1. Racionalis tortek integralasa

cel: $\int \frac{P(x)}{Q(x)} \, dx$ $(x \in I, Q(x) \neq 0)$ P, Q polinom

minden ilyen tort felbonthato elemi tortek linearis kombinaciojara

Alaptort tipusok

1

$$\int \left(\frac{1}{(ax+b)^n} \right) dx \ (n \in \mathbb{N}^+, a, b \neq 0)$$

pl

$$\int \frac{1}{(3x-2)^5} \, dx = \frac{1}{3} \int (3x-2)' \cdot (3x-2)^5 = \frac{1}{3} \cdot \frac{(3x-2)^{-5+1}}{-5+1} + C$$

$$\int \frac{1}{7x+5} \, dx \ \left(x > -\frac{5}{7} \right) = \frac{1}{7} \int \frac{(7x+5)'}{7x+5} = \frac{1}{7} \ln|7x+5| + C = \frac{1}{7} \ln(7x+5) + C$$

2

$$\int \frac{Ax+B}{ax^2+bx+c} \, dx \ (A, B, a, b, c \in \mathbb{R}, a \neq 0 \text{ es nincs valos gyoke a nevezonek})$$

pl

$$\int \frac{3x-1}{x^2+4x+7} \, dx, \quad D = 16 - 28 < 0, \quad \left(x^2+4x+7 \right)' = 2x+4$$

$$\frac{3}{2} \int \frac{2x+\frac{2}{3}}{x^2+4x+7} \, dx = \frac{3}{2} \int \frac{2x+4-4+\frac{2}{3}}{x^2+4x+7} \, dx = \frac{3}{2} \int \frac{\left(x^2+4x+7 \right)'}{x^2+4x+7} \, dx = \frac{3}{2} \cdot \frac{10}{3} \int \frac{1}{x^2+4x+7} \, dx =$$

$$= \frac{3}{2} \ln|x^2+4x+7| - 5 \int \frac{1}{(x+2)^2+3} \, dx = \frac{3}{2} \ln(x^2+4x+7) - \frac{5}{3} \int \frac{1}{1 + \left(\frac{x+2}{\sqrt{3}} \right)^2} \, dx =$$

$$= \frac{3}{2} \ln(x^2+4x+7) - \frac{5}{3} \frac{\arctan\left(\frac{x+2}{\sqrt{3}} \right)}{\frac{1}{\sqrt{3}}} + C$$

3

$$\int \frac{Ax+B}{\left(ax^2+bx+c \right)^n} \, dx \quad \left(A, B, a, b, c \in \mathbb{R}, a \neq 0, b^2 - 4a < 0, n \in \mathbb{N}, n \geq 2 \right)$$

2. Racionalis tortek felbontasa

a

$$\int \frac{7x+1}{x^2-6x+8}\,\mathrm{d}x, \quad D = 36 - 32 = 4$$

1. lepes, nevezo fakotorizacioja

$$x^2 - 6x + 8 = (x-2)(x-4)$$

$$\int \frac{7x+1}{(x-2)(x-4)}\,\mathrm{d}x$$

1. modszer, egyenlo egyutthatok

$$\frac{7x+1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4} \quad (x \in (2,4))$$

$$7x + 1 = A(x-4) + B(x-2) \quad (\forall x \in \mathbb{R})$$

$$7x + 1 = (A+B)x + (-4A - 2B)$$

$$x^1 \text{ egyutthatoi} : A + B = 7$$

$$x^2 \text{ egyutthatoi} : -4A - 2B = 1$$

$$A = -\frac{15}{2}, B = 7 + \frac{15}{2} = \frac{29}{2}$$

2. modszer, ertekadas

$$\text{Ha } x = 4 \implies 29 = 2B \implies B = \frac{29}{2}$$

$$\text{Ha } x = 2 \implies 15 = -2A \implies A = -\frac{15}{2}$$

$$\implies \int \left(\frac{-\frac{15}{2}}{x-2} + \frac{\frac{29}{2}}{x-4} \right) \mathrm{d}x = \frac{29}{2} \int \frac{1}{x-4}\,\mathrm{d}x - \frac{15}{2} \int \frac{1}{x-2}\,\mathrm{d}x =$$

$$= \frac{29}{2} \ln|x-4| - \frac{15}{2} \ln|x-2| + C \underset{2<x<4}{=} \frac{29}{2} \ln(4-x) - \frac{15}{2} \ln(x-2) + C$$

b

$$\int \frac{3x - 5}{x^2 + 2x + 1}\, \mathrm{d}x = \int \frac{3x - 5}{(x + 1)^2}\, \mathrm{d}x$$

$$\frac{3x - 5}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}$$

$$(3x - 5) = A(x - 1) + B$$

$$A = 3, B = -8$$

$$\int \frac{3}{x + 1} - \frac{8}{(x + 1)^2}\, \mathrm{d}x = 3\int \frac{(x + 1)'}{x + 1}\, \mathrm{d}x - 8\int (x + 1)'(x + 1)^{-2}\, \mathrm{d}x = 3\ln|x + 1| - 8\frac{(x + 1)^{-1}}{-1} + C =$$

$$= 3\ln(x + 1) + \frac{8}{x + 1} + C$$

megjegyzes:

$$\frac{3x - 5}{(x + 1)^2} = \frac{3x + 3 - 8}{(x + 1)^2} = \frac{3}{x + 1} - \frac{8}{(x + 1)^2}$$

c

$$\int \frac{x^3 + x^2 - x + 3}{x^2 - 1}\, \mathrm{d}x \quad x \in (-1, 1)$$

Ha $\int \frac{P(x)}{Q(x)}\, \mathrm{d}x : \deg(P) \geq \deg(Q) \Longrightarrow$ polinomosztas

$$\int \frac{x(x^2 - 1) + (x^2 - 1) + 4}{x^2 - 1}\, \mathrm{d}x = \int \left(x + 1 + \frac{4}{x^2 - 1} \right) \mathrm{d}x = \frac{x^2}{2} + x + 4\int \frac{1}{(x - 1)(x + 1)}\, \mathrm{d}x,$$

$$\int \frac{1}{(x - 1)(x + 1)}\, \mathrm{d}x = \frac{1}{2}\int \frac{(x + 1) - (x - 1)}{(x - 1)(x + 1)}\, \mathrm{d}x = \frac{1}{2}\int \frac{1}{x - 1}\, \mathrm{d}x - \frac{1}{2}\int \frac{1}{x + 1}\, \mathrm{d}x = \frac{1}{2}\ln|x - 1| - \frac{1}{2}\ln|x + 1| + C =$$

$$\underset{-1 < x < 1}{=} \frac{1}{2}\ln(1 - x) - \frac{1}{2}\ln(x - 1) + C$$

d

$$\int \frac{1}{x^3 + 4x}\,\mathrm{d}x = \int \frac{1}{x(x^2 + 4)}\,\mathrm{d}x \quad (D < 0)$$

$$\int \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$1 = A(x^2 + 4) + x(Bx + C)$$

$$1 = (A + B)x^2 + (C)x + (4A)$$

$$x^2 : A + B = 0$$

$$x^1 : C = 0$$

$$x^0 : 4A = 0$$

$$A = \frac{1}{4}, \; B = -\frac{1}{4}$$

$$\int \left(\frac{\frac{1}{4}}{x} + \frac{-(\frac{1}{4})x + 0}{x^2 + 4} \right)\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{x}\,\mathrm{d}x - \frac{1}{4}\frac{1}{2}\int \frac{2x}{x^2 + 4}\,\mathrm{d}x =$$

$$= \frac{1}{4}\ln(x) - \frac{1}{8}\ln(x^2 + 4) + C$$

e

$$\int \frac{x^3 + 9x - 9}{x^2(x^2 + 9)}\,\mathrm{d}x$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + B}{x^2 + 9} = \text{hazi feladat}$$

$$\int \frac{x(x^2 + 9) - (9 + x^2) + x^2}{x^2(x^2 + 9)}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x - \int \frac{1}{x^2}\,\mathrm{d}x + \int \frac{1}{x^2 + 9}\,\mathrm{d}x \underset{x>0}{=} \ln x - \frac{x^1}{-1} + \frac{1}{9}\frac{\arctan\left(\frac{x}{3}\right)}{\frac{1}{3}} + C =$$

$$= \ln x + \frac{1}{x} + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

hazi: orai peldak barmelyike

hazi: 1