Group Actions and the Fundamental Group

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0 Introduction

It can be understood that two connected bands are inseparable without creating a break in one of them. This is intuitively obvious, however, mathematically proving such a thing becomes increasingly arduous. Once such a concept is extended in complexity, generalised to greater dimensions or given in abstract form - a mathematical presence is required. *Fundamental groups* allow mathematical proof of such a thing by considering paths on each of the bands. This paper gives the mathematical tools and theorems required to explore relationships between topological spaces and transformations of topological spaces. Section 1 outlines the theory required to construct a fundamental group, whose construction is given in Section 2. Following this a variety of heuristic and mathematically rigorous examples will be explored. Section 3 then expands the concept of fundamental groups from topological spaces to transformation groups, detailing Rhodes' findings in [7] to construct what is known today as the *Rhodes group.* Theorems surrounding Rhodes groups will be given with proof and subsequently implemented in an array of examples.

A precursory knowledge in algebraic structures, group theory, metric spaces and topology is beneficial (but not fundamental) to the reader's understanding.

1 Algebraic Topology

Topology is the mathematical study concerning the preservation of properties through deformations, twistings and stretchings of spatial objects. It is used to abstract the inherent connectedness of objects without considering their detailed forms, where the objects of topology are formally defined as topological spaces. Combining this with the study of abstract algebra forms algebraic topology, allowing the global properties of spaces to be analysed with algebra; this forms a highly applicable field of mathematics with pertinence in robotics, molecular biology and computer sciences.

1.1 Introduction to Topology

A topological space (X, τ) is a set X endowed with a topology τ - similar to the notion of a set M being endowed with a metric d to form a metric space (M, d). The elements of τ are subsets of X and by definition constitute all open subsets of X. Elements of τ follow these axioms:

- The empty set and X itself are open;
- Arbitrary unions of open sets are open;
- Finitely many intersections of open sets is open.

Complements of open sets are, by definition, closed; from this definition the notion of a both open and closed set (often described as clopen) is much more easily understood than in studies such as metric spaces. By notational convention the topology τ of the topological space (X, τ) is omitted so that the topological space is simply denoted X.

Definition 1.1.1. Let X be a collection of points $x \in X$ and τ_i a function assigning each point in X a collection of non-empty subsets $\tau_i(x)$ of X. Each element of $\tau_i(x)$ is known as a *neighbourhood* of x. Should the following axioms hold, then $\tau = \cup\tau_i$ is known as a *neighbourhood topology* (or simply, topology):

- For all neighbourhoods τ_i of x, $x \in \tau_i$;
- If τ_i is a subset of X and contains a neighbourhood of x then τ_i is a neighbourhood of X;
- The intersection of any two neighbourhoods of x is also a neighbourhood of x;
- All neighbourhood τ_i of x contain a neighbourhood $\tau_i' \subset \tau_i$ of x such that τ_i is a neighbourhood of any point in τ_i'.

Hence, X endowed with the topology τ is known as a *topological space*, (X, τ).

Without any prior context, a 'natural' topology (in an informal sense) is simply the topology best suited to the situation. In a Euclidean space $X \subseteq \mathbb{R}^n$ the open sets of X are usually constructed from topological balls, which are any subsets of X homeomorphic to a (open or closed) Euclidean ball - as defined by the study of metric spaces. Thus the natural topology τ of $X \subseteq \mathbb{R}^n$ is the set of all open balls. Throughout this paper the topology associated with the space will be assumed natural.

The concept of continuity of a function in a topological space cannot be defined by the $\epsilon - \delta$ definition as used in metric spaces since there is no notion of distance in a topological space; we are left to define such a thing with the only information we have from the topological space, the topology itself, and with that - the open sets.

Definition 1.1.2. A function $f : X \to Y$ between topological spaces (X, τ), (Y, ν) is said to be *continuous* if for every $V \in \nu$, the inverse image defined by,

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\},$$

is an element of τ. That is, $f^{-1}(V) \subseteq X$ is open for all open sets $V \subseteq Y$.

1.2 Paths in Topological Spaces

Let X be a topological space and $I \subseteq \mathbb{R}$ be the unit interval. A continuous function $\sigma : I \to X$ is a path in X with initial point $\sigma(0)$ and terminal point $\sigma(1)$, known as the end points of σ. The reverse path $\bar{\sigma} : I \to X$ of σ is found by following the path σ backwards; it has initial point $\sigma(1)$ and terminal point $\sigma(0)$ and is defined,

$$\bar{\sigma}(s) = \sigma(1 - s). \tag{1}$$

Figure 1: A path σ with its corresponding reverse path, $\bar{\sigma}$.

The reverse path is analogous to the concept of an inverse. A path σ is called a loop if $\sigma(0) = \sigma(1) = x_0$, in which case the notion of initial and terminal points is replaced simply by the path's base point, x_0. A special case loop is the constant path ε_{x_0} defined such that $\forall s \in I$, $\varepsilon_{x_0}(s) = x_0$. This is analogous to the concept of an identity element.

Proposition 1.2.1. *The product of two paths σ, τ in X with $\sigma(1) = \tau(0)$ given by,*

$$\sigma * \tau(s) = \begin{cases} \sigma(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ \tau(2s-1) & \text{if } \frac{1}{2} < s \leqslant 1, \end{cases}$$

is well-defined and continuous by the Gluing Lemma, given as Lemma 7.3 of [4]. Hence, a path $\sigma * \tau$ with initial point $\sigma(0)$ and terminal point $\tau(1)$ is defined.

The Gluing Lemma, often referred to as the *Pasting Lemma*, is crucial in constructing the fundamental group as it allows the concatenation of two paths to form another. The product of paths $\sigma * \tau$ can be thought of heuristically as travelling first along the path σ at twice speed and then along τ at twice speed. It follows directly that $\sigma * \bar{\sigma}$ and $\bar{\sigma} * \sigma$ are loops based at $\sigma(0)$ and $\sigma(1)$ respectively; in particular, these loops are the constant paths ε_{x_0} and ε_{x_1}, respectively.

For any two points in X for which a path exists between an equivalence relation can be constructed. A path component $X_0 \subseteq X$ is an equivalence class under this equivalence relation and the collection of all such path components is denoted $\pi_0(X)$.

Definition 1.2.2. A topological space X is said to be *path connected* if the only path component is X itself, meaning a path always exists between any two points in X.

Definition 1.2.3. A topological space with a distinguished point is known as a *pointed space*; the distinguished point will be used as a initial point from which paths begin.

1.3 Wedge Sums

In set theory the disjoint union, $\sqcup$, is a modified union operation which distinguishes elements from their origin set by indexing. It is a simple concept to which the following example sufficiently describes.

Example 1.3.1. Suppose $U_0 = \{a, b, c\}$, $U_1 = \{a, b, d\}$. To distinguish elements of each set denote $U_0' = \{a_0, b_0, c_0\}$ and $U_1' = \{a_1, b_1, d_1\}$. Then,

$$U_0 \sqcup U_1 = U_0' \cup U_1' = \{a_0, b_0, c_0, a_1, b_1, d_1\}.$$

Applying the notion of the disjoint union to topological spaces gives rise to a space formed by equipping the natural topology (in this case known as the disjoint union topology) to the disjoint union of the underlying sets.

Definition 1.3.2. For all binary relations $\mathcal{R}$ on a space X, there exists a unique smallest equivalence relation that contains $\mathcal{R}$, known as the *equivalence closure*.

Definition 1.3.3. For pointed spaces X and Y, with base points x_0 and y_0 respectively, the wedge sum, $\vee$, is a *one-point union.* It is the quotient space,

$$X \vee Y = (X \sqcup Y)/\sim,$$

of the disjoint union of X and Y, formed by the identification $x_0 \sim y_0$ such that $\sim$ is the equivalence closure of the relation $\{(x_0, y_0)\}$. The wedge sum forms a new pointed space, for which the base point is (x_0, y_0). This can be generalised to an arbitrary number of spaces; heuristically the wedge sum joins several topological spaces at a single point, with the equivalence closure defining all base points to coalescence as a single base point.

1.4 Product Spaces

Definition 1.4.1. For groups G and H, a *word* (of length n) in G and H is a product of the form $s_1 s_2 \ldots s_n$ such that s_i is an element of either G or H for all $i \in [1, n]$. This form can be reduced; if s_i is an identity element it can be simply removed. Furthermore, if s_i, s_{i+1} are both elements of G (or H) they can be replaced by the product of s_i and s_{i+1} (since groups are closed under multiplication). The reduced form gives all words to be alternating products of elements in G and H, namely, $g_1 h_1 g_2 h_2 ... g_n h_n$ (where either of g_1 and h_n are allowed be the identity). The group of all words under this reduced form is denoted $G * H$ and is known as the *free product* of G and H.

Definition 1.4.2. Suppose H is a subgroup and N is a normal subgroup of some group G. If $G = HN = \{hn \mid h \in H,\, n \in N\}$ and $H \cap N = \{e\}$ for identity element e then G is said to be a *semidirect product* of H and N, written,

$$G = H \ltimes N.$$

Definition 1.4.3. Given any two groups G, H and a group homomorphism,

$$\zeta : G \to \mathrm{Aut}(H),$$

$$\zeta(g) = \zeta_g : H \to H \text{ for } g \in G,$$

where $\mathrm{Aut}(H)$ is the set of all automorphisms of H, hence the *semidirect product of G and H with respect to α*, denoted $G \ltimes_\zeta H$, is defined by the following:

- As a set $G \ltimes_\zeta H$ is the Cartesian product $G \times H$;

- Multiplication of elements in $G \ltimes_\zeta H$ is determined by $(g,h)(g',h') = (gg', h\zeta_g(h'))$ for $g, g' \in G$ and $h, h' \in H$.

The following Theorem, given by Robinson in [8] allows isomorphisms to be constructed between groups and their subgroups, which will be a highly applicable tool when calculating Rhodes Groups in Section 3.4.

Theorem 1.4.4. *Consider a group G for which N is a normal subgroup and H a subgroup. Suppose $\zeta : H \to aut(N)$ is a group homomorphism. Then N, H and ζ determine G up to isomorphism, namely,*

$$G \cong H \ltimes_\zeta N.$$

When applying theory of semidirect products with respect to a given homomorphism ζ in Section 3.4 to calculate Rhodes groups, the notation $\ltimes_\zeta$ will be simply expressed $\ltimes$ with ζ explicitly defined.

1.5 Group Actions and Orbit Spaces

Definition 1.5.1. A *group action* of a group G, for some set X, is a map $\phi : G \times X \to X$ satisfying for all $x \in X$:

- $\phi(e, x) = x$, for the identity element $e \in G$;

- $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$.

Group G is said to *act* on X. An alternative way of thinking of the group action ϕ is as a group homomorphism from G to the group of permutations of X. Collectively (X, G) is understood to be a *transformation group* of X.

The following theorem given by Emmy Noether [6] in her 1927 paper, describes the relationship between group homomorphisms, quotients and subgroups and is known as the *First Isomorphism Theorem.*

Theorem 1.5.2. *Suppose $\psi : G \to H$ is a group homomorphism between groups G and H. Then,*

- *The kernel of ψ is a normal subgroup of G;*
- *The image of ψ is a subgroup of H, and*
- *The image of ψ is isomorphic to the quotient group $G/ker(\psi)$.*

In particular, if ψ is surjective then we have H and $G/ker(\psi)$ isomorphic.

Definition 1.5.3. The *orbit* of a fixed point $x_0 \in X$ is defined as the set,

$$Gx_0 = \{gx_0 \mid g \in G\} \subseteq X,$$

which is the collection of all points in X to which x_0 can be transformed to by elements in G. The set of all orbits $\{Gx \mid x \in X\}$ is denoted X/G.

An element $g \in G$ is said to *fix* $x \in X$ if $gx = x$. Furthermore $X_0 \subseteq X$ is said to *fixed* under G if $gx = x$, $\forall g \in G$, $x \in X_0$.

Definition 1.5.4. For a point $x_0 \in X$, the *stabiliser* G_{x_0} of X is the set of all homeomorphisms $g \in G$ that fix x_0. Group G acts *freely* on X if the stabiliser of every point in X is the trivial group, $\mathbb{I} = \{e\}$. That is, $g = e$ is the only element fixing *all* points in X.

The defining properties of a group ensures the set of all orbit spaces forms a partition of X. An equivalence relation $\sim$ can be defined for points $x, x' \in X$ satisfying $gx = x'$ for some $g \in G$. This gives the orbits to be the equivalence classes under this relation, with two points being equivalent if their orbits coincide.

2 THE FUNDAMENTAL GROUP

The fundamental group $\pi_1(X, x_0)$ of a pointed space can be defined and used to analyse the space by considering all loops for a given base point. Fundamental groups allow mathematically rigorous proofs of conjectures which would otherwise be arduous to prove, such as the inseparability of two connected bands (without creating a break in a band), however intuitively obvious such a thing is.

2.1 HOMOTOPIES AND GENERATING THE FUNDAMENTAL GROUP

Definition 2.1.1. The paths σ, τ in X with initial points $\sigma(0) = \tau(0) = x_0$ and terminal points $\sigma(1) = \tau(1) = x_1$ are *homotopic* if there exists a continuous function $H : I^2 \to X$ such that,

$$\begin{aligned} H(s,0) &= \sigma(s), \\ H(s,1) &= \tau(s), \\ H(0,t) &= x_0, \\ H(1,t) &= x_1, \end{aligned} \tag{2}$$

for all $s, t \in I$. This function is a *homotopy* between paths and is written $H : \sigma \sim \tau$.

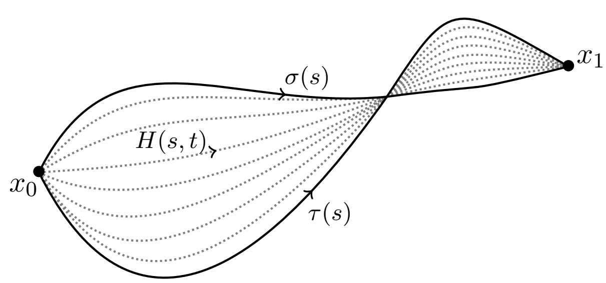


Figure 2: A homotopy H between paths σ and τ with end points x_0 and x_1.

Example 2.1.2. For any path σ with initial point x_0, a homotopy $H : \sigma * \bar{\sigma} \sim \varepsilon_{x_0}$ exists.

Choose,

$$H(s,t) = \begin{cases} \sigma(2(1-t)s), & \text{for } 0 \leqslant s \leqslant \frac{1}{2}, \\ \sigma(2(1-t)(1-s)), & \text{for } \frac{1}{2} < s \leqslant 1. \end{cases}$$

Since the formulae coincide at $s = 1/2$, H is well-defined and by the Gluing Lemma H is also continuous. Simple substitution shows (2) is satisfied and hence, the homotopy holds. Suppose σ has terminal point x_1. Then a similar homotopy exists to show $\bar{\sigma} * \sigma \sim \varepsilon_{x_1}$.

Example 2.1.3. For any loop σ with base point x_0, there exists homotopies such that $\varepsilon_{x_0} * \sigma \sim \sigma \sim \sigma * \varepsilon_{x_0}$. Define the function $H : \varepsilon_{x_0} * \sigma \sim \sigma$ by,

$$H(s,t) = \begin{cases} x_0, & \text{for } 0 \leqslant s \leqslant (1-t)/2, \\ \sigma(\frac{s-(1-t)/2}{1-(1-t)/2}), & \text{for } (1-t)/2 < s \leqslant 1. \end{cases}$$

With a similar argument as in Example 2.1.2, H forms a homotopy. A similar function can be defined to show there also exists a homotopy $H' : \sigma \sim \sigma * \varepsilon_{x_0}$, giving the required result.

An equivalence relation $\sim$ on homotopic paths in X can be constructed, with the equivalence class, or homotopy class, $[\sigma]$ being the collection of all paths homotopic to σ. Define the binary operation $*$ by $[\sigma] * [\tau] = [\sigma * \tau]$ with the product of paths defined in Proposition 1.2.1. The collection of all homotopy classes of loops base point x_0 in X forms a group $\pi_1(X, x_0)$ under this binary operation.

From the definition of reverse paths (1) it follows immediately that,

$$[\sigma]^{-1} = [\bar{\sigma}]. \tag{3}$$

As a result from Example 2.1.3, for any loop σ with base point x_0,

$$[\varepsilon_{x_0}][\sigma] = [\varepsilon_{x_0} * \sigma] = [\sigma] = [\sigma * \varepsilon_{x_0}] = [\sigma][\varepsilon_{x_0}]. \tag{4}$$

Group axioms for $\pi_1(X, x_0)$ can be easily checked, with inverse and identity $[\varepsilon_{x_0}]$ defined in (3) and (4) respectively. Hence the fundamental group $\pi_1(X, x_0)$ of the topological space X, with base point x_0 is defined.

Definition 2.1.4. If a topological space is path connected and every pair of paths between two points can be continuously transformed into one another (in particular, by a homotopy) then the space is said to be *simply connected*.

Proposition 2.1.5. *The fundamental group of a path connected pointed space is trivial if and only if the space is simply connected.*

Proof. The proof of this comes almost immediately from the definition of both trivial fundamental groups and simply connected spaces. Suppose the fundamental group of the pointed space X with base point x_0 is trivial. That is, $\pi_1(X, x_00 = \mathbb{I} = \{[\varepsilon_{x_0}]\}$. Therefore any loop in X is homotopic to ε_{x_0}, which in turn means all loops are homotopic.

Conversely, for simply connected spaces all loops σ with base point x_0 are homotopic to the constant path, ε_{x_0}, meaning $[\varepsilon_{x_0}] = [\sigma]$ and hence, $\pi_1(X, x_0) = \{[\varepsilon_{x_0}]\} = \mathbb{I}$. □

Definition 2.1.6. Consider topological spaces X and Y and suppose $\gamma : I \to X$ is a path in X and $p : Y \to X$ is continuous. Then $\tilde{\gamma} : I \to Y$ is a *lift* of γ if $\tilde{\gamma} \circ p = \gamma$. In this instance $\tilde{\gamma}$ is a path in Y; essentially a lift is a continuous function $\tilde{\gamma}$ such that the following diagram commutes.

$$\begin{tikzcd} I \arrow[r, "\gamma"] \arrow[dr, dotted, "\tilde{\gamma}"'] & X \\ & Y \arrow[u, "p"'] \end{tikzcd}$$

Figure 3: Commutatitivity diagram for a lift $\tilde{\gamma}$ of a path γ.

It is said that p has the *path-lifting property* if for all paths γ in X there exists a lift $\tilde{\gamma}$ in Y.

The Path-Lifting Theorem given by Kosniowski (1980) [5, Theorem 16.4] applies path lifts to calculate the fundamental group of a circle, described in Example 2.2.3. These results allow fundamental groups to be calculated by considering a lift of the space; this method is also applied in Example 2.2.11.

2.2 CALCULATING FUNDAMENTAL GROUPS

In the following examples the notation $\pi_1(X, x_0)$ for the fundamental group of the pointed space X with base point x_0 will be simplified to $\pi_1(X)$ since we are interested in fundamental groups up to isomorphism, unless otherwise stated.

Example 2.2.1. Consider the pointed space $\mathbb{R}$ with arbitrary points $x, x' \in \mathbb{R}$. Then the path defined by $\gamma(s) = (1-s)x + sx'$ proves the space is path connected since the path holds for all $x, x' \in X$. Consider also the path $\sigma(s) = (1-s)y + ty'$ between arbitrary points $y, y' \in \mathbb{R}$; the function defined by $H(s,t) = (1-t)\gamma(s) + t\sigma(s)$ satisfies the homotopy criteria, meaning all paths are homotopic. Hence the space is simply connected, meaning,

$$\pi_1(\mathbb{R}) \cong \mathbb{I}.$$

Example 2.2.2. Consider a unit closed 2-disc, $D^2 = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leqslant 1\}$.

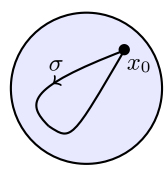


Figure 4: The 2-disc, D^2, in the Euclidean plane with loop σ on its surface.

Clearly in this space any loop σ can be continuously contracted, by homotopy, to the constant path ε_{x_0}. Meaning of course D^2 is simply connected and hence has a trivial fundamental group. This is easily generalised to an n-disc of arbitrary radius, including the whole Euclidean space $\mathbb{R}^n$.

The circle, S^1, is perhaps the most simple space with a non-trivial fundamental group, giving the circle great intrinsic value from which applications immediately stem. The fundamental group $\pi_1(S^1)$ is deduced by results from the Path-Lifting Theorem [5, Theorem 16.4] and the Monodromy Theorem [5, Corollary 16.6] given by Kosniowski (1980).

Example 2.2.3. Choose $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and consider a loop σ in S^1 with a fixed base point. Let $p : \mathbb{R} \to S^1$ be defined as the exponential map $x \mapsto \exp(2\pi i x)$. Then the Path-Lifting Theorem admits a unique lift $\tilde{\sigma} : I \to \mathbb{R}$ of σ such that $p \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = 0$. We note that $p \circ \tilde{\sigma}(1) = 1$ and hence, $\tilde{\sigma}(1) \in p^{-1}(1) = \mathbb{Z}$. Thus we define the degree of σ as $deg(\sigma) = \tilde{\sigma}(1)$. Combined with the Monodromy Theorem which states that two homotopic loops in S^1 have the same degree, there exists a group isomorphism $\phi : \pi_1(S^1) \to \mathbb{Z}$ defined by $\phi([\sigma]) = deg(\sigma)$. Hence we arrive at the conclusion, $\pi_1(S^1) \cong \mathbb{Z}$, the additive group of integers.

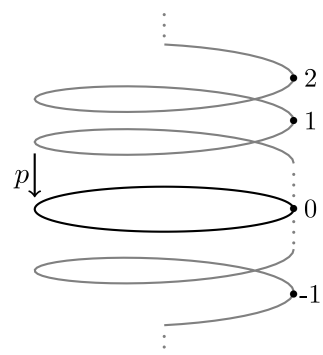


Figure 5: Pictorial representation of the Path-Lifting Theorem. From a fixed base point, looping around the circle anticlockwise and clockwise n times corresponds to the integer values n and $-n$ respectively. The exponential function as defined is denoted p.

Definition 2.2.4. A subspace A of the topological space X is known as a *retract* of X if a continuous map $r : X \to A$ exists such that $r(a) = a$ for all $a \in A$. This map is known as a *retraction* of X onto A.

Furthermore, A is known as a *deformation retract* of X if there exists a continuous map $H : X \times I \to X$, known as a *deformation retraction*, such that for all $x \in X$ and $a \in A$,

$$H(x, 0) = x,$$

$$H(x, 1) \in A,$$

$$H(a, 1) = a.$$

Clearly a deformation retract is a special case of a homotopy equivalence. Should the third condition be extended to $H(a,t) = a$ for all $t \in I$ then H is a *strong deformation retract*.

Heuristically, deformation retracts (continuously) shrink spaces into one of their subspaces. Application of this allows fundamental groups of spaces to be associated with their deformation retracts.

Example 2.2.5. Consider the surface of an uncapped cylinder $\mathcal{C}$ of unit height (without loss of generality) as our pointed space. The circle is a deformation retract of this by the deformation retraction $H : \mathcal{C} \times [0,1] \to \mathcal{C}$ defined as,

$$H((x,y,z),t) = (x,y,(1-t)z).$$

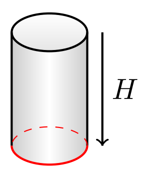


Figure 6: Deformation retract of a cylinder onto its circular base.

This deforms the cylinder by compressing its height; the cylinder is therefore homotopically equivalent to a circle,

$$\pi_1(\mathcal{C}) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Example 2.2.6. Suppose our pointed space is a unit (without loss of generality) Möbius strip, $\mathcal{M}$; although a more complicated space it can be constructed from a square $[0,1]\times[0,1]$ with the identification $(x,0) \sim (1-x,1)$, for $x \in [0,1]$, illustrated by the following figure.

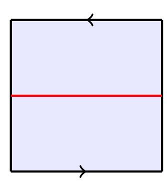


Figure 7: The Möbius strip is made by joining the identified edges so arrows match direction.

In geometric topology, any closed surface can be represented by an even sided polygon, known is a fundamental polygon, through pairwise identification of its edges. Figure 7 is such a fundamental polygon; the intersecting line (drawn in red on Figure 7), $\{(x, 1/2) \mid x \in [0, 1]\}$, forms a circle as a subspace of $\mathcal{M}$ after constructing the Möbius strip. A strong deformation retraction, $H : \mathcal{M} \times [0, 1] \to \mathcal{M}$, can be made from $\mathcal{M}$ to this circle by the mapping,

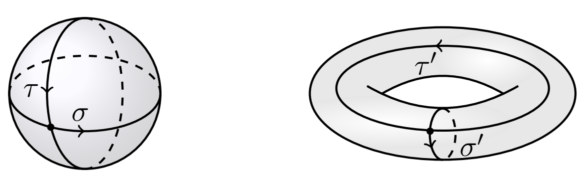
$$H((x, y), t) = (t/2 + (1 - t)x, y).$$

Similarly to Example 2.2.5 this gives the necessary homotopy between $\mathcal{M}$ and S^1, meaning,

$$\pi_1(\mathcal{M}) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Examples 2.2.6 and 2.2.5 are shown to have the same fundamental groups as a circle even though neither of the spaces are homeomorphic to a circle. Fundamental groups gives no indication to the twist in the Möbius strip space; this can be explained by the fact fundamental groups consider the similarity of loops, not spaces.

Example 2.2.7. To gain a better perspective of fundamental groups it is worth heuristically considering the sphere S^2 and the torus $T^2 = S^1 \times S^1$. Neither T^2 nor S^2 are deformation retracts of subspaces which have a known fundamental group (a torus is, however, a retract of a circle - although this says nothing about its fundamental group since it is not a *deformation* retract).



(a) The sphere S^2. (b) The torus T^2.

Figure 8: A sphere and torus embedded in $\mathbb{R}^3$, each with a pair of loops.

From Figure 8(a) it can be intuitively reasoned that any two paths σ, τ on a sphere's surface can be transformed into one another and then continuously contracted to a point.

This results in every path being homotopic to the constant path, meaning S^2 is simply connected. Hence the fundamental group of S^2 must be the trivial group, $\mathbb{I}$.

This does not apply for paths on a torus; in particular, consider the paths σ', τ' drawn in Figure 8(b). The path σ' cannot be continuously transformed into τ' since σ' would need to be temporarily cut, nor can either path be contracted to a point. This implies the torus has a richer set of distinguishable loops on its surface than previous examples.

Although not rigorous, it can be reasoned a torus has a set of loops homotopic to σ' and a set homotopic to τ'. Since each loop is homotopic to a circle, the Path-Lifting Theorem depicted in Figure 5 can be applied to each circle. Combined with the fact a torus is made from the Cartesian product of circles it is reasonable to expect the fundamental group of T^2 equates to the cross product of the fundamental group of two circles, namely,

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

It transpires, much like the case of $\pi_1(S^1)$, that each element of $\mathbb{Z} \times \mathbb{Z}$ has its first component corresponding to the number of times the loop traverses the length of the torus with the second component corresponding to the number of winds around the torus.

The intuition for the fundamental group of a torus gives motivation for Theorem 2.2.8, which proves the result. Theorems 2.2.8 and 2.2.12 allow fundamental groups to be calculated for increasingly complex spaces by considering the known fundamental groups of spaces from which they are created.

If a space is constructed by a Cartesian product of spaces for which the fundamental groups are known, the following theorem given proof by Hatcher (2002) [2, Proposition 1.12] allows simple calculation of its fundamental group.

Theorem 2.2.8. *Suppose X and Y are pointed, path connected spaces. Then there exists an isomorphism between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Example 2.2.9. Example 2.2.1 can now be generalised to $\mathbb{R}^n$, since,

$$\pi_1(\mathbb{R}^n) \cong \underbrace{\pi_1(\mathbb{R}) \times \cdots \times \pi_1(\mathbb{R})}_{n \text{ times}} \cong \mathbb{I}^n \cong \mathbb{I}.$$

In extension to this it occurs the fundamental group of any subset of $\mathbb{R}^n$ which has a straight line connecting any two points, known as a convex subset, is trivial. The theorem can also be applied gives another proof for the fundamental group of a cylinder, since it is constructed from the cross product of a circle with a convex subset of $\mathbb{R}$.

Example 2.2.10. Consider the n-dimensional torus,

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}},$$

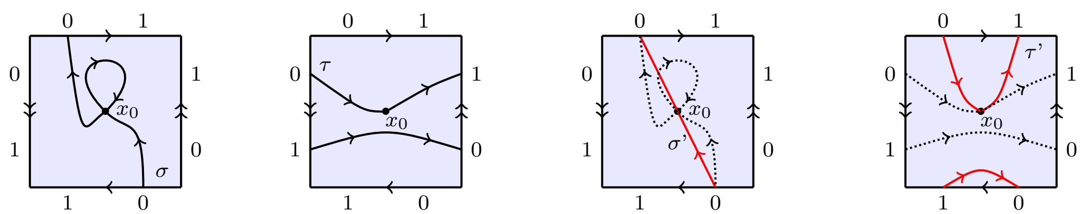
such that $n \in \mathbb{N}$. Similarly to Example 2.2.9, applying Theorem 2.2.8 with induction on n to T^n and by use of the fundamental group $\pi_1(S^1) = \mathbb{Z}$ we find $\pi_1(T^n) = \mathbb{Z}^n$. This case is trivial for $n = 1$ since $T^1 = S^1$; for $n = 2$, the intuition used for the torus in Example 2.2.7 is justified.

Example 2.2.11. A less intuitive topological space is the real projective plane, $\mathbb{RP}^2$, whose set of loops are not so easily visualised.

The real projective plane $\mathbb{RP}^2$ can be realised as the quotient space of a sphere S^2 for identified antipodal points. Namely, $x \sim -x$ for all $x \in S^2$, giving $\mathbb{RP}^2 = S^2/\sim$. We will now construct a path lifting function p by similar methodology used to equate the fundamental group of a circle (as in Example 2.2.3). Let $p : S^2 \to \mathbb{RP}^2$ be the quotient map defined by $\sim$ and suppose $\mathbb{RP}^2$, S^2 have base points x_0, x_0' respectively. Consider a non-trivial loop σ in $\mathbb{RP}^2$ (meaning σ is not homotopic to the constant loop, ε_{x_0}). The loop σ can now be *lifted* to S^2 by $\tilde{p}$; we note that the preimage of x_0 in $\mathbb{RP}^2$ under p is exactly x_0 and $-x_0$, meaning the end points of the lifted path $\tilde{\sigma}$ in S^2 will either both be at x_0' or initially at x_0' and terminating at $-x_0'$ (without loss of generality when relabelling x_0' as $-x_0'$). If, as in the first case, both end points are at x_0' then $\tilde{\sigma}$ is understood as a loop in

S^2, and since S^2 is simply connected we find $\tilde{\sigma}$ to be homotopic to the constant loop $\varepsilon_{x_0'}$. This homotopy can induce a similar homotopy between σ and ε_{x_0}, meaning σ is trivial - contradicting the non-triviality assumption of loop σ. Therefore $\tilde{\sigma}(0) = x_0'$ and $\tilde{\sigma}(1) = -x_0'$. For these end points there exists no homotopy between $\tilde{\sigma}$ and $\varepsilon_{x_0'}$ without the fixed end points being moved. Now consider travelling around the non-trivial loop σ twice, namely, $\sigma * \sigma = 2\sigma$. Hence the path is lifted to S^2 with both end points of $\widetilde{2\sigma}$ at x_0' (again without loss of generality), meaning $\widetilde{2\sigma}$ is homotopic to the constant loop ε_{x_0}. Similarly as before, this homotopy induces a homotopy between 2σ and $\varepsilon_{x_0'}$, meaning the loop 2σ is trivial. Since the above holds for any loop σ in $\mathbb{RP}^2$, the definition of the two classes of loops in $\mathbb{RP}^2$ is concluded, giving the result $\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}_2$.

Interestingly, $\mathbb{RP}^2$ has a finite non-trivial fundamental group. The loops in the fundamental group can be visualised by the fundamental polygon which associates both sides of a unit square with opposite directions from the identification $(x, 0) \sim (1 - x, 1)$ and $(0, y) \sim (1, 1 - y)$ for $x, y \in [0, 1]$ to form $\mathbb{Z}_2$, illustrated by the following figure.



(a) A loop σ in $\mathbb{Z}_2$. (b) A loop τ in $\mathbb{Z}_2$. (c) Equivalent loop to σ. (d) Equivalent loop to τ.

Figure 9: Real projective plane visualisation with arbitrarily chosen basepoint x_0.

The values 0 and 1 in Figure 9 indicate where paths continue upon meeting the boundary, as expected in a cyclic space. Figures 9(a) and 9(b) show two distinct loops σ and τ to which the loops σ' and τ' in Figures 9(c) and 9(d) are respectively homotopic.

The loop σ' is also homotopic to a loop passing only through the value 1 (as proven by the antipodal quotient map p). The bottom line segment of τ' can be contracted further until only the top segment remains, which can also then be further contracted to the origin - meaning this type of loop is equivalent to ε_{x_0}.

This theory can be generalised to $\mathbb{RP}^n$ for all $n > 1$ with the same antipodal map which is generated by $\mathbb{Z}_2$. Therefore,

$$\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2.$$

A method of calculating fundamental groups of spaces expressed as a wedge sum of spaces is by the Seifert-van Kampen Theorem, often shorted to van Kampen's Theorem. A simple version of this theorem is given by the following, with the more generalised case proven by van Kampen [3].

Theorem 2.2.12 (Seifert-van Kampen)**.** *Consider the pointed spaces X, U, and V and suppose $X = U \vee V$ such that U, V are path-connected open spaces and $U \cap V$ is simply connected. Then for the free product $*$ as defined in Definition 1.4.1,*

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V).$$

Example 2.2.13. Consider the n-sphere, S^n such that $n \geqslant 2$. Although the fundamental group of such a space could easily be shown to be trivial (through the fact S^n is simply connected) the example outlines the usage of van Kampen's Theorem.

Denote a pair of antipodal points $\mathcal{N}, \mathcal{S} \in S^n$ and assume the base point of loops is at a point other than these. Consider the open subspaces $U = S^n \backslash \{\mathcal{N}\}$ and $V = S^n \backslash \{\mathcal{S}\}$. We find U and V are homeomorphic to the n-disc, D^n, meaning both are path connected spaces and in particular are simply connected by Example 2.2.2, giving their fundamental groups both to be the trivial group, $\mathbb{I}$. The intersection $U \cap V = S^n \backslash \{\mathcal{N}, \mathcal{S}\}$ is clearly path connected also, so by van Kampen's Theorem,

$$\pi_1(S^n) \cong \pi_1(U) * \pi_1(V) \cong \mathbb{I} * \mathbb{I} \cong \mathbb{I}.$$

Van Kampen's Theorem is inapplicable for the case $n = 1$ since the intersection of U and V would not be path connected (and hence not simply connected).

Example 2.2.14. Consider the wedge product of the pointed spaces S^2 and S^1, meeting at a single point x_0. These are both path connected spaces and their intersection is trivially simply connected since $S^2 \cap S^1 = \{x_0\}$. Hence,

$$\pi_1(S^2 \vee S^1, x_0) \cong \pi_1(S^2, x_0) * \pi_1(S^1, x_0) \cong \mathbb{I} * \mathbb{Z} = \mathbb{Z}.$$

Example 2.2.15. Consider the *figure of 8* pointed space $\mathcal{S}$ with base point x_0 at a location illustrated by the following figure.

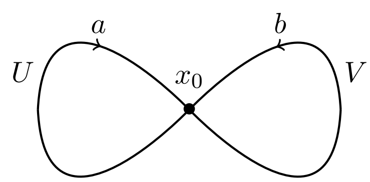


Figure 10: Figure of 8.

This space can be considered as a wedge sum of two circles U and V which form a new base point x_0, meaning van Kampen's Theorem can be applied since U and V are path connected open spaces which both contain x_0 and the intersection $U \cap V = \{x_0\}$ is (trivially) path connected. Hence van Kampen's Theorem gives,

$$\pi_1(\mathcal{S}) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

The points marked a and b in Figure 10 are generators of subsets U, V of $\mathcal{S}$. An example *word* as in Definition 1.4.1 is $a^{-1}b^{-3}a^2 \in \mathbb{Z} * \mathbb{Z}$, representing a loop which (in order) turns once around U anticlockwise then clockwise 3 times around V and finally 2 turns clockwise around U.

This can be extended to n circles meeting at a single base point (known as a *bouquet* of n circles), denoted $\mathcal{S}^n$ for $n > 1$ and by repeated use of van Kampen's Theorem we find $\pi_1(\mathcal{S}^n) \cong \mathbb{Z}^{n_*}$, where $\mathbb{Z}^{n_*}$ denotes n free products of $\mathbb{Z}$ with itself.

This example will be met again when considering Rhodes groups.

3 The Rhodes Group

The fundamental group of a transformation group was first introduced by F. Rhodes [7] in 1965; Rhodes used $\sigma(X, x_0, G)$ to denote the fundamental group of the transformation group (X, G), where X is a pointed space with base point x_0 and G is a group of self-mapping homeomorphisms on X. This fundamental group is now commonly known as the *Rhodes group* and in this paper is denoted $\pi_1^G(X, x_0)$ to more clearly indicate its association to fundamental groups. It is conceptually an extended fundamental group, however it does not consider loops on the surface of a topological space but rather paths whose terminal points are mapped to from their initial points by group actions of G.

3.1 Generating the Rhodes Group

Unless stated otherwise, henceforth assume G is a group of self-mapping homeomorphisms of the pointed space X which has base point x_0.

Definition 3.1.1. A path $\gamma : I \to X$ is of *phase* $g \in G$ if $\gamma(1) = gx_0$.

Proposition 3.1.2. *Consider the paths γ and γ' of phase g and g' respectively. These paths induce a path $\gamma * g\gamma'$ of phase gg' defined by,*

$$(\gamma * g\gamma')(s) = \begin{cases} \gamma(2s), & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ g\gamma'(2s-1), & \text{if } \frac{1}{2} < s \leqslant 1. \end{cases} \tag{5}$$

Similarly to the homotopy relation used in Section 2.1 to generate the fundamental group of a pointed space, Rhodes used a homotopy relation to construct the Rhodes group, which we will now prove is indeed a group. Let $\sim$ to denote the homotopy relation between two paths of the same phase with equal initial points. Denote the homotopy class of a path γ with phase g by $[\gamma : g]$ and let $\pi_1^G(X, x_0)$ denote the collection of all such homotopy classes. Two paths γ and γ' of distinct phases g and g' respectively have distinct homotopy classes even if $gx_0 = g'x_0$.

Theorem 3.1.3. *The set $\pi_1^G(X, x_0)$ equipped with the binary operation,*

$$[\gamma : g][\gamma' : g'] = [\gamma * g\gamma' : gg'], \tag{6}$$

defined in (5)*, forms a group.*

Proof. Suppose $\gamma_0 \sim \gamma_1$ are paths of phase g and $\gamma_0' \sim \gamma_1'$ are paths of phase g'. Then $\gamma_0 * g\gamma_1$ and $\gamma_0' * g'\gamma_1'$ are of the same phase gg' and hence $\gamma_0 * g\gamma_1 \sim \gamma_0' * g'\gamma_1'$, meaning the binary operation (6) is a well-defined rule of composition for homotopy classes of the prescribed phase. Group axioms will now be checked.
Associativity: let $[\gamma_i : g_i] \in \pi_1^G(X, x_0)$ for $i = 0, 1, 2$.

$$\begin{aligned}([\gamma_0 : g_0][\gamma_1 : g_1])[\gamma_2 : g_2] &= [\gamma_0 * g_0\gamma_1 : g_0g_1][\gamma_2 : g_2],\\ &= [\gamma_0 * g_0\gamma_1 * g_0g_1\gamma_2 : g_0g_1g_2],\\ &= [\gamma_0 : g_0][\gamma_1 * g_1\gamma_2 : g_1g_2],\\ &= [\gamma_0 : g_0]([\gamma_1 : g_1][\gamma_2 : g_2]),\end{aligned}$$

giving the associativity of $*$.
Identity: Let $e \in G$ be the identity element of G (which exists, since G is a group). We find $[\gamma * g\varepsilon_{x_0} : ge] = [\gamma : g] = [\varepsilon_{x_0} * e\gamma : eg]$ due to the fact $\gamma * g\varepsilon_{x_0}$, γ and $\varepsilon_{x_0} * e\gamma$ are homotopic paths of phase g, with homotopies similar to those described in Example 2.1.3. Therefore for any $[\gamma : g] \in \pi_1^G(X, x_0)$,

$$\begin{aligned}[\gamma : g][\varepsilon_{x_0} : e] &= [\gamma * g\varepsilon_{x_0} : ge],\\ &= [\varepsilon_{x_0} * e\gamma : eg],\\ &= [\varepsilon_{x_0} : e] * [\gamma : g],\\ &= [\gamma : g].\end{aligned}$$

Hence $[\varepsilon_{x_0} : e]$ is the identity element for $\pi_1^G(X, x_0)$.
Inverse: For $[\gamma : g] \in \pi_1^G(X, x_0)$ consider $[g^{-1}\bar{\gamma} : g^{-1}]$, which always exists since all elements of G have inverses by definition and all paths have respective reverse paths. We note that for

any $g \in G$, $g\varepsilon_{x_0}$ is of phase e since $g\varepsilon_{x_0}(0) = gx_0$ and $g\varepsilon_{x_0}(1) = gx_0 = egx_0$. Furthermore, $\gamma * \bar{\gamma} \sim \varepsilon_{x_0}$ by Example 2.1.2 and hence,

$$\begin{aligned}
[\gamma : g][g^{-1}\bar{\gamma} : g^{-1}] &= [\gamma * gg^{-1}\bar{\gamma} : gg^{-1}], \\
&= [\gamma * \bar{\gamma} : e], \\
&= [\varepsilon_{x_0} : e], \\
&= [g^{-1}\varepsilon_{x_0} : e], \\
&= [g^{-1}(\bar{\gamma} * \gamma) : e], \\
&= [g^{-1}\bar{\gamma} * g^{-1}\gamma : g^{-1}g], \\
&= [g^{-1}\bar{\gamma} : g^{-1}][\gamma : g],
\end{aligned}$$

giving the inverse $[g^{-1}\bar{\gamma} : g^{-1}]$ to any $[\gamma : g] \in \pi_1^G(X, x_0)$. Hence $\pi_1^G(X, x_0)$ is a group under the binary operation (6). □

A special case arises from this if $G = \mathbb{I} = \{e\}$, the group containing only the identity mapping. Then $[\gamma : g] = [\gamma : e]$ for all $[\gamma : g] \in \pi_1^{\mathbb{I}}(X, x_0)$, meaning all elements of $\pi_1^{\mathbb{I}}(X, x_0)$ are simply loops in X with base point x_0. It then follows directly a bijective homomorphism exists between $\pi_1^{\mathbb{I}}(X, x_0)$ and $\pi_1(X, x_0)$ by the projection map $p^{(1)} : \pi_1^{\mathbb{I}}(X, x_0) \mapsto \pi_1(X, x_0)$ defined as $p^{(1)}([\gamma : e]) = [\gamma]$.

Theorem 3.1.4. *The fundamental group $\pi_1(X, x_0)$ is a normal subgroup of $\pi_1^G(X, x_0)$.*

Proof. Since $\mathbb{I} \subseteq G$ and $\pi_1^{\mathbb{I}}(X, x_0) \cong \pi_1(X, x_0)$, we find $\pi_1(X, x_0)$ must be a subgroup of $\pi_1^G(X, x_0)$. Consider the kernel of the projection map $p^{(2)} : \pi_1^G(X, x_0) \to G$ defined by $p^{(2)} : [\gamma : g] \mapsto g$. We note that for any $[\gamma : g]$, $[\gamma' : g'] \in \pi_1^G(X, x_0)$,

$$p^{(2)}([\gamma : g][\gamma' : g']) = p^{(2)}([\gamma * g\gamma' : gg']) = gg' = p^{(2)}([\gamma : g])p^{(2)}([\gamma' : g']), \tag{7}$$

$$p^{(2)}([\varepsilon_{x_0} : e]) = e, \tag{8}$$

and finally,

$$p^{(2)}([\gamma : g]^{-1}) = p^{(2)}([g^{-1}\gamma : g^{-1}]) = g^{-1} = (p^{(2)}([\gamma : g]))^{-1}. \tag{9}$$

Hence (7), (8) and (9) show that $p^{(2)}$ is a homomorphism. Consider now,

$$ker(p^{(2)}) = \{[\gamma : e] \mid \gamma \in \pi_1(X, x_0)\} = \pi_1^{\mathbb{I}}(X, x_0) = \pi_1(X, x_0).$$

Hence by the First Isomorphism Theorem (Theorem 1.5.2) gives the kernel of a homomorphism to be a normal subgroup, giving our result. □

Corollary 3.1.5. *If the path connected pointed space X has a trivial fundamental group then its Rhodes group is isomorphic to the group which acts on it, namely $\pi_1^G(X) \cong G$.*

Proof. Assume the path connected pointed space X has a trivial fundamental group. The projection map $p^{(2)}$ defined in the proof of Theorem 3.1.4 gives a homomorphism between $\pi_1^G(X, x_0)$ and G. The image of $p^{(2)}$ is G itself meaning, by the First Isomorphism Theorem, $\pi_1^G(X)$ and $G/ker(p^{(2)})$ are isomorphic; since the kernel of $p^{(2)}$ is $\pi_1(X) = \mathbb{I}$ by assumption we hence have $G/ker(p^{(2)}) = G/\mathbb{I} = G$, acquiring our result. □

The fundamental group of (X, G) is only dependent on the choice of path component $X_0 \in \pi_0(X)$ and the elements $g \in G$ such that $\forall x_0 \in X_0$, $gx_0 \in X_0$. We will denote $G_0 \subseteq G$ to be the collection of all such elements g. It follows immediately $\pi_1^G(X, x_0) \cong \pi_1^{G_0}(X_0, x_0)$. Henceforth we assume $\pi_0(X) = \{X\}$ and $G_0 = G$ (unless otherwise stated).

3.2 Rhodes Groups and Product Spaces

Rhodes [7, Theorem 8] considered the effect of the Cartesian product on transformation groups, formulating the following theorem.

Theorem 3.2.1. *Consider for pointed spaces X, Y (with base points x_0 and y_0 respectively) the transformation groups (X, G), (Y, H). A transformation group $(X \times Y, G \times H)$ can be constructed by the Cartesian product $\times$ such that $(g, h)(x, y) = (gx, hy)$ for $g \in G$, $h \in H$, $x \in X$ and $y \in Y$, where $X \times Y$ is a pointed space with base point (x_0, y_0). We find the*

Rhodes group of such a space is isomorphic to the Cartesian product of the Rhodes groups of the spaces from which it is constructed, namely,

$$\pi_1^{G\times H}(X \times Y) \cong \pi_1^G(X) \times \pi_1^H(Y).$$

From this we can consider a single group G acting simultaneously on both X and Y, which is not identical to the case of $G = H$ in Theorem 3.2.1. There is a subtle difference: in general $\pi_1^G(X) \times \pi_1^G(Y)$ has a greater cardinality than $\pi_1^G(X \times Y)$ since, for $[\gamma : g] \in \pi_1^G(X)$ and $[\sigma : g'] \in \pi_1^G(Y)$, we find $([\gamma : g], [\sigma : g']) \in \pi_1^G(X \times Y)$ if and only if $g = g'$. The following lemma does however give a result for such a case, whose proof comes immediately.

Lemma 3.2.2. *Consider the group action G acting independently on each of the pointed spaces X and Y (with base points x_0 and y_0 respectively). Consider the Cartesian product of these transformation groups,*

$$\pi_1^G(X, x_0) \times \pi_1^G(Y, y_0) = \\ \{([\gamma : g], [\sigma : h]) \mid g, h \in G \textit{ and for appropriate paths } \gamma, \sigma \textit{ in } X, Y \textit{ respectively}\},$$

and let the subgroup $\Pi < \pi_1^G(X, x_0) \times \pi_1^G(Y, y_0)$ be such that $g = h$ in this definition, that is,

$$\Pi = \{([\gamma : g], [\sigma : g]) \mid g \in G \textit{ and for appropriate paths } \gamma, \sigma \textit{ in } X, Y \textit{ respectively}\}.$$

Then the fundamental group of the transformation group $(X \times Y, G)$ is isomorphic to Π, namely,

$$\pi_1^G(X \times Y, (x_0, y_0)) \cong \Pi.$$

If a fixed point $x_0 \in X$ under action G exists then the loop γ with base point x_0 will be mapped to another loop by G which is also based at x_0, since $gx_0 = x_0$. We therefore find G maps $\pi_1(X, x_0)$ to $\pi_1^G(X, x_0)$ particularly by homomorphisms $g \in G$. This motivates the following theorem, which forms a relation between Rhodes groups and fundamental groups through semidirect products.

Theorem 3.2.3. *For the path connected pointed space X, if there exists an $x_0 \in X$ such that $Gx_0 = \{x_0\}$ then,*

$$\pi_1^G(X, x_0) \cong G \ltimes \pi_1(X, x_0),$$

where the generator of G acts on $\pi_1(X, x_0)$ by mapping between homotopy classes of $\pi_1(X, x_0)$.

Proof. Suppose $Gx_0 = \{x_0\}$ for some $x_0 \in X$ and define the map $\zeta : \pi_1^G(X, x_0) \to Aut(\pi_1(X, x_0))$, where $\zeta([\gamma : g]) = \zeta_g : \pi_1(X, x_0) \to \pi_1(X, x_0)$ which itself is defined by $\zeta_g([\gamma]) = [g\gamma]$. Clearly ζ_g is well defined since $g\gamma$ is a loop with base point x_0 in X since $g\gamma(0) = gx_0 = x_0$ and $g\gamma(1) = gx_0 = x_0$. We find ζ_g to be a group homomorphism since for homotopy classes $[\gamma], [\gamma'] \in \pi_1(X, x_0)$,

$$\begin{aligned}
\zeta_g([\gamma][\gamma']) &= [g(\gamma * \gamma')] \\
&= [g\gamma * g\gamma'] \\
&= [g\gamma][g\gamma'] \\
&= \zeta_g([\gamma]) * \zeta_g([\gamma']).
\end{aligned}$$

Since G is a subgroup of $\pi_1^G(X, x_0)$ by binary operation $g * g' = gg'$ for all $g, g' \in G$ and $\pi_1(X, x_0)$ is a normal subgroup of $\pi_1^G(X, x_0)$ by Theorem 3.1.4 we find G, $\pi_1(X, x_0)$ and ζ determine $\pi_1^G(X, x_0)$ up to isomorphism by Theorem 1.4.4, giving the required result. □

3.3 Rhodes Groups and Orbit Spaces

In Theorem 1 of Rhodes' paper on fundamental groups of transformation groups [7], Rhodes proved the following theorem which allows Rhodes groups to be discussed without specifying a base point for path connected spaces.

Theorem 3.3.1. *If x_1 belongs to the path connected component X_0 of x_0, and γ is a path from x_0 to x_1 then γ induces an isomorphism,*

$$\gamma_* : \pi_1^G(X, x_0) \to \pi_1^G(X, x_1).$$

Theorem 3.3.2. *Let (X, G) be a transformation group; if $x_1 \in GX_0$ where X_0 is the path connected component of x_0 then $\pi_1^G(X, x_0)$ and $\pi_1^G(X, x_1)$ are isomorphic.*

Proof. In view of Theorem 3.3.1 it is sufficient to prove that if $x_1 = gx_0$ then $\pi_1^G(X, x_0)$ and $\pi_1^G(X, x_1)$ are isomorphic [9]. Let $g \in G$ such that $gx_0 = x_1$ and define the map $g_* : \pi_1^G(X, x_0) \to \pi_1^G(X, x_1)$ by,

$$g_*([\gamma : h]) = [g\gamma : ghg^{-1}].$$

By definition γ is phase h, meaning $\gamma(0) = x_0$ and $\gamma(1) = hx_0$. Noting that $g\gamma(0) = gx_0 = x_1$ by assumption and $g\gamma(1) = ghx_0 = gh(g^{-1}g)x_0 = ghg^{-1}x_1$, it is clear to see g_* is well defined. Furthermore, g_* is a homomorphism since,

$$\begin{aligned}
g_*([\gamma : h][\gamma' : h']) &= g_*([\gamma * h\gamma' : hh']) \\
&= [g(\gamma * h\gamma') : ghh'g^{-1}] \\
&= [g\gamma * gh\gamma' : ghh'g^{-1}] \\
&= [g\gamma : ghg^{-1}][g\gamma' : gh'g^{-1}] \\
&= g_*([\gamma : h])g_*([\gamma' : h']).
\end{aligned}$$

To consider the injectivity of g_* suppose $g_*([\gamma : h]) = g_*([\gamma' : h'])$, implying $[g\gamma : ghg^{-1}] = [g\gamma' : gh'g^{-1}]$. By the converse of Proposition 3.1.2, $ghg^{-1} = gh'g^{-1}$ and hence $h = h'$. This implies $g\gamma$ is homotopic to $g\gamma'$ and hence γ is homotopic to γ'. Therefore $[\gamma : h] = [\gamma' : h']$, meaning g_* is indeed injective. Furthermore, g_* is surjective since for any $[\gamma' : h'] \in \pi_1^G(X, x_1)$ the choice of $[g^{-1}\gamma : h'] \in \pi_1^G(X, x_0)$ gives $g_*([g^{-1}\gamma' : g^{-1}h'g]) = [\gamma' : h']$. Therefore g_* is an isomorphism between $\pi_1^G(X, x_0)$ and $\pi_1^G(X, x_1)$. □

Rhodes [7, Theorem 4] proved the general case of the following theorem based on results given by Armstrong's paper on fundamental groups of orbits spaces [1]. Similarly to Example 2.2.11 on the fundamental group of the real projective plane the Path Lifting Theorem is implemented in its proof.

Theorem 3.3.3. *Suppose G acts freely on X. Then $\pi_1^G(X, x_0) \cong \pi_1(X/G, x_0)$.*

3.4 Calculating Rhodes Groups

Example 3.4.1. Consider any group G of self-mapping homeomorphisms of the pointed space $X = \mathbb{R}^n$. Since the fundamental group of this space is trivial by Example 2.2.9, Corollary 3.1.5 can be applied to deduce $\pi_1^G(\mathbb{R}^n) \cong G$.

Example 3.4.2. Consider the pointed space $X = S^1$ with base point x_0 and suppose G is a reflection in the antipodal line through x_0. This group is order 2 meaning G can be generated with $\mathbb{Z}_2$.

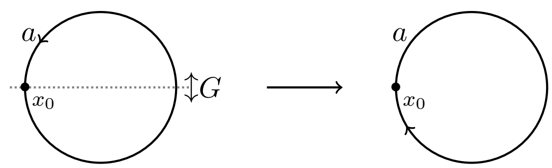


Figure 11: Circle with reflection in antipidoal line passing point x_0.

Since $Gx_0 = \{x_0\}$, Theorem 3.2.3 can be applied to give,

$$\pi_1^G(X) = \pi_1^{\mathbb{Z}_2}(S^1) \cong \mathbb{Z}_2 \ltimes \pi_1(S^1) = \mathbb{Z}_2 \ltimes \mathbb{Z}.$$

The generator of $\mathbb{Z}_2$ acts on $\mathbb{Z}$ by mapping a point $a \in \mathbb{Z}$ to $-a$, reversing direction of rotation - as illustrated in Figure 11.

The following example illustrates the difference in the Rhodes group of a circle depending on the group acting upon the pointed space.

Example 3.4.3. Consider again for the pointed space $X = S^1$, a group G which acts on S^1 by rotating the space π radians. We find there are no fixed points under this action unlike Example 3.4.2, meaning Theorem 3.2.3 cannot be applied. We note, however, G acts freely on X allowing us to apply Theorem 3.3.3. To do so we must understand what the quotient group S^1/G is, which can be easily found to be the real projective line $\mathbb{RP}^1$ which in turn is homeomorphic to the circle itself, hence giving,

$$\pi_1^G(S^1) \cong \pi_1(S^1/G) \cong \pi_1(\mathbb{RP}^1) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Such a rotation can be applied to the generalised case of the n-sphere S^n, from which we find $S^n/G = \mathbb{RP}^n$ by the antipodal map which sends a point in S^n to its antipodal opposite. Hence by the generalised case of Example 2.2.11 on the fundamental group of the real projective plane,

$$\pi_1^G(S^n) \cong \pi_1(S^n/G) \cong \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2.$$

Example 3.4.4. Consider the pointed space T^2 (the torus) with base point x_0 and suppose group G acts on this space by reflecting T^2 in one of its lines of symmetry, depicted by the following figure which illustrates T^2 by associating the two pairs of opposite sides of a square and identifying the four corners.

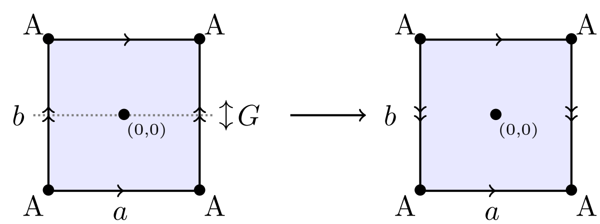


Figure 12: Fundamental polygon of torus with reflection in line of symmetry.

By 'gluing' first one pair of opposite sides of the square's surface a cylinder is made; gluing the second pair connects the ends of this cylinder, creating a torus. The point at which these two pairs of sides intersect occurs at each corner of the square, giving reason for the identification. It may be noted, then, that there are 2 fixed points; namely the four corners (which are identified to be a single point) and the origin. Consider now the orbit space of $x_0 = (0,0)$, which easily evaluates to $Gx_0 = \{x_0\}$. Since this is a fixed point Theorem 3.2.3 may be applied. We note G is order 2, meaning G can be generated as $\mathbb{Z}_2$. Hence,

$$\pi_1^G(T^2, x_0) \cong \mathbb{Z}_2 \ltimes \pi_1(T^2, x_0) \cong \mathbb{Z}_2 \ltimes (\mathbb{Z} \times \mathbb{Z}),$$

since the fundamental group of a torus is known by Example 2.2.10 to be the Cartesian product of a pair of additive groups of integers. The generator of $\mathbb{Z}_2$ acts on $\mathbb{Z} \times \mathbb{Z}$ by mapping point $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to $(a, -b)$, as illustrated in Figure 12.

Example 3.4.5. Consider again the torus T^2 with base point $x_0 = (0, 0)$ (labelled in the following figure), but choose the group G to act on T^2 by rotating the space through $\frac{\pi}{2}$ radians. This group action can be generated by $G = \mathbb{Z}_4$ since the order of G is 4.

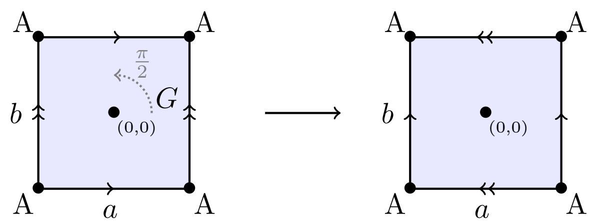


Figure 13: Fundamental polygon of torus with rotation of $\frac{\pi}{2}$ radians.

Again x_0 is a fixed point which allows application of Theorem 3.2.3 to give,

$$\pi_1^G(T^2, x_0) \cong \mathbb{Z}_4 \ltimes \pi_1(T^2, x_0) \cong \mathbb{Z}_4 \ltimes (\mathbb{Z} \times \mathbb{Z}).$$

The generator of $\mathbb{Z}_4$ acts on $\mathbb{Z} \times \mathbb{Z}$ by mapping a point $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to $(b, -a)$, as illustrated in Figure 13.

Example 3.4.6. We will now combine the group actions used in Examples 3.4.4 and 3.4.5 to reflect and then rotate (by $\frac{\pi}{2}$ radians) the pointed space $X = T^2$. This group is order 8, and is generated by the dihedral group $D_4 =< rk, r \mid k^2 = r^4 = e, (rk)^2 = e >$. This notation uses k as the reflection and r as the rotation, explaining why $r^4 = k^2 = e$ and $(rk)^2 = e$. Once again, Theorem 3.2.3 may be applied when choosing the base point x_0 of T^2 to be the origin of its fundamental polygon, as in Figure 13, since this point is fixed under G. Therefore,

$$\pi_1^G(T^2, x_0) \cong D_4 \ltimes \pi_1(T^2, x_0) \cong D_4 \ltimes \mathbb{Z} \times \mathbb{Z},$$

where the generator of D_4 acts on $\mathbb{Z} \times \mathbb{Z}$ by mapping a point $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to $(-b, -a)$.

Example 3.4.7. Consider the figure of 8 pointed space $X = \mathcal{S}$ with base point x_0 (depicted in Figure 10) and suppose the group action G reflects the space in the figure's horizontal line of symmetry. Group G is order 2, and is therefore generated by $\mathbb{Z}_2$. We find the base point is a fixed point under this group action so by application of Theorem 3.2.3 we find,

$$\pi_1^G(\mathcal{S}) \cong \mathbb{Z}_2 \ltimes \mathbb{Z} * \mathbb{Z}. \tag{10}$$

The generator of $\mathbb{Z}_2$ acts on $\mathbb{Z} * \mathbb{Z}$ by mapping $(a, b) \in \mathbb{Z} * \mathbb{Z}$ to (a^{-1}, b^{-1}). So the example word $a^{-1}b^{-3}a^2$ given in Example 2.2.15 will be mapped to $a^1b^3a^{-2}$.

Now consider the same space with the group action G instead reflecting the space in the figure of 8's vertical line of symmetry through x_0. Again we find the Rhodes group of $\mathcal{S}$ to be as given in (10), however the generator of $\mathbb{Z}_2$ will now act on $\mathbb{Z} * \mathbb{Z}$ by mapping $(a, b) \in \mathbb{Z} * \mathbb{Z}$ to (b, a), meaning our example word will be mapped to $b^{-1}a^{-3}b^2$.

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