1. Racionalis tortek integralasa

cel: $\int \frac{P(x)}{Q(x)}\,\mathrm{d}x \ (x \in I, Q(x) \neq 0)\ P, Q$ polinom

minden ilyen tort felbonthato elemi tortek linearis kombinaciojara

Alaptort tipusok

1

$$\int \left(\frac{1}{(ax+b)^n}\right) \mathrm{d}x \ (n \in \mathbb{N}^+, a, b \neq 0)$$

pl

$$\int \frac{1}{(3x-2)^5}\,\mathrm{d}x = \frac{1}{3}\int (3x-2)' \cdot (3x-2)^5 = \frac{1}{3} \cdot \frac{(3x-2)^{-5+1}}{-5+1} + C$$

$$\int \frac{1}{7x+5}\,\mathrm{d}x \ \left(x > -\frac{5}{7}\right) = \frac{1}{7}\int \frac{(7x+5)'}{7x+5} = \frac{1}{7}\ln|7x+5| + C = \frac{1}{7}\ln(7x+5) + C$$

2

$$\int \frac{Ax+B}{ax^2+bx+c}\,\mathrm{d}x \ (A, B, a, b, c \in \mathbb{R}, a \neq 0 \text{ es nincs valos gyoke a nevezonek})$$

pl

$$\int \frac{3x-1}{x^2+4x+7}\,\mathrm{d}x, \quad D = 16 - 28 < 0, \quad \left(x^2+4x+7\right)' = 2x+4$$

$$\frac{3}{2}\int \frac{2x+\frac{2}{3}}{x^2+4x+7}\,\mathrm{d}x = \frac{3}{2}\int \frac{2x+4-4+\frac{2}{3}}{x^2+4x+7}\,\mathrm{d}x = \frac{3}{2}\int \frac{\left(x^2+4x+7\right)'}{x^2+4x+7}\,\mathrm{d}x = \frac{3}{2} \cdot \frac{10}{3}\int \frac{1}{x^2+4x+7}\,\mathrm{d}x =$$

$$= \frac{3}{2}\ln\left|x^2+4x+7\right| - 5\int \frac{1}{(x+2)^2+3}\,\mathrm{d}x = \frac{3}{2}\ln\left(x^2+4x+7\right) - \frac{5}{3}\int \frac{1}{1+\left(\frac{x+2}{\sqrt{3}}\right)^2}\,\mathrm{d}x =$$

$$= \frac{3}{2}\ln\left(x^2+4x+7\right) - \frac{5}{3}\frac{\arctan\left(\frac{x+2}{\sqrt{3}}\right)}{\frac{1}{\sqrt{3}}} + C$$

3

$$\int \frac{Ax+B}{\left(ax^2+bx+c\right)^n}\,\mathrm{d}x \quad \left(A, B, a, b, c \in \mathbb{R}, a \neq 0, b^2 - 4a < 0, n \in \mathbb{N}, n \geq 2\right)$$

2. Racionalis tortek felbontasa

a

$$\int \frac{7x+1}{x^2-6x+8}\,\mathrm{d}x, \quad D = 36 - 32 = 4$$

1. lepes, nevezo fakotorizacioja

$$x^2 - 6x + 8 = (x-2)(x-4)$$

$$\int \frac{7x+1}{(x-2)(x-4)}\,\mathrm{d}x$$

1. modszer, egyenlo egyutthatok

$$\frac{7x+1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} \quad (x \in (2,4))$$
$$7x+1 = A(x-4) + B(x-2) \quad (\forall x \in \mathbb{R})$$
$$7x+1 = (A+B)x + (-4A-2B)$$
$$x^1 \text{ egyutthatoi} : A + B = 7$$
$$x^2 \text{ egyutthatoi} : -4A - 2B = 1$$
$$A = -\frac{15}{2}, B = 7 + \frac{15}{2} = \frac{29}{2}$$

2. modszer, ertekadas

$$\text{Ha } x = 4 \Longrightarrow 29 = 2B \Longrightarrow B = \frac{29}{2}$$
$$\text{Ha } x = 2 \Longrightarrow 15 = -2A \Longrightarrow A = -\frac{15}{2}$$

$$\Longrightarrow \int \left(\frac{-\frac{15}{2}}{x-2} + \frac{\frac{29}{2}}{x-4} \right) \mathrm{d}x = \frac{29}{2} \int \frac{1}{x-4}\,\mathrm{d}x - \frac{15}{2} \int \frac{1}{x-2}\,\mathrm{d}x =$$
$$= \frac{29}{2} \ln|x-4| - \frac{15}{2} \ln|x-2| + C \underset{2<x<4}{=} \frac{29}{2} \ln(4-x) - \frac{15}{2} \ln(x-2) + C$$

b

$$\int \frac{3x-5}{x^2+2x+1}\,\mathrm{d}x = \int \frac{3x-5}{(x+1)^2}\,\mathrm{d}x$$

$$\frac{3x-5}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$(3x-5) = A(x-1) + B$$

$$A = 3, B = -8$$

$$\int \frac{3}{x+1} - \frac{8}{(x+1)^2}\,\mathrm{d}x = 3\int \frac{(x+1)'}{x+1}\,\mathrm{d}x - 8\int (x+1)'(x+1)^{-2}\,\mathrm{d}x = 3\ln|x+1| - 8\frac{(x+1)^{-1}}{-1} + C =$$

$$= 3\ln(x+1) + \frac{8}{x+1} + C$$

megjegyzes:

$$\frac{3x-5}{(x+1)^2} = \frac{3x+3-8}{(x+1)^2} = \frac{3}{x+1} - \frac{8}{(x+1)^2}$$

c

$$\int \frac{x^3+x^2-x+3}{x^2-1}\,\mathrm{d}x \qquad x \in (-1,1)$$

Ha $\int \frac{P(x)}{Q(x)}\,\mathrm{d}x : \deg(P) \geq \deg(Q) \implies$ polinomosztas

$$\int \frac{x(x^2-1)+(x^2-1)+4}{x^2-1}\,\mathrm{d}x = \int \left(x+1+\frac{4}{x^2-1}\right)\mathrm{d}x = \frac{x^2}{2} + x + 4\int \frac{1}{(x-1)(x+1)}\,\mathrm{d}x,$$

$$\int \frac{1}{(x-1)(x+1)}\,\mathrm{d}x = \frac{1}{2}\int \frac{(x+1)-(x-1)}{(x-1)(x+1)}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x-1}\,\mathrm{d}x - \frac{1}{2}\int \frac{1}{x+1}\,\mathrm{d}x = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C =$$

$$\underset{-1<x<1}{=} \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(x-1) + C$$

d

$$\int \frac{1}{x^3+4x}\,\mathrm{d}x = \int \frac{1}{x(x^2+4)}\,\mathrm{d}x \qquad (D<0)$$

$$\int \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$1 = A(x^2+4) + x(Bx+C)$$

$$1 = (A+B)x^2 + (C)x + (4A)$$

$$x^2 : A+B=0$$

$$x^1 : C=0$$

$$x^0 : 4A=0$$

$$A=\frac{1}{4},\ B=-\frac{1}{4}$$

$$\int \left(\frac{\frac{1}{4}}{x} + \frac{-\left(\frac{1}{4}\right)x+0}{x^2+4}\right)\mathrm{d}x = \frac{1}{4}\int \frac{1}{x}\,\mathrm{d}x - \frac{1}{4}\frac{1}{2}\int \frac{2x}{x^2+4}\,\mathrm{d}x =$$

$$= \frac{1}{4}\ln(x) - \frac{1}{8}\ln(x^2+4) + C$$

e

$$\int \frac{x^3+9x-9}{x^2(x^2+9)}\,\mathrm{d}x$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+B}{x^2+9} = \text{hazi feladat}$$

$$\int \frac{x(x^2+9)-(9+x^2)+x^2}{x^2(x^2+9)}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x - \int \frac{1}{x^2}\,\mathrm{d}x + \int \frac{1}{x^2+9}\,\mathrm{d}x \underset{x>0}{=} \ln x - \frac{x^1}{-1} + \frac{1}{9}\frac{\arctan\left(\frac{x}{3}\right)}{\frac{1}{3}} + C =$$

$$= \ln x + \frac{1}{x} + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

hazi: orai peldak barmelyike
hazi: 1