Simulation of the Galton Board

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Introduction

The Galton board is a classical device consisting of several levels of pegs. When many balls are dropped from the top, each ball randomly bounces left or right at each level. In older mechanical versions, this is achieved by nails that deflect the balls so that, at the end, all balls accumulate in different columns at the bottom (see Figure 1).

The purpose of the Galton board is to visualize random processes and their probability distributions. If this experiment were repeated infinitely many times, the resulting distribution of ball positions would follow a binomial distribution, which converges to a normal distribution as the number of levels increases. This phenomenon is a direct consequence of the Central Limit Theorem (CLT), which states that the sum of many independent random events tends to follow a normal distribution. In this experiment, we compare the empirical distribution of simulated ball drops with the theoretical normal and binomial distributions.

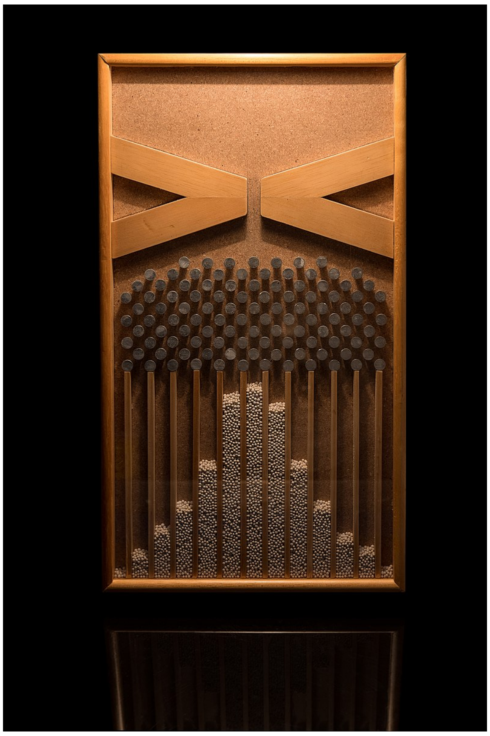

Figure 1: Schematic view of a Galton board.

Methodology

In the simulation, each ball passes through n levels. At each level, it has a probability of 0.5 to move right or stay. After all n levels, the number of right moves k determines the final bin where the ball lands. Repeating this experiment for N balls gives the empirical probability mass function (PMF).

The theoretical probability of a ball ending in bin k follows the binomial law:

$$P(X = k) = \binom{n}{k}(0.5)^n.$$

For large n, the binomial can be approximated by the normal distribution

$$\mathcal{N}\left(\mu = \frac{n}{2}, \sigma^2 = \frac{n}{4}\right).$$

To quantify the deviation between the simulated and theoretical distributions, we compute the Mean Squared Error (MSE):

$$\mathrm{MSE}(p, q) = \frac{1}{n+1}\sum_{k=0}^{n}(p_k - q_k)^2,$$

where p_k are the simulated probabilities and q_k are the theoretical ones (binomial or normal).

Results

The experiment was carried out for several configurations of $n \in \{5, 10, 20, 50\}$ and $N \in \{500, 2000, 10000, 50000\}$. For each configuration, the empirical PMF, the theoretical binomial, and the normal approximation were plotted and compared numerically.

Table 1 summarizes typical results.

Levels (n)	Balls (N)	MSE (empirical vs normal)
5	500	1.71×10^{-4}
5	2000	7.61×10^{-5}
5	10000	6.41×10^{-5}
5	50000	8.66×10^{-5}
10	500	1.16×10^{-4}
10	2000	2.30×10^{-6}
10	10000	1.95×10^{-5}
10	50000	6.86×10^{-6}
20	500	5.79×10^{-5}
20	2000	5.48×10^{-6}
20	10000	9.51×10^{-6}
20	50000	8.40×10^{-7}
50	500	2.59×10^{-5}
50	2000	4.37×10^{-6}
50	10000	2.57×10^{-6}
50	50000	3.66×10^{-7}

Table 1: Mean squared error (MSE) between empirical and normal distributions for different board levels n and number of balls N.

As expected, the empirical distribution becomes smoother and more symmetric as N increases. Likewise, for larger n, the binomial curve becomes indistinguishable from the normal approximation (see Figure 2 and 3 as examples). The MSE values decrease with both parameters, confirming convergence predicted by the Central Limit Theorem.

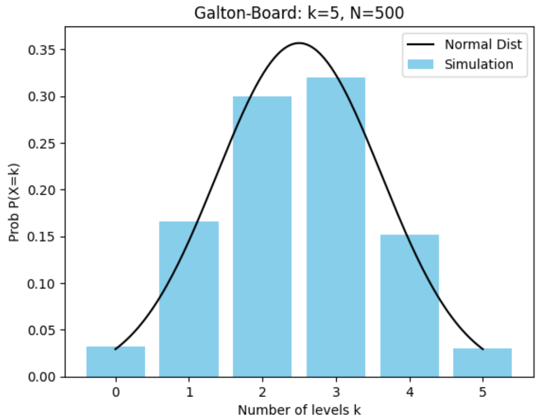

Figure 2: Galton Board with low amount of levels and balls.

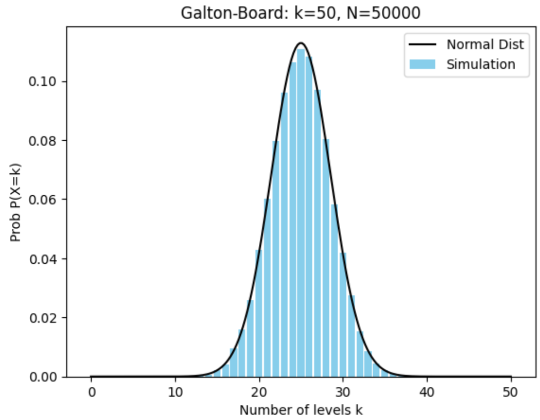

Figure 3: Galton Board with high amount of levels and balls.

Conclusion

The simulation of the Galton board successfully demonstrates the Central Limit Theorem. With increasing numbers of levels and balls, the empirical distribution of ball positions converges toward the theoretical binomial and normal distributions. The decreasing mean squared error quantitatively confirms this convergence. Even a simple random left–right process can approximate the continuous Gaussian law when repeated many times.