Competition and multiscaling in evolving networks

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Abstract

The rate at which nodes in a network increase their connectivity depends on their fitness to compete for links. For example, in social networks some individuals acquire more social links than others, or on the www some webpages attract considerably more links than others. We find that this competition for links translates into multiscaling, i.e. a fitness dependent dynamic exponent, allowing fitter nodes to overcome the more connected but less fit ones. Uncovering this fitter-gets-richer phenomena can help us understand in quantitative terms the evolution of many competitive systems in nature and society.

PACS numbers: 5.65+b, 89.75-k, 89.75Fb, 89.75Hc.

Typeset using REVTEX

The complexity of many systems can be attributed to the interwoven web in which their constituents interact with each other. For example, the society is organized in a social web, whose nodes are individuals and links represent various social interactions, or the www forms a complex web whose nodes are documents and links are URLs. While for a long time these networks have been modeled as completely random [1, 2], recently there is increasing evidence that they in fact have a number of generic non-random characteristics, obeying various scaling laws or displaying short length-scale clustering [3–16].

A generic property of these complex systems is that they constantly evolve in time. This implies that the underlying networks are not static, but continuously change through the addition and/or removal of new nodes and links. Such evolving networks characterize the society, thanks to the birth and death of nodes and their constant acquisition of social links; or characterize the www, where the number of nodes increases exponentially and the links connecting them are constantly modified [3, 6, 7]. Consequently, in addressing these complex systems we have to uncover the dynamical forces that act at the level of individual nodes, whose cumulative effect determines the system's large-scale topology. A first step in this direction was the introduction of the scale-free model [8], that incorporates the fact that network evolution is driven by at least two coexisting mechanisms: (1) growth, implying that networks continuously expand by the addition of new nodes that attach to the nodes already present in the system; (2) preferential attachment, mimicking the fact that a new node links with higher probability to the nodes that already have a large number of links. With these two ingredients the scale-free model predicts the emergence of a power-law connectivity distribution, observed in many systems [3, 8–10], ranging from the Internet to citation networks. Furthermore, extensions of this model, including rewiring [11] or aging [12, 13] have been able to account for more realistic aspects of the network evolution, such as the existence of various scaling exponents or cutoffs in the connectivity distribution.

Despite its success in predicting the large-scale topology of real networks, the scale-free model neglects an important aspect of competitive systems: not all nodes are equally successful in acquiring links. The model predicts that all nodes increase their connectivity in

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time as $k_i(t) = (t/t_i)^\beta$, where $\beta = 1/2$ and t_i is the time at which node i has been added into the system. Consequently, the oldest nodes will have the highest number of links, since they had the longest timeframe to acquire them. Since new nodes attach preferentially to more connected nodes, highly connected nodes will continue to acquire further links at a higher rate at the expense of the smaller nodes, a rich-gets-richer phenomena which is responsible for the power-law tail of the connectivity distribution. Furthermore, if two nodes arrive at the same time, apart from some statistical fluctuations, at any time they will approximately have the same number of links.

On the other hand numerous examples convincingly indicate that in real systems a node's connectivity and growth rate does not depends on its age alone. For example, in social systems not everybody makes friends at the same rate: some individuals are better in turning a random meeting into a lasting social link than others. On the www some documents through a combination of good content and marketing acquire a large number of links in a very short time, easily overtaking websites that have been around for much longer time [17]. Also, in Hollywood some actors in a very short timeframe build a movie portfolio and a collection of links that easily surpasses many actors in business for much longer time. Finally, some research papers in a short timeframe acquire a very large number of citations, much in excess of the majority of their contemporary or even older publications. In all these examples we see a similar pattern: some nodes acquire links at a rate much higher than other nodes in the system. We tend to associate these differences with some intrinsic quality of the nodes, such as the social skills of an individual, the content of a web page, the talent of an actor or the content of a scientific article. We will call this the node's fitness, describing it's ability to compete for links at the expense of other nodes. While such competition for links in real systems is well documented, it has not been incorporated in the current network models. In this paper we take a first step in this direction by proposing a simple model that allows us to investigate this competitive aspect of real networks in quantitative terms. Assuming that the existence of a fitness modifies the preferential attachment to compete for links, we find that different fitness translates into multiscaling in the dynamical

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evolution: while the connectivity of individual nodes will still follow a power-law in time, i.e. $k_i(t) \sim t^{\beta_i}$, the dynamical exponent, β_i, will depend on the fitness of the node. We develop the continuum model for this competitive evolving network, allowing us to calculate analytically β and derive a general expression for the connectivity distribution. We find that the analytical predictions are in excellent agreement with the results obtained from numerical simulations.

The fitness model – The examples discussed above indicate that nodes have different ability (fitness) to compete for links. To account for these differences we introduce a fitness parameter, η_i, that we assign to each node, and assume that it is unchanged in time (i.e. η_i represents a quenched noise) [19]. Starting with a small number of nodes, at every timestep we add a new node i with fitness η_i, where η is chosen from the distribution $\rho(\eta)$. Each new node i has m links that are connected to the nodes already present in the system. We assume that the probability Π_i that a new node will connect to a node i already present in the network depends on the connectivity k_i and on the fitness η_i of that node, such that

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}. \tag{1}$$

This generalized preferential attachment [8] incorporates in the simplest possible way that fitness and connectivity jointly determine the rate at which new links are added to a given node, i.e. even a relatively young node with a few links can acquire links at a high rate if it has a large fitness parameter. To address the scaling properties of this model we first develop a continuum theory, allowing us to predict the connectivity distribution [8, 11, 12]. A node i will increase its connectivity k_i at a rate that is proportional to the probability (1) that a new node will attach to it, giving

$$\frac{\partial k_i}{\partial t} = m \frac{\eta_i k_i}{\sum_j k_j \eta_j}. \tag{2}$$

The factor m accounts for the fact that each new node adds m links to the system. If $\rho(\eta) = \delta(\eta - 1)$, i.e. all fitness are equal, (2) reduces to the scale-free model, which predicts that $k_i(t) \sim t^{1/2}$ [8]. In order to solve (2) we assume that similarly to the scale-free model

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the time evolution of k_i follows a power-law, but there is multiscaling in the system, i.e. the dynamic exponent depends on the fitness η_i,

$$k_{\eta_i}(t, t_0) = m \left(\frac{t}{t_0} \right)^{\beta(\eta_i)}, \tag{3}$$

where t_0 is the time at which the node i was born. The dynamic exponent $\beta(\eta)$ is bounded, i.e. $0 < \beta(\eta) < 1$ because a node always increases the number of links in time $(\beta(\eta) > 0)$ and $k_i(t)$ cannot increases faster than t $(\beta(\eta) < 1)$. We first calculate the mean of the sum $\sum_j \eta_j k_j$ over all possible realization of the quenched noise $\{\eta\}$. Since each node is born at a different time t_0, the sum over j can be written as an integral over t_0

$$
\begin{aligned}
< \sum_j \eta_j k_j > &= \int d\eta \rho(\eta)\, \eta \int_1^t dt_0 \; k_\eta(t, t_0) \\
&= \int d\eta \; \eta \rho(\eta) m \frac{(t - t^{\beta(\eta)})}{1 - \beta(\eta)}. \tag{4}
\end{aligned}
$$

Since $\beta(\eta) < 1$, in the $t \to \infty$ limit $t^{\beta(\eta)}$ can be neglected compared to t, thus we obtain

$$< \sum_j \eta_j k_j > \overset{t \to \infty}{=} Cmt(1 + O(t^{-\epsilon})), \tag{5}$$

where

$$
\begin{aligned}
\epsilon &= (1 - \max_\eta \beta(\eta)) > 0, \\
C &= \int d\eta \rho(\eta) \frac{\eta}{1 - \beta(\eta)}. \tag{6}
\end{aligned}
$$

Using (5), and the notation $k_\eta = k_{\eta_i}(t, t_0)$ the dynamic equation (2) can be written as

$$\frac{\partial k_\eta}{\partial t} = \frac{\eta k_\eta}{Ct}, \tag{7}$$

which has a solution of form (3), given that

$$\beta(\eta) = \frac{\eta}{C}, \tag{8}$$

thereby confirming the self-consistent nature of the assumption (3). To complete the calculation we need to determine C from (6) after substituting $\beta(\eta)$ with η/C,

$$1 = \int_0^{\eta_{max}} d\eta \rho(\eta) \frac{1}{\frac{C}{\eta} - 1}, \tag{9}$$

where η_{max} is the maximum possible fitness in the system [20]. Apparently (9) is a singular integral. However, since $\beta(\eta) = \eta/C < 1$ for every value of η, we have $C > \eta_{max}$, thus the integration limit never reaches the singularity. Note also that, since $\sum_j \eta_j k_j \le \eta_{max} \sum_j k_j = 2mt\eta_{max}$, we have, using (5), that $C \le 2\eta_{max}$.

Finally, we can calculate the connectivity distribution $P(k)$, which gives the probability that a node has k links. If there is a single dynamic exponent β, the connectivity distribution follows the power-law $P(k) \sim k^\gamma$, where the connectivity exponent is given by $\gamma = 1/\beta + 1$. However, in this model we have a spectrum of dynamic exponents $\beta(\eta)$, thus $P(k)$ is given by a weighted sum over different power-laws. To find $P(k)$ we need to calculate the cumulative probability that for a certain node $k_\eta(t) > k$,

$$P(k_\eta(t) > k) = P\left(t_0 < t\left(\frac{m}{k}\right)^{C/\eta}\right)$$
$$= t\left(\frac{m}{k}\right)^{\frac{C}{\eta}}. \tag{10}$$

Thus the connectivity distribution, i.e. the probability that a node has k links, is given by the integral

$$P(k) = \int_0^{\eta_{max}} d\eta \frac{\partial P(k_\eta(t) > k)}{\partial t} =$$
$$\propto \int d\eta \rho(\eta) \frac{C}{\eta} \left(\frac{m}{k}\right)^{\frac{C}{\eta}+1}. \tag{11}$$

Scale-free model– Given the fitness distribution $\rho(\eta)$, the continuum theory allows us to predict both the dynamics, described by the dynamic exponent $\beta(\eta)$ (Eqs. (8) and (9)), and the topology, characterized by the connectivity distribution $P(k)$ (Eq. (11)). To demonstrate the validity of our predictions, in the following we calculate these quantities for two different $\rho(\eta)$ functions. As a first application let us consider the simplest case, corresponding to the scale-free model, when all fitnesses are equal. Thus we have $\rho(\eta) = \delta(\eta - 1)$, which, inserted in (9), gives $C = 2$, which represents the largest possible value of C. Using (8) we obtain $\beta = 1/2$ and from (11) we get $P(k) \propto k^{-3}$, the known scaling of the scale-free model. Thus

the scale-free model represents an extreme case of the fitness model considered here, the connectivity exponent taking up the largest possible value of γ.

Uniform fitness distribution– The behavior of the system is far more interesting, however, when nodes with different fitness compete for links. The simplest such case, which already offers nontrivial multiscaling, is obtained when $\rho(\eta)$ is chosen uniformly from the interval $[0, 1]$. The constant C can be determined again from (9), which gives

$$\exp(-2/C) = 1 - 1/C, \tag{12}$$

whose solution is $C^* = 1.255$. Thus, according to (8), each node will have a different dynamic exponent, given by $\beta(\eta) \sim \frac{\eta}{C^*}$. Using (11) we obtain

$$P(k) \propto \int_0^1 d\eta \frac{C^*}{\eta} \frac{1}{k^{1+C^*/\eta}} \sim \frac{k^{-(1+C^*)}}{\log(k)}, \tag{13}$$

i.e. the connectivity distribution follows a generalized power-law, with an inverse logarithmic correction.

To check the predictions of the continuum theory we performed numerical simulations of the discrete fitness model, choosing fitness with equal probability from the interval $[0, 1]$. Most important is to test the validity of the ansatz (3), for which we recorded the time evolution of nodes with different fitness η. As Fig. 1 shows, we find that $k_i(t)$ follows a power-law for all η, and the scaling exponent, $\beta(\eta)$, depends on η, being larger for nodes with larger fitness. Eq. (6) predicts that the sum $< \sum_i \eta_i k_i > /mt \to C^*$ in the $t \to \infty$ limit, where C^* is given by (12) as $C^* = 1.255$. Indeed as the inset in Fig. 1 shows, the discrete network model indicates that this sum converges to the analytically predicted value. Figure 1 allows us to determine numerically the exponent $\beta(\eta)$, and compare it to the prediction (8). As the inset in Fig. 2 indicates, we obtain excellent agreement between the numerically determined exponents and the prediction of the continuum theory. Finally, in Fig. 2 we show the agreement between the prediction (13) and the numerical results for the connectivity distribution $P(k)$.

An interesting feature of the numerically determined connectivity distribution (Fig. 2) is the appearance of a few nodes that have higher number of links than predicted by the

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connectivity distribution. Such highly connected hubs, appearing as a horizontal line with large k on the log-log plot, are present in many systems, including the www [3] or the metabolic network of a cell [18], clearly visible if we do not use logarithmic binning. This indicates, that the appearance of a few "super hubs", i.e. nodes that have connections in excess to that predicted by a power-law, is a generic feature of competitive systems.

Discussion – The above results offer interesting insights into the evolution of nodes in a competitive environment. The model studied by us reflects the basic properties of many real systems in which the nodes compete for links with other nodes, thus a node can acquire links only at the expense of the other nodes. The competitive nature of the model is guaranteed by the fact that the new node has only a fixed number of links, m, which, as the system grows, are distributed between an increasing number of nodes. Thus nodes that are already in the system have to compete with a linearly increasing number of other nodes for a link. In the scale-free model, where each node has the same fitness, all nodes increase their connectivity following the same scaling exponent $\beta = 1/2$. In contrast, we find that when we allow different fitness, multiscaling emerges and the dynamic exponent depends on the fitness parameter, η. This allows nodes with a higher fitness to enter the system at a later time and overcome nodes that have been in the system for a much longer timeframe. Such nodes with a higher fitness correspond to people with higher social skills; to websites with better content or services; or to articles that by report some important discovery. What is interesting, however, is that despite the significant differences in their fitness, all nodes will continue to increase their connectivity following a power-law in time. Thus, our results indicate that the fitter wins by following a power-law time dependence with a higher exponent than its less fit peers [21].

Beyond the conceptual importance of these results, the predictions of the model could be verified on networks for which the dynamic evolution of the nodes can be extracted, such as the science citation index or the actor network (for both of which the date at which a node is added to the system is recorded). Such measurement could also offer an independent determination of the fitness parameters and $\rho(\eta)$, which would allow the simultaneous testing

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of (3), (8) and (11).

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[19] Note that in some real systems the fitness can change with time, for example a research field can slowly close down or an actor can suspend acting diminishing the ability of the corresponding nodes to compete for links [12, 13].

[20] Equation (9) can be also derived from the normalization condition $2k_0 t = \sum_{j \epsilon N(t)} k_j$, a "mass conservation" law, giving the total number of links in the network at time t.

[21] In many systems, such as the www, nodes can acquire links through rewiring or the appearance of new internal links [11]. Such processes can be included in the present model, enhancing further the connectivity of the fittest nodes.
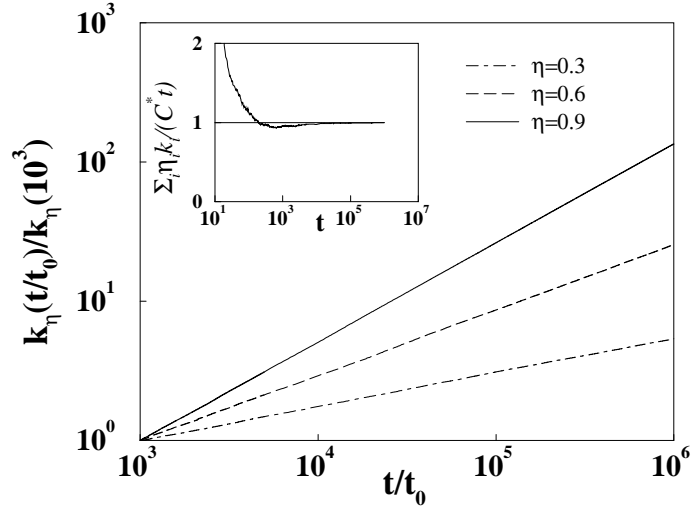
FIG. 1. Time dependence of the connectivity, $k_\eta(t)$, for nodes with fitness $\eta = 0.3$, 0.6 and 0.9. Note that $k_\eta(t)$ follows a power-law in each case and the dynamic exponent $\beta(\eta)$, given by the slope of $k(t)$, increase with η. While in the simulation the fitness of the nodes have been drawn uniformly, between $[0, 1]$, in the figure we show only the connectivity of three nodes with selected fitness. In the simulation we used $m = 2$ and the shown curves represent averages over 20 runs. *Inset*: Asymptotic convergence of $(\sum_{i=1}^{t} \eta_i k_i)/t$ to the analytically predicted limit $C^* = 1.255$, shown as an horizontal line (see Eq. (12)).
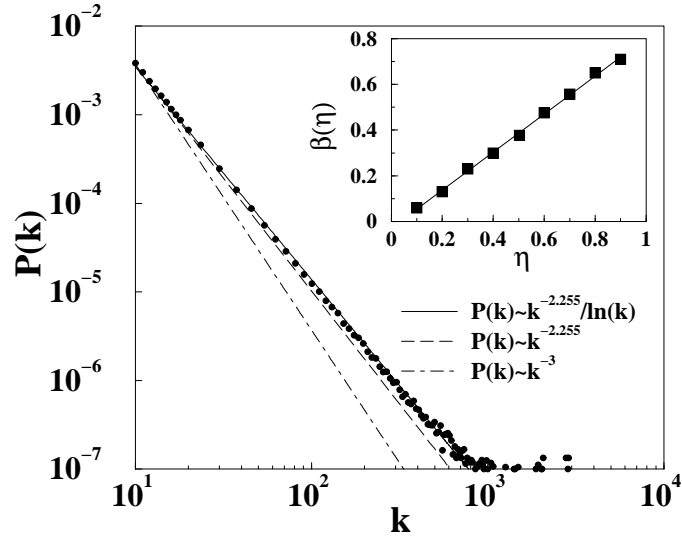
FIG. 2. Connectivity distribution in the fitness model, obtained for a network with $m = 2$ and $N = 10^6$ nodes. The upper solid line that goes along the circles provided by the numerical simulations corresponds to the theoretical prediction (13), with $\gamma = 2.25$. The dashed line corresponds to a simple fit $P(k) \sim k^{-2.255}$ without the logarithmic correction, while the long-dashed curve correspond to $P(k) \sim k^{-3}$, as predicted by the scale-free model, in which all fitness are equal. *Inset*: The dependence of the dynamic exponent $\beta(\eta)$ on the fitness parameter η in the case of a uniform $\rho(\eta)$ distribution. The squares were obtained from the numerical simulations while the solid line corresponds to the analytical prediction $\beta(\eta) = \eta/1.255$.