Structures Lab
Curvature and. Deflection in Elastic Beams

Louis Zeng (lz406)
Lab Group 94
Fitzwilliam College
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- **Summary**

An elastic beam has been numbered at equally distanced 14 stations, with 5 cm gap between adjacent stations. Curvature is measured at each 14 stations when it's under an applied load of 80 N by means of a digital curvature gauge. Bending moment diagram & defection of the beam at each station is obtained through curvature readings. Bending stiffness of the beam then be evaluated through equation B=M/k, comparison is made between the calculated value and the theoretical value gained from equation B = EI. (E = elastic modulus of the material, I = second moment of area of the beam). The measured deflection is also compared with the theoretical value, by solving the differential equation: $\varkappa = d\psi/ds$ and $\psi = dy/ds$ for small angle deflection ψ. Potential sources of error cause the differences in theoretical and measured are evaluated and analysed.

- **Apparatus and Method**

1. The curvature gauge which is calibrated with a horizontal metal block and a wheel with known radius.
2. The beam curvature with & without applied load was firstly measured with the calibrated curvature gauge.
3. Deflection of beam at a station was measured with the calibrated deflection gauge.
4. Bending stiffness of the beam, B of the beam is measured from the gradient of the M against κ graph.

- **Results, Observations and Calculations**

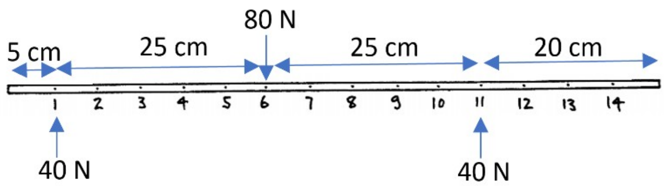
Initial Calibration of Curvature Gauge

Radius of Circle	$60\ cm$
Gauge curvature reading	$\frac{1}{60\text{ cm}} = 0.0167\text{ rad/cm}$
Measure Curvature Reading	0.01565 rad/cm
Percentage Error in curvature reading	$\frac{0.01565-0.0167}{0.0167} \times 100 = -6.3\%$

The percentage error value of -6.3% shows that the curvature gauge will provide a value of 6.3% lower than the true curvature.

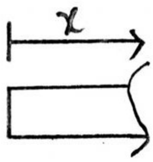
- **Increase in Curvature due to Applied Load**

Initial curvature values at all stations without loading (ϰ initial) are measured through the curvature gauge. When the 80 N loading was applied at station 6, and the final curvature readings from all stations (ϰ final) can be measured. The change of curvature due to the 80 N load is calculated as ϰ = ϰ initial - ϰ final (notice sign convention). The change in curvature ϰ due to load, is plotted against station number (see appendix).



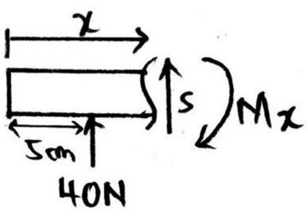
Free Body Diagram of Beam with Applied Load

- Bending Moment Diagram (with respect to length x)



For $0 < x < 5\ cm$, Bending Moment $M_X = 0$.

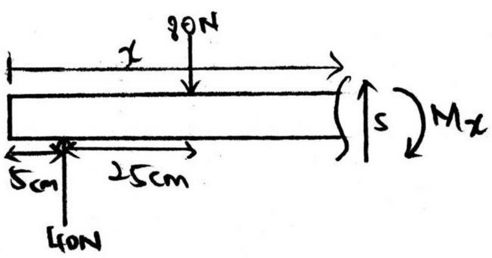
(Station 0 to 1)



For 5 cm < x < 30cm, S = −40 N.

$M_X = Sx + 40 \times 5 = (200 - 40x)$ Ncm

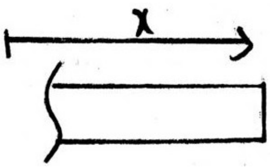
(Station 1 to 6)



For $30\ cm < x < 55\ cm, S = 40\ N$

$M_X = 40 \times 5 + Sx - 80 \times 30 = (40x\text{-}2200)$ N*cm (Stations 6 to 11)

At x = 30 cm (position of point load 80 N), $M_X = -1000$Ncm



For x > 55 cm, Shear Force =0, $M_X = 0$.

A graph of Bending Moment against Station number is plotted (Appendix Figure 5). The shape of the graph is similar to the graph of κ against Station number as mentioned before.

Experimental and Theoretical Values for Bending Stiffness of Beam

Using equation M = Bκ, where B is the bending stiffness of beam, κ is the curvature. Plotting a graph of M against κ gives a gradient of B. The measured value of B from the gradient of graph can be obtained from Figure 6 of the Appendix. The experimental value of B is 250000 Ncm*2 or 25 Nm*2.

The theoretical B value is calculated from the equation B = EI, where E is the Young's Modulus for aluminium alloy and I= bd*3/12 is the second-moment area of the beam. Obtaining the Young's Modulus of Aluminium Alloy from the Structures Databook, which is E = 70 GPa.

Our experimental value is reasonably close to the theoretical value, with an error of 7.8% which is relatively small. (< 10%)

$$\mathrm{B} = \mathrm{EI} = \frac{70 \times 10^{9} \times 40 \times 10^{-3} \times (4.63 \times 10^{-3})^{3}}{12} = 23.2 \text{ Nm}$$

- ## Calculation of $\Delta\psi$ and ψ values

With the calculation of simplified κ values, a plot of κ against station number can be done (Appendix Figure 6). Using the knowledge of $\kappa = \frac{d\psi}{ds} \approx \frac{\Delta\psi}{\Delta s}$, $\Delta\psi \approx \kappa\Delta s$, where $\psi = \int \kappa \, ds$. The graph of ψ against s is the area under the curve of κ against s.

Given an arbitrary shaped graph of κ against s, the value ψ can be obtained by using rectangular stripes to approximate the area. Summing up all the area of rectangles will give the value ψ.

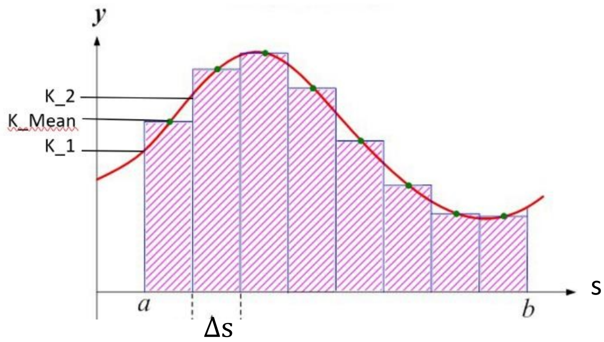


Figure 2: Numerial Integration Graph

Using midpoint rule, the height of each rectangle stripes is the κ_mean value, which is the average value between κ_n and κ_{n+1}. Therefore, the area of one rectangle is approximated to be κ_mean × Δs, where Δs is the width of each rectangle stripes. The general formula for all the rectangle stripes, which means the total area is calculated as:

$$\sum_{n=1}^{n-1} \Delta s \times \kappa\_mean$$

Since Δψ between each station is obtained, one value of ψ is needed in order to calculate all the values of ψ. From inspection, $\psi = 0$ at the point of loading (station 6). Therefore, all values of ψ can be worked out to the left and right from station 6. Graph of κ against s and ψ against s is labelled as figure 7 and figure 8 in Appendix.

From inspection of the triangular graph plotted, the results can be further confirmed that a rectangular mid-point method to calculate the area is very accurate as the linear properties of graph will cancel off the error of rectangular stripes for each station.

- ## Computation of Δy and y values

Understanding small angle theorem, $\sin\psi \approx \psi \approx \frac{dy}{ds}$ $\Delta y = \psi\Delta s$. Therefore $y = \int \psi \, ds$. From figure

8, the shape of the graph is no longer in linear condition, so it is not very accurate to use the midpoint rule in this case however still acceptable. Using of numerical integration again to solve for y, given the boundary condition where the y = 0 at both supports (station 1 and 11). This result will give the deflection at each station due to the applied load at station 6. Graph of y against s is plotted (Figure 9 Appendix). Our measured deflection at station 3, 6 and 9 are as below:

mm Final Value/mm	Deflection (Theoretical)/mm		
3	0	-4.72	4.7
6	-0.01	-8.23	8.25
9	0	-4.82	4.7

Table 2: Measured Initial and Final Value of Deflection Gauge and Theoretical Deflection

• Discussion

The deflection of station 3,6 and 9 due to the 80 N applied load are 4.72mm, 8.22mm and 4.82mm respectively. These values are calculated from the initial and final reading of the deflection gauge. The experimental values are reasonably close to the theoretical deflection when compared.

However, after adding the calibration factor of the curvature gauge (-6.3%), the error between the theoretical and experimental value for deflection increases. In fact, after including the calibration factor in the experimental deflection,

Station	Before calibration/mm	After Calibration/mm
3	4.72	$4.72 \pm \frac{6.3}{100} \times 4.72 = 4.72 \pm 0.30$
6	8.22	8.22 ± 0.52
9	4.82	4.82 ± 0.30

Table 3: Deflection Before and After Calibration

As compared with the calculated value, the values after calibration lies within the uncertainty range. Therefore, the overall experimental deflection is still considered to be accurate.

Some sources of error in the experiment were listed as below:

- Errors in the calibration of curvature gauge, and the calibration of deflection gauge (which is not considered in this case).
- The inaccurate position of curvature gauge at each station. The slight difference of value may be caused by parallax error when locating the curvature gauge at each station.
- The self-weight of the Aluminium Alloy was not considered where the uniform distributed load may cause deflection at each station in this case.
- The using of $g = 10ms^{-2}$, which is not really accurate when calculating Bending Moment, M and may cause inaccuracy in determining the value bending stiffness, B.

• Conclusion

1. The curvature of the elastic beam was measured through experimental method
2. The bending stiffness of beam was determined from the gradient of M- κ graph and compared with the theoretical value.
3. The deflection angle and vertical deflection displacement were analysed and calculated through numerical integration method. The measured deflection at certain stations was compared with the numerical integration result and it has been shown to lie within the uncertainty range when considering the calibration factor.
4. Some sources of error in this experiment have been evaluated and explained.

Station	Initial Curvature	Final Curvature	κ due to load	M	κ "simplified" = M/B	$\kappa_{mean}\Delta s = \Delta\psi$	ψ	$\psi_{mean}\Delta s = \Delta y$	y
	rad/cm	rad/cm	rad/cm	Ncm	rad/cm	rad	rad	cm	cm
1	-0.0008	-0.0009	-0.0001	0	0		0.05		0
						-2×10^{-3}		0.245	
2	-0.00095	-0.00175	-0.0008	-200	-8×10^{-4}		0.048		0.245
						-6×10^{-3}		0.225	
3	-0.0017	-0.0033	-0.0016	-400	-1.6×10^{-3}		0.042		0.47
						-0.01		0.185	
4	-0.0014	-0.00395	-0.00255	-600	-2.4×10^{-3}		0.032		0.655
						-0.014		0.125	
5	-0.00125	-0.00455	-0.0033	-800	-3.2×10^{-3}		0.018		0.78
						-0.018		0.045	
6	-0.001	-0.00495	-0.00395	-1000	-4×10^{-3}		0		0.825
						-0.018		-0.045	
7	-0.0009	-0.0042	-0.0033	-800	-3.2×10^{-3}		-0.018		0.78
						-0.014		-0.125	
8	-0.0008	-0.0032	-0.0024	-600	-2.4×10^{-3}		-0.032		0.655
						-0.01		-0.185	
9	-0.0008	-0.0025	-0.0017	-400	-1.6×10^{-3}		-0.042		0.47
						-6×10^{-3}		-0.225	
10	-0.0009	-0.00165	-0.00075	-200	-8×10^{-4}		-0.048		0.245
						-2×10^{-3}		-0.245	
11	-0.0007	-0.0008	-0.0001	0	0		-0.05		0
						0		-0.25	
12	-0.00045	-0.00045	0	0	0		-0.05		-0.25
						0		-0.25	
13	-0.00045	-0.00045	0	0	0		-0.05		-0.5
						0		-0.25	
14	-0.0006	-0.00075	-0.00015	0	0		-0.05		-0.75

Figure 3: List of all values of k, φ and y.

Test for deflection at station A.......

(6) Initial: - 0.01
Final: - 8.23

(9) Initial: 0
Final: -4.82

(mm)

(3) 0
-4.72 (mm)

-ψ ψ
0

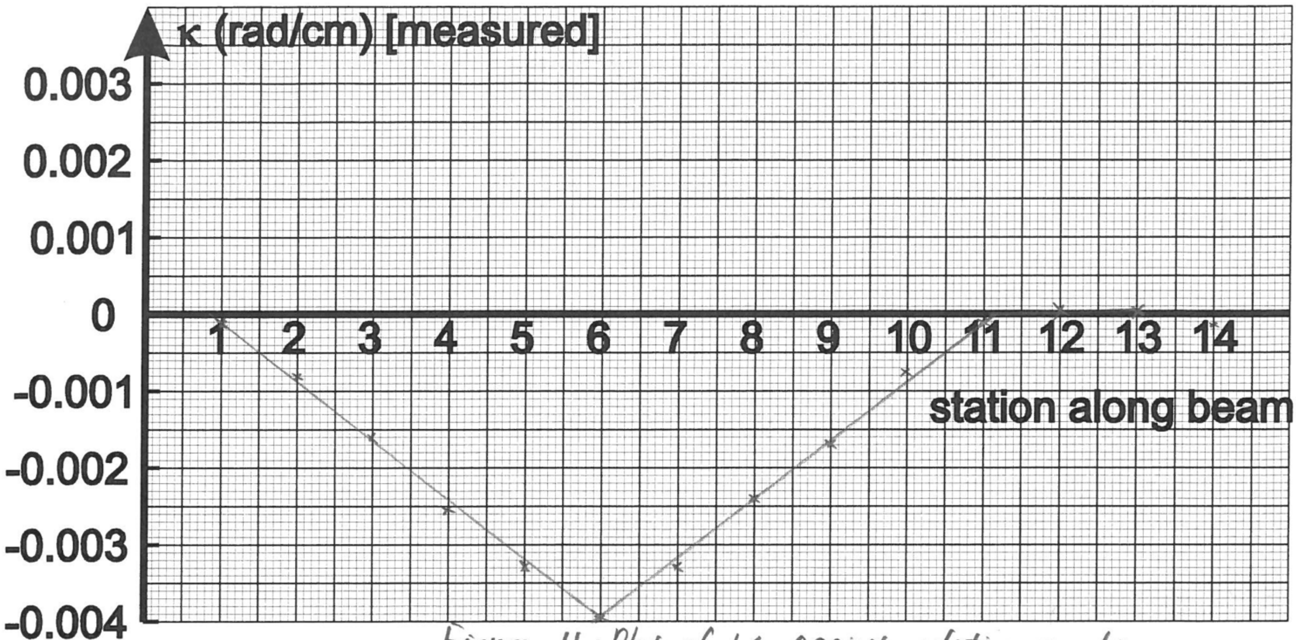


Figure 4: Plot of K against station number

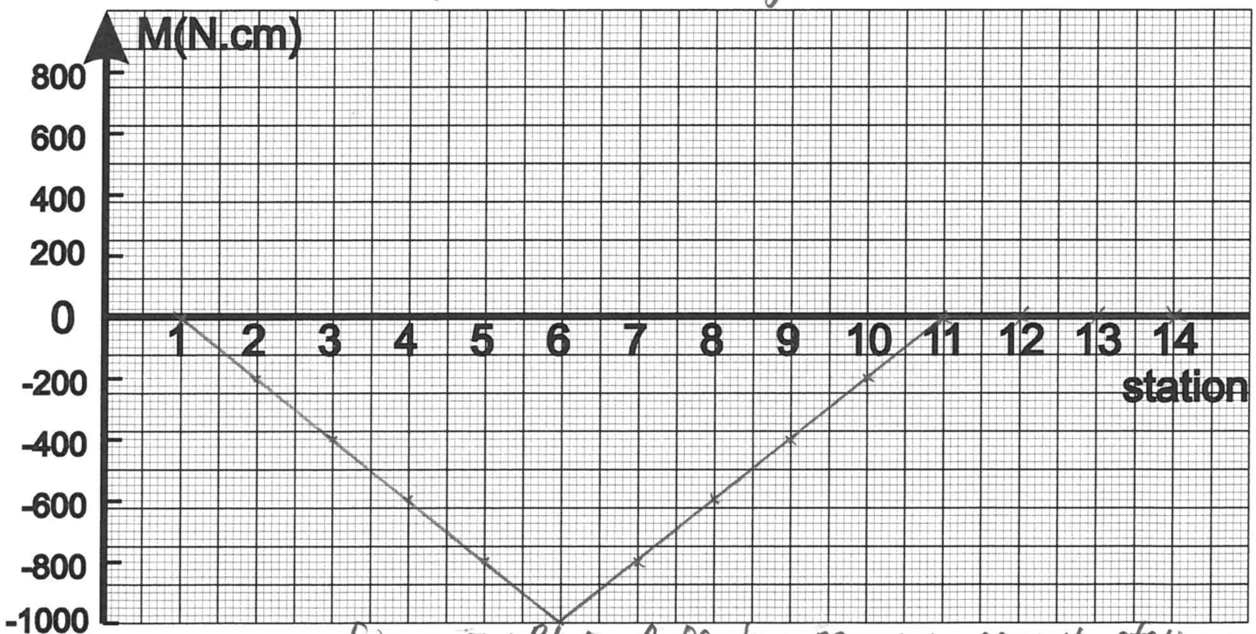


Figure 5: Plot of Bending moment against station number

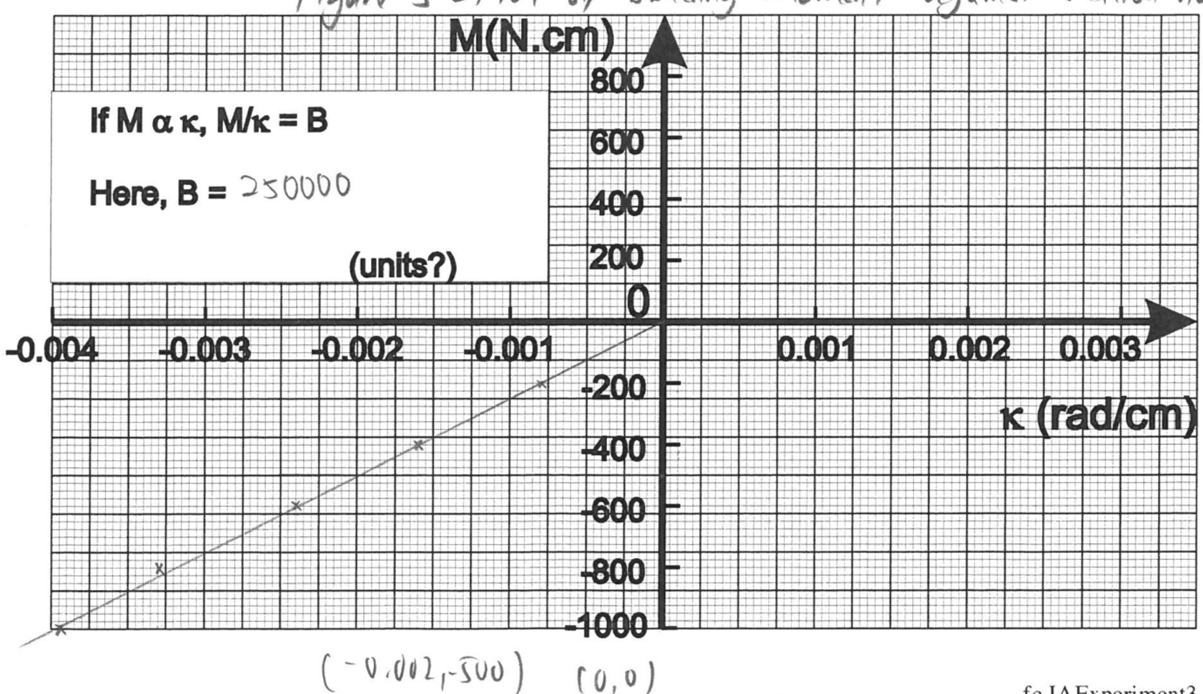


Figure 6: plot of Bending moment againt K.

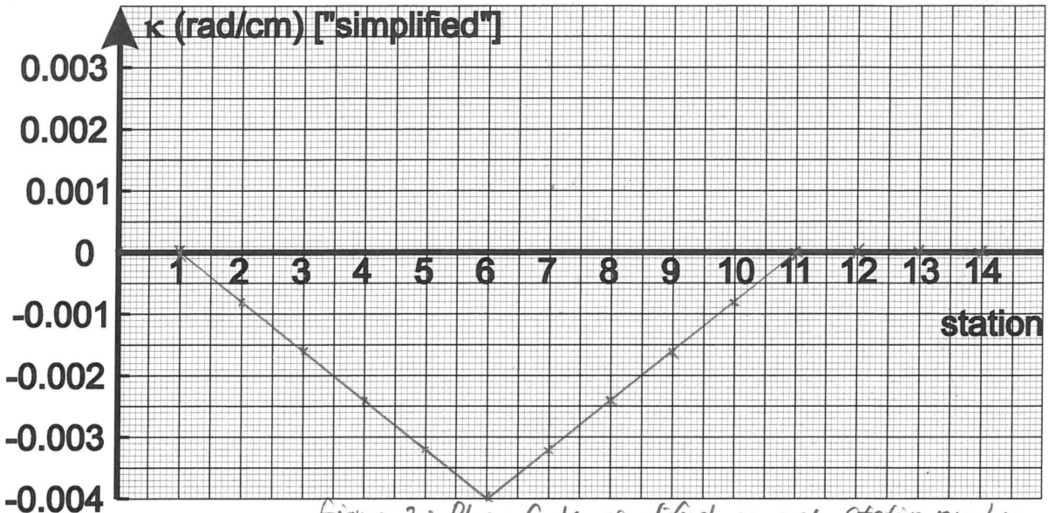


Figure 7: Plot of κ simplified against station number.

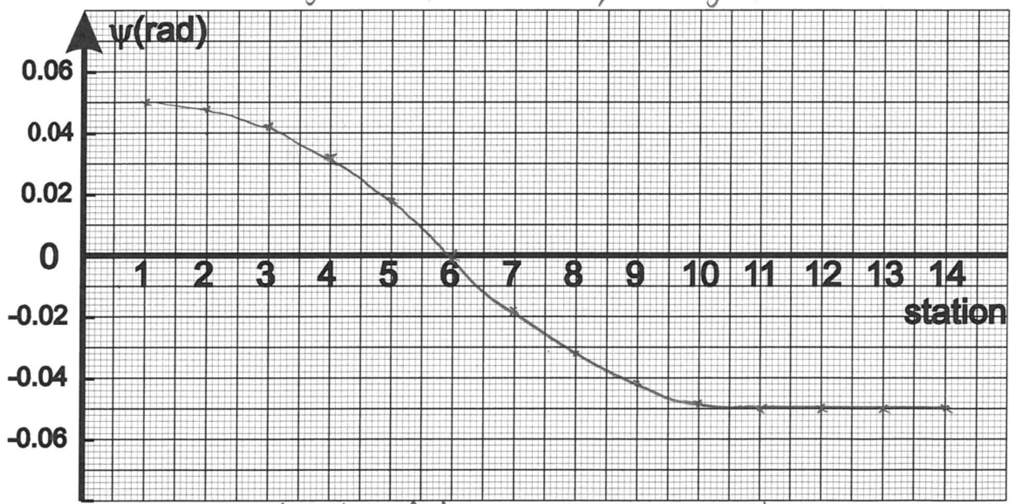


Figure 8: plot of ψ against station number

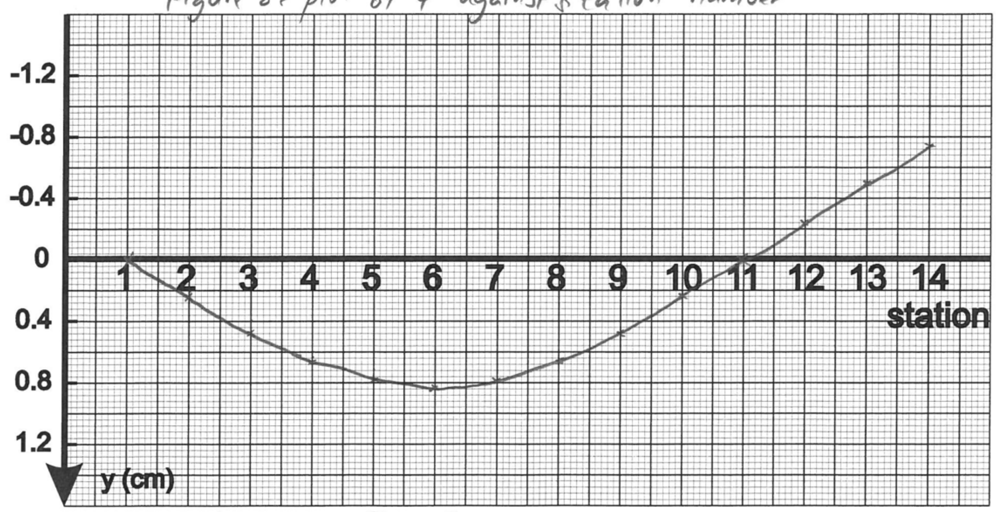


Figure 9: Plot of deflection y against station number