Developing Heuristic Algorithms for Graph Optimization Problems using Tree Decompositions

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Overview

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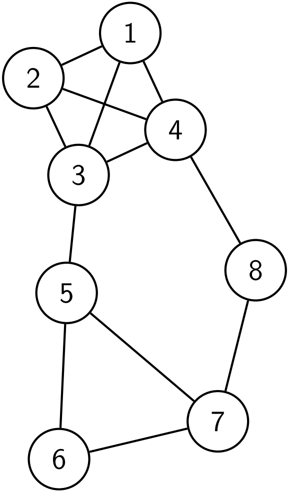
Graph

- A graph $G = (V, E)$
  - vertices $V(G)$: 1, 2, 3, ...
  - edges $E(G)$: $\{1, 2\}$, $\{3, 5\}$, $\{7, 8\}$, ...



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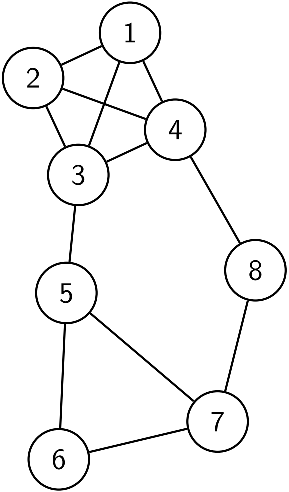

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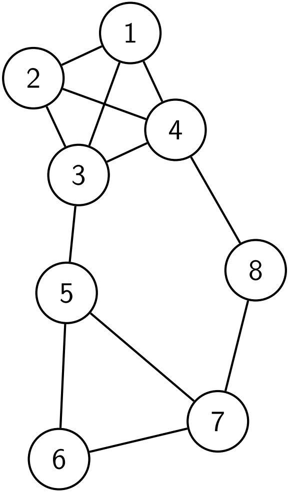
- For all $v \in V(G)$, $N(v)$ is the set of vertices u such that $uv \in E(G)$
  - $N(3) = \{1, 2, 4, 5\}$, $N(8) = \{4, 7\}$



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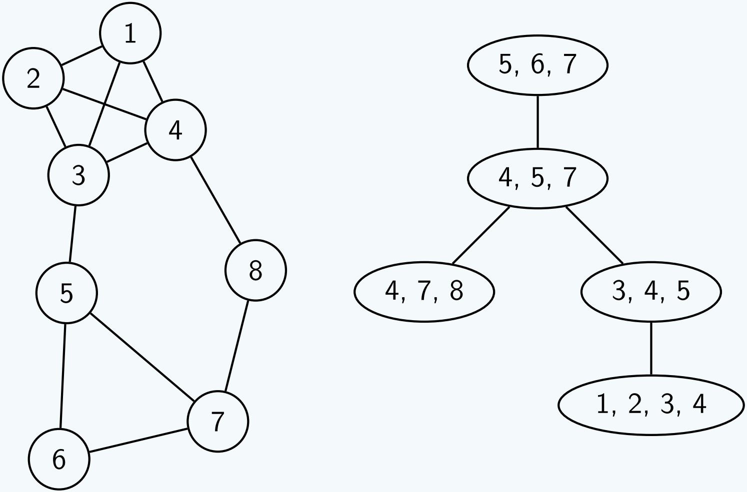
Tree Decomposition

Definition (Tree decomposition Cygan et al. 2015)

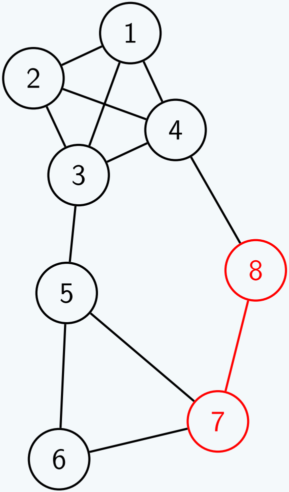
A *tree decomposition* of graph G is a pair $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, such that the following three conditions hold:

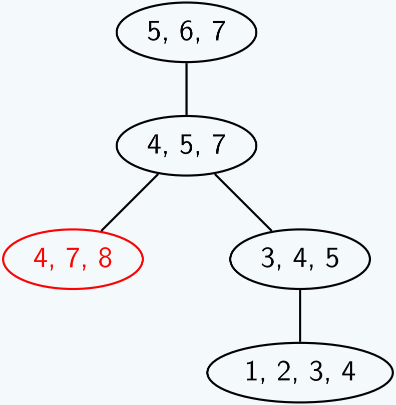
1. $\bigcup_{t\in V(T)} X_t = V(G)$: every vertex $v \in V(G)$ is contained in some bag X_t for $t \in V(T)$.
2. For every $\{u, v\} \in E(G)$, there exists a node t of $V(T)$ such that bag X_t contains both u and v.
3. For every $v \in V(G)$, the set $T_v = \{t \in V(T) : v \in X_t\}$ induces a connected subtree of T.

Example (Property 1: $\bigcup_{t \in V(T)} X_t = V(G)$: every vertex $v \in V(G)$ is contained in some bag X_t for $t \in V(T)$)

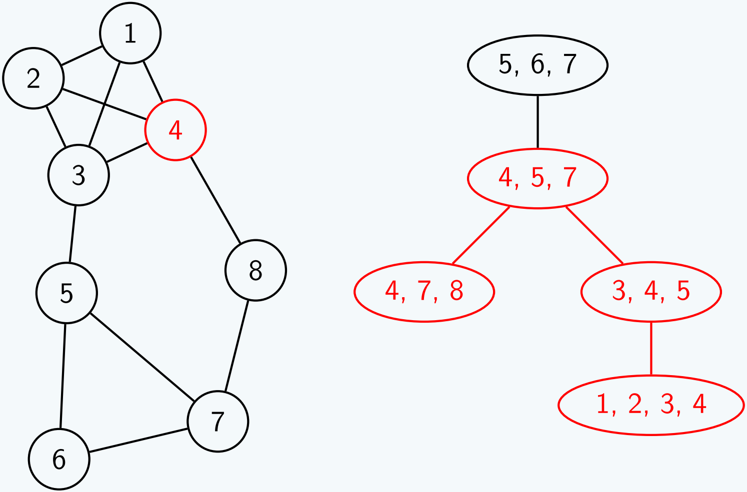


Example (Property 2: For every $\{u, v\} \in E(G)$, there exists a node t of $V(T)$ such that bag X_t contains both u and v)





Example (Property 3: For every $v \in V(G)$, the set $T_v = \{t \in V(T) : v \in X_t\}$ induces a connected subtree of T)



Treewidth

Definition (Width of a tree decomposition)

The *width of a tree decomposition* $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ equals $max_{t \in V(T)} |X_t| - 1$.

Definition (Treewidth of a graph)

The *treewidth* of a graph G – denoted by $t = tw(G)$ – is the minimum possible width of a tree decomposition of G.

The Treewidth Problem

Theorem (Arnborg, Corneil, and Proskurowski 1987)

Given a graph G and integer t, deciding if G has a treewidth of at most t is $\mathcal{NP}$-complete.

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  - See Bodlaender 2005 for an extensive overview
- PACE 2017: Dell et al. 2018
  - Construct an algorithm to compute tree decompositions
  - See for example Tamaki 2019; Strasser 2017; Bannach and Berndt 2019

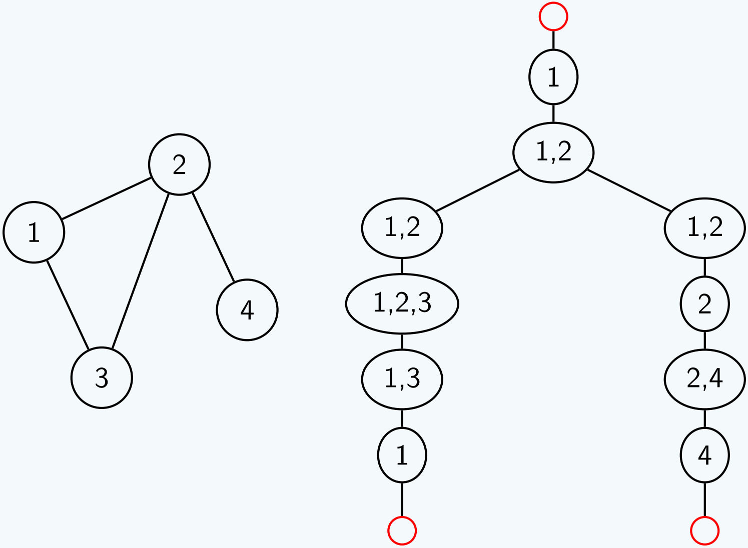
Nice Tree Decomposition

Definition (Nice Tree Decomposition)

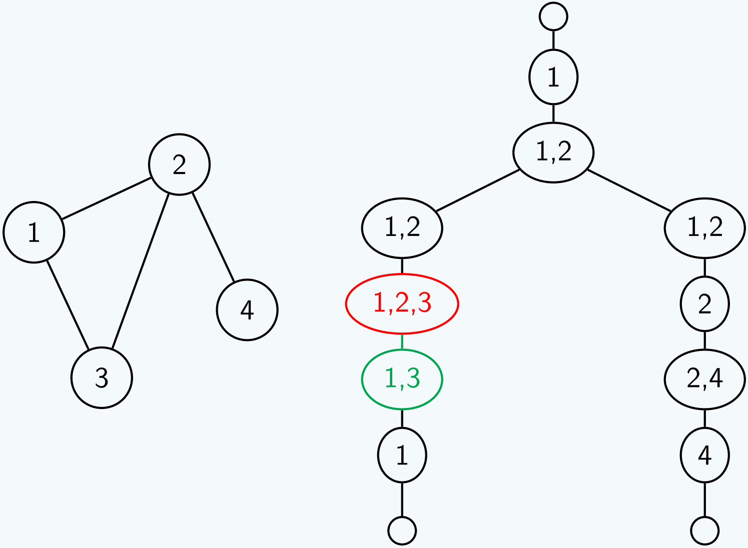
A rooted tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of graph G is *nice* if the following conditions hold:

- $X_r = \emptyset$ and $X_l = \emptyset$ for every leaf l of $V(T)$.
- Every non-leaf node of T is one of the following types:
  1. Introduce node: a node t with exactly 1 child t' such that $X_t = X_{t'} \cap \{v\}$ for some vertex $v \notin X_{t'}$.
  2. Forget node: a node t with exactly 1 child t' such that $X_t = X_{t'} \setminus \{v\}$ for some vertex $v \in X_{t'}$.
  3. Join node: a node t with exactly two children t_1, t_2 such that $X_t = X_{t_1} = X_{t_2}$

Example (Root and leaves)



Example (Introduce node)



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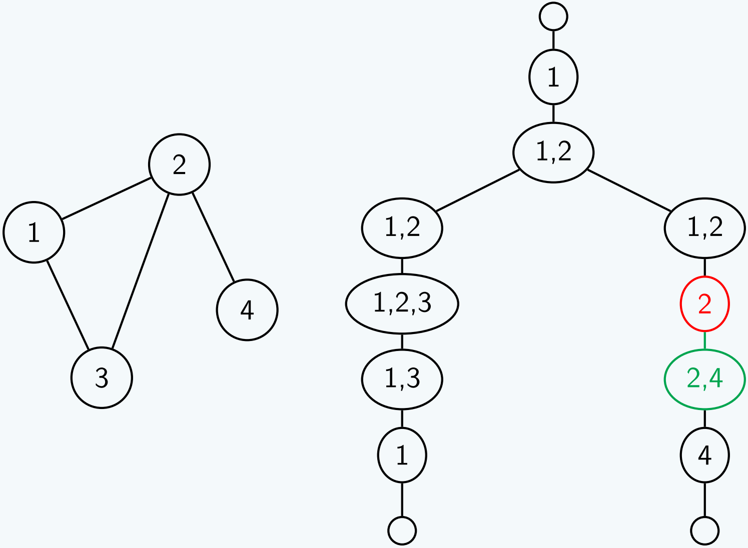

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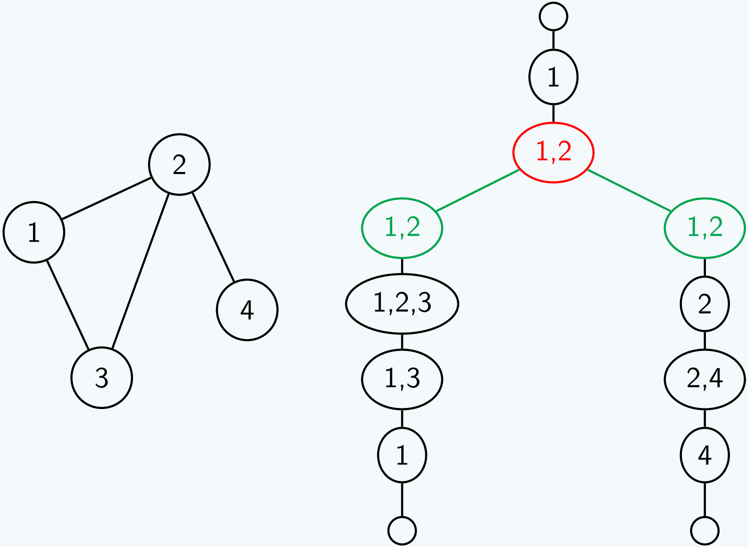

Dynamic Programming using Nice Tree Decompositions

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➡ Goal of my thesis

Overview

Maximum Happy Vertices Problem (Zhang and Li 2015)

Definition (Happy and Unhappy Vertices)

Given a graph G and a colouring $c : V(G) \to \{1 \ldots k\}$. A vertex $v \in V(G)$ is said to be *happy* if and only if $c(v) = c(v')$ for all vertices $v' \in N(v)$, otherwise vertex v is said to be *unhappy*.

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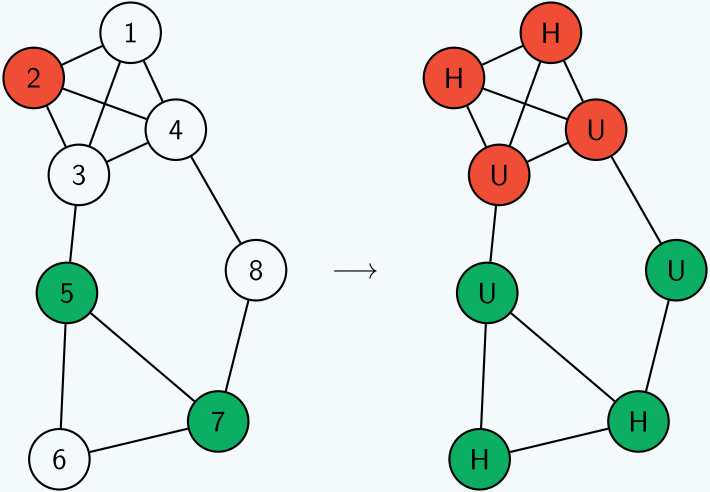
Definition (k-MHV)

An instance of the MHV problem, in which the number of colours k is a constant.

Example



Applications of Maximum Happy Vertices

1. Modeling social networks: *homophyly*
   - People tend to be similar to their friends
   - Elements in a small community share remarkable common features
   - The feature values are represented by a colour
   - See Easley, Kleinberg, et al. 2010; Li and Peng 2011; Li and Peng 2012 for more information

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2. Clustering
   - The vertices represent data points
   - The edges represent strong connections between the data points
   - The colours represent the different clusters

Complexity of the Maximum Happy Vertices Problem

Theorem (Zhang and Li 2015)

The k-MHV problem is $\mathcal{NP}$-complete for every constant $k \geq 3$.

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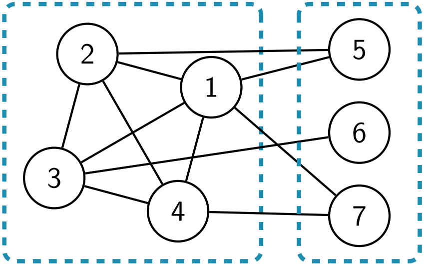

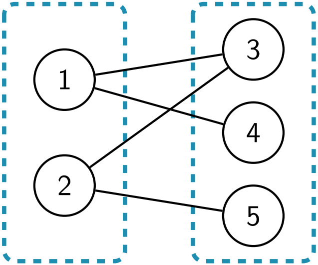

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Theorem (Agrawal 2017)

The MHV problem when parametrized by the number of happy vertices is $\mathcal{W}[1]$-hard.

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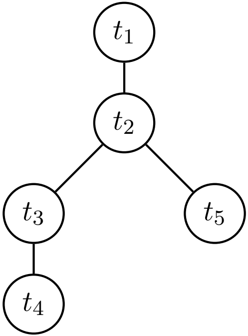
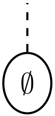
- Compute a fixed number of colour functions at each node
- Colour functions at node t contain information about $X_{t'}$ for all descendants t' of t
  - Descendants of t_3: $\{t_4\}$
  - Descendants of t_2: $\{t_3, t_4, t_5\}$



Leaf Nodes

- Leaf nodes have empty bags
  - ⇒ Pass initial colour function to parent



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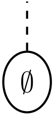
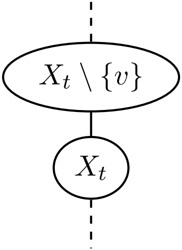


Forget Nodes

- Vertex v is 'forgotten' from the tree decomposition
- Sub tree contains the same vertices

  Pass colour functions of child to parent



Introduce Nodes

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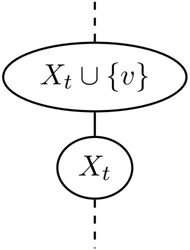
  - v must receive a colour



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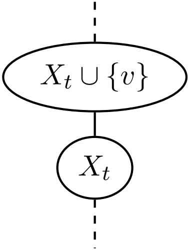

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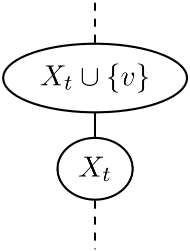

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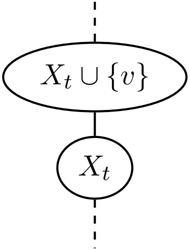
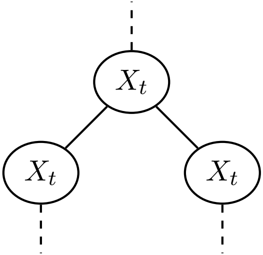


Two approaches:

1. For each colour function of child, give v the colour that results in the most happy vertices
2. For each colour function of child, give v all the possible colours and select the best colour functions

Join Nodes

- The left and right children pass different colour functions to the join node
- Merge all colour functions c_l of the left child with all colour functions c_r of the right child
- Iterate over all vertices $v \in X_t$ and assign it a colour



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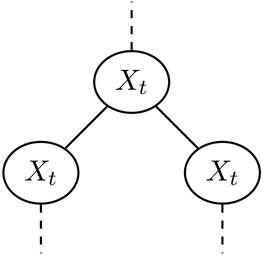

Order to iterate over the vertices in X_t?

1. Static: random, greatest degree first, smallest degree first
2. Dynamic: vertex connected to most/fewest already coloured vertices, vertex with fewest differently coloured neighbours

Evaluation method

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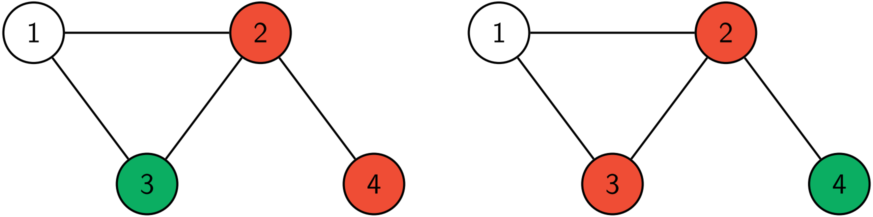

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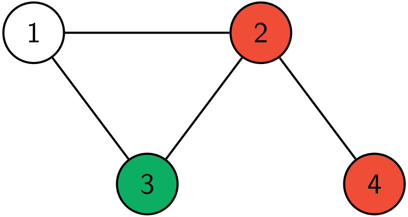

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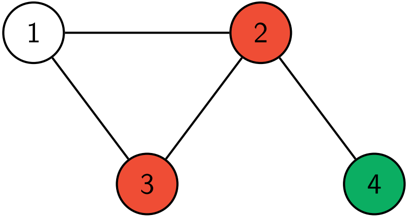

No vertex is happy, but both vertex 1 and 3 are happy if vertex 1 is coloured red

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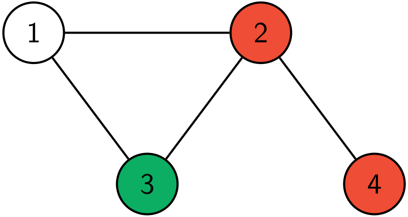
How to decide what the best colour function is?

1. Count the number of happy vertices
2. Count the number of happy and potentially happy vertices



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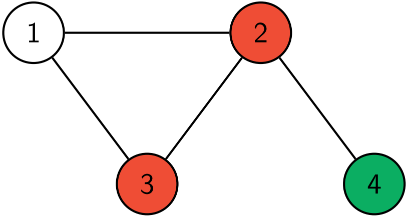

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Future Work

Algorithm improvement:

- Implement more techniques to handle the different types of nodes
- Apply kernelization on the input graph H. Gao and W. Gao 2018

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Experimental analysis:

- Compare algorithm performance against other, existing heuristic algorithms
- Compare the influence on both solution quality and running time for different tree decompositions
- Compare the retrieved colouring functions of the heuristic approach and exact algorithm at each node t of the tree decomposition
- Apply *framework* on other problems

Thank you for your attention!

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