Computational Finance : CW1

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. Exercise 1

i. $I(T) = \int_0^t B(t)\,dB(t)$ with $B(t) \sim \mathcal{N}(0,t)$. What is $\mathbb{V}[I(T)]$?

From the lectures, we know that $\boxed{I(T) = \frac{1}{2}[B(T)^2 - T]}$

Also, $\mathbb{V}[I(T)] = \mathbb{E}[I(T)^2] - \mathbb{E}[I(T)]^2$

- $I(T)^2 = \frac{1}{4}[B(T)^4 + T^2 - 2TB(T)^2]$

  so $\mathbb{E}[I(T)^2] \underset{\mathbb{E} \text{ is linear}}{=} \frac{1}{4}\left(\mathbb{E}[B(T)^4] + \mathbb{E}[T^2] - 2\mathbb{E}[TB(T)^2]\right)$

  with : $\mathbb{E}[B(T)^4] \underset{B(T)\sim\mathcal{N}(0,T)}{=} 3T^2$, $\mathbb{E}[T^2] \underset{T^2 \text{ is a constant}}{=} T^2$, $\mathbb{E}[TB(T)^2] = T\,\mathbb{V}[B(T)] = T^2$

  so $\mathbb{E}[I(T)^2] = \frac{1}{4}(3T^2 + T^2 - 2T^2) = \boxed{\frac{1}{2}T^2}$

- $\mathbb{E}[I(T)] = \frac{1}{2}\left(\mathbb{E}[B(T)^2] - \mathbb{E}[T]\right) = \frac{1}{2}(T - T) = \boxed{0}$

Finally, $\boxed{\mathbb{V}[I(T)] = \frac{1}{2}T^2}$

ii. $A = \int_{t_{i-1}}^{t_i}\left[\int_{t_{i-1}}^{s} dB(y)\right] dB(s)$ with $0 < t_{i-1} < s < t_i$

- $\int_{t_{i-1}}^{s} dB(y) = [B(y)]_{t_{i-1}}^{s} = \boxed{B(s) - B(t_{i-1})}$

Hence, $A = \int_{t_{i-1}}^{t_i}[B(s) - B(t_{i-1})]\,dB(s) = \underbrace{\int_{t_{i-1}}^{t_i} B(s)\,dB(s)}_{= C} - \underbrace{B(t_{i-1})\int_{t_{i-1}}^{t_i} dB(s)}_{= D}$

- $D = B(t_{i-1})[B(s)]_{t_{i-1}}^{t_i} = \boxed{B(t_i)B(t_{i-1}) - B(t_{i-1})^2}$

- $C = \int_0^{t_i} B(s)\,dB(s) - \int_0^{t_{i-1}} B(s)\,dB(s) = \frac{1}{2}[B(t_i)^2 - t_i] - \frac{1}{2}[B(t_{i-1})^2 - t_{i-1}]$

  $= \boxed{\frac{1}{2}[B(t_i)^2 - B(t_{i-1})^2 + t_{i-1} - t_i]}$

Finally, $A = \int_{t_{i-1}}^{t_i}\left[\int_{t_{i-1}}^{s} dB(y)\right] dB(s) = C - D$

$= \frac{1}{2}[B(t_i)^2 - B(t_{i-1})^2 + t_{i-1} - t_i] + B(t_{i-1})^2 - B(t_i).B(t_{i-1})$

$= \boxed{\frac{1}{2}[B(t_i)^2 + B(t_{i-1})^2 + t_{i-1} - t_i] - B(t_i).B(t_{i-1})}$

iii. $J_N = \sum_{k=0}^{N-1} f(t_k)[M(t_{k+1}) - M(t_k)]$ with $\left| \begin{array}{l} 0 < t_1 < t_2 < \dots < t_N < \dots \\ M(\cdot) \text{ is a martingale} \end{array} \right.$

J_N is a martingale $\Leftrightarrow \mathbb{E}[J_{N+1} - J_N | F_N] = 0$
with F_N the filtration / information generated by $M(t_k), k = 0, \dots, N$

- $J_{N+1} - J_N = \sum_{k=0}^{N} f(t_k)[M(t_{k+1}) - M(t_k)] - \sum_{k=0}^{N-1} f(t_k)[M(t_{k+1}) - M(t_k)]$

$= \boxed{f(t_N)[M(t_{N+1}) - M(t_N)]}$

- $\mathbb{E}[J_{N+1} - J_N | F_N] = \mathbb{E}[f(t_N)[M(t_{N+1}) - M(t_N)] | F_N]$ f_N is a constant not generated by $M(t_k)$

$= f(t_N) . \mathbb{E}[M(t_{N+1}) - M(t_N) | F_N]$

$= f(t_N) . 0$ because $M(\cdot)$ is a martingale

$= \boxed{0}$ so J_N is a martingale

Exercise 2

We consider : μ = constant$_1$ > 0, σ = constant$_2$ > 0, $S(0)$ = constant$_3$

and $B(t) \sim \mathcal{N}(0, t)$

i. $Y(t) = \frac{1}{3} B(t)^3$

With Ito's formula : $dY(t) = \frac{\partial Y(t)}{\partial t} dt + \frac{\partial Y(t)}{\partial B(t)} dB(t) + \frac{1}{2} \cdot \frac{\partial^2 Y(t)}{\partial B(t)^2} (dB(t))^2$

· $\frac{\partial Y}{\partial t} = 0$, $\frac{\partial Y}{\partial B} = B(t)^2$, $\frac{\partial^2 Y}{\partial B^2} = 2B(t)$ and $(dB)^2 = dt$

Hence, $\boxed{dY(t) = B(t)^2 dB(t) + B(t) dt}$

ii. $Y(t) = B(t)^2 - t$

$\frac{\partial Y}{\partial t} = -1$, $\frac{\partial Y}{\partial B} = 2B(t)$, $\frac{\partial^2 Y}{\partial B^2} = 2$ and $(dB)^2 = dt$

Hence, $dY(t) = -dt + 2B(t) dB(t) + dt = \boxed{2B(t) dB(t)}$

iii. $Y(t) = e^{B(t)}$

$\frac{\partial Y}{\partial t} = 0$, $\frac{\partial Y}{\partial B} = e^{B(t)}$, $\frac{\partial^2 Y}{\partial B^2} = e^{B(t)}$ and $(dB)^2 = dt$

Hence, $dY(t) = e^{B(t)} dB(t) + \frac{1}{2} e^{B(t)} dt = \boxed{e^{B(t)} \left[\frac{1}{2} dt + dB(t) \right]}$

iv. $Y(t) = e^{B(t) - \frac{1}{2} t}$

$\frac{\partial Y}{\partial t} = -\frac{1}{2} Y(t)$, $\frac{\partial Y}{\partial B} = Y(t)$, $\frac{\partial^2 Y}{\partial B^2} = Y(t)$ and $(dB)^2 = dt$

Hence, $dY(t) = -\frac{1}{2} Y(t) dt + Y(t) dB(t) + \frac{1}{2} Y(t) dt = \boxed{e^{B(t) - \frac{1}{2} t} \cdot dB(t)}$

v. $Y(t) = e^{[\mu - \frac{1}{2}\sigma^2]t + \sigma B(t)}$

$\frac{\partial Y}{\partial t} = (\mu - \frac{1}{2}\sigma^2) Y(t)$, $\frac{\partial Y}{\partial B} = \sigma Y(t)$, $\frac{\partial^2 Y}{\partial B^2} = \sigma^2 Y(t)$, $(dB)^2 = dt$

Hence, $dY(t) = (\mu - \frac{1}{2}\sigma^2) Y(t) dt + \sigma Y(t) dB(t) + \frac{1}{2}\sigma^2 Y(t) dt$

$= (\mu dt + \sigma dB(t)) Y(t)$

$= \boxed{(\mu dt + \sigma dB(t)) e^{[\mu - \frac{1}{2}\sigma^2]t + \sigma B(t)}}$

vi. $Y(t) = \ln(S(t))$ and $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t)$

$$dY(t) = 0\,dt + \frac{1}{S(t)} dS(t) + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)(dS(t))^2$$

$$(dS(t))^2 = (\mu S(t)\,dt)^2 + (\sigma S(t)\,dB(t))^2 + 2\mu\sigma\, S(t)^2\, dB(t)\,dt$$

$$= \boxed{\sigma^2 S(t)^2\,dt}$$

we only keep terms as big as dt: | $dt^2 = 0$, $dB(t)\,dt = 0$, $dB(t)^2 = dt$

Hence, $dY(t) = \mu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt = \boxed{[\mu - \frac{1}{2}\sigma^2]\,dt + \sigma dB(t)}$

vii. $Y(t) = \frac{1}{S(t)}$ with $\left| \begin{array}{l} S(t) = S(0)\, e^{[\mu - \frac{1}{2}\sigma^2]t + \sigma B(t)} \\ \frac{1}{S(t)} = \frac{1}{S(0)}\, e^{[\frac{1}{2}\sigma^2 - \mu]t - \sigma B(t)} \end{array}\right.$

$$dY(t) = 0\,dt - \frac{1}{S(t)^2} dS(t) + \frac{1}{2}\left(\frac{2}{S(t)^3}\right)(dS(t))^2$$

- $dS(t) = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial B} dB + \frac{1}{2}\frac{\partial^2 S}{\partial B^2}(dB)^2$

$$= [\mu - \tfrac{1}{2}\sigma^2]\, S(t)\,dt + \sigma S(t)\,dB(t) + \tfrac{1}{2}\sigma^2 S(t)\,dt$$

$$= \boxed{S(t)\,[\mu dt + \sigma dB(t)]}$$

- $\boxed{(dS(t))^2 = \sigma^2 S(t)^2\,dt}$

Hence, $dY(t) = -\frac{1}{S(t)}[\mu dt + \sigma dB(t)] + \frac{1}{S(t)}\sigma^2 dt$

$$= \frac{1}{S(t)}[\sigma^2 dt - \sigma dB(t) - \mu dt]$$

$$= \frac{1}{S(t)}[(\sigma^2 - \mu)dt - \sigma dB(t)]$$

$$= \boxed{\frac{1}{S(0)}[(\sigma^2 - \mu)dt - \sigma dB(t)]\, e^{[\frac{1}{2}\sigma^2 - \mu]t - \sigma B(t)}}$$

Exercise 3

i. Value of European Call Option (buying): $C_T = max(0, S_T - K)$ — assumption: same strike

Value of European Put Option (selling): $P_T = max(0, K - S_T)$

K: strike price (prespecified price)

S_T: price of stock at T

We want to show: $\mathbb{E}[C_T] = \mathbb{E}[P_T]$

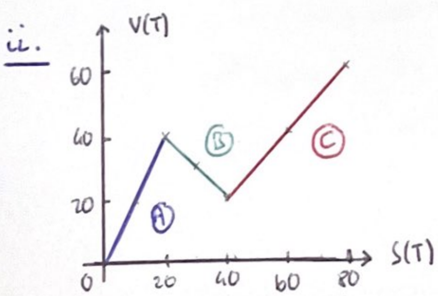
$$\Leftrightarrow \left| \begin{array}{ll} \mathbb{E}[S_T - K] = \mathbb{E}[0] & \text{if } S_T - K > 0 \\ \mathbb{E}[0] = \mathbb{E}[K - S_T] & \text{if } K - S_T > 0 \\ \mathbb{E}[S_T - K] = \mathbb{E}[K - S_T] & \text{if } K - S_T = 0 \end{array} \right.$$

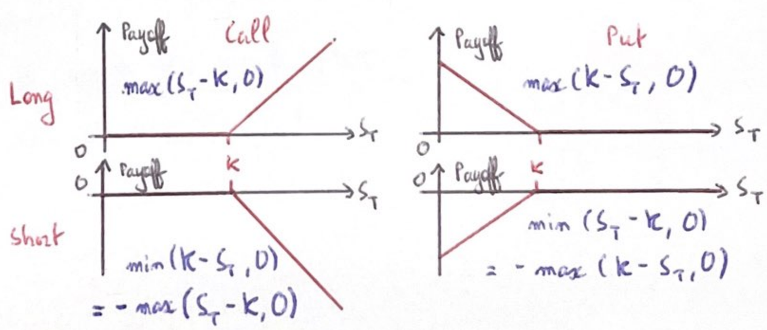
$\mathbb{E}$ is linear $\Leftrightarrow$ $\mathbb{E}[S_T] = \mathbb{E}[K]$

$\Leftrightarrow$ (K is a constant) $\boxed{K = \mathbb{E}[S_T] = \text{Forward price } (F)}$ (no arbitrage)

Hence, the values of a European call and European put options (with the same strike and underlying asset and no arbitrage) are equal if and only if the strike (K) is equal to the forward price (F).

ii.





The portfolio (π) is a linear combinations of put and calls options. It must match the graph above.

Finally, $$\boxed{V_T^{\pi} = \underbrace{2 . max(0, S_T)}_{\text{Long call for Ⓐ}} - \underbrace{3 . max(0, S_T - 20)}_{\text{short call for Ⓑ}} + \underbrace{2 . max(0, S_T - 40)}_{\text{Long call for Ⓒ}}}$$

iii. Profit is defined as : P = Payoff - Premium paid (cost of entry)

- Buy call with £35 strike and one year maturity :

$$C_1 = \max(0, S(1) - K)$$

$$P_{call} = C_1 - \text{premium paid}$$

$$= (S(1) - \overset{\text{strike price}}{K}) - \text{premium paid}$$

$$= ([40, 45] - 35) - 4 = [5, 10] - 4$$

$$= \boxed{[1, 6]\ £}$$

↑ "one year call options are trading for £4"

- Buy the asset :

$$P_{asset} = \underbrace{S(1)}_{\text{price of the stock in one year}} - \overset{\text{current price of the stock}}{S(0)} = [40, 45] - 35 = \boxed{[5, 10]\ £}$$

In terms of pure profit, the asset looks more interesting as there is a £4 difference with the profit generated by buying the call (for the same value of the stock price in one year).

However, the call may be more interesting as it is possible to buy several of them given the purchase cost (£4) and this translates into a better return on investment.

$$R_{call} = \frac{P_{call}}{\text{premium paid}} = \frac{[1, 6]}{4} = \boxed{[25, 150]\ \%}$$

$$R_{asset} = \frac{P_{asset}}{S(0)} = \frac{[5, 10]}{35} = \boxed{[14, 29]\ \%}$$

Thus, if the profit is to be one-off and immediate, buying the asset should be preferred. On the other hand, if the profit can be multiple and does not need to be immediate, it is more favourable to buy one (or more) call with a strike of £35 and one year maturity.

To grow our money (being sure of the price in the future) it is therefore more interesting to buy calls since the purchase of 1 asset corresponds approximately to the purchase of 8 calls which yield [8, 48] £ as opposed to the asset which yields [5, 10] £.