van meg egy ilyen tipus amit nem beszeltunk meg

$$\int R(\sin x, \cos x)\,\mathrm{d}x;$$

cheat kodok:

$$\sin 2x = 2\sin x\cos x$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\operatorname{tg}\frac{x}{2}\cos^2\frac{x}{2} = \underline{\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}$$

$$1+\operatorname{tg}^2\alpha = 1+\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha} \Longrightarrow \cos^2\alpha = \frac{1}{1+\operatorname{tg}^2\alpha}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1-\operatorname{tg}^2\frac{x}{2}\right) = \underline{\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}$$

a)

$$\int \frac{1}{\sin x}\,\mathrm{d}x \qquad x\in(0,\pi)$$

$$u = \operatorname{tg}\frac{x}{2} \Longrightarrow x = 2\arctan u \Longrightarrow x' = \frac{2}{1+u^2}$$

$$\int \frac{1}{\frac{2u}{1+u^2}}\frac{2}{1+u^2}\,\mathrm{d}u = \int \frac{1}{u}\,\mathrm{d}u = \underline{\ln\left|\operatorname{tg}\frac{x}{2}\right| + C}$$

b)

$$\int \frac{1+\sin x}{1-\cos x}\,\mathrm{d}x \qquad x\in(0,\pi)$$

$$u = \operatorname{tg}\frac{x}{2} > 0 \Longrightarrow x = 2\arctan = g(u) > 0 \Longrightarrow g\uparrow \Longrightarrow g'(u) = \frac{2}{1+u^2} \Longrightarrow \exists g^{-1}(x) = \operatorname{tg}\frac{x}{2}$$

$$\int \frac{1+\frac{2u}{1+u^2}}{1-\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\,\mathrm{d}u = \int \frac{1+u^2+2u}{1+u^2-1+u^2}\frac{2}{1+u^2}\,\mathrm{d}u = \int \frac{u^2+2u+1}{u^2(u^2+1)}\,\mathrm{d}u = \int \frac{1}{u^2-1}\,\mathrm{d}u + \int \frac{2u+1}{u^2(u^2+1)}\,\mathrm{d}u =$$

$$= \arctan u + \int \frac{1}{u^2}\,\mathrm{d}u + 2\int \frac{1}{u^2(u^2+1)}\,\mathrm{d}u - \int \frac{1}{1+u^2}\,\mathrm{d}u =$$

$$= -\frac{1}{u} + 2\int \frac{1}{u}\,\mathrm{d}u - \int \frac{2u}{u^2+1}\,\mathrm{d}u = -\frac{1}{u} + 2\ln u - \ln(u^2+1) \Longrightarrow \underline{-\frac{1}{\operatorname{tg}\frac{x}{2}} + 2\ln\left(\operatorname{tg}\frac{x}{2}\right) - \ln\left(1+\operatorname{tg}^2\frac{x}{2}\right) + C}$$

megjegyzes

$$\frac{u^2+2u+1}{u^2(u^2+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{Cu+D}{u^2+1}$$

masik mod

$$\int \frac{1+\sin x}{1-\cos x}\,\mathrm{d}x = \int \frac{(1+\sin x)(1+\cos x)}{(1-\cos x)(1+\cos x)}\,\mathrm{d}x = \int \frac{1+\cos x+\sin x+\sin x\cos x}{1-\cos^2 x}\,\mathrm{d}x =$$

$$= \int \frac{1}{\sin^2 x}\,\mathrm{d}x + \int \frac{\cos x}{\sin x}\,\mathrm{d}x + \int \frac{1}{\sin x}\,\mathrm{d}x + \int \frac{\cos x}{\sin x}\,\mathrm{d}x =$$

$$= -\operatorname{ctg} x + \int (\sin x)'(\sin x)^{-2}\,\mathrm{d}x + \ln\left(\operatorname{tg}\frac{x}{2}\right) + \int \frac{(\sin x)'}{\sin x}\,\mathrm{d}x =$$

$$= -\operatorname{ctg} x + \frac{(\sin x)^{-1}}{-1} + \ln\left(\operatorname{tg}\frac{x}{2}\right) + \ln|\sin x| + C = \underline{-\frac{\cos x + 1}{\sin x} + \ln\left(\operatorname{tg}\frac{x}{2}\right) + \ln(\sin x) + C}$$

nem nez ki ugyanyugy a ket megoldas de szorgalmikent bemutathato hogy ekvivalensek

harmadik mod

$$\int \frac{1+\sin x}{1-\cos x}\,\mathrm{d}x = (\text{felszogre teres}) = \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}\,\mathrm{d}x =$$

$$= \int \frac{1}{2\sin^2\frac{x}{2}}\,\mathrm{d}x + \int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x = \underline{-\operatorname{ctg}\frac{x}{2} + 2\ln\left(\sin\frac{x}{2}\right) + C}$$

hatarozott integral

a)

$$\int_{10}^{66} \frac{1}{x - \sqrt[3]{x-2} - 2} \, \mathrm{d}x$$

$$x \in [10, 66] \implies f \in R[10, 66]$$

$$u = \sqrt[3]{x-2} \implies x = u^3 + 2 = g(u) \implies g'(u) = 3u^2 > 0$$

$$\text{Ha} \quad 10 \leq x \leq 66 \implies 8 \leq x - 2 \leq 64 \implies \sqrt[3]{8} \leq \sqrt[3]{x-2} \leq \sqrt[3]{64}$$

$$\text{tehat } 2 \leq u \leq 4$$

$$\int_2^4 \frac{1}{u^3 + 2 - u - 2} 3u^2 \, \mathrm{d}u = \int_2^4 \frac{3u^2}{u(u^2-1)} \, \mathrm{d}u = 3 \int_2^4 \frac{3u^2}{u(u-1)(u+1)} \, \mathrm{d}u = 3 \int_2^4 \frac{u}{(u-1)(u+1)} \, \mathrm{d}u =$$

$$= \frac{3}{2} \int_2^4 \frac{(u-1)+(u+1)}{(u-1)(u+1)} \, \mathrm{d}u = \frac{3}{2} \int_2^4 \frac{1}{u+1} + \frac{1}{u-1} \, \mathrm{d}u = \frac{3}{2} [\ln|u+1| + \ln|u-1|]_2^4 =$$

$$= \frac{3}{2} (\ln|5| + \ln|3| - \ln|3| - \ln|1|) = \underline{\frac{3}{2} \ln 5}$$

b)

$$\int_1^e \frac{\sin(\ln x)}{x} \, \mathrm{d}x$$

$$f \in C[1, e] \implies f \in R[1, e]$$

$$\int_1^e \frac{\sin(\ln x)}{x} \, \mathrm{d}x = \int_1^e (\ln x)' \cdot \sin(\ln x) \, \mathrm{d}x = [-\cos(\ln x)]_1^e = -\cos(\ln e) + \cos(\ln 1) = \underline{1 - \cos 1}$$

vagy ha nem veszed eszre:

$$\int_1^e \frac{\sin(\ln x)}{x} \, \mathrm{d}x$$

$$u = \ln x$$

$$\text{ha } x = 1 \implies u = \ln 1 = 0$$

$$\text{ha } x = e \implies u = \ln e = 1$$

$$x = e^u \implies x' = e^u$$

$$\int_1^e \frac{\sin(u)}{e^u} \cdot e^u \, \mathrm{d}u = \int_0^1 \sin u \, \mathrm{d}u = [-\cos u]_0^1 = -\cos 1 + \cos 0 = \underline{1 - \cos 1}$$

c)

$$\int_{-2}^{\sqrt{3}-2} \frac{\mathrm{d}x}{x^2 + 4x + 5}$$

$$f \in C\left[-2, \sqrt{3} - 2\right] \implies f \in R\left[-2, \sqrt{3} - 2\right]$$

$$\int_{-2}^{\sqrt{3}-2} \frac{\mathrm{d}x}{x^2 + 4x + 5} = \int_{-2}^{\sqrt{3}-2} \frac{1}{(x+2)^2 + 1} \, \mathrm{d}x = \left[\frac{\arctan(x+2)}{1}\right]_{-2}^{\sqrt{3}-2} = \arctan\sqrt{3} - \arctan 0 = \frac{\pi}{3}$$

d)

$$\int_3^4 \frac{1}{x^2-3x+2}\,\mathrm{d}x$$

$$f \in C[3,4] \Longrightarrow f \in R[3,4]$$

$$\int_3^4 \frac{1}{x^2-3x+2}\,\mathrm{d}x = \int_3^4 \frac{1}{(x-2)(x-1)}\,\mathrm{d}x = \int_3^4 \frac{(x-1)-(x-2)}{(x-2)(x-1)}\,\mathrm{d}x = \int_3^4 \frac{1}{x-2} - \frac{1}{x-1}\,\mathrm{d}x = [\ln|x-2| - \ln|x-1|]_3^4 =$$

$$= \left[\ln\frac{|x-2|}{|x-1|}\right]_3^4 = \ln\frac{2}{3} - \ln\frac{1}{2} = \ln\left(\frac{2}{3}\cdot\frac{2}{1}\right) = \underline{\ln\frac{4}{3}}$$