数学分析作业

第2章第4节

1.(1)

$$(1-\frac{1}{n})^n(1+\frac{1}{n})^n=(1-\frac{1}{n^2})^n$$
$$0<1-(1-\frac{1}{n^2})^n=\frac{1}{n^2}(1+(1-\frac{1}{n^2})+(1-\frac{1}{n^2})^2+\cdots+(1-\frac{1}{n^2})^{n-1})<\frac{1}{n}$$
$$\text{由夹逼定理得}\lim_{n\to\infty}(1-(1-\frac{1}{n^2})^n)=0$$
$$\therefore \lim_{n\to\infty}(1-\frac{1}{n^2})^n=1$$
$$\therefore \lim_{n\to\infty}(1-\frac{1}{n})^n=\frac{1}{e}$$

1.(4)

$$(\frac{n^2-1}{n^2})^n<(\frac{n^2+1}{n^2})^n<(\frac{n^2}{n^2-1})^n$$
$$\text{由夹逼定理得}\lim_{n\to\infty}(\frac{n^2+1}{n^2})^n=1$$

1.(5)

$$(1+\frac{1}{n})^n(1-\frac{1}{n^2})^n=(1+\frac{1}{n}-\frac{1}{n^2}-\frac{1}{n^3})^n<(1+\frac{1}{n}-\frac{1}{n^2})^n<(1+\frac{1}{n})^n$$
$$\text{由夹逼定理得}\lim_{n\to\infty}(1+\frac{1}{n}-\frac{1}{n^2})^n=e$$

4.

$$x_1\in[1,\frac{3}{2}],\text{设}x_n\in[1,\frac{3}{2}],\text{则}x_{n+1}\in[\sqrt{2},\frac{3}{2}]\subset[1,\frac{3}{2}],\text{归纳可得}x_n\in[1,\frac{3}{2}]\text{有界.}$$
$$\text{对}x_{n+1}=\frac{1}{2}(x_n+\frac{2}{x_n})\text{两边求极限得}\lim_{n\to\infty}x_n=\frac{\lim\limits_{n\to\infty}x_n}{2}+\frac{1}{\lim\limits_{n\to\infty}x_n}$$
$$\text{解得}\lim_{n\to\infty}=\pm\sqrt{2}$$
$$\text{又}\because x_n\in[1,\frac{3}{2}]\therefore\lim_{n\to\infty}=\sqrt{2}$$
$$\text{同理得}x_1=-2\text{时}\lim_{n\to\infty}=-\sqrt{2}$$

6.(1)

$$x_n<y_n\Rightarrow x_n<\sqrt{x_ny_n}<\frac{x_n+y_n}{2}<y_n$$
$$\{x_n\}\text{单调递增},\{y_n\}\text{单调递减},\text{因此都有极限}$$
$$\text{设}x_n\to A(n\to\infty),y_n\to B(n\to\infty)$$
$$A=\sqrt{AB},B=\frac{A+B}{2}$$
$$\therefore A=B,\text{即}\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$$

6.(2)

$$\text{设}x_1'=a,y_1'=b,x_{n+1}'=\frac{2x_n'y_n'}{x_n'+y_n'},y_{n+1}'=\frac{x_n'+y_n'}{2}$$
$$x_{n+1}'<x_n'<y_{n+1}'<y_n$$
$$\{x_n'\}\text{和}\{y_n'\}\text{单调递减,因此都有极限}$$
$$\text{设}x_n'\to A'(n\to\infty),y_n'\to B'(n\to\infty)$$
$$A'=\frac{A'+B'}{2}$$
$$\therefore A'=B',\text{即}\lim_{n\to\infty}x_n'=\lim_{n\to\infty}y_n'=c$$
$$\text{又}\because x_n'\le x_n\le y_n',x_n'\le y_n\le y_n'$$
$$\therefore\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=c$$

9.

$$\exists\varepsilon>0,\forall N\in N_+,\exists n\ge m\ge N:|x_n-x_m|>\varepsilon$$
$$\text{设}a_n=x_m,\exists b_n=x_n,|x_n-x_m|>\varepsilon$$
$$\{a_n\},\{b_n\}\text{都有界,存在收敛子列}\{a_n'\},\{b_n'\}$$
$$|\lim_{n\to\infty}a_n'-\lim_{n\to\infty}b_n'|>\varepsilon$$
$$\therefore\text{存在}\{a_n'\},\{b_n'\}\text{收敛于不同的极限.}$$

11.

$$\text{取}x_1\in S,x_{n+1}=\frac{x_n+a}{2}\in(x_n,a)$$
$$\{x_n\}\text{有界且单增,因此}\{x_n\}\text{收敛.}$$
$$\lim_{n\to\infty}x_n=\frac{\lim_{n\to\infty}x_n+a}{2}$$
$$\therefore\lim_{n\to\infty}x_n=a$$

12.

$$a_1<a_n<b_1,\{a_n\}\text{有界且单调递增,因此}\{a_n\}\text{有极限.}$$
$$\text{设}a_n\to\xi(n\to\infty),\text{因为}\lim_{n\to\infty}(b_n-a_n)=0,\text{所以}b_n\to\xi(n\to\infty)$$

利用 $e=\sum_{n=0}^{\infty}\frac{1}{n!}\left(:=\lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}\right)$ 证明 e 是无理数.

$$x_n=\frac{1}{n}+\frac{1}{n(n+1)}+\frac{1}{n(n+1)(n+2)}+\cdots$$
$$x_{n+1}=nx_n-1<x_n\Rightarrow x_n<\frac{1}{n-1}$$
$$\therefore\forall n>2:x_n<1$$
$$\therefore\forall n>2:x_n=(n-1)!e-(n-1)!\sum_{i=0}^{n-1}\frac{1}{i!}\in(0,1)$$
$$C=(n-1)!\sum_{i=0}^{n-1}\frac{1}{i!}\in\mathbb{N}$$
$$\therefore\forall n>2:(n-1)!e\in(C,C+1)\notin\mathbb{N}$$

第2章第4节

13.(1)

$$
\text{设}n > m
$$
$$
x_n - x_m \le M|q|^m + M|q|^{m+1} + M|q|^{m+2} + \cdots + M|q|^n \le M\frac{|q|^m - |q|^{n+1}}{1-q} \le \frac{M|q|^m}{1-q}
$$
$$
\forall\varepsilon > 0, \exists N(\varepsilon) = \lceil \log_{|q|}\frac{(1-|q|)\varepsilon}{M} \rceil, \forall n > m > N(\varepsilon) : |x_n - x_m| < \varepsilon
$$
$$
\text{由}\texttt{Cauchy}\text{收敛原理得}\{x_n\}\text{收敛.}
$$

14.(1)

不一定.反例:$x_n = \ln n$.

14.(2)

$$
\forall\varepsilon > 0, \exists N(\varepsilon) = \lceil 1 - \log_2\varepsilon \rceil, \forall n > m > N(\varepsilon) : |x_n - x_m| < |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|
$$
$$
|x_n - x_m| < \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m} < \frac{1}{2^{m-1}} < \varepsilon
$$
$$
\therefore \{x_n\}\text{是基本数列.}
$$

15.

$$
\lim_{k\to\infty} \texttt{diam}\, A_k = 0 \Leftrightarrow \forall\varepsilon > 0, \exists N(\varepsilon), \forall n \ge m \ge N(\varepsilon) : |x_n - x_m| < \varepsilon \Rightarrow \{x_n\}\text{是基本数列.}
$$

16.

反证法.假设$\exists\varepsilon > 0, \forall N, \exists n \ge m \ge N : |x_n - x_m| > \varepsilon.$
也就是说, $\forall m, \exists n : |x_n - x_{f(n)}|$.设$|x_n| \le A$, 那么$|x_{f^{\lceil \frac{|A|+|x_1|}{\varepsilon} \rceil}(n)}| > A$,矛盾.
因此$\forall\varepsilon > 0, \exists N(\varepsilon), \forall n \ge m \ge N(\varepsilon) : |x_n - x_m| \le \varepsilon$,即$\{x_n\}$是基本数列,因此$\{x_n\}$收敛.
(这里$f^1(x) = f(x), f^n(x) = f^{n-1}(f(x))$)

第3章第1节

1.(2)

$$
|\sqrt{x} - 2| < \varepsilon \Leftrightarrow \varepsilon^2 - 4\varepsilon + 4 < x < \varepsilon^2 + 4\varepsilon + 4
$$
$$
\forall 0 < \varepsilon < 1, \exists\delta = 4\varepsilon - \varepsilon^2, \forall x \in B_\delta(4) : |\sqrt{x} - 2| < \varepsilon
$$

2.(1)

设$x_n = 1 + \frac{1}{n} \to 1(n\to\infty)$, $\frac{x_n^2-1}{2x_n^2-x_n-1} = \frac{2n+1}{3n+2} \to \frac{2}{3}(n\to\infty)$
$\therefore \lim_{x\to 1}\frac{x^2-1}{2x^2-x-1} = \frac{2}{3}$

2.(6)

$$
\begin{aligned}
\lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2} &= \lim_{x\to 0}\frac{\binom{n}{0}(mx)^0-\binom{m}{0}(nx)^0}{x^2}+\lim_{x\to 0}\frac{\binom{n}{1}(mx)^1-\binom{m}{1}(nx)^1}{x^2}\cdots \\
&= \frac{nm(n-m)}{2}+\lim_{x\to 0}C_1\frac{1}{x}+\lim_{x\to 0}C_2\frac{1}{x^2}\cdots \\
&= \frac{nm(n-m)}{2}
\end{aligned}
$$

2.(7)

设$x_n=a+\frac{\pi}{n}\to 1(n\to\infty),\frac{\sin x_n-\sin a}{x_n-a}=\frac{\sin\frac{\pi}{n}\cos a}{\frac{\pi}{n}}+\frac{\sin a(\cos\frac{\pi}{n}-1)}{\frac{\pi}{n}}$
$\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cos a\to\cos a(n\to\infty),\frac{\sin a(\cos\frac{\pi}{n}-1)}{\frac{\pi}{n}}=\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\frac{\sin a(\cos\frac{\pi}{n}-1)}{\sin\frac{\pi}{n}}=-\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\tan\frac{\pi}{2n}\sin a\to 0(n\to\infty).$
$\therefore\lim_{x\to a}\frac{\sin x_n-\sin a}{x_n-a}=\cos a$

2.(9)

$\frac{\cos x-\cos 3x}{x^2}=4\frac{\cos x-\cos^3 x}{x^2}=4\frac{\cos x\sin^2 x}{x^2}$
设$x_n=\frac{\pi}{n},\frac{\cos x_n-\cos 3x_n}{x_n^2}=4\cos\frac{\pi}{n}(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}})^2\to 4(n\to\infty)$
$\therefore\lim\limits_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=4$

3.(1)

$\frac{1}{x}\le[\frac{1}{x}]\le\frac{1}{x}+1,1\le x[\frac{1}{x}]\le 1+x$
$\lim\limits_{x\to 0}(1+x)=1$
$\therefore\lim\limits_{x\to\infty}x[\frac{1}{x}]=1$

5.(1)

$$
\begin{gathered}
\lim_{x\to 0^+}=+\infty \\
\lim_{x\to 1^-}=\frac{1}{2} \\
\lim_{x\to 1^+}=1 \\
\lim_{x\to 2^-}=4 \\
\lim_{x\to 2^+}=4
\end{gathered}
$$

5.(4)

$$
\begin{gathered}
\lim_{x\to\frac{1}{n}^-}(\frac{1}{x}-\left[\frac{1}{x}\right])=0 \\
\lim_{x\to\frac{1}{n}^+}(\frac{1}{x}-\left[\frac{1}{x}\right])=1
\end{gathered}
$$

8.

$$
\begin{gathered}
\forall\varepsilon>0,\exists\delta(\varepsilon)>0,\forall x\in B_{\delta(\varepsilon)}(a):|f(x)-A|<\varepsilon\Rightarrow \\
\forall\varepsilon>0,\exists\delta'(\varepsilon)=\sqrt{a}-\sqrt{a-\delta(\varepsilon)},\forall x\in B_{\delta'(\varepsilon)}(\sqrt{a}):|f(x^2)-A|<\varepsilon
\end{gathered}
$$

9.(1)

$$
\begin{gathered}
\forall\varepsilon>0,\exists\delta(\varepsilon)>0,\forall x\in B_{\delta(\varepsilon)}(0):|f(x^3)-A|<\varepsilon\Rightarrow \\
\forall\varepsilon>0,\exists\delta'(\varepsilon)=\delta(\varepsilon)^3,\forall x\in B_{\delta'(\varepsilon)}(0):|f(x)-A|<\varepsilon
\end{gathered}
$$

9.(2)

不成立.反例:

$$f(x)=\begin{cases}\dfrac{1}{x+1},x>0\\0,x=0\\\dfrac{1}{-1+x},x<0\end{cases}\tag{1}$$

证明Dirichlet函数有以下解析表达式$D(x)=\lim_{m\to\infty}\left\{\lim_{n\to\infty}\left[\cos(\pi m!x)\right]^{2n}\right\}$.

$$\begin{aligned}\lim_{n\to\infty}\left[\cos(\pi m!x)\right]^{2n}&=[\cos(\pi m!x)=\pm1]\\&=[m!x\in\mathbb{Z}]\end{aligned}$$

$$\text{如果}x\in\mathbb{Q},\exists p,q\in\mathbb{Z},x=\frac{p}{q}$$

$$\forall m>q:m!x\in\mathbb{Z},\text{此时}[m!x\in\mathbb{Z}]=1$$

$$D(x)=\lim_{m\to\infty}\left\{\lim_{n\to\infty}\left[\cos(\pi m!x)\right]^{2n}\right\}=1$$

$$\text{如果}x\in\mathbb{R}\backslash\mathbb{Q},\text{那么}m!x\text{仍然是无理数},\text{此时}[m!x\in\mathbb{Z}]=0$$

$$D(x)=\lim_{m\to\infty}\left\{\lim_{n\to\infty}\left[\cos(\pi m!x)\right]^{2n}\right\}=0$$