University of Washington
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Homework 4 Problem 2

Problem 2 (10 points):

Let $G = (V, E)$ be a directed graph with lengths assigned to the edges. Let $\delta(u, v)$ denote the shortest path distance from u to v. Prove that for all vertices $u, v, w \in V$:

$$\delta(u, w) \leq \delta(u, v) + \delta(v, w).$$

You may assume that the graph is strongly connected, so that there is a path between every pair of vertices.

Answer:

Since $\delta(u, w)$ is the shortest path from u to w, there is only two possibilities: either (1) v on that path, or (2) v not on that path.
For (1), since $\delta(u, v)$ and $\delta(v, w)$ is all shortest path, let any $v' \neq v$ and v' not on $\delta(u, w)$, and then $dist(u, w) = dist(u, v') + dist(v', w)$, since $\delta(u, v)$ and $\delta(v, w)$ is the shortest path, there must be $\delta(u, v') + \delta(v', v) \geq \delta(u, v)$ and $\delta(v, v') + \delta(v', w) \geq \delta(v, w)$, hence there $!\exists v'$ that $\delta(u, v') + \delta(v', w) = \delta(u, v') + 2 \times \delta(v', v) + \delta(v', w) \leq \delta(u, v) + \delta(v, w)$, therefore when v on shortest path of $u - w$, $\delta(u, w) = \delta(u, v) + \delta(v, w)$.
For (2), since v not on the shortest path between u, w, then that naturally means $\delta(u, w) < \delta(u, v) + \delta(v, w)$.