# Homework 4

**Note:** The due date is February 29 (Thursday). The goal of this homework consists of two. First, I expect you to extend your knowledge of the ARMA class to the ARIMA class from a modeling perspective. Second, I expect you to practice the procedure of the unit-root tests for a given time series. Any questions about R/RStudio, email the teaching assistant. All problems below will carry equal weight.

**Problem 1.** Recall that the difference operator $\Delta$ is defined by $\Delta X_t = (I - B)X_t = X_t - X_{t-1}$. Likewise, $\Delta^k X_t = (I - B)^k X_t$, $k \geq 1$ and $\Delta_s X_t = (I - B^s)X_t = X_t - X_{t-s}$, $s \geq 1$. Do the following.

(a) Suppose $X_t = \beta_1 + \beta_2 t + \ldots + \beta_k t^r + Y_t$ with $\beta_k \neq 0$, where $\{Y_t\}$ is a stationary process. Show that $\Delta^k Y_t$ is stationary for $k \geq 1$ and $\Delta^k X_t$ is stationary for $k \geq r$. If needed, you can denote the ACVFs of $\{X_t\}$ and $\{Y_t\}$ as $\gamma_X(h)$ and $\gamma_Y(h)$, respectively.

(b) Let $X_t = a + bt + S_t + Y_t$, where $a, b$ are constants, $S_t$ is a seasonal component with period 12, and $\{Y_t\}$ is a stationary process with mean zero. Denote the ACVF of $\{Y_t\}$ as $\gamma_Y(h)$. Show that $\Delta_{12} X_t$ is stationary; Express its ACVF in terms of $\gamma_Y(h)$.

**Problem 2.** Consider the time series data `gnp` from the R package `astsa`. Do the following.

(a) Take a suitable preliminary transformation of the series, and produce its time plot; In the following parts, work with the transformed series.

(b) Leave out the last 5 observations. Denote these samples as test data. The remaining observations are your training data. Fit a quadratic trend to the series using regression with the training data; Produce a time plot and a correlogram of the residuals obtained after removing the trend from the series;

(c) Fit an ARMA$(p, q)$ model to the residual of the regression with an order $p, q$ determined by an information criterion; Include the output; Produce the sample ACF and PACF of the residuals of the ARMA$(p, q)$ model; Check the assumptions of white noise and normality for the residuals;

(d) Forecast the transformed time series for 5 steps into the future; Compute the mean squared forecast error (MSFE) by using the test data; Provide the outputs;

Are we forecasting trainigndata here?

or residuals?

If residuals then what does it make sense that we fit a model to the residuals and use it to model the original series

(e) Now, fit an ARIMA$(p, d, q)$ model to the training data; Forecast the transformed time series for 5 steps into the future by using this model; Compute the MSFE again; Which model is preferred in terms of minimal MSFE? Include the output;

**Problem 3.** Do the following.

(a) Use `set.seed(99)` to generate the following random walk with drift:

include.drift = TRUE
allowdrift = TRUE

$$X_t = -0.2t + 0.8 \sum_{s=1}^{t} Z_s, \quad t = 1, \ldots, 100,$$

is q=100 or less here?

where $\{Z_t\}$ is IID standard normal; Include a time plot of the series and sample ACF and PACF of the series.

allowmean ?

Should we inclide?

Are we using ARMA(pq) or ARMA(pdq)?

(b) Use the function `auto.arima` with suitable inputs to recover this model. That is, can you find an outcome of the model indicating the random walk with drift?

(c) Go through the testing procedure for unit roots with significance level $\alpha = 0.05$ (for all steps). Indicate the conclusion at each step of the procedure; Check if the testing result corresponds to the used model;

**Problem 4.** Do the following real data applications.

(a) Consider the time series `Raotbl3$lc` of real consumption expenditure from the United Kingdom starting in 1966:4 until 1991:2 in the R package `urca`. Produce a time plot of the series; Go through the testing procedure for unit roots discussed in class taking $k = 3$ for the number of lagged series differences to include in the regression; Indicate the conclusion at each step of the procedure.

(b) Repeat the testing procedure with the series in (a) with a smaller lag, $k = 2$; Check if a different choice of lag affects the conclusion;

(c) Go through the testing procedure with the transformed `gnp` used in Problem 1. Here, use the entire samples. Use the lag $p$ determined for the ARMA model. If your ARMA model contains the MA part, use `selectlags`, contained to `ur.df` function; Check if the conclusion of this problem corresponds to the preference of the model (i.e. trend stationary or non-stationary) in Problem 1. (e);

Do we need to fit anther model here or use the one we trained in 2c?

**Problem 1.** Recall that the difference operator $\Delta$ is defined by $\Delta X_t = (I - B)X_t = X_t - X_{t-1}$. Likewise, $\Delta^k X_t = (I - B)^k X_t$, $k \geq 1$ and $\Delta_s X_t = (I - B^s)X_t = X_t - X_{t-s}$, $s \geq 1$. Do the following.

(a) Suppose $X_t = \beta_1 + \beta_2 t + \ldots + \beta_r t^r + Y_t$ with $\beta_k \neq 0$, where $\{Y_t\}$ is a stationary process. Show that $\Delta^k Y_t$ is stationary for $k \geq 1$ and $\Delta^k X_t$ is stationary for $k \geq r$. If needed, you can denote the ACVFs of $\{X_t\}$ and $\{Y_t\}$ as $\gamma_X(h)$ and $\gamma_Y(h)$, respectively. [Hint: Think trend and $Y_t$ separately. Use binomial theorem in terms of $Y_t$.]

(b) Let $X_t = a + bt + S_t + Y_t$, where $a, b$ are constants, $S_t$ is a seasonal component with period 12, and $\{Y_t\}$ is a stationary process with mean zero. Denote the ACVF of $\{Y_t\}$ as $\gamma_Y(h)$. Show that $\Delta_{12} X_t$ is stationary; Express its ACVF in terms of $\gamma_Y(h)$.

(a) $X_t = \beta_1 + \beta_2 t + \ldots + \beta_r t^r + Y_t$, $\beta_K \neq 0$

$\{Y_t\}$ stationary.

Show $\Delta^K Y_t$ stationary for $K \geq 1$

$\boxed{\text{By induction}}$

Case $K=1$ $\quad \Delta Y_t = y_t - y_{t-1}$

i) $\mathbb{E}(\Delta y_t) = \mathbb{E}(y_t) - \mathbb{E}(y_{t-1})$ both constant as $y_t$ stationary

ii) $\mathrm{Var}(\Delta y_t) = \gamma_Y(h=0)$ constant as $y_t$ stationary

iii) $\mathrm{Cov}(\Delta y_t, \Delta y_{t+h}) = \mathrm{Cov}(y_t - y_{t-1}, y_{t+h} - y_{t+h-1})$

$= \mathbb{E}[(y_t - y_{t-1})(y_{t+h} - y_{t+h-1})] - \mathbb{E}[y_t - y_{t-1}]\mathbb{E}[y_{t+h} - y_{t+h-1}]$

$= \mathbb{E}[y_t y_{t+h}] - \mathbb{E}[y_t y_{t+h-1}] - \mathbb{E}[y_{t-1} y_{t+h}] + \mathbb{E}[y_{t-1} y_{t+h-1}]$

$- \{[\mathbb{E}(y_t) - \mathbb{E}(y_{t-1})][\mathbb{E}(y_{t+h}) - \mathbb{E}(y_{t+h-1})]\}$

$= \mathbb{E}[y_t y_{t+h}] - \mathbb{E}[y_t y_{t+h-1}] - \mathbb{E}[y_{t-1} y_{t+h}] + \mathbb{E}[y_{t-1} y_{t+h-1}]$

$- \mathbb{E}(y_t)\mathbb{E}(y_{t+h}) + \mathbb{E}(y_t)\mathbb{E}(y_{t+h-1}) + \mathbb{E}(y_{t-1})\mathbb{E}(y_{t+h})$

$- \mathbb{E}(y_{t-1})\mathbb{E}(y_{t+h-1})$

$= \mathrm{Cov}(y_t, y_{t+h}) - \mathrm{Cov}(y_t y_{t+h-1}) - \mathrm{Cov}(y_{t-1}, y_{t+h}) + \mathrm{Cov}(y_{t-1} y_{t+h-1})$

Given $\{y_t\}$ stationary, each covariance is constant or only depends on the lag $h$

$\Rightarrow \Delta^K y_t$ is stationary for $K=1$

Lets proceed to prove it for $K>1$, $K \in \mathbb{N}$

Case $k=n+1$ given $k=n$

Assume $\Delta^n y_t$ stationary and we know:

$$\Delta^{n+1} y_t = (I-B)^{n+1} y_t = (I-B)(I-B)^n y_t = \Delta(\Delta^n y_t)$$

i) $\mathbb{E}[\Delta(\Delta^n y_t)] = \mathbb{E}(\Delta^n y_t - \Delta^n y_{t-1}) = \mathbb{E}(\Delta^n y_t) - \mathbb{E}(\Delta^n y_{t-1})$ both constant by assumption

ii) $Var[\Delta(\Delta^n y_t)] = Var[\Delta^n y_t] + Var[\Delta^n y_{t-1}] + 2\,\gamma_{\Delta^n y_t}(h=1)$

both constant by assumption

iii) $Cov[\Delta(\Delta^n y_t)] = Cov(\Delta^n y_t - \Delta^n y_{t-1}, \Delta^n y_{t+h} - \Delta^n y_{t+h-1})$

$$= \mathbb{E}\left[(\Delta^n y_t - \Delta^n y_{t-1})(\Delta^n y_{t+h} - \Delta^n y_{t+h-1})\right] - \left[\mathbb{E}(\Delta^n y_t - \Delta^n y_{t-1})\,\mathbb{E}(\Delta^n y_{t+h} - \Delta^n y_{t+h-1})\right]$$

$$= \mathbb{E}(\Delta^n y_t \Delta^n y_{t+h}) - \mathbb{E}(\Delta^n y_t \Delta^n y_{t+h-1}) - \mathbb{E}(\Delta^n y_{t-1} \Delta^n y_{t+h}) + \mathbb{E}(\Delta^n y_{t-1} \Delta^n y_{t+h-1}) - \left\{\left[\mathbb{E}(\Delta^n y_t) - \mathbb{E}(\Delta^n y_{t-1})\right]\left[\mathbb{E}(\Delta^n y_{t+h}) - \mathbb{E}(\Delta^n y_{t+h-1})\right]\right\}$$

$$= \mathbb{E}(\Delta^n y_t \Delta^n y_{t+h}) - \mathbb{E}(\Delta^n y_t \Delta^n y_{t+h-1}) - \mathbb{E}(\Delta^n y_{t-1} \Delta^n y_{t+h}) + \mathbb{E}(\Delta^n y_{t-1} \Delta^n y_{t+h-1}) - \mathbb{E}(\Delta^n y_t)\mathbb{E}(\Delta^n y_{t+h}) + \mathbb{E}(\Delta^n y_t)\mathbb{E}(\Delta^n y_{t+h-1}) + \mathbb{E}(\Delta^n y_{t-1})\mathbb{E}(\Delta^n y_{t+h}) - \mathbb{E}(\Delta^n y_{t-1})\mathbb{E}(\Delta^n y_{t+h-1})$$

$$= Cov(\Delta^n y_t, \Delta^n y_{t+h}) - Cov(\Delta^n y_t, \Delta^n y_{t+h-1}) - Cov(\Delta^n y_{t-1}, \Delta^n y_{t+h}) + Cov(\Delta^n y_{t-1}, \Delta^n y_{t+h-1})$$

Given our assumption all these Covariances are either constant or only depend on lag $h$

Conclusion: By induction we have proven $\Delta^k y_t$ is stationary for $k \geq 1$, $k \in \mathbb{N}$

Show $\Delta^k X_t$ stationary for $k \geq r$

with $X_t = \beta_1 + \beta_2 t + \dots + \beta_r t^r + Y_t$

An expression for the k-th differencing operator can be obtained from binomial theorem

$$\Delta^k X_t = (1-B)^k X_t = \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell X_{t-\ell}$$ Ruppert & Matteson (2015)

Let $m_t = \sum_{i=0}^{r} \beta_i t^i$ then $\Delta^k X_t = (1-B)^k m_t + (1-B)^k Y_t$

Trend

$$\Delta m_t = \sum_{i=0}^{r} \beta_i t^i - \sum_{i=0} \beta_i (t-1)^i$$ by binomial theorem $(t-1)^i = \sum_{j=0}^{i} \binom{i}{j} t^j (-1)^{i-j}$

$$= \sum_{i=0}^{r} \beta_i t^i - \sum_{i=0}^{r} \beta_i \sum_{j=0}^{i} \binom{i}{j} t^j (-1)^{i-j}$$

$$= \sum_{i=0}^{r} \beta_i t^i - \sum_{i=0}^{r} \beta_i \underbrace{\binom{i}{i} t^i (-1)^{i-i}}_{t^i} - \sum_{i=1}^{r} \beta_i \sum_{j=0}^{i-1} \binom{i}{j} t^j (-1)^{i-j}$$

$$= -\sum_{i=0}^{r-1} \beta_{i+1} \sum_{j=0}^{i} \binom{i+1}{j} t^j (-1)^{i-j+1}$$

$$= -\sum_{j=0}^{r-1} t^j \sum_{i=j}^{r-1} \beta_{i+1} \binom{i+1}{j} (-1)^{i-j+1}$$

Let $\tilde{\beta}_j = -\sum_{i=j}^{r-1} \beta_{i+1} \binom{i+1}{j} (-1)^{i-j+1}$ then $\Delta m_t = \sum_{j=0}^{r-1} \tilde{\beta}_j t^j$

a polynomial of at most $(r-1)$ degrees.

Case $0 \leq k \leq r$

Thinking recursively,

$\Delta^k m_t$ is a polynomial of order $(r-k)$ when $0 \leq k < r$

We know that the expectation of any polynomial in time $t$, by linearity of expectations, is equal to the summation of the expectation of each individual term, all of which are not random variables, rather, constants equal to the time period $t$, e.g., $\mathbb{E}(\sum_{i=0}^{r-1} \alpha_i t^i) = \sum_i \alpha_i \mathbb{E}(t^i) = \sum_i \alpha_i t^i$

Thus any differentiation $\Delta^k m_t$ with $0 \leq k < r$ will have an expected value dependent on time $t$, and therefore not stationary

**Case $k=r$**

Since $\Delta^{r-1} m_t$ is a polynomial of degree 1 then $\Delta^r m_t = \Delta(\Delta^{r-1} m_t)$ is a polynomial of degree 0, a constant $\beta_k (k!)$ where $k=r$ which have constant expectation and zero variance and covariance, thus, are stationary

**Case $k>r$**

Since $\Delta^r m_t$ is a constant, therefore, $\Delta^k m_t = 0$ $\forall k > r$, also stationary

Now, $\Delta^k X_t = (1-B)^k m_t + (1-B)^k Y_t$

i) $\mathbb{E}(\Delta^k X_t) = \mathbb{E}(\Delta^k m_t) + \mathbb{E}(\Delta^k Y_t)$

$\rightarrow$ constant as both are stationary

ii) $Var(\Delta^k X_t) = Var(\Delta^k m_t) + Var(\Delta^k Y_t) + 2\, Cov(\underbrace{\Delta^k m_t}_{\text{constant}}, \Delta^k y_t)$

$\rightarrow$ constant as $\{m_t\}, \{Y_t\}$ stationary

iii) $Cov(\Delta^k X_t, \Delta^k X_{t+h})$

$= Cov(\Delta^k m_t + \Delta^k Y_t, \Delta^k m_{t+h} + \Delta^k Y_{t+h})$

Given $k \geq r \Rightarrow \Delta^k m_t$ and $\Delta^k m_{t+h}$ are constant

$= Cov(\Delta^k y_t, \Delta^k y_{t+h})$

which we already know only depends on lagh

$\Rightarrow \Delta^k X_t$ stationary

(b) Let $X_t = a + bt + S_t + Y_t$, where $a, b$ are constants, $S_t$ is a seasonal component with period 12, and $\{Y_t\}$ is a stationary process with mean zero. Denote the ACVF of $\{Y_t\}$ as $\gamma_Y(h)$. Show that $\Delta_{12}X_t$ is stationary; Express its ACVF in terms of $\gamma_Y(h)$.

(b)

$$\begin{aligned}\Delta_{12} X_t &= X_t - X_{t-12} \\ &= a + bt + S_t + Y_t - (a + b(t-12) + S_{t-12} + Y_{t-12}) \\ &= 12b + \underbrace{(S_t - S_{t-12})}_{=0} + (Y_t - Y_{t-12})\end{aligned}$$

i) $\mathbb{E}(\Delta_{12} X_t) = 12b + \underbrace{\mathbb{E}(Y_t) - \mathbb{E}(Y_{t-12})}_{=0} = 12b$ ✓ constant, independent of time $t$

ii) $Var(\Delta_{12} X_t) = Var(Y_t - Y_{t-12}) = \underbrace{Var(Y_t) + Var(Y_{t-12})}_{= m,\ m \in \mathbb{R} \text{ constant}} \underbrace{- 2\gamma_Y(h=12)}_{\text{Doesnt depend on time } t}$

✓ constant, independent of time $t$

iii) $\gamma(\Delta_{12} X_t) = Cov(\Delta_{12} X_t, \Delta_{12} X_{t+h})$

$$= Cov\left[12b + \underbrace{(S_t - S_{t-12})}_{=0} + (Y_t - Y_{t-12}),\ 12b + \underbrace{(S_{t+h} - S_{t+h-12})}_{=0} + (Y_{t+h} - Y_{t+h-12})\right]$$

given $12b \in \mathbb{R}$, constant

$$= Cov\left[Y_t - Y_{t-12},\ Y_{t+h} - Y_{t+h-12}\right]$$

$$= \gamma_Y(h) - \gamma_Y(h-12) - \gamma_Y(h+12) + \gamma_Y(h)$$

✓ only depends on time $h$

# ORIE5550_HW4_Markdown

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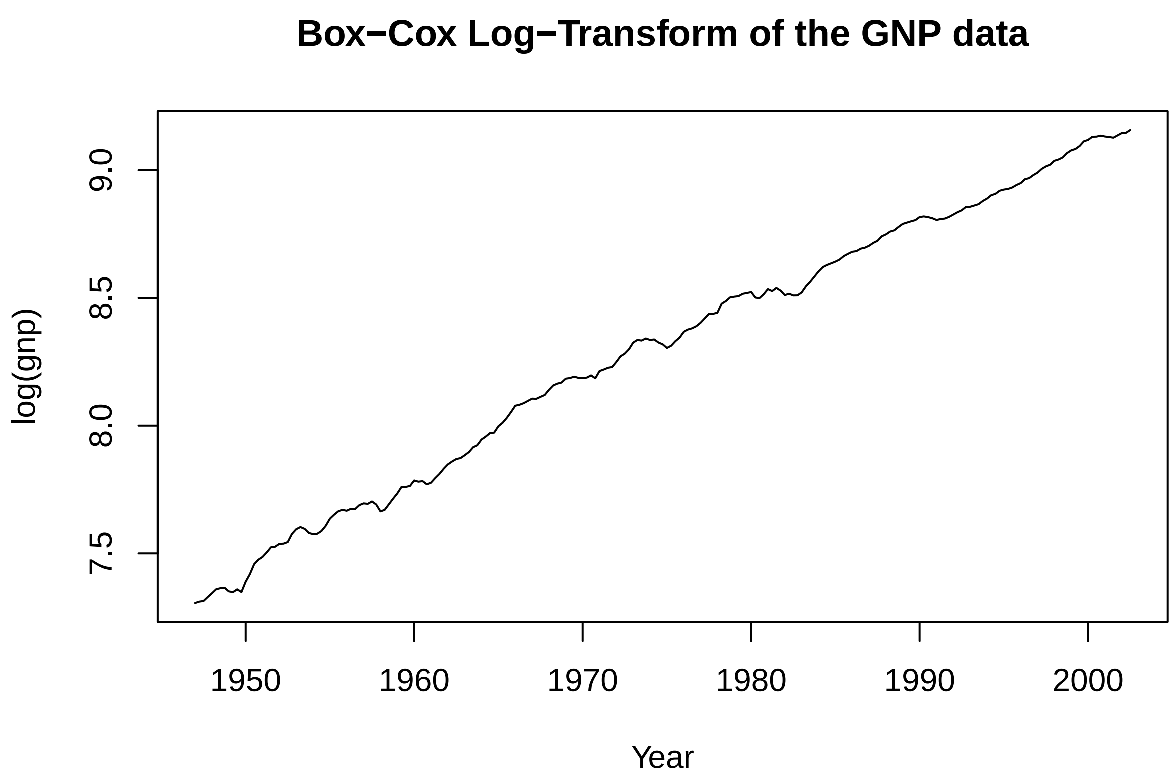
## Question 2

Consider the time series data gnp from the R package astsa.

#### (a) Take a suitable preliminary transformation of the series, and produce its time plot; In the following parts, work with the transformed series.

```
gnp_data <- ts(gnp, start = c(1947, 1), end = c(2002, 3) , frequency = 4)
gnp_data = log(gnp_data)
plot(gnp_data, type="l", main="Box-Cox Log-Transform of the GNP data",
     ylab = "log(gnp)", xlab = "Year")
```

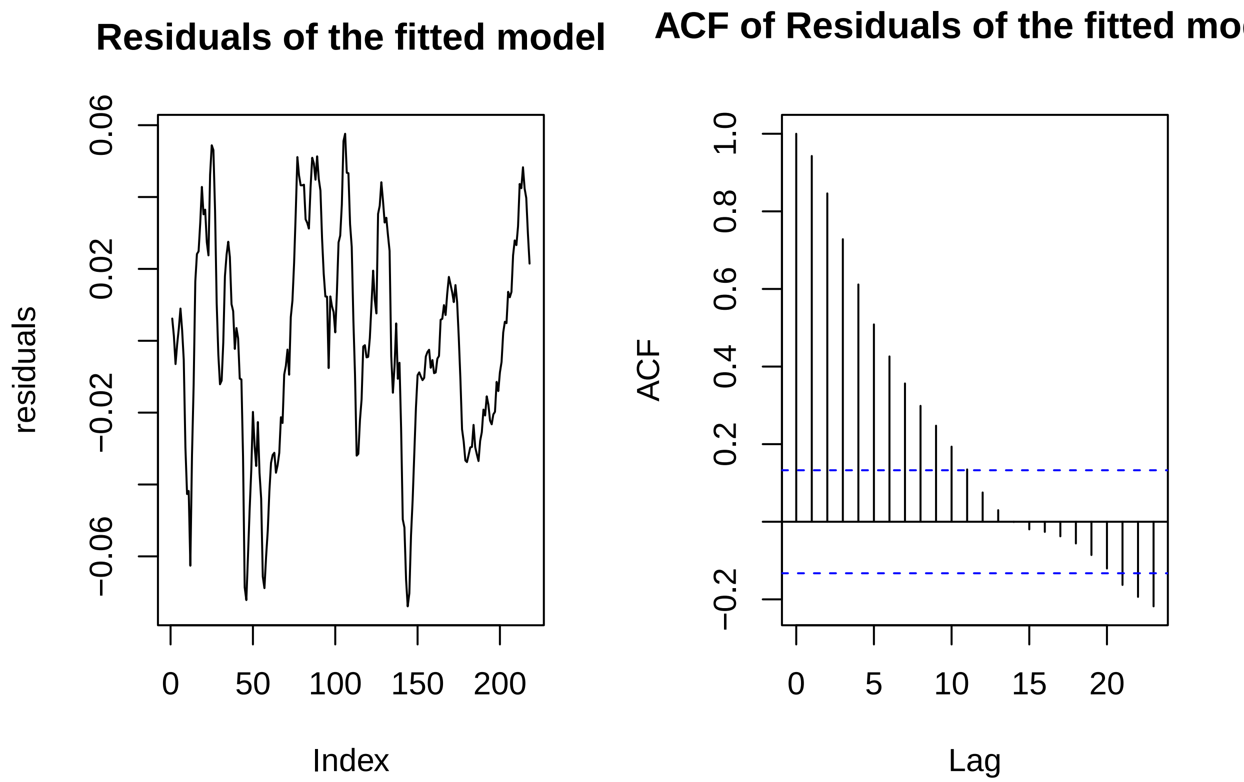


**(b) Leave out the last 5 observations. Denote these samples as test data. The remaining observations are your training data. Fit a quadratic trend to the series using regression with the training data; Produce a time plot and a correlogram of the residuals obtained after removing the trend from the series**

```r
# Leave out the last 5 observations as test data
training_data <- head(gnp_data, -5)
test_data <- tail(gnp_data, 5)

tt <- seq(1, length(training_data), by=1)
tt2 <- tt^2

fitModel <- lm(training_data ~ tt + tt2)
summary(fitModel)
```

```
##
## Call:
## lm(formula = training_data ~ tt + tt2)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.073915 -0.021840 -0.000729  0.024088  0.057608
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  7.289e+00  6.264e-03 1163.78   <2e-16 ***
## tt           1.025e-02  1.321e-04   77.63   <2e-16 ***
## tt2         -8.755e-06  5.840e-07  -14.99   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.03054 on 215 degrees of freedom
## Multiple R-squared:  0.9967, Adjusted R-squared:  0.9966
## F-statistic: 3.225e+04 on 2 and 215 DF,  p-value: < 2.2e-16
```

```r
residuals <- fitModel$residuals

par(mfrow = c(1, 2))
plot(residuals, type="l", main="Residuals of the fitted model")
acf(residuals, main="ACF of Residuals of the fitted model")
```




**(c) Fit an ARMA(p,q) model to the residual of the regression with an order p,q determined by an information criterion; Include the output; Produce the sample ACF and PACF of the residuals of the ARMA(p,q) model; Check the assumptions of white noise and normality for the residuals**

```
auto.arima(residuals,max.p=10,max.q=10,ic="aic",allowmean = FALSE) # AIC
```

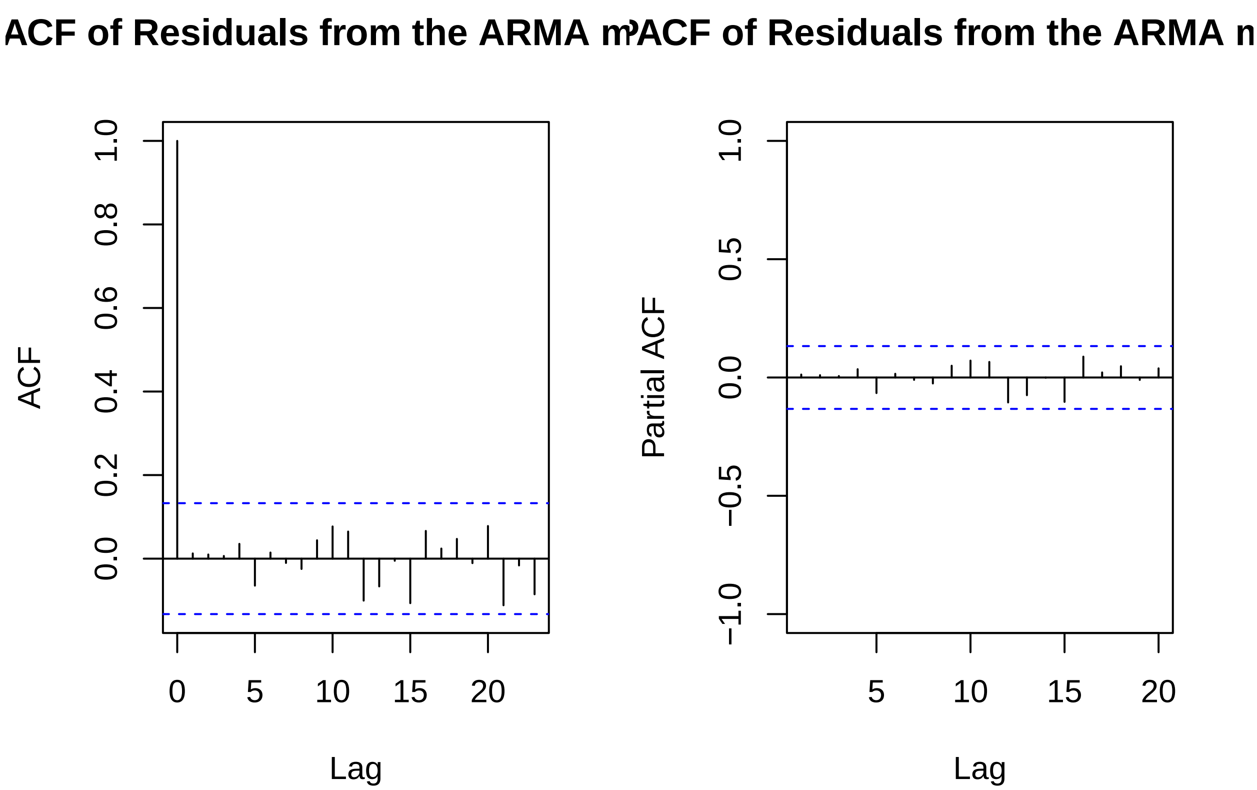
```
## Series: residuals
## ARIMA(1,0,3) with zero mean
##
## Coefficients:
##          ar1     ma1     ma2     ma3
##       0.8499  0.3880  0.3237  0.1553
## s.e.  0.0430  0.0744  0.0774  0.0733
##
## sigma^2 = 8.49e-05:  log likelihood = 713.16
## AIC=-1416.32   AICc=-1416.03   BIC=-1399.39
```

```
auto.arima(residuals,max.p=10,max.q=10,ic="bic",allowmean = FALSE) # BIC
```

```
## Series: residuals
## ARIMA(2,0,0) with zero mean
##
```

```
## Coefficients:
##          ar1      ar2
##       1.3015  -0.3812
## s.e.  0.0623   0.0624
##
## sigma^2 = 8.638e-05:  log likelihood = 710.3
## AIC=-1414.61   AICc=-1414.5   BIC=-1404.46
```

```r
arma.model <- arima(residuals, order=c(1,0,3), include.mean = FALSE, method = "ML")
par(mfrow = c(1, 2))
acf(arma.model$residuals, main="ACF of Residuals from the ARMA model")
pacf(arma.model$residuals, lag.max=20,
     ylim=c(-1, 1), main = "PACF of Residuals from the ARMA model") # PACF
```

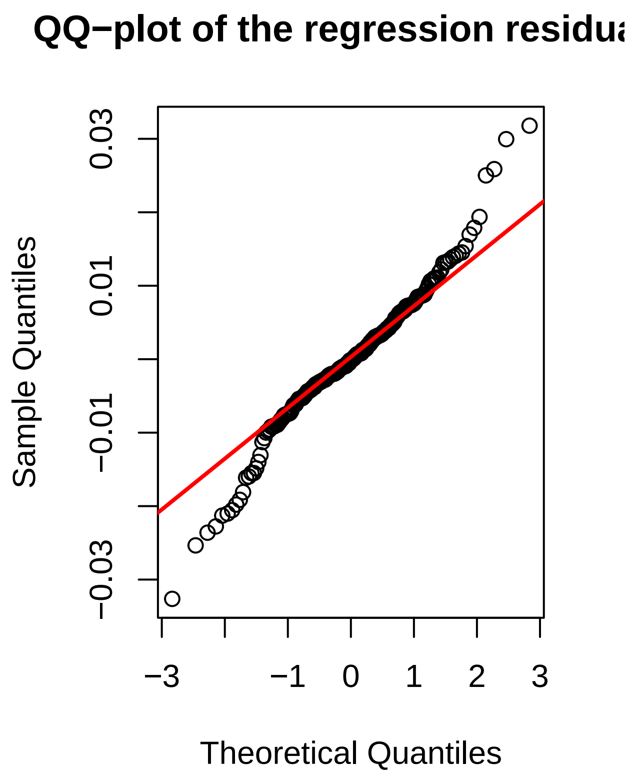



```r
# QQ plot
qqnorm(arma.model$residuals, main = "QQ-plot of the regression residuals")
qqline(arma.model$residuals, col = "red", lwd = 2)

# Shapiro-Wilks test. Ho is normality
shapiro.test(arma.model$residuals)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  arma.model$residuals
## W = 0.97059, p-value = 0.0001647
```

```
#Box-Ljung test
Box.test(arma.model$residuals, lag=20, type="Ljung-Box")
```

```
##
##  Box-Ljung test
##
## data:  arma.model$residuals
## X-squared = 13.588, df = 20, p-value = 0.8508
```




We favor the ARIMA(1,0,3) due to its higher log-likelihood and lower information criteria.

From the Box-Ljung test, we cannot reject the no-autocorrelation among residuals up to lag 20, in accordance to the assumptions of white noise. The QQ-plot of the residuals exhibit heavy tails and we cannot reject the null hypothesis of normality for the Shapiro-Wilks test, thus, evidence suggests our residuals are not normally distributed; nevertheless, this is not a violation to white noise assumptions as there's no particular requirement that these are normally distributed.

**(d) Forecast the transformed time series for 5 steps into the future; Compute the mean squared forecast error (MSFE) by using the test data; Provide the outputs.**

```
arma.model <- arima(residuals, order=c(1,0,3), include.mean = FALSE, method = "ML")

h = 5
arma.forecast <- predict(arma.model, h)
round(arma.forecast$pred, 3)
```

```
## Time Series:
## Start = 219
## End = 223
## Frequency = 1
## [1] 0.016 0.012 0.010 0.008 0.007
```

```r
round(arma.forecast$se, 3)
```

```
## Time Series:
## Start = 219
## End = 223
## Frequency = 1
## [1] 0.009 0.015 0.019 0.023 0.025
```

```r
forecasted_residuals <- matrix(0, nrow = 1, ncol = 5)
forecasted_residuals[1,1] <- as.numeric(arma.forecast$pred)[1]
forecasted_residuals[1,2] <- as.numeric(arma.forecast$pred)[2]
forecasted_residuals[1,3] <- as.numeric(arma.forecast$pred)[3]
forecasted_residuals[1,4] <- as.numeric(arma.forecast$pred)[4]
forecasted_residuals[1,5] <- as.numeric(arma.forecast$pred)[5]

estimated_coeffs <- matrix(0, nrow= 1, ncol = 3)
estimated_coeffs[1,1] <- fitModel$coefficients[1]
estimated_coeffs[1,2] <- fitModel$coefficients[2]
estimated_coeffs[1,3] <- fitModel$coefficients[3]

time_matrix <- matrix(0, nrow = 5, ncol = 3)
time_matrix[, 1] <- 1
time_matrix[, 2] <- 219:223
time_matrix[, 3] <- time_matrix[, 2]^2
time_matrix <-t(time_matrix)

trend.forecast = estimated_coeffs %*% time_matrix
ts.forecast.pred <- trend.forecast + forecasted_residuals
ts.forecast.pred
```

```
##          [,1]     [,2]     [,3]     [,4]     [,5]
## [1,] 9.130672 9.133126 9.137347 9.142227 9.147312
```

```r
lower <- ts.forecast.pred - (qnorm(0.975) * forecasted_residuals)
upper <- ts.forecast.pred + (qnorm(0.975) * forecasted_residuals)

msfe <- mean( (ts.forecast.pred - as.numeric(test_data))^2 )
msfe
```

```
## [1] 3.667099e-05
```

**(e) Now, fit an ARIMA(p,d,q) model to the training data; Forecast the transformed time series for 5 steps into the future by using this model; Compute the MSFE again; Which model is preferred in terms of minimal MSFE? Include the output**

```r
auto.arima(training_data,max.p=10,max.q=10, max.d=2, ic="aic",allowmean = FALSE) # AIC
```

```
## Series: training_data
## ARIMA(1,1,0) with drift
##
## Coefficients:
##          ar1   drift
##       0.3473  0.0084
## s.e.  0.0636  0.0010
##
## sigma^2 = 9.275e-05:  log likelihood = 700.81
## AIC=-1395.63   AICc=-1395.51   BIC=-1385.49
```

```r
auto.arima(training_data,max.p=10,max.q=10, max.d=2, ic="bic",allowmean = FALSE) # BIC
```

```
## Series: training_data
## ARIMA(1,1,0) with drift
##
## Coefficients:
##          ar1   drift
##       0.3473  0.0084
## s.e.  0.0636  0.0010
##
## sigma^2 = 9.275e-05:  log likelihood = 700.81
## AIC=-1395.63   AICc=-1395.51   BIC=-1385.49
```

```r
arma.model.2e <- arima(training_data, order=c(1,1,0), include.mean = FALSE, method = "ML")

h = 5
arma.forecast.2e <- predict(arma.model.2e, h)
round(arma.forecast.2e$pred, 3)
```

```
##       Qtr1  Qtr2  Qtr3  Qtr4
## 2001             9.128 9.128
## 2002 9.127 9.127 9.127
```

```r
msfe <- mean( (arma.forecast.2e$pred - as.numeric(test_data))^2 )
msfe
```

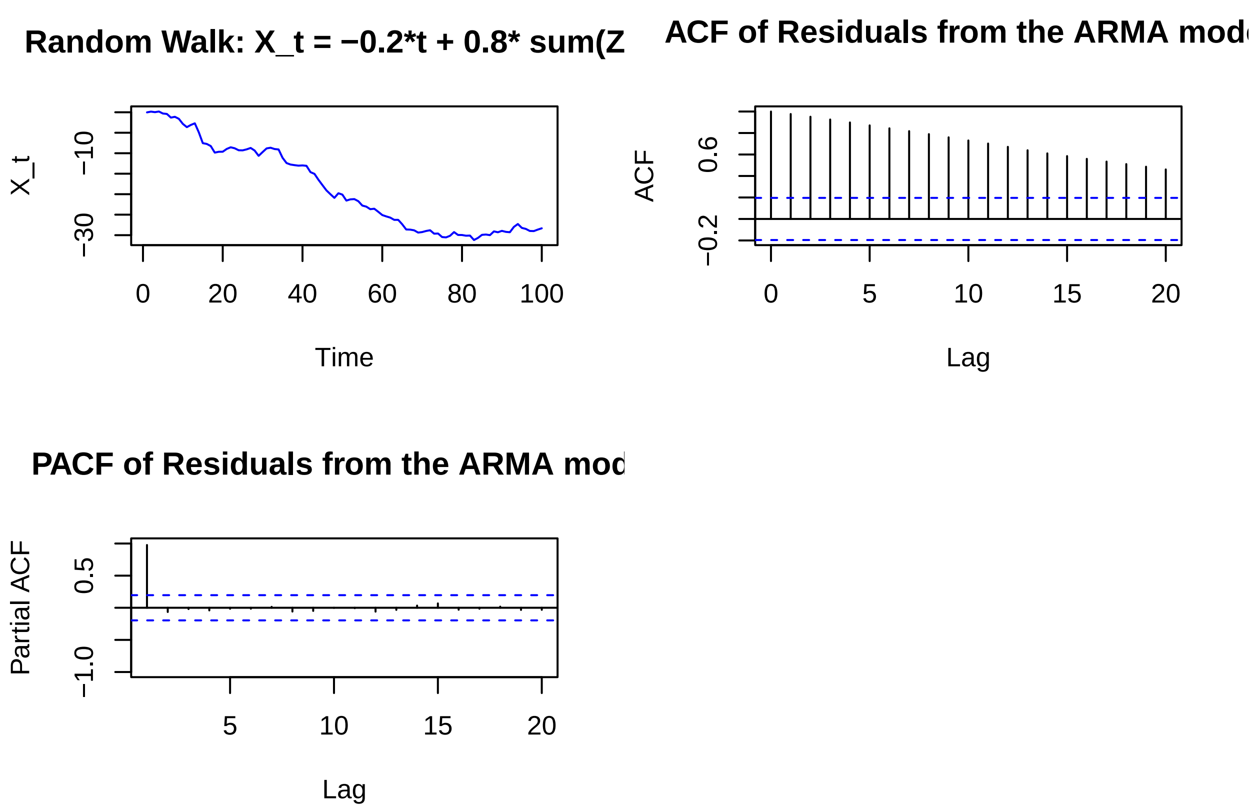
```
## [1] 0.0003259941
```

We can see that the ARMA(p,q) model minimizes the MSFE better than ARMA(p,d,q), for a forecasting horizon of 5 periods. This can be an unexpected result, but we can re conciliate these results by recognizing the fact that the test data has a somewhat of a linear monotonically increasing trend: $\{9.12, 9.13, 9.14, 9.15, 9.16\}$. This behavior is well captured by a trend model, as opposed to data with fluctuations which we believe the ARIMA model should be better at capturing. Additionally, the test data sample is small with only 5 observations, thus, making the measurement of the models performance challenging.

## Question 3

**(a) Use set.seed(99) to generate the following random walk with drift: $X_t = -0.2t + 0.8\sum_{s=1}^{t} Z_s$ with $t = 1, ..., 100$ where $Z_t$ is IID standard normal; Include a time plot of the series and sample ACF and PACF of the series.**

```
set.seed(99)
n_steps <- 100
Z_t <- rnorm(n_steps)
t <- seq_len(n_steps)
X_t <- -1 * 0.2 * t + 0.8 * cumsum(Z_t)

par(mfrow = c(2,2))
plot(t, X_t, type = 'l', col = 'blue', xlab = 'Time', ylab = 'X_t',
     main = 'Random Walk: X_t = -0.2*t + 0.8* sum(Z_t)')
acf(X_t, main="ACF of Residuals from the ARMA model")
pacf(X_t, lag.max=20, ylim=c(-1, 1), main = "PACF of Residuals from the ARMA model") # PACF
```




**(b) Use the function auto.arima with suitable inputs to recover this model. That is, can you find an outcome of the model indicating the random walk with drift?**

```r
auto.arima(X_t,max.p=20,max.q=20, max.d=2, ic="aic", allowdrift=TRUE) # AIC
```

```
## Series: X_t
## ARIMA(0,1,1) with drift
##
## Coefficients:
##          ma1    drift
##       0.2996  -0.2830
## s.e.  0.1001   0.0905
##
## sigma^2 = 0.492:  log likelihood = -104.4
## AIC=214.8   AICc=215.06   BIC=222.59
```

```r
auto.arima(X_t,max.p=20,max.q=20, max.d=2, ic="bic", allowdrift=TRUE) # BIC
```

```
## Series: X_t
## ARIMA(0,1,1) with drift
##
## Coefficients:
##          ma1    drift
##       0.2996  -0.2830
## s.e.  0.1001   0.0905
##
## sigma^2 = 0.492:  log likelihood = -104.4
## AIC=214.8   AICc=215.06   BIC=222.59
```

Thus, suggesting that the series is integrated of order 1 or random walk with a drift parameter. Given that we know the true model of the data, we can compare the estimations with the true parameters; for instance, the drift parameter is $\approx -0.28$ close to the true value of $-0.2$ and the variance of the model is close to the true variance parameter of $(0.8)^2 \approx 0.64$.

**(c) Go through the testing procedure for unit roots with significance level $\alpha = 0.05$ (for all steps). Indicate the conclusion at each step of the procedure; Check if the testing result corresponds to the used model**

```r
ur.gt <- ur.df(X_t, lags=0, type='trend')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt)
##
```

```
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.16040 -0.42329  0.03346  0.41717  1.42773
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.545465   0.146325  -3.728 0.000327 ***
## z.lag.1      0.001874   0.025610   0.073 0.941817
## tt           0.005879   0.009311   0.631 0.529242
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7154 on 96 degrees of freedom
## Multiple R-squared:  0.04302,    Adjusted R-squared:  0.02308
## F-statistic: 2.158 on 2 and 96 DF,  p-value: 0.1212
##
##
## Value of test-statistic is: 0.0732 6.7048 2.1578
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -4.04 -3.45 -3.15
## phi2  6.50  4.88  4.16
## phi3  8.73  6.49  5.47
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a trend by testing if $\beta_2 = 0$ given $\pi = 0$ with the null $\phi_3 : (\pi, \widetilde{\beta_1}, \widetilde{\beta_2}) = (0, \widetilde{\beta_1}, 0)$. Similarly, we cannot reject this null at a significance level of 5%. Therefore, we take $\beta_2 = 0$ and proceed to fit another model without the trend term.

```r
ur.gt <- ur.df(X_t, lags=0, type='drift')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression drift
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.1635 -0.4045  0.0486  0.4480  1.4270
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.536400   0.145167  -3.695 0.000364 ***
## z.lag.1     -0.013696   0.006899  -1.985 0.049940 *
## ---
```

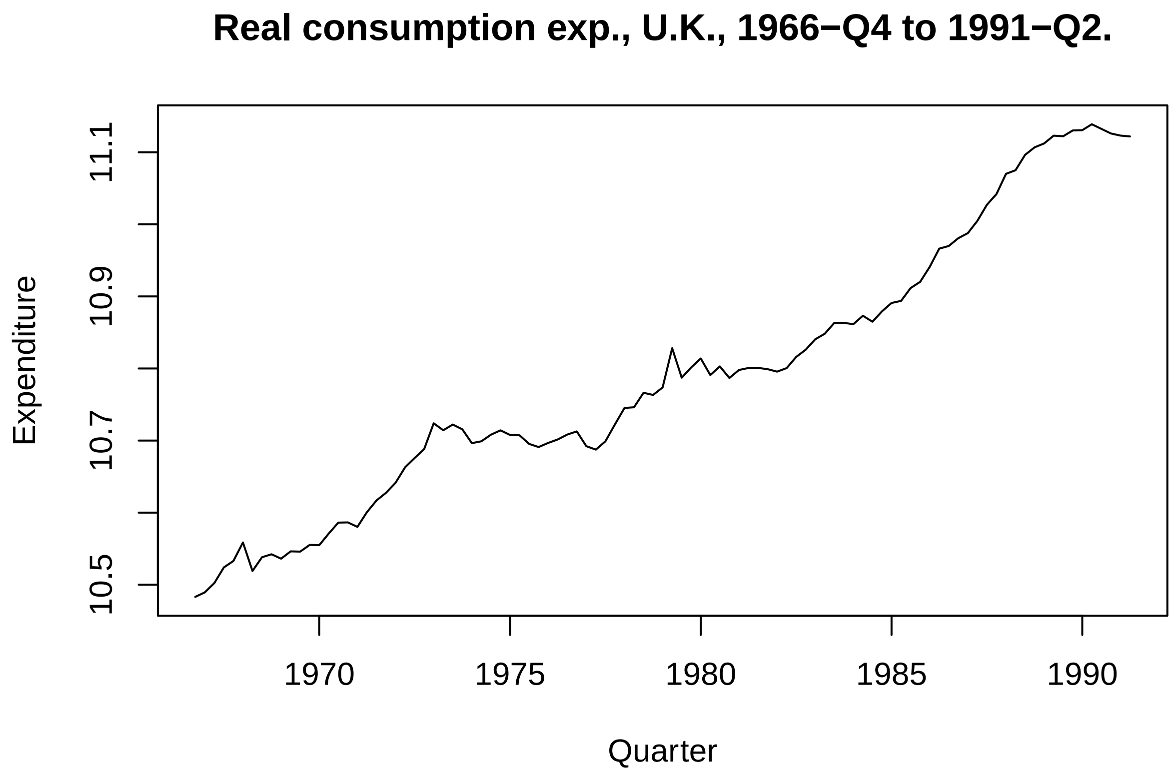
```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7132 on 97 degrees of freedom
## Multiple R-squared:  0.03904,    Adjusted R-squared:  0.02914
## F-statistic: 3.941 on 1 and 97 DF,  p-value: 0.04994
##
##
## Value of test-statistic is: -1.9853 9.9193
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.51 -2.89 -2.58
## phi1  6.70  4.71  3.86
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a drift by testing if $\beta_1 = 0$ given $\pi = 0$ with the null $\phi_1 : (\pi, \widetilde{\beta_1}) = (0, 0)$. In this case, we reject this null at a significance level of 5% and conclude that a drift exists. We proceed to test whether $\pi = 0$, excluding the $\beta$'s. By checking the t-statistic reported, we can state that $\pi$ is significantly different then zero at a 5% confidence level, and conclude there is not a unit root.

# Question 4

Raotbl3, description: This dataset contains the time series used by Darryl Holden and Roger Perman in their article: "Unit Roots and Cointegration for the Economist"

```
data(Raotbl3)
attach(Raotbl3)
lc_TS <- ts(lc, start = c(1966, 4), end = c(1991, 2), frequency = 4)
plot(lc_TS, type="l", main="Real consumption exp., U.K., 1966-Q4 to 1991-Q2.",
     ylab = "Expenditure", xlab = "Quarter")
```




**(a) Consider the time series Raotbl3$lc of real consumption expenditure from the United Kingdom starting in 1966:4 until 1991:2 in the R package urca. Produce a time plot of the series; Go through the testing procedure for unit roots discussed in class taking $k = 3$ for the number of lagged series differences to include in the regression; Indicate the conclusion at each step of the procedure.**

```
ur.gt <- ur.df(lc_TS, lags=3, type='trend')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.044714 -0.006525  0.000129  0.006225  0.045353
##
```

```
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.7976591  0.3547775   2.248   0.0270 *
## z.lag.1     -0.0758706  0.0338880  -2.239   0.0277 *
## tt           0.0004915  0.0002159   2.277   0.0252 *
## z.diff.lag1 -0.1063957  0.1006744  -1.057   0.2934
## z.diff.lag2  0.2011373  0.1012373   1.987   0.0500 .
## z.diff.lag3  0.2998586  0.1020548   2.938   0.0042 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01307 on 89 degrees of freedom
## Multiple R-squared:  0.1472, Adjusted R-squared:  0.09924
## F-statistic: 3.071 on 5 and 89 DF,  p-value: 0.01325
##
##
## Value of test-statistic is: -2.2389 3.7382 2.5972
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -4.04 -3.45 -3.15
## phi2  6.50  4.88  4.16
## phi3  8.73  6.49  5.47
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a trend by testing if $\beta_2 = 0$ given $\pi = 0$ with the null $\phi_3 : (\pi, \widetilde{\beta_1}, \widetilde{\beta_2}) = (0, \widetilde{\beta_1}, 0)$. Similarly, we cannot reject this null at a significance level of 5%. Therefore, we take $\beta_2 = 0$ and proceed to fit another model without the trend term.

```r
ur.gt <- ur.df(lc_TS, lags=3, type='drift')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression drift
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.047547 -0.007071  0.000265  0.007731  0.046880
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.0123237  0.0851358   0.145   0.8852
## z.lag.1     -0.0007356  0.0079043  -0.093   0.9261
## z.diff.lag1 -0.1433015  0.1016454  -1.410   0.1620
## z.diff.lag2  0.1615256  0.1020242   1.583   0.1169
```

```
## z.diff.lag3  0.2585280  0.1027364   2.516   0.0136 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01337 on 90 degrees of freedom
## Multiple R-squared:  0.09747,    Adjusted R-squared:  0.05735
## F-statistic:  2.43 on 4 and 90 DF,  p-value: 0.05335
##
##
## Value of test-statistic is: -0.0931 2.8806
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.51 -2.89 -2.58
## phi1  6.70  4.71  3.86
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a drift by testing if $\beta_1 = 0$ given $\pi = 0$ with the null $\phi_1 : (\pi, \widetilde{\beta_1}) = (0, 0)$. Similarly, we cannot reject this null at a significance level of 5%. Therefore, we take $\beta_1 = 0$ and proceed to fit another model without trend nor drift terms.

```r
ur.gt <- ur.df(lc_TS, lags=3, type='none')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression none
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.047220 -0.007276  0.000229  0.007674  0.046921
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## z.lag.1      0.0004083  0.0001695   2.409   0.0180 *
## z.diff.lag1 -0.1444994  0.1007615  -1.434   0.1550
## z.diff.lag2  0.1599782  0.1009153   1.585   0.1164
## z.diff.lag3  0.2568572  0.1015353   2.530   0.0131 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0133 on 91 degrees of freedom
## Multiple R-squared:  0.2546, Adjusted R-squared:  0.2218
## F-statistic:  7.77 on 4 and 91 DF,  p-value: 1.967e-05
##
##
```

```
## Value of test-statistic is: 2.4089
##
## Critical values for test statistics:
##      1pct  5pct 10pct
## tau1 -2.6 -1.95 -1.61
```

We can reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we conclude there is no unit root.

**(b) Repeat the testing procedure with the series in (a) with a smaller lag, k = 2; Check if a different choice of lag affects the conclusion.**

```
ur.gt <- ur.df(lc_TS, lags=2, type='trend')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.042654 -0.006960  0.000573  0.007325  0.048921
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.5900161  0.3634079   1.624    0.108
## z.lag.1     -0.0558308  0.0347023  -1.609    0.111
## tt           0.0003664  0.0002218   1.652    0.102
## z.diff.lag1 -0.0755700  0.1047334  -0.722    0.472
## z.diff.lag2  0.1555229  0.1044548   1.489    0.140
##
## Residual standard error: 0.01366 on 91 degrees of freedom
## Multiple R-squared:  0.05993,    Adjusted R-squared:  0.01861
## F-statistic:  1.45 on 4 and 91 DF,  p-value: 0.2239
##
##
## Value of test-statistic is: -1.6089 5.3138 1.3643
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -4.04 -3.45 -3.15
## phi2  6.50  4.88  4.16
## phi3  8.73  6.49  5.47
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a trend by testing if $\beta_2 = 0$ given $\pi = 0$ with the null $\phi_3 : (\pi, \widetilde{\beta_1}, \widetilde{\beta_2}) = (0, \widetilde{\beta_1}, 0)$. Similarly, we cannot reject this null at a significance level of 5%. Therefore, we take $\beta_2 = 0$ and proceed to fit another model without the trend term.

```
ur.gt <- ur.df(lc_TS, lags=2, type='drift')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression drift
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.044303 -0.007044  0.001045  0.007454  0.049709
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  6.412e-03  8.590e-02   0.075    0.941
## z.lag.1     -1.105e-05  7.969e-03  -0.001    0.999
## z.diff.lag1 -1.069e-01  1.040e-01  -1.029    0.306
## z.diff.lag2  1.299e-01  1.043e-01   1.246    0.216
##
## Residual standard error: 0.01379 on 92 degrees of freedom
## Multiple R-squared:  0.03174,    Adjusted R-squared:  0.0001708
## F-statistic: 1.005 on 3 and 92 DF,  p-value: 0.3941
##
##
## Value of test-statistic is: -0.0014 6.4846
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.51 -2.89 -2.58
## phi1  6.70  4.71  3.86
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a drift by testing if $\beta_1 = 0$ given $\pi = 0$ with the null $\phi_1 : (\pi, \widetilde{\beta_1}) = (0, 0)$. We reject this null at a significance level of 5% and conclude that a drift exists. We proceed to test whether $\pi = 0$, excluding the $\beta$'s. By checking the t-statistic reported, we cannot reject $\pi$ is significantly different from zero, and conclude $\pi = 0$ and there is a unit root.

Therefore, there is a change in conclusions from a different lag choice.

**(c) Go through the testing procedure with the transformed gnp used in Problem 1. Here, use the entire samples. Use the lag p determined for the ARMA model. If your ARMA model contains the MA part, use selectlags, contained to ur.df function; Check if the conclusion of this problem corresponds to the preference of the model (i.e. trend stationary or non-stationary) in Problem 1. (e);**

```
auto.arima(gnp_data,max.p=10,max.q=10,ic="aic",allowmean = FALSE) # AIC
```

```
## Series: gnp_data
## ARIMA(1,1,0) with drift
##
## Coefficients:
##          ar1   drift
##       0.3467  0.0083
## s.e.  0.0627  0.0010
##
## sigma^2 = 9.136e-05:  log likelihood = 718.61
## AIC=-1431.22   AICc=-1431.11   BIC=-1421.01
```

```
auto.arima(gnp_data,max.p=10,max.q=10,ic="bic",allowmean = FALSE) # BIC
```

```
## Series: gnp_data
## ARIMA(1,1,0) with drift
##
## Coefficients:
##          ar1   drift
##       0.3467  0.0083
## s.e.  0.0627  0.0010
##
## sigma^2 = 9.136e-05:  log likelihood = 718.61
## AIC=-1431.22   AICc=-1431.11   BIC=-1421.01
```

For this result, we see that $p = 1$ and according to the lectures material, we select $k = p - 1 = 0$ for the ADF test.

```
ur.gt <- ur.df(gnp_data, lags=0, type='trend')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
```

```
## -0.037285 -0.004751  0.000541  0.005980  0.028789
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.2157840  0.1141539   1.890   0.0600 .
## z.lag.1     -0.0279548  0.0155053  -1.803   0.0728 .
## tt           0.0002174  0.0001292   1.682   0.0939 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01009 on 219 degrees of freedom
## Multiple R-squared:  0.02325,    Adjusted R-squared:  0.01433
## F-statistic: 2.607 on 2 and 219 DF,  p-value: 0.07606
##
##
## Value of test-statistic is: -1.8029 52.303 2.6068
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -3.99 -3.43 -3.13
## phi2  6.22  4.75  4.07
## phi3  8.43  6.49  5.47
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a trend by testing if $\beta_2 = 0$ given $\pi = 0$ with the null $\phi_3 : (\pi, \widetilde{\beta_1}, \widetilde{\beta_2}) = (0, \widetilde{\beta_1}, 0)$. Similarly, we cannot reject this null at a significance level of 5%. Therefore, we take $\beta_2 = 0$ and proceed to fit another model without the trend term.

```r
ur.gt <- ur.df(gnp_data, lags=0, type='drift')
summary(ur.gt)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression drift
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.036450 -0.005358  0.000236  0.005875  0.030495
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.024555   0.010570   2.323   0.0211 *
## z.lag.1     -0.001957   0.001273  -1.537   0.1256
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Residual standard error: 0.01013 on 220 degrees of freedom
## Multiple R-squared:  0.01063,    Adjusted R-squared:  0.006132
## F-statistic: 2.364 on 1 and 220 DF,  p-value: 0.1256
##
##
## Value of test-statistic is: -1.5374 76.4037
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.46 -2.88 -2.57
## phi1  6.52  4.63  3.81
```

We cannot reject the null $\tau : \pi = 0$ at a significance level of 5%. Therefore, we proceed to the existence of a drift by testing if $\beta_1 = 0$ given $\pi = 0$ with the null $\phi_1 : (\widetilde{\pi, \beta_1}) = (0, 0)$. We reject this null at a significance level of 5% and conclude that a drift exists. We proceed to test whether $\pi = 0$, excluding the $\beta$'s. By checking the t-statistic reported, we cannot reject $\pi$ is significantly different from zero, and conclude $\pi = 0$ and there is a unit root. Same conclusion as in $4.b$)

Our conclusion for problem $2.e.$ is that gnp_data looks like an ARIMA(1,1,0) with a drift. Which means that the series is integrated or order(1), thus, not stationary.

Our conclusion from $4.c.$ is that the $\beta_1 \neq 0$ and $pi = 0$, thus, there exists a unit root and there exists a drift.

So, we conclude that the models agree that the data is not stationary and has a drift.