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Course: Inverse Problem

Home assignement

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1 Exercise 1:

1. . Let's write the problem to the form

$$y = Ax + n \tag{1}$$

We have: $\forall j = 1, \cdots, 150$, $y_j = g(t_j) + n_j = \int_{t_0}^{t_j} f(s)ds + n_j$. That integral can be approximated usin gtrapezoidal rule as:

$$\begin{aligned} \int_{t_0}^{t_j} f(s)ds &= \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} f(s)ds & (2) \\ &= \sum_{k=0}^{j-1} \frac{f(t_{k+1}) + f(t_k)}{2}(t_{k+1} - t_k) & (3) \\ &= \sum_{k=0}^{j-1} \frac{f(t_{k+1}) + f(t_k)}{2}\frac{1}{150} & (4) \\ &= \frac{1}{300}\sum_{k=0}^{j-1}(f(t_{k+1}) + f(t_k)) & (5) \end{aligned}$$

then $\forall j = 1, \cdots, 150$, we have:

$$y_j = \frac{1}{300}\sum_{k=0}^{j-1}(f(t_{k+1}) + f(t_k)) + n_j \tag{6}$$

Expanding equation (6) for every y_j yields to

$$\begin{cases} y_1 = 1/300(f(t_1) + f(t_0)) + n_1 \\ y_2 = 1/300(f(t_2) + 2f(t_1) + f(t_0)) + n_2 \\ y_3 = 1/300(f(t_3) + 2f(t_2) + 2f(t_1) + f(t_0)) + n_3 \\ \qquad \vdots \\ y_{150} = 1/300(f(t_{150}) + 2f(t_{149}) + \cdots + 2f(t_1) + f(t_0)) + n_{150} \end{cases} \tag{7}$$

We can then generalize equations (7) into matrix from as:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{149} \\ y_{150} \end{pmatrix} = \frac{1}{300} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & 2 & \cdots & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix} \begin{pmatrix} f(t_0) \\ f(t_1) \\ f(t_2) \\ \vdots \\ f(t_{149}) \\ f(t_{150}) \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_{149} \\ n_{150} \end{pmatrix} \tag{8}$$

This yields to:

$$y = Ax + n \tag{9}$$

with $A = \frac{1}{300} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & 2 & \cdots & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix} \in \mathbb{R}^{150\times151}$

2. Characterize $ker(A)$ and $Ran(A)^{\perp}$

   We have:

$$ker(A) = \{x \in \mathbb{R}^{1\times151}, |Ax = 0_{\mathbb{R}^{1\times150}}\} \tag{10}$$

Let be $x = (x_0, x_1, \cdots, x_{150})^T \in ker(A)$. then, $Ax = 0_{\mathbb{R}^{1\times150}}$ i.e.

$$\frac{1}{300} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & 2 & \cdots & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{149} \\ x_{150} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} x_0 + x_1 = 0 \\ x_0 + 2x_1 + x_2 = 0 \\ x_0 + 2x_1 + 2x_2 + x_3 = 0 \\ \qquad \vdots \\ x_0 + 2x_1 + \cdots + 2x_{149} + x_{150} = 0 \end{cases}$$

In the first equation we can deduce that $x_0 = -x_1$. Substituying this back in all the equqtions gives

$$\begin{cases} x_0 = -x_1 \\ x_1 = x_1 \\ x_2 = -x_1 \\ x_3 = x_1 \\ \qquad \vdots \\ x_{150} = -x_1 \end{cases} \tag{11}$$

i.e. $\forall j = 0, \cdots, 75, \quad x_{2j} = -x_1$ and $\forall j = 1, \cdots, 75 \quad x_{2j-1} = x_1$ then we can rewrite x as $x = x_1(-1, 1, -1, 1, \cdots, -1)^T$. We then deduce that the kernel of A is spanned by $(-1, 1, -1, 1, \cdots, -1)^T$ i.e.

$$ker(A) =< (-1, 1, -1, 1, \cdots, -1)^T > \tag{12}$$

$$Ran(A)^{\perp} = \{u \in \mathbb{R}^{150}, < u, Av >= 0 \quad \forall v \in \mathbb{R}^{151}\} \tag{13}$$

Where $< \cdot >$ is the euclidian inner product.

3. Use Matlab to compute the singular value decomposition and the Moore–Penrose pseudoinverse of A. Plot the singular values of A
We have for our case

$$\begin{aligned} y = Ax \implies & A^T y = A^T A x \\ \implies & (A^T A)^{-1} A^T y = x \end{aligned}$$

ie the Moore–Penrose pseudoinverse of A for our case is given by:

$$A^+ = (A^T A)^{-1} A^T$$

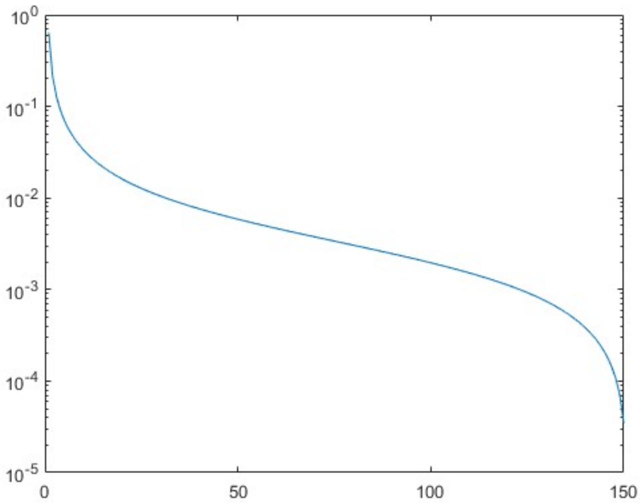


Figure 1: Singular values of A

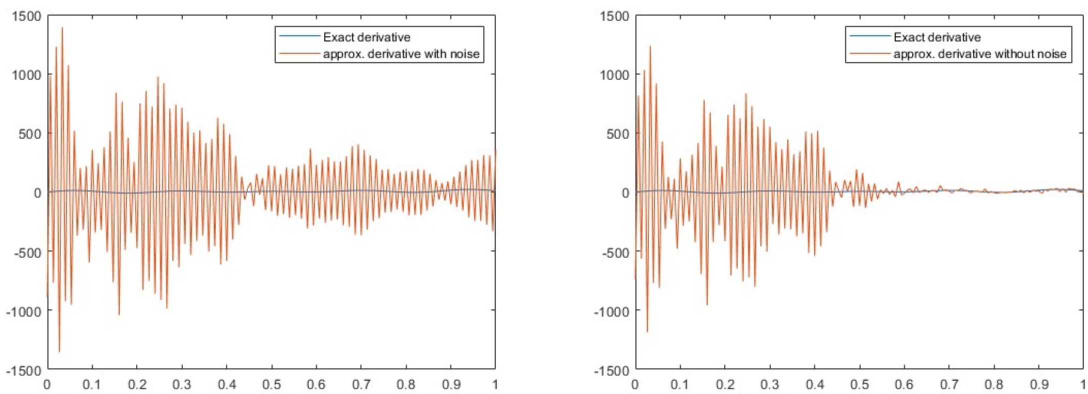
4. Consider the linear system when

$$g(t) = \cos(\pi t) \sin^2(4\pi t) + 5t^2 \tag{14}$$

Therefore, the exact derivative of g is given by:

$$g'(t) = -\pi \sin(\pi t) \sin^2(4\pi t) + 8\pi \cos(\pi t) \cos(4\pi t) \sin(4\pi t) + 10t \tag{15}$$

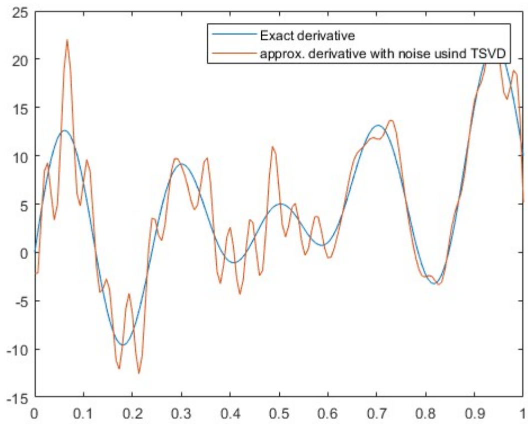
Plotting the minimum norm solution with noise and without noise yield the figures below:



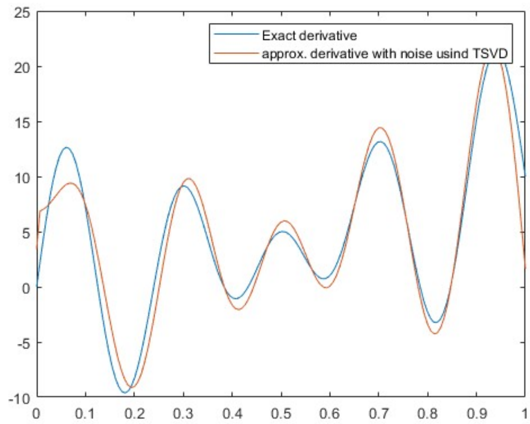
(a) Approximation with noise (b) Approximation without noise

We observe from the plots above that the use of the pseudo inverse does not give a reasonable results. Meanwhile, the approximation with the data without noise is becoming very close to the exact solution on the interval $[0.5 \quad 1]$.

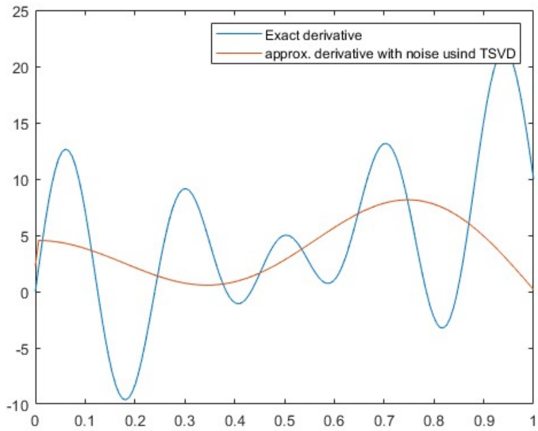
5. Alternative solution using TSVD (Truncated Singular Value Decomposition) in the noisy data. By using TSVD with a eigevalues treshold of 0.006, 0.03 and 0.11 we have the results below:



(a) treshold = 0.006



(b) treshold = 0.03



(c) treshold = 0.11

Among the 3 cuts-off, the treshold 0.03 gives the best reconstruction.

2 Exercise2

1. We have the system:

$$u_t = u_{xx} - u \quad \text{on} \quad (0,\pi)\times\mathbb{R}^+ \tag{16}$$
$$u(0,\cdot) = u(\pi,\cdot) = 0 \quad \text{on} \quad \mathbb{R}^+ \tag{17}$$
$$u(\cdot,0) = f \quad \text{on} \quad (0,\pi) \tag{18}$$

We also have: $x_j = jh,\ j = 0, 1\cdots, 100,\ h = \dfrac{\pi}{100}$ and $U(t) = \begin{pmatrix} U_1(t) & \cdots & U_{99}(t)\end{pmatrix}^T = \begin{pmatrix} u(x_1,t) & \cdots & u(x_{99},t)\end{pmatrix}^T$

Using the finite element methods we can write $\forall j = 1,\cdots,99$:

$$u_{xx}(x_j,t) \approx \frac{u(x_{j+1},t) - 2u(x_j,t) + u(x_{j-1},t)}{h^2} \tag{19}$$
$$= \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2} \tag{20}$$

Now, let's deal with the boundary conditions. We have: $\forall x \in \mathbb{R}^+, u(x,0) = f$ ie $U(0) = F$ where $F = (f(x_1), f(x_2), \cdots, f(x_{99}))^T$. Also, $\forall t \in (0,\pi), \quad u(0,t) = u(\pi,t) = 0 \implies U_0(t) = U_{100}(t) = 0$ Replacing all this, for $j = 2 \cdots 98$ the initial system yields:

$$U_j'(t) = \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2} - U_j(t) \tag{21}$$

$$= \frac{U_{j+1}(t) - (2+h^2)U_j(t) + U_{j-1}(t)}{h^2} \tag{22}$$

and

$$U_1'(t) = \frac{U_2(t) - 2U_1(t)}{h^2} - U_1(t) = \frac{U_2(t) - (2+h^2)U_1(t)}{h^2} \tag{23}$$

$$U_{99}'(t) = \frac{-2U_{99}(t) + U_{98}(t)}{h^2} - U_{99}(t) = \frac{-(2+h^2)U_{99}(t) + U_{98}(t)}{h^2} \tag{24}$$

The discretize system is then:

$$U'(t) = BU(t) \quad \forall t \in \mathbb{R}^+ \tag{25}$$

$$U(0) = F \tag{26}$$

Where $B \in \mathbb{R}^{99\times 99}$ and

$$B = \frac{1}{h^2}\begin{pmatrix} b & 1 & & & \\ 1 & b & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & b & 1 \\ & & & 1 & b \end{pmatrix} \tag{27}$$

Where $b = -(2+h^2)$ The solution of the ODE in equation (26) is then:

$$U(T) = \exp(TB)F = AF \tag{28}$$

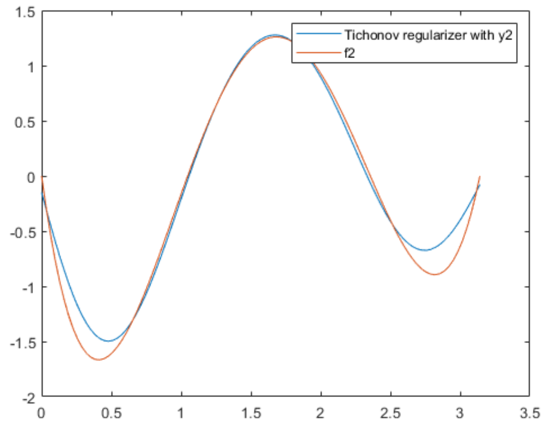
2. Let's solve the inverse problem for y_1 and y_2 using the Tichonov regularizer.

   We have:

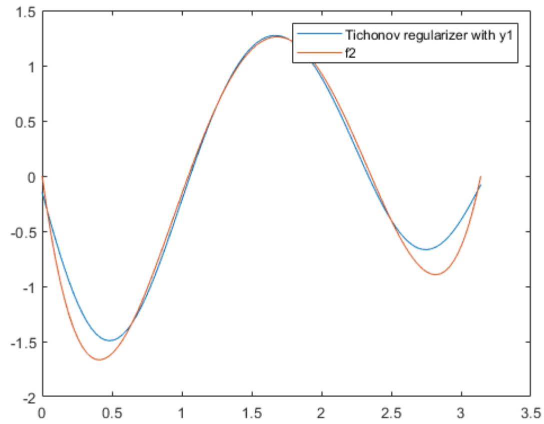
$$F_i = \underset{F}{argmin}\|AF - y_i\|^2 + \alpha\|F\|^2 \quad i = 1,2 \tag{29}$$

   Taking the sequence $\alpha_n = (10^{-n})_{n\in\mathbb{N}}$, We found on Matlab that the alpha which satisfy closely the dispregnancy rule is $\alpha = 10^{-2}$ for both measurement.

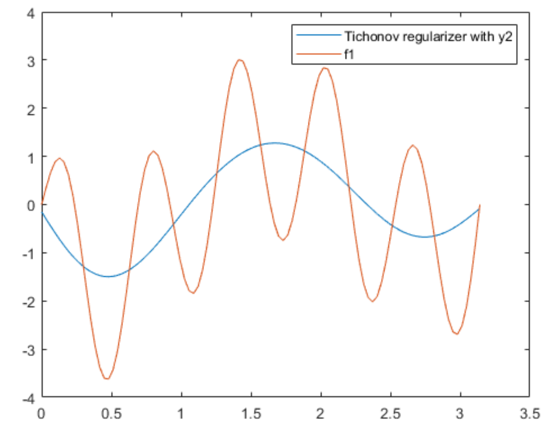
3. let's call $f_1(x)$ and $f_2(x)$ the two vectors corresponding to the initial heat distribution ie $f_1(x) = \frac{1}{10}x(3x-\pi)(4x-3\pi)(x-\pi) + 2\sin(10x)$ and $f_2(x) = \frac{1}{10}x(3x-\pi)(4x-3\pi)(x-\pi)$. It is not possible to distinguish which vector is for which data. In fact, using that value of $\alpha = 0.01$, we plotted the measurements data y_1 and y_2 vs F_1 and F_2 obtained with the Tichonov regularizer. Below is the result



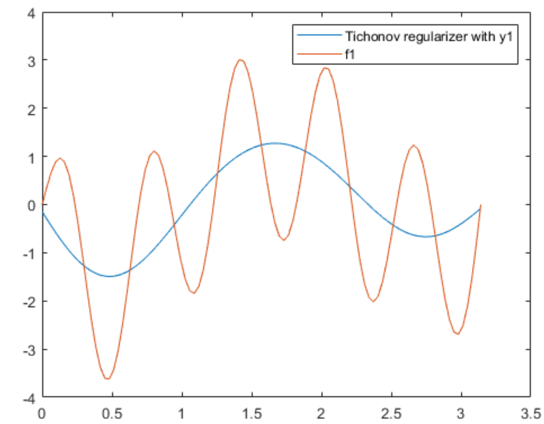
(a) F_2 vs f_2



(b) F_1 vs f_2



(c) F_2 vs f_1



(d) F_1 vs f_1

We Observe from the plots that the Tichonov regularizer built with either measurements y_1 or y_2 fit very well function f_2 and fit badly function f_1 therefore, it is not possible to distinguish from which function the measurements belong to. This is somehow surprising but can be intuitively explained by the fact that the initial temperature of the heat may not be too much varying.