University of Washington
Department of Computer Science and Engineering
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Homework 10, Problem 4

Problem 4 (10 Points):

(Kleinberg-Tardos, Page 513, Problem 17). You are given a directed graph $G = (V, E)$ with weights w_e on its edges $e \in E$. The weights can be negative or positive. The *Zero-Weight-Cycle Problem* is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that the Zero-Weight-Cycle problem is NP-Complete. (Hint: Hamiltonian PATH)

Answer:

1) Zero-Weight-Cycle problem is in NP
Since the verification to check the sum of cycle edge weight to be zero can be done in polynomial time by traversing the cycle once, Zero-Weight-Cycle problem is in NP.

2) Subset sum $\leq_P$ Zero-Weight-Cycle problem
Given a subset sum with a set S, for each element in S, construct a graph G with edge cost equal to element value: if $S = s_1, s_2, s_3 \cdots s_n$, then the graph should have 2n vertices from $v_1 \cdots v_{2n}$. And for each element s_i in S, create a pair of two vertices v_i and v_{2*i} with edge cost of (v_i, v_{2*i}) to be s_i. And for any other pair of vertices v_a and v_b which $a \neq 2b$ or $b \neq 2a$, set the edge (v_a, v_b) cost to be 0.

By doing so, if it is possible to correctly find the Zero-Weight-Cycle on graph G, say the set of result C is $c_1, c_2 \cdots c_n$, then in order to solve the original subset sum with sum equal to k, as long as there is a edge of -k in any cycle of set C, then we can get the cost of each element in subset whose sum equal to k by keeping any non-zero edge cost and remove of others. Since if the cycle has cost zero, by removing the edge with cost of -k, we can get a path whose edge cost sum to k. And removing the irrelevant edges, the rest is the subset sum to k of the original problem.

It has been proven that Subset Sum problem is an NP complete problem (Kleinberg page 492 8.23). And according to Kleinberg page 453 8.2: Suppose Y $\leq_P$ X. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time. And also since 1) and 2), Zero-Weight-Cycle problem is NP-Complete.