0.1 Matrix Exponential

We consider the matrix exponential

$$e^{-Mt}, \quad \text{where} \quad M = \begin{bmatrix} \lambda & \alpha \\ -\alpha & \lambda \end{bmatrix}.$$

It is expected that

$$e^{-Mt} = \begin{bmatrix} e^{-\lambda t}\cos(\alpha t) & -e^{-\lambda t}\sin(\alpha t) \\ e^{-\lambda t}\sin(\alpha t) & e^{-\lambda t}\cos(\alpha t) \end{bmatrix}. \tag{1}$$

Proof. First, we decompose M into two parts

$$M = A + B = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}. \tag{2}$$

The exponential of the diagonal part can be expanded using a Taylor series, yielding

$$e^{-At} = \sum_{n=0}^{\infty} \frac{(-At)^n}{n!} = I - At + \frac{1}{2}(-At)^2 + \dots \tag{3}$$

$$= \begin{bmatrix} e^{-\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{bmatrix}. \tag{4}$$

For the off-diagonal part B, eigenvalue diagonalization is necessary. Thus, we have

$$B = S\Lambda S^{-1}, \tag{5}$$

where $S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{bmatrix}$, and $\Lambda = \begin{bmatrix} -i\alpha & 0 \\ 0 & i\alpha \end{bmatrix}$.

Similarly, the exponential of the off-diagonal matrix B should be

$$\begin{aligned} e^{-Bt} &= e^{-S\Lambda S^{-1}} \\ &= \sum_{n=0}^{\infty} \frac{(-S\Lambda S^{-1}t)^n}{n!} \\ &= I - S\Lambda S^{-1}t + \frac{1}{2}S\Lambda^2 S^{-1}t^2 - \frac{1}{6}S\Lambda^3 S^{-1}t^3 + \dots \\ &= S(I + (-\Lambda t) + \frac{1}{2}(-\Lambda t)^2 + \frac{1}{6}(-\Lambda t)^3 + \dots)S^{-1} \\ &= Se^{-\Lambda t}S^{-1} \end{aligned}$$

By inserting the matrices from (5), we have

$$\begin{aligned} e^{-Bt} &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{i\alpha t} & 0 \\ 0 & e^{-i\alpha t} \end{bmatrix} \cdot \begin{bmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(e^{i\alpha t} + e^{-i\alpha t}) & \frac{i}{2}(e^{i\alpha t} - e^{-i\alpha t}) \\ \frac{i}{2}(-e^{i\alpha t} + e^{-i\alpha t}) & \frac{1}{2}(e^{i\alpha t} + e^{-i\alpha t}) \end{bmatrix}. \end{aligned}$$

Using Euler's formula

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{6}$$

$$\sin(x) = -i\frac{e^{ix} - e^{-ix}}{2}, \tag{7}$$

it follows that

$$e^{-Bt} = \begin{bmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{bmatrix}. \tag{8}$$

Therefore, the desired exponential is

$$e^{-Mt} = e^{-At} \cdot e^{-Bt} = \begin{bmatrix} e^{-\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{bmatrix} \cdot \begin{bmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} e^{-\lambda t}\cos(\alpha t) & -e^{-\lambda t}\sin(\alpha t) \\ e^{-\lambda t}\sin(\alpha t) & e^{-\lambda t}\cos(\alpha t) \end{bmatrix}. \tag{10}$$

□