Review of Topology

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1 Introduction

This note is to give a basic introduction to topology, including point set topology (which will contains many basic topologies and their properties), and algebraic topology. We assume readers have familiarity of topology.

Part I
Point Set Topology

2 Topological Space

Topological spaces is a set endowed with a special structure, called the **topological structure**. Let X be a set. A **topology on** X or the topological structure on X is a collection $\mathcal{T}$ of subsets of X, called **open sets**, satisfying

(i) $\varnothing, X \in \mathcal{T}$.

(ii) $\bigcup\limits_{\alpha \in A} U_\alpha \in \mathcal{T}$ for $U_\alpha \in \mathcal{T}$.

(iii) $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$ for $U_i \in \mathcal{T}$.

We will write $(X, \mathcal{T})$ to mean a topological space, with X to be the base set and $\mathcal{T}$ is the topology on X. If there is no ambiguity, we will omit $\mathcal{T}$ and use X to mean the topological space, and say "X is a topological space".

We next introduce some basic concetps in topology. Assume X is a topological space and $p \in X$, $S \subseteq X$.

(i) A **neighbourhood of** p is a subset which contains an open subset containing p. Similarly, a **neighbourhood of** S is a subsett which contains an open subset containing S. Be aware that some authors use an open subset containing p or S as a neighbourhood of p or S. In our definition, we will use **an open neighbourhood** to say an open subset containing p or S.

(ii) S is said to be closed if $X - S$ is open (where $X - S$ is the set difference).

(iii) The **interior of** S, denoted by $\text{Int}S$, is the union of all open subsets of X contained in S.

(iv) The **exterior of** S, denoted by $\text{Ext}S$, is the union of all open subsets of X contained in $X - S$.

(v) The **closure of** S, denoted by $\bar{S}$, is the intersection of all closed subsets of X containing S.

(vi) The **boundary of** S, denoted by ∂S, is the set of all points of X that are in neither $\text{Int}S$ nor $\text{Ext}S$.

(vii) A point $p \in S$ is said to be an **isolated point of** S if p has an open neighbourhood $U \subseteq X$ such that $U \cap S = \{p\}$.

(viii) A point $p \in S$ (not necessarily in S) is said to be a **limit point of** S if every neighbourhood of p contains at least one point of S other than p.

(ix) S is said to be **dense in** X if $\bar{S} = X$, or equivalently if every nonempty open subset of X contains at least one point of S.

(x) S is said to be **nowhere dense in** X if $\bar{S}$ contains no nonempty open subset.

One of the most important concepts in topology is continuity and convergence. Now let X, Y be topological spaces.

(i) A map $F : X \to Y$ is said to be **continuous** if for every open subset $U \subseteq Y$, the preimage $F^{-1}(U)$ is an open subset of X.

(ii) A continuous bijective map $F : X \to Y$ with continuous inverse is called a **homeomorphism**. If there exists a homeomorphism from X to Y, we say that X and Y are **homeomorphic**.

(iii) A continuous map $F : X \to Y$ is said to be a **local homeomorphism** if every point $p \in X$ has an open neighbourhood $U \subseteq X$ such that $F(U)$ is open in Y and F restricts to a homeomorphism from U to $F(U)$.

(iv) Given a sequence $(p_i)_{i=1}^{\infty}$ of points in X and a point $p \in X$, the sequence is said to **converge to** p if for every neighbourhood U of p, there exists a positive integer N such that $p_i \in U$ for all $i \geq N$. In this case, we write $p_i \to p$ or $\lim_{i\to\infty} p_i = p$.

Remark. Let $F : X \to Y$ be a map between topological spaces. Then the continuity of F is equivalent to each of the following:

(a) For every subset $A \subseteq X$, $F(\bar{A}) \subseteq \bar{F(A)}$.

(b) For every subset $B \subseteq Y$, $F^{-1}(\mathrm{Int} B) \subseteq \mathrm{Int} F^{-1}(B)$.

Remark. Let X, Y and Z be topological spaces. The following maps are continuous:

(a) The **identity map** $\mathrm{Id}_X : X \to X$ by $x \mapsto x$ for all $x \in X$.

(b) Any **constant map** $F : X \to Y$ such that $F(x) = F(y)$ for all $x, y \in X$.

(c) Any composition $G \circ F$ of continuous maps $F : X \to Y$ and $G : Y \to Z$.

Remark. Let X, Y be topological spaces. Suppose $F : X \to Y$ is continuous and $p_i \to p$ in X. Then $F(p_i) \to F(p)$ in Y.

Example (Discrete Space)**.** If X is an arbitrary set, the **discrete topology** on X is a topology defined by claiming every point of X to be open. Then every subset of X is open in this topology. Any space with this topology is called a **discrete space**.

Example (Metric Space)**.** A **metric space** is a set X endowed with a **metric** $d : X \times X \to \mathbb{R}$ satisfying the following properties:

(i) Positivity: $d(x, y) \geq 0$, with equality if and only if $x = y$.

(ii) Symmetry: $d(x, y) = d(y, x)$.

(iii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

If X is a metric space, $x \in X$ and $r \geqslant 0$, the **open ball of radius** r **around** x is the set

$$B(x, r) = \{y \in X : d(x, y) \leqslant r\}$$

and the **closed ball of radius** r **around** x is

$$\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}\,.$$

The **metric topology on** X is defined by declaring each open ball to be open.

A subset $S \subseteq X$ of a metric space can inherit the metric topology of X, and hence is a metric space. We use the following standard terminology for metric spaces:

(i) A subset $S \subseteq X$ is **bounded** if there exists a positive number M such that $d(x, y) \leq M$ for all $x, y \in X$. In this case, the **diameter of** S is defined as $\operatorname{diam} S = \sup \{d(x, y) : x, y \in S\}$.

(ii) A sequence of point $(x_i)_{i=1}^{\infty}$ in X is a **Cauchy sequence** if for every $\varepsilon \geqslant 0$, there exists an integer N such that $i, j \geq N$ implies $d(x_i, x_j) \leqslant \varepsilon$.

(iii) A metric space X is said to be complete if every Cauchy sequence in X converges to a point of X.

Example (Euclidean Space)**.** For each integer $n \geq 1$, the set $\mathbb{R}^n$ of ordered n-tuples of real numbers is called n**-dimensional Euclidean Space**. We use x, (x_i), or $(x_1, \ldots, x_n)$ to denote a point in $\mathbb{R}^n$, where each x_i is a **coordinate or component of** x. For each $x \in \mathbb{R}^n$, the **Euclidean norm of** x is a non-negative real number defined by

$$\|x\| = \sqrt{(x_1)^2 + \ldots + (x_n)^2}$$

and for $x, y \in \mathbb{R}^n$, the **Euclidean metric** is defined by

$$d(x, y) = \|x - y\|$$

Under this metric, $\mathbb{R}^n$ becomes a complete metric space. The topology induced by this metric is called the **Euclidean topology**.

Example (Complex Eucliden Space)**.** If we work in a finite-dimensional complex vector space (a vector space over the complex number field $\mathbb{C}$), we can regard this space as a real vector space. For this purpose, we call the ***n*-dimensional complex Euclidean space** the set $\mathbb{C}^n$ of ordered n-tuples of complex numbers. This space can be identified with $\mathbb{R}^{2n}$ via the correspondence

$$(x_1 + iy_1, \ldots, x_n + iy_n) \leftrightarrow (x_1, \ldots, x_n, y_1, \ldots y_n)$$

Actually, this process is called the **realification** of a finite-dimensional complex vector space: Since the real line $\mathbb{R}$ can be imbedded into the complex plane $\mathbb{C}$, every complex vector space is not only a module over $\mathbb{C}$, but also a module over $\mathbb{R}$, that is, we have multiplication by real numbers in this space. Thus, we can discard the complex module structure, and consider only the real module structure. After realification, the dimension of this vector space over $\mathbb{R}$ is two times as that of this space over $\mathbb{C}$. If $\{e_1, \ldots, e_n\}$ is a set of $\mathbb{C}$-basis, then $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$ is a set of $\mathbb{R}$-basis. If under $\{e_1, \ldots, e_n\}$, an element has the coordinate expression $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$, then under $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$, this element has the coordinate expression $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Example (Subsets of Euclidean Space)**.** Since each complex vector space $\mathbb{C}^n$ can be regarded as $\mathbb{R}^{2n}$, we will use the real Euclidean space in this example. Every subset of $\mathbb{R}^n$ becomes a metric space, and thus a topological space, when equipped with the Euclidean space. Whenever we mention such a subset, it is always assumed to have this metric topology unless otherwise specified. It is a complete metric space if and only if it is a closed subset of $\mathbb{R}^n$. Here are some standard subsets of Euclidean spaces that we work with frequently:

(i) The **unit interval** is the subset I of $\mathbb{R}$ defined by

$$I = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$$

(ii) The **(open) unit ball of dimension** n is the subset $\mathbb{B}^n$ of $\mathbb{R}^n$ defined by

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$$

(iii) The **closed unit ball of dimension** n is the subset $\bar{\mathbb{B}}^n$ of $\mathbb{R}^n$ defined by

$$\bar{\mathbb{B}}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

When $n = 2$, we also use terms like **(open) unit disk** and **closed unit**

disk for $\mathbb{B}^2$ and $\bar{\mathbb{B}}^2$.
For $n \geq 0$, the **(unit)** n**-sphere** is the subset $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ defined by

$$\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$$

Sometimes it is useful to think of an odd-dimensional sphere $\mathbb{S}^{2n+1}$ as a subset of $\mathbb{C}^{n+1}$, by means of the usual identification of $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. The **(unit) circle** is the 1-sphere $\mathbb{S}^1$, considered either as a subset of $\mathbb{R}^2$ or as a subset of C.

3 Hausdorff Space

Topological spaces allow us to describe a wide variety of concepts of ”spaces”. But for the purposes of manifold theory, arbitrary topological spaces are far too general, because they can have some unpleasant properties, as the next example illustrates.

Example. Let X be any set. $\mathcal{T} = \{\varnothing, X\}$ is called the **trivial topology**. When X is equipped with this topology, every sequence in X converges to every point of X, and every map from a topological space into X is continuous.

To avoid pathological cases like this, which result when X does not have sufficiently many open subsets, we often restrict our attention to topological spaces satisfying the following special condition. A topological space X is said to be a **Hausdorff space** if for every pair of distinct points $p, q \in X$, there exist disjoint open subsets $U, V \subseteq X$ such that $p \in U$ and $q \in V$.

Remark. (i) Every metric space is Hausdorff space in the metric topology.

(ii) Let X be a Hausdorff space, then each singleton is closed, and each converging sequence has a unique limit point.

4 Base and Countability

If $\mathcal{T}_1, \mathcal{T}_2$ are two topologies on X such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is **weaker**, or **coarser** than $\mathcal{T}_2$, and $\mathcal{T}_2$ is **stronger**, or **finer** than $\mathcal{T}_1$. Clearly the trivial topology is the weakest topology on X, while the discrete topology is the strongest. Clearly, the intersection of any family of topologies on X is still a topology, then for any $\mathcal{E} \subseteq \mathcal{P}(X)$, there is a unique weakest topology $\mathcal{T}(\mathcal{E})$ on X that contains $\mathcal{E}$, namely the intersection of all topologies on X containing $\mathcal{E}$:

$$\mathcal{T}(\mathcal{E}) = \bigcap_{\substack{\mathcal{T} \text{ is a topology on } X \\ \mathcal{E} \subseteq \mathcal{T}}} \mathcal{T}$$

It is called the topology **generated by** $\mathcal{E}$, and $\mathcal{E}$ is sometimes called a **subbasis** for $\mathcal{T}(\mathcal{E})$. It can be checked that each element in $\mathcal{T}(\mathcal{E})$ can be written as an arbitrary union of finite intersections of elements in $\mathcal{E}$.

A collection $\mathcal{B}$ of open subsets of X is said to be a **basis for the topology of** X if every open subset of X is the union of some collection of elements of $\mathcal{B}$. Or equivalently, if there exist a subbasis $\mathcal{E}$ for this topology, each basis element is either $\varnothing$, X, or can be written as a finite intersection of elements in $\mathcal{E}$.

More generally, suppose X is merely a set, and $\mathcal{B}$ is a collection of subsets of X satisfying the following conditions:

(i) $X = \bigcup_{B \in \mathcal{B}} B$.

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the collection of all unions of elements of $\mathcal{B}$ is a topology on X, called the **topology generated by** $\mathcal{B}$, and $\mathcal{B}$ is a basis for this topology. And a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X is a basis for this topology if and only if elements in $\mathcal{B}$ satisfies the above two conditions.

If X is a topological space and $p \in X$, a **neighbourhood basis at** p is a collection $\mathcal{B}_p$ of neighbourhoods of p such that every neighbourhood of p contains at least one $B \in \mathcal{B}_p$.

A topological space X is said to be **first-countable** if there is a countable neighbourhood basis at each point, and **second-countable** if there is a countable basis for this topology. Since a countable basis for X contains a countable neighbourhood basis at each point, second-countability implies first-countability.

The next lemma expresses the most important properties of first-countable spaces. To say that a sequence is **eventually in a subset** means that all but finitely many terms of the sequence are in the subset.

Lemma (Sequence lemma)**.** Let X be a first-countable space, let $A \subseteq X$ be any subset, and let $x \in X$.

(i) $x \in \bar{A}$ if and only if x is a limit of a sequence of points in A.

(ii) $x \in \mathrm{Int} A$ if and only if every sequence in X converging to x is eventually in A.

(iii) A is closed in X if and only if A contains every limit of every convergent sequence of points in A.

(iv) A is open in X if and only if every sequence in X converging to a point of A is eventually in A.

Remark. Every metric space is first-countable.

Remark. The set of all open balls in $\mathbb{R}^n$ whose radii are rational and whose centers have rational coordinates is a countable basis for the Euclidean topology, and thus $\mathbb{R}^n$ is second countable.

One of the most important properties of second-countable spaces is expressed in the following proposition. Let X be a topological space. A **cover of** X is a collection $\mathcal{U}$ of subsets of X whose union is X; it is called an **open cover** if each of the sets in $\mathcal{U}$ is open. A **subcover of** $\mathcal{U}$ is a subcollection of $\mathcal{U}$ that is still a cover.

Proposition. Let X be a second-countable topological space. Every open cover of X has a countable subcover.

5 Subspace

Probably the simplest way to obtain new topological spaces from old ones is by taking subsets of other spaces. If X is a topological space and $S \subseteq X$ is an arbitrary subset, we define the **subspace topology on** S (Some authors use **relative topology**) by claiming a subset $U \subseteq X$ to be open in S if and only if there exists an open subset $V \subseteq X$ such that $U = V \cap S$. A subset of S that is open or closed in the subspace topology is sometimes said to be **relaatively open** or **relatively closed in** S, to make it clear that we do not mean open or closed as a subset of X. Any subset of X endowed with the subspace topology is said to be a **subspace of** X. Whenever we treat a subset of a topological space as a space in its own right, we always assume that it has the subspace topology unless otherwise specified.

If X, Y are topological spaces, a continuous injective map $F : X \to Y$ is called a **topological embedding** if it is a homeomorphism onto its image $F(X) \subseteq Y$ in the subspace topology.

Proposition (Properties of the Subspace Topology)**.** Let X be a topological space and let S be a subspace of X.

(i) Characteristic Property: If Y is a topological space, a map $F : Y \to S$ is continuous if and only if $\iota_S \circ F : Y \to X$ is continuous, where $\iota_S : S \hookrightarrow X$ is the inclusion map.

(ii) The subspace topology is the unique topology on S for which the characteristic property holds.

(iii) A subset $K \subseteq S$ is closed in S if and only if there exists a closed subset $L \subseteq X$ such that $K = L \cap S$.

(iv) The inclusion map $\iota_S : S \hookrightarrow X$ is a topological embedding.

(v) If Y is a topological space and $F : X \to Y$ is continuous, then $F|_S : S \to Y$ is continuous.

(vi) If $\mathcal{B}$ is a basis for the topology of X, then $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$ is a basis for the subspace topology on S.

(vii) If X is Hausdorff, then so is S.

(viii) If X is first-countable, then so is S.

(ix) If X is second-countable, then so is S.

If X, Y are topological spaces and $F : X \to Y$ is a continuous map, then the restriction of F to every subspace of X is continuous. We naturally ask the converse question: If we know that the restriction of F to certain subspaces of

X is continuous, is F itself continuous? The next two propositions express two somewhat different answers to this question.

Lemma (Continuity is Local Property)**.** Continuity is a local property, in the following sense: if $F : X \to Y$ is a map between topological spaces such that every point $p \in X$ has a neighbourhood U on which the restriction $F|_U$ is continuous, then F is continuous.

Lemma (Gluing Lemma for Continuous Maps)**.** Let X, Y be topological spaces, and suppose one of the following conditions holds:

(i) $B_1, \ldots, B_n$ are finitely many closed subsets of X whose union is X.

(ii) $\{B_\alpha\}_{\alpha \in A}$ is a collection of open subsets of X whose union is X.

Suppose that for all i (the first case), or α (the second case) we are given continuous maps $F_i : B_i \to Y$, or $F_\alpha : B_\alpha \to Y$ that agree on overlaps: $F_i|_{B_i \cap B_j} = F_j|_{B_i \cap B_j}$, or $F_\alpha|_{B_\alpha \cap B_\beta} = F_\beta|_{B_\alpha \cap B_\beta}$. Then there exists a unique continuous map $F : X \to Y$ whose restriction to each B_i is equal to F_i , or to B_α equal to F_α.

Remark. Let X be a topological space, and suppose X admits a countable open cover $\{U_i\}$ such that each set U_i is second-countable in the subspace topology. Then X is second-countable.

6 Product Space

If X is any set and $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$ is a family of maps from X into some topological spaces Y_α, there is a unique weakest topology $\mathcal{T}$ on X that makes all f_α continuous; it is called the **weak topology** generated by $\{f_\alpha\}_{\alpha \in A}$. Namely, $\mathcal{T}$ is the topology generated by sets of the form $f_\alpha^{-1}(U_\alpha)$ where $\alpha \in A$ and U_α is open in Y_α.

The most important example of this construction is the Cartesian product of topological spaces. If $\{X_\alpha\}_{\alpha \in A}$ is any family of topological spaces, the **product topology** on $X = \prod_{\alpha \in A} X_\alpha$ is the weak topology generated by the coordinate maps $\pi_\alpha : X \to X_\alpha$. When we consider a Cartesian product of topological spaces, we always endow it with the product topology unless otherwise specified. The subbasis for this topology is $\{\pi_\alpha(U_\alpha) : \alpha \in A, U_\alpha \text{ is an open subset in } X_\alpha\}$, and a basis for the product topology is given by sets of the form $\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ where $n \in \mathbb{N}$ and U_{α_i} is open in X_{α_i}, for $1 \leq i \leq n$. These basis elements can also be written as $\prod_{\alpha \in A} U_\alpha$ where $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$.

Proposition (Properties of the Finite Product Topology)**.** Suppose $(X_\alpha)_{\alpha \in A}$ is an indexed family of topological spaces, and let $X = \prod_{\alpha \in A} X_\alpha$ be their product space.

(i) Characteristic Property: If B is a topological space, a map $F : B \to X$ is continuous if and only if each of its component function $F_\alpha = \pi_\alpha \circ F : B \to X_\alpha$ is continuous.

(ii) The product topology is the unique topology on X for which the characteristic property holds.

(iii) Each coordinate map $\pi_\alpha : X \to X_\alpha$ is continuous.

(iv) Given any continuous maps $F_\alpha : X_\alpha \to Y_\alpha$, the **product map** $\prod_{\alpha \in A} F_\alpha : \prod_{\alpha \in A} X_\alpha \to \prod_{\alpha \in A} Y_\alpha$ is continuous, where

$$(\prod_{\alpha \in A} F_\alpha)((x_\alpha)_{\alpha \in A}) = (F_\alpha(x_\alpha))_{\alpha \in A}$$

(v) If S_α is a subspace of X_α for each $\alpha \in A$, the product topology and the subspace topology on $\prod_{\alpha \in A} S_\alpha \subseteq \prod_{\alpha \in A} X_\alpha$ coincide.

(vi) For any $\alpha \in A$ and any choice of points $a_\beta \in X_\beta$ for $\beta \neq \alpha$, the map $x \mapsto (a_\beta, \ldots, x, \ldots, a_{\beta'})$ is a topological embedding of X_α into X.

(vii) If $\mathcal{B}_{\alpha_i}$ is a basis for the topology of X_{α_i} for $i = 1, \ldots, k$, then the collection

$$\mathcal{B} = \{B_{\alpha_1} \times \ldots \times B_{\alpha_k} : B_{\alpha_i} \in \mathcal{B}_{\alpha_i}\}$$

is a basis for the product topology on X.

(viii) Every product of Hausdorff spaces is Hausdorff.

(ix) Every product of first-countable spaces is first-countable.

(x) Every product of second-countable spaces is second-countable.

7 Disjoint Union Space

Another simple way of building new topological spaces is by taking disjoint unions of other spaces. From a set-theoretic point of view, the disjoint union is defined as follows: If $(X_\alpha)_{\alpha\in A}$ is an indexed family of sets, their **disjoint union** is the set

$$\coprod_{\alpha\in A} X_\alpha = \{(x,\alpha) : \alpha \in A, x \in X_\alpha\}.$$

For each α, there is a canonical injective map $\iota_\alpha : X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ given by $\iota_\alpha(x) = (x,\alpha)$, and the images of these maps for different values of α are disjoint. Typically, we implicitly identify X_α with its image in the disjoint union, thereby regarding X_α as a subset of $\coprod_{\alpha\in A} X_\alpha$. The α in the notation (x,α) should be thought of as a "tag" to indicate which set x comes from, so that the subsets corresponding to different values of α are disjoint, even if some or all of the origin sets X_α were identical.

Given an indexed family of topological spaces $(X_\alpha)_{\alpha\in A}$, we define the **disjoint union topology** on $\coprod_{\alpha\in A} X_\alpha$ by claiming a subset of $\coprod_{\alpha\in A} X_\alpha$ to be open if and only if its intersection with each X_α is open in X_α.

Proposition (Properties of the Disjoint Union Topology)**.** Suppose $(X_\alpha)_{\alpha\in A}$ is an indexed family of topological spaces, and $\coprod_{\alpha\in A} X_\alpha$ is equipped with the disjoint union topology.

(i) Characteristic Property: If Y is a topological space, a map

$$F : \coprod_{\alpha\in A} X_\alpha \to Y$$

is continuous if and only if $F \circ \iota_\alpha$ is continuous for each $\alpha \in A$.

(ii) The disjoint union topology is the unique topology on $\coprod_{\alpha\in A} X_\alpha$ for which the characteristic property holds.

(iii) A subset of $\coprod_{\alpha\in A} X_\alpha$ is closed if and only if its intersection with each X_α is closed.

(iv) Each injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ is a topological embedding.

(v) Each disjoint union of Hausdorff spaces is Hausdorff.

(vi) Each disjoint union of first-countable spaces is first-countable.

(vii) Each disjoint union of second-countable spaces is second-countable.

8 Quotient Space and Quotient Topology

If X is a topological space, Y is a set, and $\pi : X \to Y$ is a surjective map, the **quotient topology on Y determined by π** is defined by claiming a subset $U \subseteq Y$ to be open if and only if $\pi^{-1}(U)$ is open in X. If X and Y are topological spaces, a map $\pi : X \to Y$ is called a **quotient map** if it is surjective and continuous and Y has the quotient topology determined by π.

The following construction is the most common way of producing quotient maps. Suppose $\sim$ is an equivalence relation on X, then for each $x \in X$, the **equivalence class of** x, denoted by $[x]$, is the set of all $y \in X$ such that $y \sim x$. The set of all equivalence classes is a **partition of** X: a collection of disjoint nonempty subsets whose union is X. It can be shown that giving a equivalence relation on X is equivalent to giving a partition of X.

Suppose X is a topological space and $\sim$ is an equivalence relation on X. Let $X/\sim$ denote the set of equivalence classes in X, and let $\pi : X \to X/\sim$ be the natural projection sending each point to its equivalence class. Endowed with the quotient topology determined by π, the space $X/\sim$ is called the **quotient space of X determined by $\sim$**. For example, suppose X and Y are topological spaces, $A \subseteq Y$ is a closed subset, and $f : A \to X$ is a continuous map. The relation $a \sim f(a)$ for all $a \in A$ generates an equivalence relation on $X \coprod Y$, whose quotient space is denoted by $X \cup_f Y$ and called an **adjunction space**. It is said to be deformed by **attaching Y to X along f**.

If $\pi : X \to Y$ is a map, a subset $U \subseteq X$ is said to be **saturated with respect to** π if U is the entire preimage of its image: $U = \pi^{-1}(\pi(U))$. Given $y \in Y$, the **fiber of π over y** is the set $\pi^{-1}(y)$. Thus, a subset of X is saturated if and only if it is a union of fibers.

Proposition (Properties of Quotient Maps)**.** Let $\pi : X \to Y$ be a quotient map.

(i) Characteristic Property: If B is a topological space, a map $F : Y \to B$ is continuous if and only if $F \circ \pi : X \to B$ is continuous.

(ii) The quotient topology is the unique topology on Y for which the characteristic property holds.

(iii) A subset $K \subseteq Y$ is closed if and only if $\pi^{-1}(K)$ is closed in X.

(iv) If π is injective, then it is a homeomorphism.

(v) If $U \subseteq X$ is a saturated open or closed subset, then the restriction $\pi|_U : U \to \pi(U)$ is a quotient map.

(vi) Any composition of π with another quotient map is again a quotient map.

Remark. Let X, Y be topological spaces, and suppose that $F : X \to Y$ is a surjective continuous map. Then the following are equivalent:

(i) F is a quotient map.

(ii) F takes saturated open subsets to open subsets.

(iii) F takes saturated closed subsets to closed subsets.

The next two properties of quotient maps play important roles in topology, and have equally important generalizations in smooth manifold theory.

Theorem (Passing to the Quotient)**.** Suppose $\pi : X \to Y$ is a quotient map, B is a topological space, and $F : X \to B$ is a continuous map that is constant on each fiber of π. Then there exists a unique continuous map $\tilde{F} : Y \to B$ such that $F = \tilde{F} \circ \pi$.

Theorem (Uniqueness of Quotient Space)**.** If $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ are quotient maps that are constant on each other's fibers, then there exists a unique homeomorphism $\varphi : Y_1 \to Y_2$ such that $\varphi \circ \pi_1 = \pi_2$.

9 Open and Closed Maps

A map $F : X \to Y$ is said to be an **open map** if for every open subset $U \subseteq X$, the image set $F(U)$ is open in Y, and a **closed map** if for every closed subset $K \subseteq X$, the image $F(K)$ is closed in Y. Continuous maps may be open, closed, both, or neither, as can be seen by examing simple examples involving subsets of the plane.

Remark. (i) Suppose $X_1, \ldots, X_k$ are topological spaces, then each projection $\pi_i : X = X_1 \times \ldots \times X_k \to X_i$ is an open map.

(ii) Let $(X_\alpha)_{\alpha \in A}$ be an indexed family of topological spaces, then each injection $\iota_\alpha : X_\alpha \to \coprod_{\alpha \in A} X_\alpha$ is both open and closed.

(iii) Every local homeomorphism is an open map.

(iv) Every bijective local homeomorphism is a homeomorphism.

(v) Suppose $q : X \to Y$ is an open quotient map, then Y is Hausdorff if and only if the set $\mathcal{R} = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

(vi) Let X, Y be topological spaces, and let $F : X \to Y$ be a map. Then

   (a) F is closed if and only if for every $A \subseteq X$, $F(\bar{A}) \supseteq F(\bar{A})$.

   (b) F is open if and only if for every $B \subseteq Y$, $F^{-1}(\text{Int}B) \sqsubseteq \text{Int}F^{-1}B$

The most important classes of continuous maps in topology are the homeomorphisms, quotient maps, and topological embeddings. Obviously, it is necessary for a map to be bijective in order for it to be a homeomorphism, suejective for it to be a quotient map, and injective for it to be a topological embedding. However, even when a continuous map is known to satisfy one of these necessary set-theoretic conditions, it is not always easy to tell whether it has the desired topological property. One simple sufficient condition is that it be either an open or a closed map, as the next theorem shows.

Theorem. Suppose X, Y are topological spaces, and $F : X \to Y$ is a continuous map that is either open or closed.

(i) If F is surjective, then it is a quotient map.

(ii) If F is injective, then it is a topological embedding.

(iii) If F is bijective, then it is a homeomorphism.

10 Connecctedness

A topological space X is said to be **disconnected** if it is the union of two disjoint nonempty open subsets, and it is **connected** otherwise. Equivalently, X is connected if and only if the only subsets that are both open and closed are ε and X itself. If X is any topological space, a **connected subset of** X is a subset that is a connected space when endowed with the subspace topology. For example, the nonempty connected subsets of $\mathbb{R}$ are the singletons and the intervals. A maximal connected subset of X is called a **(connected) component of** X.

Proposition (Properties of Connected Sapce)**.** Let X, Y be topological spaces.

(i) If $F : X \to Y$ is continuous and X is connected, then $F(X)$ is connected.

(ii) Every connected subsets of X is contained in a single component of X.

(iii) A union of connected subspaces of X with a point in common is connected.

(iv) The components of X are disjoint nonempty closed subsets whose union is X, thus they form a partition of X.

(v) If S is a subset of X that is both open and closed, then S is a union of components of X.

(vi) Every finite product of connected spaces is connected.

(vii) Every quotient space of a connected space is connected.

Closely related to connectedness is path connectedness. If X is a topological space and $p, q \in X$, a **path in** X **from** p **to** q is a continuous map $f : I \to X$ such that $f(0) = p, f(1) = q$. If for every pair of points $p, q \in X$ there exists a path in X from p to q, then X is said to be **path-connected**. The **path components of** X are its maximal path-connected subsets.

Proposition (Properties of Path-Connected Space)**.** (i) Properties of Connected Space holds with "connected" replaced by "path-connected", and "component" by "path-component" throughout.

(ii) Every path-connected space is connected.

For most topological spaces we treat, including all manifolds, connectedness and path-connectedness turn out to be equivalent. The link between the two concepts is provided by the following notion. A topological space is said to be **locally path-connected** if it admits a basis of path-connected open subsets.

Proposition. Let X be a locally path-connected topological space.

(i) The components of X are open in X.

(ii) The path-components of X are equal to its components.

(iii) X is connected if and only if it is path-connected.

(iv) Every open subset of X is locally path-connected.

11 Compactedness

A topological space X is said to be **compact** if every open cover of X has a finite subcover. A **compact subset** of a topological space is one that is a compact space in the subspace topology. For example, it is a consequence of the Heine-Borel theorem that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proposition. Let X, Y be topological spaces.

(i) If $F : X \to Y$ is continuous and X is compact, then $F(X)$ is compact.

(ii) If X is compact and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains its maximum and minimum values on X.

(iii) Any union of finitely many compact subspaces of X is compact.

(iv) If X is Hausdorff and K, L are disjoint compact subsets of X, then there exists disjoint open subsets $U, V \subseteq X$ such that $K \subseteq U, L \subseteq V$.

(v) Every closed subset of a compact space is compact.

(vi) Every compact subset of a Hausdorff space is closed.

(vii) Every compact subset of a metric space is bounded.

(viii) Every finite product of compact spaces is compact.

(ix) Every quotient of a compact space is compact.

For maps between metric spaces, there are several variants of continuity that are useful, especially in the context of compact spaces. Suppose (X_1, d_1) and (X_2, d_2) are metric spaces, and $F : X_1 \to X_2$ is a map. Then F is said to be **uniformly continuous** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X_1$, $d_1(x, y) < \delta$ implies $d_2(F(x), F(y)) < \varepsilon$. It is said to be **Lipschitz continuous** if there is a positive constant C such that $d_2(F(x), F(y)) \leqslant C d_1(x, y)$ for all $x, y \in X_1$. Any such C is called a **Lipschitz constant for** F. We say that F is **locally Lipschitz continuous** if every point $x \in X_1$ has an open neighbourhood on which F is Lipschitz continuous.

Remark. For maps between metric spaces, Lipschitz continuity implies uniform continuity, which implies continuity, and Lipschitz continuity implies local Lipschitz continuity, which implies continuity. We use two examples to show these implications are not reversible: let $f, g : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$ and $g(x) = x^2$. Then f is uniformly continuous but not locally or globally Lipschitz continuous, and g is Locally Lipschitz continuous but not uniformly continuous or globally Lipschitz continuous.

Proposition. Suppose (X_1, d_1) and (X_2, d_2) are metric spaces and $F : X_1 \to X_2$ is a map. Let K be any compact subset of X_1.

(i) If F is continuous, then $F|_K$ is uniformly continuous.

(ii) If F is locally Lipschitz continuous, then $F|_K$ is Lipschitz continuous.

We next give equivalent criterions for compactness in a metric space. $E \subseteq X$ is a **totally bounded** subset if, for every $\varepsilon > 0$, E can be covered by finitely many balls of radius ε. Clearly, each totally bounded set is bounded, but the converse is false in general.

Theorem. If E is a subset of a metric space (X, d), the following are equivalent:

(i) E is complete and totally bounded.

(ii) (The bolzano-Weierstrass Property) Every sequence in E has a subsequence that converges to a point of E.

(iii) (The Heine-Borel Property) Each open cover of E has a finite subcover.

The next lemma expresses one of the most important properties of compact spaces.

Lemma (Closed Map Lemma)**.** Suppose X is a compact space, Y is a Hausdorff space, and $F : X \to Y$ is a continuous map.

(i) F is a closed map.

(ii) If F is injective, it is a topological embedding.

(iii) If F is surjective, it is a quotient map.

(iv) If F is bijective, it is a homeomorphism.

If X and Y are topological spaces, a map $F : X \to Y$ is said to be **proper** if for every compact set $K \subseteq Y$, the preimage $F^{-1}(K)$ is compact. Here are some useful sufficient conditions for a map to be proper.

Proposition. Suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map.

(i) If X is compact and Y is Hausdorff, then F is proper.

(ii) IF F is a closed map with compact fibers, then F is proper.

(iii) If F is a topological embedding with closed image, then F is proper.

(iv) If Y is Hausdorff and F has a continuous left inverse, then F is proper.

(v) If F is proper and $A \subseteq X$ is a subset that is saturated with respect to F, then $F|_A : A \to F(A)$ is proper.