$\boxed{2}$ $X_t = 2Z_t + Z_{t-1}$

$$Z_t \begin{cases} 2 & \text{prob } 1/3 \\ -1 & \text{prob } 2/3 \end{cases}$$

a) See R code section

b) $\mathbb{E}(Z_t) = 2\,\frac{1}{3} + (-1)\,\frac{2}{3} = 0$

$$\mathrm{Var}(Z_t) = \frac{1}{3}(2^2) + \frac{2}{3}(-1)^2 = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$$

$$\mathbb{E}(X_t) = 2\,\mathbb{E}(Z_t) + \mathbb{E}(Z_{t-1}) = 0$$

$$\mathbb{E}(X_t^2) = \mathrm{Var}(X_t) = 4\,\mathrm{Var}(Z_t) + \mathrm{Var}(Z_{t-1})$$

$$= 4\,(2) + 2 = 10$$

c)

$$\mathbb{E}(X_t X_{t-1}) = \mathbb{E}\left[(2Z_t + Z_{t-1})(2Z_{t-1} + Z_{t-2})\right]$$

$$= \mathbb{E}\left[4 Z_t Z_{t-1} + 2 Z_t Z_{t-2} + 2 Z_{t-1}^2 + Z_{t-1} Z_{t-2}\right]$$

$$= \underbrace{4\,\mathbb{E}(Z_t)\,\mathbb{E}(Z_{t-1})}_{0} + \underbrace{2\,\mathbb{E}(Z_t)\,\mathbb{E}(Z_{t-2})}_{0} + \underbrace{2\,\mathbb{E}(Z_{t-1}^2)}_{2\cdot 2} + \underbrace{\mathbb{E}(Z_{t-1})\mathbb{E}(Z_{t-2})}_{0}$$

$$= 4$$

$$\mathrm{Corr}(X_t, X_{t-1}) = \frac{\mathbb{E}(X_t X_{t-1}) - \mathbb{E}(X_t)\,\mathbb{E}(X_{t-1})}{\sqrt{\mathbb{E}(X_t^2) - [\mathbb{E}(X_t)]^2}\,\sqrt{\mathbb{E}(X_{t-1}^2) - [\mathbb{E}(X_{t-1})]^2}} = \frac{4}{\sqrt{10}\,\sqrt{10}} = \frac{2}{5}$$

d)

$$X_t \begin{cases} 6 & \text{prob } 1/9 = \left(\frac{1}{3}\cdot\frac{1}{3}\right) \\ 3 & \text{prob } 2/9 = \left(\frac{1}{3}\cdot\frac{2}{3}\right) \\ 0 & \text{prob } 2/9 = \left(1/3 \cdot 2/3\right) \\ -3 & \text{prob } 4/9 = \left(2/3 \cdot 2/3\right) \end{cases}$$

3 $X_t = \alpha t + Z_t$ and $Z_t \sim N(0, \sigma_z^2)$ iid noise, $t = 1, 2, \dots, T$, $\alpha \in \mathbb{R}$

a) See R code section

** For the R section, I will use $T = 100$, $\alpha = 2$, $\sigma_z^2 = 2$ (instead of the generalization developed here) **

b) $$\mathbb{E}(X_t) = \mathbb{E}(\alpha t) + \mathbb{E}(Z_t) = \alpha t$$

$$\mathbb{E}(X_t^2) = \mathbb{E}\left[(\alpha t + Z_t)^2\right] = \mathbb{E}\left[(\alpha t)^2\right] + \mathbb{E}\left[2\alpha t Z_t\right] + \mathbb{E}\left[Z_t^2\right]$$

$$= (\alpha t)^2 + \sigma_z^2$$

c) $$\mathbb{E}(X_t X_{t-1}) = \mathbb{E}\left[(\alpha t + Z_t)(\alpha(t-1) + Z_{t-1})\right] = \mathbb{E}\left[(\alpha t + Z_t)(\alpha t - \alpha + Z_{t-1})\right]$$

$$= \mathbb{E}\left[(\alpha t)^2 - \alpha^2 t + \alpha t Z_{t-1} + \alpha t Z_t - \alpha Z_t + Z_t Z_{t-1}\right]$$

$$= (\alpha t)^2 - \alpha^2 t = \alpha^2 t(t-1)$$

$$\mathrm{Corr}(X_t, X_{t-1}) = \frac{\mathbb{E}(X_t X_{t-1}) - \mathbb{E}(X_t)\,\mathbb{E}(X_{t-1})}{\sqrt{\mathbb{E}(X_t^2) - [\mathbb{E}(X_t)]^2}\sqrt{\mathbb{E}(X_{t-1}^2) - [\mathbb{E}(X_{t-1})]^2}}$$

$$= \frac{\alpha^2 t(t-1) - (\alpha t)[\alpha(t-1)]}{\sqrt{(\alpha t)^2 + \sigma_z^2 - (\alpha t)^2}\sqrt{[\alpha(t-1)]^2 + \sigma_z^2 - [\alpha(t-1)]^2}}$$

$$= \frac{\alpha^2 t(t-1) - (\alpha t)[\alpha(t-1)]}{\sigma_z^2}$$

$\boxed{4}$ $X_t = X_{t-1} + Z_t$ with $Z_t \sim N(1, 2^2)$, $X_0 = 0$

a) Let $\{W_t\}$ iid noise $N(0, 2^2)$

$$X_t = X_{t-1} + Z_t = X_{t-2} + Z_{t-1} + Z_t \ (\ldots \text{so on} \ldots)$$

$$= \sum_{s=1}^{t} Z_s = \sum_{s=1}^{t} \Big(\underbrace{Z_s - \mathbb{E}(Z_s)}^{W_t} + \underbrace{\mathbb{E}(Z_s)}^{1} \Big)$$

$$= \sum_{s=1}^{t} W_s + t$$

b) See R code section

c) $\mathbb{E}(X_t) = t + \mathbb{E}\left(\sum_{s=1}^{t} W_s\right) = t + \sum_{s=1}^{t} \mathbb{E}(W_s) = \underline{t}$

$$\mathbb{E}(X_t^2) = \mathbb{E}\left[(X_{t-1} + Z_t)^2\right] = \mathbb{E}\left\{\left[t + \sum_{s=1}^{t} W_s\right]^2\right\}$$

$$= \mathbb{E}\left[t^2 + 2t\sum_{s=1}^{t} W_s + \left(\sum_{s=1}^{t} W_s\right)^2\right]$$

$$= t^2 + 2t\,\mathbb{E}\left(\sum_{s=1}^{t} W_s\right) + \mathbb{E}\left[\left(\sum_{s=1}^{t} W_s\right)^2\right]$$

$$= t^2 + \mathbb{E}\left[\left(\sum_{s=1}^{t} W_s\right)^2\right]$$

$$= t^2 + \mathbb{E}\left[\sum_{s=1}^{t} W_s^2 + 2\sum_{j=1}^{t}\sum_{i=1}^{j-1} W_i W_j\right]$$

$$= t^2 + \left[\underbrace{\sum_{s=1}^{t} \mathbb{E}(W_s^2)}_{t \cdot 4} + 2 \sum_{j=1}^{t} \sum_{i=1}^{j-1} \underbrace{\mathbb{E}(W_i W_j)}_{0} \right]$$

$$= t^2 + 4t \quad = \underline{t(t+4)}$$

d) $$\mathbb{E}(X_t X_{t-1}) = \mathbb{E}\left[\left(t + \sum_{s=1}^{t} W_s\right)\left(t-1 + \sum_{s=1}^{t-1} W_s\right)\right]$$

$$= \mathbb{E}\left[X_{t-1} X_{t-2} + X_{t-1} Z_{t-1} + Z_t X_{t-2} + Z_t Z_{t-1}\right]$$

$$= \mathbb{E}\left[t(t-1) + t \sum_{s=1}^{t-1} W_s + (t-1) \sum_{s=1}^{t} W_s + \sum_{s=1}^{t} W_s \sum_{s=1}^{t-1} W_s\right]$$

$$= t(t-1) + t \sum_{s=1}^{t-1} \mathbb{E}(W_s) + (t-1) \sum_{s=1}^{t} \mathbb{E}(W_s) + \mathbb{E}\left(W_t \left(\sum_{s=1}^{t-1} W_s\right)^2\right)$$

$$= t(t-1) + \mathbb{E}(W_t) \cdot \mathbb{E}\left(\sum_{s=1}^{t-1} W_s\right)^2 \quad = \underline{t(t-1)}$$

$$\mathrm{Corr}(X_t X_{t-1}) = \frac{\mathbb{E}(X_t X_{t-1}) - \mathbb{E}(X_t)\, \mathbb{E}(X_{t-1})}{\sqrt{\mathbb{E}(X_t^2) - [\mathbb{E}(X_t)]^2} \sqrt{\mathbb{E}(X_{t-1}^2) - [\mathbb{E}(X_{t-1})]^2}}$$

$$= \frac{t(t-1) - t(t-1)}{\sqrt{t(t+4) - t^2} \sqrt{(t-1)(t+3) - (t-1)^2}}$$

$$= 0$$