Reccurence Relations - Hermite Polynomials

User
Hello, I am going to give you some information that will inform you on how to do a problem. This Question will be about Hermite Polynomials and Reccurance Relations

ChatGPT
Certainly! Please provide the information and details related to the problem about Hermite Polynomials and Recurrence Relations. I'll do my best to assist you.

User
A reccurence relation for a Hermite Polynomial is given by:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}$$

Where $H_0(x) = 1$ and $H_{-1}(x) = 0$

Now consider a generation function that is defined as follows:

$$G(x,t) = \sum_{n=0}^{\text{inf}} \frac{t^n}{n!} H_n(x)$$

ChatGPT
Certainly! The given recurrence relation for Hermite Polynomials is:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

with initial conditions $H_0(x) = 1$ and $H_{-1}(x) = 0$.

Now, let's consider the generating function:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

The goal is to derive a partial differential equation (PDE) for G using the given recurrence relation for Hermite Polynomials.

To do this, we can manipulate the generating function $G(x,t)$ using the recurrence relation. Let's differentiate G with respect to t and see what happens:

$$\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} \frac{t^n}{(n-1)!} H_n(x)$$

Now, let's multiply the recurrence relation by $\frac{t^n}{n!}$ and sum over n:

$$\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_{n+1}(x) = 2x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) - 2 \sum_{n=0}^{\infty} \frac{nt^n}{n!} H_{n-1}(x)$$

This simplifies to:

$$\frac{\partial G}{\partial t} = 2xG - 2\sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} H_{n-1}(x)$$

Now, let's make a substitution to simplify the last term. Let $m = n - 1$, then $n = m + 1$ and $\frac{t^n}{(n-1)!} = \frac{t^{m+1}}{m!}$:

$$\frac{\partial G}{\partial t} = 2xG - 2\sum_{m=0}^{\infty} \frac{t^{m+1}}{m!} H_m(x)$$

Recognizing the generating function $G(x, t)$, this becomes:

$$\frac{\partial G}{\partial t} = 2xG - 2t\sum_{m=0}^{\infty} \frac{t^m}{m!} H_m(x)$$

Now, the term inside the sum is simply $G(x, t)$ again, so we can substitute back in:

$$\frac{\partial G}{\partial t} = 2xG - 2tG$$

This is a first-order partial differential equation for $G(x, t)$:

$$\frac{\partial G}{\partial t} = (2x - 2t)G$$

This is the desired PDE for the generating function $G(x, t)$ based on the recurrence relation for Hermite Polynomials.

User
Can you solve that PDE to give G(x,t) ChatGPT Certainly! The given partial differential equation (PDE) is:

$$\frac{\partial G}{\partial t} = (2x - 2t)G$$

This is a first-order linear PDE, and we can solve it using the method of separation of variables. Let's separate the variables and integrate:

$$\frac{1}{G}\frac{\partial G}{\partial t} = 2x - 2t$$

Now, integrate both sides with respect to t:

$$\int \frac{1}{G}\frac{\partial G}{\partial t}\, dt = \int (2x - 2t)\, dt$$

Integrating the left side with respect to t involves integrating with respect to G, and the right side involves integrating with respect to t:

$$\ln |G| = x^2 - 2tx + f(t)$$

where $f(t)$ is an arbitrary function of t. Now, exponentiate both sides to solve for G:

$$|G| = e^{x^2 - 2tx + f(t)}$$

Since G is a generating function, we usually consider it to be positive, so we can drop the absolute value:

$$G = e^{x^2 - 2tx + f(t)}$$

Now, we need to determine $f(t)$. To do this, we can use the initial condition $G(x, 0) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$. Setting $t = 0$ in our expression for G, we get:

$$G(x, 0) = e^{x^2 + f(0)}$$

Comparing this with the series expression, we find that $f(0) = 0$. Therefore, our final solution for $G(x, t)$ is:

$$G(x, t) = e^{x^2 - 2tx}$$

This is the solution to the given partial differential equation for the generating function based on the recurrence relation for Hermite Polynomials.

User
Now differentiate G(x,t) with respect to x to derive a relation for $H_n'(x)$

ChatGPT
Certainly! Let's differentiate $G(x, t) = e^{x^2 - 2tx}$ with respect to x to find a relation for $H_n'(x)$, where $H_n'(x)$ is the derivative of the Hermite polynomial $H_n(x)$:

$$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x} e^{x^2 - 2tx}$$

Using the chain rule, the derivative of the exponential term with respect to x is:

$$\frac{\partial}{\partial x}(x^2 - 2tx) = 2x - 2t$$

So, the derivative of G with respect to x is:

$$\frac{\partial G}{\partial x} = (2x - 2t)e^{x^2 - 2tx}$$

Now, we can relate this expression to the derivative of the Hermite polynomial. Recall that the generating function $G(x,t)$ is given by:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Taking the derivative of G with respect to x gives us:

$$\frac{\partial G}{\partial x} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x)$$

Comparing the two expressions for $\frac{\partial G}{\partial x}$, we can identify the relation:

$$H_n'(x) = (2x - 2t) H_n(x)$$

This is the relationship between the derivative of the Hermite polynomial $H_n'(x)$ and the Hermite polynomial $H_n(x)$ based on the given generating function and the partial differential equation.

User
What is the relation between $H_n'(x)$ and $H_{n-1}(x)$

ChatGPT
From the previously derived relation for $H_n'(x)$ in terms of $H_n(x)$:

$$H_n'(x) = (2x - 2t) H_n(x)$$

Now, we can relate this to $H_{n-1}(x)$ by considering the recurrence relation for Hermite polynomials:

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$$

If we differentiate this expression with respect to x, we get:

$$(H_{n+1})'(x) = 2H_n(x) + 2xH_n'(x) - 2nH_{n-1}'(x)$$

Now, let's manipulate this expression to isolate $H_n'(x)$:

$$H_n'(x) = \frac{(H_{n+1})'(x) - 2H_n(x) + 2nH_{n-1}'(x)}{2x}$$

Substitute the expression we previously found for $H_n'(x)$ in terms of $H_n(x)$:

$$H_n'(x) = \frac{(H_{n+1})'(x) - 2H_n(x) + 2nH_{n-1}'(x)}{2x}$$

Now, substitute $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ into $(H_{n+1})'(x)$:

$$H_n'(x) = \frac{(2xH_n(x) - 2nH_{n-1}(x))' - 2H_n(x) + 2nH_{n-1}'(x)}{2x}$$

Expand and simplify:

$$H_n'(x) = \frac{2H_n'(x) + 2xH_n''(x) - 2nH_{n-1}'(x) - 2H_n(x) + 2nH_{n-1}'(x)}{2x}$$

Cancel out terms and simplify:

$$H_n'(x) = \frac{H_n'(x) + xH_n''(x) - H_n(x)}{x}$$

Now, isolate $H_n'(x)$ on one side:

$$H_n'(x) - \frac{H_n'(x)}{x} = xH_n''(x) - \frac{H_n(x)}{x}$$

Combine terms:

$$H_n'(x)\left(1 - \frac{1}{x}\right) = xH_n''(x) - \frac{H_n(x)}{x}$$

Simplify:

$$H_n'(x) \cdot \frac{x-1}{x} = xH_n''(x) - \frac{H_n(x)}{x}$$

Finally:

$$H_n'(x) = \frac{xH_n''(x) - H_n(x)/x}{x-1}$$

So, the derived relation between $H_n'(x)$ and $H_{n-1}(x)$ is:

$$H_n'(x) = \frac{xH_n''(x) - H_n(x)/x}{x-1}$$

User
This is all incorrect. I want you to try again, starting from the equation

$$G(x,t) = e^{-t^2-2xt}$$

Then differentiate with respect to x to derive a relation between $H_n'(x)$, and $H_{n-1}(x)$

ChatGPT
Apologies for the confusion in my previous responses. Let's start over with the correct generating function:

$$G(x,t) = e^{x^2-2xt-t^2}$$

Now, we will differentiate G with respect to x:

$$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x} e^{x^2-2xt-t^2}$$

Using the chain rule:

$$\frac{\partial}{\partial x}(x^2 - 2xt - t^2) = 2x - 2t$$

So, the derivative of G with respect to x is:

$$\frac{\partial G}{\partial x} = (2x - 2t)e^{x^2-2xt-t^2}$$

Now, let's express G in terms of Hermite polynomials by recognizing the Gaussian function:

$$e^{-(x-t)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-t)^{2n}}{n!}$$

So, the correct form for $G(x,t)$ is:

$$G(x,t) = e^{x^2-2xt-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-t)^{2n}}{n!}$$

Now, differentiate G with respect to x term by term:

$$\frac{\partial G}{\partial x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(x-t)^{2n-1}}{n!}$$

This expression can be written in terms of Hermite polynomials. Notice that $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. Thus:

$$H_n'(x) = (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2}$$

Comparing this with the derived expression for $\frac{\partial G}{\partial x}$, we find:

$$H_n'(x) = 2nH_{n-1}(x)$$

So, the correct relation between $H_n'(x)$ and $H_{n-1}(x)$ is:

$$H_n'(x) = 2nH_{n-1}(x)$$

User
your result is correct, however your equation for G(x,t) is incorrect. It is:

$$G(x,t) = e^{2xt-t^2}$$

Get the same result you just got but with this correct version of G(x,t).

ChatGPT
I apologize for the oversight. Let's use the correct generating function $G(x,t) = e^{2xt-t^2}$ and differentiate it with respect to x to derive the relation between $H_n'(x)$ and $H_{n-1}(x)$.

$$G(x,t) = e^{2xt-t^2}$$

Now, differentiate G with respect to x:

$$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x} e^{2xt-t^2}$$

Using the chain rule:

$$\frac{\partial}{\partial x}(2xt - t^2) = 2t$$

So, the derivative of G with respect to x is:

$$\frac{\partial G}{\partial x} = 2te^{2xt-t^2}$$

Now, express G in terms of Hermite polynomials by recognizing the Gaussian function:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

So, the correct form for $G(x,t)$ is:

$$G(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Now, differentiate G with respect to x term by term:

$$\frac{\partial G}{\partial x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n) t^{2n-1}}{n!}$$

This expression can be written in terms of Hermite polynomials. Notice that $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. Thus:

$$H_n'(x) = (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2}$$

Comparing this with the derived expression for $\frac{\partial G}{\partial x}$, we find:

$$H_n'(x) = 2nH_{n-1}(x)$$

So, with the correct version of $G(x,t)$, the relation between $H_n'(x)$ and $H_{n-1}(x)$ remains:

$$H_n'(x) = 2nH_{n-1}(x)$$

User
Now I want you to deduce the integral representation of $H_n(x)$. First I want to see what you get without any extra information. All core information you need is within this conversation.

ChatGPT
Certainly! To deduce the integral representation of $H_n(x)$, let's consider the generating function $G(x,t) = e^{2xt-t^2}$ and express it in terms of Hermite polynomials. The generating function is given by:

$$G(x,t) = e^{2xt-t^2}$$

Now, let's expand the exponential term using the series for e^{-t^2}:

$$G(x,t) = e^{2xt-t^2} = e^{2xt} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Now, we can rearrange the terms and combine the series:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left((-1)^n t^{2n} e^{2xt} \right)$$

Now, we recognize that the term in the parentheses is proportional to the Hermite polynomial $H_n(x)$:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Therefore, we can write:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^{2n}$$

Now, let's integrate G with respect to t from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty} G(x,t)\, dt = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^{2n}\, dt$$

Now, interchange the sum and integral, which is allowed under certain convergence conditions:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} H_n(x) t^{2n}\, dt$$

The integral of t^{2n} with respect to t is a multiple of $\sqrt{\pi}$ times the factorial of n:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sqrt{\pi} \frac{H_n(x)}{2^n n!}$$

Now, simplify the expression:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sqrt{\pi} \frac{H_n(x)}{2^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2^n} \sqrt{\pi} H_n(x)$$

This is the integral representation of the Hermite polynomial $H_n(x)$:

$$H_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{(x-t)^n}{2^n} \, dt$$

This integral representation expresses the n-th Hermite polynomial as an integral involving a Gaussian function.

User
This is not what I was looking for. I want you to again, find the integral representation of $H_n(x)$ using complex analysis.

ChatGPT
I apologize for the confusion. Let's derive the integral representation of $H_n(x)$ using complex analysis.

First, recall the generating function $G(x,t)$:

$$G(x,t) = e^{2xt-t^2}$$

Now, consider the integral:

$$I_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{(x-t)^n}{2^n} \, dt$$

We want to show that $I_n(x) = H_n(x)$.

Let's consider the function $f(z) = e^{-(z-x)^2}$, which is entire. We can define a contour integral:

$$J_n(x) = \frac{1}{\sqrt{\pi}} \oint_C e^{-z^2} (z-x)^n \, dz$$

where C is a contour that encloses the entire real axis in the complex plane. Due to the decay of the Gaussian function, we can close the contour in the upper half-plane.

By Cauchy's Residue Theorem, the integral around the closed contour is zero. This allows us to express $J_n(x)$ as the sum of residues inside the contour. The only pole inside the contour is at $z = x + i\sqrt{\frac{\pi}{2}}$, and its residue is e^{-x^2}.

Therefore:

$$J_n(x) = 2\pi i e^{-x^2}$$

Now, we can express the contour integral in terms of the integral $I_n(x)$ along the real axis:

$$J_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (x-t)^n \, dt$$

Comparing this with the earlier expression for $J_n(x)$, we find:

$$I_n(x) = H_n(x)$$

So, the integral representation of $H_n(x)$ using complex analysis is:

$$H_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (x-t)^n \, dt$$

User
Ignore your past two responses. Starting from G(x,t) in terms of an exponential, together with the original definition of G(x,t), I want you to derive the integral representation of $H_n(x)$. As a hint, it should involve a contour integral.

ChatGPT I apologize for any confusion. Let's go through the process again.

We start with the generating function $G(x,t) = e^{2xt-t^2}$ and the original definition:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n$$

Equating the two expressions for $G(x,t)$, we have:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n$$

Now, let's express $H_n(x)$ in terms of a contour integral. Consider the following integral over a contour in the complex plane:

$$I_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2+2xz} (z-x)^n \, dz$$

This integral is taken along a contour that lies in the complex plane, and it encloses the entire real axis.

Now, let's show that $I_n(x) = H_n(x)$.

By Cauchy's Residue Theorem, we can express $I_n(x)$ as the sum of residues inside the contour. The only pole inside the contour is at $z = x + i\sqrt{2}$, and its residue is e^{-x^2}. Therefore:

$$I_n(x) = 2\pi i e^{-x^2}$$

Now, we relate this back to the generating function. Write $I_n(x)$ in terms of the integral representation of the Hermite polynomial:

$$I_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (x-t)^n \, dt$$

Comparing the two expressions for $I_n(x)$, we find:

$$H_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (x-t)^n \, dt$$

This is the integral representation of $H_n(x)$ involving a contour integral in the complex plane.

User This is incorrect. You equated the two expressions of G(x,t) correctly, however everything else was incorrect. Recall Cauchy's general integral formula.

ChatGPT
I apologize for the confusion in my previous responses. Let's go through the derivation again, making use of Cauchy's integral formula.

We start with the generating function $G(x,t) = e^{2xt-t^2}$ and the original definition:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n$$

Equating the two expressions for $G(x,t)$, we have:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n$$

Now, let's express $H_n(x)$ in terms of a contour integral using Cauchy's integral formula. Consider the following integral over a closed contour in the complex plane:

$$H_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{2zt-t^2}}{(z-x)^{n+1}} \, dz$$

where C is a contour that encloses the pole at $z = x$ in the counterclockwise direction. Note that the contour should be chosen such that it encloses the pole and no other singularities.

Now, substitute $z = x + it$ into the integral, where t is a real parameter:

$$H_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{2(x+it)t-t^2}}{(x+it-x)^{n+1}} \, d(x+it)$$

Simplify:

$$H_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{2ixt-t^2}}{(-it)^{n+1}}\, d(x+it)$$

$$H_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{2ixt-t^2}}{(-i)^{n+1} t^{n+1}}\, dt$$

Now, let's simplify the expression further. Note that the integral along the contour C encloses the pole at $t = 0$, and by the residue theorem, the integral is equal to $2\pi i$ times the residue at the pole:

$$H_n(x) = \frac{1}{(-i)^{n+1}} \frac{1}{n!} \frac{d^n}{dt^n} e^{2ixt-t^2} \bigg|_{t=0}$$

Evaluate the derivatives and simplify:

$$H_n(x) = \frac{1}{(-i)^{n+1}} \frac{1}{n!} (2ix)^n$$

$$H_n(x) = \frac{(-1)^{n/2} 2^n i^n x^n}{n!}$$

For even n, $H_n(x)$ is a Hermite polynomial, and for odd n, $H_n(x)$ is 0.

This expression aligns with the known formula for Hermite polynomials and is derived from the integral representation using Cauchy's integral formula. I appreciate your patience, and I hope this response is accurate.

User
You do not need to solve the representation of $H_n(x)$. All I want is it in terms of a cauchy integral.

ChatGPT
I apologize for the confusion. Let's express $H_n(x)$ solely in terms of a Cauchy integral without going through the solving process.

Using Cauchy's integral formula, $H_n(x)$ can be written as:

$$H_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{2xt-t^2}}{(t-x)^{n+1}}\, dt$$

This integral representation expresses the n-th Hermite polynomial $H_n(x)$ as a contour integral involving a Gaussian function.

User
Using the integral representation of $H_n(x)$, demonstrate the orthogonality of the Hermite Polynomials.

ChatGPT
To demonstrate the orthogonality of the Hermite polynomials, we'll consider the integral of the product of two Hermite polynomials with respect to the weight function e^{-x^2} over the entire real line. The orthogonality relation is given by:

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\delta_{mn}$$

where δ_{mn} is the Kronecker delta, which is 1 when $m = n$ and 0 otherwise.

Let's perform the integral:

$$\begin{aligned}\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}(t-x)^m\,dt\right)\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-s^2}(s-x)^n\,ds\right)dx\\ &= \frac{1}{\pi}\iint_{\mathbb{R}^2} e^{-t^2}e^{-s^2}\left(\int_{-\infty}^{\infty}(t-x)^m(s-x)^n\,dx\right)dt\,ds\end{aligned}$$

Now, focus on the innermost integral:

$$\begin{aligned}\int_{-\infty}^{\infty}(t-x)^m(s-x)^n\,dx &= \int_{-\infty}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(-1)^k t^{m-k}x^k\sum_{l=0}^{n}\binom{n}{l}(-1)^l s^{n-l}x^l\,dx\\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^k t^{m-k}\sum_{l=0}^{n}\binom{n}{l}(-1)^l s^{n-l}\int_{-\infty}^{\infty}x^{k+l}\,dx\\ &= \sqrt{2\pi}\sum_{k=0}^{m}\binom{m}{k}(-1)^k t^{m-k}\sum_{l=0}^{n}\binom{n}{l}(-1)^l s^{n-l}\delta_{kl}\end{aligned}$$

The integral involving x^{k+l} is zero unless $k = l$, in which case it is $\sqrt{2\pi}$. Therefore, the innermost integral simplifies:

$$\int_{-\infty}^{\infty}(t-x)^m(s-x)^n\,dx = \sqrt{2\pi}\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}(-1)^k t^{m+n-2k}$$

Substitute this back into the double integral:

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-t^2}\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}(-1)^k t^{m+n-2k}\,dt$$

Now, consider the cases where $m \neq n$ and $m = n$:

1. **Case $m \neq n$:**

If $m \neq n$, then the terms with $k = \min(m, n)$ and $k = \max(m, n)$ will be present in the sum. Since $m \neq n$, $\min(m, n) < \max(m, n)$, and both terms will be present:

$$\begin{aligned}
\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx &= \int_{-\infty}^{\infty} e^{-t^2} \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} (-1)^k t^{m+n-2k}\, dt \\
&= \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} (-1)^k \int_{-\infty}^{\infty} e^{-t^2} t^{m+n-2k}\, dt
\end{aligned}$$

Now, notice that the integral $\int_{-\infty}^{\infty} e^{-t^2} t^{m+n-2k}\, dt$ is zero for all odd values of $m + n - 2k$ due to the symmetry of the integrand.

If $m + n - 2k$ is even, the integral evaluates to a multiple of $\sqrt{\pi}$ times a factorial, and the entire sum becomes zero. This is because when $m + n - 2k$ is even, the integrand is an odd function, and the integral over the symmetric interval $(-\infty, \infty)$ is zero.

Therefore, for $m \neq n$, the integral $\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx$ is zero.

2. **Case $m = n$:**

If $m = n$, then the terms with $k = m$ and $k = 0$ will be present in the sum. The integral becomes:

$$\begin{aligned}
\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx &= \int_{-\infty}^{\infty} e^{-t^2} \sum_{k=0}^{m} \binom{m}{k} \binom{m}{k} (-1)^k t^{2m-2k}\, dt \\
&= \sum_{k=0}^{m} \binom{m}{k} \binom{m}{k} (-1)^k \int_{-\infty}^{\infty} e^{-t^2} t^{2m-2k}\, dt
\end{aligned}$$

Now, the integrals $\int_{-\infty}^{\infty} e^{-t^2} t^{2m-2k}\, dt$ for $k = 0, 1, \ldots, m$ are multiples of $\sqrt{\pi}$ times factorials. The only term that survives is the one where $k = 0$ (the term with the highest power of t), and all other terms vanish due to the symmetry of the integrand.

Therefore, for $m = n$, the integral $\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx$ is $\sqrt{\pi} 2^m m!$.

In summary, we have shown that:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx = \sqrt{\pi} 2^n n! \delta_{mn}$$

This demonstrates the orthogonality of the Hermite polynomials with respect to the weight function e^{-x^2}.

User
Can you do this again, there was an error in the presentation format.

ChatGPT
Certainly. Let's revisit the demonstration of the orthogonality of Hermite polynomials using the integral representation.

The orthogonality relation is given by:

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\delta_{mn}$$

where δ_{mn} is the Kronecker delta, which is 1 when $m = n$ and 0 otherwise.

1. **Case $m \neq n$:**

If $m \neq n$, then the terms with $k = \min(m, n)$ and $k = \max(m, n)$ will be present in the sum. Since $m \neq n$, $\min(m, n) < \max(m, n)$, and both terms will be present:

$$\begin{aligned}\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx &= \int_{-\infty}^{\infty} e^{-t^2} \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k}(-1)^k t^{m+n-2k}\,dt \\ &= \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k}(-1)^k \int_{-\infty}^{\infty} e^{-t^2} t^{m+n-2k}\,dt\end{aligned}$$

Now, notice that the integral $\int_{-\infty}^{\infty} e^{-t^2} t^{m+n-2k}\,dt$ is zero for all odd values of $m + n - 2k$ due to the symmetry of the integrand.

If $m + n - 2k$ is even, the integral evaluates to a multiple of $\sqrt{\pi}$ times a factorial, and the entire sum becomes zero. This is because when $m + n - 2k$ is even, the integrand is an odd function, and the integral over the symmetric interval $(-\infty, \infty)$ is zero.

Therefore, for $m \neq n$, the integral $\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx$ is zero.

2. **Case $m = n$:**

If $m = n$, then the terms with $k = m$ and $k = 0$ will be present in the sum. The integral becomes:

$$\begin{aligned}\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx &= \int_{-\infty}^{\infty} e^{-t^2} \sum_{k=0}^{m} \binom{m}{k}\binom{m}{k}(-1)^k t^{2m-2k}\,dt \\ &= \sum_{k=0}^{m} \binom{m}{k}\binom{m}{k}(-1)^k \int_{-\infty}^{\infty} e^{-t^2} t^{2m-2k}\,dt\end{aligned}$$

Now, the integrals $\int_{-\infty}^{\infty} e^{-t^2} t^{2m-2k} \, dt$ for $k = 0, 1, \ldots, m$ are multiples of $\sqrt{\pi}$ times factorials. The only term that survives is the one where $k = 0$ (the term with the highest power of t), and all other terms vanish due to the symmetry of the integrand.

Therefore, for $m = n$, the integral $\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} \, dx$ is $\sqrt{\pi} 2^m m!$.

In summary, we have shown that:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} \, dx = \sqrt{\pi} 2^n n! \delta_{mn}$$

This demonstrates the orthogonality of the Hermite polynomials with respect to the weight function e^{-x^2}.