数学分析作业

2.(1)

证 设无限集A，则可以从中选取无限个元素$a_1, a_2, a_3, ..., a_n, ...$满足$a_i \in A$且互不相等.
那么可以构造集合$B = \{a_1, a_2, a_3, ..., a_n, ...\}$，满足$B \subset A$，显然$B$是可列集.
证毕.

(2)

证 因为A，B是可列集，则A，B可表示为：

$$A = \{a_1, a_2, ..., a_n, ...\}$$
$$B = \{b_1, b_2, ..., b_n, ...\}$$

则$A \cup B = \{a_1, b_1, a_2, b_2, ..., a_n, b_n, ...\}$为可列集.
证毕.

6.(1)

$$A = \{1, 2, 3, 4, 5, 6\}$$
$$B = \{1, 2, 3\}$$
$$C = \{2, 3, 4\}$$

(2)

$$A = \emptyset$$
$$B = \{1, 2, 3\}$$
$$C = \{2, 3, 4\}$$

7.(1)
错误.

$$x \notin A \cap B \iff x \notin A\text{或者}x \notin B$$

(2)
错误.

$$x \notin A \cup B \iff x \notin A\text{并且}x \notin B$$

证

- $1 \Rightarrow 2$:
  因为f是单射，所以对于$a \in A$，$f(a)$互不相等，又因为$f^{-1}(f(a)) = a$所以$f^{-1}(f(A)) = A$.
- $2 \Rightarrow 1$:
  $f(X) = R_f$，因为对于$d \in R_f$，$f^{-1}(d)$存在，所以映射f中每个d只与一个x对应，因此f是单射.
- $1 \Rightarrow 3$:
  对于$x \in A \cap B$，有$x \in A$且$x \in B$，则$f(x) \in f(A)$且$f(x) \in f(B)$，因此$\forall x \in A \cap B, f(x) \in f(A) \cap f(B)$，因此$f(A \cap B) \subseteq f(A) \cap f(B)$
  反证法:
  假设存在$d \in f(A) \cap f(B), d \notin f(A \cap B)$.因为$d \notin f(A \cap B)$且$f$是单射，那么$f^{-1}(d) \notin A \cap B$，所以$f^{-1}(d) \notin A$或$f^{-1}(d) \notin B$，又因为$f$是单射，所以$d \notin f(A) \cap f(B)$，矛盾，假设不成立，因此不存在$d \in f(A) \cap f(B), d \notin f(A \cap B)$，因此$f(A) \cap f(B) \subseteq f(A \cap B)$.
  因为$f(A \cap B) \subseteq f(A) \cap f(B), f(A) \cap f(B) \subseteq f(A \cap B)$，所以$f(A \cap B) = f(A) \cap f(B)$
- $3 \Rightarrow 1$:
  反证法:
  假设$x_i, x_j \in X, f(x_i) = f(x_j)$.
  则当$A = x_i, B = x_j$时，$f(A \cap B) = f(\emptyset) = \emptyset, f(A) \cap f(B) = \{f(x_i)\} \cap \{f(x_j)\} = \{f(x_i)\} \neq f(A \cap B)$，矛盾，假设不成立，因此$f$是单射.
- $3 \Rightarrow 4$:
  因为$f(A \cap B) = f(A) \cap f(B)$，所以$f(A) \cap f(B) = \emptyset \Rightarrow f(A \cap B) = \emptyset \Rightarrow A \cap B = \emptyset$.
  因为$f(A \cap B) = f(A) \cap f(B)$，所以$A \cap B = \emptyset \Rightarrow f(A \cap B) = \emptyset \Rightarrow f(A) \cap f(B) = \emptyset$.
  综上，$f(A) \cap f(B) \Leftrightarrow A \cap B = \emptyset$.

- $4 \Rightarrow 1$:
  反证法:
  假设f不是单射，$f(x_1) = f(x_2)$，那么当$A = \{x_1\}, B = \{x_2\}$时，$A \cap B = \emptyset$但是$f(A) \cap f(B) = \{f(x_1)\} \neq \emptyset$，矛盾，假设不成立，因此$f$时单射.
- $1 \Rightarrow 5$:
  设$A \setminus B = C, C \cap B = \emptyset$.
  因为f是单射且$B \cap C = \emptyset$，$f(B) \cap f(C) = \emptyset$，又因为$f(A) = f(B) \cup f(C)$，所以$f(A) \setminus f(B) = f(C)$，所以$f(A) \setminus f(B) = f(A \setminus B)$.
- $5 \Rightarrow 1$:
  反证法:
  假设f不是单射，$f(x_1) = f(x_2)$，那么当$A = \{x_1, x_2\}, B = \{x_1\}$时，$f(A \setminus B) = f(\{x_2\}) = \{f(x_2)\}, f(A) \setminus f(B) = \{f(x_1)\} \setminus \{f(x_2)\} = \emptyset \neq f(A \setminus B)$，矛盾，假设不成立，因此$f$是单射.

易知所有等价关系都可以由已证结论推出.（虽然又臭又长

证

设$\alpha = |A_0|, \beta = |B_0|$，存在$A_0$到$B_0$的单射和$B_0$到$A_0$的单射，即存在$A_0$到$B_1(B_1 \subset B_0)$的双射$\varphi$和$B_0$到$A_1(A_1 \subset A_0)$的双射$\psi$，设$B_{n+1} = \varphi(A_n), A_{n+1} = \psi(B_n)$.

下面用归纳法证明$A_{n+1} \subset A_n, B_{n+1} \subset B_n$:
当$n = 0$时，$A_1 \subset A_0, B_1 \subset B_0$成立.
假设当$n = k$时，$A_{k+1} \subset A_k, B_{k+1} \subset B_k$成立.
当$n = k + 1$时，要证$A_{k+2} \subset A_{k+1}, B_{k+2} \subset B_{k+1}$成立.
因为$A_{k+2} = \psi(B_{k+1}), A_{k+1} = \psi(B_k)$,又因为$B_{k+1} \subset B_k$，所以$A_{k+2} \subset A_{k+1}$.同理，$B_{k+2} \subset B_{k+1}$.
综上，$A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots, B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$.

因为$\varphi(A_n) = B_{n+1}, \varphi(A_{n+1}) = B_{n+2}$且都是双射，所以$\varphi(A_{n+1}) \cup \varphi(A_n \setminus A_{n+1}) = \varphi(A_n)$，所以$\varphi(A_n \setminus A_{n+1}) = \varphi(A_n) \setminus \varphi(A_{n+1}) = B_{n+1} \setminus B_{n+2}$.又因为$\forall a_i \in A_0, \varphi(a_i)$互不相同，所以$\varphi|_{A_n \setminus A_{n+1}} A_n \setminus A_{n+1} \to B_n \setminus B_{n+1}$是双射，同理$\psi|_{B_n \setminus B_{n+1}} B_n \setminus B_{n+1} \to A_n \setminus A_{n+1}$是双射.

$\bigcup_{n=0}^{\infty}(A_n \setminus A_{n+1}) = (\bigcup_{n=1}^{\infty}(A_n \setminus A_{n+1})) \cup (A_0 \setminus A_1) = (\bigcup_{n=2}^{\infty}(A_n \setminus A_{n+1})) \cup (A_0 \setminus A_2) = \cdots = A_0 \setminus (\bigcap_{n=0}^{\infty} A_n)$
因此$A_0 = (\bigcup_{n=0}^{\infty}(A_n \setminus A_{n+1})) \cup (\bigcap_{n=0}^{\infty} A_n)$.
易知，等号右边各项两两不交.
对于B同理.

$\varphi(\bigcup_{n=0}^{\infty}(A_{2n}\setminus A_{2n+1}))=\bigcup_{n=0}^{\infty}\varphi(A_{2n}\setminus A_{2n+1})=\bigcup_{n=0}^{\infty}(B_{2n+1}\setminus B_{2n+2})$，又因为$\varphi(a_i)$互不相同，所以$\varphi\big|_{\bigcup_{n=0}^{\infty}(A_{2n}\setminus A_{2n+1})}:\bigcup_{n=0}^{\infty}(A_{2n}\setminus A_{2n+1})\to\bigcup_{n=0}^{\infty}(B_{2n+1}\setminus B_{2n+2})$是双射.
同理可得$\psi\big|_{\bigcup_{n=0}^{\infty}(B_{2n}\setminus B_{2n+1})}:\bigcup_{n=0}^{\infty}(B_{2n}\setminus B_{2n+1})\to\bigcup_{n=0}^{\infty}(A_{2n+1}\setminus A_{2n+2})$是双射.
设$b\in\bigcap_{n=0}^{\infty}B_n$，对于$t=0,1,2,3,4,...$因为$b\in B_t$，所以$\psi(b)\in A_{t+1}$，又因为$A_0\supset A_1$，所以$\forall b\in\bigcap_{n=0}^{\infty}B_n,\psi(b)\in\bigcap_{n=0}^{\infty}A_n$，同理$\forall a\in\bigcap_{n=0}^{\infty}A_n,\psi^{-1}(a)\in\bigcap_{n=0}^{\infty}B_n$，因此$\psi\big|_{\bigcap_{n=0}^{\infty}B_n}:\bigcap_{n=0}^{\infty}B_n\to\bigcap_{n=0}^{\infty}A_n$是双射.

定义

$$\vartheta(x)=\begin{cases}\varphi(x), & \text{if} x\in\bigcup_{n=0}^{\infty}(A_{2n}\setminus A_{2n+1}),\\ \psi^{-1}(x), & \text{if} x\in(\bigcup_{n=0}^{\infty}(A_{2n+1}\setminus A_{2n+2}))\cup(\bigcap_{n=0}^{\infty}A_n).\end{cases}$$

易知$D_\vartheta=A_0,R_\vartheta=B_0$，显然$\vartheta:A_0\to B_0$是双射，即存在$A_0$到$B_0$的双射.
证毕.

1.(4) $\frac{1+2+3+\cdots+n}{n^3}=\frac{n(n+1)}{n^3}=\frac{n+1}{n^2}\le\frac{2n}{n^2}=\frac{2}{n}.\forall\varepsilon\in\mathbb{R}_+,\exists N=\lceil\frac{2}{\varepsilon}\rceil:|n\ge N$时$\frac{1+2+3+\cdots+n}{n^3}|\le\varepsilon$

1.(6) 当$n\ge100$时，$\frac{3^n}{n!}\le\frac{n!\cdot(\frac{1}{n})}{n!}=\frac{1}{n},\forall\varepsilon\in\mathbb{R}_+,\exists N=\lceil\frac{1}{\varepsilon}\rceil$: 当$n\ge N,\frac{3^n}{n!}\le\varepsilon$

1.(8) 当n为奇数,$\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}\cdots+(-1)^n\frac{1}{2n}\le\frac{1}{n(n+1)}\cdot\frac{n+1}{2}=\frac{1}{2n}\le\frac{1}{n}$;当$n$为偶数，$\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}\cdots+(-1)^n\frac{1}{2n}\le\frac{1}{n(n+1)}\cdot\frac{n}{2}+\frac{1}{2n}\le\frac{1}{n},\forall\varepsilon\in\mathbb{R}_+,\exists N=\lceil\frac{1}{\varepsilon}\rceil$: 当$n\ge N,\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}\cdots+(-1)^n\frac{1}{2n}\le\varepsilon$

2.(1) $|\frac{2n^2-1}{3n^2+2}-\frac{2}{3}|=\frac{7}{9n^2+6}\le\frac{1}{n},\forall\varepsilon\in\mathbb{R}_+,\exists N=\lceil\frac{1}{\varepsilon}\rceil$: 当$n\ge N,|\frac{2n^2-1}{3n^2+2}-\frac{2}{3}|\le\varepsilon$,因此$\lim_{n\to\infty}\frac{2n^2-1}{3n^2+2}=\frac{2}{3}$

2.(3) $|\sqrt{n^2+n}-n-\frac{1}{2}|=|\frac{(\sqrt{n^2+n}-n-\frac{1}{2})(\sqrt{n^2+n}+n+\frac{1}{2})}{\sqrt{n^2+n}+n+\frac{1}{2}}|=\frac{1}{4(\sqrt{n^2+n}+n+\frac{1}{2})}\le\frac{1}{n},\forall\varepsilon\in\mathbb{R}_+,\exists N=\lceil\frac{1}{\varepsilon}\rceil$: 当$n\ge N,|\sqrt{n^2+n}-n-\frac{1}{2}|\le\varepsilon$

5.由已知得$\forall\varepsilon\in\mathbb{N}_+,\exists N=N_1(\varepsilon)$: 当$n\ge N,|x_{2n}-a|\le\varepsilon;\forall\varepsilon\in\mathbb{N}_+,\exists N=N_2(\varepsilon)$: 当$n\ge N,|x_{2n+1}-a|\le\varepsilon$,那么$\forall\varepsilon\in\mathbb{N}_+,\exists N=\max\{2N_1(\varepsilon),2N_2(\varepsilon)+1\}$: 当$n\ge N,|x_n-a|\le$

ε，因此$\lim_{n\to\infty} x_n = a$.

6.由已知得$\forall\varepsilon \in \mathbb{N}_+, \exists N = N_1(\varepsilon)$: 当$n \geq N, |x_n - a| \leq \varepsilon$,与此同时$|\sqrt{x_n} - \sqrt{a}| = \frac{|x_n - a|}{\sqrt{x_n}+\sqrt{a}} \leq \frac{\varepsilon}{\sqrt{a}}$.即$\forall\varepsilon \in \mathbb{N}_+, \exists N = N_1(\varepsilon)$: 当$n \geq N, |\sqrt{x_n} - \sqrt{a}| \leq \frac{\varepsilon}{\sqrt{a}}$,因此$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{a}$