(Hf) 1. Számítsa ki az alábbi sorozatok határértékét:

a) $$\lim_{n\to+\infty} \frac{n^3-2n-1}{-3n^3+n+3} = \lim_{n\to+\infty} \cancel{\frac{n^3}{n^3}} \cdot \frac{1-\underbrace{\frac{2}{n^2}}_{\to 0}-\underbrace{\frac{1}{n^3}}_{\to 0}}{-3+\underbrace{\frac{1}{n^2}}_{\to 0}+\underbrace{\frac{3}{n^3}}_{\to 0}} = -\frac{1}{3}$$

b) $$\lim_{n\to+\infty} \frac{(n+1)^3+(n-1)^3}{n^3+1} = \lim_{n\to+\infty} \frac{n^3\left(1+\frac{1}{n}\right)^3+n^3\left(1-\frac{1}{n}\right)^3}{n^3+1} =$$

$$= \lim_{n\to+\infty} \cancel{\frac{n^3}{n^3}} \cdot \frac{\left(1+\frac{1}{n}\right)^3+\left(1-\frac{1}{n}\right)^3}{1+\frac{1}{n^3}} = \frac{(1+0)^3+(1-0)^3}{1+0} = 2$$

2. Konvergensek-e a köv. sorozatok? Ha igen, akkor mi a határértékük?

a) $$\lim_{n\to+\infty} \left(\sqrt{n^2+3n+1}-2n\right) = \lim_{n\to+\infty} \left(\sqrt{n^2\left(1+\frac{3}{n}+\frac{1}{n^2}\right)}-2n\right) =$$

$$= \lim_{n\to+\infty} \left(n\sqrt{1+\frac{3}{n}+\frac{1}{n^2}}-2n\right) = \lim_{n\to+\infty} \left(n\left(\sqrt{1+\frac{3}{n}+\frac{1}{n^2}}-2\right)\right) =$$

$$= (+\infty)\left(\sqrt{1+0+0}-2\right) = (+\infty)(-1) = -\infty.$$

b) $$\lim_{n\to+\infty} n\left(n-\sqrt{n^2+1}\right) = \lim_{n\to+\infty} n\left(n-\sqrt{n^2\left(1+\frac{1}{n^2}\right)}\right) =$$

$$= \lim_{n\to+\infty} n\left(n-n\sqrt{1+\frac{1}{n^2}}\right) = \lim_{n\to+\infty} n^2\left(1-\sqrt{1+\frac{1}{n^2}}\right) = (+\infty)\left(1-\sqrt{1+0}\right) =$$

$$= (+\infty)\cdot 0 = ?$$

Nem jó módszer

$$\lim_{n\to+\infty} n\left(n-\sqrt{n^2+1}\right) = \lim_{n\to+\infty} \frac{n\left(n-\sqrt{n^2+1}\right)\left(n+\sqrt{n^2+1}\right)}{n+\sqrt{n^2+1}} =$$

$$= \lim_{n\to+\infty} \frac{n\left(n^2-(n^2+1)\right)}{n+\sqrt{n^2+1}} = \lim_{n\to+\infty} \frac{-n}{n+n\sqrt{1+\frac{1}{n^2}}} = \lim_{n\to+\infty} \cancel{\frac{n}{n}} \cdot \frac{-1}{1+\sqrt{1+\frac{1}{n^2}}} =$$

$$= \lim_{n\to+\infty} \frac{-1}{1+\sqrt{1+\frac{1}{n^2}}} = \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}.$$

A sorozat konvergens, és határértéke $\frac{1}{2}$.

Hf 3. Számítsa ki az alábbi határértékeket:

a) $\dfrac{2^n+2^{-n}}{2^{-n}+3^n} = \dfrac{2^n+\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n+3^n} = \dfrac{2^n}{3^n}\cdot\dfrac{1+\left(\frac{1}{4}\right)^n}{\left(\frac{1}{6}\right)^n+1} = \left(\frac{2}{3}\right)^n\dfrac{1+\left(\frac{1}{4}\right)^n}{\left(\frac{1}{6}\right)^n+1} \xrightarrow[n\to+\infty]{}$

$(q^n \to 0$, ha $|q|<1)$ $\xrightarrow[n\to+\infty]{} 0\cdot\dfrac{1+0}{0+1} = \underline{\underline{0}}$

b) $\dfrac{n2^{n+1}+3^{2n}}{9^{n-1}+3^n} = \dfrac{2n\cdot 2^n+9^n}{\frac{1}{9}\cdot 9^n+3^n} = \dfrac{\cancel{9^n}}{\cancel{9^n}}\cdot\dfrac{2n\left(\frac{2}{9}\right)^n+1}{\frac{1}{9}+\left(\frac{1}{3}\right)^n} \xrightarrow[n\to+\infty]{}$

$(q^n\to 0$ és $n\cdot q^n\to 0$, ha $|q|<1)$ $\xrightarrow[n\to+\infty]{} \dfrac{2\cdot 0+1}{\frac{1}{9}+0} = \underline{\underline{9}}.$

c) $\sqrt{\dfrac{(-2)^n+5^n}{5^{n+1}+n^5}} = \sqrt{\dfrac{(-2)^n+5^n}{5\cdot 5^n+n^5}} = \sqrt{\dfrac{\cancel{5^n}}{\cancel{5^n}}\cdot\dfrac{\left(-\frac{2}{5}\right)^n+1}{5+n^5\cdot\left(\frac{1}{5}\right)^n}} \xrightarrow[n\to+\infty]{}$

$(q^n\to 0$ és $n^5\cdot q^n\to 0$, ha $|q|<1)$ $\xrightarrow[n\to+\infty]{} \sqrt{\dfrac{0+1}{5+0}} = \dfrac{1}{\sqrt{5}} = \underline{\underline{\dfrac{\sqrt{5}}{5}}}$

d) $\dfrac{(-3)^n+n^3}{n!+5^n} = \dfrac{(-3)^n}{n!}\cdot\dfrac{1+n^3\left(-\frac{1}{3}\right)^n}{1+\frac{5^n}{n!}} \xrightarrow[n\to+\infty]{}$ $\left(\dfrac{a^n}{n!}\to 0\right.$, ha $a\in\mathbb{R}$,

$n^3\cdot q^n\to 0$, ha $|q|<1)$ $\xrightarrow[n\to+\infty]{} 0\cdot\dfrac{1+0}{1+0} = \underline{\underline{0}}$

e) $\sqrt[n]{2^n+n^2+1} = \sqrt[n]{2^n\left(1+n^2\cdot\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^n\right)} = 2\cdot\sqrt[n]{1+n^2\cdot\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^n} \xrightarrow[n\to+\infty]{}$

$(q^n\to 0$ és $n^2q^n\to 0$, ha $|q|<1$, ezért $x_n := 1+n^2\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^n \to 1 > 0 \Rightarrow \sqrt[n]{x_n}\to 1)$

$\xrightarrow[n\to+\infty]{} 2\cdot 1 = \underline{\underline{2}}$

f) $\sqrt[n]{n\cdot 3^n+n^3+(-1)^n} = \sqrt[n]{n3^n\left(1+n^2\left(\frac{1}{3}\right)^n+\frac{1}{n}\cdot\left(-\frac{1}{3}\right)^n\right)} =$

$= \sqrt[n]{n}\cdot 3\cdot\sqrt[n]{1+n^2\left(\frac{1}{3}\right)^n+\frac{1}{n}\left(-\frac{1}{3}\right)^n} \xrightarrow[n\to+\infty]{}$ $(\sqrt[n]{n}\to 1$; $q^n\to 0$ és $n^2q^n\to 0$,

ha $|q|<1$, ezért $x_n := 1+n^2\left(\frac{1}{3}\right)^n+\frac{1}{n}\left(-\frac{1}{3}\right)^n \to 1+0+0\cdot 0 = 1 > 0 \Rightarrow \sqrt[n]{x_n}\to 1)$

$\xrightarrow[n\to+\infty]{} 1\cdot 3\cdot 1 = \underline{\underline{3}}$