Graph Structure

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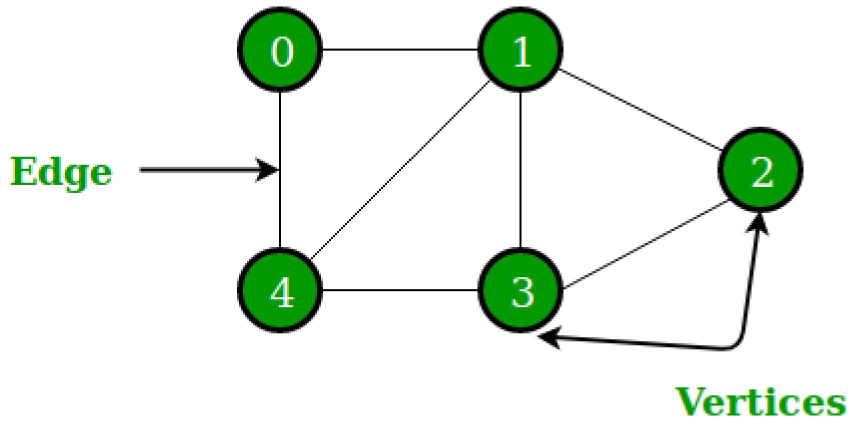
- Terminologies
- Graph representation
- Graph traversal
- Spanning tree
- Shortest path

Graph

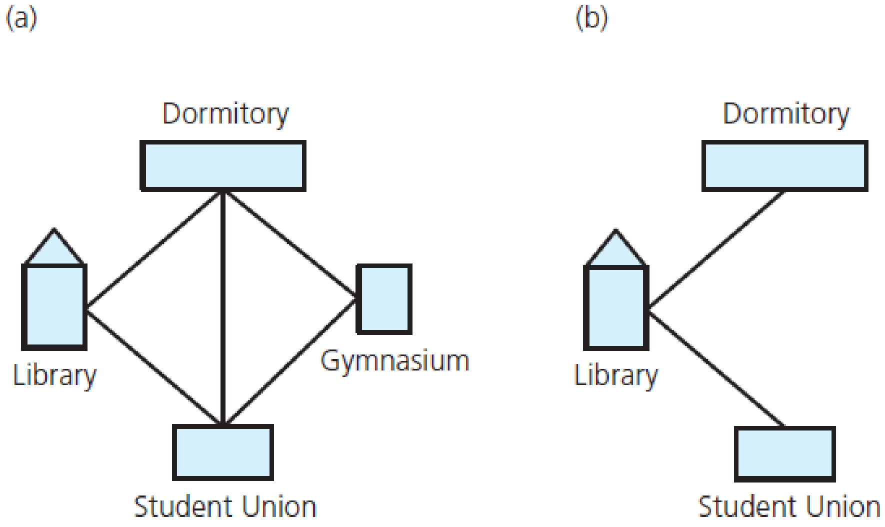
- A graph consists of **a finite set of vertices** (or nodes) and **set of edges** which connect a pair of nodes.
- G = {V, E}
  - V: set of vertices. $V = \{v_1, v_2, \ldots, v_n\}$
  - E: set of edges. $E = \{e_1, e_2, \ldots, e_m\}$
- Example:
  - V = {0, 1, 2, 3, 4}
  - E = {01, 04, 12, 13, 14, 23, 34}



Terminologies

Terminologies

- A **subgraph** consists of a subset of a graph's vertices and a subset of its edges.
  - G'= {V', E'} is a subgraph of G = {V, E} if V' ⊆ V, E' ⊆ E



(a) A campus map as a graph;
(b) a subgraph

Terminologies

- **Vertex**: also called a **node**.
- **Edge**: connects two vertices.
- **Loop** (*self-edge*): An edge of the form (v, v).
- **Adjacent**: two vertices are **adjacent** if they are joined by an edge.

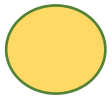
Terminologies

- **Path**: A sequence of edges that begins at one vertex and ends at another vertex.
  - If all vertices of a path is distinct, the path is **simple**.
- **Cycle**: A path that starts and ends at the same vertex and does not traverse the same edge more than once.
- **Acyclic** graph: A graph with no cycle.

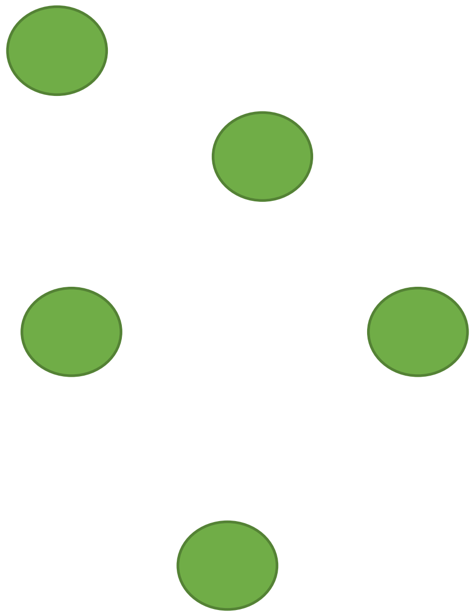
Terminologies

- **Null graph**: A graph having no edges

- **Trivial graph**: A graph with only one vertex.



trivial graph

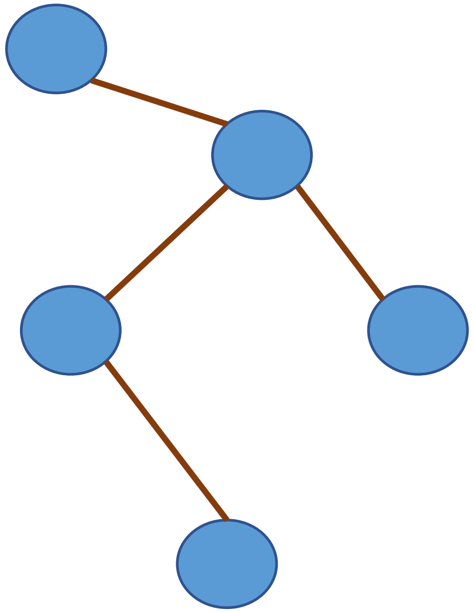


null graph

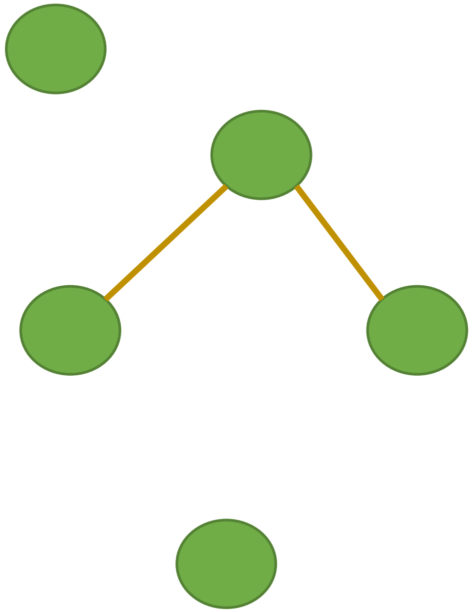
Terminologies

- **Connected graph:** A graph in which each pair of **distinct vertices** has a **path** between them.
- **Disconnected graph:** A graph that contains two vertices not connected.
- **Complete graph:** A graph in which each pairs of **distinct vertices** has an **edge** between them
- Graph cannot have duplicate edges between vertices.
  - **Multigraph**: does allow multiple edges

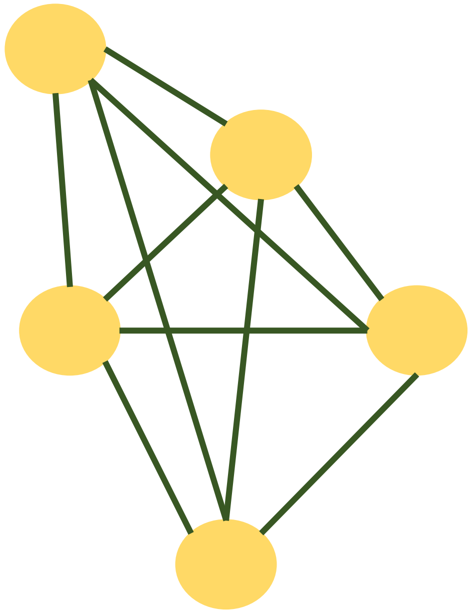
Terminologies



connected graph



disconnected graph



complete graph

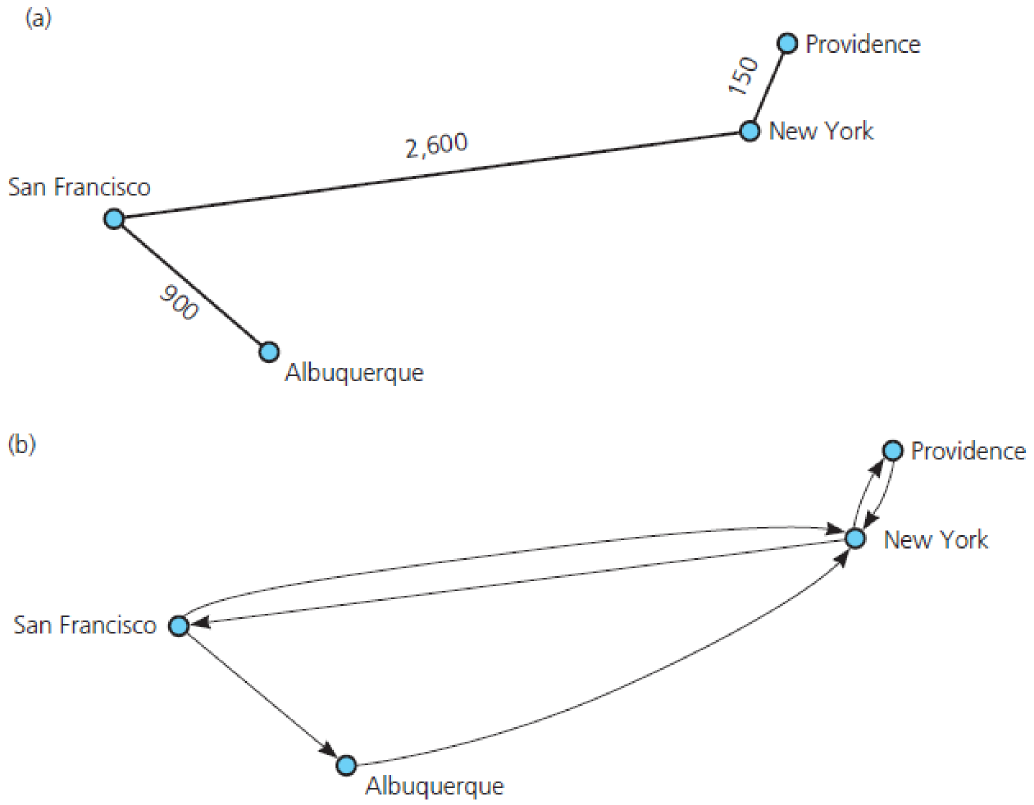
Terminologies

- **Undirected graph**: the graph in which edges do not indicate a direction.
- **Directed graph, or digraph**: a graph in which each edge has a direction.
- **Weighted graph**: a graph with numbers (weights, costs) assigned to its edges.

Terminologies

(a): undirected graph

(b): directed graph



Graph Representation

Graph Representation

- Adjacency Matrix

- Adjacency List

Adjacency Matrix

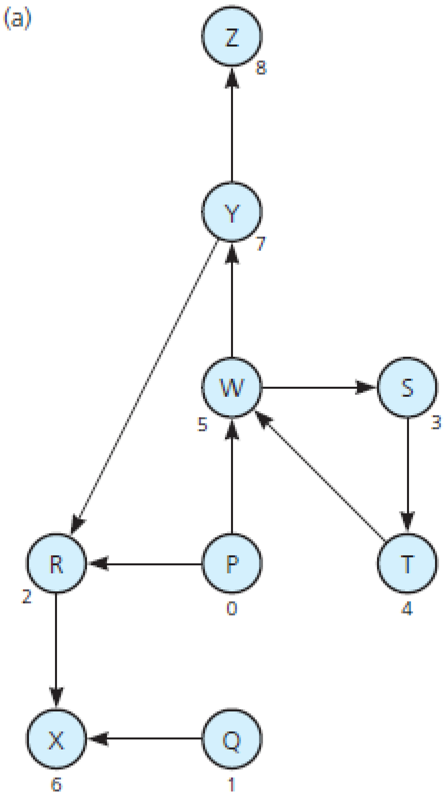
A[*n*][*n*] with *n* is the number of vertices.

- $A[i][j] = \begin{cases} 1\ if\ \ there\ is\ an\ edge(i,j) \\ 0\ if\ \ there\ is\ no\ edge\ (i,j) \end{cases}$

- $A[i][j] = \begin{cases} w & with\ w\ is\ the\ weight\ of\ edge(i,j) \\ \infty & if\ there\ is\ no\ edge\ (i,j) \end{cases}$

Adjacency Matrix

(a)

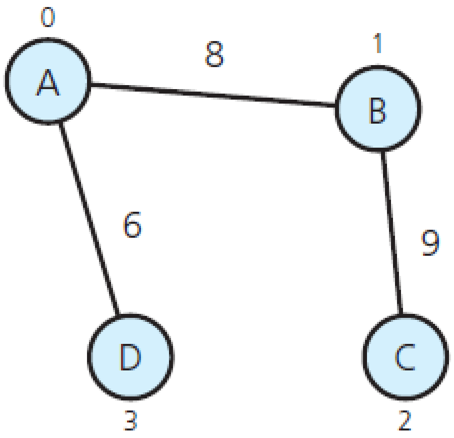


(b)

		0	1	2	3	4	5	6	7	8
		P	Q	R	S	T	W	X	Y	Z
0	P	0	0	1	0	0	1	0	0	0
1	Q	0	0	0	0	0	0	1	0	0
2	R	0	0	0	0	0	0	1	0	0
3	S	0	0	0	0	1	0	0	0	0
4	T	0	0	0	0	0	1	0	0	0
5	W	0	0	0	1	0	0	0	1	0
6	X	0	0	0	0	0	0	0	0	0
7	Y	0	0	1	0	0	0	0	0	1
8	Z	0	0	0	0	0	0	0	0	0

Adjacency Matrix

(a)



(b)

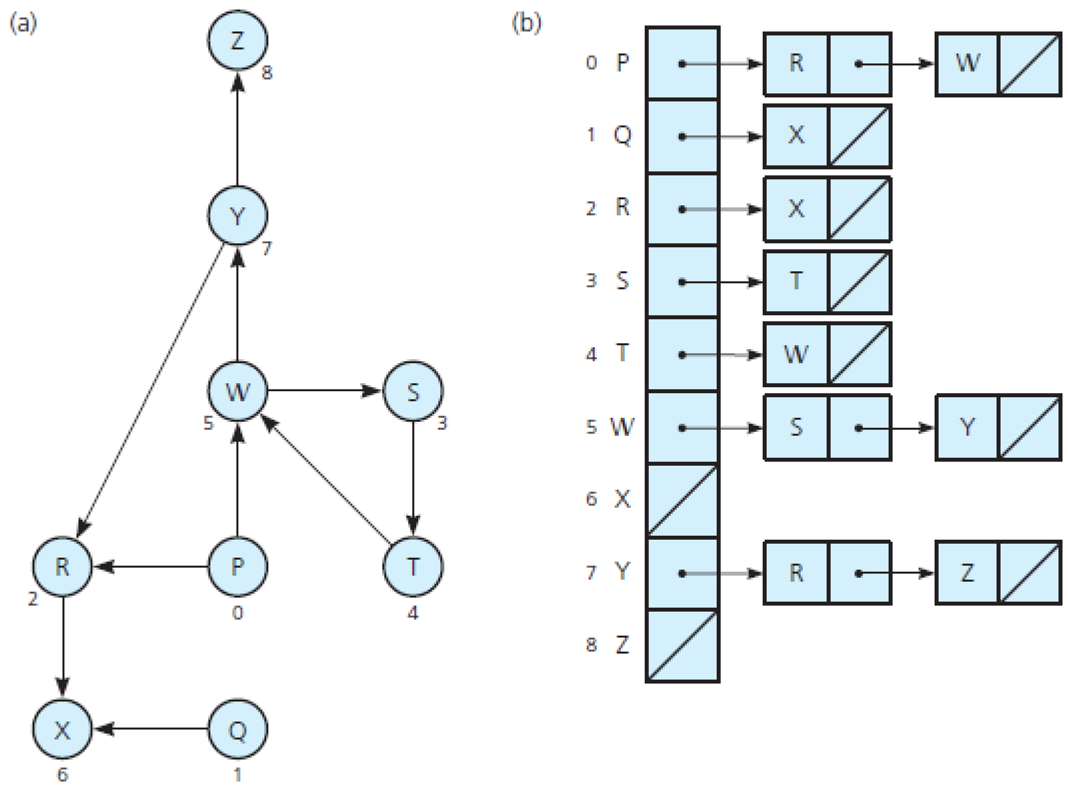
		0 A	1 B	2 C	3 D
0	A	∞	8	∞	6
1	B	8	∞	9	∞
2	C	∞	9	∞	∞
3	D	6	∞	∞	∞

Adjacency List

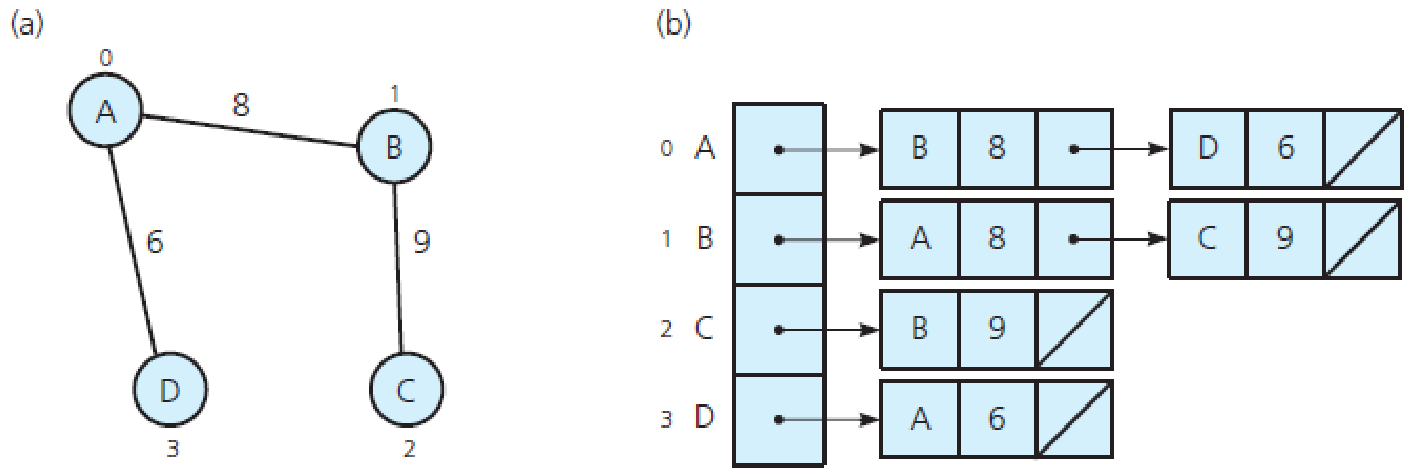
- A graph with *n* vertices has *n* linked chains.

- The i^{th} linked chain has a node for vertex *j* if and only if having edge (*i*,*j*).

Adjacency List



Adjacency List



Graph Traversal

Graph Traversal

- Visits (all) the vertices that it can reach.

- **Connected component** is subset of vertices visited during traversal that begins at given vertex.

Depth-First Search

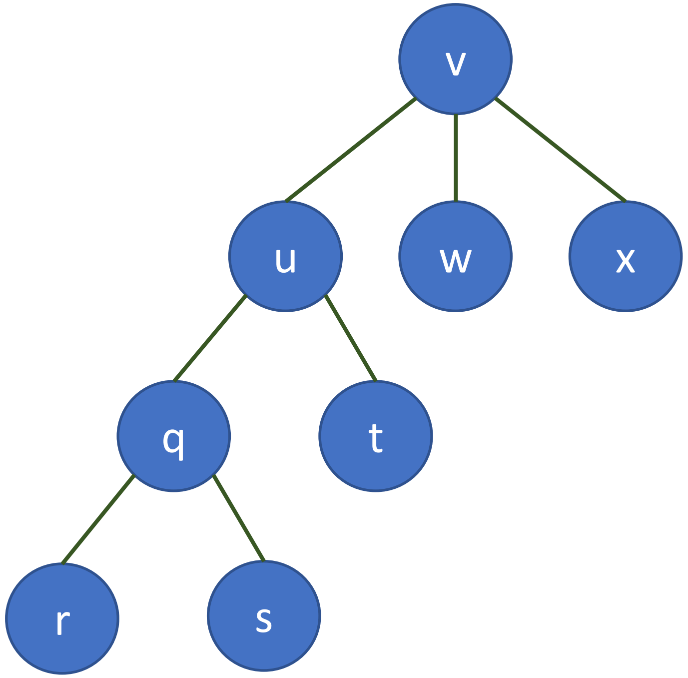
- Goes as far as possible from a vertex before backing up.

```
DFS(v: vertex)
{
      Mark v as visited
      for (each unvisited vertex u adjacent to v)
            DFS(u)
}
```

Depth-First Search

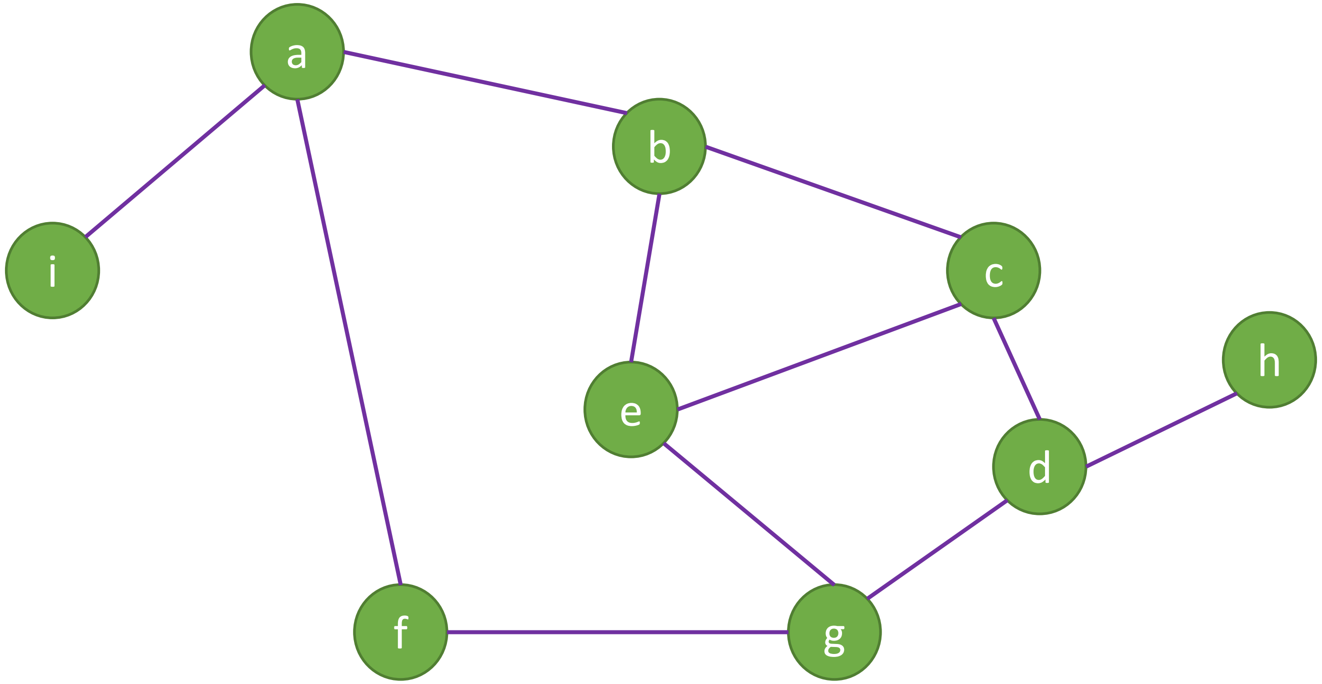
```
DFS(v: vertex)
      s = new empty stack
      s.push(v)
      Mark v as visited
      while (s is not empty)  {
            if (no unvisited vertices are adjacent to the vertex on the top of the stack)
                  s.pop()
            else {
                  s.push(u)
                  Marked u as visited
            }
      }
```

Depth-First Search



v - u - q - r - s - t - w - x

Depth-First Search



DFS starts at ***a***:

DFS starts at ***e***:

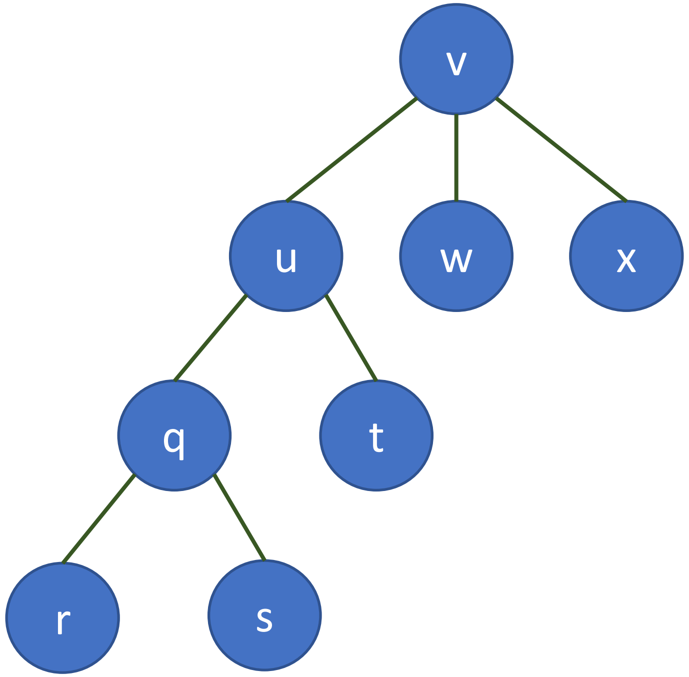
Breadth-First Search

- Visits all vertices adjacent to vertex before going forward.
- Breadth-first search uses a **queue**.

Breadth-First Search

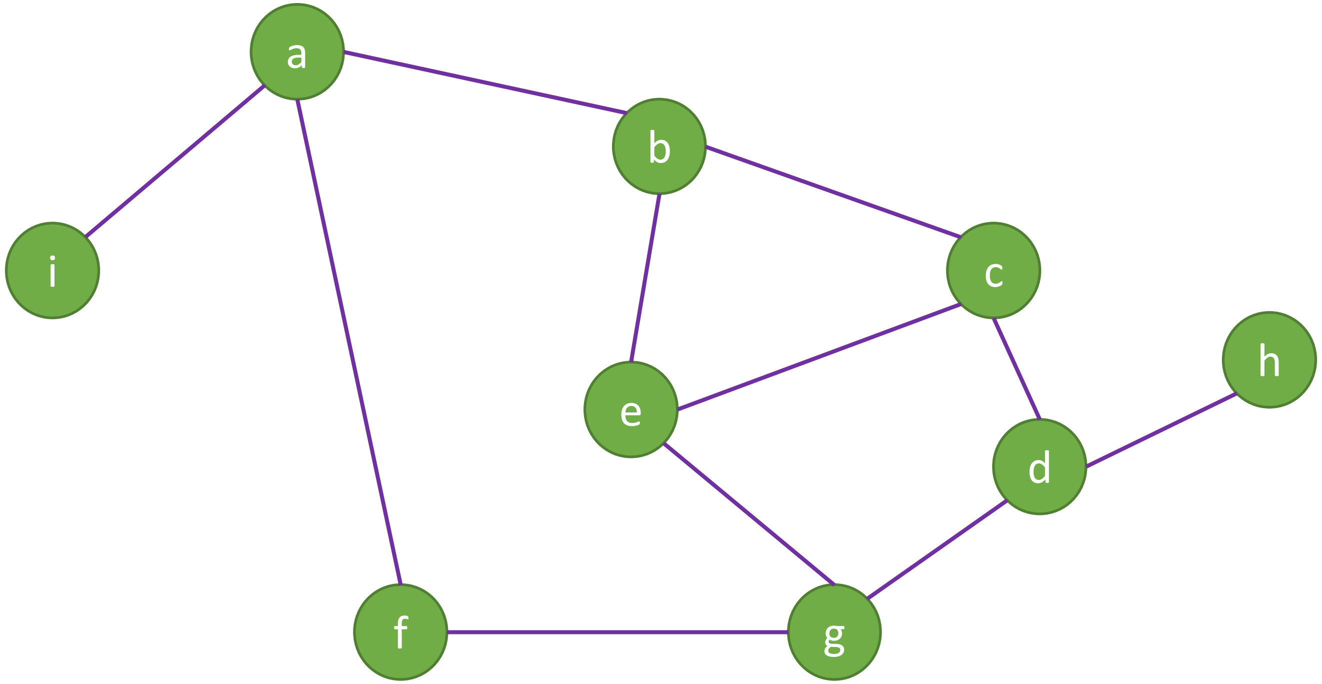
```
BFS(v: Vertex)
    q = a new empty queue
    q.enqueue(v)
    Mark v as visited
    while (q is not empty){
        w = q.dequeue()
        for (each unvisited vertex u adjacent to w){
            Mark u as visited
            q.enqueue(u)
        }
    }
```

Breadth-First Search



v - u - w - x - q - t - r - s

Breadth-First Search



BFS starts at ***a***:

BFS starts at ***e***:

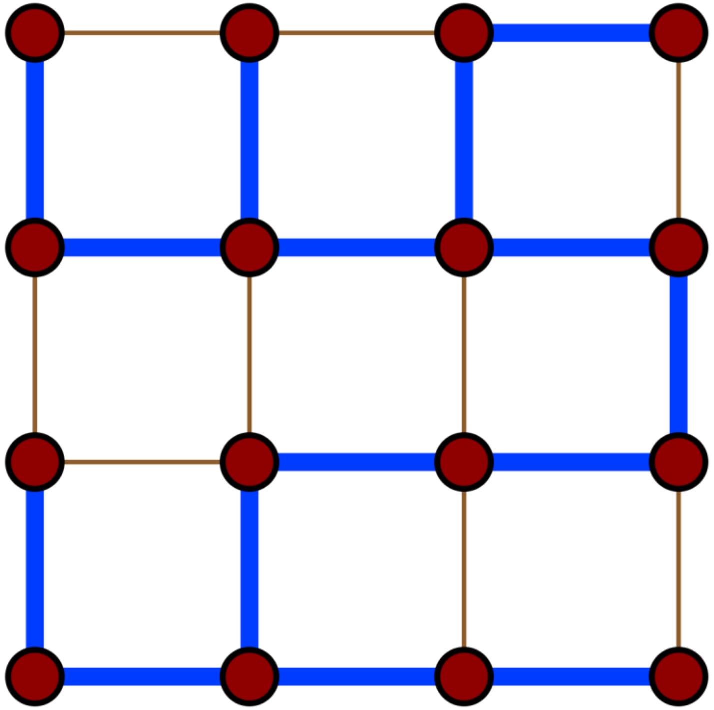
Minimum Spanning Tree

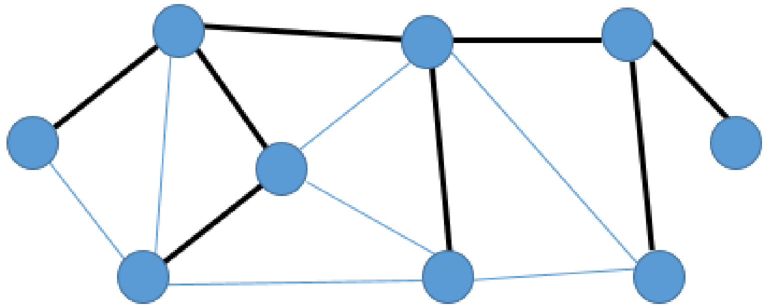
Spanning Tree

- A spanning tree
  - is a **subgraph** of undirected graph G
  - has **all** the vertices covered with **minimum** possible number of edges.

- does not have cycles
- cannot be disconnected.

Spanning Tree



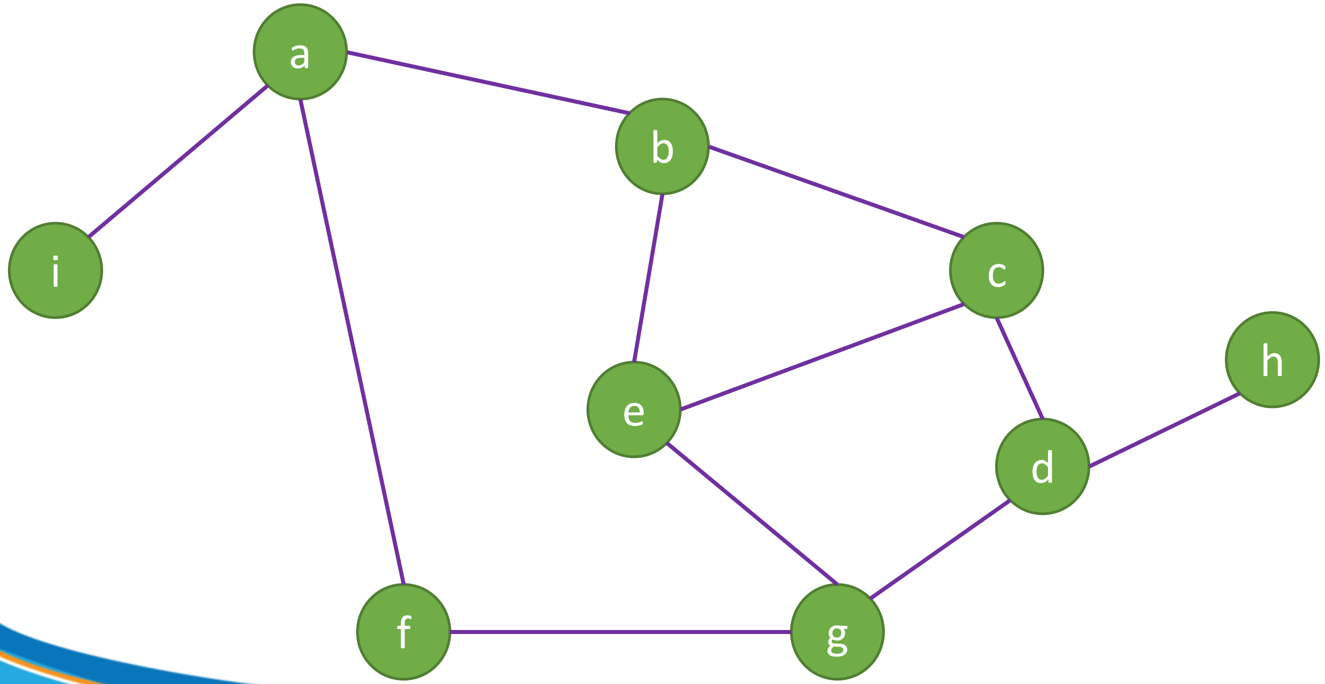


Spanning Tree

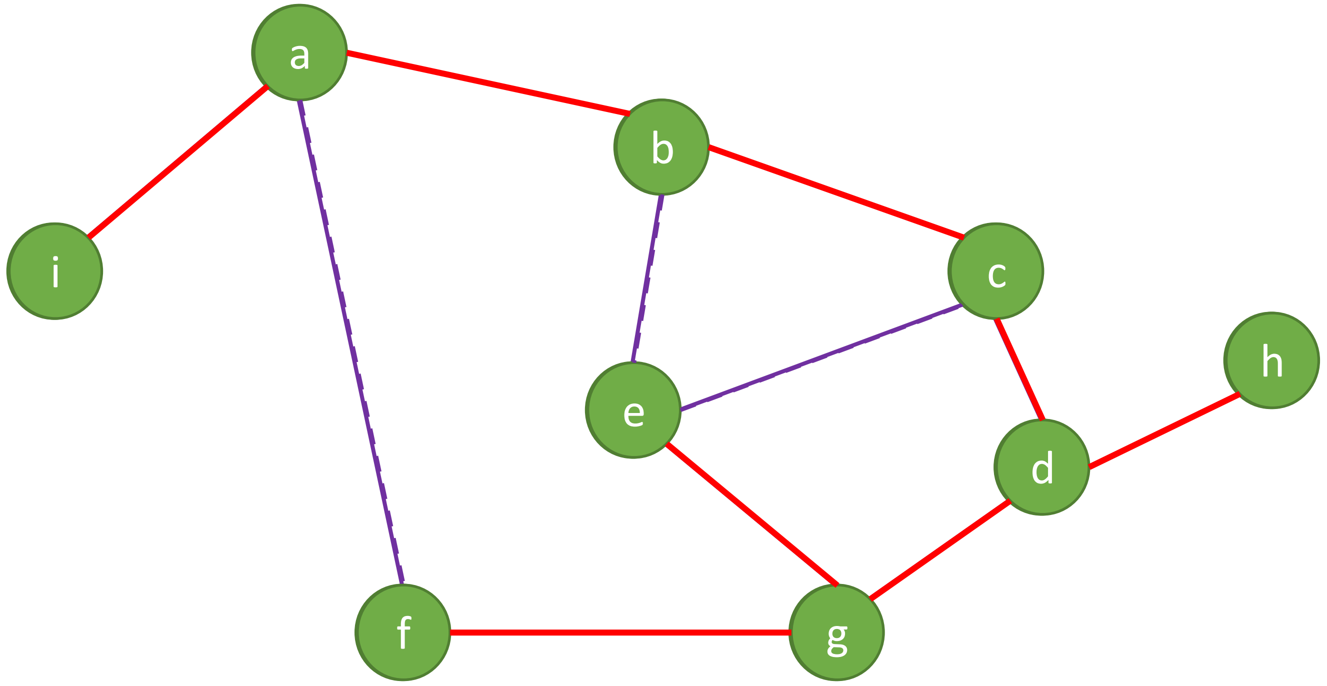
- A connected graph G can have **more than one** spanning tree.
- All possible spanning trees of graph G, **have the same** number of **edges** and **vertices**.
- The spanning tree **does not have any cycle** (loops).
- The spanning tree is **minimally connected**.
- The spanning tree is **maximally acyclic**.

Spanning Tree

- Depth-first-search spanning tree
- Breadth-first-search spanning tree

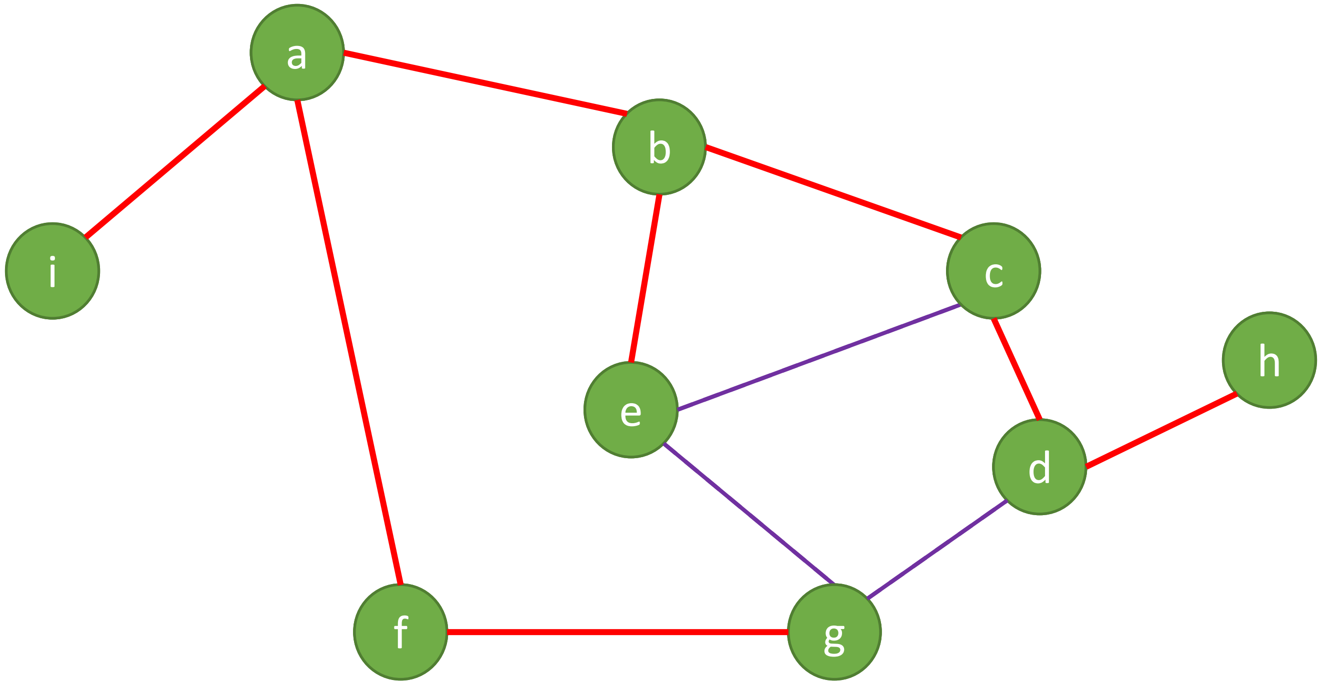


Spanning Tree



DFS spanning tree

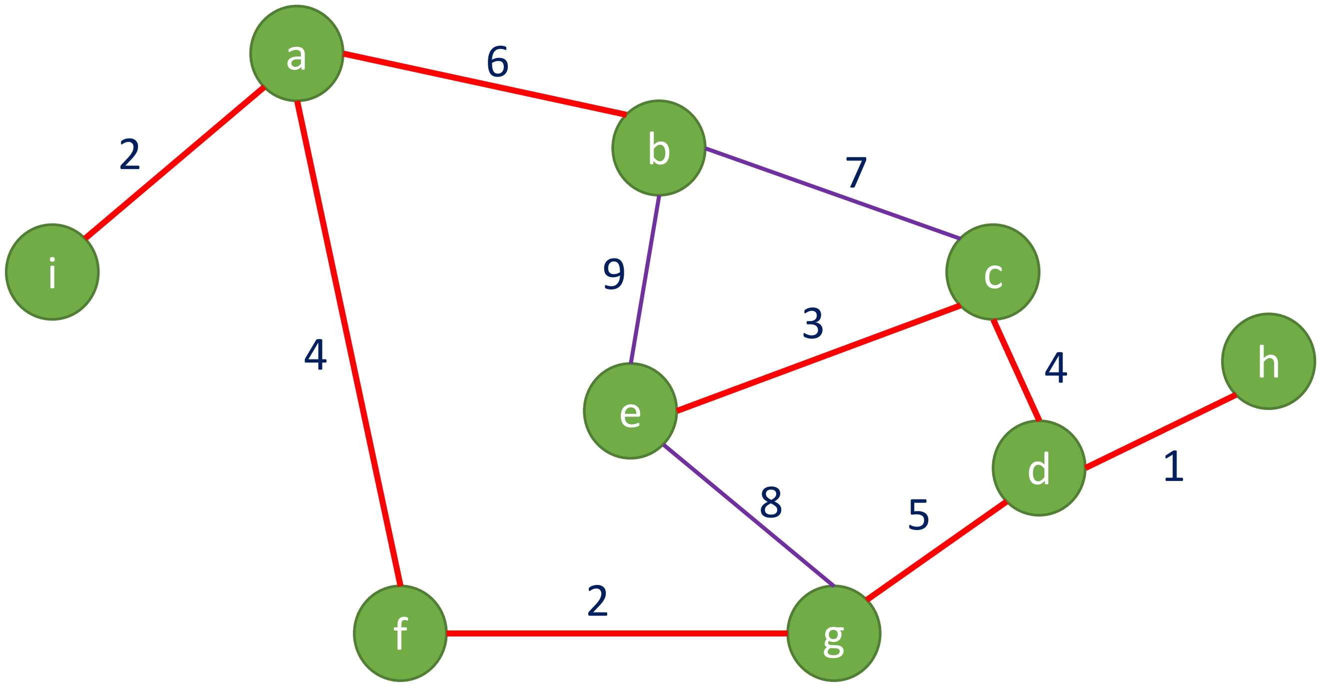
Spanning Tree



BFS spanning tree

Minimum Spanning Tree

- A minimum spanning tree is a spanning tree that has **minimum weight** than all other spanning trees of the same graph.



Prim's Minimum Spanning Tree

- Begins with any vertex.
- Initially, the tree T contains only the starting vertex.
- At each stage,
  - Select the least cost edge *e*(v, u) with *v* in T and *u* not in T.
  - Add *u* and *e* to T

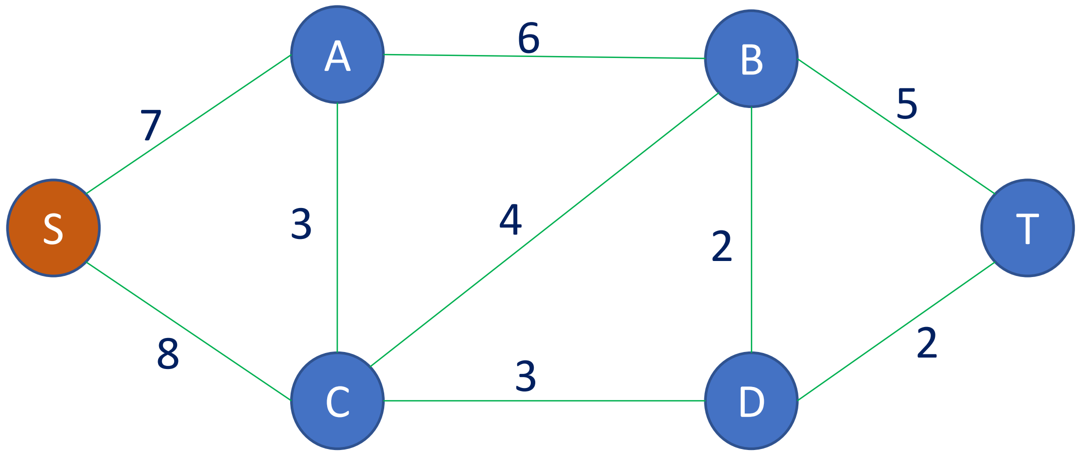
Prim's Minimum Spanning Tree

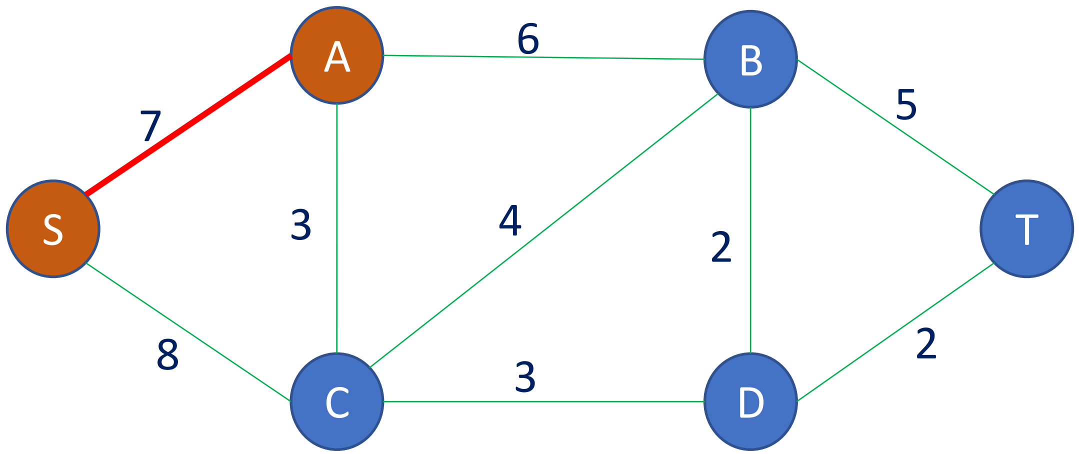
```
primAlgorithm(v: Vertex)
   Mark v as visited and include it in the minimum spanning tree
   while (there are unvisited vertices)
   {
         Find the least-cost edge e(v, u) from a visited vertex v to some unvisited vertex u
         Mark u as visited
         Add the vertex u and the edge e(v, u) to the minimum spanning tree
   }
```

Prim's Minimum Spanning Tree

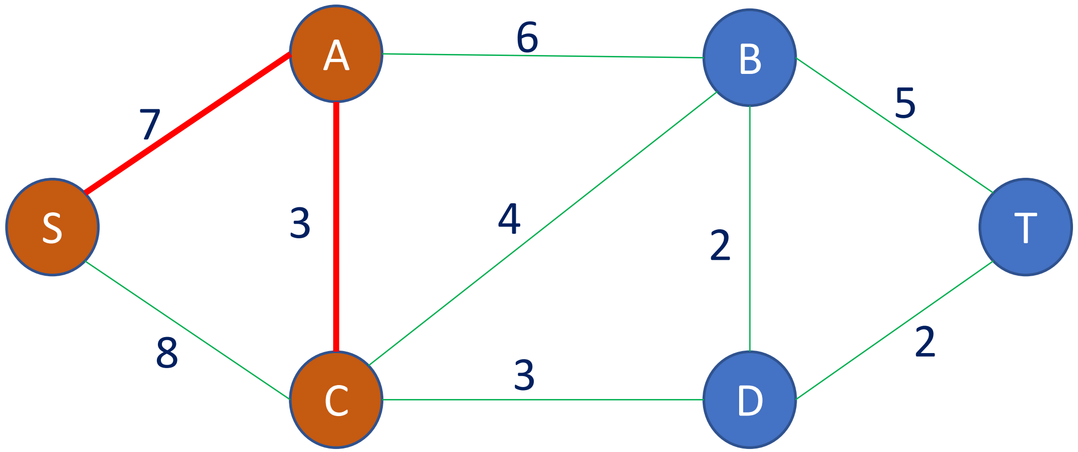
```
PrimSpanningTree(matrix[N][N], source)
{
   for v = 0 to N-1 {
         length[v] = matrix[source][v]
         parent[v] = source }
   Mark source //Add source to the spanning tree
   for step = 1 to N-1 {
         Find the vertex v such that length[v] is smallest
and v is not in spanning tree
         Mark v
         for all vertices u not in vertexSet
               if (length[u] > matrix[v][u]){
                    length[u] = matrix[v][u]
                    parent[u] = v }
   }
}
```

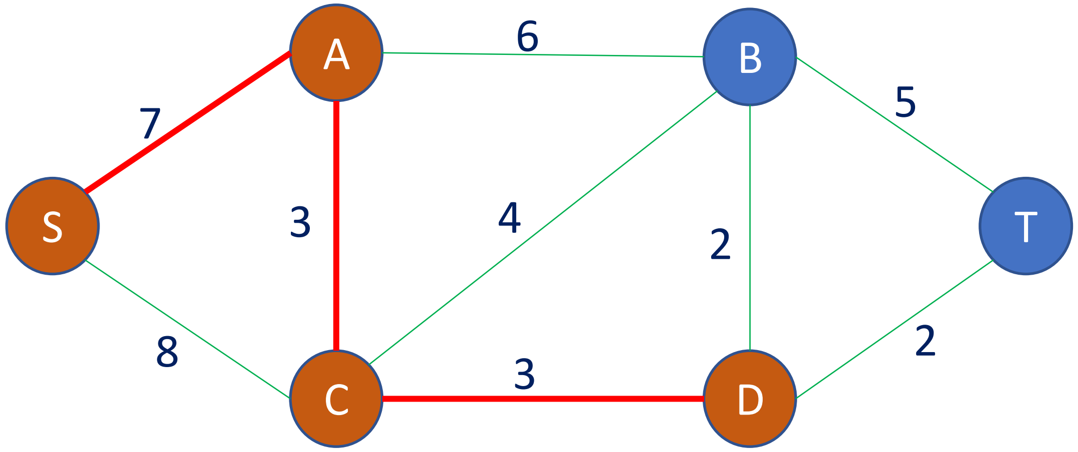
Prim's Minimum Spanning Tree





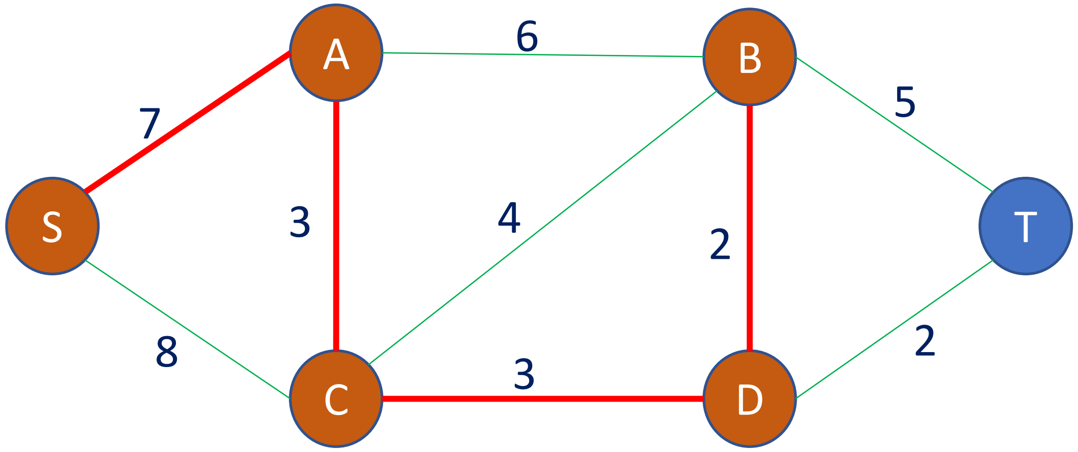
Prim's Minimum Spanning Tree

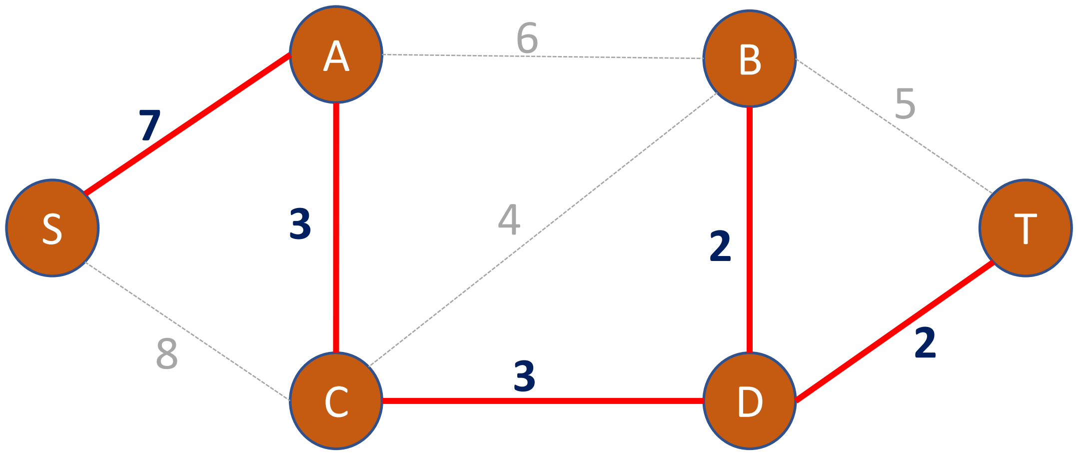




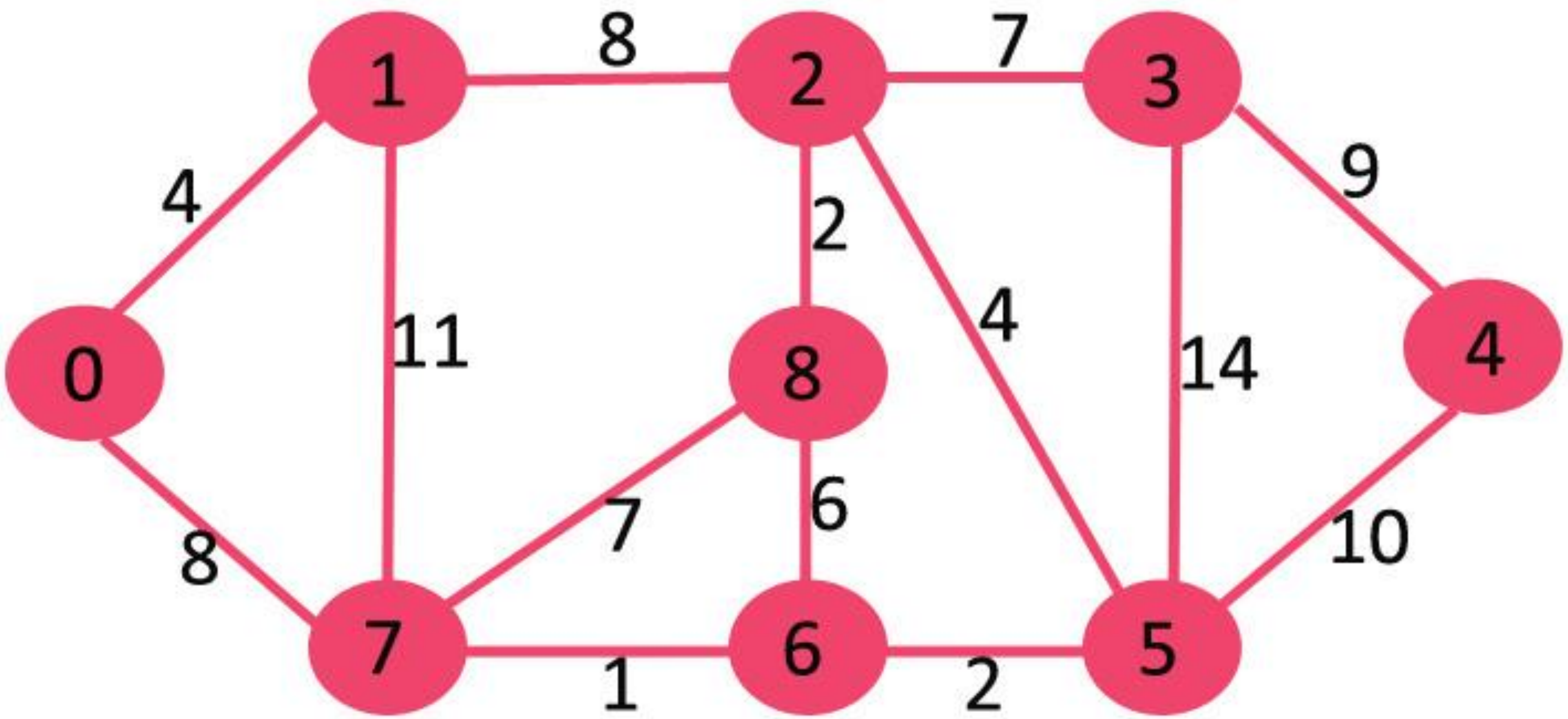
fit@hcmus

Prim’s Minimum Spanning Tree

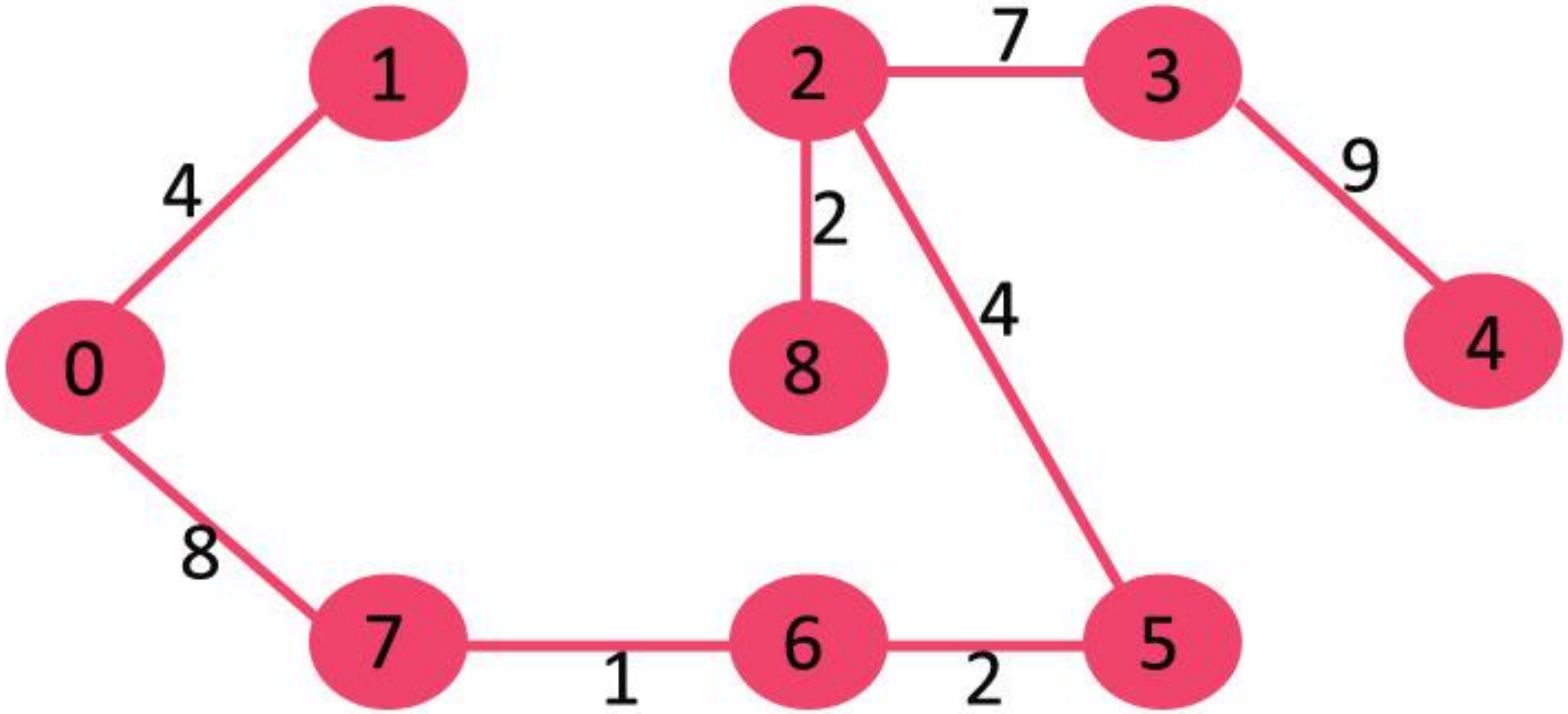




Example



Example



Shortest Path

Dijkstra's Shortest Path Algorithm

- Given a graph and a source vertex in the graph, find shortest paths from source to all vertices in the given graph.
- **Dijkstra's** algorithm is very **similar** to **Prim's** algorithm for minimum spanning tree.
- This algorithm is applicable to graphs with **non-negative weights** only.

Dijkstra's Shortest Path Algorithm

shortestPath(matrix[N][N], source, length[])

Input:

matrix[N][N]: adjacency matrix of Graph *G* with *N* vertices

source: the *source* vertex

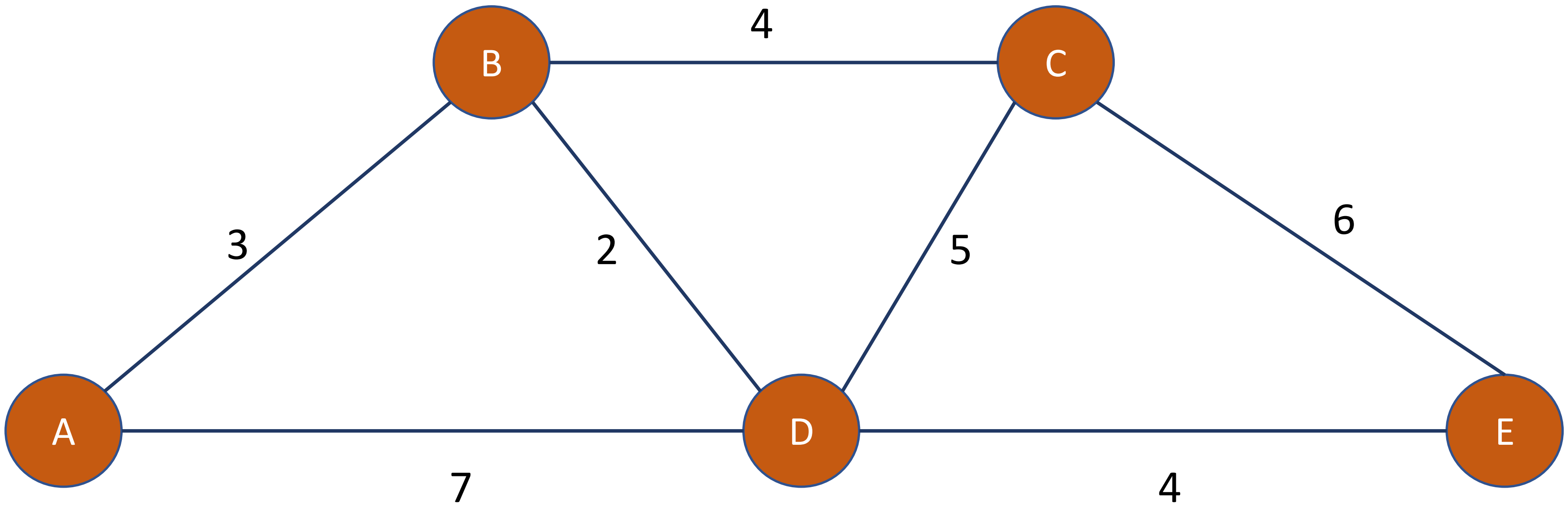
Output:

length[]: the length of the shortest path from *source* to all *vertices* in G.

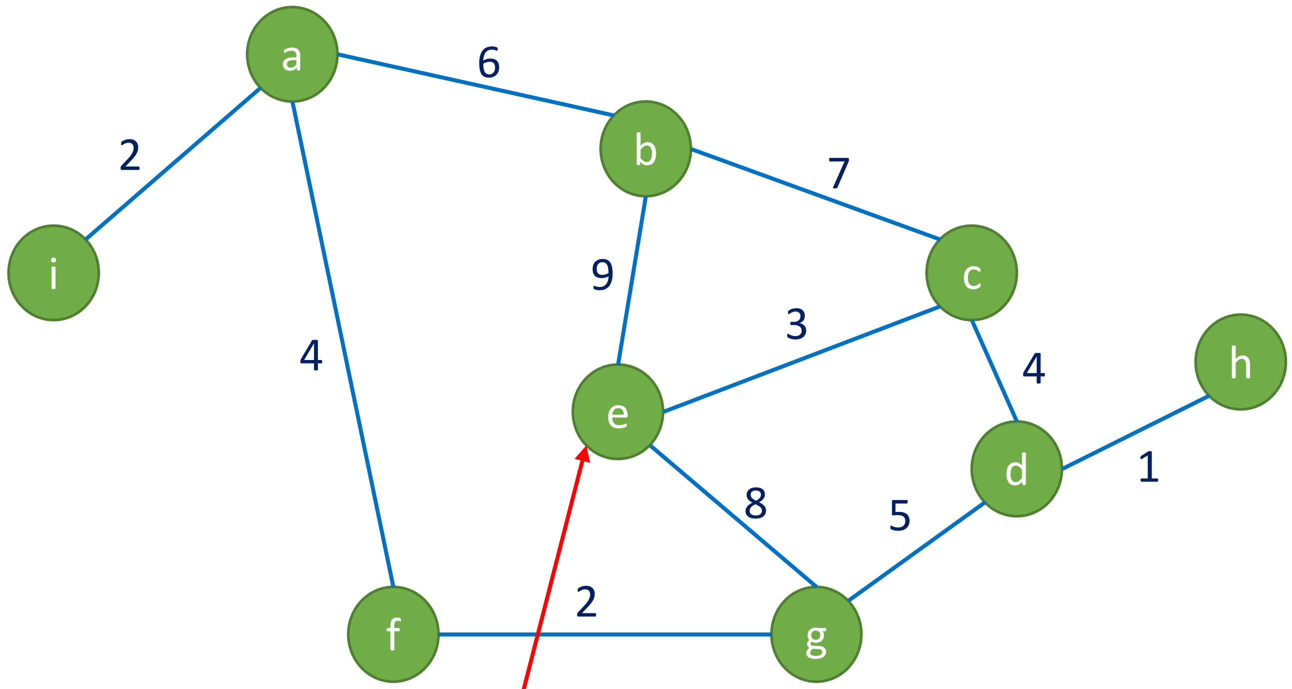
Dijkstra's Shortest Path Algorithm

```
shortestPath(matrix[N][N], source, length[])
{
   for v = 0 to N-1
         length[v] = matrix[source][v]
   length[source] = 0 //why?
   for step = 1 to N {
         Find the vertex v such that length[v] is smallest
and v is not in vertexSet
         Add v to vertexSet
         for all vertices u not in vertexSet
               if (length[u] > length[v] + matrix[v][u]){
                     length[u] = length[v] + matrix[v][u]
                     parent[u] = v }
   }
}
```

Example



Example



source = e

step = 0

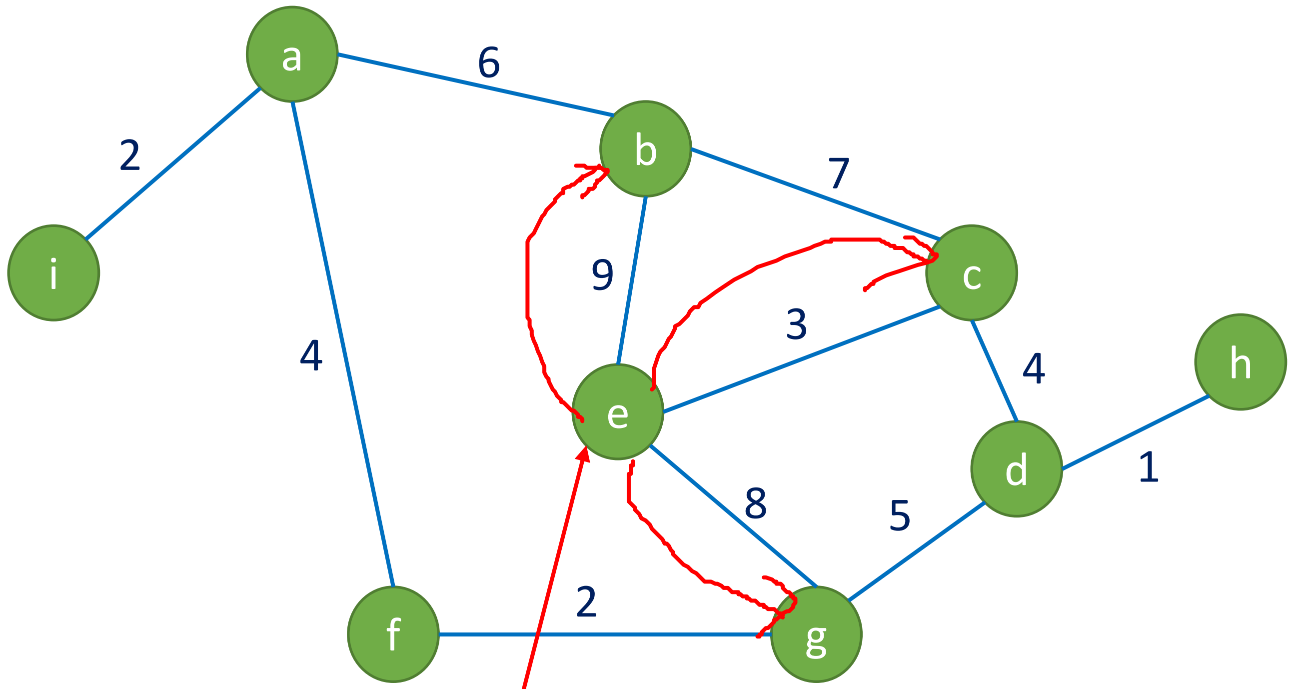
```
Initialize:
    for v = 0 to N-1
        length[v] = matrix[source][v]
    length[source] = 0 //why?
```

V = {}

non-V = {a, b, c, d, e, f, g, h, i}

	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	∞	9	3	∞	0	∞	8	∞	∞
Parent[i]		e	e		e		e		

Example



source = e

step = 1

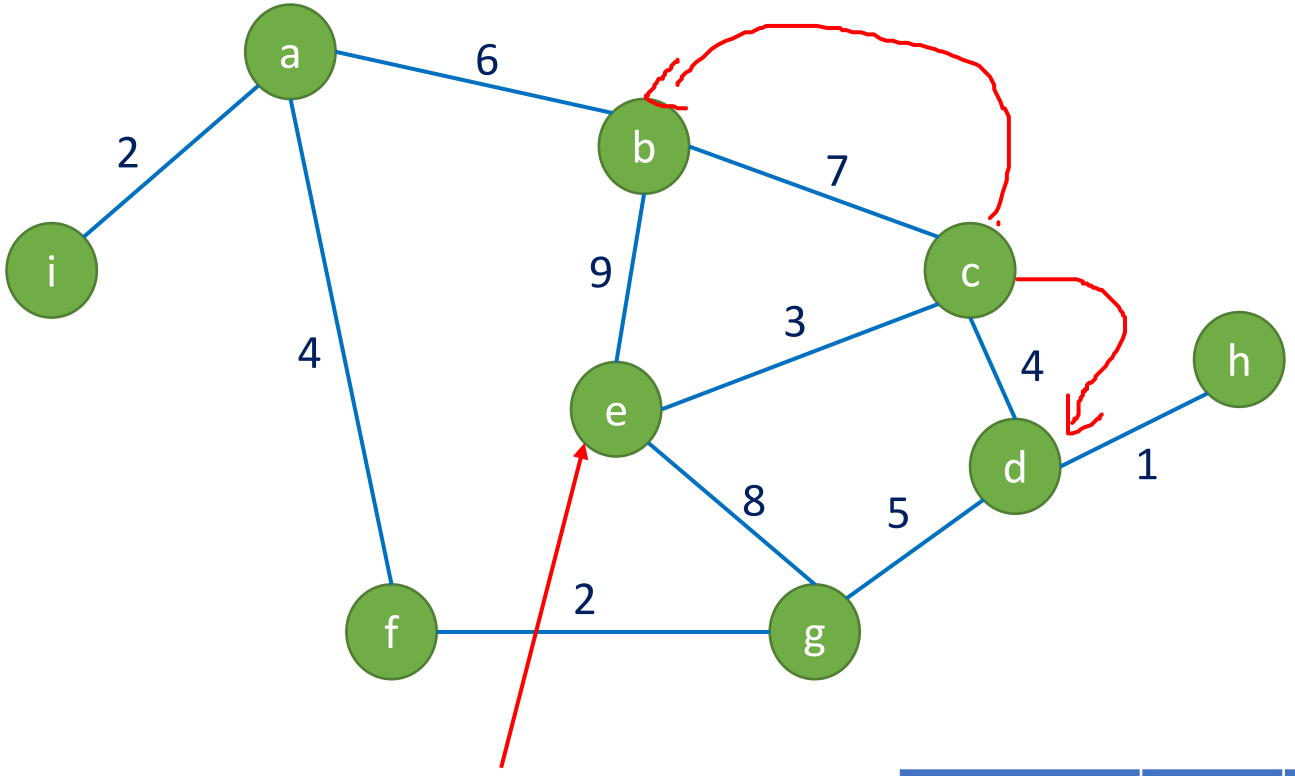
```
Loop:
for step = 1 to N {
   Find vertex v: length[v] smallest, v not in V
   Add v to V
   for all vertices u not in V
      if (length[u] > length[v] + matrix[v][u]){
         length[u] = length[v] + matrix[v][u]
         parent[u] = v }
}
```

V = {e}

non-V = {a, b, c, d, f, g, h, i}

	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	∞	9	3	∞	0	∞	8	∞	∞
Parent[i]		e	e		e		e		

Example



source = e

step = 2

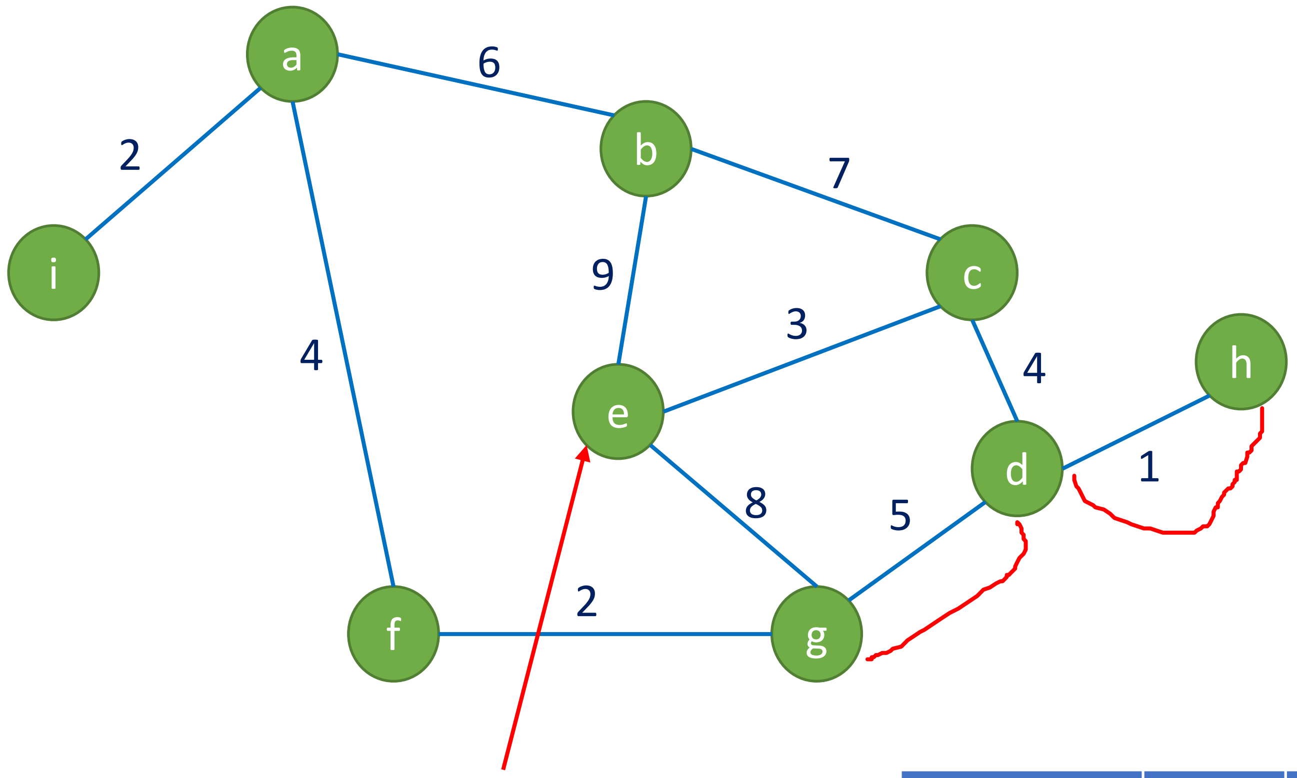
```
Loop:
for step = 1 to N {
    Find vertex v: length[v] smallest, v not in V
    Add v to V
    for all vertices u not in V
        if (length[u] > length[v] + matrix[v][u]){
            length[u] = length[v] + matrix[v][u]
            parent[u] = v }
}
```

V = {e, c}

non-V = {a, b, d, f, g, h, i}

	a	b	c	d	e	f	g	h	i
Index	0	1	**2**	3	4	5	6	7	8
Length[i]	∞	9	**3**	7	0	∞	8	∞	∞
Parent[i]		e	**e**	c	e		e		

Example



source = e
step = 3

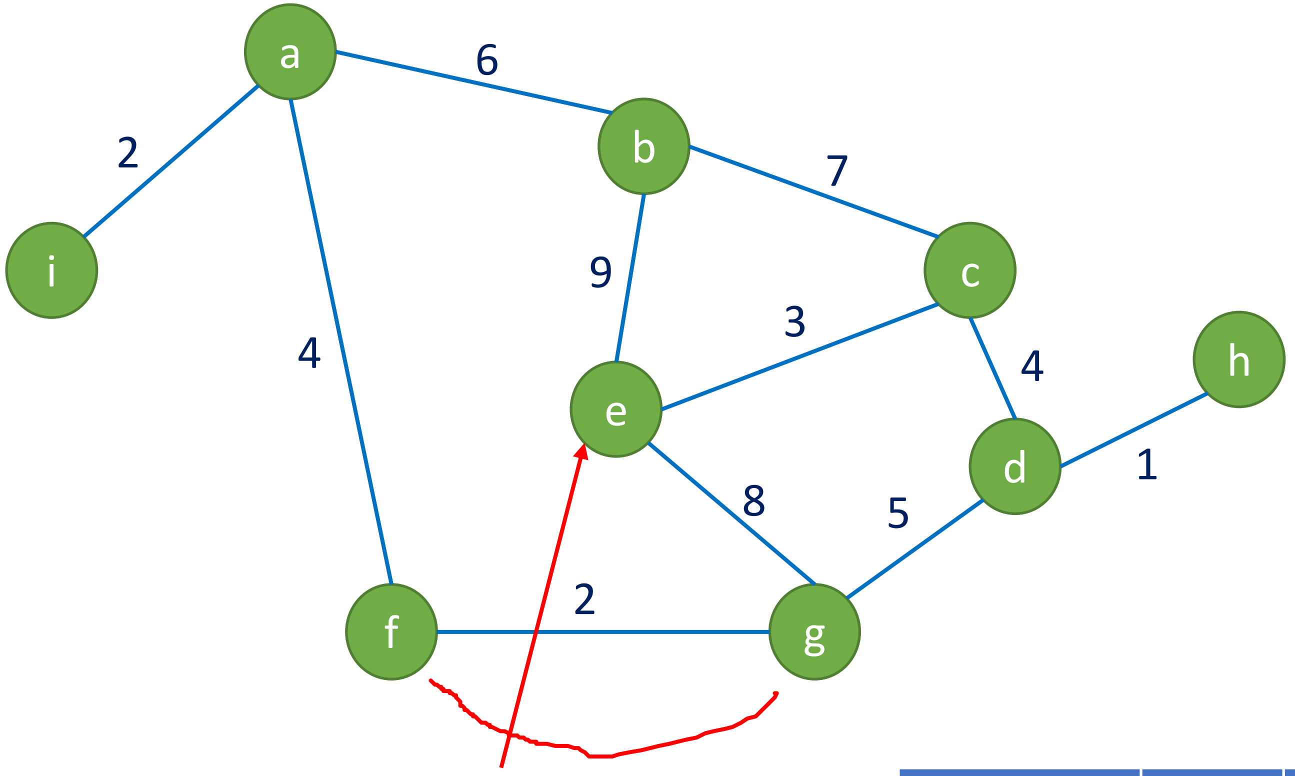
```
Loop:
for step = 1 to N {
    Find vertex v: length[v] smallest, v not in V
    Add v to V
    for all vertices u not in V
        if (length[u] > length[v] + matrix[v][u]){
            length[u] = length[v] + matrix[v][u]
            parent[u] = v }
}
```

V = {e, c, d}

non-V = {a, b, f, g, h, i}

	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	∞	9	3	7	0	∞	8	8	∞
Parent[i]		e	e	c	e		e	d	

Example



source = e
step = 4

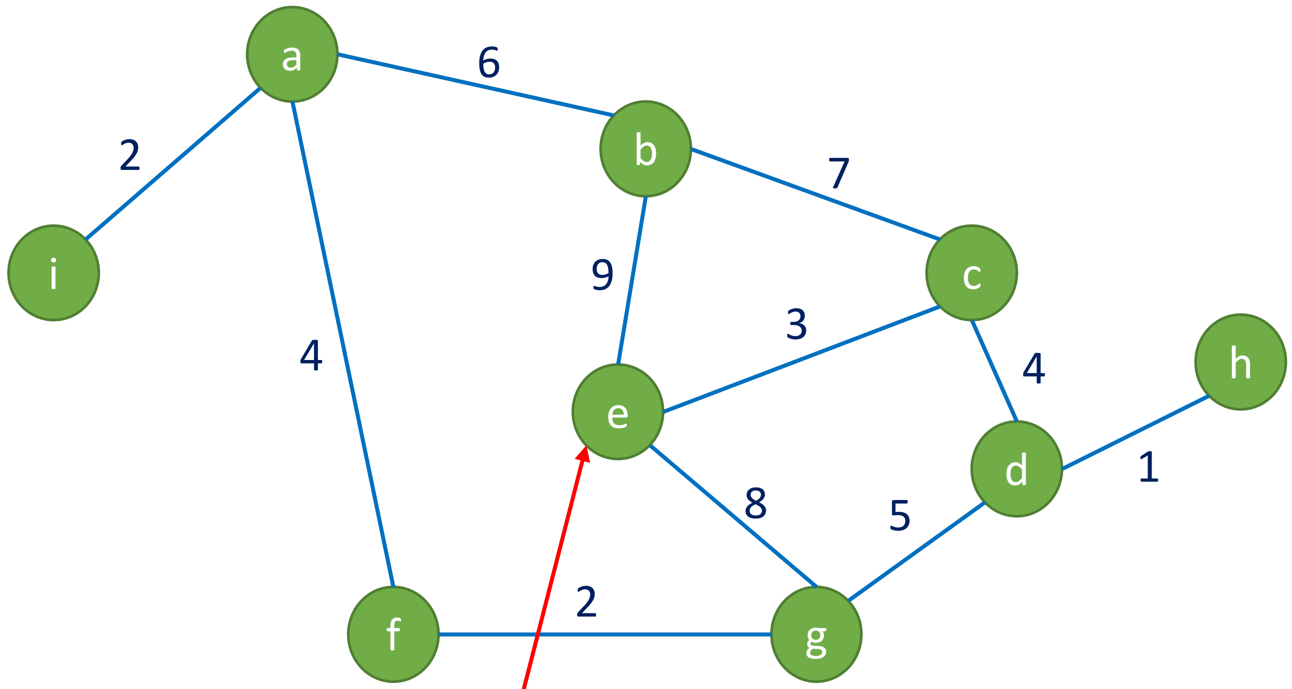
```
Loop:
for step = 1 to N {
   Find vertex v: length[v] smallest, v not in V
   Add v to V
   for all vertices u not in V
      if (length[u] > length[v] + matrix[v][u]){
         length[u] = length[v] + matrix[v][u]
         parent[u] = v }
}
```

V = {e, c, d, g}

non-V = {a, b, f, h, i}

	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	∞	9	3	7	0	10	8	8	∞
Parent[i]		e	e	c	e	g	e	d	

Example



source = e
step = 5

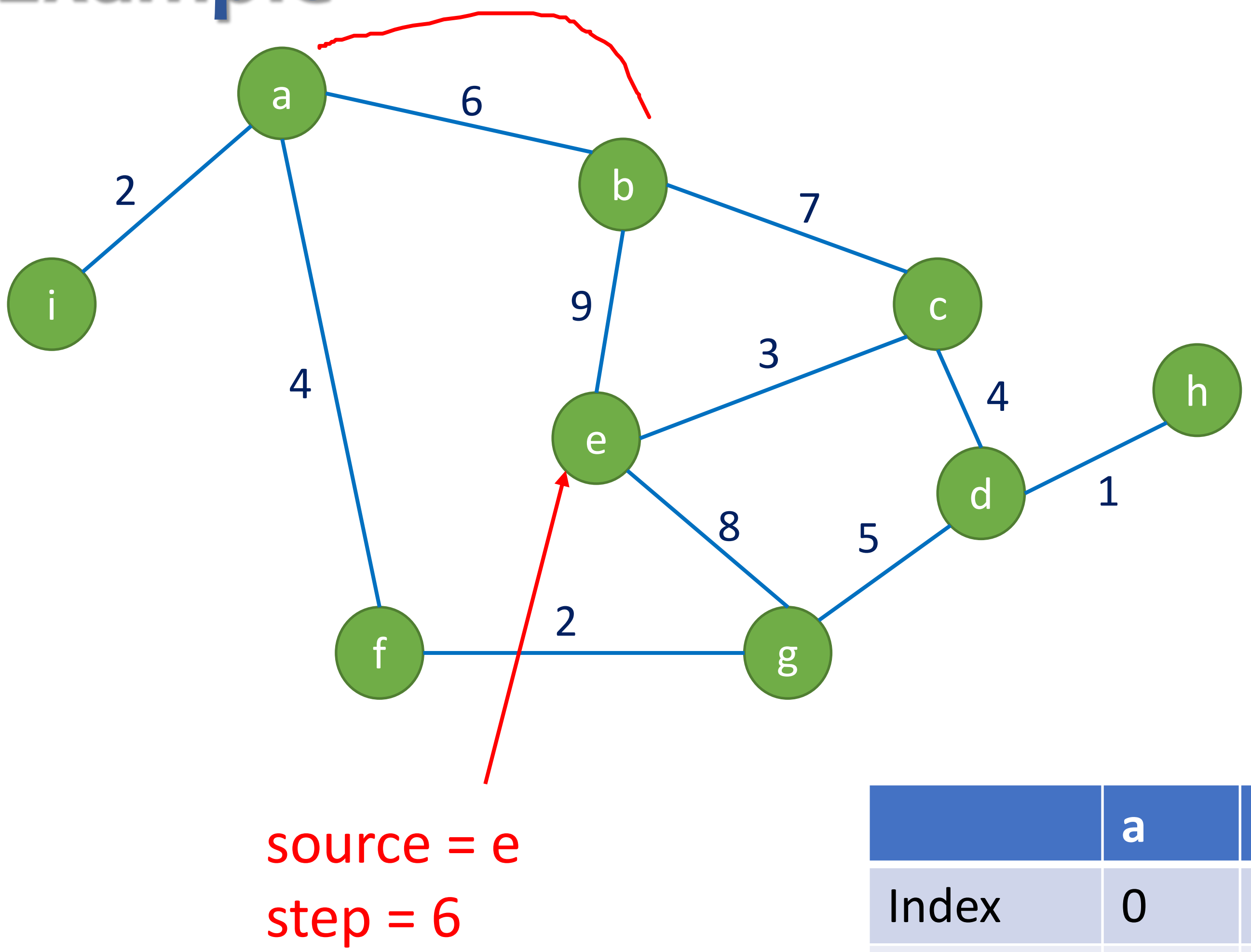
```
Loop:
for step = 1 to N {
    Find vertex v: length[v] smallest, v not in V
    Add v to V
    for all vertices u not in V
        if (length[u] > length[v] + matrix[v][u]){
            length[u] = length[v] + matrix[v][u]
            parent[u] = v }
}
```

V = {e, c, d, g, h}

non-V = {a, b, f, i}

	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	∞	9	3	7	0	10	8	8	∞
Parent[i]		e	e	c	e	g	e	d	

Example



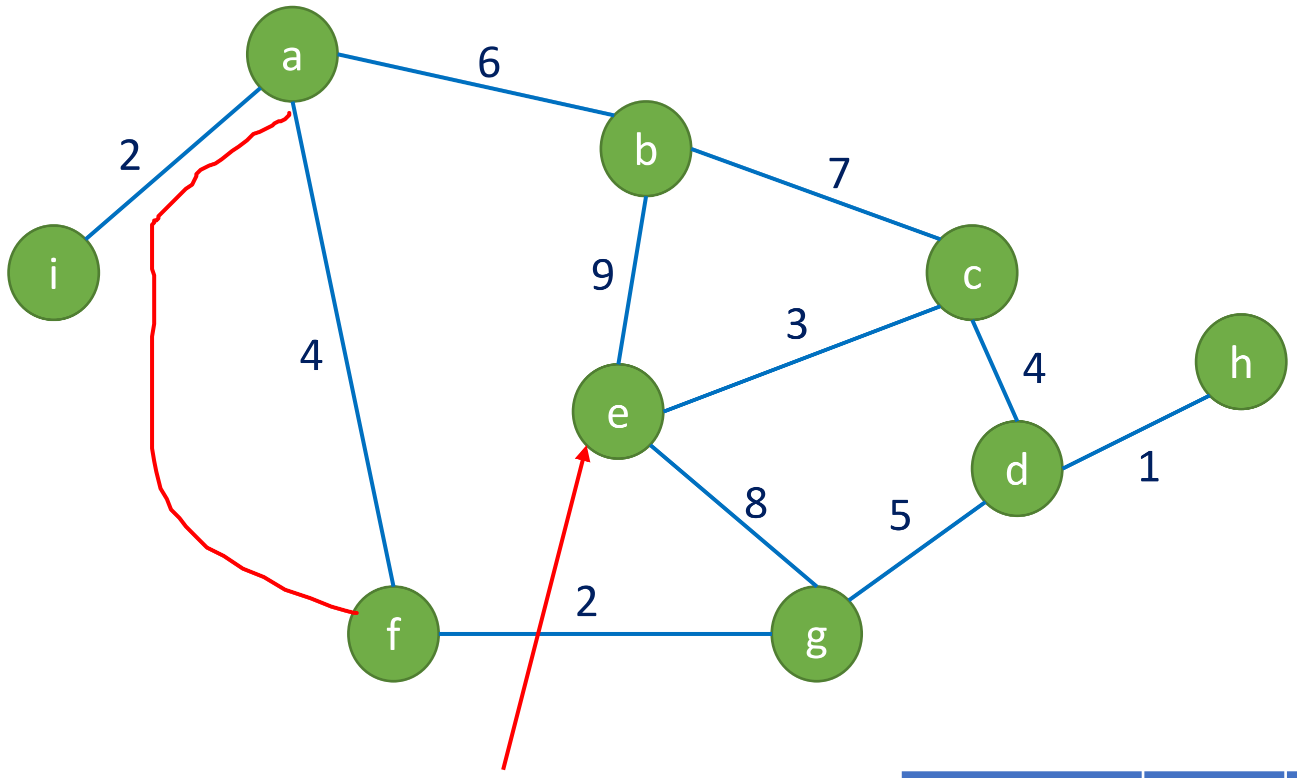

```
Loop:
for step = 1 to N {
   Find vertex v: length[v] smallest, v not in V
   Add v to V
   for all vertices u not in V
      if (length[u] > length[v] + matrix[v][u]){
         length[u] = length[v] + matrix[v][u]
         parent[u] = v }
}
```

V = {e, c, d, g, h, b}

non-V = {a, f, i}

	a	b	c	d	e	f	g	h	i
Index	0	**1**	2	3	4	5	6	7	8
Length[i]	15	**9**	3	7	0	10	8	8	∞
Parent[i]	b	**e**	e	c	e	g	e	d	

Example



source = e

step = 7

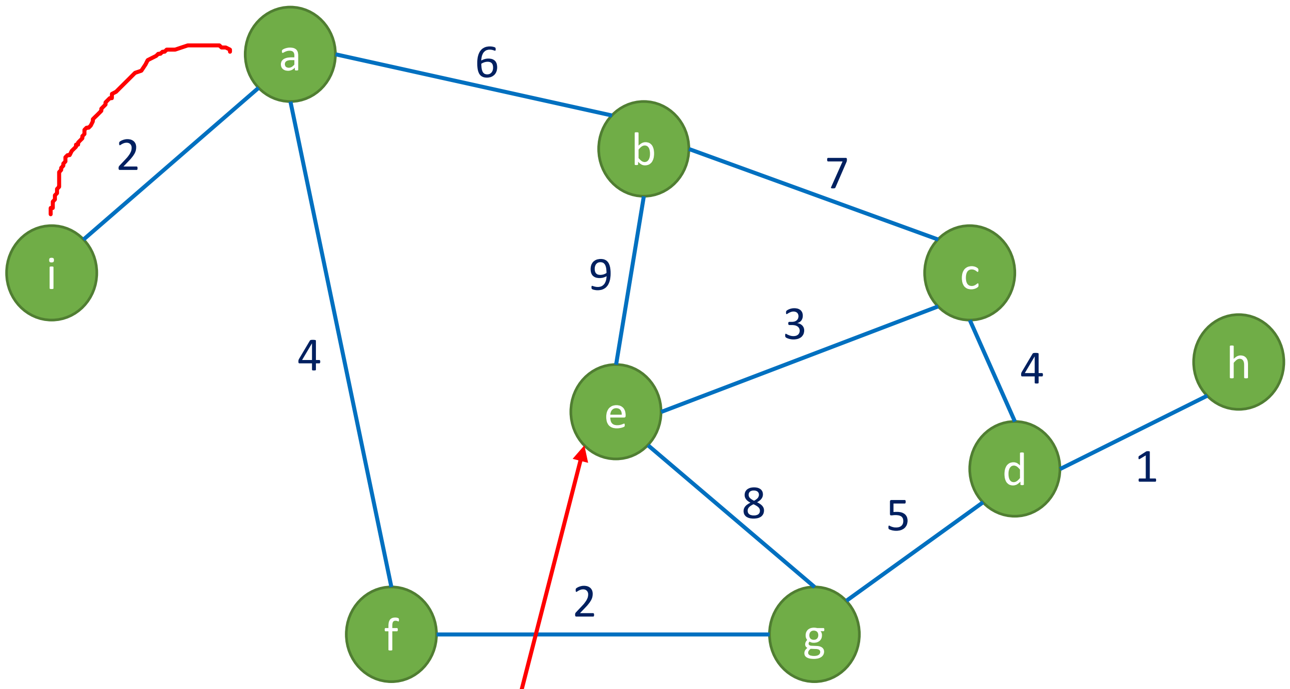
```
Loop:
for step = 1 to N {
   Find vertex v: length[v] smallest, v not in V
   Add v to V
   for all vertices u not in V
      if (length[u] > length[v] + matrix[v][u]){
         length[u] = length[v] + matrix[v][u]
         parent[u] = v }
}
```

V = {e, c, d, g, h, b, f}

non-V = {a, i}

	a	b	c	d	e	**f**	g	h	i
Index	0	1	2	3	4	**5**	6	7	8
Length[i]	14	9	3	7	0	**10**	8	8	∞
Parent[i]	f	e	e	c	e	**g**	e	d	

Example



source = e
step = 8

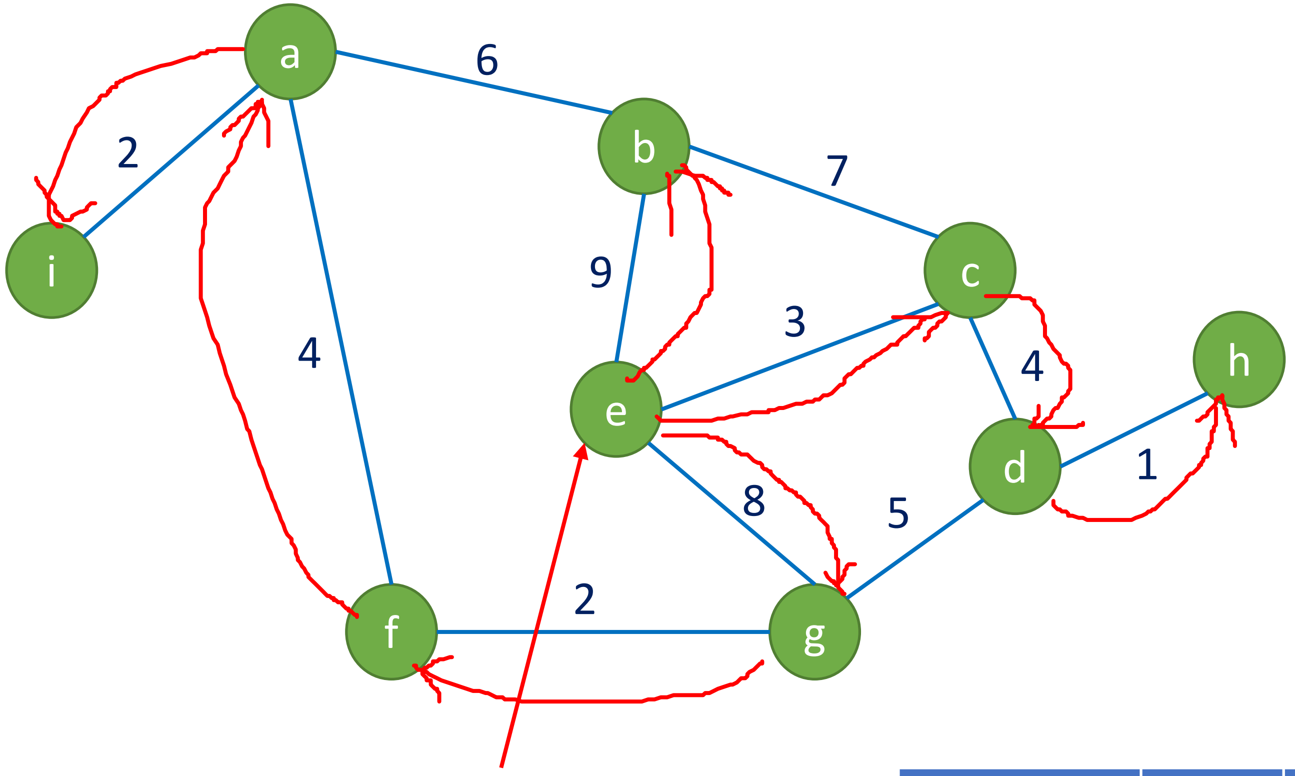
```
Loop:
for step = 1 to N {
   Find vertex v: length[v] smallest, v not in V
   Add v to V
   for all vertices u not in V
      if (length[u] > length[v] + matrix[v][u]){
         length[u] = length[v] + matrix[v][u]
         parent[u] = v }
}
```

V = {e, c, d, g, h, b, f, a}

non-V = {i}

	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	14	9	3	7	0	10	8	8	16
Parent[i]	f	e	e	c	e	g	e	d	a

Example



source = e

step = 9

```
Loop:
for step = 1 to N {
   Find vertex v: length[v] smallest, v not in V
   Add v to V
   for all vertices u not in V
      if (length[u] > length[v] + matrix[v][u]){
         length[u] = length[v] + matrix[v][u]
         parent[u] = v }
}
```

V = {e, c, d, g, h, b, f, a, i}

non-V = {}

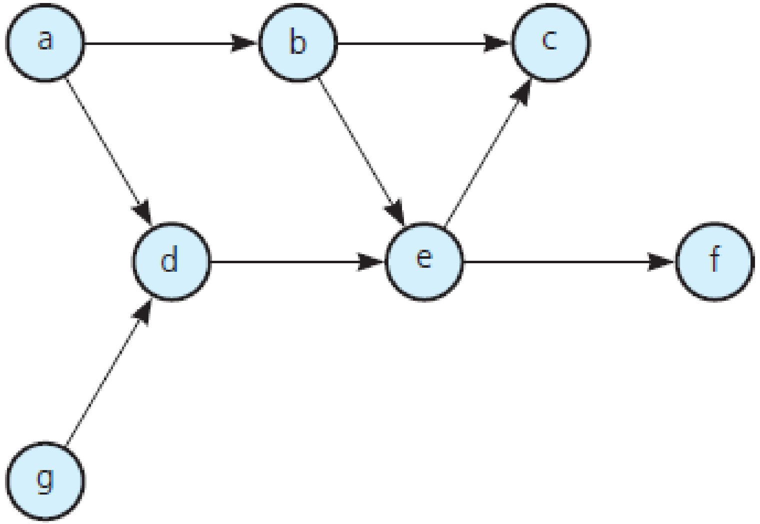
	a	b	c	d	e	f	g	h	i
Index	0	1	2	3	4	5	6	7	8
Length[i]	14	9	3	7	0	10	8	8	16
Parent[i]	f	e	e	c	e	g	e	d	a

Questions and Answers

Topological sorting

Directed acyclic graph

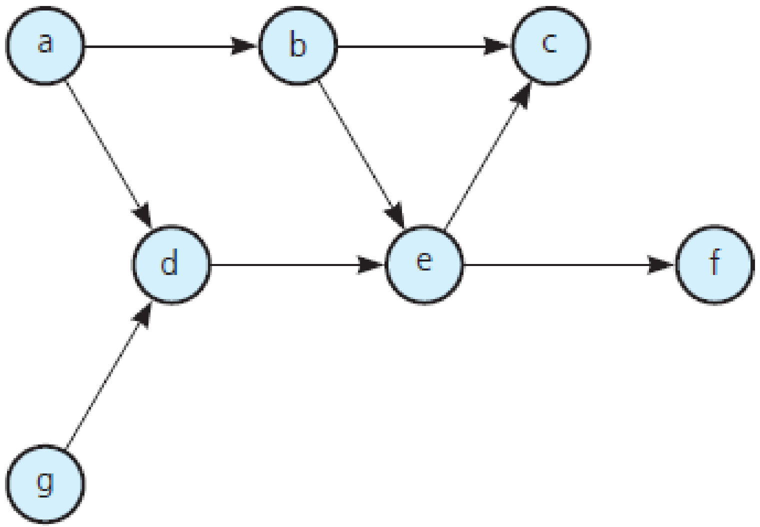
- dag: directed acyclic graph



Topological sorting

- Topological order: order to take to satisfy all the prerequisites.
- Topological sorting: arranging the vertices in topological order.

- Prerequisite structure of the courses.



Topological sorting

```
topSort1(theGraph: Graph, aList: List)
    n  = number of vertices in theGraph
    for (step = 1 to n)
    {
        Select a vertex v that has no successors
        aList.addHead(v)
        Remove from the Graph vertex v and its edges
    }
```

Topological sorting

```
L: Empty list that will contain the sorted elements
S: Stack of all nodes with no incoming edge
while S is not empty
    remove a node n from S //n = S.pop()
    add n to tail of L
    for each node m with an edge e from n to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S // S.push(m)
if graph has edges then
    return error   //(graph has at least one cycle)
else
    return L   //(a topologically sorted order)
```

Topological sorting

- Try this one