Algorithm Efficiency

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A review on algorithm

What is Algorithm?

- An algorithm is
  - a finite sequence of well-defined steps (statements, often called instructions or commands)
  - that provides the solution to a problem.

Algorithm

- Give some examples of algorithms.

An Example

- Input: No
- Output: what do you think about the output?

- Step 1. Assign sum = 0. Assign i = 0.
- Step 2.
  - Assign i = i + 1
  - Assign sum = sum + i
- Step 3. Compare i with 10
  - if i < 10, back to step 2.
  - otherwise, if i ≥ 10, go to step 4.
- Step 4. return sum

Characteristics of Algorithms

- Finiteness
  - For any input, the algorithm must terminate after a finite number of steps.
- Correctness
  - Always correct. Give the same result for different run time.
- Definiteness
  - All steps of the algorithm must be precisely defined.
- Effectiveness
  - It must be possible to perform each step of the algorithm correctly and in a finite amount of time.

Algorithm Efficiency

- The two factors of Algorithm Efficiency are:
  - **Time Factor**: Time is measured by counting the number of key operations.
  - **Space Factor**: Space is measured by counting the maximum memory space required by the algorithm.

Measuring Efficiency of Algorithms

- Can we compare two algorithms (in time factor) like this?
  - Implement those algorithms (into programs)
  - Calculate the execution time of those programs
  - Compare those two time values.

Measuring Efficiency of Algorithms

- Comparison of algorithms should focus on **significant differences** in efficiency

- Difficulties with comparing programs instead of algorithms
  - How are the algorithms coded?
  - What computer should you use?
  - What data should the programs use?

Measuring Efficiency of Algorithms

- Employ mathematical techniques that analyze algorithms **independently** of specific implementations, computers, or data.

Execution Time of Algorithm

- Derive an algorithm’s time requirement as a function of **the problem size**

- base on the **key operations**:
  - Comparisons
  - Assignments
- For exmaple
  - Algorithm *A* requires $n^2/5$ time unit to solve a problem of size *n*.
  - Algorithm *B* requires 5 x n time unit to solve a problem of size *n*.

Execution Time of Algorithm

- Traversal of linked nodes – example:

```
Node<ItemType>* curPtr = headPtr;          ← 1 assignment
while (curPtr != nullptr)                  ← n + 1 comparisons
{
   cout << curPtr->getItem() < endl;       ← n writes
   curPtr = curPtr->getNext();             ← n assignments
}  // end while
```

  - Assignment: *a* time units.
  - Comparison: *c* time units.
  - Write: *w* time units.
- Displaying data in linked chain of *n* nodes requires time proportional to *n*

Execution Time of Algorithm

- Nested loops

```
for (i = 1 through n)
    for (j = 1 through i)
        for (k = 1 through 5)
            Task T
```

  - Task *T* requires *t* time units.

Previous Example

- **Step 1.** Assign sum = 0. Assign i = 0.
- **Step 2.**
  - Assign i = i + 1
  - Assign sum = sum + i
- **Step 3.** Compare ***i*** with ***10***
  - if i < 10, back to step 2.
  - otherwise, if i ≥ 10, go to step 4.
- **Step 4.** Return sum

How many
- Assignments?
- Comparisons?

Another Example

- **Step 1.** Assign sum = 0. Assign i = 0.
- **Step 2.**
  - Assign i = i + 1
  - Assign sum = sum + i
- **Step 3.** Compare *i* with ***n***
  - if i < ***n***, back to step 2.
  - otherwise, if i ≥ ***n***, go to step 4.
- **Step 4.** Return sum

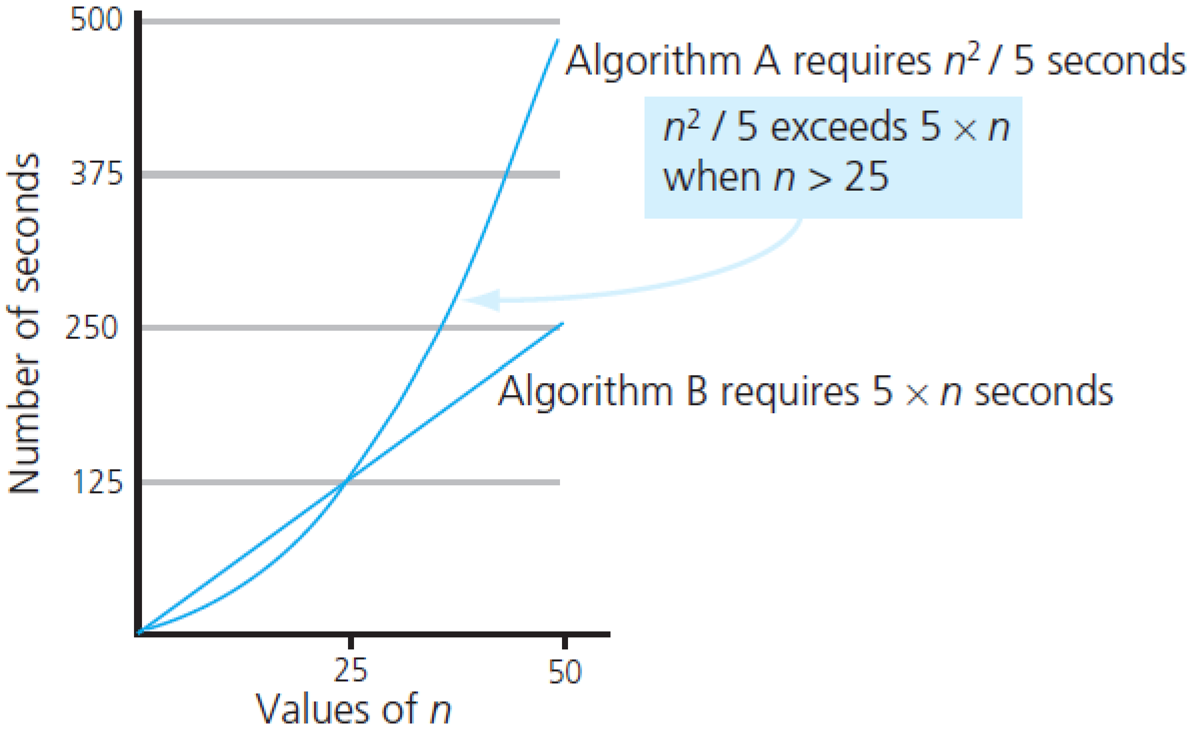
How many
- Assignments?
- Comparisons?

Algorithm Growth Rates

- Measure algorithm's time requirement as a function of problem size
- Compare algorithm efficiencies for **large problems**
- Look only at **significant differences**.

Algorithm Growth Rates

- Time requirements as a function of the problem size *n*



Analysis and Big O Notation

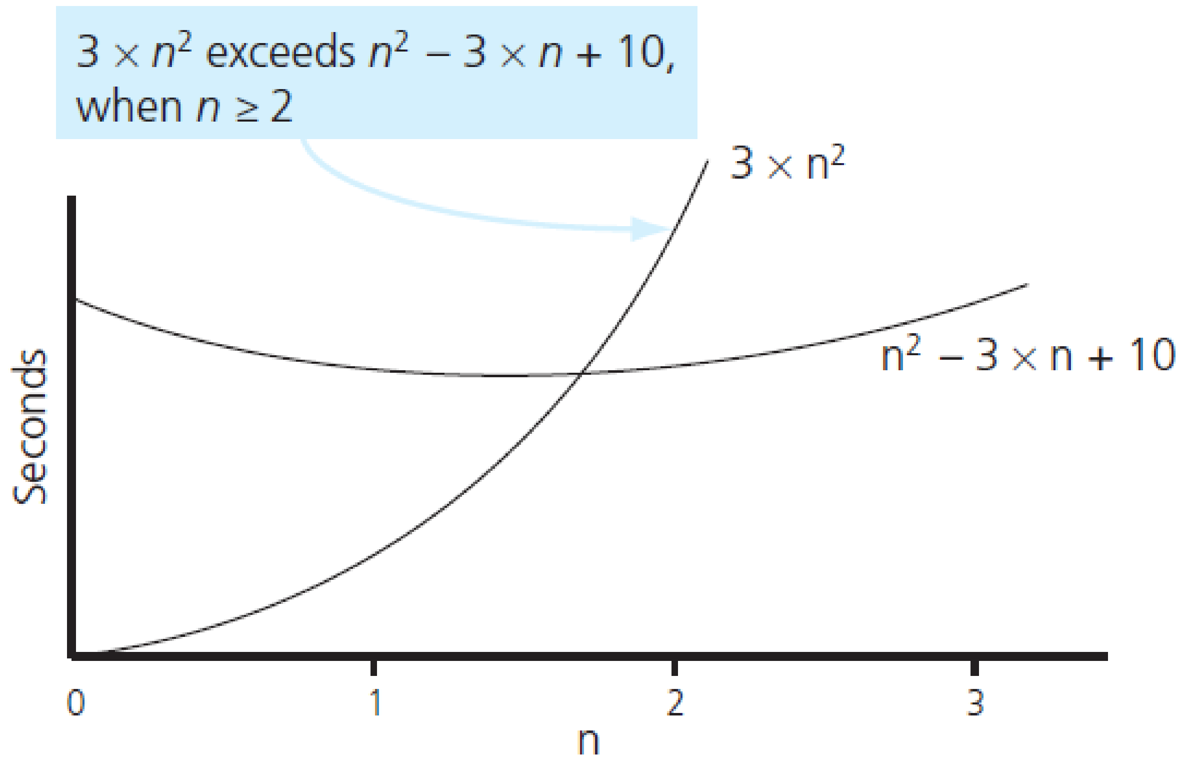
Big O Notation

- Definition:
  - Algorithm A is order f (n)
    - Denoted **O(f (n))**
  - If constants **k** and $\mathbf{n_0}$ exist
  - Such that A requires **no more** than **k × f (n)** time units to solve a problem of size $\mathbf{n \geq n_0}$.

Example

- An algorithm requires $\mathbf{n^2 - 3n + 10}$ (time units). What is the order of algorithm?
  - Hint: Find the values ***k*** va $\boldsymbol{n_0}$.
- When $1 < n$
  - $1 < n^2$
  - $10 < 10n^2$, $-3n < 0$
  - $n^2 – 3n + 10 < 11n^2$
- $\mathbf{O(n^2)}$, $n_0 = 1$, $k = 11$

Example



The graphs of $\mathbf{3 \times n^2}$ and $\mathbf{n^2 - 3 \times n + 10}$

Another Example

- How about the order of an algorithm requiring **$(n + 1) \times (a + c) + nw$** time units?
  - a, c, w are constants and greater than 0
- $(n + 1) \times (a + c) + nw = (a + c + w)n + a + c$
- When $1 < n$
  - $a + c < a + c + w$, $a + c + w < (a + c + w)n$
  - $a + c < (a + c + w)n$
  - $(a + c + w)n + a + c < 2(a + c + w)n$
- **O(n)**, $n_0 = 1$, $k = 2(a + c + w)$

Another Example

- Another algorithm requires $\mathbf{n^2 + 3n + 2}$ time units. What is the order of this algorithm?
- When $1 < n$
  - $n < n^2$, $1 < n^2$
  - $3n < 3n^2$, $2 < 2n^2$
  - $n^2 + 3n + 2 < 6n^2$
- $\mathbf{O(n^2)}$, $n_0 = 1$, $k = 6$

Common Growth-Rate Functions

- f(n) =
    - 1: Constant
    - $\log_2 n$: Logarithmic
    - n: Linear
    - $n \times \log_2 n$: Linearithmic
    - n^2: Quadratic
    - n^3: Cubic
    - 2^n: Exponential

Common Growth-Rate Functions

- Order of growth of some common functions

$$O(1) < O(\log_2 n) < O(n) < O(n \times \log_2 n) < O(n^2) < O(n^3) < O(2^n)$$

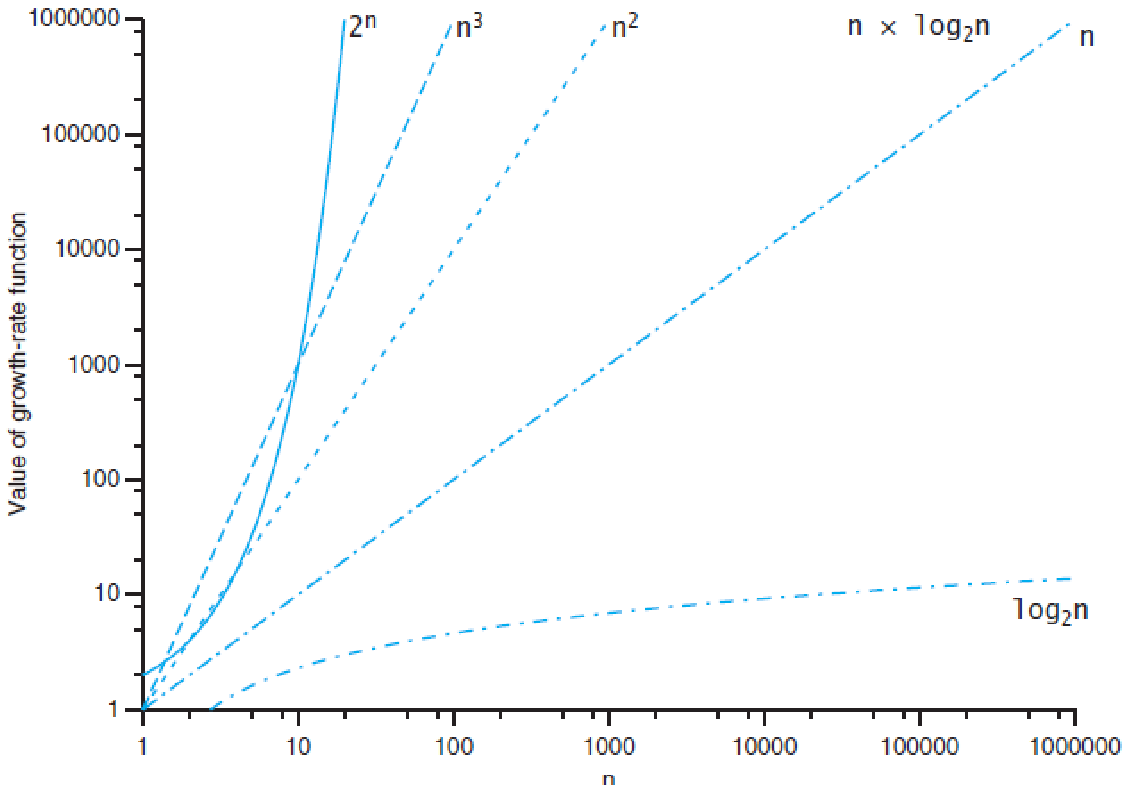
Common Growth-Rate Functions

- A comparison of growth-rate functions in **tabular form**

Function	n = 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n \times \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Common Growth-Rate Functions

- A comparison of growth-rate functions in **graphical form**



Properties of Growth-Rate Functions

- Ignore low-order terms
- Ignore a multiplicative constant in the high-order term
- $O(f(n)) + O(g(n)) = O(f(n) + g(n))$

Some Useful Results

- **Constant Multiplication**:
  - If f(n) is O(g(n)) then **c**.f(n) is O(g(n)), where **c** is a constant.

- **Polynomial Function:**
  - $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is $O(x^n)$.

Some Useful Results

- **Summation Function:**
  - If $f_1(n)$ is $O(g_1(n))$ and $f_2(n)$ is $O(g_2(n))$
  - Then $f_1(n) + f_2(n)$ is $O(\max(g_1(n), g_2(n)))$

- **Multiplication Function:**
  - If $f_1(n)$ is $O(g_1(n))$ and $f_2(n)$ is $O(g_2(n))$
  - Then $f_1(n) \times f_2(n)$ is $O(g_1(n) \times g_2(n))$

Quiz

Are these functions of order O(x)?

a) $f(x) = 10$
b) $f(x) = 3x + 7$
c) $f(x) = 2x^2 + 2$

Quiz

What are the order of the following functions?

- $f(n) = (2 + n) * (3 + \log_2 n)$
- $f(n) = 11 * \log_2 n + n/2 – 3542$
- $f(n) = n * (3 + n) – 7 * n$
- $f(n) = \log_2(n^2) + n$

Notes

- Use like this:
  - $f(x)$ is $O(g(x))$, or
  - $f(x)$ is of order $g(x)$, or
  - $f(x)$ has order $g(x)$

Algorithm Efficiency

Algorithm Efficiency

- Best case

- Worst case

- Average case

An Algorithm to Analyze

- Input:
- Output:

- **Step 1.** Set the first integer the temporary maximum value (temp_max).
- **Step 2.** Compare the current value with the temp_max.
  - If it is greater than, assign the current value to temp_max.
- **Step 3.** If there is other integer in the list, move to next value. Back to step 2.
- **Step 4.** If there is no more integer in the list, stop.
- **Step 5.** return temp_max (the maximum value of the list).

Another Algorithm to Analyze

- Input:
- Output:

- Step 1. Assign ***i*** = 0
- Step 2. While ***i*** < ***n*** and ***x*** ≠ $\boldsymbol{a_i}$, increase ***i*** by 1.

```
while (i < n and x ≠ a_i)
    i = i + 1
```

- Step 3.
  - If ***i*** < ***n***, return ***i***.
  - Otherwise (i >= n), return **-1** to tell that ***x*** does not exist in list ***a***.

Another Algorithm to Analyze

- Use comparisons for counting.

- Worst case:
  - When it occurs?
  - How many operations?

- Best case:
  - When it occurs?
  - How many operations?

Another Algorithm to Analyze

- Use comparisons for counting.

- Average case:
  - If x is found at position i^{th}, the number of comparisons is $2i + 1$.
  - The average number of comparisons is:

$$\frac{3+5+7+..+(2n+1)}{n} = \frac{2(1+2+3+...+n)+n}{n} = \frac{2\frac{n(n+1)}{2}+n}{n} = n+2$$

Keeping Your Perspective

- If problem size always small, ignore an algorithm's efficiency
- Weigh trade-offs between algorithm's time and memory requirements
- Compare algorithms for both style and efficiency

Exercises

Exercise

- Propose an algorithm to calculate the value of *S* defined below. What order does the algorithm have?

$$S = 1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n!}$$

- How many comparisons, assignments are there in the following code fragment with the size *n*?

```
sum = 0;
for (i = 0; i < n; i++)
{
    cin >> x;
    sum = sum + x;
}
```

Exercise

How many assignments are there in the following code fragment with the size *n*?

```
for (i = 0; i < n ; i++)
      for (j = 0; j < n; j++)
      {
            C[i][j] = 0;
            for (k = 0; k < n; k++)
                  C[i][j] = C[i][j] + A[i][k]*B[k][j];
      }
```

Exercise

- Give the order of growth (as a function of N) of the running time of the following code fragment:

```
int sum = 0;
for (int n = N; n > 0; n /= 2)
    for (int i = 0; i < n; i++)
        sum++;
```

Exercise

- Give the order of growth (as a function of N) of the running time of the following code fragment:

```
int sum = 0;
for (int i = 1; i < N; i *= 2)
    for (int j = 0; j < i; j++)
        sum++;
```

Exercise

- Give the order of growth (as a function of N) of the running time of the following code fragment:

```
int sum = 0;
for (int i = 1; i < N; i *= 2)
    for (int j = 0; j < N; j++)
        sum++;
```

Questions and Answers