Lecture 1 Mean-Variance Theory and Capital Asset Pricing Model

Sep 28, 2023

Mean-variance utility

- Economists want agents (people, firms, or institutions) to have as many choices as possible and to guide agents to make optimal choices.
- These choices reflect asset owners' preferences which are ultimately a collection of trade-offs such as portfolio weights, saving and assumption.
- The preferences are represented by utility, an index numerically describing preferences in the sense that decisions that are made by ranking or maximizing utilities fully coincide with the asset owner's underlying preferences.
- The utility function is defined as a function of wealth W, $U(W)$.

Mean-variance utility

- Expected utility combines probabilities of outcomes with how investors feel about these outcomes

$$U = E[U(W)] = \sum_s p_s U(W_s)$$

- The subscript s denotes outcome s happening, so expected utility multiplies the probability of it happening, p_s, with the utility of the event in that state, $U(W_s)$.
- To make choices, the asset owner maximizes expected utility:

$$\max_\theta E[U(W)]$$

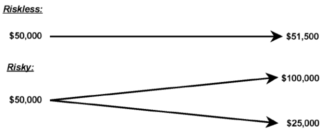
where θ are choice (or control) variables, such as asset allocation decisions (portfolio weights).

Mean-variance utility

- Realized returns differ over time time and across different assets
  - We need a systematic way to characterize these differences and the distribution of returns
- The mean $E(r_t)$ (often denoted μ) provides a measure of the typical return
  - What to expect on average
- The variance $\text{Var}(r_t)$ (often denote σ^2) provides a measure of dispersion
  - How risky is the particular asset
- If returns were normally distributed, $E(r_t)$ and $\text{Var}(r_t)$ are all we need to know to fully describe the distribution of r_t.

Risky vs. Risk-Free Asset

- Suppose that an investor has the choice of investing $50,000 in a risk-free investment or a risky investment
  - The riskless investment yield a certain payoff of $51,500
  - The risky investment will either half or double in value, with equal probability



- How should she decide which of these investments to take? Need to look at the risk-return tradeoff.

Risky vs. Risk-Free Asset

- Calculate the expected return for each investment
  - The return on the risk free investment is:

$$r_f = \frac{51,500}{50,000} - 1 = 3\%$$

  - The expected return on the risky investment is:

$$E(r) = \frac{1}{2} \cdot \underbrace{\left(\frac{100-50}{50}\right)}_{100\%} + \frac{1}{2} \cdot \underbrace{\left(\frac{25-50}{50}\right)}_{-50\%} = 25\%$$

- Calculate the risk premium on the risky investment
  - The risk premium is defined as the expected excess return:

$$E\left(r^e\right) = E\left(r - r_f\right) = E(r) - r_f = 22\%$$

Risky vs. Risk-Free Asset

- Calculate the riskiness of the investments, The measure we will use for now is the return variance, or equivalently the standard deviation
  - For the risk-free asset, the variance equals zero.
  - For the risky investment the return variance equals:

$$\sigma^2(r) = \frac{1}{2} \cdot \left[(1.00 - 0.25)^2 + (-0.50 - 0.25)^2\right] = 0.5625$$

    and the return standard-deviation equals:

$$\sigma(r) = \sqrt{0.5625} = 0.75 = 75\%$$

- If asset returns were normally distributed, this would be a perfect measure of risk.
- If returns are not normal (as is the case here), the variance is generally not a perfect measure of riskiness.

Mean-variance utility

- We need to quantify our attitudes, or preferences, over risk and return.
- Mean-variance utility (Markowitz, 1952) is given by:

$$U = E\left(r_p\right) - \frac{\gamma}{2}\,\mathrm{var}\left(r_p\right)$$

  where r_p is the return of the investor's portfolio and γ is coefficient of risk aversion.
    - Asset owners care only about means (which they like), and variances (which they dislike).
    - Define bad times as low means and high variances.
- Levy and Markowitz (1972) proved that any expected utility function can be approximated by a mean-variance utility function:

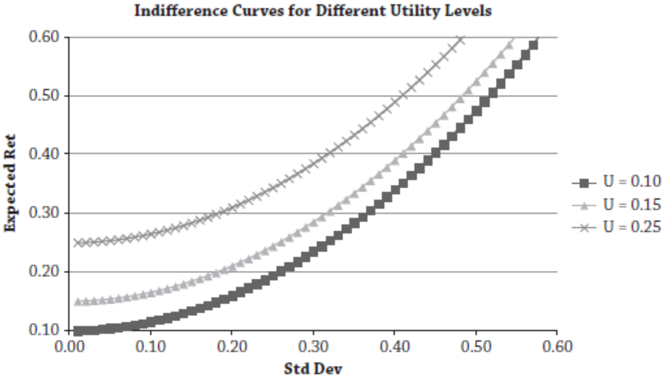
$$E\left[U\left(1+r_p\right)\right] \approx U\left(1+E\left(r_p\right)\right) + \frac{1}{2}U''\left(1+E\left(r_p\right)\right)\mathrm{var}\left(r_p\right)$$

  where $U''(\bullet) < 0$ denotes the second derivative of the concave utility function.

Indifference curve

- Indifference curve represents one particular level of utility.
- Three different indifference curve for different utility levels for an asset owner with a risk aversion of $\gamma = 3$ in **mean-standard** deviation space.

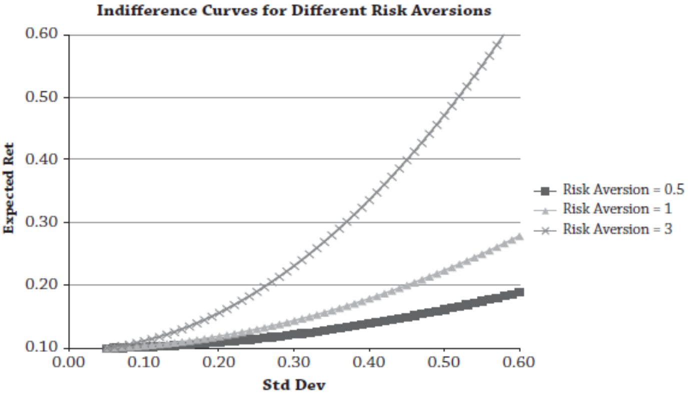
Figure: Indifference Curves for Different Utility Levels



Indifference curve

- The more risk averse an investor, the steeper the slope of his indifference curves.

Figure: Indifference Curves for Different Risk Aversion



Choosing a Portfolio of a Risky and Risk-Free Assets

- Of course, we don't usually have a binary choice like this this
  - We can hold a portfolio of the risky and risk-free asset.
  - What does the optimal portfolio of the risky and risk-free asset look like?

- We need to calculate the return on a portfolio p consisting of a risky asset and the risk-free asset:
  - r_A : return on (risky) asset A
  - $E(r_A)$: expected risky rate of return
  - σ_A : standard deviation
  - r_f : risk-free rate
  - w : fraction of portfolio p invested in asset A

- Working with the fraction w mirrors the payoff of $1 investment because everything scales

Choosing a Portfolio of a Risky and Risk-Free Assets

- The return on a portfolio with fraction w invested in the risky security and $(1-w)$ in the risk-free asset is:

$$\begin{aligned} r_p &= wr_A + (1-w)\cdot r_f \\ r_p &= r_f + w\underbrace{(r_A - r_f)}_{r_A^e} \\ E(r_p) &= r_f + wE(r_A^e) \end{aligned} \tag{1}$$

- The risk (variance) of this combined portfolio is::

$$\begin{aligned} \sigma_p^2 &= E\left[(r_p - E(r_p))^2\right] \\ &= E\left[(wr_A - wE(r_A))^2\right] \\ &= w^2 E\left[(r_A - E(r_A))^2\right] \\ &= w^2\sigma_A^2 \end{aligned} \tag{2}$$

Capital Allocation Line

- Combining (1) and (2) gives the Capital Allocation Line (CAL):

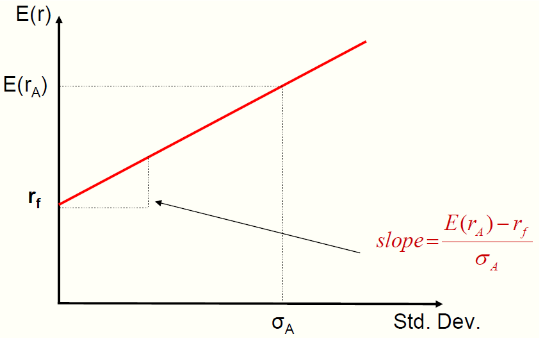
$$E(r_p) = r_f + \underbrace{\left[\frac{E(r_A) - r_f}{\sigma_A}\right]}_{\text{price of risk}} \underbrace{\sigma_p}_{\text{amount of risk}}$$

- Defines all the possible portfolios obtained by combining the risky and riskless asset.

- The price of risk represents the return premium per unit of portfolio risk (standard deviation) which is also called the Sharpe Ratio.

Capital Allocation Line

- Capital Allocation Line

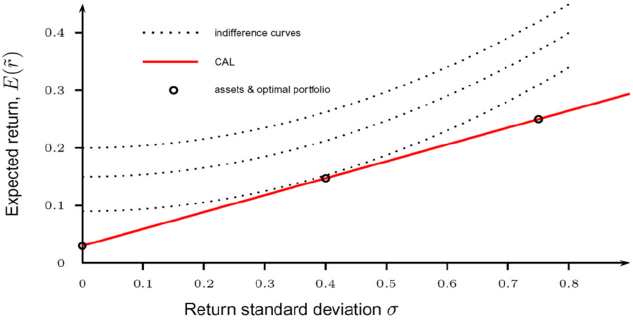


- The CAL shows all of the risk-return combinations that are possible from a portfolio of one risky-asset and the risk-free asset.

- The slope of the CAL is the Sharpe Ratio.

Which Portfolio?

- Which risk-return combination along the CAL do you want?



Which Portfolio?

- Mathematically, the optimal portfolio is the solution to the following problem:

$$U^* = \max_w U(r_p) = \max_w E(r_p) - \frac{1}{2} A \sigma_p^2$$

where,

$$E(r_p) = r_f + wE(r_A - r_f) \quad \sigma_p^2 = w^2 \sigma_A^2$$

- Combining these two equations we get:

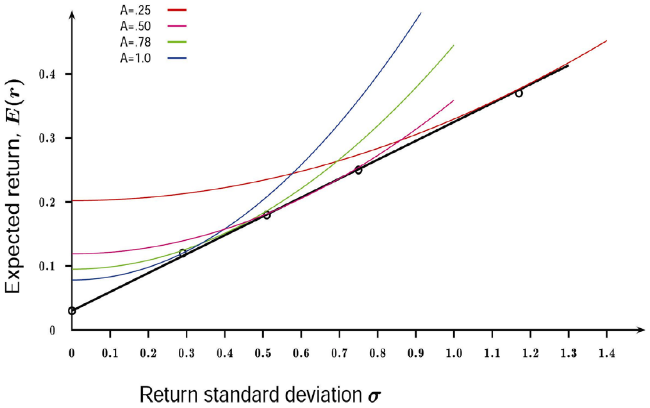
$$\max_w U(r_p) = \max_w \left(r_f + wE(r_A - r_f) - \frac{1}{2} A w^2 \sigma_A^2 \right)$$

And the solution is:

$$\left. \frac{dU}{dw} \right|_{w=w^*} = 0 \Rightarrow w^* = \frac{E(r_A - r_f)}{A\sigma_A^2}$$

Which Portfolio?

- Different levels of risk aversion leads to different choices



Which Portfolio?

- Now that we understand how to allocate capital between a risky asset and the riskfree asset, we need to show that it really is true that there is a single optimal risky portfolio.

- First, we'll ask the question:
    - How should you combine two risky securities in your portfolio?
    - To answer this question, we will plot the possible set of expected returns and standard deviations for different combinations of two risky assets
    - We will define the Minimum Variance Frontier, or the set of portfolios with the lowest variance for a given expected return

- To begin, we need some basic portfolio mathematics and intuition for combining two risky assets.

Portfolio Mathematics

- Portfolio made up of N assets $i = 1, 2, \ldots, N$
  - Portfolio weight for asset i:

$$w_i \equiv \frac{\$ \text{ amount invested in asset } i}{\text{Total } \$ \text{ cost of port folio}}$$

  - By definition: $w_1 + w_2 + \ldots + w_N = 1$
  - Long asset i : $w_i > 0$
  - Short asset i : $w_i < 0$

- Portfolio returns are linear in the portfolio weights:

$$r_p = w_1 r_1 + w_2 r_2 + \ldots + w_n r_N$$

A Portfolio of Two Risky Assets

- The expected return for the portfolio $r_p = w_A r_A + w_B r_B$ is:

$$E(r_p) = w \cdot E(r_A) + (1 - w) \cdot E(r_B)$$

where $w \equiv w_A$ denotes the fraction that is invested in asset A, with the remainder $w_B = (1 - w)$ invested in asset B.

- The variance of the portfolio is:

$$\begin{aligned} \sigma_p^2 &= E\left[(r_p - E(r_p))^2\right] \\ &= w^2\sigma_A^2 + (1 - w)^2\sigma_B^2 + 2w(1 - w)\,\mathrm{cov}(r_A, r_B) \end{aligned}$$

or, since $\rho_{AB} = \mathrm{cov}(r_A, r_B) / (\sigma_A \cdot \sigma_B)$,

$$\sigma_p^2 = w^2\sigma_A^2 + (1 - w)^2\sigma_B^2 + 2w(1 - w)\rho_{AB}\sigma_A\sigma_B.$$

Case $\rho_{AB} = 1$

- When $\rho = 1$ we can simplify the variance equation as:

$$\begin{aligned}\sigma_p^2 &= w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_A\sigma_B \\ &= (w\sigma_A + (1-w)\sigma_B)^2 \\ \sigma_p &= |w\sigma_A + (1-w)\sigma_B|\end{aligned}$$

- Also, recall that no matter what the value of ρ :

$$E(r_p) = w \cdot E(r_A) + (1-w) \cdot E(r_B).$$

- When $\rho = 1$, it is possible to find a perfect hedge, or a "synthetic" risk-free security with $\sigma_p = 0$

Case $\rho_{AB} = 1$

- As an example, consider asset A from before and another risky asset B :

Asset	$E(r)$	$\sigma(r)$
A	25%	75%
B	10%	25%

- We can easily build a table for various values of w :

w	$E(r_p)$	σ_p
-.5	2.5%	0.0%
0	10%	25%
0.5	17.5%	50.0%
1	25.0%	75.0%
1.5	32.5%	100.0%

Case $\rho_{AB} = 1$

- To find one of perfect hedge portfolios, set the risk to zero and solve for w :

$$\sigma_p = w\sigma_A + (1-w)\sigma_B = 0$$
$$\Rightarrow w = -\sigma_B / (\sigma_A - \sigma_B) = -0.5$$

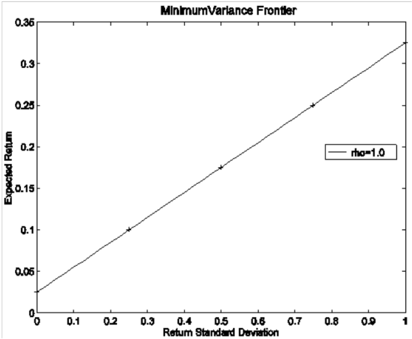
- Plugging this value for w into the expected return equation:

$$E(r_p) = 0.25w + 0.10(1-w)$$
$$= (0.25)(-0.5) + (0.10)(1.5) = 2.5\%$$

- The "synthetic" risk-free security yields 2.5%.

Case $\rho_{AB} = 1$

- Case $\rho_{AB} = 1$:



- The picture looks very similar to the case with one risky and one riskless asset.
- Because the two assets are perfectly correlated, it is possible to construct a "synthetic" riskless asset, or a perfect hedge.

Case $\rho_{AB} = -1$

- When $\rho = -1$, it is again possible to simplify the variance equation:

$$\begin{aligned}
\sigma_p^2 &= w^2\sigma_A^2 + (1-w)^2\sigma_B^2 - 2w(1-w)\sigma_A\sigma_B \\
&= (w\sigma_A - (1-w)\sigma_B)^2 \\
\sigma_p &= |w\sigma_A - (1-w)\sigma_B|
\end{aligned}$$

- To find the perfect hedge for $\rho = -1$, set the risk to zero $\sigma_p = 0$, and solve for w :

$$\sigma_p = w\sigma_A - (1-w)\sigma_B = 0$$
$$\Rightarrow w = \sigma_B / (\sigma_A + \sigma_B) = 0.25$$

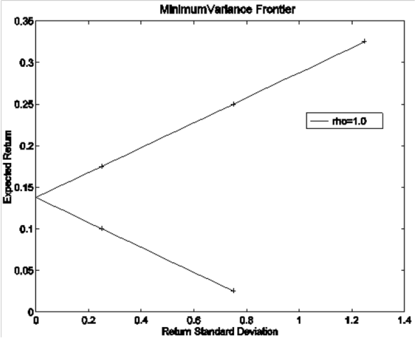
- Plugging this value for w into the expected return equation:

$$\begin{aligned}
E(r_p) &= 0.25w + 0.10(1-w) \\
&= (0.25)(0.25) + (0.10)(0.75) = 13.75\%
\end{aligned}$$

- We have created a "synthetic" risk-free security with a 13.75% return.

Case $\rho_{AB} = -1$

- As before, let's plot the resulting expected returns and standard deviations for different weights, $-0.5 \leq w \leq 1.5$



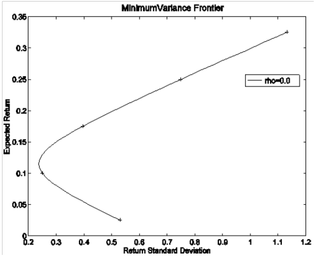
- Because the two assets are perfectly (negatively) correlated, we can again create a "synthetic" riskless asset.
- In this case, some combinations are dominated, or inefficient. Which ones?

Case $\rho_{AB} = 0$

- Finally consider the case $\rho = 0$:

$$E(r_p) = w \cdot E(r_A) + (1 - w) \cdot E(r_B)$$
$$\sigma_p^2 = w^2 \sigma_A^2 + (1 - w)^2 \sigma_B^2$$

- Plotting $E(r_p)$ and σ_p for different values of $-0.5 \leq w \leq 1.5$ now yields the following picture



- There is some reduction (diversification) of the risk, though not as much as for $\rho = -1$.
- Which portfolios are efficient? All portfolios above the minimum variance portfolio.

Minimum Variance Portfolio with Two Risky Assets

- Mathematically, the minimum variance portfolio is the solution to the following problem:

$$\min_{w} \sigma_p^2 = \min_{w} \left[w^2 \sigma_A^2 + (1-w)^2 \sigma_B^2 + 2w(1-w)\operatorname{cov}(r_A, r_B) \right]$$

- The solution is:

$$\left.\frac{d\sigma_p^2}{dw}\right|_{w=w^*} = 0 \Rightarrow w^* = \frac{\operatorname{Var}(r_B) - \operatorname{Cov}(r_A, r_B)}{\operatorname{Var}(r_A) + \operatorname{Var}(r_B) - 2\operatorname{Cov}(r_A, r_B)}$$

- For the three cases analyzed above:
  - $\rho_{AB} = 1 \Rightarrow w^* = -0.5$ and $\sigma_{\min} = 0$
  - $\rho_{AB} = -1 \Rightarrow w^* = 0.25$ and $\sigma_{\min} = 0$
  - $\rho_{AB} = 0 \Rightarrow w^* = 0.1$ and $\sigma_{\min} \approx 0.237$

Mean-variance frontiers

- Mean-variance frontiers depict the best set of portfolios that an investor can obtain in terms of means and variance.

- Minimize variance with constraints:

$$\min_{\{w_i\}} \text{var}\left(r_p\right)$$
$$\text{subject to } E\left(r_p\right) = \mu^* \text{ and } \sum_i w_i = 1$$

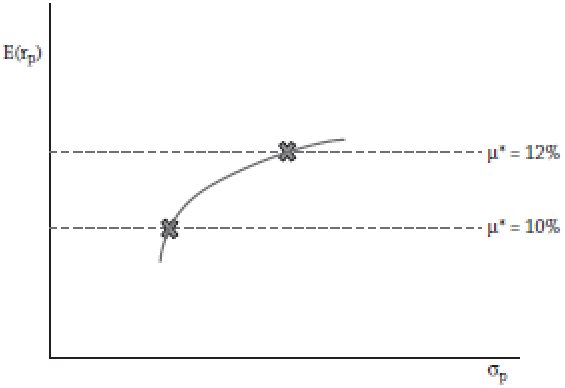
  where the portfolio weight for asset i is w_i.

- The combination of portfolio weights w_i that minimize the portfolio variance subject to two constraints:
  - The expected return on the portfolio is equal to a target return, μ^*.
  - The portfolio must be a valid portfolio.

Mean-variance frontiers

- The mean-variance frontier is a locus of points, where each point denotes the minimum variance achievable for each expected return.

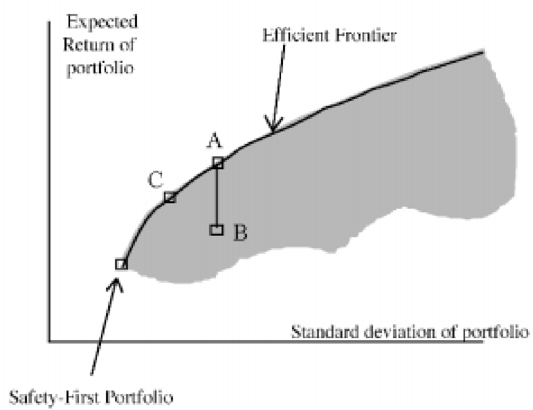
Figure: Mean-variance frontier



Mean-variance frontiers

- The feasible set of portfolios constructed from individual securities: The Risk-Return Diagram. The portfolios A and C are efficient, whereas B is dominated by A.

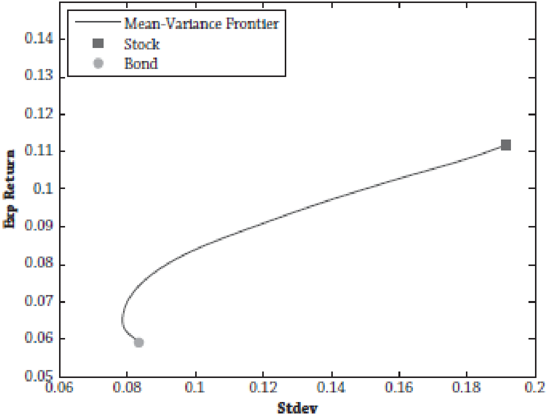
Figure: Efficient Mean-variance frontier



Mean-variance frontiers

- A stock and bond Mean-variance Frontier with U.S. data from January 1926 to December 2011.

Figure: Mean-variance frontier



Mean-variance frontiers

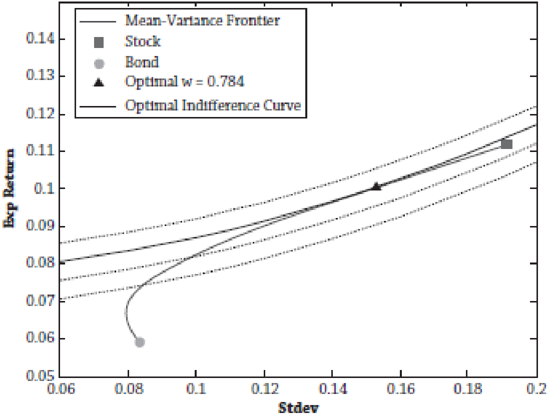
- Bonds lie on an inefficient part of the mean-variance frontier.
- By holding some equities, we can strictly increase the portfolio's expected return while lowering the portfolio's volatility.
- Equity diversifies a bond portfolio because equities have a low correlation with bonds; the correlation of equities with bonds in this sample is just 11%.
- Diversification benefits are measured by covariance correlations.

$$\begin{aligned} \text{var}(r_p) &= w_b^2 \text{var}(r_b) + w_s^2 \text{Var}(r_s) + 2 w_b w_s \text{cov}(r_b, r_s) \\ &= w_b^2 \text{var}(r_b) + w_s^2 \text{Var}(r_s) + 2 w_b w_s \rho_{b,s} \sigma_b \sigma_s \end{aligned}$$

Mean-variance optimization

- Maximizing utility is equivalent to finding the highest possible indifference curve.
- The highest indifference curve is tangent to the mean-variance frontier.

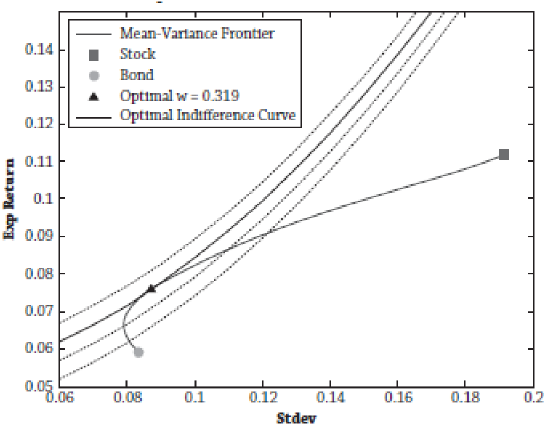
Figure: Optimal Asset Choice for Risk Aversion = 2



Mean-variance optimization

- The optimal portfolio holdings depend on the asset owner's degree of risk aversion.

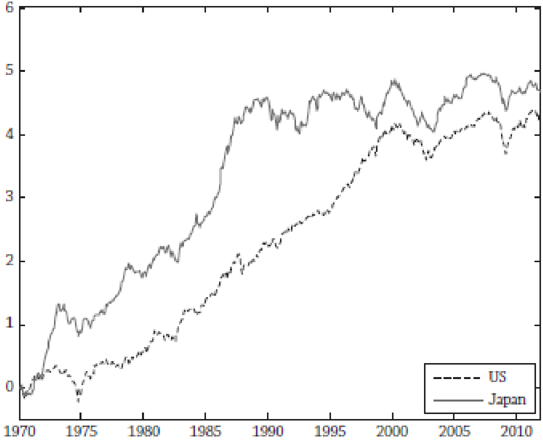
Figure: Optimal Asset Choice for Risk Aversion = 7



Mean-variance optimization

- The cumulated returns of U.S. and Japanese equities from January 1970 to December 2011.

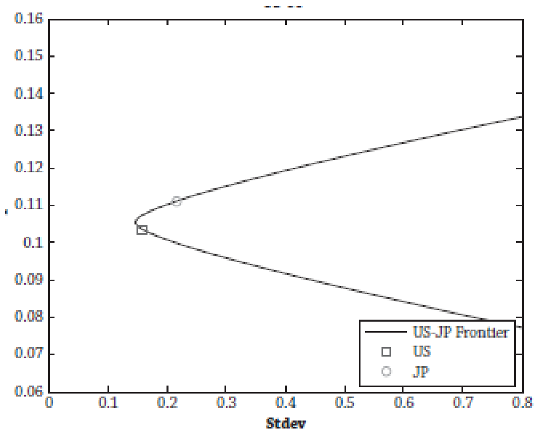
Figure: Cumulated returns



Mean-variance optimization

- The mean-variance frontier of US and JP equities from January 1970 to December 2011.

Figure: US-JP MVF

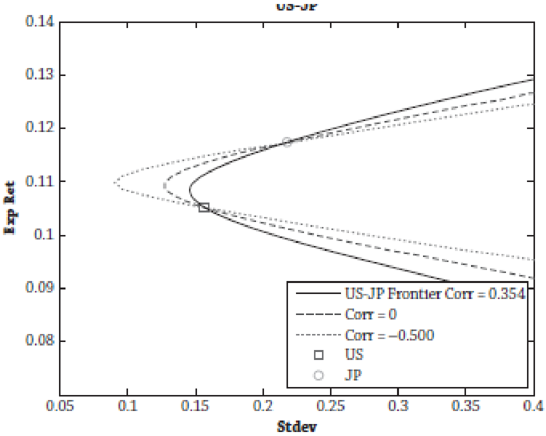


Mean-variance optimization

- Large diversification benefits correspond to low correlations.

$$\begin{aligned}\operatorname{var}(r_p) &= w_{US}^2 \operatorname{var}(r_{US}) + w_{JP}^2 \operatorname{var}(r_{JP}) + 2 w_{US} w_{JP} \operatorname{cov}(r_{US}, r_{JP}) \\ &= w_{US}^2 \operatorname{var}(r_{US}) + w_{JP}^2 \operatorname{var}(r_{JP}) + 2 w_{US} w_{JP} \rho_{US,JP} \sigma_{US} \sigma_{JP}\end{aligned}$$

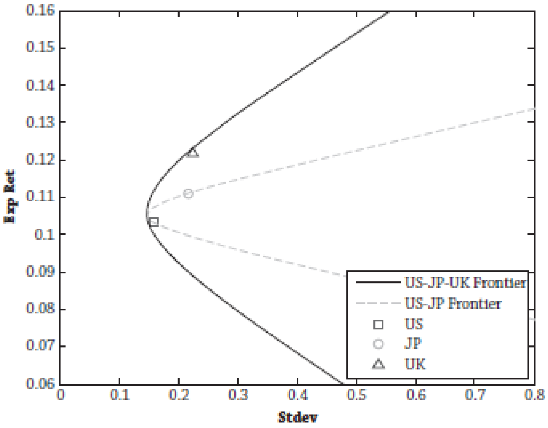
Figure: US-JP MVF



Mean-variance optimization

- More than two assets.

Figure: US-JP-UK MVF



Mean-variance optimization

- The frontier has expanded.
- All individual assets lie inside the frontier.
- Individual assets are dominated: diversified portfolios on the frontier do better than assets held individually.
- Diversification removes **asset-specific** risk and reduces the **overall** risk of the portfolio.

Mean-variance optimization

- **Two-fund theorem**: The entire efficient frontier can be generated from only two portfolios (funds).
  - Let $\mathbf{w}^1 = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1)$ be a solution to the Markowitz problem for a given expected rate of return $\bar{r}^1$, and $\mathbf{w}^2 = (\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2)$ be a solution to the Markowitz problem for a different given expected rate of return $\bar{r}^2$.
  - For any number α, the new portfolio $\alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$ is itself a solution to the Markowitz problem for expected rate of return $\alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$.
  - As α varies, $\bar{r} = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$ takes on all feasible values for expected rate of return; thus all solutions to the Markowitz problem can be constructed.

Mean-variance optimization

- The addition of a risk-free asset expands the investor's opportunities considerably which has no variance.
- Sharpe ratio (also known as the Sharpe index, the Sharpe measure, and the reward-to-variability ratio) measures the performance of an investment (e.g., a security or portfolio) compared to a risk-free asset, after adjusting for its risk.
  - It is defined as the difference between the returns of the investment and the risk-free return, divided by the standard deviation of the investment (i.e., its volatility).
  - It represents the additional amount of return that an investor receives per unit of increase in risk.

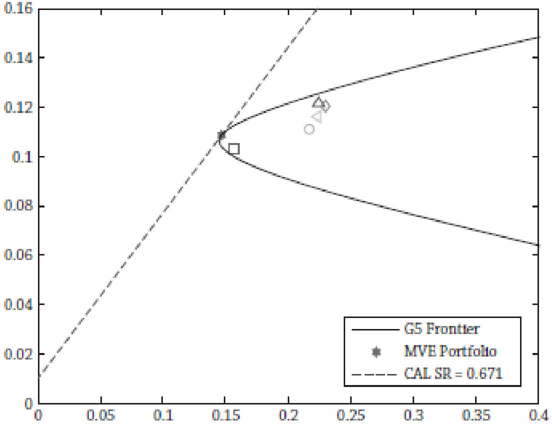
Mean-variance optimization

- When there is a risk-free asset, the investor proceeds in two steps:
    - Find the best risky asset portfolio. This is called the mean-variance efficient (MVE) portfolio, or tangency portfolio, and is the portfolio of risky assets that maximizes the Sharpe ratio.
    - Mix the best risky asset portfolio with the risk-free asset. This changes the efficient set from the frontier into a wider range of opportunities. The efficient set becomes a capital allocation line (CAL).
- The new efficient frontier is the line connecting the point $(0, r_f)$ to the unique point F (MVE,on the old efficient frontier) yielding a line tangent to the old efficient frontier.

Mean-variance optimization

- How to find MVE and CAL.
- The slope of the CAL represents the portfolio's Sharpe ratio.
- Since the CAL is tangent at the MVE, it represents the maximum Sharpe ratio that can be obtained by the investor.

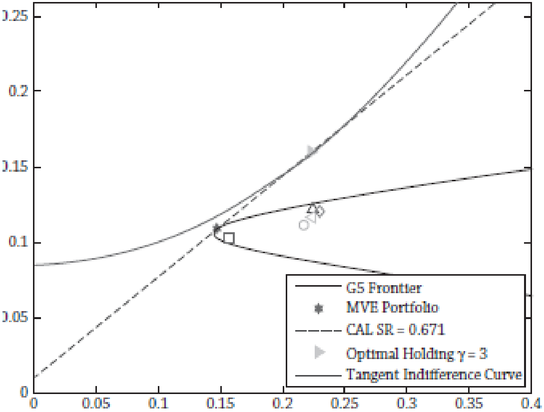
Figure: Asset Allocation G5 with Risk-Free Asset



Mean-variance optimization

- Finding the optimal combination of the MVE with the risk-free asset is equivalent to finding the point at which the highest possible indifference curve touches the CAL.
- The tangency point is the investor's optimal portfolio.

Figure: Asset Allocation G5 with Risk-Free Asset



Capital Asset Pricing Model - CAPM

- We assume an idealized framework for an open market place
  - All the risky assets refer to (say) all the tradeable stocks available to all.
  - In addition we have a risk-free asset (for borrowing and/or lending in unlimited quantities) with interest rate r_f.
  - All information is available to all such as covariances, variances, mean rates of return of stocks and so on.
  - Everyone is a risk-averse rational investor who uses the same financial engineering mean-variance portfolio theory from Markowitz.
- Everyone has the same assets to choose from, the same information about them, and the same decision methods, then:
  - Everyone has the same efficient frontier with risky assets (without risk-free asset).
  - Everyone has the same unique efficient fund (Mean-Variance Efficient portfolio, MVE) F of risky assets, hence the same capital asset line (CAL).
  - Everyone's optimal portfolio choice is a mixture of the risk-free asset and F (All that differs are the weights for the mixture due to the different risk-averse coefficients for different people).
- This efficient fund used by all is called the **market portfolio** and is denoted by M, and CAL is denoted as capital market line (CML)

Capital Asset Pricing Model - CAPM

- The weight for any asset in the market portfolio is given by its capital value (total worth of its shares) divided by the total capital value of the whole market (all assets together).
- The market portfolio must consist of all the risky assets.

Capital Asset Pricing Model - CAPM

- Let $(\sigma_M, \bar{r}_M)$ denote the point corresponding to the market portfolio M, all portfolios chosen by a rational investor will have a point $(\sigma, \bar{r})$ that lies on the so-called capital market line

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma$$

which is the efficient frontier for investment.

- Then the expected return of any efficient portfolio which is on the capital market line, $\bar{r}_p^e$, satisfies

$$\bar{r}_p^e = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma_p^e$$

where $(\bar{r}_M - r_f)/\sigma_M$ is called **the price of risk**.

- **What about the expected return of any portfolio (maybe inefficient)?**

Capital Asset Pricing Model - CAPM

- The Capital Asset Pricing Model

Theorem

For any asset i

$$E[r_i] - r_f = \beta_i \left(E[r_M] - r_f\right)$$

where

$$\beta_i = \frac{\sigma_{M,i}}{\sigma_M^2}$$

is called the beta of asset i. This beta value serves as an important measure of risk for individual assets (portfolios) that is different from σ_i^2: it measures the nondiversifiable part of risk.
More generally, for any portfolio $p = (\alpha_1, \ldots, \alpha_n)$ of risky assets, its beta can be computed as a weighted average of individual asset betas:

$$E[r_p] - r_f = \beta_p \left(E[r_M] - r_f\right)$$

where

$$\beta_p = \frac{\sigma_{M,p}}{\sigma_M^2} = \sum_{i=1}^{n} \alpha_i \beta_i .$$

Capital Asset Pricing Model - CAPM

- For the efficient portfolio

$$\bar{r}_p^e - r_f = \frac{\sigma_p^e}{\sigma_M}(\bar{r}_M - r_f) = \sigma_p^e \frac{\bar{r}_M - r_f}{\sigma_M}.$$

- Form CAPM, for any portfolio

$$\bar{r}_i - r_f = \rho_{M,i}\frac{\sigma_i}{\sigma_M}(\bar{r}_M - r_f) = \rho_{M,i}\sigma_i \frac{\bar{r}_M - r_f}{\sigma_M}$$

  where $\rho_{M,i}$ is the correlation coefficient between r_M and r_i

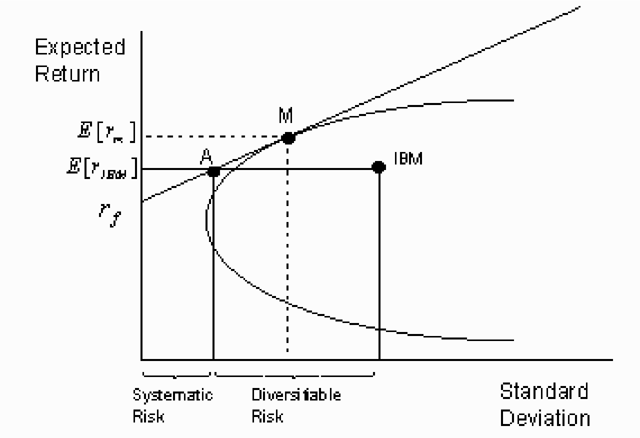
$$\rho_{M,i} = \frac{\sigma_{M,i}}{\sigma_M \sigma_i}.$$

- If σ_i is the asset i's total risk, then $\rho_{M,i}\sigma_i$ is the **Systematic risk** while $(1 - \rho_{M,i})\sigma_i$ is the **unsystematic part**.
- If portfolio i is on the capital market line, $\rho_{M,i} = 1$.

Capital Asset Pricing Model - CAPM

- Let's show why assets that are not perfectly correlated with M do not fall on the CML by using an investment in IBM stock as an example.

Figure: Asset not perfectly correlated with M



Capital Asset Pricing Model - CAPM

- There are two ways to receive an expected return of $E[r_{IBM}]$: simply buy shares in IBM, or buy portfolio A.
- For a risk averse investor, portfolio A is preferred to an investment solely in IBM since it produces the same return with less risk.
- It is impossible to earn an expected return of $E[r_{IBM}]$ incurring less risk than that of portfolio A.
- The total risk of IBM can therefore be decomposed into two parts:
  - **Systematic risk**: the minimum risk required to earn that expected return
  - **Diversifiable risk**: the portion of the risk that can be eliminated, without sacrificing any expected return , simply by diversifying.
- Investors are rewarded for bearing this systematic risk, but they are not rewarded for bearing diversifiable risk, because it can easily be eliminated at no cost.

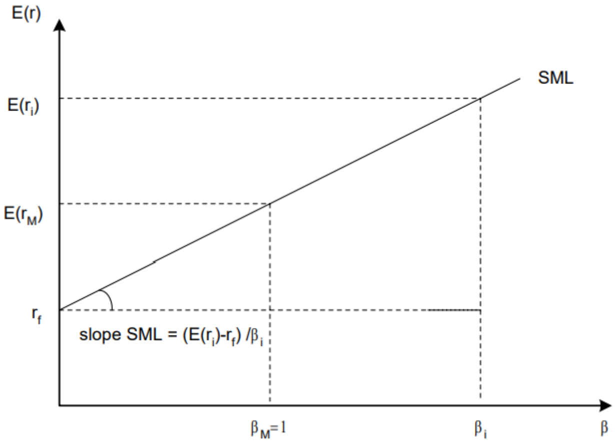
Capital Asset Pricing Model - CAPM

- For a given asset i, σ_i^2 tells us the risk associated with its own fluctuations about its mean rate of return, but not with respect to the market portfolio.
- We can view β_i as a measure of **nondiversifiable risk**, the correlated-with-the-market part of risk that we can't reduce by diversifying which is sometimes called **market or systematic risk**.
- It is not true in general that higher beta value β_i implies higher variance σ_i^2, but of course a higher beta value does imply a higher expected rate of return: you are rewarded (via a high expected rate of return) for taking on risk that can't be diversified away.

Capital Asset Pricing Model - CAPM

- Security market line is the relationship between expected return on an individual security and the beta of the security.

Figure: Security Market Line



Capital Asset Pricing Model - CAPM

- It is useful to compare the security market line to the capital market line.
  - The CML graphs the risk premiums of efficient portfolios (i.e., portfolios composed of the market and the risk-free asset) as a function of portfolio standard deviation. This is appropriate because standard deviation is a valid measure of risk for efficiently diversified portfolios that are candidates for an investor's overall portfolio.
  - The SML, in contrast, graphs individual asset risk premiums as a function of asset risk. The relevant measure of risk for individual assets held as parts of well-diversified portfolios is not the asset's standard deviation or variance; it is, instead, the contribution of the asset to the portfolio variance, which we measure by the asset's beta. The SML is valid for both efficient portfolios and individual assets.

Capital Asset Pricing Model - The Black CAPM

- The Black CAPM does not assumed that the investors can lend and borrow at a common risk-free rate.
- The market portfolio is still a mean-variance efficient portfolio.
- There exists a portfolio Z, which is named as **zero-beta portfolio** which has zero covariance with respect to the market portfolio.
- The Black CAPM takes the form as

$$E[r_i] = E[r_Z] + \beta_i (E[r_M] - E[r_Z]) \tag{3}$$

- The Black CAPM can be rewritten as

$$E[r_i] - r_f = E[r_Z] - r_f + \beta_i (E[r_M] - E[r_Z]) \tag{4}$$

- Note that $E[r_Z] - r_f > 0$ and $E[r_M] - E[r_Z] < E[r_M] - r_f$ which means that equation (4) is flatter than SML.

Arbitrage pricing theory

- In finance, arbitrage pricing theory (APT) is a multi-factor model for asset pricing which relates various macro-economic (systematic) risk variables to the pricing of financial assets.
- The APT model states that if asset returns follow a factor structure then the following relation exists between expected returns and the factor sensitivities:

$$\mathbb{E}\left(r_j\right) = r_f + \beta_{j1} RP_1 + \beta_{j2} RP_2 + \cdots + \beta_{jK} RP_K$$

  where β_{jk} is the sensitivity of the jth asset to factor k, also called factor loading, RP_k is the risk premium of the kth factor, r_f is the risk-free rate.
- That is, the expected return of an asset j is a linear function of the asset's sensitivities to the K factors.

Arbitrage pricing theory

- From APT, an asset's returns can be forecasted with the linear relation of an asset's expected returns and the macroeconomic factors that affect the asset's risk.
- The risky asset returns are said to follow a factor intensity structure if they can be expressed as:

$$r_j = a_j + \beta_{j1} f_1 + \beta_{j2} f_2 + \cdots + \beta_{jn} f_n + \epsilon_j$$

  where a_j is a constant for asset j, f_n is a systematic factor (return), and ϵ_j is the risky asset's idiosyncratic random shock with mean zero.
- Idiosyncratic shocks are assumed to be uncorrelated across assets and uncorrelated with the factors.

Arbitrage pricing theory

- Arbitrage is the practice of the simultaneous purchase and sale of an asset on different exchanges, taking advantage of slight pricing discrepancies to lock in a risk-free profit for the trade.
- In the APT, arbitrage is not a risk-free operation, but it does offer a high probability of success.
- What the arbitrage pricing theory offers traders is a model for determining the theoretical fair market value of an asset.
- Having determined that value, traders then look for slight deviations from the fair market price, and trade accordingly.

APT Assumptions

- The basic assumptions necessary for the APT are:
  - All securities have finite expected returns and variances
  - Some agent(s) can form well diversified portfolios
  - There are no market frictions (taxes, transaction costs, etc.)
- These assumptions are considerably weaker than what we needed for the CAPM
- The central idea behind the APT is to price assets relative to one another
  - The resulting restrictions on the prices will be based on no-arbitrage

No Arbitrage

- Absence of arbitrage in financial markets precludes the existence of any security with a zero price and a strictly positive payoff
  - Also, no portfolio can be created with this property
  - This implies that two securities, or portfolios, with the same payoffs must have the same price
  - No "free lunch"
- In an efficiently functioning financial market arbitragen opportunities should not exist
  - At least not for very long
  - This same no-arbitrage principle is also used extensively in the pricing of options and other derivative instruments
- Unlike the equilibrium arguments underlying the CAPM, this no-arbitrage rule only requires one smart investor

Arbitrage - Example

- Suppose that there are only two possible and equally likely states of nature for inflation and real interest rates: high or low
- Suppose that the four securities A, B, C and D are all currently selling for \$100, and that the known payoffs in each of the four possible states are:

State/ Stock	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
Int. Rate	5%	5%	0%	0%
Inflation	10%	0%	10%	0%
Prob.	0.25	0.25	0.25	0.25
Apex (A)	-20	20	40	60
Bull (B)	0	70	30	-20
Crush (C)	90	-20	-10	70
Dreck (D)	15	23	15	36

Arbitrage - Example

- Using the standard formulas, the expected returns, standard deviations, and correlations are:

Stock	Current Price	Expected Return(%)	Standard Dev. (%)	Correlation Matrix			
				A	B	C	D
A	100	25.00	29.58	1.00	-0.15	-0.29	0.68
B	100	20.00	33.91	-0.15	1.00	-0.87	-0.38
C	100	32.50	48.15	-0.29	-0.87	1.00	0.22
D	100	22.25	8.58	0.68	-0.38	0.22	1.00

- There is a simple arbitrage opportunity lurking in these numbers.

Arbitrage - Example

- Consider the return/payoff of an equally weighted portfolio of A, B and C, and compare this with the return/payoff of D:

State/ Port.	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
Portfolio	23.33	23.33	20.00	36.67
D	15	23	15	36

- The return/payoff on the portfolio made up of A, B and C is higher than D in all states of nature
- This is an arbitrage opportunity
- What would happen to the price of D in a well functioning market?

Specifying Risks

- We will assume that we know the probabilities of each of the different states of nature that can occur and what will happen in each of these different states
  - Of course, we don't know which state will actually occur
  - Factor models provide a convenient framework for formally operationalizing this
- Factor models provide a convenient framework for realistically describing how security returns move with economy wide risks, and in turn with one another.
- A factor model is a multivariate statistical/mathematical model for returns (return generating process).
- The sources of co-movement are called factors (systematic risks).

Specifying Risks

- The sensitivities of the assets to the different factors are called factor loadings (factor betas or factor sensitivities).
- The single-index model that we used to simplify the calculation of covariances and correlations is a one-factor model
- In the absence of arbitrage, we can price assets relative to one another based on their comovements with the factors which is the basic idea behind the APT.

Factor Model

- A K-factor model is formally defined by:

$$r_i = b_{i,0} + b_{i,1} f_1 + b_{i,2} f_2 + \ldots + b_{i,K} f_K + e_i$$

- The f_j 's represent the K common factors that affect most assets
  - Examples of macroeconomic factors might be economic growth, interest rates, inflation
- $b_{i,j}$ is the factor loading of asset i with respect to the j 'th factor
  - This tells you how much the asset's return goes up/down when the factor is one unit higher/lower than expected
- e_i accounts for the idiosyncratic risk of asset i
  - For example, e_i is likely negative when a firm loses a big contract
  - The factor model assumes that $\text{cov}\left(e_i, e_h\right) = 0$ for $i \neq h$

Factor Model

- It is often convenient to write the K-factor model in terms of the factor surprises $\tilde{f}_j$:

$$r_i = a_i + b_{i,1}\tilde{f}_1 + b_{i,2}\tilde{f}_2 + \ldots + b_{i,K}\tilde{f}_K + e_i$$

- By definition $E\left(\tilde{f}_j\right) = 0$
  - Instead of defining a factor directly as economic growth, it is defined as the deviation of economic growth from what was expected
- The intercept a_i in this representation is equal to $E\left(r_i\right)$
  - Why?
- Sometimes we will also assume that $\text{cov}\left(\tilde{f}_j, \tilde{f}_k\right) = 0$ for $j \neq k$

Factor Model

- Suppose that two factors have been identified for the U.S. economy: the growth rate of industrial production (IP) and the inflation rate (Inf).
- Industrial production is expected to grow at 4%, along with an inflation rate of 6%.
- A stock with a beta of 1.0 for IP and 0.4 for Inf is currently expected to provide an annual rate of return of 14%.
- If industrial production actually grows by 5% over the next year, while the inflation rate turns out to be 7%, what is your revised best estimate of the return on the stock?

Factor Model

- We know that $E(IP) = 4\%$, $b_{IP} = 1$, $E(\text{Inf}) = 6\%$, $b_{\text{Inf}} = 0.4$, and $E(r_i) = 14\%$.
- The actual realized factor values and surprises are:

$$\tilde{f}_{IP} = 0.05 - 0.04 = 0.01$$
$$\tilde{f}_{\text{Inf}} = 0.07 - 0.06 = 0.01$$

- Consequently, our best guess as to the return on the stock conditional on the actual realized industrial production growth rate (IP) and the inflation rate (Inf) is:

$$\begin{aligned} E\left(r_i \mid \tilde{f}_{IP}, \tilde{f}_{Inf}\right) &= 0.14 + 1 \cdot 0.01 + 0.4 \cdot 0.01 \\ &= 15.4\% \end{aligned}$$

  - Is this necessarily what the return on the stock will actually turn out to be?
  - There is still the idiosyncratic risk, e_i

Factor Model

- The factor model has important implications about asset return variances and covariances.
- Consider a two-factor model:

$$\begin{aligned}\operatorname{var}(r_i) &= \operatorname{var}(b_{i,1}f_1 + b_{i,2}f_2 + e_i) \\ &= b_{i,1}^2 \operatorname{var}(f_1) + b_{i,2}^2 \operatorname{var}(f_2) + 2 \cdot b_{i,1} \cdot b_{i,2} \cdot \operatorname{cov}(f_1, f_2) + \sigma_{e,i}^2\end{aligned}$$

- If the factors are uncorrelated:

$$\operatorname{var}(r_i) = b_{i,1}^2 \operatorname{var}(f_1) + b_{i,2}^2 \operatorname{var}(f_2) + \sigma_{e,i}^2$$

  - $b_{i,1}^2 \operatorname{var}(f_1) + b_{i,2}^2 \operatorname{var}(f_2)$ represents the systematic variance
  - $\sigma_{e,i}^2$ is the idiosyncratic variance
  - How does this expression compare to that for the single-index model?

Factor Model

- The general formula with K factors:

$$\text{var}(r_i) = \sum_{j=1}^{K} \sum_{k=1}^{K} b_{i,j} \cdot b_{i,k} \cdot \sigma_{j,k} + \sigma_{e,i}^2$$

  - $\sigma_{j,k}$ for $j \neq k$ denotes the covariance between the j 'th and k 'th factors
  - $\sigma_{j,j} \equiv \sigma_j^2$ denotes the variance of the j 'th factor
  - The systematic variance is given by $\sum_{j=1}^{K} \sum_{k=1}^{K} b_{i,j} \cdot b_{i,k} \cdot \sigma_{j,k}$
  - The idiosyncratic variance is $\sigma_{e,i}^2$

- If the factors are uncorrelated the formula simplifies to:

$$\text{var}(r_i) = \sum_{k=1}^{K} b_{i,k}^2 \cdot \sigma_k^2 + \sigma_{e,i}^2$$

Factor Model

- Now consider the covariance between stocks i and j for the two-factor model:

$$\begin{aligned}
\operatorname{cov}(r_i, r_j) =& \operatorname{cov}(b_{i,1} f_1 + b_{i,2} f_2 + e_i, b_{j,1} f_1 + b_{j,2} f_2 + e_j) \\
=& b_{i,1} b_{j,1} \operatorname{var}(f_1) + b_{i,2} b_{j,2} \operatorname{var}(f_2) + \\
& + (b_{i,1} b_{j,2} + b_{j,1} b_{i,2}) \operatorname{cov}(f_1, f_2)
\end{aligned}$$

- If the factors are uncorrelated:

$$\operatorname{cov}(r_i, r_j) = b_{i,1} b_{j,1} \operatorname{var}(f_1) + b_{i,2} b_{j,2} \operatorname{var}(f_2)$$

Diversified Portfolios

- The APT implies that only systematic risk should be rewarded
  - The idiosyncratic (non-systematic) risk can be diversified away
- The idiosyncratic variance of a portfolio with portfolio weights equals to w is:

$$\sum_{i=1}^{\mathrm{n}} w_i^2 \sigma_{e,i}^2 \leq \left(\max_i \sigma_{e,i}^2 \right) \sum_{i=1}^{n} w_i^2$$

- If the holdings are spread widely over the n assets (so that all the portfolio weights are close to $1/\mathrm{n}$) then the sum of squared portfolio weights approaches zero as n goes to infinity.
- As long as there is an upper bound on the idiosyncratic variances of the individual assets, the idiosyncratic variance of any well-spread portfolio will be near zero.
- Therefore, given a strict factor model and many assets, the idiosyncratic returns contain only diversifiable risk.

Diversified Portfolios

- A diversified portfolio is a portfolio that carries no idiosyncratic risk
  - For a K-factor model the return on a well diversified portfolio is given by:

$$r_p = E(r_p) + b_{p,1}\tilde{f}_1 + \ldots + b_{p,K}\tilde{f}_K$$

  - Note the actual return depends on the specific factor model and the corresponding factor surprises $\tilde{f}_j$
  - We will assume that investors can form such well diversified portfolios

Arbitrage - Example

- The APT pricing equation (which we will develop both intuitively and more formally) states that:

$$E(r_i) = \lambda_0 + \lambda_1 b_{i,1} + \ldots + \lambda_K b_{i,K}$$

  - The λ_j s are called factor risk premia and represent the extra return for an extra unit of the j th risk
  - There is one λ_j for each of the systematic risk factors, plus one additional λ_0
  - If there is a risk-free asset, then $\lambda_0 = r_f$
  - **If this equation is not satisfied for all well diversified portfolios, there is an arbitrage opportunity.**

Arbitrage - Example

- Suppose that Apple (AAPL) is currently (time 0) selling for \$100 per share
- Suppose also that Apple is going to pay a liquidating dividend in exactly one year from now (time 1), and this is the only future payment that Apple will ever make
- The dividend that Apple will pay is uncertain, and depends on how well the economy is doing
  - If the economy is in an expansion the dividend will be \$140
  - If the economy is in a recession the dividend will only be \$100
  - Assume that the two states are equally likely, so that the expected cash flow from Apple equals $E\left(CF_1^{AAPL}\right) = \$120$
- Note this setup corresponds to our previous definition of risk
  - We know exactly what will happen to Apple in each scenario, but we don't know which scenario will actually occur.

Arbitrage - Example

- For Apple, we have

	AAPL
Boom Payoff ($\Pr = 0.5$)	140
Bust Payoff ($\Pr = 0.5$)	100
$E\left(CF_1\right)$	120
Time 0 Price	100
Discount Rate	20%

- Given Apple's current price of \$100, investors are applying a discount rate of 20% to Apple's expected cash-flows
  - The rate that equates the time 1 expected cash flow $E\left(CF_1^{AAPL}\right) = \$120$ to the current price of \$100.
  - Alternatively, we may say that the expected return on Apple is 20%.
- In general, the expected return on an asset is equivalent to the discount rate that investors are applying to the expected future cash-flows
  - Expected returns, or equivalently discount rates, are determined by market forces and supply and demand

Arbitrage - Example

- Now let's consider a second stock, Starbucks (SBUX), which like Apple, is going to pay a liquidating dividend exactly one year from now:

	AAPL	SBUX
Boom Payoff ($\mathrm{Pr} = 0.5$)	140	160
Bust Payoff ($\mathrm{Pr} = 0.5$)	100	80
$E\left(CF_1\right)$	120	120
Time 0 Price	100	?
Discount Rate	20%	?

- Since Apple and Starbucks have the same expected cash-flows, one might naturally think that the price for Starbucks would also be \$100.

- But, that is not necessarily the case ...

Arbitrage - Example

- Looking a bit closer at the cash-flows for Starbucks, we see that even though the expected value is the same as for Apple, the distribution of the cash-flows for Starbucks is arguably worse
- Starbucks' payoff is lower in the bust/recession state, and higher in the boom/expansion state
- The rest of our portfolio is likely to do poorly during the recession (job prospects are likely to be poor as well), so we need the cash more then
- In the expansion the rest of our portfolio will probably do well (job opportunities are also likely to be better), so an extra dollar isn't worth as much then
- If Apple and Starbucks were selling at the same price, we would therefore want to buy Apple.
- Consequently, to induce investors to buy all of the outstanding shares, the current price of Starbucks must be lower than $100.

Arbitrage - Example

- Let's assume that investors are only willing to buy up all of Starbucks' shares if the current price of Starbucks is \$90

- The discount rate (or equivalently the expected return) for Starbucks is therefore:

$$E\left(r_{SBUX}\right) = \frac{E\left(CF_1^{SBUX}\right) - P_{SBUX}}{P_{SBUX}} = \frac{120 - 90}{90} = 0.333$$

	AAPL	SBUX
Boom Payoff $(\mathrm{Pr} = 0.5)$	140	160
Bust Payoff $(\mathrm{Pr} = 0.5)$	100	80
$E\left(CF_1\right)$	120	120
Time 0 Price	100	90
Discount Rate	20%	33.3%

- Now let's see how all of this relates to the APT equations

Arbitrage - Example

- Let's rely on the same idea behind the NBER (National Bureau of Economic Research) business cycle indicator to construct our factor.
- The NBER indicator is one if the economy is in an expansion, and zero if the economy is in a recession.
- Recall that we need the unexpected component of the business cycle for the factor.
- Following our example, let's assume that there is a 50/50 chance that at time 1 we will be in an expansion/recession, so that the expected value of the indicator is 0.5.
- This means that the surprise in the business-cycle factor has a value of $0.5 = 1 - 0.5$ in expansions, and $-0.5 = 0 - 0.5$ in recessions.

Arbitrage - Example

- Let's now calculate the factor loadings $b_{AAPL,BC}$ and $b_{SBUX,BC}$.
- Run a time series regression of Apple returns on the surprise in the factor:

$$r_{AAPL,t} = a_{AAPL} + b_{AAPL,BC} \cdot \tilde{f}_{BC,t} + e_{AAPL,t}.$$

- Since we only have two data points, we can fit the line perfectly:

$$\begin{aligned} 0.40 &= a_{AAPL} + b_{AAPL,BC} \cdot 0.5 \quad \text{(boom)} \\ 0.00 &= a_{AAPL} + b_{AAPL,BC} \cdot -0.5 \quad \text{(bust)} \end{aligned}$$

- Solving these two equations yields $a_{AAPL} = 0.20$ and $b_{AAPL,BC} = 0.4$.
- Solving the corresponding two equations for Starbucks yields $a_{SBUX} = 0.33$ and $b_{SBUX,BC} = 0.89$.
- The factor loadings $b_{AAPL,BC}$ and $b_{SBUX,BC}$ tell us how much systematic business cycle risk Apple and Starbucks are exposed to.
- The intercepts correspond to the expected returns $E(r_{AAPL}) = a_{AAPL} = 0.20$ and $E(r_{SBUX}) = a_{SBUX} = 0.33$.

Arbitrage - Example

- Finally, let's calculate the factor risk premia λ_0 and λ_{BC}
- To determine how investors price the risks we need the APT pricing equation:

$$E(r_i) = \lambda_0 + \lambda_{BC} \cdot b_{i,BC}$$

- λ_{BC} represents the price of business cycle risk
  - How much more investors discount the cash flows as a result of having one extra unit of business cycle factor risk
- λ_0 is the required return for a security with no risk, the time value of money.
- Solving these two equations based on our previous estimates for $b_{AAPL,BC}$ and $b_{SBUX,BC}$, yields $\lambda_0 = 0.0909$ and $\lambda_{BC} = 0.2727$.

$$0.20 = \lambda_0 + \lambda_{BC} \cdot 0.40$$
$$0.33 = \lambda_0 + \lambda_{BC} \cdot 0.89$$

Arbitrage - Example

- Arbitrage opportunities arise when the price of risk isn't consistent across all assets
  - Specifically, if the APT pricing equation:

    $$E(r_i) = \lambda_0 + \lambda_{BC} \cdot b_{i,BC}$$

    isn't satisfied for all assets.

- To illustrate, let's augment the previous example to include a risk-free asset with a return of 5%
  - Since $\lambda_0 = 0.0909$ based on the equations for Apple and Starbucks, we know that it is possible to combine Apple and Starbucks to create a synthetic risk-free portfolio with a return of 9.09%.
  - Borrowing money at 5% to invest in this synthetic risk-free portfolio therefore represents an arbitrage opportunity.

Arbitrage - Example

- To determine how much of Apple and Starbucks we have to buy/sell to construct this risk-free portfolio, we need to solve:

$$b_{p,BC} = w_{AAPL} \cdot b_{AAPL,BC} + (1 - w_{AAPL}) \cdot b_{SBUX,BC} = 0 \tag{5}$$

- Why is this portfolio risk-free?

$$\begin{aligned}
\text{Var}(r_p) &= \text{Var}(r_{APPL}) + \text{Var}(r_{SBUX}) + 2\,\text{Cov}(r_{APPL}, r_{SBUX}) \\
&= \left[w_{APPL}^2 b_{APPL,BC}^2 + w_{SBUX}^2 b_{SBUX,BC}^2 + 2 w_{APPL} w_{SBUX}\right] \text{Var}\left(\tilde{f}_{BC}\right) \\
&= \left(w_{APPL} b_{APPL,BC} + w_{SBUX} b_{SBUX,BC}\right)^2 \text{Var}\left(\tilde{f}_{BC}\right)
\end{aligned}$$

- Solving (5) yields:

$$w_{AAPL} = 1.8182 \quad w_{SBUX} = (1 - w_{AAPL}) = -0.8182.$$

- We can create an arbitrage portfolio by investing \$1.8182 in Apple, shorting \$0.8182 worth of Starbucks, and borrowing \$1 (or equivalently shorting \$1 worth of the risk-free asset)
  - This portfolio requires zero initial investment, but it has a positive payoff in all states

Arbitrage - Example

- To verify that this works, lets look at the return on the stock portion of the portfolio in the two possible states:

$$w_{AAPL} \cdot (140/100) + (1 - w_{AAPL}) \cdot (160/90) = 1.0909 \quad \text{(boom)}$$
$$w_{AAPL} \cdot (100/100) + (1 - w_{AAPL}) \cdot (80/90) = 1.0909 \quad \text{(bust)}$$

- The payoff from the zero-investment arbitrage portfolio is therefore $1.0909 - 1.05 = 0.0409$ in both the boom and bust states
  - So this portfolio is indeed risk-free
  - We can scale this up as much as we like
  - For a \$1 million investment in this long-short portfolio, we would get a risk-free payoff of $\$40,900$
  - We have created a "money pump" ...
- This works because the systematic risk isn't priced consistently across the different assets.

APT Glossary

- **Factors** (f_k's): Economy wide, or systematic, risks that impact the returns on most assets
- **Factor loadings** ($b_{i,k}$'s): How much a given asset i moves (on average) when the k 'th factor moves by one unit (or 1%)
- **Factor risk-premia** (λ_k's): The effect on the expected return (or discount-rate) of a one unit increase in the sensitivity to the k 'th factor
- **Arbitrage opportunity**: Positive payoff at zero cost
- **Arbitrage portfolio**: A well-diversified zero-cost portfolio with no systematic risk
  - Such portfolios may be used to spot arbitrage opportunities, and assess whether the systematic risks (factors) are priced consistently across all assets.

APT Summary

- The APT can be used in place of the CAPM for:
  - Calculating expected returns and cost of capital
  - Performance evaluation
  - Risk management
- Unlike the CAPM, the APT does not tell us what the systematic risks that drive the returns are
- The APT relies on a statistical factor model for describing the systematic risks and the co-movements among returns
  - The usefulness of the APT depends on getting the "right" factors
- So, how do you determine the factors?

APT Practical Implementation: Conditional Factor Models

- There is clear empirical support for time-varying means and variances in asset returns which leds to time-varying (or dynamic) factor models of returns

$$r_{i,t+1} = b_{i,t,0} + b_{i,t,1} f_{t+1,1} + b_{i,t,2} f_{t+1,2} + \ldots + b_{i,t,K} f_{t+1,K} + e_{i,t+1}$$

APT Practical Implementation: Macroeconomic Approach

- Treat the factors $f_{t+1,k}$ as the primitives
- Run time series regression to estimate $b_{i,t,k}$ for asset i recursively
- The factors might include macroeconomic variables like inflation and GDP growth
- These variables should be able to capture all the systematic risks in the economy
- The Chen, Roll and Ross (CRR) model is one of the first examples of this approach

APT Practical Implementation: "Fundamental" Approach

- Treat the factor loadings $b_{i,t,k}$ as the primitives
- Run cross sectional regression to estimate $f_{t+1,k}$ for factor k
- The loadings are inferred from "fundamental" information about the characteristics of the securities
- The corresponding factors must be constructed from indices based on these characteristics
- The Fama-French 3 -factor model is an example of this approach

APT Practical Implementation: Statistical Approach

- Treat both the factors and the loadings as unobservable/latent
- Principal Components Analysis (PCA) provides a statistical procedure for identifying the "best" set of factors and factor loadings based on a sample of historical returns
- The resulting factors will be portfolios, or linear combinations, of the different assets
- However, the resulting factors are often hard to interpret from an economic perspective

Test for CAPM - Main Issues

- How can we test CAPM model?
- Are those tests reliable?

Test for CAPM - CAPM is in an Ex-ante Form

- The basic CAPM model is

$$E(r_i) = r_f + \beta\left[E(r_m) - r_f\right]$$

- If lending and borrowing at the risk free rate is not possible or there is no risk free rate, then the CAPM becomes

$$E(r_i) = E(r_Z) + \beta_i\left[E(r_m) - E(r_Z)\right]$$

- These models are in an expectation form which are supposed to be about future values.
- An expectation means that we are thinking about the situation before the uncertainty is realized. We call this ex-ante.

Test for CAPM - CAPM is tested using Ex-post Data

- On the other hand, we usually perform tests of CAPM models using realized (historical) data. These values are said to be ex-post values. How can we justify using ex-post data to test ex-ante model?
- Solution: using the law of large number, we should be able to estimate consistently (unbiased) the ex-ante expectation using sample mean of ex-post data.
- CAPM is a two period model, then we impose an important assumption: **The CAPM holds period-by-period.**

Test for CAPM -Time Series

- For asset i,using

$$r_{it} - r_{ft} = \alpha_i + \beta_i \left(r_{mt} - r_{ft} \right) + \varepsilon_{it} \quad t = 1, \ldots, T \tag{6}$$

  to estimate α_i and β_i.

- If the CAPM is valid
  - Intercept term α_i should be zero for all assets.
  - The CAPM implies that the intercept estimate, $\hat{\alpha}_i$, should not be statistically significantly different from zero for all individual assets or portfolios.
- The estimates of α_i, however, could introduce bias and/or inefficiencies in the tests, because the residual errors of the time-series market model are correlated within certain groups.

Test for CAPM -Time Series

- To reduce this problem, Black, Jensen, and Scholes (1972) (hereafter BJS) used a method of grouping stocks into portfolios.
- If stocks are assigned toportfolios randomly, then the portfolio betas will tend to be clustered about one.
- The maximum possible cross-sectional dispersion in the sizes of the portfolio betas was desired.
- BJS used the individual stock betas estimated by equation (6) to rank the stocks and then formed 10 portfolios of stocks that were grouped into risk deciles.

Test for CAPM -Time Series

- The stocks in the high (low) beta portfolio will tend to have positive (negative) measurement errors.
- The stocks will be in the high (low) beta decile either because their beta has been estimated correctly and they belong in this decile, or, because their beta has been overestimated (underestimated) and the stock belongs in a lower (higher) decile, resulting in a net positive (negative) measurement error in the portfolio.
- These measurement errors cause α_i to be biased in the opposite direction. That is, the intercept estimate of the high (low) beta portfolio tends to be negatively (positively) biased.
- To avoid this selection bias, BJS calculated the rate of return on each portfolio in the following sixth year (that is, the year following the five-year period during which individual betas were calculated and the decile portfolios formed).

Test for CAPM - Cross Section

- Cross-sectional tests of the CAPM examine whether actual returns of assets are cross-sectionally linearly related with their actual betas.
- These tests estimate the intercept and slope coefficient of the following cross-sectional regression model:

$$\bar{r}_i - \bar{r}_f = \gamma_0 + \gamma_1 \hat{\beta}_i + \varepsilon_i, \quad i = 1, \ldots, N$$

  where $\bar{r}_i - \bar{r}_f$ denotes the average excess rate of return for asset i over a particular time period, and $\hat{\beta}_i$ is the beta of asset i estimated with the time-series market model.
- Cross-sectional tests of the CAPM are usually performed with a two-pass methodology.
  - In the first pass, betas are estimated from the market model using time-series return observations.
  - In the second pass, these estimated betas are used as the regressor in cross-sectional regressions.

Test for CAPM - Cross Section

- If CAPM is valid, the following finding should hold:
  - The intercept term, γ_0, should not be significantly different from zero.
  - The slope coefficient on the beta, γ_1, should be positive and equal to the market risk premium $(\bar{r}_m - \bar{r}_f)$
  - Beta should be the only variable that explains returns of risky assets. When other variables such as idiosyncratic risk or firm characteristics are added into the cross-sectional regression equation, these variables should have no explanatory power. That is, the coefficient on these variables should not be significantly different from zero.

Black-Jensen-Scholes (1972) - Portfolio Construction

- Data: 1926-1965 NYSE stocks, $E(r_m)$ are the returns on the NYSE Index
- Portfolio Construction
  - Start with 1926-1930 (monthly data, 60 months), run time series regression for each stock i to get the eatimator of β_i, $\hat{\beta}_i$

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,year}\left(r_{mt} - r_{ft}\right) + \epsilon_{it} \tag{7}$$

    which is estimated by OLS regression, $t = 1, ..., 60$,
  - In (7), the index *year* can take values $1931, 1932, \ldots, 1965$, $i = 1, 2, \ldots, N$, N is the number of securities. Here we take $year = 1931$.
  - Rank securities by $\hat{\beta}_{i,1931}$ and form into portfolios $1, 2, \ldots, 10$.

Black-Jensen-Scholes (1972) - Portfolio Returns Calculation

- Portfolio returns calculation
  - Calculate monthly returns for each of the 12 months of 1931 for each of the 10 portfolios

$$\bar{r}_{pt} = \frac{1}{N_p} \sum_{j=1}^{N_p} r_{jt} \tag{8}$$

where N_p is the number of securities in p-th portfolio, $p = 1, \ldots, 10$ is the portfolio index, and $t = (year - 1931) \times 12 + 1, \ldots, (year - 1931) \times 12 + 12$ is the month index.

- Recalculate $\hat{\beta}_{i,1932}$ using $1927 - 1931$ period, and repeat the previous steps to get the monthly returns for 1932 whiich are $\bar{r}_{p13}, \ldots, \bar{r}_{p24}$. (Rolling regression.)
- Finally, we can get 12 monthly returns per year for 35 years for each portfolio, $\bar{r}_{p1}, \ldots, \bar{r}_{p420}$, where $p = 1, \ldots, 10$.

Black-Jensen-Scholes (1972) - Time Series Test

- Compute the estimate of β_p for each portfolio, $\hat{\beta}_p$, by regressing $\bar{r}_{pt}$ on r_{mt} (totally 10 time series regression):

$$\bar{r}_{pt} - r_{ft} = \alpha_p + \beta_p (r_{mt} - r_{ft}) + e_{pt} \tag{9}$$

  for $t = 1, 2, \ldots, 420$.

- If the CAPM is valid
  - The intercept term α_p should be zero for all assets.
  - The intercept estimate, $\hat{\alpha}_i$, should not be statistically significantly different from zero for all individual aeests or portfolios.
- BJS (1972) undertake time-series tests of the CAPM by estimating α_p and β_p in equation (9) for each of the 10 portfolios for the entire 35 year (420 month) sample period.

$$H_0 : \alpha_p = 0, \quad p = 1, 2, \ldots, 10 \tag{10}$$

Black-Jensen-Scholes (1972) - Time Series Test

- Time-Series Test Results by Black, Jensen, and Scholes (1972)

TABLE 1.1 Time-Series Test Results by BJS (1972)

β-Sorted Portfolio	$\bar{r}_p - \bar{r}_f$	$\hat{\beta}_p$	$\hat{\alpha}_p(\times 100)$	R^2
high	0.0213	1.561	$-0.083(-0.43)$	0.963
2	0.0177	1.384	$-0.194(-1.99)$	0.988
3	0.0171	1.248	$-0.065(-0.76)$	0.988
4	0.0163	1.163	$-0.017(-0.25)$	0.991
5	0.0145	1.057	$-0.054(-0.87)$	0.992
6	0.0137	0.923	0.059(0.79)	0.983
7	0.0126	0.853	0.046(0.71)	0.985
8	0.0115	0.753	0.081(1.18)	0.979
9	0.0109	0.629	0.197(2.31)	0.956
low	0.0091	0.499	0.201(1.87)	0.898

* *t*-statistics in parentheses from two-tailed tests. Source: Black, Fischer, Michael C. Jensen, and Myron Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in Studies in the Theory of Capital Markets, ed. Michael C. Jensen. (New York: Praeger, 1972) 79-121.

Black-Jensen-Scholes (1972) - Time Series Test

- $\hat{\alpha}_p$ and $\hat{\beta}_p$ are inversely related which suggests that high-beta stocks tend to earn returns less than expected and low-beta stocks tended to earn more than expected.
- In the last point,**the returns that the stocks tend to earn** (eatimated by the time series regression) is

$$\hat{\alpha}_p + (\bar{r}_p - \bar{r}_f)\,\hat{\beta}_p,$$

  and **the returns expected from CAPM model** should be $(\bar{r}_p - \bar{r}_f)\,\hat{\beta}_p$. ($\bar{r}_p$ and $\bar{r}_f$ are defined in the next slide - Cross-Section Test.
- The CAPM does not hold, because
  - The intercept estimates $\hat{\alpha}_p$ of portfolios 2 and 9, ($\hat{\alpha}_2$) and ($\hat{\alpha}_9$) are significantly different from zero at a 5 percent significance level.
  - The intercept estimate of the lowest beta portfolio ($\hat{\alpha}_{10}$) is also significantly different from zero at a 10 percent significance level.

Black-Jensen-Scholes (1972) - Cross-Sectional Test

- For the entire sample, calculate mean portfolio returns for $p = 1, 2, \ldots, 10$ and the mean risk free rate as

$$\bar{r}_p = \frac{1}{420} \sum_{t=1}^{420} r_{pt}, \quad \bar{r}_f = \frac{1}{420} \sum_{t=1}^{420} r_{ft}. \tag{11}$$

- Do cross sectional regression for the portfolios (Regress $\bar{r}_p$ against $\hat{\beta}_p$ to estimate the ex-post SML)

$$\bar{r}_p - \bar{r}_f = \gamma_0 + \gamma_1 \hat{\beta}_p + e_p, \quad p = 1, 2, \ldots, 10. \tag{12}$$

- If the CAPM is valid, the following finding should hold:
  - The intercept term, γ_0, should not be significantly different from zero.
  - The slope coefficient on the beta, γ_1, should be positive and equal to the market risk premium $(\bar{r}_m - \bar{r}_f)$.

Black-Jensen-Scholes (1972) - Cross-Sectional Test

- Cross-Sectional Test Results by Black, Jensen, and Scholes (1972)

TABLE 1.2 Cross-Sectional Test Results by Black, Jensen, and Scholes (1972)

Test period	$\hat{\gamma}_0$	$t(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$\bar{r}_m - \bar{r}_f$ $(= \gamma_1)$	$t(\gamma_1 - \hat{\gamma}_1)$
1931.01-1965.12	0.0036	6.52	0.0108	0.0142	6.53
1931.01-1939.09	−0.0080	−4.45	0.0304	0.0220	−4.91
1939.10-1948.06	0.0044	3.20	0.0107	0.0149	3.23
1948.07-1957.03	0.0078	7.40	0.0033	0.0112	7.98
1957.04-1965.12	0.0102	18.89	−0.0012	0.0088	19.61

* *t*-statistics are from two-tailed tests. Source: F. Black, M. Jensen, and M. Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in Studies in the Theory of Capital Markets, ed. M. C. Jensen (New York: Praeger, 1972), 79-121.

Black-Jensen-Scholes (1972) - Cross-Sectional Test

- For three of the four subperiods and for the entire sample period, the estimate of γ_0 is significantly greater than zero and the estimate of γ_1 is significantly less than $(\bar{r}_m - \bar{r}_f)$.
- The intercept and slope terms in equation (12) are, relative to the CAPM, too high and too low, respectively.
- On average low-beta stocks earn more than the CAPM suggests, and high-beta stocks earn less than the CAPM suggests.
- BJS argue that these results are to be expected if the zero-beta version of the CAPM.

Fama and MacBeth (1973) - Portfolio Formation

- Data: January 1926-June 1968 NYSE stocks, $E(r_m)$ are the returns on the NYSE Index
- Portfolio Formation: The initial portfolios were formed by grouping all stocks into 20 portfolios based on their ranked $\hat{\beta}_i$
  - Over a 4 or 7-year formation period, $\hat{\beta}_i$ are estimated for each stock (the first formation period is 1926 - 1929)

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,year}\left(r_{mt} - r_{ft}\right) + \epsilon_{it} \tag{13}$$

    which is estimated by OLS regression, $t = 1926.01, ..., 1929.12$,
  - In (13), the index *year* denotes the formation period which takes values $26-29$, $27-33$, $31-37$, $35-41$,$39-45$,$43-49$, $47-53$, $51-57$, $55-61$, $i = 1, 2, \ldots, N$, N is the number of securities. Here we take $year = 26-29$.
  - Rank securities by $\hat{\beta}_{i,26-29}$ and form into portfolios $1, 2, \ldots, 20$.

Fama and MacBeth (1973) - Initial Beta Estimation

- Initial Beta Estimation:
  - Over subsequent 5 -year estimation periods (1930-1934) following the formation period, $\hat{\beta}_i$ are re-estimated for each stock:
  $$r_{it} - r_{ft} = \alpha_i + \beta_{i,year.m}\left(r_{mt} - r_{ft}\right) + \epsilon_{it} \tag{14}$$
  which is estimated by OLS regression, $t = 1930.01, ..., 1934.12$,
  - In (14), the index $year.m$ denotes the last month of the estimation period which takes values 1934.12, 1938.12, 1942.12, 1946.12,1950.12,1954.12, 1958.12, 1962.12, 1966.12, $i = 1, 2, \ldots, N$, N is the number of securities. Here we take $year.m = 1934.12$.
  - $\hat{\beta}_p$ is re-computed by averaging over each of the 20 portfolios using $\hat{\beta}_{i,year.m}$. Here we denote $\hat{\beta}_p$ as $\hat{\beta}_{p,year.m}$. For example
  $$\hat{\beta}_{p,1934.12} = \frac{1}{N_p}\sum_{i=1}^{N_p}\hat{\beta}_{i,1934.12} \tag{15}$$

Fama and MacBeth (1973) - Beta Recomputation

- Beta Recomputation:
  - Over another final subsequent 4-year testing period (such 1935-1938), $\hat{\beta}_{p,t}$ were reaveraged monthly from 1935.01 - 1935.12 (without re-computing the $\hat{\beta}_{i,1934.12}$-component) to **allow for de-listing of firms**.
  - The individual $\hat{\beta}_{i,year.m}$ components were annually re-computed from the beginning of the estimation period to the end of the current year of the testing period by regression

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,1935.12}\left(r_{mt} - r_{ft}\right) + \epsilon_{it} \tag{16}$$

    where $t = 1930.01, \ldots, 1935.12$.
  - Then $\hat{\beta}_{i,1935.12}$ is used to compute $\hat{\beta}_{p,t}$ from 1936.01 - 1936.12.
  - Rolling to get $\hat{\beta}_{p,t}$ from 1937.01 - 1938.12.
- Repeat the above steps, we can collect $\hat{\beta}_{p,t}$ and portfolio returns r_{pt} from 1935.01 - 1968.06, totally 402 monthly data.

Fama and MacBeth (1973) - Fama-MacBech Regression

- Step 1: Estimating the following cross-sectional regression (CSR) equation for each month t using 20 portfolios of NYSE-listed stocks:

  $$r_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_{pt-1} + \gamma_{2t}\hat{\beta}^2_{pt-1} + \gamma_{3t}\bar{s}^2_{pt-1}\left(\hat{\epsilon}_i\right) + \eta_{pt}, \quad p = 1, \ldots, 20$$

  to obtain the estimates coefficients, $\hat{\gamma}_{0t}, \hat{\gamma}_{1t}, \hat{\gamma}_{2t}, \hat{\gamma}_{3t}$.
- We define $\hat{s}^2_{it-1}\left(\hat{\epsilon}_i\right)$ as

  $$\hat{s}^2_{it-1}\left(\hat{\epsilon}_i\right) = \frac{1}{dim\left(\hat{\epsilon}_i\right)}\hat{\epsilon}_i'\hat{\epsilon}_i$$

  where $\hat{\epsilon}_i = \left(\hat{\epsilon}_{i1}, \cdots, \hat{\epsilon}_{it}, \cdots\right)'$ which can be obtained from (16).
- Then we have

  $$\bar{s}^2_{pt-1}\left(\hat{\epsilon}_i\right) = \frac{1}{N_p}\sum \hat{s}_{it-1}\left(\hat{\epsilon}_i\right).$$

Fama and MacBeth (1973) - Fama-MacBech Regression

- Step 2: Testing
  - Use the averages of these estimated values $\left(\bar{\hat{\gamma}}_0, \bar{\hat{\gamma}}_1, \bar{\hat{\gamma}}_2, \bar{\hat{\gamma}}_3\right)$ as the ultimate estimates of the risk premiums, $\gamma_0, \gamma_1, \gamma_2, \gamma_3$.
  - t-statistics for testing the null hypothesis that $\gamma_{\mathrm{j}} = 0$ for $j = 0, 1, 2, 3$ are

$$t\left(\bar{\hat{\gamma}}_j\right) = \frac{\bar{\hat{\gamma}}_j}{s\left(\bar{\gamma}_j\right)/\sqrt{T}}$$

where $s\left(\bar{\hat{\gamma}}_i\right)$ is the standard deviation of the estimated gamma coefficient $\hat{\gamma}_{it}$, and T (402 month) is the number of the estimated gamma coefficients.

Fama and MacBeth (1973) - Fama-MacBech Regression

- The hypotheses to be tested are
  - Linearity: $\gamma_{2t} = 0$. The relationship between risk and return is linear. If the hypothesis is true, then the coefficient on the squared beta term should not be significantly different from zero.
  - Market beta is the only relevant measure of risk: $\gamma_{3t} = 0$. If this hypothesis is true, then the coefficient on the non- β variables should not be significantly different from zero.
  - The relationship between risk and return is positive: $\gamma_{1t} > 0$. If this hypothesis is true, then the coefficient on the beta variable should be significantly greater than zero.

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	0.0020	0.0114	−0.0026	0.0516		
	(0.55)	(1.85)	(−0.86)	(1.11)		
1935/1945	0.0011	0.0118	0.0009	0.0817		
	(0.13)	(0.94)	(−0.14)	(0.94)		
1946/1955	0.0017	0.0209	−0.0076	−0.0378		
	(0.44)	(2.39)	(−2.16)	(−0.67)		
1956.06/1968	0.0031	0.0034	−0.0000	0.0960		
	(0.59)	(0.34)	(−0.00)	(1.11)		

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	0.0049	0.0105	−0.0008			
	(1.92)	(1.79)	(−0.29)			
1935/1945	0.0074	0.0079	0.0040			
	(1.39)	(0.65)	(0.61)			
1946/1955	0.0002	0.0217	−0.0076			
	(−0.07)	(2.51)	(−2.83)			
1956.06/1968	0.0069	0.0040	0.0013			
	(1.56)	(0.42)	(0.29)			

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	0.0054	0.0072		0.0198		
	(2.10)	(2.20)		(−0.46)		
1935/1945	0.0017	0.0104		0.0841		
	(0.26)	(1.41)		(1.05)		
1946/1955	0.0110	0.0075		−0.1052		
	(3.78)	(1.47)		(−1.89)		
1956.06/1968	0.0042	0.0041		0.0633		
	(1.28)	(0.96)		(0.79)		

Fama and MacBeth (1973)

- For the entire period and two of the three subperiods, $\overline{\hat{\gamma}}_2$ was not significantly different from zero.
- $\overline{\hat{\gamma}}_3$ was not significantly different from zero in any of the three subperiods or the entire period.
- Then the risk-return relationship is linear and that nonsystematic risk is unimportant in asset pricing.

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	**0.0061**	**0.0085**			0.0048	0.0130
	(3.24)	(2.57)			(2.55)	(4.28)
1935/1945	0.0017	0.0104			0.0037	0.0195
	(0.26)	(1.41)			(0.82)	(2.54)
1946/1955	0.0110	0.0075			0.0078	0.0103
	(3.78)	(1.47)			(3.31)	(2.60)
1956.06/1968	0.0042	0.0041			0.0034	0.0095
	(1.28)	(0.96)			(1.39)	(2.92)

Fama and MacBeth (1973)

- For the entire period, γ_0 and γ_1 are significantly positive which seems to be consistent with both the traditional and zero-beta versions of the CAPM.
- $\overline{\hat{\gamma}}_0 - \bar{r}_f$ is positive and is significant for the entire period and period 1946/1955 which appears that the empirical version of the CAPM has an intercept above r_f as in BJS (1972).
- It can be seen that $\bar{r}_m - \bar{r}_f$ is notably greater than $\overline{\hat{\gamma}}_1$ over the entire test period and in each of the subperiods.
- From the above, FM's test results, like those of BJS, are consistent with the zero-beta version of the CAPM.

Fama and MacBeth (1973)

- FM regression is a cross-sectional test which is different from cross-sectional test of BJS (1972) in some aspects.
- The data used in BJS (1972) is $\left(r_{pt}, \hat{\beta}_p\right)$, and $\left(r_{pt}, \hat{\beta}_{pt}\right)$ in FM (1973).
- BJS (1972) first compute time series average of r_{pt}, $\bar{r}_p$, then run the cross-sectional regression by regressing $\bar{r}_p$ on $\hat{\beta}_p$.
- FM (1973) first run cross-sectional regression by regressing r_{pt} on $\hat{\beta}_{pt-1}$ for each t then average the cross-sectional results such as coefficients estimation and other regression results (R-squared, adjusted R-squared).