PDE Note

Shuo Wang

Feb. 2024

Contents

0.1 Introduction

This is a lecture note to Partial Difference Equations, the reference of which is Lawrence C. Evans's book: Partial Difference Equations.

Chapter 1

Notation and Important Results

1.1 Basic Notations

1.1.1 Notation for Matrices

1. We write $A = ((a_{ij}))$ to mean A is an $m \times m$ matrix with the (i, j)-th entry a_{ij}. A diagonal matrix is denoted by $\mathrm{diag}(d_1, \ldots, d_n)$.

2. $\mathbb{M}^{m\times m} = \{$space of real $m \times m$ matrices$\}$. $\mathbb{S}^n = \{$space of real, symmetric $n \times n$ matrices$\}$.

3. $\mathrm{tr}A$ = trace of the matrix A.

4. $\det A$ = determinant of the matrix A.

5. $\mathrm{cof}A$ = cofactor matrix of A.

6. A^T = transpose of the matrix A.

7. If $A = ((a_{ij}))$ and $B = ((b_{ij}))$ are $m \times m$ matrices, then

$$A : B = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}$$

$$\|A\| = (A : A)^{1/2} = (\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2)^{1/2}$$

8. If $A \in \mathbb{S}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the corresponding quadratic form is $x \cdot Ax = \sum_{i,j=1}^{n} a_{ij}x_ix_j$.

9. If $A, B \in \mathbb{S}^n$, we write

$$A \geqslant B$$

to mean that $A - B$ is semi-positive definite. In particular, under this notation,

$$A \geqslant \theta I$$

means $x \cdot Ax \geqslant \theta \|x\|^2$ for all $x \in \mathbb{R}^n$.

1.1.2 Geometric Notations

1. $\mathbb{R}^n$ = n-dimensional real Euclidean spaces, $\mathbb{R} = \mathbb{R}^1$. $\mathbb{S}^{n-1} = \partial B(0,1)$ = $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.
2. $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ = i-th standard coordinate vector.
3. $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ = open upper half-space. $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.
4. A point in $\mathbb{R}^{n+1}$ will often be denoted as $(x, t) = (x_1, \dots, x_n, t)$, and we usually interpret $t = x_{n+1}$ as time. A point $x \in \mathbb{R}^n$ will sometimes be written $x = (x', x_n)$ for $x' = (x_1, \dots, x_{n-1})$.
5. U, V and W usually denote open subsets of $\mathbb{R}^n$. We write

$$V \Subset U$$

to mean $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact, and say V is **compactly contained in** U.

6. $U_T = U \times (0, T]$.
7. $\Gamma_T = \bar{U}_T - U_T$ = parabolic boundary of U_T.
8. $\bar{B}(x, r) = \{y \in \mathbb{R} : \|x - y\| \leqslant r\}$ = closed ball in $\mathbb{R}^n$ with center x and radius $r > 0$.
9. $B(x, r)$ = (open) ball with center x and radius $r > 0$.
10. $C(x, t; r) = \left\{y \in \mathbb{R}^n, s \in \mathbb{R} : \|x - y\| \leqslant r, t - r^2 \leqslant s \leqslant t\right\}$ = closed cylinder with top center (x, t) and radius $r > 0$, and height r^2.
11. $\alpha(n)$ = volume of unit ball $B(0,1)$ in $\mathbb{R}^n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$. $n\alpha(n)$ = surface area of unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$.

1.1.3 Notation for Functions

1. If $u : U \to \mathbb{R}$, we write

$$u(x) = u(x_1, \dots, x_n)(x \in U)$$

We say u is **smooth** provided u is infinitely differentiable.

2. If u, v are two functions, we write

$$u \equiv v$$

to mean that u is identically equal to v. We use the notation

$$u := v$$

to define u as equaling v. The support of a function u is denoted

$$\mathrm{spt} u$$

3. $u^+ = \max(u, 0), u^- = -\min(u, 0), u = u^+ - u^-, |u| = u^+ + u^-$. The **sign** function is

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

4. If $u : U \to \mathbb{R}^m$, we write

$$u(x) = (u^1(x), \ldots, u^m(x))(x \in U)$$

The function u^k is the k-th **component** of u.

5. If Σ is a smooth $(n-1)$-dimensional surface in $\mathbb{R}^n$, we write

$$\int_\Sigma f dS$$

for the integral of f over Σ, with respect to $(n-1)$-dimensional surface measure. If C is a curve in $\mathbb{R}^n$, we denote by

$$\int_C f dl$$

the integral of f over C with respect to arclength.

6. $⨍_E f dy := \dfrac{1}{m(E)} \int_E f dm$ = average of f over E, provided $m(E) > 0$.

7. For a Lipschitz continuous function $u : U \to \mathbb{R}$, we write

$$\mathrm{Lip}[u] := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|}.$$

8. The convolution of the functions f, g is denoted as

$$f * g$$

1.1.4 Notation for Derivatives

Assume $u : U \to \mathbb{R}$, $x \in U$

1. $\partial_i = \dfrac{\partial}{\partial x_i}$ $\partial_i u(x) = \dfrac{\partial u}{\partial x_i}(x) = \lim\limits_{h\to 0}\dfrac{u(x+he-i)-u(x)}{h}$, provided this limit exists.

2. $\partial_{i,j}u = \dfrac{\partial^2 u}{\partial x_i \partial x_j}, \partial_{i,j,k}u = \dfrac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}$, etc.

3. A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_i is a nonnegative integer, is called a **multiindex** of order
$$|\alpha| = \alpha_1 + \ldots + \alpha_n.$$

4. Given a multiindex α, define $D^\alpha u(x) := \dfrac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\ldots \partial x_n^{\alpha_n}} = \partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}$.

5. If k is a nonnegative integer,
$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\},$$
is the set of all partial derivatives of order k. Assigning some ordering to the various partial derivatives, we can also regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$.
$$\left\|D^k u\right\| = (\sum_{|\alpha|=k}\|D^\alpha u\|^2)^{1/2}.$$

6. (special cases): If $k = 1$, we regard the elements of Du as being arranged in a vector:
$$Du := (\partial_1 u, \dots, \partial_n u) = \textbf{ gradient vector}$$
Therefore $Du \in \mathbb{R}^n$. We will sometimes also write
$$u_r := \frac{x}{\|x\|}\cdot Du$$
for the radial derivative of u.
If $k = 2$, we regard the elements of D^2u as being arranged in a matrix:
$$D^2 u := \begin{pmatrix} \partial_{1,1}u & \ldots & \partial_{1,n}u \\ & \ddots & \\ \partial_{n,1}u & \ldots & \partial_{n,n}u \end{pmatrix} = \textbf{ Hessian matrix}$$
Therefore $D^2 u \in \mathbb{S}^n$.
$$\triangle u = \sum_{i=1}^n \partial_{i,i}u = \text{div}Du = \text{tr}D^2u = \textbf{ Laplancian} \text{ of } u.$$

1.1.5 Function Spaces

1. $C(U) = \{u : U \to \mathbb{R} : u \text{ continuous}\}$. $C(\bar{U}) = \{u \in C(U) : u$ is uniformly continuous on bounded subsets of $C^k(U) = \{u : U \to \mathbb{R} : u$ is $k-$ times continuously differentiable$\}$. $C^k(\bar{U}) =$
$\left\{u \in C^k(U) : D^\alpha u \text{ continuously extends to } \bar{U} \text{ for each multiindex } \alpha, |\alpha| \leqslant k\right\}$.
$C^\infty(U) = \{u : U \to \mathbb{R} : u \text{ is infinitely differentiable}\} = \bigcap\limits_{k=0}^{\infty} C^k(U)$. $C^\infty(\bar{U}) =$
$\bigcap\limits_{k=0}^{\infty} C^k(\bar{U})$. $C_c(U), C_c^k(U)$, etc, denote these functions in $C(U), C^k(U)$, etc.
with **compact support**. $L^p(U) = \left\{u : U \to \mathbb{R} : u \text{ is measurable}, \|u\|_p < \infty\right\}$.
$L^\infty(U) = \{u : U \to \mathbb{R} : u \text{ is measurable}, \|u\|_\infty < \infty\}$. $L^p_{\text{loc}} = \{u : U \to \mathbb{R} : u \in L^p(V) \text{ for each } V \Subset U\}$.

1.1.6 Vector-Valued Function

1. If now $m textgreater 1$ and $u : U \to \mathbb{R}$, $u = (u^1, \ldots, u^n)$, we define

$$D^\alpha u = (D^\alpha u^1, \ldots, D^\alpha u^n) \text{ for each multiindex } \alpha.$$

Then

$$D^k u = \{D^\alpha u : |\alpha| = k\}$$

and

$$\left\|D^k u\right\| := (\sum_{|\alpha|=k} \|D^\alpha u\|^2)^{1/2}$$

2. In the special case $k = 1$, we write

$$Du := \begin{pmatrix} \partial_1 u^1 & \ldots & \partial_n u^1 \\ & \ldots & \\ \partial_1 u^m & \ldots & \partial_n u^m \end{pmatrix} = \textbf{ gradient matrix } .$$

3. If $m = n$, we have

$$\text{div} u := \text{tr} Du = \sum_{i=1}^{n} \partial_i u^i = \textbf{ divergence } \text{ of } u.$$

4. The space $C(U; \mathbb{R}^m), L^p(U; \mathbb{R}^m)$, etc. consists of those functions $u : U \to \mathbb{R}$, $u = (u^1, \ldots, u^m)$, with $u^i \in C(U), L^p(U)$, etc.

Chapter 2

L^p Spaces

L^p spaces are a class of Banach spaces of functions whose norms are defined in terms of integrals and which generalize the L^1 spaces. They play a central role in modern analysis.

2.1 Basic Theory of L^p Spaces

Chapter 3

Sobolev Spaces

This chapter mostly develops the theory of Sobolev spaces, which turns out often to be the proper settings in which to apply ideas of functional analysis to glean information concerning PDE. Our overall point of view of eventual applications to rather wide classes of PDEs is the following: Our purpose will be to take various specific PDE and to recast them abstractly as operators acting on appropriate linear spaces. We can symbolically write

$$A : X \to Y,$$

where the operator A encodes the structure of the PDEs, including possibly boundary conditions, etc., and X, Y are spaces of functions. The great advantage is that once our PDE problem has been suitably interpreted in this form, we can often employ the general and elegant principles of functional analysis to study the solvability of various equations involving A. We will later see that the really hard work is not so much the invocation of functional analysis, but rather finding the "right" spaces X, Y and the "right" abstract operators A. Sobolev spaces are designed precisely to make all this work out properly, and so these are usually the proper choices for X, Y.

3.1 Hölder Spaces

Assume $U \subseteq \mathbb{R}^n$ is open and $0 < \gamma \leqslant 1$. A function $u : U \to \mathbb{R}$ is **Lipschitz continuous** if

$$|u(x) - u(y)| \leqslant C\,|x - y| \tag{3.1}$$

for all $x, y \in U$ and some constant C. This relation provides a uniform modulus of continuity which can generalize to functions that are **Hölder continuous with exponent γ**:

$$|u(x) - u(y)| \leqslant C\,|x - y|^{\gamma} \tag{3.2}$$

for all $x, y \in U$, some $0 < \gamma \leqslant 1$, and some constant C.

If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\bar{U})} := \sup_{x \in U} |u(x)|\,.$$

The γ-th Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|} \right\},$$

and the γ-th Hölder norm is

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}.$$

The **Hölder space**

$$C^{k,\gamma}(\bar{U})$$

consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$$

is finite.

$C^{k,\gamma}(\bar{U})$ consists of those functions u that are k-times continuously differentiable and whose k-th partial derivatives are bounded and Hölder continuous with exponent γ. Such functions are well-behaved and $C^{k,\gamma}(\bar{U})$ possesses a good mathematical structure:

Theorem (Hölder Spaces are Function Spaces). $C^{k,\gamma}(\bar{U})$ is a Banach space.

Proof. We write $\|\cdot\|$ for $\|\cdot\|_{C^{k,\gamma}(\bar{U})}$, for simplicity. i: We check that $\|\cdot\|$ is a norm:

1. It is obvious that $\|\cdot\| \geqslant 0$. If $\|u\| = 0$ for some u, then $\|u\|_{C(\bar{U})} = 0$, which implies that $u \equiv 0$ on $\bar{U}$.

2. $\|\lambda u\| = |\lambda| \, \|u\|$ for $\lambda \in \mathbb{R}$ is obvious.

3.

$$\begin{aligned}
\|u + v\| &= \sum_{|\alpha| \leqslant k} \|D^\alpha u + D^\alpha v\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u + D^\alpha v]_{C^{0,\gamma}(\bar{U})} \\
&\leqslant \sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| \leqslant k} \|D^\alpha v\|_{C(\bar{U})} + \sum_{|\alpha| = k} \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) + v(x) - u(y) - v(y)|}{|x - y|} \right\} \\
&\leqslant \sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| \leqslant k} \|D^\alpha v\|_{C(\bar{U})} \\
&\quad + \sum_{|\alpha| = k} \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|} \right\} + \sum_{|\alpha| = k} \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|v(x) - v(y)|}{|x - y|} \right\} \\
&= \|u\| + \|v\|
\end{aligned}$$

The first inequality follows from the fact that $\|\cdot\|_{C(\bar{U})}$ is a norm.

4. We choose a Cauchy sequence $\{u_n\}$ from $C^{k,\gamma}(\bar{U})$, then $\sum_{|\alpha|\leqslant k} \|D^\alpha u_m - D^\alpha u_n\|_{C(\bar{U})} \to 0$ and $\sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})} \to 0$ as $m, n \to \infty$.
   Especially,
$$\|u_m - u_n\|_{C^k(\bar{U})} := \sup_{|\alpha|\leqslant k} \sup_{x\in U} |D^\alpha(u_m - u_n)(x)| \to 0$$
   Since $(C^k(\bar{U}), \|\cdot\|_{C^k(\bar{U})})$ is a Banach space, there exist $u \in C^k(\bar{U})$ such that $u_n \to u$ in $C^k(\bar{U})$.
   $[D^\alpha u_n - D^\alpha u]_{C^{0,\gamma}(\bar{U})} = \sup_{\substack{x,y\in U\\ x\neq y}} \dfrac{|(D^\alpha u_n(x) - D^\alpha u(x)) - (D^\alpha u_n(y) - D^\alpha u(y))|}{|x-y|} \to$
   0 as $n \to \infty$, since $D^\alpha u_n$ converges uniformly to $D^\alpha u$.
   It remains to show $u \in C^{k,\gamma}(\bar{U})$: for any $x, y \in U, x \neq y$,
$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} \leqslant \limsup_{n\to\infty} \frac{|D^\alpha u_n(x) - D^\alpha u_n(y)|}{|x-y|^\gamma} \leqslant \limsup_{n\to\infty} [D^\alpha u_n]_{C^{0,\gamma}(\bar{U})} < \infty$$
   since each Cauchy sequence is bounded.

□

3.2 Sobolev Spaces

3.2.1 Weak Derivatives

$\mathcal{C}_c^\infty(U)$ is the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$, with compact support in U. Functions in $\mathcal{C}_c^\infty(U)$ are also called **test functions**.

Suppose $u, v \in L^1(U)$ and α is a multiindex. We say that v is the **α-th weak partial derivative** of u, and denote
$$D^\alpha u = v$$
if
$$\int_U u D^\alpha \phi dm = (-1)^{|\alpha|} \int_U v \phi dm \tag{3.3}$$
holds for all test functions $\phi \in \mathcal{C}_c^\infty(U)$. The motivation for the definition of the weak derivative is that for functions in $C^k(U)$, the above formula holds for any test functions. We hope to generalize the partial derivatives for functions without enough smoothness, and the above definition becomes natural.

Lemma (Uniqueness of the Weak Derivative)**.** A weak α-th partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.

The proof use the fact that $\mathcal{C}_c^\infty(U)$ is dense in $L^1(U)$.

Now Fix $1 \leqslant p \leqslant \infty$ and let k be a nonnegative integer. The **Sobolev Space**
$$W^{k,p}(U)$$

consists of all integrable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leqslant k$, $D^\alpha u$ exists in the weak sense, and belongs to $L^p(U)$. We identify functions in $W^{k,p}(U)$ which agree a.e. If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U)$$

The norm of $u \in W^{k,p}(U)$ is defined as

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \Big(\sum\limits_{|\alpha|\leqslant k} \int_U \|D^\alpha u\|^p \, dm \Big)^{1/p} & 1 \leqslant p < \infty \\ \sum\limits_{|\alpha|\leqslant k} \operatorname{esssup}_U |D^\alpha u| & p = \infty \end{cases}$$

$H^k(U)$ is a Hilbert space equipped with the scalar product

$$\langle u, v \rangle_{H^1} = \sum_{|\alpha|\leqslant k} \langle D^\alpha u, D^\alpha v \rangle_{L^2}$$

The associated norm $\|u\|_{H^1} = \sum\limits_{|\alpha|\leqslant k} \|D^\alpha u\|_2^2$ is equivalent to the $W^{k,2}$ norm. We use the $W^{k,2}$ norm in $H^k(U)$.
The convergence for a sequence of functions is defined to be the convergence under the norm in $W^{k,p}(U)$. We also write

$$u_m \to u \text{ in } W^{k,p}_{\text{loc}}(U)$$

to mean

$$u_m \to u \text{ in } W^{k,p}(V)$$

for each $V \Subset U$. We denote by

$$W_0^{k,p}(U)$$

the closure of $\mathcal{C}_c^\infty(U)$ in $W^{k,p}(U)$. Thus $u \in W_0^{k,p}(U)$ if and only if there exist functions $u_m \in \mathcal{C}_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. It is customary to write

$$H_0^k(U) = W_0^{k,p}(U)$$

Remark. When $n = 1$ and U is an open interval in $\mathbb{R}$, $u \in W^{1,p}(U)$ if and only if u equals a.e. an absolutely continuous function whose ordinary derivative (which exsits a.e.) belongs to L^p.

Proof. □

Example (Unbounded Function in Sobolev Spaces). We consider the Sobolev space $W^{1,p}(B(0,1))$ and functions of the form $u(x) = \|x\|^{-\alpha}, x \in B(0,1), x \neq 0$. Note that u is smooth away from 0, with

$$\partial_i u = \frac{-\alpha x_i}{\|x\|^{\alpha+2}}, \|D^\alpha u(x)\| = \frac{|\alpha|}{\|x\|^{\alpha+1}} \text{ when } x \neq 0$$

Let $\phi \in \mathscr{C}_c^\infty(B(0,1))$ and $\varepsilon > 0$, then

$$\int_{B(0,1)-B(0,\varepsilon)} u\partial_i\phi dm = -\int_{B(0,1)-B(0,\varepsilon)} \phi\partial_i u dm + \int_{\partial B(0,\varepsilon)} u\phi v^i dS$$

If $\alpha + 1 < n$, $Du \in L^1(B(0,1))$ and

$$\left|\int_{\partial B(0,\varepsilon)} u\phi v^i dS\right| \leqslant \|\phi\|_\infty \int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha} dS \leqslant C\varepsilon^{n-1-\alpha} \to 0$$

Thus $\int_U u\partial_i\phi dm = -\int_U \phi\partial_i u dm$ for all $\phi \in \mathscr{C}_c^\infty(B(0,1))$, provided that $0 \leqslant \alpha < n-1$. $Du \in L^p$ if and only if $(\alpha+1)p < n$. We conclude that $u \in W^{k,p}(B(0,1))$ if and only if $\alpha < \dfrac{n-p}{p}$.

Let $\{r_k\}$ be a countable and dense subset of $U = B(0,1)$. For $(\alpha+1)p < n$, consider

$$u_k(x) = 2^{-k}\|x - r_k\|^{-\alpha} \in W^{k,p}(B(0,1))$$

and set

$$u(x) = \sum_{k=1}^\infty u_k(x) = \sum_{k=1}^\infty 2^{-k}\|x - r_k\|^{-\alpha}$$

then $u \in W^{1,p}(B(0,1))$ and is unbounded on each open subset of $B(0,1)$. This example illustrates that functions in a Sobolev space can be unbounded on each open subset of its domain.

The next result shows that many of the rules for ordinary derivatives also apply to the weak derivatives.

Theorem (Properties of Weak Derivatives)**.** Assume $u, v \in W^{k,p}(U)$, $|\alpha| \leqslant k$, then

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi-indices α, β with $|\alpha| + |\beta| \leqslant k$.

2. For every $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leqslant k$.

3. If V is an open subset of U, then $u \in W^{k,p}(V)$.

4. IF $\zeta \in \mathscr{C}_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and **Leibniz's rule** holds:

$$D^\alpha(\zeta u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta\zeta D^{\alpha-\beta}u$$

   where $\binom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$.

In the proof of the last property, we use an identity for multiindices

$$\binom{\beta}{\sigma-\gamma}+\binom{\beta}{\sigma}=\binom{\alpha}{\sigma}$$

The next result shows that Sobolev spaces have a good mathematical structure that make it a function space.

Theorem (Sobolev Spaces as Function Spaces)**.** For each positive integer k and $1 \leqslant k \leqslant \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.

Proof. i: $\|\cdot\|_{W^{k,p}(U)}$ is a norm: It is clear that

$$\|\lambda u\|_{W^{k,p}(U)} = |\lambda| \, \|u\|_{W^{k,p}(U)}$$

and

$$\|u\|_{W^{k,p}(U)} = 0 \text{ if and only if } u = 0 \text{ a.e.}$$

Assume $u, v \in W^{k,p}(U)$. If $1 \leqslant p < \infty$, applying Minkowski's inequality twice gives

$$\begin{aligned}
\|u+v\|_{W^{k,p}(U)} &\leqslant \Big(\sum_{|\alpha|\leqslant k} \|D^\alpha u + D^\alpha v\|_p^p\Big)^{1/p} \\
&\leqslant \Big(\sum_{|\alpha|\leqslant k} (\|D^\alpha u\|_p + \|D^\alpha v\|_p)^p\Big) \\
&\leqslant \Big(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|_p^p\Big)^{1/p} + \Big(\sum_{|\alpha|\leqslant k} \|D^\alpha v\|_p^p\Big)^{1/p} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}
\end{aligned}$$

ii: assume $\{u_m\}$ is a Cauchy sequence in $W^{k,p}(U)$. For each $|\alpha| \leqslant k$, $\{D^\alpha u_m\}$ is a Cauchy sequence in $L^p(U)$. Then there exists functions $u_\alpha \in L^p(U)$ such that

$$D^\alpha u_m \to u_\alpha \text{ in } L^p(U)$$

for each $|\alpha| \leqslant k$. Especially,

$$u_m \to u := u_{(0,\dots,0)} \text{ in } L^p(U)$$

It remains to check $D^\alpha u = u_\alpha$, which completes the proof that $u \in W^{k,p}(U)$ and that $W^{k,p}(U)$ is complete.

$$\begin{aligned}
\int_U u D^\alpha \phi dm &= \lim_{m\to\infty} \int_U u_m D^\alpha \phi dm \\
&= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi dm \\
&= (-1)^{|\alpha|} \int_U u_\alpha \phi dm
\end{aligned}$$

for any $\phi \in \mathcal{C}_c^\infty(U)$. □

3.3 Approximation

3.3.1 Convolution

Later we will see the technique of mollification comes from the convolution, we now pause for a while to introduce the convolution: Let f, g be measurable functions on $\mathbb{R}^n$. The **convolution** of f, g is the function $f * g$ defined by

$$f * g(x) = \int f(x-y)g(y)dy$$

for all x such that the integral exists. We now need the measurability of K defined by $K(x,y) = f(x-y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ for a measurable function f. We see $K = f \circ s$ where $s(x,y) = x - y$; since s is continuous, K is Borel measurable if f is. Since for each Lebesgue measurable function $\tilde{f}$ on $\mathbb{R}^n$, there exists a Borel measurable function f on $\mathbb{R}^n$ such that $f = \tilde{f}$ $m-$ a.e. x for the Lebesgue measure m on $\mathbb{R}^n$, we can thus assume in the definition of the convolution that $K : (x, y) \mapsto f(x-y)$ is Lebesgue measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

Proposition. Properties of the Convolution Assuming all the integrals in question exist:

1. $f * g = g * f$.
2. $(f * g) * h = f * (g * h)$.
3. For $z \in \mathbb{R}^n$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$.
4. If A is the closure of $\{x + y : x \in \operatorname{spt}(f), y \in \operatorname{spt}(g)\}$, then $\operatorname{spt}(f * g) \subseteq A$.

Theorem (Young's Inequality). 1. If $f \in L^1$ and $g \in L^p$ with $1 \leqslant p \leqslant \infty$, then $f * g(x)$ exists a.e. x, $f * g \in L^p$, and $\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$.

2. If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, then $f * g(x)$ exists for every x, $f * g$ is bounded and uniformly continuous, and $\|f * g\|_u \leqslant \|f\|_p \|g\|_q$. If $1 < p < \infty$ (so that $1 < q < \infty$ also), then $f * g \in C_0(\mathbb{R}^n)$.

3. Suppose $1 \leqslant p, q, r \leqslant \infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$. If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$.

Proof. We use the Minkowski's inequality for integrals:

$$\|f * g\|_p = \left\| \int f(y) g(\cdot - y) dy \right\|_p \leqslant \int |f(y)| \, \|\tau_y g\|_p \, dy = \|f\|_1 \|g\|_p$$

□

One of the most important properties of convolution is that, rough speaking, $f * g$ is as smooth as either f or g, because formally we have

$$\partial^\alpha (f * g)(x) = \partial^\alpha \int f(x-y) f(y) dy = \int \partial^\alpha f(x-y) g(y) dy = (\partial^\alpha f) * g(x),$$

and similarly $\partial^\alpha (f * g) = f * (\partial^\alpha g)$. To make this precise, one needs only to impose conditions on f and g so that differentiation under the integral sign is legitimate. One such result is the following:

Proposition. If $f \in L^1$, $g \in C^k$, and $\partial^\alpha g$ is bounded for $|\alpha| \leqslant k$, then $f * g \in C^k$ and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leqslant k$.

The following theorem underlies many of the important applications of convolutions on $\mathbb{R}^n$. We introduce a bit of notation that will be used frequently hereafter: If η is any function that on $\mathbb{R}^n$ and $\varepsilon > 0$, we set

$$\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x).$$

We observe that if $\eta \in L^1$, then $\int \eta_\varepsilon$ is independent of ε. Moreover, the "mass" of η_ε becomes concentrated at the origin as $t \to 0$.

Theorem. Suppose $\eta \in L^1$ and $\int \eta(x)dm = a$.

1. If $f \in L^p(1 \leqslant p < \infty)$, then $f * \eta_\varepsilon \to af$ in the L^p norm as $\varepsilon \to 0$.
2. If f is bounded and uniformly continuous, then $f * \eta_\varepsilon \to af$ uniformly as $\varepsilon \to 0$.
3. If $f \in L^\infty$ and f is continuous on an open set U, then $f * \eta_\varepsilon \to af$ uniformly on compact subset of U as $\varepsilon \to 0$.

Proof. We have

$$\begin{aligned} f * \eta_\varepsilon(x) - af(x) &= \int (f(x-y) - f(x))\eta_\varepsilon(y)dy \\ &= \int (f(x-\varepsilon y) - f(x))\eta(y)dy \\ &= \int (\tau_{\varepsilon y}f(x) - f(x))\eta(y)dy \end{aligned}$$

Applying Minkowski's inequality for integrals gives:

$$\|f * \eta_\varepsilon - af\|_p \leqslant \int \|\tau_{\varepsilon y}f - f\|_p \, |\eta(y)| \, dy.$$

Then dominated convergence with $2\|f\|_p$ being the dominating function gives the desired convergence result in L^p case.
The second case is similar to the above argument except that we replace the $\|\cdot\|_p$ by $\|\cdot\|_u$. Note that the uniform continuity of f gives $\|\tau_{\varepsilon y}f - f\|_u \to 0$ as $\varepsilon \to 0$.
The proof for the third case is omitted. □

If we impose slightly stronger conditions on η, we can also show that $f * \eta_\varepsilon \to af$ a.e. for $f \in L^p$. The device in the following proof of breaking up an integral into pieces corresponding to the dyadic intervals $[2^k, 2^{k+1}]$ and estimating each piece separately is a standard trick of the trade in Fourier analysis.

Theorem. Suppose $|\eta(x)| \leqslant C(1 + \|x\|^{-n-\epsilon})$ for some $C, \epsilon > 0$ (which implies that $\eta \in L^1$), and $\int \eta(x)dm = a$. If $f \in L^p(1 \leqslant p < \infty)$, then $f * \eta_\varepsilon \to af(x)$ as $\varepsilon \to 0$ for every x in the Lebesgue set of f— in particular, for a.e. x, and for every x at which f is continuous.

Proof. If x is in the Lebesgue set of f, for every $\delta > 0$ there exists $\kappa > 0$ such that

$$\int_{\|y\|<r} |f(x-y) - f(x)|\, dy \leqslant \delta r^n \text{ for } r \leqslant \kappa.$$

Let us set

$$I_1 = \int_{\|y\|\leqslant\kappa} |f(x-y) - f(x)|\, |\eta_\varepsilon(y)|\, dy,$$
$$I_2 = \int_{\|y\|>\kappa} |f(x-y) - f(x)|\, |\eta_\varepsilon(y)|\, dy.$$

We claim that I_1 is bounded by $A\delta$ where A is independent of ε, whereas $I_2 \to 0$ as $\varepsilon \to 0$. Since

$$|f * \eta_\varepsilon(x) - af(x)| \leqslant I_1 + I_2,$$

We will have

$$\limsup_{\varepsilon\to 0} |f * \eta_\varepsilon(x) - af(x)| \leqslant A\delta$$

and since δ is arbitrary, this will complete the proof.

To estimate I_1, Let K be the integer such that $2^K \leqslant \kappa/\varepsilon < 2^{K+1}$ if $\kappa/\varepsilon \geqslant 1$, and $K = 0$ if $\kappa/\varepsilon < 1$. We view the ball $\|y\| \leqslant \kappa$ as the union of annuli $2^{-k}\kappa \leqslant \|y\| < 2^{1-k}\kappa (1 \leqslant k \leqslant K)$ and the ball $\|y\| \leqslant 2^{-K}\kappa$. On the k-th annulus we use the estimate

$$|\eta_\varepsilon(y)| \leqslant C\varepsilon^{-n} \left|\frac{y}{\varepsilon}\right|^{-n-\epsilon} \leqslant C\varepsilon^{-n} (\frac{2^{-k}\kappa}{\varepsilon})^{-n-\epsilon},$$

and on the ball $\|y\| \leqslant 2^{-K}\kappa$ we use the estimate $|\eta_\varepsilon(y)| \leqslant C\varepsilon^{-n}$. Thus

$$\begin{aligned}
I_1 &\leqslant \sum_{k=1}^{K} C\varepsilon^{-n} (\frac{2^{-k}\kappa}{\varepsilon})^{-n-\epsilon} \int_{2^{-k}\kappa \leqslant \|y\| < 2^{1-k}\kappa} |f(x-y) - f(x)|\, dy \\
&+ C\varepsilon^{-n} \int_{\|y\|<2^{-K}\kappa} |f(x-y) - f(x)|\, dy \\
&\leqslant C\delta \sum_{k=1}^{K} (2^{1-k}\kappa)^n \varepsilon^{-n} (\frac{2^{-k}\kappa}{\varepsilon})^{-n-\epsilon} + c\delta\varepsilon^{-n}(2^{-K}\kappa)^{-n} \\
&= 2^n C\delta (\frac{\kappa}{\varepsilon})^{-\epsilon} \sum_{k=1}^{K} 2^{k\epsilon} + C\delta (\frac{2^{-K}\kappa}{\varepsilon})^{-n} \\
&= 2^n C\delta (\frac{\kappa}{\varepsilon}^{-\epsilon}) \frac{2^{(K+1)\epsilon} - 2^\epsilon}{2^\epsilon - 1} + C\delta (\frac{2^{-K}\kappa}{\varepsilon})^n \\
&\leqslant 2^n C(2^\epsilon(2^\epsilon - 1)^{-1} + 1)\delta
\end{aligned}$$

As for I_2, if p' is the conjugate exponent to p and χ is the characteristic function of $\{y : \|y\| \geqslant \kappa\}$, by Hölder's inequality we have

$$\begin{aligned} I_2 &\leqslant \int_{\|y\|\geqslant\kappa} (|f(x-y)| + |f(x)|)\,|\eta_\varepsilon(y)|\,dy \\ &\leqslant \|f\|_p \,\|\chi\eta_\varepsilon\|_{p'} + |f(x)|\,\|\chi\eta_\varepsilon\|_1 , \end{aligned}$$

so it suffices to show that for $1 \leqslant q \leqslant \infty$, $\|\chi\eta_\varepsilon\|_q \to 0$ as $\varepsilon \to 0$. If $q = \infty$,

$$\|\chi\eta_\varepsilon\|_\infty \leqslant C\varepsilon^{-n}(1 + (\kappa/\varepsilon))^{-n-\epsilon} = C\varepsilon^{-\epsilon}(\varepsilon+\kappa)^{-n-\epsilon} \leqslant C\kappa^{-n-\epsilon}\varepsilon^{\epsilon}.$$

If $q < \infty$,

$$\begin{aligned} \|\chi\eta_\varepsilon\|_q^q &= \int_{\|y\|\geqslant\kappa} \varepsilon^{-nq}\left|\eta(\varepsilon^{-1}y)\right|^q dy = \varepsilon^{n(1-q)}\int_{\|y\|\geqslant\kappa/\varepsilon} |\eta(y)|^q\,dy \\ &\leqslant C_1\varepsilon^{n(1-q)}\int_{\kappa/\varepsilon}^{\infty} r^{n-1-(n+\epsilon)q}dr = C_2\varepsilon^{n(1-q)}(\frac{\kappa}{\varepsilon})^{n-(n+\epsilon)q} = C_3\varepsilon^{\epsilon q}. \end{aligned}$$

In either case, $\|\chi\eta_\varepsilon\|_q$ is dominated by ε^{ϵ}, so we are done. □

In most of the applications of the proceeding two theorems one has $a = 1$, although the case $a = 0$ is also useful. If $a = 1$, $\{\eta_\varepsilon\}_{\varepsilon>0}$ is called an **approximation identity**, as it furnishes an approximation to the identity operator on L^p by convolution operators. The construction is useful for approximating L^p functions by functions having specific regularity properties. For example, we have the following important result:

Proposition. $\mathscr{C}_c^\infty$ is dense in L^p.

3.3.2 Mollifiers: Smoothing by Convolution

We need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions, and the method of mollifiers provides the tool.

If $U \in \mathbb{R}^n$ is open and $\varepsilon > 0$, we write

$$U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\} = \left\{x \in U : \bar{B}(x,\varepsilon) \subseteq U\right\}$$

Now define a smooth function $\eta \in C^\infty(U)$ on U by

$$\eta(x) := \begin{cases} C\exp\left(\dfrac{1}{\|x\|^2 - 1}\right), & \|x\| < 1 \\ 0, & \|x\| \geqslant 1 \end{cases}$$

the constant C is selected to normalize η: $\int_{\mathbb{R}^n} \eta dm = 1$. Then for each $\varepsilon > 0$, set

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$$

We call η the **mollifier**. The function η_ε are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\varepsilon dm = 1, \operatorname{spt}(\eta_\varepsilon) \subseteq \bar{B}(0,\varepsilon).$$

If $f \in L^1_{\text{loc}}(U)$, we define its mollification

$$f^\varepsilon := \eta_\varepsilon * f \text{ in } U_\varepsilon$$

That is,

$$f * \varepsilon(x) = \int_U \eta_\varepsilon(x-y)f(y)dy = \int_{\bar{B}(0,\varepsilon)} \eta_\varepsilon(y)f(x-y)dy$$

for $x \in U_\varepsilon$.

We summarize our result of convolution to give some properties of the mollification of a locally integrable function: From $\int \eta dm = 1$, we know the family $\{\eta_\varepsilon\}_{\varepsilon>0}$ is an approximation identity. Each η_ε is a $\mathcal{C}_c^\infty$ function, we know $f^\varepsilon \in C^\infty(U_\varepsilon)$ and $\partial^\alpha f^\varepsilon = f * (\partial^\alpha \eta_\varepsilon)$. Obviously, $|\eta| \leqslant C(1+\|x\|)^{-n-\epsilon}$ for some $C, \epsilon > 0$, we have $f^\varepsilon \to f$ as $\varepsilon \to 0$ for every x in the Lebesgue set of f—- in particular, for a.e. x. If $f \in C(U)$, $f^\varepsilon \to f$ uniformly on compact subsets of U as $\varepsilon \to 0$. Finally, for $1 \leqslant p < \infty$, if $f \in L^p(U)$, then $f^\varepsilon \to f$ in the L^p norm. This gives

Theorem (Properties of Mollifiers).

1. $f^\varepsilon \in C^\infty(U_\varepsilon)$.
2. $f^\varepsilon \to f$ a.e. as $\varepsilon \to 0$.
3. If $f \in C(U)$, then $f^\varepsilon \to f$ uniformly on compact subsets of U.
4. If $1 \leqslant p < \infty$ and $f \in L^p_{\text{loc}}(U)$, then $f^\varepsilon \to f$ in $L^p_{\text{loc}}(U)$.

The following result means the weak derivative of the mollifier equals the mollifier of the weak derivative.

Proposition. Let $U_\varepsilon \subseteq U$ be as before. Assume $u \in L^1_{\text{loc}}(U)$ admits a weak derivative $D^\alpha u$ for some multiindex α. Then

$$D^\alpha(\eta_\varepsilon * u)(x) = \eta_\varepsilon * D^\alpha u(x) \text{ for all } x \in U_\varepsilon$$

In other words,

$$D^\alpha(u^\varepsilon) = (D^\alpha u)^\varepsilon$$

Proof.

$$\begin{aligned}
D^\alpha(\eta_\varepsilon * u)(x) &= \int_U D_x^\alpha \eta_\varepsilon(x-y)u(y)dy \\
&= (-1)^{|\alpha|} \int_U D_y^\alpha \eta_\varepsilon(x-y)u(y)dy \\
&= (-1)^{|\alpha|+|\alpha|} \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy \\
&= \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy \\
&= \eta_\varepsilon * u(x)
\end{aligned}$$

□

3.3.3 Interior Approximation by Smooth Functions

Fix a positive integer k and $1 \leqslant p < \infty$. We have the following result:

Theorem (Local Approximation by Smooth Functions). Assume that $u \in W^{k,p}(U)$ for some $1 \leqslant p < \infty$, and set

$$u^\varepsilon = \eta_\varepsilon * u \text{ in } U_\varepsilon.$$

Then

1. $u^\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$

2. $u^\varepsilon \to u$ in $W^{k,p}_{\text{loc}}(U)$, as $\varepsilon \to 0$.

Proof. We give the proof for the second statement. We claim that if $|\alpha| \leqslant k$, then

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \text{ in } U_\varepsilon$$

the ordinary α-th partial derivative of the smooth function u^ε is the ε-mollification of the α-th weak partial derivative of u. To verify this, let $x \in U_\varepsilon$,

$$\begin{aligned}
D^\alpha u^\varepsilon(x) &= \int_U \eta_\varepsilon(x-y)u(y)dy \\
&= \int_U D^\alpha_x \eta_\varepsilon(x-y)u(y)dy \\
&= (-1)^{|\alpha|} \int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)dy \\
&= (-1)^{|\alpha|+|\alpha|} \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy \\
&= \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy \\
&= (\eta_\varepsilon * D^\alpha u)(x).
\end{aligned}$$

Now choose an open set $V \Subset U$. Then $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(V)$ as $\varepsilon \to 0$, for each $|\alpha| \leqslant k$. Consequently,

$$\|u^\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \leqslant k} \|D^\alpha u^\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\varepsilon \to 0$. □

3.3.4 Approximation by Smooth Functions

Next we show that we can find smooth functions which approximate in $W^{k,p}(U)$ and not just in $W^{k,p}_{\text{loc}}(U)$. In this subsection, we make no assumption about the smooth of ∂U.

Theorem (Global Approximation by Smooth Functions). Assume U is bounded, and suppose as well that $u \in W^{k,p}(U)$ for some $1 \leqslant p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_m \to u \text{ in } W^{k,p}(U)$$

We don't assert that $u_m \in C^\infty(\bar{U})$. The proof of the theorem use **partition of unity** technique.

Proof. We have $U = \bigcup_{i=1}^{\infty} U_i$, where

$$U_i := \{x \in U : \operatorname{dist}(x, \partial U) > 1/i\}$$

and we write $V_i := U_{i+3} - \bar{U}_{i+1}$. Choose any open set $V_0 \Subset U$ so that $U = \bigcup_{i=0}^{\infty} V_i$. Let $\{\zeta_i\}_{i=0}^{\infty}$ be a smooth partition of unity subordinate to $\{V_i\}_{i=0}^{\infty}$. Then for any $u \in W^{k,p}(U)$, each $\zeta_i u \in W^{k,p}(U)$ and $\operatorname{spt}(\zeta_i u) \subseteq V_i$.

Fix $\delta > 0$. Choose $\varepsilon_i > 0$ small enough that $u^i := \eta_{\varepsilon_i} * (\zeta_i u)$ satisfies

$$\begin{cases} \left\|u^i - \zeta_i u\right\|_{W^{k,p}(U)} \leqslant \dfrac{\delta}{2^{i+1}} & i = 0, 1, \dots \\ \operatorname{spt} u^i \subseteq W_i & i = 1, \dots \end{cases}$$

for $W_i := U_{i+4} - \bar{U}_i \supseteq V_i$ for $i = 1, \dots$.

$v := \sum_{i=0}^{\infty} u^i \in C^\infty(U)$. We have for each $V \Subset U$

$$\begin{aligned} \|v - u\|_{W^{k,p}(V)} &\leqslant \sum_{i=0}^{\infty} \left\|u^i - \zeta_i u\right\|_{W^{k,p}(U)} \\ &\leqslant \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \\ &= \delta \end{aligned}$$

Taking supremum over sets $V \Subset U$ gives $\|u - v\|_{W^{k,p}(U)} \leqslant \delta$. □

3.3.5 Global Approximation by Smooth Functions

We now ask when it is possible to approximate a function $u \in W^{k,p}(U)$ by functions in $C^\infty(\bar{U})$, rather than only $C^\infty(U)$. Such an approximation requires some conditions to exclude ∂U being wild geometrically.

Theorem (Global Approximation buy Functions Smooth Up to the Boundary). Assume U is bounded and ∂U is C^1. Suppose $u \in W^{k,p}(U)$ for some $1 \leqslant p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{U})$ such that

$$u_m \to u \text{ in } W^{k,p}(U).$$

3.4 Extension

Our goal next is to extend functions in the Sobolev space $W^{k,p}(U)$ to become functions in the Sobolev space $W^{k,p}(\mathbb{R}^n)$. This can be subtle. Observe for instance that our extending $u \in W^{k,p}(U)$ to be zero in $\mathbb{R}^n - U$ will not in general works, as we may thereby create such a bad continuity along ∂U that the extend function no long has a weak first partial derivative. We must instead invent a way to extend u which "preserves the weak derivatives across ∂U". Next we suppose $1 \leqslant p \leqslant \infty$.

Theorem (**Extension Theorem**). Assume U is bounded and ∂U is C^1. Select a bounded open set V such that $U \Subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$:

1. $Eu = u$ a.e. in U.
2. Eu has compact support within V.
3. $$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \|u\|_{W^{1,p}(U)},$$

   the consant C depending only on p, U, V.

We call Eu an extension of u to $\mathbb{R}^n$. Since V is bounded, the extension Eu has compact support.

3.5 Traces

In this section, we discuss the possibility of assigning "boundary values" along ∂U to a function $u \in W^{1,p}(U)$, assuming that ∂U is C^1. Now if $u \in C(\bar{U})$, then clearly u has values on ∂U in the usual sense. The problem is that a typical function $u \in W^{k,p}(U)$ is not in general continuous and, even worse, is only defined a.e. in U. Since ∂U has n-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "u restricted to ∂U". The notion of a **trace operator** resolves the problem. For this section, assume $1 \leqslant p < \infty$.

Theorem (**Trace Theorem**). Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that

1. $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$.
2. $$\|Tu\|_{L^p(U)} \leqslant C(p, U) \|u\|_{W^{1,p}(U)},$$

   for each $u \in W^{1,p}(U)$.

We call Tu the **trace** of u along ∂U.

Theorem (Trace-zero Functions in $W^{1,p}$). Assume U is bounded and ∂U is C^1. Suppose furthermore that $u \in W^{1,p}(U)$. Then

$$u \in W_0^{1,p}(U) \text{ if and only if } Tu = 0 \text{ on } \partial U.$$

3.6 Sobolev Inequalities

our goal in this section is to discover embeddings of various Sobolev spaces into others. The crucial analytic tools here will be certain so-called "Sobolev-type inequalities", which we will prove below for smooth functions. These will then establish the estimates for arbitrary functions in the various relevant Sobolev spaces, since smooth functions are dense.

We first consider the Sobolev space $W^{1,p}(U)$ and ask the following question: if a function u belongs to $W^{1,p}(U)$, does u automatically belong to certain other spaces? The answer is "yes", but which other spaces depends upon whether $1 \leqslant p < n$, $p = n$, $n < p < \infty$.

3.6.1 Gagliardo-Nirenberg-Sobolev Inequality

If $1 \leqslant p < n$, the **Sobolev conjugate** of p is

$$p^* := \frac{np}{n-p}.$$

Then $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}, p^* > p$.

Theorem (G-N-S Inequality). Assume $1 \leqslant p < \infty$. There exists a constant C, depending only on p, n, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Note in this theorem we need u to have compact support, as the example $u \equiv 1$ shows. Remarkably the constant does not depend upon the size of the support of u.

Proof. First assume $p = 1$: Since u has compact support, for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} \partial_i u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) dy_i;$$

$$|u(x)| \leqslant \int_{-\infty}^{\infty} \|Du(x_1, \ldots, y_i, \ldots, x_n)\| \, dy_i.$$

$$|u(x)|^{\frac{n}{n-1}} \leqslant \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \|Du(x_1, \ldots, y_i, \ldots, x_n)\| \, dy_i \right)^{\frac{1}{n}}.$$

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,dx_1 &\leqslant \int_{-\infty}^{\infty} \prod_{i=1}^{n} \big(\int_{-\infty}^{\infty} \|Du\|\,dy_i\big)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{\infty} \|Du\|\,dy_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \|Du\|\,dy_i\right)^{\frac{1}{n-1}} dx_1 \\ &\leqslant \left(\int_{-\infty}^{\infty} \|Du\|\,dy_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|Du\|\,dx_1 dy_i\right)^{\frac{1}{n-1}},\end{aligned}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\,dx_1 dx_2 \leqslant \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|Du\|\,dx_1 dy_2\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{\substack{i=1\\ i\neq 2}}^{n} I_i^{\frac{1}{n-1}} dx_2,$$

for

$$I_1 := \int_{-\infty}^{\infty} \|Du\|\,dy_1, \quad I_i := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|Du\|\,dx_1 dy_i (i = 3, \dots, n)$$

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\,dx_1 dx_2 \leqslant \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|Du\|\,dx_1 dy_2\right)^{\frac{1}{n-1}} &\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|Du\|\,dy_1 dx_2\right)^{\frac{1}{n-1}} \\ &\prod_{i=3}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|Du\|\,dx_1 dx_2 dy_i\right)\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} &\leqslant \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \|Du\|\,dx_1 \dots dy_i \dots dx_n\right)^{\frac{1}{n-1}} \\ &= \left(\int_{\mathbb{R}^n} \|Du\|\,dx\right)^{\frac{n}{n-1}}\end{aligned}$$

Consider now the case that $1 < p < n$. use $v := |u|^{\gamma}$, where $\gamma > 1$ is to be selected. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\leqslant \int_{\mathbb{R}^n} \left\|D|u|^{\gamma}\right\| dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} \|Du\|\,dx \\ &\leqslant \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} \|Du\|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

We choose γ so that $\dfrac{\gamma n}{n-1} = (\gamma - 1)\dfrac{p}{p-1}$:

$$\gamma := \frac{p(n-1)}{n-p} > 1$$

in which case $\dfrac{\gamma n}{n-1} = (\gamma - 1)\dfrac{p}{p-1} = \dfrac{np}{n-p} = p^*$, and

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \leqslant C \left(\int_{\mathbb{R}^n} \|Du\|^p \, dx \right)^{\frac{1}{p}}$$

□

Theorem (Estimates for $W^{1,p}, 1 \leqslant p < n$). Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leqslant p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate

$$\|u\|_{L^{p^*}(U)} \leqslant C(p, n, U) \|u\|_{W^{1,p}(U)},$$

Proof. Since ∂U is C^1, there exists an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$, such that

1. $\bar{u} = u$ in U,
2. $\bar{u}$ has compact support,
3. $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \|u\|_{W^{1,p}(U)}$.

Because $\bar{u}$ has compact support, there exist function $u_m \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that

$$u_n \to \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n).$$

By G-N-S inequality, $\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$ for all $l, m \geqslant 1$. Thus $u_m \to \bar{u}$ in $L^{p^*}(\mathbb{R}^n)$ as well. Then

$$\begin{aligned} \|u\|_{L^{p^*}(U)} &\leqslant \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} = \lim_{n\to\infty} \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leqslant \lim_{n\to\infty} \|Du_m\|_{L^p(\mathbb{R}^n)} \\ &= \|D\bar{u}\|_{L^p(\mathbb{R}^n)} \leqslant \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \|u\|_{W^{1,p}(U)}. \end{aligned}$$

□

Theorem (Estimate for $W_0^{1,p}, 1 \leqslant p < n$). Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leqslant p < n$. Then we have the estimate

$$\|u\|_{L^q(U)} \leqslant C(p, q, n, U) \|Du\|_{L^p(U)}$$

for each $1 \leqslant q \leqslant p^*$. In particular, for all $1 \leqslant p \leqslant \infty$,

$$\|u\|_{L^p(U)} \leqslant C \|Du\|_{L^p(U)}.$$

This estimate is sometimes called **Poincare's inequality**. The difference with the second theorem is that only the gradient of u appears on the RHS of the inequality. In view of this estimate, on $W_0^{1,p}(U)$ the norm $\|Du\|_{L^p(U)}$ is equivalent to $\|u\|_{W^{1,p}(U)}$, if U is bounded.

Proof. Since $u \in W_0^{1,p}(U)$, there exist function $u_m \in \mathscr{C}_c^\infty(U)$ converging to u in $W^{1,p}(U)$. Extend each function u_m to be 0 on $R^n - \bar{U}$ and apply G-N-S inequality to discover $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} = \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C\|D\bar{u}_m\|_{L^p(\mathbb{R}^n)} = C\|Du_m\|_{L^p(\mathbb{R}^n)}$. Letting $m \to \infty$, $\|u\|_{L^{p^*}(U)} \leqslant C\|Du\|_{L^p(U)}$.

Let $qr = p^*$,

$$\begin{aligned}\int_U |u|^q\,dx &\leqslant \left(\int_U |u|^{qr}\,dx\right)^{\frac{1}{r}}\left(\int_U 1dx\right)^{\frac{r-1}{r}} = m(U)^{\frac{r-1}{r}}\left(\int_U |u|^{qr}\right)^{\frac{1}{r}} \\ &\leqslant m(U)^{\frac{r-1}{r}}\left(\Big(\int_U |u|^{p^*}\Big)^{\frac{1}{p^*}}\right)^q \leqslant Cm(U)^{\frac{r-1}{r}}\|Du\|_{L^p(U)}^q\end{aligned}$$

$$\|u\|_{L^q(U)} \leqslant C\|Du\|_{L^p(U)}.$$

Since $p^* = \dfrac{n}{n-p}p > p$, the above inequality implies

$$\|u\|_{L^p(U)} \leqslant \|Du\|_{L^p(U)}.$$

□

3.6.2 Morrey's Inequality

Next, Morrey's inequality tells us that if $n < p < \infty$, then a $W^{1,p}$ function is a.e. Hölder continuous.

Theorem (Morrey's inequality). Assume $n < p \leqslant \infty$. Then there exists a constant $C(p,n)$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leqslant C(p,n)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, where $\gamma := 1 - n/p$.

Proof. Fix any point $w \in \partial B(0,1)$. Then if $0 < s < r$,

$$\begin{aligned}|u(x+sw) - u(x)| &= \left|\int_0^s \frac{d}{dt}u(x+tw)dt\right| \\ &= \left|\int_0^s Du(x+tw)\cdot w dt\right| \\ &= \int_0^s \|Du(x+tw)\|\,dt\end{aligned}$$

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)|\,dS(w) \leqslant \int_0^s \int_{\partial B(0,1)} \|Du(x+tw)\|\,dS(w)dt.$$

$$\begin{aligned}\int_0^s \int_{\partial B(0,1)} \|Du(x+tw)\|\, dS(w)dt &= \int_0^s \int_{\partial B(x,t)} \frac{\|Du(y)\|}{t^{n-1}} dS(y)dt \\ &= \int_{\bar{B}(x,s)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} dy \\ &\leqslant \int_{\bar{B}(x,r)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} dy\end{aligned}$$

where we put $y = x + tw$, $t = \|x - y\|$.

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)|\, dS(w) = \frac{1}{s^{n-1}} \int_{\partial B(x,s)} |u(z) - u(x)|\, dS(z)$$

for $z = x + sw$. We then obtain

$$\int_{\partial B(x,s)} |u(z) - u(x)|\, dS(z) \leqslant s^{n-1} \int_{\bar{B}(x,r)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} dy.$$

$$\int_{\bar{B}(x,r)} |u(y) - u(x)|\, dy \leqslant \frac{r^n}{n} \int_{\bar{B}(x,r)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} dy$$

We thus proved the following:

$$\fint_{\bar{B}(x,r)} |u(x) - u(y)|\, dy \leqslant C(n) \int_{\bar{B}(x,r)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} dy$$

for each ball $\bar{B}(x,r) \subseteq \mathbb{R}^n$. Now fix $x \in \mathbb{R}^n$.

$$\begin{aligned}|u(x)| &\leqslant \fint_{B(x,1)} |u(x) - u(y)|\, dy + \fint_{B(x,1)} |u(y)|\, dy \\ &\leqslant C \int_{B(x,1)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} dy + C\, \|u\|_{L^p(\bar{B}(x,1))} \\ &\leqslant C \left(\int_{\mathbb{R}^n} \|Du\|^p \right)^{1/p} \left(\int_{\bar{B}(x,1)} \frac{1}{\|x-y\|^{(n-1)\frac{p}{p-1}}} dy \right)^{\frac{p-1}{p}} + C\, \|u\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\, \|u\|_{W^{1,p}(\mathbb{R}^n)}\end{aligned}$$

The last inequality holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$. Since $x \in \mathbb{R}^n$ is arbitrary, it follows that

$$\sup_{\mathbb{R}^n} \leqslant C\, \|u\|_{W^{1,p}}\, (\mathbb{R}^n)$$

Next, choose any two points $x, y \in \mathbb{R}^n$ and write $r := \|x - y\|$. Let $W :=$

$\bar{B}(x,r) \cap \bar{B}(y,r)$. Then

$$\begin{aligned}
\fint_W |u(x) - u(z)| &\leqslant C \fint_{\bar{B}(x,r)} |u(x) - u(z)|\, dz \\
&\leqslant C \left(\int_{\bar{B}(x,r)} \|Du\|^p \, dz \right)^{1/p} \left(\int_{\bar{B}(x,r)} \frac{dz}{\|x - z\|^{\frac{(n-1)p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&\leqslant C \left(r^{n-(n-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \\
&= C r^{1-\frac{n}{p} \|Du\|_{L^p(\mathbb{R}^n)}}
\end{aligned}$$

$$|u(x) - u(y)| \leqslant C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} = C \|x - y\|^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} .$$

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{\|x - y\|^{1-n/p}} \right\} \leqslant C \|Du\|_{L^p(\mathbb{R}^n)} .$$

□

Remark. A slight variant of the proof shows that

$$|u(x) - u(y)| \leqslant C r^{1-\frac{n}{p}} \left(\int_{\bar{B}(x,2r)} \|Du(z)\|^p \, dz \right)^{1/p}$$

We say u^* is a **version** of a given function u provided

$$u = u^* \text{ a.e.}$$

The following theorem allows us to identify a function $u \in W^{1,p}$ with $p > n$ with its continuous version.

Theorem (Estimate for $W^{1,p}, n < p \leqslant \infty$). Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leqslant \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{U})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leqslant C(p, n, U) \|u\|_{W^{1,p}(U)}$$

For the proof, we first extend to $W^{1,p}(\mathbb{R}^n)$, then approximate a $W^{1,p}(\mathbb{R}^n)$ function by a sequence of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ functions. Morrey's inequality tells us that the sequence is a Cauchy sequence in the Hölder space $C^{0,1-n/p}(\mathbb{R}^n)$, we thus can find a limit function u^* in $C^{0,1-n/p}(\mathbb{R}^n)$.

3.6.3 General Sobolev Inequalities

Theorem (General Sobolev Inequalities). Let U be an open bounded subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$.

1. if
$$k < \frac{n}{p},$$
then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Moreover,
$$\|u\|_{L^q(U)} \leqslant C(k,p,n,U)\, \|u\|_{W^{k,p}(U)}$$

2. If
$$k > \frac{n}{p},$$
then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})$, where
$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \frac{n}{p} \notin \mathbb{Z} \\ \text{and positive number } < 1, & \frac{n}{p} \in \mathbb{Z} \end{cases}$$
$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})} \leqslant C(k,p,n,\gamma,U)\, \|u\|_{W^{k,p}(U)}.$$

3.7 Compactness

We first review several results about compactness, and they are of fundamental importance to analysis:

Theorem (**Tychonoff's Theorem**)**.** If $\{X_\alpha\}_{\alpha\in A}$ is any family of compact topological spaces, then the product space $X = \prod_{\alpha\in A} X_\alpha$ is compact.

We now turn to the Arzelà-Ascoli theorem, which has to do with compactness in spaces of continuous mappings. Before the introduction of this result, we need to talk about some terminologies: If X is a topological spaces and $\mathcal{F} \subseteq C(X)$, $\mathcal{F}$ is called **equicontinuous at** $x \in X$ if for every $\varepsilon > 0$ there is a neighbourhood U of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U$ and all $f \in \mathcal{F}$, and $\mathcal{F}$ is called **equicontinuous** if it is equicontinuous at each point $x \in X$. Also, $\mathcal{F}$ is said to be **pointwise bounded** if $\{f(x) : f \in \mathcal{F}\}$ is a bounded subset of C for each $x \in X$. A topological space X is called σ-**compact** if it is a countable union of compact subsets.

Theorem (**Arzelà-Ascoli Theorem**)**.**

1. Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X)$ is compact.

2. Let X be a σ-compact LCH space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, there exist $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.

We have seen from G-N-S inequality implies the embedding of $W^{1,p}(U)$ into $L^{p^*}(U)$ for $1 \leqslant p < n, p^* = \dfrac{np}{n-p}$. The theorem in this section will tell us that $W^{1,p}(U)$ is in fact compactly embedded in $L^q(U)$ for $1 \leqslant q < p^*$.

Let X and Y be Banach spaces, $X \subseteq Y$. We say that X is **compactly embedded in** Y, denoted as

$$X \Subset Y,$$

if

1. $\|u\|_Y \leqslant C \|u\|_X \; (u \in X)$ for some constant C.
2. each bounded sequence in X is precompact in Y.

These two conditions means that the embedding operator $\iota : X \to Y$ is a compact operator. More precisely, the second condition means that if $\{u_k\}_{k=1}^\infty$ is a sequence in X with $\sup_k \|u_k\|_X < \infty$, then some subsequence $\{u_{k_j}\}_{j=1}^\infty$ converges in Y to some limit u:

$$\lim_{j\to\infty} \left\|u_{k_j} - u\right\|_Y = 0.$$

Theorem (**Rellich-Kondrachov Compactness Theorem**)**.** Assume U is an open bounded subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $1 \leqslant p < n$. Then

$$W^{1,p}(U) \Subset L^q(U)$$

for each $1 \leqslant q < p^*$.

Remark.

1. This theorem means that if $\{u_m\}$ is a bounded sequence in $W^{1,p}(U)$, i.e. $\sup_m \|u_m\|_{W^{1,p}(U)} \leqslant M$ for some $M > 0$, then these exists a subsequence $\{u_{m_j}\}$ and a function $u \in L^q(U)$ such that

$$\left\|u_{m_j} - u\right\|_{L^q(U)} \to 0.$$

2. When $p = n$, $W^{1,n}(U) \Subset L^p(U)$ for any $1 \leqslant p < \infty$. This is because $p^* = \frac{np}{n-p} \to \infty$ when $p < n$ and $p \to n$. Applying R-K compactness theorem gives the result. In particular, when $p = n$:

$$W^{1,n}(U) \Subset$$

3. Since U is bounded, $m(U) < \infty$. We know $L^p(U) \subseteq L^n(U)$ when $p > n$. This implies that $W^{1,p}(U) \subseteq W^{1,n}(U)$. Combining these results, we get when $n < p \leqslant \infty$:

$$W^{1,p}(U) \subseteq W^{1,n}(U) \Subset L^p(U).$$

4. we know in all cases, $1 \leqslant p \leqslant \infty$:

$$W^{1,p}(U) \Subset L^p(U).$$

5. Even if we do not assume ∂U to be C^1, we still have

$$W_0^{1,p}(U) \Subset L^p(U).$$

3.8 Additional Topics

3.8.1 Poincare's Inequality

We dente $(u)_U := \fint_U u dy$ = average of u over U.

Theorem (**Poincare's Inequality**). Let U be an open bounded and connected subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $1 \leqslant p \leqslant \infty$. Then there exists a constant C such that

$$\|u - (u)_U\|_{L^p(U)} \leqslant C(n,p,U)\,\|Du\|_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$.

Proof. Were the estimate false, there exist for each integer $k > 0$ a function $u_k \in W^{k,p}(U)$ satisfying

$$\|u_k - (u_k)_U\|_{L^p(U)} > k\,\|Du_k\|_{L^p(U)}\,.$$

Renormalize by defining

$$v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}$$

Then $(v_k)_U = 0$ and $\|v_k\|_{L^p(U)} = 1$ and thus $\|Dv_k\|_{L^p(U)} < \frac{1}{k}$. This shows that $\{v_k\}$ is a bounded sequence in $W^{1,p}(U)$. By the R-K compactness theorem, we can find a subsequence $\{v_{k_j}\}$ of $\{v_k\}$ and a function $v \in L^p(U)$ such that $v_{k_j} \to v$ in $L^p(U)$. We easily get $\|v\|_{L^p(U)} = 1$, since norm is a continuous function. We next prove if $\|v_{k_j} - v\|_{L^p(U)} \to 0$, then $(v)_U = 0$:

$$\left|\int_U v\right| \leqslant \left|\int_U v - v_{k_j}\right| + \left|\int_U v_{k_j}\right| \leqslant \Big(\int_U |v - v_{k_j}|^p\Big)^{1/p}\Big(\int_U 1\Big)^{1/q} = m(U)^{1/q}\,\|v - v_{k_j}\|_{L^p(U)} \to 0$$

On the other hand, for any $\phi \in \mathcal{C}_c^\infty(U)$,

$$\int_U v\partial_i\phi dm = \lim_{k_j\to\infty}\int_U v_{k_j}\partial_i\phi dm = -\lim_{k_j\to\infty}\int_U \partial_i v_{k_j}\phi dm$$
$$\left|\int_U \partial_i v_{k_j}\phi dm\right| \leqslant \|\partial_i v_{k_j}\|_{L^p(U)}\,\|\phi\|_{L^q(U)} \to 0$$

Hence $Dv = 0$ a.e. in U, v = const. in U. $(v)_U = 0$ implies that $v = 0$ a.e. in U, but $\|v\|_{L^p(U)} = 1$, which is a contradiction.

□

Theorem (**Poincare's Inequality for a Ball**). Assume $1 \leqslant p \leqslant \infty$. Then there exists a constant C such that

$$\left\|u - (u)_{\bar{B}(x,r)}\right\|_{L^p(\bar{B}(x,r))} \leqslant C(n,p)r\,\|Du\|_{L^p(\bar{B}(x,r))}$$

for each ball $\bar{B}(x,r) \subseteq \mathbb{R}^n$ and each function $f \in W^{1,p}(B(x,r))$

Proof. For $U = B(0,1)$, this follows from the Poincare's inequality. If $u \in W^{1,p}(B(x,r))$, set

$$v(y) = u(x + ry), y \in B(0,1),$$

then $v \in W^{1,p}(B(0,1))$;

$$\left\| v - (v)_{\bar{B}(0,1)} \right\|_{L^p(\bar{B}(0,1))} \leqslant C \left\| Dv \right\|_{L^p(\bar{B}(0,1))}$$

□

3.8.2 Difference Quotients

Assume $u : U \to \mathbb{R}$, $u \in L^1_{\text{loc}}(U)$ and $V \Subset U$. Then the **i-th difference quotient of size** h is

$$D + i^h u(x) := \frac{u(x + he_i) - u(x)}{h}$$

$(1 \leqslant i \leqslant n\ )$ for $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < \text{dist}(V, \partial U)$.

$$D^h u := (D^h_1 u, \ldots, D^h_n u).$$

Theorem (Difference Quotients and Weak Derivatives).

1. Suppose $1 \leqslant p < \infty$ and $u \in W^{1,p}(U)$. Then for each $V \Subset U$

$$\left\| D^h u \right\|_{L^p(V)} \leqslant C \left\| Du \right\|_{L^p(U)}$$

for some constant C and all $0 < |h| \leqslant \frac{1}{2}\text{dist}(V, \partial U)$.

2. Assume $1 < p < \infty$, $u \in L^p(V)$, and there exists a constant C such that

$$\left\| D^h u \right\|_{L^p(V)} \leqslant C$$

for all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$. Then $u \in W^{1,p}(V)$, with $\left\| Du \right\|_{L^p(V)} \leqslant C$.

we prove the "integration-by-parts" formula for difference quotients which will be use in the proof:

$$\int_V u(D^h_i \phi) dm = -\int_V (D^{-h}_i u)\phi dm$$

$$\begin{aligned}
\int_V u(D^h_i \phi) dm &= \int_V u(x)\left(\frac{\phi(x + he_i) - \phi(x)}{h}\right) \\
&= \frac{1}{h}\left(\int_{V+he_i} u(x - he_i)\phi(x) dm - \int_V u(x)\phi(x) dm \right) \\
&= -\frac{1}{h}\left(\int_V u(x)\phi(x) dm - \int_V u(x - he_i)\phi(x) \right) \\
&= -\int_V \left(\frac{u(x) - u(x - he_i)}{h}\right)\phi(x) dm \\
&= -\int_V (D^{-h}_i u)\phi dm
\end{aligned}$$

The third equality follows from ϕ has compact support in V, and thus we can make h small enough so that $\operatorname{spt}(\phi) + he_i \subseteq V$.

And a fact from functional analysis used in the proof is that: Let X be a reflexive Banach space, and let $\{x_n\}$ be any sequence which is bounded in norm. Then we can choose a subsequence $\{x_{n_j}\}$ which converges weakly to an element of X.

3.8.3 Lipschitz functions and $W^{1,\infty}$

Theorem (Characterization of $W^{1,\infty}$). Let U be an open bounded subset with a C^1 boundary. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$.

Remark.

1. A slight variant of the proof shows that for any open set U, $u \in W^{1,\infty}_{\text{loc}}(U)$ if and only if u is locally Lipschitz continuous in U.
2. There is a gap in the proof: the mollifier for a $W^{1,\infty}(\mathbb{R}^n)$ function satisfies the following properties:

$$\begin{cases} u^\varepsilon \in C^\infty(\mathbb{R}^n) \\ u^\varepsilon \to u \text{ uniformly as } \varepsilon \to 0 \\ \|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leqslant \|Du\|_{L^\infty(\mathbb{R}^n)} \end{cases}$$

The last property follows from the fact the weak derivative of the mollifier equals the mollifier of the weak derivative. Hence,

$$\begin{aligned} &Du^\varepsilon = D(\eta_\varepsilon * u) = \eta_\varepsilon * (Du) = (Du)^\varepsilon \\ &\|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n)} = \|\eta_\varepsilon * Du\|_{L^\infty(\mathbb{R}^n)} \leqslant \|\eta_\varepsilon\|_{L^1(\mathbb{R}^n)} \|Du\|_{L^\infty(\mathbb{R}^n)} = \|Du\|_{L^\infty(\mathbb{R}^n)} . \end{aligned}$$

3.8.4 Differentiability a.e.

A vector-valued function $u : U \to \mathbb{R}^m$ is differentiable at $x \in U$ if there exists $A(x) \in R^{m\times n}$ such that

$$u(y) = u(x) + A(x) \cdot (y - x) + o(\|y - x\|) \text{ as } y \to x$$

In other words,

$$\lim_{y\to x} \frac{\|u(y) - u(x) - A(x)(y - x)\|}{\|y - x\|} = 0$$

It is easy to check that $A(x)$, if it exists, is unique. We hence write

$$Du(x)$$

for $A(x)$ and call $A(x)$ the **gradient** of u at x.

Theorem (Differentiability a.e.). Assume $u \in W^{1,p}_{\text{loc}}(U)$ for some $n < p \leqslant \infty$. Then u is differentiable a.e. in U, and its gradient equals its weak derivative a.e.

Theorem (Rademacher's Theorem). Let u be a locally Lipschitz continuous in U. Then u is differentiable a.e. in U.

3.8.5 Hardy's Inequality

Theorem (**Hardy's Inequality**). Assume $n \geqslant 3$ and $r > 0$. Suppose that $u \in H^1(\bar{B}(0,r))$. Then $\frac{u}{\|x\|} \in L^2(\bar{B}(0,r))$, and

$$\int_{\bar{B}(0,r)} \frac{u^2}{\|x\|} dm \leqslant C \int_{\bar{B}(0,r)} \|Du\|^2 + \frac{u^2}{r^2} dm.$$

Proof. Asssume $u \in C^\infty(\bar{B}(0,r))$.

$$D(\frac{1}{\|x\|}) = -\frac{x}{\|x\|^3},$$
$$x \cdot D(\frac{1}{\|x\|}) = -\frac{1}{\|x\|},$$
$$\frac{1}{\|x\|^2} = -\frac{x}{\|x\|} \cdot D(\frac{1}{\|x\|}),$$
$$\operatorname{div}(\frac{1}{\|x\|}\frac{x}{\|x\|}u^2) = D(\frac{1}{\|x\|})\frac{x}{\|x\|}u^2 + \frac{1}{\|x\|}\operatorname{div}(\frac{x}{\|x\|})u^2 + \frac{2x}{\|x\|^2} \cdot (uDu).$$

$$\begin{aligned}\operatorname{div}(\frac{x}{\|x\|}) &= \frac{1}{\|x\|}\operatorname{div}x + x \cdot D(\frac{1}{\|x\|}) \\ &= \frac{1}{\|x\|}n + x \cdot (-\frac{1}{\|x\|^2}\frac{x}{\|x\|}) \\ &= \frac{n}{\|x\|} - \frac{1}{\|x\|} \\ &= \frac{n-1}{\|x\|}\end{aligned}$$

Also note in the proof, Cauchy's inequality with ε is used repeatedly:

$$ab \leqslant \varepsilon a^2 + \frac{b^2}{4\varepsilon}(a, b > 0, \varepsilon > 0)$$

$$\begin{aligned}\int_{\bar{B}(0,r)} \frac{u^2}{\|x\|^2} dm &= -\int_{\bar{B}(0,r)} u^2 D(\frac{1}{\|x\|}) \cdot \frac{x}{\|x\|} dx \\ &= \int_{\bar{B}(0,r)} 2uDu \cdot \frac{x}{\|x\|^2} + (n-1)\frac{u^2}{\|x\|^2} dm - \int_{\partial B(0,r)} u^2 \nu \cdot \frac{x}{\|x\|^2} dS.\end{aligned}$$

$$(2-n)\int_{\bar{B}(0,r)} \frac{u^2}{\|x\|^2} dm = 2\int_{\bar{B}(0,r)} uDu \cdot \frac{x}{\|x\|^2} dm - \frac{1}{r}\int_{\partial B(0,r)} u^2 dS$$

$$\int_{\bar{B}(0,r)} \frac{u^2}{\|x\|^2} dm \leqslant C\int_{\bar{B}(0,r)} \|Du\|^2 dm + \frac{C}{r}\int_{\partial B(0,r)} u^2 dS$$

$$r\int_{\partial B(0,r)} u^2 dS = \int_{\bar{B}(0,r)} \operatorname{div}(xu^2)dx = \int_{\bar{B}(0,r)} nu^2 + 2uDu \cdot x dx$$
$$\leqslant C\int_{\bar{B}(0,r)} u^2 + r^2 \|Du\|^2 \, dm.$$

$$\frac{1}{r}\int_{\partial B(0,r)} u^2 dS \leqslant C\int_{\bar{B}(0,r)} \|Du\|^2 + \frac{u^2}{r^2} dm.$$

□

3.8.6 Fourier Transform Methods

Theorem (**Characterization of** H^k **by Fourier Transform**). Let k be a nonnegative integer. Then a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if

$$(1 + \|\xi\|^k)\hat{u} \in L^2(\mathbb{R}^n).$$

Moreover, there exists a positive constant C such that

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leqslant \left\|(1 + \|\xi\|^k)\hat{u}\right\|_{L^2(\mathbb{R}^n)} \leqslant C\|u\|_{H^k(\mathbb{R}^n)}$$

for each $u \in H^k(\mathbb{R}^n)$.

Assume $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$. Then $u \in H^s(\mathbb{R}^n)$ if $(1 + \|\xi\|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For noninteger s, we set

$$\|u\|_{H^s(\mathbb{R}^n)} := \left\|(1 + \|\xi\|^s)\hat{u}\right\|_{L^2(\mathbb{R}^n)}.$$

3.9 Other Spaces of Functions

3.9.1 The Space H^{-1}

We denote by $H^{-1}(U)$ the dual space to $H_0^1(U)$, and by $\langle , \rangle$ the pairing between $H \in (U)$ and $H_0^1(U)$. We have the inclusion relation

$$H_0^1(U) \subseteq L^2(U) \subseteq H^{-1}(U).$$

For $f \in H^{-1}(U)$, we consider the operator norm:

$$\|f\|_{H^{-1}(U)} := \sup\left\{\langle f, u\rangle : u \in H_0^1(U), \|u\|_{H_0^1(U)} \leqslant 1\right\}.$$

Theorem (**Characterization of** H^{-1}). 1. Assume $f \in H^{-1}(U)$. Then there exist functions $f^0, f^1, \dots, f^n$ in $L^2(U)$ such that

$$\langle f, v\rangle = \int_U f^0 v + \sum_{i=1}^n f^i \partial_i v dm \quad (v \in H_0^1(U)).$$

2.
$$\|f\|_{H^{-1}(U)} = \inf\left\{\left(\int_U \sum_{i=0}^{n} \left|f^i\right|^2 dm\right)^{1/2} : f \text{ satisfies (}i\text{) for } f^0, f^1, \ldots, f^n \in L^2(U)\right\}.$$

3.
$$\langle v, u\rangle_{L^2(U)} = \langle v, u\rangle$$

for all $u \in H_0^1(U), v \in L^2(U) \subseteq H^{-1}(U)$.

We write $f = f^0 - \sum_{i=1}^{n} \partial_i f^i$ whenever (i) holds.

$H_0^1(U)$ is a Hilbert space and $f \in H^{-1}(U) = (H_0^1(U))^*$; thus, by Riesz Representation theorem, there exists $u \in H_0^1(U)$ such that

$$\langle f, v\rangle = \langle u, v\rangle_{H_0^1(U)} = \int_U \left(uv + \sum_{i=1}^{n}(\partial_i u)(\partial_i v)\right) dm.$$
$$\|f\|_{H^{-1}(U)} = \|u\|_{H_0^1(U)}$$

3.9.2 Spaces Involving Time

Let X be a real Banach Space, with norm $\|\cdot\|_X$. The space

$$L^p(0, T; X)$$

consists of all strongly measurable functions $u : [0, T] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty$$

for $1 \leqslant p < \infty$ and

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{esssup}_{0\leqslant t\leqslant T} \|u(t)\| < \infty.$$

The space

$$C([0, T]; X)$$

consists of all continuous functions $u : [0, T] \to X$ with

$$\|u\|_{C([0,T];X)} := \max_{0\leqslant t\leqslant T} \|u(t)\| < \infty.$$

Let $u \in L^1(0, T; X)$. We say $v \in L^1(0, T; X)$ is the **weak derivative of** u , written as

$$u' := v,$$

if

$$\int_0^T \phi'(t)u(t)dt = -\int_0^T \phi(t)v(t)dt$$

for all real valued test functions $\phi \in \mathcal{C}_c^\infty(0,T)$. The Sobolev space

$$W^{1,p}(0,T;X)$$

consists of all functions $u \in L^p(0,T;X)$ such that u' exists in the weak sense and belongs to $L^p(0,T;X)$. And

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|_X^p + \|u'(t)\|_X^p \right)^{1/p} & 1 \leqslant p < \infty \\ \operatorname*{esssup}_{0\leqslant t\leqslant T} \left(\|u(t)\|_X + \|u'(t)\|_X \right) & p = \infty \end{cases}$$

We write $H^1(0,T;X) = W^{1,2}(0,T;X)$.

Example. $U \subseteq \mathbb{R}^n$ is open, $X = L^q(U)$ for $1 \leqslant q \leqslant \infty$. $u = u(x,t) \in L^p(0,T;L^q(U))$.

$$\|u\|_{L^p(0,T;L^q(U))} = \left(\int_0^T \|u\|_{L^q(U)}^p \, dt \right)^{1/p} = \left(\int_0^T \big(\int_U |u|^q \, dm \big)^{p/q} dt \right)^{1/p}$$
$$\|u\|_{C(0,T;L^2(U))} = \max_{0\leqslant t\leqslant T} \|u(\cdot,t)\|_{L^2(U)}.$$

$u : [0,T] \to X$ is continuous at $t_0 \in (0,T)$ if $u(t) \longrightarrow u(t_0)$ in X as $t \longrightarrow t_0$, i.e. $\|u(t) - u(t_0)\|_X \to 0$ as $t \to t_0$.

Theorem (Calculus in Abstract Space). Let $u \in W^{1,p}(0,T;X)$ for some $1 \leqslant p \leqslant \infty$. Then

1. $u \in C([0,T;X])$.
2. $u(t) = u(s) + \int_s^t u'(\tau)d\tau$, for all $0 \leqslant s \leqslant t \leqslant T$.
3. $\max_{0\leqslant t\leqslant T} \|u(t)\|_X \leqslant C(T) \|u\|_{W^{1,p}(0,T;X)}$.

Theorem (More Calculus). Suppose $u \in L^2(0,T;H_0^1(U))$, with $u' \in L^2(0,T;H^{-1}(U))$.

1. $u \in C([0,T];L^2(U))$
2. The mapping

$$t \mapsto \|u(t)\|_{L^2(U)}^2$$

is absolutely continuous, with

$$\frac{d}{dt} \|u(t)\|_{L^2(U)}^2 = 2 \langle u'(t), u(t) \rangle$$

for a.e. $0 \leqslant t \leqslant T$.

3. $\max\limits_{0\leqslant t\leqslant T} \|u(t)\|_{L^2(U)} \leqslant C(T)\big(\|u\|_{L^2(0,T;H_0^1(U))} + \|u'\|_{L^2(0,T;H^{-1}(U))} \big)$.

Theorem (Mapping into Better Space). Assume U is open bounded subset with C^1 boundary. Take m to be a nonnegative integer. Suppose $u \in L^2(0,T;H^{m+2}(U))$, with $u' \in L^2(0,T;H^m(U))$.

1. $u \in C([0,T];H^{m+1}(U))$.
2. $\max\limits_{0\leqslant t\leqslant T} \|u(t)\|_{H^{m+1}(U)} \leqslant C(T,U,M)\big(\|u\|_{L^2(0,T;H^{m+2}(U))} + \|u'\|_{L^2(0,T;H^m(U))} \big)$.

Chapter 4

Second-Order Elliptic Equations

In this chapter, we will study the solvability of uniformly elliptic, second-order PDEs, subject to Dirichlet boundary conditions:

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where U is an open bounded subsest of $\mathbb{R}^n$ and $u : \bar{U} \to \mathbb{R}$ is the unknown function, and L denoted a second-order PDE operator having either the **divergence form**:

$$Lu = -\sum_{ij=1}^{n} \partial_j (a^{ij}(x)\partial_i u) + \sum_{i=1}^{n} b^i(x)\partial_i u + c(x)u,$$

which is most natural for energy methods based upon integration by parts, or the **nondivergence form**:

$$Lu = -\sum_{ij=1}^{n} a^{ij}(x)\partial_{ij} + \sum_{i=1}^{n} b^i(x)u + c(x)u,$$

which is most appropriate for maximal principle techniques. The uniform ellipticity means $a^{ij} = a^{ji}$ and there exists a constant $\theta > 0$ such that

$$\sum_{ij=1}^{n} a^{ij}(x)\xi_i\xi_j \geqslant \theta \left\| x \right\|^2$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}$.

4.0.1 Weak Solution

We will assume $a^{ij}, b^i, c \in L^\infty(U)$ and $f \in L^2(U)$.

The bilinear form $B[,]$ associated with the divergene form operator L is defined as

$$B[u,v] := \int_U \sum_{ij=1}^{n} a^{ij}\partial_i u \partial_j v + \sum_{i=1}^{n} b^i \partial_i u v + cuvdm$$

for $u, v \in H_0^1(U)$. Then $u \in H_0^1(U)$ is a **weak solution** of the Dirichlet boundary value problem if

$$B[u,v] = \langle f, v\rangle_{L^2(U)}$$

for all $v \in H_0^1(U)$.

Let us consider another Dirichlet boundary value problem:

$$\begin{cases} Lu = f^0 - \sum_{i=1}^{n} \partial_i f^i & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where $f^i \in L^2(U)$. The RHS $\tilde{f} := f^0 - \sum_{i=1}^{n} \partial_i f^i \in H^{-1}(U)$. Then $u \in H_0^1(U)$ is a weak solution of the above Dirichlet boundary value problem if

$$B[u,v] = \langle f, v\rangle_{H^{-1}(U)\times H_0^1(U)}$$

for all $v \in H_0^1(U)$, where $\langle f, v\rangle_{H^{-1}(U)\times H_0^1(U)} = \int_U f^0 v + \sum_{i=1}^{n} f^i \partial_i v dm$.

We assume temporarily that H is a Hilbert space, with the dual space H^*. The following Lax-Milgram theorem in Hilbert space thoery is important in that it underlies the existence of the weak solution we're considering.

Theorem (**Lax-Milgram Theorem**). Assume B is a bilinear form on H, such that there exist constants $\alpha, \beta > 0$

$$|B[u,v]| \leqslant \alpha \|u\|_H \|v\|_H \quad (u, v \in H)$$
$$B[u,u] \geqslant \beta \|u\|_H^2 \quad (u \in H)$$

Then for any $f \in H^*$, there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v\rangle_H$$

for all $v \in H$.

Note that if B is symmetric, then $(u, v) = B[u,v]$ is a new inner product on H, to which we can directly apply the Riesz Representation THeorem. Hence, the significance of the Lax-Milgram Theorem liesin that it does not require the symmetry of B.

We next deduce the energy estimate for the bilinear form B associated with the Dirichlet boudary value problem $Lu = 0$:

$$B[u,v] = \int_U \sum_{ij=1}^{n} a^{ij}\partial_i u \partial_j v + \sum_{i=1}^{n} b^i \partial_i u v + cuvdm$$

Theorem (Energy Estimate). There exis constants $\alpha, \beta > 0$ and $\gamma \geqslant 0$ such that

$$|B[u,v]| \leqslant \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$
$$\beta \|u\|^2_{H_0^1(U)} \leqslant B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

for all $u, v \in H_0^1(U)$.

Proof.

$$\begin{aligned} |B[u,v]| &\leqslant \sum_{ij=1}^{n} \left\|a^{ij}\right\|_{L^\infty(U)} \int_U \|Du\| \|Dv\| \, dm \\ &+ \sum_{i=1}^{n} \left\|b^i\right\|_{L^\infty(U)} \int_U \|Du\| |v| \, dm + \|c\|_{L^\infty(U)} \int_U |u| |v| \, dm \\ &\leqslant \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \end{aligned}$$

$$\begin{aligned} \theta \int_U \|Du\|^2 \, dm &\leqslant \int_U \sum_{ij=1}^{n} a^{ij} \partial_i u \partial_j u dm \\ &= B[u,u] - \int_U \sum_{i=1}^{n} b^i \partial_i u u + c u^2 dm \\ &\leqslant B[u,u] + \sum_{i=1}^{n} \left\|b^i\right\|_{L^\infty(U)} \int_U \|Du\| |u| \, dm + \|c\|_{L^\infty(U)} \int_U u^2 dm \end{aligned}$$

By Cauchy Inequality with ε, if we choose $\varepsilon \sum_{i=1}^{n} \left\|b^i\right\|_{L^\infty(U)} < \dfrac{\theta}{2}$:

$$\frac{\theta}{2} \int_U \|Du\|^2 \, dm \leqslant B[u,u] + C \int_U u^2 dm$$

It follows from the Poincaré's inequality that

$$\beta \|u\|^2_{H_0^1(U)} \leqslant B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

□

Theorem (First Existence Theorem for Weak Solution). There exists a $\gamma \geqslant 0$ such that for each $\mu \geqslant \gamma$ and each $f \in L^2(U)$, $Lu + \mu u = f$ has a unique weak solution $u \in H_0^1(U)$.

Proof. By energy estimate, there exists $\gamma \geqslant 0$ such that for each $\mu \geqslant \gamma$, the bilinear form $B_\mu[u,v] := B[u,v] + \mu \langle u, v\rangle_{L^2(U)}$ corresponding to $L_\mu u := Lu + \mu u$ satisfies the hypothesis of the Lax-Milgram Theorem. For each $f \in L^2(U)$, $\langle f, v, \rangle := \langle f, v\rangle_{L^2(U)}$ is a linear form on $L^2(U)$ and thus on $H_0^1(U)$. The Lax-Milgram Theorem applies to give the result. □

The operator L^*, which is the formal adjoint of L, is

$$L^* v := -\sum_{ij=1}^{n} \partial_i (a^{ij} \partial_j v) - \sum_{i=1}^{n} b^i \partial_i v + (c - \sum_{i=1}^{n} \partial_i b^i) v,$$

if $b^i \in C(\bar{U})$. The adjoint bilinear form

$$B^* : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$$

is defined by

$$B^*[v, u] := B[u, v]$$

for all $u, v \in H_0^1(U)$. We say that $v \in H_0^1(U)$ is a weak soolution of the adjoint Dirichlet boundary value problem $L^* v = f$ provided

$$B^*[v, u] = \langle f, u \rangle_{L^2(U)}$$

for all $u \in H_0^1(U)$.

We now introduce compact operators and the Fredholm alternative. Let X, Y be Banach spaces. A bounded linear operator

$$K : X \to Y$$

is called a **compact operator** if for the unit ball $\bar{B}(0, 1)$ in X, $K(\bar{B}(0, 1))$ is precompact in Y.

Theorem (**Fredholm Alternative**). Let $K : H \to H$ be a compact operator on the Hilbert space H. Then

1. $N(I - K)$ is finite dimensional,
2. $R(I - K)$ is closed,
3. $R(I - K) = N(I - K^*)^\perp$,
4. $N(I - k) = 0$ if and only if $R(I - K) = H$,
5. $\dim N(I - K) = \dim N(I - K^*)$.

Hence,

1. Precisely one and only one of the following holds:
   (α) for each $f \in H$, $u - Ku = f$ has a unique solution.
   (β) $u - Ku = 0$ has a nonzero solution.
2. Should (β) hold, $\dim N(I - K) = \dim N(I - K^*)$.
3. $u - Ku = f$ has a solution if and only if $f \in N(I - K^*)^\perp$.

We now use the Fredholm Alternative to prove the existence of weak solutions:

Theorem (Second Existence Theorem for Weak solutions). 1. Precisely one and only of the following holds:

(α) for each $f \in L^2(U)$, $Lu = f$ has a unique weak solution.

(β) $Lu = 0$ has a weak solution $u \neq 0$

2. Should (β) hold, $\dim\left\{u \in H_0^1(U) : Lu = 0\right\} = \dim\left\{v \in H_0^1(U) : L^*v = 0\right\}$
3. $Lu = f$ has a weak solution if and only if $\langle f, v\rangle_{L^2(U)} = 0$ for all $v \in \left\{v \in H_0^1(U) : L^*v = 0\right\}$.

Proof. **Step one: The euqivalent formulation of $Lu = f$.** By the first existence theorem, there exists $\gamma \geqslant 0$ such that for each $\mu \geqslant \gamma$, $Lu + \mu u = g$ has a unique weak solution in $H_0^1(U)$ for each $g \in L^2(U)$. That is, there exists a linear operator

$$L_\mu^{-1} : L^2(U) \to H_0^1(U) \subseteq L^2(U)$$

Then the following are equivalent:

1. $Lu = f$ has a unique weak solution.
2. $L_\mu u = \mu u + f$ has a unique weak solution.
3. $u = L_\mu^{-1}(\mu u + f)$ has a unique weak solution.
4. $u - Ku = h$ has a unique weak solution, where $Ku := \mu L_\mu^{-1} u$ and $h := L_\mu^{-1} f$.

Stwp two: K is a compact operator. This follows from the energy estimate that if $u \in H_0^1(U)$ is the weak solution of $Lu + \mu u = g$,

$$\beta \left\|u\right\|_{H_0^1(U)}^2 \leqslant B[u, u] + \mu \left\|u\right\|_{L^2(U)}^2 = \langle g, u\rangle_{L^2(U)} \leqslant \left\|g\right\|_{L^2(U)} \left\|u\right\|_{L^2(U)} \leqslant \left\|g\right\|_{L^2(U)} \left\|u\right\|_{H_0^1(U)}$$

Then

$$\left\|Kg\right\|_{H_0^1(U)} \leqslant C \left\|g\right\|_{L^2(U)}.$$

Rellich-Kondrachov compactness theorem applies to gives the result.

Step three: Conclusion. Apply the Fredholm Alternative to K gives:

1. Precisely one and only one of the following holds:

(α) for each $h \in L^2(U)$, $u - Ku = h$ has a unique weak solution in $H_0^1(U)$, which is equivalent to $Lu = f$ has a unique weak solution in $H_0^1(U)$.

(β) $u - Ku = 0$ has nonzero solutions in $H_0^1(U)$, which is equivalent to $Lu = 0$ has nonzero solutions.

2. Should (β) hold, $\dim N(I - K) = \dim N(I - K^*)$. But by definition, we have

$$N(I - K) = \left\{u \in H_0^1(U) : Lu = 0\right\}$$
$$N(I - K^*) = \left\{v \in H_0^1(U) : L^*v = 0\right\}$$

3. $u - Ku = h$ has a solution if and only if $\langle h, v\rangle_{L^2(U)} = 0$ for all $v \in H_0^1(U)$ solving $v - K^*v = 0$. But then

$$\langle h, v\rangle_{L^2(U)} = \frac{1}{\mu}\langle Kf, v\rangle_{L^2(U)} = \frac{1}{\mu}\langle f, K^*v\rangle_{L^2(U)} = \frac{1}{\mu}\langle f, v\rangle$$

□

Now we investigate the spectrum of a compact operator.

Theorem (**Spectrum of a Compact Operator**). Assume the Hilbert space H is infinite-dimensional and K is a compact operator on H. Then

1. $0 \in \sigma(K)$.
2. $\sigma(K) - \{0\} = \sigma_p(K) - \{0\}$.
3. $\sigma(K) - \{0\}$ is either finite or a sequence tending to 0.

Theorem (**Third Existence Theorem for Weak Solution**). There exists an at most countable set $\Sigma \in \mathbb{R}$ such that $Lu = \lambda u + f$ has a unique weak solution in $H_0^1(U)$ for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$. If, moreover, Σ is infinite, then Σ is a nondecreasing sequence. Σ is called the (**real spectrum**) of the operator L.

Proof. Choose γ as in the energy estimate and WLOG $\gamma > 0$. We assume $\lambda > -\gamma$.

By the Fredholm alternative, the following are equivalent:

1. $Lu = \lambda u + f$ has a unique weak solution.
2. $Lu = lambdau$ has a unique weak solution $u \equiv 0$.
3. $Lu + \gamma u = (\gamma + \lambda)u$ has a unique weak solution $u \equiv 0$.
4. $u = L_\gamma^{-1}(\gamma + \lambda)u = \dfrac{\gamma + \lambda}{\gamma} Ku$ has a unique weak solution $u \equiv 0$.
5. $\dfrac{\gamma}{\gamma + \lambda}$ is not an eigenvalue of K.

The Spectrum Theorem of compact operators applies to give the desired result. □

Theorem (**Boundedness of the Inverse**). If $\lambda \notin \Sigma$, there exists a constant C such that

$$\|u\|_{L^2(U)} \leqslant C(\lambda, U, a^{ij}, b^i, c)\, \|f\|_{L^2(U)}$$

whenever $f \in L^2(U)$ and u is the unique weak solution of $Lu = \lambda u + f$

Proof. Proof by contradiction, we can find $\{f_k\} \subset L^2(U)$ and $\{u_k\} \subset H_0^1(U)$ such that $Lu_k = \lambda u_k + f_k$, but

$$\|u_k\|_{L^2(U)} > k\, \|f_k\|_{L^2(U)}$$

After normalizing u_k, we find $f_k \xrightarrow{L^2(U)} 0$. $\{u_k\}$ is bounded in $H_0^1(U)$ by the energy estimate. Thus there exists a subsequence $\{u_{k_j}\} \subseteq \{u_k\}$ such that $u_{k_j} \overset{H_0^1(U)}{\rightharpoonup} u$ and $u_{k_j} \xrightarrow{L^2(U)} u$. Then u is a weak solution of $Lu = \lambda u$. $\lambda \notin \Sigma$ implies $u \equiv 0$, which contradicts $\|u\|_{L^2(U)} = 1$. $\square$

Example. We consider the solution $u = -\triangle^{-1} f \in H_0^1(U)$ of the Poisson equation $-\triangle u = f$ for $f \in L^2(U)$. The associated bilinear form can be calculated as follows:

$$B[u,v] = \int_U Du \cdot Dv dm.$$

$$B[u,u] = \int_U \|Du\|^2 dm = \langle f,u\rangle_{L^2(U)} \leqslant \|f\|_{L^2(U)} \|u\|_{L^2(U)} \leqslant \|f\|_{L^2(U)} \|Du\|_{L^2(U)}.$$

$$\left\|\triangle^{-1} f\right\|_{H_0^1(U)} = \|u\|_{H_0^1(U)} \leqslant C \|f\|_{L^2(U)}$$

Since $H_0^1(U) \Subset L^2(U)$, we know $-\triangle^{-1}$ is a compact operator.