Rendezzük át a $2x^3 + 5x^2 + 3x + 1$ polinomot $(x+1)$ hatványai szerint! A feladat általánosításaként mutassuk meg, hogy ha P egy legfeljebb n-edfokú polinom és $a \in \mathbb{R}$ tetszőleges, akkor minden $x \in \mathbb{R}$ esetén

$$P(x) = \sum_{k=0}^{n} \frac{P^k(a)}{k!}(x-a)^k.$$

$$P(x) \coloneqq 2x^3 + 5x^2 + 3x + 1 \quad (x \in \mathbb{R}) \quad \text{és} \quad a \coloneqq -1$$

$P'(x) = 6x^2 + 10x + 3$
$P''(x) = 12x + 10$
$P'''(x) = 12$
$P^{(4)}(x) = 0$

$P(-1) = 1$
$P'(-1) = -1$
$P''(-1) = -2$
$P'''(-1) = 12$
$P^{(4)}(-1) = 0$

Mivel $P \in D^4(\mathbb{R})$

$$\forall x \in \mathbb{R} \setminus \{-1\} \quad \text{ponthoz} \quad \exists \xi \quad \text{az} \quad a = -1 \quad \text{és} \quad x \quad \text{között}:$$

$$P(x) = T_{-1,4}(P,x) = P(x) - \sum_{k=0}^{3} \frac{P^{(k)}(-1)}{k!}(x+1)^k = \frac{P^{(4)}(-1)}{4!}(x+1)^4 = 0$$

$$P(x) = \sum_{k=0}^{3} \frac{P^{(k)}(-1)}{k!}(x+1)^k = P(-1) + \frac{P'(-1)}{1!}(x+1) + \frac{P''(-1)}{2!}(x+1)^2 + \frac{P'''(-1)}{3!}(x+1)^3 =$$

$$= 1 - (x+1) - (x+1)^2 + 2(x+1)^3$$

$$P(x) = 2x^3 + 5x^2 + 3x + 1 = 1 - (x+1) - (x+1)^2 + 2(x+1)^3$$

1/a

$$\lim_{x \to 0+0} \sin x \cdot \ln x$$

$$\lim_{x \to 0+0} \sin x \cdot \ln x = \lim_{x \to 0+0} \frac{\ln x}{\frac{1}{\sin x}} = \frac{0}{0} = L'H = \lim_{x \to 0+0} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = \lim_{x \to 0+0} -\frac{\sin^2 x}{x \cos x} = -\frac{x^2}{1 \cdot x} = -x \to 0$$

1/b

$$\lim_{x \to +\infty} e^{x-x^2} \cdot \ln(x^2 - x + 1)$$

$$\lim_{x \to +\infty} e^{x-x^2} \cdot \ln(x^2 - x + 1) = \lim_{x \to +\infty} \frac{\ln(x^2 - x + 1)}{e^{x^2-x}} = \frac{+\infty}{+\infty} = L'H = \lim_{x \to +\infty} \frac{\frac{2x+1}{x^2+x-1}}{e^{x^2-x}(2x-1)} =$$

$$= \lim_{x \to +\infty} \frac{1}{e^{x^2-x}(x^2+x-1)} = 0$$

1/c

$$\lim_{x\to 0+0}\left(\frac{1}{\arctan x}-\frac{1}{x}\right)$$

$$\lim_{x\to 0+0}\left(\frac{1}{\arctan x}-\frac{1}{x}\right)=\lim_{x\to 0+0}\frac{x-\arctan x}{x\arctan x}=\frac{0}{0}=L'H=\frac{1-\frac{1}{x^2+1}}{\arctan x+\frac{x}{x^2+1}})=\frac{0}{0}=L'H=$$

$$=\lim_{x\to 0+0}\frac{-\frac{2x}{(x^2+1)^2}}{\frac{1}{x^2+1}+\frac{(x^2+1)-2x^2}{(x^2+1)^2}}=\lim_{x\to 0+0}\frac{\frac{2x}{(x^2+1)^2}}{\frac{1}{x^2+1}+\frac{1-x^2}{(x^2+1)^2}}=\lim_{x\to 0+0}\frac{\frac{2x}{(x^2+1)^2}}{\frac{2}{(x^2+1)^2}}=\lim_{x\to 0+0}\frac{2x}{2}=\lim_{x\to 0+0}x=0$$

1/d

$$\lim_{x\to 0+0}(\cos x)^{\frac{1}{x^2}}$$

$$\lim_{x\to 0+0}(\cos x)^{\frac{1}{x^2}}=\lim_{x\to 0+0}\exp\left(\ln\left((\cos x)^{\frac{1}{x^2}}\right)\right)=\lim_{x\to 0+0}\exp\left(\frac{1}{x^2}\ln(\cos x)\right)$$

Vizsgáljuk: $\lim_{x\to 0+0}=\frac{\ln(\cos x)}{x^2}$

$$\lim_{x\to 0+0}=\frac{\ln(\cos x)}{x^2}=\frac{0}{0}=L'H=\lim_{x\to 0+0}\frac{\frac{-\sin x}{\cos x}}{2x}=\lim_{x\to 0+0}-\frac{\sin x}{2x\cos x}=\frac{0}{0}=L'H=\lim_{x\to 0+0}-\frac{\cos x}{2\cos x+2x(-\sin x)}=-\frac{1}{2}$$

Ekkor

$$\lim_{x\to 0+0}\exp\left(-\frac{1}{2}\right)=e^{-\frac{1}{2}}$$

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Írja fel az

$$\sqrt[3]{1+x}\qquad (x\in(-1,+\infty))$$

függvény 0 pont körüli második Taylor-polinomját, $T_{0,2}(f,x)$-et. Adjon becslést az

$$\left|f(x)-T_{0,2}(f,x)\right|$$

hibára a $\left[0,\frac{1}{4}\right]$ intervallumon.

$f(x)=\sqrt[3]{1+x}$
$f'(x)=\frac{1}{3}(1+x)^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{(1+x)^2}}$
$f''(x)=-\frac{2}{9}(1+x)^{-\frac{5}{3}}=-\frac{2}{9\sqrt[3]{(1+x)^5}}$
$f'''(x)=\frac{10}{27}(1+x)^{-\frac{8}{3}}=\frac{10}{27\sqrt[3]{(1+x)^8}}$

$f(0)=1$
$f'(0)=\frac{1}{3}$
$f''(0)=-\frac{2}{9}$
$f'''(0)=\frac{10}{27}$

$$T_{0,2}(f,x)=f(0)+f'(0)x+f''(0)x^2=1+\frac{1}{3}x-\frac{2}{9}x^2$$

$$R_2(x)=\frac{f'''(x)(\xi)}{3!}x^3=\frac{10}{27}$$

not doin allat