Fall 2022 MATH1205H Homework XXVI

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December 24, 2022

Exercise 1.

No.Because we can change the order of σ sequence.

Exercise 2.

$$Ax = \lambda x$$

$$\begin{cases} 2x + y &= \lambda x \\ 4x + 2y &= \lambda y \end{cases}$$

$$\begin{cases} \lambda_1 &= 4 \\ \lambda_2 &= 0 \end{cases}$$

$$A^T A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$$

$$\begin{cases} \lambda_1' &= 25 \\ \lambda_2' &= 0 \end{cases}$$

$$\begin{cases} \sigma_1 &= 5 \\ \sigma_2 &= 0 \end{cases}$$

Exercise 3.

$$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{cases} 2x + y &= \lambda x \\ x + y &= \lambda y \end{cases}$$

$$\begin{cases} \lambda_1 = \dfrac{3-\sqrt{5}}{2} & v_1 = \sqrt{\dfrac{2}{5+\sqrt{5}}} \begin{bmatrix} 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}^T & u_1 = \dfrac{Av_1}{\sqrt{\lambda_1}} & = \sqrt{\dfrac{2}{5+\sqrt{5}}} \begin{bmatrix} -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \\ \lambda_2 = \dfrac{3+\sqrt{5}}{2} & v_2 = \sqrt{\dfrac{2}{5-\sqrt{5}}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \end{bmatrix}^T & u_2 = \dfrac{Av_2}{\sqrt{\lambda_2}} & = \sqrt{\dfrac{2}{5-\sqrt{5}}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \end{bmatrix} \end{cases}$$

$$\begin{cases} \sigma_1 = \dfrac{\sqrt{5}-1}{2} \\ \sigma_2 = \dfrac{\sqrt{5}+1}{2} \end{cases}$$

$$A = \begin{bmatrix} \sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ -\sqrt{\frac{2}{5+\sqrt{5}}}\frac{1+\sqrt{5}}{2} & \sqrt{\frac{2}{5-\sqrt{5}}}\frac{\sqrt{5}-1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}-1}{2} & 0 \\ 0 & \frac{\sqrt{5}+1}{2} \end{bmatrix} \begin{bmatrix} -\sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}}\frac{\sqrt{5}+1}{2} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}}\frac{\sqrt{5}-1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ -\sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}-1}{2} & 0 \\ 0 & \frac{\sqrt{5}+1}{2} \end{bmatrix} \begin{bmatrix} -\sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \end{bmatrix}$$

Exercise 4.

$$A = \sum_{i\in[n]} \sigma_i u_i v_i^T$$

Let $\sigma_i = \frac{1}{\|v_i\|^2}(\forall i \in [n])$

$$Av_j = \sum_{i\in[n]} \sigma_i u_i (v_i^T v_j)$$

$v_1, ..., v_n$ are orthonormal, so

$$v_i^T v_j = \begin{cases} \|v_i\|^2 & i = j \\ 0 & i \neq j \end{cases}$$

Therefore,

$$Av_i = u_i(\forall i \in [n])$$
$$A = U \begin{bmatrix} \frac{1}{\|v_1\|^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\|v_2\|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\|v_n\|^2} \end{bmatrix} V^T$$

Exercise 5.

sufficiency

$$\forall \lambda \text{ is an eigenvalue with a corresponding eigenvector } v : Sv = \lambda v, \lambda v^T v = v^T S v$$
$$\lambda = \frac{v^T S v}{v^T v} \geq 0$$

So it's semidefinite.

neccessity

$$S = \lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T (\forall i \in [n] : \lambda_i \geq 0)$$

Then

$$x^T S x = \sum_{i\in[n]} \lambda_i x^T v_i v_i^T x = \sum_{i\in[n]} \lambda_i (v_i^T x)^2 \geq 0$$