# Multiple Object Tracking

## Random Sets


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# Random Finite Sets

Compare to Single Object Tracking (SOT), in Multiple Object Tracking (MOT) the censors produces measurements for several objects. Let's say at a given time $t_k$ the state space contains 2 objects, at time $t_{k+1}$ it contains 10 objects at time $t_{k+2}$ it contains none objects, and so on and so forth. To be able to model this kind of situation, we need a random feature that contains many random vectors as time evolves. This can be seen as:

$$
\begin{aligned}
&t_k : \{X^1, X^2\} \\
&t_{k+1} : \{X^1, X^2, \cdots, X^{10}\} \\
&t_{k+2} : \emptyset
\end{aligned}
$$

Such sets are called Random Finite Sets (RFSs):

**Definition: RFSs**

A random finite set is any set on the form:

$$\{X^1, X^2, \cdots, X^N\} \tag{1}$$

where $X^i$ are random vectors and $N$ is a random integer

# Why RFSs?

The main objective in building a MOT algorithm is to derive the posterior probability $P(X_k|Z_{1:k})$. This can be done by using the Chapman-Kolmogorov Equations. For SOT, we have:

1. Prediction:

$$P(X_k|Z_{1:k-1}) = \int P(X_k|X_{k-1})P(X_{k-1}|Z_{1:k-1})dX_{k-1} \tag{2}$$

2. Update

$$P(X_k|Z_{1:k}) = \frac{P(Z_k|X_k)P(X_k|Z_{1:k-1})}{\int P(X_k|X_{k-1})P(X_{k-1}|Z_{1:k-1})dX_{k-1}} \tag{3}$$

In Multiple Object Tracking, the state model $X_k$ and the measurement model $Z_k$ are now sets containing random vectors.

# Distribution of RFSs: Multi Objects PDFs

**Definition**

The multi-object pdf $p_{\mathbf{X}}(X)$ of a RFS $\mathbf{X}$ is the pdf of that set. It is used to describe its distribution therefore, it's a positive function on sets and its integrate to one. The multi-object pdf is invariant to the order in the given set i.e.

$$p_{\mathbf{X}}(\{X^1, X^2\}) = p_{\mathbf{X}}(\{X^2, X^1\})$$

**Examples:** If $X \sim \mathcal{N}(0,1)$ and $\mathbf{X} = \{X\}$ then

$$p_{\mathbf{X}}(X) = \begin{cases} \mathcal{N}(z,0,1) & \text{if} \quad \mathbf{X} = \{z\} \\ 0 & \text{if} \quad |\mathbf{X}| \neq 1 \end{cases} \tag{4}$$

If $X^1 \sim unif(0,1)$ and $X^2 \sim unif(1,2)$ are independent and $\mathbf{X} = \{X^1, X^2\}$, then

$$p_{\mathbf{X}}(\mathbf{X}) = \begin{cases} p_1(z^1)p_2(z^2) + p_1(z^2)p_2(z^1) & \text{if} \quad \mathbf{X} = \{z^1, z^2\} \\ 0 & \text{if} \quad |\mathbf{X}| \neq 2 \end{cases} \tag{5}$$

where $p_1$ and $p_2$ are the distributions of $X^1$ and $X^2$

# Interpretation of the Multi objects pdf as in $\mathbb{R}$

Let $a$ be a real number, $\epsilon$ a number close to zero, and $X$ a real random variable. It's well-known that

$$P(a < X < a + \epsilon) = \int_a^{a+\epsilon} f_X(x)dx \approx \epsilon f_X(a) \tag{6}$$

## For Random sets

Similarly, if $X = \{X^1, \ldots, X^n\}$ is an RFS and $\Delta X^1, \cdots, \Delta X^n$ are small,

$$p_X(\{X^1, \ldots, X^n\}) \times \Delta X^1 \times \cdots \times \Delta X^n \tag{7}$$

is the probability that each of the disjoints intervals $(X^1, X^1 + \Delta X^1), \cdots, (X^n, X^n + \Delta X^n)$ contain exactly one element.

# Important Formula: Convolution

If $X = X^1 \cup X^2$, where $X^1$ and $X^2$ are independent random sets, then $X$ has the multi-object pdf:

$$P_X(X) = \sum_{Y \subseteq X} P_{X^1}(Y) P_{X^2}(X \backslash Y) \tag{8}$$

**Eg:** $X = \{1.5, 2.7\}$

**Convolution Formula:**

If $X = X^1 \cup \cdots \cup X^n$ where $X^i$ are independent random finite sets, then:

$$P_X(X) = \sum_{X^1 \uplus \cdots \uplus X^n = X} \prod_{i=1}^{n} P_{X^i}(X^i) \tag{9}$$

The summation is done over all mutually disjoint (eventually empty) sets

**Eg:** $X = X^1 \cup X^2 \cup X^3$, find $P_X(\{4\})$

# Set Integrals and Expected value:

Let $f(X)$ be a real valued function of a finite set $X$.

**Set Integral**

The set integral of $f(X)$ concentrated in a region $S$ of $\eta_0$ is:

$$\begin{aligned}\int_S f(X)\delta x &:= \sum_{n\geq 0} \frac{1}{n!} \int_{S\times\cdots\times S} f(\{X^1, X^2, \cdots, X^n\}) dX^1 \cdots dX^n \\ &= f(\emptyset) + \int_S f(\{X^1\})dX^1 + 1/2 \int_{S\times S} f(\{X^1, X^2\}) dX^1 dX^2 + \cdots\end{aligned}$$

**NB:** Set integrals are linear on $f$ but not on $S$ even if the sets are disjoint.
**Eg:** If $X \sim \mathcal{N}(0,1)$ and $\mathbf{X} = \{X\}$ show that

$$p_{\mathbf{X}}(\{X\}) = \begin{cases} \mathcal{N}(z, 0, 1) & \text{if} \quad \mathbf{X} = \{z\} \\ 0 & \text{if} \quad |\mathbf{X}| \neq 1 \end{cases} \tag{10}$$

is a multi object pdf

# Set Integrals and Expected value:

**Expected value:**

The expectation of $f(X)$ is given by:

$$\begin{aligned}\mathbb{E}(f(X)) &= \int_{\eta_0} f(X)P_X(X)\delta X \\ &= \sum_{n\geq 0} \frac{1}{n!} \int_{S\times\cdots\times S} f(\{X^1, X^2, \cdots, X^n\})P_X(\{X^1, X^2, \cdots, X^n\})dX^1 \cdots dX^n\end{aligned}$$

**NB:** If $X$ is a random set, its expectation $\mathbb{E}(X)$ is not define. In fact, it's not possible to average sets, the sum

$$\{0.3, 0.2\} + \{0.6\} + \{1, 5, 8.5, 5\}$$

is not define.
Another interesting aspect of RFS is its **cardinality distribution** which can be expressed in term of expected value.

# Set Integrals and Expected value

**Cardinality Distribution**

The cardinality distribution of a RFS $X \sim P_X(\cdot)$ is:

$$P_X(n) = P(|X| = n)$$

Its hold that:

$$\begin{aligned} P(|X| = n) &= \mathbb{E}(\delta_{n-|X|}) \quad \text{where } \delta \text{ is the Kronecker delta function} \\ &= \sum_{i \geq 0} \frac{1}{i!} \int \delta_{n-i} P_X(\{X^1, X^2, \cdots, X^i\}) dX^1 \cdots dX^i \\ &= \frac{1}{n!} \int P_X(\{X^1, X^2, \cdots, X^n\}) dX^1 \cdots dX^n \end{aligned}$$

**Eg:** Find the cardinality distribution of the distribution in equation 8 (It's a Bernouilli distribution)

# Belief Mass Function and the p.g.fl.s

Other alternative of describing a Random Finite set is by using the belief mass function and the probability generating functionals (p.g.fl.s).

### Belief mass function

If $X$ is a RFS, and $S$ a region of $\eta_0$, then

$$\beta_X(S) = P(X \subseteq S)$$

is called the belief mass function of $X$

### p.g.fl.s

They are very useful for deriving filtering recursions

# Some Famous multi object pdfs in MOT

Some Multi objects pdfs are very useful in MOT mostly:

- The Poisson Point Process (PPP)
- The Bernouilli RFS
- Multi Bernouilli RFS
- Multi Bernouilli Mixtures RFS

**The PPP distribution:**

The multi object pdf of a PPP $X$ is:

$$P_X(x) = \exp(-\int \lambda(x)dx) \prod_{x \in X} \lambda(x)$$

Where $\lambda(x)$ is the intensity function of $X$

The PPPs are mostly used to model clutter detections in the entire object state space $\mathbb{R}^{nx}$. It's also used to model appearing object.

# Some Famous multi object pdfs in MOT

**Bernoulli RFSs:**

A benoulli RFS $X$ also called a Bernouilli process has the multi object pdf:

$$P_X(X) = \begin{cases} 1 - r & \text{if} \quad X = \emptyset \\ rP_X(X) & \text{if} \quad X = \{X\} \\ 0 & \text{if} \quad |X| > 1 \end{cases}$$

Where $0 \leq r \leq 1$ and $P_X(X)$ is a pdf.

It's easy to show that

$$P_X(|X| = n) = \begin{cases} 1 - r & \text{if} \quad n = 0 \\ r & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$$

Bernoulli RFSs are commonly used to model measurements from a single object.

# Some Famous multi object pdfs in MOT

Let be $X_1, \cdots, X_N$ independants Bernoulli RFSs with multiobjects pdfs $P_{X_1}(X_1), \cdots, P_{X_N}(X_N)$ respectively

## Multi-Bernoulli RFSs

then $X = \cup_{i=1}^{N} X_i$ is a multi Bernoulli (MB) RFS also called multi Bernoulli process with multi object pdf

$$P_X(x) = \sum_{\uplus_{i=1}^{N} X_i = X} \prod_{j=1}^{N} P_{X_j}(X_j) \quad \text{(Convolution formula)} \tag{11}$$

## MB mixture RFSs

If $P_{X_i}^{h}(X_i)$ are Bernoulli multi object pdfs for $i = 1, \cdots, N$ and $h = 1, \cdots, H$ Then $X$ is a MB mixture (MBM) RFS if it has th multi object pdf

$$P_X(X) = \sum_{h=1}^{H} W_h P_X^h(X) \tag{12}$$

# Some Famous multi object pdfs in MOT

Where $P_X^h(X)$ is the multi Bernoulli pdf

$$P_X^h(X) = \sum_{\uplus_{i=1}^{N} X_i = X} \prod_{j=1}^{N} P_{X_j}^h(X_j) \tag{13}$$

And $W_i$ are non negatives weights summing to one.
MBM RFS are mainly used to represent the posterior distribution odf the set of detected object.
Having all these notion gathered, we can now attack the bayesian filtering as well as deriving some recursives formulas.

# Filtering and Recursive formula:

We are interested on derving the measurement model $Z_k$ given the state $X_k$. We can write

$$Z_k = O_k \cup C_k$$

We denotes by $g_k(O_k|X_k) = P(O_k|X_k)$ and $P^D(x)$ will be the probability that object $x$ is detected.

**Case 1:** $X_k = \emptyset$

$$g_k(O|X_k) = \begin{cases} 1 & \text{if} \quad O = \emptyset \\ 0 & \text{else} \end{cases}$$

**Case 2:** $X_k = \{x\}$

$$g_k(O|\{x\}) = \begin{cases} 1 - P^D(x) & \text{if} \quad O = \emptyset \\ P^D(x) g_k(z|x) & \text{if} \quad O = \{z\} \\ 0 & \text{if} \quad |O| > 1 \end{cases}$$

# Filtering and Recursive formula:

**Remark:** The set of measurements for a single object is a Bernoulli RFS. The set of measurements for multiple objects is therefore a MB RFS. In fact, Suppose the given state is $X_k = \{X_k^1, \cdots, X_k^{n_k}\}$ and $O_k(X_k^i)$ the RFS of measurement from the state $X_k^i$ then,

$$O_k = O_k(X_k^1) \cup \cdots \cup O_k(X_k^{n_k}) \tag{14}$$

$O_k(X_k^i)$ are independant Bernoulli RFS ie

$$O_k(X_k^i)|X_k^i \sim g_k(\cdot|\{X_k^i\})$$

## multi object measurement model

By applying the convolution formula, we obtain the general **multi object measurement model** for the state $X_k = \{X^1, \cdots, X^{n_k}\}$:

$$g_k(O_k|\{X^1, \cdots, X^{n_k}\}) = \sum_{O^1 \uplus \cdots \uplus O^{n_k} = O_k} \prod_{i=1}^{n_k} g_k(O^i|\{X^i\}) \tag{15}$$

where $O^i = O_k(X^i)$

# Filtering and Recursive formula:

Since we have the distribution of $O_k|X_k$, Let move to the distribution of the complete measurement model:
**Recall:** $Z_k = C_k \cup O_k$ where $C_k$ and $O_k$ are independant. So, we can apply the covolution formula to derive the distribution of $Z_k|X_k$.

$$P(Z_k|X_k) = \sum_{C \uplus O = Z_k} P_{C_k}(C) g_k(O|X_k) \tag{16}$$

$P_{C_k}$ is the clutter distribution and its follows a poisson RFS distribution. Applying equation 15 yield

**Complete measurement model**

$$P(Z_k|X_k) = \sum_{C \uplus O^1 \uplus \cdots \uplus O^{n_k} = O_k} P_{C_k}(C) \prod_{i=1}^{n_k} g_k(O^i|\{X^i\}) \tag{17}$$

# Filtering and Recursive formula:

**Eg:** $X_k = \{X\}$ and $Z_k = \{Z\}$.

$$\begin{aligned} P(Z_k|X_k) &= \sum_{C \uplus O = Z_k} P_{C_k}(C) g_k(O|\{X\}) \\ &= P_{C_k}\{Z\} g_k(\emptyset|\{X\}) + P_{C_k}\{\emptyset\} g_k(\{Z\}|\{X\}) \\ &= \exp(-\bar{\lambda}_C)\lambda_C(Z)(1 - P^D(X)) + \exp(-\bar{\lambda}_C) P^D(X) g_k(Z|X) \end{aligned}$$

# Standard Motion model

Objects can appear/disappear as the time evolves.
Given $X_{k-1}$, we assume: $X_k = S_k \cup b_k$ where $S_k$ and $b_k$ are independant.

1. $S_k$ are objects present also at time $k-1$
2. $b_k$ are objects that have appeared since time $k-1$

Let's figure out the distribution of the surviving object given $X_{k-1}$

# Surviving Objects

We set $\pi_k(S_k|X_{k-1})$ the distribution of surviving objects
Denotes by $P^S(x)$ the probability that the object with state $x$ survives.

**case1: $X_{k-1} = \emptyset$**

$$\pi_k(S|\emptyset) = \begin{cases} 1 & \text{if} \quad S = \emptyset \\ 0 & \text{else} \end{cases}$$

**case2: $X_{k-1} = \{x\}$**

$$\pi_k(S|\{x\}) = \begin{cases} 1 - P^S(x) & \text{if} \quad S = \emptyset \\ P^S(x)\pi_k(z|x) & \text{if} \quad S = \{z\} \\ 0 & \text{if} \quad |S| > 1 \end{cases}$$

Using the same techniques as in the derivation of the derivation model distribution, we get

# Complete motion model: ie $X_k|X_{k-1}$

**case3: $X_{k-1} = \{X^1, \cdots, X^{n_{k-1}}\}$**

$$\pi_k(S_k|\{X^1, \cdots, X^n\}) = \sum_{S^1 \uplus \cdots \uplus S^{n_{k-1}} = S_k} \prod_{i=1}^{n_{k-1}} \pi_k(S^i|\{X^i\}) \tag{18}$$

**Recall:** $X_k = S_k \cup b_k$. Convolution formula yields:

$$P(X_k|X_{k-1}) = \sum_{b \uplus S = X_k} P_{b_k}(b)\pi_k(S|X_{k-1}) \tag{19}$$

$P_{b_k}(b)$ is the birth model and we assume it to follow a Poisson RFS i.e.

$$P_{b_k}(b) = \exp\big(-\int \lambda_b(b')db'\big) \prod_{b \in b} \lambda_b(b)$$

Having $X_{k-1} = \{X^1_{k-1}, \cdots, X^{n_{k-1}}_{k-1}\}$ we can write

$$X_k = b_k \cup S_k(X^1_{k-1}) \cup \cdots \cup S_k(X^{n_{k-1}}_{k-1})$$

# Complete motion model: ie $X_k|X_{k-1}$

We derive using the convolution formula the complete motion model as:

**Complete model**

$$P(X_k|X_{k-1}) = \sum_{b \uplus S^1 \uplus \cdots \uplus S^{n_{k-1}} = X_k} P_{b_k}(b) \prod_{i=1}^{n_{k-1}} \pi_k(S^i|\{X_{k-1}^i\})$$

Where $S^i = S(X_{k-1}^i)$

# PHD filtering

**Motivation:**

Recall the Chapman Kolmogorv equations for RFS:

$$P(X_k|Z_{1:k-1}) = \int P(X_k|X_{k-1})P(X_{k-1}|Z_{1:k-1})dX_{k-1}$$

$$P(X_k|Z_{1:k}) = \frac{P(Z_k|X_k)P(X_k|Z_{1:k-1})}{\int P(X_k|X_{k-1})P(X_{k-1}|Z_{1:k-1})dX_{k-1}}$$

we already know how to get the RFS distribution of $X_k|X_{k-1}$ and $Z_k|X_k$. What about $X_k|Z_{1:k-1}$ and $X_{k-1}|Z_{1:k-1}$?

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