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Note: Show your work on all problems. Anytime necessary, you can use Matlab or other software to calculate integrals. However, you do need to write down the integrals that you are computing. A total of 50 points are possible.

1. [5 pts.] Suppose the distribution of Y conditional on X is $N(\mu = X, \sigma^2 = X^2)$ and that the distribution of X is Beta$(\alpha = 1, \beta = 2)$. Find $E(Y)$, $Var(Y)$, and Cov(X, Y).

$Y|X \sim N(X, X^2)$, $X \sim Beta(1,2)$

$\cdot\ E(Y) = E(E(Y|X)) = E(X) = \frac{1}{1+2} = \boxed{\frac{1}{3}}$

$\cdot\ Var(Y) = Var(E(Y|X)) + E(Var(Y|X))$

$Var(E(Y|X)) = Var(X) = \frac{1\cdot 2}{(1+2)^3(1+2+1)} = \frac{1}{18}$

$E(Var(Y|X)) = E(X^2) = Var(X) + (E(X))^2 = \frac{1}{18} + \left(\frac{1}{3}\right)^2 = \frac{1}{6}$

$\Rightarrow Var(Y) = \frac{1}{18} + \frac{1}{6} = \boxed{\frac{2}{9}}$

$\cdot\ Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = E(E(XY|X)) = E(X\,E(Y|X)) = E(X^2) = \frac{1}{6}$

$\Rightarrow cov(X,Y) = \frac{1}{6} - \frac{1}{3}\cdot\frac{1}{3} = \boxed{\frac{1}{18}}$

2. **[5 pts.]** Prove the central limit theorem for a sequence of χ^2 random variables with one degree of freedom. Specifically, let $X_1, X_2, \cdots$ be an iid sequence of χ^2 random variables with 1 degree of freedom. Show that $T_n = \sum_{i=1}^n X_i$, shifted and scaled appropriately, converges in distribution to the normal distribution. Prove this **without citing** the central limit theorem.

$X_1, X_2, \ldots \overset{iid}{\sim} \chi^2_{(1)}$, $T_n = \sum_{i=1}^n X_i$

Consider $Z_n = \dfrac{\frac{1}{n} T_n - \mu}{\sigma/\sqrt{n}} = \dfrac{T_n - n\mu}{\sqrt{n}\,\sigma}$

Since $X_i \sim \chi^2_{(1)}$, we have $\mu = 1$ and $\sigma^2 = 2 \Rightarrow \sigma = \sqrt{2}$

So, $Z_n = \dfrac{T_n - n}{\sqrt{2n}} = \dfrac{1}{\sqrt{2n}} \sum_{i=1}^n X_i - \dfrac{n}{\sqrt{2n}} = \dfrac{1}{\sqrt{2n}} \sum_{i=1}^n X_i - \sqrt{\dfrac{n}{2}}$

Therefore, $M_{Z_n}(t) = e^{-t\sqrt{\frac{n}{2}}} \cdot \left[M_{X_1}\left(\frac{t}{\sqrt{2n}}\right)\right]^n = e^{-t\sqrt{\frac{n}{2}}} \cdot \left[\left(\dfrac{1}{1 - 2\frac{t}{\sqrt{2n}}} \right)^{1/2} \right]^n$

X_i are iid $\qquad X_1 \sim \chi^2_{(1)}$

$= e^{-t\sqrt{\frac{n}{2}}} \cdot \left(1 - \sqrt{\frac{2}{n}}\, t\right)^{-n/2} \Rightarrow \log(M_{Z_n}(t)) = -t\sqrt{\frac{n}{2}} + \frac{-n}{2}\left[\log\left(1 - \sqrt{\frac{2}{n}}\, t\right)\right]$

$= -t\sqrt{\frac{n}{2}} - \frac{n}{2}\left[-t\sqrt{\frac{2}{n}} - \frac{\cancel{2} t^2}{\cancel{2} n} + O\left(\frac{t^3}{n^{3/2}}\right)\right]$

taylor expansion

$= \cancel{-t\sqrt{\frac{n}{2}}} + \cancel{t\sqrt{\frac{n}{2}}} + \frac{t^2}{2} + O\left(\frac{t^3}{n^{1/2}}\right) = \frac{t^2}{2} + O\left(\frac{t^3}{n^{1/2}}\right)$

So, $\log(M_{Z_n}(t)) \xrightarrow[n \to \infty]{} \frac{t^2}{2}$, therefore,

$M_{Z_n}(t) \xrightarrow[n \to \infty]{} e^{t^2/2} \Rightarrow Z_n \xrightarrow{D} N(0,1)$

$$\Rightarrow \boxed{\dfrac{T_n - n}{\sqrt{2n}} \xrightarrow{D} N(0,1)}$$

3. **[5 pts.]** Suppose that we have two independent random samples $X_1, \cdots, X_n \sim$ exponential(θ) and $Y_1, \cdots, Y_m \sim$ exponential(θ). Show that the random variable

$$T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}.$$

has a Beta distribution with parameters $\alpha = \cancel{m}\ n$ and $\beta = \cancel{n}.\ m$

Let $X_n = \sum_{i=1}^n X_i$, $Y_m = \sum_{i=1}^m Y_i$, so: $T = \dfrac{X_n}{X_n + Y_m}$

let $U = X_n + Y_m$, so: $\begin{cases} T = \dfrac{X_n}{U} \\ U = X_n + Y_m \end{cases} \Rightarrow \begin{cases} X_n = TU \\ Y_m = U - TU = U(1-T) \end{cases}$

Then, $J^{-1} = \begin{bmatrix} U & T \\ -U & 1-T \end{bmatrix} \Rightarrow \det(J^{-1}) = U(1-\cancel{T}) + \cancel{UT} = U$

$$\Rightarrow f_{T,U}(t,u) = f_{X_n,Y_m}(tu, u(1-t))\cdot u \underset{\uparrow}{=} f_{X_n}(tu)\cdot f_{Y_m}(u(1-t))\cdot u$$

⫫ random samples

But since $X_1, \dots, X_n \sim exp(\theta)$, then $X_n = \sum_{i=1}^n X_i \sim gamma(n, \theta)$

Similarly, $Y_m = \sum_{i=1}^m Y_i \sim gamma(m, \theta)$, therefore:

$$f_{T,U}(t,u) = \frac{1}{\Gamma(n)\theta^n}(tu)^{n-1} e^{\frac{-tu}{\theta}} \cdot \frac{1}{\Gamma(m)\theta^n}(u(1-t))^{m-1} e^{\frac{-u(1-t)}{\theta}} \cdot u$$

$$= \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} \cdot t^{n-1}(1-t)^{m-1} u^{n+m-1} e^{-u/\theta}$$

$$\Rightarrow f_T(t) = \frac{1}{\Gamma(n)\Gamma(m)\theta^{n+m}} t^{n-1}(1-t)^{m-1} \cdot \underbrace{\int_0^\infty u^{n+m-1} e^{-u/\theta}\, du}_{gamma(n+m,\, \theta)}$$

$$= \frac{\Gamma(n+m)\cancel{\theta^{n+m}}}{\Gamma(n)\Gamma(m)\cancel{\theta^{n+m}}} t^{n-1}(1-t)^{m-1}$$

$$= \boxed{\frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} t^{n-1}(1-t)^{m-1}} \Longrightarrow \boxed{T \sim Beta(n, m)}$$

4. [5 pts.] Let $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ denote the order statistics of a random sample of size 5 from the exponential distribution with parameter $\beta = 1$. Show that $Z_1 = X_{(2)}$ and $Z_2 = X_{(4)} - X_{(2)}$ are independent.

From a theorem seen in class, $f_{X_{(1)},\dots,X_{(5)}}(x_1,\dots,x_5) = 5!\, f_{X_1}(x_1)\cdots f_{X_5}(x_5)$.

Since $X_1,\dots,X_5 \overset{iid}{\sim} \exp(1)$, we have:

$$f_{X_{(1)},\dots,X_{(5)}}(x_1,\dots,x_5) = 5!\, e^{-x_1} e^{-x_2} e^{-x_3} e^{-x_4} e^{-x_5} = 120\, e^{-(x_1+x_2+x_3+x_4+x_5)} \quad (*)$$

Let $\begin{cases} Z_1 = X_{(2)} \\ Z_2 = X_{(4)} - X_{(2)} \end{cases} \Longrightarrow \begin{cases} X_{(2)} = Z_1 \\ X_{(4)} = Z_1 + Z_2 \end{cases} \Rightarrow J^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \det(J^{-1}) = 1$

$$\Rightarrow f_{Z_1,Z_2}(u,v) = f_{X_{(2)},X_{(4)}}(u, u+v)\cdot 1$$

But we have: $f_{X_{(2)},X_{(4)}}(x_2,x_4) = \int_0^\infty \int_0^\infty \int_0^\infty f_{X_{(1)},\dots,X_{(5)}}(x_1,\dots,x_5)\, dx_1\, dx_3\, dx_5$

$$\overset{(*)}{=} 120\, e^{-(x_2+x_4)} \cdot \underbrace{\int_0^\infty e^{-x_5}}_{=1} \underbrace{\int_0^\infty e^{-x_3}}_{=1} \underbrace{\int_0^\infty e^{-x_1}}_{=1}\, dx_1\, dx_3\, dx_5$$

(all 3 are kernel of exp(1))

$$= 120\, e^{-(x_2+x_4)}$$

Therefore, $f_{Z_1,Z_2}(u,v) = 120\, e^{-(u+u+v)} = \boxed{(120\, e^{-2u})\cdot(e^{-v})}$

As a result, $f_{Z_1,Z_2}(u,v)$ can be "splited" into two parts, one depending only on u and the other only on v. Therefore, $\boxed{Z_1 \perp\!\!\!\perp Z_2}$.

5. **[5 pts.]** Let X be a continuous random variable with the density

$$f(x) = h(x)c(\theta)e^{\theta x} I_{(a<x<b)},$$

such that $f(x) \to 0$ as $x \downarrow a$ or $x \uparrow b$. Furthermore, let $h(x)$ be differentiable with its derivative denoted by $h'(x)$. Suppose that $g(x)$ is an absolutely continuous function such that $E(|g'(X)|) < \infty$. Show that

$$E\left[\left(\frac{h'(X)}{h(X)} + \theta\right) g(X)\right] = -E\left[g'(X)\right].$$

[Hint: The given conditions ensure that $f(x)g(x) \to 0$ as $x \downarrow a$ or $x \uparrow b$. Moreover, it is possible that $a = -\infty$ and $b = \infty$, in which case the condition $f(x)g(x) \to 0$ is also satisfied. You can assume these statements without proof.] $*$

First, we have: $f'_X(x) = c(\theta) \cdot \left[h'(x) e^{\theta x} + \theta e^{\theta x} h(x)\right] \cdot 1\!\!1_{(a<x<b)}$

$= c(\theta) e^{\theta x}\left[h'(x) + \theta h(x)\right] = h(x) c(\theta) e^{\theta x}\left[\frac{h'(x)}{h(x)} + \theta\right]$

$= \left(\frac{h'(x)}{h(x)} + \theta\right) f_X(x)$ $*$

Moreover, we have: $E\left[\left(\frac{h'(X)}{h(X)} + \theta\right) g(X)\right] = \int_a^b g(x)\left(\frac{h'(x)}{h(x)} + \theta\right) f_X(x)\, dx$

$\overset{*}{=} \int_a^b g(x) f'_X(x)\, dx$, Integration by parts: $u = g(x)$ $v' = f'(x)$

$u' = g'(x)$ $v = f(x)$

$= f(x) g(x)\Big|_a^b - \int_a^b g'(x) f(x)\, dx$

$= \lim_{x \to b} f(x) g(x) - \lim_{x \to a} f(x) g(x) - E(g'(X))$

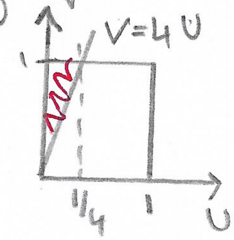
$\overset{*}{=} 0 - 0 - E(g'(X)) = \boxed{-E(g'(X))}$

Since $A, B, C \overset{iid}{\sim} \text{Beta}(2,2)$ then: $f_A(a) = 6a(1-a)$, $f_B(b) = 6b(1-b)$, $f_C(c) = 6c(1-c)$

6. **[5 pts.]** Consider the quadratic equation $Ax^2 + Bx + C = 0$. Let A, B, and C be i.i.d. $Beta(\alpha = 2, \beta = 2)$. Obtain the probability that this equation has real roots. Calculate a final numerical value for your probability.

The equation has real roots iff $B^2 - 4AC \geq 0 \Leftrightarrow B^2 \geq 4AC$.

Let $V = B^2$ and $U = AC$, so we want to find $P(V \geq 4U)$

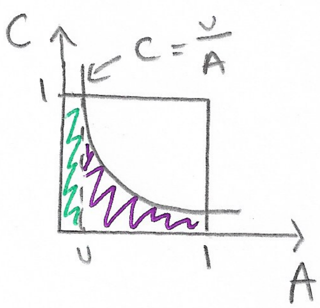


So, $P(V \geq 4U) = \int_0^{1/4} \int_{4u}^{1} f_{U,V}(u,v)\,dv\,du \underset{U \perp V}{=} \int_0^{1/4} \int_{4u}^{1} f_U(u) \cdot f_V(v)\,dv\,du$

Now, Let's find $f_V(v)$ and $f_U(u)$:

① $f_V(v)$: $B^2 = V$, $0 < v < 1$. $F_V(v) = P(V \leq v) = P(B \leq \sqrt{v}) = F_B(\sqrt{v})$

$\Rightarrow f_V(v) = \frac{1}{2\sqrt{v}} f_B(\sqrt{v}) = \frac{1}{2\sqrt{v}} \cdot 6\sqrt{v}(1-\sqrt{v}) = \boxed{3(1-\sqrt{v})}$



② $f_U(u)$: $AC = U$, $0 < u < 1$, $F_U(u) = P(U \leq u) = P\left(C \leq \frac{u}{A}\right)$

$= \underbrace{\int_0^u \int_0^1 f_{A,C}(a,c)\,dc\,da}_{①} + \underbrace{\int_u^1 \int_0^{u/a} f_{A,C}(a,c)\,dc\,da}_{②}$

$f_{A,C}(a,c) \underset{A \perp C}{=} f_A(a) \cdot f_C(c) = 6a(1-a) \cdot 6c(1-c) = 36\,a(1-a)\,c(1-c)$

① $= 36 \int_0^u a(1-a) \int_0^1 c(1-c)\,dc\,da = 36 \int_0^u a(1-a) \cdot \frac{-c^2(2c-3)}{6}\Big|_0^1 da = 6\int_0^u a(1-a)\,da = -u^2(2u-3)$

② $= 36 \int_u^1 a(1-a) \int_0^{u/a} c(1-c)\,dc\,da \overset{*}{=} \cdots = 6\left[2u^3 \cdot \frac{1}{a}\Big|_u^1 + (3u^2+2u^3)\ln(a)\Big|_u^1 - 3u^2(1-u)\right]$

$= 6\left[5u^3 - 5u^2 - u^2\ln(u)(3+2u)\right]$

$\Rightarrow F_U(u) = -u^2(2u-3) + 30u^3 - 30u^2 - \ln(u)(18u^2 + 12u^3)$

$\Rightarrow f_U(u) = F_U'(u) \overset{*}{=} \boxed{-36u\left[2 - 2u - (1+u)\ln(u)\right]}.$

$\Rightarrow P(V \leq 4U) = \int_0^{1/4} f_U(u) \cdot \int_{4u}^{1} 3(1-\sqrt{v})\,dv\,du \overset{*}{=} \int_0^{1/4} f_U(u) \cdot (16u^{3/2} - 12u + 1)\,du$

$= \int_0^{1/4} \left[-36u\left[2-2u+(1+u)\ln(u)\right]\right] \cdot \left[16u^{3/2} - 12u + 1\right] du = \boxed{0.1467}$

$*$ = "busy work": easy but long integration and derivation.

7. **[5 pts.]** Let X_i be independent Poisson random variables with mean a_i, and $S_n = X_1 + \cdots + X_n$. Prove that if $\lim_{n\to\infty} \sum_{i=1}^n a_i = \infty$, then $S_n/E(S_n)$ converges to 1 in probability.

$S_n = \sum_{i=1}^n X_i \sim \text{Poisson}\left(\sum_{i=1}^n a_i\right)$ ⊛ , we suppose $\sum_{i=1}^n a_i = E(S_n) \xrightarrow[n\to\infty]{} \infty$

We need to show that $\frac{S_n}{E(S_n)} \xrightarrow{P} 1$ but since 1 is a constant,

$\Leftrightarrow \frac{S_n}{E(S_n)} \xrightarrow{D} 1 \Leftrightarrow S_n \xrightarrow{D} E(S_n)$.

Now, let's analyse the cdf of S_n, we are going to show that it is a degenerate distribution at $y = E(S_n)$.

$$F_{S_n}(y) \overset{⊛}{=} \sum_{x=0}^{\lfloor y \rfloor} e^{-\sum_{i=1}^n a_i} \cdot \frac{\left(\sum_{i=1}^n a_i\right)^x}{x!} = \sum_{x=0}^{\lfloor y \rfloor} e^{-E(S_n)} \frac{(E(S_n))^x}{x!}$$

If $y \geq E(S_n)$: Since $E(S_n) = \sum_{i=1}^n a_i \xrightarrow[n\to\infty]{} \infty$, $y \geq E(S_n) \Rightarrow y \xrightarrow[n\to\infty]{} \infty$

Therefore, $F_{S_n}(y) \xrightarrow[n\to\infty]{} \sum_{x=0}^{\infty} e^{-E(S_n)} \frac{(E(S_n))^x}{x!} = 1$

If $y < E(S_n)$: Since $E(S_n) = \sum_{i=1}^n a_i \xrightarrow[n\to\infty]{} \infty$, $y < E(S_n) \Rightarrow \lfloor y \rfloor = c \in \mathbb{N}$

Thus, since $0 \leq x \leq \lfloor y \rfloor$, then $0 \leq x \leq c$, so $e^{-E(S_n)} \frac{(E(S_n))^x}{x!} \leq e^{-E(S_n)} \cdot \frac{(E(S_n))^c}{c!}$

Clearly $c \not\to \infty$ and since $E(S_n) = \sum_{i=1}^n a_i \xrightarrow[n\to\infty]{} \infty$,

then $\frac{1}{c!} \cdot \frac{[E(S_n)]^c}{e^{E(S_n)}} \xrightarrow[n\to\infty]{} 0$ [e^x goes faster to 0 than x^c when $x \to \infty$]

Therefore, $\sum_{x=0}^{\lfloor y \rfloor} e^{-E(S_n)} \frac{(E(S_n))^x}{x!} \xrightarrow[n\to\infty]{} 0$ [each term of the sum goes to 0, so the finite sum goes to 0]

As a result, $F_{S_n}(y) \xrightarrow[n\to\infty]{} \begin{cases} 1, & y \geq E(S_n) \\ 0, & y < E(S_n) \end{cases} \Rightarrow S_n \xrightarrow{D} E(S_n) \Rightarrow \boxed{\frac{S_n}{E(S_n)} \xrightarrow{P} 1}$

8. [4 pts.] A random variable X has the moment generating function

$$M_X(t) = \frac{1}{4}e^{2t} + \frac{1}{4}e^{5t} + \frac{1}{2}e^{8t}.$$

Give the pdf/pmf (whichever is relevant) for the random variable X.

We first notice that this mgf doesn't match any of the known mgfs, therefore, the distribution of X is probably a "made-up" distribution.

Suppose X is discrete, then:

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } x} e^{tx} \cdot f_X(x) = \frac{1}{4}e^{2t} + \frac{1}{4}e^{5t} + \frac{1}{2}e^{8t}$$

After trial and error, we find that the following distribution of X works:

$X = x$	2	5	8
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

, therefore, the pmf of X is:

$$\boxed{P(X=x) = \begin{cases} \frac{1}{4}, & x = 2, 5 \\ \frac{1}{2}, & x = 8 \\ 0 & \text{otherwise} \end{cases}}$$

9. [4 pts.] Explain how you would use Monte Carlo integration to approximate the following integral:

$$I = \int_{-\infty}^{\infty} \cos(x) e^{-x^2}\, dx.$$

Give a step-by-step description.

① Transform the integral to get bounds from 0 to 1:

$$I = \int_{-\infty}^{\infty} \cos x \, e^{-x^2} dx = 2 \int_0^{\infty} \cos x \, e^{-x^2} dx$$

$\uparrow$ even fct

$Y = \frac{x}{x+1} \Rightarrow dy = \frac{1}{(x+1)^2} dx$

$x = \frac{y}{1-y} \Rightarrow dx = \frac{1}{(1-y)^2} dy$

$$\Rightarrow I = \int_0^1 \frac{2\cos\left(\frac{y}{1-y}\right)}{(1-y)^2} e^{-\left(\frac{y}{1-y}\right)^2} dy$$

② If $X \sim$ uniform$(0,1)$, then we know that $E(g(X)) = \int_0^1 g(x)\,dx$

Moreover, if $X_1, \dots, X_n \overset{iid}{\sim}$ uniform$(0,1)$, by WLLN, $\frac{1}{n}\sum_{i=1}^{n} g(X_i) \xrightarrow{P} E(g(X_1))$

Therefore, $\frac{1}{n}\sum_{i=1}^{n} g(X_i) \xrightarrow{P} \int_0^1 g(x)\,dx$. So we can approximate the integral by generating n uniform(0,1) independent random variables, summing each value through the function g and the divide by $\frac{1}{n}$.

In this case, $g(x) = \frac{2\cos\left(\frac{x}{1-x}\right)}{(1-x)^2} e^{-\left(\frac{x}{1-x}\right)^2}$. ③ Here's the R code to compute the sum:

```
n = 1000 000   # the larger the n, the better the approximation.
x <- runif(n, 0, 1) # generating n uniform(0,1) values
S = 0
for(i in 1:n){
  S <- S + g(x[i]) # we sum the values of g(x) with each x[i] being an uniform(0,1) value
}
Print((1/n)*S). # We divide the result by 1/n to stick to the WLLN.
```

(real value of I) = 1.3804 , (Approx value of I with n = 1000 000) = 1.3809

10. An insurance company has many clients. If a client has a claim, the size of the claim X in thousands of dollars has a distribution with pdf

$$f_X(x) = \begin{cases} e^{1/2}e^{-x} & \text{for } x \geq 1/2 \\ 0 & \text{Otherwise,} \end{cases}$$

with cdf $F_X(x) = 1 - e^{1/2}e^{-x}$ for $x \geq 1/2$ and zero otherwise. Assuming that all claims are iid and there are five claims, compute the following:

(a) **[3 pts.]** The probability that the median claim is at most \$1500. Leave your answer as an integral. No need to simplify any of the quantities.

5 is odd, so the median claim is $X_{(\frac{5+1}{2})} = X_{(3)}$

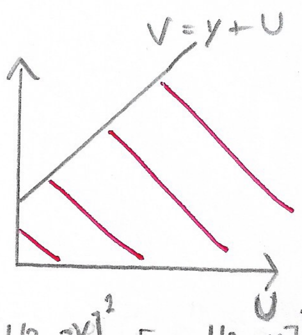
$$P\left(X_{(3)} \leq \frac{3}{2}\right) = \int_{1/2}^{3/2} f_{X_{(3)}}(x)\,dx = \int_{1/2}^{3/2} \frac{5!}{(3-1)!(5-3)!} f_X(x)\, F_X^{3-1}(x)\left[1 - F_X(x)\right]^{5-3}$$

$$= \int_{1/2}^{3/2} \frac{5!}{4} e^{1/2}e^{-x}\left[1 - e^{1/2}e^{-x}\right]^2 \cdot \left[e^{1/2}e^{-x}\right]^2 = \boxed{30 \int_{1/2}^{3/2} \left[e^{1/2}e^{-x}\right]^3 \cdot \left[1 - e^{1/2}e^{-x}\right]^2 dx}$$

(b) **[4 pts.]** The cdf of the interquartile range. Leave your final answer as an integral. No need to simplify any of the quantities.

Let $Y = X_{(4)} - X_{(2)}$ be the interquartile range. $(y > 0)$.

$$F_Y(y) = P(Y \leq y) = P(X_{(4)} - X_{(2)} \leq y) = P(X_{(4)} \leq y + X_{(2)})$$



Let $U = X_{(2)}$ and $V = X_{(4)}$ for simplicity. So:

$$F_Y(y) = \int_0^\infty \int_0^{y+u} f_{U,V}(u,v)\,dv\,du \overset{U \perp V}{=} \int_0^\infty \int_0^{y+u} f_U(u) \cdot f_V(v)\,dv\,du$$

But we have: $f_V(v) = f_{X_{(4)}}(v) = \frac{5!}{3!} e^{1/2}e^{-v}\left[1 - e^{1/2}e^{-v}\right]^3\left[e^{1/2}e^{-v}\right] = 20\left[e^{1/2}e^{-v}\right]^2 \cdot \left[1 - e^{1/2}e^{-v}\right]^3$

$f_U(u) = f_{X_{(2)}}(u) = \frac{5!}{3!} e^{1/2}e^{-u}\left[1 - e^{1/2}e^{-u}\right] \cdot \left[e^{1/2}e^{-u}\right]^3 = 20\left[e^{1/2}e^{-u}\right]^4 \cdot \left[1 - e^{1/2}e^{-u}\right]$

Therefore:

$$\boxed{F_Y(y) = 400 \int_0^\infty \left(e^{1/2}e^{-u}\right)^4\left(1 - e^{1/2}e^{-u}\right) \int_0^{y+u} \left(e^{1/2}e^{-v}\right)^2\left(1 - e^{1/2}e^{-v}\right)^3 dv\,du}$$