bemelegites

$$\int_0^\pi e^{-x}\cos^2 x\,\mathrm{d}x = \int_0^\pi e^{-x}\frac{1+\cos 2x}{2}\,\mathrm{d}x = \frac{1}{2}\int_0^\pi e^{-x}\,\mathrm{d}x + \frac{1}{2}\int_0^\pi e^{-x}\cos x\,\mathrm{d}x =$$

$$= \frac{1}{2}[e^{-x}]_0^\pi + \frac{1}{2}\int_0^\pi e^{-x}\cos x\,\mathrm{d}x = \frac{1}{2}\left(1-\frac{1}{e^\pi}\right) + \frac{1}{2}\int_0^\pi e^{-x}\cos x\,\mathrm{d}x = \frac{e^\pi-1}{2e^\pi} + \frac{1}{2}\int_0^\pi e^{-x}\cos x\,\mathrm{d}x$$

$$\int_0^\pi e^{-x}\cos x\,\mathrm{d}x = \int_0^\pi e^{-x}\cos 2x\,\mathrm{d}x = -e^{-x}\cos 2x - \int(-e^{-x})(\cos 2x)'\,\mathrm{d}x =$$

$$= [-e^{-x}\cos 2x]_0^\pi - \int_0^\pi(-e^{-x})(\cos 2x)'\,\mathrm{d}x = 1 - \frac{\cos 2\pi}{e^\pi} - 2\int_0^\pi e^{-x}\sin 2x\,\mathrm{d}x =$$

$$= 1 - \frac{1}{e^\pi} - 2\left[[-e^{-x}\sin 2x]_0^\pi - \int_0^\pi(-e^{-x}\cos 2x\cdot 2\,\mathrm{d}x)\right](-e^{-x})' = \frac{e^\pi-1}{e^\pi} - 2\left[0 - 4\int_0^\pi e^{-x}\cos 2x\,\mathrm{d}x\right] \Longrightarrow$$

$$\Longrightarrow \int_0^\pi e^{-x}\cos x\,\mathrm{d}x = \frac{1-e^\pi}{7e^\pi} \Longrightarrow \text{Eredeti integral: } \frac{e^\pi-1}{2e^\pi} + \frac{1-e^\pi}{14e^\pi} = \underline{\frac{e^\pi-3}{7e^\pi}}$$

Alkalmazasok

1. Terulet

1. pelda

$$y = x-1,\ y^2 = 2x+6,\ \text{kozrefogott kozos terulet?}$$

1

$$y = \pm\sqrt{2x+6}$$

$$(x-1)^2 = 2x+6 \Longrightarrow x^2-2x+1-2x-6=0 \Longrightarrow x^2-4x-5=0 \Longrightarrow (x-5)(x+1)=0$$

2

$$T_1 = \int_{-3}^{-1}\sqrt{2x+6}\,\mathrm{d}x \qquad u=\sqrt{2x+6},\ x=\frac{u^2-6}{2},\ x'=u$$

$$\int_{-3}^{-1}\sqrt{2x+6}\,\mathrm{d}x = \int_0^2 uu\,\mathrm{d}u\left[\frac{u^3}{3}\right]_0^2 = \frac{8}{3}$$

$$T_3 = \int_{-1}^{5}\left[\sqrt{2x+6}-(x-1)\right]\mathrm{d}x \qquad u=\sqrt{2x+6},\ x=\frac{u^2-6}{2},\ x'=u$$

$$\int_2^4\left(u-\frac{u^2-6}{2}+1\right)u\,\mathrm{d}u = \int_2^4\left(u^2-\frac{1}{2}u^3+4u\right)\mathrm{d}u = \left[\frac{u^3}{3}-\frac{u^4}{8}+2u^2\right]_2^4 = \frac{4^3}{3}-\frac{4^3}{2}+2\cdot 4^2-\frac{8}{3}+2-8 =$$

$$= \frac{56}{3}+26-32 = \frac{56}{3}-6 = \underline{\frac{38}{3}}$$

Tehat:

$$T = 2\cdot\frac{8}{3}+\frac{38}{3} = \frac{54}{3} = \underline{\underline{18}}$$

2. pelda

milyen aranyu reszekre osztja a $y^2 = 2x$ az $x^2 + y^2 = 8$ teruletet?

$$x^2 + y^2 = 8 = \left(2\sqrt{2}\right)^2$$

$$T = \frac{T_1 + T_2}{T_{\text{felkor}} - (T_1 + T_2)}$$

metszespontok

$$x^2 + 2x = 8 \Longleftrightarrow x^2 + 2x - 8 = 0 \Longrightarrow x_1 = 2, x_2 = -4 \quad \text{(-4 nem felel meg)}$$

$$T_1 = \int_0^2 \sqrt{2x}\,\mathrm{d}x = \sqrt{2}\int_0^2 \sqrt{x}\,\mathrm{d}x = \sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \frac{2\sqrt{2}}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2\sqrt{2}}{3}\cdot 2\sqrt{2} = \frac{8}{3}$$

$$T_{\text{felkor}} = \frac{\pi\left(\sqrt{8}\right)^2}{2} = 4\pi$$

$$T_2 = T_{OAC} - T_{OAB} = \frac{T_{\text{kor}}}{8} - \frac{2\cdot 2}{2} = \frac{8\pi}{8} - 2 = \pi - 2$$

A keresett arany:

$$T = \frac{T_1 + T_2}{T_{\text{felkor}} - (T_1 + T_2)} = \frac{\frac{8}{3} + \pi - 2}{4\pi - \left(\frac{8}{3} + \pi - 2\right)} = \frac{\frac{2}{3} + \pi}{4\pi - \pi - \frac{2}{3}} = \frac{\frac{2}{3} + \pi}{3\pi - \frac{2}{3}} = \underline{\underline{\frac{3\pi + 2}{9\pi - 2}}}$$

2. forgastest terfogata/felszine

$$0 \le f \in R[a,b] \Longrightarrow \exists V = \pi\int_a^b f^2(x)\,\mathrm{d}x$$

1. pelda

$$f(x) = \sin^2 x \qquad x \in [0,\pi]$$

$$\exists V \underset{f\in C[0,\pi]\Longrightarrow f\in R[0,\pi]}{=} \pi\int_0^\pi f^2(x)\,\mathrm{d}x = \pi\int_0^\pi \sin^4 x\,\mathrm{d}x = \pi\int_0^\pi \left(\sin^2 x\right)^2\mathrm{d}x = \pi\int_0^\pi \left(\frac{1-\cos 2x}{2}\right)^2\mathrm{d}x =$$

$$= \frac{\pi}{4}\int_0^\pi \left(1 - 2\cos 2x + \cos^2 2x\right)\mathrm{d}x = \frac{\pi}{4}\int_0^\pi \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)\mathrm{d}x = \frac{\pi}{4}\left[\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{2}\right]_0^\pi =$$

$$= \frac{\pi}{4}\left(\frac{3\pi}{2} - \sin 2\pi + \frac{\sin 4\pi}{8} - 0\right) = \frac{\pi}{4}\cdot\frac{3\pi}{2} = \underline{\underline{\frac{3\pi^2}{8}}}$$

3. ivhossz

$$f \in C^1[a,b] \Longrightarrow \exists l = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,\mathrm{d}x$$

1. pelda

$$f(x) := \frac{2(x-1)^{\frac{3}{2}}}{3} \qquad 2 \leq x \leq 5$$

$$f \in C^1 \in [2,5] \Longrightarrow \exists l = \int_2^5 \sqrt{1 + \left[\left(\frac{2(x-1)^{\frac{3}{2}}}{3}\right)'\right]^2}\,\mathrm{d}x = \int_2^5 \sqrt{1 + \left(\frac{2}{3} \cdot \frac{3}{2} \cdot (x+1)^{\frac{1}{2}} \cdot 1\right)^2}\,\mathrm{d}x = \int_2^5 \sqrt{1 + x - 1}\,\mathrm{d}x =$$

$$= \int_2^5 \sqrt{x}\,\mathrm{d}x = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^5 = \frac{2}{3}\left(5\sqrt{5} - 2\sqrt{2}\right)$$

4. hatarertek

$$\lim_{n \to +\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right) =?$$

$$\lim_{n \to +\infty} \frac{1}{n} \cdot \left(\underbrace{\frac{1}{n + \frac{1}{n}} + \frac{1}{n + \frac{2}{n}} + \ldots + \frac{1}{1 + \frac{k}{n}} + \ldots + \frac{1}{1 + \frac{n}{n}}}_{\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}}\right)$$

$$\lim_{n \to +\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} 1 + \frac{k}{n} \underset{\text{ahol }f(x)=\frac{1}{1+x}}{=} \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \lim_{n \to +\infty} s(f, \tau_n)$$

$$f \in C[0,1] \Longrightarrow f \in R[0,1] \wedge \lim_{n \to +\infty} (s_n, f) = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x = [\ln(x+1)]_0^1 = \ln 2 - \ln 1 = \underline{\ln 2}$$