Racionalis helyettesitesek

1 tipus

$$\int R(e^x)\,\mathrm{d}x,$$

pl

1

$$\int \frac{e^{3x}}{e^x-2}\,\mathrm{d}x \qquad (x\in\mathbb{R})$$

Masodik helyettesitesi szabaly:

TFH $I, J \subset \mathbb{R}$ nyílt intervallumok,
$f : I \to \mathbb{R}, g : J \to I$ bijekció,
$g \in D(J)$,

$g'(x) \neq 0\ (\forall x \in J)$

és az $f \circ g \cdot g' : J \to \mathbb{R}$ függvényeknek van primitív függvénye.
Ekkor az f függvénynek is van primitív függvénye és

$$\int f(x)\,\mathrm{d}x \underset{x=g(t)}{=} \int f(g(t))\cdot g'(t)\,\mathrm{d}t_{|\ t=g^{-1}(x)} \qquad (x\in I)$$

masodik helyettesitei szabaly szerint
legyen

$$t := e^x$$

$$\int \frac{e^{3x}}{e^x-2}\,\mathrm{d}x \underset{\substack{t:=e^x>0 \\ x=\ln t=g(t) \\ g'(t)=\frac{1}{t}>0\Longrightarrow g\uparrow\Longrightarrow\exists g^{-1}=e^x}}{=} \int \frac{t^3}{t+2}\cdot\frac{1}{t}\,\mathrm{d}t\ _{|\ t=e^x=g^{-1}(x)}$$

$$= \int \frac{t^2}{t+2}\,\mathrm{d}t = \int \frac{t^2-4+4}{t+2}\,\mathrm{d}t = \int \frac{(t-2)(t+2)+4}{t+2}\,\mathrm{d}t = \int \left(t-2+\frac{4}{t+2}\right)\mathrm{d}t = \frac{t^2}{2}-2t+4\int\frac{(t+2)'}{t+2}\,\mathrm{d}t =$$
$$= \frac{t^2}{2}-2t+4\ln|t+2|+C = \frac{t^2}{2}-2t+4\ln(t+2)+C$$

vissza x-re

$$\int \frac{e^{3x}}{e^x+2}\,\mathrm{d}x = \underline{\frac{e^{2x}}{2}-2e^x+4\ln(e^x+2)+C}$$

Egyezmenyes rovid jeloles:

$$x = \ln t \underset{()'}{\Longrightarrow} 1\,\mathrm{d}x = \frac{1}{5}\,\mathrm{d}t$$

2

$$\int \frac{4}{e^{2x}-4}\,\mathrm{d}x$$

$$e^{2x} > 4 \Longrightarrow 2x > \ln 4 \Longleftrightarrow x > \frac{1}{2}\ln 4 = \ln 2$$

$$e^{2x} = t > 4$$

$$(x > \ln 2) \mid \Longrightarrow 2x = \ln t \Longrightarrow \frac{1}{2}\ln t = g(t)\ (t > 4) \Longrightarrow g'(t) = \frac{1}{2t} > 0\ (\forall t > 4) \Longrightarrow g \uparrow (4; +\infty) \Longrightarrow$$

$$\Longrightarrow \exists g^{-1}(x) = e^{2x}\ (x > \ln 2)\ \text{ es g bijektiv}$$

$$= \int \frac{4}{t-4}\cdot\frac{1}{2t}\,\mathrm{d}t = \int \frac{2}{t(t-4)}\,\mathrm{d}t = \frac{2}{4}\int \frac{t-(t-4)}{t(t-4)}\,\mathrm{d}t = \frac{1}{2}\int \frac{1}{t-4}\,\mathrm{d}t - \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln|t-4| - \frac{1}{2}\ln|t| + C \underset{t>4}{=} =$$

$$= \frac{1}{2}\ln(t-4) - \frac{1}{2}\ln t + C = \frac{1}{2}\ln\frac{t-4}{t} + C$$

$$\int \frac{4}{e^{2x}-4}\,\mathrm{d}x = \frac{1}{2}\ln\frac{e^{2x}-4}{e^{2x}} + C$$

2 tipus

$$\int R\left(x;\ \sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x;$$

ilyenkor

$$t := \sqrt[n]{\frac{ax+b}{cx+d}}$$

pl

1

$$\int \frac{1}{1+\sqrt{x}}\ (x>0) \underset{\substack{t=\sqrt{x}>0 \\ \Longrightarrow x=t^2=g(t)\ (t>0) \Longrightarrow g'(t) \Longrightarrow t>0\ \text{ ha } (t>0) \\ \Longrightarrow g\uparrow(0,+\infty) \Longrightarrow \exists g^{-1}(x)=\sqrt{x}\ (x>0)}}{=} \int \frac{1}{1+t}2t\,\mathrm{d}t$$

$$\Longrightarrow 2\int \frac{t}{t+1}\,\mathrm{d}t = 2\int \frac{t+1-1}{t+1}\,\mathrm{d}t = 2\int\left(1-\frac{1}{t+1}\right)\mathrm{d}t = 2\int 1\,\mathrm{d}t - 2\int \frac{1}{1+t}\,\mathrm{d}t = 2t - 2\ln \mathsf{abs}\ (t+1) + C =$$

$$\underset{t>0}{=} 2t - 2\ln(t+1) + C$$

$$\int \frac{1}{1+\sqrt{x}} = \underline{2\sqrt{x} - 2\ln\left(\sqrt{x}+1\right) + C}$$

2

$$\int x\sqrt{5x+3}\,\mathrm{d}x\ \left(x > -\frac{3}{5}\right)$$

$$\int x\sqrt{5x+3}\,\mathrm{d}x \underset{\substack{t:=\sqrt{5x+3}>0, \Longrightarrow x=\frac{t^2-3}{5}=g(t)\ (t>0) \\ g'(t)=\frac{2}{5}t>0 \Longrightarrow\ \text{ ha}(t>0) \Longrightarrow g\uparrow(0,+\infty) \Longrightarrow R_g=\left(-\frac{3}{5},+\infty\right); \\ g\ \text{ bijekcio } (0,+\infty)\ \text{ es } a\left(-\frac{3}{5};+\infty\right)\ \text{ kozott}}}{=} \int \frac{t^2-3}{5}\cdot t\cdot\frac{2}{5}t\,\mathrm{d}t_{|\ t=\sqrt{5x+3}}$$

az uj integral:

$$\frac{2}{25}\int (t^2-3)t^2\,\mathrm{d}t = \frac{2}{25}\int (t^4-3t^2)\,\mathrm{d}t = \frac{3}{25}\left(\frac{t^5}{5}-t^3\right)+C$$

$$\Longrightarrow \int x\sqrt{5x+3}\,\mathrm{d}x = \underline{\frac{2}{125}\left(\sqrt{5x+3}\right)^5 - \frac{3}{25}\left(\sqrt{5x+3}\right)^3 + C}$$

3

$$\int \frac{1}{x^2}\sqrt[3]{\frac{x+1}{x}}\,\mathrm{d}x\ (x>0) \underset{\substack{t:=\sqrt[3]{\frac{x+1}{x}}=\sqrt[3]{1+\frac{1}{x}}>\sqrt[3]{1+0}\text{ ha } x>0 \\ \Longrightarrow t^3=1+\frac{1}{x}\Longrightarrow x=\frac{1}{t^3-1}=g(t)\ (t>1)\Longrightarrow \\ \Longrightarrow g'(t)=-\frac{1}{\left(t^3-1\right)^2}\cdot(3t^2)=-\frac{3t^2}{\left(t^3-1\right)^2}<0\ (\forall t>0)\Longrightarrow \\ \Longrightarrow g\downarrow(1;+\infty),\text{ es } R_g=(0,+\infty)=D_f=I, \\ \exists g^{-1}(x)=\sqrt[3]{\frac{x+1}{x}}}}{=} = \int \frac{1}{\left(\frac{1}{t^3-1}\right)^2}\cdot t\cdot\frac{-3t^2}{\left(t^3-1\right)^2}\,\mathrm{d}t_{|_{t=\sqrt[3]{\frac{x+1}{x}}}}$$

$$= -3\int t^3\,\mathrm{d}t = -3\left[\frac{t^4}{4}+C\right]_{t=\sqrt[3]{\frac{x+1}{x}}} = \underline{-\frac{3}{4}\left(\sqrt[3]{\frac{x+1}{x}}\right)^4 + C}$$

megjegyzes

$$-\int\left(1+\frac{1}{x}\right)'\left(1+\frac{1}{x}\right)^{\frac{1}{3}}\mathrm{d}x = -\frac{\left(1+\frac{1}{x}\right)^{\frac{1}{3}+1}}{\frac{1}{3}+1}+C = \underline{-\frac{3}{4}\left(\sqrt[3]{\frac{x+1}{x}}\right)^4 + C}$$

4

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,\mathrm{d}x \underset{\substack{t:=\sqrt[6]{x}>\sqrt[6]{0}=0 \\ \Longrightarrow x=t^6=g(t)\Longrightarrow g'(t)=6t^5>0\ \text{ ha}(t>0)\Longrightarrow g\uparrow(0;+\infty)\ \text{ es }\ R_g=(0,+\infty)}}{=} \int \frac{1}{t^3+t^2}6t^5\,\mathrm{d}t_{|\ \sqrt[6]{x}} = 6\int \frac{t^3}{t+1}\,\mathrm{d}t_{|\ t=\sqrt[6]{x}}$$

uj integral:

$$\int \frac{t^3+1^3-1}{t+1}\,\mathrm{d}t = \int \frac{(t+1)\left(t^2-t+1\right)-1}{t+1}\,\mathrm{d}t = \int\left(t^2-t+1-\frac{1}{t+1}\right)\mathrm{d}t = \frac{t^3}{3}-\frac{t^2}{2}+t-\ln\left(\underbrace{t+1}_{+}\right)+C$$

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,\mathrm{d}x = \underline{\frac{\left(\sqrt[6]{x}\right)^3}{3}-\frac{\left(\sqrt[6]{x}\right)^3}{2}+\sqrt[6]{x}-\ln\left(\sqrt[6]{x}+1\right)+C}$$

hazi a, b, c
gyakorlo 2, 3b