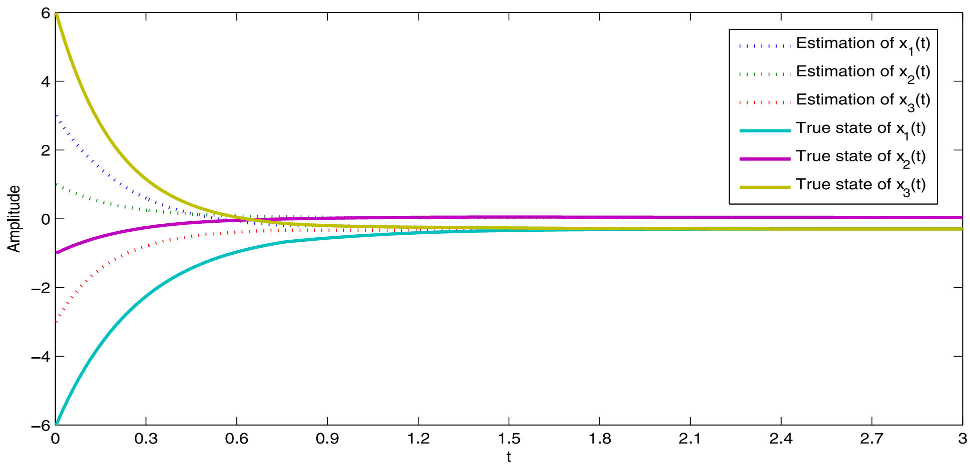

Fig. 5. Responses of the true state $\mathbf{x}(t)$ and its estimation $\hat{\mathbf{x}}(t)$ in the plane for $d = 0.9$ in Example 2.

Set $\mathbf{\Gamma}^- = diag\{0, 0, 0\}\mathbf{\Gamma}^+ = diag\{0.5, 0.5, 0.5\}$, $\mathbf{F} = 0.4 * \mathbf{I}$, $\mathbf{J} = [-1, 0.2, -1.5]^T$, the activation functions are set as $g_1(x(s)) = 0.25(|x(s)+1| - |x(s)-1|)$, $g_2(x(s)) = 0.25(|x(s)+1| - |x(s)-1|)$, and $g_3(x(s)) = 0.25(|x(s)+1| - |x(s)-1|)$, and the nonlinear disturbance is taken as $\tilde{\mathbf{g}}(t, \mathbf{x}(t)) = 0.4cos(\mathbf{x}(t))$.

The maximal allowable time-delay upper bounds d and the corresponding state estimator gain matrices $\mathbf{K}$ obtained from the reference [45], Theorems 3.1 and 3.2 are listed in Table 2. Thus, it is clear to show that our approach is more effective than the one given in [45].

When $\mu = 0.3$, $\sigma = 0.9649$, $\delta_1 = 0.1576$, $\delta_2 = 0.9706$, $\delta_3 = 0.9572$, $\varepsilon = 0.4854$, $x = 0.8003$, $y = 0.1419$, $l = 0.4218$ and $m = 0.9157$, the maximal allowable time-delay upper bound $d = 0.9$ is obtained by using Theorem 3.1. Then, by solving the LMIs (9), the corresponding state estimator gain matrix can be found as

$$\mathbf{K} = \mathbf{T}^{-1}\mathbf{U} = \begin{bmatrix} 0.2888 & 0.0070 & -0.0006 \\ 0.0362 & 0.2792 & -0.0007 \\ 0.0118 & 0.0014 & 0.2889 \end{bmatrix}.$$

In this situation, the simulation results are displayed in Figs. 5–7. Fig. 5 shows the true state $\mathbf{x}(t)$ and its estimation $\hat{\mathbf{x}}(t)$ with initial conditions $[-6, -1, 6]^T$ and $[3, 1, -3]^T$, respectively; the response of the error states $\mathbf{r}(t)$ with initial condition $[-5, -1, 5]^T$ is also given in Fig. 6. Fig. 7 presents clearly that the error system (6) is globally asymptotically stable. In conclusion, the simulation results imply that the designed estimator is effective.

Example 3. Last, we consider the DNNs in (6) with parameters listed in [9]. Parameters are given as follows:

$$\mathbf{W}_0 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, \mathbf{W}_1 = \begin{bmatrix} 0.8 & 0.4 & -0.3 \\ 0 & 0.4 & 0.2 \\ 0.1 & -0.5 & -0.5 \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} -0.5 & 0.6 & 0 \\ 0.2 & 0.4 & -0.3 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $\mathbf{\Gamma}^- = diag\{-0.5, -0.5, -0.5\}$, $\mathbf{\Gamma}^+ = diag\{0.5, 0.5, 0.5\}$, $\mathbf{F} = 0.4 * \mathbf{I}$, $\mathbf{J} = [-1, 0.2, -1.5]^T$, the activation functions are set as $g_1(x(s)) = 0.25(|x(s)+1| - |x(s)-1|)$, $g_2(x(s)) = 0.25(|x(s)+1| - |x(s)-1|)$, and $g_3(x(s)) = 0.25(|x(s)+1| - |x(s)-1|)$, and the nonlinear disturbance is taken as $\tilde{\mathbf{g}}(t, \mathbf{x}(t)) = 0.4cos(\mathbf{x}(t))$.

In order to illustrate further the feasibility and effectiveness of the proposed methods, the time-varying delay is chosen as $d(t) = 0.4 + 0.4sin(t)$. For $\mu = 0.4$, $d = 0.8$, $\sigma = 0.7922$, $\delta_1 = 0.9595$, $\delta_2 = 0.6557$, $\delta_3 = 0.0357$, $\varepsilon = 0.8491$, $x = 0.9340$, $y = 0.6787$, $l = 0.7577$ and $m = 0.7431$, by solving the LMIs (9), the corresponding state estimator gain matrix can be found

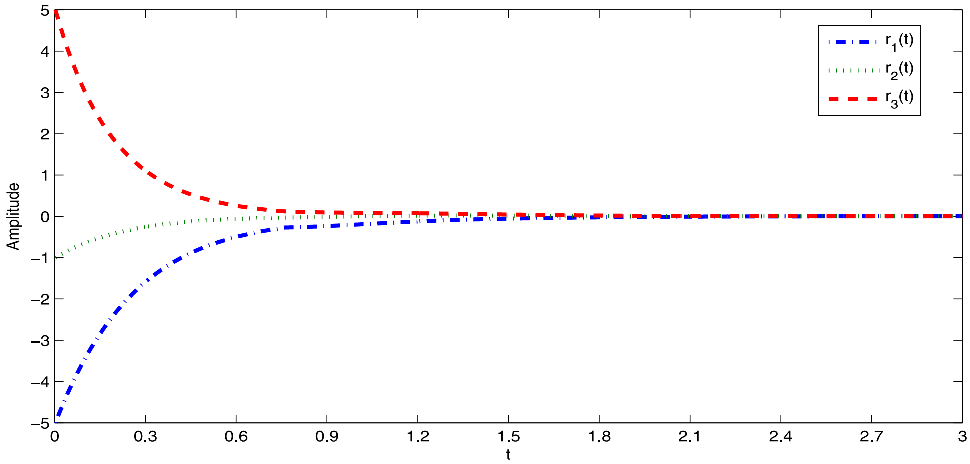

Fig. 6. Phase trajectory of model of $\mathbf{r}(t)$ in the plane for $d = 0.9$ in Example 2.

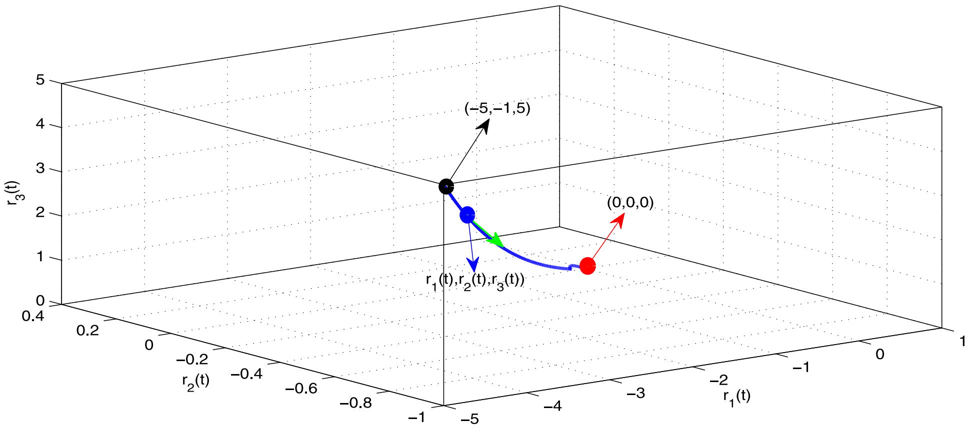

Fig. 7. Phase trajectory of model of $\mathbf{r}(t)$ in the space for $d = 0.9$ in Example 2.

Table 3
Maximal allowable time-delay upper bounds d in Example 3.

Method	[9]	Theorem 3.1	Theorem 3.2
d	0.80	0.83	0.87
K	$\begin{bmatrix} 0.3929 & 0.0841 & 0.0517 \\ 0.0838 & 0.2209 & -0.1437 \\ 0.0577 & -0.1443 & 0.4633 \end{bmatrix}$	$\begin{bmatrix} 0.0917 & -0.0006 & 0.0250 \\ 0.0268 & 0.0626 & -0.0231 \\ 0.0252 & -0.0313 & 0.1815 \end{bmatrix}$	$\begin{bmatrix} 0.2900 & 0.0876 & 0.0574 \\ 0.0742 & 0.2416 & -0.0209 \\ -0.0046 & -0.0201 & 0.2997 \end{bmatrix}$

as

$$\mathbf{K} = \mathbf{T}^{-1}\mathbf{U} = \begin{bmatrix} 0.1326 & 0.0098 & 0.0363 \\ 0.0327 & 0.0863 & -0.0365 \\ 0.0165 & -0.0411 & 0.2366 \end{bmatrix}.$$

The simulation results are shown in Figs. 8–10. Among them, Fig. 8 represents the true state $\mathbf{x}(t)$ and its estimation $\hat{\mathbf{x}}(t)$ with initial conditions $[-1, -0.5, 1]^T$ and $[2, 0.5, -2]^T$, respectively. The response of the error states $\mathbf{r}(t)$ with initial condition $[-2, 0.5, 2]^T$ is also given in Fig. 9. Fig. 10 shows clearly that the error system (6) is globally asymptotically stable.

In addition, the maximal allowable time-delay upper bounds d and the corresponding state estimator gain matrices $\mathbf{K}$ of this paper and the result of [9] are listed in Table 3. From this table, we can observe that the our results given this paper are better.

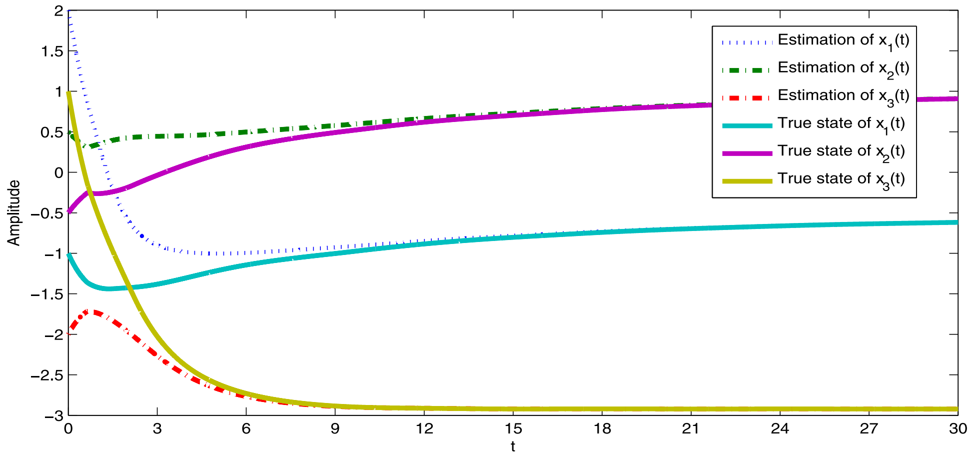

Fig. 8. Responses of the true state $\mathbf{x}(t)$ and its estimation $\hat{\mathbf{x}}(t)$ in the plane for $d = 0.8$ in Example 3.

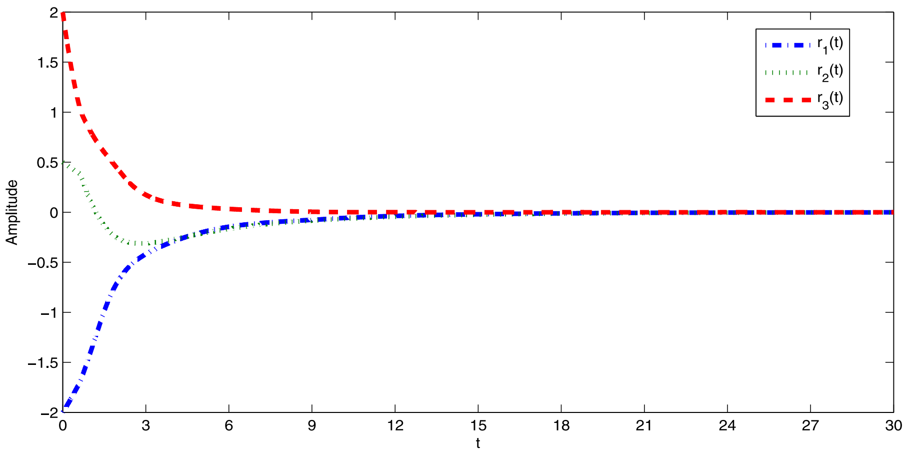

Fig. 9. Phase trajectory of model of $\mathbf{r}(t)$ in the plane for $d = 0.8$ in Example 3.

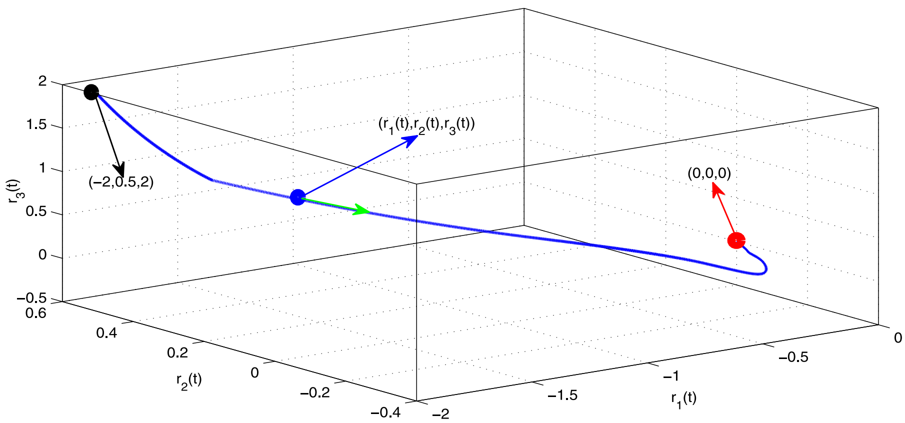

Fig. 10. Phase trajectory of model of $\mathbf{r}(t)$ in the space for $d = 0.8$ in Example 3.

5. Conclusions

We have studied in this paper the state estimation problem for a class of NNs with time-varying delay by constructing a more effective LKF and proposed a general and complete DPA to derive delay-dependent stability criterion, which guarantees successfully the existence of conceivable state estimator for DNNs. The obtained stability results can be calculated by using a convex optimization algorithms subject to LMIs. In addition, the desired estimator gain matrix can be obtained by utilizing a MWII. Furthermore, we have developed a new activation function dividing approach by introducing an adjustable parameter Δ. Finally, we have illustrated the feasibility and effectiveness of the designed methods by three numerical simulation examples. The aforementioned results have the potential to stimulate further study of DNNs. Meanwhile, we expect that our results can be applied to other delayed systems.

Appendix A

$$\begin{aligned}
\varpi_1(t) &= \frac{1}{d(t)}\int_{t-d(t)}^{t}\mathbf{r}(s)ds,\ \varpi_2(t) = \frac{1}{\sigma d - d(t)}\int_{t-\sigma d}^{t-d(t)}\mathbf{r}(s)ds,\ \varpi_3(t) = \frac{1}{\varsigma d}\int_{t-d}^{t-\sigma d}\mathbf{r}(s)ds,\\
\varpi_4(t) &= \frac{1}{\sigma d}\int_{t-\sigma d}^{t}\mathbf{r}(s)ds,\ \varpi_5(t) = \frac{1}{d(t)-\sigma d}\int_{t-d(t)}^{t-\sigma d}\mathbf{r}(s)ds,\ \varpi_6(t) = \frac{1}{d-d(t)}\int_{t-d}^{t-d(t)}\mathbf{r}(s)ds,\\
\xi(t) &= [\mathbf{r}^T(t), \dot{\mathbf{r}}^T(t), \mathbf{r}^T(t-\sigma d(t)), \mathbf{r}^T(t-d(t)), \mathbf{r}^T(t-\sigma d), \mathbf{r}^T(t-\sigma d-\varsigma d(t)), \mathbf{r}^T(t-d), \mathbf{f}^T(\mathbf{r}(t)),\\
&\quad \mathbf{f}^T(\mathbf{r}(t-d(t))), \mathbf{f}^T(\mathbf{r}(t-\sigma d)), \mathbf{f}^T(\mathbf{r}(t-d)), \varpi_1^T(t), \varpi_2^T(t), \sigma d\varpi_4^T(t), \varpi_3^T(t), -\mathbf{h}^T(t,\mathbf{r}(t))]^T,\\
\tilde{\xi}(t) &= [\mathbf{r}^T(t), \dot{\mathbf{r}}^T(t), \mathbf{r}^T(t-\sigma d(t)), \mathbf{r}^T(t-\sigma d), \mathbf{r}^T(t-d(t)), \mathbf{r}^T(t-\sigma d-\varsigma d(t)), \mathbf{r}^T(t-d), \mathbf{f}^T(\mathbf{r}(t)),\\
&\quad \mathbf{f}^T(\mathbf{r}(t-\sigma d)), \mathbf{f}^T(\mathbf{r}(t-d(t))), \mathbf{f}^T(\mathbf{r}(t-d)), \varpi_4^T(t), \varpi_5^T(t), \varpi_6^T(t), \varsigma d\varpi_3^T(t), -\mathbf{h}^T(t,\mathbf{r}(t))]^T.
\end{aligned}$$

Appendix B

$$\begin{aligned}
&\Xi_{1,1} = \mathbf{R}_1 + \mathbf{R}_3 - \frac{2}{\sigma d}\mathbf{R}_5 + \sigma d\overrightarrow{\mathbf{X}_1} - y\overrightarrow{\mathbf{TW}_0} - y\overrightarrow{UC} + \varepsilon\mathbf{F}^T\mathbf{F},\\
&\Xi_{1,2} = \mathbf{P} + \mathbf{\Gamma}^{+}\mathbf{H} - \mathbf{\Gamma}^{-}\mathbf{R} + \mathbf{G}\mathbf{\Gamma} + \sigma d\mathbf{X}_2 - x\mathbf{W}_0\mathbf{T}^T - x\mathbf{C}^T\mathbf{U}^T - y\mathbf{T},\ \Xi_{1,3} = \sigma d\mathbf{X}_3,\\
&\Xi_{1,4} = -\frac{1}{\sigma d}\mathbf{R}_5 + \sigma d\mathbf{X}_4,\ \Xi_{1,5} = \sigma d\mathbf{X}_5 + \varsigma d\mathbf{Y}_1^T,\ \Xi_{1,6} = \sigma d\mathbf{X}_6,\ \Xi_{17} = \sigma d\mathbf{X}_7,\\
&\Xi_{1,8} = \sigma d\mathbf{X}_8 + y\mathbf{TW}_1 - l\mathbf{W}_0\mathbf{T} - l\mathbf{C}^T\mathbf{U},\ \Xi_{1,9} = \sigma d\mathbf{X}_9 + y\mathbf{TW}_2 - m\mathbf{W}_0\mathbf{T} - m\mathbf{C}^T\mathbf{U}^T,\\
&\Xi_{1,10} = \sigma d\mathbf{X}_{10},\ \Xi_{1,11} = \sigma d\mathbf{X}_{11},\ \Xi_{1,12} = \sigma d\mathbf{X}_{12} + \frac{3}{\sigma d}\mathbf{R}_5,\ \Xi_{1,13} = \sigma d\mathbf{X}_{13},\ \Xi_{1,14} = \sigma d\mathbf{X}_{14} - \mathbf{X}_1^T,\\
&\Xi_{1,15} = \sigma d\mathbf{X}_{15} - \varsigma d\mathbf{Y}_1^T,\ \Xi_{1,16} = \sigma d\mathbf{X}_{16} + y\mathbf{U},\ \Xi_{2,2} = \sigma d\mathbf{R}_5 + \varsigma d\mathbf{R}_6 + \frac{(\sigma d)^2}{2}\mathbf{R}_7 + \frac{(\varsigma d)^2}{2}\mathbf{R}_8 - x\overrightarrow{\mathbf{T}},\\
&\Xi_{2,5} = \varsigma d\mathbf{Y}_2^T,\ \Xi_{2,8} = \mathbf{R} - \mathbf{H} + \mathbf{G} + x\mathbf{TW}_1 - l\mathbf{T}^T,\ \Xi_{2,9} = x\mathbf{TW}_2 - m\mathbf{T}^T,\ \Xi_{2,14} = -\mathbf{X}_2^T,\ \Xi_{2,15} = \varsigma d\mathbf{Y}_2^T,\\
&\Xi_{2,16} = x\mathbf{U},\ \Xi_{3,3} = -(1-\sigma\mu)\mathbf{R}_3,\ \Xi_{3,5} = \varsigma d\mathbf{Y}_3^T,\ \Xi_{3,14} = -\mathbf{X}_3^T,\ \Xi_{3,15} = -\varsigma d\mathbf{Y}_3^T,\ \Xi_{4,4} = -\frac{5}{\sigma d}\mathbf{R}_5,\\
&\Xi_{4,5} = -\frac{1}{\sigma d}\mathbf{R}_5 + \varsigma d\mathbf{Y}_4^T,\ \Xi_{4,12} = \frac{4}{\sigma d}\mathbf{R}_5,\ \Xi_{4,13} = \frac{3}{\sigma d}\mathbf{R}_5,\ \Xi_{4,14} = -\mathbf{X}_4^T,\ \Xi_{4,15} = -\varsigma d\mathbf{Y}_4^T,\\
&\Xi_{5,5} = -\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_4 - \frac{3}{\sigma d}\mathbf{R}_5 - \frac{2}{\varsigma d}\mathbf{R}_6 + \varsigma d\overrightarrow{\mathbf{Y}_5}^T,\ \Xi_{5,7} = \frac{1}{\varsigma d}\mathbf{R}_6 + \varsigma d\mathbf{Y}_7,\ \Xi_{5,8} = \varsigma d\mathbf{Y}_8,\\
&\Xi_{5,9} = \varsigma d\mathbf{Y}_9,\ \Xi_{5,10} = \varsigma d\mathbf{Y}_{10},\ \Xi_{5,11} = \varsigma d\mathbf{Y}_{11},\ \Xi_{5,12} = \varsigma d\mathbf{Y}_{12},\ \Xi_{5,13} = \frac{4}{\sigma d}\mathbf{R}_5 + \varsigma d\mathbf{Y}_{13},\\
&\Xi_{5,14} = -\mathbf{X}_5^T + \varsigma d\mathbf{Y}_{14},\ \Xi_{5,15} = \frac{3}{\varsigma d}\mathbf{R}_6 - \varsigma d\mathbf{Y}_5^T + \varsigma d\mathbf{Y}_{15},\ \Xi_{5,16} = \varsigma d\mathbf{Y}_{16},\ \Xi_{6,6} = -(1-\varsigma\mu)\mathbf{R}_4,\\
&\Xi_{6,14} = -\mathbf{X}_6^T,\ \Xi_{6,15} = -\varsigma dY_6^T,\ \Xi_{7,7} = -R_2 - \frac{3}{\varsigma d}\mathbf{R}_6,\ \Xi_{7,14} = -\mathbf{X}_7^T,\ \Xi_{7,15} = \frac{4}{\varsigma d}\mathbf{R}_6 - \varsigma d\mathbf{Y}_7^T,\\
&\Xi_{8,8} = \mathbf{Z}_1 + l\overrightarrow{\mathbf{TW}_1},\ \Xi_{8,9} = l\mathbf{TW}_2 + m\mathbf{W}_1^T\mathbf{T},\ \Xi_{8,14} = -\mathbf{X}_8^T,\ \Xi_{8,15} = -\varsigma d\mathbf{Y}_8^T,\ \Xi_{8,16} = lU,\\
&\Xi_{9,9} = m\overrightarrow{\mathbf{TW}_2},\ \Xi_{9,14} = -\mathbf{X}_9^T,\ \Xi_{9,15} = -\varsigma d\mathbf{Y}_9^T,\ \Xi_{9,16} = mU,\ \Xi_{10,10} = -\mathbf{Z}_1 + \mathbf{Z}_2,\ \Xi_{10,14} = -\mathbf{X}_{10}^T,\\
&\Xi_{10,15} = -\varsigma d\mathbf{Y}_{10}^T,\ \Xi_{11,11} = -\mathbf{Z}_2,\ \Xi_{11,14} = -\mathbf{X}_{11}^T,\ \Xi_{11,15} = -\varsigma d\mathbf{Y}_{11}^T,\ \Xi_{12,12} = -\frac{7}{\sigma d}\mathbf{R}_5,\ \Xi_{12,14} = -\mathbf{X}_{12}^T,\\
&\Xi_{12,15} = -\varsigma d\mathbf{Y}_{12}^T,\ \Xi_{13,13} = -\frac{7}{\sigma d}\mathbf{R}_5,\ \Xi_{13,14} = -\mathbf{X}_{13}^T,\ \Xi_{13,15} = -\varsigma d\mathbf{Y}_{13}^T,\ \Xi_{14,14} = -\overrightarrow{\mathbf{X}}_{14},\ \Xi_{14,15} = -\mathbf{X}_{15} - \varsigma d\mathbf{Y}_{14}^T,\\
&\Xi_{14,16} = -\mathbf{X}_{16},\ \Xi_{15,15} = -\varsigma d\overrightarrow{\mathbf{Y}}_{15} - \frac{7}{\varsigma d}R_6,\ \Xi_{15,16} = -\varsigma d\mathbf{Y}_{16},\ \Xi_{15,16} = -\varsigma d\mathbf{Y}_{16},\ \Xi_{16,16} = -\varepsilon\mathbf{I},\ \varsigma = 1-\sigma,\\
&\mathbf{\Pi}_{11}^i = -\mathbf{\Pi}_1^i\mathbf{M}_1^i - \mathbf{\Pi}_1^i\mathbf{M}_5^i,\ \mathbf{\Pi}_{14}^i = \mathbf{\Pi}_1^i\mathbf{M}_5^i,\ \mathbf{\Pi}_{18}^i = \mathbf{\Pi}_2^i\mathbf{M}_1^i + \mathbf{\Pi}_2^i\mathbf{M}_5^i,\ \mathbf{\Pi}_{19}^i = -\mathbf{\Pi}_2^i\mathbf{M}_5^i,\\
&\mathbf{\Pi}_{44}^i = -\mathbf{\Pi}_1^i\mathbf{M}_2^i - \mathbf{\Pi}_1^i\mathbf{M}_5^i - \mathbf{\Pi}_1^i\mathbf{M}_6^i,\ \mathbf{\Pi}_{45}^i = \mathbf{\Pi}_1^i\mathbf{M}_6^i,\ \mathbf{\Pi}_{48}^i = -\mathbf{\Pi}_2^i\mathbf{M}_5^1,\ \mathbf{\Pi}_{49}^i = \mathbf{\Pi}_2^i\mathbf{M}_2^i + \mathbf{\Pi}_2^i\mathbf{M}_5^i + \mathbf{\Pi}_2^1\mathbf{M}_6^i,
\end{aligned}$$

$$\mathbf{\Pi}^i_{4,10} = -\mathbf{\Pi}^i_2\mathbf{M}^i_6,\ \mathbf{\Pi}^i_{55} = -\mathbf{\Pi}^i_1\mathbf{M}^i_3 - \mathbf{\Pi}^i_1\mathbf{M}^i_6 - \mathbf{\Pi}^i_1\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{57} = \mathbf{\Pi}^1_i\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{59} = -\mathbf{\Pi}^i_2\mathbf{M}^i_6,$$
$$\mathbf{\Pi}^i_{5,10} = \mathbf{\Pi}^i_2\mathbf{M}^i_3 + \mathbf{\Pi}^i_2\mathbf{M}^i_6 + \mathbf{\Pi}^i_2 M^i_7,\ \mathbf{\Pi}^i_{5,10} = -\mathbf{\Pi}^i_2\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{77} = -\mathbf{\Pi}^i_1\mathbf{M}^i_4 - \mathbf{\Pi}^i_1\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{7,10} = -\mathbf{\Pi}^i_2\mathbf{M}^i_7,$$
$$\mathbf{\Pi}^i_{7,11} = \mathbf{\Pi}^i_2\mathbf{M}^i_4 + \mathbf{\Pi}^i_2\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{88} = -2\mathbf{M}^i_1 - 2\mathbf{M}^i_5,\ \mathbf{\Pi}^i_{89} = 2\mathbf{M}^i_5,\ \mathbf{\Pi}^i_{99} = -2\mathbf{M}^i_2 - 2\mathbf{M}^i_5 - 2\mathbf{M}^i_6,$$
$$\mathbf{\Pi}^i_{9,10} = 2\mathbf{M}^i_5,\ \mathbf{\Pi}^i_{10,10} = -2\mathbf{M}^i_3 - 2\mathbf{M}^i_6 - 2\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{10,11} = 2\mathbf{M}^i_7,\ \mathbf{\Pi}^i_{11,11} = -2\mathbf{M}^i_4 - 2\mathbf{M}^i_7,$$

where $\mathbf{\Pi}^1_1 = 2\mathbf{\Gamma}^-[\mathbf{\Gamma}^- + \Delta(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-)]$, $\mathbf{\Pi}^1_2 = 2\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-)$, $\mathbf{\Pi}^2_1 = 2\mathbf{\Gamma}^+[\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-)]$, $\mathbf{\Pi}^2_2 = \mathbf{\Gamma}^+ + \mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-)$. The other terms of $\mathbf{\Xi}$ and $\mathbf{\Pi}^i$ are all $\mathbf{0}$.

Appendix C

$$\Theta^1_1(t) = -2\Big[\mathbf{f}(\mathbf{r}(t)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{L}^+ - \mathbf{\Gamma}^-))\mathbf{r}(t)\Big]^T\mathbf{M}^1_1\Big[\mathbf{f}(\mathbf{r}(t)) - \mathbf{\Gamma}^-\mathbf{r}(t)\Big],$$
$$\Theta^1_2(t) = -2\Big[\mathbf{f}(\mathbf{r}(t-d(t))) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))\mathbf{r}(t-d(t))\Big]^T\mathbf{M}^1_2\Big[\mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{\Gamma}^-\mathbf{r}(t-d(t))\Big],$$
$$\Theta^1_3(t) = -2\Big[\mathbf{f}(\mathbf{r}(t-\sigma d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))\mathbf{r}(t-\sigma d)\Big]^T\mathbf{M}^1_3\Big[\mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{\Gamma}^-\mathbf{r}(t-\sigma d)\Big],$$
$$\Theta^1_4(t) = -2\Big[\mathbf{f}(\mathbf{r}(t-d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))\mathbf{r}(t-d)\Big]^T\mathbf{M}^1_4\Big[\mathbf{f}(\mathbf{r}(t-d))) - \mathbf{\Gamma}^-\mathbf{r}(t-d)\Big],$$
$$\Theta^1_5(t) = -2\Big[\mathbf{f}(\mathbf{r}(t)) - \mathbf{f}(\mathbf{r}(t-d(t))) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))(\mathbf{r}(t) - \mathbf{r}(t-d(t)))\Big]^T\mathbf{M}^1_5$$
$$\times\Big[\mathbf{f}(\mathbf{r}(t)) - \mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{\Gamma}^-(\mathbf{r}(t) - \mathbf{r}(t-d(t)))\Big],$$
$$\Theta^1_6(t) = -2\Big[\mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{f}(\mathbf{r}(t-\sigma d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))(\mathbf{r}(t-d(t)) - \mathbf{r}(t-\sigma d))\Big]^T\mathbf{M}^1_6$$
$$\times\Big[\mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{\Gamma}^-(\mathbf{r}(t-d(t)) - \mathbf{r}(t-\sigma d))\Big],$$
$$\Theta^1_7(t) = -2\Big[\mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{f}(\mathbf{r}(t-d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))(\mathbf{r}(t-\sigma d) - \mathbf{r}(t-d))\Big]^T\mathbf{M}^1_7$$
$$\times\Big[\mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{f}(\mathbf{r}(t-d)) - \mathbf{\Gamma}^-(\mathbf{r}(t-\sigma d) - \mathbf{r}(t-d))\Big],$$

Appendix D

$$\tilde{\mathbf{\Xi}}_{1,4} = -\frac{1}{\sigma d}\mathbf{R}_5 + \sigma d\mathbf{X}_4 + \varsigma d\mathbf{Y}^T_1,\ \tilde{\mathbf{\Xi}}_{1,5} = \sigma d\mathbf{X}_5,\ \tilde{\mathbf{\Xi}}_{1,9} = \sigma d\mathbf{X}_9,$$
$$\tilde{\mathbf{\Xi}}_{1,10} = \sigma d\mathbf{X}_{10} + y\mathbf{T}\mathbf{W}_2 - m\mathbf{W}_0\mathbf{T} - m\mathbf{C}^T\mathbf{U}^T,\ \tilde{\mathbf{\Xi}}_{1,12} = \sigma d\mathbf{X}_{12} + \frac{3}{\sigma d}\mathbf{R}_5 - \sigma d\mathbf{X}^T_1,\ \tilde{\mathbf{\Xi}}_{1,14} = \sigma dX_{14},$$
$$\tilde{\mathbf{\Xi}}_{1,15} = \sigma d\mathbf{X}_{15} - \mathbf{Y}^T_1,\ \tilde{\mathbf{\Xi}}_{2,4} = \varsigma d\mathbf{Y}^T_2,\ \tilde{\mathbf{\Xi}}_{2,10} = x\mathbf{T}\mathbf{W}_2 - m\mathbf{T}^T,\ \tilde{\mathbf{\Xi}}_{2,12} = -\sigma d\mathbf{X}^T_2,\ \tilde{\mathbf{\Xi}}_{2,15} = -\mathbf{Y}^T_2,$$
$$\tilde{\mathbf{\Xi}}_{3,4} = \varsigma d\mathbf{Y}^T_3,\ \tilde{\mathbf{\Xi}}_{3,12} = -\sigma d\mathbf{X}^T_3,\ \tilde{\mathbf{\Xi}}_{3,15} = -\mathbf{Y}^T_3,\ \tilde{\mathbf{\Xi}}_{4,4} = -\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_4 - \frac{3}{\sigma d}\mathbf{R}_5 - \frac{2}{\varsigma d}\mathbf{R}_6 + \varsigma d\overrightarrow{\mathbf{Y}^T_4},$$
$$\tilde{\mathbf{\Xi}}_{4,5} = -\frac{1}{\sigma d}\mathbf{R}_5 + \varsigma d\mathbf{Y}_5,\ \tilde{\mathbf{\Xi}}_{4,6} = \varsigma d\mathbf{Y}_6,\ \tilde{\mathbf{\Xi}}_{4,7} = \varsigma d\mathbf{Y}_7,\ \tilde{\mathbf{\Xi}}_{4,8} = \varsigma d\mathbf{Y}_8,\ \tilde{\mathbf{\Xi}}_{4,9} = \varsigma d\mathbf{Y}_9,\ \tilde{\mathbf{\Xi}}_{4,10} = \varsigma d\mathbf{Y}_{10},$$
$$\tilde{\mathbf{\Xi}}_{4,11} = \varsigma d\mathbf{Y}_{11},\ \tilde{\mathbf{\Xi}}_{4,12} = \frac{4}{\sigma d}\mathbf{R}_5 - \sigma d\mathbf{X}^T_4 + \varsigma d\mathbf{Y}_{12},\ \tilde{\mathbf{\Xi}}_{4,13} = \frac{3}{\varsigma d}\mathbf{R}_5 + \varsigma d\mathbf{Y}_{13},\ \tilde{\mathbf{\Xi}}_{4,14} = \varsigma d\mathbf{Y}_{14},$$
$$\tilde{\mathbf{\Xi}}_{4,15} = \varsigma d\mathbf{Y}_{15} - \mathbf{Y}^T_4,\ \tilde{\mathbf{\Xi}}_{4,16} = \varsigma d\mathbf{Y}_{16},\ \tilde{\mathbf{\Xi}}_{5,5} = -\frac{3}{\varsigma d}\mathbf{R}_5 - \frac{2}{\varsigma d}\mathbf{R}_6,\ \tilde{\mathbf{\Xi}}_{5,7} = -\frac{1}{\varsigma d}\mathbf{R}_6,\ \tilde{\mathbf{\Xi}}_{5,12} = \sigma d\mathbf{X}^T_5,$$
$$\mathbf{\Xi}_{5,13} = \frac{4}{\varsigma d}\mathbf{R}_5,\ \tilde{\mathbf{\Xi}}_{5,14} = \frac{3}{\varsigma d}\mathbf{R}_6,\ \tilde{\mathbf{\Xi}}_{5,15} = -\mathbf{Y}^T_5,\ \tilde{\mathbf{\Xi}}_{6,12} = -\sigma d\mathbf{X}^T_6,\ \tilde{\mathbf{\Xi}}_{6,15} = -\mathbf{Y}^T_6,\ \tilde{\mathbf{\Xi}}_{7,12} = -\sigma d\mathbf{X}^T_7,\ \tilde{\mathbf{\Xi}}_{7,14} = \frac{4}{\varsigma d}\mathbf{R}_6,$$
$$\tilde{\mathbf{\Xi}}_{7,15} = -\varsigma d\mathbf{Y}^T_7,\ \tilde{\mathbf{\Xi}}_{8,10} = l\mathbf{T}\mathbf{W}_2 + m\mathbf{W}^T_1\mathbf{T},\ \tilde{\mathbf{\Xi}}_{8,12} = -\sigma d\mathbf{X}^T_8,\ \mathbf{\Xi}_{8,15} = -\mathbf{Y}^T_8,\ \tilde{\mathbf{\Xi}}_{9,9} = -\mathbf{Z}_1 + \mathbf{Z}_2,\ \tilde{\mathbf{\Xi}}_{9,12} = -\sigma d\mathbf{X}^T_9,$$
$$\tilde{\mathbf{\Xi}}_{9,15} = -\mathbf{Y}^T_9,\ \tilde{\mathbf{\Xi}}_{10,10} = m\overrightarrow{\mathbf{T}\mathbf{W}_2},\ \tilde{\mathbf{\Xi}}_{10,12} = -\sigma d\mathbf{X}^T_{10},\ \tilde{\mathbf{\Xi}}_{10,15} = -\mathbf{Y}^T_{10},\ \tilde{\mathbf{\Xi}}_{10,16} = m\mathbf{U},\ \tilde{\mathbf{\Xi}}_{11,12} = -\sigma d\mathbf{X}^T_{11},$$
$$\tilde{\mathbf{\Xi}}_{11,15} = -\mathbf{Y}^T_{11},\ \tilde{\mathbf{\Xi}}_{12,13} = -\mathbf{X}_{13},\ \tilde{\mathbf{\Xi}}_{12,14} = -\mathbf{X}_{14},\ \tilde{\mathbf{\Xi}}_{12,15} = -\mathbf{X}_{15} - \mathbf{Y}^T_{12},\ \tilde{\mathbf{\Xi}}_{12,16} = -\mathbf{X}_{16},\ \tilde{\mathbf{\Xi}}_{13,13} = -\frac{7}{\varsigma d}\mathbf{R}_5,$$
$$\tilde{\mathbf{\Xi}}_{13,15} = -\mathbf{Y}^T_{13},\ \tilde{\mathbf{\Xi}}_{14,14} = -\frac{7}{\varsigma d}\mathbf{R}_6,\ \tilde{\mathbf{\Xi}}_{14,15} = -\mathbf{Y}^T_{14},\ \tilde{\mathbf{\Xi}}_{15,15} = -\overrightarrow{\mathbf{Y}}_{15},\ \tilde{\mathbf{\Xi}}_{15,16} = -\mathbf{Y}_{16},\ \tilde{\mathbf{\Xi}}_{15,16} = -\varsigma d\mathbf{Y}_{16}.$$

Appendix E

$$
\begin{aligned}
\Theta_2^2(t) &= -2[\mathbf{f}(\mathbf{r}(t-\sigma d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))\mathbf{r}(t-\sigma d)]^T\mathbf{M}_2^1[\mathbf{f}(\mathbf{r}(t-\sigma d)) - \Gamma^-\mathbf{r}(t-\sigma d)],\\
\Theta_3^2(t) &= -2[\mathbf{f}(\mathbf{r}(t-d(t))) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))\mathbf{r}(t-d(t))]^T\mathbf{M}_3^1[\mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{\Gamma}^-\mathbf{r}(t-d(t))],\\
\Theta_5^2(t) &= -2[\mathbf{f}(\mathbf{r}(t)) - \mathbf{f}(\mathbf{r}(t-\sigma d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))(\mathbf{r}(t) - \mathbf{r}(t-\sigma d))]^T\mathbf{M}_5^1\\
&\quad\times[\mathbf{f}(\mathbf{r}(t)) - \mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{\Gamma}^-(\mathbf{r}(t) - \mathbf{r}(t-\sigma d))],\\
\Theta_6^2(t) &= -2[\mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{f}(\mathbf{r}(t-d(t))) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))(\mathbf{r}(t-\sigma d) - \mathbf{r}(t-d(t)))]^T\mathbf{M}_6^1\\
&\quad\times[\mathbf{f}(\mathbf{r}(t-\sigma d)) - \mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{\Gamma}^-(\mathbf{r}(t-\sigma d) - \mathbf{r}(t-d(t)))],\\
\Theta_7^2(t) &= -2[\mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{f}(\mathbf{r}(t-d)) - (\mathbf{\Gamma}^- + \mathbf{\Delta}(\mathbf{\Gamma}^+ - \mathbf{\Gamma}^-))(\mathbf{r}(t-d(t)) - \mathbf{r}(t-d))]^T\mathbf{M}_7^1\\
&\quad\times[\mathbf{f}(\mathbf{r}(t-d(t))) - \mathbf{f}(r(t-d)) - \mathbf{\Gamma}^-(\mathbf{r}(t-d(t)) - \mathbf{r}(t-d))].
\end{aligned}
$$

Appendix F

To handle the integral terms, we have used different methods from the ones in Refs. [19,24]. For example, Ref. [19] wrote,

$$
\begin{aligned}
&-\int_{t-h_U}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)ds - \int_{t-h_U}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Q}_5\dot{\mathbf{x}}(s)ds\\
&= -\int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)ds - \int_{t-h_U}^{t-h(t)}\dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)ds - \int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Q}_5\dot{\mathbf{x}}(s)ds - \int_{t-h_U}^{t-h(t)}\dot{\mathbf{x}}^T(s)\mathbf{Q}_5\dot{\mathbf{x}}(s)ds\\
&= -\int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)ds - \int_{t-h_U}^{t-h(t)}\dot{\mathbf{x}}^T(s)\mathbf{Q}_4\dot{\mathbf{x}}(s)ds + \mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) - \mathbf{x}^T(t-h(t))\mathbf{P}_1\mathbf{x}(t-h(t)) - 2\int_{t-h_U}^{t}\mathbf{x}^T(s)\mathbf{P}_1\dot{\mathbf{x}}(s)ds,\\
&\quad -\int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Q}_5\dot{\mathbf{x}}(s)ds - \int_{t-h_U}^{t-h(t)}\dot{\mathbf{x}}^T(s)\mathbf{Q}_5\dot{\mathbf{x}}(s)ds + \mathbf{x}^T(t-h(t))\mathbf{P}_2\mathbf{x}(t-h(t)) - \mathbf{x}^T(t-h_U)\mathbf{P}_2\mathbf{x}(t-h_U)\\
&\quad -2\int_{t-h_U}^{t-h(t)}\mathbf{x}^T(s)\mathbf{P}_2\dot{\mathbf{x}}(s)ds\\
&= \mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) + \mathbf{x}^T(t-h(t))(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{x}(t-h(t)) - \mathbf{x}^T(t-h_U)\mathbf{P}_2\mathbf{x}(t-h_U)\\
&\quad -\int_{t-h(t)}^{t}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}^T\begin{bmatrix}\mathbf{Q}_4 & \mathbf{P}_1\\ * & \mathbf{Q}_5\end{bmatrix}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}ds - \int_{t-h_U}^{t-h(t)}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}^T\begin{bmatrix}\mathbf{Q}_4 & \mathbf{P}_2\\ * & \mathbf{Q}_5\end{bmatrix}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}ds\\
&\le \mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) + \mathbf{x}^T(t-h(t))(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{x}(t-h(t)) - \mathbf{x}^T(t-h_U)\mathbf{P}_2\mathbf{x}(t-h_U),
\end{aligned}
$$

where

$$\mathbf{x}^T(t)\mathbf{P}_1\mathbf{x}(t) - \mathbf{x}^T(t-h(t))\mathbf{P}_1\mathbf{x}(t-h(t)) - 2\int_{t-h_U}^{t}\mathbf{x}^T(s)\mathbf{P}_1\dot{\mathbf{x}}(s)ds = 0,$$

$$\mathbf{x}^T(t-h(t))\mathbf{P}_2\mathbf{x}(t-h(t)) - \mathbf{x}^T(t-h_U)\mathbf{P}_2\mathbf{x}(t-h_U) - 2\int_{t-h_U}^{t-h(t)}\mathbf{x}^T(s)\mathbf{P}_2\dot{\mathbf{x}}(s)ds = 0,$$

$$\begin{bmatrix}\mathbf{Q}_4 & \mathbf{P}_1\\ * & \mathbf{Q}_5\end{bmatrix} > 0, \begin{bmatrix}\mathbf{Q}_4 & \mathbf{P}_2\\ * & \mathbf{Q}_5\end{bmatrix} > 0.$$

(inequalities (29)–(31) in Ref. [19])

In Ref. [24], the following inequality holds:

$$
\begin{aligned}
&-\int_{t-h_U}^{t}\dot{\mathbf{x}}^T(s)\mathbf{S}_1\dot{\mathbf{x}}(s)ds - \int_{t-h_U}^{t}\dot{\mathbf{x}}^T(s)\mathbf{S}_2\dot{\mathbf{x}}(s)ds\\
&= -\int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{S}_1\dot{\mathbf{x}}(s)ds - \int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{S}_1\dot{\mathbf{x}}(s)ds + \zeta^T(t)\mathbf{\Xi}_4\zeta(t) - 2\int_{t-h(t)}^{t}\mathbf{x}^T(s)Z_1\dot{\mathbf{x}}(s)ds\\
&\quad -\int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{S}_2\dot{\mathbf{x}}(s)ds - \int_{t-h(t)}^{t}\dot{\mathbf{x}}^T(s)\mathbf{S}_2\dot{\mathbf{x}}(s)ds + \zeta^T(t)\mathbf{\Xi}_5\zeta(t) - 2\int_{t-h_U}^{t-h(t)}\mathbf{x}^T(s)Z_2\dot{\mathbf{x}}(s)ds\\
&= \zeta^T(t)\mathbf{\Xi}_4\zeta(t) + \zeta^T(t)\mathbf{\Xi}_5\zeta(t)
\end{aligned}
$$

$$
-\int_{t-h(t)}^{t}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}^T\begin{bmatrix}\mathbf{S}_1 & \mathbf{Z}_1\\ \mathbf{Z}_1 & \mathbf{S}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}ds-\int_{t-h_U}^{t-h(t)}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}^T\begin{bmatrix}\mathbf{S}_1 & \mathbf{Z}_2\\ \mathbf{Z}_2 & \mathbf{S}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}(s)\\ \dot{\mathbf{x}}(s)\end{bmatrix}ds
\leq \zeta^T(t)\Xi_4\zeta(t)+\zeta^T(t)\Xi_5\zeta(t),
$$

where

$$
\zeta^T(t)\Xi_4\zeta(t)-2\int_{t-h(t)}^{t}\mathbf{x}^T(s)Z_1\dot{\mathbf{x}}(s)ds=0,\ \zeta^T(t)\Xi_5\zeta(t)-2\int_{t-h_U}^{t-h(t)}\mathbf{x}^T(s)Z_2\dot{\mathbf{x}}(s)ds=0,
$$

$$
\zeta^T(t)=[\mathbf{x}^T(t),\mathbf{x}^T(t-h(t)),\mathbf{x}^T(t-h_U),\dot{\mathbf{x}}^T(t),\dot{\mathbf{x}}^T(t-h(t)),\dot{\mathbf{x}}^T(t-h_U),\mathbf{f}^T(\mathbf{x}(t)),\mathbf{f}^T(\mathbf{x}(t-h(t))),\mathbf{f}^T(\mathbf{x}(t-h_U))],
$$

$$
\Xi_4=\mathbf{e}_1\mathbf{Z}_1\mathbf{e}_1^T-\mathbf{e}_2\mathbf{Z}_1\mathbf{e}_2^T,\ \Xi_5=\mathbf{e}_2\mathbf{Z}_2\mathbf{e}_2^T-\mathbf{e}_3\mathbf{Z}_2\mathbf{e}_3^T,
$$

$$
\mathbf{e}_1^T=[I_n,\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{8}],\ \mathbf{e}_2^T=[\mathbf{0},I_n,\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{7}],\ \mathbf{e}_3^T=[\mathbf{0},\mathbf{0},I_n,\underbrace{\mathbf{0},\cdots,\mathbf{0}}_{6}],
$$

$$
\begin{bmatrix}\mathbf{S}_1 & \mathbf{Z}_1\\ \mathbf{Z}_1 & \mathbf{S}_2\end{bmatrix}>\mathbf{0},\ \begin{bmatrix}\mathbf{S}_1 & \mathbf{Z}_2\\ \mathbf{Z}_2 & \mathbf{S}_2\end{bmatrix}>\mathbf{0}.
$$

(inequality (8) in Ref. [24])

According to the above-mentioned inequalities and the inequalities (20)–(22), we can see that the authors in Refs. [19,24] added 2 new variables to deal with integral terms in their papers, which may increase the complexity of the computation. On the other hand, the additional constraints $\begin{bmatrix}\mathbf{Q}_4 & \mathbf{P}_1\\ * & \mathbf{Q}_5\end{bmatrix}>0$ and $\begin{bmatrix}\mathbf{Q}_4 & \mathbf{P}_2\\ * & \mathbf{Q}_5\end{bmatrix}>0$, $\begin{bmatrix}\mathbf{S}_1 & \mathbf{Z}_1\\ \mathbf{Z}_1 & \mathbf{S}_2\end{bmatrix}>\mathbf{0}$ and $\begin{bmatrix}\mathbf{S}_1 & \mathbf{Z}_2\\ \mathbf{Z}_2 & \mathbf{S}_2\end{bmatrix}>\mathbf{0}$ were also introduced in Refs. [19,24], respectively. By contrast, our method is clear and easier to understand.

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