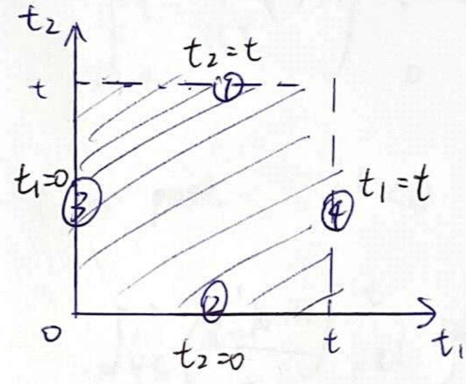
$$\lim_{t \to \infty} \frac{1}{2t} \int_0^t dt_1 \int_0^t dt_2 \, \langle u(t_1) u(t_2) \rangle$$

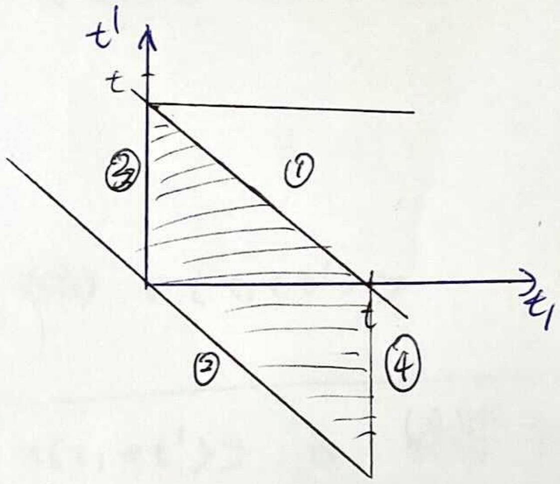
change variable

$t_2 = t_1 + t'$

New variable $\rightarrow$ $\boxed{\begin{array}{l} t' = t_2 - t_1 \\ t_1 = t_1 \end{array}}$

original integral





① $t_2 = t = t_1 + t' \Rightarrow -t_1 + t = t'$

② $t_2 = 0 = t_1 + t' \Rightarrow t_1 = -t'$

③ $t_1 = 0 = t_1 \Rightarrow t_1 = 0$

④ $t_1 = t = t_1 \Rightarrow t_1 = t$

Jacobian
$$\begin{vmatrix} \frac{dt'}{dt_1} & \frac{dt'}{dt_2} \\ \frac{dt_1}{dt_1} & \frac{dt_1}{dt_2} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

New integral

$$\iint_{?} dt_1 \, dt' \, \langle u(t_1) u(t_1 + t') \rangle \, |\text{Jacobian}|$$

$|-1| = 1$

With the black integral graph, we can split parallelogram into 2 parts

Upper part: $t' > 0$

$$\int_0^t dt' \int_0^{t-t'} dt_1 \langle u(t_1)\, u(t_1+t') \rangle$$

lower part $t' < 0$

$$\int_{-t}^0 dt' \int_{-t'}^{t} dt_1 \langle u(t_1)\, u(t_1+t') \rangle$$

We assume that $\langle u(t_1)\, u(t_1+t')\rangle$ is time-homogenous

$\langle u(t_1)\, u(t_1+t')\rangle = \langle u(0)\, u(t')\rangle$. Then, we can define a function

$$f(t') := \langle u(t_1)\, u(t_1+t')\rangle$$

~~Therefore $f(t')$~~

Additional symmetry condition is $f(t') = f(-t')$. ~~It yields that~~

Under the symmetry condition, the outer integral $\int_0^t dt'$ and $\int_{-t}^0 dt'$ are equivalent !

Knowing that upper part requires $t' > 0$ and lower part requires $t' < 0$, $\int_{-t'}^{t} dt_1$ and $\int_0^{t-t'} dt_1$ are also equivalent !

Hence, 2 integral are the same ! $\square$

Using the assumption that velocity process $u(x)\cdot u(x+t)$ is time - homogenous

$$\int_0^t dt' \int_0^{t-t'} dt_1 \ \langle u(t_1)\, u(t_1+t') \rangle$$

$$= \int_0^t dt' \int_0^{t-t'} dt_1 \ \langle u(0)\, u(0+t') \rangle$$

$$\cancel{= t \int_0^{t-t'} dt_1 \ \langle u(0)\, u(t')}$$

$$= t-t' \int_0^t dt' \ \langle u(0)\, u(t') \rangle$$

finally, Let's plug it back into the limit term

$$\lim_{t\to\infty} \frac{1}{t} (t-t') \int_0^t dt' \ \langle u(0)\, u(t') \rangle$$

$$= \cancel{t/t} \int_0^\infty \langle u(t_0)\, u(t_0+t') \rangle dt'$$

$\square$