Homework for Linear Algebra (XXV)

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In the following $\boldsymbol{V}$ is a finite dimensional vector space with dual space $\boldsymbol{V}'$.

Exercise 1. Prove that every $L \in \boldsymbol{V}'$ is either surjective or the zero function (i.e., $L(\boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \boldsymbol{V}$).

Exercise 2. Show that the dual transformation of the *identity* linear transformation on $\boldsymbol{V}$, i.e., $T(\boldsymbol{v}) = \boldsymbol{v}$ for every $\boldsymbol{v} \in \boldsymbol{V}$, is the identity linear transformation on $\boldsymbol{V}'$, i.e., $T'(L) = L$ for every $L \in \boldsymbol{V}'$.

Exercise 3. Assume $L_1, \dots, L_n$ is a basis for $\boldsymbol{V}'$. Show that there exists a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ for $\boldsymbol{V}$ whose dual basis is precisely $L_1, \dots, L_n$.

Hint: We suggest three approaches, and the second and third ones are by your brilliant TAs.

(i) Induction on n, and for $n \geqslant 2$ consider $\boldsymbol{U} = \mathrm{Ker}(L_1) \subseteq \boldsymbol{V}$ and the restrictions of $L_2, \dots, L_n$ on $\boldsymbol{U}$. Thereby the *restriction* of $L_i : \boldsymbol{V} \to \mathbb{R}$ on $\boldsymbol{U}$ is the function $L_i \restriction_{\boldsymbol{U}} : \boldsymbol{U} \to \mathbb{R}$ defined by

$$L_i \restriction_{\boldsymbol{U}} (\boldsymbol{u}) = L_i(\boldsymbol{u})$$

for every $\boldsymbol{u} \in \boldsymbol{U}$. Prove that $L_2 \restriction_{\boldsymbol{U}}, \dots, L_n \restriction_{\boldsymbol{U}}$ is a basis for $\boldsymbol{U}'$ and then apply the induction hypothesis.

(ii) Consider the dual space of $\boldsymbol{V}'$, i.e., $\boldsymbol{V}''$. For every $\boldsymbol{v} \in \boldsymbol{V}$ let $\phi_{\boldsymbol{v}}$ be the function defined by

$$\phi_{\boldsymbol{v}}(L) = L(\boldsymbol{v})$$

for every $L \in \boldsymbol{V}'$. Show $\phi_{\boldsymbol{v}} \in \boldsymbol{V}''$. Then prove that the function $\Phi : \boldsymbol{V} \to \boldsymbol{V}''$ defined by

$$\Phi(\boldsymbol{v}) = \phi_{\boldsymbol{v}}$$

is a bijective linear transformation from $\boldsymbol{V}$ to $\boldsymbol{V}''$.

Finally for the dual basis $\phi_1, \dots, \phi_n$ of $L_1, \dots, L_n$ (which is basis for $\boldsymbol{V}''$) consider

$$\Phi^{-1}(\phi_1), \dots, \Phi^{-1}(\phi_n).$$

(iii) Choose an arbitrary basis $\tilde{\boldsymbol{v}}_1, \dots, \tilde{\boldsymbol{v}}_n$ for $\boldsymbol{V}$ and the corresponding dual basis $\tilde{L}_1, \dots, \tilde{L}_n$. Then let M be the change of basis matrix M from $\tilde{L}_1, \dots, \tilde{L}_n$ to $L_1, \dots, L_n$. Try to apply M in a clever way to $\tilde{\boldsymbol{v}}_1, \dots, \tilde{\boldsymbol{v}}_n$.

Exercise 4. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two finite dimensional vector spaces and $T \in \boldsymbol{T}(\boldsymbol{V}, \boldsymbol{W})$. Prove that

$$T' = \mathbf{0} \iff T = \mathbf{0}.$$

Exercise 5. Let A be an $m \times n$ matrix. Prove that $\mathrm{rank}(A) = 1$ if and only if there are $\boldsymbol{u} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}.$$