1

a

$$\int \frac{x \cdot \sqrt[3]{\ln(x^2+1)}}{x^2+1}\,\mathrm{d}x \qquad (x>0)$$

$$\left(\ln(x^2+1)\right)' = \frac{2x}{x^2+1}$$

$$\frac{1}{2}\int \frac{2x}{x^2+1}\left(\ln(x^2+1)\right)^{\frac{1}{3}}\,\mathrm{d}x = \frac{1}{2}\int \left(\ln(x^2+1)\right)'\left(\ln(x^2+1)\right)^{\frac{1}{3}}\,\mathrm{d}x = \frac{1}{2}\frac{\ln(x^2+1)^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{1}{2}\cdot\frac{3}{4}\sqrt[3]{\left(\ln(x^2+1)\right)^4} + C$$

b

$$\int \frac{(3-\sin x)^2}{1+\cos 2x}\,\mathrm{d}x \qquad x\in\left(0,\frac{\pi}{2}\right)$$

$$\int \frac{9-6\sin x+\sin^2 x}{\sin^2 x+\cos^2 x+\cos^2 x-\sin^2 x}\,\mathrm{d}x = \int \frac{9-6\sin x+\sin^2 x}{2\cos^2 x}\,\mathrm{d}x = \frac{9}{2}\int \frac{1}{\cos^2 x}\,\mathrm{d}x - 3\int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x + \frac{1}{2}\int \frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x =$$

$$= \frac{9}{2}\tan x - \frac{3}{\cos x} + \frac{1}{2}\int \frac{1-\cos^2 x}{\cos^2 x}\,\mathrm{d}x = \frac{9}{2}\tan x + \frac{3}{\cos x} + \frac{1}{2}\tan x - \frac{1}{2}x + C$$

c

$$\int x\arctan\frac{1}{x}\,\mathrm{d}x \qquad (x>0)$$

$$\int \left(\frac{x^2}{2}\right)'\arctan\frac{1}{x}\,\mathrm{d}x = \frac{x^2}{2}\arctan\frac{1}{x} - \int \frac{x^2}{2}\cdot\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x}\right)^2\,\mathrm{d}x =$$

$$= \frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}\underbrace{\int \frac{x^2}{x^2+1}\,\mathrm{d}x}_{\int \frac{x^2-1+1}{x^2+1}\,\mathrm{d}x} = \frac{x^2}{2}\arctan\frac{1}{x} + \frac{1}{2}x - \frac{1}{2}\arctan x + C$$

2

$$\int \frac{1}{x-6\sqrt{x+13}}\,\mathrm{d}x \qquad x\in(0,+\infty)$$

$$u=\sqrt{x+13}, \quad x=u^2-13, \quad x'=2u$$

$$\int \frac{1}{u^2-6u+13}\cdot 2u\,\mathrm{d}u = \int \frac{2u-6+6}{u^2-6u+13}\,\mathrm{d}u = \int \frac{2u-6}{u^2-6u+13}\,\mathrm{d}u + \int \frac{6}{u^2-6u+13}\,\mathrm{d}u =$$

$$= \ln(u^2-6u+13) + \frac{6}{4}\int \frac{1}{\left(\frac{u-3}{2}\right)^2+1} = \ln(u^2-6u+13) + \frac{3}{2}\frac{\arctan\left(\frac{u-3}{2}\right)}{\frac{1}{2}} + C \Longrightarrow$$

$$\Longrightarrow \ln\left(x+13-6\sqrt{x+13}+13\right) + 3\arctan\left(\frac{\sqrt{x+13}-3}{2}\right) + C$$

3

a

$$y = x^2 - 6x; \quad y = 4x - x^2$$

b

$$y = \sqrt{x}; \quad y = \sqrt{2x-1}; \quad y = 0$$

$$\sqrt{x} = \sqrt{2x-1} \Longrightarrow x = 2x - 1 \Longrightarrow x = 1$$

$$\int_0^1 \sqrt{x}\,\mathrm{d}x - \int_{\frac{1}{2}}^1 \sqrt{2x-1}\,\mathrm{d}x = \left[\frac{\frac{x^3}{2}}{\frac{3}{2}}\right]_0^1 - \frac{1}{2}\left[\frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\frac{1}{2}}^1 = \frac{2}{3}\left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) - \frac{1}{2}\cdot\frac{2}{3}\left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

vagy

$$H = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \land y^2 \le x \le \frac{y^2+1}{2}\right\}$$

$$T(H) = \int_0^1 \left(\frac{y^2+1}{2} - y^2\right)\mathrm{d}y = \frac{1}{2}\int_0^1 (y^2 + 1 - 2y^2)\,\mathrm{d}y = \frac{1}{2}\int_0^1 (1 - y^2)\,\mathrm{d}y = \frac{1}{2}\left[y - \frac{y^3}{3}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

4

$$f(y) = \sqrt{\frac{4e^{2x} + 13e^x}{e^{2x} + 3e^x - 10}} \qquad x \in [\ln 3, \ln 4]$$

$$0 \le f \land f \in C[\ln 3, \ln 4] \land \text{nevezo nem } 0 \Longrightarrow f \in R[\ln 3, \ln 4] \Longrightarrow \exists V = \pi \int_{\ln 3}^{\ln 4} f^2(x)\,\mathrm{d}x$$

$$\pi \int_{\ln 3}^{\ln 4} \frac{4e^{2x} + 13e^x}{e^{2x} + 3e^x - 10}\,\mathrm{d}x$$

$$u = e^x, \quad x = \ln u, \quad x' = \frac{1}{u}$$

$$\frac{V}{\pi} = \int_3^4 \frac{4u^2 + 13u}{u^2 + 3u - 10}\frac{1}{u}\,\mathrm{d}u = \int_3^4 \frac{4u + 13}{u^2 + 3u - 10}\,\mathrm{d}u = \int_3^4 \frac{4u+13}{(u-2)(u+5)}\,\mathrm{d}u$$

$$A = 3,\ B = 1$$

$$\int_3^4 \frac{3}{u-2} + \frac{1}{u+5}\,\mathrm{d}u = 3[\ln|t-2|]_3^4 + [\ln|t+5|]_3^4 = 3(\ln 3 - \ln 1) + (\ln 9 - \ln 8) = \ln 8 + \ln 9 - \ln 8 = \ln 9$$