The universal covering group of SO(3) and its applications in representation theory

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摘要

我们首先考虑李群，它具有两个兼容的结构：群结构和光滑流形结构. 我们研究两个重要的李群：$SU(2)$ 和 $SO(3)$，并证明 $SU(2)$ 是 $SO(3)$ 的万有复叠群. 使用这个结果和 $SO(3)$ 的有限子群我们可以找到 $SU(2)$ 的有限子群，Mckay 对应告诉我们 $SU(2)$ 这五种类型的子群与五类单李代数的 Dynkin 图之间有一一对应关系。然后我们研究李群及其对应的李代数之间的关系，并使用 $SU(2)$ 和 $SO(3)$ 来检查一个一般命题，我们将提供该命题的证明：一个李群及其万有复叠群有相同的李代数。最后，我们利用 $SU(2)$ 和 $SO(3)$ 之间的双重复叠关系以及 $SU(2)$ 的不可约表示来找到 $SO(3)$ 的不可约表示。

关键字： 万有复叠群；二重复叠；迈凯对应；李群；李代数；表示.

Abstract

We first consider Lie groups, which have two compatible structures: group structure and smooth manifold structure. We study two Lie groups: $SU(2)$ and $SO(3)$, then we prove $SU(2)$ is the universal covering group of $SO(3)$. Using this result together with the finite subgroups of $SO(3)$ we can find the finite subgroups of $SU(2)$ and Mckay correspondence tells us a one-to-one correspondence between these five types of subgroups of $SU(2)$ and the Dynkin diagrams of five types of simple Lie algebra. Then we study the relations between Lie groups and their corresponding Lie algebras and use $SU(2)$ and $SO(3)$ to check a general proposition of which we will supply a proof: A Lie group and its universal covering group have the same Lie algebra. Finally, we use the double covering relations between $SU(2)$ and $SO(3)$ and the irreducible representations of $SU(2)$ to find the irreducible representations of $SO(3)$.

Key Word: Universal Covering Group; Double Cover; Mckay Correspondence; Lie Group; Lie Algebra; Representation.

1 Introduction

Mathematicians have developed mature theories about manifolds and Lie groups, which have played significant roles in many branches of mathematics. Lie theory, including Lie groups and Lie algebras, was first introduced by the Norwegian mathematician Sophus Lie, who tried to simplify differential equations and geometry problems. For each Lie group, we have corresponding Lie algebra, the space of left-invariant vector fields, which is a nonassociative algebra satisfying several conditions such as the Jacobi identity. We can subtract these properties and form the axioms of the Lie algebra in algebraic settings. We similarly define algebraic concepts for Lie algebras, among which semisimple Lie algebras are almost the most essential and exciting types of finite-dimensional Lie algebras. Serre's relations are among the essential relations satisfied by semisimple Lie algebras. We can derive the theory of infinite-dimensional Lie algebras by Serre's relations and the generalized Cartan matrix.

In representation theory, we can consider the modules of a particular kind of algebraic structure that are equivalent to the representations of that structure. Hence, we can use the module to characterize some important representations: semisimple modules correspond to completely reducible representations, and simple modules correspond to irreducible representations. Since many representations are completely reducible, we can uniquely decompose these representations into irreducible representations. Actually, in some sense, we can view irreducible representations as atoms of the area of Chemistry.

2 SO(3) and SU(2)

2.1 Background

2.1.1 Smooth manifold

Definition 2.1. *A topological space is a base set X together with a collection of subsets $\Gamma(X)$ of X which is called the topology of X, such that the following conditions hold:*
(1) $\phi, X \in \Gamma(X)$,
(2) $\cap_{i=1}^{n} X_i \in \Gamma(X)$ for finite many $X_i \in \Gamma(X)$,
(3) $\cup_{\alpha} X_{\alpha} \in \Gamma(X)$ for arbitrary $X_{\alpha} \in \Gamma(X)$.
The elements in $\Gamma(X)$ are called open subsets of X.

Definition 2.2. *Given that M is a topological space, M is a topological manifold of dimension n if the following conditions hold:*
(1) M is Hausdorff: for every pair of distinct points in M, disjoint open subsets exist separating these two points.
(2) M is second-countable: M has a countable base for its topology,
(3) M is locally Euclidean of dimension n: every point has an open neighbourhood which is homeomorphic to an open subset of the Euclidean space of dimension n.[1]

Because the neighbourhood is homeomorphic to its image in the Euclidean space, where coordinates can represent elements, the open neighbourhood and the corresponding homeomorphism are called the manifold coordinate chart. We collect all the coordinate charts such that the union of the domains of the charts covers M, then such collection is an atlas for M. Since on $\mathbb{R}^n$, there are definitions for differentiability to a particular order and smoothness (component functions have continuous partial derivatives of all orders), we can use the condition of locally

Euclidean manifolds to define the smooth condition of a manifold:

Definition 2.3. *A topologically manifold M is a smooth manifold if for every pair of two charts (U, φ) and (V, ψ) of the manifold, $\varphi \circ \psi^{-1}$ is smooth at every point in $U \cap V$. Then the chart on M is smooth, the atlas is smooth, and (U, φ) is smoothly compatible with (V, ψ). The maximal smooth atlas is a smooth atlas that no other smooth atlases properly contain. Suppose we can collect all smooth charts which are smoothly compatible with every chart in a fixed atlas $\mathcal{A}$, this is the maximal smooth atlas determined by $\mathcal{A}$. The manifold M equipped with a maximal smooth atlas is called a smooth manifold, and the maximal smooth atlas is the smooth structure of M[1].*

We can use the smooth structure on manifolds and the smooth condition of maps between Euclidean spaces to define the smoothness of maps between smooth manifolds: for a map between two smooth manifolds M and N, for any point in the domain and the corresponding image point in the codomain, we choose respectively smooth charts containing them such that the domain of the second chart contains the image under f of the domain of the first chart, and if the composite map between Euclidean spaces which uses the homeomorphism in the first chart to pull from Euclidean space back to M, then uses f to transfer to N and finally uses the homeomorphism in the second chart to push forward from N to Euclidean space is smooth at that point, then f is said to be smooth at that point in M. We can similarly define smooth maps between two smooth manifolds with smoothness at one point.

Definition 2.4. *A Lie group is a group and a smooth manifold simultaneously, such that the multiplication and the inverse map are smooth.*

Given a smooth manifold M with smooth structure $\mathcal{A}$ and a nonzero open subset $V \subseteq M$, we give V the atlas:

$$\mathcal{A}_V = \{(U \cap V, \phi|_V) : (U, \phi) \in \mathcal{A}\} \tag{1}$$

We can check that $\mathcal{A}_V$ constructed above is a maximal smooth structure on V, and the manifold obtained in this way is an open submanifold of M.

The vector space $M(k \times l, \mathbb{R})$ of all $k \times l$ matrices with entries in $\mathbb{R}$ is a kl-dimensional smooth manifold since the matrix addition and multiplication are smooth, similarly the vector space $M(k \times l, \mathbb{C})$ is a $2kl$-dimensional smooth manifold. We consider the determinant map from $M(n, \mathbb{R})$ to $\mathbb{R}$, $GL(n, \mathbb{R})$ is the set where the determinant map takes nonzero values, hence $GL(n, \mathbb{R})$ is a nonzero open submanifold of $M(n, \mathbb{R})$. Similarly, $GL(n, \mathbb{C})$ is a nonzero open submanifold of $M(n, \mathbb{C})$.

2.1.2 Cartan's theorem

Giving a group G, if a subset H is closed under the inherited multiplication and inverse map, then is it a subgroup of G. We generalize the definition of the subgroup to that of the Lie subgroup for a given Lie group.

Definition 2.5. *(Lie subgroup) A subset H of a Lie group G is a Lie subgroup if it has a smooth structure such that H itself is a Lie group and an immersed submanifold of G, here the immersed submanifold means a subset H of G with a smooth structure such that the inclusion map from H to G is smooth.[1]*

We quote the theorem without proof:

Theorem 2.6. *(Cartan's theorem) A topologically closed subgroup of a Lie group is a Lie subgroup.*

Next, we mainly focus on matrix group $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, and Cartan's theorem tells us that most of the subgroups of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ we have encountered before such as $O(n)$, $SO(n)$, $U(n)$, and $SU(n)$ are Lie subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$.

2.1.3 Lie groups and Lie algebras

Lie algebras are first derived as the vector space of the left-invariant vector fields; alternatively, we can view them as the tangent space at the identity of the Lie groups. We first define Lie algebras:

Definition 2.7. *A finite dimensional real or complex vector space* $\mathfrak{g}$ *is called a Lie algebra if it is equipped with a bilinear multiplication* $[.,.]$ *which is called the Lie bracket:* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that:*

$$\text{(1),(skew-symmetric) } [x,x] = 0 \text{ for any } x \in \mathfrak{g} \tag{2}$$

$$\text{(2),(Jacobi identity) } [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \text{ for any } x,y,z \in \mathfrak{g} \tag{3}$$

To make the relationship between Lie groups and Lie algebras clear, we first introduce the concept of the tangent space and tangent vector:

Definition 2.8. *Given a smooth manifold* M*, a tangent vector* v *at a point* $p \in M$ *that is also called a derivation at* $p \in M$ *is a linear map and satisfies*

$$v(fg) = f(p)v(g) + g(p)v(f), \text{ for all } f,g \in C^\infty(M) \tag{4}$$

Here $C^\infty(M)$ *is the space of all smooth functions on* M*, the set of all tangent vectors at* p *is called the tangent space of* M *at* p *and is denoted by* T_pM.

On $\mathbb{R}^n$, the tangent space at each point $a \in \mathbb{R}^n$ is a n-dimensional vector space with basis $\{\frac{\partial}{\partial x^i}|_a\}$ where $\frac{\partial}{\partial x^i}|_a$ is defined by

$$\frac{\partial}{\partial x^i}|_a f = \frac{\partial f}{\partial x^i}(a) \tag{5}$$

Given a smooth map $F : M \to N$ between two smooth manifolds M and N, for each point $p \in M$ let us consider a linear map

$$dF_p : T_pM \to T_{F(p)}N \tag{6}$$

which is called the differential of F at $p \in M$ and is defined by the following equation:

$$dF_p(v)(f) = v(f \circ F), \text{ for } v \in T_pM \text{ and } f \in C^\infty(N) \tag{7}$$

The differential of F sends a tangent vector of M at p to a tangent vector of N at $F(p)$. On M, consider a smooth atlas (U, ϕ) containing p, since $\phi(U) \subset \mathbb{R}^n$, and assume the component of the coordinate representation of ϕ is $\{x^i\}_{i=1}^n$, we see $\{\frac{\partial}{\partial x^i}|_{\phi(p)}\}$ is a basis of $T_{\phi(p)}\mathbb{R}^n$, since ϕ is a diffeomorphism from $U \subset M$ onto its image, the differential of ϕ at each point of U is an isomorphism, so the pullback of $\{\frac{\partial}{\partial x^i}|_{\phi(p)}\}$ under $d\phi_p$ is a basis of T_pM.

Definition 2.9. *The tangent bundle TM of a smooth manifold M is the disjoint union of tangent spaces at all points of M:*

$$TM = \coprod_{p \in M} T_pM \tag{8}$$

The projection map $\pi : TM \to M$ sends a tangent vector to the point where it is tangent:

$$\pi : v \in T_pM \mapsto p \tag{9}$$

A vector field X on M is a right inverse of π, i.e.

$$\pi \circ X = Id_M \tag{10}$$

Given two vector fields X, Y on M, the Lie bracket of X, Y is a vector field, denoted by $[X, Y]$ and defined by:

$$[X, Y] : C^\infty(M) \to C^\infty(M) \tag{11}$$

$$[X,Y]f = XYf - YXf \tag{12}$$

A vector field on a Lie group G is said to be left-invariant if it satisfies the following conditions:

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}, \text{ for all } g, g^{'} \in G \tag{13}$$

Here L_g means left multiplication of G by the element $g \in G$. [1]

Proposition 2.10. *The vector space of all left-invariant smooth vector fields on a Lie group G is a Lie algebra in the algebraic setting we have defined before and is denoted by $Lie(G)$ to indicate its relation with G. Moreover, the map $Lie(G) \to T_e(G)$, $X \mapsto X_e$ is a vector space isomorphism.[1]*

The above proposition tells us we can identify the Lie algebra of a Lie group with its tangent space at the identity. The value of a left-invariant vector field at the identity of the Lie group determines the vector field; we can determine its value at the identity, which is a tangent vector at the tangent space at the identity. Conversely, giving a tangent vector of the tangent space at the identity, we can use the left-invariance of the vector field and the differential of the left multiplication to determine the value of the vector field at any point since each differential is a linear isomorphism and different vectors in the tangent space of the identity give different vectors of the target tangent space. It is just the one-to-one correspondence between left-invariant vector fields and tangent vectors in the tangent space at the identity. Now we turn our concern to the main topic: the matrix Lie group $GL(n,\mathbb{R})$ ($GL(n,\mathbb{C})$) and its specific closed subgroups. We first consider the matrix algebra $M(n,\mathbb{R})$ ($M(n,\mathbb{C})$) consisting of real (complex) matrices of order n, it becomes a n^2-dimensional real (complex) Lie algebra under the Lie bracket defined as follows:

$$[A,B] = AB - BA \tag{14}$$

Bilinearity and antisymmetry are easy to check, and some calculations show the Jacobi identity holds in $M(n,\mathbb{R})$ $(M(n,\mathbb{C}))$. We denote this Lie algebra by $\mathfrak{gl}(n,\mathbb{R})$ $(\mathfrak{gl}(n,\mathbb{C}))$. The above notation about the matrix algebra makes us wonder about its relation with the general linear groups $GL(n,\mathbb{R})$ $(GL(n,\mathbb{C}))$, and the next result guarantees our assumption:

Theorem 2.11. *(The Lie algebra of the general linear group). The composite of the maps:*

$$Lie(GL(n,\mathbb{R})) \to T_{I_n}GL(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R}) \tag{15}$$

gives a Lie algebra isomorphism betweem $Lie(GL(n,\mathbb{R}))$ and the matrix algebra $\mathfrak{gl}(n,\mathbb{R})$.

The composite of the maps:

$$Lie(GL(n,\mathbb{C})) \to T_{I_n}GL(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C}) \tag{16}$$

gives a Lie algebra isomorphism between $Lie(GL(n,\mathbb{C}))$ and the matrix algebra $\mathfrak{gl}(n,\mathbb{C})$.[1]

Now we wonder about the relation of the Lie algebra of Lie subgroups with the Lie algebra of the Lie group, but we need the knowledge of integral curves and exponential maps to make things clear:

Definition 2.12. *An integral curve of a vector field X on a smooth manifold M is a differentiable curve $\gamma : I \to M$ where I is an open interval of $\mathbb{R}$, such that:*

$$\gamma^{'}(t) = X_{\gamma(t)}, \text{ for all } t \in I. \tag{17}$$

A flow is a collection of integral curves satisfying certain conditions, to be precise, we define a global flow on M to be a one-parameter group action, i.e. a continuous map $\theta : \mathbb{R} \times M \to M$ satisfies[1]:

$$\theta(t,\theta(s,p)) = \theta(t+s,p), \text{ for all } s,t \in \mathbb{R}, p \in M \tag{18}$$

$$\theta(0,p) = p, \text{ for all } s,t \in \mathbb{R}, p \in M \tag{19}$$

Given the global flow (the one-parameter group action) θ, we introduce the following notations:

$$\theta_t(p) = \theta(t, p) = \theta^{(p)}(t). \tag{20}$$

Here $\theta^{(p)} : \mathbb{R} \to M$ is the unique integral curve starting at $p \in M$: $\theta^{(p)}(0) = p$, and $\theta_t : M \to M$ is a map sending $p \in M$ to the point at the time t of the integral curve $\theta^{(p)}$. For a smooth flow θ, the infinitesimal generator X of θ is defined by:

$$p \in M \mapsto X_p = \theta^{(p)\prime}(0) \tag{21}$$

We only consider Lie groups; at this time, we can state the following result as follows:

Theorem 2.13. *(The fundamental theorem of flows). Let $\theta : \mathbb{R} \times G \to G$ be a smooth global flow on a Lie group G. The infinitesimal generator X of θ is a smooth vector field on G, and each curve $\theta^{(p)}$ is an integral curve of X.*

Conversely, let X be a smooth vector field on a Lie group G. There is a unique smooth global flow $\theta : \mathbb{R} \times G \to G$ whose infinitesimal generator is X. [1]

The integral curve on a Lie group G is a special kind of curve which is a one-parameter subgroup of G, which is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \to G$.

Theorem 2.14. *(Characterization of one-parameter subgroups). The one-parameter subgroups of a Lie group G are the integral curves of left-invariant vector fields starting at the identity, and the domains of these curves are $\mathbb{R}$. The one-parameter subgroup determined by the vector field X is said to be the one-parameter subgroup generated by X. [1]*

The left-invariant vector field is determined by its value at the identity; hence by the theorem, we get a one-to-one correspondence between the tangent space at the identity $T_e(G)$ and the set of one-parameter subgroups of G.

Proposition 2.15. *The exponential map* exp *of a Lie group G is a map from its Lie algebra to the group* G *defined as follows:*

$$X \mapsto \gamma(1), \textit{ for any } X \in Lie(G) \tag{22}$$

γ *is the the integral curve of* X *starting at the identity. Then for any* $X \in Lie(G)$, $\gamma(s) = exp(sX)$ *is the one-parameter subgroup of* G *generated by* X. *For the general linear group* $GL(n, \mathbb{R})$, *the exponential map can be calculated as follows [1]:*

$$exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \textit{ for any } A \in \mathfrak{gl}(n, \mathbb{R}) \tag{23}$$

The next results tell us how to calculate the Lie algebra of a Lie subgroup:

Proposition 2.16. *Suppose* G *is a Lie group and* H *is a Lie subgroup of* G *with Lie algebra* $\mathfrak{g} = Lie(G)$ *and* $\mathfrak{h} = Lie(H)$ *respectively. Then* $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$, *and the exponential map of* H *is the restriction to* $\mathfrak{h}$ *of the exponential map of* G, *and [1]*

$$\mathfrak{h} = \{X \in \mathfrak{g} : exp(tX) \in H \textit{ for all } t \in \mathbb{R}\} \tag{24}$$

$GL(n, \mathbb{C})$ can be embedded into $GL(2n, \mathbb{R})$ as a Lie subgroup, then the exponential map in $GL(n, \mathbb{C})$ can be calculated as (23) by Proposition 2.16.

2.1.4 Covering space

We introduce some concepts from algebraic topology to make the discussion between $SU(2)$ and $SO(3)$ clear. We use X to denote a topological space without otherwise stated.

Definition 2.17. *A path in* X *is a continuous map* $f : I \to X$, *where* $I = [0, 1]$ *in* $\mathbb{R}$.

We can state the idea of continuously deforming a path to another path with the two endpoints fixed as follows:

Definition 2.18. *A homotopy of paths in X is a family of paths $f_t : I \to X$, such that the following two conditions are satisfied:*
(1) The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ do not depend on t.
(2) $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.
The two paths f_0 and f_1 associated in this way via a homotopy f_t is said to be homotopic and we denote this relation by $f_0 \simeq f_1$. [2]

The relation of homotopy with fixed endpoints in X is an equivalence between paths, so we form the classes of paths under the equivalence and denote by $[f]$ the class of the path f, and call it the homotopy class of f. As the name algebraic topology shows, we want to study the topological properties of spaces from the algebraic point of view. The first thing to do is to endow the homotopy classes of paths in X with an algebraic structure, and the idea works. The group structure will occur if we properly define the binary operation on the set of the homotopy classes of paths.

Definition 2.19. *Given two paths f, g which satisfy $f(1) = g(0)$, there is a product path (denoted by $f \cdot g$) which traverses first f and then g and is defined by the formula [2]:*

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} \tag{25}$$

This definition is well defined since it preserves homotopy classes, i.e. if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via f_t and g_t respectively, and if $f(1) = g(0)$ so that $f_0 \cdot g_0$ can be defined, then $f_t \cdot g_t$ is defined and gives a homotopy $f_0 \cdot g_0 \simeq f_1 \cdot g_1$. The path f with the same endpoints $f(0) = f(1) = x_0$ is a loop and x_0 is the basepoint of the loop, we denote the homotopy classes of loops with the same basepoint x_0 by $\pi_1(X, x_0)$.

Theorem 2.20. *$\pi_1(X, x_0)$ is a group with respect to the product we have defined and it is called the fundamental group of X at x_0. Different choices of basepoints in a path component of X give isomorphic groups. [2]*

Remark: If X is path-connected, we drop x_0 and denote the group by $\pi_1(X)$ and call it the fundamental group of X.

Definition 2.21. . *X is simply-connected if it is path-connected and $\pi_1(X)$ is trivial.*

Remark: We can show a space is simply connected if and only if there exists only one homotopy class of paths connecting two arbitrarily chosen points $x_0, x_1 \in X$.[2]

To better illustrate the motivations of the next concept, we give a classic example:

Problem 2.22. *Every continuous homomorphism $\Phi : \mathbb{R} \to \mathbb{S}^1$ is of the form $x \mapsto exp(i\lambda x)$ for some $\lambda \in \mathbb{R}$.*

Proof. We view $\mathbb{R}$ as an additive group and $\mathbb{S}^1$ as a multiplicative group. There exist $\lambda \in \mathbb{R}$ such that $\Phi(1) = exp(i\lambda)$. Since Φ is a homomorphism, we have by definition that $\Phi(x_1 + x_2) = \Phi(x_1)\Phi(x_2)$, and it can be easily checked that $\Phi(0) = 1$ and $\Phi(\frac{1}{n}) = exp(\frac{i\lambda}{n})$ for $n \in Z_{>0}$ or $n \in Z_{<0}$, hence $\Phi(r) = \exp(i\lambda r)$ for $r \in \mathbb{Q}$. By continuity, we have $\Phi(x) = exp(i\lambda x)$ for $x \in \mathbb{R}$. $\square$

We now consider the coordinates of $\mathbb{S}^1$, we see without loss of generality that $\Phi : \mathbb{R} \to \mathbb{S}^1$ can be expressed as $p : s \mapsto (cos(\lambda s), sin(\lambda s))$ for some $\lambda \in \mathbb{R}$. If we carefully choose an open covering $\{U_\alpha\}$ of $\mathbb{S}^1$, then for each α, $p^{-1}(U_\alpha)$

is a disjoint union of open subsets in $\mathbb{R}$ the restriction of p to each of which is a homomorphism. A generalization of this is what is called covering space.

Definition 2.23. *A covering space of a topological space X is a topological space $\tilde{X}$ together with a continuous surjective map $p : \tilde{X} \to X$ such that: there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped by p homeomorphically onto U_α. A universal covering space is a simply-connected covering space.[2]*

Now we consider not only topological properties but also smooth structures, so if both the base space and the covering space have a smooth structure, we need the following:

Definition 2.24. *Give two smooth manifolds M and $\tilde{M}$ together with the topological covering map $p : \tilde{M} \to M$, if the map is smooth surjective and each $p^{-1}(U_\alpha)$ is mapped diffeomorphically onto U_α, then $\tilde{M}$ is said to be a smooth covering manifold of M. [1]*

We next state several important results about smooth covering spaces:

Proposition 2.25. *(Covering spaces of smooth manifolds). Suppose M is a connected smooth manifold, and $p : \tilde{M} \to M$ is a topological covering map. Then $\tilde{M}$ is a topological manifold of the same dimension, which has a unique smooth structure such that p is a smooth covering map.[1]*

Theorem 2.26. *(Existence of a universal covering manifold). If M is a connected smooth manifold, then there is a unique simply connected smooth manifold $\tilde{M}$ up to diffeomorphism together with the smooth covering map, called the universal covering manifold of M.[1]*

Theorem 2.27. *(Existence of the universal covering group). Let G be a connected Lie group; then there is a simply connected Lie group $\tilde{G}$ together with a smooth* *covering map $p : \tilde{G} \to G$ which is a Lie group homomorphism, $\tilde{G}$ is called the universal covering group of G.[1]*

Theorem 2.28. *(Uniqueness of the universal covering group). The universal covering group for any connected Lie group G is unique up to Lie group isomorphism.[1]*

2.2 The special orthogonal group $SO(3)$

$SO(3)$ is the special orthogonal group whose elements are 3×3 orthogonal matrices with determinant 1. The following theorem tells us how to handle its elements:

Theorem 2.29. *Every matrix in $SO(3)$ is a rotation in $\mathbb{R}^3$: it is a linear operator in $\mathbb{R}^3$ and fixes a nonzero vector u and rotates the plane P orthogonal to the line spanned by u.*

Proof. From linear algebra, we know that a 3×3 orthogonal matrix with determinant 1 has an eigenvalue equal to 1, and we take the corresponding eigenvector whose length is equal to 1 to be u, moreover, choose the other two unit vectors v and w orthogonal to each other in P so that u, v, and w form an orthonormal basis in $\mathbb{R}^3$ and have the same orientation as the standard basis in $\mathbb{R}^3$.

Since there are two unit vectors corresponding to the eigenvalue 1, we choose one so that from the head of the vector, the matrix rotates P counterclockwise. We call such u the pole of the rotation. In $\mathbb{R}^2$, the corresponding matrix of the rotation has the form:

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

or

$$\begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$$

according to the determinant is 1 or -1, where θ is rotation angle. M restricts to a rotation on P, hence has the above form concerning the orthonormal basis v and w in P. And u is an eigenvector corresponding to the eigenvalue 1, so M is similar to the following matrix concerning u, v and w:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

□

Remark: We denote the rotation by (u, θ), where u is the pole of the rotation, and θ is the rotation angle on the plane orthogonal to the axis. So that (u, θ) and $(-u, \theta)$ may represent different rotations.

2.3 The special unitary group $SU(2)$

The elements of the special unitary group $SU(2)$ are 2×2 complex unitary matrices with determinant 1, and they have the following form:

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

Let $a = x_1 + ix_2, b = x_3 + ix_4$, we see they have the following form:

$$\begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}$$

and the determinant is

$$|a|^2 + |b|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 = 1$$

.

We prove that $SU(2)$ and $\mathbb{S}^3$ are diffeomorphic smooth manifolds.

Proposition 2.30. *As smooth manifolds, $SU(2)$ and $\mathbb{S}^3$ are diffeomorphic, which means they have the same topological structure and smooth structure.*

Proof. We consider the algebra of quaternions $\mathbb{H}$ over $\mathbb{R}$. Let $\mathbb{H} = \mathbb{C}\times\mathbb{C}$(regarded as a real vector space), and define a bilinear product $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$ as follows:

$$(a, b)(c, d) = (ac - d\bar{b}, \bar{a}d + cb), \quad a, b, c, d \in \mathbb{C}$$

With this product, $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$ which is called the algebra of quaternions. For each $p = (a, b) \in \mathbb{H}$, define $p^* = (\bar{a}, -b)$. We consider the following basis $\{\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$ of $\mathbb{H}$ by

$$\mathbb{1} = (1, 0), \mathbb{i} = (i, 0), \mathbb{j} = (0, 1), \mathbb{k} = (0, -i)$$

It is straightforward to check that with this basis, the following equations hold in $\mathbb{H}$:

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -1, \mathbb{1}q = q\mathbb{1} = q(q \in \mathbb{H}),$$

$$\mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}, \mathbb{j}\mathbb{k} = -\mathbb{k}\mathbb{j} = \mathbb{i}, \mathbb{k}\mathbb{i} = -\mathbb{i}\mathbb{k} = \mathbb{j},$$

$$\mathbb{1}^* = \mathbb{1}, \mathbb{i}^* = -\mathbb{i}, \mathbb{j}^* = -\mathbb{j}, \mathbb{k}^* = -\mathbb{k}.$$

A quaternion p is called real if $p^* = p$, imaginary if $p^* = -p$, here real numbers can be identified with real quaternions by $x \mapsto x\mathbb{1}$. There are several steps to establish the result of the proposition:

Step I, We first show that $(pq)^* = q^*p^*$ for $p, q \in \mathbb{H}$:

Suppose $p = (a, b), q = (c, d)$, then

$$\begin{aligned}(pq)^* &= (ac - d\bar{b}, \bar{a}d + cb)^* \\ &= (\bar{a}\bar{c} - \bar{d}b, -\bar{a}d - cb) \\ &= q^*p^*\end{aligned}$$

Step II, $< p, q >= \frac{1}{2}(p^*q + q^*p)$ is an inner product on $\mathbb{H}$, and the associated norm satisfies $|pq| = |p||q|$:

It is easy to check that $< .,. >$ defined as above is symmetric and bilinear over $\mathbb{R}$. Suppose $p = (a, b)$, then $< p, p >= a\bar{a} + b\bar{b} = |a|^2 + |b|^2$, which says that $< .,. >$ is positive definite. At last,

$$\begin{aligned}|pq|^2 &= |ac - d\bar{b}|^2 + |\bar{a}d + cb|^2 \\ &= |ac|^2 + |ad|^2 + |bc|^2 + |bd|^2 \\ &= (|a|^2 + |b|^2)(|c|^2 + |d|^2) \\ &= |p|^2|q|^2\end{aligned}$$

Step III, each nonzero quaternion in $\mathbb{H}$ has a two-sided multiplicative inverse given by $p^{-1} = |p|^{-2}p^*$, showing that the algebra of quaternions is a divisible algebra: We have $pp^* = |p|^2$.

Step IV, the set $\mathbb{H}^*$ of nonzero quaterions is a Lie group under quaternionic multiplication:

It is easy to check $\mathbb{H}^*$ is a group. As a vector space, $\mathbb{H} = \mathbb{R}\mathbb{1} \oplus \mathbb{R}\mathbb{i} \oplus \mathbb{R}\mathbb{j} \oplus \mathbb{R}\mathbb{k}$, we thus write $p = (x_1 + ix_2, x_3 - ix_4) = x_1\mathbb{1} + x_2\mathbb{i} + x_3\mathbb{j} + x_4\mathbb{k} = (x_1, x_2, x_3, x_4)$. Consider the continuous map $p \in \mathbb{H} \mapsto |p| \in \mathbb{R}$, we see $\mathbb{H}^*$ is the open subset where the above map takes nonzero values; hence $\mathbb{H}^* \subseteq \mathbb{H}$ is an open submanifold of $\mathbb{H}$. The condition of the smoothness of the inverse map in the definition of a Lie group is redundant, we only need to check the smoothness of the multiplication map: Suppose $p = (x_1, x_2, x_3, x_4), q = (y_1, y_2, y_3, y_4)$, then

$$\begin{aligned}pq = (&x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4, x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3, \\ &x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2, x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1),\end{aligned}$$

this shows the multiplication map is smooth, hence $\mathbb{H}^*$ is a Lie group.

Step V, let $\mathcal{S} \subseteq \mathbb{H}^*$ to be the set of unit quaternions. Then $\mathcal{S}$ is a Lie subgroup, diffeomorphic to $SU(2)$: $\mathcal{S}$ is the preimage of the singleton $\{1\} \in \mathbb{R}$ under the map $(p \in \mathbb{H} \mapsto |p| \in \mathbb{R})$, so $\mathcal{S}$ is a closed subset of $\mathbb{H}^*$, hence is a smooth manifold by Cartan's theorem. We write

$$\mathcal{S} = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

We see $\mathcal{S}$ is the unit sphere $\mathbb{S}^3$ in $\mathbb{H} = \mathbb{R}^4$, and we construct a map between $\mathcal{S}$ and $SU(2)$:

$$\phi : SU(2) \to \mathcal{S} \tag{26}$$

$$\begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} \mapsto (x_1, x_2, x_3, x_4) \tag{27}$$

We see the coordinate representation of ϕ is the identity map, and ϕ is a group homomorphism. Thus $\mathcal{S} = \mathbb{S}^3$ is diffeomorphic to $SU(2)$. $\square$

2.4 The relations between SU(2) and SO(3)

To state that $SU(2)$ is the double cover of $SO(3)$: (cf. [3]) Let V be the subspace of $M_2(\mathbb{C})$ consisting of skew-Hermitian traceless matrices, the following matrices form a basis of V:

$$\{A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\}$$

$$V = \mathbb{R}A_1 \oplus \mathbb{R}A_2 \oplus \mathbb{R}A_3 \tag{28}$$

define inner product on V by

$$< A, B >= \frac{1}{2} tr(AB) \tag{29}$$

we see $\{A_1, A_2, A_3\}$ forms an orthonormal basis with respect to the inner product. Let $SU(2)$ act on V by conjugation and denote this action by ϕ, since conjugation action keeps the trace of matrix, ϕ is well-defined. It can be checked ϕ_U preserves the inner product for each $U \in SU(2)$ hence it is orthogonal. Since $SU(2)$ is connected and

$$\phi : SU(2) \to O(3) \tag{30}$$

$$U \mapsto \Phi_U \tag{31}$$

is a continuous homomorphism, ϕ maps $SU(2)$ into the connected component $SO(3)$ of $O(3)$ since $I \in SO(3)$. ϕ is surjective and ker$(\Phi) = \pm I$, we have a 2:1 map:

$$\Phi : SU(2) \to O(3) \tag{32}$$

$$U \mapsto \Phi_U \tag{33}$$

We thus obtain the exact sequence:

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1 \tag{34}$$

In order to make our discussion clearer, we view $SU(2)$ as $\mathbb{S}^3$ and $SO(3)$ as $\mathbb{RP}^3$, the 3-dimensional real projective space. Take $V_i = \{[x_1, x_2, x_3, x_4] : x_i \neq 0\} \subseteq \mathbb{RP}^3$, as an open covering of $\mathbb{RP}^3$. The preimage of each V_i is a disjoint union of two open subset $U_{i,1} = \{(x_1, x_2, x_3, x_4) : x_i > 0\}$ and $U_{i,2} = \{(x_1, x_2, x_3, x_4) : x_i < 0\}$ of $\mathbb{S}^3$. It is easy to check that each $U_{i,j}, \{1 \leq i \leq 4, 1 \leq j \leq 2\}$ is mapped homeomorphically by ϕ onto V_i. So $SU(2) = \mathbb{S}^3$ is a covering space of $SO(3) = \mathbb{RP}^3$, precisely, double covering space. Since $SU(2)$ is simply-connected, we see $SU(2)$ is the universal covering space of $SO(3)$. Other methods prove $SU(2)$ is the double cover of $SO(3)$, such as the Clifford algebra. For reference, see [4, 5].

Proposition 2.25 and Theorem 2.26 tell us that as the universal covering space of $SO(3)$, $SU(2)$ has a unique smooth structure such that it is the universal covering manifold of $SO(3)$. We construct the smooth structure of $SU(2)$ as follows: choose any point $x \in SU(2)$, let U be an evenly covered open subset of $SO(3)$ containing $p(x)$, we may assume that U is the domain of a smooth coordinate chart (U, ϕ). If V is the component of $p^{-1}(U)$ containing x, and let $\tilde{\phi} = \phi \circ p|_V : V \to \mathbb{R}^n$, then $(V, \tilde{\phi})$ is a chart on $SU(2)$. We can check (from the explanation that $SU(2)$ is the double cover of $SO(3)$) that the unique structure on $SU(2)$ as the universal covering manifold constructed above coincides with the smooth structure diffeomorphic to $\mathcal{S} = \mathbb{S}^3$.

We can construct the multiplication map and the inverse map in the universal covering group by the lifting criterion for covering maps, which coincide with the group structure in $SU(2)$ inherited from $GL(2, \mathbb{C})$. We thus have proved that $SU(2)$ is the universal covering group of $SO(3)$.

Proposition 2.15 and Proposition 2.16 tell us how to calculate Lie algebras of some Lie subgroups of the general linear group:

$$\mathfrak{su}(2) = Lie(SU(2)) = \{A \in \mathfrak{gl}(2, \mathbb{C}) : trace(A) = 0, A \text{ is skew-Hermitian}\} \tag{35}$$

$$\mathfrak{so}(3) = Lie(SO(3)) = \{A \in \mathfrak{gl}(3, \mathbb{R}) : A \text{ is skew-symmetric}\} \tag{36}$$

We show the calculation:

I, suppose $A \in Lie(SU(2))$, then

$$det(exp(tA)) = exp((trace(A))t) = 1$$
$$trace(A) = 0$$

$$exp(tA)exp(tA)^* = exp(t(A + A^*)) = I$$
$$\text{differentiating at } t = 0 : A + A^* = 0$$

Conversely, $A \in \{A \in \mathfrak{gl}(2, \mathbb{C}) : trace(A) = 0, A \text{ is skew-Hermitian}\}$, then

$exp(tA) \in SU(2)$. Similar calculations give the Lie algebra of the special orthogonal group $SO(3)$.

Now we can use these results to characterize the Lie algebra of $SU(2)$ and $SO(3)$, first we can take

$$\{E_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, E_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, E_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\}$$

as basis of $\mathfrak{su}(2)$, and similarly we can take

$$\{F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, F_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\}$$

as basis of $\mathfrak{so}(3)$. we have by an easy computation that $[E_1, E_2] = E_3, [E_2, E_3] = E_1, [E_3, E_1] = E_2$, so $ad_{E_1} = F_1, ad_{E_2} = F_2, ad_{E_3} = F_3$ with respect to $\{E_1, E_2, E_3\}$. Since $[F_1, F_2] = F_3, [F_2, F_3] = F_1, [F_3, F_1] = F_2$, we see the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic via the isomorphism ad.

We have seen $SO(3)$ and its universal covering group $SU(2)$ have isomorphic Lie algebras, and we now prove a claim which helps us have a generalized proposition:

Problem 2.31. *A Lie group and its universal covering group have isomorphic Lie algebras.*

Proof. Step I We first prove a claim:

Claim 2.32. *Let $F : G \to H$ be a Lie group homomorphism between Lie groups G and H and a local diffeomorphism. The induced homomorphism between Lie algebras $F_* : Lie(G) \to Lie(H)$ is a Lie algebra isomorphism.*

Proof. The induced homomorphism F_* is defined as follows: choose a left-invariant vector field $X \in Lie(G)$, the value of the Lie algebra $F_*(X) \in Lie(H)$ at the identity is defined to be $dF_e(X_e)$. Since a left-invariant vector field is uniquely determined by its value at the identity, $F_*(X)$ is well-defined. Since the Lie algebra of a Lie group is linearly isomorphic to the tangent space at the identity of the Lie group, we only need to show that the differential dF_e of F is a linear isomorphism between the tangent spaces T_eG and T_eH. It is obvious since F is a local diffeomorphism. $\square$

Step II Take G as a Lie group together with its universal covering group $\phi : \tilde{G} \to G$. Because the universal covering map is a Lie group homomorphism and a local diffeomorphism, by step I we know that G and its universal covering group $\tilde{G}$ have isomorphic Lie algebras. $\square$

3 Finite subgroups of $SU(2)$

Our next goal is to find the finite subgroups of $SU(2)$ using the double cover of $SO(3)$ and finite subgroups of $SO(3)$. Take a finite subgroup F of $SO(3)$ and assume F^* is its preimage under the double cover $\phi : SU(2) \to SO(3)$, suppose $\Gamma \subseteq SU(2)$ is a finite subgroup of $SU(2)$, we have:

Theorem 3.1. *(Finite subgroups of $SU(2)$). Let $\Gamma \subseteq SU(2)$ then $\Gamma = F^*$ for some finite subgroup $F \subseteq SO(3)$ if and only if Γ is not a cyclic group of odd order.[6]*

So we only need to investigate the finite subgroups of $SO(3)$. We use Theorem 2.1 to realize elements in $SO(3)$ as rotations in $\mathbb{R}^3$ about the rotation axis in $\mathbb{R}^3$. We denote a rotation in $SO(3)$ by (u, θ), and the result about finite subgroups of $SO(3)$ follows this. We supply proof.

Theorem 3.2. *(Finite subgroups of $SO(3)$). A finite subgroup of $SO(3)$ is one of the following five types: the cyclic group, the dihedral group, the rotation group of* *the Platonic solid: the tetrahedron, the cube, the octahedron, the dodecahedron, the icosahedron.[7]*

Remark: the tetrahedron is self-dual, while the cube and the octahedron are dual, and the dodecahedron and the icosahedron are dual. Two polyhedrons A, and B, are dual means that the faces of A are in one-to-one correspondence to the vertices of B and the vertices of A are in one-to-one correspondence to the faces of B. We can check that the dual polyhedron has the same rotation group, so the cube has the same rotation group as the octahedron. The dodecahedron has the same rotation group as the icosahedron. To introduce the Macky Correspondence, we supple the proof of this important theorem.

Proof. Let $F \subseteq SO(3)$ be a finite subgroup of $SO(3)$, and collect poles of elements of F except the identity element and denote this set by E. Let F act on E, and we get several orbits of poles: $Ob_1, Ob_2, \ldots, Ob_s$, such that $E = \coprod_{i=1}^{s} Ob_i$(disjoint union). Denote by O_i the number of poles in Ob_i and by $Stab_i$ the stabilizer of an element in Ob_i and by r_i the number of elements in $Stab_i$. Count the number of poles in E in two ways we get:

$$2(|F|-1) = \sum_{i=1}^{s} \sum_{p \in Ob_i} (r_i - 1) = \sum_{i=1}^{s} O_i * (r_i - 1) \tag{37}$$

$$\sum_{i=1}^{s} \frac{1}{r_i} = s - 2 + \frac{2}{|F|} \tag{38}$$

since $r_i \geqslant 2$, we only have $s = 2$ or $s = 3$, there are in total five cases:

I: $s = 2$ and $r_1 = r_2 = |F|$ and $O_1 = O_2 = 1$, this is the cyclic group C_n.

II: $s = 3$ and $r_1 = r_2 = 2, r_3 = n$ and $O_1 = O_2 = n$ and $O_3 = 2$,(|F|=2n), this is the dihedral group $< X, Y, Z | X^2 = Y^2 = Z^n = XYZ = 1 >= D_n$.

III: $s = 3$ and $r_1 = 2, r_2 = 3, r_3 = 3$, this is the rotation group of tetrahedron:$< X, Y, Z | X^2 = Y^3 = Z^3 = XYZ = 1 >= A_4$.

IV: $s = 3$ and $r_1 = 2, r_2 = 3, r_3 = 4$, this is the rotation group of cube or octahedron:$< X, Y, Z | X^2 = Y^3 = Z^4 = XYZ = 1 >= S_4$.
V: $s = 3$ and $r_1 = 2, r_2 = 3, r_3 = 5$, this is the rotation group of dodecahedron or icosahedron:$< X, Y, Z | X^2 = Y^3 = Z^5 = XYZ = 1 >= A_5$.

$\square$

4 Mckay correspondence

We introduce $D(p, q, r) =< X, Y, Z | X^p = Y^q = Z^r = XYZ = 1 >$ and denote by $\tilde{D}(p, q, r)$ the inverse image under the double cover ϕ. We have proved that

$$D(1, n, n) = C_n, D(2, 2, n) = D_n, D(2, 3, 3) = A_4, D(2, 3, 4) = S_4, D(2, 3, 5) = A_5$$

4.1 Lie algebras and their Dynkin diagrams

Let us pause for a while and introduce the necessary knowledge of Lie algebras. Many contents in this subsection can be found in [8].

Definition 4.1. *A finite dimensional real or complex vector space* $\mathfrak{g}$ *is called a Lie algebra if it is equipped with a bilinear multiplication* $[.,.]$ *which is called the Lie bracket:* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that:*

(1),(skew-symmetric) $[x, x] = 0$ *for any* $x \in \mathfrak{g}$ (39)

(2),(Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for any* $x, y, z \in \mathfrak{g}$ (40)

A subspace closed under the Lie bracket is a Lie subalgebra, and an ideal of a Lie algebra is defined similarly to the definition of the ideal in a ring. A simple Lie algebra is a Lie algebra that has no nontrivial proper ideals and whose dimension is larger than 1. We consider a subalgebra consisting of all the finite linear combinations of elements of the form $[x, y]$ where $x, y \in \mathfrak{g}$; it is called the derived algebra of $\mathfrak{g}$ and is denoted by $[\mathfrak{g}, \mathfrak{g}]$. Using derived algebra, we can construct a descending sequence of subalgebras as follows:

$$\mathfrak{g}^{(0)} = \mathfrak{g} \tag{41}$$

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}], \text{ for } k \in \mathbb{N} \tag{42}$$

then $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$. A Lie algebra is said to be a semisimple Lie algebra if it has no solvable ideals, where a solvable ideal is an ideal of $\mathfrak{g}$ and is a solvable Lie algebra if itself is considered as a Lie algebra. It is easy to see that a simple Lie algebra is semisimple and it can be proved that a semisimple Lie algebra can be decomposed uniquely as a direct sum of simple ideals. [8]

There are also concepts about homomorphism and isomorphism which are both defined similarly, and we study one important type of endmorphism of a Lie algebra $\mathfrak{g}$ which is called the adjoint map:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \tag{43}$$

$$ad_x(y) = [x, y] \tag{44}$$

Using the adjoint map, we can define an associative symmetric bilinear form on $\mathfrak{g}$ which is the Killing form:

$$K(x, y) = Tr(ad_x ad_y) \text{ for } x, y \in \mathfrak{g} \tag{45}$$

The Jordan-Chevalley decomposition tells us that a linear transformation in a finite-dimensional vector space can be uniquely decomposed into its semisimple part and nilpotent part.[9] If the Jordan-Chevalley decomposition of ad_x has zero nilpotent part or ad_x is a semisimple endomorphism, then $x \in \mathfrak{g}$ is said to be a semisimple element. Next, we only talk about the semisimple Lie algebras over an algebraically closed field of characteristic 0 and consider its subalgebras $\mathfrak{h}$ consisting of semisimple elements which equals its centralizer $C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{h}\}$, such a subalgebra is called a Cartan subalgebra. A standard result in linear algebra tells us that a collection of commuting semisimple endomorphism can be simultaneously diagonalizable. We see $\mathfrak{h}$ has a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \bigoplus \underset{\alpha\in\Phi}{\oplus} \mathfrak{g}_\alpha, \tag{46}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : ad_h(x) = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is called the root space of $\mathfrak{g}$ and $\Phi = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$ is called the root system of $\mathfrak{g}$. We summarize the main properties of the Cartan subalgebra and the root system:[8]

(1) The Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate.

Using the property (1), and Riesz representation theorem, we can identity $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$: for each $\psi \in \mathfrak{h}^*$, there exist a unique $h_\psi \in \mathfrak{h}$ such that:

$$\psi(h) = K(h_\psi, h) \text{ for all } h \in \mathfrak{h} \tag{47}$$

We can use the above relation to define an inner product on $\mathfrak{h}^*$:

$$(\gamma, \delta) = K(h_\gamma, h_\delta) \tag{48}$$

(2) $\alpha, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$, and if $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta \neq 0$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

(3) Each $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ is one dimensional and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$.

(4) Φ spans $\mathfrak{h}^*$.

By (4), we can choose a basis Δ consisting of roots in Φ, and we can check (cf. [8]) that any root in Φ can be written as a $\mathbb{Q}$-linear combination of elements in Δ. We tensor the $\mathbb{Q}$-subspace $E_{\mathbb{Q}}$ of $\mathfrak{h}^*$ spanned by all the roots in Φ with $\mathbb{R}$, obtaining a $\mathbb{R}$-space E. Since $\mathfrak{h}$ consists of semisimple elements, we can check

$$(\lambda, \mu) = \sum_{\alpha\in\Phi} \alpha(h_\lambda)\alpha(h_\mu) = \sum_{\alpha\in\Phi} (\alpha, \lambda)(\alpha, \mu), \quad \text{for } \lambda, \mu \in \mathfrak{h}^*, \tag{49}$$

hence we have

$$(\lambda, \lambda) = \sum_{\alpha \in \Phi} (\alpha, \lambda)^2 \tag{50}$$

This shows the inner product is positive definite, hence:

(5) E is a Euclidean space with inner product defined above.

(6) Φ spans E and 0 does not belong to E.

(7) If $\alpha \in \Phi$, then $\mathbb{R}\alpha \cap \Phi = \pm\alpha$.

(8) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{\alpha,\alpha}\alpha \in \Phi$.

(9) If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Motivated by the property of (6)-(9), we call a finite subset Φ in a Euclidean space E satisfying (6)-(9) a root system of E and abbreviate the notation $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ as $\langle \beta, \alpha \rangle$. Note that $\langle \beta, \alpha \rangle$ is only linear in the first variable β. Fix a root system Φ of E, a subset Δ of Φ is said to be a base of E if it satisfies the following two conditions:

(1), Δ is a basis of E.

(2), every root β in Φ can be written as a integral linear combination of elements in Δ where the coefficients are all positive or negative.

Elements in Δ are called simple roots. For a fixed root system Φ, we can always find a basis of E (cf. [8]). Now we fix a basis Δ and order its elements as $\alpha_1, \alpha_2, \ldots, \alpha_l$, then we call the integral matrix of order l with entries $\langle \alpha_i, \alpha_j \rangle$ the Cartan matrix of Φ. Using the Cartan matrix, we can draw the Dynkin diagram of Φ which is a graph having l vertices corresponding to the simple root in Δ, the i-th joined to the j-th ($i \neq j$) by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. The Dynkin diagram of a semisimple Lie algebra is a union of connected graphs each of which corresponds to a simple ideal of the semisimple Lie algebra.

One of the most important relations satisfied by semisimple Lie algebras is Serre's relations which can help us recover the corresponding semisimple Lie algebra via a root system that satisfies these relations:

Theorem 4.2. *(Serre's relation). Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ, fix a base $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ of Φ. Using the nondegeneracy of the Killing form on $\mathfrak{h}$, for each $\alpha_i \in \Delta$, we can find a unique h_{α_i} such that $\alpha_i(h) = K(h_{\alpha_i}, h)$ for $h \in \mathfrak{h}$, now let $h_i = \frac{2h_{\alpha_i}}{(\alpha_i,\alpha_i)}$, then we have $\langle \alpha_i, \alpha_j \rangle = \alpha_i(h_j)$. By the property (3) of the root system, find $x_i \in \mathfrak{g}_{\alpha_i}$, we can find a unique $y_i \in \mathfrak{g}_{-\alpha}$ such that $[x_i, y_i] = h_i$. Then $\mathfrak{g}$ is generated by$\{x_i, y_i, h_i : 1 \le i \le l\}$ with the generators satisfying the following relations:*

$$(S_1) : [h_i, h_j] = 0, \qquad \text{for } 1 \le i, j \le l \tag{51}$$

$$(S_2) : [x_i, y_i] = h_i, [x_i, y_j] = 0, \qquad \text{for } i \ne j \tag{52}$$

$$(S_3) : [h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j, [h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j \tag{53}$$

$$(S_{ij}^+) : (ad_{x_i})^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0 \qquad \text{for } i \ne j \tag{54}$$

$$(S_{ij}^-) : (ad_{y_i})^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0 \qquad \text{for } i \ne j \tag{55}$$

Conversely, fix a root system Φ and its base $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$. Suppose $\mathfrak{g}$ is a Lie algebra generated by $\{x_i, y_i, h_i : 1 \le i \le l\}$ with these generators satisfying the relations $(S_1), (S_2), (S_3), (S_{ij}^+), (S_{ij}^-)$, then $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra whose Cartan subalgebra is spanned by h_i and the corresponding root system is Φ.[8]

4.2 Generalized Cartan matrix and its associated Dynkin diagram

We follow the idea of generalized Cartan matrix and Kac-Moody algebra in this subsection, and many contents can be found in [10].

First, let us consider a complex integral $n \times n$ matrix $A = (a_{ij})$ of rank l and it is said to be a generalized Cartan matrix if the following conditions are satisfied:

$$(C_1) : a_{ii} = 2 \text{ for } 1 \leq i \leq n \tag{56}$$

$$(C_2) : a_{ij} \leq 0 \text{ for } i \neq j \tag{57}$$

$$(C_3) : a_{ij} = 0 \text{ implies } a_{ji} = 0 \tag{58}$$

The associated Kac-Moody algebra $\mathfrak{g}(A)$ is a complex Lie algebra with generators $x_i, y_i, h_i (1 \leq i \leq n)$ and the defining relations [10]:

$$[h_i, h_j] = 0, \qquad \text{for } 1 \leq i, j \leq n \tag{59}$$

$$[x_i, y_i] = h_i, [x_i, y_j] = 0 \text{ for } i \neq j \tag{60}$$

$$[h_i, x_j] = a_{ij} x_j, [h_i, y_j] = -a_{ij} y_j \tag{61}$$

$$(ad_{x_i})^{1-a_{ij}}(x_j) = 0 \qquad \text{for } i \neq j \tag{62}$$

$$(ad_{y_i})^{1-a_{ij}}(y_j) = 0 \qquad \text{for } i \neq j \tag{63}$$

We introduce Serre's relations via which the Kac-Moody algebra is defined, although the corresponding matrix is a generalized Cartan matrix. We will omit many contents about this algebra since many of them is unnecessary for our discussion at this time, and readers who are interested in this algebra can read [10].

Our next goal is to classify the generalized Cartan matrix, and from now on, we abbreviate the generalized Cartan matrix as GCM. We have the following important classification theorem:

Theorem 4.3. *(Classification theorem for GCM). We use the notation $\alpha \geq 0$ for a n-dimensional real column vector if all its component ≥ 0, and $\alpha \leq 0$ or $\alpha = 0$ is defined similarly. Then a GCM A is defined to be of finite type, affine type, indefinite type if there exist a real column vector $\alpha \geq 0$ such that $A\alpha \geq 0, = 0, \leq 0$ respectively. Then every GCM is of one and only one type.[10]*

We introduce the Dynkin diagrams of a GCM (cf. [10]): assume $A = (a_{ij})$ is a GCM, the associated Dynkin diagram is a graph such that:

(1) if $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|,$ the vertices i and j are connected by $|a_{ij}|$ lines and these lines are equipped with an arrow pointing to i if $|a_{ij}| \geq 1.$

(2) if $a_{ij}a_{ji} > 4,$ the vertices i and j are connected by a bold-faced line equipped with an ordered pair of integers$|a_{ij}|, |a_{ji}|.$

We can check that the Dynkin diagram of a GCM of finite type is the Dynkin diagram of a simple Lie algebra $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$, hence a GCM is of finite type if and only if it is a Cartan matrix.

4.3 Mckay correspondence

Definition 4.4. *Given positive integers $p, q, r,$ we define the graph $Y(p, q, r)$ as follows:*

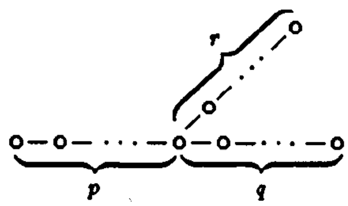

Figure 1: The graph Y(p,q,r)

Problem 4.5. *Let A be a GCM whose Dynkin diagram is the graph $Y(p, q, r)$ with $p \geq q \geq r$, Set $c = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$, show that A is of finite (resp. affine or indefinite) type if and only if $c > 1$ (resp. $c = 1$, $c < 1$). [10]*

Proof. We first show that A is of affine type if and only if $c = 1$:
We write the corresponding GCM concerning the graph $Y(p, q, r)$ as follows: Let

the first $p-1$ rows and columns correspond to the left $p-1$ vertices of the graph. The following $q-1$ rows and columns correspond to the right $q-1$ vertices of the graph, the following $r-1$ rows and columns correspond to the above $r-1$ vertices of the graph and the last row and column correspond to the branch point of the graph. Then the generalized Cartan matrix has the following form:

$$\begin{bmatrix} A_p & 0 & 0 & \alpha_p \\ 0 & A_q & 0 & \alpha_q \\ 0 & 0 & A_r & \alpha_r \\ \alpha_p' & \alpha_q' & \alpha_r' & 2 \end{bmatrix}$$

where A_p, A_q, A_r are matrices of order $p-1$, $q-1$ and $r-1$ respectively and have the following form:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

and α_p, α_q and α_r are column vectors of order $p-1$, $q-1$ and $r-1$ respectively and have the following form:

$$\begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ -1 \end{bmatrix}$$

Furthermore, a GCM A is of affine type if and only if there exists a vector u with positive entries such that Au is the zero vector. We assume there exists such a vector u:

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_{p-1} \\ b_1 \\ b_2 \\ \dots \\ b_{q-1} \\ c_1 \\ c_2 \\ \dots \\ c_{r-1} \\ d \end{bmatrix}$$

So $Au = 0$ is equivalent to the following systems:

$$\begin{cases} a_i = ia_1 & 1 \leq i \leq p-1 \\ b_j = jb_1 & 1 \leq j \leq q-1 \\ c_k = kc_1 & 1 \leq i \leq r-1 \\ d = pa_1 \\ d = qb_1 \\ d = rc_1 \\ -a_{p-1} - b_{q-1} - c_{r-1} + 2d = 0 \end{cases} \tag{64}$$

which is equivalent to $c = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

And a GCM A is of finite type if and only if it is a Cartan matrix, and there are in total five type simply-laced Dynkin diagrams:

$A_l = Y(1, \frac{l+1}{2}, \frac{l+1}{2})$

$D_l = Y(2, 2, l-2)$

$E_6 = Y(2, 3, 3)$

$E_7 = Y(2, 3, 4)$

$E_8 = Y(2, 3, 5)$

We see in all cases that $c = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

From the classification theorem of the GCM, a GCM is of one and only one of the following types: finite type, affine type and indefinite type. Hence from the discussion above, we have a GCM is of indefinite type if and only if $c < 1$ $\square$

We know there is a bijection between the affine Dynkin diagram of type ADE and the Dynkin diagram of type ADE, hence, a bijection between the simple Lie algebra of type ADE and the affine Kac-Moody algebra of type ADE. Let $Y \in \{A_l, D_l, E_6, E_7, E_8\}$, and let $\tilde{Y}$ be the corresponding affine Dynkin diagram. If Y has l vertices, then $\tilde{Y}$ has $l+1$ vertices and index them by $I = \{0, 1, \ldots, l\}$, where $I - \{0\}$ index the vertices of Y. Let $\{n_i | i \in I\}$ be the vertex labels on the affine diagram $\tilde{Y}$, i.e., the entries of the eigenvector of the corresponding GCM of the graph $\tilde{Y}$, scaled so that $n_0 = 1$. It can be checked case-by-case that the labels of the graph $\tilde{Y}(p, q, r)$ satisfy:

$$\sum_{i=0}^{l} n_i^2 = \frac{4}{(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1)} = 2|D(p, q, r)|. \tag{65}$$

The number of all irreducible representations of a finite group G equals the number of conjugacy classes of G. We denote the number of conjugacy classes by $c(G)$. We can read the information of all representations of G from the character table since every finite-dimensional representation of a finite group is completely reducible, and the corresponding character uniquely determines the representation. The next important theorem relates the affine dynkin diagram of type ADE and the finite subgroups of $SU(2)$. For more details about the Macky correspondence,see [11, 12]

Theorem 4.6. *(Mckay Correspondence). Let Γ be a nontrival subgroup of $SU(2)$ and keep the notation as before. $A(\Gamma)$ is the $(l+1) \times (l+1)$ matrix and there exists a complex simple Lie algebra $\mathfrak{g}$ of rank l and of type ADE, unique up to isomorphism, and an ordering of the simple positive roots of the associated affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ together with a bijection $\{\gamma_j\} \mapsto \{\alpha_j\}$, $\gamma_j \mapsto \alpha_j$ such that*

$$A(\Gamma) = 2I - C(\tilde{\mathfrak{g}}).$$

Here I is the $(l+1) \times (l+1)$ identity matrix. Moreover, the correspondence $\Gamma \mapsto \mathfrak{g}$ sets up a bijection between the set of isomorphism classes of nontrivial finite subgroups of $SU(2)$ and the isomorphism classes of complex simple Lie algebra of type ADE.

5 The relationship between representations of $SU(2)$ and representations of $SO(3)$

Let us start with the following diagram:

$$\begin{tikzcd} SU(2) \arrow[r, "\phi"] \arrow[dr, "\gamma"'] & SO(3) \arrow[d, "\tilde{\gamma}"] \\ & GL(V) \end{tikzcd}$$

where ϕ denotes the double covering map, γ is the representation of $SU(2)$ and $\tilde{\gamma}$ is the representation of $SO(3)$, if $\tilde{\gamma}$ gives an irreducible representation of $SO(3)$, then the composite $\tilde{\gamma} \circ \phi$ gives an irreducible representation of $SU(2)$. Conversely, if ker(ϕ) $\subseteq$ ker(γ), the representation γ of $SU(2)$ can factor through $SO(3)$ thus giving an irreducible representation of $SO(3)$. Since ker(ϕ) $= \{\pm I\}$, when $\gamma(-I) = I$, we get an irreducible representation of $SO(3)$ induced by the corresponding representation of $SU(2)$. More precisely, given $R \in SO(3)$, there exists one and only one pair of elements $\{T, -T\} \subseteq SU(2)$, such that

$$\tilde{\gamma}(R)(v) = \gamma(\pm T)(v) \text{ for all } v \in V \tag{66}$$

We continue our discussion, but we need some representation theory of $SU(2)$: consider the symmetric algebra $S(\mathbb{C}^2) = \oplus_{n=0}^{\infty} S^n(\mathbb{C}^2)$ where $S^n(\mathbb{C}^2)$ is the homogeneous component of the symmetric algebra. We know that $S(\mathbb{C}^2)$ is the same as the polynomial algebra in two variables over $\mathbb{C}^2$, so each homogeneous component can be viewed as a space of homogeneous polynomial of the same degree. The natural action of $SU(2)$ given by the right multiplication of a matrix on the row vector in $\mathbb{C}^2$ can be extended to the action of $SU(2)$ on $S^n(\mathbb{C}^2)$, so each component gives a representation π_n of $SU(2)$, and the following result can be found in many references:

Theorem 5.1. *Each representation π_n is irreducible and the set of irreducible representations $\{\pi_n\}$ for $n \in \mathbb{Z}_{\geq 0}$ gives all the irreducible representation of $SU(2)$ up to isomorphism.[13]*

We denote the representation space $S^n(\mathbb{C}^2)$ by V_n, and since $-I \in SU(2)$ acts as $(-1)^n$ on V_n, we know if and only if n is an even number, V_n gives an irreducible representation of $SO(3)$, however, the dimension of V_n at this time is $n + 1$, which is an odd number. The representation of $SO(3)$ can be related to spherical harmonics. We now give an introduction to the spherical harmonics [14]:

We consider the Euclidean space $\mathbb{R}^n$, and use E_k to denote the space of homogeneous polynomials of degree k with complex coefficients and F_k to denote the restriction of E_k to $\mathbb{S}^{n-1}$. We have

$$\mathbb{C}[x_1, \ldots, x_n] = \oplus_{k \in \mathbb{N}} E_k \tag{67}$$

Since on $\mathbb{S}^{n-1}$, we have $||x||^2 = \sum_{i=1}^{n} x_i^2 = 1$, we have the following:

$$E_k \cong F_k \supset F_{k-2} \supset F_{k-4} \supset \cdots \tag{68}$$

Let O_n acts on $\mathbb{R}^n$ by the left multiplication then we get representation spaces E_k and F_k of O_n. If F_k is equipped with a Hermitian inner product, from the representation theory, we can construct a unitary Hermite inner product using the Haar integral and the translation-invariance of the normalized Haar measure. So F_k is a unitary representation of $O(n)$. Moreover, using the unitary inner product, we can see that for a $O(n)$-invariant subspace $W \subseteq V$, the orthogonal complement $W^{\perp}$ is also $O(n)$-invariant, so we can also consider the orthogonal complement as a representation space. Now, we consider H_k, the orthogonal complement of F_{k-2} in F_k. Letting $F_{-1} = F_{-2} = 0$, inductively we have the decomposition of F_k:

$$F_k = H_k \bigoplus H_{k-2} \bigoplus \ldots \bigoplus H_1, \text{ if } k \equiv 1 (mod 2) \tag{69}$$

$$F_k = H_k \bigoplus H_{k-2} \bigoplus \ldots \bigoplus H_0, \text{ if } k \equiv 0 (mod 2) \tag{70}$$

We now explain why H_k is called the spherical harmonic of $\mathbb{S}^{n-1}$. We denote the subspace of all harmonic polynomials in E_k by L_k, and we have the following result. [15]

Lemma 5.2. *Every $f \in E_k$ can be uniquely written in the form*

$$f = f_k + |x|^2 f_{k-2} + \ldots + |x|^{2l} f_{k-2l} + \ldots + |x|^{2m} f_{k-2m} \tag{71}$$

where $m = [\frac{k}{2}]$ and $f_i \in L_i$.

Using the above lemma, we see when restricted on $\mathbb{S}^{n-1}$, the length of elements $||x|| = 1$, so we see each $f \in F_k$ can be uniquely written in the form

$$f = \tilde{f}_k + \tilde{f}_{k-2} + \ldots + \tilde{f}_{k-2l} + \ldots + \tilde{f}_{k-2m}$$

where $\tilde{f}_i \in F_i$ is a harmonic polynomial $f_i \in L_i$ restricted to $\mathbb{S}^{n-1}$ where the decomposition (70) is exactly the decomposition

$$F_k = H_k \bigoplus H_{k-2} \bigoplus H_{k-4} \bigoplus \cdots \tag{72}$$

So $H_k = L_k|_{\mathbb{S}^{n-1}}$, which means that each elements of H_k is a harmonic polynomial of degree k restricted to $\mathbb{S}^{n-1}$.

From linear algebra, eigenvalues of a unitary matrix all lie in $\mathbb{S}^1$. It can be unitarily similar to a diagonal matrix, and the unitary matrix can be in $SU(n)$ so matrix in $SU(2)$ is similar to the matrix of the following type:

$$e(t) = \begin{bmatrix} exp(it) & 0 \\ 0 & exp(-it) \end{bmatrix}$$

for some $t \in \mathbb{R}$. An orthogonal matrix can be orthogonally similar to the following type of matrix:

$$diag(A_1, A_2, \ldots, A_s, I_p, -I_q)$$

where I is the identity matrix and each A_i have the following form:

$$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$ and the orthogonal matrix can be in $SO(3)$. So each matrix in $SO(3)$ is similar to a matrix in the following form:

$$R(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(t) & -sin(t) \\ 0 & sin(t) & cos(t) \end{bmatrix}$$

We can construct a group homomorphism between $SU(2)$ and $SO(3)$ as follows:

$$\begin{aligned} \phi : & SU(2) \to SO(3) \\ & e(t) \mapsto R(2t) \end{aligned} \tag{73}$$

(Why do we restrict ϕ only to elements of the form $e(t) \in SU(2)$? Because every matrix A in $SU(2)$ is similar via an element in $SU(2)$ to a matrix of the form $e(t)$, and since ϕ is a group homomorphism, we have $\phi(A) = \phi(e(t))$.)

Let P_m be the complex space of homogeneous polynomials of degree m in three variables, and similar to the case of $SU(2)$, we have an action of $SO(3)$ on P_m via:

$$(Af)(x) = f(xA), \text{ for } A \in SO(3), x \in \mathbb{R}^3, f \in P_m \tag{74}$$

Unlike the $SU(2)$ case, each P_m for $m \geq 2$ is not irreducible, for example, $f = x_1^2 + x_2^2 + x_3^2$ generate a 1 dimensional invariant subspace of P_m which can be checked as follows: consider $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then f can be expressed with respect to the standard inner product $< \cdot, \cdot >$ in Euclidean space $\mathbb{R}^3$ as $f(x) =< x, x >$. Given any element in $A \in SO(3)$,

$$\begin{aligned}(A \cdot f)(x) &= f(x \cdot A) \\ &=< x \cdot A, x \cdot A > \\ &=< x, x >= f(x)\end{aligned}$$

since orthogonal matrix R preserves the inner product $< \cdot, \cdot >$.

Let $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ denote the Laplacian operator on $\mathbb{R}^3$, we have a subspace $\mathbb{R}_m$ of P_m where $\Re_m = \{f \in P_m : \triangle\ f = 0\}$ consisting of harmonic polynomials. Since a homogeneous polynomial can be uniquely determined by its restriction on $\mathbb{S}^2$, we restrict functions in $\Re_m$ to $\mathbb{S}^2$ and get the spherical harmonics of degree $m + 1$.

Lemma 5.3. *dim*P_m=$\frac{1}{2}(m+1)(m+2)$ *and dim*$\Re_m$=$2m+1$.*[13]*

Proof. The first is a purely combinatorial question to count the number of pairs (p, q, r) such that $p, q, r \in \mathbb{Z}_{\geq 0}$ and that $p + q + r = m$. We omit the proof of the second claim. $\square$

Lemma 5.4. *The action of the Laplace-operator on the space of smooth functions on $\mathbb{R}^3$ commutes with the action of $SO(3)$, hence $\Re_m$ is an $SO(3)$-invariant subspace of P_m.[13]*

Proof. Assume

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Let $p = p(x, y, z)$ be a smooth function, denote

$$\begin{aligned} \tilde{p}(x, y, z) &= (A \cdot p)(x, y, z) \\ &= p(ax + dy + gz, bx + ey + hz, cx + fy + iz). \end{aligned}$$

We set $p_{ij} := \frac{\partial^2 p}{\partial x_i \partial x_j}$, where $1 \leq i, j \leq 3$.
First, $(A \cdot (\triangle\, p)) = A \cdot (p_{11} + p_{22} + p_{33}) = \tilde{p}_{11} + \tilde{p}_{22} + \tilde{p}_{33}$.
Second,

$$\begin{aligned} \triangle\, \tilde{p} =& (a^2 + d^2 + g^2)\tilde{p}_{11} + (b^2 + e^2 + h^2)\tilde{p}_{22} + (c^2 + f^2 + i^2)\tilde{p}_{33} \\ &+ 2(ab + de + gh)\tilde{p}_{12} + 2(ac + df + gi)\tilde{p}_{13} + 2(bc + ef + hi)\tilde{p}_{23} \end{aligned}$$

We denote the column vectors of A by v_1, v_2, v_3, then

$$\begin{aligned} \triangle\, \tilde{p} =& < v_1, v_1 > \tilde{p}_{11} + < v_2, v_2 > \tilde{p}_{22} + < v_3, v_3 > \tilde{p}_{33} \\ &+ 2 < v_1, v_2 > \tilde{p}_{12} + 2 < v_1, v_3 > \tilde{p}_{13} + 2 < v_2, v_3 > \tilde{p}_{23} \end{aligned}$$

where $< \cdot, \cdot >$ denote the usual inner product in Euclidean space.

Since $A \in SO(3)$, the column vectors form an orthonormal basis, and the result follows. $\square$

We now pause for a while and introduce one of the most important concepts in representation theory of groups:

Definition 5.5. *Given a representation (ρ, V) of a group G, i.e. a associative algebra homomorphism ρ between the the group algebra $\mathbb{C}[G]$ to $End_{\mathbb{C}}(V)$, the character of ρ is defined as follows:*

$$\mathcal{X}_{\rho}(g) = trace(\rho(g)). \tag{75}$$

Remark: From the property of trace that similar transformation or matrix in a finite-dimensional linear space have the same trace, we can see the character is a class function that takes the same values on elements in a conjugacy class and isomorphic representations give the same character since there exists an invertible intertwining map between the two spaces.

Given the irreducible representation (π_m, V_m) of $SU(2)$ we have discussed earlier, the irreducible representation π_m where $\pi_m(-I) = I$ of $SU(2)$ corresponds to an irreducible representation $\tilde{\pi}_m$ of $SO(3)$ which is induced by π_m, and this is the case if and only if m is an even number, and since the pair of elements $\pm e(t) \in SU(2)$ corresponds to $R(2t) \in SO(3)$, their action on V_m are the same, especially the characters of the corresponding representations of them are the same, i.e. $\mathcal{X}_{\pi_m}(\pm e(t)) = \mathcal{X}_{\tilde{\pi}_m}(R(2t))$. Since the character is a class function and every matrix in $SU(2)$ is unitarily similar to one $e(t)$ via a unitary matrix in $SU(2)$, every matrix in $SU(2)$ is conjugate to one matrix of the form $e(t)$, we only need to know the value of the character on elements of the form $e(t)$. Similar arguments show that to study the character of the representation $\tilde{\pi}_m$ of $SO(3)$, it suffices to know the value of $\mathcal{X}_{\tilde{\pi}_m}$ on elements of the form $R(t) \in$ SO(3).

The above arguments tell us that it suffices to know the value of $\mathcal{X}_{\pi_m}$ on elements of the form $e(t) \in SU(2)$ to study the representation of SO(3). A basis of V_m can be taken as follows: $\{z_1^k z_2^{m-k} : 0 \leq k \leq m\}$, and it can be checked $\mathcal{X}_{\pi_m}(e(t)) = \sum_{k=0}^{m} exp(i(m-2k)t)$ is linear combination of $exp(ikt)$, $|k| \leq m.$ Now it is enough to give the next proposition:

Proposition 5.6. *The space $\mathfrak{R}_m$ of harmonic polynomials of degree m is an irreducible $SO(3)$-module.[13]*

The irreducible representations of $SU(2)$ where $-I$ acts as the identity are in one-to-one correspondence to the irreducible representations of $SO(3)$, and the irreducible representation of $SU(2)$ are V_m which is the space of homogeneous polynomials in two variables. When m is even, $-I \in SU(2)$ acts as an identity, and this gives an irreducible representation of $SO(3)$, and all of the irreducible representations of $SO(3)$ are isomorphic to one of V_m (m even). Since $dim\mathfrak{R}_m = 2m + 1 = dimV_m$, we have found all irreducible representations of $SO(3)$ up to isomorphism.

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6 Appendices

None.

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