Michele_Lotto_Exercises

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Michele Lotto Exercises

4) 3.21
5) 5.22

List 4 - Exercise 3.21

If A flips $n+1$ and B flips n fair coins, show that the probability that A gets more heads than B is $\frac{1}{2}$.

Hint: Condition on which player has more heads after each has flipped n coins. (There are three possibilities.)

Solution

List 8 - Exercise 5.22

Let U be a uniform $(0,1)$ random variable, and let $a < b$ be constants.

(a) Show that if $b > 0$, then bU is uniformly distributed on $(0,b)$, and if $b < 0$, then bU is uniformly distributed on $(b,0)$.

(b) Show that $a+U$ is uniformly distributed on $(a,1+a)$.

(c) What function of U is uniformly distributed on (a,b)?

(d) Show that $min(U,1-U)$ is a uniform $(0,1/2)$ random variable.

(e) Show that $max(U,1-U)$ is a uniform $(1/2,1)$ random variable.

Solution

Let $U \sim U(0,1)$ with PDF:

$$f_U(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and CDF:

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

a.1) We have to prove that if $b > 0$ then $Y = bU \sim U(0,b)$

First we calculate the interval extremes:

- if $U = 0$ then $Y = 0$
- if $U = 1$ then $Y = b$

We calculate the CDF:

$F_Y(y) = P(Y \leq y) = P(bU \leq y) = P(U \leq \frac{y}{b}) = F_U(\frac{y}{b}) = \frac{y}{b}$

Given that $\frac{\partial F}{\partial y} = \frac{1}{b}$ we derive the PDF:

$$f_Y(y) = \begin{cases} \frac{1}{b}, & y \in [0, b] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(0, b)$ by definition.

a.2) We have to prove that if $b < 0$ then $Y = bU \sim U(b, 0)$

First we calculate the interval extremes:

- if $U = 0$ then $Y = 0$
- if $U = 1$ then $Y = b$

We calculate the CDF:

$F_Y(y) = P(Y \leq y) = P(bU \leq y) = P(U \geq \frac{y}{b}) = 1 - F_U(\frac{y}{b}) = 1 - \frac{y}{b}$

Given that $\frac{\partial F}{\partial y} = \frac{-1}{b}$ we derive the PDF:

$$f_Y(y) = \begin{cases} \frac{-1}{b}, & y \in [b, 0] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(b, 0)$ by definition.

b) We have to prove that $Y = a + U \sim U(a, 1 + a)$

First we calculate the interval extremes:

- if $U = 0$ then $Y = a$
- if $U = 1$ then $Y = 1 + a$

We calculate the CDF:

$F_Y(y) = P(Y \leq y) = P(a + U \leq y) = P(U \leq y - a) = F_U(y - a) = y - a$

Given that $\frac{\partial F}{\partial y} = 1$ we derive the PDF:

$$f_Y(y) = \begin{cases} 1, & y \in [a, 1 + a] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(a, 1 + a)$ by definition.

d) We have to prove that $Y = min(U, 1 - U) \sim U(0, 1/2)$

We distinguish two cases:

1) if $U \geq \frac{1}{2}$ then the min is $Y = 1 - U$
2) if $U \leq \frac{1}{2}$ then the min is $Y = U$

1)

First we calculate the interval extremes:

- if $U = \frac{1}{2}$ then $Y = \frac{1}{2}$
- if $U = 1$ then $Y = 0$

We calculate the CDF:

$F_Y(y) = P(Y \leq y) = P(1-U \leq y) = P(-U \leq y-1) = P(U \geq -y+1) = 1-P(U \leq 1-y) = 1-F_U(1-y) = 1-1+y = y$

Given that $\frac{\partial F}{\partial y} = 1$ we derive the PDF:

$$f_Y(y) = \begin{cases} 1, & y \in [0, \frac{1}{2}] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(0, \frac{1}{2})$ by definition.

2)

First we calculate the interval extremes:

- if $U = 0$ then $Y = 0$
- if $U = \frac{1}{2}$ then $Y = \frac{1}{2}$

We calculate the CDF:

$F_Y(y) = P(Y \leq y) = P(U \leq y) = F_U(y) = y$

Given that $\frac{\partial F}{\partial y} = 1$ we derive the PDF:

$$f_Y(y) = \begin{cases} 1, & y \in [0, \frac{1}{2}] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(0, \frac{1}{2})$ by definition.

c) We have to find which function $f(x)$ such that $f(U) \sim U(a,b)$

Let $Y \sim U(a,b)$ with CDF:

$$f_Y(y) = \begin{cases} 0, & y < a \\ \frac{y-a}{b-a}, & y \in [a,b] \\ 1, & y > b \end{cases}$$

Given that $p = \frac{y-a}{b-a}$ is a probability, we know that p takes values between 0 and 1. p can be seen as a Standard Uniform $(p = U)$. distribution.

$$U = \frac{y-a}{b-a} \Rightarrow U \times (b-a) = y - a \Rightarrow U \times (b-a) + a = y$$

The function is that transform a Standard Uniform in a $U(a,b)$ is indeed $f(U) = U \times (b-a) + a$

e) We have to prove that $Y = max(U, 1-U) \sim U(\frac{1}{2}, 1)$

We distinguish two cases:

1) if $U \geq \frac{1}{2}$ then the max is $Y = U$

2) if $U \le \frac{1}{2}$ then the max is $Y = 1 - U$

1)

First we calculate the interval extremes:

- if $U = 1$ then $Y = 1$
- if $U = \frac{1}{2}$ then $Y = \frac{1}{2}$

We calculate the CDF:

$F_Y(y) = P(Y \le y) = P(U \le y) = F_U(y) = y$

Given that $\frac{\partial F}{\partial y} = 1$ we derive the PDF:

$$f_Y(y) = \begin{cases} 1, & y \in [\frac{1}{2}, 1] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(\frac{1}{2}, 1)$ by definition.

2)

First we calculate the interval extremes:

- if $U = 0$ then $Y = 1$
- if $U = \frac{1}{2}$ then $Y = \frac{1}{2}$

We calculate the CDF:

$F_Y(y) = P(Y \le y) = P(1-U \le y) = P(-U \le y-1) = P(U \ge -y+1) = 1-P(U \le 1-y) = 1-F_U(1-y) = 1 - 1 + y = y$

Given that $\frac{\partial F}{\partial y} = 1$ we derive the PDF:

$$f_Y(y) = \begin{cases} 1, & y \in [\frac{1}{2}, 1] \\ 0, & \text{otherwise} \end{cases}$$

which in turn is a $U(\frac{1}{2}, 1)$ by definition.