Contents

1 Introduction to Nuclear Physics

Week 1 provided a foundational understanding of nuclear physics, focusing on the fundamental principles, units, and equations essential for describing nuclear phenomena. Here are the key insights and how they interrelate:

1. **Introduction to Nuclear Forces and Models**:

- The range of nuclear forces is derived from the energy-mass relationship:

$$R \sim \frac{\hbar c}{\Delta E} \sim \frac{\hbar}{m_X c}$$

- Understanding nuclear phenomenology requires grasping the properties of nuclei, such as stability, structure, and reactions.
- Various nuclear models (liquid-droplet, Fermi gas, shell, and collective models) offer different perspectives on nuclear behavior, each highlighting different aspects of nuclear forces and interactions.

2. **Nuclear Physics Units**:

- The unit of atomic mass (u) and energy (MeV) are fundamental for measuring nuclear properties. These units help in quantifying nuclear masses and binding energies, crucial for understanding nuclear stability and reactions.

$$1 \text{ u} = 1 \text{ g}/N_A = 1.66 \times 10^{-27} \text{ kg} = 931.5 \text{ MeV}/c^2$$

- The masses of proton, neutron, and electron in energy units:

$$m_p c^2 = 938.27231 \text{ MeV}, \quad m_n c^2 = 939.56563 \text{ MeV}, \quad m_e c^2 = 0.51099906 \text{ MeV}$$

- Other useful constants:

$$\hbar c = 197.3269602 \text{ MeV} \cdot \text{fm}, \quad hc = 1239.84 \text{ MeV} \cdot \text{fm}, \quad \frac{e^2}{4\pi\varepsilon_0} = \alpha\hbar c$$

3. **Special Relativity (SR) and Relativistic Kinematics**:

- SR relations, like time dilation and length contraction, are crucial for high-energy nuclear processes where relativistic effects cannot be ignored:

$$t = \gamma\tau, \quad L = \frac{L_0}{\gamma}$$

- The relativistic energy-momentum relationship is essential for calculating particle energies and understanding reaction dynamics:

$$E^2 = m^2c^4 + p^2c^2$$

- Other important relativistic formulas:

$$E = \gamma mc^2, \quad T = E - mc^2 = (\gamma - 1)mc^2$$

- Invariant mass relationships:

$$x^\mu x_\mu = c^2t^2 - |\vec{x}|^2, \quad p^\mu p_\mu = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2c^2$$

$$M_X^2 c^4 = \left(\sum_{i=1}^{n} E_i\right)^2 - \left|\sum_{i=1}^{n} \vec{p}_i\right|^2 c^2$$

4. **Quantum Mechanics**:

- The Schrödinger equation describes non-relativistic quantum systems:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t)$$

- The Klein-Gordon equation for spin-0 particles:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(t,\mathbf{r}) = 0$$

- The Dirac equation for spin-1/2 particles:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

5. **Cross Sections and Decay Rates**:

- Cross sections measure the probability of nuclear reactions:

$$R = \Phi_a \cdot N_b \cdot \sigma, \quad N_b = n_b s d, \quad n_b = \frac{\rho N_{Av}}{A}$$

- Cross-section formula:

$$\sigma = \frac{R}{\Phi_a \cdot N_b}$$

- Exponential decay law describes spontaneous nuclear transformations:

$$P(t) = P_0 e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{T_{1/2}}, \quad \tau = \frac{1}{\lambda}$$

6. **Fermi's Golden Rule and Feynman Diagrams**:

- Fermi's Golden Rule relates the transition rates between quantum states to the interaction matrix elements:

$$\Gamma(i \to f) = \lambda_f = \frac{2\pi}{\hbar} \left|M_{fi}\right|^2 \rho(E_f)$$

- Matrix element for a potential interaction:

$$M_{fi} \equiv \langle \Psi_f | \widehat{H} | \Psi_i \rangle = \int d^3\vec{r}\, V(\vec{r}) e^{i\vec{q}\cdot\vec{r}/\hbar}, \quad \vec{q} = \vec{p}_f - \vec{p}_i$$

- Reaction rate:

$$R = \sigma \Phi_a N_b = \frac{2\pi}{\hbar} \left|\mathcal{M}(q^2)\right|^2 \rho(E_f) N_a N_b$$

The interplay of these topics highlights the multi-faceted nature of nuclear physics, combining classical mechanics, special relativity, and quantum mechanics to describe and predict nuclear phenomena.

2 Nuclear Forces and Models

Week 2 delved deeper into specific nuclear phenomena, such as scattering, stability, and the distribution of mass and charge within nuclei. Key insights include:

1. **Rutherford Scattering**:

- Rutherford scattering describes the deflection of particles by the Coulomb force, with the cross-section formula linking impact parameters and scattering angles:

$$a = \frac{Z_1 Z_2 e^2}{E_{\text{cm}}^2}, \quad b = a \cot \frac{\theta}{2}$$

- The Rutherford cross-section formula:

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \left(\frac{Z_1 Z_2}{4E_{\text{kin}}}\right)^2 \sin^{-4}\frac{\theta}{2} = (\alpha\hbar c)^2 \left(\frac{Z_1 Z_2}{4E_{\text{kin}}}\right)^2 \sin^{-4}\frac{\theta}{2}$$

- The Mott cross-section formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right) \cong \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \cos^2\left(\frac{\theta}{2}\right)$$

2. **Nuclear Stability**:

- The stability of nuclei is influenced by the balance of nuclear forces, the even-odd rules, and magic numbers.
- The stability line and isobar nuclides illustrate the role of proton-neutron ratios in determining nuclear stability, guiding predictions about which nuclei are stable or radioactive.

3. **Nuclear Size and Charge Distribution**:

- The distribution of mass and charge within nuclei affects nuclear reactions and properties.
- The de Broglie wavelength and form factors help quantify these distributions:

$$\lambda = \frac{h}{p}$$

- The charge distribution formula describes how nuclear charge density varies with distance from the nucleus:

$$\rho(r) = \rho_0 \left(1 + e^{(r-R)/a}\right)^{-1}$$

- Understanding these distributions is crucial for predicting nuclear behavior in reactions and decays.

4. **Nuclear size and distribution of mass and charge**:

- Nuclear force radius ¿ Mass radius ≈ Charge radius.
- de Broglie wavelength: $\lambda = \dfrac{h}{p} = \dfrac{hc}{\sqrt{E_{\text{kin}} \cdot (E_{\text{kin}} + 2mc^2)}} = \dfrac{1239.84 \text{ MeV} \cdot \text{fm}}{\sqrt{E_{\text{kin}} \cdot (E_{\text{kin}} + 2mc^2)}}$.
- Relationship between cross section and charge distribution: $\dfrac{d\sigma}{d\Omega} = \left(\dfrac{d\sigma}{d\Omega}\right)_{\text{point-like}} \left|F(q^2)\right|^2$
- Form factor is the Fourier transform of the normalized charge distribution:

$$F\left(q^2\right) \equiv \int e^{i\vec{q}\cdot\vec{r}/\hbar} \rho(\vec{r}) d^3\vec{r}$$

  This relationship is crucial because it allows us to infer the charge distribution $\rho(\vec{r})$ within a nucleus by performing an inverse Fourier transform of the form factor $F(q^2)$. This inverse relationship is essential for understanding the spatial distribution of charge and mass within the nucleus, providing insight into its structure and stability.

- Charge distribution: $\rho(r) = \dfrac{\rho_0}{1 + e^{(r-R)/a}}$, $\rho_0 \approx 0.17 \text{ fm}^{-3}$, $R = R_0 A^{1/3}$, $a = 0.54$ fm.
- Skin depth (Depth of $0.9\rho - 0.1\rho$): $t = 4 \ln 3 \times a = 2.4$ fm
- Mass and charge distribution can have different radii: $R_M \approx r_M \cdot A^{1/3} = 1.4 A^{1/3}$ fm, $R_Q \approx r_Q \cdot A^{1/3} = 1.2 A^{1/3}$ fm.
- The nuclear mass density is constant:

$$\rho_N = \frac{M}{\frac{4}{3}\pi R_M^3} = \frac{Au}{\frac{4}{3}\pi r_M^3 A} = \frac{3u}{4\pi r_M^3} \sim 10^{17} \text{ kg/m}^3.$$

3 Shell Model and Nuclear Structure

In Week 3, we focused on understanding the concept of binding energy, the theory of nuclear model and its implications for nuclear stability and reactions.

3.1 Nucleon Binding Energy

Here attaches the binding energy per nucleon verse A.

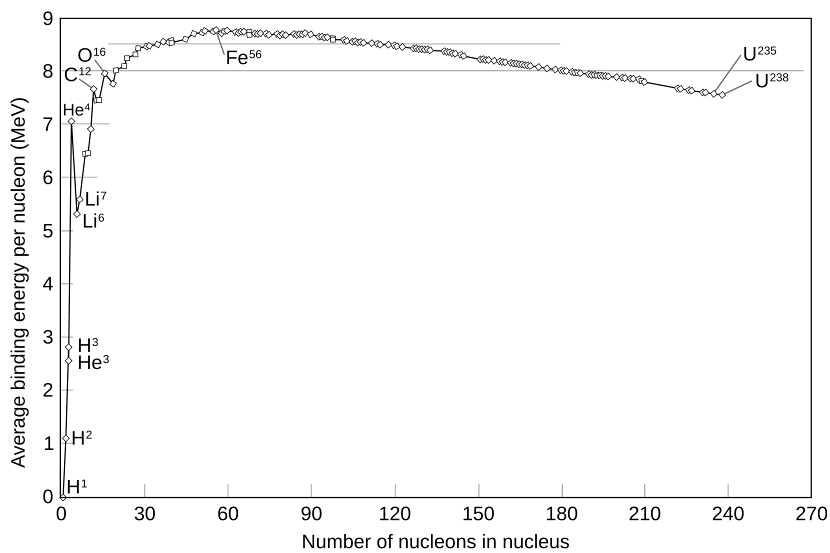


Figure 1: From `https://en.wikipedia.org/wiki/Nuclear_binding_energy`.

Stable Region:

- The graph of BE/A shows a peak around $A \approx 56$ (iron), indicating that nuclei around this mass number are the most stable. The binding energy per nucleon is maximized here, which implies that these nuclei are more stable and require more energy to break apart.

Nuclear Fission:

- Occurs in heavy nuclei (e.g., uranium-235). The binding energy per nucleon decreases as the nucleus becomes heavier, making it less stable. During fission, the nucleus splits into smaller fragments, releasing a large amount of energy due to the difference in BE/A before and after fission.

Nuclear Fusion:

- Occurs in light nuclei (e.g., hydrogen isotopes). Fusion results in a nucleus with a higher BE/A, releasing energy. For instance, fusion of hydrogen isotopes to form helium releases energy as the resulting nucleus has a higher binding energy per nucleon.

Constant Plateau:

- For nuclei with mass numbers significantly larger than iron, BE/A remains relatively constant between $7 - 9$ MeV. This indicates that adding more nucleons does not significantly change the binding energy per nucleon, though it may affect stability.

3.2 Liquid Drop Model and Semi-Empirical Mass Formula

By using the Liquid Drop Model, we can get the binding energy $E_B(Z, A)$ of a nucleus with atomic number Z and mass number A estimated using the semi-empirical mass formula:

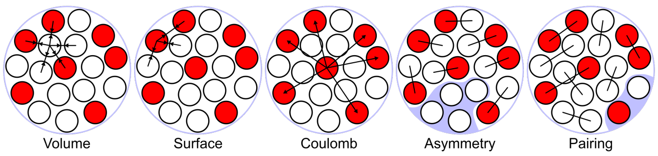


Figure 2: From `https://en.wikipedia.org/wiki/Semi-empirical_mass_formula`.

$$E_B(Z, A) = a_v A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_{\text{Asym}} \frac{(A-2Z)^2}{A} + \delta(A, Z)$$

where:

- $a_v = 15.56\,\text{MeV}$ (volume term), $a_s = 17.23\,\text{MeV}$ (surface term),
- $a_c = 0.697\,\text{MeV}$ (Coulomb term), $a_{\text{Asym}} = 23.285\,\text{MeV}$ (Asymmetry term),
- $\delta(A, Z)$ is the pairing term, given by:

$$\delta(A, Z) = \begin{cases} +12\,A^{-1/2} & \text{for even } Z \text{ and even } N \\ 0 & \text{for odd } A \\ -12\,A^{-1/2} & \text{for odd } Z \text{ and odd } N \end{cases}$$

Limitations of the SEMF:

- Neutron and proton separation energies show deviations at N or $Z = 2, 8, 20, 28, 50, 82, 126, \cdots$
- Spin and parity of nuclei do not fit into the liquid drop model.
- Magnetic moments of nuclei are incompatible with drops.
- Actual value of nuclear density is unpredicted.
- Values of the SEMF coefficients except Coulomb and Asymmetry are completely empirical.

3.3 Applications of Nuclear Physics

Mirror Nuclei

Mirror nuclei are these which have numbers of protons and neutrons exchanged one with respect to the other like ${}^{A}_{Z}X_{N}$ is a mirror nucleus of ${}^{A}_{N}Y_{Z}$.

Different nuclear masses between mirror nuclei are resulted from p-n difference and the different Coulomb terms. For the atomic mass difference,

$$M(A, Z+1) - M(A, Z) = \Delta E_c + m_p + m_e - m_n$$

Q-Value

In a reaction: $\mathbf{A} \to \mathbf{B} + \mathbf{C}$, the reaction Q-value is calculated as:

$$Q \equiv (M_A - M_B - M_C) \cdot c^2 = (BE_C + BE_B - BE_A) \cdot c^2$$

- If $Q > 0$, reaction releases energy, reaction can proceed without additional energy.
- If $Q < 0$, reaction requires energy to proceed.

Alpha Decay Q-Value

The Q-value (energy released) in α-decay is:

$$\begin{aligned} Q &= B_\alpha - [B(A,Z) - B(A-4, Z-2)] \\ &= B_\alpha - \left[\frac{\partial B}{\partial A} \cdot 4 + \frac{\partial B}{\partial Z} \cdot 2\right] \\ &= 28.3\,\mathrm{MeV} - 4a_v + \frac{8}{3}a_s\frac{1}{A^{1/3}} + 4a_c\frac{Z}{A^{1/3}}\left[1 - \frac{Z}{3A}\right] - 4a_A\left[1 - 2\frac{Z}{A}\right]^2 \end{aligned}$$

Mass Parabola

Rearrange the terms of SEMF according to the powers of Z

$$M(A,Z)c^2 = \left[m_n - a_V + a_A + \frac{a_S}{A^{1/3}}\right] \cdot A - \left[4a_A + \left(m_n - m_p\right)\right] \cdot Z + \left[\frac{4a_A}{A} + \frac{a_C}{A^{1/3}}\right] \cdot Z^2 \pm \delta(A,Z)$$

This quadratic function in Z is called Mass Parabolas

$$M(A,Z)c^2 = aA + bZ + cZ^2 \pm \delta(A,Z)$$

For different situation,

$$\begin{aligned} &\text{odd-A} && \Rightarrow \delta = 0 \Rightarrow \text{one parabola only} \\ &\text{even-A} && \Rightarrow \pm\delta \quad\ \Rightarrow \text{two parabolas} \end{aligned}$$

For odd A, we have stable valley position:

$$Z_0 = \frac{4a_A + (m_n - M_H)}{2\left(\frac{4a_A}{A} + \frac{a_c}{A^{1/3}}\right)}.$$

Note that to a good approximation we can neglect $(m_n - M_H)$ compared to $4a_A$, a good approximation we get:

$$Z_0 = \frac{A}{2}\frac{1}{\left[1 + \varepsilon A^{2/3}\right]}$$

where $\varepsilon = \dfrac{a_C}{4a_A} = \dfrac{0.697}{4 \times 23.285} = 7.48 \times 10^{-3}$.

Neutron drip lines

We first consider the single neutron separation energy

$$\begin{aligned} S &= [M(A-1,Z) + m - M(A,Z)]c^2 \\ &= -[B(A-1,Z) - B(A,Z)] \\ &= \frac{\partial B}{\partial A} \pm \delta \\ &= \left(a_V - a_A - \frac{2}{3}a_S A^{-1/3}\right) + \left(\frac{a_C}{3A^{4/3}} + \frac{4a_A}{A^2}\right)Z^2 \pm \delta. \end{aligned}$$

For the neutron drip line. By putting $S_n = 0$ we have

$$Z_{\text{drip}} = \sqrt{\frac{a_A + \frac{2}{3}a_S A^{-1/3} - a_V}{\frac{1}{3}a_C A^{-4/3} + 4a_A A^{-2}}}.$$

4 Nuclear Decay Processes

4.1 Nuclear Spin & Parity

In classical mechanics, angular momentum $\mathbf{L}$ is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position vector and $\mathbf{p}$ is the linear momentum. In quantum mechanics, angular momentum is represented by operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, which follow the commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

The total angular momentum operator is given by:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Eigenstates of angular momentum are characterized by quantum numbers l and m_l, where $l(l+1)\hbar^2$ is the eigenvalue of $\hat{L}^2$, and $m_l\hbar$ is the eigenvalue of $\hat{L}_z$.

Nuclear spin I is the vector sum of the total angular momentum j_i of the individual nucleons:

$$\mathbf{I} = \sum_i \mathbf{j}_i, \quad \text{where} \quad \mathbf{j}_i = \mathbf{l}_i + \mathbf{s}_i$$

In many nuclei, nucleons pair off in such a way that their angular momenta cancel each other out. Therefore, the nuclear spin I is often determined by the unpaired nucleon(s). The spin quantum number I can take integer or half-integer values depending on whether the mass number A is even or odd.

- **Even-A nuclei:** The nuclear spin I is an integer, and for even-even nuclei (even number of protons and neutrons), $I = 0$ in the ground state.
- **Odd-A nuclei:** The nuclear spin I is a half-integer, and the spin is typically determined by the unpaired nucleon.

Parity is an intrinsic property of the nucleus, reflecting the behavior of the nuclear wavefunction under spatial inversion $\mathbf{r} \to -\mathbf{r}$. The parity of a nucleus is the product of the intrinsic parities of the individual nucleons, each of which contributes $(-1)^l$ where l is the orbital angular momentum quantum number. Thus, for a nucleus with an unpaired nucleon, the total parity is $P = (-1)^l$.

- **Even parity:** $l = 0, 2, 4, \ldots$
- **Odd parity:** $l = 1, 3, 5, \ldots$

For even-even nuclei, the ground state always has even parity. For odd-A nuclei, the parity is determined by the unpaired nucleon.

4.2 Electric and Magnetic Moments in Nuclei

Nuclei have electric moments due to the distribution of charge within them.

- **Electric Monopole Moment:** Represents the total charge of the nucleus.
- **Electric Dipole Moment:** Typically zero for nuclei with definite parity because the charge distribution is symmetric.
- **Electric Quadrupole Moment:** Measures the deviation of the nuclear charge distribution from spherical symmetry. It provides information about the shape of the nucleus, where a positive quadrupole moment indicates a prolate (elongated) shape, and a negative moment indicates an oblate (flattened) shape.

The magnetic moment of a nucleus arises from both the orbital motion and intrinsic spin of protons and neutrons.

- **Magnetic Dipole Moment:** Given by $\mu = g_I m_I \mu_N$, where g_I is the nuclear g-factor, m_I is the magnetic quantum number, and μ_N is the nuclear magneton.
- **Intrinsic Spin Contributions:** The magnetic moment also has contributions from the intrinsic spin of the nucleons. For example, the proton and neutron each have intrinsic magnetic moments due to their internal quark structure.

4.3 Fermi Gas Model and Quantum Statistics

Quantum particles are classified as either fermions or bosons based on their spin:

- **Fermions:** Particles with half-integer spin ($s = 1/2, 3/2, \ldots$), which obey Fermi-Dirac statistics and the Pauli exclusion principle, preventing them from occupying the same quantum state.
- **Bosons:** Particles with integer spin ($s = 0, 1, 2, \ldots$), which obey Bose-Einstein statistics and can occupy the same quantum state.

In the context of nuclear physics, protons and neutrons are fermions, meaning they fill up available quantum states starting from the lowest energy levels.

The Fermi gas model treats nucleons (protons and neutrons) as a collection of non-interacting fermions confined within a potential well. This model assumes that nucleons occupy the lowest available energy states up to the Fermi energy E_F at absolute zero temperature.

- **Fermi Energy E_F:** The highest occupied energy level at zero temperature, typically around 33 MeV for nucleons.
- **Kinetic Energy:** The average kinetic energy per nucleon is given by $\langle E_{\text{kin}} \rangle = \frac{3}{5} E_F$, approximately 20 MeV.

The Fermi gas model is useful for understanding the gross properties of nuclei, such as the distribution of nucleon energies and the effects of the Pauli exclusion principle on nuclear structure.

5 Nuclear Reactions and Energy Generation

5.1 Key Concepts of Nuclear Force

- **Definition and Origin**: Nuclear force, also known as the strong nuclear force, is the interaction responsible for binding protons and neutrons together within the nucleus. It originates from the strong interaction between quarks within nucleons, mediated by the exchange of gluons. Unlike the strong force acting within a hadron, nuclear force is a residual interaction between quarks localized in different nucleons.
- **Role of Nuclear Force**: The nuclear force is essential in overcoming the electrostatic repulsion between protons and in maintaining the stability of nuclei. A thorough understanding of nuclear force would allow for accurate predictions of nuclear properties.
- **Theoretical Challenges**: Nuclear force cannot yet be calculated directly from Quantum Chromodynamics (QCD) due to the complexity of multi-nucleon systems and the empirical nature of our current understanding. Theoretical models are largely built on experimental data and phenomenological approaches.

5.2 Fundamental Properties of Nuclear Forces

1. **Finite Range**:
   - **Short-Range Repulsion ("Hard Core")**: At very short distances (0.6 fm), the nuclear force becomes repulsive, preventing nucleons from coming too close.
   - **Intermediate-Range Attraction**: The force is strongly attractive at distances of about 1 fm, which is crucial for binding nucleons together.
   - **Long-Range Vanishing**: The force rapidly decreases and becomes negligible at distances greater than 2 fm.
2. **Spin-Spin Interaction**: Nuclear force depends on the spins of the interacting nucleons. The force can be either attractive or repulsive depending on the spin alignment (parallel or antiparallel).
3. **Non-Central Tensor Force**: In addition to the central force, nuclear force includes a tensor component that depends on the relative orientation of the spins and positions of the nucleons. This tensor force is responsible for the non-spherical shape of the deuteron.
4. **Spin-Orbit Coupling**: The interaction between the spin and orbital angular momentum of nucleons plays a significant role in the structure of the nucleus, particularly in determining the energy levels of nucleons.
5. **Charge Independence**: The nuclear force is nearly identical for proton-proton, neutron-neutron, and proton-neutron interactions. This property, known as charge independence, implies that the nuclear force does not depend on the electric charge of the nucleons, which is supported by the near-identical properties of mirror nuclei.

5.3 Exchange Interaction in Nuclear Forces

- **Concept of Exchange Force**: Proposed by Yukawa in 1934, the exchange force model describes nuclear force as arising from the exchange of virtual particles, specifically mesons, between nucleons. The exchange of these particles mediates the force, with the range of the force inversely proportional to the mass of the exchanged meson.
- **Yukawa's Model for Nuclear Force**: The model predicts that the potential between two nucleons decreases exponentially with distance, described by the Yukawa potential:

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-r/R}}{r}$$

  where g is the coupling constant, and R is the characteristic range of the force, related to the mass of the exchanged meson.
- **Range of Nuclear Force**: The range of the nuclear force is approximately 1.4 fm, which corresponds to the exchange of pions (mesons with a mass of about 140 MeV). The discovery of pions validated Yukawa's model, as they are responsible for the long-range part of the nuclear interaction.

6 Applications and Advanced Topics

Week 6 covers the structure of atomic nuclei, focusing on the nuclear shell model. This summary explores the evidence supporting the existence of nuclear shells, the development of the shell model, its applications in predicting nuclear properties, and the limitations that arise when dealing with deformed nuclei.

6.1 The Many-Body Nuclear Problem

The nucleus is a complex many-body system where interactions between nucleons (protons and neutrons) are challenging to model accurately. The Schrödinger equation for such a system is:

$$\hat{H} = \sum_{i=1}^{A} -\frac{\hbar^2}{2m_i}\nabla_i^2 + \sum_{i>j}^{A} V_{ij}(\mathbf{r}_i, \mathbf{r}_j)$$

where A is the number of nucleons, ∇_i^2 is the Laplacian operator, and V_{ij} represents the interaction potential between nucleons i and j. The complexity of this equation for nuclei with more than two nucleons necessitates the use of model theories to approximate nuclear behavior.

Evidence for Nuclear Shell Structure:One of the key motivations for the nuclear shell model is the observation of "magic numbers" in nuclear properties. Magic numbers (2, 8, 20, 28, 50, 82, 126) correspond to nucleon numbers where nuclei exhibit enhanced stability. Several pieces of evidence support the existence of nuclear shells:

Binding Energy Curves: The binding energy per nucleon BE/A shows discontinuities at magic numbers, indicating extra stability for nuclei with closed shells. The liquid-drop model, which describes the general trends in binding energy, fails to account for these discontinuities, highlighting the need for a shell model.

Abundance and Cross-Section Data: Nuclei with magic numbers tend to be more abundant and exhibit lower neutron capture cross-sections, suggesting greater stability. This is consistent with the idea of closed nuclear shells, where additional nucleons would find it more difficult to bind or be captured.

Excited State Energies: Magic nuclei have higher first excited state energies compared to their neighbors. This indicates that it is harder to excite these nuclei from their ground state, further supporting the existence of a shell structure.

6.2 Nuclear Shell Model: Structure and Properties

The nuclear shell model was developed to explain magic numbers and other nuclear properties. It assumes that each nucleon moves independently in a mean-field potential created by interactions with other nucleons. This is known as the independent-particle model.

The primitive shell model considers a central potential $V(r)$ such as:

$$V(r) = \begin{cases} -V_0 & \text{if } r \leq R \\ 0 & \text{if } r > R \end{cases}$$

This potential is a square well, where nucleons occupy discrete energy levels. However, this model initially failed to explain the observed magic numbers. To account for the observed magic numbers, the model incorporates spin-orbit coupling, which splits the energy levels depending on the total angular momentum $j = l + s$. The full potential is then:

$$V(r) = V_{\text{central}}(r) + V_{\text{ls}}(r)\mathbf{L}\cdot\mathbf{S}$$

where **L** and **S** are the orbital and spin angular momentum operators, respectively. This adjustment successfully predicts the magic numbers and aligns with experimental data.

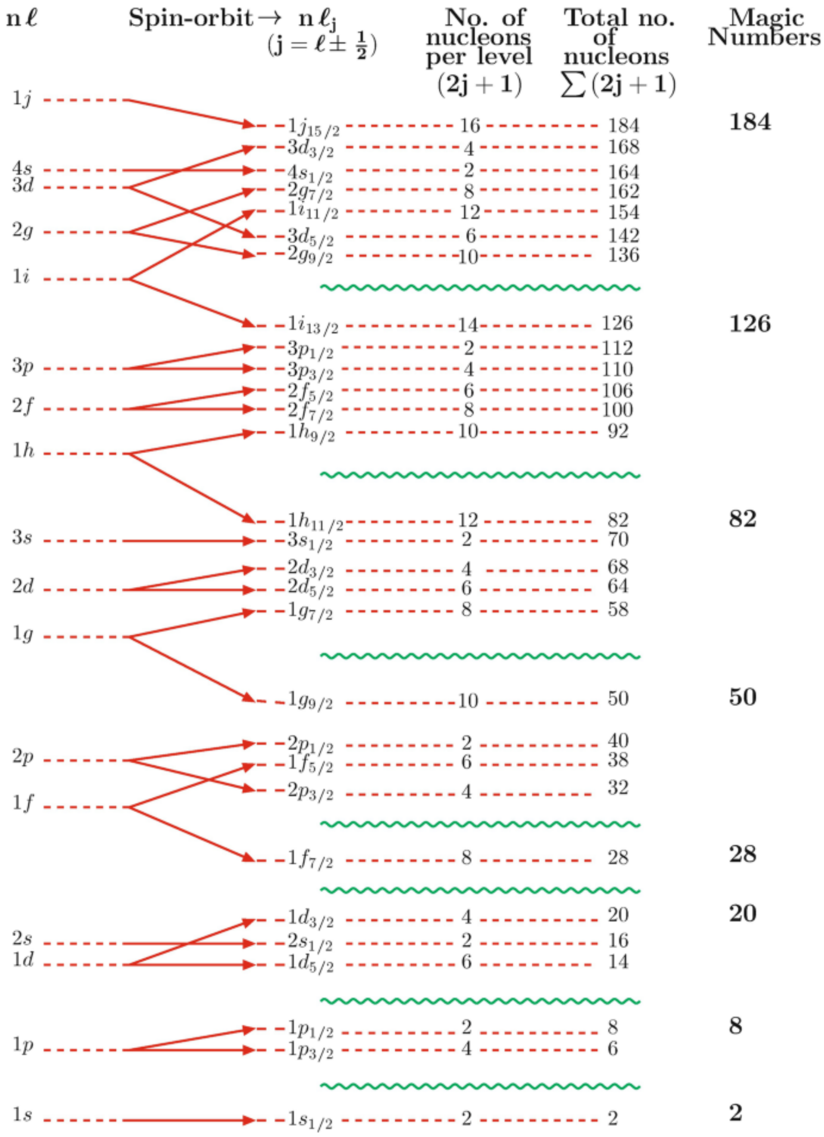


Figure 3: Shell model energy level.

6.3 Nuclear Spin and Parity

The shell model predicts the spin J and parity π of a nucleus based on the unpaired nucleons. For even-even nuclei, the ground state spin and parity are always $J^\pi = 0^+$. For nuclei with unpaired nucleons, the spin and parity are determined by the quantum numbers of the unpaired nucleon, given by:

$$\pi = (-1)^\ell$$

where ℓ is the orbital angular momentum quantum number.

6.4 Magnetic Moments of Nucleons

The magnetic dipole moment μ of a nucleus is related to the spin and orbital angular momenta of the nucleons. For a single nucleon, the magnetic moment is given by:

$$\mu = (g_\ell \ell + g_s s)\,\mu_N$$

where g_ℓ and g_s are the gyromagnetic ratios for orbital and spin contributions, respectively, and μ_N is the nuclear magneton.

The shell model provides a framework for calculating these moments, particularly for nuclei with unpaired nucleons.

6.5 Challenges and Limitations of the Shell Model

While the shell model is powerful, it has limitations:

- **Deformed Nuclei**: The model assumes spherical symmetry, which breaks down for nuclei far from closed shells. These nuclei may become deformed, leading to collective behaviors such as vibrations and rotations.
- **Odd-Odd Nuclei**: Predicting the spin and parity of odd-odd nuclei (where both the proton and neutron numbers are odd) is more challenging due to the interaction between the unpaired proton and neutron.

6.6 Collective Model: Vibrations and Rotations

For nuclei that are not near closed shells, collective models become important. These models describe the nucleus as a system where groups of nucleons act together to produce collective excitations.

Nuclear Vibrations:

Nuclear vibrations can be described as oscillations of the nuclear surface, characterized by a quantum number λ. Quadrupole vibrations ($\lambda = 2$) are common in even-even nuclei and are associated with low-lying excited states.

Nuclear Rotations:

In deformed nuclei, the entire nucleus can rotate, producing rotational spectra with energy levels:

$$E(J) = \frac{\hbar^2}{2\mathcal{I}} J(J+1)$$

where $\mathcal{I}$ is the moment of inertia and J is the rotational quantum number. The rotational bands are characterized by the spacing between these energy levels.

The nuclear shell model, with its incorporation of spin-orbit coupling, provides a robust framework for understanding the structure of atomic nuclei, particularly those near magic numbers. It explains many nuclear properties, including spin, parity, and magnetic moments, while also laying the groundwork for more complex models that account for collective nuclear behavior. Despite its limitations, the shell model remains a cornerstone of nuclear physics, offering insights into the fundamental nature of matter at the nuclear level.

Aspect	Liquid-Drop Model	Fermi-Gas Model	Shell Model
Essential Features	• Treats the nucleus as a drop of incompressible nuclear fluid. • Accounts for surface tension and volume effects. • Explains nuclear binding using the semi-empirical mass formula (SEMF). • Relies on macroscopic properties similar to a liquid.	• Models the nucleus as a collection of non-interacting fermions (protons and neutrons) confined in a potential well. • Emphasizes the quantum statistical properties of nucleons (Pauli exclusion principle). • Nucleons move independently without mutual interaction.	• Describes the nucleus as nucleons moving in an average potential. • Considers quantum mechanical energy levels with strong spin-orbit coupling. • Focuses on the independent-particle motion and the concept of magic numbers corresponding to closed shells. • Includes a mean-field approach.
Successes	• Accurately predicts nuclear binding energies, nuclear masses, fission, and fusion processes. • Describes the valley of stability and nuclear saturation. • Explains nuclear symmetry energy.	• Explains nuclear level density and excited state spectra. • Provides insight into the distribution of nucleons within the nucleus and their individual energies. • Useful in predicting neutron capture rates and high-energy nuclear reactions.	• Successfully explains nuclear magic numbers, ground-state spins, and parities. • Accurately predicts shell structure and the ordering of energy levels. • Provides a quantum mechanical explanation for nuclear stability and low-lying excited states.
Limitations	• Fails to explain the occurrence of magic numbers, shell structure, and individual nucleon behavior. • Does not account for quantum effects or the microscopic nature of nucleon interactions. • Less accurate for small nuclei and nuclei far from the valley of stability.	• Cannot explain magic numbers or detailed nuclear shell structure. • Assumes a uniform potential well, neglecting nucleon-nucleon interactions. • Does not account for collective phenomena like nuclear deformation.	• Computationally complex and intensive, especially for nuclei with a large number of nucleons. • Limited in describing nuclear deformation and collective motion such as vibrations and rotations. • Struggles with predicting the properties of odd-odd nuclei and deformed nuclei.

Table 1: Comparison of Nuclear Models: Liquid-Drop Model, Fermi-Gas Model, and Shell Model

7 Fundamentals of Radioactive Decay

The decay of a radioactive substance follows an exponential law, where the number of undecayed particles N decreases over time according to:

$$N(t) = N_0 e^{-\lambda t}$$

where:

- N_0 is the initial number of radioactive nuclei,
- λ is the decay constant, defined as the probability per unit time that a nucleus will decay,
- t is the time elapsed.

Similarly, the activity $A(t)$ of a radioactive substance, which is the rate of decay or disintegrations per second (dps), is given by:

$$A(t) = A_0 e^{-\lambda t}$$

The activity decreases over time in the same manner as the number of undecayed nuclei. The SI unit of activity is the Becquerel (Bq), where 1 Bq = 1 dps. In practice, the Curie (Ci) is also often used:

$$1\,\text{Ci} = 3.7 \times 10^{10}\,\text{Bq}$$

The mean lifetime τ of a radioactive particle is the average time a nucleus remains undecayed, and is inversely related to the decay constant λ:

$$\tau = \frac{1}{\lambda}$$

A more commonly used measure is the half-life $t_{1/2}$, which is the time it takes for half of the radioactive nuclei to decay. It is related to the decay constant by:

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

This exponential nature of decay means that after each half-life, half of the remaining material will decay, regardless of the amount of material present.

The specific activity (SA) is a measure of the activity per unit mass of the substance:

$$SA = \frac{A}{m}$$

where A is the activity in Becquerels and m is the mass in kilograms. Specific activity is crucial for understanding the concentration of radioactivity in a material, particularly in cases where isotopes with long half-lives are involved, as they tend to have low specific activity.

Many radioactive elements decay into other unstable elements, forming a decay chain. The Bateman equation describes the activity of each element in the chain over time. The Bateman solution for a two-nuclide system (where N_1 decays into N_2, which also decays) is given by:

$$N_2(t) = \frac{\lambda_1 N_1(0)}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)$$

This formula generalizes for more complex decay chains and is critical for understanding the long-term behavior of nuclear decay products.

One common application of radioactive decay is in radiocarbon dating, which utilizes the half-life of Carbon-14 (^{14}C), around 5730 years. The age of an object containing carbon-based materials can be estimated from the ratio of ^{14}C to ^{12}C isotopes remaining.

The decay rate of ^{14}C is used to solve for the time t that has passed since the death of the organism, based on the remaining fraction of ^{14}C.

8 Alpha Decay: Mechanism and Theory

Alpha decay is the process through which an unstable nucleus emits an α particle, which is essentially a helium nucleus (^{4}He) composed of 2 protons and 2 neutrons. The α particle is tightly bound with a binding energy of approximately 28.3 MeV.

In alpha decay, a large nucleus asymmetrically splits, resulting in a smaller, more stable nucleus (the daughter nucleus) and the emitted α particle. This is considered spontaneous emission because it occurs without external input, provided the process is energetically favorable. For an α particle to escape the nucleus, it must overcome the Coulomb barrier that arises from the repulsion between the positively charged nucleus and the α particle.

$$Q_\alpha(A, Z) = B_D + B_\alpha - B_p = B_D(A-4, Z-2) + 28.3\text{ MeV} - B_P(A, Z)$$
$$Q_\alpha(A, Z) = (M_P - M_D - M_\alpha)c^2$$

Given the much larger mass of the daughter nucleus compared to the α particle, the majority of the kinetic energy is carried by the α particle:

$$E_\alpha = \frac{Q}{1+\frac{m_\alpha}{m_\mathrm{D}}} \approx Q$$

The Gamow theory combines quantum tunneling with the frequency of α particle formation to explain the decay process. According to this model, the decay rate is determined by:

- The probability of α particle formation inside the nucleus,
- The frequency at which the α particle approaches the nuclear surface,
- The probability of tunneling through the Coulomb barrier.

8.1 Mathematical Approach to Decay Rate

The decay rate, λ, is expressed as:

$$\lambda = PfT$$

where P is the probability of α particle formation, f is the frequency of attempts to escape, and T is the tunneling probability through the Coulomb barrier.

The tunneling probability, T, is given by:

$$T = e^{-2G}$$

where G is the Gamow factor, which quantifies the difficulty of tunneling through the barrier. The Gamow factor G is expressed as:

$$G = 2Z\alpha\sqrt{\frac{2m_\alpha c^2}{E_\alpha}}\left[\cos^{-1}\left(\sqrt{\frac{R}{b}}\right) - \sqrt{\frac{R}{b}\left(1-\frac{R}{b}\right)}\right]$$

Here, Z is the charge of the daughter nucleus, m_α is the mass of the α particle, Q_α is the energy released in the decay, R is the nuclear radius of the daughter, and b is the barrier distance.

The Coulomb potential that the alpha particle experiences outside the nucleus is:

$$V_c(R) = \frac{2Ze^2}{4\pi\varepsilon_0 R} = \frac{2Z\alpha\hbar c}{R}$$

where α is the fine-structure constant ($\alpha \approx 1/137$). The potential represents the energy barrier that the α particle must tunnel through. When we have

$$\frac{R}{b} = \frac{E_\alpha}{V(R)} \ll 1,$$

we can use the apporximation $G \approx Z\alpha\pi\sqrt{\frac{2m_\alpha c^2}{E_\alpha}}$, hence give the formula of decay rate in:

$$\lambda = \frac{P}{2R}\sqrt{\frac{2(E_\alpha+U)}{m_a}}e^{-2Z\alpha\pi\sqrt{\frac{2m_\alpha c^2}{E_\alpha}}}$$

where R is the radius of the daughter.

The Geiger-Nuttall law relates the half-life of an alpha-emitting nucleus to the energy of the emitted α particle. The law can be expressed as:

$$\ln t_{1/2} \propto a + \frac{b}{\sqrt{E_\alpha}}$$

This shows that even a small increase in the Q-value (energy of the emitted α particle) can result in a large decrease in the half-life. For example:

- ^{232}Th, with $Q_\alpha = 4.08$ MeV, has a half-life of 1.4×10^{10} years,
- ^{218}Th, with $Q_\alpha = 9.85$ MeV, has a half-life of 10^{-7} seconds.

This variation shows that alpha decay is extremely sensitive to the energy released during the process.

8.2 Selection Rules in Nuclear Transitions

Selection Rules of α-decay for $X \longrightarrow Y + \alpha$:
Since α has $J^P = 0^+$

$$
\begin{aligned}
J_X &= J_Y + J_\alpha + L_\alpha \\
J_X &= J_Y + L_\alpha
\end{aligned}
$$

l_α can take values from:

$$|J_X - J_Y| \leq J_Y + l_\alpha \leq J_X + J_Y$$

Parity is conserved in decay, orbital wavefunction has $P = (-1)^{l_\alpha}$

$$\text{X, Y same parity} \Rightarrow l_\alpha \text{ must be even}$$

$$\text{X, Y opposite parity} \Rightarrow l_\alpha \text{ must be odd}$$

For example, X and Y are not even-even nuclei in their ground states. The shell model predicts both have $J^P = 0^+ \Rightarrow l_\alpha = 0$. If X has $J^P = 0^+$, the states of Y which can be formed in decay are:

$$J^P = 0^+, 1^-, 2^+, 3^-, 4^+, \ldots$$

9 Beta Decay: Types and Selection Rules

Conversion of p-n due to the weak nuclear force

1. From the nucleon perspective:

$$
\begin{aligned}
\beta^- :&\quad n \rightarrow p + e^- + \bar{\nu}_e \\
\beta^+ :&\quad p \rightarrow n + e^+ + \nu_e \\
EC :&\quad p + e^- \rightarrow n + \nu_e
\end{aligned}
$$

2. From the nucleus perspective:

$$
\begin{aligned}
\beta^- :&\quad {}^A_Z X \rightarrow {}_{Z+1}{}^A_1 Y + e^- + \bar{\nu}_e \\
\beta^+ :&\quad {}^A_Z X \rightarrow {}^A_{Z-1} Y + e^+ + \nu_e \\
EC :&\quad {}^A_Z X + e^- \rightarrow {}^A_{Z-1} Y + \nu_e
\end{aligned}
$$

3. From the quark perspective:

$$
\begin{aligned}
\beta^- :&\quad d \rightarrow u + e^- + \bar{\nu}_e \\
\beta^+ :&\quad u \rightarrow d + e^+ + \nu_e \\
EC :&\quad u + e^- \rightarrow d + \nu_e
\end{aligned}
$$

9.1 Conditions Governing Beta Decay

After introducing the neutrino, beta decay becomes a three-body problem. The total energy of beta decay is defined as:

$$E_0 = T_R + T_\beta + T_\nu$$

where:

- T_R is the recoil energy of the nucleus.
- T_β is the kinetic energy of the beta particle (electron or positron).
- T_ν is the kinetic energy of the neutrino.

Beta decay can occur in the following forms:

- In β^- decay, an electron and an antineutrino are emitted:

$$^A_Z X \rightarrow {}^A_{Z+1} Y + e^- + \bar{\nu}_e$$

- In β^+ decay, a positron and a neutrino are emitted:

$$^A_Z X \rightarrow {}^A_{Z-1} Y + e^+ + \nu_e$$

- In electron capture (EC), a proton captures an orbital electron, emitting a neutrino:

$$^{A}_{Z}X + e^- \to {}^{A}_{Z-1}Y + \nu_e$$

- Since beta decay is a three-body process, the energy distribution is not unique.
- For EC decay, the captured electron must come from the inner orbitals, and the energy distribution is unique.
- Neutron-rich nuclei undergo β^- decay, while proton-rich nuclei can undergo β^+ decay or EC.

The energy released in beta decay is the sum of the kinetic energies $T_\beta + T_\nu$.

- For β^- Decay:
  The energy released in β^- decay is:

$$E_0(\beta^-) = (m_X - m_Y - m_e)c^2$$

  which can be approximated as:

$$E_0(\beta^-) \approx (M_X - M_Y)c^2 = \Delta(Z, A) - \Delta(Z+1, A)$$

  Thus, the condition for β^- decay to occur is:

$$E_0(\beta^-) > 0 \iff \Delta(Z, A) > \Delta(Z+1, A)$$

- For β^+ Decay:
  The energy released in β^+ decay is:

$$E_0(\beta^+) = (m_X - m_Y - m_{e^+})c^2$$

  which can be approximated as:

$$E_0(\beta^+) \approx (M_X - M_Y - 2m_e)c^2 = \Delta(Z, A) - \Delta(Z-1, A) - 2m_e c^2$$

  Thus, the condition for β^+ decay to occur is:

$$E_0(\beta^+) > 0 \iff \Delta(Z, A) > \Delta(Z-1, A) + 2m_e c^2$$

- The energy conservation equation for EC decay differs from β^+ decay, as EC requires the binding energy of the captured electron to be considered. The energy released is:

$$E_0(\text{EC}) = (m_X + m_e - \epsilon_i - m_Y)c^2$$

  which can be approximated as:

$$E_0(\text{EC}) \approx (M_X - M_Y)c^2 - \epsilon_i = \Delta(Z, A) - \Delta(Z-1, A) - \epsilon_i$$

  Thus, the condition for EC decay to occur is:

$$M_X(Z, A) - M_Y(Z-1, A) > \frac{\epsilon_i}{c^2}$$

  where ϵ_i is the binding energy of the captured electron.

9.2 Beta-Decay and Mass Parabola

$$M(Z,A) = \alpha - \beta Z + \gamma Z^2 \pm \frac{\delta}{A^{1/2}}$$

where the parameters are given as:

$$\begin{aligned}
\alpha &= Am_n - a_V A + a_C A^{2/3} + a_A A \\
\beta &= 4a_A + m_n - m_p - m_e \\
\gamma &= \frac{a_C}{A^{1/3}} + \frac{4a_a}{A} \\
\delta &= \begin{cases} -11.2 & \text{for even } Z, N \text{ (even } A), \\ 0 & \text{for odd } A, \\ +11.2 & \text{for odd } Z, N \text{ (even } A), \end{cases}
\end{aligned}$$

To decide the minimal of M, we let $\frac{dM}{dZ}|_{Z=Z_0} = 0$, and we will get

$$Z_0 = \underbrace{\frac{4a_A + m_n - m_p - m_e}{a_C A^{2/3} + 4a_A}}_{<1} \cdot \frac{A}{2}$$

9.3 Fermi Theory of Beta Decay

The Fermi theory of beta decay provides a framework to describe the weak interaction responsible for beta processes. It calculates the decay rate by considering the matrix elements of the interaction Hamiltonian and the density of final states available to the decay products.

The differential decay rate dw is given by Fermi's golden rule:

$$dw = \frac{2\pi}{\hbar} |\mathcal{M}_{fi}|^2 \rho_f(E_0)$$

where:

- $\mathcal{M}_{fi}$ is the matrix element of the interaction Hamiltonian between the initial and final states.
- $\rho_f(E_0)$ is the density of final states at the energy E_0.

The matrix element $\mathcal{M}_{fi}$ is approximated as:

$$\mathcal{M}_{fi} = \langle \psi_f | \mathcal{H}_{\text{int}} | \psi_i \rangle \approx \left(\frac{G_w}{V} \right) |\mathcal{M}_f|$$

The density of final states $\rho_f(E_0)$ is related to the differential number of states dn by:

$$\rho_f(E_0) = \frac{dn}{dE_0}$$

The differential number of states dn is given by:

$$dn = \frac{V d\Omega |\vec{k}|^2 d|\vec{k}|}{(2\pi)^3}$$

Substituting for $|\vec{k}|^2 d|\vec{k}|$, we obtain:

$$|\vec{k}|^2 d|\vec{k}| = \frac{\sqrt{E^2 - m^2c^4}}{(\hbar c)^3} E dE$$

For positron (β^+) decay, the differential decay rate dw can be expressed as:

$$dw = \frac{G_w^2 |\mathcal{M}_F|^2}{2\pi^3 \hbar^7 c^6} S_0(E_e) dE_e$$

where $S_0(E_e)$ is the phase space factor given by:

$$S_0(E_e) = \sqrt{E_e^2 - m_e^2 c^4} \cdot \sqrt{(E_0 - E_e)^2 - m^2c^4} \cdot (E_0 - E_e) \cdot E_e$$

- V: Normalization volume.
- $\vec{k}$: Momentum of the emitted particle.
- $d\Omega$: Differential solid angle.
- G_w: Fermi coupling constant.

- $\mathcal{M}_f$: Reduced matrix element.
- E: Energy of the emitted particle.
- m: Mass of the emitted particle.
- c: Speed of light.
- $\hbar$: Reduced Planck constant.
- E_e: Energy of the emitted positron.
- m_e: Mass of the electron.
- E_0: Total available energy in the decay.

The Coulomb interaction between the emitted positron and the nucleus modifies the density of electron states. This modification is accounted for by the Fermi function $F(Z, E)$, also known as the Fermi screening factor.

The phase space factor is modified as:

$$S_0 \longrightarrow S_{\text{SCCE}} = F(Z, E) \cdot S_0(E_e)$$

The Fermi function $F(Z, E)$ is approximately given by:

$$F(Z, E) \approx 2\pi\eta$$

where η is the Sommerfeld parameter, defined as:

$$\eta = \frac{Ze^2}{4\pi\epsilon_0\hbar v}$$

The Fermi theory of beta decay provides a comprehensive approach to calculate the decay rates by considering the matrix elements of the weak interaction and the available phase space for the decay products. Coulomb corrections are essential for accurately describing the interactions between the emitted particles and the nucleus, especially for processes involving charged particles like positrons in β^+ decay.

9.4 Selection Rules in Nuclear Transitions

Selection Rules for Beta Decay:

The selection rules for beta decay are determined by changes in angular momentum and parity between the parent and daughter nuclei. These rules are based on the properties of the emitted particles (electron/positron and neutrino/antineutrino). The two main types of transitions are:

1. Fermi (F) Transition:

- The total spin of the electron and neutrino pair is $s = 0$, meaning their spins are aligned.
- The change in nuclear spin between the parent and daughter nucleus is $\Delta J = 0$.
- No change in parity occurs.

2. Gamow-Teller (G-T) Transition:

- The total spin of the electron and neutrino pair is $s = 1$, meaning their spins are anti-aligned.
- The change in nuclear spin is $\Delta J = 1$.
- There is no change in parity.

For forbidden transitions, the angular momentum of the emitted particles is non-zero ($l \neq 0$). These transitions are classified by the orbital angular momentum l of the emitted particles:

First Forbidden Transition ($l = 1$):

- Allowed changes in angular momentum: $\Delta J = 0, 1, 2$.
- There must be a change in parity between the parent and daughter nuclei.

Second Forbidden Transition ($l = 2$):

- Allowed changes in angular momentum: $\Delta J = 2, 3$.
- If $\Delta J = 1$, parity must remain unchanged.

General Rules:

- For transitions where $\Delta J = n$, the parity change must be opposite, and the allowed angular momentum difference is $n, n+1$.
- Higher forbidden transitions, where $l > 2$, have lower probabilities because the orbital angular momentum of the emitted particles increases.

10 Gamma Decay: Characteristics and Selection Rules

Gamma-decay refers to the process where an excited nucleus emits electromagnetic radiation to de-excite to a lower energy state. This process typically follows nuclear reactions such as alpha or beta decay.

10.1 General Characteristics

- No change in N or Z; only energy levels are involved.
- Emitted gamma radiation ranges from 1 keV to 10 MeV.
- The half-life of gamma transitions ranges from 10^{-17} s to 100 s.
- Both radiation and conversion electron emissions are involved.

Gamma Decay Energy: The energy difference between initial and final states is:

$$E_0 = E_i - E_f$$

and the relation between the transition energy, recoil energy, and photon energy is given by:

$$E_0 = T_R + E_\gamma$$

Recoil Nucleus Kinetic Energy:

$$T_R = \frac{1}{2} m_R v_R^2 = \frac{E_0^2}{2 m_R c^2}$$

- The recoil energy of the nucleus in gamma decay is negligible.
- Gamma rays have energies typically between 1 keV and 10 MeV.
- For high-energy gamma rays around 50 to 100 MeV, recoil effects can be significant.

10.2 Weisskopf Approximation in Shell Model

- Electric multipole transition rate:

$$\lambda_E(L) = \frac{2(L+1)}{\epsilon_0 L[(2L+1)!!]^2} \left(\frac{3}{L+3}\right)^2 e^2 R^{2L} \left(\frac{\omega}{c}\right)^{2L+1}$$

which can be rewritten as:

$$\lambda_E(L) = \frac{2(L+1)}{L[(2L+1)!!]^2} \left(\frac{3}{L+3}\right)^2 \frac{e^2}{4\pi\epsilon_0} \frac{1}{\hbar c} (KR)^{2L} \omega$$

- Magnetic multipole transition rate:

$$\lambda_M(L) = 10 \frac{2(L+1)}{L[(2L+1)!!]^2} \left(\frac{3}{L+3}\right)^2 \frac{e^2}{4\pi\epsilon_0} \frac{1}{\hbar c} (KR)^{2L} \omega \left(\frac{\hbar}{m_p c R}\right)^2$$

- Since $kR \sim 5 \times 10^{-2} \ll 1$, the transition rate decreases as the angular momentum carried by the photon increases.
- Relative multipole transition rates:

$$\frac{\lambda_E(L+1)}{\lambda_E(L)} \approx \frac{\lambda_M(L+1)}{\lambda_M(L)} \approx (kR)^2 \approx 2.5 \times 10^{-3}$$

- Relative multipole transition ratios:

$$\frac{\lambda_M(L)}{\lambda_E(L)} = 10 \left(\frac{\hbar}{m_p c R}\right)^2 \approx 4 \times 10^{-3}$$

- In general, $\lambda_M(L) \sim \lambda_E(L+1)$, which means the magnetic dipole transition is comparable to the electric quadrupole transition.

Radiation Type	Name	$l = \Delta I$	$\Delta\pi$
E1	Electric dipole	1	Yes
M1	Magnetic dipole	1	No
E2	Electric quadrupole	2	No
M2	Magnetic quadrupole	2	Yes
E3	Electric octupole	3	Yes
M3	Magnetic octupole	3	No
E4	Electric hexadecapole	4	No
M4	Magnetic hexadecapole	4	Yes

Table 2: Different types of radiation, their corresponding names, $l = \Delta I$, and parity change $\Delta\pi$.

10.3 Selection Rules in Nuclear Transitions

Based on the angular momentum and parity of the gamma transition, the allowed selection rules for multipole transitions are shown in the table below (e.g., transitions from $0 \to 0$ cannot occur):

- According to the selection rules, pay attention to the types of transitions that can or cannot occur. For example, if both electric and magnetic transitions are possible, the lower-order transition dominates.

10.4 Doppler Effect in Nuclear Reactions

The Doppler shift refers to the change in the observed frequency (or energy) of gamma radiation due to the relative motion between the source and the observer. The energy shift can be expressed as:

$$\Delta E = E_0(1 - \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta})$$

where E_0 is the energy of the gamma photon at rest. This effect can broaden the gamma-ray spectrum when there is motion along the line of sight.

10.5 Linewidth and Lifetime in Nuclear Decay

The natural linewidth of a gamma-ray transition arises due to the uncertainty principle, given by:

$$\Gamma = \Delta E = \frac{\hbar}{\tau}$$

where ΔE is the uncertainty in energy (the linewidth) and Δt is the lifetime of the excited state. For a short-lived excited state, the natural linewidth is larger. This phenomenon limits the resolution of gamma-ray spectra.

10.6 Mössbauer Effect and Recoil-Free Gamma Emission

The Mössbauer effect describes the recoil-free emission and absorption of gamma rays by nuclei in a solid. In this effect, the entire crystal lattice absorbs the recoil momentum, leading to no energy loss, which allows for extremely high-resolution gamma-ray spectroscopy. The key condition for the Mössbauer effect to occur is that the gamma-ray energy must be small enough such that the recoil energy is negligible, allowing the gamma photon to be emitted or absorbed without losing energy due to nuclear recoil.

$$E_\gamma = \Delta E \pm \frac{\Delta E^2}{2N_A M c^2}$$

where E_γ is the energy of the gamma photon, m is the mass of the nucleus, and c is the speed of light.

11 Nuclear Fission and Fusion Processes

11.1 Fission Process in Liquid Drop Model

Under certain conditions, such as the absorption of a neutron, the nucleus can become excited and start to oscillate or deform from its spherical shape. When the elongation reaches a critical point, the repulsive Coulomb forces between the two halves of the nucleus become stronger than the surface tension (binding forces), causing the nucleus to split.

$$\Delta E = (E_s + E_C) - (E_s + E_C)_{\text{SEMF}} = \frac{\epsilon^2}{5}(2a_s A^{2/3} - a_c Z^2 A^{-1/3})$$

If $\Delta E < 0$, i.e. $\dfrac{Z^2}{A} \geq \dfrac{2a_s}{a_C} \approx 47$, the deformation is energetically favorable and spontaneous fission can occur, inquality is satisfied for nuclei with $Z > 116$, $A > 270$.

For more detailed math, we start from when a nucleus elongates without changing its density, let the ellipsoid be defined by semi-major and semi-minor axes $a = R(1+\epsilon)$ and $b = R(1+\epsilon)^{-1/2}$ so that

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi ab^2$$

Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the ellipse such that $a > b$. For prolate spheroid, it rotates about the $x-$axis, whereas the surface is given by

$$y = \frac{b}{a}\sqrt{a^2-x^2}, \quad \frac{dy}{dx} = -\frac{bx}{a\sqrt{a^2-x^2}}, ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}dx = a\frac{\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\frac{x^2}{a^2}}}{\sqrt{a^2-x^2}}dx$$

$$\begin{aligned} S &= \int_{-a}^{a} 2\pi y ds \\ &= 4b\pi \int_0^a \sqrt{1-\left(1-\frac{b^2}{a^2}\right)\frac{x^2}{a^2}}dx \\ &= 4b\pi\left[\frac{x}{2}\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\frac{x^2}{a^2}} + \frac{a^2}{2\sqrt{a^2-b^2}}\sin^{-1}\frac{x\sqrt{a^2-b^2}}{a^2}\right]_0^a \\ &= 2\pi b\left(b + \frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right) \end{aligned}$$

Then we have

$$S' = 2\pi b\left(b + \frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right) = 4\pi R^2(1 + \frac{2\epsilon^2}{5} + O(\epsilon^{5/2}))$$

The Surface Term would increase due to the increased surface area.

$$E_s = a_s c^2 A^{2/3}(1 + \frac{2}{5}\epsilon^2 + \cdots)$$

The Coulomb Term would be affected because elongation changes the distribution of protons, potentially increasing Coulomb repulsion.

Using the formula for the electrostatic energy of a ellipsoid $E_C = \dfrac{1}{8\pi\varepsilon_0}\displaystyle\int_{V_1}\int_{V_2}\frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|}dV_1dV_2$ where $V_{1,2}$ is the ellipsoid volume $\{(x,y,z)|\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} \leq 1\}$

To calculate this, we consider the ellipsoid divided into infinitely thin coaxial ellipsoidal shells, each having the same eccentricity ratios. The semi-axes of a thin shell are given by:

$$a' = ka, \quad b' = kb, \quad c' = kc \quad \text{where} \quad 0 \leq k \leq 1$$

The potential dU created by a thin ellipsoidal shell at a point within the ellipsoid can be calculated in the ellipsoidal coordinates:

$$\begin{aligned} x^2 &= \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 &= \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-a^2)(b^2-c^2)} \\ z^2 &= \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)} \end{aligned}$$

In this coordinate, we have the Possion's equation

$$\nabla^2\phi = \frac{4\sqrt{\varphi(\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\frac{\partial}{\partial\lambda}\left(\sqrt{\varphi(\lambda)}\frac{\partial\phi}{\partial\lambda}\right) + \frac{4\sqrt{\varphi(\mu)}}{(\mu-\lambda)(\mu-\nu)}\frac{\partial}{\partial\mu}\left(\sqrt{\varphi(\mu)}\frac{\partial\phi}{\partial\mu}\right) + \frac{4\sqrt{\varphi(\nu)}}{(\nu-\lambda)(\nu-\mu)}\frac{\partial}{\partial\nu}\left(\sqrt{\varphi(\nu)}\frac{\partial\phi}{\partial\nu}\right)$$

where $\varphi(s) = \sqrt{(s^2+a^2)(s^2+b^2)(s^2+c^2)}$. A natural trial solution is a function $\phi(\lambda)$ that only depends on λ, but not on μ or ν. In this case,

$$dU = E\int_0^\infty \frac{d\lambda}{2\sqrt{\varphi(\lambda)}}$$

E is a constant.To see the relationship between λ and $r = \sqrt{x^2+y^2+z^2}$, rewrite the coordinates as:

$$\frac{x^2}{1+\frac{a^2}{\lambda}}+\frac{y^2}{1+\frac{b^2}{\lambda}}+\frac{z^2}{1+\frac{c^2}{\lambda}}=\lambda$$

It is evident that when λ is very large, from the above equation we know $\lambda \rightarrow x^2+y^2+z^2=r^2$. Then we obtain:

$$U \rightarrow \frac{2E}{r}$$

When $r \rightarrow \infty$, the potential U at a point far from the charged ellipsoid due to charge Q is:

$$U \rightarrow \frac{Q}{4\pi\epsilon r}$$

Comparing these two equations, we obtain $E=\dfrac{Q}{8\pi\epsilon}$. At a certain k, $E=\dfrac{Q(k+dk)-Q(k)}{8\pi\varepsilon_0}$, then we have

$$dU=\frac{dq}{8\pi\varepsilon_0}\int_0^\infty \frac{d\lambda}{\sqrt{(\lambda+a^2)(\lambda+b^2)(\lambda+c^2)}}=\frac{dq}{4\pi\epsilon_0}\cdot\frac{1}{\sqrt{a^2-c^2}}\cdot F(m,\phi_0)$$

where $dq=\dfrac{dq}{dk}dk=4\pi k^2\rho abcdk$ is the charge of the thin shell, and $F(m,\phi_0)$ is the incomplete elliptic integral of the first kind, defined as:

$$F(m,\phi_0)=\int_0^{\phi_0}\frac{d\phi}{\sqrt{1-m^2\sin^2\phi}}$$

where $m=\dfrac{\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}}$, $\phi_0=\sin^{-1}\sqrt{1-\dfrac{c^2}{a^2}}$. The total electrostatic potential energy E of the ellipsoid is obtained by integrating over the entire volume of the ellipsoid where $b=c$, that gives $m=1$ and $\phi_0=\sin^{-1}\sqrt{1-\dfrac{b^2}{a^2}}$, we have $F(1,\phi_0)=\tanh^{-1}(1-\dfrac{b^2}{a^2})$,

$$E_c=\int_0^1\frac{(4\pi ab^2)^2}{3}\cdot\frac{\rho^2}{4\pi\epsilon_0}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{a^2-b^2}}\cdot\ln\left(\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right)k^4dk$$

After performing the integration, the final expression for the electrostatic self-energy E of the ellipsoid is:

$$E_c=\frac{3}{5}\cdot\frac{Q^2}{4\pi\epsilon_0}\cdot\frac{1}{2\sqrt{a^2-b^2}}\cdot\ln\left(\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right)$$

where $Q=\rho\cdot\dfrac{4\pi}{3}ab^2$ is the total charge of the ellipsoid, so we have:

$$E_C=\frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{2\sqrt{a^2-b^2}}\ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\approx\frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}(1-\frac{\epsilon^2}{5}+O(\epsilon^{5/2}))$$

Hence the change to the total energy is

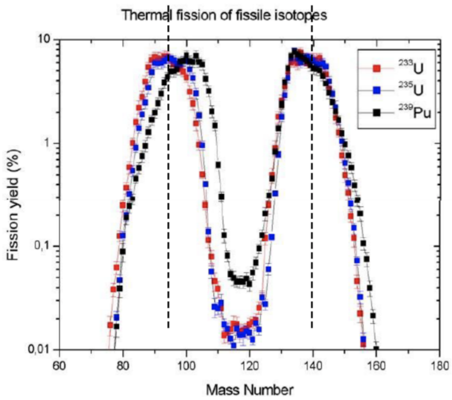
$$\Delta E=(E_s+E_C)-(E_s+E_C)_{\text{SEMF}}=\frac{\epsilon^2}{5}(2a_sc^2A^{2/3}-a_cc^2Z^2A^{-1/3})$$

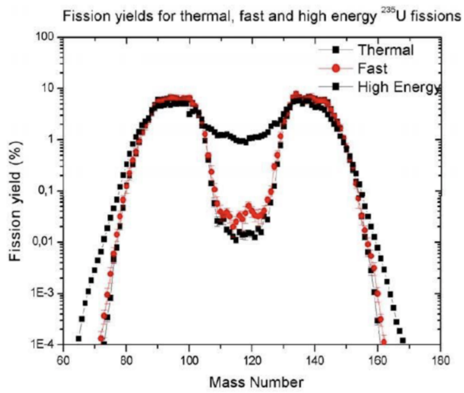
If $\Delta E<0$, i.e. $\dfrac{Z^2}{A}\geq\dfrac{2a_s}{a_C}\approx 49$, the deformation is energetically favorable and spontaneous fission can occur.
Given that uranium requires about 6 MeV of excitation energy to fission, ^{238}U is more likely induced a fission.

11.2 Types of Nuclear Fission

- Spontaneous Fission
  - The activation energy determines the probability of spontaneous fission, large nuclei $\approx$ 6 MeV.
  - Possible but low probability for large mass fragments. 10^6 less likely for ^{238}U than alpha decay.
- Induced Fission
  -Supply the energy by neutron capture.
  -The nucleus may be excited to a state above the fission barrier and therefore split up.

11.3 Fission Fragment Mass Distribution





- Fission into almost equal mass fragments is less probable than the maximum yield by 600.
- Fission induced by fast particles shows a different mass distribution, with more yield around $A/2$.

11.4 Chain Reactions in Nuclear Fission

Fission chain reactions occur when the neutrons produced by fission go on to induce further fission events, sustaining the reaction. The condition for a sustained chain reaction is that the number of neutrons produced exceeds the number lost:

$$k = \frac{\text{neutrons in one generation}}{\text{neutrons in the previous generation}}$$

If $k > 1$, the reaction is supercritical and the number of fissions grows exponentially. If $k = 1$, the reaction is critical and proceeds at a constant rate.

11.5 Nuclear Fusion: Mechanisms and Applications

$$\begin{aligned}
&p +^2 \mathrm{H} \rightarrow^3 \mathrm{He} + \gamma + 5.49\ \mathrm{MeV}\\
&{}^2\mathrm{H} +^2 \mathrm{H} \rightarrow^4 \mathrm{He} + \gamma + 23.85\ \mathrm{MeV}\\
&{}^2\mathrm{H} +^2 \mathrm{H} \rightarrow^3 \mathrm{He} + n + 3.27\ \mathrm{MeV}\\
&{}^2\mathrm{H} +^2 \mathrm{H} \rightarrow^3 \mathrm{H} + p + 4.03\ \mathrm{MeV}\\
&{}^2\mathrm{H} +^3 \mathrm{H} \rightarrow^4 \mathrm{He} + n + 17.6\ \mathrm{MeV}\\
&{}^2\mathrm{H} +^3 \mathrm{He} \rightarrow^4 \mathrm{He} + p + 18.36\ \mathrm{MeV}
\end{aligned}$$

11.6 Fusion Reactions in Stars

In the core of stars, fusion occurs predominantly through the proton-proton chain reaction at temperatures of $T \approx 10^7$ K:

$$\begin{gathered}
p + p \rightarrow D + e^+ + \nu_e + 1.44\ \mathrm{MeV}\\
p + D \rightarrow {}^3\mathrm{He} + \gamma + 5.49\ \mathrm{MeV}\\
{}^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + p + p + 12.86\ \mathrm{MeV}
\end{gathered}$$

This chain converts hydrogen into helium, releasing 26.7 MeV of energy per reaction. In stars with higher temperatures, the carbon-nitrogen-oxygen (CNO) cycle dominates: ${}^{12}_{6}\mathrm{C} \rightarrow {}^{13}_{7}\mathrm{N} \rightarrow {}^{13}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} \rightarrow {}^{15}_{8}\mathrm{O} \rightarrow {}^{15}_{7}\mathrm{N} \rightarrow {}^{12}_{6}\mathrm{C}$.

$$\begin{aligned}
{}^{12}_{6}\mathrm{C} + p &\rightarrow^{13}_{7} \mathrm{N} + \gamma + 1.95\,\mathrm{MeV}\\
{}^{13}_{7}\mathrm{N} &\rightarrow^{13}_{6} \mathrm{C} + \mathrm{e}^+ + \nu_\mathrm{e} + 1.20\,\mathrm{MeV}\\
{}^{13}_{6}\mathrm{C} + p &\rightarrow^{14}_{7} \mathrm{N} + \gamma + 7.54\,\mathrm{MeV}\\
{}^{14}_{7}\mathrm{N} + p &\rightarrow^{15}_{8} \mathrm{O} + \gamma + 7.35\,\mathrm{MeV}\\
{}^{15}_{8}\mathrm{O} &\rightarrow^{15}_{7} \mathrm{N} + \mathrm{e}^+ + \nu_\mathrm{e} + 1.73\,\mathrm{MeV}\\
{}^{15}_{7}\mathrm{N} + p &\rightarrow^{12}_{6} \mathrm{C} + 2{}^4_2\mathrm{He} + 4.96\,\mathrm{MeV}
\end{aligned}$$

Net result conversion of 4 protons to heliem, 2 e^+, 2 ν_e.