Lecture 1 Mean-Variance Theory and Capital Asset Pricing Model

Tao Zeng

SOE & AFR, Zhejiang University

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Mean-variance utility

- Economists want agents (people, firms, or institutions) to have as many choices as possible and to guide agents to make optimal choices.
- These choices reflect asset owners' preferences which are ultimately a collection of trade-offs such as portfolio weights, saving and assumption.
- The preferences are represented by utility, an index numerically describing preferences in the sense that decisions that are made by ranking or maximizing utilities fully coincide with the asset owner's underlying preferences.
- The utility function is defined as a function of wealth W, $U(W)$.

Mean-variance utility

- Expected utility combines probabilities of outcomes with how investors feel about these outcomes

$$U = E[U(W)] = \sum_s p_s U(W_s)$$

- The subscript s denotes outcome s happening, so expected utility multiplies the probability of it happening, p_s, with the utility of the event in that state, $U(W_s)$.
- To make choices, the asset owner maximizes expected utility:

$$\max_\theta E[U(W)]$$

  where θ are choice (or control) variables, such as asset allocation decisions (portfolio weights).

Mean-variance utility

- Mean-variance utility (Markowitz, 1952) is given by:

$$U = E(r_p) - \frac{\gamma}{2}\,\mathrm{var}(r_p)$$

  where r_p is the return of the investor's portfolio and γ is coefficient of risk aversion.
  - Asset owners care only about means (which they like), and variances (which they dislike).
  - Define bad times as low means and high variances.
- Levy and Markowitz (1972) proved that any expected utility function can be approximated by a mean-variance utility function:

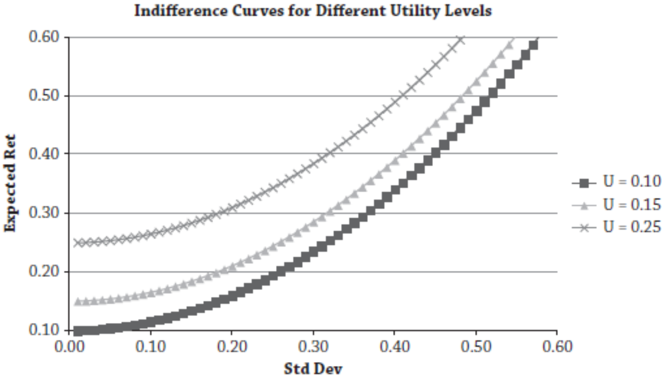
$$E[U(1+r_p)] \approx U(1+E(r_p)) + \frac{1}{2}U''(1+E(r_p))\,\mathrm{var}(r_p)$$

  where $U''(\bullet) < 0$ denotes the second derivative of the concave utility function.

Mean-variance utility

- Indifference curve represents one particular level of utility.
- Three different indifference curve for different utility levels for an asset owner with a risk aversion of $\gamma = 3$ in **mean-standard** deviation space.

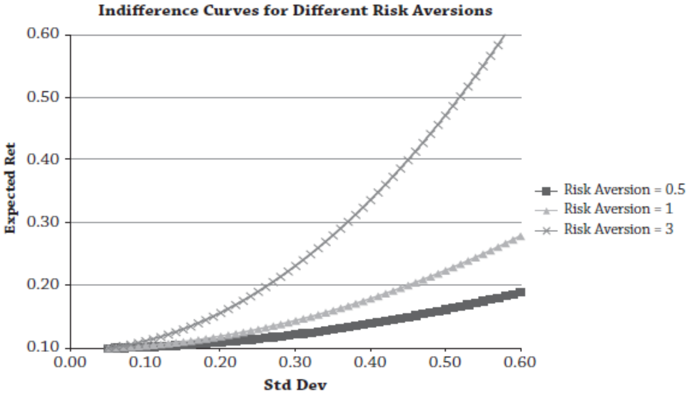
Figure: Indifference Curves for Different Utility Levels



Mean-variance utility

- The more risk averse an investor, the steeper the slope of his indifference curves.

Figure: Indifference Curves for Different Risk Aversion



Mean-variance frontiers

- Mean-variance frontiers depict the best set of portfolios that an investor can obtain in terms of means and variance.

- Minimize variance with constraints:

$$\begin{gathered} \min_{\{w_i\}} \operatorname{var}(r_p) \\ \text{subject to } E(r_p) = \mu^* \text{ and } \sum_i w_i = 1 \end{gathered}$$

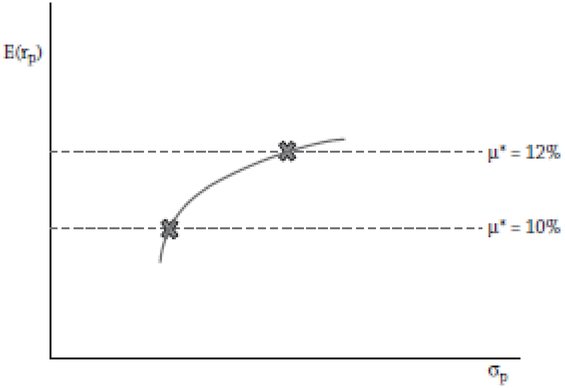
  where the portfolio weight for asset i is w_i.

- The combination of portfolio weights w_i that minimize the portfolio variance subject to two constraints:
  - The expected return on the portfolio is equal to a target return, μ^*.
  - The portfolio must be a valid portfolio.

Mean-variance frontiers

- The mean-variance frontier is a locus of points, where each point denotes the minimum variance achievable for each expected return.

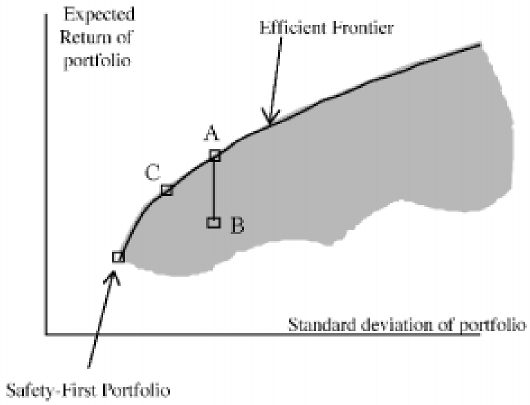
Figure: Mean-variance frontier



Mean-variance frontiers

- The feasible set of portfolios constructed from individual securities: The Risk-Return Diagram. The portfolios A and C are efficient, whereas B is dominated by A.

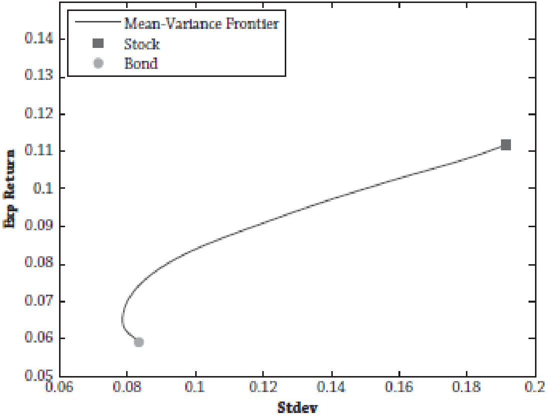
Figure: Efficient Mean-variance frontier



Mean-variance frontiers

- A stock and bond Mean-variance Frontier with U.S. data from January 1926 to December 2011.

Figure: Mean-variance frontier



Mean-variance frontiers

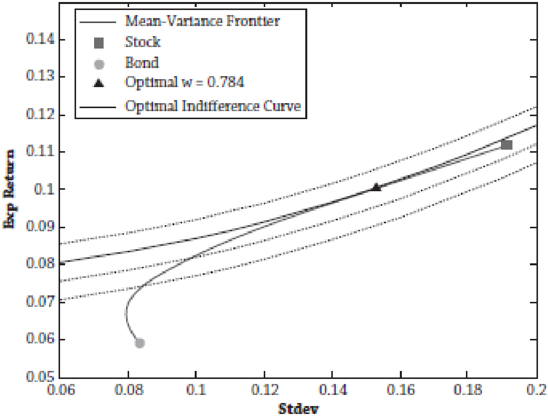
- Bonds lie on an inefficient part of the mean-variance frontier.
- By holding some equities, we can strictly increase the portfolio's expected return while lowering the portfolio's volatility.
- Equity diversifies a bond portfolio because equities have a low correlation with bonds; the correlation of equities with bonds in this sample is just 11%.
- Diversification benefits are measured by covariance correlations.

$$\begin{aligned} \text{var}(r_p) &= w_b^2 \, \text{var}(r_b) + w_s^2 \, \text{Var}(r_s) + 2 w_b w_s \, \text{cov}(r_b, r_s) \\ &= w_b^2 \, \text{var}(r_b) + w_s^2 \, \text{Var}(r_s) + 2 w_b w_s \rho_{b,s} \sigma_b \sigma_s \end{aligned}$$

Mean-variance optimization

- Maximizing utility is equivalent to finding the highest possible indifference curve.
- The highest indifference curve is tangent to the mean-variance frontier.

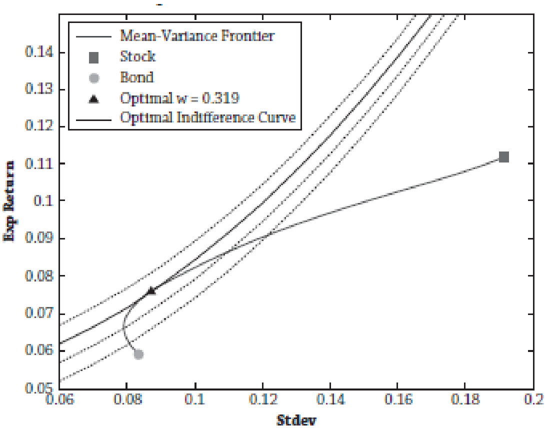
Figure: Optimal Asset Choice for Risk Aversion = 2



Mean-variance optimization

- The optimal portfolio holdings depend on the asset owner's degree of risk aversion.

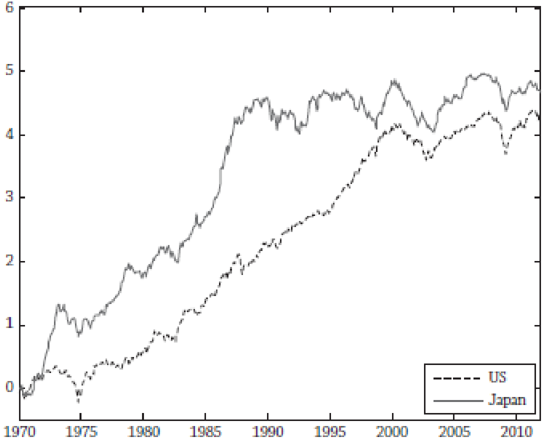
Figure: Optimal Asset Choice for Risk Aversion = 7



Mean-variance optimization

- The cumulated returns of U.S. and Japanese equities from January 1970 to December 2011.

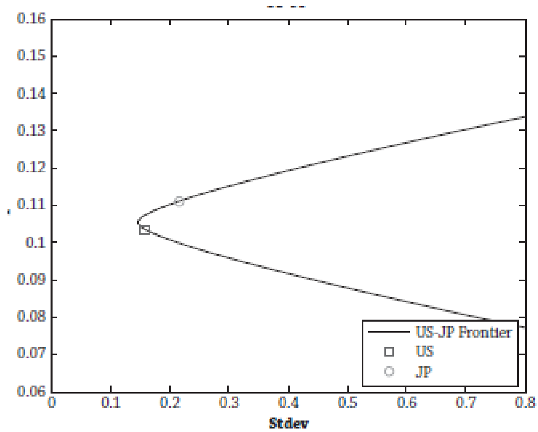
Figure: Cumulated returns



Mean-variance optimization

- The mean-variance frontier of US and JP equities from January 1970 to December 2011.

Figure: US-JP MVF

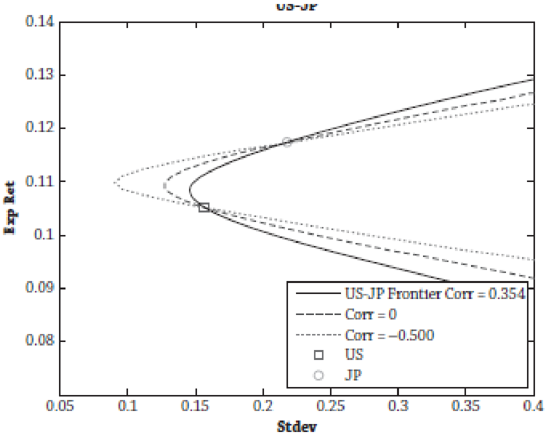


Mean-variance optimization

- Large diversification benefits correspond to low correlations.

$$
\begin{aligned}
\text{var}(r_p) &= w_{US}^2 \,\text{var}(r_{US}) + w_{JP}^2 \,\text{var}(r_{JP}) + 2 w_{US} w_{JP} \,\text{cov}(r_{US}, r_{JP}) \\
&= w_{US}^2 \,\text{var}(r_{US}) + w_{JP}^2 \,\text{var}(r_{JP}) + 2 w_{US} w_{JP} \rho_{US,JP} \sigma_{US} \sigma_{JP}
\end{aligned}
$$

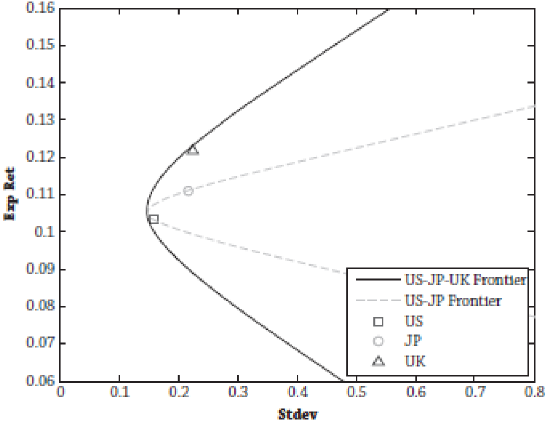
Figure: US-JP MVF



Mean-variance optimization

- More than two assets.

Figure: US-JP-UK MVF



Mean-variance optimization

- The frontier has expanded.
- All individual assets lie inside the frontier.
- Individual assets are dominated: diversified portfolios on the frontier do better than assets held individually.
- Diversification removes **asset-specific** risk and reduces the **overall** risk of the portfolio.

Mean-variance optimization

- **Two-fund theorem**: The entire efficient frontier can be generated from only two portfolios (funds).
  - Let $\mathbf{w}^1 = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1)$ be a solution to the Markowitz problem for a given expected rate of return $\bar{r}^1$, and $\mathbf{w}^2 = (\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2)$ be a solution to the Markowitz problem for a different given expected rate of return $\bar{r}^2$.
  - For any number α, the new portfolio $\alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$ is itself a solution to the Markowitz problem for expected rate of return $\alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$.
  - As α varies, $\bar{r} = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$ takes on all feasible values for expected rate of return; thus all solutions to the Markowitz problem can be constructed.

Mean-variance optimization

- The addition of a risk-free asset expands the investor's opportunities considerably which has no variance.
- Sharpe ratio (also known as the Sharpe index, the Sharpe measure, and the reward-to-variability ratio) measures the performance of an investment (e.g., a security or portfolio) compared to a risk-free asset, after adjusting for its risk.
    - It is defined as the difference between the returns of the investment and the risk-free return, divided by the standard deviation of the investment (i.e., its volatility).
    - It represents the additional amount of return that an investor receives per unit of increase in risk.

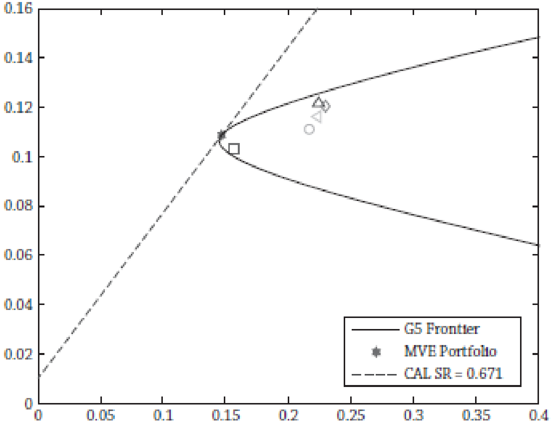
Mean-variance optimization

- When there is a risk-free asset, the investor proceeds in two steps:
  - Find the best risky asset portfolio. This is called the mean-variance efficient (MVE) portfolio, or tangency portfolio, and is the portfolio of risky assets that maximizes the Sharpe ratio.
  - Mix the best risky asset portfolio with the risk-free asset. This changes the efficient set from the frontier into a wider range of opportunities. The efficient set becomes a capital allocation line (CAL).
- The new efficient frontier is the line connecting the point $(0, r_f)$ to the unique point F (MVE,on the old efficient frontier) yielding a line tangent to the old efficient frontier.

Mean-variance optimization

- How to find MVE and CAL.
- The slope of the CAL represents the portfolio's Sharpe ratio.
- Since the CAL is tangent at the MVE, it represents the maximum Sharpe ratio that can be obtained by the investor.

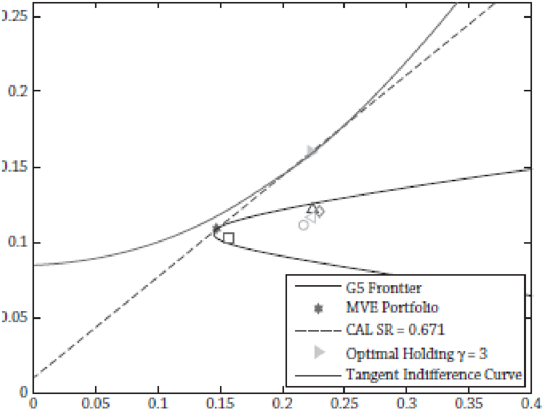
Figure: Asset Allocation G5 with Risk-Free Asset



Mean-variance optimization

- Finding the optimal combination of the MVE with the risk-free asset is equivalent to finding the point at which the highest possible indifference curve touches the CAL.
- The tangency point is the investor's optimal portfolio.

Figure: Asset Allocation G5 with Risk-Free Asset



Capital Asset Pricing Model - CAPM

- We assume an idealized framework for an open market place
  - All the risky assets refer to (say) all the tradeable stocks available to all.
  - In addition we have a risk-free asset (for borrowing and/or lending in unlimited quantities) with interest rate r_f.
  - All information is available to all such as covariances, variances, mean rates of return of stocks and so on.
  - Everyone is a risk-averse rational investor who uses the same financial engineering mean-variance portfolio theory from Markowitz.
- Everyone has the same assets to choose from, the same information about them, and the same decision methods, then:
  - Everyone has the same efficient frontier with risky assets (without risk-free asset).
  - Everyone has the same unique efficient fund (Mean-Variance Efficient portfolio, MVE) F of risky assets, hence the same capital asset line (CAL).
  - Everyone's optimal portfolio choice is a mixture of the risk-free asset and F (All that differs are the weights for the mixture due to the different risk-averse coefficients for different people).
- This efficient fund used by all is called the **market portfolio** and is denoted by M, and CAL is denoted as capital market line (CML)

Capital Asset Pricing Model - CAPM

- The weight for any asset in the market portfolio is given by its capital value (total worth of its shares) divided by the total capital value of the whole market (all assets together).
- The market portfolio must consist of all the risky assets.

Capital Asset Pricing Model - CAPM

- Let $(\sigma_M, \bar{r}_M)$ denote the point corresponding to the market portfolio M, all portfolios chosen by a rational investor will have a point $(\sigma, \bar{r})$ that lies on the so-called capital market line

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma$$

  which is the efficient frontier for investment.

- Then the expected return of any efficient portfolio which is on the capital market line, $\bar{r}_p^e$, satisfies

$$\bar{r}_p^e = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma_p^e$$

  where $(\bar{r}_M - r_f)/\sigma_M$ is called **the price of risk**.

- **What about the expected return of any portfolio (maybe inefficient)?**

Capital Asset Pricing Model - CAPM

- The Capital Asset Pricing Model

Theorem

For any asset i

$$E\left[r_i\right] - r_f = \beta_i \left(E\left[r_M\right] - r_f\right)$$

where

$$\beta_i = \frac{\sigma_{M,i}}{\sigma_M^2}$$

is called the beta of asset i*. This beta value serves as an important measure of risk for individual assets (portfolios) that is different from* σ_i^2*: it measures the nondiversifiable part of risk.*
More generally, for any portfolio $p = (\alpha_1, \ldots, \alpha_n)$ *of risky assets, its beta can be computed as a weighted average of individual asset betas:*

$$E\left[r_p\right] - r_f = \beta_p \left(E\left[r_M\right] - r_f\right)$$

where

$$\beta_p = \frac{\sigma_{M,p}}{\sigma_M^2} = \sum_{i=1}^{n} \alpha_i \beta_i .$$

Capital Asset Pricing Model - CAPM

- For the efficient portfolio

$$\bar{r}_p^e - r_f = \frac{\sigma_p^e}{\sigma_M}\left(\bar{r}_M - r_f\right) = \sigma_p^e \frac{\bar{r}_M - r_f}{\sigma_M}.$$

- Form CAPM, for any portfolio

$$\bar{r}_i - r_f = \rho_{M,i}\frac{\sigma_i}{\sigma_M}\left(\bar{r}_M - r_f\right) = \rho_{M,i}\sigma_i \frac{\bar{r}_M - r_f}{\sigma_M}$$

  where $\rho_{M,i}$ is the correlation coefficient between r_M and r_i

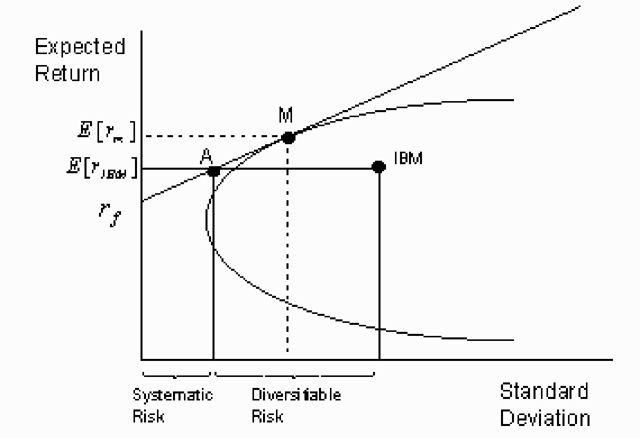
$$\rho_{M,i} = \frac{\sigma_{M,i}}{\sigma_M \sigma_i}.$$

- If σ_i is the asset i's total risk, then $\rho_{M,i}\sigma_i$ is the **Systematic risk** while $\left(1 - \rho_{M,i}\right)\sigma_i$ is the **unsystematic part**.
- If portfolio i is on the capital market line, $\rho_{M,i} = 1$.

Capital Asset Pricing Model - CAPM

- Let's show why assets that are not perfectly correlated with M do not fall on the CML by using an investment in IBM stock as an example.

Figure: Asset not perfectly correlated with M



Capital Asset Pricing Model - CAPM

- There are two ways to receive an expected return of $E[r_{IBM}]$: simply buy shares in IBM, or buy portfolio A.
- For a risk averse investor, portfolio A is preferred to an investment solely in IBM since it produces the same return with less risk.
- It is impossible to earn an expected return of $E[r_{IBM}]$ incurring less risk than that of portfolio A.
- The total risk of IBM can therefore be decomposed into two parts:
  - **Systematic risk**: the minimum risk required to earn that expected return
  - **Diversifiable risk**: the portion of the risk that can be eliminated, without sacrificing any expected return , simply by diversifying.
- Investors are rewarded for bearing this systematic risk, but they are not rewarded for bearing diversifiable risk, because it can easily be eliminated at no cost.

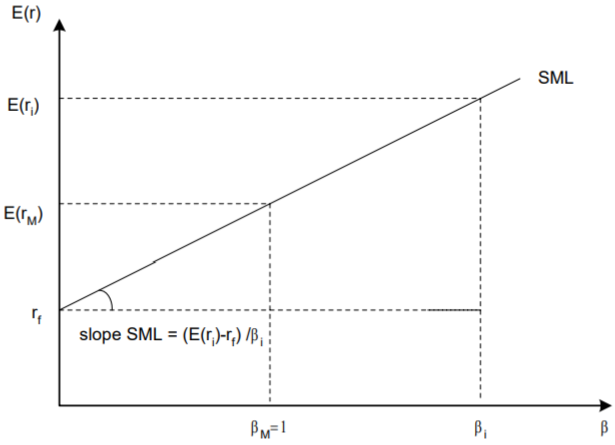
Capital Asset Pricing Model - CAPM

- For a given asset i, σ_i^2 tells us the risk associated with its own fluctuations about its mean rate of return, but not with respect to the market portfolio.
- We can view β_i as a measure of **nondiversifiable risk**, the correlated-with-the-market part of risk that we can't reduce by diversifying which is sometimes called **market or systematic risk**.
- It is not true in general that higher beta value β_i implies higher variance σ_i^2, but of course a higher beta value does imply a higher expected rate of return: you are rewarded (via a high expected rate of return) for taking on risk that can't be diversified away.

Capital Asset Pricing Model - CAPM

- Security market line is the relationship between expected return on an individual security and the beta of the security.

Figure: Security Market Line



Capital Asset Pricing Model - CAPM

- It is useful to compare the security market line to the capital market line.
  - The CML graphs the risk premiums of efficient portfolios (i.e., portfolios composed of the market and the risk-free asset) as a function of portfolio standard deviation. This is appropriate because standard deviation is a valid measure of risk for efficiently diversified portfolios that are candidates for an investor's overall portfolio.
  - The SML, in contrast, graphs individual asset risk premiums as a function of asset risk. The relevant measure of risk for individual assets held as parts of well-diversified portfolios is not the asset's standard deviation or variance; it is, instead, the contribution of the asset to the portfolio variance, which we measure by the asset's beta. The SML is valid for both efficient portfolios and individual assets.

Capital Asset Pricing Model - The Black CAPM

- The Black CAPM does not assumed that the investors can lend and borrow at a common risk-free rate.
- The market portfolio is still a mean-variance efficient portfolio.
- There exists a portfolio Z, which is named as **zero-beta portfolio** which has zero covariance with respect to the market portfolio.
- The Black CAPM takes the form as

$$E[r_i] = E[r_Z] + \beta_i (E[r_M] - E[r_Z]) \tag{1}$$

- The Black CAPM can be rewritten as

$$E[r_i] - r_f = E[r_Z] - r_f + \beta_i (E[r_M] - E[r_Z]) \tag{2}$$

- Note that $E[r_Z] - r_f > 0$ and $E[r_M] - E[r_Z] < E[r_M] - r_f$ which means that equation (**??**) is flatter than SML.

Arbitrage pricing theory

- In finance, arbitrage pricing theory (APT) is a multi-factor model for asset pricing which relates various macro-economic (systematic) risk variables to the pricing of financial assets.
- The APT model states that if asset returns follow a factor structure then the following relation exists between expected returns and the factor sensitivities:

$$\mathbb{E}\left(r_j\right) = r_f + \beta_{j1} RP_1 + \beta_{j2} RP_2 + \cdots + \beta_{jK} RP_K$$

  where β_{jk} is the sensitivity of the jth asset to factor k, also called factor loading, RP_k is the risk premium of the kth factor, r_f is the risk-free rate.
- That is, the expected return of an asset j is a linear function of the asset's sensitivities to the K factors.

Arbitrage pricing theory

- From APT, an asset's returns can be forecasted with the linear relation of an asset's expected returns and the macroeconomic factors that affect the asset's risk.
- The risky asset returns are said to follow a factor intensity structure if they can be expressed as:

$$r_j = a_j + \beta_{j1} f_1 + \beta_{j2} f_2 + \cdots + \beta_{jn} f_n + \epsilon_j$$

  where a_j is a constant for asset j, f_n is a systematic factor (return), and ϵ_j is the risky asset's idiosyncratic random shock with mean zero.
- Idiosyncratic shocks are assumed to be uncorrelated across assets and uncorrelated with the factors.

Arbitrage pricing theory

- Arbitrage is the practice of the simultaneous purchase and sale of an asset on different exchanges, taking advantage of slight pricing discrepancies to lock in a risk-free profit for the trade.
- In the APT, arbitrage is not a risk-free operation, but it does offer a high probability of success.
- What the arbitrage pricing theory offers traders is a model for determining the theoretical fair market value of an asset.
- Having determined that value, traders then look for slight deviations from the fair market price, and trade accordingly.

Test for CAPM - Main Issues

- How can we test CAPM model?
- Are those tests reliable?

Test for CAPM - CAPM is in an Ex-ante Form

- The basic CAPM model is

$$E(r_i) = r_f + \beta [E(r_m) - r_f]$$

- If lending and borrowing at the risk free rate is not possible or there is no risk free rate, then the CAPM becomes

$$E(r_i) = E(r_Z) + \beta_i [E(r_m) - E(r_Z)]$$

- These models are in an expectation form which are supposed to be about future values.
- An expectation means that we are thinking about the situation before the uncertainty is realized. We call this ex-ante.

Test for CAPM - CAPM is tested using Ex-post Data

- On the other hand, we usually perform tests of CAPM models using realized (historical) data. These values are said to be ex-post values. How can we justify using ex-post data to test ex-ante model?
- Solution: using the law of large number, we should be able to estimate consistently (unbiased) the ex-ante expectation using sample mean of ex-post data.
- CAPM is a two period model, then we impose an important assumption: **The CAPM holds period-by-period.**

Test for CAPM -Time Series

- For asset i,using

$$r_{it} - r_{ft} = \alpha_i + \beta_i \left(r_{mt} - r_{ft} \right) + \varepsilon_{it} \quad t = 1, \ldots, T \tag{3}$$

  to estimate α_i and β_i.

- If the CAPM is valid
  - Intercept term α_i should be zero for all assets.
  - The CAPM implies that the intercept estimate, $\hat{\alpha}_i$, should not be statistically significantly different from zero for all individual assets or portfolios.
- The estimates of α_i, however, could introduce bias and/or inefficiencies in the tests, because the residual errors of the time-series market model are correlated within certain groups.

Test for CAPM -Time Series

- To reduce this problem, Black, Jensen, and Scholes (1972) (hereafter BJS) used a method of grouping stocks into portfolios.

- If stocks are assigned toportfolios randomly, then the portfolio betas will tend to be clustered about one.

- The maximum possible cross-sectional dispersion in the sizes of the portfolio betas was desired.

- BJS used the individual stock betas estimated by equation (**??**) to rank the stocks and then formed 10 portfolios of stocks that were grouped into risk deciles.

Test for CAPM -Time Series

- The stocks in the high (low) beta portfolio will tend to have positive (negative) measurement errors.
- The stocks will be in the high (low) beta decile either because their beta has been estimated correctly and they belong in this decile, or, because their beta has been overestimated (underestimated) and the stock belongs in a lower (higher) decile, resulting in a net positive (negative) measurement error in the portfolio.
- These measurement errors cause α_i to be biased in the opposite direction. That is, the intercept estimate of the high (low) beta portfolio tends to be negatively (positively) biased.
- To avoid this selection bias, BJS calculated the rate of return on each portfolio in the following sixth year (that is, the year following the five-year period during which individual betas were calculated and the decile portfolios formed).

Test for CAPM - Cross Section

- Cross-sectional tests of the CAPM examine whether actual returns of assets are cross-sectionally linearly related with their actual betas.
- These tests estimate the intercept and slope coefficient of the following cross-sectional regression model:

$$\bar{r}_i - \bar{r}_f = \gamma_0 + \gamma_1 \hat{\beta}_i + \varepsilon_i, \quad i = 1, \ldots, N$$

  where $\bar{r}_i - \bar{r}_f$ denotes the average excess rate of return for asset i over a particular time period, and $\hat{\beta}_i$ is the beta of asset i estimated with the time-series market model.
- Cross-sectional tests of the CAPM are usually performed with a two-pass methodology.
  - In the first pass, betas are estimated from the market model using time-series return observations.
  - In the second pass, these estimated betas are used as the regressor in cross-sectional regressions.

Test for CAPM - Cross Section

- If CAPM is valid, the following finding should hold:
  - The intercept term, γ_0, should not be significantly different from zero.
  - The slope coefficient on the beta, γ_1, should be positive and equal to the market risk premium $(\bar{r}_m - \bar{r}_f)$
  - Beta should be the only variable that explains returns of risky assets. When other variables such as idiosyncratic risk or firm characteristics are added into the cross-sectional regression equation, these variables should have no explanatory power. That is, the coefficient on these variables should not be significantly different from zero.

Black-Jensen-Scholes (1972) - Portfolio Construction

- Data: 1926-1965 NYSE stocks, $E\left(r_m\right)$ are the returns on the NYSE Index
- Portfolio Construction
  - Start with 1926-1930 (monthly data, 60 months), run time series regression for each stock i to get the eatimator of β_i, $\hat{\beta}_i$

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,year}\left(r_{mt} - r_{ft}\right) + \epsilon_{it} \tag{4}$$

    which is estimated by OLS regression, $t = 1, ..., 60$,
  - In (**??**), the index *year* can take values $1931, 1932, \ldots, 1965$, $i = 1, 2, \ldots, N$, N is the number of securities. Here we take $year = 1931$.
  - Rank securities by $\hat{\beta}_{i,1931}$ and form into portfolios $1, 2, \ldots, 10$.

Black-Jensen-Scholes (1972) - Portfolio Returns Calculation

- Portfolio returns calculation
  - Calculate monthly returns for each of the 12 months of 1931 for each of the 10 portfolios

$$\bar{r}_{pt} = \frac{1}{N_p} \sum_{j=1}^{N_p} r_{jt} \tag{5}$$

where N_p is the number of securities in p-th portfolio, $p = 1, \ldots, 10$ is the portfolio index, and $t = (year - 1931) \times 12 + 1, \ldots, (year - 1931) \times 12 + 12$ is the month index.

- Recalculate $\hat{\beta}_{i,1932}$ using $1927 - 1931$ period, and repeat the previous steps to get the monthly returns for 1932 whiich are $\bar{r}_{p13}, \ldots, \bar{r}_{p24}$. (Rolling regression.)
- Finally, we can get 12 monthly returns per year for 35 years for each portfolio, $\bar{r}_{p1}, \ldots, \bar{r}_{p420}$, where $p = 1, \ldots, 10$.

Black-Jensen-Scholes (1972) - Time Series Test

- Compute the estimate of β_p for each portfolio, $\hat{\beta}_p$, by regressing $\bar{r}_{pt}$ on r_{mt} (totally 10 time series regression):

$$\bar{r}_{pt} - r_{ft} = \alpha_p + \beta_p \left(r_{mt} - r_{ft} \right) + e_{pt} \tag{6}$$

  for $t = 1, 2, \ldots, 420$.
- If the CAPM is valid
  - The intercept term α_p should be zero for all assets.
  - The intercept estimate, $\hat{\alpha}_i$, should not be statistically significantly different from zero for all individual aeests or portfolios.
- BJS (1972) undertake time-series tests of the CAPM by estimating α_p and β_p in equation (**??**) for each of the 10 portfolios for the entire 35 year (420 month) sample period.

$$H_0 : \alpha_p = 0, \quad p = 1, 2, \ldots, 10 \tag{7}$$

Black-Jensen-Scholes (1972) - Time Series Test

- Time-Series Test Results by Black, Jensen, and Scholes (1972)

TABLE 1.1 Time-Series Test Results by BJS (1972)

β-Sorted Portfolio	$\bar{r}_p - \bar{r}_f$	$\hat{\beta}_p$	$\hat{\alpha}_p(\times 100)$	R^2
high	0.0213	1.561	$-0.083(-0.43)$	0.963
2	0.0177	1.384	$-0.194(-1.99)$	0.988
3	0.0171	1.248	$-0.065(-0.76)$	0.988
4	0.0163	1.163	$-0.017(-0.25)$	0.991
5	0.0145	1.057	$-0.054(-0.87)$	0.992
6	0.0137	0.923	0.059(0.79)	0.983
7	0.0126	0.853	0.046(0.71)	0.985
8	0.0115	0.753	0.081(1.18)	0.979
9	0.0109	0.629	0.197(2.31)	0.956
low	0.0091	0.499	0.201(1.87)	0.898

* *t*-statistics in parentheses from two-tailed tests. Source: Black, Fischer, Michael C. Jensen, and Myron Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in Studies in the Theory of Capital Markets, ed. Michael C. Jensen. (New York: Praeger, 1972) 79-121.

Black-Jensen-Scholes (1972) - Time Series Test

- $\hat{\alpha}_p$ and $\hat{\beta}_p$ are inversely related which suggests that high-beta stocks tend to earn returns less than expected and low-beta stocks tended to earn more than expected.
- In the last point,**the returns that the stocks tend to earn** (eatimated by the time series regression) is

$$\hat{\alpha}_p + (\bar{r}_p - \bar{r}_f)\,\hat{\beta}_p,$$

  and **the returns expected from CAPM model** should be $(\bar{r}_p - \bar{r}_f)\,\hat{\beta}_p$. ($\bar{r}_p$ and $\bar{r}_f$ are defined in the next slide - Cross-Section Test.
- The CAPM does not hold, because
  - The intercept estimates $\hat{\alpha}_p$ of portfolios 2 and 9, ($\hat{\alpha}_2$) and ($\hat{\alpha}_9$) are significantly different from zero at a 5 percent significance level.
  - The intercept estimate of the lowest beta portfolio ($\hat{\alpha}_{10}$) is also significantly different from zero at a 10 percent significance level.

Black-Jensen-Scholes (1972) - Cross-Sectional Test

- For the entire sample, calculate mean portfolio returns for $p = 1, 2, \ldots, 10$ and the mean risk free rate as

$$\bar{r}_p = \frac{1}{420} \sum_{t=1}^{420} r_{pt}, \quad \bar{r}_f = \frac{1}{420} \sum_{t=1}^{420} r_{ft}. \tag{8}$$

- Do cross sectional regression for the portfolios (Regress $\bar{r}_p$ against $\hat{\beta}_p$ to estimate the ex-post SML)

$$\bar{r}_p - \bar{r}_f = \gamma_0 + \gamma_1 \hat{\beta}_p + e_p, \quad p = 1, 2, \ldots, 10. \tag{9}$$

- If the CAPM is valid, the following finding should hold:
  - The intercept term, γ_0, should not be significantly different from zero.
  - The slope coefficient on the beta, γ_1, should be positive and equal to the market risk premium $(\bar{r}_m - \bar{r}_f)$.

Black-Jensen-Scholes (1972) - Cross-Sectional Test

- Cross-Sectional Test Results by Black, Jensen, and Scholes (1972)

TABLE 1.2 Cross-Sectional Test Results by Black, Jensen, and Scholes (1972)

Test period	$\hat{\gamma}_0$	$t(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$\bar{r}_m - \bar{r}_f$ $(= \gamma_1)$	$t(\gamma_1 - \hat{\gamma}_1)$
1931.01-1965.12	0.0036	6.52	0.0108	0.0142	6.53
1931.01-1939.09	−0.0080	−4.45	0.0304	0.0220	−4.91
1939.10-1948.06	0.0044	3.20	0.0107	0.0149	3.23
1948.07-1957.03	0.0078	7.40	0.0033	0.0112	7.98
1957.04-1965.12	0.0102	18.89	−0.0012	0.0088	19.61

* *t*-statistics are from two-tailed tests. Source: F. Black, M. Jensen, and M. Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in Studies in the Theory of Capital Markets, ed. M. C. Jensen (New York: Praeger, 1972), 79-121.

Black-Jensen-Scholes (1972) - Cross-Sectional Test

- For three of the four subperiods and for the entire sample period, the estimate of γ_0 is significantly greater than zero and the estimate of γ_1 is significantly less than $(\bar{r}_m - \bar{r}_f)$.
- The intercept and slope terms in equation (**??**) are, relative to the CAPM, too high and too low, respectively.
- On average low-beta stocks earn more than the CAPM suggests, and high-beta stocks earn less than the CAPM suggests.
- BJS argue that these results are to be expected if the zero-beta version of the CAPM.

Fama and MacBeth (1973) - Portfolio Formation

- Data: January 1926-June 1968 NYSE stocks, $E(r_m)$ are the returns on the NYSE Index
- Portfolio Formation: The initial portfolios were formed by grouping all stocks into 20 portfolios based on their ranked $\hat{\beta}_i$
  - Over a 4 or 7-year formation period, $\hat{\beta}_i$ are estimated for each stock (the first formation period is 1926 - 1929)

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,year}(r_{mt} - r_{ft}) + \epsilon_{it} \tag{10}$$

    which is estimated by OLS regression, $t = 1926.01, ..., 1929.12$,
  - In (**??**), the index *year* denotes the formation period which takes values $26-29$, $27-33$, $31-37$, $35-41$,$39-45$,$43-49$, $47-53$, $51-57$, $55-61$, $i = 1, 2, \ldots, N$, N is the number of securities. Here we take $year = 26-29$.
  - Rank securities by $\hat{\beta}_{i,26-29}$ and form into portfolios $1, 2, \ldots, 20$.

Fama and MacBeth (1973) - Initial Beta Estimation

- Initial Beta Estimation:
  - Over subsequent 5 -year estimation periods (1930-1934) following the formation period, $\hat{\beta}_i$ are re-estimated for each stock:

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,year.m} (r_{mt} - r_{ft}) + \epsilon_{it} \tag{11}$$

    which is estimated by OLS regression, $t = 1930.01, ..., 1934.12$,
  - In (**??**), the index $year.m$ denotes the last month of the estimation period which takes values 1934.12, 1938.12, 1942.12, 1946.12,1950.12,1954.12, 1958.12, 1962.12, 1966.12, $i = 1, 2, \ldots, N$, N is the number of securities. Here we take $year.m = 1934.12$.
  - $\hat{\beta}_p$ is re-computed by averaging over each of the 20 portfolios using $\hat{\beta}_{i,year.m}$. Here we denote $\hat{\beta}_p$ as $\hat{\beta}_{p,year.m}$. For example

$$\hat{\beta}_{p,1934.12} = \frac{1}{N_p} \sum_{i=1}^{N_p} \hat{\beta}_{i,1934.12} \tag{12}$$

Fama and MacBeth (1973) - Beta Recomputation

- Beta Recomputation:
  - Over another final subsequent 4-year testing period (such 1935-1938), $\hat{\beta}_{p,t}$ were reaveraged monthly from 1935.01 - 1935.12 (without re-computing the $\hat{\beta}_{i,1934.12}$-component) to **allow for de-listing of firms**.
  - The individual $\hat{\beta}_{i,year.m}$ components were annually re-computed from the beginning of the estimation period to the end of the current year of the testing period by regression

$$r_{it} - r_{ft} = \alpha_i + \beta_{i,1935.12} \left(r_{mt} - r_{ft} \right) + \epsilon_{it} \tag{13}$$

    where $t = 1930.01, \ldots, 1935.12$.
  - Then $\hat{\beta}_{i,1935.12}$ is used to compute $\hat{\beta}_{p,t}$ from 1936.01 - 1936.12.
  - Rolling to get $\hat{\beta}_{p,t}$ from 1937.01 - 1938.12.
- Repeat the above steps, we can collect $\hat{\beta}_{p,t}$ and portfolio returns r_{pt} from 1935.01 - 1968.06, totally 402 monthly data.

Fama and MacBeth (1973) - Fama-MacBech Regression

- Step 1: Estimating the following cross-sectional regression (CSR) equation for each month t using 20 portfolios of NYSE-listed stocks:

  $$r_{pt} = \gamma_{0t} + \gamma_{1t}\hat{\beta}_{pt-1} + \gamma_{2t}\hat{\beta}^2_{pt-1} + \gamma_{3t}\bar{s}^2_{pt-1}\left(\hat{\epsilon}_i\right) + \eta_{pt}, \quad p = 1, \ldots, 20$$

  to obtain the estimates coefficients, $\hat{\gamma}_{0t}, \hat{\gamma}_{1t}, \hat{\gamma}_{2t}, \hat{\gamma}_{3t}$.
- We define $\hat{s}^2_{it-1}\left(\hat{\epsilon}_i\right)$ as

  $$\hat{s}^2_{it-1}\left(\hat{\epsilon}_i\right) = \frac{1}{dim\left(\hat{\epsilon}_i\right)}\hat{\epsilon}_i'\hat{\epsilon}_i$$

  where $\hat{\epsilon}_i = \left(\hat{\epsilon}_{i1}, \cdots, \hat{\epsilon}_{it}, \cdots\right)'$ which can be obtained from (**??**).
- Then we have

  $$\bar{s}^2_{pt-1}\left(\hat{\epsilon}_i\right) = \frac{1}{N_p}\sum \hat{s}_{it-1}\left(\hat{\epsilon}_i\right).$$

Fama and MacBeth (1973) - Fama-MacBech Regression

- Step 2: Testing
  - Use the averages of these estimated values $\left(\bar{\hat{\gamma}}_0, \bar{\hat{\gamma}}_1, \bar{\hat{\gamma}}_2, \bar{\hat{\gamma}}_3\right)$ as the ultimate estimates of the risk premiums, $\gamma_0, \gamma_1, \gamma_2, \gamma_3$.
  - t-statistics for testing the null hypothesis that $\gamma_{\mathrm{j}} = 0$ for $j = 0, 1, 2, 3$ are

$$t\left(\bar{\hat{\gamma}}_j\right) = \frac{\bar{\hat{\gamma}}_j}{s\left(\bar{\gamma}_j\right)/\sqrt{T}}$$

where $s\left(\bar{\hat{\gamma}}_i\right)$ is the standard deviation of the estimated gamma coefficient $\hat{\gamma}_{it}$, and T (402 month) is the number of the estimated gamma coefficients.

Fama and MacBeth (1973) - Fama-MacBech Regression

- The hypotheses to be tested are
  - Linearity: $\gamma_{2t} = 0$. The relationship between risk and return is linear. If the hypothesis is true, then the coefficient on the squared beta term should not be significantly different from zero.
  - Market beta is the only relevant measure of risk: $\gamma_{3t} = 0$. If this hypothesis is true, then the coefficient on the non- β variables should not be significantly different from zero.
  - The relationship between risk and return is positive: $\gamma_{1t} > 0$. If this hypothesis is true, then the coefficient on the beta variable should be significantly greater than zero.

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	0.0020	0.0114	-0.0026	0.0516		
	(0.55)	(1.85)	(-0.86)	(1.11)		
1935/1945	0.0011	0.0118	0.0009	0.0817		
	(0.13)	(0.94)	(-0.14)	(0.94)		
1946/1955	0.0017	0.0209	-0.0076	-0.0378		
	(0.44)	(2.39)	(-2.16)	(-0.67)		
1956.06/1968	0.0031	0.0034	-0.0000	0.0960		
	(0.59)	(0.34)	(-0.00)	(1.11)		

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	0.0049	0.0105	−0.0008			
	(1.92)	(1.79)	(−0.29)			
1935/1945	0.0074	0.0079	0.0040			
	(1.39)	(0.65)	(0.61)			
1946/1955	0.0002	0.0217	−0.0076			
	(−0.07)	(2.51)	(−2.83)			
1956.06/1968	0.0069	0.0040	0.0013			
	(1.56)	(0.42)	(0.29)			

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	0.0054	0.0072		0.0198		
	(2.10)	(2.20)		(−0.46)		
1935/1945	0.0017	0.0104		0.0841		
	(0.26)	(1.41)		(1.05)		
1946/1955	0.0110	0.0075		−0.1052		
	(3.78)	(1.47)		(−1.89)		
1956.06/1968	0.0042	0.0041		0.0633		
	(1.28)	(0.96)		(0.79)		

Fama and MacBeth (1973)

- For the entire period and two of the three subperiods, $\overline{\hat{\gamma}}_2$ was not significantly different from zero.
- $\overline{\hat{\gamma}}_3$ was not significantly different from zero in any of the three subperiods or the entire period.
- Then the risk-return relationship is linear and that nonsystematic risk is unimportant in asset pricing.

Fama and MacBeth (1973)

- Results

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\bar{\hat{\gamma}}_0 - \bar{r}_f$	$\bar{r}_m - \bar{r}_f$
1935.06/1968	**0.0061**	**0.0085**			0.0048	0.0130
	(3.24)	(2.57)			(2.55)	(4.28)
1935/1945	0.0017	0.0104			0.0037	0.0195
	(0.26)	(1.41)			(0.82)	(2.54)
1946/1955	0.0110	0.0075			0.0078	0.0103
	(3.78)	(1.47)			(3.31)	(2.60)
1956.06/1968	0.0042	0.0041			0.0034	0.0095
	(1.28)	(0.96)			(1.39)	(2.92)

Fama and MacBeth (1973)

- For the entire period, γ_0 and γ_1 are significantly positive which seems to be consistent with both the traditional and zero-beta versions of the CAPM.
- $\overline{\hat{\gamma}}_0 - \bar{r}_f$ is positive and is significant for the entire period and period 1946/1955 which appears that the empirical version of the CAPM has an intercept above r_f as in BJS (1972).
- It can be seen that $\bar{r}_m - \bar{r}_f$ is notably greater than $\overline{\hat{\gamma}}_1$ over the entire test period and in each of the subperiods.
- From the above, FM's test results, like those of BJS, are consistent with the zero-beta version of the CAPM.

Fama and MacBeth (1973)

- FM regression is a cross-sectional test which is different from cross-sectional test of BJS (1972) in some aspects.
- The data used in BJS (1972) is $\left(r_{pt}, \hat{\beta}_p\right)$, and $\left(r_{pt}, \hat{\beta}_{pt}\right)$ in FM (1973).
- BJS (1972) first compute time series average of r_{pt}, $\bar{r}_p$, then run the cross-sectional regression by regressing $\bar{r}_p$ on $\hat{\beta}_p$.
- FM (1973) first run cross-sectional regression by regressing r_{pt} on $\hat{\beta}_{pt-1}$ for each t then average the cross-sectional results such as coefficients estimation and other regression results (R-squared, adjusted R-squared).