Poisson Random Variable:

- Describe a given number of events occurring in a fixed interval of time if these events occur with a known average rate and independently of the time since the last event.

Theorem 3.8

Perform n Bernoulli trials. In each trial, let the probability of success be α/n, where $\alpha > 0$ is a constant and $n > \alpha$. Let the random variable K_n be the number of successes in the n trials. As $n \to \infty$, $P_{K_n}(k)$ converges to

Proof: Theorem 3.8

We first note that K_n is the binomial $(n, \alpha n)$ random variable with PMF

$$P_{K_n}(k) = \binom{n}{k} (\alpha/n)^k \left(1 - \frac{\alpha}{n}\right)^{n-k}. \tag{1}$$

For $k = 0, \ldots, n$, we can write

$$\boxed{C(n,k) = \frac{n!}{(n-k)!\,k!}}$$

$$P_K(k) = \frac{\boxed{n(n-1)\cdots(n-k+1)}}{n^k} \frac{\alpha^k}{k!} \left(1 - \frac{\alpha}{n}\right)^{n-k}. \tag{2}$$

Notice that in the first fraction, there are k terms in the numerator. The denominator is n^k, also a product of k terms, all equal to n. Therefore, we can express this fraction as the product of k fractions, each of the form $(n-j)/n$. As $n \to \infty$, each of these fractions approaches 1. Hence,

$$\lim_{n \to \infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} = 1. \tag{3}$$

Furthermore, we have

$$\left(1 - \frac{\alpha}{n}\right)^{n-k} = \frac{\left(1 - \frac{\alpha}{n}\right)^n}{\left(1 - \frac{\alpha}{n}\right)^k}. \tag{4}$$

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

As n grows without bound, the denominator approaches 1 and, in the numerator, we recognize the identity $\lim_{n\to\infty}(1 - \alpha/n)^n = e^{-\alpha}$. Putting these three limits together leads us to the result that for any integer $k \geq 0$,

$$\lim_{n \to \infty} P_{K_n}(k) = \begin{cases} \alpha^k e^{-\alpha}/k! & k = 0, 1, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

Section 3.4

Cumulative Distribution Function (CDF)

累计分布函数

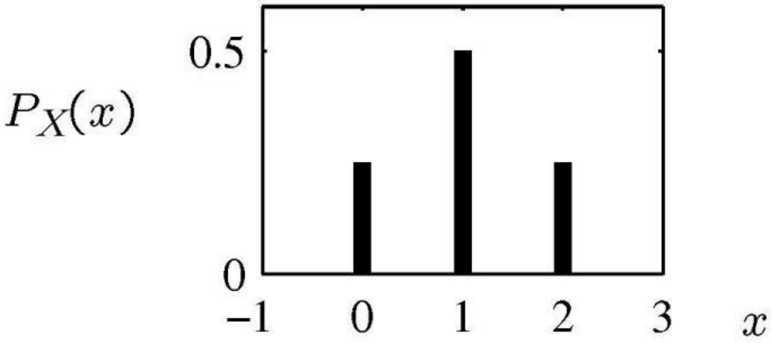
Definition 3.10 Cumulative Distribution Function (CDF)

The cumulative distribution function *(CDF) of random variable X is*

$$F_X(x) = \mathsf{P}[X \le x].$$

Example 3.21 Problem

In Example 3.5, random variable X has PMF

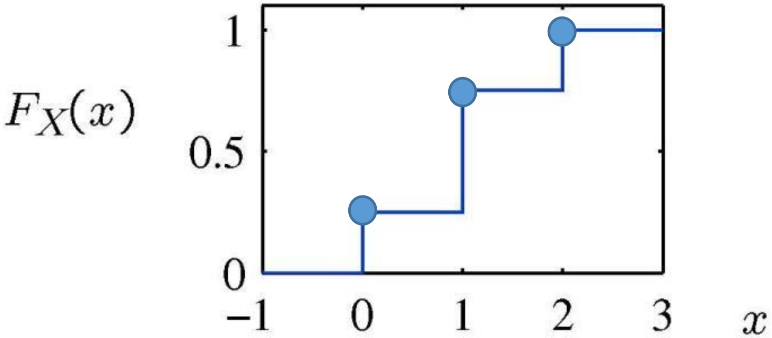


$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Find and sketch the CDF of random variable X.

Example 3.21 Solution

Referring to the PMF $P_X(x)$, we derive the CDF of random variable X:



$$F_X(x) = \mathrm{P}[X \le x] = \begin{cases} 0 & x < 0, \\ 1/4 & 0 \le x < 1, \\ 3/4 & 1 \le x < 2 \\ 1 & x \ge 2. \end{cases}$$

Keep in mind that at the discontinuities $x = 0$, $x = 1$ and $x = 2$, the values of $F_X(x)$ are the upper values: $F_X(0) = 1/4$, $F_X(1) = 3/4$ and $F_X(2) = 1$. Math texts call this the *right hand limit* of $F_X(x)$.

Theorem 3.2

For any discrete random variable X with range $S_X = \{x_1, x_2, \ldots\}$ satisfying $x_1 \le x_2 \le \ldots$,

(a) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.

(b) For all $x' \ge x$, $F_X(x') \ge F_X(x)$. Monotonically Increasing

(c) For $x_i \in S_X$ and ϵ, an arbitrarily small positive number,

$$F_X(x_i) - F_X(x_i - \epsilon) = P_X(x_i).$$

(d) $F_X(x) = F_X(x_i)$ for all x such that $x_i \le x < x_{i+1}$.

3.4 Comment: Theorem 3.2

Each property of Theorem 3.2 has an equivalent statement in words:

(a) Going from left to right on the x-axis, $F_X(x)$ starts at zero and ends at one.

(b) The CDF never decreases as it goes from left to right.

(c) For a discrete random variable X, there is a jump (discontinuity) at each value of $x_i \in S_X$. The height of the jump at x_i is $P_X(x_i)$.

(d) Between jumps, the graph of the CDF of the discrete random variable X is a horizontal line.

Example 3.22 Problem

In Example 3.9, let the probability that a circuit is rejected equal $p = 1/4$. The PMF of Y, the number of tests up to and including the first reject, is the geometric (1/4) random variable with PMF

$$P_Y(y) = \begin{cases} (1/4)(3/4)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

What is the CDF of Y?

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$$P_Y(y) = \begin{cases} (1/4)(3/4)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

What is the CDF of Y?

$$\mathrm{F}_Y(y) = P[Y \leq y]$$

$\lfloor \quad \rfloor$ Floor function

$$= \sum_{i=1}^{\lfloor y \rfloor} P_Y(i) = \sum_{i=1}^{\lfloor y \rfloor} \frac{1}{4}\left(\frac{3}{4}\right)^{i-1} = 1 - \left(\frac{3}{4}\right)^{\lfloor y \rfloor}$$