Faculty of Science, Technology, Engineering and Mathematics
M347 Mathematical statistics

M347

TMA 03

Covers Block 3 **See module website for the cut-off date.**

Submitting your assignment

You can submit your TMA either by post or electronically using the online TMA/EMA service. Please read the guidance in the 'Assessment' area of the M347 website.

This TMA

This TMA covers Block 3 of M347. You will probably find it best to answer the questions associated with each unit soon after completing that unit, rather than waiting until you have completed the block.

Each TMA is marked out of 25, with the marks allocated to each part of each question indicated in brackets in the margin. Your overall score for the TMA will be the sum of your marks for all questions in the TMA, which is then converted into a percentage score. Your tutor's reply will give you feedback on how well you answer the TMA. Being a formative assessment, your mark on this TMA counts only towards achieving the required threshold for completion of assignments to be able to pass the module, but does not contribute to your final mark for the module.

Your work should only include your answers to the questions: do not include material which was not asked for in the questions. You should, however, include your working. If you don't, and you make a mistake, then your tutor will not be able to provide any feedback about where you went wrong. Leaving out your working may also cost you marks if it is specifically asked for.

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5.1

This assignment covers Block 3.

Question 1 (Unit 8) – 9 marks

Let independent positive random variables $X_1, X_2, \dots, X_n$ be modelled by a gamma distribution with parameters $a = 2$ (known) and $\theta > 0$ (unknown), so that for $i = 1, 2, \dots, n$,

$$f(x_i|\theta) = \theta^2 x_i \exp(-\theta x_i) \quad \text{on } x_i > 0.$$

(a) Show that $L(\theta)$, the likelihood for θ based on observed data $\boldsymbol{x} = (x_1\ x_2\ \cdots\ x_n)^T$, can be written

$$L(\theta) \propto \theta^{2n} \exp(-\theta n \overline{x}).$$ *[2]*

(b) Suppose that a Gamma(a, b) prior is specified for θ. Show that the posterior $f(\theta|\boldsymbol{x})$ can be written

$$f(\theta|\boldsymbol{x}) \propto \theta^{a+2n-1} \exp\{-(b + n\overline{x})\theta\}.$$ *[2]*

(c) The posterior corresponds to which distribution for θ? *[1]*

(d) The prior mean is $E(\theta) = a/b$; the MLE, $\widehat{\theta}$, of θ under the Gamma$(2, \theta)$ model turns out to be $\widehat{\theta} = 2/\overline{x}$; and the posterior mean turns out to be

$$E(\theta|\boldsymbol{x}) = \frac{a + 2n}{b + n\overline{x}}.$$

Show that $E(\theta|\boldsymbol{x})$ can be written

$$E(\theta|\boldsymbol{x}) = t\,\widehat{\theta} + (1 - t)\,E(\theta),$$

where $0 < t < 1$, and give the expressions for t and $1 - t$. *[4]*

Question 2 (Unit 9) – 9 marks

You showed in Exercise 9.10 in Subsection 3.3 that for the 0–1 loss function

$$L(d, \theta) = \begin{cases} 1 & \text{if } |d - \theta| > \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

with $\alpha > 0$,

$$E[L(d, \theta)\,|\,\boldsymbol{x}] = 1 + F(d - \alpha\,|\,\boldsymbol{x}) - F(d + \alpha\,|\,\boldsymbol{x}),$$

where $F(\theta|\boldsymbol{x})$ is the posterior distribution function. Suppose that the posterior density for $\theta|\boldsymbol{x}$ turns out to be symmetric about a point θ_0. This implies (and you need not check this) that

$$F(\theta|\boldsymbol{x}) = 1 - F(2\theta_0 - \theta\,|\,\boldsymbol{x}) \quad \text{for all } \theta \in \Omega.$$

(a) Obtain an alternative expression for $E[L(d, \theta)\,|\,\boldsymbol{x}]$ by applying the symmetry condition on the posterior to the final term in the expression for $E[L(d, \theta)\,|\,\boldsymbol{x}]$. *[2]*

(b) Confirm that

$$\frac{d}{dd}E[L(d,\theta)\,|\,\boldsymbol{x}] = f(d-\alpha\,|\,\boldsymbol{x}) - f(2\theta_0 - d - \alpha\,|\,\boldsymbol{x}),$$

where $f(\theta|\boldsymbol{x})$ is the posterior density function. Hence show that when $d = \theta_0$,

$$\frac{d}{dd}E[L(d,\theta)\,|\,\boldsymbol{x}] = 0. \qquad [2]$$

(c) Now assume that the posterior density $f(\theta|\boldsymbol{x})$ is differentiable and unimodal, with mode at θ_0. This means that $f'(\theta|\boldsymbol{x}) > 0$ for $\theta < \theta_0$ and $f'(\theta|\boldsymbol{x}) < 0$ for $\theta > \theta_0$. Show that the second derivative of $E[L(d,\theta)\,|\,\boldsymbol{x}]$ is positive at $d = \theta_0$. *[3]*

(d) Summarise, in words, the result that you have proved in this question. Pay particular attention to any differences from the result proved in Subsection 3.3.1. *[2]*

Question 3 (Unit 10) – 7 marks

Suppose that

$$X|\mu \sim \text{Bernoulli}\left(\frac{1}{1+e^{-(\mu+2)}}\right)$$

and a standard normal $N(0,1)$ prior is specified for μ. In a trial, the value $x = 1$ is observed. It can be shown that the posterior for μ when X is observed to be $x = 1$ is

$$f(\mu\,|\,x=1) \propto \left(1+e^{-(\mu+2)}\right)^{-1} e^{-\mu^2/2} \quad \text{on } \mathbb{R}.$$

This cannot be sampled directly. Suppose that the Metropolis–Hastings algorithm is to be used to simulate samples from the target posterior density $f(\mu\,|\,x=1)$.

Parameters have distributions in Bayesian inference, so it makes sense to refer to unknown parameters as random variables. In order to simulate samples from the posterior distribution of parameter μ, consider the sequence of random variables, $\mu_1, \mu_2, \ldots$, so that μ_t is the variable for the tth iteration of the Metropolis–Hastings algorithm. Let μ'_t be its simulated value. Suppose that the proposal distribution for $\mu_{t+1}\,|\,\mu_t = \mu'_t$, which has density denoted by $q(\mu_{t+1}|\mu'_t)$, is $N(\mu'_t, 2)$.

(a) Let $\mu'_1 = 0$ be the starting value for the Metropolis–Hastings algorithm. What distribution will the algorithm use to generate the candidate value μ^* for μ_2? *[1]*

(b) Explain why, in this case, the acceptance probability for μ^* can be written

$$\alpha(\mu^*|\mu'_1) = \min\left(\frac{f(\mu^*\,|\,x=1)}{f(\mu'_1\,|\,x=1)}, 1\right). \qquad [2]$$

(c) Suppose that the candidate value generated from the distribution in part (a) is $\mu^* = 1.2$. Calculate the acceptance probability for this candidate value. *[2]*

(d) Suppose that u is simulated from $U(0,1)$ so that $u = 0.81$. What is the value of μ'_2? From what distribution will the candidate value μ^{**} for μ_3 be generated? *[2]*