Mathematical Methods

Linear Vector Spaces

A linear vector space $\mathbb{V}$ is defined over a field $\mathbb{F}$ (think of vector spaces as being composite objects of both vectors and a field). If they're defined over $\mathbb{R}$ it's a real vector space, if $\mathbb{C}$, complex. The field consists of scalars, so the vectors themselves are neither complex nor real. A *vector* is an element of $\mathbb{V}$.

They have the following requirements:

- A definite rule representing $|V\rangle + |W\rangle$
- A definite rule for $a|V\rangle$ where $a \in \mathbb{F}$
- Closure under addition, $|V\rangle + |W\rangle \in \mathbb{V}$
- Distributive scalar multiplication for vectors, $a(|V\rangle + |W\rangle) = a|V\rangle + a|W\rangle$
- Distributive scalar multiplication for scalars $(a + b)|V\rangle = a|V\rangle + b|V\rangle$
- Associative scalar multiplication, $a(b|V\rangle) = ab|V\rangle$
- Commutative addition, $|V\rangle + |W\rangle = |W\rangle + |V\rangle$
- Associative addition, $|V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$
- Null vector, $|V\rangle + |0\rangle = |V\rangle$
- Inverse that can create null vector under addition, $|V\rangle + |-V\rangle = |0\rangle$

Linear independence is the property for a vector to not be written in terms of other vectors in the same set.

Any $|V\rangle \in \mathbb{F}^n$ can be written as a linear combination of n linearly independent vectors $|1\rangle \ldots |n\rangle$. Further, a set of n linear independent vectors in $\mathbb{F}^n$ is called a basis. We get that

$$|V\rangle = \sum_{i=1}^{n} v_i |i\rangle$$

Where $|i\rangle$ are the basis vectors. v_i are thus the components of the vector $|V\rangle$. Addition of vectors of the same basis $|i\rangle$, add their components, i.e.

$$|V\rangle + |W\rangle = \sum_{i=1}^{n} (v_i + w_i)|i\rangle$$

and scalar multiplication is similar

$$a|V\rangle = \sum_{i=1}^{n} a v_i |i\rangle$$

Since both functions and matrices can be examples of vector spaces, a vector space requires no definition of length or direction for the elements.

Vector spaces can be a space of functions from a set to a field. Let $\mathbb{F}$ be any field, and let S be a non-empty set. Let $\mathbb{V}$ be the set of all functions from the S into $\mathbb{F}$. any function f of $\mathbb{V}$ will thus satisfy all the vector space requirements.

Exercise 1.1.2: Show that vectors of the form $(a, b, 1)$ do not form a vector space

They don't obey closure under addition

$$(a, b, 1) + (c, d, 1) \neq (a + c, b + d, 1)$$

Exercise 1.1.4

Given

$$\begin{aligned} |1\rangle &= (1, 1, 0) \\ |2\rangle &= (1, 0, 1) \\ |3\rangle &= (0, 1, 1) \end{aligned}$$

We can show that these are linearly independent by attempting to solve the equation for non-zero a_i

$$\sum_{i=1} a_i |i\rangle = 0$$

from which we get that

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_1 + a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}$$

we can now solve a_1 and a_2 in terms of a_3

$$\begin{aligned} a_1 &= -a_3 \\ a_2 &= -a_3 \end{aligned}$$

substituting into $a_1 + a_2 = 0$

$$\begin{aligned} -2a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

which finally gives us

$$a_1 = a_2 = a_3 = 0$$

Thus the vectors are linearly independent, as there is no non-zero solution.

Inner Product Space

Recall the arrow vector dot product

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

This obeys linearity, positive semi-definiteness and also symmetry. We can create a generalized scalar product or inner product between any vectors $|V\rangle$ and $|W\rangle$, denoted by $\langle V|W\rangle$, which "spits out" a scalar. Generally it's defined on $\mathbb{C}$. It obeys the following axioms

- $\langle V|W\rangle = \langle W|V\rangle^*$ skew-symmetry (* denotes the complex conjugation)
- $\langle V|V\rangle \geq 0 \iff |V\rangle = |0\rangle$ positive semi-definiteness
- $\langle V|(a|W\rangle + b|Z\rangle)\rangle = \langle V|aW + bZ\rangle = a\langle V|W\rangle + b\langle V|Z\rangle$ linearity in ket

If a vector space has an inner product, it's called an inner product space. Just to note how we can combine

What if the first factor in the product is a linear superposition, i.e., what is $\langle aW + bZ|V\rangle$? - given the first action we get anti-linearity:

$$\begin{aligned}\langle aW + bZ|V\rangle &= \langle V|aW + bZ\rangle^* \\ &= (a\langle V|W\rangle + b\langle V|Z\rangle)^* \\ &= a^*\langle V|W\rangle^* + b^*\langle V|Z\rangle^* \\ &= a\langle W|V\rangle^* + b\langle Z|V\rangle^*\end{aligned}$$

Were the superposition in the second factor we wouldn't have the conjugates. This is an asymmetry.

Two vectors are orthogonal/perpendicular if their inner product vanishes.

$\sqrt{\langle V|V\rangle} = \|V\|$ is the norm/length of a vector. A normalized vector has unit norm.

A set of pairwise orthogonal basis vectors of unit norm are an orthonormal basis.

The inner product is defined verbosely as

$$\langle V|W\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i^* w_j \langle i|j\rangle$$

For an orthonormal basis we get that

$$\langle i|j\rangle = \delta_{ij}$$

Which means that $v_i^* w_j \neq 0 \Leftrightarrow i = j$ thus collapsing the sum as follows

$$\begin{aligned}\langle V|W\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i^* w_j \delta_{ij} \\ &= \sum_{i=1}^{n} v_i^* w_i\end{aligned}$$

We refer to $\langle V|V\rangle$ as the norm squared, as follows

$$\langle V|V\rangle = \sum_{i}^{n} |v_i|^2 \geq 0$$

We can write ket vectors in this basis like

$$|X\rangle = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The inner product $\langle V|W\rangle$ is then the matrix product of the transpose conjugate of $|V\rangle$ with $|W\rangle$, i.e. $\mathbf{V}^\dagger\mathbf{W}$ which is the following

$$\langle V|W\rangle = \begin{bmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Dual Spaces and Dirac Notation

Column vectors are manifestations of an abstract vector $|V\rangle$ in a basis. Each row vector we can associate an abstract vector $\langle W|$. Associated with every ket $|V\rangle$ is a column vector, and taking the transpose conjugate of this, gives us a row vector. The abstract bra associated with this row vector is called $\langle V|$. There are *two* vector spaces, the ket space and a dual space of bras, with a ket for every bra and the other way around.

Inner products are defined between bra's and ket's.

There is a basis of vectors $|i\rangle$ for expanding ket's, and a similar basis $\langle i|$ for expanding bra's.

For a basis column/row vector, the i'th row or column is a 1 whilst all the others are 0's.

We can describe the associations like

$$|V\rangle \leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \leftrightarrow \begin{bmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{bmatrix} \leftrightarrow \langle V|$$

We can find the j'th component of a vector, recalling

$$|V\rangle = \sum_i^n v_i |i\rangle$$

which gives us

$$\begin{aligned} \langle j|V\rangle &= \sum_i^n v_i \langle j|i\rangle \\ &= v_j \end{aligned}$$

We can use this result and construct $|V\rangle$ as

$$|V\rangle = \sum_i |i\rangle\langle i|V\rangle$$

Where $\langle i|V\rangle$ gives us the i'th component v_i and scalar multiply it with $|i\rangle$.

Adjoint

The adjoint operation is in this case a transpose conjugate, but the adjoint is a generalized operation. Taking the adjoint of a linear equation relating bra's to ket's, replace every ket with its bra (and vice versa), and complex conjugate all coefficients.

We can see this easily given

$$|V\rangle = \sum_i |i\rangle\langle i|V\rangle$$

taking the adjoint gives us

$$\langle V| = \sum_i \langle V|i\rangle\langle i|$$

Gram-Schmidt and dimensionality

Given a set linearly independent basis vectors, we can orthonormalize them using the Gram-Schmidt method.

Let $|I\rangle, |II\rangle, \ldots$ be a linearly independent basis. We calculate the first vector of our orthonormalized basis as

$$|1\rangle = \frac{|I\rangle}{\||I\|}$$

this means that

$$\langle 1|1\rangle = \frac{\langle I|I\rangle}{\||I\|^2} = 1$$

and we can now orthonormalize our last vector

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle$$

from which you can get the basis vector by

$$|2\rangle = \frac{1}{\||2'\|}|2'\rangle$$

As for dimensionality,

> *The dimensionality of a space equals $n_\perp$, the maximum number of mutually orthogonal vectors in it.*

Any mutually orthogonal set is linearly independent.

Subspaces

Given a vector space $\mathbb{V}$, a subset of its elements that form a vector space is called a subspace. A subspace i of n_i dimensionality is denoted $\mathbb{V}_i^{n_i}$.

Given two subspaces, their sum can be found

$$\mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{m_j} = \mathbb{V}_k^{m_k}$$

Where $\mathbb{V}_k^{m_k}$ contains all elements of $\mathbb{V}_i^{n_i}$ and $\mathbb{V}_j^{m_j}$, and all the possible linear combinations of them.

A subset of a vector space, is a subspace iff for each pair of vectors a, b in the subset and each scalar c in the field, the vector $ca + b$ has closure.

Given $\mathbb{V}$ over the field $\mathbb{F}$. Any intersection of any collection of subspaces of $\mathbb{V}$ is also a subspace of $\mathbb{V}$.

Linear Operators

A linear operator Ω or linear transformation T is any function from $\mathbb{V}_1$ to $\mathbb{V}_2$ such that

$$T(cv_1 + v_2) = c(Tv_1) + Tv_2$$

An operator Ω takes a vector $|V\rangle \in \mathbb{V}$ and transforms it to another $|V'\rangle$ (and vice versa for bra's). We're interested mainly in operators for which $\Omega|V\rangle \in \mathbb{V}$.

Linear operators obey the following rules:

- $\Omega a|V\rangle = a\Omega|V\rangle$
- $\Omega(\alpha|V_i\rangle + \beta|V_j\rangle) = \alpha\Omega|V_i\rangle + \beta\Omega|V_j\rangle$

And the same rules for bra's, except the order is reversed.

The simplest operator is the identity operator $\hat{I}$. It works as follows

$$\forall|V\rangle,\ \hat{I}\,|V\rangle = |V\rangle$$

or

$$\forall\langle V|,\ \langle V|\hat{I} = \langle V|$$

We have other simple interesting operators, namely the rotation operator over $\mathbb{V}^3(R)$. E.g. $\hat{R}(\frac{1}{2}\pi e_x)$ which rotates a vector about the e_x unit vector by $\frac{1}{2}\pi$. An example

$$\hat{R}\left(\frac{1}{2}\pi e_x\right)|2\rangle = |3\rangle$$

This operator is linear.

The product of two operators means their operation carried out in sequence, i.e.

$$\Lambda\Omega|V\rangle = \Lambda(\Omega|V\rangle) = \Lambda|\Omega V\rangle$$

We denote the ket vector obtained by the action of Ω on $|V\rangle$ as $|\Omega V\rangle$. The order is important.

The commutator defined below is non-zero

$$\Omega\Lambda - \Lambda\Omega \equiv [\Omega, \Lambda]$$

as these two operators do not commute. Two useful commutation identities

$$[\Omega, \Lambda\theta] = \Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta$$
$$[\Lambda\Omega, \theta] = \Lambda[\Omega, \theta] + [\Lambda, \theta]\Omega$$

The inverse of Ω is Ω^{-1} and satisfies

$$\Omega\Omega^{-1} = \Omega^{-1}\Omega = \hat{I}$$

Matrix Elements of Linear Operators

Linear operators can be represented in a basis by a set of n^2 numbers, written as a $n \times n$ matrix - these are called the matrix elements in that basis.

If the basis vectors suffer a change

$$\Omega|i\rangle = |i'\rangle$$

then any vector in this space undergoes a change that is readily calculable

$$\Omega|V\rangle = \Omega \sum_i v_i |i\rangle = \sum_i v_i \Omega|i\rangle = \sum_i v_i \Omega|i'\rangle$$

When we say $|i'\rangle$ is known, we mean that its components in the original basis are known,

$$\langle j|i'\rangle = \langle j|\Omega|i\rangle \equiv \Omega_{ji}$$

The n^2 number, Ω_{ij}, are the matrix elements of Ω in this basis. If

$$\Omega|V\rangle = |V'\rangle$$

then we can expand the terms of the transformed $|V'\rangle$ as

$$v'_i = \sum_j \Omega_{ij} v_j$$

Just for good measure, we can show this in matrix form as

$$\begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{bmatrix} = \begin{bmatrix} \langle 1|\Omega|1\rangle & \cdots & \langle 1|\Omega|n\rangle \\ \vdots & \ddots & \\ \langle n|\Omega|1\rangle & & \langle n|\Omega|n\rangle \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Projection Operator

Remember that the "outer product" $|V\rangle\langle W|$ produces an $n \times n$ matrix, whilst the inner or dot product $\langle V|W\rangle$ produces a scalar. The projection operator is defined as follows

$$\hat{I} = \sum_i |i\rangle\langle i| = \sum_i \mathcal{P}_i$$

The object $\mathcal{P}_i = |i\rangle\langle i|$ is the projection operator for the ket $|i\rangle$. The above equation denotes the identity operator $\hat{I}$ of i as summation over projection operators for i.

The next definition encapsulates the usage of the projection operator, that is

$$\mathcal{P}_i|V\rangle = |i\rangle\langle i|V\rangle = |i\rangle v_i$$

It projects any component v_i out as a ket in the direction i. $|V\rangle$ is thus the sum of all of those kets. Notice that it works similarly on bras, just by reversing the terms

$$\langle V|\mathcal{P}_i = \langle V|i\rangle\langle i| = v_i^*\langle i|$$

For the following equation

$$\mathcal{P}_i\mathcal{P}_j = |i\rangle\langle i|j\rangle\langle j| = \delta_{ij}\mathcal{P}_j$$

We note that, has $\mathcal{P}_j$ already been applied to a $|V\rangle$, further applications of $\mathcal{P}$ to $|\mathcal{P}_j V\rangle$ yields an unchanged $|P_j V\rangle$ (hence the Kronecker delta). We also see that, if $i \neq j$ for $\delta_{ij}\mathcal{P}_j$, we yield 0 (meaning that if you project a vector onto one basis, projecting it again yields nothing, as there's no information left of the other bases).

Any projection operator has a 1 at the i'th element on the diagonal of the matrix representation. $\sum_i^n \mathcal{P}_i = \hat{I}$, whilst if we sum it only over i corresponding to a subspace's basis vectors, we get an operator that projects a vector into the given subspace. For example, projecting an $\mathbb{R}^n$ vector into $\mathbb{R}^{n-1}$

$$|V'\rangle = \sum_i^{n-1} \mathcal{P}_i|V\rangle$$

Matrices Corresponding to Products of Operators

Matrix for a product of operators

$$(\Omega\Lambda)_{ij} = \langle i|\Omega\Lambda|j\rangle = \sum_k \Omega_{ik}\Lambda_{kj}$$

The matrix-form of the operator product is simply the product of the matrix-forms.

Operator Adjoint

Given a ket

$$\Omega|V\rangle = |\Omega V\rangle$$

Its corresponding bra is

$$\langle \Omega V| = \langle V|\Omega^\dagger$$

$\Omega^\dagger$ is the adjoint of Ω, and is the operator that can act on the bra vector. Its matrix-form is the conjugate transpose of Ω's matrix-form. We also see that

$$\Omega_{ij}^\dagger = \Omega_{ji}^*$$

The adjoint of a product of operators is the product of each individual adjoint in reverse.

Hermitian, Anti-Hermitian and Unitary operators An operator is Hermitian if

$$\Omega^\dagger = \Omega$$

An operator is anti-Hermitian if

$$\Omega^\dagger = -\Omega$$

An operator can be decomposed into Hermitian and anti-Hermitian parts as

$$\Omega = \frac{\Omega + \Omega^\dagger}{2} + \frac{\Omega - \Omega^\dagger}{2}$$

An operator $\hat{U}$ is unitary if

$$\hat{U}\hat{U}^\dagger = \hat{I}$$

This means that the adjoint is the inverse of it (and vice versa).

$$\hat{U}^\dagger\hat{U} = \hat{I}$$

Unitary operators preserve the inner product between the vectors they act on.

Section not finished

Groups

The set of *invertible* linear operators with the operation of composition provides a nice example of a group.

A group consists of the following

1. A set G
2. A rule that associates each pair of elements (x, y) in G with xy in G in such a way that
    a. $x(yz) = (xy)z$ for all x, y and z in G - associativity.
    b. there is an element $e \in G$ such that $ex = xe = x$ for every x in G
    c. to each element x in G there corresponds an element x^{-1} in G such that $xx^{-1} = x^{-1}x = e$

In general, groups requires totality, associativity, identity and invertibility, but not necessarily commutativity. We will briefly look at abelian groups below.

We're already acquainted with the identity operator $\hat{I}$ that abides by such requirements. Thus the set $\mathbb{V}$ together with $\hat{I}$ is a group. $n \times n$ invertible matrices together with matrix multiplication are also a group. A group of set G with an operator $(\cdot)$ is non-abelian if

$$(\exists a, b \in G) \ni a \cdot b \neq b \cdot a$$

Thus both examples are non-abelian. An example of abelian (commutative) groups would be $|V\rangle \in \mathbb{V}$ together with the operation of vector addition.

Isomorphism

Any injective linear transformation T from $\mathbb{V}_1$ onto $\mathbb{V}_2$ is called an isomorphism of $\mathbb{V}_1$ onto $\mathbb{V}_2$. If there exists such isomorphism, we can say that $\mathbb{V}$ is isomorphic to $\mathbb{V}_2$.

$\mathbb{V}_1$ is trivially isomorphic to itself, the identity operator $\hat{I}$ being an isomorphism of $\mathbb{V}_1$ onto itself. If $\mathbb{V}_1$ is isomorphic to $\mathbb{V}_2$ with Ω, then $\mathbb{V}_2$ is isomorphic onto $\mathbb{V}_1$ with Ω^{-1}, thus we say that $\mathbb{V}_1$ and $\mathbb{V}_2$ are isomorphic.

Isomorphic vector spaces are often regarded as "the same", but the vectors and operations in either vector space may be different.

Homeomorphism

A map $T : U \to \tilde{U}$ between the topological spaces U and $\tilde{U}$ is homeomorphic, if T has the properties that

- it is a bijection
- it is continuous
- the inverse T^{-1} is also continuous

Topological Space

From the Hausdorff neighborhood definition:

Given a set X along with some point $x \in X$, and a function U which assigns each point x a non-empty collection of subsets of X, $U(x)$. The elements in $U(x)$ are called neighborhoods of x. The function U is called a "neighborhood topology" if it satisfies that

- if N is a neighborhood of x i.e. $N \in U(x)$, then $x \in N$.
- if N is a subset of X and includes a neighborhood of x, then N is a neighborhood of x i.e. $\forall P$ where $P \supset N$ we get that $P \in U(x)$
- $(N_1 \cap N_2) \in U(x)$

More intuitively, a neighborhood N (of x) is a subset of X that includes an open set V that also includes x. This with the above axioms, means that N is some "region" around x such that you can move in any direction away from x without leaving N.

The set X together with these axioms for $U : X \to N$ is a topological space.

Topological Manifolds

Given a topological space M. We call it a topological *manifold* if it satisfies that

- It's a Hausdorff space i.e. $\forall x_1, x_2 \in M$ there exists neighborhoods of the points N_1 and N_2 for which $N_1 \cap N_2 = \emptyset$ (disjoint)
- It's second-countable
- It's locally Euclidean

Eigenvalues and Eigenkets

Each operator has a vector of its own called an eigenvector, which can be related to the original vector by some scalar. If we have that

$$\Omega|V\rangle = \omega|V\rangle$$

then $|V\rangle$ is the eigenvector of Ω, and the eigenvalue is ω. For example, the only eigenvalue for the identity operator is 1. All vectors are its eigenvectors.

The collection of eigenvectors for an operator, is called an eigenspace (associated with the eigenvalue).

The Characteristic Equation

From 1.8.8, we get the characteristic equation

$$\sum_{m=0}^{n} c_m \omega^m = 0$$

and the characteristic polynomial

$$P^n(\omega) = \sum_{m=0}^{n} c_m \omega^m$$

The roots of P^n are the eigenvalues. Any operator in $\mathbb{V}^n(C)$ has n eigenvalues. For Hermitian and unitary operators, we can easily find the corresponding eigenvectors if we know the eigenvalues. The characteristic equation is

$$\det(\Omega - \omega\hat{I}) = 0$$

We can calculate it for the identity operator as

$$\det(\hat{I} - \omega\hat{I}) = \det\begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & 1-\omega & 0 \\ 0 & 0 & 1-\omega \end{vmatrix} = 0$$

If you calculate the determinant, you get

$$(-\omega + 1)^3 = 0$$

If we plug in $\omega = 1$ (which is the only solution) we get

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which yields the three equations

$$\begin{aligned} 0 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Corresponding to each index. This means that any vector of the form

$$x_i|i\rangle$$

Is acceptable. Thus we get that the ω_i are all 1. We often normalize the eigenvectors such that

$$|\omega = 1\rangle \leftrightarrow \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is a popular convention: labeling the eigenvectors by their eigenvalue.

If an eigenvalue represents more than one eigenvector, we call it *degeneracy*. This is an example of degeneracy.

Hermitian operators

The eigenvalues of Hermitian operators are always real!

To every Hermitian operator Ω there exists a basis consisting of its orthonormal eigenvectors. It is diagonal in this eigenbasis and has its eigenvalues as its diagonal entries.

Generalizations to infinite dimensions

Blablabla.

Hilbert Spaces

A Hilbert Space is an inner product space with the properties/requirements

- It's defined over a square-integrable set
- The sets are separable

Review of Classical Mechanics

Lagrangian Mechanics

Lagrangian formalism of mechanics, the problem of a particle in a potential $V(x)$ is posed in a simple way

> *Given that the particle is at x_i and x_f at times t_i and t_f respectively, what is it that distinguishes the actual trajectory $x_{cl}(t)$ from all other paths?*

Lagrangian mechanics attempts to determine in one stroke an entire path for a particle, as opposed to Newtonian mechanics that is concerned with infinitesimals.

We can create a few steps for this problem

1. Define a function $\mathscr{L}$ called the Lagrangian using

$$\mathscr{L} = T - V$$

Giving us a function $\mathscr{L} = (x, \dot{x}, t)$

2. For all paths $x(t)$ connecting (x_i, t_i) and (x_f, t_f) calculate the action as

$$S[x(t)] = \int_{t_i}^{t_f} \mathscr{L}(x, \dot{x})\, dt$$

3. We will then attempt to minimize $S[x(t)]$, because of the principle of least action.

I won't derive the Euler-Lagrange equations here, but we can find a minimized path given

$$d_t(\partial_{\dot{x}_i} \mathscr{L}) = \partial_{x_i} \mathscr{L}$$

Systems often use generalized coordinates q_i.

The Hamiltonian Formalism

In the Lagrangian formalism, the independent variables are generalized coordinates. Momentum is a derived quantity that we define as

$$p_i = \frac{\partial \mathscr{L}}{\dot{q}_i}$$

The Hamiltonian formalism is similar, but exchanges the role of $\dot{q}$ and p. Instead of $\mathscr{L}(q, \dot{q})$ we yield $\mathcal{H}(q, p)$ which generates the equations of motion and $\dot{q}$ thus becomes the derived quantity as

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

We can effect such a change with the Legendre Transform. Simply, given

$$u(x) = d_x f$$

We can define a function

$$g(u) = x(u)u - f(x(u))$$

thus giving us

$$d_u g = d_u x \cdot u + x(u) - d_x f \cdot d_u x = x(u)$$

We can generalize this given $f = f(x_1, x_2, \ldots, x_n)$ as

$$g(u_1, \ldots, u_j, x_{j+1}, \ldots, x_n) = \sum_{j=1}^{j} \partial_{x_i} f x_i - f(x_1, \ldots, x_n)$$

We see that

$$\partial_{u_i} g = x_i$$

which will become useful in a second.

The Lagrangian formalism describes the state of a system as a path in n-dimensional configuration space. Conversely, the Hamiltonian mechanics describes the state as a point in $2n$-dimensional phase space with the coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$. For a given $\mathcal{H}$, only one path passes through a given point in phase space.

With all this in mind, we can apply the Legendre transform to the Lagrangian to obtain the "corresponding" Hamiltonian

$$\mathcal{H}(q,p) = \sum_i q_i p_i - \mathscr{L}(q_i, \dot{q}_i)$$

Hamilton's canonical equations are

$$\frac{\partial \mathcal{H}}{p_i} = \dot{q}_i$$

and

$$-\frac{\partial \mathcal{H}}{\partial q_i} = \dot{p}_i$$

A relation we will use later is that given

$$T = \frac{1}{2} m \dot{x}^2,\ V = V(x)$$

We get the canonical momentum

$$p = \partial_{\dot{x}} \mathscr{L} = m\dot{x}$$

Which we can invert to $x_i(p)$ (instead of $p(x_i)$) as

$$\dot{x} = \frac{p}{m}$$

Which yields

$$\mathcal{H}(x,p) = \frac{p^2}{2m} + V(x)$$

Cyclic Coordinates, Poisson Brackets and Canonical Transformation

Not done.

All Is Not Well with Classical Mechanics

Particles and Waves in Classical Physics

The double-slit experiment with photons can be summarized as follows:

Every incoming photon of energy E and momentum p has a wave function Ψ associated with it, which is a plane wave with $\omega = \frac{E}{\hbar}$ and $k = \frac{p}{\hbar}$. This wave interferes with itself and forms the oscillating pattern $|\Psi(x)|^2$ along the screen, which gives the probability that the given photon will arrive at x. A

given photon arrives at some definite x and does not reveal the probability distribution, however, if we wait till several photons described with the same Ψ have arrived, the number at any x will become proportional to the probability function.

Similarly, if any intense monochromatic beam is incident, many photons, all described by the same wave function, arrive at the same time and all along the screen. Their intensity intensity distribution then assumes the shape of the probability distribution right away.

A wave is associated with *each* photon, not a beam of photons. A monochromatic beam is one where every photon has the same Ψ and thus same probability distribution.

Classical electromagnetic wave theory predictions fall in line with such monochromatic beams and hides the probability associated with each photon, but discretized light reveals this truth.

Matter Waves (de Broglie Waves)

De Broglie conjectured that any particle with p and x would exhibit wave-like behavior. This was verified for electrons by Davisson and Germer experimentally.

This includes billiard balls and baseballs! The reason we do not notice this, is because the probability amplitudes are so small for such macroscopic objects. For example, the wave-length associated with a 1 gram pellet would be about 10^{-26} cm - clearly unnoticeable.

The Postulates of Quantum Mechanics

Postulates of Non-Relativistic Quantum Mechanics

1. The state of a particle is given by a vector $|\Psi\rangle$ in a Hilbert space.
2. The *independent* variables x and p are given by Hermitian operators $\hat{x}$ and $\hat{p}$ with the following elements in the Eigenbasis of $\hat{x}$

   $$\langle x|\hat{x}|x_0\rangle = x\delta(x-x')$$
   $$\langle x|\hat{p}|x_0\rangle = -i\hbar\delta'(x-x')$$

   The operators corresponding to dependent variables $\omega(x,p)$ are given Hermitian operators

   $$\Omega(\hat{x},\hat{p}) = \omega(x\to\hat{x}, p\to\hat{p})$$

3. If the particle is in a state $|\Psi\rangle$, measurement of the variable corresponding to Ω will yield one of the eigenvalues ω with a probability

   $$P(\omega) \propto |\langle\omega|\psi\rangle|^2$$

   The state of the system will "collapse" to $|\omega\rangle$ after measurement/interaction.
4. A state vector $|\Psi\rangle$ obeys the Schrödinger Equation (SE)

$$i\hbar\partial_t|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

Where $\hat{H}(\hat{x},\hat{p}) = \mathcal{H}(x \to \hat{x}, p \to \hat{p})$ is the "quantum" Hamiltonian operator.