数学分析作业

第3章第1节

1.(6)

$$\forall\varepsilon\in(0,1),\exists X(\varepsilon)=\ln\frac{1}{\varepsilon},\forall x>X:|e^{-x}-0|<\varepsilon\Rightarrow\lim_{x\to\infty}e^{-x}=0$$

1.(8)

$$\forall G>0,\exists X(G)=\min\{-G,-2\},\forall x<1-G:\frac{x^2}{x+1}<\frac{x^2-1}{x+1}=x-1<-G$$
$$\Rightarrow\lim_{x\to-\infty}\frac{x^2}{x+1}=-\infty$$

2.(10)

$$\lim_{x\to0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2\cos x}$$
$$\lim_{x\to0}\frac{x^2\cos x}{1-\cos x}=\lim_{x\to0}(\frac{x^2}{2\sin^2\frac{x}{2}}-x^2)=2$$
$$\lim_{x\to0}\frac{\tan x-\sin x}{x^3}=\frac{1}{2}$$

3.(2)

$$\sqrt{x}>\ln x$$
$$x^{\frac{1}{x}}\geq1$$
$$x^{\frac{1}{x}}=e^{\frac{\ln x}{x}}\leq e^{\frac{1}{\sqrt{x}}}$$
$$\lim_{x\to\infty}e^{\frac{1}{\sqrt{x}}}=1$$
$$\lim_{x\to\infty}x^{\frac{1}{x}}=1$$

4.(2)

$$\text{当}x>e^{(k+1)^2}\text{时},k+1<\sqrt{\ln x}=\frac{\ln x}{\sqrt{\ln x}}<\frac{\ln x}{\ln\ln x}\Rightarrow(k+1)\ln\ln x<\ln x\Rightarrow\ln^{k+1}x<x$$
$$\text{因此当}x>e^{(k+1)^2}\text{时}\frac{\ln^k x}{x}<\frac{1}{\ln x}\to0(x\to\infty)$$
$$\text{又因为当}x>2\text{时}\frac{\ln^k x}{x}>0,\text{因此}\lim_{x\to\infty}\frac{\ln^k x}{x}=0$$

5.(2)

$$\forall\varepsilon>0,\exists\sigma=\frac{1}{\log_2(\frac{2+\varepsilon}{\varepsilon})},\forall0<x<\sigma:|\frac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}-1|=|\frac{2}{2^{\frac{1}{x}}-1}|<\varepsilon$$
$$\lim_{x\to0^+}\frac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}=1$$

$$\forall\varepsilon>0,\exists\sigma=\frac{1}{\log_2(\frac{2+\varepsilon}{\varepsilon})},\forall-\sigma<x<0:|\frac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}-(-1)|=|\frac{2(2^{\frac{1}{x}})}{2^{\frac{1}{x}}-1}|<\varepsilon$$
$$\lim_{x\to0^-}\frac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}=-1$$

6.(1)

$$\forall\varepsilon>0,\exists X(\varepsilon)=\frac{1}{\varepsilon},\forall|x|>X(\varepsilon):|\frac{\sin x}{x}-0|<|\frac{1}{x}|<\varepsilon$$
$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$

6.(4)

$$\forall G>0,\exists X(G)=\max\{1+\ln G,10\},\forall x>X(G):(1+\frac{1}{x})^{x^2}=((1+\frac{1}{x})^{x+1})^{\frac{x^2}{x+1}}>e^{\frac{x^2}{x+1}}>e^{x-1}>G$$
$$\lim_{x\to\infty}(1+\frac{1}{x})^{x^2}=+\infty$$

6.(6)

$$设a_n=\frac{1}{n+\frac{1}{2}}\to0^+(n\to\infty),b_n=\frac{1}{n+\frac{1}{3}}\to0^+(n\to\infty).$$
$$A=\lim_{n\to\infty}(\frac{1}{a_n}-\left[\frac{1}{a_n}\right])=\lim_{n\to\infty}(n+\frac{1}{2}-n)=\frac{1}{2}$$
$$B=\lim_{n\to\infty}(\frac{1}{b_n}-\left[\frac{1}{b_n}\right])=\lim_{n\to\infty}(n+\frac{1}{3}-n)=\frac{1}{3}$$
$$A\neq B\Rightarrow 极限不存在.$$

10.(2)

$$\exists\varepsilon>0,\forall N\in\mathbb{N}^+,\exists n>N:|x_n|>\varepsilon$$

10.(4)

$$\exists G>0,\forall N\in\mathbb{N}^+,\exists n>N:x_n<G$$

10.(6)

$$\exists G>0,\forall X\in\mathbb{R},\exists x>X:f(x)>-G$$

13.

必要性:

$$\forall\varepsilon>0,\exists X(\varepsilon),\forall x>X(\varepsilon):|f(x)-A|<\varepsilon$$
$$\forall G>0,\exists N(G),\forall n>N(G):x_n>G$$
$$\Rightarrow\forall\varepsilon>0,\exists N_1(\varepsilon)=N(X(\varepsilon)),\forall n>N_1(\varepsilon):x_n>X(\varepsilon),|f(x_n)-A|<\varepsilon$$

充分性:

用反证法证明.

$$假设\exists\varepsilon>0,\forall X,\exists x>X:|f(x)-A|>\varepsilon$$
$$取x_1=1,可以找到一个x_n>x_{n-1}+1,使得|f(x_n)-A|>\varepsilon_0,此时\{f(x_n)\}不收敛.$$
$$矛盾,假设不成立,因此\lim_{x\to+\infty}f(x)=A存在.$$

14.(2)

$$\lim_{x\to x_0^+} f(x)\text{存在} \Leftrightarrow \forall\varepsilon>0,\exists\sigma_2(\varepsilon),\forall x_0<x_1,x_2<x_0+\sigma_2(\varepsilon):|f(x_1)-f(x_2)|\le\varepsilon$$

必要性:

$$\begin{gathered}\exists A\in\mathbb{R},\forall\varepsilon>0,\exists\sigma_1(\varepsilon),\forall x_0<x<x_0+\sigma_1(\varepsilon):|f(x)-A|<\varepsilon\\ \Rightarrow|f(x_1)-f(x_2)|=|f(x_1)-A+A-f(x_2)|<|f(x_1)-A|+|f(x_2)-A|<2\varepsilon\\ \Rightarrow\forall\varepsilon>0,\exists\sigma_2(\varepsilon)=\sigma_1(\frac{\varepsilon}{2}),\forall x_0<x_1,x_2<x_0+\sigma_2(\varepsilon):|f(x_1)-f(x_2)|\le\varepsilon\end{gathered}$$

充分性:

$$\begin{gathered}\text{对于任意数列}\{x_n\},x_n\in D,x_n\to x_0^+(n\to\infty)\\ \forall\varepsilon>0,\exists N(\varepsilon),\forall n_1,n_2>N:|f(x_{n_1})-f(x_{n_2})|<\varepsilon\\ \Rightarrow\{f(x_n)\}\text{是基本数列,即}\{f(x_n)\}\text{收敛}\Rightarrow\lim_{x\to x_0^+}f(x)\text{存在}\end{gathered}$$

15.

反证法.

$$\begin{gathered}\text{假设}f(x_0)=B\ne A,\text{则}f(2^n x_0)=B.\\ \Rightarrow\exists\varepsilon=|A-B|,\forall X\in(0,+\infty),\exists x=2^{\lceil\log_2\frac{X}{x_0}\rceil}x_0:f(x)=B,|f(x)-A|\ge\varepsilon\\ \text{与}\lim_{x\to+\infty}f(x)=A\text{矛盾,假设不成立,因此}f(x)\equiv A,x\in(0,+\infty)\end{gathered}$$

第3章第2节

1.(2)

在$(0,+\infty)$

$$\begin{gathered}\forall x_0\in(0,+\infty),\forall\varepsilon>0,\exists\sigma=\varepsilon\sqrt{x_0},\forall x(x>0,0<|x-x_0|<\sigma):|\sqrt{x}-\sqrt{x_0}|=\frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}}<\frac{|x-x_0|}{\sqrt{x_0}}<\varepsilon\\ \lim_{x\to x_0}\sqrt{x}=\sqrt{x}\\ \text{当}x_0=0,\forall\varepsilon>0,\exists\sigma=\varepsilon^2,\forall x(0<x<\sigma):|\sqrt{x}-0|<\sqrt{\sigma}=\varepsilon\\ \lim_{x\to0^+}\sqrt{x}=\sqrt{0}\\ \text{综上},y=\sqrt{x}\text{在定义域内连续.}\end{gathered}$$

1.(3)

$$
\begin{aligned}
|\frac{\sin x}{x}-\frac{\sin x_0}{x_0}| &= |\frac{x_0\sin x - x\sin x_0}{xx_0}| \\
&= |\frac{x_0\sin x - x_0\sin x_0 + x_0\sin x_0 - x\sin x_0}{xx_0}| \\
&= |\frac{x_0(\sin x-\sin x_0)+(x_0-x)\sin x_0}{xx_0}| \\
&\le |\frac{2x_0\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2}}{xx_0}| + |\frac{(x_0-x)\sin x_0}{xx_0}| \\
&\le |1-\frac{x_0}{x}|(2+\frac{|\sin x_0|}{x_0})
\end{aligned}
$$

$$
\forall x_0\in(0,+\infty),\forall\varepsilon>0,\exists\sigma=\min\{\frac{x_0}{1-\frac{\varepsilon}{2+\frac{|\sin x_0|}{x_0}}}-x_0, x_0-\frac{x_0}{1+\frac{\varepsilon}{2+\frac{|\sin x_0|}{x_0}}}\},\forall x(|x-x_0|<\sigma):|\frac{\sin x}{x}-\frac{\sin x_0}{x_0}|<\varepsilon
$$

$$
\text{当}x_0=0,\lim_{x\to0}\frac{\sin x}{x}=1
$$

$$
\text{综上},y\text{在定义域内连续.}
$$

2.(2)

$(2k\pi-\frac{\pi}{2},2k\pi+\frac{\pi}{2}),k\in\mathbb{Z}$

2.(4)

$(-1,1)\cup[n,n+1),n\in\mathbb{N}^+$

2.(6)

$(2k\pi,2k\pi+\pi),k\in\mathbb{Z}$

4.

(1) 不能. 反例:$x_0=0,f(x)=x,g(x)=\begin{cases}0,x=0\\1,x\neq0\end{cases}$

(2) 不能. 反例:$x_0=0,f(x)=\begin{cases}x,x\neq0\\1,x=0\end{cases},g(x)=\begin{cases}\frac{1}{x},x\neq0\\1,x=0\end{cases}$

6.(1)

$$
\forall x_0\in(a,b),\exists\delta=\min\{\frac{x_0-a}{2},\frac{b-x_0}{2}\}:x_0\in(a+\delta,b-\delta)\subset[a+\delta,b-\delta]
$$

$$
\text{因为}f\text{在}[a+\delta,b-\delta]\text{单调,所以}\lim_{x\to x_0}f(x)=f(x_0).
$$

$$
\text{即}\forall x_0\in(a,b):\lim_{x\to x_0}f(x)=f(x_0),\text{即}f(x)\text{在}(a,b)\text{连续.}
$$

6.(2)

不能.反例:$a=1,b=2,f(x)=\begin{cases}x,x\in(1,2)\\0,x=1\text{或}x=2\end{cases}$

7.

$$
\lim_{x\to x_0}f(x)^{g(x)}=\lim_{x\to x_0}e^{g(x)\ln f(x)}=e^{\lim\limits_{x\to x_0}g(x)\ln\lim\limits_{x\to x_0}f(x)}=e^{\beta\ln\alpha}=\alpha^\beta
$$

7.(3)

$$\lim_{x\to a}(\frac{\sin x}{\sin a})^{\frac{1}{x-a}}=\lim_{x\to a}(\frac{\sin x}{\sin a})^{\frac{1}{\sin x-\sin a}\frac{\sin x-\sin a}{x-a}}=(\lim_{x\to a}(\frac{\sin x}{\sin a})^{\frac{1}{\sin x-\sin a}})^{\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}}$$
$$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a}=\lim_{x\to a}\cos\frac{x+a}{2}=\cos a$$
$$\text{设}y=\frac{x-\sin a}{\sin a},\text{则}\lim_{x\to a}(\frac{\sin x}{\sin a})^{\frac{1}{\sin x-\sin a}}=\lim_{x\to\sin a}(\frac{x}{\sin a})^{\frac{1}{x-\sin a}}=\lim_{y\to 0}(y+1)^{\frac{1}{y\sin a}}=e^{\frac{1}{\sin a}}$$
$$\lim_{x\to a}(\frac{\sin x}{\sin a})^{\frac{1}{x-a}}=e^{\frac{\cos a}{\sin a}}$$

7.(5)

$$\lim_{n\to\infty}\tan^n(\frac{\pi}{4}+\frac{1}{n})=\frac{\lim\limits_{n\to\infty}(1+\tan\frac{1}{n})^n}{\lim\limits_{n\to\infty}(1-\tan\frac{1}{n})^n}$$
$$\lim_{n\to\infty}(1+\tan\frac{1}{n})^n=\lim_{n\to\infty}(1+\tan\frac{1}{n})^{\frac{1}{\tan\frac{1}{n}}\cdot n\tan\frac{1}{n}}=e^{\lim\limits_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}\cos\frac{1}{n}}}=e$$
$$\text{同理}\lim_{n\to\infty}(1-\tan\frac{1}{n})^n=\frac{1}{e}$$
$$\lim_{n\to\infty}\tan^n(\frac{\pi}{4}+\frac{1}{n})=e^2$$

8.(2)

$x=0$ 是第二类间断点, $x\in\mathbb{Z}\backslash\{0\}$ 是跳跃间断点.

8.(4)

$x=\frac{n}{2},n\in\mathbb{Z}$ 是跳跃间断点.

8.(6)

$x=0$ 是可去间断点.

8.(10)

$x\in\mathbb{Q}\backslash\{x|x=\frac{1}{n},n\in\mathbb{Z}\backslash\{0\}\}$

9.

反证法.
假设设 $f(x_0)=a,f(y_0)=b,x_0\neq y_0,a\neq b,x_n=\sqrt{x_{n-1}},y_n=\sqrt{y_{n-1}}$.显然$f(x_n)=a,f(y_n)=b,x_n\to 1(n\to\infty),y_n\to 1(n\to\infty)$
由Heine定理, $\lim\limits_{x\to 1}f(x)=\lim\limits_{n\to\infty}f(x_n)=a=\lim\limits_{n\to\infty}f(y_n)=b$ 与$a\neq b$矛盾.假设不成立,因此$f(x)$在$(0,+\infty)$上为常数函数.

第3章第3节

1.(4)

$x\to 0+,a=1,\alpha=\frac{1}{8}$
$x\to +\infty,a=1,\alpha=\frac{1}{2}$

1.(7)

$x\to 0+,a=1,\alpha=\frac{1}{2}$

1.(9)

$x \to 0, a = -\frac{3}{2}, \alpha = 2$

1.(10)

$x \to 0, a = 1, \alpha = 1$

2.(1)

$\ln^k x$,x^α,a^x,$[x]!$,x^x

$\frac{\ln^k x}{x^\alpha} = (\frac{\ln x}{x^{\frac{\alpha}{k}}})^k = (\frac{\frac{2k}{\alpha}\ln x^{\frac{\alpha}{2k}}}{x^{\frac{\alpha}{k}}})^k \le (\frac{\frac{2k}{\alpha}(x^{\frac{\alpha}{2k}}-1)}{x^{\frac{\alpha}{k}}})^k \to 0(x \to +\infty)$

$\frac{x^\alpha}{\alpha^x} = e^{a\ln x - x\ln a} < e^{2a(\sqrt{x}-1)-x\ln a} \to 0(x \to +\infty)$

设$n \le x < n+1$, $\frac{a^x}{n!} = C\frac{a^{[n]-[a]}}{n!/[a]!}(C\text{是一个常数}) \le C\frac{a^{[n]-[a]}}{([a]+1)^{[n]-[a]}} \to 0(x \to +\infty)$

设$n \le x < n+1$, $\frac{n!}{x^x} \le \frac{n!}{n^n} \le \frac{1}{n} \to 0(x \to +\infty)$

2.(2)

$(\frac{1}{x})^{-\frac{1}{x}}, \frac{1}{[\frac{1}{x}]!}, a^{-\frac{1}{x}}, x^\alpha, \ln^{-k}\frac{1}{x}$

理由同(1)

3.(1)

$\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1+2x^2}}{\ln(1+3x)} = \lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1+2x^2}}{3x}\cdot\frac{3x}{\ln(1+3x)} = \lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt[3]{1+2x^2}}{3x} = \lim\limits_{x\to 0}\frac{(\frac{x}{2}+1+o(x))-(1+o(x))}{3x} = \frac{1}{6}$

3.(2)

$\lim\limits_{x\to 0^+}\frac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}} = \lim\limits_{x\to 0^+}\frac{1-\cos x}{(1-\cos\sqrt{x})(1+\sqrt{\cos x})} = \lim\limits_{x\to 0^+}\frac{\frac{x^2}{2}}{(\frac{x}{2})(1+\sqrt{\cos x})} = 0$

3.(4)

$\lim\limits_{x\to\infty}(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}) = \lim\limits_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}$=1

3.(6)

$\lim\limits_{x\to a}\frac{x^\alpha-a^\alpha}{x-a} = a^\alpha\lim\limits_{x\to a}\frac{(\frac{x}{a})^\alpha-1}{x-a} = a^\alpha\lim\limits_{x\to a}\frac{e^{\alpha(\ln x-\ln a)}-1}{x-a} = a^\alpha\lim\limits_{x\to a}\frac{\alpha(\ln x-\ln a)}{x-a} = a^{\alpha-1}\lim\limits_{x\to a}\frac{\alpha\ln\frac{x}{a}}{\frac{x}{a}-1} = \alpha a^{\alpha-1}$

3.(8)

$\lim\limits_{x\to a}\frac{\ln x-\ln a}{x-a} = \frac{1}{a}\lim\limits_{x\to a}\frac{\ln\frac{x}{a}}{\frac{x}{a}-1} = \frac{1}{a}$

3.(11)

$\lim\limits_{n\to\infty}n(\sqrt[n]{x}-1) = \lim\limits_{n\to\infty}n(e^{\frac{\ln x}{n}}-1) = \lim\limits_{n\to\infty}n(\frac{\ln x}{n}) = \ln x$

设对每个自然数n, 数集$A_n \subset [0,1]$是有限集, 而且$A_i \cap A_j = \emptyset,\ \forall i,j \in \mathbb{N}_+,\ i \neq j$. 定义函数$f(x) = \begin{cases} 1/n, & \texttt{if } x \in A_n, \\ 0, & \texttt{if } x \in [0,1] \setminus (\cup_{i=1}^{\infty} A_i). \end{cases}$.对每个$a \in [0,1]$, 求$\lim_{x\to a} f(x)$. (提示: 仿照Riemann函数的讨论)

$\forall\varepsilon > 0$,取$n = \lfloor \frac{1}{\varepsilon} \rfloor, \exists\delta = \min\{s|x \in \cup_{i\in[n]} A_i, s = |x-a|\}, \forall x \in [0,1](|x-a| < \delta) : |f(x)| \leq \frac{1}{n+1} \leq \varepsilon$.因此$\lim_{x\to a} f(x) = 0$

第3章第4节

1.

$$\text{由于} \lim_{x\to+\infty} f(x) = A, \exists X, \forall x \geq X, |f(x) - A| \leq 1$$
$$\text{由于} f(x) \text{在} [a,\infty) \text{连续}, \forall x \in [a,X], m \leq f(x) \leq M.$$
$$\text{因此} \min\{A-1, m\} \leq f(x) \leq \max\{A+1, M\}, \text{即} f(x) \text{在} [a,+\infty] \text{有界}.$$

6.

$$\text{设} f(x) = x - a\sin x - b.$$
$$f(0) = -b < 0, f(a+b+1) = 1 + a(1 - \sin(a+b)) > 0, f(a+b+1)\cdot f(0) < 0$$
$$\text{因为} f(x) \text{连续, 由零点存在定理}, f(x) \text{在} (0, a+b+1) \text{上至少有一个零点, 即} x = a\sin x + b \text{至少有一个正根}.$$

10.

$$\text{设} g(x) = f(x+1) - f(x)$$
$$f(0) = f(2) \Rightarrow f(2) - f(1) + f(1) - f(0) = 0 \text{即} g(1) + g(0) = 0.$$
$$\text{若} g(0) = 0, \text{则} g(x) \text{在} [0,1] \text{上存在零点}.$$
$$\text{若} g(0) \neq 0, \text{则} g(1)\cdot g(0) = -g^2(0) < 0. \text{因为} g(x) \text{连续, 由零点存在定理}, g(x) \text{在} [0,1] \text{上有零点}.$$
$$\text{因此} \exists x, y \in [0,2], y - x = 1, s.t. f(x) = f(y).$$

设函数$f: \mathbb{R} \to \mathbb{R}$且存在一个常数$q(0 < q < 1)$使得$\forall x, y \in \mathbb{R}:\ |f(x) - f(y)| < q|x-y|$, 则称$f$是$\mathbb{R}$上一个压缩映射. 证明: $\exists! \xi \in \mathbb{R}:\ f(\xi) = \xi$.

取$x = x_0, \lim\limits_{y\to x_0} |f(y) - f(x_0)| = 0$,因此$f(x)$是连续函数.
设$g(x) = x - f(x)$
设$x < y$,则$g(x) - g(y) = x - y - (f(x) - f(y)) > x - y - |f(x) - f(y)| > (1-q)(x-y) > 0$,因此$g(x)$单调递增.
任取$x_0 \in \mathbb{R}$,若$g(x_0) = 0$,则$\xi = x_0$
若$g(x_0) < 0$,由于$|g(x) - g(y)| > |x-y| - |f(x) - f(y)| > (1-q)|x-y|$,那么$g(x_0 + \frac{-g(x_0)}{1-q}) > g(x_0) + (-g(x_0)) = 0$,由零点存在定理,存在$\xi \in (x_0, x_0 + \frac{-g(x_0)}{1-q}) s.t. g(\xi) = 0$.
若$g(x_0) > 0$,同理可知存在$\xi \in \mathbb{R} s.t. g(\xi) = 0$

第3章第4节

8.(2)

取$\varepsilon=2,\forall\delta>0:x_1=\sqrt{2k\pi-\frac{\pi}{2}},x_1=\sqrt{2k\pi},x_2-x_1=\frac{\frac{\pi}{2}}{\sqrt{2k\pi-\frac{\pi}{2}}+\sqrt{2k\pi}}<\sqrt{\frac{\pi}{8k}}$,当$k=\lceil\frac{\pi}{8\varepsilon^2}\rceil$时,$|x_2-x_1|<\delta$,但$|f(x_1)-f(x_2)|=1$,因此$\sin x^2$在$(0,1)$上不一致连续.

$\forall\varepsilon>0,\exists\delta=\frac{\varepsilon}{1+A},\forall x_1,x_2\in[1,A](|x_1-x_2|<\delta):|\sin x_1^2-\sin x_2^2|=|2\sin\frac{x_1^2-x_2^2}{2}\cos\frac{x_1^2+x_2^2}{2}|\le|x_1^2-x_2^2|<(1+A)|x_2-x_1|\le\varepsilon$.因此在$[1,A]$上一致连续.

8.(3)

$\forall\varepsilon>0,\exists\delta=\varepsilon^2,\forall x_1,x_2\in[0,+\infty)(|x_1-x_2|<\delta,x_1>x_2):|\sqrt{x_1}-\sqrt{x_2}|\le|\sqrt{x_1-x_2}|<\varepsilon$,因此在$[0,+\infty)$一致连续

8.(4)

$\forall\varepsilon>0,\exists\delta=\varepsilon,\forall x_1,x_2\in[1,+\infty)(|x_1-x_2|<\delta,x_1>x_2):|\ln x_1-\ln x_2|=\ln\frac{x_1}{x_2}<\ln\frac{x_2+\delta}{x_2}<\frac{x_2+\delta}{x_2}-1\le\varepsilon$,因此在$[0,+\infty)$一致连续.

9.

设$l_1(\theta),l_2(\theta)$表示弦与x轴的角度为θ时P将弦分成的两段长度.
设$l(\theta)=l_1(\theta)-l_2(\theta),l(0)+l(\pi)=0$
若$l(0)=0$,则l存在零点.若$l(0)\ne0$,则$l(0)l(\pi)=-l^2(0)<0$,由零点存在定理,l在$(0,\pi)$存在零点.
因此$\exists\theta_0\in[0,\pi],l_1(\theta_0)=l_2(\theta_0)$

11.

$\forall\varepsilon>0,\exists\delta>0,\forall x_1,x_2\in(a,b)(|x_1-x_2|\le\delta):|f(x_1)-f(x_2)|\le\varepsilon$
取$\varepsilon=\varepsilon_0$,此时$\delta=\delta_0$.设$|f(\frac{a+b}{2})|=y_0$
则$\forall n\in\mathbb{N}(\frac{a+b}{2}+n\delta_0<b):|f(\frac{a+b}{2}+n\delta_0)|=|(f(\frac{a+b}{2}+n\delta_0)-f(\frac{a+b}{2}+(n-1)\delta_0))+(f(\frac{a+b}{2}+(n-1)\delta_0)-f(\frac{a+b}{2}+(n-2)\delta_0))+\cdots+(f(\frac{a+b}{2}+\delta_0)-f(\frac{a+b}{2}))+f(\frac{a+b}{2})|\le|(f(\frac{a+b}{2}+n\delta_0)-f(\frac{a+b}{2}+(n-1)\delta_0))|+|(f(\frac{a+b}{2}+(n-1)\delta_0)-f(\frac{a+b}{2}+(n-2)\delta_0))|+\cdots+|(f(\frac{a+b}{2}+\delta_0)-f(\frac{a+b}{2}))|+|f(\frac{a+b}{2})|\le n\varepsilon_0+y_0$
同理,$\forall n\in\mathbb{Z}^-(\frac{a+b}{2}+n\delta_0>a):|f(\frac{a+b}{2}+n\delta_0)|\le|n|\varepsilon_0+y_0$
那么,$\forall x\in(a,b),\exists n\in Z:|x-(\frac{a+b}{2}+n\delta_0)|\le\delta_0$,则$|f(x)|\le(|n|+1)\varepsilon+y_0$,即$f(x)$在$(a,b)$上有界

12.(1)

证:
$\forall\varepsilon>0,\exists\delta_1(\varepsilon)>0,\forall x_1,x_2\in D,(|x_1-x_2|\le\delta_1(\varepsilon)):|f(x_1)-f(x_2)|<\varepsilon$
$\forall\varepsilon>0,\exists\delta_2(\varepsilon)>0,\forall x_1,x_2\in D,(|x_1-x_2|\le\delta_2(\varepsilon)):|g(x_1)-g(x_2)|<\varepsilon$
则$\forall\varepsilon>0,\exists\delta(\varepsilon)=\min\{\delta_1(\frac{\varepsilon}{2}),\delta_2(\frac{\varepsilon}{2})\},\forall x_1,x_2\in D,(|x_1-x_2|<\delta(\varepsilon)):|f(x_1)+g(x_1)-f(x_2)-g(x_2)|\le|f(x_1)-f(x_2)|+|g(x_1)-g(x_2)|<\varepsilon$
因此$f+g$在D上一致连续

12.(2)

反例$:f(x)=g(x)=x$

14.

设$g(x)=f(x)-\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}=\frac{(f(x)-f(x_1))+(f(x)-f(x_2))+\cdots+(f(x)-f(x_n))}{n}$
若$f(x_1)=f(x_2)=\cdots=f(x_n)$,那么$\xi=x_1$
否则,设$x_u=\max\{x_1,x_2,...,x_n\},x_v=\min\{x_1,x_2,...,x_n\}$,则$f(x_u)>0,f(x_v)<0$,由零点存在定理,$\exists\xi\in(x_u,x_v)$(或$(x_v,x_u)$)$:f(\xi)=0$

得证.

15.

$\forall \varepsilon > 0, \exists X(\varepsilon), \forall x_1, x_2 > X(\varepsilon) : |f(x_1) - f(x_2)| < \varepsilon$
那么$\forall \varepsilon > 0, \exists \delta, \forall x_1, x_2 \in [a, +\infty)(|x_1 - x_2| < \delta)$: 在$[a, X(\varepsilon) + \delta]$上$f(x)$一致连续因此对于$x_1, x_2 \in [a, X(\varepsilon) + \delta](|x_1 - x_2| < \delta)$满足$|f(x_1) - f(x_2)| < \varepsilon$,而$[X(\varepsilon), +\infty)$上显然也满足.因此$f(x)$在$[a, +\infty)$上一致连续.