improprius integralok

def Legyen $[a,b) \subset \mathbb{R}$ intervallum es TFH $f : [a,b) \to \mathbb{R}$ fuggvenyre $f \in R[a,c] \quad (\forall a \leq c < b)$

Ekkor f impropriusan integralhato $[a,b)$ ha $\exists$ es veges $\lim_{t \to b-0} \int_a^t f(x)\,\mathrm{d}x$

jeloles

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t \to b-0} \int_a^t f(x)\,\mathrm{d}x$$

megjegyzes

- ugyanigy: $(a,b]$-n

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to a+0} \int_t^b f(x)\,\mathrm{d}x$$

- sot: (a,b)-re tetszoleges $c \in (a,b)$-vel:

$$\int_a^b f(x)\,\mathrm{d}x = \underbrace{\int_a^c f(x)\,\mathrm{d}x}_{(a,c]} + \underbrace{\int_c^b f(x)\,\mathrm{d}x}_{[c,b)}$$

- $(-\infty, b], [a, +\infty), (-\infty, +\infty)$-re mind mukodik ez

- nem korlatos fuggveny eseten bizonyos fokig tudjuk csak kezelni, pl: $\frac{1}{x}$

$$\int_0^1 \frac{1}{x}\,\mathrm{d}x \text{ mukodik, mert minden } [t,1] \text{ -on korlatos}$$

szamoljuk ki

$$\int_0^1 \frac{1}{x}\,\mathrm{d}x = \lim_{t \to 0+0} \int_t^1 \frac{1}{x}\,\mathrm{d}x = \lim_{t \to 0+0} [\ln(x)]_{x=t}^1 = \lim_{t \to 0+0} (\ln(1) - \ln(t)) = 0 - (-\infty) = \underline{\underline{+\infty}}$$

feladatok

1

$$\int_1^0 \frac{1}{x^\alpha}\,\mathrm{d}x, \quad \alpha \in \mathbb{R}$$

megjegyes Ha $f \in R[a,b]$ akkor impropriusan is integralhato az (a,b)-n es az improprius integrallal azonos (ez abbol kovetkezik hogy az integralfuggveny folytonos)

nezzuk $(0,1]$-en impropriusan (bizonyos α-ra $\in R[0,1]$, de nem erdekel)

$$\lim_{t\to 0+0} \int_t^1 \underbrace{\frac{1}{x^\alpha}}_{x^{-\alpha}}\,\mathrm{d}x \underset{\alpha\neq 1}{=} \lim_{t\to 0+0} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_{x=t}^{1} = \lim_{t\to 0+0}\left(\frac{1}{1-\alpha} - \frac{t^{1-\alpha}}{1-\alpha}\right) = \frac{1}{1-\alpha} - \frac{1}{1-\alpha}\cdot \lim_{t\to 0+0}\left(t^{1-\alpha}\right) =$$

$$\underset{\alpha\neq 1\Longrightarrow\beta\neq 0}{=} \begin{cases} \frac{1}{1-\alpha}, & \text{ha } \alpha < 1 \\ +\infty, & \text{ha } \alpha > 1 \quad \text{vagy} \quad \alpha = 1 \end{cases}$$

$$1-\alpha = \beta < 0 \Longleftrightarrow 1 < \alpha \Longrightarrow +\infty$$
$$1-\alpha = \beta > 0 \Longleftrightarrow 1 > \alpha \Longrightarrow 0$$

hazi

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x = \begin{cases} \frac{1}{\alpha-1}, & \text{ha } \alpha > 1 \\ +\infty, & \text{ha } \alpha \leq 1 \end{cases}$$

Megjegyzes

emlekezzunk:

- $$1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{n=1}^{+\infty} = +\infty$$

- $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty \quad \text{konvergens csak nem tudjuk mennyi az osszege}$$

altalanosan:

- $$\sum_{n=1} \frac{1}{n^\alpha} = \begin{cases} \text{divergens} \Longleftrightarrow \alpha \leq 1 \\ \text{konvergens} \Longleftrightarrow \alpha > 1 \end{cases}$$

$$\int_1^{+\infty} f(x)\,\mathrm{d}x \quad \text{vagy} \quad \sum_{n=1}^{+\infty} f(n)$$

$$\int_1^{+\infty} f(x)\,\mathrm{d}x < \sum_{n=1}^{+\infty} f(n) < f(1) + \int_1^{+\infty} f(x)\,\mathrm{d}x$$

$$\text{integral konvergens} \Longrightarrow \text{osszeg konvergens}$$

altalanosan

$$f : (1, +\infty) \to \mathbb{R} \;\; \text{fv monoton csokkeno es } f \geq 0$$

Ekkor

$$\int_1^{+\infty} f(x)\,\mathrm{d}x \text{ konvergens} \Longleftrightarrow \sum_{n=1}^{+\infty} f(n) \text{ konvergens}$$

2

$$\int_0^{\pi} \frac{1}{\alpha + \cos(x)}\,\mathrm{d}x, \;\; \text{ahol} \;\; a > 1 \;\; \text{valos parameter}$$

$a > 1$ miatt $a + \underbrace{\cos(x) > 0}_{\in[-1,1]} \Longrightarrow \frac{1}{a+\cos(x)} \in C[0,\pi] \Longrightarrow \in R[0,\pi]$

emlekezzunk:

- $$\int R(\sin x, \cos x), R \text{ racionalis tort tuggveny tipusu}$$

  $$\cos(x) = \frac{\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

- $$\sin(x) = \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})}{\sin^2(\frac{x}{2}) + \cos^2(\frac{x}{2})} = \frac{2\tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

$t = \operatorname{tg}(\frac{x}{2})$ helyettesitessel

$$\int R(\sin x, \cos x)\,\mathrm{d}x = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

mert

$$t = \tan\left(\frac{x}{2}\right) \Longleftrightarrow x = g(t) = 2\arctan(t) \Longrightarrow g'(t) = \frac{2}{1+t^2}$$

ha

$$x = 2\arctan(t) \in [0, \pi]$$

$$\arctan(t) \in \left[0, \frac{\pi}{2}\right]$$

$$t \in [0, +\infty)$$

igy a feladatnal

$$\int_0^\pi \frac{1}{a+\cos(x)}\,\mathrm{d}x \neq \int_0^{+\infty} \frac{1}{a+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

nincs ilyen szabaly

helyette hasznalhatnank imprortius integralt es akkor sem mukodne, szoval vesszuk Reimann integralkent es vesszuk es akkor mar mukodni fog a helyettesitesi szabaly

$$\int_0^\pi \frac{1}{a+\cos(x)}\,\mathrm{d}x = \lim_{u\to\pi-0}\int_0^u \frac{1}{a+\cos(x)}\,\mathrm{d}x = \int_0^u \frac{1}{a+\cos(x)}\,\mathrm{d}x = \int_0^{\tan\left(\frac{u}{2}\right)} \frac{1}{a+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

mert

$$x = 2\arctan(t) \in [0,u],$$
$$\arctan(t) \in \left[0, \frac{u}{2}\right],$$
$$t \in \left[\tan(0), \underline{\underline{\tan\left(\frac{u}{2}\right)}}\right]$$

$$= \int_0^{\tan\left(\frac{u}{2}\right)} \underbrace{\frac{2}{a(1+t^2)+(1-t^2)}}_{\frac{2}{(a+1)+(a-1)t^2}}\,\mathrm{d}t = \frac{2}{a+1}\int_0^{\tan\left(\frac{u}{2}\right)} \frac{1}{1+\left(\sqrt{\frac{a-1}{a+1}}\cdot t\right)^2}\,\mathrm{d}t = \frac{2}{a+1}\cdot\left[\frac{\arctan\left(\sqrt{\frac{a-1}{a+1}}\cdot t\right)}{\sqrt{\frac{a-1}{a+1}}}\right]_{t=0}^{\tan\left(\frac{u}{2}\right)} =$$

$$= \frac{2}{\sqrt{a^2-1}}\cdot\arctan\left(\sqrt{\frac{a-1}{a+1}}\cdot\tan\left(\frac{u}{2}\right)\right)$$

tehat

$$\int_0^\pi \frac{1}{a+\cos(x)}\,\mathrm{d}x = \lim_{u\to\pi-0}\left(\frac{2}{\sqrt{a^2-1}}\cdot\arctan\left(\sqrt{\frac{a-1}{a+1}}\cdot\tan\left(\frac{u}{2}\right)\right)\right)$$

$$\lim_{u\to\pi-0}\left(\frac{2}{\sqrt{a^2-1}}\cdot\underbrace{\arctan\left(\sqrt{\frac{a-1}{a+1}}\cdot\underbrace{\tan\left(\frac{u}{2}\right)}_{+\infty}\right)}_{\frac{\pi}{2}}\right) = \frac{\pi}{\sqrt{a^2-1}}$$

masik megoldas: (Newton-Leibniz tetel improprius integralokra tetel)

TFH $a, b \in \overline{\mathbb{R}}, a < b$ es
- $f \in R[u,v]$ minden $a < u < v < b$
- f-nek $\exists F$ primitiv fuggvenye (a,b)-n

Ekkor

$$\int_a^b f(x)\,\mathrm{d}x \;\; \text{improprius integral konvergens} \iff \exists F(a) := \lim_{\infty+0} F \land \exists F(b) := \lim_{b-0} F$$