# Fall 2022 MATH1205H Homework XXV

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## Exercise 1.

If $L$ is not the zero function,

$$\exists v \in V : L(v) \neq 0$$

So

$$\forall x \in \mathbb{R} : x = \frac{x}{L(v)} L(v) = L(\frac{x}{L(v)} v)$$

Therefore, $L$ is surjective.

## Exercise 2.

$$\forall v \in V : T'(L)(v) = L(I(v)) = L(v)$$

Therefore, $\forall L \in V' : T'(L) = L$

## Exercise 3.

Choose an arbitrary basis $\tilde{v}_1, ..., \tilde{v}_n$ and the corresponding dual basis $\tilde{L}_1, ..., \tilde{L}_n$.

$$\exists M = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{n \times n} : \begin{bmatrix} \tilde{L}_1 & \cdots & \tilde{L}_n \end{bmatrix} M = \begin{bmatrix} L_1 & \cdots & L_n \end{bmatrix}$$

$$L_i = \begin{bmatrix} \tilde{L}_1 & \cdots & \tilde{L}_n \end{bmatrix} x_i$$

Suppose

$$(M^{-1})^T = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}$$

So

$$x_i \cdot y_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Let

$$v_i = \begin{bmatrix} \tilde{v}_1 & \cdots & \tilde{v}_n \end{bmatrix} y_i$$

Then

$$L_i(v_i) = x_i \cdot y_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

So the corresponding dual basis for $v_1, ..., v_n$ is precisely $L_1, ..., L_n$

# Exercise 4.

## $T = 0 \Rightarrow T' = 0$

$$\forall v \in V : T(v) = 0$$

Therefore

$$\forall L \in W', \forall v \in V : T'(L)(v) = L(Tv) = L(0) = 0$$
$$\forall L \in W' : T'(L) = 0$$

So $T' = 0$

## $T' = 0 \Rightarrow T = 0$

$$\forall L \in W', \forall v \in V : L(Tv) = T'(L)(v) = 0$$

If $T \neq 0$, i.e., $\exists v_0 \in V : Tv_0 \neq 0$

$$T'((Tv_0)')(v_0) = (Tv_0)'(Tv_0) = 1$$

which contradicts with above.So $T = 0$

# Exercise 5.

## sufficiency

If

$$A = uv^T = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \cdots & u_mv_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

Without loss of generosity, we suppose that $v_1 \neq 0$.So

$$\forall i \geq 2 : a_i = \frac{v_i}{v_1} a_1 \in \text{span}\{a_1\}$$

So $a_1$ is a basis for $C(A)$ and $\text{rank}(A) = 1$

## neccessity

$\text{rank}(A) = 1$ and suppose $u$ is a basis of $C(A)$, then

$$A = \begin{bmatrix} v_1u & v_2u & \cdots & v_mu \end{bmatrix} (\prod_{i \in [m]} v_i \neq 0)$$

So

$$A = u \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} = uv^T$$