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By **interpolation** we understand the following problem:

Given:
- An interval $I \subset \mathbb{R}$,
- A set F of functions $f : I \to \mathbb{R}$,
- Pairwise distinct points $x_i \in I$ and values f_i, $i = 0, \ldots, n$.

Find:
- $f \in F$ such that $f(x_i) = f_i$ for $i = 0, \ldots, n$.

- For interpolation we require the strict equality $f(x_i) = f_i$ at each point whereas for an approximation problem we require $f(x_i) \approx f_i$ only.
- Typical questions about interpolation problems are existence, uniqueness and effective computation.

Polynomial Interpolation

- We take F as the set of polynomials of degree at most n.
- The simplest case is linear interpolation.

$$p(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) = f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$$

- The first expression is the prototype of Newton's form of the interpolation polynomial.
- The second that of Lagrange's form.

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Definition 5.1.1 (Lagrange Polynomials)

For pairwise different points $x_i \in \mathbb{R}$, $i = 0, \ldots, n$ we define the **Lagrange polynomials** as:

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \qquad i = 0, 1, \ldots n.$$

Observe that L_i is a polynomial of degree n and satisfies

$$L_i(x_k) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

This property implies that the polynomial

$$p(x) := \sum_{i=0}^{n} f_i L_i(x)$$

satisfies the interpolation condition

$$p(x_k) = f_k, \qquad k = 0, \ldots, n.$$

Example

Let the following values be given:

x_i	1	2	3	4
f_i	6	5	2	-9

Then the corresponding interpolating polynomial can be defined as

$$\begin{aligned} p_3(x) &= 6L_0(x) + 5L_1(x) + 2L_2(x) - 9L_3(x) \\ &= 6\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 5\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \\ &\quad +2\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} - 9\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \\ &= -x^3 + 5x^2 - 9x + 11 \end{aligned}$$

Theorem 5.1.2 (Existence and Uniqueness for Polynomial Interpolation)

There is exactly one polynomial p of degree atmost n which passes through the data points (x_i, f_i), $0 \leq i \leq n$ (with pairwise different x_i).

Proof.

- Existence: Is proved by the explicit formula

$$p(x) := \sum_{i=0}^{n} f_i L_i(x)$$

  as shown above.

- Uniqueness:
  - Assume there is another interpolating polynomial q of degree atmost n.
  - Then the difference $d := p - q$ is a polynomial of degree atmost n with $n+1$ zeros, namely x_i.
  - But by a theorem of Calculus this implies that d is the zero-polynomial, hence $p = q$.

□

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Theorem 5.2.1

Conditions:

- *Let f be a function in $C^{n+1}[a, b]$.*
- *Let p be a polynomial of degree $\leq n$ that interpolates the function f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in the interval $[a, b]$.*

$$\Rightarrow$$

To each x in $[a, b]$ there corresponds a point ξ_x in (a, b) such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

Example

If the function $f(x) = \sin(x)$ is approximated by a polynomial of degree 9 that interpolates f at 10 points in the in the interval $[0, 1]$, how large is the error on this interval?

Obviously:

$$|f^{(10)}(\xi_x)| \leq 1 \qquad \text{and} \qquad \prod_{i=0}^{9} |x - x_i| \leq 1.$$

So for all $x \in [0, 1]$

$$|\sin x - p(x)| \leq \frac{1}{10!} < 2.8 \cdot 10^{-7}.$$

Example: Linear interpolation ($n = 1$)

Let $[a, b] = [x_0, x_1]$ and $h = x_1 - x_0$. Then

$$f(x) - p(x) = \frac{1}{2!} f''(\xi_x)(x - x_0)(x - x_1), \qquad x_0 < \xi_x < x_1$$

and an computation gives

$$\|f - p\|_\infty := \sup\{|f(x) - p(x) \mid x \in [a, b]\} \leq \frac{M_2}{8} h^2 \quad \text{with } M_2 = \|f''\|_\infty.$$

Example: Quadratic interpolation ($n = 2$)

Equally spaced points

$$a = x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h = b.$$

Now we have for $x \in [x_0, x_2]$

$$f(x) - p(x) = \frac{1}{3!} f'''(\xi_x)(x - x_0)(x - x_1)(x - x_2), \qquad x_0 < \xi_x < x_2$$

and hence

$$\|f - p\|_\infty \leq \frac{M_3}{9\sqrt{3}} h^3, \quad \text{with } M_3 = \|f'''\|_\infty.$$

If interpolating polynomials p_n of higher and higher degree are constructed, we might expect that these polynomials will converge to f uniformly, that is

$$\|f - p_n\|_\infty \to 0 \qquad \text{as} \qquad n \to \infty.$$

For many continous functions this is not true:

Example: Runge, 1901

Let

$$f(x) = \frac{1}{1+x^2}\,, \qquad -5 \le x \le 5$$

and

$$x_k^{(n)} = -5 + k\frac{10}{n}, \qquad k = 0, 1, 2\ldots, n$$

equally spaced nodes. Then one can prove that

$$\lim_{n\to\infty} |f(x) - p_n(x)| = \begin{cases} 0 & \text{if} \quad |x| < 3.633\ldots \\ \infty & \text{if} \quad |x| > 3.633\ldots \end{cases}$$

Therefore

$$\|f - p_n\|_\infty \to \infty \qquad \text{as} \qquad n \to \infty.$$

One may wonder if the equidistribution of the nodes can be blamed for the non-convergence.

One can prove the following result:

Theorem 5.2.2

If f is a continous function on $[a, b]$, then there is a system of nodes such that the interpolation polynomials p_n satisfy

$$\|f - p_n\|_\infty \to 0 \qquad as \qquad n \to \infty.$$

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- We derive another form of the interpolating polynomial which is particularly useful if the nodes are extended during the computation.
  - For Example if the computed result is not yet satisfactory

- Instead of the lagrange polynomials we use the following basis functions for given pairwise different points $x_i \in \mathbb{R}$, $i = 0, \dots, n$:

$$Q_k(x) := \prod_{i=0}^{k-1} (x - x_i) = (x - x_0) \cdots (x - x_{k-1})$$

  (An empty product is set to be one. $Q_0(x) = 1$)

- We want to solve the interpolation problem by the polynomial

$$p(x) := \sum_{k=0}^{n} a_k Q_k(x).$$

- We have to determine the coefficients from the system of equations:

$$f_i = p(x_i) = \sum_{k=0}^{n} a_k Q_k(x_i) = \sum_{k=0}^{i} a_k Q_k(x_i), \qquad i = 0, \ldots, n.$$

- The last equality is a consequence of the particular definition of the Q_k: $Q_k(x_i) = 0$ if $k > i$.

- Hence this system is triangular and can be solved directly by forward substitution.

There is an even more practical way to compute these coefficients by using Divided Differences.

Definition 5.3.1 (Divided Differences, Newton Form)

Let $x_i \in \mathbb{R}$ be pairwise different points and $f_i \in \mathbb{R}$, $i = 0, \dots, n$.

- For any subset $T \subseteq \{x_0, \dots, x_n\}$ we denote by $f[T]$ the leading coefficient of the interpolating polynomial p with $p(t) = f_i$ for $t = x_i \in T$.
- These values are called **divided differences**.
- By the interpolating polynomial in **Newton form** we understand the representation:

$$p(x) := \sum_{k=0}^{n} f[x_0, \dots, x_k] Q_k(x).$$

- It follows inductively that the partial polynomials

$$p_m(x) := \sum_{k=0}^{m} f[x_0, \ldots, x_k] Q_k(x)$$

  satisfy $p_m(x_i) = f_i$ for $i = 0, \ldots, m$.
- This means that the interpolation polynomial $p(x)$ in Newtons form from 1.3.1 solves the interpolation problem.

To compute the coefficients we have the following theorem which also justifies the notation divided differences:

Theorem 5.3.2

The divided differences satisfy the recursion formula

$$f[t_0, t_1, \ldots, t_m] = \frac{f[t_1, \ldots, t_m] - f[t_0, \ldots, t_{m-1}]}{t_m - t_0}.$$

Proof.

1. Define the following polynomials:

$$\begin{aligned} p &: \text{ polynomial interpolating at } t = t_0, t_1, \ldots, t_m \\ q &: \text{ polynomial interpolating at } t = t_0, \ldots, t_{m-1} \\ r &: \text{ polynomial interpolating at } t = t_1, \ldots, t_m \end{aligned}$$

2. Then we obtain the equation

$$p(x) = r(x) + \frac{x - t_m}{t_m - t_0}(r(x) - q(x)) =: h(x).$$

3. The polynomial h has degree at most m.

4. Further we have $h(t_i) = r(t_i)$ for $i = 1, \ldots m$ and $h(t_0) = q(t_0)$, so in any case $h(t_i) = p(t_i)$.

5. By the uniqeness of the interpolating polynomial we conclude: $p = h$.
6. Now equating the leading coefficients of p and h we obtain the theorem.

$\square$

If a table of function values $(x_i, f(x_i))$ is given, we can construct from it a table of divided differences.

$$\begin{array}{cc|ccc}
x_0 & f_0 & f[x_0, x_1] & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] \\
x_1 & f_1 & f[x_1, x_2] & f[x_1, x_2, x_3] & \\
x_2 & f_2 & f[x_2, x_3] & & \\
x_3 & f_3 & & &
\end{array}$$

Example

Compute a divided difference table for these function values:

x	3	1	5	6
$f(x)$	1	-3	2	4

We have

x_i	f_i			
3	1	2	$-\frac{3}{8}$	$\frac{7}{40}$
1	-3	$\frac{5}{4}$	$\frac{3}{20}$	
5	2	2		
6	4			

The corresponding Newton polynomial is therefore

$$p(x) = 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) + \frac{7}{40}(x-3)(x-1)(x-5)$$

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Definition 5.4.1

A function $S : [a, b] \to \mathbb{R}$ is called a **spline function** of degree $k \geq 1$ with respect to a subdivision Δ, if

- $S_{|_{[x_{i-1}, x_i]}}$ $(i = 1, \ldots, n)$ is a polynomial of degree k,
- $S \in C^{(k-1)}([a, b])$.

- A spline function consists of polynomial pieces on subintervals joint together with certain continuity conditions.

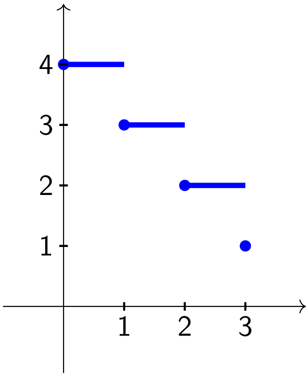
- Let

  $$\Delta : \quad a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

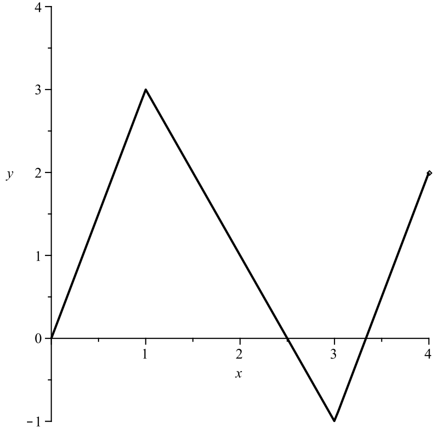
  be a subdivision of the interval $[a, b]$. The points x_i are called **knots**.

- Hence a spline function is a piecewise polynomial of degree k having continous derivatives of all orders up to $k - 1$.

A spline of degree 0 is a piecewise constant function.



A spline of degree 1 is a piecewise linear function.



The most widely used splines in practise are cubic splines, that is $k = 3$. This means that each

$$p_i := S_{|_{[x_{i-1}, x_i]}}, \qquad 1 \leq i \leq n$$

is a polynomial of degree 3.

We now show how to compute an interpolating cubic spline.

- We assume that a subdivision Δ and values f_i, $i = 0, \ldots, n$ are given.
- We look for a cubic spline function satisfying the interpolation condition:

$$S(x_i) = f_i, \quad i = 0, \ldots, n$$

1. We define numbers $z_i = S''(x_i),\ 0 \leq i \leq n$.
   These numbers are well defined because S is twice continously differentiable.

2. Since each p_i is a polynomial of degree 3, p_i'' is a linear function:

$$p_i''(x) = z_{i-1} + \frac{z_i - z_{i-1}}{h_i}(x - x_{i-1}) \qquad \text{with} \quad h_i := x_i - x_{i-1}.$$

3. If this is integrated twice we obtain

$$p_i(x) = a_i(x - x_{i-1})^3 + b_i(x - x_{i-1})^2 + c_i(x - x_{i-1}) + d_i$$

4. The coefficients a_i and b_i are derived from p_i''.

5. The interpolation condition implies: $d_i = f_{i-1}$.

6. We have to satisfy:

$$f_i = S(x_i) = p_i(x_i) = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i$$

Solving for c_i we obtain:

$$c_i = \frac{f_i - f_{i-1}}{h_i} - a_i h_i^2 - b_i h_i$$

Collecting these information we get:

$$\boxed{\begin{aligned} a_i &= \frac{1}{6h_i}(z_i - z_{i-1}) \\ b_i &= \frac{1}{2} z_{i-1} \\ c_i &= \frac{f_i - f_{i-1}}{h_i} - \frac{1}{6} h_i (2z_{i-1} + z_i) \\ d_i &= f_{i-1} \end{aligned}}$$

7. We have to assure that at the inner knots the left and right values of the first derivative coincide. Hence we have to satisfy:

$$p_i'(x_i) = p_{i+1}'(x_i) \Leftrightarrow 3a_i h_i^2 + 2b_i h_i + c_i = c_{i+1}, \quad i = 1, \ldots, n-1.$$

8. By inserting the above expressions in the box and sorting for the unknowns z_i:

$$h_i z_{i-1} + 2(h_i + h_{i+1}) z_i + h_{i+1} z_{i+1} = 6 \left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i} \right) \qquad i = 1, \ldots, n-1.$$

These are $n-1$ equations in the $n+1$ unknowns $z_0, \ldots, z_n$.

9. We can select the values of z_0 and z_n arbitrarily and solve the remaining equations.

There are several systematic ways to determine z_0 and z_n by additional conditions:

1. One simply requires: $z_0 = 0$ and $z_n = 0$. The resulting spline is called a **natural spline**.

2. One prescribes values $S'(x_0)$ and $S'(x_n)$.

3. One prescribes additional function values $S(0.5(x_0 + x_1))$ and $S(0.5(x_{n-1} + x_n))$.

4. One requires that S can be continued periodically.

In the case of a natural spline the linear system is of the form

$$\begin{pmatrix} u_1 & h_2 & & & & \\ h_2 & u_2 & h_3 & & & \\ & h_3 & u_3 & h_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & u_{n-2} & h_{n-1} \\ & & & & h_{n-1} & u_{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{pmatrix}$$

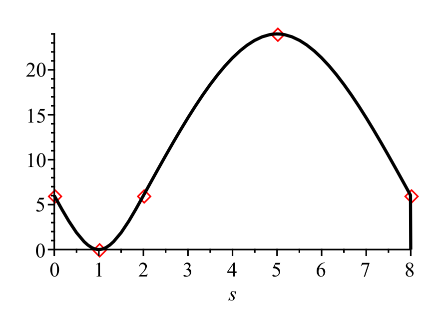
where

$$\boxed{\begin{aligned} h_i &= x_i - x_{i-1} \\ u_i &= 2(h_i + h_{i+1}) \\ w_i &= \frac{6}{h_i}(f_i - f_{i-1}) \\ v_i &= w_{i+1} - w_i \end{aligned}}$$

Example

Determine the interpolating cubic natural spline for the following data:

x	0	1	2	5	8
f	6	0	6	24	6



The computed interpolating spline

To set up the system of equations to determine the z_i we may collect the relevant values in the following table:

i	x_i	h_i	u_i	f_i	$\frac{w_i}{6}$	$\frac{v_i}{6}$
0	0			6		
1	1	1	4	0	-6	12
2	2	1	8	6	6	0
3	5	3	12	24	6	-12
4	8	3		6	-6	

From this we obtain the system of equations:

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 8 & 3 \\ 0 & 3 & 12 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 72 \\ 0 \\ -72 \end{pmatrix}$$

with the solution (extended by $z_0 = 0$ and $z_4 = 0$):

$$z = (0, 18, 0, -6, 0).$$

From this one computes the following representation of the poynomial-pieces:

$$\begin{aligned} p_1(x) &= 3x^3 - 9x + 6, \quad \text{in } [0,1] \\ p_2(x) &= -3(x-1)^3 + 9(x-1)^2, \quad \text{in } [1,2] \\ p_3(x) &= -\frac{1}{3}(x-2)^3 + 9(x-2) + 6, \quad \text{in } [2,5] \\ p_4(x) &= \frac{1}{3}(x-5)^3 - 3(x-5)^2 + 24, \quad \text{in } [5,8] \end{aligned}$$