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基+附.

集合. Ω. A, B, 事件, $A, B \subset \Omega$.

$A \cap B$, $A \cup B = \{w \mid w \in A$ 或 $w \in B\}$,

$\{w \mid w \in A$ 且 $w \in B\}$.

$A \setminus B$, $A \triangle B$.

C.

$(A \cup B) \cup C = A \cup (B \cup C)$ ......

$(A \cap B)^c = A^c \cup B^c$, $(A^c)^c = A$.

$(A \cup B)^c = A^c \cap B^c$

1. 证明:

$$A = \bigcap_{n \in \mathbb{N}} \left\{ x \in \mathbb{R} : x > \frac{n}{n+1} \right\} = \{ x \in \mathbb{R} : x \geq 1 \}.$$

$[1, +\infty)$

x. $\forall n \in \mathbb{N}$, $x \in \left(\frac{n}{n+1}, +\infty\right)$.

若 $x \geq 1$, $x \in A$, 另一方面, $x < 1$, $x \notin A$,

I 为指标集. $i\in I$. A_i 一族集合

$$\bigcap_{i\in I} A_i = \{w \mid \forall i\in I,\ w\in A_i\}.$$

$$\bigcup_{i\in I} A_i = \{w \mid \exists i\in I,\ s.t\ w\in A_i\}.$$

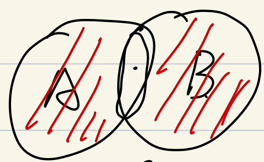
(x,y). $A_{x_0} = \{(x,y) \mid x=x_0\}$.

2.

差. $A\setminus B = \{w \mid w\in A$ 且 $w\notin B\}$. 不交换.

$A\triangle B = (A\setminus B)\cup(B\setminus A)$. 对称差.

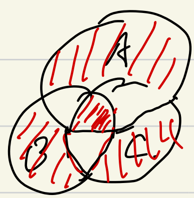
$B\setminus A$ 与 $A\setminus B$ 不相交



$A\cap B$

(1). $(A\triangle B)\triangle C = A\triangle(B\triangle C)$.

A B^c,



(2). $(A\triangle B)\cap C = (A\cap C)\triangle(B\cap C)$.

证明:

$$(A\cap C)\cap(B\cap C)^c \cup (A\cap C)^c\cap(B\cap C).$$

$$=[A\cap C\cap(B^c\cup C^c)]\cup[(A^c\cup C^c)\cap(B\cap C)]$$

$= [A \cap B^c \cap C] \cup [A^c \cap B \cap C]$

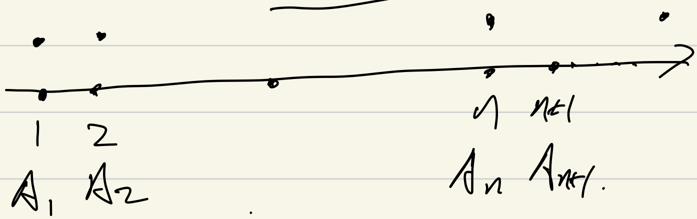
$= [(A \cap B^c) \cup (A^c \cap B)] \cap C$

$= (A \triangle B) \triangle C.$

(3). 略

3. $\{A_n, n \in N\}$ 是可列个集合构成的集合类

(i) $B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \iff$ B中元素属于无穷多的 A_n.



A_k

$B = \bigcap_{n=1}^{\infty} B_n$. $\quad B_n \supset B_{n+1}$ $\quad X \subset X \cup Y$.

$\forall \omega \in B$, $\forall n \in N^+$, $\omega \in B_n$.

$\exists k \geq n$, s.t. $\omega \in A_k$,

一方面,

另一方面, $\forall n \in N^+$, $\exists k \geq n$ s.t $\omega \in A_k$, B_n.

$\bigcap_{n=1}^{\infty} B_n$.

(iii). 至多不属于有限个 A_n.

$$C=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k, \quad =\bigcup_{n=1}^{\infty}B_n.$$

B与C的关系? $C\subset B$.

$\{A_n, n\geq 1\}$, $A_n\supset A_{n+1}$, 单调降.

令 $A=\bigcap_{n=1}^{\infty}A_n$, $\lim_{n\to\infty}A_n=A$. $A_n\downarrow A$

$B=\limsup_{n\to\infty}A_n$ $\{A_n\}$的上极限. $C=\liminf_{n\to\infty}A_n$ 下极限

映射. $f: X\to Y$,

集合 集合.

$\forall x\in X$, $\exists^{\text{唯一}} y\in Y$, s.t $f(x)=y$.

$f:\mathbb{R}\to\mathbb{R}$.

单射, 满射,

单: $\forall x_1\neq x_2$, $x_1,x_2\in X$, $f(x_1)\neq f(x_2)$

满: $\forall y\in Y$, $\exists x\in X$ s.t $f(x)=y$.

$f(X) = \{ f(x) \mid x \in X \}$ $\quad f': X \to f(X)$

$f(\phi) = \phi$ $\quad f'(x) = f(x)$

附1. $f: X \to Y$, $\quad 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$.

1. f 单射. $\quad 1 \Leftarrow 2,\ 1 \Leftarrow 3, \dots\dots$

2. $\forall A \subset X$, $f^{-1}(f(A)) = A$.

f^{-1} 双射, $\quad y \mapsto x$.

$x \mapsto y$

$f^{-1}(B) = \{ f^{-1}(y) \mid y \in B \}$.

3. A, B 为 X 子集, $f(A \cap B) = f(A) \cap f(B)$

$f': X \to f(X)$

$f'(x) = f(x)$

$1 \Rightarrow 3$, f' 双射, $f'(A \cap B) = f'(A) \cap f'(B)$.

$A \cup B = A \setminus B \cup A \cap B \cup B \setminus A$.

$3 \Rightarrow 1$, 若不然, $\exists\ x_1, x_2$ s.t. $f(x_1) = f(x_2) = y_0$.

$A = \{x_1\}$ $\quad B = \{x_2\}$ $\quad f(A \cap B) = f(A) \cap f(B)$.

$\phi = \{y\}$, 矛盾.

4 $\Leftarrow$ 7 1.

5 $\Leftarrow$ 7 1, 1 $\Rightarrow$ 7 5, 5 $\Rightarrow$ 7 1, 反证.

C-B-S 定理:

α, β 是两个基数, $\alpha \le \beta$ 且 $\beta \le \alpha$, 则 $\alpha = \beta$

step 1. A_0, B_0 集合, B_0 能得与 A_0 双射. A_0 B_0 双射,

那么 B_0 与 A_0 建立双射.

step2. $\varphi: A_0 \to B_1 \subset B_0$

$\psi: B_0 \to A_1 \subset A_0$

设 $B_{n+1} = \varphi(A_n)$, $A_{n+1} = \psi(B_n)$, $n = 1, 2, \cdots$

$\{A_n\}$, $\{B_n\}$ 单调降.

$B_2 = \varphi(A_1)$, $A_2 = \varphi(B_1)$.

证明:

$B_2 = \varphi(A_1) \subset \varphi(A_0) = B_1$. $\Rightarrow B_2 \subset B_1$

$A_2 = \psi(B_1) \subset \psi(B_0) = A_1$ $\Rightarrow A_2 \subset A_1$

$B_{k+1} \subset B_k$, $A_{k+1} \subset A_k$.

$B_{k+2} = \varphi(A_{k+1}) \subset \varphi(A_k) = B_{k+1}$. 归纳 ✓.

step 3. f 是单射, $X\to Y$, X 的子集上, f 也是单射,

$f|_A$, φ, ψ,

$n\in\mathbb{N}$, $\varphi_0=\varphi|_{A_0\setminus A_1}$, $A_0\setminus A_1\to B_1\setminus B_2=B_1\setminus\varphi(A_1)$

$X\supset A\supset B$. $f(A\setminus B)=f(A)\setminus f(B)$

$\varphi(A_0\setminus A_1)=\varphi(A_0)\setminus\varphi(A_1)=B_1\setminus B_2$ 满射.

✓.

step 4. $A_0=\tilde{A}'\cup\tilde{A}''\cup\tilde{A}$ $\tilde{A}$ 不交,

$\tilde{A}',\tilde{A}'',\tilde{A}$ 不同于任何偶的 A_{2n}.

A' 奇

step 5.

① $\varphi|_{A'}: A'\to B''$ 双射.

②

③. $\psi|_B: B\to A$ 双射.

$B=B_0\setminus(B'\cup B'')$ $A=A_0\setminus(A'\cup A'')$

$\psi|_B$ $\bigcap_{n=0}^{\infty}B_n=(\bigcap_{n=1}^{\infty}B_n)$

$$\psi(B_0 \setminus B' \cup B'') = \psi(B_0) \setminus \psi(B' \cup B'')$$

$$= A_1 \setminus (\psi(B') \cup \psi(B''))$$

$$= A_1 \setminus A'' \cup \psi(B'')$$

$$= A_1 \setminus A'' \cup (A' \setminus (A_0 \setminus A_1))$$

化简

$$= A_0 \setminus A'' \cup A' = A.$$

step 6: Θ 单, Θ 双射. $\Theta(A') = B''$

$$\Theta(A'' \cup A) = \psi^{-1}(A'' \cup A)$$

$$= B' \cup B.$$