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Homework 1 Problem 2

Problem 2 (10 points):

Show that the stable matching problem may have an exponential number of solutions. To be specific, show that for every n, there is an instance of stable matching on sets M and W with $|M| = |W| = n$ where there are at least c^n stable matchings, for some $c > 1$. (Hint: Suppose you have an instance of size n with k solutions, show that you can create an instance of size $2n$ with k^2 solutions.)

Answer:

Base case (i = 2, n = 2):
$\exists\ M_2, W_2 : |M_2| = |W_2| = 2 = n_2, k_2 = k;$

Induction Hypothesis:
Assume there are M_i, W_i satisfy that:
(1) $|M_i| = |W_i| = 2 \times |M_{i/2}| = 2 \times |W_{i/2}| = n;$
(2) concatenate two duplicated (and relabeling) M_i and W_i to form M_{2i} and W_{2i}; And $\forall w \in W_{i/2}$ and $m \in M_{i/2}$, append the duplicated w and m to the end of their preference list. Then the $k_i = k^{\frac{n}{2}}$

Example:
when i = 2
$M_2 =$

$$\begin{bmatrix} w_0 & w_1 \\ w_1 & w_0 \end{bmatrix}$$

$W_2 =$

$$\begin{bmatrix} m_1 & m_0 \\ m_0 & m_1 \end{bmatrix}$$

then after concatenation, when i = 2

$M_4 =$

$$\begin{bmatrix} w_0 & w_1 & w_2 & w_3 \\ w_1 & w_0 & w_3 & w_2 \\ w_2 & w_3 & w_0 & w_1 \\ w_3 & w_2 & w_1 & w_0 \end{bmatrix}$$

and
$W_4 =$

$$\begin{bmatrix} m_1 & m_0 & m_3 & m_2 \\ m_0 & m_1 & m_2 & m_3 \\ m_3 & m_2 & m_1 & m_0 \\ m_2 & m_3 & m_0 & m_1 \end{bmatrix}$$

when n is even:
Since M_i, W_i was concatenated by two duplicates of $M_{i/2}, W_{i/2}$, and since any cross duplicates matching will not be stable (each party can improve because the duplicates are lower preference of original, original are lower preference duplicates) each concatenated sub-part are independent when they combines to the instance size as a whole. The $k_i = k_{i/2}^2$

Inducted Steps
when i = 4, n = 4:
$M_4, W_4 : |M_4| = |W_4| = 4 = n_4, k_4 = k_2^2 = k^2;$

when i = 8, n = 8:
$M_8, W_8 : |M_8| = |W_8| = 8 = n_8, k_8 = k_4^2 = k^4;$

......
when i = n, n = n:
$M_n, W_n : |M_n| = |W_n| = n, k_n = k_{n/2}^2 = k^{\frac{n}{2}};$

since
$k_n = k^{\frac{n}{2}} = \sqrt{k}^n$
$\exists c > 1$ to make $k^{\frac{n}{2}} >= c^n$
assume k = 2,$\forall n > 1$, $k_n = 2^{\frac{n}{2}} = \sqrt{2}^n >> 1.01^n$
Therefore, when n is even, there are at least c^n stable matchings, for some $c > 1$.

when n is odd:
Assume $|M| = |W| = n$, consider $\forall w \in W$ appending w' to its preference list, and $\forall m \in M$ appending m' to its preference list, and w', m' has each other on top of preference. After appending, the $|W'| = |W| + 1 =$ n + 1 and $|M'| = |M| + 1$, while k does not change because w', m' will not affected the match result of $\forall m \in M$ and $\forall w \in W$, for the same reason that if $\exists m$ or $w \in M, W$ match with w' or m', either m or w' (or w and m') can improve its match (non-stable), the final match won't happen to cross w with m' or m with w', the $k_m = k_{m-1}$.

Example:

when n = 3:

$W_3 =$

$$\begin{bmatrix} w_0 & w_1 & w_2 \\ w_1 & w_0 & w_2 \\ w_2 & w_0 & w_1 \\ w_2 & w_1 & w_0 \end{bmatrix}$$

$M_4 =$

$$\begin{bmatrix} w_1 & w_0 & w_2 \\ w_0 & w_1 & w_2 \\ w_2 & w_1 & w_0 \\ w_2 & w_0 & w_1 \end{bmatrix}$$

Therefore, k of $|M| = |N| = n$ should equal to $|M| = |N| = n+1$, so we have $k_n = k^{\frac{n-1}{2}}$; Similarly, $\exists c > 1$ to make $k^{\frac{n-1}{2}} >= c^n$
assume k = 2,$\forall n > 1$, $k_n = 2^{\frac{n-1}{2}} = \sqrt{2}^{n-1} >> 1.01^n$
Therefore, when n is odd, there are at least c^n stable matchings, for some $c > 1$.