Lecture 3 Bivariate Portfolio Analysis and Fama-French Three Factor Model

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Bivariate Portfolio Analysis - Introduction

- The univariate portfolio analysis is designed to assess the cross-sectional relation between two variables without accounting for the effects of any other variables.
- The bivariate portfolio analysis is very similar to univariate portfolio analysis, except in bivariate portfolio analysis there are two sort variables.
- There are two types of sorting procedures, independent and dependent, that are commonly employed in bivariate portfolio analysis.
- The bivariate independent-sort portfolio analysis is designed to assess the crosssectional relations between two sort variables, which we refer to as $X1$ and $X2$, and an outcome variable Y.

Bivariate Independent-sort Portfolio Analysis - Breakpoints

- In bivariate independent-sort portfolio analysis, portfolios are formed by sorting on two variables independently, in each period, two sets of breakpoints will be calculated:
    1. The first set of breakpoints corresponds to values of the first sort variable $X1$.
    2. The second set of breakpoints corresponds to values of the second sort variable $X2$ and is calculated completely independently of the breakpoints for $X1$.
- The fact that the sorts are independent means:
    - It makes no difference which sort variable is considered the first sort variable, and which is considered the second.
    - Switching the order will have no effect on the results of the analysis.

Bivariate Independent-sort Portfolio Analysis - Breakpoints

- The breakpoints (percentiles) used to form the groups for the first sort variable are denoted as $B1_{j,t}\,(p1_j)$ for $j \in \{1, 2, \ldots, n_{P1} - 1\}$ which are calculated as

$$B1_{j,t} = \text{Pctl}_{p1_j}\left(\{X1_t\}\right)$$

- The breakpoints (percentiles) for the second sort variable are $B2_{k,t}\,(p2_k)$ for $k \in \{1, 2, \ldots, n_{P2} - 1\}$

$$B2_{k,t} = \text{Pctl}_{p2_k}\left(\{X2_t\}\right)$$

- Let the two sorted variables be beta $(X1 = \beta)$ and market capitalization $(X2 = MktCap)$
  - Divide the sample into three groups based on $\beta\,(n_{P1} = 3)$ using the 30th and 70th percentiles
  - Divide the sample into four groups based on $MktCap$ $(n_{P2} = 4)$ using the 25th, 50th, and 75th percentiles.

Bivariate Independent-sort Portfolio Analysis - Breakpoints

- Bivariate Independent-Sort Breakpoints

t	$B1_{1,t}$	$B1_{2,t}$	$B2_{1,t}$	$B2_{2,t}$	$B2_{3,t}$
1988	0.18	0.66	9.65	34.83	159.85
1989	0.17	0.70	9.77	37.04	184.14
1990	0.23	0.86	6.53	25.90	149.54
1991	0.24	0.85	9.70	41.56	223.70
1992	0.26	0.97	16.21	62.88	284.60
1993	0.29	0.92	22.48	78.32	345.67
1994	0.36	0.97	20.72	72.17	304.26
1995	0.27	0.90	26.03	91.38	382.85
1996	0.32	0.90	28.54	102.21	438.55
1997	0.26	0.72	32.62	119.62	521.35
1998	0.41	0.94	28.78	106.66	509.71
1999	0.15	0.53	33.79	128.21	623.06
2000	0.25	0.85	24.36	102.25	610.68

Bivariate Independent-sort Portfolio Analysis - Breakpoints

- Bivariate Independent-Sort Breakpoints (Continuted)

t	$B1_{1,t}$	$B1_{2,t}$	$B2_{1,t}$	$B2_{2,t}$	$B2_{3,t}$
2001	0.31	0.95	34.09	142.62	717.07
2002	0.33	0.89	33.28	130.81	635.43
2003	0.38	0.97	70.06	270.53	1054.81
2004	0.63	1.36	90.64	334.90	1308.59
2005	0.60	1.30	96.99	349.11	1410.58
2006	0.61	1.40	108.61	406.64	1592.96
2007	0.56	1.18	92.95	352.91	1513.62
2008	0.57	1.14	38.11	197.13	879.62
2009	0.65	1.45	68.54	307.83	1352.40
2010	0.76	1.33	95.47	420.44	1850.99
2011	0.82	1.38	80.78	393.16	1771.85
2012	0.76	1.31	105.41	485.94	2034.15

Bivariate Independent-sort Portfolio Analysis - Breakpoints

- If the sort variables are highly positively correlated, then this may result in a large number of entities being put into the portfolio that holds entities with high values of both sort variables as well as the portfolio holding entities with low values of both sort variables.
- If the sort variables are highly positively correlated, portfolios that hold entities with low values of one sort variable and high values of the other will contain relatively fewer entities.

Bi-Independent-sort Portfolio Analysis - Portfolio Formation

- Let $P_{j,k,t}$ be the portfolios for period t where the first subscript indicates the group of the first sort variable and the second subscript indicates that of the second sort variable.
- $P_{j,k,t}$ can be defined as

$$P_{j,k,t} = \{i \mid B1_{j-1,t} \leq X1_{i,t} \leq B1_{j,t}\} \cap \{i \mid B2_{k-1,t} \leq X2_{i,t} \leq B2_{k,t}\}$$

for $j \in \{1, 2, \ldots, n_{P1}\}$, $k \in \{1, 2, \ldots, n_{P2}\}$, where $B1_{0,t} = B2_{0,t} = -\infty$, $B1_{n_{p,t},t} = B2_{np,t} = \infty$, and $\cap$ is the intersection operator.

- If there is positive correlation between $X1$ and $X2$
  - Portfolios that contain entities with high (or low) values of both sort variables having a disproportionately large number of entities
  - Portfolios comprised entities with low values of one sort variable and high values of the other contains fewer entities.

Bi-Independent-sort Portfolio Analysis - Portfolio Formation

- Bivariate Independent-Sort Number of Stocks per Portfolio

t		$\beta 1$	$\beta 2$	$\beta 3$	t		$\beta 1$	$\beta 2$	$\beta 3$
1988	MktCap 1	736	468	217	1998	MktCap 1	842	557	252
	MktCap 2	539	585	297		MktCap 2	622	667	361
	MktCap 3	335	683	403		MktCap 3	368	709	574
	MktCap 4	95	538	788		MktCap 4	149	708	794

- There is a positive cross-sectional relation between β and *MktCap* .
- The portfolios that hold entities with high (low) values of β and high (low) values of *MktCap* having a large number of stocks.
- The portfolios that hold entities with low values of one of the sort variables and high values of the other contain relatively few stocks.

Bi-Indep-sort Portfolio Analysis - Average Portfolio Values

- The average value of the outcome variable for portfolio $P_{j,k,t}$ is

$$\bar{Y}_{j,k,t} = \frac{\sum_{i \in P_{j,k,t}} W_{i,t} Y_{i,t}}{\sum_{i \in P_{j,k,t}} W_{i,t}}$$

  for $j \in \{1, 2, \ldots, n_{P1}\}$ and $k \in \{1, 2, \ldots, n_{P2}\}$, where the summations in both the numerator and denominator are taken over all entities in portfolio $P_{j,k,t}$.

- For each of the n_{P1} groups of the first sort variable $X1$, $\bar{Y}_{j,Diff,t}$ denotes the difference in average Y value of the portfolio that holds the entities with the highest and lowest values of the second sort variable $X2$ at time period t:

$$\bar{Y}_{j,Diff,t} = \bar{Y}_{j,n_{P2},t} - \bar{Y}_{j,1,t}$$

  for $j \in \{1, \ldots, n_{p_1}\}$.

Bi-Indep-sort Portfolio Analysis - Average Portfolio Values

- The average value of $\bar{Y}$ across all groups of sort variable $X1$ and within the k th group of sort variable $X2$ is defined as

$$\bar{Y}_{Avg,k,t} = \frac{\sum_{j=1}^{n_{P1}} \bar{Y}_{j,k,t}}{n_{P1}}.$$

for $k \in \{1, 2, \ldots, n_{P2}, \text{ Diff }\}$.

- For each group of the second sort variable $X2$, the difference in average Y value between the portfolio with the highest and lowest values of the first sort variable $X1$ is

$$\bar{Y}_{Diff,k,t} = \bar{Y}_{n_{P1},k,t} - \bar{Y}_{1,k,t}.$$

- The average value of $\bar{Y}$ across all groups of the second sort variable $X2$ and within the j th group of the first sort variable $X1$, giving

$$\bar{Y}_{j,Avg,t} = \frac{\sum_{k=1}^{n_{P2}} \bar{Y}_{j,k,t}}{n_{P2}}$$

for $j \in \{1, 2, \ldots, n_{P1}, \text{ Diff }\}$.

Bi-Indep-sort Portfolio Analysis - Average Portfolio Values

- We have

$$
\begin{aligned}
\bar{Y}_{Avg,Diff,t} &= \frac{\sum_{j=1}^{n_{P1}} \bar{Y}_{j,Diff,t}}{n_{P1}} = \frac{1}{n_{P1}} \sum_{j=1}^{n_{P1}} \left(\bar{Y}_{j,n_{P2},t} - \bar{Y}_{j,1,t} \right) \\
&= \frac{1}{n_{P1}} \sum_{j=1}^{n_{P1}} \bar{Y}_{j,n_{P2},t} - \frac{1}{n_{P1}} \sum_{j=1}^{n_{P1}} \bar{Y}_{j,1,t} = \bar{Y}_{Avg,n_{P2},t} - \bar{Y}_{Avg,1,t}
\end{aligned}
$$

- $\bar{Y}_{Avg,n_{P2},t}$ denote the returns of portfolio which is formulated by holding portfolios $P_{j,n_{P2},t}$ with equal weight $1/n_{P1}$ for $j = 1, \cdots, n_{P1}$. It is denoted by $P_{Avg,n_{P2},t}$.
- $\bar{Y}_{Avg,1,t}$ denote the returns of portfolio which is formulated by holding portfolios $P_{j,1,t}$ with equal weight $1/n_{P1}$ for $j = 1, \cdots, n_{P1}$. It is denoted by $P_{1,n_{P2},t}$.

Bi-Indep-sort Portfolio Analysis - Average Portfolio Values

- The averages of averages is defined by

$$\bar{Y}_{Avg,Avg,t} = \frac{\sum_{j=1}^{n_{P1}} \bar{Y}_{j,Avg,t}}{n_{P1}} = \frac{\sum_{k=1}^{n_{p2}} \bar{Y}_{Avg,k,t}}{n_{P2}} = \frac{\sum_{j=1}^{n_{P1}} \sum_{k=1}^{n_{p2}} \bar{Y}_{j,k,t}}{n_{P1} \times n_{P2}}.$$

Bi-Indep-sort Portfolio Analysis - Average Portfolio Values

- Bivariate Independent-Sort Portfolio Excess Returns

$t/t+1$		$\beta 1$	$\beta 2$	$\beta 3$	β Diff	β Avg
1988/1989	MktCap 1	1.13	-0.96	1.69	0.56	0.62
	MktCap 2	-2.02	-1.49	-10.49	-8.47	-4.67
	MktCap 3	4.48	3.62	5.45	0.97	4.52
	MktCap 4	9.92	14.45	17.08	7.15	13.82
	MktCap Diff	8.80	15.40	15.39	6.59	**13.20**
	MktCap Avg	3.38	3.90	3.43	**0.05**	3.57

Bi-Indep-sort Portfolio Analysis - Summarizing the Results

- The final calculation required to complete the bivariate dependent-sort portfolio analysis is the **time-series means** of the periodic average values along with the corresponding standard errors, t-statistics, and p-values for each of the portfolios.
- Bivariate Independent-Sort Portfolio Excess Returns

	Coefficients	$\beta 1$	$\beta 2$	$\beta 3$	β Diff	β Avg
MktCap 1	Excess return	20.96	23.68	21.20	0.24	21.95
		(6.12)	(6.81)	(3.62)	(0.06)	(5.66)
MktCap 2	Excess return	11.07	13.45	11.49	0.41	12.00
		(4.13)	(4.03)	(4.98)	(0.13)	(5.20)
MktCap 3	Excess return	8.86	10.48	8.51	−0.36	9.28
		(3.01)	(5.50)	(5.18)	(−0.11)	(5.52)
MktCap 4	Excess return	6.45	8.99	8.64	2.19	8.03
		(3.85)	(6.74)	(4.00)	(1.17)	(5.37)
MktCap Diff	Excess return	−14.51	−14.69	−12.55	1.95	−13.92
		(−4.81)	(−4.37)	(−2.16)	(0.60)	(−3.67)
MktCap Avg	Excess return	11.84	14.15	12.46	0.62	12.82
		(4.90)	(6.59)	(5.14)	(0.25)	(6.29)

Bi-Indep-sort Portfolio Analysis - Summarizing the Results

- The average annual excess return for the portfolio that holds low-*MktCap* (*MktCap* 1) and low-β (β1) stocks is 20.96%, with a t-statistic of 6.12.
- Within the low-β group (β1), the difference in average excess return between the high-MktCap and low-MktCap portfolios is -14.51% per year (t-statistic $= -4.81$).

Bi-Indep-sort Portfolio Analysis - Intepreting the Results

- Bivariate Independent-Sort Portfolio Excess Returns

	Coefficients	$\beta 1$	$\beta 2$	$\beta 3$	β Diff	β Avg
MktCap 1	Excess return	20.96	23.68	21.20	0.24	21.95
		(6.12)	(6.81)	(3.62)	(0.06)	(5.66)
MktCap 2	Excess return	11.07	13.45	11.49	0.41	12.00
		(4.13)	(4.03)	(4.98)	(0.13)	(5.20)
MktCap 3	Excess return	8.86	10.48	8.51	−0.36	9.28
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MktCap 4	Excess return	6.45	8.99	8.64	2.19	8.03
		(3.85)	(6.74)	(4.00)	(1.17)	(5.37)
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		(−4.81)	(−4.37)	(−2.16)	(0.60)	(−3.67)
MktCap Avg	Excess return	11.84	14.15	12.46	0.62	12.82
		(4.90)	(6.59)	(5.14)	(0.25)	(6.29)

Bi-Indep-sort Portfolio Analysis - Intepreting the Results

- The differences in average Y values between portfolios with high values of $X1$ and low values of $X1$ ($X1$ Diff portfolios) indicate whether a cross-sectional relation between $X1$ and Y exists after controlling for the effect of $X2$.
- The differences in average Y values between portfolios with high values of $X2$ and low values of $X2$ ($X2$ Diff portfolios) indicate whether a cross-sectional relation between $X2$ and Y exists after controlling for the effect of $X1$.
- For each β (here β is $X1$) quantile, high-MktCap stocks have significantly lower average returns (Y) than low-*MktCap* stocks, since the *MktCap* (here *MktCap* is $X2$) difference portfolio (*MktCap* Diff) generate negative average returns with t-statistics all greater than 2.
- Examination of the relation between β and future stock returns presents no evidence of such a relation after controlling for the effect of MktCap as, within each of the four MktCap groups.

Bi-Indep-sort Portfolio Analysis - Intepreting the Results

- Bivariate Independent-Sort Portfolio Excess Returns

	Coefficients	$\beta 1$	$\beta 2$	$\beta 3$	β Diff	β Avg
MktCap 1	Excess return	20.96	23.68	21.20	0.24	21.95
		(6.12)	(6.81)	(3.62)	(0.06)	(5.66)
MktCap 2	Excess return	11.07	13.45	11.49	0.41	12.00
		(4.13)	(4.03)	(4.98)	(0.13)	(5.20)
MktCap 3	Excess return	8.86	10.48	8.51	−0.36	9.28
		(3.01)	(5.50)	(5.18)	(−0.11)	(5.52)
MktCap 4	Excess return	6.45	8.99	8.64	2.19	8.03
		(3.85)	(6.74)	(4.00)	(1.17)	(5.37)
MktCap Diff	Excess return	−14.51	−14.69	−12.55	1.95	−13.92
		(−4.81)	(−4.37)	(−2.16)	(0.60)	(−3.67)
MktCap Avg	Excess return	11.84	14.15	12.46	0.62	12.82
		(4.90)	(6.59)	(5.14)	(0.25)	(6.29)

Bi-Indep-sort Portfolio Analysis - Intepreting the Results

- In some cases, looking at the $X1$ difference portfolio within the different groups of $X2$ may give differing indications for different $X2$ groups.
  - There may be a statistically significant relation between $X1$ and Y among entities with low values of $X2$.
  - But this relation may not exist, or may even take the opposite sign, for entities with high $X2$ values.
  - It is instructive to examine the $X1$ difference portfolio for the average $X2$ group which indicates whether, for the average group of $X2$, there is a relation between $X1$ and Y.
- The β difference portfolio for the average *MktCap* group has the average returns 0.62% with t-statistics 0.25 which is statistically insignificant,then we can't detect a relation between β and excess returns.
- The *MktCap* difference portfolio for the average β group has the average returns -13.92% with t-statistics -3.67 which is statistically significant, then we can detect a relation between *MktCap* and excess returns.

Bi-Dep-sort Portfolio Analysis - Introduction

- In the dependent-sort analysis, breakpoints for the second sort variable are formed within each group of the first sort variable.
- Dependent-sort analysis is used when the objective is to understand the relation between $X2$ and Y conditional on $X1$.
- The relation between $X1$ and Y is not examined in dependent-sort analysis, $X1$ is used only as a control variable.
- In the dependent-sort analysis, it is extremely important to distinguish which independent variable is the control variable, $X1$, and which variable is part of the relation of interest, $X2$.
- Unlike independent-sort analysis, in the dependent-sort analysis, the order of sorting is critically important.

Bi-Dep-sort Portfolio Analysis - Breakpoints

- Letting n_{P1} be the number of groups based on $X1$, and $p1_j$, $j \in \{1, \ldots, n_{P1} - 1\}$ be the percentiles used to calculate the breakpoints, the breakpoints for $X1$ are calculated exactly as in independent-sort analysis.
- Based on the breakpoints $B1_{j,t}$, define the breakpoints for the second sort variable as

$$B2_{j,k,t} = \mathrm{Pctl}_{p2_k}\left(\{X2_t \mid B1_{j-1,t} \leq X1_t \leq B1_{j,t}\}\right)$$

  where $j \in \{1, \ldots, n_{P1}\}$, $k \in \{1, \ldots, n_{P2} - 1\}$, $p2_k$ is the percentile for the k th breakpoint based on the second sort variable, n_{P2} is the number of groups to be formed based on the second sort variable $X2$, $B1_{0,t} = -\infty$, $B1_{n_{P1},t} = \infty$.
- $\{X2_t \mid B1_{j-1,t} \leq X1_t \leq B1_{j,t}\}$ is the set of values of $X2$ across all entities in the sample with values of $X1$ that are between $B1_{j-1,t}$ and $B1_{j,t}$ inclusive.

Bi-Dep-sort Portfolio Analysis - Portfolio Formation

- In dependent-sort analysis, because sorting based on the second sort variable $X2$ is done within each group of entities formed by the first sort, correlation between the sort variables does not play a role in determining an appropriate number of breakpoints.
- In general, the portfolio holding stocks in group j of the first sort variable $X1$ and group k of the second sort variable $X2$ is

  $$P_{j,k,t} = \{i \mid B1_{j-1,t} \leq X1_{i,t} < B1_{j,t}\} \cap \{i \mid B2_{j,k-1,t} \leq X2_{i,t} < B2_{j,k,t}\}$$

  for $j \in \{1, 2, \ldots, n_{P1}\}$, $k \in \{1, 2, \ldots, n_{P2}\}$.
- When the sample used to calculate the breakpoints is the same as the sample that is sorted into portfolios, the percentage of entities in any given portfolio is easily calculated from the percentiles used to calculate the breakpoints.

Bi-Dep-sort Portfolio Analysis - Average Portfolio Values

- For each of the $n_{P1} \times n_{P2}$ portfolios, the procedure for calculating the average dependent variable value is identical to the procedure in the independent-sort analysis.
- The calculation of the difference in averages between group n_{P2} and group one of the second sort variable, for each of the groups of the first sort variable $\left(\bar{Y}_{j,Diff,t}\right)$, as well as the calculation of the average portfolio value for each $X2$ group across all $X1$ groups $\left(\bar{Y}_{\text{Avg},k,t}\right)$, for $k \in \{1, 2, \ldots, n_{P2}, \text{Diff}\}$, remain unchanged.
- In dependent-sort analysis, we do not calculate the differences in mean values between groups n_{P1} and group one of the first sort variable $X1$.
- It is because that the dependent-sort analysis is only designed to assess the relation between the second sort variable $X2$ and the outcome variable Y.

Bi-Dep-sort Portfolio Analysis - Average Portfolio Values

- Bivariate Dependent-Sort Mean Values (**Crossectional mean**)

$t/t+1$		$\beta 1$	$\beta 2$	$\beta 3$	β Avg
1988/1989	*Mkt* Cap 1	−3.02	−0.24	−3.63	−2.29
	MktCap 2	3.22	−1.67	1.44	1.00
	MktCap 3	−0.91	3.96	12.38	5.14
	MktCap 4	5.75	13.61	19.98	13.11
	MktCap Diff	8.77	13.84	23.61	15.41
1989/1990	*MktCap* 1	−20.67	−32.48	−27.18	−26.78
	MktCap 2	−34.95	−38.05	−33.72	−35.58
	MktCap 3	−31.57	−30.87	−25.74	−29.39
	MktCap 4	−27.67	−19.88	−15.97	−21.17
	MktCap Diff	−7.00	12.60	11.21	5.60

Bi-Dep-sort Portfolio Analysis - Summarizing the Results

- The procedure for summarizing the results for each of the **time series** of portfolio average values in a bivariate dependent-sort portfolio analysis is identical to that for univariate portfolio analysis and bivariate independent-sort portfolio analysis.
- For each portfolio, the time-series mean and inferential statistics are calculated.
- Bivariate Dependent-Sort Portfolio Results - Summary

	Coefficient	$\beta 1$	$\beta 2$	$\beta 3$	β Avg
MktCap 1	Excess return	27.82	20.52	16.89	21.74
		(7.29)	(6.18)	(5.56)	(6.95)
MktCap 2	Excess return	14.05	12.65	9.30	12.00
		(4.82)	(4.05)	(5.06)	(5.34)
MktCap 3	Excess return	10.67	10.21	8.46	9.78
		(3.62)	(6.21)	(4.87)	(6.20)
MktCap 4	Excess return	8.84	8.79	8.67	8.77
		(4.17)	(6.55)	(3.46)	(5.62)
MktCap Diff	Excess return	-18.98	-11.73	-8.22	-12.98
		(-6.08)	(-3.74)	(-2.66)	(-4.50)

Bi-Dep-sort Portfolio Analysis - Interpreting the Results

- The only relation we are interested in understanding is the relation between $X2$ and Y after controlling for $X1$.
- Interpretation of the results will focus on the difference portfolios for the second sort variable.
- Statistically significant differences indicate a cross-sectional relation between $X2$ and Y after controlling for $X1$.
- There is a strong negative cross-sectional relation between *MktCap* and future portfolio returns, as within each β group, the difference in returns between the portfolio comprised high- *Mkt* Cap stocks and that of low- *Mkt* Cap stocks is negative, economically large, and highly statistically significant.

Bi-Dep-sort Portfolio Analysis - Interpreting the Results

- It can be seen that, in this case, the results of the bivariate dependent-sort portfolio analysis are qualitatively the same as those of the independent-sort analysis.
- While it is usually the case that both types of bivariate-sort analyses produce similar results, this is not necessarily the case and it is standard to check the robustness of any results using both sorting methodologies.

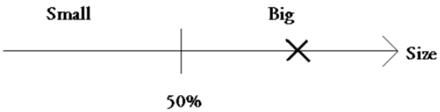
Fama-French Three Factor Model

Introduction

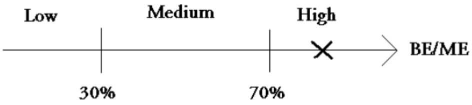
- The cross-section of average returns on U.S. common stocks shows little relation to either the market βs of the Sharpe (1964)-Lintner (1965) asset pricing model.
- Size and book-to-market equity, do a good job explaining the cross-section of average returns on NYSE, Amex, and NASDAQ stocks for the 1963-1990 period (Fama and French, 1992).
- Different from Fama and French (1992), the time-series regression of BJS (1972) instead of the cross-sections regressons of Fama and MacBech (1973) is used.
- The time-series regression slopes are factor loadings that can be interpreted as risk-factor sensitivities for stocks.

Fama-French Three Factor Model - Independent Variables

- The stocks are grouped by the median into two size groups, small and big (S and B).

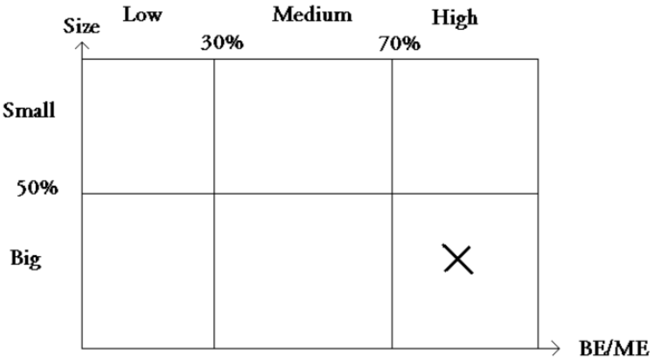


- The stocks are also divided into three book-to-market groups; low, medium and high (L, M and H), where the lowest 30% is the low group, the middle 40% is the medium group and the highest 30% is the high group.



Fama-French Three Factor Model - Independent Variables

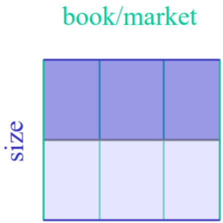
- There are three BE/ME groups and only two size groups, then any given stock will be present in one size group and in one book-tomarket group.



Fama-French Three Factor Model - Independent Variables

- Six portfolios are constructed $\mathrm{S/L, S/M, S/H, B/L, B/M}$ and $\mathrm{B/H}$, that is Small/Low, Small/Medium, Small/High, Big/Low, Big/Medium and Big/High.
- The return of the mimicking portfolio for size is denoted by SMB (small minus big) which is the difference of each month between the average return for the three small portfolios (S/L, S/M, $\mathrm{S/H}$) and the average return for the three big portfolios $(\mathrm{B/L, B/M, B/H})$

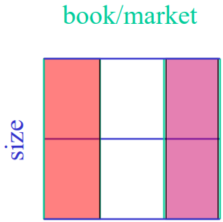
$$SMB = \frac{R_{\mathrm{S/L}} + R_{\mathrm{S/M}} + R_{\mathrm{S/H}}}{3} - \frac{R_{\mathrm{B/L}} + R_{\mathrm{B/M}} + R_{\mathrm{B/H}}}{3}.$$



Fama-French Three Factor Model - Independent Variables

- The return of the mimicking portfolio for BE/ME denoted by HML (high minus low) is the difference of each month between the average return for the two high BE/ME portfolios (S/H, $\mathrm{B/H}$) and the average return of the two low BE/ME portfolios ($\mathrm{S/L}, \mathrm{B/L}$)

$$HML = \frac{R_{\mathrm{S/H}} + R_{\mathrm{B/H}}}{2} - \frac{R_{\mathrm{S/L}} + R_{\mathrm{B/L}}}{2}.$$



- Note that the two medium portfolios $\mathrm{S/M}$ and $\mathrm{B/M}$ are not included in the HML portfolio.

Fama-French Three Factor Model - Dependent Variables

- The excess return on 25 portfolios, created from the factors size and BE/ME, are used as dependent variables in the time-series regressions.
- The 25 portfolios are constructed in the same manner as the six size-BE/ME portfolios which are formed in June each year t by size and BE/ME.

	Low	2	3	4	High
Small					
2					
3					
4					
Big					×

Fama-French Three Factor Model - Dependent Variables

- Size is measured in the end of June year t and BE/ME is measured in December year $t-1$.
- Five size groups and five BE/ME groups are created and the 25 portfolios are formed by a 5×5 matrix of these two categories.
- The dependent variables are then the excess returns of the 25 portfolios from July of year t to June year $t+1$.
- Regressions are run for each one of the 25 portfolios.

FF 3 Factor Model - Regression using Execss Market Return

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME

Book-to-market equity (BE/ME)

	Low	2	3	4	High	Low	2	3	4	High
Size			b					$t(b)$		
Small	1.40	1.26	1.14	1.06	1.08	26.33	28.12	27.01	25.03	23.01
2	1.42	1.25	1.12	1.02	1.13	35.76	35.56	33.12	33.14	29.04
3	1.36	1.15	1.04	0.96	1.08	42.98	42.52	37.50	35.81	31.16
4	1.24	1.14	1.03	0.98	1.10	51.67	55.12	46.96	37.00	32.76
Big	1.03	0.99	0.89	0.84	0.89	51.92	61.51	43.03	35.96	27.75

FF 3 Factor Model - Regression using Execss Market Return

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME Book-to-market equity (BE/ME)										
	Low	2	3	4	High	Low	2	3	4	High
Size			R^2					$s(e)$		
Small	0.67	0.70	0.68	0.65	0.61	4.46	3.76	3.55	3.56	3.92
2	0.79	0.79	0.76	0.76	0.71	3.34	2.96	2.85	2.59	3.25
3	0.84	0.84	0.80	0.79	0.74	2.65	2.28	2.33	2.26	2.90
4	0.89	0.90	0.87	0.80	0.76	2.01	1.73	1.84	2.21	2.83
Big	0.89	0.92	0.84	0.79	0.69	1.66	1.35	1.73	1.95	2.69

FF 3 Factor Model - Regression using Execss Market Return

- The excess return on the market portfolio of stocks, $RM - RF$, captures common variation in stock returns.
- The market leaves much variation in stock returns that might be explained by other factors.
  - The only R^2 values near 0.9 are for the big-size low-book-to-market portfolios.
  - For small-size and high-BE/ME portfolios, R^2 values less than 0.8 or 0.7.

FF 3 Factor Model - Regression using Size and BE/ME

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME										
Book-to-market equity (BE/ME)										
	Low	2	3	4	High	Low	2	3	4	High
Size			s					$t(s)$		
Small	1.93	1.73	1.63	1.59	1.67	22.52	21.38	21.88	22.30	22.16
2	1.52	1.46	1.35	1.18	1.40	17.23	17.68	17.08	15.47	16.42
3	1.28	1.12	1.05	0.93	1.16	14.43	13.89	13.42	12.13	13.45
4	0.86	0.82	0.77	0.72	0.95	10.16	9.64	9.29	8.57	10.02
Big	0.28	0.35	0.22	0.29	0.44	3.70	4.39	2.79	3.69	5.02

FF 3 Factor Model - Regression using Size and BE/ME

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME										
	Book-to-market equity (BE/ME)									
	Low	2	3	4	High	Low	2	3	4	High
Size			h					$t(h)$		
Small	-0.95	-0.57	-0.35	-0.18	0.01	-9.72	-6.19	-4.10	-2.20	0.16
2	-1.23	-0.66	-0.38	-0.16	0.00	-12.25	-7.02	-4.20	-1.82	0.05
3	-1.09	-0.65	-0.31	-0.11	-0.01	-10.84	-7.07	-3.43	-1.23	-0.12
4	-1.11	-0.65	-0.36	-0.11	-0.01	-11.43	-6.69	-3.80	-1.12	-0.09
Big	-1.07	-0.65	-0.42	-0.06	0.08	-12.46	-7.07	-4.64	-0.66	0.81

FF 3 Factor Model - Regression using Size and BE/ME

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME										
	Book-to-market equity (BE/ME)									
	Low	2	3	4	High	Low	2	3	4	High
Size			R^2					$s(e)$		
Small	0.65	0.60	0.60	0.60	0.59	4.57	4.31	3.98	3.79	4.01
2	0.59	0.53	0.49	0.42	0.44	4.68	4.41	4.20	4.06	4.53
3	0.51	0.43	0.37	0.31	0.35	4.71	4.31	4.19	4.10	4.60
4	0.43	0.30	0.24	0.18	0.23	4.53	4.55	4.40	4.48	5.06
Big	0.34	0.18	**0.08**	**0.04**	**0.06**	4.02	4.27	4.20	4.19	4.69

FF 3 Factor Model - Regression using Size and BE/ME

- In the absence of competition from the market portfolio, *SMB* and *HML* typically capture substantial time-series variation in stock returns; 20 of the 25 R^2 values are above 0.2 and eight are above 0.5.
- For the portfolios in the larger-size quintile, *SMB* and *HML* leave common variation in stock returns that is picked up by the market portfolio.
  - The R^2 for the three larger-size quintile are 0.08, 0.04 and 0.06.
  - The corresponding R^2 for regressing on the excess return on the market portfolio alone are 0.84, 0.79 and 0.69.

FF 3 Factor Model - Regression using $RM - RF$, Size and BE/ME

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + +sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME Book-to-market equity (BE/ME)										
	Low	2	3	4	High	Low	2	3	4	High
Size			b					$t(b)$		
Small	1.04	1.02	0.95	0.91	0.96	39.37	51.80	60.44	59.73	57.89
2	1.11	1.06	1.00	0.97	1.09	52.49	61.18	55.88	61.54	65.52
3	1.12	1.02	0.98	0.97	1.09	56.88	53.17	50.78	54.38	52.52
4	1.07	1.08	1.04	1.05	1.18	53.94	53.51	51.21	47.09	46.10
Big	0.96	1.02	0.98	0.99	1.06	60.93	56.76	46.57	53.87	38.61

FF 3 Factor Model - Regression using $RM - RF$, Size and BE/ME

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + +sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME										
					Book-to-market equity (BE/ME)					
	Low	2	3	4	High	Low	2	3	4	High
Size			s					$t(s)$		
Small	1.46	1.26	1.19	1.17	1.23	37.92	44.11	52.03	52.85	50.97
2	1.00	0.98	0.88	0.73	0.89	32.73	38.79	34.03	31.66	36.78
3	0.76	0.65	0.60	0.48	0.66	26.40	23.39	21.23	18.62	21.91
4	0.37	0.33	0.29	0.24	0.41	12.73	11.11	9.81	7.38	11.01
Big	-0.17	-0.12	-0.23	-0.17	-0.05	-7.18	-4.51	-7.58	-6.27	-1.18

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + +sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME										
	Book-to-market equity (BE/ME)									
	Low	2	3	4	High	Low	2	3	4	High
Size			h					$t(h)$		
Small	-0.29	0.08	0.26	0.40	0.62	-6.47	2.35	9.66	15.53	22.24
2	-0.52	0.01	0.26	0.46	0.70	-14.57	0.41	8.56	17.24	24.80
3	-0.38	0.00	0.32	0.51	0.68	-11.26	-0.05	9.75	16.88	19.39
4	-0.42	0.04	0.30	0.56	0.74	-12.51	1.04	8.83	14.84	17.09
Big	-0.46	0.00	0.21	0.57	0.76	-17.03	0.09	5.80	18.34	16.24

FF 3 Factor Model - Regression using $RM - RF$, Size and BE/ME

- Regressions of excess stock returns (in percent) on the excess stock-market return, July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + +sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME Book-to-market equity (BE/ME)										
	Low	2	3	4	High	Low	2	3	4	High
Size			R^2					$s(e)$		
Small	0.94	0.96	0.97	0.97	0.96	1.94	1.44	1.16	1.12	1.22
2	0.95	0.96	0.95	0.95	0.96	1.55	1.27	1.31	1.16	1.23
3	0.95	0.94	0.93	0.93	0.93	1.45	1.41	1.43	1.32	1.52
4	0.94	0.93	0.91	0.89	0.89	1.46	1.48	1.49	1.63	1.88
Big	0.94	0.92	0.88	0.90	0.83	1.16	1.32	1.55	1.36	2.02

FF 3 Factor Model - Regression using $RM - RF$, Size and BE/ME

- The three stock-market factors capture strong common variation in stock returns.
- The market βs for stocks are all more than 38 standard errors from 0.
- *SMB*, the mimicking return for the size factor, clearly captures shared variation in stock returns that is missed by the market and by *HML*:
    - Almost all the t-statistics on the *SMB* slopes for stocks are greater than 4.
    - The slopes on *SMB* for stocks are related to size, that is, the slopes on *SMB* decrease monotonically from smaller to bigger size quintiles in every book-to-market quintile.

FF 3 Factor Model - Regression using $RM - RF$, Size and BE/ME

- HML clearly captures shared variation in stock returns, related to book-to-market equity, that is missed by the market and by SMB:
  - Almost all the t-statistics on the HML slopes for stocks are greater than 2.
  - In every size quintile of stocks, the HML slopes increase monotonically from strong negative values for the lowest BE/ME quintile to strong positive values for the highest BE/ME quintile.
- Adding SMB and HML to the regressions results in large increases in R^2.
- In general. adding SMB and HML to the regressions collapses the βs for stocks toward 1.0:
  - Low βs move up toward 1.0 and high βs move down.
  - This behavior is due to correlation between the market and SMB or HML.
  - Although SMB and HML are almost uncorrelated (-0.08), the correlations between $RM - RF$ and the SMB and HML returns are 0.32 and -0.38.

FF 3 Factor Model - The cross-section of average returns

- Test how well the average premiums for the three stock-market factors explain the cross-section of average returns stocks.
- The a verage-return tests center on the intercepts in the time-series regressions.
- The intercepts in the time-series regressions of excess returns on the mimicking portfolio returns should be indistinguishable from 0.

FF 3 Factor Model - The cross-section of average returns

- Intercepts from excess stock return regressions for 25 stock portfolios formed on size and book-to-market equity: July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME										
	Book-to-market equity (BE/ME)									
	Low	2	3	4	High	Low	2	3	4	High
Size			a					$t(a)$		
Small	-0.22	0.15	0.30	0.42	0.54	-0.90	0.73	1.54	2.19	2.53
2	-0.18	0.17	0.36	0.39	0.53	-1.00	1.05	2.35	2.79	3.01
3	-0.16	0.15	0.23	0.39	0.50	-1.12	1.25	1.82	3.20	3.19
4	-0.05	-0.14	0.12	0.35	0.57	-0.50	-1.50	1.20	2.91	3.71
Big	-0.04	-0.07	-0.07	0.20	0.21	-0.49	-0.95	-0.70	1.89	1.41

FF 3 Factor Model - The cross-section of average returns

- When the excess market return is the only explanatory variable in the time-series regressions, the intercepts show the size effect of Banz (1981)
    - The intercepts for the smallest-size portfolios exceed those for the biggest by 0.22% to 0.37% per month except in the lowest- BE/ME quintile.
- The intercepts are also related to book-to-market equity
    - In every size quintile, the intercepts increase with BE/ME; the intercepts for the highest BE/ME quintile exceed those for the lowest by 0.25% to 0.76% per month.
    - These results parallel the evidence in Fama and French (1992a) that, used alone, market βs leave the cross-sectional variation in average stock returns that is related to size and book-to-market equity.
- A regression of average return on β yields a slope of -0.22 with a standard error of 0.31.

FF 3 Factor Model - The cross-section of average returns

- Intercepts from excess stock return regressions for 25 stock portfolios formed on size and book-to-market equity: July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME

	Book-to-market equity (BE/ME)									
	Low	2	3	4	High	Low	2	3	4	High
Size			a					$t(a)$		
Small	0.24	0.46	0.49	0.53	0.55	0.97	1.92	2.24	2.52	2.49
2	0.52	0.58	0.64	0.58	0.64	2.00	2.40	2.76	2.61	2.56
3	0.52	0.61	0.52	0.60	0.66	2.00	2.58	2.25	2.66	2.61
4	0.69	0.39	0.50	0.62	0.79	2.78	1.55	2.07	2.51	2.85
Big	0.76	0.52	0.43	0.51	0.44	3.41	2.23	1.84	2.20	1.70

FF 3 Factor Model - The cross-section of average returns

- The two-factor regression intercepts are, however, large (around 0.5% per month) and close to or more than two standard errors from 0.
- Intercepts that are similar in size support the conclusion from the cross-section regressions in Fama and French (1992a) that size and book-to-market factors explain the strong differences in average returns across stocks.
- But the large intercepts also say that *SMB* and *HML* do not explain the average premium of stock returns over one-month bill returns.

FF 3 Factor Model - The cross-section of average returns

- Intercepts from excess stock return regressions for 25 stock portfolios formed on size and book-to-market equity: July 1963 to December 1991, 342 months.

$$R(t) - RF(t) = a + b[RM(t) - RF(t)] + sSMB(t) + hHML(t) + e(t)$$

Dependent variable: Excess returns on 25 stock portfolios formed on size and BE/ME

	Book-to-market equity (BE/ME)									
	Low	2	3	4	High	Low	2	3	4	High
Size			a					$t(a)$		
Small	-0.34	-0.12	-0.05	0.01	0.00	-3.16	-1.47	-0.73	0.22	0.14
2	-0.11	-0.01	0.08	0.03	0.02	-1.24	-0.20	1.04	0.51	0.34
3	-0.11	0.04	-0.04	0.05	0.05	-1.42	0.47	-0.47	0.71	0.56
4	0.09	-0.22	-0.08	0.03	0.13	1.07	-2.65	-0.99	0.33	1.24
Big	0.21	-0.05	-0.13	-0.05	-0.16	3.27	-0.67	-1.46	-0.69	-1.41

FF 3 Factor Model - The cross-section of average returns

- Intercepts close to 0 say that the regressions that use $RM - RF$, SMB, and HML to absorb common time-series variation in returns do a good job explaining the cross-section of average stock returns.
- In the three-factor regressions, the stock portfolios produce slopes on $RM - RF$ close to 1.
- The average market risk premium (0.43% per month) then absorbs the similar strong positive intercepts observed in the regressions of stock returns on SMB and HML.
- The size and book-to-market factors can explain the differences in average returns across stocks, but the market factor is needed to explain why stock returns are on average above the one-month bill rate.