We shall consider two different models for the earth magnetic field.

**Dipole approximation**

The magnetic field is given by

$$\vec{B}\left(\vec{r}\right)=\frac{\mu_0}{4\pi}\left[\frac{3\left(\vec{m}\cdot\vec{s}\right)\vec{s}}{s^5}-\frac{\vec{m}}{s^3}\right]\quad \text{with}\quad \vec{s}=\vec{r}-\vec{R}$$

Here $\vec{r}$ is the distance between the ISS and the center of the Earth, $\vec{m}$ is the Earth's magnetic dipole and $\vec{R}$ is the distance between the center of the Earth and the placement of the magnetic dipole that best fits the data. These two vectors form a set of 6 parameters to be determined.

**Multipolar expansion**

We represent the magnetic field as

$$\vec{B}=-\vec{\nabla}V=-\left(\frac{\partial V}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial V}{\partial\phi}\hat{e}_\phi+\frac{1}{r\cos\phi}\frac{\partial V}{\partial\theta}\hat{e}_\theta\right)$$

where

$$V=a\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left[g_n^m\cos\left(m\theta\right)+h_n^m\sin\left(m\theta\right)\right]P_n^m\left(\sin\phi\right)$$

Here we follow the World Magnetic Model [1], with $a=6,3712\times10^6$ m, $\phi$ is latitude and $\theta$ is longitude. The $g_n^m$ and $h_n^m$ are the set of $N\left(N+3\right)$ parameters ($N$ being the order of the expansion) to be determined to fit the data. In order to convert these expressions to cartesian coordinates we have to use

$$\begin{aligned}
\hat{e}_r &= \cos\phi\cos\theta\hat{e}_x+\cos\phi\sin\theta\hat{e}_y+\sin\phi\hat{e}_z\\
\hat{e}_\phi &= -\sin\phi\cos\theta\hat{e}_x-\sin\phi\sin\theta\hat{e}_y+\cos\phi\hat{e}_z\\
\hat{e}_\theta &= -\sin\theta\hat{e}_x+\cos\theta\hat{e}_y
\end{aligned}$$

The $P_n^m$ are the associated Legendre polynomials with the normalisation used in the WMM ([1] and [2], Ap.B), e.g., up to order 3:

$$\begin{aligned}
V &= \frac{a^3}{r^2}\left\{g_1^0\sin\phi+\left[g_1^1\cos\left(\theta\right)+h_1^1\sin\left(\theta\right)\right]\cos\phi\right\}+\\
&+\frac{a^4}{r^3}\left\{g_2^0\left(1-\frac{3}{2}\cos^2\phi\right)+\sqrt{3}\left[g_2^1\cos\left(\theta\right)+h_2^1\sin\left(\theta\right)\right]\sin\phi\cos\phi+\frac{\sqrt{3}}{2}\left[g_2^2\cos\left(2\theta\right)+h_2^2\sin\left(2\theta\right)\right]\cos^2\phi\right.\\
&+\frac{a^5}{r^4}\left\{g_3^0\sin\phi\left(1-\frac{5}{2}\cos^2\phi\right)+\sqrt{\frac{3}{2}}\left[g_3^1\cos\left(\theta\right)+h_3^1\sin\left(\theta\right)\right]\cos\phi\left(2-\frac{5}{2}\cos^2\phi\right)+\right.\\
&\left.+\sqrt{\frac{15}{2}}\left[g_3^2\cos\left(2\theta\right)+h_3^2\sin\left(2\theta\right)\right]\sin\phi\cos^2\phi+\sqrt{\frac{15}{8}}\left[g_3^3\cos\left(3\theta\right)+h_3^3\sin\left(3\theta\right)\right]\cos^3\phi\right\}
\end{aligned}$$

### The position coordinates

We have acess to the ISS position coordinates in the form of altitude $h$, latitude $\phi$ and longitude $\theta$, which can be converted to the ECEF frame by

$$\begin{aligned} x &= [h + s(\phi)] \cos\phi \cos\theta \\ y &= [h + s(\phi)] \cos\phi \sin\theta \\ z &= [h + s(\phi)] \sin\phi \end{aligned}$$

where

$$s(\phi) = \frac{A^2}{\sqrt{A^2 \cos^2\phi + B^2 \sin^2\phi}}$$

is the equation for the reference elipsoid with

$$A = 6378137 \text{ m} \qquad \text{and} \qquad B = 6356752.314245 \text{ m}$$

### Conversion to the LVLH reference frame

Now we have to convert the expected magnetic field, which is expressed in an ECEF reference frame to the LVLH reference frame of the ISS. Denoting by capital letters the latter reference frame, we can in a fairly good approximation write

$$\begin{aligned} \hat{e}_X &= \frac{\vec{r}_i \times (\vec{r}_i \times \vec{r}_{i-1})}{r_i \|\vec{r}_i \times \vec{r}_{i-1}\|} = \frac{(\vec{r}_i \cdot \vec{r}_{i-1})\,\vec{r}_i - r_i^2 \vec{r}_{i-1}}{r_i \|\vec{r}_i \times \vec{r}_{i-1}\|} \\ \hat{e}_Y &= \frac{\vec{r}_i}{r_i} \\ \hat{e}_Z &= \frac{\vec{r}_i \times \vec{r}_{i-1}}{\|\vec{r}_i \times \vec{r}_{i-1}\|} \end{aligned}$$

where $\vec{r}_i$ is the position of the ISS at a certain instant $t$ and $\vec{r}_{i-1}$ is its position at the instant imediatly before.

Hence the components of the magnetic field in the LVLH reference frame are

$$\begin{aligned} B_X &= \frac{(\vec{r}_i \cdot \vec{r}_{i-1}) \left(\vec{B}_i \cdot \vec{r}_i\right) - r_i^2 \left(\vec{B}_i \cdot \vec{r}_{i-1}\right)}{r_i \|\vec{r}_i \times \vec{r}_{i-1}\|} \\ B_Y &= \frac{\vec{B}_i \cdot \vec{r}_i}{r_i} \\ B_Z &= \frac{\vec{B}_i \cdot (\vec{r}_i \times \vec{r}_{i-1})}{\|\vec{r}_i \times \vec{r}_{i-1}\|} \end{aligned}$$

### Conversion to the raspberry-pi frame

In order to compare the predicted values of the magnetic field with the measured ones we have now to convert to the raspberry-pi frame. But since the raspberry-pi is going to be fixed with respect to the ISS, this transformation is simply given by a rotation matrix:

$$\begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix} \begin{bmatrix} B_X \\ B_Y \\ B_Z \end{bmatrix}$$

In the end, for each instant $i$ our model predicts a magnetic field $(B_a, B_b, B_c)$ which depends on the parametes of the model (the $\vec{m}$ and $\vec{R}$ in the displaced dipole model and the $g_n^m$ and $h_n^m$ in the centered multipole model) plus $\alpha$, $\beta$ and $\gamma$, which we have no way to know since we do not know the exact placement of the raspberry-pi inside the ISS.

**Determination of the parameters**

Finaly we want to determine the parameters by the method of least squares, that is we want to minimize

$$\sum_i \left[ (B_a^{\text{exp}} - B_a)^2 + (B_b^{\text{exp}} - B_b)^2 + (B_c^{\text{exp}} - B_c)^2 \right]$$

where $B_a^{\text{exp}}$, $B_b^{\text{exp}}$ and $B_c^{\text{exp}}$ are the measured values of the magnetic field.

**References**

[1] P. Alken et all, *International Geomagnetic Reference Field: the thirteenth generation*, Earth Planets and Space, **73**, 49 (2021)

[2] R. Merrill, M. McElhinny and P. McFadde, *The magnetic Field of the Earth*, Academic Press (1996)