Homework 1 Solution

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EL9343 - Data Structure and Algorithm

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Exercise 1. Prove the *Transitivity* property of $\Theta(.)$, i.e., $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$

Proof.

From $f(n) = \Theta(g(n))$, we can get

$$\exists c_1 > 0, c_2 > 0 \, and \, n_0 > 0, \forall n \geq n_0 \, , c_1 g(n) \leq f(n) \leq c_2 g(n)$$

From $g(n) = \Theta(h(n))$, we can get

$$\exists c_3 > 0, c_4 > 0 \, and \, n_1 > 0, \forall n \geq n_1 \, , c_3 h(n) \leq g(n) \leq c_4 h(n)$$

With $c_1 g(n) \leq f(n)$ and $c_3 h(n) \leq g(n)$, we could conclude that, $\exists \ c_1 c_3 > 0$, $c_2 c_4 > 0$ $and \, n_3 \geq max(n_0, n_1)$, where $\forall n \geq n_3, c_1 c_3 h(n) \leq f(n) \leq c_2 c_4 h(n) \Rightarrow f(n) = \Theta(h(n))$ $\square$

Exercise 2. Problem 3-1 in CLRS Text Book.

a. If $k \geq d, p(n) = \sum_{i=0}^{d} a_i n^i \leq \sum_{i=0}^{d} a_i n^d \leq \sum_{i=0}^{d} a_i n^k, c = \sum_{i=0}^{d} \Rightarrow p(n) = O(n^k)$

b. If $k \leq d, p(n) = \sum_{i=0}^{d} a_i n^i = \sum_{i=0}^{k} a_i n^i + \sum_{i=k+1}^{d} a_i^i \geq a_k n^k, c = a_k \Rightarrow p(n) = \Omega(n^k)$

c. Same as (a) and (b), choose $c_1 = \sum_{i=0}^{k} \, and \, c_2 = a_k$, we get $c_1 n_k \leq n_k \leq c_2 n_k \Rightarrow p(n) = \Theta(n^k)$

d. If $k > d, p(n) = \sum_{i=0}^{d} a_i n^i < \sum_{i=0}^{d} a_i n^d < \sum_{i=0}^{d} a_i n^k, c = \sum_{i=0}^{d} \Rightarrow p(n) = o(n^k)$

e. If $k < d, p(n) = \sum_{i=0}^{d} a_i n^i = \sum_{i=0}^{k} a_i n^i + \sum_{i=k+1}^{d} a_i^i > a_k n^k, c = a_k \Rightarrow p(n) = \omega(n^k)$

Exercise 3. Problem 3-2 in CLRS Text Book.

A	B	O	o	Ω	ω	Θ
$lg^k n$	n^ϵ	Yes	Yes	No	No	No
n^k	c^n	Yes	Yes	No	No	No
$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
2^n	$2^{n/2}$	No	No	Yes	Yes	No
n^{lgc}	c^{lgn}	Yes	No	Yes	No	Yes
$lg(n!)$	$lg(n^n)$	Yes	No	Yes	No	Yes

Exercise 4. Problem 3-4 (a) (b) (g), (h) in CLRS Text Book.

Either Give examples which negate the proposition or prove it using Equations

a. False, for instance, $f(n) = n$ and $g(n) = n^2$

b. False, for instance, $f(n) = n$ and $g(n) = n^2$, $min(f(n), g(n)) = n$ while $f(n) + g(n) > n$, violates $f(n) + g(n) = \Theta(n)$

g. False, If $f(n) = \Theta(f(n/2))$, $f(n) = O(f(\frac{n}{2})) \leq cf(\frac{n}{2})$, $n \geq n_0 \geq 0$.
Take $f(n) = 2^n$, $c \geq 2^{\frac{n}{2}}$, which is not a constant satisfying the proposition.

h.$\exists$ c and $n_0, \forall n >= n_0, f(n)) < cf(n)$, $f(n) + o(f(n)) <= (c+1)f(n) \Rightarrow f(n) + o(f(n)) = O(f(n))$, and $f(n) + o(f(n)) >= f(n) + b$, as f(n) is positive, so we can conclude $f(n) + o(f(n)) = \Theta(f(n))$

Exercise 5. Use the substitution method to show that the solution to $T(n) = T(\alpha n) + T((1-\alpha)n) + 10$, with $0 < \alpha < 1$, is $\Theta(n)$, then use the substitution method to prove that.

Suppose $T(n) = \Theta(n)$ is True, $T(n) = \Omega(n) \Rightarrow T(n) \geq c_1 n$, then

$$\begin{aligned} T(n+1) &= T(\alpha(n+1)) + T((1-\alpha)(n+1)) + 10 \\ &\geq c_1\alpha(n+1) + c_1(1-\alpha)(n+1) + 10 \\ &\geq c_1(n+1) + 10 \\ &= \Omega(n+1) \end{aligned}$$

It is proved that $T(n) = \Omega(n)$ is True.

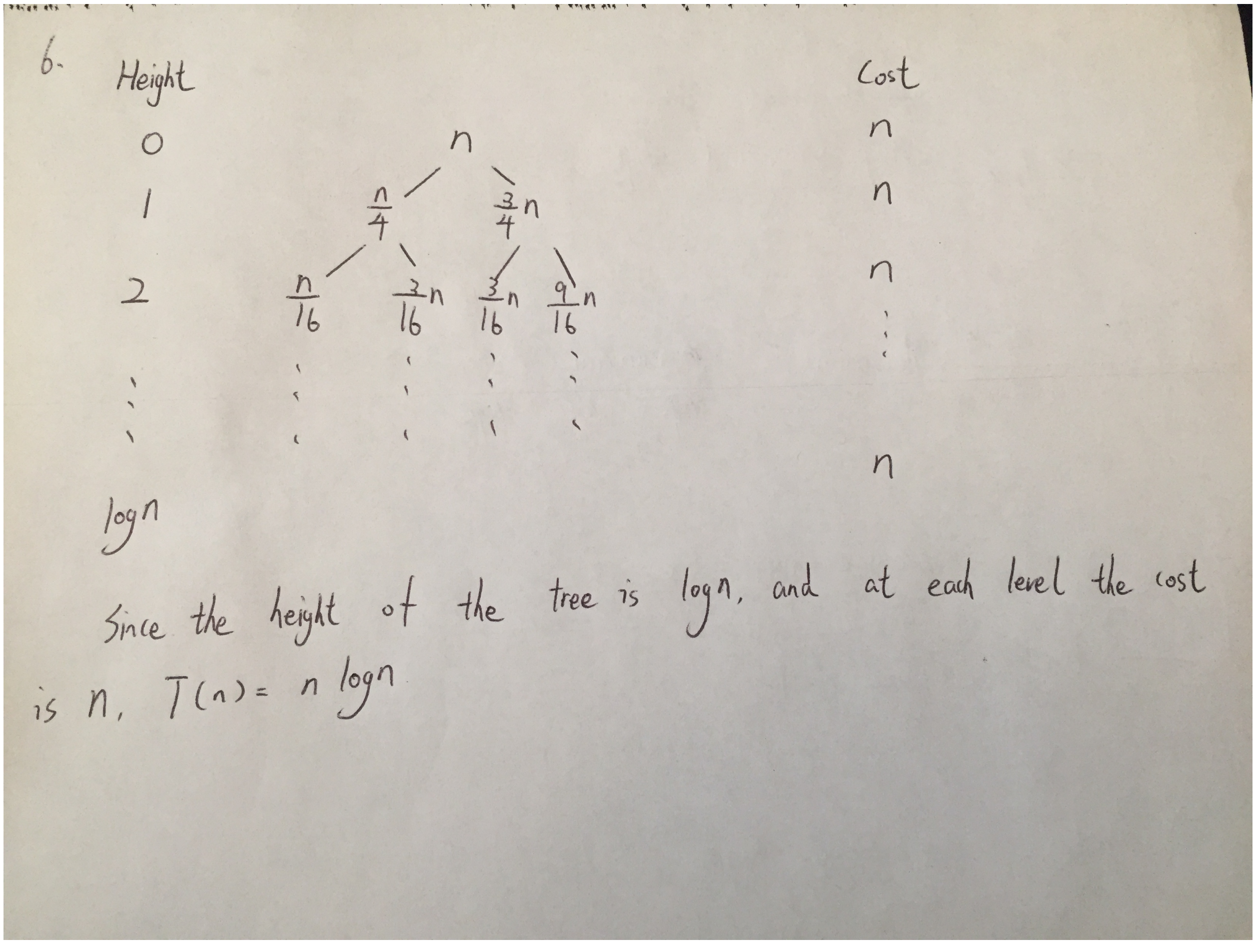
If $T(n) = O(n) \Rightarrow \exists c_2 \geq 0, \forall n > n_2, T(n) \leq c_2 n$, then

$$\begin{aligned} T(n+1) &\leq c_2\alpha(n+1) + c_2(1-\alpha)(n+1) + 10 \\ &\leq c_2\alpha(n+1) + c_2(n+1) - c_2\alpha(n+1) + 10 \\ &\leq c_2(n+1) + 10 \\ &= O(n+1) \end{aligned}$$

It is also proved that $T(n) = O(n) \Rightarrow T(n) = \Theta(n)$

Exercise 6. First use the iteration method to solve the recurrence,then use the substitution method to verify your solution.

$$T(n) = T(\frac{n}{4}) + T(\frac{3n}{4}) + n$$



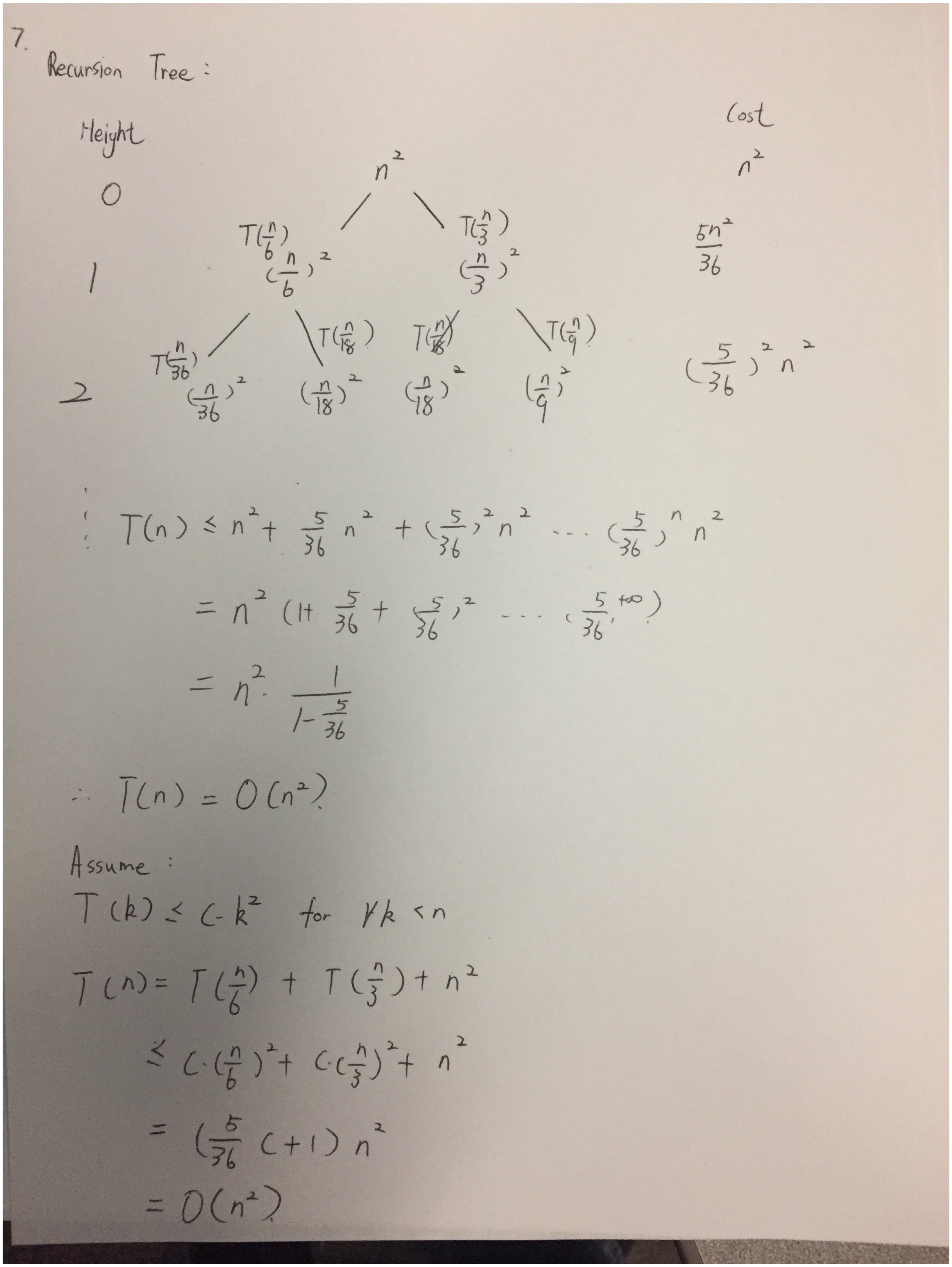
Since the height of the tree is logn, and at each level the cost is n, T(n)= n logn

Assume $T(n) \leq c * n * logn + b, \forall n < n_0$

$$\begin{aligned}
T(k) &= T(\frac{3k}{4}) + T(\frac{k}{4}) + k \\
&\leq c\frac{k}{4}log\frac{k}{4} + c\frac{3k}{4}log\frac{3k}{4} + k \\
&= (c\frac{k}{4} + c\frac{3k}{4})log_k + c\frac{3k}{4}log_3 - (c\frac{k}{4} + c\frac{3k}{4})log_4 + k \\
&= ck(log_k + \frac{3}{4}log_3 - log_4 + c) \\
&\leq cklog_k
\end{aligned}$$

So $T(n) = O(nlog_n)$

Exercise 7.



7\. Recursion Tree:

Height

Cost

0

n^2

n^2

1

$T(\frac{n}{6})$ $(\frac{n}{6})^2$

$T(\frac{n}{3})$ $(\frac{n}{3})^2$

$\frac{5n^2}{36}$

2

$T(\frac{n}{36})$ $(\frac{n}{36})^2$

$T(\frac{n}{18})$ $(\frac{n}{18})^2$

$T(\frac{n}{18})$ $(\frac{n}{18})^2$

$T(\frac{n}{9})$ $(\frac{n}{9})^2$

$(\frac{5}{36})^2 n^2$

$$T(n) \leq n^2 + \frac{5}{36} n^2 + (\frac{5}{36})^2 n^2 \cdots (\frac{5}{36})^n n^2$$

$$= n^2 (1 + \frac{5}{36} + (\frac{5}{36})^2 \cdots (\frac{5}{36})^{+\infty})$$

$$= n^2 \cdot \frac{1}{1 - \frac{5}{36}}$$

$$\therefore T(n) = O(n^2)$$

Assume:

$T(k) \leq c \cdot k^2$ for $\forall k < n$

$$T(n) = T(\frac{n}{6}) + T(\frac{n}{3}) + n^2$$

$$\leq c \cdot (\frac{n}{6})^2 + c \cdot (\frac{n}{3})^2 + n^2$$

$$= (\frac{5}{36} c + 1) n^2$$

$$= O(n^2)$$

Exercise 8. Solving the recurrence:

$$T(n) = 9T(n^{\frac{1}{3}}) + \log^2(n)$$

Substitute n = 3^m and $S(m) = T(n) = T(3^m)$, we could get

$$S(m) = 9S(\frac{m}{3}) + log^2(3^m)$$
$$S(m) = 9S(\frac{m}{3}) + m^2 log^2 3$$

With Master theory, $m^{\log_3 9} = m^2$, so $S(m) = \Theta(m^2 \log_m)$,

$$\begin{aligned} S(m) = T(3^m) &= \Theta(m^2 \log_m) \\ &= \Theta(\log_3(3^m)^2 log(log(3^m))) \end{aligned}$$

As $n = 3^m$, $T(n) = log^2 n log(logn)$

Exercise 9. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Make the bounds as tight as possible, and justify your answers.

a. $T(n) = 2T(\frac{n}{3}) + n^{\frac{1}{2}} logn$

Using Master's theory, $T(n) = aT(\frac{n}{b}) + f(n)$, where a = 2, b = 3 and f(n) = $n^{\frac{1}{2}} log_n$, $T(n) = \Theta(n^{log_3 2})$

$$\begin{aligned} \frac{f(n)}{n^{log_b a + \epsilon}} &= \frac{n^{\frac{1}{2}} log(n)}{n^{log_3 2 + \epsilon}} \\ &= \frac{log(n)}{n^{log_3 2 - 0.5 + \epsilon}} \\ &= \frac{log(n)}{n^{0.13 + \epsilon}} \end{aligned}$$

For any $\epsilon + 0.13 > 1$, according to L'Hospital's Rule,

$$\begin{aligned} \lim_{n \to \infty} \frac{f(n)}{n^{log_b a + \epsilon}} &= \lim_{n \to \infty} \frac{log(n)}{n^{\epsilon + 0.13}} \\ &= 0 \end{aligned}$$

So $T(n) = \Theta(n^{log_3 2})$

b. $T(n) = 25T(\frac{n}{5}) + n^2$

Using Master's theory, $T(n) = aT(\frac{n}{b}) + f(n)$, where a = 25, b = 5 and f(n) = n^2, $n^{\log_a b} = n^2$, therefore, $T(n) = \Theta(n^2 logn)$

c. $T(n) = 4T(\frac{n}{2}) + n2\!\!\sqrt{n}$

Using Master's theory, $T(n) = aT(\frac{n}{b}) + f(n)$, where a = 4, b = 2 and f(n) = $n^{2.5}$, $n^{\log_a b} = n^2$, therefore, $T(n) = \Theta(n^{2.5})$

d. $T(n) = T(n-2) + \frac{1}{n}$

$$\begin{aligned}
T(n) &= T(n-2) + \frac{1}{n} \\
&= T(n-4) + \frac{1}{n-2} + \frac{1}{n} \\
&= T(n-6) + \frac{1}{n-4} + \frac{1}{n-2} + \frac{1}{n} \\
&= T(0) + \sum_{i=1}^{n} \frac{1}{2i} \\
&= \Theta(ln(n))
\end{aligned}$$