

EL9343

Data Structure and Algorithm

Lecture 4: Introduction to sorting II: HeapSort, Quicksort

Instructor: Yong Liu

Last Lecture

- ▶ Divide-and-conquer algorithms
 - ▶ maximum subarray
- ▶ Insertion sort
 - ▶ Design approach: Incremental
 - ▶ Sorts in place: Yes
 - ▶ Best case: $\Theta(n)$
 - ▶ Worst case: $\Theta(n^2)$

Introduction to Sorting: Merge Sort

▶ Bubble Sort

- ▶ Design approach: Incremental
- ▶ Sorts in place: Yes
- ▶ Running time: $\Theta(n^2)$

▶ Merge Sort

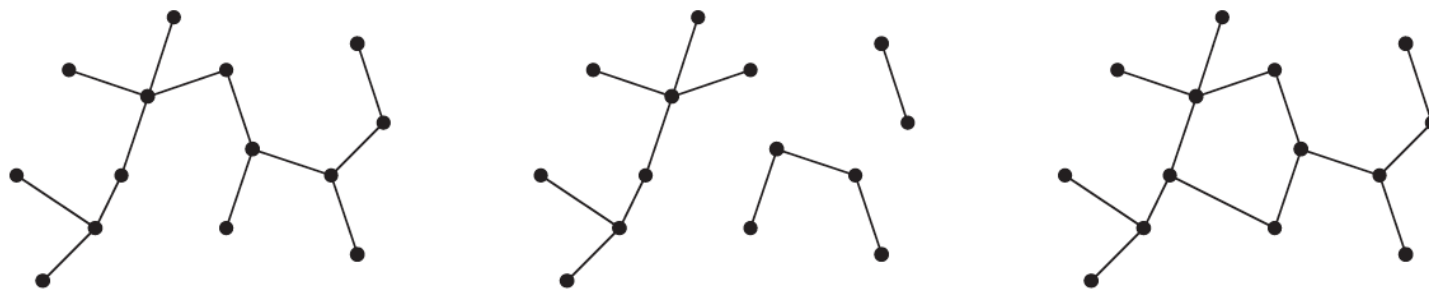
- ▶ Design approach: divide and conquer
- ▶ Sorts in place: No
- ▶ Running time: $\Theta(n \log n)$

Introduction to Sorting: HeapSort

- ▶ So far we've talked about several algorithms to sort an array of numbers
- ▶ What is the advantage of merge sort?
 - ▶ Answer: $O(n \lg n)$ worst-case running time
- ▶ What is the advantage of insertion sort?
 - ▶ Answer: sorts in place
 - ▶ Also: When array “nearly sorted”, runs fast in practice
- ▶ Next on the agenda: *Heapsort*
 - ▶ Combines advantages of both previous algorithms

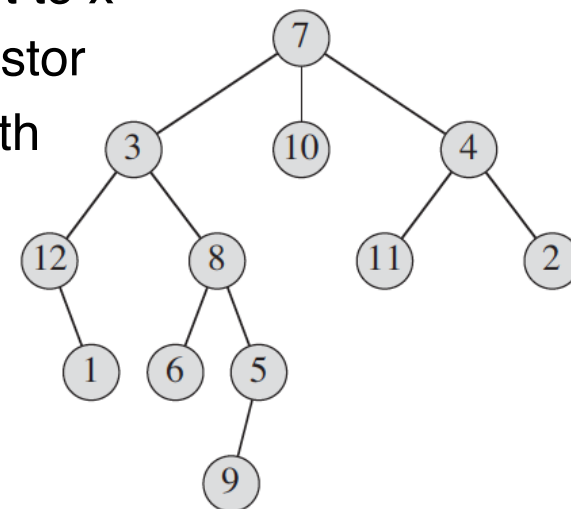
Data Structure: Tree

- ▶ **(Free) Tree**: connected, acyclic, undirected graph
- ▶ **Forest**: acyclic, undirected graph, possibly disconnected



- ▶ **Rooted Tree**: a free tree with special **root** node

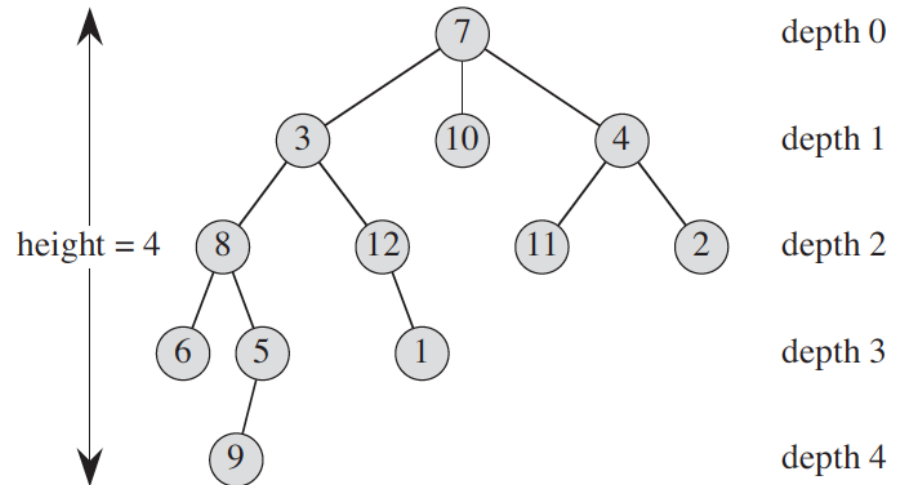
- **Ancestor** of node x: any node on the path from root to x
- **Descendant** of node x: any node with x as its ancestor
- **Parent** of node x: node immediately before x on path from root
- **Child** of node x: any node with x as its parent
- **Siblings** of node x: nodes sharing parent with x
- **Leaf/external node**: without child



- ▶ 5 ● **Internal node**: with at least one child

Data Structure: Tree

- ▶ **degree** of x: number of children
- ▶ **depth** of x: length of the simple path from root to x
- ▶ **level** of a tree: all nodes at the same depth
- ▶ **height** of x: length of the longest simple path from x downward to some leaf node
- ▶ **height of a tree**: height of root
- ▶ **Ordered Tree**: rooted tree in which children of each node are ordered



Special Types of Trees

- ▶ **Binary Tree (recursive def.)**

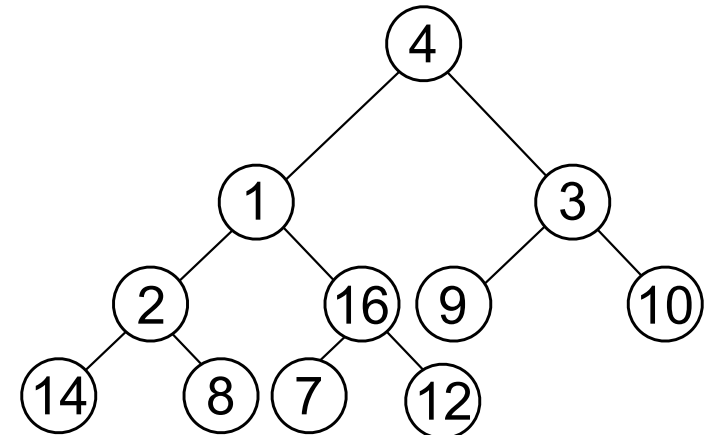
- ▶ contains no node
- ▶ root node, left subtree (binary), right subtree (binary)

- ▶ **Full binary tree**

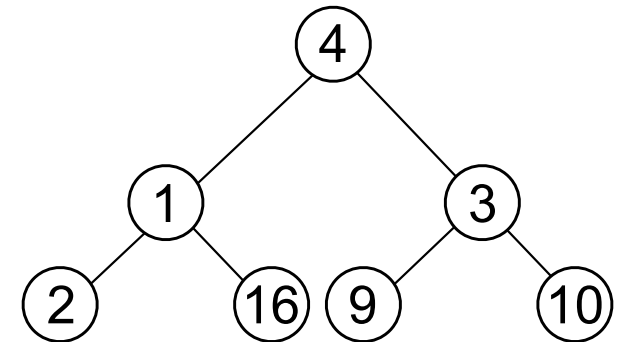
- ▶ A binary tree in which each node is either a leaf or has degree exactly 2.

- ▶ **Complete binary tree**

- ▶ A binary tree in which all leaves are on the same level and all internal nodes have degree 2.




Full binary tree

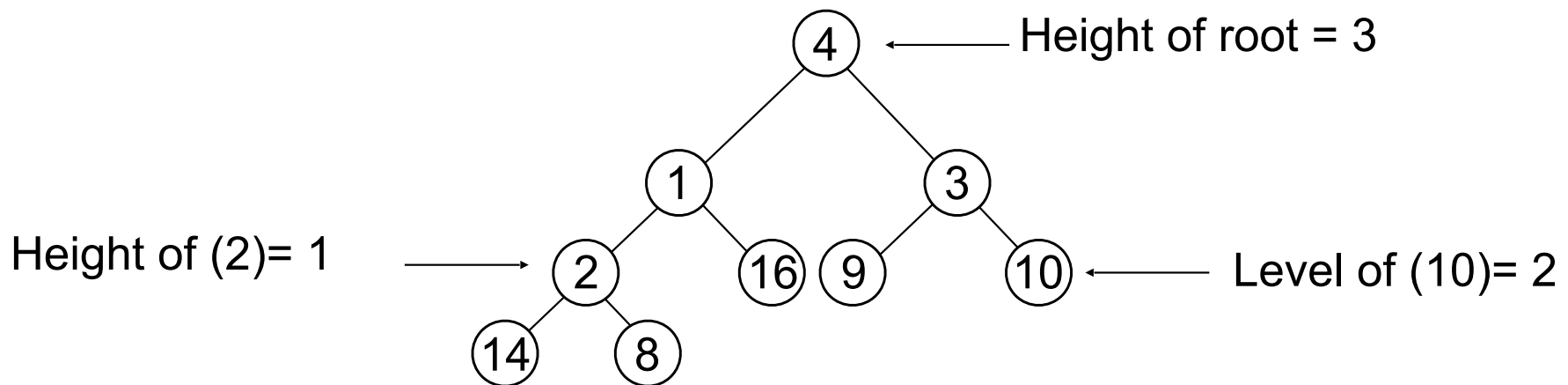


Complete binary tree

Useful Properties

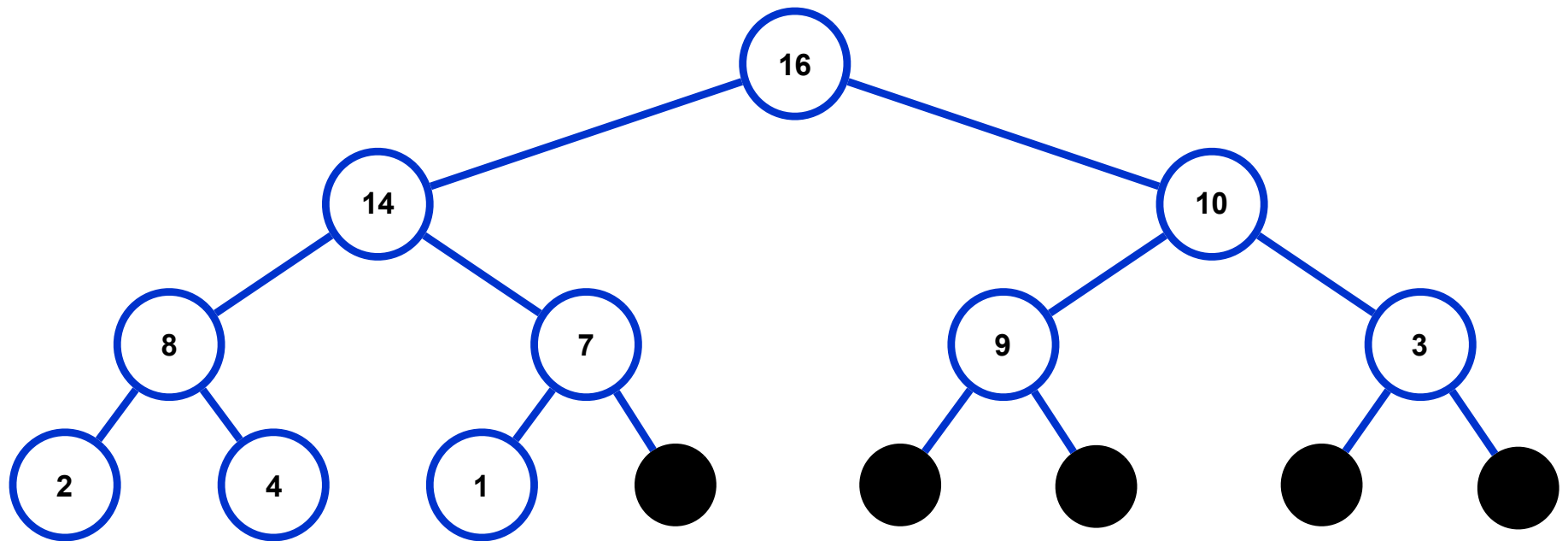
- ▶ There are at most 2^l nodes at level l of a binary tree
- ▶ A binary tree with depth d has at most $2^{d+1}-1$ nodes
- ▶ A binary tree with n nodes has depth at least $\lfloor \lg n \rfloor$


$$n \leq \sum_{l=0}^d 2^l = \frac{2^{d+1} - 1}{2 - 1} = 2^{d+1} - 1$$



Data Structure: Heap

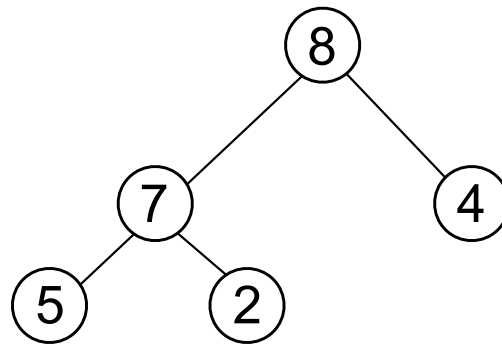
- ▶ A **heap** can be seen as a complete binary tree:



- ▶ The book calls them “nearly complete” binary trees; can think of unfilled slots as null pointers

The Heap Data Structure

- ▶ A **heap** can be seen as a complete binary tree with the following two properties:
 - ▶ **Structural property:** all levels are full, except possibly the last one, which is filled from **left to right**
 - ▶ **Order (heap) property:** for any node x : $\text{Parent}(x) \geq x$



Heap

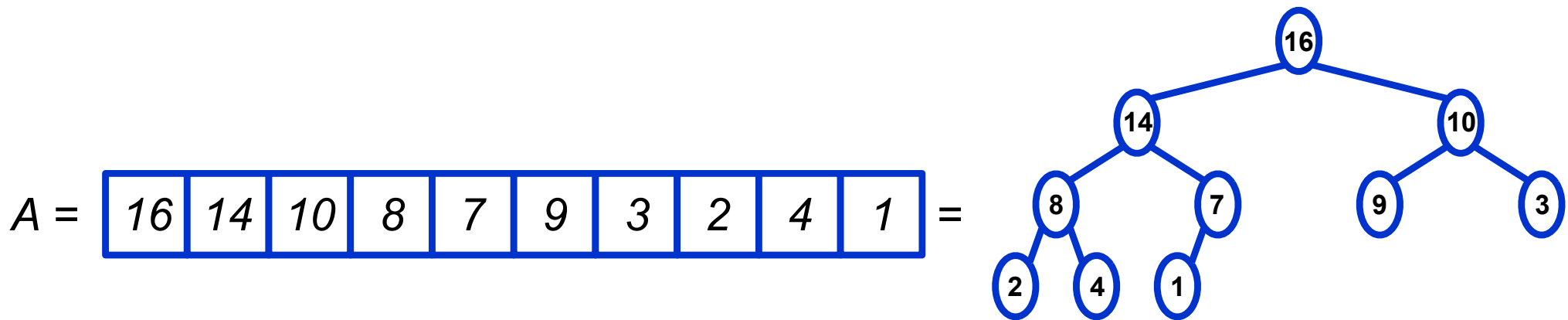
From the heap property, it follows that:

“The root is the maximum element of the heap!”

A heap is a binary tree that is filled in order

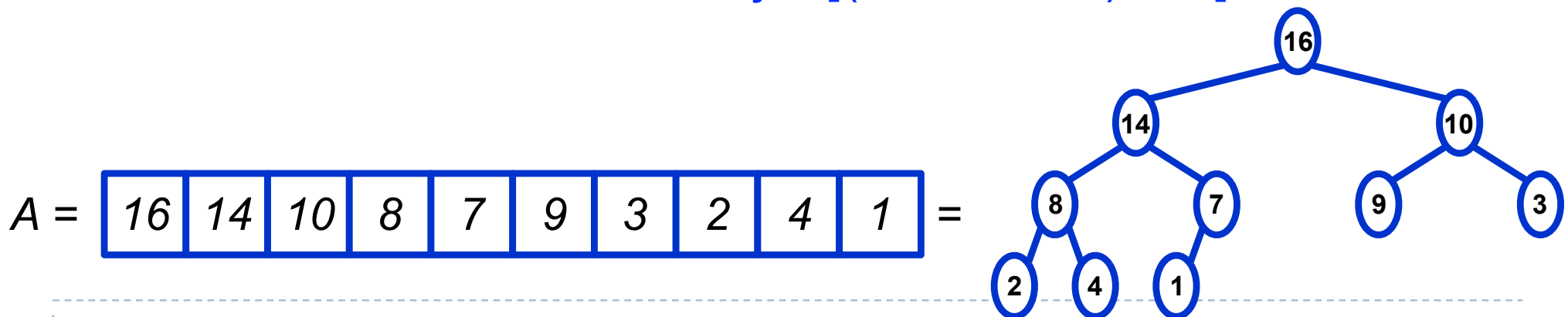
The Heap Data Structure

- ▶ In practice, heaps are usually implemented as arrays:



Array Representation of Heaps

- ▶ A heap can be stored as an array A .
 - ▶ Root of tree is $A[1]$
 - ▶ Node i is $A[i]$
 - ▶ Left child of node $i = A[2i]$
 - ▶ Right child of node $i = A[2i + 1]$
 - ▶ Parent of node $i = A[\lfloor i/2 \rfloor]$
 - ▶ $\text{Heapsize}[A] \leq \text{length}[A]$
- ▶ The elements in the subarray $A[(\lfloor n/2 \rfloor + 1) .. n]$ are leaves



Heap Types

- ▶ **Max-heaps** (largest element at root), have the *max-heap property*:

- ▶ For all nodes i , excluding the root:

$$A[\text{PARENT}(i)] \geq A[i]$$

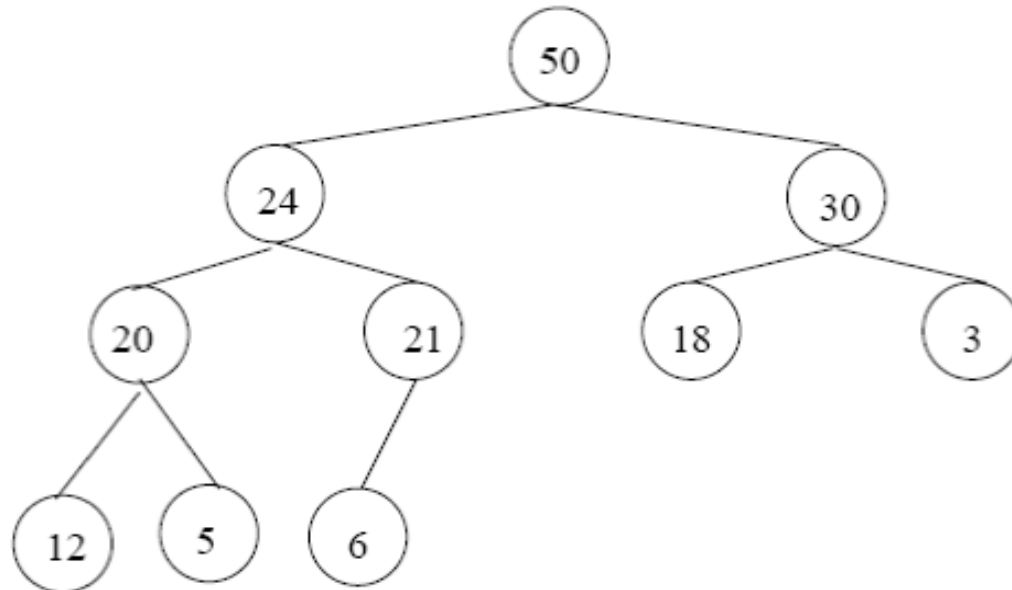
- ▶ **Min-heaps** (smallest element at root), have the *min-heap property*:

- ▶ For all nodes i , excluding the root:

$$A[\text{PARENT}(i)] \leq A[i]$$

Adding/Deleting Nodes

- ▶ New nodes are always inserted at the bottom level (left to right)
- ▶ Nodes are removed from the bottom level (right to left)



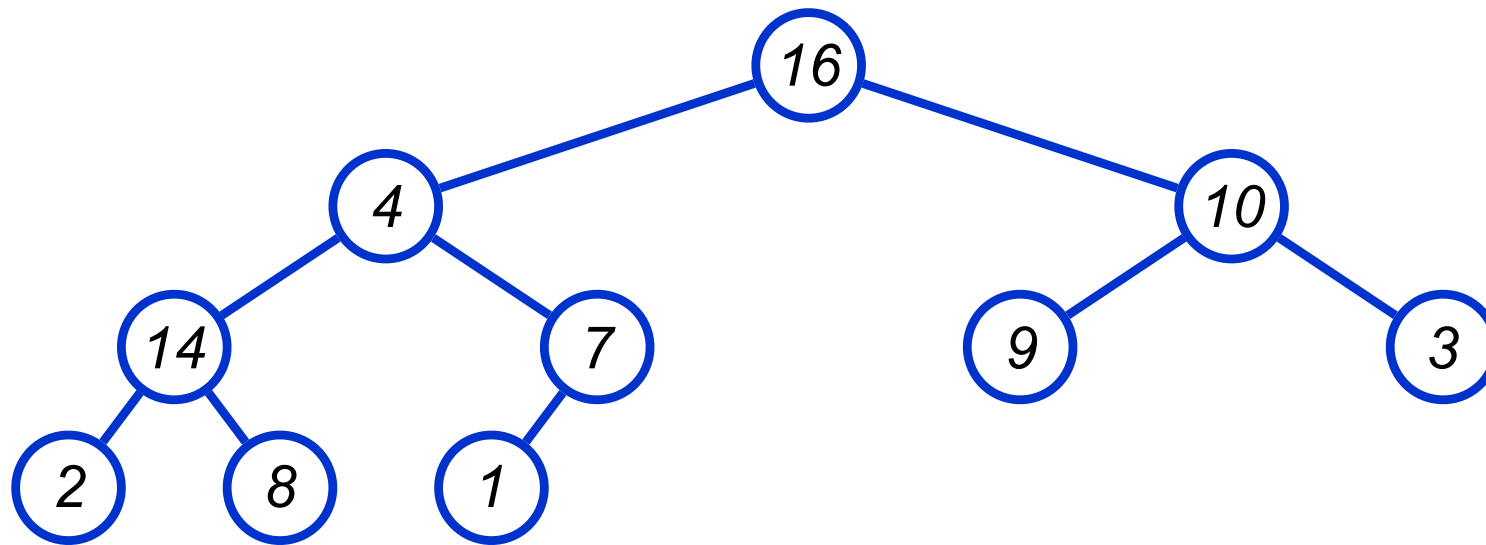
Operations on Heaps

- ▶ Maintain/Restore the max-heap property
 - ▶ MAX-HEAPIFY
- ▶ Create a max-heap from an unordered array
 - ▶ BUILD-MAX-HEAP
- ▶ Sort an array in place
 - ▶ HEAPSORT
- ▶ Priority queues

Heap Operations: MAX-HEAPIFY

- ▶ Maintain the max-heap property: **MAX-HEAPIFY**
- ▶ Suppose a node is smaller than a child
 - ▶ Left and Right subtrees of i are max-heaps
- ▶ To eliminate the violation:
 - ▶ Exchange with larger child
 - ▶ Move down the tree
 - ▶ Continue until node is not smaller than children

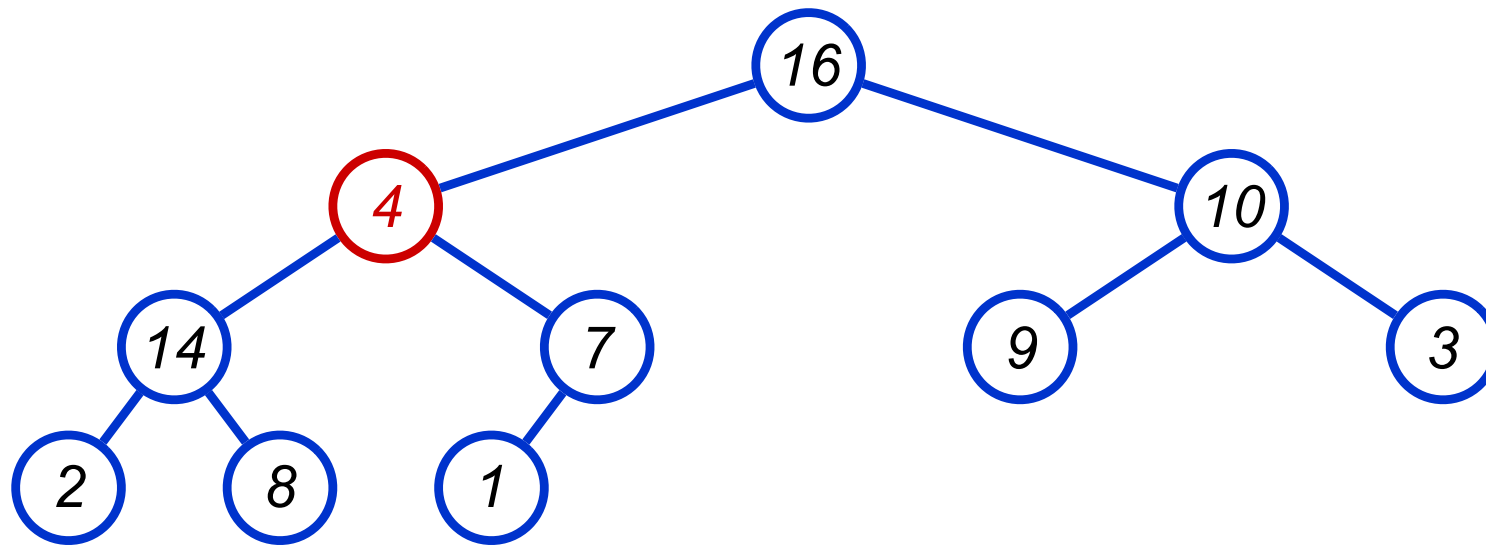
MAX-HEAPIFY Example



$A =$

16	4	10	14	7	9	3	2	8	1
----	---	----	----	---	---	---	---	---	---

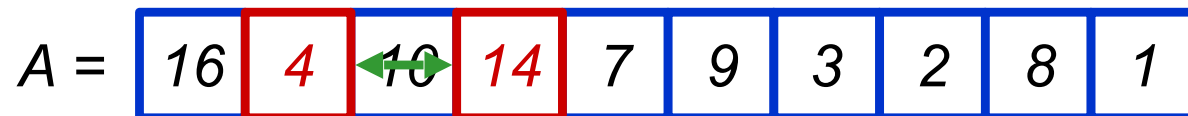
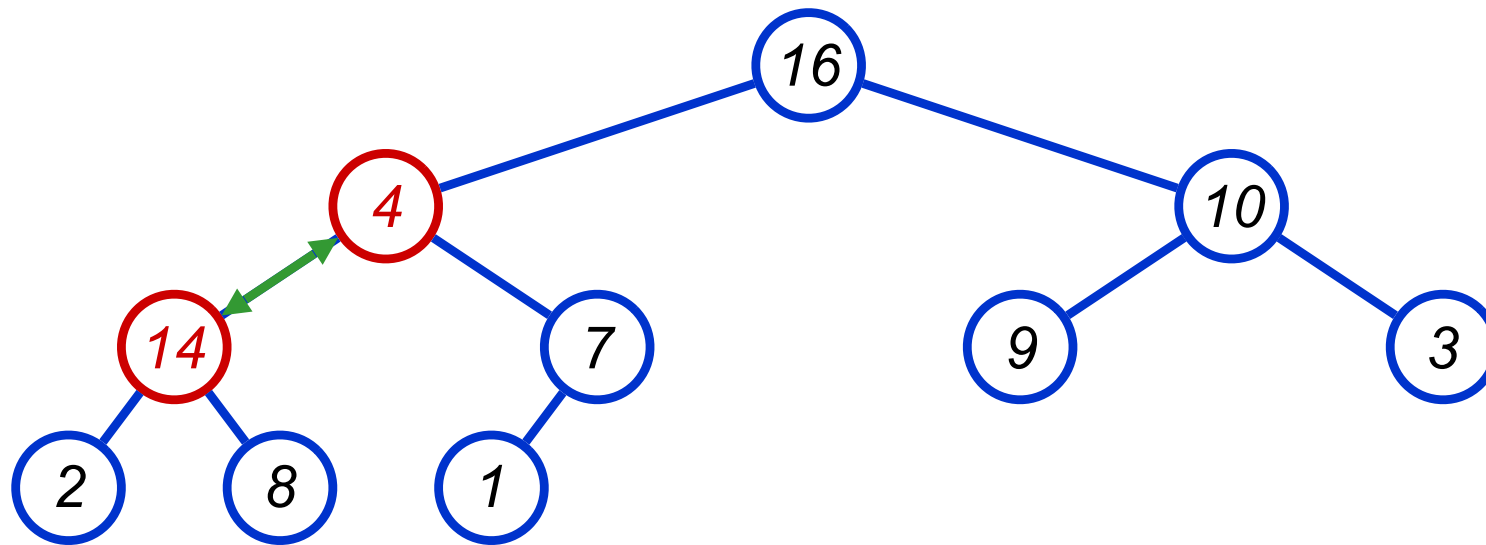
MAX-HEAPIFY Example



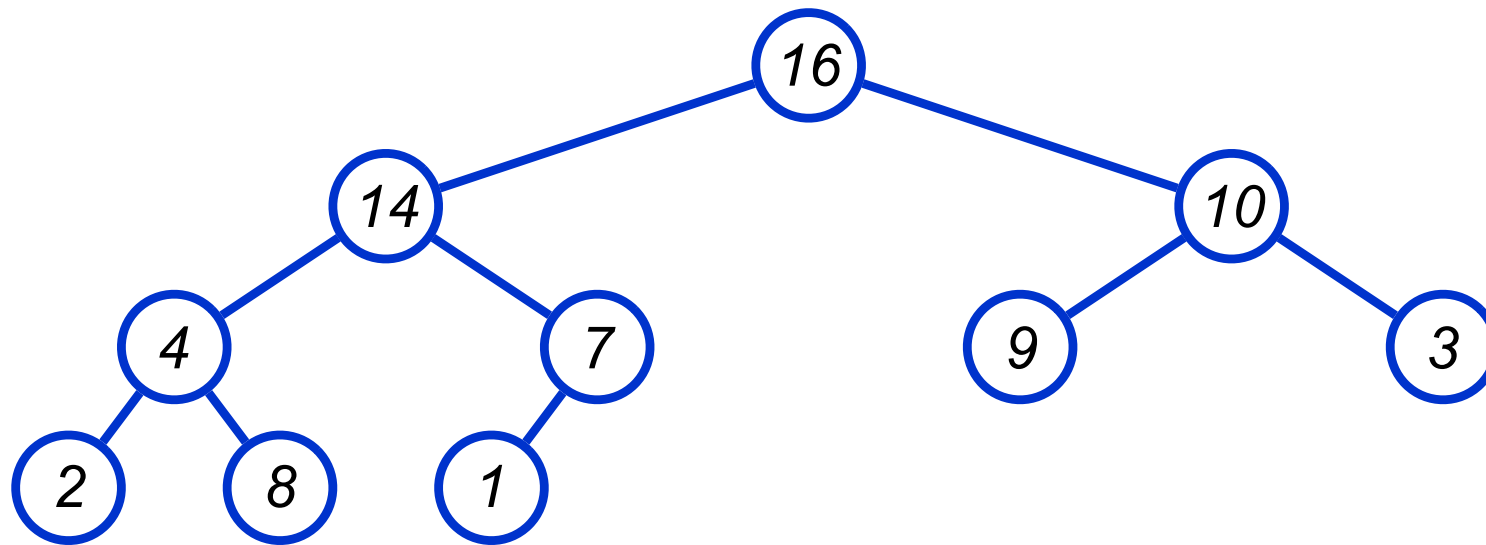
$A =$

16	4	10	14	7	9	3	2	8	1
----	---	----	----	---	---	---	---	---	---

MAX-HEAPIFY Example



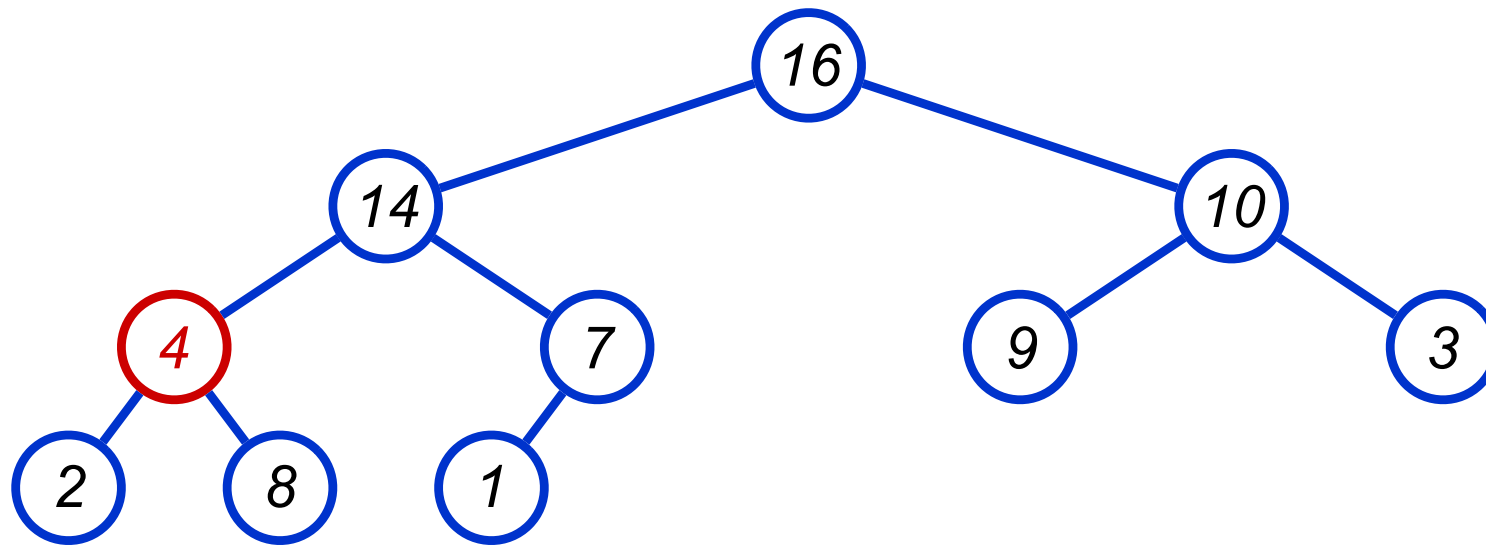
MAX-HEAPIFY Example



$A =$

16	14	10	4	7	9	3	2	8	1
----	----	----	---	---	---	---	---	---	---

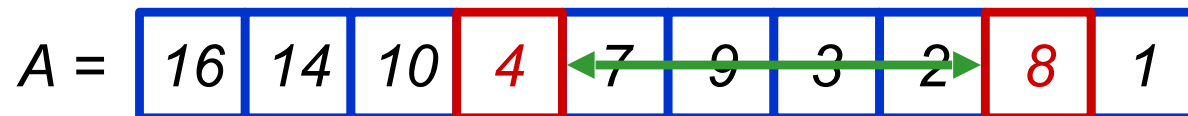
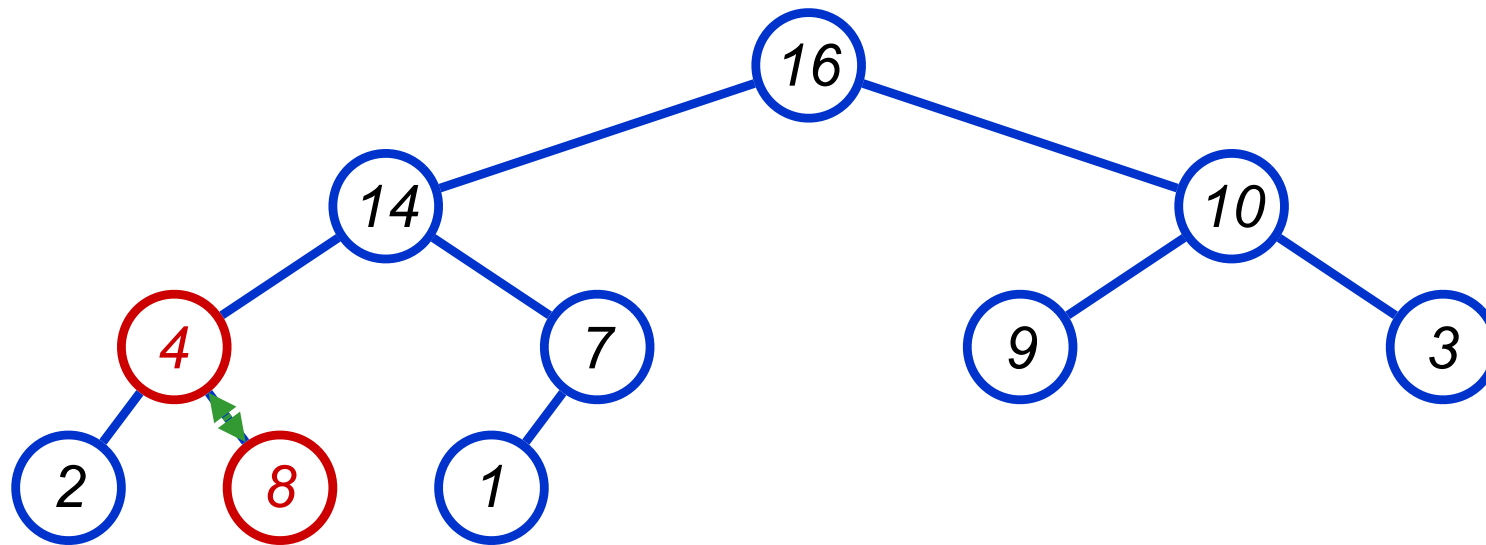
MAX-HEAPIFY Example



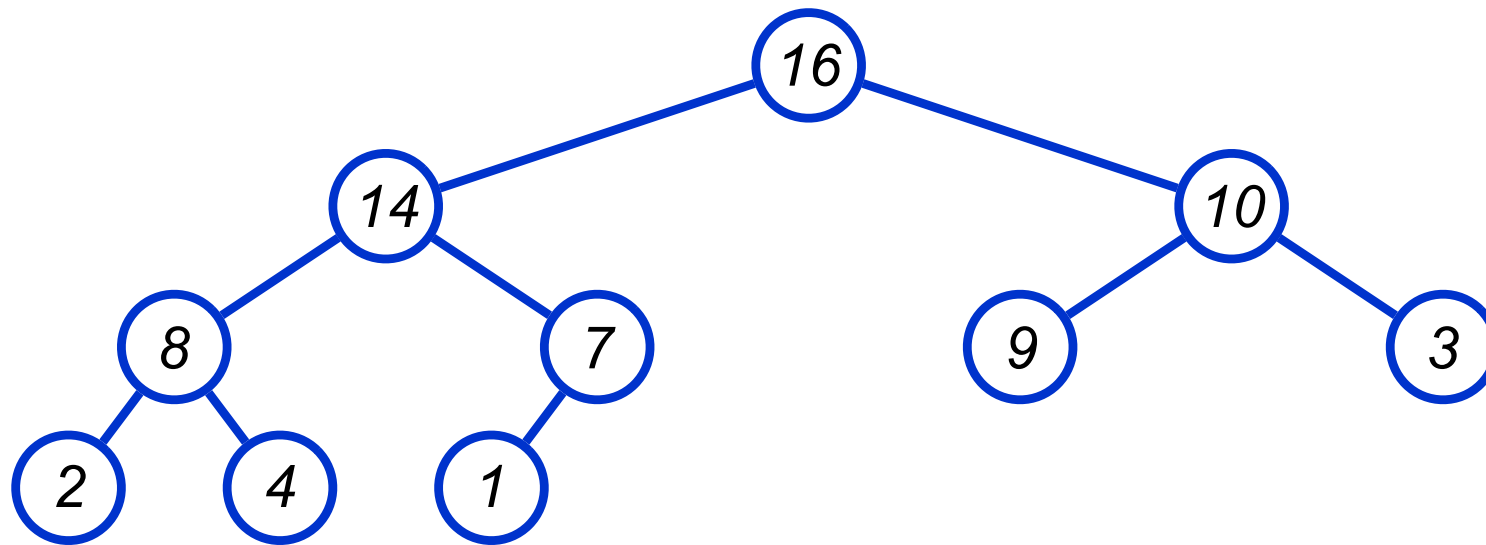
$A =$

16	14	10	4	7	9	3	2	8	1
----	----	----	---	---	---	---	---	---	---

MAX-HEAPIFY Example



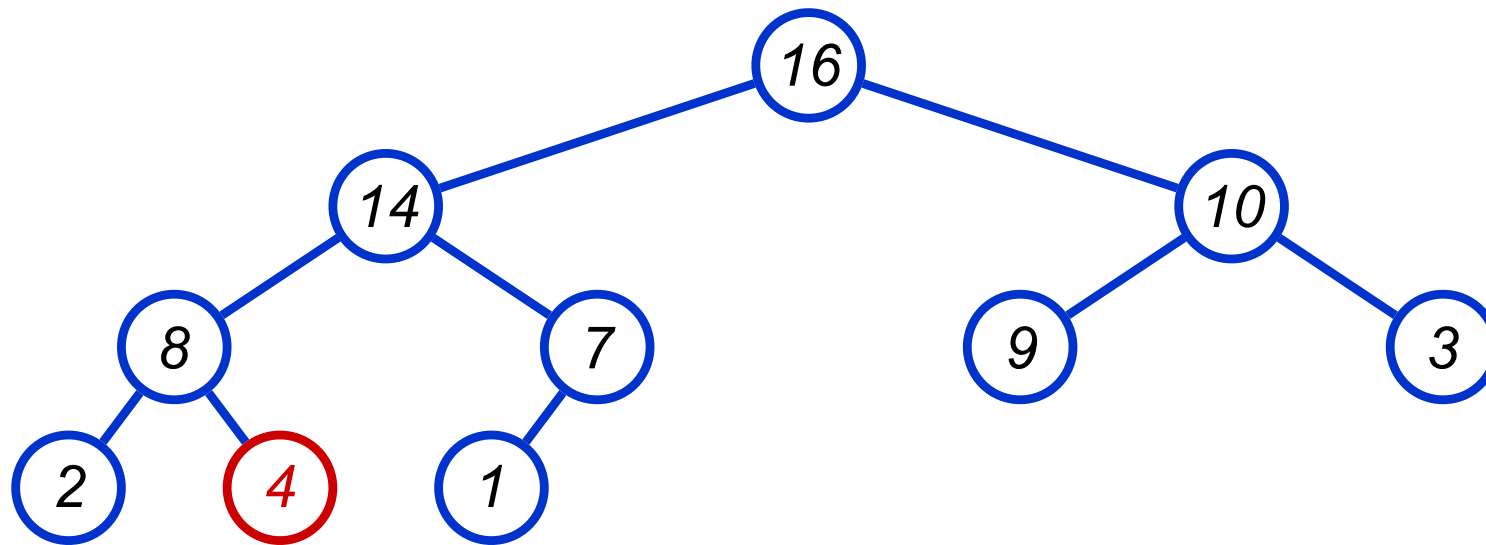
MAX-HEAPIFY Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

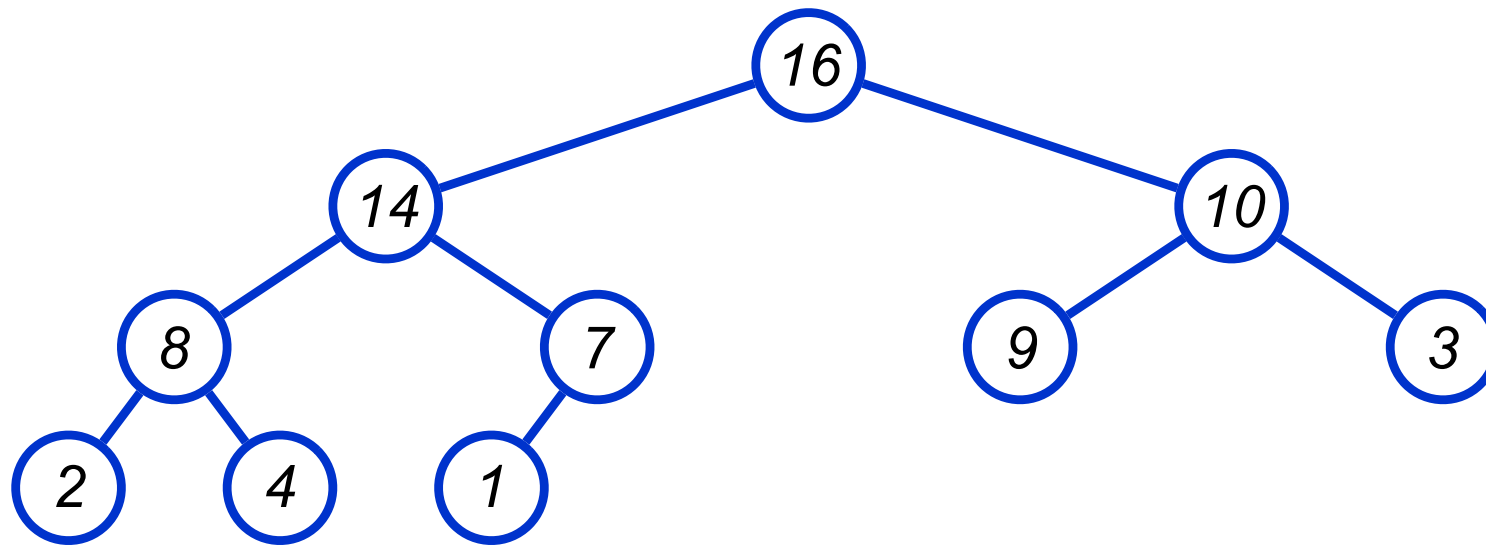
MAX-HEAPIFY Example



$A =$

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----	----	----	---	---	---	---	---	---	---

MAX-HEAPIFY Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

Heap Operations: MAX-HEAPIFY

```
Max-Heapify(A, i)
{
    l = Left(i); r = Right(i);
    if (l <= heap_size(A) && A[l] > A[i])
        largest = l;
    else
        largest = i;
    if (r <= heap_size(A) && A[r] > A[largest])
        largest = r;
    if (largest != i)
        Swap(A, i, largest);
        Heapify(A, largest);
}
```

Assumptions:

- ▶ Left and Right subtrees of i are max-heaps
- ▶ $A[i]$ may be smaller than its children

Analyzing MAX-HEAPIFY - Informal

- ▶ Intuitively
 - ▶ It trace a path from the root to a leaf (longest path length: h)
 - ▶ At each level, it makes exactly 2 comparisons
 - ▶ Total number of comparison is $2h$
 - ▶ Running time is $O(h)$ or $O(\lg n)$
- ▶ Running time of MAX-HEAPIFY is $O(\lg n)$
- ▶ Can be written in terms of the height of the heap, as being $O(h)$
 - ▶ Since the height of the heap is $\lfloor \lg n \rfloor$

Analyzing MAX-HEAPIFY - Formal

- ▶ Fixing up relationships between i , l , and r takes $\Theta(1)$ time
- ▶ *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
 - ▶ Answer: $2n/3$ (worst case: bottom row 1/2 full)
- ▶ So time taken by MAX-HEAPIFY is given by
 - ▶ $T(n) \leq T(2n/3) + \Theta(1)$

Analyzing MAX-HEAPIFY - Formal

- ▶ So we have
 - ▶ $T(n) \leq T(2n/3) + \Theta(1)$
- ▶ By case 2 of the Master Theorem,
 - ▶ $T(n) = O(\lg n)$
- ▶ Thus, MAX-HEAPIFY takes logarithmic time

Building a Heap

- ▶ We can build a heap in a bottom-up manner by running MAX-HEAPIFY on successive subarrays
 - ▶ Convert an array $A[1 \dots n]$ into a max-heap ($n = \text{length}[A]$)
 - ▶ The elements in the subarray $A[(\lfloor n/2 \rfloor + 1) \dots n]$ are leaves
 - ▶ Apply MAX-HEAPIFY on elements between 1 and $\lfloor n/2 \rfloor$

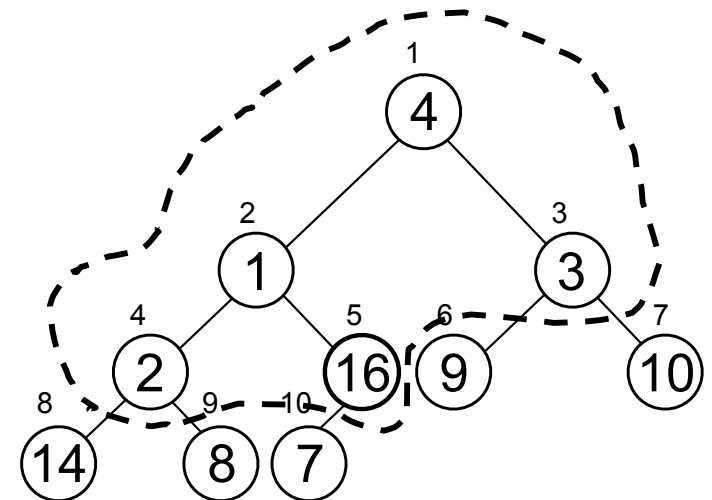
Alg: BUILD-MAX-HEAP(A)

$n = \text{length}[A]$

for $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1

do MAX-HEAPIFY(A, i, n)

A:

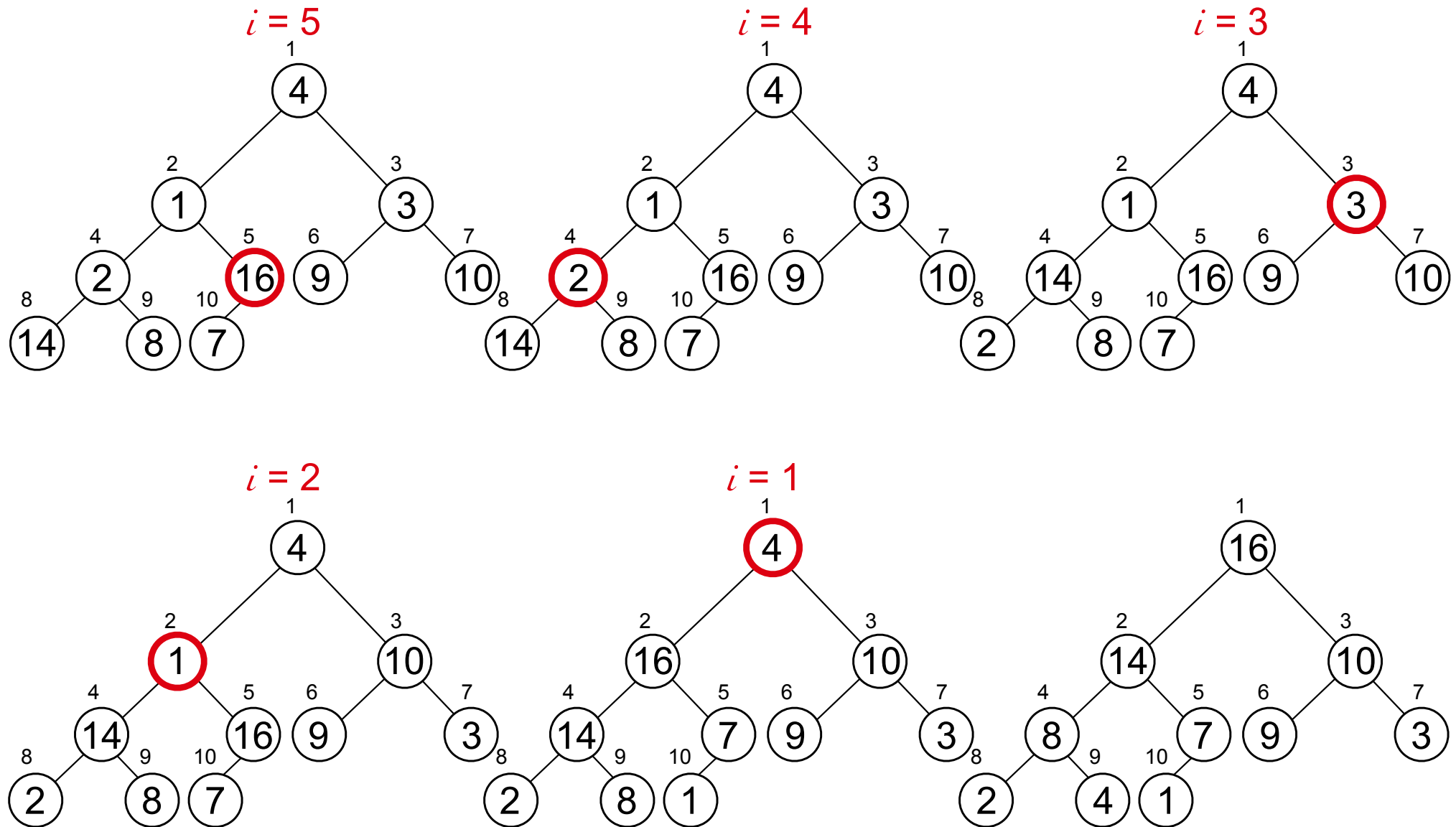


4	1	3	2	16	9	10	14	8	7
---	---	---	---	----	---	----	----	---	---

Example:

A

4	1	3	2	16	9	10	14	8	7
---	---	---	---	----	---	----	----	---	---



Analyzing BUILD MAX HEAP

Alg: BUILD-MAX-HEAP(A)

1. $n = \text{length}[A]$

2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1

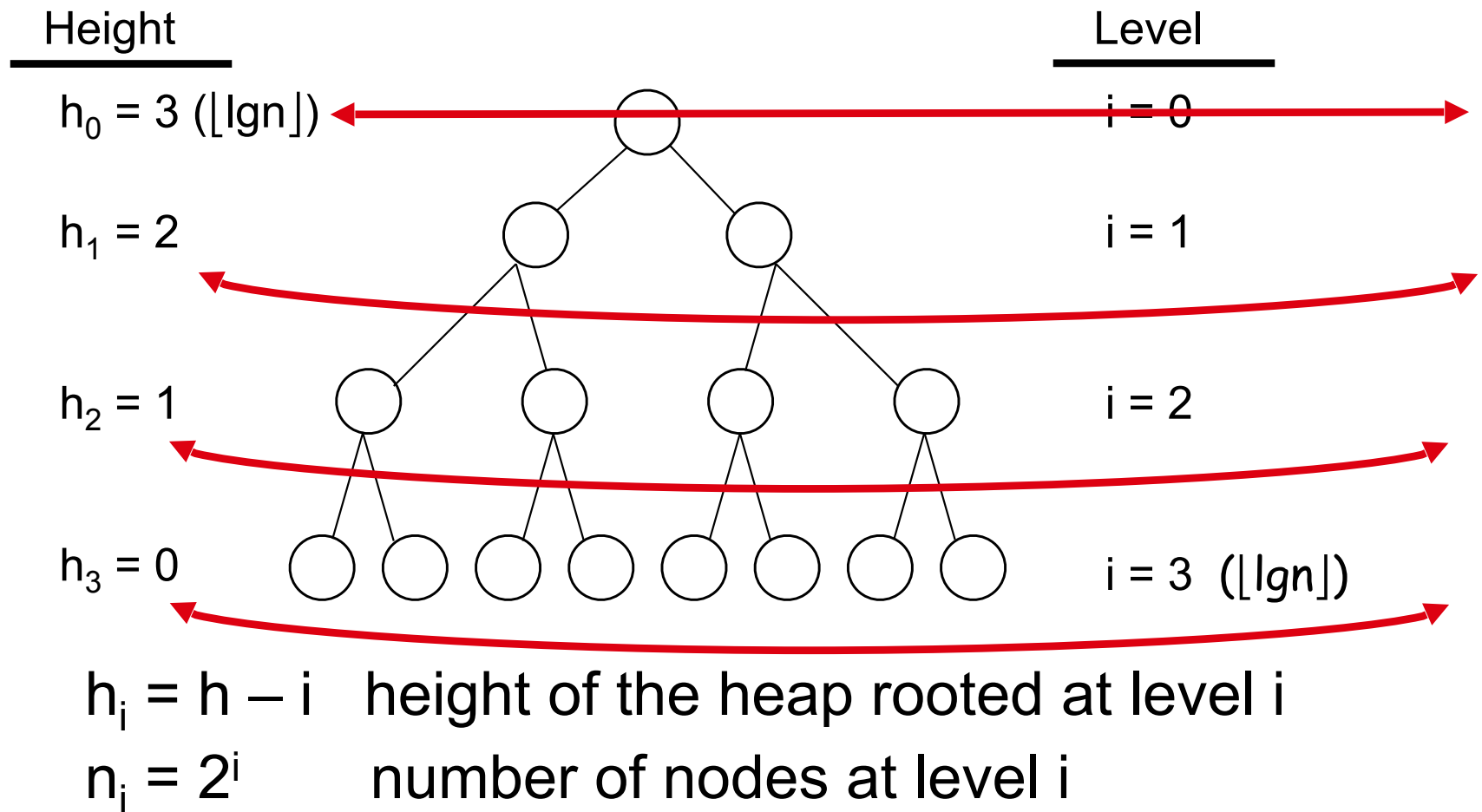
3. **do** MAX-HEAPIFY(A, i, n)

$O(\lg n)$ } $O(n)$

- ▶ Each call to MAX-HEAPIFY takes $O(\lg n)$ time
- ▶ There are $O(n)$ such calls (specifically, $\lfloor n/2 \rfloor$)
- ▶ Thus the running time is $O(n \lg n)$
 - ▶ *Is this a correct asymptotic upper bound?* **YES**
 - ▶ *Is this an asymptotically tight bound?* **NO**
- ▶ A tighter bound is $O(n)$

Running Time of BUILD MAX HEAP

- ▶ HEAPIFY takes $O(h) \Rightarrow$ the cost of HEAPIFY on a node i is proportional to the height of the node i in the tree



Running Time of BUILD MAX HEAP

$$\begin{aligned} T(n) &= \sum_{i=0}^h n_i h_i && \text{Cost of HEAPIFY at level } i * \text{number of nodes at that level} \\ &= \sum_{i=0}^h 2^i (h - i) && \text{Replace the values of } n_i \text{ and } h_i \text{ computed before} \\ &= \sum_{i=0}^h \frac{h - i}{2^{h-i}} 2^h && \text{Multiply by } 2^h \text{ both at the nominator and denominator and} \\ & && \text{write } 2^i \text{ as } \frac{1}{2^{-i}} \\ &= 2^h \sum_{k=0}^h \frac{k}{2^k} && \text{Change variables: } k = h - i \\ &\leq n \sum_{k=0}^{\infty} \frac{k}{2^k} && \text{The sum above is smaller than the sum of all elements to } \infty \\ & && \text{and } h = \lg n \\ &= O(n) && \text{The sum above is 2} \end{aligned}$$

Running time of BUILD-MAX-HEAP: $T(n) = O(n)$

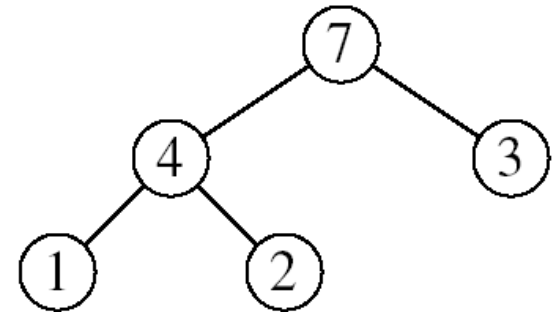
Heapsort

- ▶ **Goal:**

- ▶ Sort an array using heap representations

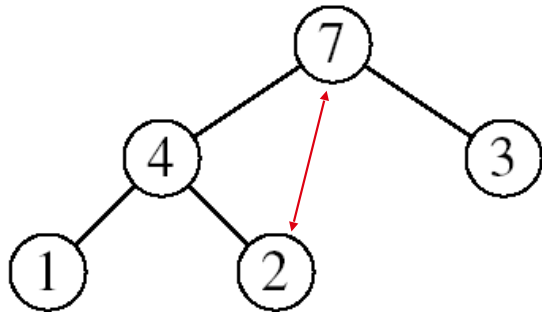
- ▶ **Idea:**

- ▶ Build a **max-heap** from the array
 - ▶ Swap the root (the maximum element) with the last element in the array
 - ▶ “Discard” this last node by decreasing the heap size
 - ▶ Call MAX-HEAPIFY on the new root
 - ▶ Repeat this process until only one node remains

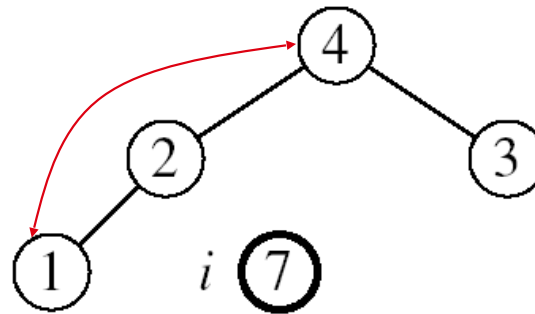


Example

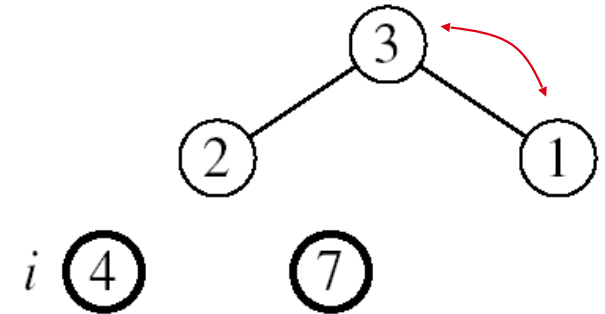
$A = [7, 4, 3, 1, 2]$



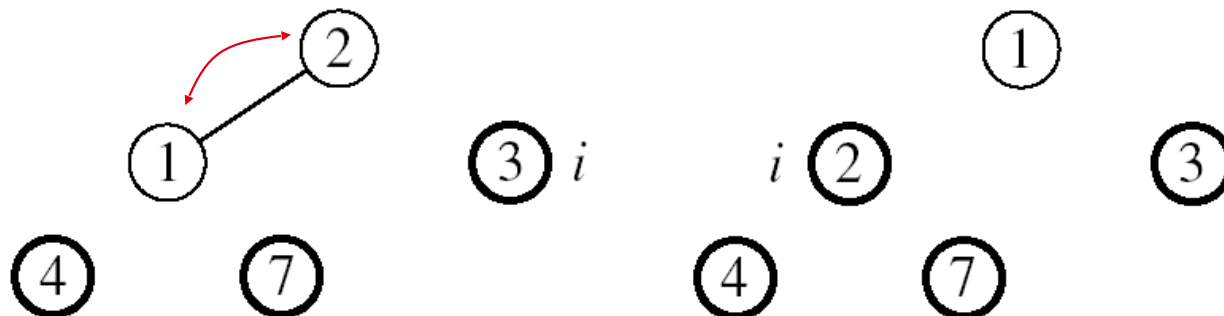
MAX-HEAPIFY(A, 1, 4)



MAX-HEAPIFY(A, 1, 3)



MAX-HEAPIFY(A, 1, 2)



MAX-HEAPIFY(A, 1, 1)

A

1	2	3	4	7
---	---	---	---	---

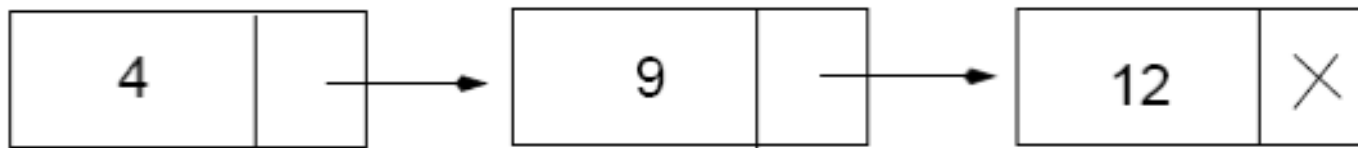
Analyzing Heapsort

1. **BUILD-MAX-HEAP**(A) $O(n)$
 2. **for** $i \leftarrow \text{length}[A]$ **downto** 2
 3. **do** exchange $A[1] \leftrightarrow A[i]$
 4. **MAX-HEAPIFY**(A, 1, $i - 1$) $O(\lg n)$
- } $n-1$ times

- ▶ The call to **BUILD-MAX-HEAP** takes $O(n)$ time
- ▶ Each of the $n - 1$ calls to **MAX-HEAPIFY** takes $O(\lg n)$ time
- ▶ Thus the total time taken by **HeapSort**
 $= O(n) + (n - 1) O(\lg n) = O(n) + O(n \lg n) = O(n \lg n)$

Priority Queues

- ▶ The heap data structure is incredibly useful for implementing *priority queues*, which maintains a set of elements.
- ▶ Properties of priority queues
 - ▶ Each element is associated with a value (priority)
 - ▶ The key with the highest (or lowest) priority is extracted first



Operations on Priority Queues

- ▶ Max-priority queues support the following operations:
 - ▶ **INSERT(S, x)**: inserts element x into set S
 - ▶ **EXTRACT-MAX(S)**: removes and returns element of S with largest key
 - ▶ **MAXIMUM(S)**: returns element of S with largest key
 - ▶ **INCREASE-KEY(S, x, k)**: increases value of element x's key to k (Assume $k \geq x$'s current key value)

HEAP-MAXIMUM

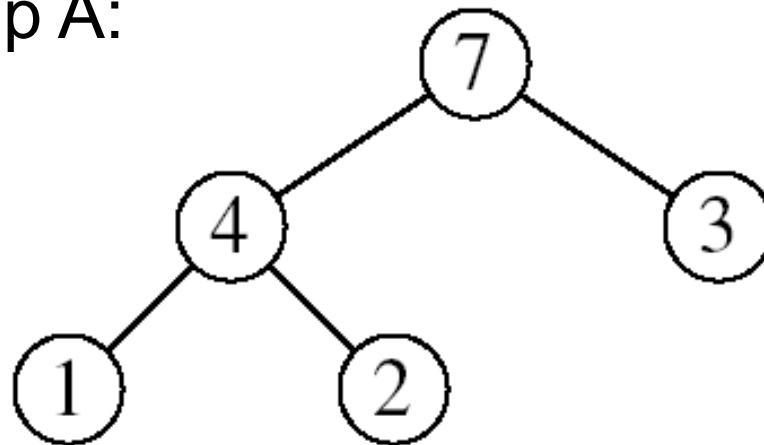
Goal: Return the largest element of the heap

Alg: HEAP-MAXIMUM(A)

Running time: $O(1)$

1. **return** A[1]

Heap A:

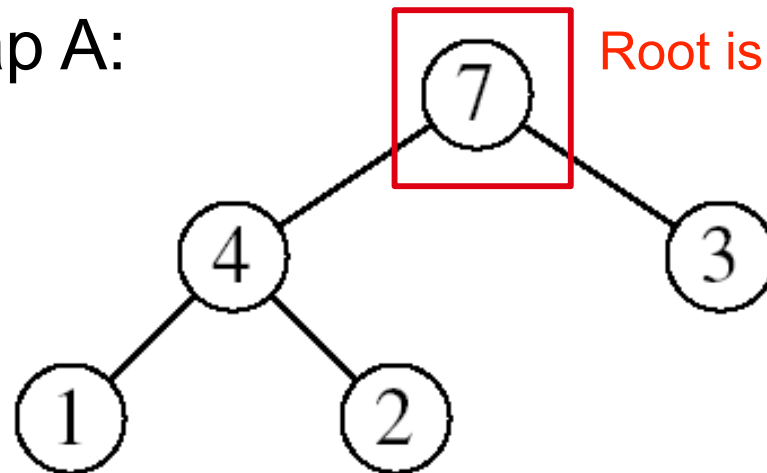


Heap-Maximum(A) returns 7

HEAP-EXTRACT-MAX

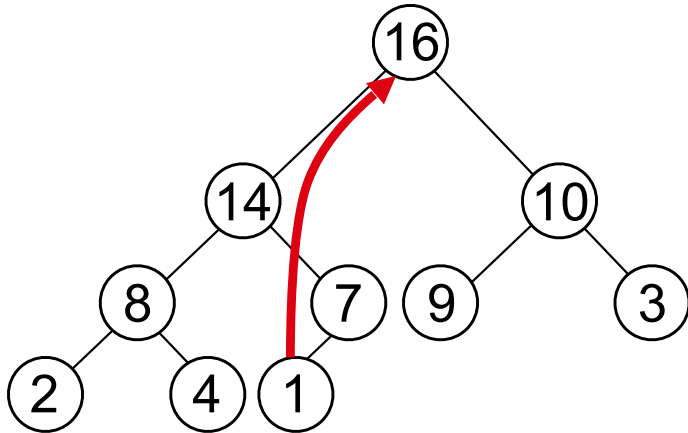
- ▶ Goal
 - ▶ Extract the largest element of the heap (i.e., return the max value and also remove that element from the heap)
- ▶ Idea
 - ▶ Exchange the root element with the last
 - ▶ Decrease the size of the heap by 1 element
 - ▶ Call MAX-HEAPIFY on the new root, on a heap of size $n-1$

Heap A:

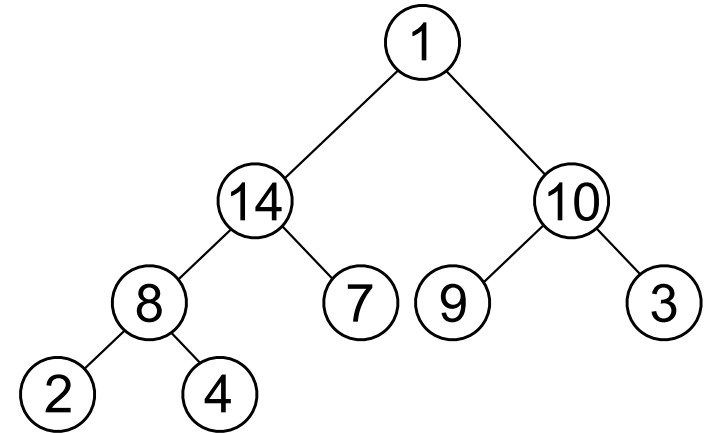


Root is the largest element

Example: HEAP-EXTRACT-MAX

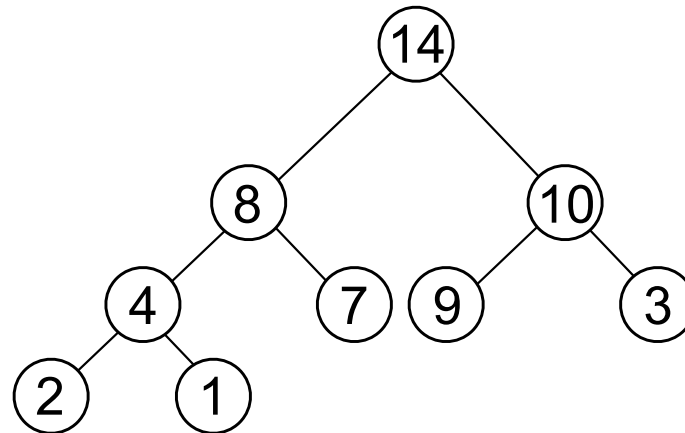


max = 16



Heap size decreased with 1

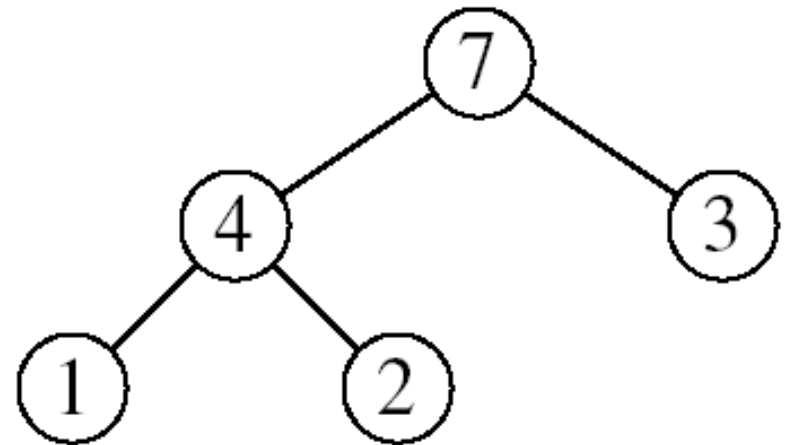
Call MAX-HEAPIFY(A, 1, n-1)



HEAP-EXTRACT-MAX

Alg: HEAP-EXTRACT-MAX(A, n)

1. **if** $n < 1$
2. **then error** “heap underflow”
3. $\text{max} \leftarrow A[1]$
4. $A[1] \leftarrow A[n]$
5. MAX-HEAPIFY($A, 1, n-1$)
6. **return** max



remakes heap

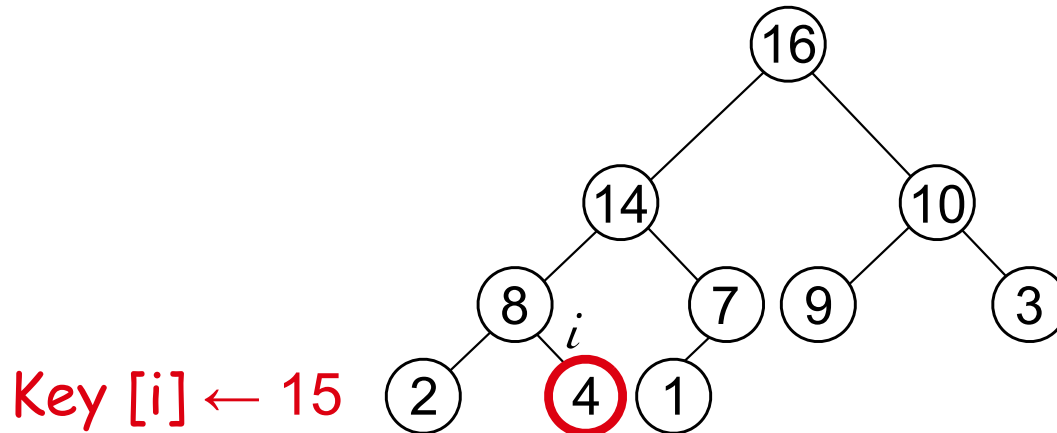
HEAP-INCREASE-KEY

▶ Goal

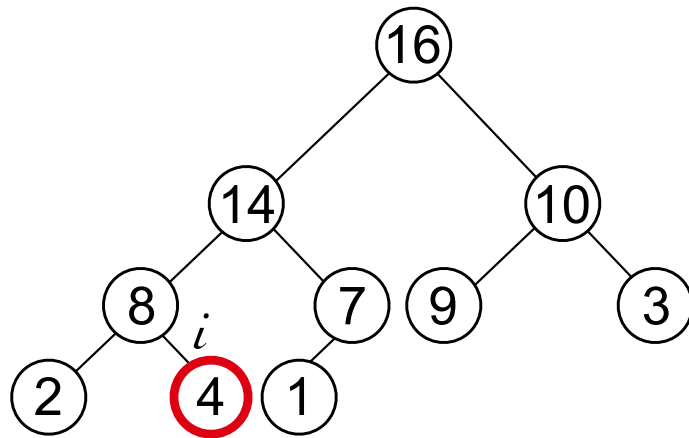
- ▶ Increases the key of an element i in the heap

▶ Idea

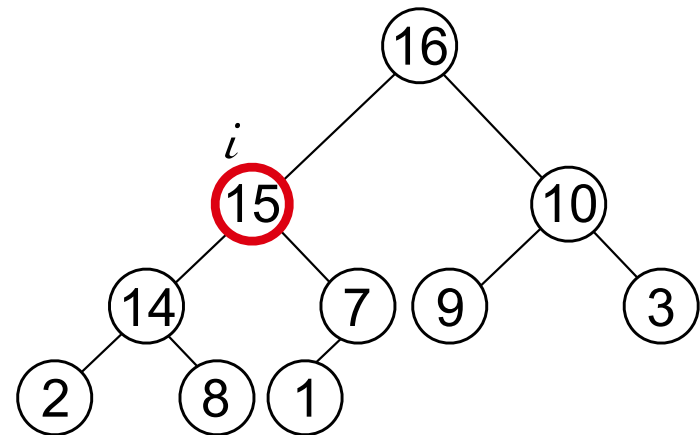
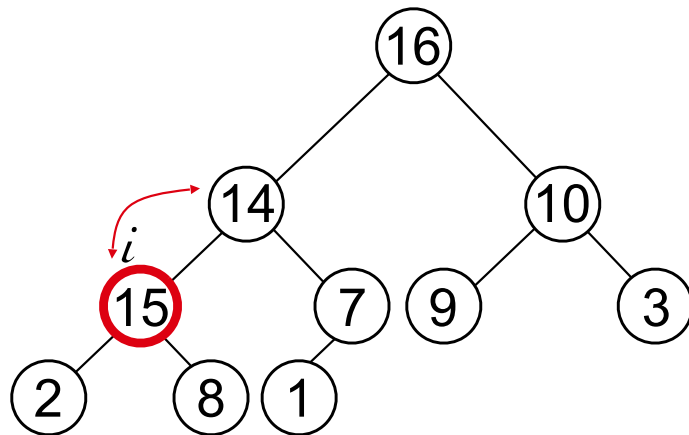
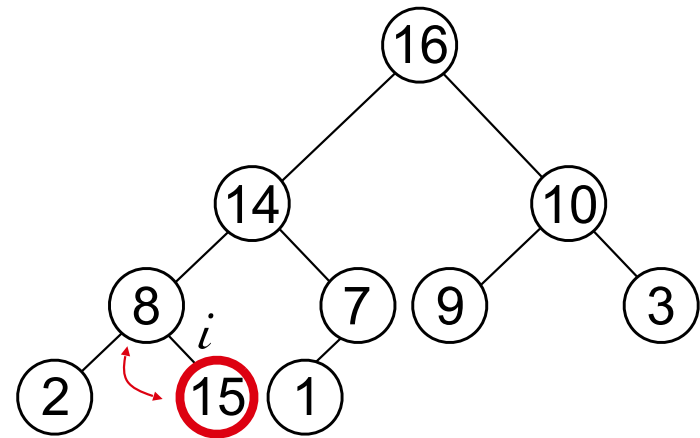
- ▶ Increment the key of $A[i]$ to its new value
- ▶ If the max-heap property does not hold anymore:
traverse a path toward the root to find the proper place
for the newly increased key



Example: HEAP-INCREASE-KEY



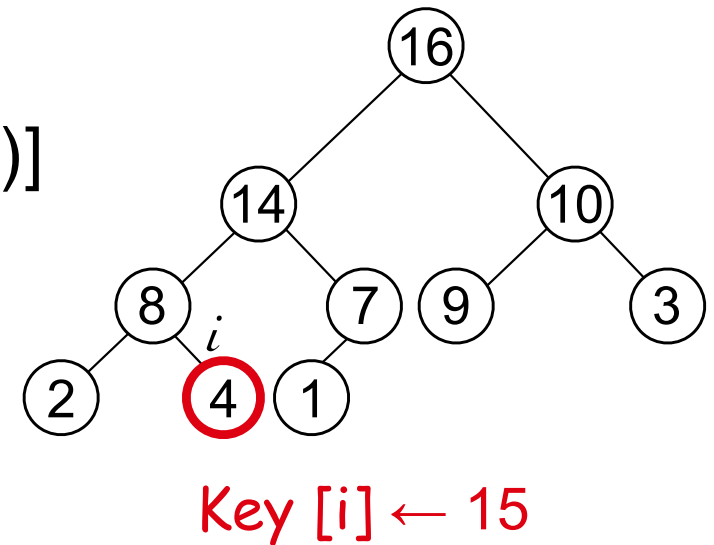
Key [i] \leftarrow 15



Analyzing HEAP-INCREASE-KEY

Alg: HEAP-INCREASE-KEY(A, i, key)

1. **if** $\text{key} < A[i]$
2. **then error** “new key is smaller than current key”
3. $A[i] \leftarrow \text{key}$
4. **while** $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
5. **do** exchange $A[i] \leftrightarrow A[\text{PARENT}(i)]$
6. $i \leftarrow \text{PARENT}(i)$



► Running time: $O(\lg n)$

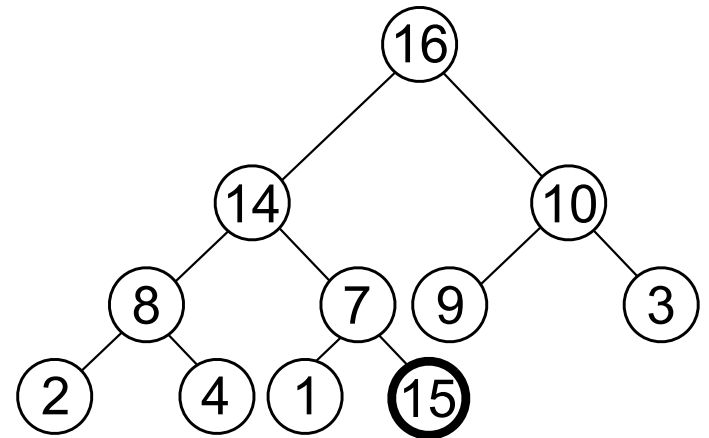
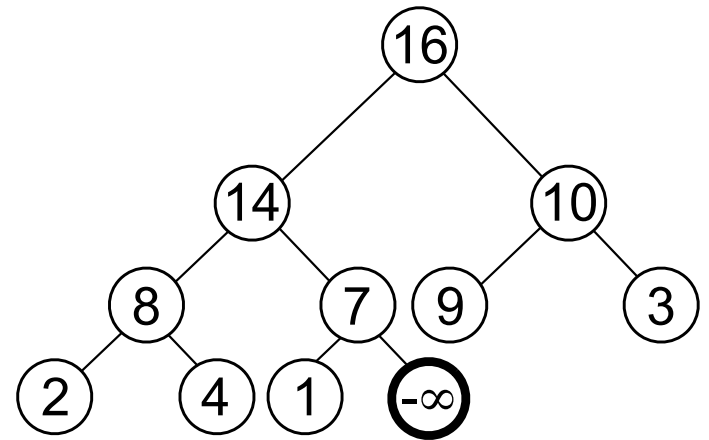
MAX-HEAP-INSERT

▶ Goal

- ▶ Inserts a new element into a max-heap

▶ Idea

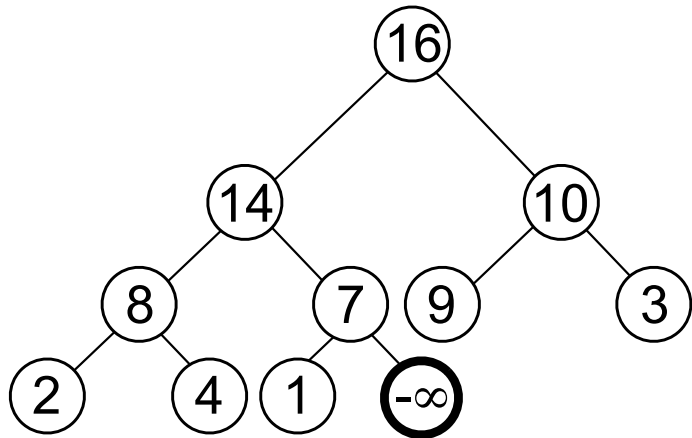
- ▶ Expand the max-heap with a new element whose key is $-\infty$
- ▶ Calls HEAP-INCREASE-KEY to set the key of the new node to its correct value and maintain the max-heap property



Example: MAX-HEAP-INSERT

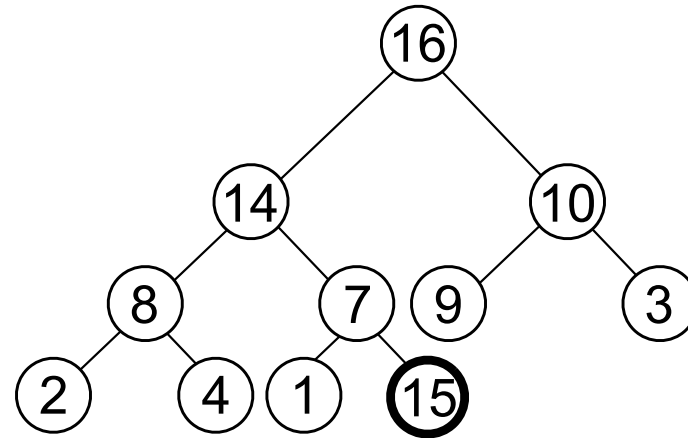
Insert value 15:

- Start by inserting $-\infty$

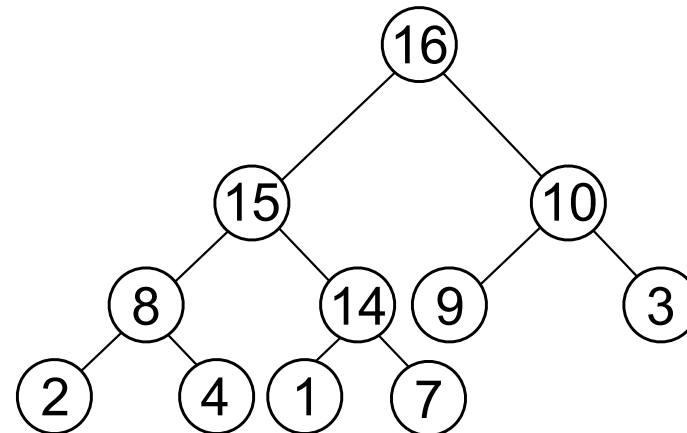
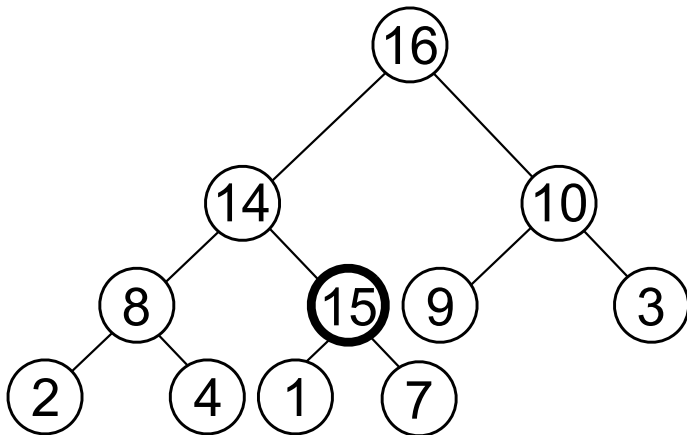


Increase the key to 15

Call HEAP-INCREASE-KEY on $A[11] = 15$



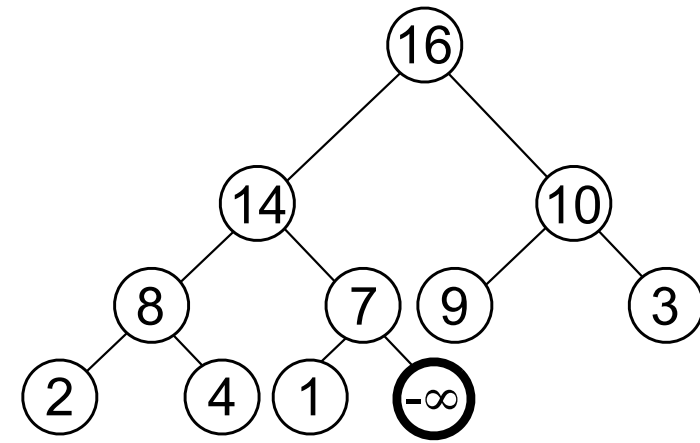
The restored heap containing the newly added element



Analyzing MAX-HEAP-INSERT

Alg: MAX-HEAP-INSERT(A , key , n)

1. $heap-size[A] \leftarrow n + 1$
2. $A[n + 1] \leftarrow -\infty$
3. HEAP-INCREASE-KEY(A , $n + 1$, key)



Running time: $O(\lg n)$

Summary

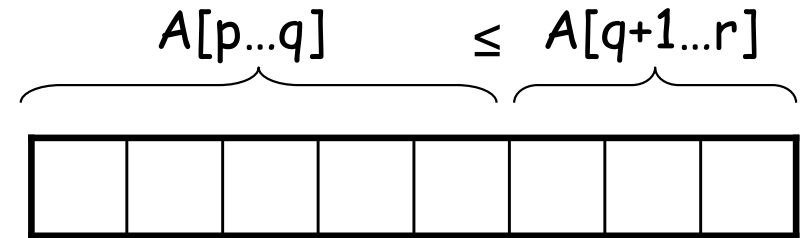
- ▶ We can perform the following operations on heaps:
 - ▶ MAX-HEAPIFY $O(\lg n)$
 - ▶ BUILD-MAX-HEAP $O(n)$
 - ▶ HEAP-SORT $O(n \lg n)$
 - ▶ MAX-HEAP-INSERT $O(\lg n)$
 - ▶ HEAP-EXTRACT-MAX $O(\lg n)$
 - ▶ HEAP-INCREASE-KEY $O(\lg n)$
 - ▶ HEAP-MAXIMUM $O(1)$

Quicksort: Brief Review

- ▶ Sorts in place
- ▶ Sorts $O(n \lg n)$ in the average case
- ▶ Sorts $O(n^2)$ in the worst case
 - ▶ But the worst case doesn't happen often (more on this later...)

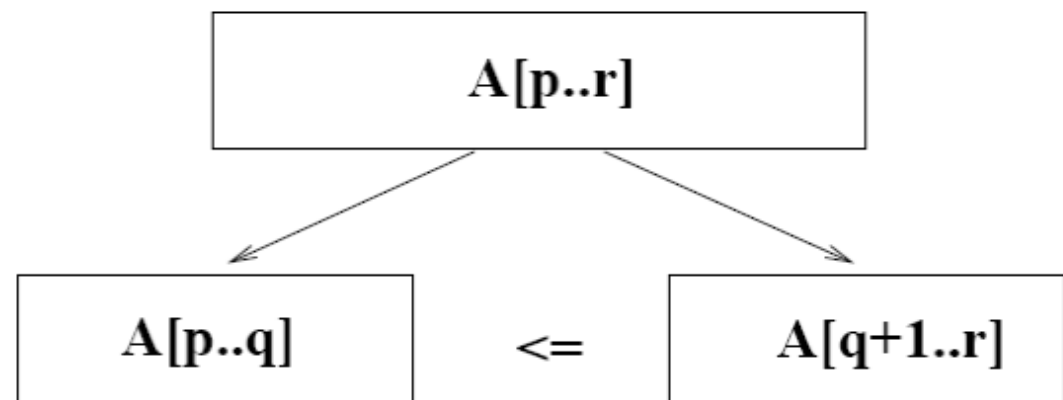
Quicksort

Sort an array $A[p..r]$

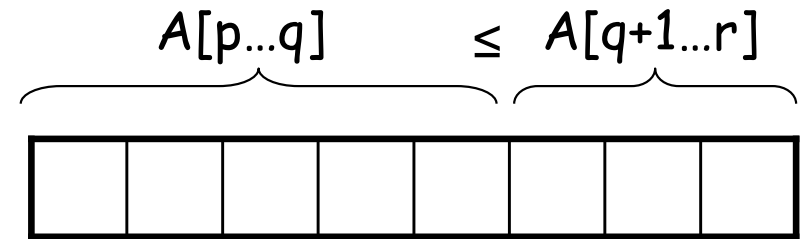


► Divide

- Partition the array A into 2 subarrays $A[p..q]$ and $A[q+1..r]$, such that each element of $A[p..q]$ is smaller than or equal to each element in $A[q+1..r]$
- Need to find index q to partition the array



Quicksort



► Conquer

- Recursively sort $A[p \dots q]$ and $A[q+1 \dots r]$ by calls to Quicksort

► Combine (unlike merge sort)

- Trivial: the arrays are sorted in place
- No additional work is required to combine them
- The entire array is now sorted

Quicksort

Alg.: QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{PARTITION}(A, p, r)$

QUICKSORT (A, p, q)

QUICKSORT (A, q+1, r)

Initially: $p=1, r=n$

Recurrence: $T(n) = T(q) + T(n - q) + f(n)$

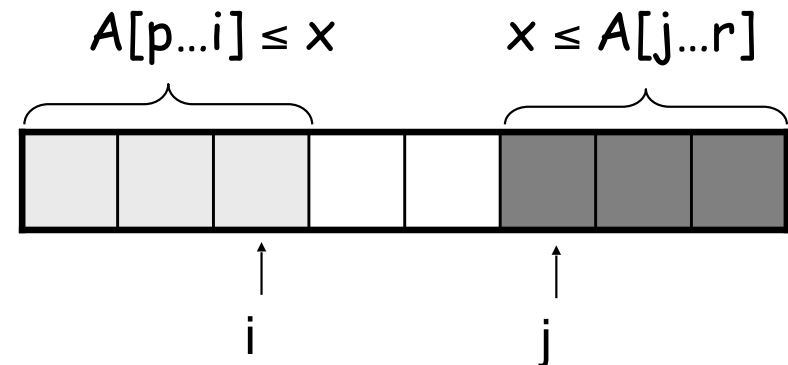
 $f(n)$ depends on partition()

Partition

- ▶ Clearly, all the action takes place in the **PARTITION()** function
 - ▶ Rearranges the subarray in place
 - ▶ End result:
 - ▶ Two subarrays
 - ▶ All values in first subarray \leq all values in second
 - ▶ Returns the index of the “pivot” element separating the two subarrays
- ▶ *How should you implement this?*

Partition

- ▶ Choosing PARTITION()
 - ▶ There are different ways to do this
 - ▶ Each has its own advantages/disadvantages
- ▶ Hoare partition
 - ▶ Select a pivot element x around which to partition
 - ▶ Starts from both ends
 - ▶ Grows two regions
 - ▶ $A[p \dots i] \leq x$
 - ▶ $x \leq A[j \dots r]$



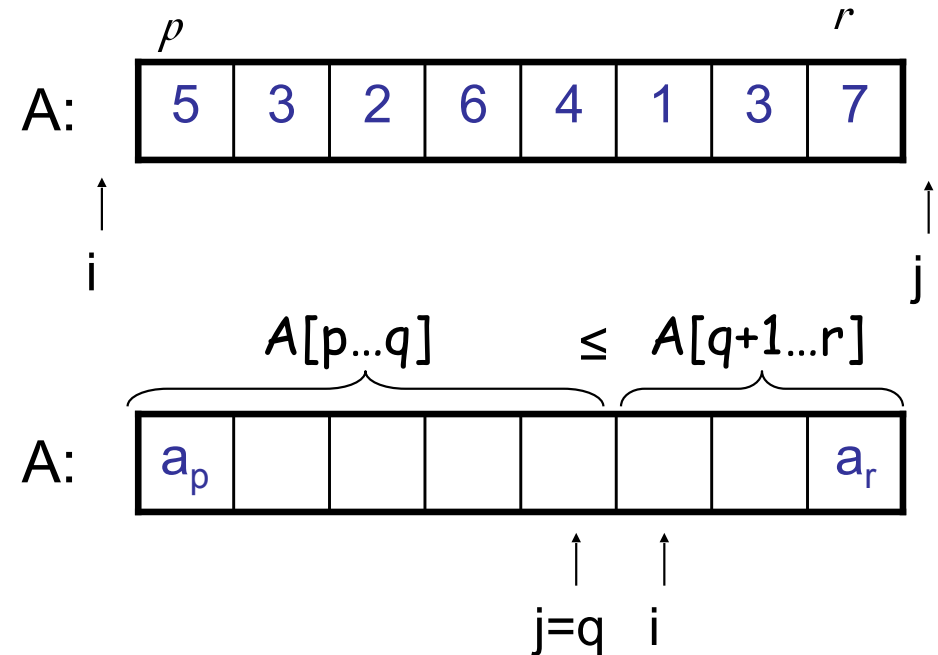
Partition in Words

- ▶ Partition(A, p, r):
 - ▶ Select an element to act as the “pivot” (*which?*)
 - ▶ Grow two regions, A[p..i] and A[j..r]
 - ▶ All elements in A[p..i] \leq pivot
 - ▶ All elements in A[j..r] \geq pivot
 - ▶ Increment i until A[i] \geq pivot
 - ▶ Decrement j until A[j] \leq pivot
 - ▶ Swap A[i] and A[j]
 - ▶ Repeat until i \geq j
 - ▶ Return j

Partition Code

Alg. PARTITION (A , p , r)

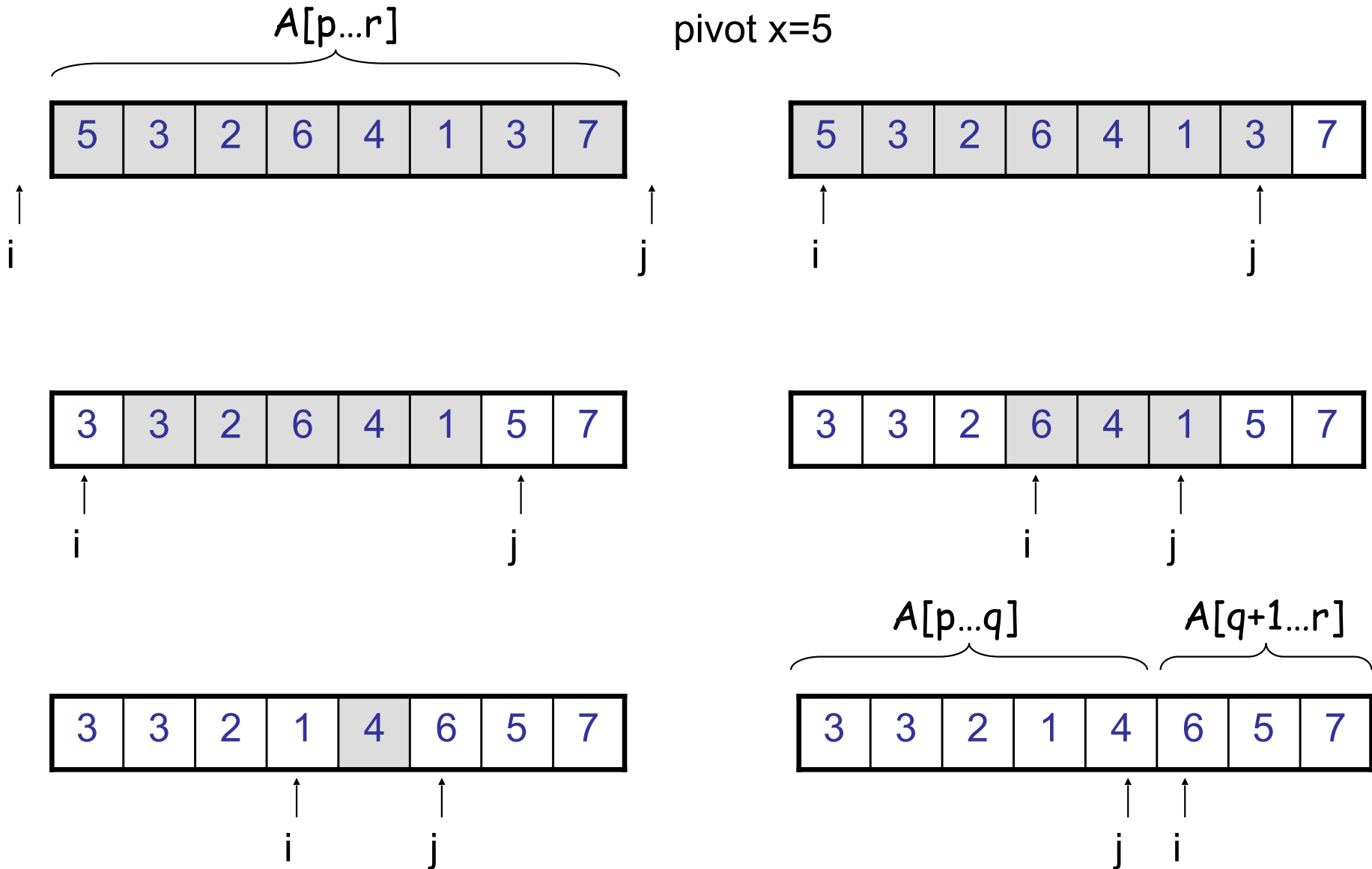
1. $x \leftarrow A[p]$
2. $i \leftarrow p - 1$
3. $j \leftarrow r + 1$
4. **while** TRUE
5. **do repeat** $j \leftarrow j - 1$
6. **until** $A[j] \leq x$
7. **do repeat** $i \leftarrow i + 1$
8. **until** $A[i] \geq x$
9. **if** $i < j$
10. **then** exchange $A[i] \leftrightarrow A[j]$
11. **else return** j



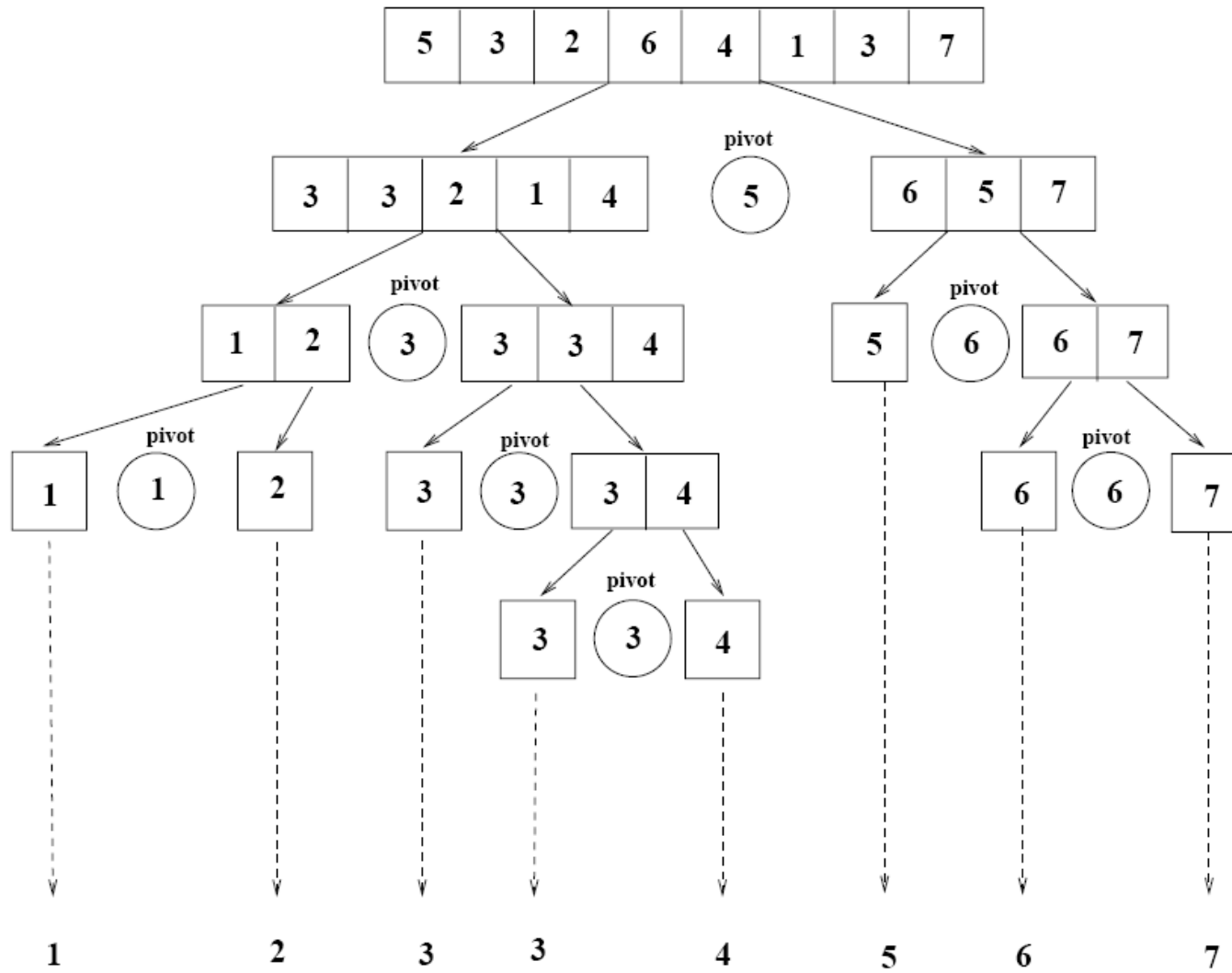
Each element is
visited once!

Running time: $\Theta(n)$
 $n = r - p + 1$

Example



Example



Quicksort: Recurrence

Alg.: QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{PARTITION}(A, p, r)$

QUICKSORT (A, p, q)

QUICKSORT (A, q+1, r)

Initially: $p=1, r=n$

Recurrence: $T(n) = T(q) + T(n - q) + n$

Analyzing Quicksort

- ▶ *What will be the worst case for the algorithm?*
 - ▶ Partition is always unbalanced
- ▶ *What will be the best case for the algorithm?*
 - ▶ Partition is perfectly balanced
- ▶ *Which is more likely?*
 - ▶ The latter, except...
- ▶ *Will any particular input elicit the worst case?*
 - ▶ Yes: Already-sorted input

Analyzing Quicksort: Worst Case Partitioning

▶ Worst-case partitioning

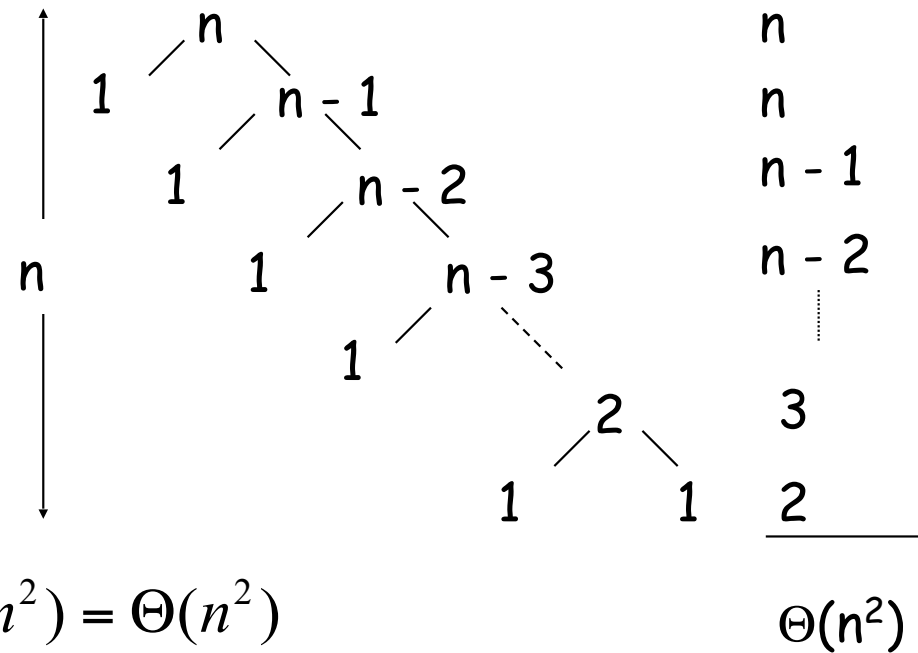
- ▶ One region has one element and the other has $n - 1$ elements
- ▶ Maximally unbalanced

▶ Recurrence: $q=1$

$$T(n) = T(1) + T(n - 1) + n,$$

- ▶ $T(1) = \Theta(1)$

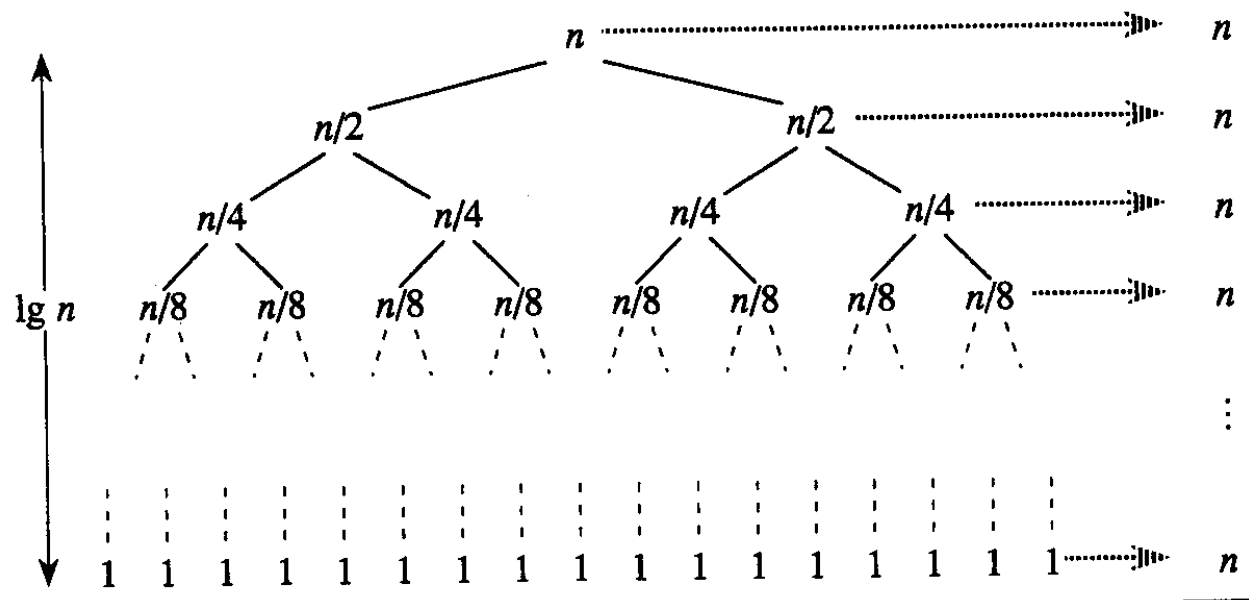
- ▶ $T(n) = n + \left(\sum_{k=1}^n k \right) - 1 = \Theta(n) + \Theta(n^2) = \Theta(n^2)$



When does the worst case happen?

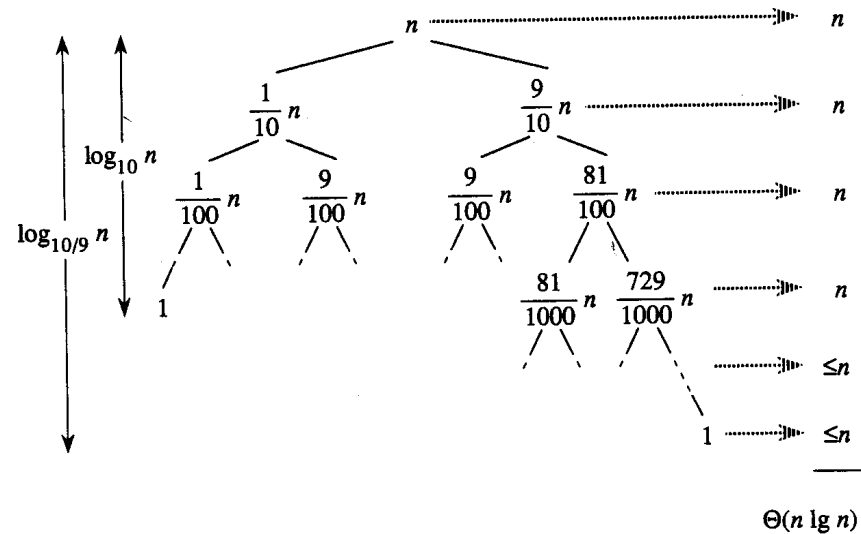
Analyzing Quicksort: Best Case Partitioning

- ▶ Best-case partitioning
 - ▶ Partitioning produces two regions of size $n/2$
- ▶ Recurrence: $q=n/2$
 - ▶ $T(n) = 2T(n/2) + \Theta(n)$
 - ▶ $T(n) = \Theta(n \lg n)$ (Master theorem)



Case Between Worst and Best

- ▶ An intuitive explanation/example:
 - ▶ Suppose that partition() always produces a 9-to-1 split. This looks quite unbalanced!
 - ▶ The recurrence is thus:
 - ▶ $T(n) = T(9n/10) + T(n/10) + n$
 - ▶ *How deep will the recursion go?*



- Using the recursion tree:

$$\text{longest path: } Q(n) \leq n \sum_{i=0}^{\log_{10/9} n} 1 = n(\log_{10/9} n + 1) = c_2 n \lg n$$

$$\text{shortest path: } Q(n) \geq n \sum_{i=0}^{\log_{10} n} 1 = n \log_{10} n = c_1 n \lg n$$

$$\text{Thus, } Q(n) = \Theta(n \lg n)$$

How does partition affect performance?

- **Any splitting of constant proportionality** yields $\Theta(n \lg n)$ time !!!

- Consider the $(1 : n - 1)$ splitting:

ratio = $1/(n - 1)$ not a constant !!!

- Consider the $(n/2 : n/2)$ splitting:

ratio = $(n/2)/(n/2) = 1$ it is a constant !!

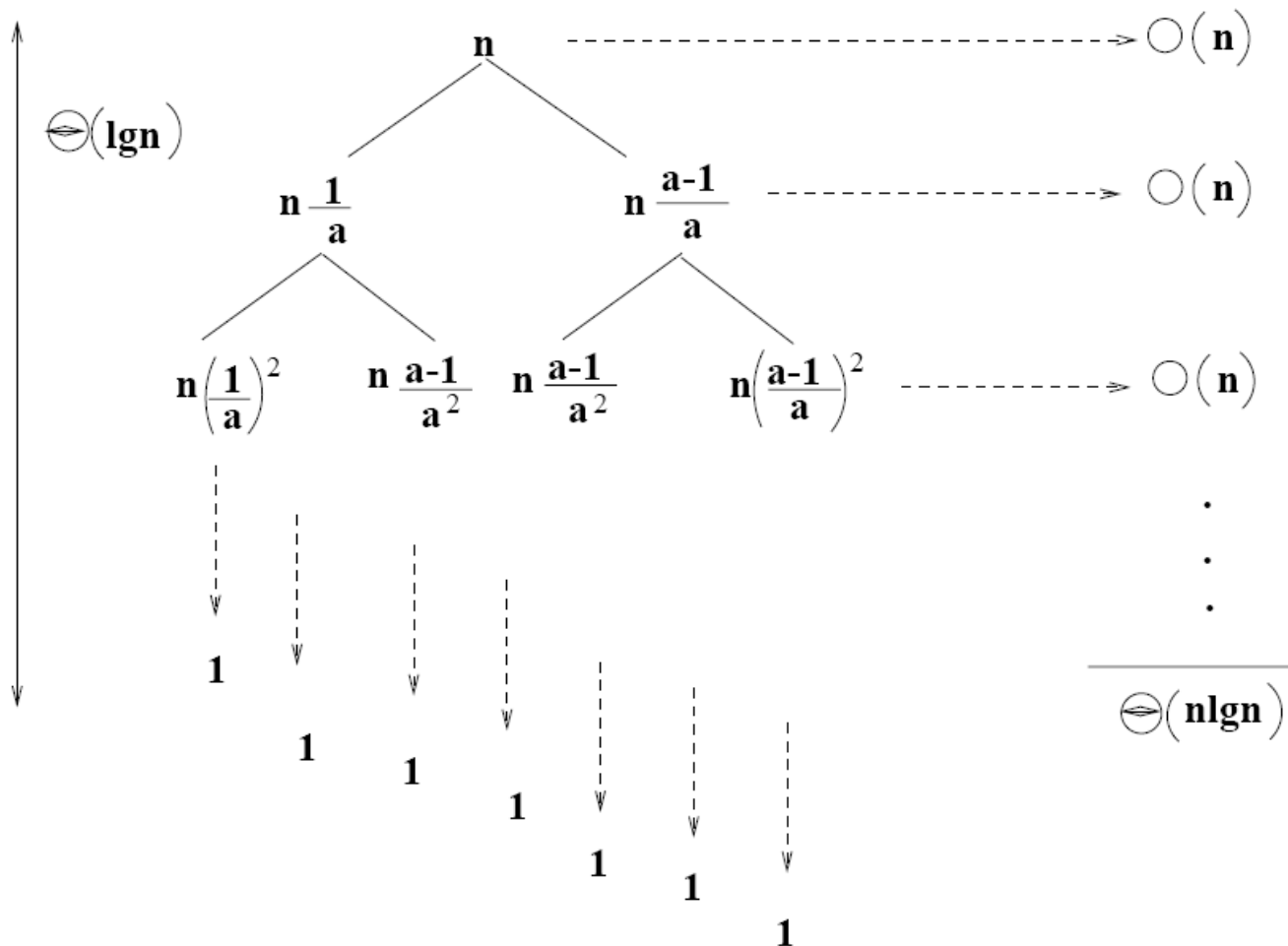
- Consider the $(9n/10 : n/10)$ splitting:

ratio = $(9n/10)/(n/10) = 9$ it is a constant !!

How does partition affect performance?

- Any $((a-1)n/a : n/a)$ splitting:

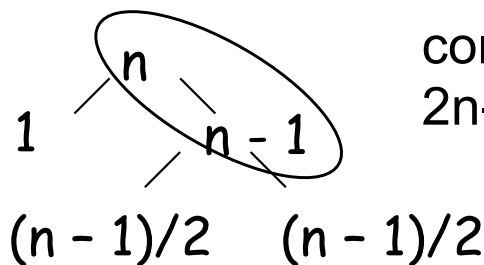
ratio= $((a-1)n/a)/(n/a) = a-1$ it is a constant !!



Analyzing Quicksort: Average Case Partitioning

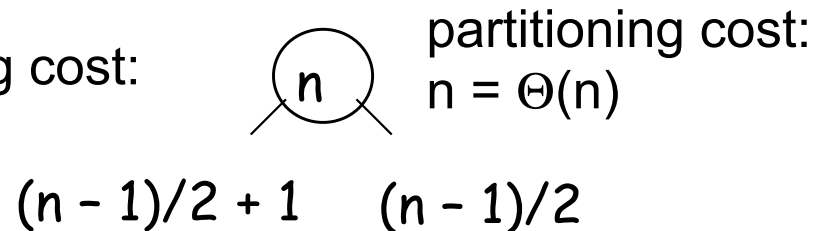
► Average case

- All permutations of the input numbers are equally likely
- On a random input array, we will have a **mix** of well balanced and unbalanced splits
- Good and bad splits are randomly distributed across throughout the tree



Alternate of a good
and a bad split

combined partitioning cost:
 $2n-1 = \Theta(n)$



Nearly well
balanced split

Running time of Quicksort when levels alternate between good and bad splits is $O(n \lg n)$

Randomizing Quicksort

- ▶ Randomly permute the elements of the input array before sorting
- ▶ OR ... modify the PARTITION procedure
 - ▶ At each step of the algorithm we exchange element $A[p]$ with an element chosen at random from $A[p..r]$
 - ▶ The pivot element $x = A[p]$ is equally likely to be any one of the $r - p + 1$ elements of the subarray

Randomizing PARTITION

Alg.: RANDOMIZED-PARTITION(A, p, r)

$i \leftarrow \text{RANDOM}(p, r)$

exchange $A[p] \leftrightarrow A[i]$

return PARTITION(A, p, r)

Randomizing Quicksort

Alg.:RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{RANDOMIZED-PARTITION}(A, p, r)$

 RANDOMIZED-QUICKSORT(A, p, q)

 RANDOMIZED-QUICKSORT($A, q + 1, r$)

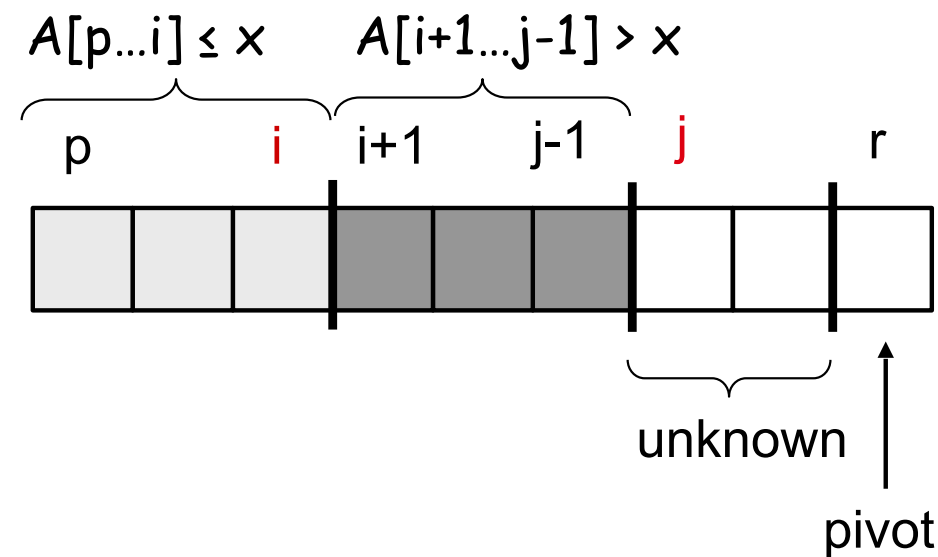
Revisit Partitioning

- ▶ Hoare's partition
 - ▶ Select a pivot element x around which to partition
 - ▶ Grows two regions
 - ▶ $A[p \dots i] \leq x$
 - ▶ $x \leq A[j \dots r]$

Another Partitioning: Lomuto's Partition

▶ Given an array A , partition the array into the following subarrays:

- ▶ A pivot element $x = A[q]$
- ▶ Subarray $A[p..q-1]$ such that each element of $A[p..q-1]$ is smaller than or equal to x (the pivot)
- ▶ Subarray $A[q+1..r]$, such that each element of $A[p..q+1]$ is strictly greater than x (the pivot)
- ▶ The pivot element is not included in any of the two subarrays



Another Partitioning: Lomuto's Partition

Alg.: PARTITION(A, p, r)

$x \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$

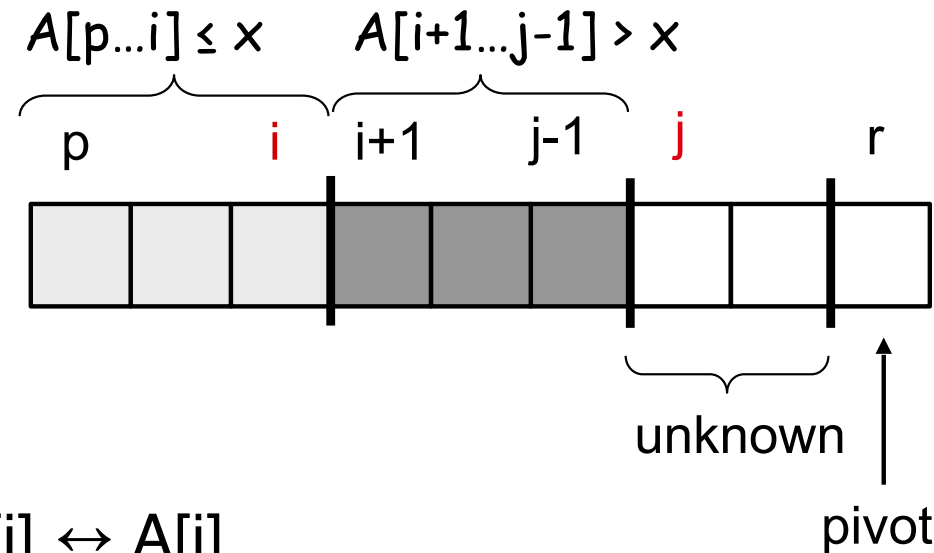
do if $A[j] \leq x$

then $i \leftarrow i + 1$

 exchange $A[i] \leftrightarrow A[j]$

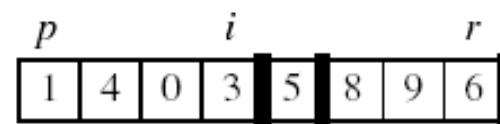
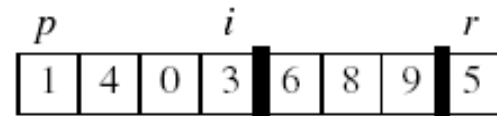
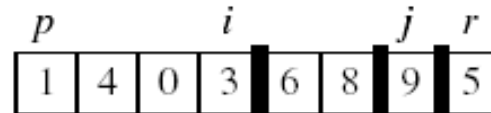
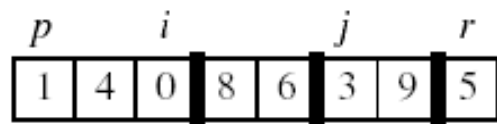
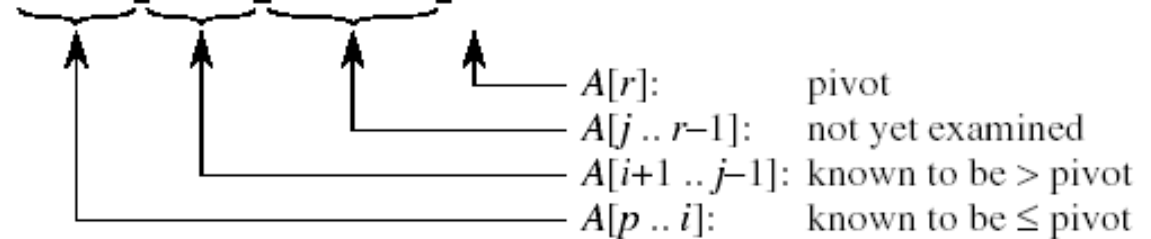
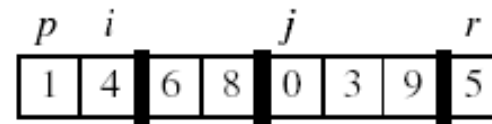
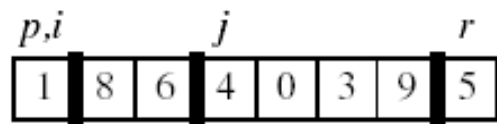
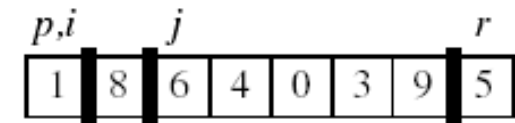
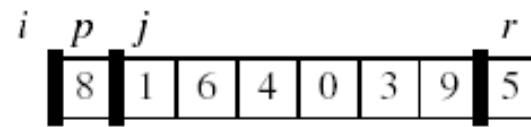
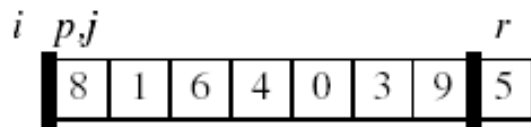
exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



- ▶ Chooses the last element of the array as a pivot
- ▶ Grows a subarray $[p..i]$ of elements $\leq x$
- ▶ Grows a subarray $[i+1..j-1]$ of elements $> x$
- ▶ Running Time: $\Theta(n)$, where $n=r-p+1$

Example



at the end,
swap pivot

Randomizing Quicksort using Lomuto's partition

Alg.:RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{RANDOMIZED-PARTITION}(A, p, r)$

RANDOMIZED-QUICKSORT($A, p, q - 1$)

RANDOMIZED-QUICKSORT($A, q + 1, r$)

The pivot is no longer included in any of the subarrays!!

Analysis of Randomized Quicksort

Alg.:RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{RANDOMIZED-PARTITION}(A, p, r)$

RANDOMIZED-QUICKSORT($A, p, q - 1$)

RANDOMIZED-QUICKSORT($A, q + 1, r$)



PARTITION is called at most n times

- ▶ (at each call a pivot is selected and never again included in future calls)

Partition

Alg.: PARTITION(A, p, r)

$x \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$

do if $A[j] \leq x$

then $i \leftarrow i + 1$

 exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$

$O(1)$ - constant

of comparisons: X_k
between the pivot and
the other elements

$O(1)$ - constant

Amount of work at call k : $c + X_k$

Average-Case Analysis of Quicksort

- ▶ Let X = total number of comparisons performed in all calls to PARTITION:
- ▶ The total work done over the **entire** execution of Quicksort is

$$O(nc + X) = O(n + X)$$

- ▶ Need to estimate $E(X)$

Indicator Random Variables

- ▶ Given a sample space S and an event A , we define the ***indicator random variable*** $I\{A\}$ associated with A :
 - ▶ $I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$
- ▶ The expected value of an indicator random variable $X_A = I\{A\}$:
 - ▶ $E[X_A] = \Pr\{A\}$
- ▶ **Proof:** $E[X_A] = E[I\{A\}] = 1 * \Pr\{A\} + 0 * \Pr\{\bar{A}\} = \Pr\{A\}$

Average-Case Analysis of Quicksort

- ▶ Let X = **total number of comparisons performed in all calls to PARTITION:**
- ▶ The total work done over the **entire** execution of Quicksort is $O(n+X)$
- ▶ Need to estimate $E(X)$

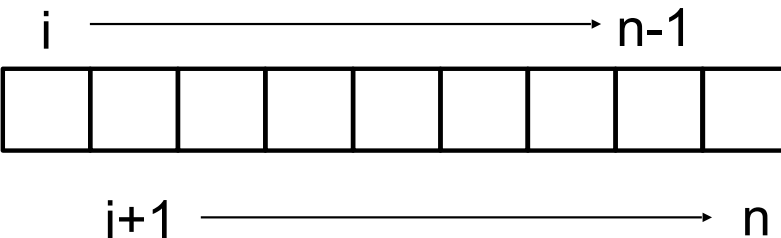
Notation

z_2	z_9	z_8	z_3	z_5	z_4	z_1	z_6	z_{10}	z_7
2	9	8	3	5	4	1	6	10	7

- ▶ Rename the elements of A as z_1, z_2, \dots, z_n , with z_i being the i -th smallest element
- ▶ Define the set $Z_{ij} = \{z_i, z_{i+1}, \dots, z_j\}$ the set of elements between z_i and z_j , inclusive

Total Number of Comparisons in PARTITION

- ▶ Define $X_{ij} = I \{z_i \text{ is compared to } z_j\}$
- ▶ Total number of comparisons X performed by the algorithm:

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij}$$


The diagram illustrates an array of size n . The array is represented as a horizontal row of 9 empty boxes. Above the first box is the index i , and above the last box is the index $n-1$. A horizontal arrow points from i to $n-1$. Below the first box is the index $i+1$, and below the last box is the index n . A horizontal arrow points from $i+1$ to n .

Expected Number of Total Comparisons in PARTITION

- Compute the **expected value of X** :

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n E[X_{ij}] \\ &\quad \text{by linearity of expectation} \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr\{z_i \text{ is compared to } z_j\} \end{aligned}$$

the expectation of X_{ij} is equal to the probability of the event
“ z_i is compared to z_j ”

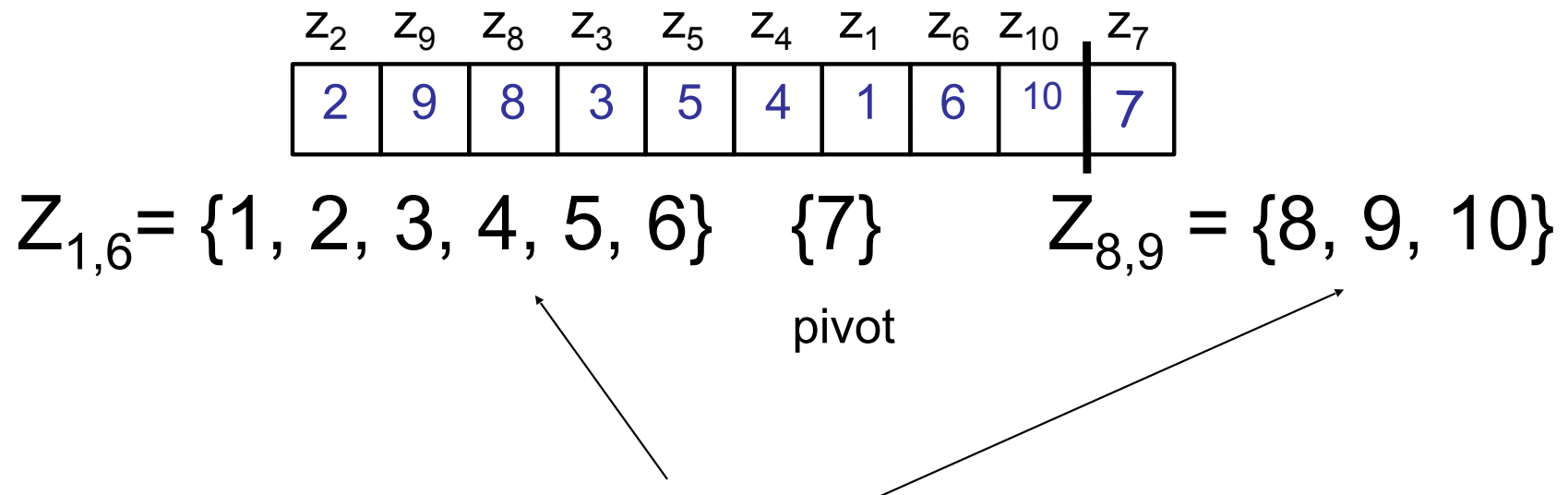
indicator random variable

Comparisons in PARTITION : Observation 1

- ▶ Each pair of elements is compared **at most once** during the entire execution of the algorithm
 - ▶ Elements are compared only to the pivot point
 - ▶ Pivot point is excluded from future calls to PARTITION

Comparisons in PARTITION : Observation 2

- Only the pivot is compared with elements in both partitions!



Elements between different partitions
are never compared!

Comparisons in PARTITION

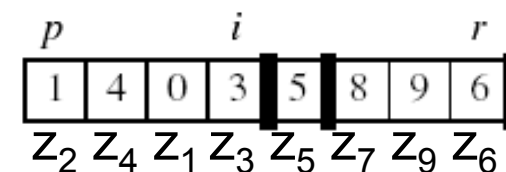
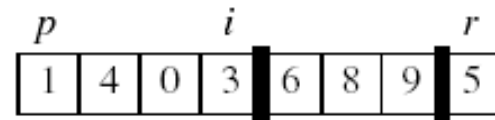
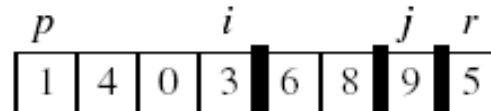
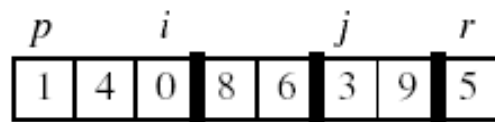
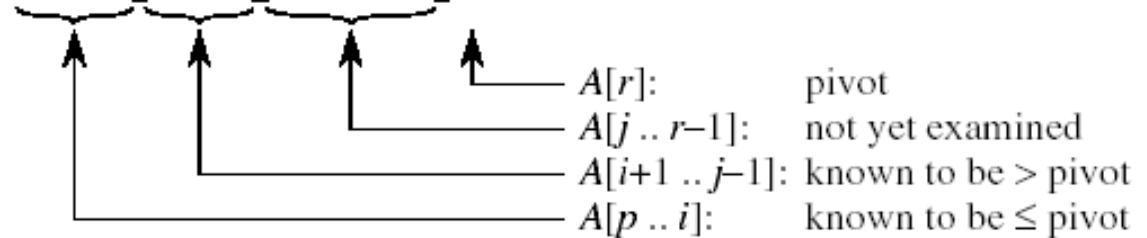
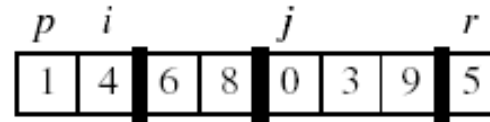
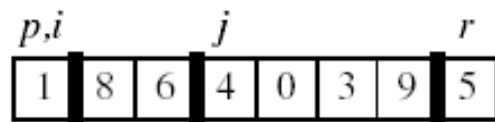
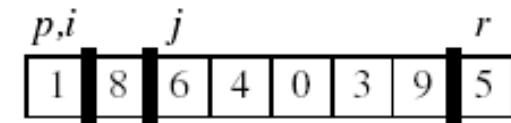
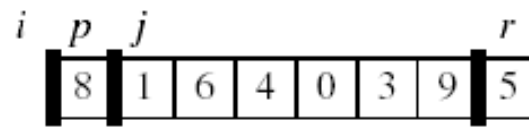
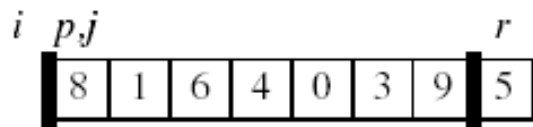
z_2	z_9	z_8	z_3	z_5	z_4	z_1	z_6	z_{10}	z_7
2	9	8	3	5	4	1	6	10	7

$$Z_{1,6} = \{1, 2, 3, 4, 5, 6\} \quad \{7\} \quad Z_{8,9} = \{8, 9, 10\}$$

$$\Pr\{z_i \text{ is compared to } z_j\}?$$

- ▶ **Case 1: pivot chosen such as: $z_i < x < z_j$**
 - ▶ z_i and z_j will never be compared
- ▶ **Case 2: z_i or z_j is the pivot**
 - ▶ z_i and z_j will be compared
 - ▶ only if one of them is chosen as pivot before any other element in range z_i to z_j

Let's See Why



z_2 will never be compared with z_6 since z_5 (which belongs to $[z_2, z_6]$) was chosen as a pivot first !

Probability of comparing z_i with z_j

$\Pr\{z_i \text{ is compared to } z_j\}$

$= \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\}$

$\Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\}$

$$= 1/(j - i + 1) + 1/(j - i + 1) = 2/(j - i + 1)$$

- ▶ There are $j - i + 1$ elements between z_i and z_j
 - ▶ Pivot is chosen randomly and independently
 - ▶ The probability that any particular element is the first one chosen is $1/(j - i + 1)$

Number of Comparisons in PARTITION

Expected number of comparisons in PARTITION:

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr\{z_i \text{ is compared to } z_j\}$$
$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{2}{j-i+1} \underset{\text{(set } k=j-i)}{=} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^n \frac{2}{k} \underset{\text{(harmonic series)}}{=} \sum_{i=1}^{n-1} O(\lg n)$$
$$= O(n \lg n)$$

⇒ Expected running time of Quicksort using RANDOMIZED-PARTITION is **$O(n \lg n)$**
