Applied Probability for Computer Science

Exercise list 1: Poisson Processes and Exponential Waiting Times – **Solutions**

Question 1

Superposition. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities λ and μ.

(a) Show that the arrivals of flying insects form a Poisson process with intensity $\lambda + \mu$.

Solution: Let N_F and N_W denote the Poisson processes of incoming flies and wasps, respectively, and let $N(t) = N_F(t) + N_W(t)$. Clearly, $N(0) = 0$ and N is non-decreasing, properties inherited directly from N_F and N_W. Arrivals of flies during $[0, s]$ are independent of arrivals during $(s, t]$ for any $0 \leq s < t$, since N_F is a Poisson process. The same is true for wasps. As a consequence, the aggregated arrivals during $[0, s]$ are independent of the aggregated arrivals during $(s, t]$. Thus, N satisfies conditions (a) and (c) of Definition 1.

Let

$$A = \{\text{one fly arrives during } (t, t+h]\}, \quad B = \{\text{one wasp arrives during } (t, t+h]\}.$$

Then, by independence of flies and wasp arrivals, $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$ and

$$\begin{aligned}\mathbb{P}[N(t+h) = n+1 | N(t) = n] &= \mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] \\ &= \lambda h + \mu h - (\lambda h)(\mu h) + o(h) = (\lambda + \mu)h + o(h),\end{aligned}$$

since $h^2 = o(h)$ and $o(h) + o(h) = o(h)$. Finally, let

$$\begin{aligned} C &= \{\text{two or more flies arrive during } (t, t+h]\}, \\ D &= \{\text{two or more wasps arrive during } (t, t+h]\}. \end{aligned}$$

Then,

$$\begin{aligned}\mathbb{P}[N(t+h) > n+1 | N(t) = n] &\leq \mathbb{P}[A \cap B] + \mathbb{P}[C \cup D] \\ &\leq (\lambda h)(\mu h) + o(h) = o(h),\end{aligned}$$

thus condition (b) of Definition 1 is also satisfied.

(b) What is the probability that the first insect landing on the plate is a wasp?

Solution: Let T_F and T_W denote the times until the first arrival of a fly and a wasp, respectively. These two times are clearly independent with $T_F \sim$Exp(λ), $T_W \sim$Exp(μ), and the required probability is

$$\mathbb{P}[T_W < T_F] = \mathbb{P}[\min\{T_W, T_F\} = T_W] = \frac{\mu}{\lambda + \mu},$$

by the properties of the exponential distribution (seen in class).

Question 2

Thinning. Insects land in the soup in the manner of a Poisson process with intensity λ, and each such insect is green with probability p, independently of the colours of all other insects. Show that the arrivals of green insects form a Poisson process with intensity λp.

Solution: Let N_I be the process of incoming insects and let N_G be the process of incoming green insects only. N_G is clearly increasing with $N_G(0) = 0$. The independence of green insect arrivals in $[0, s]$ from those in $(s, t]$ for $s < t$ is inherited from the independence for overall insect arrivals. Thus, conditions (a) and (c) of Definition 1 are satisfied. Finally,

$$\begin{aligned}
\mathbb{P}[N_G(t+h) = n+1 | N_G(t) = n] &= p\mathbb{P}[N_I(t+h) = n+1 | N_I(t) = n] + o(h) = p\lambda h + o(h) \\
\mathbb{P}[N_G(t+h) > n+1 | N_G(t) = n] &\leq \mathbb{P}[N_I(t+h) > n+1 | N_I(t) = n] = o(h).
\end{aligned}$$

Question 3

Consider a Poisson process N with intensity λ.

(a) What is the probability that there are no arrivals in the interval $(0, t]$, for $t > 0$? And if $t = 2\lambda$?

Solution: Recall that $N(t) \sim$Po(λt), thus

$$\mathbb{P}[N(t) = 0] = e^{-\lambda t} \quad \Rightarrow \quad \mathbb{P}[N(2\lambda) = 0] = e^{-2\lambda^2}$$

(b) What is the probability that there are no arrivals in the interval $(s, s+t]$, for $s, t > 0$? And if $s = 6\lambda$, $t = 2\lambda$?

Solution: Recall that the Poisson process has stationary increments, thus

$$\begin{aligned}
\mathbb{P}[N(s+t) - N(s) = 0] = \mathbb{P}[N(t) = 0] =& e^{-\lambda t} \\
\Rightarrow \quad \mathbb{P}[N(8\lambda) - N(6\lambda) = 0] =& e^{-2\lambda^2}
\end{aligned}$$

(c) What is the expected number of arrivals in the interval $(s, s+t]$, for $s, t > 0$? And if $s = t = 2\lambda$?

Solution:

$$\begin{aligned}
\mathbb{E}[N(s+t) - N(s)] = \mathbb{E}[N(t)] =& \lambda t \\
\Rightarrow \quad \mathbb{E}[N(4\lambda) - N(2\lambda)] =& 2\lambda^2
\end{aligned}$$

(d) What is the probability that the number of arrivals in the interval $(s, s+t]$, for $s, t > 0$ is exactly equal to the expected number of arrivals in that same interval? And if $s = t = 2\lambda$?

Solution: $N(s+t) - N(s)$ is a discrete random variable. Therefore, if $\mathbb{E}[N(s+t) - N(s)] = \lambda t \notin \mathbb{N}$, the required probability is zero, while for $\lambda t \in \mathbb{N}$, we have

$$\mathbb{P}\big[N(s+t) - N(s) = \mathbb{E}[N(s+t) - N(s)]\big] = \mathbb{P}[N(s+t) - N(s) = \lambda t] = \frac{(\lambda t)^{\lambda t}}{(\lambda t)!} e^{-\lambda t}.$$

Thus,

$$\mathbb{P}\big[N(4\lambda) - N(2\lambda) = \mathbb{E}[N(4\lambda) - N(2\lambda)]\big] = \mathbb{P}[N(4\lambda) - N(2\lambda) = 2\lambda^2] = \frac{(2\lambda^2)^{2\lambda^2}}{(2\lambda^2)!}e^{-2\lambda^2}$$

if $\lambda \in \mathbb{N}$ and zero otherwise.

(e) What is the probability that the number of arrivals in the interval $(s, s+t]$, for $s, t > 0$ is exactly twice the expected number of arrivals in that same interval? And if $s = t = 2\lambda$?

Solution: Once again, the required probability is zero if $\lambda t \notin \mathbb{N}$, otherwise we have

$$\mathbb{P}\big[N(s+t) - N(s) = 2\mathbb{E}[N(s+t) - N(s)]\big] = \mathbb{P}[N(s+t) - N(s) = 2\lambda t] = \frac{(\lambda t)^{2\lambda t}}{(2\lambda t)!}e^{-\lambda t}.$$

And for $\lambda \in \mathbb{N}$

$$\mathbb{P}\big[N(4\lambda) - N(2\lambda) = 2\mathbb{E}[N(4\lambda) - N(2\lambda)]\big] = \mathbb{P}[N(4\lambda) - N(2\lambda) = 4\lambda^2] = \frac{(2\lambda^2)^{4\lambda^2}}{(4\lambda^2)!}e^{-2\lambda^2}$$

(f) What is the probability of having exactly 2λ arrivals in the interval $(2\lambda, 4\lambda]$ given that 3λ arrivals occured in the interval $(0, 2\lambda]$?

Solution: Note that, given that 3λ arrivals occured in the interval $(0, 2\lambda]$, we know that $\lambda \in \mathbb{N}$. Therefore, by the independence of the increments, we have

$$\mathbb{P}[N(4\lambda) - N(2\lambda) = 2\lambda | N(2\lambda) = 2\lambda] = \mathbb{P}[N(4\lambda) - N(2\lambda) = 2\lambda] = \frac{(2\lambda^2)^{2\lambda}}{(2\lambda)!}e^{-2\lambda^2}.$$

(g) If there were exactly $2n$ arrivals in the interval $(0, T]$, what is the probability that exactly n of them occurred in the interval $(s, t]$, for $0 < s < t < T$?

Solution: We can write

$$N(T) = [N(T) - N(t)] + [N(t) - N(s)] + N(s) = [N(t) - N(s)] + [N(T) - N(t) + N(s)],$$

where $[N(T) - N(t)] \sim$Po$(\lambda(T - t))$, $[N(t) - N(s)] \sim$Po$(\lambda(t - s))$ and $[N(s)] \sim$Po(λs) are all independent. Furthermore, by the additivity of the Poisson distribution, $[N(T) - N(t) + N(s)] \sim$Po$(\lambda(T - t + s))$ is independent of $[N(t) - N(s)]$. Therefore, using the Poisson-Binomial relation property (a particular case of the Poisson-Multinomial relation for the sum of two independent Poisson random variables), we have

$$N(t) - N(s) | N(T) = 2n \sim \text{Bin}(2n, p),$$

where

$$p = \frac{\mathbb{E}[N(t) - N(s)]}{\mathbb{E}[N(t) - N(s)] + \mathbb{E}[N(T) - N(t) + N(s)]} = \frac{t - s}{T}.$$

Therefore,

$$\mathbb{P}[N(t) - N(s) = n | N(T) = 2n] = \binom{2n}{n}\left(\frac{t - s}{T}\right)^n \left(1 - \frac{t - s}{T}\right)^n.$$

(h) What is the probability for the previous point if $n = 2$, $s = 2\lambda$, $t = 3\lambda$ and $T = 5\lambda$?

Solution: In this case,

$$N(3\lambda) - N(2\lambda) | N(5\lambda) = 4 \sim \text{Bin}(4, 1/5),$$

Therefore,

$$\mathbb{P}[N(3\lambda) - N(2\lambda) = 2 | N(5\lambda) = 4] = \binom{4}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2 = \texttt{dbinom(2,4,1/5)} = 0.1536$$

Question 4

Consider a two-server system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server i are exponential random variables with rates μ_1 and μ_2 respectively. When you arrive, you find server 1 free and two customers at server 2 (customer A in service and customer B waiting in line).

(a) Find P_A, the probability that A is still in service when you move over to server 2.

Solution: Let S_1 be the time you spend at server 1 and T_A the time customer A spends at server 2. Then, $S_1 \sim$Exp(μ_1) amd $T_A \sim$Exp(μ_2). Assuming, reasonably, that the times are independent, by the properties of the exponential distribution (seen in class),

$$P_A = \mathbb{P}[S_1 < T_A] = \frac{\mu_1}{\mu_1 + \mu_2}.$$

(b) Find P_B, the probability that B is still in the system when you move over to server 2.

Solution: The event "B is in the system" can be decomposed as the union of two disjoint events: "B is still waiting" and "B is in server 2". Let $T_B \sim$Exp(μ_2) be the time customer B spends in server 2 and S_1, T_A as before. Once again, we assume independence of these times. Then

$$\begin{aligned} P_B &= \mathbb{P}[S_1 < T_A] + \mathbb{P}[T_A < S_1 < T_A + T_B] \\ &= \mathbb{P}[S_1 < T_A] + \mathbb{P}[S_1 > T_A]\mathbb{P}[S_1 < T_A + T_B | S_1 > T_A]. \end{aligned}$$

By the lack of memory property,

$$\begin{aligned} \mathbb{P}[S_1 < T_A + T_B | S_1 > T_A] &= 1 - \mathbb{P}[S_1 > T_A + T_B | S_1 > T_A] \\ &= 1 - \mathbb{P}[S_1 > T_B] = \mathbb{P}[S_1 < T_B]. \end{aligned}$$

Therefore,

$$\begin{aligned} P_B &= \mathbb{P}[S_1 < T_A] + \mathbb{P}[T_A < S_1]\mathbb{P}[S_1 < T_B] \\ &= \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2}\frac{\mu_1}{\mu_1 + \mu_2} = \frac{\mu_1}{\mu_1 + \mu_2}\left(1 + \frac{\mu_2}{\mu_1 + \mu_2}\right) \end{aligned}$$

(c) Find $\mathbb{E}[T]$, where T is the time that you spend in the system.

Hint: Write

$$T = S_1 + S_2 + W_A + W_B$$

where S_i is your service time at server i, W_A is the amount of time you wait in queue while A is being served, and W_B is the amount of time you wait in queue while B is being served.

Solution: Clearly, $S_1 \sim$Exp(μ_1) and $S_2 \sim$Exp(μ_2). And, by the lack of memory property, both W_A and W_B are exponential random variables with parameter μ_2. Therefore,

$$\begin{aligned}\mathbb{E}[T] &= \mathbb{E}[S_1 + S_2 + W_A + W_B] = \mathbb{E}[S_1] + \mathbb{E}[S_2] + \mathbb{E}[W_A] + \mathbb{E}[W_B] \\ &= \frac{1}{\mu_1} + \frac{3}{\mu_2}\end{aligned}$$

Question 5

Cars cross a certain point in the highway in accordance with a Poisson process with rate $\lambda = 3$ per minute. If an animal blindly runs across the highway, then what is the probability that it will be uninjured if the amount of time that it takes to cross the road is s seconds? (Assume that if the animal is on the highway when a car passes by, then it will be injured.) Do it for:

(a) $s = 2$

Solution: Let $N(t)$ be the number of cars passing in a time interval of t seconds. Then $N(t) \sim$Po$(3t/60)$.

Let $t = 0$ be the time (in seconds) at which the animal starts crossing the highway. If it takes s seconds to cross, it will be uninjured if and only if no cars pass in the interval $(0, s]$. Therefore, the probability that it will be uninjured is

$$\mathbb{P}[N(s) = 0] = e^{-3t/60} = e^{-t/20}$$

Therefore,

$$\mathbb{P}[N(2) = 0] = e^{-2/20} = e^{-1/10} = 0.9048$$

(b) $s = 5$

Solution:

$$\mathbb{P}[N(5) = 0] = e^{-5/20} = e^{-1/4} = 0.7788$$

(c) $s = 10$

Solution:

$$\mathbb{P}[N(5) = 0] = e^{-10/20} = e^{-1/2} = 0.6065$$

(d) $s = 20$

Solution:

$$\mathbb{P}[N(5) = 0] = e^{-20/20} = e^{-1} = 0.3679$$

Question 6

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the n-th event. Find

(a) $\mathbb{E}[S_4]$

Solution: $S_n \sim$Gamma(n, λ), therefore $\mathbb{E}[S_n] = n/\lambda$. In particular $\mathbb{E}[S_4] = 4/\lambda$

(b) $\mathbb{E}[S_4 | N(1) = 2]$

Solution: Recall that $S_4 = X_1 + X_2 + X_3 + X_4 = S_2 + X_3 + X_4$, where $X_i \overset{iid}{\sim}$Exp(λ), $i = 1, \dots, 4$ are the first interarrival times. If $N(1) = 2$, that means the first two events have occured by time $t = 1$, so only two more events must happen

$$\mathbb{E}[S_4|N(1) = 2] = \mathbb{E}[1 + X_3 + X_4] = 1 + \mathbb{E}[X_3 + X_4] = 1 + \frac{2}{\lambda},$$

since $X_3 + X_4 \sim$Gamma$(2, \lambda)$.

(c) $\mathbb{E}[N(4) - N(2)|N(1) = 3]$

Solution: The intervals $[4, 2]$ and $[0, 1]$ are disjoint. Therefore, $N(4) - N(2) \sim$Po(2λ) is independent of $N(1)$ and $\mathbb{E}[N(4) - N(2)|N(1) = 3] = \mathbb{E}[N(4) - N(2)] = 2\lambda$

Question 7

Events occur according to a Poisson process with rate λ. Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time T , where $T > 1/\lambda$. That is, if an event occurs at time t, $0 \leq t \leq T$, and we decide to stop, then we win if there are no additional events by time T, and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time T , then we lose. Also, if no events occur by time T, then we lose. Consider the strategy that stops at the first event to occur after some fixed time s, $0 \leq s \leq T$.

(a) Using this strategy, what is the probability of winning?

Solution:

$$\mathbb{P}[\text{Winning}] = \mathbb{P}[N(T) - N(s) = 1] = \lambda(T - s)e^{-\lambda(T-s)}$$

(b) What value of s maximizes the probability of winning?

Solution: Let $g(s) = \lambda(T - s)e^{-\lambda(T-s)}$. Then

$$g'(s) = \frac{dg(s)}{ds} = \lambda e^{-\lambda(T-s)}[\lambda(T - s) - 1] = 0 \quad \Leftrightarrow \quad s = T - \frac{1}{\lambda}$$

$$g''(s) = \frac{d^2g(s)}{ds^2} = \lambda^2 e^{-\lambda(T-s)}[\lambda(T - s) - 2] \quad \Rightarrow \quad g''\left(T - \frac{1}{\lambda}\right) = -\lambda^2 e^{-1} < 0.$$

Therefore, the probability of winning is maximized when $s = T - \frac{1}{\lambda}$.

(c) Show that one's probability of winning when using the preceding strategy with the value of s specified in part (b) is $1/e$.

Solution: If $s = T - \frac{1}{\lambda}$, then $T - s = 1/\lambda$

$$\mathbb{P}[\text{Winning}] = g\left(T - \frac{1}{\lambda}\right) = \frac{\lambda}{\lambda} e^{-\lambda/\lambda} = \frac{1}{e}$$

Question 8

Consider a collection $U_1, \ldots, U_n$ of independent random variables, uniformly distributed on $(0, T)$ and define $Y_1 < Y_2 < \cdots < Y_n$ obtained simply by ordering $U_1, \ldots, U_n$ in ascending order In other words, Y_1 is the smallest item, Y_2 is the second smallest and so on. Since the random variables are continuous, the probability of a tie is equal to zero. The joint density of $(Y_1, \ldots, Y_n)$ is

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = n!; \quad y_1 < y_2 < \ldots < y_n.$$

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ and arrival times $T_1, T_2, \ldots$. The conditional joint density of $(T_1, \ldots, T_n)$ given $N(T) = n$ is

$$f_{X_1,\ldots,X_n|N(T)}(t_1, \ldots, t_n) = n!; \quad t_1 < t_2 < \ldots < t_n.$$

Write an R algorithm that uses this information to simulate and plot a path of the Poisson Process on the interval $(0, T)$, for $T = 10$ and $\lambda = 0.8$

Solution: First, we simulate the number of jumps on the interval $(0, T)$. Then, conditional on the number of jumps, the arrival times can be simulated as the ordered values of i.i.d uniform random variables (since the distribution is the same).

```
# Initialize parameter and random number generator
set.seed(876)
T <- 10
lambda <- 0.8
# Simulate number of jumps
n <- rpois(1,lambda*T)
# Simulate arrival times
t <- sort(runif(n,0,T))
# Plot,
plot(c(0,t,T), c(0:n,n), type="s")
```