

M347 TMA01 R2698663

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Question 1

$$X : M_X(t) = E [e^{tX}]$$

$$Y = aX + t$$

$$M_Y(t) = E [e^{tY}]$$

$$M_Y(t) = E [e^{t(aX+b)}]$$

Separating terms, we can rewrite this as

$$M_Y(t) = E [e^{bt}e^{atX}]$$

and then, using the general fact that for any constant a and random variable Z

$$E [aZ] = aE [Z]$$

We therefore obtain

$$M_Y(t) = e^{bt} E [e^{(at)X}]$$

Which by definition of X , is equivalent to

$$M_Y(t) = e^{bt} M_X(at)$$

Question 2

2.a

$$X \sim Weibull(\beta)$$

X has a pdf of

$$f(x) = \beta x^{\beta-1} e^{-x^\beta} \text{ on } x > 0$$

Show

$$E(X^r) = \Gamma(\frac{r}{\beta} + 1)$$

By definition we have

$$E(X^r) = \int_{x=0}^{\infty} x^r f(x) dx$$

$$\Gamma(n) = \int_{x=0}^{\infty} x^{n-1} e^{-x} dx$$

Using the integration by substitution method to evaluate

$$\int f(g(x)) g'(x) dx \text{ by putting } u = g(x) \text{ and } du = g'(x) dx \text{ we get } \int f(u) du$$

Using the substitution of

$$u = x^\beta \text{ and so also } x = u^{\frac{1}{\beta}}$$

and therefore

$$du = \beta x^{\beta-1} dx$$

Checking the limits of integration, when $x = 0$ we have $u = 0$ and when $x = \infty$ we have $u = \infty$

$$E(X^r) = \int_{x=a}^b x^r \beta x^{\beta-1} e^{-x^\beta} dx$$

$$E(X^r) = \int_{x=a}^b x^r e^{-x^\beta} \beta x^{\beta-1} dx$$

$$E(X^r) = \int_{u=0}^{\infty} u^{\frac{r}{\beta}} e^{-u} du$$

Which can be written as

$$E(X^r) = \int_{u=0}^{\infty} u^{(\frac{r}{\beta}+1)-1} e^{-u} du$$

Therefore after comparing with the Gamma function $\Gamma(n)$, we get

$$E(X^r) = \Gamma(\frac{r}{\beta} + 1)$$

2.b

By definition $V(X) = E(X^2) - E(X)^2$

find $V(X)$ for *Weibull*(β) where $\beta = \frac{1}{2}$

Using the result of part a

$$E(X^2) = \Gamma(\frac{2}{\beta} + 1)$$

$$E(X)^2 = (\Gamma(\frac{1}{\beta} + 1))^2$$

$$V(X) = \Gamma(\frac{2}{\beta} + 1) - (\Gamma(\frac{1}{\beta} + 1))^2$$

$$V(X) = \Gamma(\frac{2}{\frac{1}{2}} + 1) - (\Gamma(\frac{1}{\frac{1}{2}} + 1))^2$$

$$V(X) = \Gamma(5) - \Gamma(3)^2$$

Using $\Gamma(x) = (x-1)!$ for $x > 0$, integer $x \in \mathbb{N}$

$$V(X) = 4! - 2!^2$$

$$V(X) = 24 - 4$$

Therefore

$$V(X) = 20$$

Question 3

3.a

Given the bivariate distribution (X, Y) defined as

$$f(x, y) = \frac{\lambda y^2}{\sqrt{2\pi}} e^{-(\frac{1}{2} + \lambda x)y^2} \text{ on } x > 0, y \in \mathbb{R}$$

show the marginal distribution f_X is given by

$$f_X(x) = \frac{\lambda}{2\sqrt{2}(\frac{1}{2} + \lambda x)^{\frac{3}{2}}}$$

By definition, the marginal distribution $f_X(x)$ is given by

$$f_X(x) = \int_{A_y} f(x, y) dy$$

So we have

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x, y) dy$$

$$f_X(x) = 2 \int_{y=0}^{y=\infty} f(x, y) dy ; \text{ since } f(x, y) \text{ is symmetric about } y = 0 \text{ on the term } y^2$$

Using the integration by substitution method to evaluate

$$\int f(g(x)) g'(x) dx \text{ by putting } u = g(x) \text{ and } du = g'(x) dx \text{ we get } \int f(u) du$$

Using the substitution of

$$u = g(y) = (\frac{1}{2} + \lambda x)y^2$$

$$du = g'(y) dy$$

$$du = 2(\frac{1}{2} + \lambda x)y dy$$

Checking the limits of integration, when $y = 0$ we have $u = 0$ and when $y = \infty$ we have $u = \infty$

Taking out the constant terms from the integral (terms not on y)

$$\frac{2\lambda}{\sqrt{2\pi}} \int_0^\infty y^2 e^{-(\frac{1}{2} + \lambda x)y^2} dy$$

$$\frac{2\lambda}{\sqrt{2\pi}} \int_0^\infty \frac{y^2}{2(\frac{1}{2} + \lambda x)y} e^{-(\frac{1}{2} + \lambda x)y^2} (2(\frac{1}{2} + \lambda x)y) dy$$

extracting constant terms from the integral

$$\frac{\lambda}{\sqrt{2\pi}(\frac{1}{2} + \lambda x)} \int_0^\infty \frac{y^2}{y} e^{-(\frac{1}{2} + \lambda x)y^2} (2(\frac{1}{2} + \lambda x)y) dy$$

Substituting u for $g(y)$ and du for $g'(y) dy$

$$\frac{\lambda}{\sqrt{2\pi(\frac{1}{2}+\lambda x)}} \int_0^\infty y e^{-u} du$$

Rearranging u for y , we get

$$y = \sqrt{\frac{u}{\frac{1}{2}+\lambda x}}$$

and replacing y for an expression of u , we get

$$\frac{\lambda}{\sqrt{2\pi(\frac{1}{2}+\lambda x)}} \int_0^\infty \sqrt{\frac{u}{\frac{1}{2}+\lambda x}} e^{-u} du$$

and after extracting constant terms from the integral (note x is constant)

$$\frac{\lambda}{\sqrt{2\pi(\frac{1}{2}+\lambda x)}\sqrt{\frac{1}{2}+\lambda x}} \int_0^\infty \sqrt{u} e^{-u} du$$

After collecting terms, so far we now have

$$f_X(x) = \frac{\lambda}{\sqrt{2\pi(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}} \int_0^\infty \sqrt{u} e^{-u} du$$

Now looking at the integral term, we can make use of the Gamma Function

$$\int_0^\infty u^{\frac{1}{2}} e^{-u} du$$

The Gamma function is defined as:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

So the integral we have is equivalent to $\Gamma(\frac{3}{2})$

Evaluating using $\Gamma(a+1) = a\Gamma(a)$

and also that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Then $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

So, recombining with the whole expression, we have:

$$f_X(x) = \frac{\lambda}{\sqrt{2\pi(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}} \frac{1}{2}\sqrt{\pi}$$

$$f_X(x) = \frac{\lambda}{2\sqrt{2}\sqrt{\pi}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}} \sqrt{\pi}$$

Therefore giving

$$f_X(x) = \frac{\lambda}{2\sqrt{2}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}} \text{ as required.}$$

3.b

The conditional density of $Y|X$ is defined as $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$ provided $f_X(x) > 0$

$$\frac{\frac{\lambda y^2}{\sqrt{2\pi}} e^{-(\frac{1}{2} + \lambda x)y^2}}{\lambda}$$

$$\frac{\lambda y^2}{\sqrt{2\pi}} \frac{2\sqrt{2}(\frac{1}{2} + \lambda x)^{\frac{3}{2}}}{\lambda} e^{-(\frac{1}{2} + \lambda x)y^2}$$

$$\frac{y^2 2\sqrt{2}}{\sqrt{\pi} \sqrt{2}} (\frac{1}{2} + \lambda x)^{\frac{3}{2}} e^{-(\frac{1}{2} + \lambda x)y^2}$$

$$\frac{2y^2 (\frac{1}{2} + \lambda x)^{\frac{3}{2}}}{\sqrt{\pi}} e^{-(\frac{1}{2} + \lambda x)y^2}$$

Grouping constant (those that do not depend on y) and variable terms (those that depend on y), since y is variable and x is constant

$$(\frac{(\frac{1}{2} + \lambda x)^{\frac{3}{2}}}{\sqrt{\pi}}) (2y^2 e^{-(\frac{1}{2} + \lambda x)y^2})$$

where

$2y^2 e^{-(\frac{1}{2} + \lambda x)y^2}$ is the density core, depending on the variable y and

$\frac{(\frac{1}{2} + \lambda x)^{\frac{3}{2}}}{\sqrt{\pi}}$ is a constant term depending only on x (x is constant)

3.c

Given the Marginal distribution of Y : $f_Y(y)$ and the Conditional distribution $f_{X|Y=y}(x|y)$

We can use the alternative forms of Bayes's theorem to express the conditional $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) \propto f_{X|Y}(x|y) f_Y(y)$$

This allows us to express the conditional density $f_{Y|X}$ from the marginal f_Y and conditional $f_{X|Y}$

(Since we only require the density *core*, we can omit the calculation of the scaling factor constant here, which would be needed to turn the core into a PDF).