The Open University

Faculty of Science, Technology, Engineering and Mathematics
M347 Mathematical statistics

M347

TMA 02

Covers Block 2 **See module website for the cut-off date.**

Submitting your assignment

You can submit your TMA either by post or electronically using the online TMA/EMA service. Please read the guidance in the 'Assessment' area of the M347 website.

This TMA

This TMA covers Block 2 of M347. You will probably find it best to answer the questions associated with each unit soon after completing that unit, rather than waiting until you have completed the block.

Each TMA is marked out of 25, with the marks allocated to each part of each question indicated in brackets in the margin. Your overall score for the TMA will be the sum of your marks for all questions in the TMA, which is then converted into a percentage score. Your tutor's reply will give you feedback on how well you answer the TMA. Being a formative assessment, your mark on this TMA counts only towards achieving the required threshold for completion of assignments to be able to pass the module, but does not contribute to your final mark for the module.

Your work should only include your answers to the questions: do not include material which was not asked for in the questions. You should, however, include your working. If you don't, and you make a mistake, then your tutor will not be able to provide any feedback about where you went wrong. Leaving out your working may also cost you marks if it is specifically asked for.

This assignment covers Block 2.

Question 1 (Unit 5) – 9 marks

Suppose that independent observations $x_1, x_2, \ldots, x_n$ are available from the Pareto distribution with pdf

$$f(x|\beta) = \frac{\beta}{x^{\beta+1}} \quad \text{on } x > 1,$$

with $\beta > 0$.

(a) Show that the log-likelihood is

$$\ell(\beta) = n \log \beta - (\beta + 1) \sum_{i=1}^{n} \log x_i.$$ [2]

(b) Find $\ell'(\beta)$ and hence show that the candidate MLE is

$$\widehat{\beta} = \frac{n}{\sum_{i=1}^{n} \log X_i}.$$ [2]

(c) Confirm that $\widehat{\beta}$ is indeed the MLE of β. [3]

(d) Explain why $\widehat{\beta} > 0$, as one would hope given that $\beta > 0$. [2]

Question 2 (Unit 6) – 9 marks

Again suppose that independent observations $x_1, x_2, \ldots, x_n$ are available from the Pareto distribution with pdf

$$f(x|\beta) = \frac{\beta}{x^{\beta+1}} \quad \text{on } x > 1,$$

with $\beta > 0$. You showed in Question 1 that the log-likelihood is

$$\ell(\beta) = n \log \beta - (\beta + 1) \sum_{i=1}^{n} \log x_i$$

and the MLE of β is

$$\widehat{\beta} = \frac{n}{\sum_{i=1}^{n} \log X_i}.$$

It also turns out to be the case that

$$E\{-\ell''(\beta)\} = \frac{n}{\beta^2}.$$

In this question, interest lies in testing H_0: $\beta = 1$ against H_1: $\beta \neq 1$.

(a) Obtain formulae for both Wald statistics, W_1 and W_2, in terms of $\widehat{\beta}$ (and n). [2]

(b) Show that the score statistic, S, can be written

$$S = (1 - \widehat{\beta})^2 \frac{n}{\widehat{\beta}^2}.$$ [4]

(c) What common procedure would now be followed to complete each Wald or score test to obtain the appropriate p-value? [There is no need to carry out any calculations.] What is the justification for the distribution that is used? [3]

Question 3 (Unit 7) – 7 marks

Suppose that

$$X_n = X + \frac{Z_n}{n^{1/2}},$$

where Z_n is any random variable with $E\{Z_n^2\} = cn^\alpha$, say, with $c > 0$ and $\alpha \in \mathbb{R}$ fixed, and X is any other random variable.

(a) Let $\epsilon > 0$. Use Chebyshev's inequality to show that

$$P(|X_n - X| > \epsilon) \leq \frac{c}{\epsilon^2 n^{1-\alpha}}.$$ *[4]*

(b) For what values of α does the argument in part (a) prove that X_n converges in probability to X? *[2]*

(c) For the values of α identified in part (b), what other mode of convergence of X_n to X is assured (without any further calculations)? *[1]*