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Question 1

$X : M_X(t) = E\left[e^{tX}\right]$

$Y = aX + t$

$M_Y(t) = E\left[e^{tY}\right]$

$M_Y(t) = E\left[e^{t(aX+b)}\right]$

$M_Y(t) = e^{tb}E\left[e^{atX}\right]$

$M_Y(t) = e^{bt}M_X(at)$

Question 2

2.a

$X \sim Weibull(\beta)$

X has a pdf of

$f(x) = \beta x^{\beta-1}e^{-x^{\beta}}$ on $x > 0$

Show

$E(X^r) = \Gamma(\frac{r}{\beta} + 1)$

By definition we have

$E(X^r) = \int_{x=a}^{b} x^r f(x)\ dx$

$\Gamma(n) = \int_{x=0}^{\infty} x^{n-1}e^{-x}\ dx$

Using the integration by substitution method to evaluate

$\displaystyle\int f(g(x))\ g'(x)\ dx$ by putting $u = g(x)$ and $du = g'(x)\ dx$ we get $\displaystyle\int f(u)\ du$

Using the substitution of

$u = x^{\beta}$ and so also $x = u^{\frac{1}{\beta}}$

and therefore

$du = \beta x^{\beta-1}\ dx$

Checking the limits of integration, when $x = 0$ we have $u = 0$ and when $x = \infty$ we have $u = \infty$

$$E(X^r) = \int_{x=a}^{b} x^r \beta x^{\beta-1}e^{-x^{\beta}}\ dx$$

$E(X^r) = \int_{x=a}^{b} x^r e^{-x^\beta} \beta x^{\beta-1} \ dx$

$E(X^r) = \int_{u=0}^{\infty} u^{\frac{r}{\beta}} e^{-u} \ du$

Which can be written as

$E(X^r) = \int_{u=0}^{\infty} u^{(\frac{r}{\beta}+1)-1} e^{-u} \ du$

Therefore after comparing with the Gamma function $\Gamma(n)$, we get

$E(X^r) = \Gamma(\frac{r}{\beta} + 1)$

2.b

By definition $V(X) = E(X^2) - E(X)^2$

find $V(X)$ for $Weibull(\beta)$ where $\beta = \frac{1}{2}$

Using the result of part a

$E(X^2) = \Gamma(\frac{2}{\beta} + 1)$

$E(X)^2 = (\Gamma(\frac{1}{\beta} + 1))^2$

$V(X) = \Gamma(\frac{2}{\beta} + 1) - (\Gamma(\frac{1}{\beta} + 1))^2$

$V(X) = \Gamma(\frac{2}{\frac{1}{2}} + 1) - (\Gamma(\frac{1}{\frac{1}{2}} + 1))^2$

$V(X) = \Gamma(5) - \Gamma(3)^2$

Using $\Gamma(x) = (x-1)!$ for $x > 0$, integer $x \in \mathbb{N}$

$V(X) = 4! - 2!^2$

$V(X) = 24 - 4$

Therefore

$V(X) = 20$

Question 3

3.a

Given the bivariate distribution (X, Y) defined as

$f(x,y) = \frac{\lambda y^2}{\sqrt{2\pi}} e^{-(\frac{1}{2}+\lambda x)y^2}$ on $x > 0, y \in \mathbb{R}$

show the marginal distribution f_X is given by

$f_X(x) = \frac{\lambda}{2\sqrt{2}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}$

By definition, the marginal distribution $f_X(x)$ is given by

$$f_X(x) = \int_{A_y} f(x,y)\ dy$$

So we have

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\ dy$$

$f_X(x) = 2\displaystyle\int_{y=0}^{y=\infty} f(x,y)\ dy$; since $f(x,y)$ is symmetric about $y = 0$ on the term y^2

Using the integration by substitution method to evaluate

$\displaystyle\int f(g(x))\ g'(x)\ dx$ by putting $u = g(x)$ and $du = g'(x)\ dx$ we get $\displaystyle\int f(u)\ du$

Using the substitution of

$u = g(y) = (\frac{1}{2} + \lambda x)y^2$

$du = g'(y)\ dy$

$du = 2(\frac{1}{2} + \lambda x)y\ dy$

Checking the limits of integration, when $y = 0$ we have $u = 0$ and when $y = \infty$ we have $u = \infty$

Taking out the constant terms from the integral (terms not on y)

$$\frac{2\lambda}{\sqrt{2\pi}} \int_0^\infty y^2 e^{-(\frac{1}{2}+\lambda x)y^2} dy$$

$$\frac{2\lambda}{\sqrt{2\pi}} \int_0^\infty \frac{y^2}{2(\frac{1}{2}+\lambda x)y} e^{-(\frac{1}{2}+\lambda x)y^2}\ (2(\frac{1}{2}+\lambda x)y)\ dy$$

extracting constant terms from the integral

$$\frac{\lambda}{\sqrt{2\pi}(\frac{1}{2}+\lambda x)} \int_0^\infty \frac{y^2}{y} e^{-(\frac{1}{2}+\lambda x)y^2}\ (2(\frac{1}{2}+\lambda x)y)\ dy$$

Substituting u for $g(y)$ and du for $g'(y)\ dy$

$$\frac{\lambda}{\sqrt{2\pi}(\frac{1}{2}+\lambda x)}\int_0^\infty y\ e^{-u}\ du$$

Rearranging u for y, we get

$$y = \sqrt{\frac{u}{\frac{1}{2}+\lambda x}}$$

and replacing y for an expression of u, we get

$$\frac{\lambda}{\sqrt{2\pi}(\frac{1}{2}+\lambda x)}\int_0^\infty \sqrt{\frac{u}{\frac{1}{2}+\lambda x}}\ e^{-u}\ du$$

and after extracting constant terms from the integral (note x is constant)

$$\frac{\lambda}{\sqrt{2\pi}(\frac{1}{2}+\lambda x)\sqrt{\frac{1}{2}+\lambda x}}\int_0^\infty \sqrt{u}\ e^{-u}\ du$$

After collecting terms, so far we now have

$$f_X(x) = \frac{\lambda}{\sqrt{2\pi}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}\int_0^\infty \sqrt{u}\ e^{-u}\ du$$

Now looking at the integral term, we can make use of the Gamma Function

$$\int_0^\infty u^{\frac{1}{2}}\ e^{-u}\ du$$

The Gamma function is defined as:

$$\Gamma(a) = x^{a-1}\ e^{-x}\ dx$$

So the integral we have is equivalent to $\Gamma(\frac{3}{2})$

Evaluating using $\Gamma(a+1) = a\Gamma(a)$

and also that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Then $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

So, recombining with the whole expression, we have:

$$f_X(x) = \frac{\lambda}{\sqrt{2\pi}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}\frac{1}{2}\sqrt{\pi}$$

$$f_X(x) = \frac{\lambda}{2\sqrt{2}\sqrt{\pi}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}\sqrt{\pi}$$

Therefore giving

$f_X(x) = \frac{\lambda}{2\sqrt{2}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}$ as required.

3.b

The conditional density of $Y|X$ is defined as $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$ provided $f_X(x) > 0$

$$\frac{\frac{\lambda y^2}{\sqrt{2\pi}} e^{-(\frac{1}{2}+\lambda x)y^2)}}{\frac{\lambda}{2\sqrt{2}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}}$$

$$\frac{\lambda y^2}{\sqrt{2\pi}} \frac{2\sqrt{2}(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}{\lambda} e^{-(\frac{1}{2}+\lambda x)y^2}$$

$$\frac{y^2 2\sqrt{2}}{\sqrt{\pi}\sqrt{2}}(\frac{1}{2}+\lambda x)^{\frac{3}{2}} e^{-(\frac{1}{2}+\lambda x)y^2}$$

$$\frac{2y^2(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}{\sqrt{\pi}} e^{-(\frac{1}{2}+\lambda x)y^2}$$

Grouping constant (those that do not depend on y) and variable terms (those that depend on y), since y is variable and x is constant

$$(\frac{(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}{\sqrt{\pi}})(2y^2 e^{-(\frac{1}{2}+\lambda x)y^2})$$

where

$2y^2 e^{-(\frac{1}{2}+\lambda x)y^2}$ is the density core, depending on the variable y and

$\frac{(\frac{1}{2}+\lambda x)^{\frac{3}{2}}}{\sqrt{\pi}}$ is a constant term depending only on x (x is constant)

3.c

Given the Marginal distribution of $Y: f_Y(y)$ and the Conditional distribution $f_{X|Y=y}(x|y)$

We can use the alternative forms of Bayes's theorem to express the conditional $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) \propto f_{X|Y}(x|y) f_Y(y)$$

This allows us to express the conditional density $f_{Y|X}$ from the marginal f_Y and conditional $f_{X|Y}$

(Since we only require the density *core*, we can omit the calculation of the scaling factor constant here, which would be needed to turn the core into a PDF).